# REPRESENTATIONS BY COMPLETE SEQUENCES - PAPT I (FIBONACCI) <br> V.E. HOGGATT, JR. and S.L. BASIN, FIBONACCI BIBLIOGRAPHICAL AND RESEARCH CENTER, SAN JOSE STATE COLLEGE 

## 1. INTRODUCTION

The notion of completeness was extended to sequences of integers when Hoggatt and King [1] defined a sequence $\left\{\mathrm{A}_{\mathbf{i}}\right\}_{i=1}^{\infty}$ of positive integers as a complete sequence if and only if every natural number $N$ could be represented as the sum of a subsequence, $\left\{B_{j}\right\}_{j=1}^{k}$, such that $B_{j} \equiv A_{i_{j}}$.

A necessary and sufficient condition for completeness is stated in the following Lemma, the proof of which is given by H. L. Alder [2] and J. L. Brown, Jr. [3] .

Lemma 1.1 Given any non-decreasing sequence of positive integers $\left\{\mathrm{A}_{\mathrm{i}}\right\}_{\mathrm{i}=1}^{\infty}$, not necessarily distinct, with $\mathrm{A}_{1}=1$, then there exists a sequence $\left\{\alpha_{i}\right\}_{i=1}^{k}$, where $\alpha_{i}=0$ or 1 , such that any natural number, $N$, can be represented as the $\operatorname{sum}_{p}$ of a subsequence $\left\{B_{j}\right\}_{j=1}^{k^{\prime}}$, i. e. , $N=\sum_{j=1}^{k} \alpha_{j} A_{j}$ if and only if, $\quad A_{p+1} \leq 1+\sum_{1}^{p} A_{i}, p=1,2,3, \cdots$.

The intention of this paper is to extend this past work by investigating the number of possible representations of any given natural number N as the sum of a subsequence of specific complete sequences.

## 2. THE GENERATING FUNCTION

We denote the number of distinct representations of $N$, not counting permutations of the subsequence $\left\{B_{j}\right\}_{j=i}^{k^{\prime}}$, by $R(N)$. The following combinatorial generating function yields $R(N)$ for any given subsequence $\left\{A_{i}\right\}_{i=1}^{k}$,

$$
\begin{equation*}
\Pi_{k}(x)=\prod_{i=1}^{k}\left(1+x^{A_{i}}\right) \tag{1}
\end{equation*}
$$

Now, given any subsequence $\left\{A_{i}\right\}_{i=1}^{k}$ the expansion of (1) takes the form,

$$
\begin{equation*}
\Pi_{k}(x)=\sum_{n=0}^{\sigma} R(n) x^{n} \tag{2}
\end{equation*}
$$

where

$$
\sigma=\sum_{i=1}^{k} A_{i}
$$

To illustrate this, consider the subsequence $\{2,1,3,4\}$ of the Lucas sequence $\left\{L_{n}\right\}_{0}^{+\infty}$, where $L_{n}=L_{n-1}+L_{n-2}$, and $L_{0}=2, L_{1}=1$ :

$$
\begin{align*}
& \Pi_{1}(x)=1+x^{2} \\
& \Pi_{2}(x)=\left(1+x^{2}\right)\left(1+x^{1}\right)=1+x+x^{2}+x^{3}  \tag{3}\\
& \Pi_{3}(x)=\left(1+x^{2}\right)\left(1+x^{1}\right)\left(1+x^{3}\right)=1+x+x^{2}+2 x^{3}+x^{4}+x^{5}+x^{6} \\
& \Pi_{4}(x)=1+x+x^{2}+2 x^{3}+2 x^{4}+2 x^{5}+2 x^{6}+2 x^{7}+x^{8}+x^{9}+x^{10}
\end{align*}
$$

In (3) the coefficient of $x^{n}$ is $R(n)$, the number of ways of representing the natural number, $n$, by the summation of a subsequence of these four Lucas numbers.

