PART I

GENERATING FUNCTIONS FOR PRODUCTS OF POWERS OF FIBONACCI NUMBERS*

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1. INTRODUCTION

We may define the Fibonacci numbers, F_n, by $F_0 = 0, F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. A well-known generating function for these numbers is

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad . \tag{1.1}$$

Intimately associated with the numbers of Fibonacci are the numbers of Lucas, L_n, which we may define by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. The numbers F_n and L_n may be considered as special cases of general functions first studied in great detail by Lucas [8], though as Bell [1] has observed many expansions for the Lucas functions appeared in papers of Cauchy and others prior to Lucas. Dickson [4] devotes all of one chapter (17) to recurring series and more particularly Lucas functions. Here one may find further references to the many papers on the subject which have appeared since Leonardo Pisano, or Fibonacci, first introduced the famous numbers in 1202. It would be difficult to estimate how many papers related to Fibonacci numbers have appeared since Dickson's monumental History was written, however it may be of interest to point out that a project has been initiated under the direction of Professor Vern Hoggatt, San Jose State College, San Jose, California, to collect formulas, maintain a bibliography and coordinate work on Fibonacci numbers. As part of the writer's activity with this Fibonacci Bibliographical Project the subject of generating functions for powers of the Fibonacci numbers has come in for some study, and the object of this present paper is to develop some very general generating functions for the Lucas functions.

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Riordan [10] has recently made a very interesting study of arithmetic properties of certain classes of coefficients which arose in his analysis of the generating function defined by the p-th powers of Fibonacci numbers.

$$f_p(x) = \sum_{n=0}^{\infty} f_n^p x^n , \tag{1.2}$$

where $f_n = F_{n+1}$. Golomb [5] had found essentially that for squares of Fibonacci numbers we have

$$(1 - 2x - 2x^2 + x^3) f_2(x) = 1 - x$$

and it was this which led Riordan to seek the general form of $f_p(x)$.

However, there are other simple generating functions for the numbers of Fibonacci. First of all, let us observe that we may define the Fibonacci and Lucas numbers by

$$F_n = \frac{a^n - b^n}{a - b} , \quad L_n = a^n + b^n \tag{1.3}$$

where

$$a = \frac{1}{2}(1 + \sqrt{5}) , \qquad b = \frac{1}{2}(1 - \sqrt{5}) . \tag{1.4}$$

The very general functions studied by Lucas, and generalized by Bell [1, 2], are essentially the F_n and L_n defined by (1.3) with a, b being the roots of the quadratic equation $x^2 = Px - Q$ so that $a + b = P$ and $ab = Q$. In view of this formulation it is easy to show that we also have the generating function

$$\frac{e^{ax} - e^{bx}}{a - b} = \sum_{n=0}^{\infty} \frac{x^n}{n!} F_n . \tag{1.5}$$

For many purposes this expansion is easier to consider than (1.1), and one is naturally led to ask what form of generating function holds if we put p-th powers of the

Fibonacci numbers F_n in (1.5). Similar questions arise for L_n. We shall also consider negative powers of. F_n, L_n, and suggest an analogy with the polynomials of Bernoulli and Euler.

2. GENERATING FUNCTIONS FOR LUCAS FUNCTIONS

Suppose we are given for any initial generating function, $F(x)$, say

$$F(x) = \sum_{n=0}^{\infty} A_n x^n \tag{2.1}$$

with no particular restrictions on A_n. It follows at once from this that

$$F(ax) + F(bx) = \sum_{n=0}^{\infty} A_n x^n (a^n + b^n) = \sum_{n=0}^{\infty} A_n x^n L_n \tag{2.2}$$

and

$$\frac{F(ax) - F(bx)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_n \quad . \tag{2.3}$$

This incidentally is rather like the method used by Riordan [10] to begin his study of recurrence relations for the generation function (1.2), except that we could study (1.5) as well as (1.1) in the general expansion of (2.3).

Now of course we may iterate upon the formulas (2.2) and (2.3) by making successive substitutions, replacing x by ax, or bx, and adding or subtracting, and by this iteration build up generating functions involving F_n^p and L_n^p. Thus we have from (2.2)

$$F(a^2x) + F(abx) = \Sigma A_n x^n a^n L_n \quad ,$$

$$F(abx) + F(b^2x) = \Sigma A_n x^n b^n L_n \quad ,$$

so that

$$F(a^2x) + 2F(abx) + F(b^2x) = \sum_{n=0}^{\infty} A_n x^n L_n^2 , \tag{2.4}$$

and in similar fashion

$$\frac{F(a^2x) - 2F(abx) + F(b^2x)}{(a - b)^2} = \sum_{n=0}^{\infty} A_n x^n F_n^2 . \tag{2.5}$$

Clearly we may proceed inductively to obtain a general result. We find the general relations

$$\sum_{k=0}^{p} \binom{p}{k} F(a^{p-k} b^k x) = \sum_{n=0}^{\infty} A_n x^n L_n^p , \tag{2.6}$$

and

$$(a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} F(a^{p-k} b^k x) = \sum_{n=0}^{\infty} A_n x^n F_n^p . \tag{2.7}$$

In fact we may readily combine the relations to obtain

$$(a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \cdot \sum_{j=0}^{q} \binom{q}{j} F(a^{p+q-k-j} b^{k+j} x) = \sum_{n=0}^{\infty} A_n x^n F_n^p L_n^q , \tag{2.8}$$

for any non-negative integers p, q. Thus in principle we may set down generating functions for products of powers of the Fibonacci and Lucas numbers, though the result may not usually be in the simplest form.

We obtain (1.1) when $A_n = 1$ identically; (1.5) when $A_n = 1/n!$ identically. The expansion analogous to (1.5) for L_n is

(2.9)
$$e^{ax} + e^{bx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n .$$

Proceding in the same manner as above, it is clear that we also have

(2.10)
$$F(a^m x) + F(b^m x) = \sum_{n=0}^{\infty} A_n x^n L_{mn} ,$$

and

(2.11)
$$\frac{F(a^m x) - F(b^m x)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_{mn} ,$$

which include other well-known generating functions. Consequently we have

(2.12)
$$\sum_{k=0}^{p} \binom{p}{k} F(a^{pm-km} b^{km} x) = \sum_{n=0}^{\infty} A_n x^n L_{mn}^p ,$$

with a corresponding result for F_n.

3. RECIPROCALS OF FIBONACCI NUMBERS

Landau [7] showed that a certain series of reciprocals of Fibonacci numbers could be expressed in terms of a Lambert series. In fact he showed that if we write

(3.1)
$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} ,$$

then we have

(3.2)
$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5} \left[L\left(\frac{3 - \sqrt{5}}{2}\right) - L\left(\frac{7 - 3\sqrt{5}}{2}\right) \right] .$$

The device used to obtain this is to expand $(a - b)/(a^n - b^n) = (a - b)/a^n \cdot 1/(1 - z)$, where $z = (b/a)^n$, by a power series, and then invert the order of summation in the series, this being justifiable. Landau's result is noted in Bromwich [3, p. 194, example 32], in Knopp [6, p. 279, ex. 144; p. 468, ex. 9], and in Dickson [4, p. 404]. Dickson also notes that $\sum_1^\infty 1/F_n$ was put in finite form by A. Arista.

Let us now define in general

$$R(x) = \sum_{n=1}^{\infty} A_n \frac{x^n}{F_n} . \tag{3.3}$$

Then by the same technique we have used earlier to obtain (2.2) and (2.3) we see at once that $R(x)$ satisfies a functional equation

$$R(ax) - R(bx) = (a - b) \sum_{n=1}^{\infty} A_n x^n . \tag{3.4}$$

Thus if we have

$$R(x) = \sum_{n=1}^{\infty} \frac{x^n}{F_n} \quad \text{then} \quad R(ax) - R(bx) = (a - b)\frac{x}{1 - x} \tag{3.5}$$

$$= \frac{x}{1 - x}\sqrt{5} \quad \text{for ordinary Fibonacci numbers.}$$

For $R(x)$ as defined by (3.3) we also easily verify that

$$R(a^2x) - R(b^2x) = (a - b) \sum_{n=1}^{\infty} A_n x^n L_n . \tag{3.6}$$

To use this when $A_n = 1$ we need to note the generating function for L_n which is a companion to (1.1). Let us obtain this from (2.2). Inasmuch as we have

$$a + b = 1, \quad a - b = \sqrt{5}, \quad \text{and} \quad ab = -1, \tag{3.7}$$

we find

$$\begin{aligned} F(ax) + F(bx) &= \frac{1}{1 - ax} + \frac{1}{1 - bx} \\ &= \frac{2 - (a + b)x}{1 - (a + b)x + abx} = \frac{2 - x}{1 - x - x^2} \end{aligned}$$

and so for the ordinary Lucas numbers we find

$$\frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n . \tag{3.8}$$

Sometimes this is stated in the equivalent form (since $L_0 = 2$)

$$\frac{x(2x + 1)}{1 - x - x^2} = \sum_{n=1}^{\infty} L_n x^n . \tag{3.9}$$

Thus if we have

$$R(x) = \sum_{n=1}^{\infty} \frac{x^n}{F_n} \text{ then } R(a^2x) - R(b^2x) = (a - b) \cdot \frac{x(2x + 1)}{1 - x - x^2} . \tag{3.10}$$

4. BILINEAR GENERATING FUNCTIONS

We wish to turn next to some simple results for what are called bilinear generating functions for Fibonacci and Lucas numbers. To discuss this we first introduce what we shall call a general Turán operator defined by

$$Tf = T_x f(x) = f(x + u) f(x + v) - f(x) f(x + u + v) . \tag{4.1}$$

For the Fibonacci numbers it is a classic formula first discovered apparently by Tagiuri (Cf. Dickson [4, p. 404]) and later given as a problem in the American Mathematical Monthly (Problem E 1396) that

$$T_n F_n = F_{n+u} F_{n+v} - F_n F_{n+u+v} = (-1)^n F_u F_v . \tag{4.2}$$

We may determine a general bilinear generating function for the series $\Sigma A_n x^n F_{n+u} F_{n+v}$ if we can first determine a result for $\Sigma A_n x^n F_n F_{n+u+v}$. To do this let us consider $\Sigma A_n x^n F_n F_{n+j}$.

Again, let us set as in (2.1)

$$F(x) = \sum_{n=0}^{\infty} A_n x^n$$

so that

$$x^j \cdot F(x) = \sum_{n=0}^{\infty} A_n x^{n+j} .$$

Then we find

$$a^j x^j F(ax) - b^j x^j F(bx) = \sum_{n=0}^{\infty} A_n x^{n+j}(a^{n+j} - b^{n+j})$$

and hence ultimately

$$\frac{a^j F(ax) - b^j F(bx)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_{n+j} . \tag{4.3}$$

Next we introduce F_n by the same device and we find

$$\frac{a^j F(a^2x) - b^j F(abx)}{a - b} - \frac{a^j F(abx) - b^j F(b^2x)}{a - b} = \Sigma A_n x^n F_{n+j}(a^n - b^n) ,$$

and consequently we have

$$(4.4)\qquad \frac{a^j F(a^2x) - (a^j + b^j)F(abx) + b^j F(b^2x)}{(a-b)^2} = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+j} \ .$$

Moreover, since we have the Turán expression (4.2), we have

$$(4.5)\qquad \sum_{n=0}^{\infty} A_n x^n F_{n+u} F_{n+v} = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+u+v} + F_u F_v \cdot F(-x) \ ,$$

where the relation (4.4) is used to simplify the right-hand side. In principle at least we have a way to write down explicit bilinear generating functions provided merely that we have $F(x) = \Sigma A_n x^n$ given.

It should be remarked that since we have made no special assumptions about the coefficients A_n in any of the work so far, we could apply our work to finite series just as well by supposing that $A_n = 0$ identically for $n \geq$ some value n_0.

As far as relation (4.4) is concerned, there is an alternative method. Call the series $M_j(x)$, i.e., let

$$M_j(x) = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+j} \ .$$

If it is possible to evaluate $M_0(x) = \Sigma A_n x^n F_n^2$, then one may note that $M_{j+1}(x) = M_j(x) + M_{j-1}(x)$ and so a simple formula could be written down giving M_j.

5. BERNOULLI AND EULER POLYNOMIALS

Returning to the problems presented by reciprocals of Fibonacci and Lucas numbers, it would appear to be of value to introduce some new polynomials based upon the polynomials of Bernoulli and Euler. Using a standard notation [9] we define Euler and Bernoulli polynomials by

$$(5.1)\qquad \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} = \frac{2e^{tx}}{e^t + 1} \ ,$$

and

$$\sum_{k=0}^{\infty} B_k(x)\,\frac{t^k}{k!} = \frac{t\,e^{tx}}{e^t - 1} \quad . \tag{5.2}$$

Now we have for general Lucas functions

$$\frac{1}{L_t} = \frac{1}{a^t + b^t} = \frac{1}{a^t}\,\frac{1}{C^t + 1} \qquad \text{with } C = b/a \ ,$$

so that

$$\frac{2\,a^t\,C^{tx}}{L_t} = \frac{2\,C^{tx}}{C^t + 1} \qquad \text{where } C = b/a \ . \tag{5.3}$$

Similarly we find that when $F_t = (a^t - b^t)/(a - b)$ we have

$$\frac{t\,a^t\,C^{tx}}{(b - a)F_t} = \frac{t\,C^{tx}}{C^t - 1} \qquad \text{where } C = b/a \ . \tag{5.4}$$

The similarity of (5.3) with (5.1) and (5.4) with (5.2) motivates what follows.

We define generalized Bernoulli and Euler polynomials by

$$\frac{t\,C^{tx}}{C^t - 1} = \sum_{k=0}^{\infty} B_k\,(x, C)\,\frac{t^k}{k!} \ , \tag{5.5}$$

and

$$\frac{2\,C^{tx}}{C^t + 1} = \sum_{k=0}^{\infty} E_k(x, C)\,\frac{t^k}{k!} \quad . \tag{5.6}$$

Now in fact

$$(5.7) \qquad \frac{t\,C^{tx}}{C^t - 1} = \frac{1}{\log C} \cdot \frac{t \log C \cdot e^{x(t \log C)}}{e^{t \log C} - 1} = \frac{1}{\log C} \cdot \frac{z\,e^{xz}}{e^z - 1}$$

$$= \frac{1}{\log C} \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} (\log C)^{k-1}$$

so that

$$(5.8) \qquad B_k(x, C) = (\log C)^{k-1} \cdot B_k(x) .$$

Similarly one easily finds that

$$(5.9) \qquad E_k(x, C) = (\log C)^k \cdot E_k(x) .$$

Putting these observations together we ultimately have the expansions

$$(5.10) \qquad \frac{1}{L_t} = \frac{1}{2\,a^t\,(b/a)^{tx}} \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} (\log b/a)^k ,$$

and

$$(5.11) \qquad \frac{1}{F_t} = \frac{b - a}{t\,a^t (b/a)^{tx}} \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} (\log b/a)^{k-1} .$$

Thus we also have an amusing analogy between four names:

$$(5.12) \qquad \frac{\text{BERNOULLI}}{\text{FIBONACCI}} = \frac{\text{EULER}}{\text{LUCAS}}$$

We may extend the analogy by considering the more general Bernoulli and Euler polynomials of higher order as discussed in [9] and find expansions for the reciprocals of powers of the numbers of Fibonacci and Lucas.

We have

$$\sum_{k=0}^{\infty} B_k^{(n)}(x)\frac{t^k}{k!} = \frac{t^n e^{xt}}{(e^t - 1)^n}, \tag{5.13}$$

and

$$\sum_{k=0}^{\infty} E_k^{(n)}(x)\frac{t^k}{k!} = \frac{2^n e^{xt}}{(e^t + 1)^n}, \tag{5.14}$$

and we ultimately find as before the reciprocal expansions (with $C = b/a$)

$$\frac{t^n C^{xt}}{(C^t - 1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x)\frac{t^k(\log C)^{k-n}}{k!} = \frac{t^n a^{nt} C^{tx}}{(b-a)^n F_t^n}, \tag{5.15}$$

and

$$\frac{2^n C^{xt}}{(C^t + 1)^n} = \sum_{k=0}^{\infty} E_k^{(n)}(x)\frac{t^k(\log C)^k}{k!} = \frac{2^n a^{nt} C^{tx}}{L_t^n}. \tag{5.16}$$

Consequently we have

$$\sum_{t=m}^{\infty} A_t \frac{z^t}{F_t^n} = (b-a)^n \sum_{k=0}^{\infty} B_k^{(n)}(x)\frac{(\log C)^{k-n}}{k!}\sum_{t=m}^{\infty} A_t t^{k-n}\left(\frac{z}{a^n C^x}\right)^t, \tag{5.17}$$

$$m \geq 1$$

and

$$\sum_{t=m}^{\infty} A_t \frac{z^t}{L_t^n} = 2^{-n} \sum_{k=0}^{\infty} E_k^{(n)}(x) \frac{(\log C)^k}{k!} \sum_{t=m}^{\infty} A_t t^k \left(\frac{z}{a^n C^x}\right)^t , \tag{5.18}$$

$$m \geq 0 .$$

In these, log C will be real provided, e.g. that both a and b are positive, or both negative. In case C is negative we may take principal values for the log C.

6. SOME MISCELLANEOUS FIBONACCI FORMULAE

We shall conclude our remarks here by deriving a few miscellaneous relations.

In relation (2.1) let $A_n = \binom{z}{n}$, z being any real number. Then we find by (2.2)

$$\sum_{n=0}^{\infty} \binom{z}{n} x^n L_n = (1 + ax)^z + (1 + bx)^z , \tag{6.1}$$

and by (2.3)

$$\sum_{n=0}^{\infty} \binom{z}{n} x^n F_n = \frac{(1 + ax)^z - (1 + bx)^z}{a - b} . \tag{6.2}$$

Let us examine one special instance. Let $z = r$ be any non-negative integer, and take the ordinary Fibonacci-Lucas numbers when $a = \frac{1}{2}(1 + \sqrt{5})$, $b = \frac{1}{2}(1 - \sqrt{5})$. Then we find $1 + a = a^2$, $1 + b = b^2$, whence (6.1) and (6.2) become

$$\sum_{n=0}^{r} \binom{r}{n} L_n = (1 + a)^r + (1 + b)^r = a^{2r} + b^{2r} = L_{2r} , \tag{6.3}$$

and

$$\sum_{n=0}^{r} \binom{r}{n} F_n = \frac{a^{2r} - b^{2r}}{a - b} = F_{2r} . \tag{6.4}$$

It is clear that by using the general relations previously developed here, we could go on to derive many interesting Fibonacci-Lucas number relations. As another example using the same value for A_n, we find from (2.10) that

$$\sum_{n=0}^{r} \binom{r}{n} x^n L_{mn} = (1 + a^m x)^r + (1 + b^m x)^r . \tag{6.5}$$

In this, let $x = -1$, $m = 2$, and a and b as above. Then we find

$$\sum_{n=0}^{r} \binom{r}{n} (-1)^n L_{2n} = (-1)^r L_r . \tag{6.6}$$

Also we have

$$\sum_{n=0}^{r} \binom{r}{n} (-1)^n F_{2n} = (-1)^r F_r . \tag{6.7}$$

By (4.3) we have

$$\sum_{n=0}^{\infty} A_n x^n F_{n+j} = \frac{a^j F(ax) - b^j F(bx)}{a - b} ,$$

and similarly we have

$$\sum_{n=0}^{\infty} A_n x^n L_{n+j} = a^j F(ax) + b^j F(bx) , \tag{6.8}$$

where as before in (2.1) we have $F(x) = \sum_{n=0}^{\infty} A_n x^n$.

Let $A_n = \binom{r}{n}$ and take $x = 1$. We then have for the ordinary Fibonacci-Lucas numbers

(6.9) $$\sum_{n=0}^{r} \binom{r}{n} F_{n+j} = F_{2r+j} ,$$

and

(6.10) $$\sum_{n=0}^{r} \binom{r}{n} L_{n+j} = L_{2r+j} ,$$

which are well-known recursions.

A very elegant symmetrical relation may be gotten from (2.6). In that relation we choose $A_n = \binom{r}{n}$ and set $x = a^{-p}$. The reader may easily verify that the formula then becomes, since $F(a^{p-k}b^k x) = (1 + b^k a^{-k})^r = L_k^r a^{-kr}$,

(6.11) $$\sum_{n=0}^{r} \binom{r}{n} \frac{L_n^p}{a^{pn}} = \sum_{n=0}^{p} \binom{p}{n} \frac{L_n^r}{a^{rn}} .$$

Similarly in (2.7) if we set $A_n = \binom{r}{n}$ and take $x = -a^{-p}$ we find easily that

(6.12) $$(a - b)^p \sum_{n=0}^{r} (-1)^n \binom{r}{n} \frac{F_n^p}{a^{pn}} = (a - b)^r \sum_{n=0}^{p} (-1)^n \binom{p}{n} \frac{F_n^r}{a^{rn}} .$$

And similarly we find from (2.8) that (here we take $x = a^{-p-q}$)

(6.13) $$\sum_{n=0}^{r} \binom{r}{n} a^{-n(p+q)} F_n^p L_n^q = (a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{L_{k+j}^r}{a^{r(k+j)}} .$$

With other choices of x we could give similar results. In fact with $x = -a^{-p-q}$. we have

(6.14) $$\sum_{n=0}^{r} (-1)^n \binom{r}{n} \frac{F_n^p L_n^q}{a^{n(p+q)}} = (a - b)^{r-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{F_{k+j}^r}{a^{r(k+j)}}$$

It may be of interest to note that Kelisky [12] developed some curious results involving Bernoulli, Euler, Fibonacci, and Lucas numbers. The relations he gives should be compared with those developed in the present paper. In particular, Kelisky has since written the present author that the unpublished proofs of the last collection of relations he found are somewhat similar to the methods of the present note.

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A FIBONACCI ARRAY*

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We take $u_0 = 0$, $u_1 = 1$,

$$u_{n+1} = u_n + u_{n-1} \qquad (n \geq 1) ,$$

and define

$$u_{0,n} = u_n \qquad (n = 0, 1, 2, \cdots) \tag{1}$$

as the 0-th row of the array F. We next put

$$u_{1,n} = u_{n+2} \qquad (n = 0, 1, 2, \cdots) , \tag{2}$$

the first row of F. For $r \geq 2$ we define $u_{r,n}$ by means of

$$u_{r,n} = u_{r-1,n} + u_{r-2,n} \qquad (n = 0, 1, 2, \quad) . \tag{3}$$

Thus $u_{r,n}$ is defined for all $r, n \geq 0$. It follows from the definition that

$$u_{r,n} = u_{r,n-1} + u_{r,n-2} \qquad (n \geq 2) . \tag{4}$$

Indeed, assuming the truth of (4), we get

$$\begin{aligned} u_{r+1,n} &= u_{r,n} + u_{r-1,n} \\ &= u_{r,n-1} + u_{r,n-2} + u_{r-1,n-1} + u_{r-1,n-2} \\ &= u_{r+1,n-1} + u_{r+1,n-2} . \end{aligned}$$

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The following table is easily computed

r \ n	0	1	2	3	4	5	6	7	8
0	0	1	1	2	3	5	8	13	21
1	1	2	3	5	8	13	21	34	55
2	1	3	4	7	11	18	29	47	76
3	2	5	7	12	19	31	50	81	131
4	3	8	11	19	30	49	79	128	207
5	5	13	18	31	49	80	129	209	338
6	8	21	29	50	79	129	208	337	545
7	13	34	47	81	128	209	337	546	883
8	21	55	76	131	207	338	545	883	1428

The symmetry property

(5) $$u_{r,n} = u_{n,r}$$

is easily proved by making use of (3) and (4).

We now put

(6) $$f_r(x) = \sum_{n=0}^{\infty} u_{r,n} x^n \quad (r = 0, 1, 2, \ldots) .$$

In particular, it follows from (1) and (2) that

(7) $$f_0(x) = \frac{x}{1 - x - x^2} , \quad f_1(x) = \frac{1 + x}{1 - x - x^2} ,$$

and by (3) we have also

(8) $$f_r(x) = f_{r-1}(x) + f_{r-2}(x) \qquad (r \geq 2) .$$

Using (7) and (8), we prove readily that

$$f_r(x) = \frac{u_r + u_{r+1}x}{1 - x - x^2} \qquad (r \geq 0) \quad . \tag{9}$$

Thus (6) yields

$$u_{r,n} = u_r u_{n+1} + u_{r+1} u_n \quad , \tag{10}$$

which again implies the truth of (5).

If we put

$$f(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} u_{r,n} x^r y^n \quad ,$$

then by (9)

$$f(x,y) = \sum_{r=0}^{\infty} \frac{u_r + u_{r+1}y}{1 - y - y^2} x^r = \frac{1}{1 - y - y^2}\left(\frac{x}{1 - x - x^2} + \frac{y}{1 - x - x^2}\right)$$

so that

$$f(x,y) = \frac{x + y}{(1 - x - x^2)(1 - y - y^2)} \quad . \tag{11}$$

We remark that (10) is equivalent to

$$u_{r,n} = u_r u_n + u_{r+n} \quad , \tag{12}$$

as is easily proved.

It appears from the table that

$$u_{r+1,r-1} - u_{r,r} = (-1)^r \qquad (r \geq 1) \quad . \tag{13}$$

Indeed (13) holds for $r = 1$. Then

$$u_{r+2,r} - u_{r+1,r+1} = \left(u_{r+1,r} + u_{rr}\right) - \left(u_{r+1,r} - u_{r+1,r-1}\right)$$
$$= u_{r,r} - u_{r+1,r-1} = (-1)^{r+1} .$$

Also the relation

$$u_{r+2,r-2} - u_{r,r} = (-1)^{r+1} \qquad (r \geq 2) \tag{14}$$

is suggested; the proof of (14) is similar to the proof of (13).

In the next place we have

$$u_{r+3,r-3} - u_{r,r} = (-1)^r \, 4 \qquad (r \geq 3) . \tag{15}$$

The general formula of which (13), (14), and (15) are special cases is

$$u_{r+s,r-s} - u_{r,r} = (-1)^{r-s+1} u_s^2 \qquad (r \geq s) . \tag{16}$$

Indeed it follows from (12) that

$$u_{r+s,r-s} - u_{r,r} = u_{r+s} u_{r-s} - u_r^2$$

and (16) is an easy consequence.

For a later purpose we shall require the formula

$$\sum_{r=0}^{n-1} u_{r,r} = \begin{cases} 2u_n^2 & (n \text{ even}) \\ 2u_{n+1} u_{n-1} & (n \text{ odd}) . \end{cases} \tag{17}$$

This is equivalent to

$$u_{n-1,n-1} = \begin{cases} 2(u_n^2 - u_n u_{n-2}) = 2u_n u_{n-1} & \text{(n even)} \\ 2(u_{n+1} u_{n-1} - u_{n-1}^2) = 2u_n u_{n-1} & \text{(n odd)} \end{cases},$$

which is in agreement with (10).

In connection with (17) we note that

$$\sum_{r=0}^{\infty} u_{r,r} x^r = \frac{2x}{(1+x)(1-3x+x^2)} . \tag{18}$$

Formulas of this kind are perhaps most easily proved by using the familiar representation

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} , \qquad \beta = \frac{1-\sqrt{5}}{2} .$$

To illustrate we shall evaluate

$$\sum_{r=0}^{\infty} u_{n+r,r} x^r .$$

Since by (12)

$$u_{n+r,r} = u_{n+r}u_r + u_{n+2r} = \frac{1}{5}\left[2\left(\alpha^{n+2r+1} + \beta^{n+2r+1}\right) - (-1)^r\left(\alpha^n + \beta^n\right)\right],$$

we get

$$\sum_{r=0}^{\infty} u_{n+r,r}x^r = \frac{1}{5}\left(\frac{2\alpha^{n+1}}{1-\alpha^2 x} + \frac{2\beta^{n+1}}{1-\beta^2 x} - \frac{\alpha^n + \beta^n}{1+x}\right)$$

$$= \frac{1}{5}\left(\frac{2(v_{n+1} - v_{n-1}x)}{1 - 3x + x^2} - \frac{v_n}{1+x}\right),$$

where

(19) $$v_n = \alpha^n + \beta^n .$$

It follows that

(20) $$\sum_{r=0}^{\infty} u_{n+r,r}x^r = \frac{1}{5}\,\frac{(v_{n+1} + v_{n-1})(1 - x^2) + 5v_n x}{(1+x)(1 - 3x + x^2)} .$$

When $n = 0$, (20) reduces to (18). When $n = 1, 2$ we get

(21) $$\sum_{r=0}^{\infty} u_{r,r+1}x^r = \frac{1 + x - x^2}{(1+x)(1 - 3x + x^2)} ,$$

(22) $$\sum_{r=0}^{\infty} u_{r,r+2}x^r = \frac{1 + 3x - x^2}{(1+x)(1 - 3x + x^2)} ,$$

respectively.

Returning to (11), we replace x, y by xt, yt, respectively, so that

(23) $$\sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} u_{r,n-r}x^r y^{n-r} = \frac{(x+y)t}{(1 - xt - x^2t^2)(1 - yt - y^2t^2)} .$$

Since the right member of (23) is equal to

$$\frac{x+y}{(x-y)(x^2+3xy+y^2)}\left[\frac{xy+x^2(x+y)t}{1-xt-x^2t^2}-\frac{xy+y^2(x+y)t}{1-yt-y^2t^2}\right]$$

$$= \frac{x+y}{(x-y)(x^2+3xy+y^2}\left\{\left[xy+x^2(x+y)t\right]\sum_0^\infty u_{n+1}x^n t^n\right.$$

$$\left. - \left[xy+y^2(x+y)t\right]\sum_0^\infty u_{n+1}y^n t^n\right\} ,$$

it follows that

$$(24) \qquad \sum_{r=0}^n u_{r,n-r}x^r y^{n-r} = \frac{xy(x+y)(x^n-y^n)u_{n-1}+(x+y)^2(x^{n+1}-y^{n+1})u_n}{(x-y)(x^2+3xy+y^2)} .$$

The polynomials

$$D_n = D_n(x,y) = \sum_{r=0}^n u_{r,n-r}x^r y^{n-r}$$

correspond to the secondary diagonals in the Fibonacci array. For example, we have

$$D_0 = 0, \qquad D_1 + x + y, \qquad D_2 = (x+y)^2,$$

$$D_3 = 2(x+y)^3 - 3xy(x+y),$$

$$D_4 = 3(x+y)^4 - 7xy(x+y)^2 .$$

Since

$$\frac{x^{n+1}-y^{n+1}}{x-y} = \sum_{2r\le n}(-1)^r\binom{n-r}{r}(xy)^r(x+y)^{n-2r} ,$$

we find, after a little manipulation, that (24) implies

$$D_n(x,y) = -\sum_r \left[\binom{n-r}{r} u_n - \binom{n-r}{r-1} u_{n+1}\right] (x+y)^{n-2r+2} \tag{25}$$

$$\times \frac{(x+y)^{2r} - (-1)^r (xy)^r}{(x+y)^2 + xy} .$$

In particular, if we take

$$x = \alpha = \frac{1+\sqrt{5}}{2} , \quad y = \beta = \frac{1-\sqrt{5}}{2} ,$$

(25) reduces to

$$D_n(\alpha, \beta) = \sum_r \left[\binom{n-r}{r-1} u_{n+1} - \binom{n-r}{r} u_n\right] r . \tag{26}$$

However, it is simpler to make use of (11). It is easily verified that

$$\sum_{n=0}^{\infty} D_n(\alpha,\beta)t^n = \frac{t}{(1+t)^2(1-3t+t^2)} = (1+t)^{-2} \sum_{n=0}^{\infty} u_{2n} t^n ,$$

so that

$$D_n(\alpha,\beta) = \sum_{r=0}^{n} (-1)^r (r+1) u_{2n-2r} . \tag{27}$$

It is not obvious that (26) and (27) are identical. As an instance of (27), we have

$$D_4(\alpha, \beta) = u_8 - 2u_6 + 3u_4 - 4u_2 + 5u_0 = 21 - 16 + 9 - 4 = 10.$$

In the next place we evaluate the determinant

$$\Delta(r, s; m, n) = \begin{vmatrix} u_{r,m} & u_{r,n} \\ u_{s,m} & u_{s,n} \end{vmatrix} .$$

Using (10) we get

$$\Delta(r, s; m, n) = (u_r u_{s+1} - u_{r+1} u_s)(u_{m+1} u_n - u_m u_{n+1}) .$$

Since, for $n \geq m$,

$$u_{m+1}u_n - u_m u_{n+1} = -(u_m u_{n-1} - u_{m-1}u_n) = (-1)^m (u_1 u_{n-m} - u_0 u_{n-m+1})$$

$$= (-1)^m u_{n-m},$$

it follows that

(28) $$\Delta(r, s; m, n) = (-1)^{m+r+1} u_{n-m} u_{s-r} \quad (n \geq m, \ s \geq r).$$

In particular, when $m = r$, $n = s$, (28) becomes

(29) $$\Delta(r, s; r, s) = -u_{s-r}^2 \qquad (s \geq r) .$$

Consider the symmetric matrix of order n:

(30) $$M_n = (u_{r,s}) \qquad (r, s = 0, 1, \ldots, n-1) .$$

Clearly the rank of $M_n \leq 2$ and indeed is equal to 2 for $n \geq 2$. The characteristic polynomial of M_n is given by

$$p_n(x) = x^n - \sum_{r=0}^{n-1} u_{r,r} x^{n-1} + \sum_{0 \leq r < s < n} \Delta(r, s; r, s) x^{n-2}$$

The coefficient of x^{n-1} can be found by means of (17). As for the coefficient of x^{n-2}, it follows from (29) that

$$\sum_{o\le r<s<n} \Delta(r,s;\,r,s) = -\sum_{o\le r<s<n} u_{s-r}^2 = -\sum_{r=0}^{n-2}\sum_{s=r+1}^{n-1} u_{s-r}^2$$

$$= -\sum_{r=0}^{n-2}\sum_{s=1}^{n-r-1} u_s^2 = -\sum_{r=0}^{n-1}\sum_{s=0}^{n-r-1} u_s^2 .$$

But

$$5\sum_{s=0}^{n-1} u_s^2 = \sum_{s=0}^{n-1}\left[\alpha^{2s} + \beta^{2s} - 2(-1)^s\right] = \frac{1-\alpha^{2n}}{1-\alpha^2}\;\frac{1-\beta^{2n}}{1-\beta^2} - 2\epsilon_n$$

$$= 1 - v_{2n-2} + v_{2n} - 2\epsilon_n \quad ,$$

where as above $v_n = \alpha^n + \beta^n$ and

(31) $$\epsilon_n = \begin{cases} 0 & \text{(n even)} \\ 1 & \text{(n odd)} . \end{cases}$$

Then

$$5\sum_{r=0}^{n-1}\sum_{s=0}^{n-r-1} u_s^2 = \sum_{r=0}^{n-1} (1 - v_{2n-2r-2} + v_{2n-2r} - 2\epsilon_{n-r})$$

$$= n - 2 + v_{2n} - 2\left[\frac{1}{2}(n+1)\right] = v_{2n} - 2 - \epsilon_n ,$$

so that

(32) $$\sum_{o\le r<s<n} \Delta(r,s;\,r,s) = -\frac{1}{5}(v_{2n} - 2 - \epsilon_n) .$$

Therefore, using (17) and (32), we find that the characteristic polynomial of M_n is given by

$$p_n(x) = \begin{cases} x^n - 2u_n^2 x^{n-1} - u_n^2 x^{n-2} & (n \text{ even}) \\ x^n - 2u_{n+1}u_{n-1}x^{n-1} - (u_n^2 - 1)x^{n-2} & (n \text{ odd},\ n > 1). \end{cases} \tag{33}$$

For example, we have

$$p_2(x) = x^2 - 2x - 1 \quad , \quad p_3(x) = x^3 - 6x^2 - 3x \quad ,$$

as can be verified directly.