The expansion of (1) becomes quite tedious as $k$ increases; however, we have developed a convenient algorithm for rapidly expanding (1). The representation of the factors of (1) is the foundation of this algorithm. The coefficients of $x^{n}$ in (2) will be tabulated in columns labeled $n$. The process of computing entries in this table is as follows:
(i) The first factor of (1), namely $\left(1+x^{A_{1}}\right)$, is represented by entering 1 in row 1 , column 0 and row 1 , column $A_{1}$ of our table. The remaining entries in row 1 are zero.
(ii) The entries in row 2 consist of rewriting row 1 after shifting it $A_{2}$ columns to the right.
(iii) The product $\left(1+x^{A_{1}}\right)\left(1+x^{A_{2}}\right)$ is represented in the third row as the sum of row 1 and row 2 .
The following example considers the subsequence of $\left\{L_{n}\right\}_{0}^{\infty}$ given above. The product

$$
\Pi_{k}(x)=\prod_{i=1}^{k}\left(1+x^{A_{i}}\right), \text { for } k=4
$$

and

$$
\left\{A_{i}\right\}_{i=1}^{4}=\{2,1,3,4\} \text { is given by }
$$

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\Pi_{1}(\mathrm{x})$ | 1 | 0 | 1 |  |  |  |  |  |  |  |  |  |
|  | 0 | 1 | 0 | 1 |  |  |  |  |  |  |  |  |
| $\Pi_{2}(\mathrm{x})$ | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |
|  | 0 | 0 | 0 | 1 | 1 | 1 | 1 |  |  |  |  |  |
| $\Pi_{3}(\mathrm{x})$ | 1 | 1 | 1 | 2 | 1 | 1 | 1 |  |  |  |  |  |
|  | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |  |
| $\Pi_{4}(\mathrm{x})$ | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |  |

The coefficients $R(n)$ of $\Pi_{k}(x), k=1,2,3,4$ in the above table are exactly those given in (3) and the entries in the row labeled $\Pi_{4}(x)$ are the number of ways of representing the natural numbers 0 to 10 as sums of $\{2,1,3,4\}$, not counting permutations.

Itis important to note at this point that the representations of the natural numbers 4 through 10 will change and the representations of 0 through 3 remain constant in the above table with subsequent partial products. The representations which remain invariant under subsequent partial products will be made explicit in Lemma 3.3 below.

Prior to investigating representations as sums of specific sequences, it is convenient to define the following terms:

Definition 1.1 Level - The product $\mathbb{\pi}_{k}(x)$ is defined as the $k^{\text {th }}$ level in the table.

Definition 1.2 Length - The number of terms in $\Pi_{k}(x)$ will be denoted as the length $\lambda_{k}$ of the $k^{\text {th }}$ level. From (1) it is clear that

$$
\lambda_{k}=1+\sum_{i=1}^{k} A_{i}
$$

Definition $1.3 R(n, k)$ denotes the number of representations of $n$ in the $\mathrm{k}^{\text {th }}$ level.

## 3. THE COMPLETE FIBONACCI SEQUENCE

Now that the machinery has been developed for the investigation of complete sequences, we proceed with the study of representations as sums of Fibonacci numbers.

Lemma 3.1 The length $\lambda_{k}$ is $\mathrm{F}_{\mathrm{k}+2}$.
Proof: By definition.

$$
\lambda_{k}=1+\sum_{i=1}^{k} A_{i}
$$

therefore

$$
\lambda_{k}=1+\sum_{i=1}^{k} F_{i}=F_{k+2}
$$

The following lemmas $3,2,3.3$, and 3.4 follow directly from the algorithm for expanding $\Pi_{k}(x)$.

Lemma 3.2 (Symmetry)

$$
R\binom{k}{\sum_{1} A_{i}-j, k}=R(j, k) \text { for } j=0,1,2,3, \cdots, \sum_{i=1}^{k} A_{i}
$$

Therefore,

$$
R\binom{k}{\sum_{1} F_{i}-j, k}=R(j, k) \text { for } j=0,1,2, \cdots, \sum_{1}^{k} F_{i}
$$

## Lemma 3.2F

$$
R\left(F_{k+2}-(j+1), k\right)=R(j, k), j=0,1,2,3, \cdots,\left(F_{k+2}-1\right)
$$