By means of (33) we can compute the characteristic values of M_n. In addition to $n - 2$ zeros we have

$$\begin{cases} u_n^2 \pm u_n\sqrt{u_n^2 + 1} & (n \text{ even}) \\ u_{n+1}\, u_{n-1} \pm \sqrt{u_{n+1}^2\, u_{n-1}^2 + u_n^2 - 1} & (n \text{ odd}) \ . \end{cases} \tag{34}$$

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D. W. Robinson, 'The Fibonacci Matrix Modulo m,' presented to the Mathematical Association of America, March 9, 1963.

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FIBONACCI RELATED MASTER'S THESES

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2. Charles H. King, 'Some Properties of the Fibonacci Sequence,' San Jose State College, 1960, Advisor: Prof. Verner E. Hoggatt, Jr.
3. Richard A. Hayes, 'Fibonacci and Lucas Polynomials,' San Jose State College, Advisor: Prof. Verner E. Hoggatt, Jr. (Not yet completed.)
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THE FIBONACCI MATRIX MODULO m*

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In this paper we investigate some of the arithmetical properties of the famous Fibonacci sequence by use of elementary matrix algebra. We believe the approach to be conceptual and, at least in part, novel. Thus, it is our purpose to explore the pedagogical advantages of matrix methods for problems of this kind as well as to provide a refreshing appreciation of the arithmetical properties themselves. At the conclusion of the paper we also indicate how the methods may be applied to other linear recurrent sequences.

We begin by considering the following example. Suppose that the Fibonacci sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

is reduced modulo 8:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 0,\ 5,\ 5,\ 2,\ 7,\ 1,\ 0,\ 1,\ 1,\ \cdots$$

We observe that the reduced sequence is periodic. Indeed, the 12 terms of the period form two sets of 6 terms each, the terms of the second half being 5 times the corresponding terms of the first half. We say that the Fibonacci sequence reduced modulo 8 is of period 12 and restricted period 6 with multiplier 5. Also, we observe that the multiplier is of exponent 2 modulo 8.

More generally, let $u_0, u_1, \cdots, u_n, \cdots$ be the Fibonacci sequence of integers satisfying $u_{n+2} = u_{n+1} + u_n$ for $n \geq 0$ with $(u_0, u_1) = (0, 1)$. Given any integer $m > 1$ we provide below an elementary proof of the fact that there is a positive integer n such that $(u_n, u_{n+1}) \equiv (0, 1) \pmod m$. The least such integer $\delta(m)$ is called the period of the Fibonacci sequence modulo m. The least positive integer n such that $(u_n, u_{n+1}) \equiv s(0, 1) \pmod m$, where s is some integer, is called the restricted period $\alpha(m)$ of the sequence modulo m. If $(u_{\alpha(m)}, u_{\alpha(m)+1}) \equiv s(m)(0,1)$ $\pmod m$, $0 < s(m) < m$, then $s(m)$ is called the multiplier of the Fibonacci sequence modulo m. Obviously $s(m) \equiv u_{\alpha(m)-1} \pmod m$. Finally, we denote the exponent modulo m of the multiplier $s(m)$ by $\beta(m)$.

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By direct calculation we obtain the following table:

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\alpha(m)$	3	4	6	5	12	8	6	12	15	10	12	7	24	20	12	9	12	18
$\beta(m)$	1	2	1	4	2	2	2	2	4	1	2	4	2	2	2	4	2	1
$\delta(m)$	3	8	6	20	24	16	12	24	60	10	24	28	48	40	24	36	24	18

The results of this table illustrate several interesting arithmetical properties. In fact, if (a,b) and $[a,b]$ denote the greatest common divisor and the least common multiple, respectively, of the integers a and b, then we propose to establish the following:

(i) $m \mid u_n$ if and only if $\alpha(m) \mid n$, and $m \mid u_n$, $m \mid (u_{n+1} - 1)$ if and only if $\delta(m) \mid n$;

(ii) $\delta(m) = \alpha(m)\beta(m) = (2, \beta(m))\ [\gamma(m), \alpha(m)]$, where $\gamma(2) = 1$ and $\gamma(m) = 2$ for $m > 2$;

(iii) $\alpha([m_1, m_2]) = [\alpha(m_1), \alpha(m_2)]$, and $\delta([m_1, m_2]) = [\delta(m_1), \delta(m_2)]$;

(iv) for every odd prime p there is a positive integer e(p) such that $\alpha(p^e) = \alpha(p)p^{\max(0, e-e(p))}$ and $\delta(p^e) = \delta(p)p^{\max(0, e-e(p))}$;

(v) $\alpha(p) \mid (p - (5/p))$, where $(5/p)$ is the usual Legendre symbol; furthermore, if $p \neq 5$, then $\delta(p) \mid (p-1)$ or $\delta(p) \mid 2(p+1)$.

With the possible exception of the last equation of (ii), which is due to Morgan Ward, these properties are all well known. Indeed, the fact that reduced sequences of this type are periodic was observed by J. L. Lagrange in the eighteenth century. A century later E. Lucas engaged in an extensive study of the arithmetic divisors of such sequences. These early results together with some of the later developments in the subject are reviewed in Chapter 17 of Dickson's History [6]. However, it is suggested that this general background be supplemented with at least the papers of Carmichael [3], Lehmer [11], and Ward [19]. (See also [4, 7, 8, 9, 10, 17, 20, 21].)

Furthermore, since the main purpose of this present paper is to indicate the use of matrix algebra for the study of linear recurrence relations, we also remark that such techniques are certainly not new. (See for example [1, 13, 16, 18].) In fact, some of the arithmetical properties of linear recurrent sequences have been studied by means of matrices. (See for example [2, 12, 15].) It is our aim to now indicate some of the further possibilities of this method.

We begin by introducing the main tool of our discussion. Specifically, we view the linear recurrence above as defining a mapping of the ordered pair (u_{n-1}, u_n) onto the ordered pair (u_n, u_{n+1}). Since $u_{n+1} = u_{n-1} + u_n$, it is clear that this mapping is represented by the matrix product $(u_{n-1}, u_n)U = (u_n, u_{n+1})$, where

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

Furthermore, by induction on n, we observe that $(u_n, u_{n+1}) = (0, 1)U^n$ and

$$U^n = \begin{bmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{bmatrix} .$$

Because of these results we call U the <u>Fibonacci matrix</u>.

From the foregoing it is evident that $(u_n, u_{n+1}) \equiv (0, 1) \pmod{m}$ if and only if U^n is congruent (elementwise) modulo m to the identity matrix. Thus, the study of the period of the Fibonacci sequence modulo m is equivalent to the study of the period of the sequence $I, U, U^2, \cdots, U^n,$ reduced modulo m. In particular, since there are only a finite number of distinct matrices in this reduced sequence, it follows that there are integers k and n such that U^{k+n} is congruent to U^k with $k + n > k \geq 0$. But since the determinant of U is the unit -1, this means that for some positive integer n, $U^n \equiv I \pmod{m}$. Thus, there exists a least such positive integer n, which is in fact $\delta(m)$ as defined above. Also, it is clear that every such n is an integral multiple of $\delta(m)$. That is, $U^n \equiv I \pmod{m}$ if and only if $\delta(m) | n$, which is equivalent to the second statement of (i).

By a similar argument we have $(u_n, u_{n+1}) \equiv s(0, 1) \pmod{m}$ if and only if $U^n \equiv sI \pmod{m}$. Indeed, U^n is congruent to a scalar matrix modulo m if and only if $\alpha(m) | n$, where $\alpha(m)$ is the restricted period defined above. This result is equivalent to the first part of (i).

Furthermore, we have

$$U^{\alpha(m)} \equiv s(m)\, I \pmod{m} ,$$

where the multiplier $s(m)$ is of exponent $\beta(m)$ modulo m. Since $U^{\alpha(m)\beta(m)} \equiv s(m)^{\beta(m)} I \equiv I \pmod{m}$, $\delta(m) | \alpha(m)\beta(m)$. On the other hand, since it is evident that $\alpha(m) | \delta(m)$, we have by a similar argument that $\beta(m) | \delta(m)/\alpha(m)$. Thus, $\delta(m) = \alpha(m)\beta(m)$, which establishes the first equation of (ii).

Also, since the determinant of U is -1, we have from the matrix congruence above that

$$(-1)^{\alpha(m)} \equiv (s(m))^2 \pmod{m}$$

Hence, these congruent integers have the same exponent modulo m. Specifically,

$$\frac{\gamma(m)}{(\gamma(m),\ \alpha(m))} = \frac{\beta(m)}{(2, \beta(m))} \quad ,$$

where $\gamma(m)$ is the exponent of -1 modulo m. That is,

$$\delta(m) = \alpha(m)\beta(m) = (2, \beta(m)) \frac{\gamma(m)\alpha(m)}{(\gamma(m), \alpha(m))} \;,$$

which is clearly equivalent to (ii). In particular, we observe that $\delta(m)$ is even for $m > 2$ and that $\beta(m) \mid 4$.

We now demonstrate the second equation of (iii). We first observe that if $m' \mid m$, then $U^{\delta(m)} \equiv I \pmod{m'}$ and $\delta(m') \mid \delta(m)$. Thus, since m_1 and m_2 both divide $[m_1, m_2]$, it follows that $\delta([m_1, m_2])$ is a common multiple of $\delta(m_1)$ and $\delta(m_2)$. On the other hand, suppose $\delta(m_1)$ and $\delta(m_2)$ both divide δ. Since U^{δ} is congruent to the identity matrix modulo both m_1 and m_2, the congruence is also valid modulo $[m_1, m_2]$. That is, $\delta([m_1, m_2])$ divides δ and is therefore the least common multiple of $\delta(m_1)$ and $\delta(m_2)$.

We obtain similarly the first equation of (iii). Thus, we observe that both α and δ are factorable (l.c.m. multiplicative) functions of the argument m, which suggests next the consideration of property (iv).

Therefore, let p be any odd prime and let e be any positive integer. Since $U^{\delta(p^e)} = I + p^e B$ for some matrix B, $U^{p\,\delta(p^e)} = (I + p^e B)^p \equiv I \pmod{p^{e+1}}$. That is, $\delta(p^{e+1}) \mid p\,\delta(p^e)$. But obviously $\delta(p^e) \mid \delta(p^{e+1})$. We conclude, since p is a prime, that $\delta(p^{e+1})$ is either $\delta(p^e)$ or $p\,\delta(p^e)$. In particular, $\delta(p^e)/\delta(p)$ is some non-negative power of p. Similarly, $\alpha(p^e)/\alpha(p)$ is some non-negative power of p. Recalling that for any given modulus the ratio of the period to the restricted period divides 4 and that p is odd, it is immediate from the identity

$$\frac{\alpha(p^e)}{\alpha(p)} \cdot \frac{\delta(p^e)}{\alpha(p^e)} = \frac{\delta(p^e)}{\delta(p)} \cdot \frac{\delta(p)}{\alpha(p)}$$

that $\alpha(p^e)/\alpha(p) = \delta(p^e)/\delta(p)$ and $\delta(p^e)/\alpha(p^e) = \delta(p)/\alpha(p)$.

Moreover, suppose that $\delta(p^{e+1}) \neq \delta(p^e)$. Then $U^{\delta(p^e)} = I + p^eB$ with $B \not\equiv 0 \pmod p$. Hence,

$$U^{p\,\delta(p^e)} = I + p^{e+1}B \not\equiv I \pmod{p^{e+2}} \quad .$$

That is, if $\delta(p^{e+1}) = p\,\delta(p^e)$, then $\delta(p^{e+2}) = p\,\delta(p^{e+1})$. Consequently, if $e(p)$ is the largest positive e such that $\delta(p^e) = \delta(p)$, then $\delta(p^e) = \delta(p)$ for $1 \leq e \leq e(p)$ and $\delta(p^e) = p^{e-e(p)}\delta(p)$ for $e > e(p)$. Finally, the existence of $e(p)$ is assured from a consideration of the alternative: if $U^{\delta(p)} \equiv I \pmod{p^e}$, $e = 1, 2, \cdots$, then $U^{\delta(p)} = I$, which is impossible. This completes the proof of (iv).

It is of interest to remark that a test [17] with a digital computer has shown that $e(p) = 1$ for all primes p less than 10,000. However, the problem of identifying the exceptional primes p with $e(p) > 1$ remains unsolved.

Finally, we prove property (v). For every prime p we define the <u>restricted graph</u> $R(p)$ of U modulo p to consist of the $p + 1$ points $P_0 = (0, 1), P_1 = (1,1), \cdots, P_{p-1} = (p - 1, 1)$, $P_\infty = (1, 0)$ together with the collection of all directed edges $P_i \to P_{i'}$, where $P_{i'}$ is the unique point which is linearly dependent upon the matrix product P_iU. (Contrast this with, for example, [5].) By way of illustration, $R(5)$ consists of the 1-cycle $P_2 \to P_2$ and the 5-cycle $P_0 \to P_1 \to P_3 \to P_4 \to P_\infty \to P_0$. In general, since this graph is determined by a one-to-one correspondence, it follows that $R(p)$ consists of a collection of disjoint cycles. (See for example [14] pp. 25-27.) Furthermore, it is clear that P_i belongs to a 1-cycle (or in other words is a fixed point under the correspondence) if and only if P_i is a characteristic vector of U modulo p. Moreover, suppose that P_i belongs to an α-cycle with $\alpha > 1$. Since $\{P_i, P_iU\}$ is a linearly independent set, it follows that $P_iU^\alpha \equiv sP_i \pmod p$ implies $U^\alpha \equiv sI \pmod p$, which means that $\alpha(p)|\alpha$. Thus, since obviously $\alpha|\alpha(p)$, $\alpha = \alpha(p)$. That is, $R(p)$ consists of a collection of 1-cycles and $\alpha(p)$-cycles. Consequently, $\alpha(p)|(p + 1 - t)$, where t is the number of 1-cycles of $R(p)$. But t is also the number of linearly independent characteristic vectors of U modulo p, or equivalently the number of distinct roots modulo p of the minimum polynomial $\lambda^2 - \lambda - 1$ of U. Since the discriminant of this quadratic is 5, it follows that t is 0, 1, or 2 according as the Legendre symbol $(5/p)$ is -1, 0, or 1.

That is, $\alpha(p) \mid (p - (5/p))$, which means that $U^{p-(5/p)} \equiv sI$ and $U^p \equiv sU^{(5/p)} \pmod p$, for some integer s depending upon p. Now, considering the trace of each of the matrices in this last congruence, $\operatorname{tr} U^5 \equiv 2s \pmod 5$ and $\operatorname{tr} U^p \equiv (5/p)s \pmod{p \neq 5}$. But, since $U^{-1} = U - I$ implies $U^{-p} \equiv U^p - I \pmod p$, we have $-\operatorname{tr} U^p \equiv \operatorname{tr} U^p - 2$ and $\operatorname{tr} U^p \equiv 1 \pmod p$. Therefore $U^5 \equiv 3I \pmod 5$ and

$$U^{p-(5/p)} \equiv (5/p)\, I \qquad (\operatorname{mod} p \neq 5)\ ,$$

which establishes property (v). As a corollary we obtain the well-known congruence $u_p \equiv (5/p) \pmod p$. Also, it is of interest to add that, by the quadratic reciprocity law, we have $(5/p) = 1$ if $p = 5k \pm 1$ and $(5/p) = -1$ if $p = 5k \pm 2$.

Thus, by use of the Fibonacci matrix, we have established some of the principal arithmetical properties of the sequence $0, 1, 1, 2, \cdots$. Although we may use this matrix to establish many other interesting properties and identities of the Fibonacci numbers, we feel that the foregoing is sufficient to illustrate the application of this tool (at least as far as the arithmetical properties are concerned). However, we indicate in conclusion how the idea may be readily adapted to the study of more general linear recurrent sequences.

Specifically, let $x_0, x_1, \cdots, x_n, \cdots$ be the sequence of integers satisfying the linear recurrence

$$x_{n+r} = a_1 x_{n+r-1} + \cdots + a_r x_n \ ,$$

for $n \geq 0$ where $x_0, \cdots, x_{r-1}$ and $a_1, \cdots, a_r$ are given integers. A study of this linear recurrent sequence may be made by means of the equation $X_n = X_0 A^n$, where $X_n = (x_n, \cdots, x_{n+r-1})$ and

$$A = \begin{bmatrix} 0 & \cdots & 0 & a_r \\ 1 & \cdots & 0 & a_{r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & a_1 \end{bmatrix} .$$

Indeed, the arithmetical properties of this sequence may be investigated by a generalization of the methods suggested by this present paper. In particular, if m is a positive integer such that $\{X_0, \cdots, X_{r-1}\}$ is linearly independent modulo m, then

$X_n \equiv sX_0 \pmod{m}$ if and only if $A^n \equiv sI \pmod{m}$. Furthermore, if $(m, a_r) = 1$, then the determinant of A is the unit $(-1)^{r-1}a_r$ modulo m and the sequence of powers of this matrix reduced modulo m is periodic. That is, under these assumptions, the periodic properties of the sequence of integers reduced modulo m may be identified with those of the sequence I, A, $\cdots$, A^n, $\cdots$ reduced modulo m. For example, we have that (ii) above is a special case of the equation

$$\delta(m) = \alpha(m)\beta(m) = (r, \beta(m))\,[\,\gamma(m), \alpha(m)\,],$$

where r is the order of the recurrence, $\gamma(m)$ is the exponent of the determinant of A modulo m, and $\alpha(m)$, $\beta(m)$, and $\delta(m)$ are obvious extensions of the definitions above.

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THE RELATION OF THE PERIOD MODULO TO THE RANK OF APPARITION OF m IN THE FIBONACCI SEQUENCE

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The Fibonacci sequence is defined by the recurrence relation,

(1) $$u_{n+2} = u_{n+1} + u_n\ , \qquad n = 0, 1, 2, \quad ,$$

and the initial values $u_0 = 0$ and $u_1 = 1$. Lucas [2, pp. 297-301] has shown that every integer, m, divides some member of the sequence, and also that the sequence is periodic modulo m for every m. By this we mean there is an integer, k, such that

(2) $$u_{k+n} \equiv u_n \pmod m\ , \qquad n = 0, 1, 2, \cdots\ .$$

Definition. The period modulo m, denoted by s(m), is the smallest positive integer, k, for which the system (2) is satisfied.

Definition. The rank of apparation of m, denoted by f(m), is the smallest positive integer, k, for which $u_k \equiv 0 \pmod m$.

Wall [3] has shown that

(3) $$u_n \equiv 0 \pmod m \text{ iff } f(m) | n.$$

In particular, since $u_{s(m)} \equiv u_0 \equiv 0 \pmod m$ we have

(4) $$f(m) | s(m)\ .$$

Definition. We define a function t(m) by the equation $f(m)t(m) = s(m)$.

We note that t(m) is an integer for all m. The purpose of this paper is to give criteria for the evaluation of t(m).

Now we give some results which will be needed later.

(5) $$u_{n-1}^2 = u_n u_{n-2} + (-1)^n\ .$$

This can be proved by induction, using the recurrence relation (1).

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$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } \alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}. \tag{6}$$

This is the well-known "Binet formula." It gives a natural extension of the Fibonacci sequence to negative values of n. By using the relation $\alpha^n\beta^n = (-1)^n$, we find

$$u_{-n} = (-1)^{n+1} u_n . \tag{7}$$

From this we see that the recurrence (1) holds for the extended sequence.

By solving the system

$$\alpha^k - \beta^k = (\alpha - \beta) u_k$$

$$\alpha \cdot \alpha^k - \beta \cdot \beta^k = (\alpha - \beta) u_{k+1}$$

for α^k and β^k, we obtain

$$\alpha^k = u_{k+1} - \beta u_k = (1 - \beta)u_k + u_{k-1} = \alpha u_k + u_{k-1}$$

and

$$\beta^k = u_{k+1} - \alpha u_k = (1 - \alpha) u_k + u_{k-1} = \beta u_k + u_{k-1} .$$

Then

$$(\alpha - \beta) u_{nk+r} = \alpha^{nk+r} - \beta^{nk+r} = (\alpha u_k + u_{k-1})^n \alpha^r - (\beta u_k + u_{k-1})^n \beta^r$$

By expanding and recombining we get (for $n \geq 0$)

$$u_{nk+r} = \sum_{j=0}^{n} \binom{n}{j} u_k^j u_{k-1}^{n-j} u_{r+j} .$$

Now if we set $k = f(m)$, we find

$$u_{nf(m)+r} \equiv u_{f(m)-1}^n u_r \pmod{m} . \tag{8}$$

We note that this is valid for negative as well as non-negative integers, r.

Lemma 1. $t(m)$ is the exponent to which $u_{f(m)-1}$ belongs (mod m).

Proof: Suppose $u^n_{f(m)-1} \equiv 1 \pmod{m}$. Then from (8) we have $u_{nf(m)+r} \equiv u_r \pmod{m}$ for all r. It follows from the definition of $s(m)$ that $s(m) \leq nf(m)$ and thus $u^n_{f(m)-1} \equiv 1 \pmod{m}$ implies $t(m) = s(m)/f(m) \leq n$.

Now we set $r = 1$ and $n = t(m)$ in (8) to obtain

$$u^{t(m)}_{f(m)-1} \equiv u_{t(m)f(m)+1} \equiv u_{s(m)+1} \equiv u_1 \equiv 1 \pmod{m} .$$

Thus $t(m)$ is the smallest positive n for which $u^n_{f(m)-1} \equiv 1 \pmod{m}$, that is, $u_{f(m)-1}$ belongs to $t(m)$ (mod m).

Theorem 1. For $m > 2$ we have

i) $t(m) = 1$ or 2 if $f(m)$ is even, and

ii) $t(m) = 4$ if $f(m)$ is odd.

Also, $t(1) = t(2) = 1$. Conversely, $t(m) = 4$ implies $f(m)$ is odd, $t(m) = 2$ implies $f(m)$ is even, and $t(m) = 1$ implies $f(m)$ is even or $m = 1$ or 2.

Proof. The cases $m = 1$ and $m = 2$ are easily verified. Now suppose $m > 2$ and set $n = f(m)$ in (5) to get

$$u^2_{f(m)-1} \equiv u_{f(m)}u_{f(m)-2} + (-1)^{f(m)} \equiv (-1)^{f(m)} \pmod{m} .$$

If $f(m)$ is even we have $u^2_{f(m)-1} \equiv 1 \pmod{m}$, and i) follows from Lemma 1.

If $f(m)$ is odd we have $u^2_{f(m)-1} \equiv -1 \pmod{m}$, and since $m > 2, u^2_{f(m)-1} \not\equiv 1 \pmod{m}$. This implies $u_{f(m)-1} \not\equiv \pm 1 \pmod{m}$ and then

$$u^3_{f(m)-1} \equiv u^2_{f(m)-1}u_{f(m)-1} \equiv -u_{f(m)-1} \not\equiv \pm 1 \pmod{m}.$$

Finally, $u^4_{f(m)-1} \equiv (u^2_{f(m)-1})^2 \equiv (-1)^2 \equiv 1 \pmod{m}$ and, by Lemma 1, $t(m) = 4$.

The converse follows from the fact that the cases in the direct statement of the theorem are all inclusive.

Theorem 2. Let p be an odd prime and let e be any positive integer. Then

i) $t(p^e) = 4$ if $2 \nmid f(p)$,

ii) $t(p^e) = 1$ if $2 \mid f(p)$ but $4 \nmid f(p)$,

iii) $t(p^e) = 2$ if $4 \mid f(p)$, and

iv) $t(2^e) = 2$ for $e \geq 3$ and $t(2) = t(2^2) = 1$.

Conversely, if q represents any prime, then $t(q^e) = 4$ implies $f(q)$ is odd, $t(q^e) = 2$ implies $4 | f(q)$ or $q = 2$ and $e \geq 3$, and $t(q^e) = 1$ implies $2 | f(q)$ but $4 \nmid f(q)$ or $q^e = 2$ or 4.

Proof. Wall [3, p. 527] has shown that if $p^{n+1} \nmid u_{f(p^n)}$, then $f(p^{n+1}) = pf(p^n)$. It follows by induction that $f(p^e) = p^k f(p)$, where k is some non-negative integer. We emphasize that $f(p^e)$ and $f(p)$ are divisible by the same power of 2, since this fact is used several times in the sequel without further explicit reference.

In case i), $f(p^e)$ is odd and the result is given by setting $m = p^e$ in Theorem 1.

In cases ii) and iii), $f(p^e)$ is even and we may set $m = p^e$, $n = 1$, and $r = \frac{1}{2} f(p^e)$ in (8) to get

$$u_{\frac{1}{2}f(p^e)} \equiv u_{f(p^e)-1} \quad u_{-\frac{1}{2}f(p^e)} \qquad (\text{mod } p^e)$$

which, in view of (7), is the same as

$$u_{f(p^e)-1} \; u_{\frac{1}{2}f(p^e)} \equiv (-1)^{\frac{1}{2}f(p^e)+1} \; u_{\frac{1}{2}f(p^e)} \qquad (\text{mod } p^e) \quad .$$

Now $\frac{1}{2}f(p^e) = \frac{1}{2}p^k f(p)$, where k is some non-negative integer, and we see that $f(p) \nmid \frac{1}{2}f(p^e)$. Then from (3) we have $p \nmid u_{\frac{1}{2}f(p^e)}$ so that we may divide the above congruence by $u_{\frac{1}{2}f(p^e)}$. We get

$$u_{f(p^e)-1} \equiv (-1)^{\frac{1}{2}f(p^e)+1} \qquad (\text{mod } p^e) \quad .$$

Now in case ii), $\frac{1}{2}f(p)$ is odd and so is $\frac{1}{2}f(p^e)$, and the last congruence gives $u_{f(p^e)-1} \equiv 1 \pmod{p^e}$ and thus, by Lemma 1, $t(p^e) = 1$.

In case iii) the congruence becomes

$$u_{f(p^e)-1} \equiv -1 \quad (\text{mod } p^e) \quad ,$$

since

$$\tfrac{1}{2}f(p) \quad \text{and} \quad \tfrac{1}{2}f(p^e)$$

are both even. Then

$$u^2_{f(p^e)-1} \equiv 1 \; (\text{mod } p^e)$$

and by Lemma 1 again, $t(p^e) = 2$.

In case iv) we can easily verify $t(2) = t(2^2) = 1$. That $t(2^e) = 2$ for $e \geq 3$ follows from results given by Carmichael [1, p. 42] and Wall [3, p. 527]. These results are, respectively:

A. Let q be any prime and let r be any positive integer such that $(q, r) = 1$. If $q^\lambda \mid u_n$ and $q^{\lambda+1} \nmid u_n$, then $q^{\lambda+a} \mid u_{nrq^a}$ and $q^{\lambda+a+1} \nmid u_{nrq^a}$ except when $q = 2$ and $\lambda = 1$.

B. Let q be any prime and let λ be the largest integer such that $s(q^\lambda) = s(q)$. Then $s(q^e) = q^{e-\lambda} s(q)$ for $e \geq \lambda$.

The hypotheses of A. are satisfied by $q = 2$, $\lambda = 3$, and $n = f(2^3)$, and we find that $2^{3+a} \mid u_{kf(2^3)}$ iff $2^a \mid k$. It follows from (3) that $f(2^{3+a})$ must be a multiple of $f(2^3)$, hence $f(2^{3+a}) = 2^a f(2^3)$. Since $f(2^3) = 2f(2)$ we have $f(2^e) = 2^{e-2}f(2)$ for $e \geq 3$. Now set $q = 2$ and $\lambda = 1$ in B. We get $s(2^e) = 2^{e-1}s(2)$. Thus for $e \geq 3$ we have

$$t(2^e) = \frac{s(2^e)}{f(2^e)} = \frac{2^{e-1}s(2)}{2^{e-2}f(2)} = 2 \ .$$

The converse follows from the fact that the cases in the direct statement of the theorem are all inclusive. This completes the proof.

Now we give a lemma which is needed in the proof of the next theorem.

Lemma 2. If m has the prime factorization

$$m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n} \ , \text{ then}$$

i) $s(m) = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{s(q_i^{\alpha_i})\}$, and

ii) $f(m) = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{f(q_i^{\alpha_i})\}$.

Wall has given i). The proof of ii) is as follows: Since the $q_i^{\alpha_i}$ are pairwise relatively prime, $m \mid u_k$ is equivalent to $q_i^{\alpha_i} \mid u_k$ $(i = 1, 2, \cdots, n)$, which, by (3), is equivalent to $f(q_i^{\alpha_i}) \mid k$ $(i = 1, 2, \cdots, n)$. The smallest positive k which satisfies these conditions is

$$k = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{f(q_i^{\alpha_i})\} \ ,$$

which, according to the definition of $f(m)$, gives the desired result.

Theorem 3. We have

i) $t(m) = 4$ if $m > 2$ and $f(m)$ is odd.

ii) $t(m) = 1$ if $8 \nmid m$ and $2 | f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m, and

iii) $t(m) = 2$ for all other m.

Proof: From what has already been given in Theorem 1, we see that it suffices to show that the conditions given here in ii) are both necessary and sufficient for $t(m) = 1$. Let m have the prime factorization $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$ and set

$$f(q_i^{\alpha_i}) = 2^{\gamma_i} K_i \quad (i = 1, 2, \cdots, n) \quad ,$$

where the K_i are odd integers. By Theorem 1, we may set

$$t(q_i^{\alpha_i}) = 2^{\delta_i} \quad (i = 1, 2, \cdots, n) \quad \text{where } \delta_i = 0, 1, \text{ or } 2.$$

Then $s(q_i^{\alpha_i}) = f(q_i^{\alpha_i})t(q_i^{\alpha_i}) = 2^{\gamma_i+\delta_i} K_i$ $(i = 1, 2, \cdots, n)$. From Lemma 2 we have, where K is an odd integer,

$$s(m) = \underset{1 \le i \le n}{\text{l. c. m.}} \{s(q_i^{\alpha_i})\} = 2^{\max(\gamma_i + \delta_i)}K,$$

$$f(m) = \underset{1 \le i \le n}{\text{l. c. m.}} f(q_i^{\alpha_i}) = 2^{\max \gamma_i}K, \text{ and}$$

$$t(m) = s(m)/f(m) = 2^{\max(\gamma_i + \delta_i) - \max \gamma_i}$$

Now suppose $t(m) = 1$. Then $\max(\gamma_i + \delta_i) = \max \gamma_i$. Let $\gamma_k = \max \gamma_i$. We have

$$\gamma_k \le \gamma_k + \delta_k \le \max(\gamma_i + \delta_i) = \max \gamma_i = \gamma_k ,$$

and thus $\delta_k = 0$ and $t(q_k^{\alpha_k}) = 2^{\delta_k} = 1$. It follows from Theorem 2 that $4 \nmid f(q_k^{\alpha_k})$, that is, that $\gamma_k \le 1$. Then for all i,

$$\delta_i \le \max(\gamma_i + \delta_i) = \max \gamma_i = \gamma_k \le 1 .$$

Furthermore, $\delta_i = 1$ is impossible, for $\delta_i = 1$ is the same as $t(q_i^{\alpha_i}) = 2$ which implies, by Theorem 2, that $2 \; f(q_i^{\alpha_i})$ and thus $\gamma_i \ge 1$. Then we would have

$\gamma_i + \delta_i \geq 2$, which is contrary to $\max(\gamma_i + \delta_i) \leq 1$. Thus for all i, $\delta_i = 0$ and $t(q_i^{\alpha_i}) = 2^{\delta_i} = 1$, which, by Theorem 2, is equivalent to the conditions given in ii).

Now suppose, conversely, that the conditions given in ii) are satisfied, which, as we have just seen, is equivalent to the condition $t(q_i^{\alpha_i}) = 1$ for all i. Then

$$s(q_i^{\alpha_i}) = f(q_i^{\alpha_i})\, t(q_i^{\alpha_i}) \quad \text{for all } i.$$

Then Lemma 2 gives

$$s(m) = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{s(q_i^{\alpha_i})\} = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{f(q_i^{\alpha_i})\} = f(m)$$

and thus $t(m) = s(m)/f(m) = 1$.

Our last theorem is of rather different character. Once again, we need a preliminary lemma.