Lemma 3.3 (Invariance) $\left(\mathrm{A}_{1} \leq \mathrm{A}_{2} \leq \mathrm{A}_{3} \leq \cdots \leq \mathrm{A}_{\mathrm{n}} \leq \cdots\right)$

$$
R(j, k)=R(j, \infty) \text { for } j=0,1,2,3, \cdots,\left(A_{k+1}-1\right)
$$

For the Fibonacci sequence we have,
Lemma 3.3F Since ( $\mathrm{F}_{1} \leq \mathrm{F}_{2} \leq \mathrm{F}_{3} \leq \ldots \leq \mathrm{F}_{\mathrm{n}} \leq \ldots$ )

$$
R(j, k)=R(j, \infty) \text { for } j=0,1,2, \cdots,\left(F_{k+1}-1\right)
$$

i. e., the first $F_{k+1}$ terms of $\Pi_{k}(x)$ are also the first $F_{k+1}$ terms of all subsequent partial products

$$
\Pi_{k+m}(x), m=1,2,3, \cdots
$$

Lemma 3.4 (Additive Property)

$$
R\left(A_{k+1}+j, k+1\right)=R\left(A_{k+1}+j, k\right)+R(j, k)
$$

and by symmetric property, Lemma 3.2 , it is also true that

$$
R\left(A_{k+1}+j, k+1\right)=R\left(A_{k+1}+j, k\right)+R\left(\begin{array}{l}
k \\
\sum A_{i}-j, k \\
1
\end{array}\right)
$$

for

$$
j=0,1,2,3, \cdots,\left(\begin{array}{l}
k \\
\sum A_{i}-A_{k+1} \\
1
\end{array}\right)
$$

For the Fibonacci sequence $\left\{\mathrm{F}_{\mathrm{i}}\right\}_{\mathrm{i}=1}^{\infty}$ this is:

Lemma 3.4F

$$
R\left(F_{k+1}+j, k+1\right)=R\left(\mathbb{F}_{k+1}+j, k\right)+R(j, k)
$$

and

$$
R\left(F_{k+1}+j, k+1\right)=R\left(F_{k+1}+j, k\right)+R\left(F_{k+2}-(j+1), k\right)
$$

for

$$
j=0,1,2,3, \cdots,\left(F_{k}-1\right)
$$

Lemma 3.5F

$$
\mathrm{R}\left(\mathrm{~F}_{\mathrm{k}+2}, \infty\right)=1+\mathrm{R}\left(\mathrm{~F}_{\mathrm{k}},{ }^{\infty}\right)
$$

Proof: Using Lemma 3.4 F we have,

$$
R\left(F_{k+2}, k+2\right)=R(0, k+1)+R\left(F_{k+2}, k+1\right) .
$$

But

$$
R(0, k+1)=R(0, \infty)=1 .
$$

By the symmetry property of $\Pi_{k+1}(x)$,

$$
R\left(\begin{array}{c}
\sum_{1}^{k+1} \\
A_{i}
\end{array}-j, k+1\right)=R(j, k+1)
$$

for

$$
j=0,1,2,3, \cdots, \sum_{1}^{k+1} A_{i} .
$$

Since

$$
\mathrm{F}_{\mathrm{k}+3}=1+\sum_{1}^{\mathrm{k}+1} \mathrm{~F}_{\mathrm{i}}
$$

we let

$$
j=\left(F_{k+1}-1\right)
$$

which results in

$$
R\left(F_{k+2}, k+1\right)=R\left(F_{k+1}-1, k+1\right)
$$

Also by Lemma 3.3F,

$$
R\left(F_{k+1}-1, k+1\right)=R\left(F_{k+1}-1, k\right)
$$

By symmetry,

$$
R\left(F_{k+1}-1, k\right)=R\left(F_{k}, k\right)
$$

But invariance yields

$$
R\left(F_{k}, k\right)=R\left(F_{k}, \infty\right)
$$

Therefore,

$$
R\left(F_{k+2}, \infty\right)=1+R\left(F_{k}, \infty\right)
$$

The notation $R(m)$ will be used to denote $R(m, \infty)$ in what follows. Theorem 1.

$$
\mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}}\right)=\mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}+1}\right)=\mathrm{k}+1
$$