Lemma 3. Let p be an odd prime. Then

i) $f(p) \mid (p-1)$ if $p \equiv \pm 1 \pmod{10}$,
ii) $f(p) \mid (p+1)$ if $p \equiv \pm 3 \pmod{10}$,
iii) $s(p) \mid (p-1)$ if $p \equiv \pm 1 \pmod{10}$, and
iv) $s(p) \nmid (p+1)$ but $s(p) \mid 2(p+1)$ if $p \equiv \pm 3 \pmod{10}$.

Lucas [2, p. 297] gave the following result:

$$p \mid u_{p-1} \text{ if } p \equiv \pm 1 \pmod{10} \text{ and } p \mid u_{p+1} \text{ if } p \equiv \pm 3 \pmod{10}.$$

We get i) and ii) by applying (3) to this result. Wall [3, p. 528] has given iii) and iv).

Theorem 4. Let p be an odd prime and let e be any positive integer. Then

i) $t(p^e) = 1$ if $p \equiv 11$ or $19 \pmod{20}$,
ii) $t(p^e) = 2$ if $p \equiv 3$ or $7 \pmod{20}$,
iii) $t(p^e) = 4$ if $p \equiv 13$ or $17 \pmod{20}$, and
iv) $t(p^e) \neq 2$ if $p \equiv 21$ or $29 \pmod{40}$.

Proof: Theorem 2 shows that $t(p^e)$ is independent of the value of e, hence is sufficient to consider $e = 1$ throughout the proof.

If follows from the definition of ff(p) that $p \nmid u_{f(p)-1}$ so that by Fermat's theorem,

$$u_{f(p)-1}^{p-1} \equiv 1 \pmod{p}.$$

Then, since $u_{f(p)-1}$ belongs to $t(p) \pmod{p}$, it follows that $t(p) \mid (p-1)$. Now if $p \equiv 3 \pmod{4}$ we have $4 \nmid (p-1)$ and thus $t(p) \neq 4$.

i) Here $p \equiv 3 \pmod{4}$ so $t(p) \neq 4$. Suppose $t(p) = 2$. Then, by Theorem 2, $4 \mid f(p)$. Now $p \equiv \pm 1 \pmod{10}$ and, by Lemma 3 i), $f(p)\ (p-1)$ and thus $4 \mid (p-1)$. But this is impossible when $p \equiv 3 \pmod{4}$, hence $t(p) \neq 2$ and we must have $t(p) = 1$.

ii) Again $p \equiv 3 \pmod{4}$ and $t(p) \neq 4$. Also $p \equiv \pm 3 \pmod{10}$ and it follows from Lemma 3 that $s(p) \neq f(p)$ and $t(p) = s(p)/f(p) \neq 1$. Hence $t(p) = 2$.

iii) We have just seen that $t(p) \neq 1$ when $p \equiv \pm 3 \pmod{10}$, which is here the case. Also, $f(p) \mid (p+1)$. Now $p \equiv 1 \pmod{4}$ so that $4 \nmid (p+1)$ and thus $4 \nmid f(p)$, and it follows from Theorem 2 that $t(p) \neq 2$. Hence $t(p) = 4$.

iv) Suppose $t(p) = 2$. Then by Theorem 2, $4 \mid f(p)$ and thus $8 \mid s(p)$ (since $s(p) = t(p)f(p) = 2f(p)$). Furthermore, $s(p) \mid (p-1)$ since $p \equiv \pm 1 \pmod{10}$. Then $t(p) = 2$ implies $8 \mid (p-1)$. But we have $p - 1 \equiv 20$ or $28 \pmod{40}$ which gives $p - 1 \equiv 4 \pmod{8}$, so that $8 \mid (p-1)$ is impossible. Hence $t(p) \neq 2$.

We naturally ask if anything more can be said about $t(p^e)$ for $p \equiv 1, 9, 21, 29 \pmod{40}$. The following examples show that the theorem is "complete":

$p \equiv 1 \pmod{40}$	:	$t(521) = 1,$	$t(41) = 2,$	$t(761) = 4.$
$p \equiv 9 \pmod{40}$	:	$t(809) = 1,$	$t(409) = 2,$	$t(89) = 4.$
$p \equiv 21 \pmod{40}$	:	$t(101) = 1,$	$t(61) = 4.$	
$p \equiv 29 \pmod{40}$	:	$t(29) = 1,$	$t(109) = 4.$	

Now we might ask whether there is a number, m, for which $t(p^e)$ is always determined by the modulo m residue class to which p belongs. The answer to this question is not known. We note that the principles upon which the proof of Theorem 4 is based are not applicable to other moduli.

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1. R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, Ann. of Math. 15 (1913-1914), 30-70.
2. E. Lucas, Théorie des fonctions numériques simplement périodiques, Amer. J. Math. 1 (1878), 184-240, 289-321.
3. D. D. Wall, Fibonacci series modulo m, Amer. Math. Monthly 67 (1960), 525-532.

SPECIAL NOTICE

The Fibonacci Association has on hand 14 copies of Dov Jarden, Recurrent Sequences, Riveon Lematematika, Jerusalem, Israel. This is a collection of papers on Fibonacci and Lucas numbers with extensive tables of factors extending to the 385th Fibonacci and Lucas numbers. The volume sells for $5.00 and is an excellent investment. Check or money order should be sent to Verner Hoggatt at San Jose State College, San Jose, Calif.

REFERENCES TO THE QUARTERLY

Martin Gardner, Editor, Mathematics Games, Scientific American, June, 1963 (Column devoted this issue to the helix.)

A Review of The Fibonacci Quarterly will appear in the Feb. 1963 issue of the Recreational Mathematics Magazine.

FIBONACCI NEWS

Brother U. Alfred reports that he is currently offering a one unit course on Fibonacci Numbers at St. Mary's College.

Murray Berg, Oakland, Calif., reports that he has computed phi to some 2300 decimals by dividing F_{11004} by F_{11003} on a computer. Any inquiries should be addressed to the editor.

Charles R. Wall, Ft. Worth, Texas, reports that he is working on his master's thesis in the area of Fibonacci related topics.

SORTING ON THE B-5000 -- Technical Bulletin 5000-21004P Sept., 1961, Burroughs Corporation, Detroit 32, Michigan.

This contains in Section 3 the use of Fibonacci numbers in the merging of information using three tape units instead of the usual four thus effecting considerable efficiency. (This was brought to our attention by Luanne Anglemyer and the pamphlet was sent to us by Ed Olson of the San Jose office.)

S. L. Basin, Generalized Fibonacci Numbers and Squared Rectangles, American Mathematical Monthly, pp. 372-379, April, 1963.

J. A. H. Hunter and J. S. Madachy, Mathematical Diversions, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1963.

S. K. Stein, The Intersection of Fibonacci Sequences, Michigan Math. Journal, 9 (1962), Dec., No. 4, pp. 399-402. (Correction)

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A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, The American Mathematical Monthly, Mar., 1963, pp. 289-291.

S. L. Basin, The Appearance of Fibonacci Numbers and the Q-matrix in Electrical Network Theory, Math. Magazine, Vol. 36, No. 2, March 1963, pp. 84-97.

J. Browkin and A. Schinzel, On the Equation $2^n - D = y^2$, Bulletin de l'Academie Polonaise des Sciences, Serie des sci. math., astr. et phys.--Vol. VIII, No. 5, 1960, pp. 311-318.

Georges Browkin and Andre Schnizel, Sur les nombres de Mersenne qui sont triangulaires, Comptes rendues des seances de l'Academie des sciences, t. 242, pp. 1780-1781, seance du 4 Avril 1956.

C. D. Olds, Continued Fractions, Random House (New Mathematical Library Series--part of the Monograph Project of SMSG) 1963.

This is an excellent understandable treatment of the subject at a reasonable level with many interesting topics for those devoted to the study of integers with special properties.

L. Zippin, Uses of Infinity, Random House (New Mathematical Library Series, 1962.)

This has no index which makes the Fibonacci topics harder to find but there are several interesting comments there.

Mannis Charosh, Problem Department, Mathematics Student Journal, May, 1963.

In the editorial comment following the solution of Problem 187, there is a little generalized result similar to problem B-2 of the Elementary Problems and Solutions section of the Fibonacci Quarterly, Feb., 1963.

A. Rotkiewicz, On Lucas Numbers with Two Intrinsic Prime Divisors, Bulletin de l'Academie Polonaise des Sciences, Serie des sci. math., astr. et phys.--Vol. X., No. 5, 1962.

FIBONACCI MATRICES AND LAMBDA FUNCTIONS

MARJORIE BICKNELL AND VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE, SAN JOSE, CALIF.

When we speak of a Fibonacci matrix, we shall have in mind matrices which contain members of the Fibonacci sequence as elements. An example of a Fibonacci matrix is the Q matrix as defined by King in [1], pp. 11-27, where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The determinant of Q is -1, written det Q = -1. From a theorem in matrix theory,

$$\det Q^n = (\det Q)^n = (-1)^n .$$

By mathematical induction, it can be shown that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} ,$$

so that we have the familiar Fibonacci identity

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n .$$

The lambda function of a matrix was studied extensively in [2] by Fenton S. Stancliff, who was a professional musician. Stancliff defined the lambda function $\lambda(M)$ of a matrix M as the change in the value of the determinant of M when the number one is added to each element of M. If we define $(M + k)$ to be that matrix formed from M by adding any given number k to each element of M, we have the identity

(1) $$\det (M + k) = \det M + k\,\lambda(M).$$

For an example, the determinant $\lambda(Q^n)$ is given by

$$\lambda(Q^n) = \begin{vmatrix} F_{n+1} + 1 & F_n + 1 \\ F_n + 1 & F_{n-1} + 1 \end{vmatrix} - \det Q^n$$

$$= (F_{n+1}F_{n-1} - F_n^2) + (F_{n-1} + F_{n+1} - 2F_n) - \det Q^n$$

$$= F_{n-3}$$

which follows by use of Fibonacci identities. Now if we add k to each element of Q^n, the resulting determinant is

$$\begin{vmatrix} F_{n+1} + k & F_n + k \\ F_n + k & F_{n-1} + k \end{vmatrix} = \det Q^n + kF_{n-3} .$$

However, there are more convenient ways to evaluate the lambda function. For simplicity, we consider only 3 x 3 matrices.

THEOREM. For the given general 3 x 3 matrix M, $\lambda(M)$ is expressed by either of the expressions (2) or (3). For

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} ,$$

(2) $$\lambda(M) = \begin{vmatrix} a + e - (b + d) & b + f - (c + e) \\ d + h - (g + e) & e + j - (h + f) \end{vmatrix} ;$$

or

(3) $$\lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix} .$$

Proof: This is made by direct evaluation and a simple exercise in algebra.

An application of the lambda function is in the evaluation of determinants. Whenever there is an obvious value of k such that det (M + k) is easy to evaluate, we can use equation (1) advantageously. To illustrate this fact, consider the matrix

$$M = \begin{pmatrix} 1000 & 998 & 554 \\ 990 & 988 & 554 \\ 675 & 553 & 554 \end{pmatrix} .$$

We notice that, if we add $k = -554$ to each element of M, then $\det(M+k) = 0$ since every element in the third column will be zero. From (2) we compute

$$\lambda(M) = \begin{vmatrix} 0 & 10 \\ -120 & 435 \end{vmatrix} = 1200 ;$$

and from (1) we find that

$$0 = \det M + (-554)(1200) ,$$

so that $\det M = (554)(1200)$.

Readers who enjoy mathematical curiosities can create determinants which are not changed in value when any given number k is added to each element, by writing any matrix D such that $\lambda(D) = 0$.

LEMMA: If two rows (or columns) of a matrix D have a constant difference between corresponding elements, then $\lambda(D) = 0$.

Proof: Evaluate $\lambda(D)$ directly, by (2) or (3).

For example, we write the matrix D, where corresponding elements in the first and second rows differ by 4, such that

$$\det D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1+k & 2+k & 3+k \\ 5+k & 6+k & 7+k \\ 4+k & 9+k & 8+k \end{vmatrix} = 24$$

Now, we consider other Fibonacci matrices. Suppose that we want to write a Fibonacci matrix U such that $\det U = F_n$. Now

$$\begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & d \end{vmatrix} = abd.$$

We can write $F_n = F_1F_1F_n = F_1F_2F_n = F_2F_2F_n$ for any n, and for some n we will also have other Fibonacci factorizations. Hence, $F_n = \det U$ for

$$U = \begin{pmatrix} F_1 & F_0 & F_0 \\ F_m & F_2 & F_0 \\ F_k & F_p & F_n \end{pmatrix}$$

where $F_0 = 0$. If we choose $m = k = 3$ and $p = 2$, we find that $\lambda(U) = 0$. If we choose $m = 1$ or 2, $k = 1$ or 2, and let p be an arbitrary integer, then $\lambda(U) = F_n$.

A more elegant way to write such a matrix was suggested by Ginsburg in [3], who showed that if $a = F_{2p}$, $c = b = F_{2p+1}$, $d = e = F_{2p+2}$, and $f = F_{2p+3}$, then $\det B = n$, where

$$B = \begin{pmatrix} a & b & n \\ c & d & n \\ e & f & n \end{pmatrix}.$$

Letting $n = F_m$, we can write $F_m = \det U$, where

$$U = \begin{pmatrix} F_{2p} & F_{2p+1} & F_m \\ F_{2p+1} & F_{2p+2} & F_m \\ F_{2p+2} & F_{2p+3} & F_m \end{pmatrix}.$$

Using equation (3) we have

$$\lambda(U) = \begin{vmatrix} 1 & b & F_m \\ 1 & d & F_m \\ 1 & f & F_m \end{vmatrix} + \begin{vmatrix} a & 1 & F_m \\ c & 1 & F_m \\ e & 1 & F_m \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

$$= 0 + 0 + 1/F_m (\det U) = 1$$

If we let $k = F_{m-1}$, from (1) we see that

$$\det(U + F_{m-1}) = F_m + (F_{m-1})(1) = F_{m+1} .$$

Notice the possibilities for finding Fibonacci identities using the lambda function and evaluation of determinants. As a brief example, we let $k = F_n$ and consider det $(Q^n + F_n)$, which gives us

$$\begin{vmatrix} F_{n+1} + F_n & F_n + F_n \\ F_n + F_n & F_{n-1} + F_n \end{vmatrix} = \det Q^n + F_n \lambda(Q^n)$$

or

$$\begin{vmatrix} F_{n+2} & 2F_n \\ 2F_n & F_{n+1} \end{vmatrix} = (-1)^n + F_n F_{n-3}$$

so that

$$4F_n^2 = F_{n+2}F_{n+1} - F_n F_{n-3} + (-1)^{n+1} .$$

As a final example of a Fibonacci matrix, we take the matrix R, given by

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} .$$

which has been considered by Brennan [4].

It can be shown by mathematical induction that

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}$$

The reader may verify by equation (2) and by Fibonacci identities that

$$\lambda(R^n) = \begin{vmatrix} F^2_{n-1} + F^2_{n+1} - 4F_{n-1}F_n & 2F_{n-1}F_n + 2F_nF_{n+1} - F^2_n - F^2_{n+1} \\ 3F_{n-1}F_n - F^2_n - F^2_{n+1} + F_nF_{n+1} & 2F^2_{n+1} - 3F_nF_{n+1} - F_nF_{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} F_{2n-3} & F_{2n-2} \\ -F^2_{n-2} & -F_{n-2}F_{n-1} + (-1)^n \end{vmatrix} = (-1)^n(F^2_{n-1} - F_{n-3}F_{n-2}) .$$

Here we see that the value of (R^n) is the center element of R^{n-2} multiplied by $(-1)^n$.

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1. Charles H. King, Some Properties of the Fibonacci Numbers, (Master's Thesis) San Jose State College, June, 1960.
2. From the unpublished notes of Fenton S. Stancliff.
3. Jukethiel Ginsburg, "Determinants of a Given Value," Scripta Mathematica, Vol. 18, issues 3-4, Sept.-Dec., 1952, p. 219.
4. From the unpublished notes of Terry Brennan.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-9 *Proposed by Olga Taussky, California Institute of Technology, Pasadena, California*

Find the numbers $a_{n,r}$, where $n \geq 0$ and r are integers, for which the relations

$$a_{n,r} + a_{n,r-1} + a_{n,r-2} = a_{n+1,r}$$

and

$$a_{0,r} = \delta_{o,r} = \begin{cases} 0 & r \neq 0 \\ 1 & r = 0 \end{cases}$$

hold.

H-10 *Proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey*

Show that $$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n F_{n+1} F_{n+2}}$$

H-11 *Proposed by John L. Brown, Jr., Ordnance Research Laboratory, The Pennsylvania State University, University Park, Penna.*

Find the function whose formal Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{F_n \sin nx}{n!},$$

where F_n is the nth Fibonacci number.

H-12 *Proposed by D. E. Thoro, San Jose State College, San Jose, California*

Find a formula for the nth term in the sequence:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, \ldots$$

H-13 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va., and Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Show that

$$F_n = \sum_{j=0}^{r} \binom{r}{j} F_{k-1}^{r-j} F_k^j F_{n+j-rk} .$$

H-14 *Proposed by David Zeitlin, Minneapolis, Minnesota, and F. D. Parker, University of Alaska, College, Alaska.*

Prove the Fibonacci identity

$$F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0$$

H-15 *Proposed by Malcolm H. Tallman, Brooklyn, New York*

Do there exist integers N_1, N_2, and N_3 for which the following expressions cannot equal other Fibonacci numbers?

(i) $F_n^3 - F_n^2 F_m - F_m^3 \qquad m, n \geq N_1$,

(ii) $F_n^3 + F_n^2 F_m + F_n F_m^2 \qquad m, n \geq N_2$,

(iii) $F_n^2 - 3F_m^3 \qquad m, n \geq N_3$.

H-16 *Proposed by H. W. Gould, West Virginia University, Morgantown, West Virginia*

Define the ordinary Hermite polynomials by $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$.

Show that:

(i) $$\sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} = 1 ,$$

(ii) $$\sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} F_n = 0 ,$$

(iii) $$\sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} L_n = 2e^{-x^2} ,$$

where F_n and L_n are the nth Fibonacci and nth Lucas numbers, respectively.

H-17 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Sum: $$\sum_{k=1}^{n} k^3 F_k$$

H-18 *Proposed by R. G. Buschman, Oregon State University, Corvallis, Ore.*

"Symbolic relations" are sometimes used to express identities. For example, if F_n and L_n denote, respectively, Fibonacci and Lucas numbers, then

$$(1 + L)^n \doteq L_{2n}, \qquad (1 + F)^n \doteq F_{2n}$$

are known identities, where $\doteq$ denotes that the exponents on the symbols are to be lowered to subscripts after the expansion is made.

(a) Prove $(L + F)^n = (2F)^n$.

(b) Evaluate $(L + L)^n$.

(c) Evaluate $(F + F)^n$.

(d) How can this be suitably generalized?

NOTE: On occasion there will be problems listed at the ends of the articles in the advanced and elementary sections of the magazine. These problems are to be considered as logical extensions of the corresponding problem sections and solutions for these problems will be discussed in these sections as they are received.

See, for example, "Expansion of Analytic Functions In Polynomials Associated with Fibonacci Numbers," by Paul F. Byrd, San Jose State College, in the first issue of the Quarterly, and "Linear Recurrence Relations – Part I," by James Jeske, San Jose State College, in this issue.

Solutions for problems in ISSUE ONE will appear in ISSUE THREE.

A convenient method of generating Fibonacci numbers is the alternating subtotal exchange and add procedure which is easily performed on the Olivetti Tetractys desk calculator.

The Olivetti Tetractys has two separate accumulating registers so arranged that the contents of either can be added to the other by manual commands and without rewriting numbers into the input keyboard. The detailed procedure for generating any Fibonacci sequence on the Tetractys follows:

Set automatic total lever (14) to left and set automatic accumulation lever (29) to up position.

a) Clear the registers by depressing the green and black total keys.

b) Write the value of the initial term of the sequence in the 10-key keyboard.

c) Depress the green add key. The initial value now printed on the tape in blue ink is the sequence title but is not the first term.

d) Advance the paper tape 3 lines.

e) Depress simultaneously the green sub-total key and the black add key. The initial term of the sequence is now printed on the tape in red ink.

f) Depress simultaneously the black sub-total key and the green add key. The second term of the sequence is now printed on the tape in red ink.

g) Repeat step (e) for the third term.

h) Repeat step (f) for the fourth term.

i) Continue alternating steps (e) and (f) for as many terms as desired up to the 13 decimal digit capacity of the arithmetic registers.

j) The sequence can be continued beyond the 13 digit limit by clearing the registers, step (a), and rewriting the required most significant digits of the last term obtained as a new initial value, restarting at step (b).

It takes a little practice to develop the manual knack of simultaneously depressing the adjacent sub-total and add keys. A firm push is necessary but it must not last too long or the operation will be done twice, producing an error.

In generating long sequences it is a read-out convenience to depress the non-add key after every fifth term. This provides a blank line and makes it easy to count terms.

It is difficult, but not impossible to make errors. The usual errors consist in skipping a step or doing a step twice, which amount to the same thing.

The printed tape can be checked for errors in two ways.

1) Each term approximates 1.6 times the preceding term.

2) The + symbols on the far right side of the printed tape should alternate between the two symbol columns. If two successive + signs fall in the same vertical symbol column, an error was made at that point.

(Continued on p. 88)

PART II

BEGINNERS' CORNER

EDITED BY DMITRI THORO, SAN JOSE STATE COLLEGE

DIVISIBILITY II

We shall continue our investigation of some "background material" for the beginning Fibonacci explorer. Whenever necessary, we may assume that the integers involved are not negative (or zero).

1. DEFINITIONS

Two integers a and b are <u>relatively prime</u> if their greatest common divisor (g. c. d.) is 1. When convenient we will use the customary abbreviation <u>(a, b)</u> to designate the g. c. d. of a and b. Finally, as previously implied, we shall say that n is <u>composite</u> if n has more than two divisors.

2. ILLUSTRATIONS

<u>E1</u>. If $d \mid a$ and $d \mid b$, then d $(a \pm b)$; i. e., a common divisor of two numbers is a divisor of their sum or difference.

PROOF: $d \mid a$ means there exists an integer a such that $a = a'd$. Similarly, we may write $b = b'd$. Thus $a \pm b = d\,(a' \pm b')$ which proves that $d \mid (a \pm b)$. [This follows from the <u>definition</u> of divisibility.]

<u>E2</u>. If $d = (a,b)$, then $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$; i. e., if two numbers are divided by their g. c. d., then the quotients are relatively prime. [This result is a widely used "tool."]

PROOF: Since d is the g. c. d. of a and b, $\frac{a}{d}$ and $\frac{b}{d}$ are certainly integers; let us call them a' and b', respectively. Thus $a = a'd$, $b = b'd$, and we are to show that $(a', b') = 1$. The trick is to use an <u>indirect</u> argument.

Suppose $(a', b') = d' > 1$. Then (there exist integers a'' and b'' such that) $a' = a''d'$ and $b' = b''d'$. This implies $a = a'd = a''d'd$ and $b = b'd = b''d'd$; i. e., d'd is a <u>common</u> <u>divisor</u> of a and b. But we assumed d' 1, which means $dd' > d$ — contrary to the fact that d is the <u>greatest</u> common divisor of a and b.

<u>E3</u>. If a and b are relatively prime, what can you say about the g. c. d. of $a + b$ and $a - b$? For example, $(13 + 8,\ 13 - 8) = 1$, but $(5 - 3,\ 5 - 3) = 2$. It turns out, however, that these are the only possibilities!

If $(a,b) = 1$, then $(a + b, a - b) = 1$ or 2.

PROOF: Let $(a + b, a - b) = d$. Then by E1, $d \mid [(a + b) \pm (a - b)]$; i.e., $d \mid 2a$ and $d \mid 2b$.

(i) If d is even, set $d = 2K$. Then from $2a = 2Ka'$, $2b = 2Kb'$ we have $a = Ka'$, $b = Kb'$. Therefore K must be 1 (why?), and hence $d = 2$.

(ii) Similarly if d is odd, then d would have to divide both a and b, whence $d = 1$.

Can you see what objection a pedantic reader might have to this proof?

E4. The following is a special case of a result due to Sophie Germain, a French mathematician (1776–1831).

$$n^4 + 4 \text{ is composite for } n > 1 .$$

PROOF: Unlike the preceding illustrations, here one needs to stumble onto a factorization. $n^4 + 4 = (n^2 + 2)^2 - (2n)^2 = (n^2 + 2 + 2n)(n^2 + 2 - 2n)$ does the trick, for this shows that $N = n^4 + 4$ is divisible by a number between 1 and N and hence must be composite.

E5. We now consider one of the simplest properties of Fibonacci numbers.

Two consecutive Fibonacci numbers are always relatively prime.

PROOF: Certainly this is obvious for the first few numbers: 1, 1, 2, 3, 5, 8, . Let us use an indirect argument. Suppose F_n and F_{n+1} is the first pair for which $(F_n, F_{n+1}) = d > 1$. Now examine the pair F_{n-1} and F_n. Since $F_{n+1} - F_n = F_{n-1}$, d is a divisor of F_{n-1} (for $d \mid F_{n+1}$, $d \mid F_n$ and hence, by E1, their difference). This means that F_{n-1} and F_n are not relatively prime — a contradiction to our assumption that F_n and F_{n+1} is the first such pair.

3. SOME USEFUL THEOREMS

T1. Any composite integer n has at least one prime factor.

PROOF: (i) Since n is composite, it must have at least one divisor greater than 1 and less than n.

(ii) Let d be the smallest divisor of n such that $1 < d < n$.

(iii) Suppose d is composite; let $d' \mid d$, $1 < d' < d$.

(iv) Thus we have $n = n_1 d = n_1 d' d''$; i.e., $d' \mid n$ but $d < d'$ — a contradiction to the definition of d. Therefore d must be a prime.

T2. Given $n > 1$. Suppose that the quotient q, in the division of n by a, is less than a. If n is not divisible by 2, 3, 4, , $(a - 1)$, a, then n is a prime.

PROOF: (i) If q ($q < a$) is the quotient and r the remainder in the division of n by a, we may write

$$\frac{n}{a} = q + \frac{r}{a}, \quad 0 \le r < a .$$

(ii) Assume that n is not divisible by 2, 3,$\cdots$, a but has a divisor d, $1 < d < n$. We shall show that this leads to a contradiction.
(iii) Since $d|n$, $n = dd'$, where $1 < d' < n$.
(iv) By (ii), $d' > a$ and by (i) $a > q$; hence $d' > q$ or $d' \ge q + 1$. Also $d > a$; multiplying $d' \ge q + 1$ by $d > a$, we arrive at
(v) $n = dd' > aq + a$. But by (i), $n = aq + r < aq + a$ since $r < a$. This is the desired contradiction which proves that n cannot be divisible by d, $1 < d < n$, and hence must be a prime.

T3. If $n > 1$ is not divisible by $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7, \cdots, p_k$, where p_k is the largest prime whose square does not exceed n, then n is a prime.

PROOF: Assume $a^2 \le n < (a + 1)^2$ and that n is not divisible by 2, 3, $\cdots$, $(a - 1)$, a. Then $n = dd'$ implies $d \ge a + 1$, $d' \ge a + 1$, whence $n = dd' \ge (a+1)^2$ — a contradiction. Thus n must be prime. The reader should convince himself that here (as well as in T2) it suffices to consider only prime divisors $\le \sqrt{n}$.

4. PROBLEMS

1.1 Suppose that

(i) p is the smallest prime factor of n and

(ii) $p > \sqrt[3]{n}$.

What interesting conclusion can you draw?

1.2 Prove that two consecutive Fibonacci numbers are relatively prime by using one of the identities on p. 66 (Fibonacci Quarterly, February, 1963).

1.3 Prove that if p and $p + 2$ are (twin) primes, then $p + 1$ is divisible by 6. (Assume $p > 3$.) [This problem was suggested by James Smart.]

1.4 Prove that if n is divisible by k, $1 < k < n$, then $2^n - 1$ is divisible by $2^k - 1$. For example, $2^{35} - 1 = 34\ 359\ 738\ 367$ is divisible by $2^5 - 1 = 31$ and $2^7 - 1 = 127$.

1.5 Prove that there are infinitely many primes. Hint: Assuming that p_n is the largest prime, Euclid considered the expression $N = 1 + 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_n$. Now either N is prime or N is composite. Complete his proof by investigating the consequences of each alternative.

Additional hints may be found on p. 80.

FIBONACCI FORMULAS

Maxey Brooke, Sweeny, Texas

If you have a favorite Fibonacci formula, send it to us and we will try to publish it. Some historically interesting ones are shown below.

1. Perhaps the first Fibonacci formula was developed by Simpson in 1753.
$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$
2. A very important formula was developed in 1879 by an obscure French mathematician, Aurifeuille. In fact, it is his one claim to fame.
$$L_{5n} = L_n(L_{2n} - 5F_n + 3)(L_{2n} + 5F_n + 3)$$
3. The only formula involving cubes of Fibonacci numbers given in Dickson's "History of the Theory of Numbers" is due to Lucas.
$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$$
The late Jekuthiel Ginsburg offers $F_{n+2}^3 - 3F_n^3 + F_{n-1}^3 = 3F_{3n}$.
4. The recursion formula for sub-factorials is similar to the one for Fibonacci numbers: $P_{n+1} = n(P_n + P_{n-1})$; $P_0 = 1$, $P_1 = 0$.
5. Fibonacci numbers have been related to almost every other kind of number. Here is H. S. Vandiver's relation with Bernoulli numbers.
$$\sum_{k=0}^{\overline{p-3}} B_{2k}F_{2k} \equiv \frac{1}{2} \pmod{p} \quad \text{if } p = 5a \pm 1$$
$$\sum_{k=0}^{\overline{p-3}} B_{2k}F_{2(k-1)} \equiv 1 \pmod{p} \quad \text{if } p = 5a \pm 2$$

$\overline{p-3}$ denotes the greatest integer not exceeding $(p - 3)/2$.

I think that this is a good idea.

Ed.

A PRIMER ON THE FIBONACCI SEQUENCE – PART II

S. L. BASIN AND VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

1. INTRODUCTION

The proofs of existing Fibonacci identities and the discovery of new identities can be greatly simplified if matrix algebra and a particular 2 x 2 matrix are introduced. The matrix approach to the study of recurring sequences has been used for some time [1] and the Q matrix appeared in a thesis by C. H. King [2]. We first present the basic tools of matrix algebra.

THE ALGEBRA OF (TWO-BY-TWO) MATRICES

The two-by-two matrix A is an array of four elements a, b, c, d:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The zero matrix, Z, is defined as,

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The identity matrix, I, is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix C, which is the <u>matrix sum</u> of two matrices A and B, is

$$C = A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

The matrix P, which is the <u>matrix product</u> of two matrices A and B, is

$$P = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

The determinant D(A) of matrix A is

$$D(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Two matrices are equal if and only if the corresponding elements are equal; that is,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = B ,$$

if and only if, $a = e$, $b = f$, $c = g$, $d = h$.

A SIMPLE THEOREM

The determinant, D(P), of the product, $P = AB$, of two matrices A and B is the product of the determinants D(A) and D(B)

$$D(P) = D(AB) = D(A)\,D(B)$$

The proof is left as a simple exercise in algebra.

THE Q MATRIX

The Q matrix and the determinant of Q, D(Q), are:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D(Q) = -1 .$$

If we designate $Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, then

$$Q = Q^1 = Q^0 Q = IQ = QI = QQ^0 .$$

DEFINITION: $Q^{n+1} = Q^n Q^1$, an inductive definition where $Q^1 = Q$. This is the law of exponents for matrices.

It is easily proved by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} ,$$

where F_n is the nth Fibonacci number, and the determinant of Q^n is

$$D(Q^n) = D^n(Q) = (-1)^n .$$

MORE PROOFS

We may now prove several of the identities very nicely. Let us prove identity III (given in Part I), that is,

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n .$$

Proof:

$$\text{If} \quad Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \text{ and } D(Q^n) = (-1)^n ,$$

then

$$D(Q^n) = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1} F_{n-1} - F_n^2 - (-1)^n$$

Let us prove identity VI

$$F_{2n+1} = F_{n+1}^2 + F_n^2 .$$

since

$$Q^{n+1} Q^n = Q^{2n+1},$$

then

$$Q^n\,Q^{n+1} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

$$= \begin{pmatrix} F_{n+1}F_{n+2} + F_nF_{n+1} & F_{n+1}^2 + F_n^2 \\ F_nF_{n+2} + F_{n-1}F_{n+1} & F_nF_{n+1} + F_{n-1}F_n \end{pmatrix}$$

But this is also

$$Q^{2n+1} = \begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Since these two matrices are equal we may equate corresponding elements so that

$$F_{2n+2} = F_{n+1}F_{n+2} + F_nF_{n+1} \qquad \text{(Upper Left)}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \qquad \text{(Upper Right)}$$

$$F_{2n+1} = F_nF_{n+2} + F_{n-1}F_{n+1} \qquad \text{(Lower Left)}$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n \qquad \text{(Lower Right)}$$

$$= F_n(F_{n+1} + F_{n-1})$$

If we accept identity V: $L_n = F_{n+1} + F_{n-1}$, then

$$F_{2n} = F_nL_n$$

which gives identity VIII. Return again to

$$F_{2n} = F_n(F_{n+1} + F_{n-1})$$

From $F_{k+2} = F_{k+1} + F_k$, for $k = n - 1$, one can write $F_n = F_{n+1} - F_{n-1}$, thus also

$$F_{2n} = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_n^2$$

which is identity VII.

It is a simple task to verify

$$Q^2 = Q + I$$

and

$$Q^{n+2} = Q^{n+1} + Q^n ,$$

and

$$Q^n = Q F_n + I F_{n-1} ,$$

where F_n is the nth Fibonacci number and the multiplication of matrix A, by a number q, is defined by

$$qA = q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq & bq \\ cq & dq \end{pmatrix} .$$

GENERATION OF FIBONACCI NUMBERS BY LONG DIVISION

$$\frac{1}{1 - x - x^2} = F_1 + F_2 x + F_3 x^2 + \cdots + F_n x^{n-1} + \ldots$$

In the process of long division below

$$1 - x - x^2 \overline{\big) \quad 1 \quad}$$

there is no ending. As far as you care to go the process will yield Fibonacci Numbers as the coefficients.

F_n AS A FUNCTION OF ITS SUBSCRIPT

It is not difficult to show by mathematical induction that

$$P(n): \quad F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}$$

This can be derived in many ways. P(1) and P(2) are clearly true. From $F_k = F_{k-1} + F_{k-2}$ and the inductive assumption that P(k-2) and P(k-1) are true, then

$$\text{(a)} \qquad F_{k-2} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{k-2} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-2} \right\}$$

$$\text{(b)} \qquad F_{k-1} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \right\}$$

Adding, after a simple algebra step, we get

$$F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{k-2} \left(\frac{1+\sqrt{5}}{2} + 1 \right) - \left(\frac{1-\sqrt{5}}{2} \right)^{k-2} \left(\frac{1-\sqrt{5}}{2} + 1 \right) \right\}$$

Observing that

$$\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2} \right)^2$$

$$\frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2} \right)^2$$

it follows simply that if (a) and (b) are true (P(k - 2) and P(k - 1) are true), then for $n = k$,

$$P(k): \quad F_k = F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right\}$$

The proof is complete by mathematical induction. Similarly it may be shown that

$$L_n = F_{n+1} + F_{n-1}$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

Let us now prove identity VIII

$$F_{2n} = F_n L_n$$

Proof:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]^2 - \left[\left(\frac{1-\sqrt{5}}{2}\right)^n\right]^2\right\}$$

Now factoring:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

$$F_{2n} = F_n L_n .$$

MORE IDENTITIES

XIV. $$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

XV. $$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

XVI. $$F_1^3 + F_2^3 + \cdots + F_n^3 = \frac{F_{3n+2} + (-1)^{n+1}6F_{n-1} + 5}{10}$$

XVII. $$1 \cdot F_1 + 2F_2 + 3F_3 + \cdots + nF_n = (n+1)F_{n+2} - F_{n+4} + 2$$

XVIII. $F_2 + F_4 + \qquad + F_{2n} = F_{2n+1} - 1$

XIX. $F_1 F_2 + F_2 F_3 + F_3 F_4 + \cdots + F_{n-1} F_n = \frac{1}{2}(F_{n+2} - F_n F_{n-1}) - 1$

XX. $\sum_{i=0}^{n} \binom{n}{i} F_{n-1} = F_{2n}$,

where $$\binom{n}{i} = \frac{n!}{(n-i)!\,i!} \quad \text{and} \quad m! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot m \ .$$

XXI. $F_{3n+3} = F_{n+1}^3 + F_{n+2}^3 - F_n^3$

XXII. $F_n F_m - F_{n-k} F_{m+k} = (-1)^{n-k} F_k F_{m+k-n}$

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1. J. S. Frame, "Continued fractions and matrices," Amer. Math. Monthly, Feb. 1949, p. 38.
2. Charles H. King, 'Some properties of the Fibonacci numbers,' Master's Thesis, San Jose State College, June, 1960.

REQUEST

The Fibonacci Bibliographical Research Center desires that any reader finding a Fibonacci reference send a card giving the reference and a brief description of the contents. Please forward all such information to:

Fibonacci Bibliographical Research Center,
Mathematics Department,
San Jose State College,
San Jose, Calif.

SPECIAL NOTICE

The Fibonacci Association has on hand 14 copies of Dov Jarden, Recurrent Sequences, Riveon Lematematika, Jerusalem, Israel. This is a collection of papers on Fibonacci and Lucas numbers with extensive tables of factors extending to the 385th Fibonacci and Lucas numbers. The volume sells for \$5.00 and is an excellent investment. Check or money order should be sent to Verner Hoggatt at San Jose State College, San Jose, California.

LINEAR RECURRENCE RELATIONS – PART I

JAMES A. JESKE, SAN JOSE STATE COLLEGE

1. INTRODUCTION

Most of the special sequences, which appear in The Fibonacci Quarterly, satisfy a type of equation called a recurrence relation i.e., a difference equation whose independent variable is restricted to integral values. Although there arc several good textbooks (e.g., see [1], [2], [3] or [4]) which present various methods of solution for many such equations, the beginner may not be acquainted with any of them, and in fact is likely to have more knowledge of the theory of differential equations than of that of recurrence relations.

The purpose of this series of articles is to introduce the beginner to the subject, and to derive explicit expressions for the solution of certain general, linear recurrence relations by applying a generating function transformation. The particular generating function chosen is seldom used in the treatment of recurrence relations, but for the purpose of developing general formulas it has the advantage of immediately transforming the problem to a more familiar one involving differential equations, for which there is already available a great wealth of special formulas and techniques.

2. DEFINITIONS

A linear recurrence relation of order k is an equation of the form

$$\sum_{j=0}^{k} a_{j,n}\, y_{n+j} = b_n \quad , \tag{2.1}$$

where $a_{0,n}, a_{1,n}, \cdots, a_{k,n}$ and b_n are given functions of the independent variable n over the set of consecutive non-negative integers S, and $a_{0,n}a_{k,n} \neq 0$ on S. If $b_n \equiv 0$, the relation is called homogeneous, otherwise it is said to be non-homogeneous. We may introduce the translation operator E^m, defined by

$$E^m y_n = y_{n+m} \quad (m = 0, 1, \cdots, k) \quad , \tag{2.2}$$

and thus we can write (2.1) as

$$L_k(E)\, y_n = b_n \quad , \tag{2.3}$$

where the linear operator $L_k(E)$ is

$$(2.4) \qquad L_k(E) = \sum_{j=0}^{k} a_{j,n} E^j \quad .$$

A sequence whose terms are v_n is a solution to the recurrence relation on the set S if the substitution $y_n = v_n$ reduces relation (2.3) to an identity on S.

If a set of k successive initial values $y_0, y_1, \cdots, y_{k-1}$ is given arbitrarily, equation (2.1) or (2.3) enables us to extend this set to $k+1$ successive values. Using mathematical induction, it can be easily established that the recurrence relation (2.3) over the set S of consecutive non-negative integers has one and only one solution for which the k values are prescribed.

3. A SERIES TRANSFORM

For the sequence $\{y_n\}$, $n = 0, 1, 2, \cdots$, we introduce the exponential generating function defined by tne infinite series

$$(3.1) \qquad Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} \quad ,$$

which we suppose is convergent for some positive value of t. From (3.1) we find the derived series

$$(3.2) \qquad \frac{d^jY}{dt^j} \equiv Y^{(j)}(t) = \sum_{n=0}^{\infty} y_{n+j} \frac{t^n}{n!} \qquad (j = 0, 1, \cdots, k).$$

These series of course have the same radius of convergence as (3.1), and are seen as the generating functions of the sequences $\{y_{n+j}\}$, $j = 0, 1, \cdots, k$. Now from (3.1), we have the inverse transform

$$(3.3) \qquad y_n = Y^{(n)}(0) = \left[\frac{d^n}{dt^n} Y(t)\right]_{t=0} \qquad (n = 0, 1, 2, \cdots, k) \; .$$

The relations (3.2) and (3.3) follow from known properties of Taylor series.

4. EXPLICIT SOLUTION OF A LINEAR RECURRENCE RELATION

We shall now derive the formula for the general solution to the linear homogeneous recurrence relation

$$\sum_{j=0}^{k} a_j y_{n+j} \equiv L_k(E) y_n = 0 \tag{4.1}$$

with constant coefficients. (Discussion of the non-homogeneous case will appear in the next issue of this journal.) The derivation is based on the application of the exponential generating function (3.1) which transforms the recurrence relation into a more familiar differential equation.

Multiplying both sides of (4.1) by $t^n/n!$ and summing over n from 0 to ∞, we thus obtain the <u>transformed equation</u>

$$\sum_{j=0}^{k} a_j Y^{(j)}(t) = L_k(D) Y = 0 , \quad \left(D \equiv \frac{d}{dt}\right) , \tag{4.2}$$

which is an ordinary linear differential equation of order k. Now it is well known[1] that, if $r_1, r_2, \cdots, r_k$ are k distinct roots of the characteristic equation

$$L_k(r) = 0, \tag{4.3}$$

then the general solution of (4.2) is given by

$$Y(t) = \sum_{i=1}^{k} c_i e^{r_i t} , \tag{4.4}$$

where c_i are k arbitrary constants. Application of the inverse transform (3. then yields immediately the explicit formula

[1]See for example, almost any textbook on ordinary differential equations.

(4.5) $$y_n = \sum_{i=1}^{k} c_i r_i^n$$

for the general solution of the recurrence relation (4.1).

In the case where the characteristic equations $L_k(r) = 0$ possesses m distinct roots $r_1, r_2, \cdots, r_m$ and each root r_i being of multiplicity m_i $(i = 1, \cdots, m)$, with

(4.6) $$\sum_{i=1}^{m} m_i = k ,$$

the differential equation (4. 2) is known to have the general solution

(4.7) $$Y(t) = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j ,$$

where b_{ij} are k arbitrary constants. Applying the inverse transform (3.3), we then obtain the general solution

(4.8) $$y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i - 1} b_{ij} n^j$$

to the recurrence relation (4.1).

In Part II of this article, we shall not only derive an explicit formula for the general solution of the non-homogeneous linear recurrence relation with constant coefficients, but shall also show how the method employing the exponential generating function may solve certain recurrence relations having variable coefficients.

5 EXAMPLE

The Fibonacci numbers satisfy the second-order recurrence relation

(5.1) $$F_{n+2} - F_{n+1} - F_n = 0, \qquad F_0 = 0, \quad F_1 = 1$$

or

$$(5.2)\qquad L_2(E)\,F_n = 0 \quad ,$$

where

$$(5.3)\qquad L_2(E) = E^2 - E - 1.$$

The characteristic equation $L_2(r) = 0$ has the distinct roots

$$(5.4)\qquad r_1 = (1 + \sqrt{5})/2, \qquad R_2 = (1 - \sqrt{5})/2 \quad ,$$

so that the formula (4.5) immediately yields

$$(5.5)\qquad F_n \equiv y_n = c_1 r_1^n + c_2 r_2^n \quad .$$

Now since $F_0 = 0$, $F_1 = 1$, we obtain $c_1 = -c_2 = 1/\sqrt{5}$; hence the general solution for the Fibonacci sequence is expressed by

$$(5.6)\qquad F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right] \quad .$$

We note from (4.2) that the transformed equation for (5.1) is the second-order differential equation

$$(5.7)\qquad Y'' - Y' - Y = 0, \quad Y(0) = 0, \quad Y'(0) = 1 \quad .$$

Hence the exponential generating function for the Fibonacci sequence is

$$(5.8)\qquad Y(t) = \left[e^{r_1 t} - e^{r_2 t}\right]\Big/\sqrt{5} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \quad ,$$

while the well-known ordinary generating function for this sequence is

$$(5.9) \qquad W(t) = \frac{t}{1 - t - t^2} = \sum_{n=0}^{\infty} F_n t^n \quad .$$

The two generating functions W(t) and Y(t) are related by the expression

$$(5.10) \qquad W(t) = \int_0^{\infty} e^{-z}\, Y(tz)\, dz \quad .$$

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1. L. M. Milne-Thompson, The Calculus of Finite Differences, London, 1933.
2. C. Jordan, Calculus of Finite Differences, New York, 2nd Ed., 1947.
3. S. Goldberg, Introduction to Difference Equations, New York, 1958.
4. G. Boole, Calculus of Finite Differences, New York, 4th Ed., 1926.

PROBLEM DEPARTMENT

P-1. The recurrence relation for the sequence of Lucas numbers is

$$L_{n+2} - L_{n+1} - L_n = 0 \text{ with } L_1 = 1, \ L_2 = 3 \ .$$

Find the transformed equation, the exponential generating function, and the general solution.

P-2. Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0 \ ,$$

with $y_0 = 0$, $y_1 = 0$, $y_2 = -1$.

REQUEST

Maxey Brooke would like any references suitable for a Lucas bibliography. His address is 912 Old Ocean Ave., Sweeny, Tex.

SOME FIBONACCI RESULTS USING FIBONACCI-TYPE SEQUENCES

I. DALE RUGGLES, SAN JOSE STATE COLLEGE

The elements of the Fibonacci sequence satisfy the recursion formula, $F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$ and $F_1 = 1$. Let us define an F-sequence as one for which the recursion formula $u_{n+1} = u_n + u_{n-1}$ holds for the elements u_n of the sequence.

Suppose $\{u_n\}$ and $\{v_n\}$ are two F-sequences. Then a linear combination, $\{cu_n + dv_n\}$, is also an F-sequence. If the determinant

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0 \; ,$$

then by an application of a theorem from algebra, every F-sequence can be expressed as a unique linear combination of the F-sequences $\{u_n\}$ and $\{v_n\}$.

Consider the sequence $1, \gamma, \gamma^2, \gamma^3, \cdots$. This will be an F-sequence if $\gamma^{n+1} = \gamma^n + \gamma^{n-1}$ for all integers n; that is, for γ such that $\gamma^2 = \gamma + 1$. This equation has solutions which we will denote by $\beta = \frac{1 + \sqrt{5}}{2}$ and $\alpha = \frac{1 - \sqrt{5}}{2}$. Thus, the α-sequence $1, \alpha, \alpha^2, \cdots$ and the β-sequence $1, \beta, \beta^2,$ are F-sequences. These can be extended to include negative integer exponents as well.

Since

$$\begin{vmatrix} \beta & \beta^2 \\ \alpha & \alpha^2 \end{vmatrix} = \beta - \alpha = \sqrt{5} \quad ,$$

every F-sequence can be written as a unique linear combination of the α-sequence and the β-sequence. (Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$.)

In particular this applies to the Fibonacci sequence. From the equations

$$F_1 = 1 = c\alpha + d\beta$$

$$F_2 = 1 = c\alpha^2 + d\beta^2$$

one finds that $c = -1/\sqrt{5}$ and $d = 1/\sqrt{5}$. Thus,

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = \frac{\beta^n - \alpha^n}{\sqrt{5}}$$

The F-sequence with $L_1 = 1$ and $L_2 = 3$ is known as the Lucas sequence. In the case of the Lucas sequence,

$$L_n = \beta^n + \alpha^n \quad .$$

The α- and β-sequences can be used to prove many well-known relations involving Fibonacci numbers, Lucas numbers, and general F-sequences:

1. Since

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}$$

and $L_n = \beta^n + \alpha^n$ then it follows immediately that

since $$F_n \cdot L_n = \frac{\beta^{2n} - \alpha^{2n}}{\beta - \alpha} = F_{2n} \quad .$$

2. Since $\beta^{n+1} + \beta^{n-1} = \beta^n(\beta + \beta^{-1}) = \beta^n(\beta - \alpha)$ and $\alpha^{n+1} + \alpha^{n-1} = \alpha^n(\alpha - \beta)$, it follows that $\beta^{n+1} - \alpha^{n+1} + \beta^{n-1} - \alpha^{n-1} = (\beta - \alpha)(\beta^n + \alpha^n)$; thus $L_n = F_{n+1} + F_{n-1}$. Also, $L_{n+1} + L_{n-1} = 5F_n$ can be similarly shown.

3. Let $\{u_n\}$ be an F-sequence, such that $u_n = c\alpha^n + d\beta^n$. Then the determinant

$$\begin{vmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{vmatrix}$$

can be simplified as follows:

$$\begin{aligned} \begin{vmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{vmatrix} &= \begin{vmatrix} c\alpha^{n+1} + d\beta^{n+1} & c\alpha^n + d\beta^n \\ c\alpha^n + d\beta^n & c\alpha^{n-1} + d\beta^{n-1} \end{vmatrix} \\ &= cd \begin{vmatrix} \alpha^{n+1} & \beta^n \\ \alpha^n & \beta^{n-1} \end{vmatrix} + cd \begin{vmatrix} \beta^{n+1} & \alpha^n \\ \beta^n & \alpha^{n-1} \end{vmatrix} \\ &= (-1)^{n+1} 5cd \ . \end{aligned}$$

In particular,

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n .$$

4. $F^2_{n+p} - F^2_{n-p} = F_{2n} \cdot F_{2p}$ for all p and n. Consider $F_{n+p} + F_{n-p}$. Then,

$$F_{n+p} + F_{n-p} = \frac{\beta^{n+p} - \alpha^{n+p}}{\beta - \alpha} + \frac{\beta^{n-p} - \alpha^{n-p}}{\beta - \alpha}$$

$$= \frac{\beta^n(\beta^p + \beta^{-p}) - \alpha^n(\alpha^p + \alpha^{-p})}{\beta - \alpha}$$

$$= \frac{(\beta^p + \beta^{-p})\,[\beta^n + (-1)^{p+1}\alpha^n]}{\beta - \alpha} \quad \text{since } \alpha^{-p} = (-1)^p\beta^p$$

$$= \frac{[\beta^n + (-1)^{p+1}\alpha^n]\,[\beta^p + (-1)^p\alpha^p]}{\beta - \alpha}$$

Therefore, if p is even, $F_{n+p} + F_{n-p} = F_n \cdot L_p$ and if p is odd, $F_{n+p} + F_{n-p} = L_n \cdot F_p$.

Also, $F_{n+p} - F_{n-p} = L_n \cdot F_p$ for p even and $F_{n+p} - F_{n-p} = F_n \cdot L_p$ for p odd. Thus, $F^2_{n+p} - F^2_{n-p} = F_{2n} \cdot F_{2p}$ for all p and n.

5. Let us simplify $F_3 + F_6 + \cdots + F_{3n}$. Since the α-sequence and the β-sequence are also geometric sequences it follows that

$$\beta^3 + \cdots + \beta^{3n} = \frac{\beta^3(\beta^{3n} - 1)}{\beta^3 - 1}$$

and

$$\alpha^3 + \cdots + \alpha^{3n} = \frac{\alpha^3(\alpha^{3n} - 1)}{\alpha^3 - 1} .$$

Thus,
$$F_3 + F_6 + \cdots + F_{3n} = \frac{-\beta^{3n} + \alpha^{3n} + \beta^3 - \alpha^3 - \beta^{3n+3} + \alpha^{3n+3}}{(-\alpha^3 - \beta^3)(\beta - \alpha)}$$

$$= \frac{F_{3n+3} + F_{3n} - F_3}{L_3} .$$

6. As another example consider $F_1 + 2F_2 + \cdots + nF_n$, n positive. Now

$$\beta + 2\beta^2 + \cdots + n\beta^n = \beta\left[\frac{n\beta^{n-1} - \beta^n + 1}{\alpha^2}\right]$$

$$= n\beta^{n+2} - \beta^{n+3} + \beta^3$$

since

$$1 + 2x + \cdots + nx^{n-1} = \frac{d}{dx}\left[\frac{x(x^n - 1)}{(x - 1)}\right]$$

Also, $\alpha + 2\alpha^2 + \cdots + n\alpha^n = n\alpha^{n+2} - \alpha^{n+3} + \alpha^3$. Therefore, $F_1 + 2F_2 + \cdots + nF_n = nF_{n+2} - F_{n+3} + F_3$. Note that a similar result holds for a general F-sequence.

7. Let us consider some results that utilize the binomial theorem. Since

$$\beta^n = (1 - \alpha)^n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j \alpha^j$$

and

$$\alpha^n = (1 - \beta)^n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j \beta^j \quad ,$$

it follows that

$$\beta^n - \alpha^n = \sum_{j=0}^{n} \binom{n}{j}(-1)^{j+1} (\beta^j - \alpha^j) \quad ;$$

hence,

$$F_n = \sum_{j=0}^{n} \binom{n}{j}(-1)^{j+1} F_j \quad .$$

Also,

$$L_n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j L_j$$

8. Again using the binomial theorem,

$$\alpha^{2n} = (1+\alpha)^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^j$$

and

$$\beta^{2n} = (1+\beta)^n = \sum_{j=0}^{n} \binom{n}{j} \beta^j \quad .$$

Therefore

$$F_{2n} = \sum_{j=0}^{n} \binom{n}{j} F_j \quad ;$$

also

$$L_{2n} = \sum_{j=0}^{n} \binom{n}{j} L_j \quad .$$

If $\{u_n\}$ is a general F-sequence, it also follows that

$$u_{2n} = \sum_{j=0}^{n} \binom{n}{j} u_j \quad .$$

9. As a final example to illustrate the usefulness of the α- and β-sequences in establishing Fibonacci relations we will derive the result

$$F_n = F_{n-p+1} F_p + F_{n-p} F_{p-1}$$

for all n and p. First, from

$$\beta^p \beta^{n-p-1} + \beta^{p-1} \beta^{n-p} = \beta^{n-1} + \beta^{n-1} = \beta^n(\beta - \alpha)$$

and

$$\beta^p \alpha^{n-p+1} + \beta^{p-1} \alpha^{n-p} = 0$$

we obtain

$$\beta^n = \beta^p F_{n-p+1} + \beta^{p-1} F_{n-p} .$$

Similarly, one can show that

$$\alpha^n = \alpha^p F_{n-p+1} + \alpha^{p-1} F_{n-p} .$$

It then follows that $F_n = F_p F_{n-p+1} + F_{p-1} F_{n-p}$ and if $\{u_n\}$ is an F-sequence, then

$$u_n = u_p F_{n-p+1} + u_{p-1} F_{n-p} .$$

Note that if $q = n - p + 1$, then $u_{p+q-1} = u_p F_q + u_{p-1} F_{q-1}$.

Since $\beta^n - \alpha^n = \sqrt{5}\, F_n$ and $\beta^n + \alpha^n = L_n$, it follows that

$$\beta^n = \frac{L_n + \sqrt{5}\, F_n}{2}$$

and

$$\alpha^n = \frac{L_n - \sqrt{5}\, F_n}{2} .$$

HINTS TO BEGINNERS' CORNER PROBLEMS

(See page 59)

1.1 Examine $\frac{n}{p}$.

1.2 Use identity III .

1.3 Notice that p, $p + 1$, $p + 2$ are three consecutive integers. Since $p > 3$ is an odd prime, $p + 1$ is even. Why must $p + 1$ be a multiple of 3?

1.4 $2^{5 \cdot 7} - 1 = (2^5)^7 - (1)^7 = (2^5 - 1)\left[(2^5)^6 + (2^5)^5 + \cdots + (2^5) + 1\right]$.

1.5 If N is composite, then by T1 it must have a prime factor p. This factor must be one of the following: $2, 3, 5, 7, \cdots, p_n$. Thus $p|N$ and $p|(2 \cdot 3 \cdot 5 \cdots p_n)$.

EXPLORING RECURRENT SEQUENCES

EDITED BY BROTHER U. ALFRED, ST. MARY'S COLLEGE, CALIF.

The following article constitutes the Elementary Research Department of the present issue of the Fibonacci Quarterly. Readers are requested to send their discoveries, queries, and suggestions dealing with this portion of the Quarterly to Brother U. Alfred, St. Mary's College, Calif.

Everyone who buys insurance is urged to read the fine print because it usually contains qualifications of an important nature. In a similar vein the readers of the newly created Fibonacci Quarterly should turn to the inside cover and examine the sub-title: "A journal devoted to the study of integers with special properties." This in no way indicates that the editors could not fill the pages of their magazine with material dealing exclusively with Fibonacci sequences. It does, however, provide for a measure of latitude and a certain variety in the contents while adhering to the main theme indicated by the title of the magazine. In this spirit, the "Fibonacci explorers" are invited to look into a somewhat broader topic: Recurrent Sequences.

The word "recurrent" need not frighten anyone. Recurrence simply means repetition. A sequence is a set of mathematical quantities that are ordered in the same way as the integers: 1, 2, 3,$\cdots$. Put the two ideas together and the result is a "recurrent sequence."

Perhaps the simplest example of such a sequence is the integers themselves. Let us denote the terms of our sequence by T_1, T_2, $T_3, \cdots, T_n \cdots$. In the case of the integers, the relation involved is

$$T_{n+1} = T_n + 1$$

that is, every integer is one more than the integer preceding it. This idea is readily extended to even integers and odd integers. If, for example, T_n is an even integer the next even integer is

$$T_{n+1} = T_n + 2 \quad .$$

Likewise, if T_n is an odd integer, the next odd integer is

$$T_{n+1} = T_n + 2 \quad .$$

Now look at the last two laws of recurrence. They are the same! This fact is a source of confusion to students of elementary algebra who think that if x and $x + 2$ represent consecutive even integers, something else would represent consecutive odd integers. The answer lies, of course, in the "if" portion of the proposition. If x is an odd integer, then $x + 2$ is also the next odd integer.

The natural extension of such relations which we have been considering is the arithmetic progression in which each term differs from the preceding term by a fixed quantity, a, called the common difference. Thus for this type of sequence we have

$$T_{n+1} = T_n + a ,$$

where a can be any real or complex quantity.

Another well-known type of recurrence sequence is the geometric progression in which each term is a fixed multiple, r, of the previous term. The relation in this case is

$$T_{n+1} = r\,T_n .$$

A simple example is: 2, 6, 18, 54, 162, $\cdots$, where $r = 3$, $T_1 = 2$.

We now come to the commercial. Recurrent sequences in which each term is the sum of the two preceding terms are known as Fibonacci sequences. The law of recurrence for all such sequences is

$$T_{n+1} = T_n + T_{n-1} .$$

Starting with the values of T_1 and T_2, it is possible to build up such a sequence. Thus, if $T_1 = 3$ and $T_2 = 11$, it follows that $T_3 = 14$, $T_4 = 25$, $T_5 = 39$, $\cdots$.

One can go on to variations of this idea. For example:

$$T_{n+1} = 2\,T_n + 3\,T_{n-1}$$

or

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}$$

Any one such sequence can be the subject of a great deal of investigation and research which can lead to many interesting mathematical results.

At this juncture it may be well to point out that in some instances, the law of recurrence is such that it is possible to work out an explicit mathematical expression for the nth term. In others, this is not convenient or possible. For example; if $T_1 = 1$, $T_2 = 1$, and every term is the sum of all the terms preceding it, we find directly that $T_3 = 2$, $T_4 = 4$, $T_5 = 8$, $T_6 = 16$, $T_7 = 32, \cdots$, so that a person not endowed with mathematical genius can see that the nth term is given by ...... ?

On the other hand, if $T_1 = 1$, $T_2 = 1$ and

$$T_{n+1} = T_n + T_{n-1} \quad ,$$

we have the well-known Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, ··· whose terms are such that they are not as readily expressible by a simple formula. Hence, we establish them as a standard sequence which can serve to express results found in other sequences.

EXPLORATION

A few sequences worthy of exploration have already been indicated. Other suggestions follow, and beginning readers are urged to create additional sequences of their own. Interesting mathematical results derived from such work should be communicated to the Editor of this department of the Fibonacci Quarterly. Here are a few suggestions to start you exploring:

1. Let $T_1 = a$, $T_2 = b$, where a and b are any positive numbers, and let the law of formation in the sequence be that each term is the quotient of the two preceding terms.
2. Starting with the same initial terms, let each term be the product of the two previous terms.
3. Another law: Let each odd-numbered term be the sum of the two previous terms and each even-numbered term be the difference of the two previous terms.
4. Let each odd-numbered term be the product of the two preceding terms and each even-numbered term be the quotient of the two preceding terms.
5. Starting with $T_1 = a$, $T_2 = b$, $T_3 = c$, let the law of formation be:

$$T_{n+1} = T_n + T_{n-1} - T_{n-2} \quad .$$

A Computer Investigation of a Property of the Fibonacci Sequence

Stephen P. Geller
Mathematics Department, University of Alaska
February 18, 1963

Publication of a table of the first 571 Fibonacci numbers in Recreational Mathematics Magazine (Oct. 1962) brought out the fact that the last (units) digit of the sequence is periodic with period 60, i.e., the 1, 1, 2, 3, 4, ··· sequence repeats on the last digit every 60 entires of the sequence. It also appeared that the last two are similarly periodic with a period of 300. Noting that the table had been calculated by an IBM 7090 digital computer, I resolved to set up our IBM 1620 to check out the above observations and extend to more digits. The size of our memory (20K) prohibited calculation of the terms of the sequence in their entirety, but this was not necessary since it was quite easy on this machine to truncate off all the digits of the running sums beyond those under consideration. The machine verified that the last two digits repeat every 300 times, the last three every 1500, the last four every 15000, the last five every 150,000, and finally after the computer ran for nearly three hours a repetition of the last six digits appeared at the 1,500,000th Fibonacci number. These may be written in the form:

$$F_{(n+60)} - F_n \equiv 0 \pmod{10}$$
$$F_{(n+300)} - F_n \equiv 0 \pmod{100}$$
$$F_{(n+1500)} - F_n \equiv 0 \pmod{1000}$$
$$F_{(n+15000)} - F_n \equiv 0 \pmod{10000}$$
$$F_{(n+150000)} - F_n \equiv 0 \pmod{100000}$$
$$F_{(n+1500000)} - F_n \equiv 0 \pmod{1000000}$$

There does not yet seem to be any way of guessing the next period, but perhaps a new program for the machine which will permit initialization at any point in the sequence for a test will cut down computer time enough so that more data can be gathered for conjecturing some rule for these repetition periods.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY S. L. BASIN, SAN JOSE STATE COLLEGE

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with the name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

B-9 *Proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey*

Prove
$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} F_{n+1}} = 1$$

and
$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1} F_{n+1}} = 2 ,$$

where F_n is the nth Fibonacci number.

B-10 *Proposed by Stephen Fisk, San Francisco, California*

Prove the "de Moivre-type" identity,

$$\left(\frac{L_n + \sqrt{5}\, F_n}{2}\right)^p = \frac{L_{np} + \sqrt{5}\, F_{np}}{2} ,$$

where L_n denotes the nth Lucas number and F_n denotes the nth Fibonacci number.

B-11 *Proposed by S. L. Basin, Sylvania Electronic Defense Laboratory, Mt. View, California*

Show that the hypergeometric function

$$G(x, n) = \sum_{k=0}^{n-1} \frac{2^k (n + k)!\; (x - 1)^k}{(n - k - 1)!\; (2k + 1)!}$$

generates the sequence $G\left(\frac{3}{2}, n\right) = F_{2n}$, $\qquad n = 1, 2, 3, \cdots$.

B-12 *Proposed by Paul F. Byrd, San Jose State College, San Jose, Calif.*

Show that

$$L_{n+1} = \begin{vmatrix} 3 & i & 0 & 0 & \cdots & 0 & 0 \\ i & 1 & i & 0 & \cdots & 0 & 0 \\ 0 & i & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & i & 1 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & 1 & i \\ 0 & 0 & 0 & 0 & \cdots & i & 1 \end{vmatrix}_n \qquad n \geq 1$$

where L_n is the nth Lucas number given by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, and $i = \sqrt{-1}$.

B-13 *Proposed by S. L. Basin, Sylvania Electronic Defense Laboratory, Mt. View, Calif.*

Determinants of order n which are of the form,

$$K_n(b,c,a) = \begin{vmatrix} c & a & 0 & 0 & 0 & \\ b & c & a & 0 & 0 & \cdots \\ 0 & b & c & a & 0 & \cdots \\ 0 & 0 & b & c & a & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

are known as <u>CONTINUANTS</u>.

Prove that,

$$K_n(b,c,a) = \frac{(c + \sqrt{c^2 - 4\,ab})^{n+1} - (c - \sqrt{c^2 - 4\,ab})^{n+1}}{2^{n+1}\sqrt{c^2 - 4\,ab}}$$

and show, for special values of a, b, and c, that $K_n(b,c,a) = F_{n+1}$.

B-14 *Proposed by Maxey Brooke, Sweeny, Texas, and C. R. Wall, Ft. Worth, Texas*

Show that

$$\sum_{n=1}^{\infty} \frac{F_n}{10^n} = \frac{10}{89} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n}{10^n} = \frac{10}{109} .$$

B-15 *Proposed by R. B. Wallace, Beverly Hills, Calif. and Stephen Geller, University of Alaska, College, Alaska.*

If p_k is the smallest positive integer such that

$$F_{n+p_k} \equiv F_n \bmod (10^k)$$

for all positive n, then p_k is called the period of the Fibonacci sequence relative to 10^k. Show that p_k exists for each k, and find a specific formula for p_k as a function of k.

B-16 *Proposed by Marjorie Bicknell, San Jose State College, and Terry Brennan, Lockheed Missiles & Space Co., Sunnyvale, Calif.*

Show that if

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

then

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}.$$

NOTE: On occasion there will be problems listed at the ends of the articles in the advanced and elementary sections of the magazine. These problems are to be considered as logical extensions of the corresponding problem sections and solutions for these problems will be discussed in these sections as they are received.

See, for example, "Expansion of Analytic Functions In Polynomials Associated with Fibonacci Numbers," by Paul F. Byrd, San Jose State College, in the first issue of the Quarterly, and "Linear Recurrence Relations—Part I," by James Jeske, San Jose State College, in this issue.

Solutions for problems in ISSUE ONE will appear in ISSUE THREE.

When an error is found, clear the registers and start at Step (b) with the last accurate values before the error. It is not necessary to start fresh and do the whole sequence over.

It may be noted for any simple Fibonacci sequence starting with an odd digit, that the odd probability of any most significant digit is 0.6, and the odd probability of any least significant digit is 0.66. For a sequence starting with an even digit in the LSD position, the odd probability of any MSD is still 0.6, but the odd probability of any LSD is 0.0!

FIBONACCI SEQUENCE

1<+	832040○+
<<	<<
<<	1346269+S
<<	2178309○+
<<	3524578+S
<<	5702887○+
<<	9227465+S
1+S	<<
1○+	14930352○+
2+S	24157817+S
3○+	39088169○+
5+S	63245986+S
<<	102334155○+
8○+	<<
13+S	165580141+S
21○+	267914296○+
34+S	433494437+S
55○+	701408733○+
<<	1134903170+S
89+S	<<
144○+	1836311903○+
233+S	2971215073+S
377○+	4807526976○+
610+S	7778742049+S
<<	12586269025○+
987○+	<<
1597+S	20365011074+S
2584○+	32951280099○+
4181+S	53316291173+S
6765○+	86267571272○+
<<	139583862445+S
10946+S	<<
17711○+	225851433717○+
28657+S	365435296162+S
46368○+	591286729879○+
75025+S	956722026041+S
<<	1548008755920○+
121393○+	<<
196418+S	2504730781961+S
317811○+	4052739537881○+
514229+S	6557470319842+S