Proof: (By induction) When $\mathrm{k}=1$, we observe that

$$
R\left(F_{2}\right)=R\left(F_{3}\right)=R(1)=R(2)=2 .
$$

The inductive hypothesis is

$$
R\left(F_{2 k}\right)=R\left(F_{2 k+1}\right)=k+1
$$

The inductive transition follows from:
Lemma 3.5F

$$
R\left(F_{2 k+2}, \infty\right)=1+R\left(F_{2 k}, \infty\right)=1+(k+1)
$$

and

$$
R\left(F_{2 k+3}, \infty\right)=1+R\left(F_{2 k+1}, \infty\right)=1+(k+1)
$$

The proof is now complete by mathematical induction. Proofs of the following two theorems rely on:

Lemma 3.6F
(a) $\quad R\left(F_{k+1}+F_{k-2}, k+1\right)=R\left(F_{k-1}-1, k\right)+R\left(F_{k-2}, k\right)$
and
(b)

$$
R\left(F_{k+1}+F_{k-1}, k+1\right)=R\left(F_{k-2}-1, k\right)+R\left(F_{k-1}, k\right)
$$

Proof: Using the additive property of the algorithm as stated in Lemma 3.4, we have

$$
\begin{gathered}
R\left(A_{k+1}+j, k+1\right)=R\left(A_{k+1}+j, k\right)+R(j, k) \\
j=0,1,2, \cdots\left(\begin{array}{c}
k \\
\Sigma \\
1
\end{array} A_{i}-A_{k+1}\right) .
\end{gathered}
$$

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Let $\quad j=A_{k-2}$ for (a), and $j=A_{k-1}$ for (b),
(c) $\quad R\left(A_{k+1}+A_{k-2}, k+1\right)=R\left(A_{k+1}+A_{k-2}, k\right)+R\left(A_{k-2}, k\right)$
(d) $\quad R\left(A_{k+1}+A_{k-2}, k+1\right)=R\left(A_{k+1}+A_{k-1}, k\right)+R\left(A_{k-1}, k\right)$

By symmetry (Lemma 3.2)
(e)

$$
R\left(A_{k+1}+A_{k-2}, k\right)=R\binom{k}{\left.\sum_{1} A_{i}-A_{k+1}-A_{k-2}, k\right)}
$$

$$
\begin{equation*}
R\left(A_{k+1}+A_{k-1}, k\right)=R\left(\sum_{1}^{k} A_{i}-A_{k+1}-A_{k-1}, k\right) \tag{f}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
& R\left(A_{k+1}+A_{k-2}, k+1\right)=R\left(\begin{array}{l}
k \\
\Sigma A_{i}-A_{k+1}-A_{k-2}, k \\
1
\end{array}\right)+R\left(A_{k-2}, k\right) \\
& R\left(A_{k+1}+A_{k-1}, k+1\right)=R\left(\begin{array}{c}
k \\
\Sigma \\
1
\end{array} A_{i}-A_{k+1}-A_{k-1}, k\right)+R\left(A_{k-1}, k\right)
\end{aligned}
$$

Specializing the above for the Fibonacci sequence,
(a)

$$
\begin{aligned}
R\left(F_{k+1}+F_{k-2}, k+1\right) & =R\left(F_{k-1}-1, k\right)+R\left(F_{k-2}, k\right) \\
R\left(F_{k+1}+F_{k-1}, k+1\right) & =R\left(F_{k-2}-1, k\right)+R\left(F_{k-1}, k\right) \\
\text { Theorem 2 } \quad R\left(2 F_{k}\right) & =2 R\left(F_{k-2}\right)
\end{aligned}
$$

(b)
and

$$
R\left(2 \mathrm{~F}_{2 \mathrm{k}}\right)=\mathrm{R}\left(2 \mathrm{~F}_{2 \mathrm{k}+1}\right)=2 \mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}-2}\right)=2 \mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}-1}\right)=2 \mathrm{k}
$$

Proof: Using the recurrence relation

$$
\mathrm{F}_{\mathrm{k}}=\mathrm{F}_{\mathrm{k}-1}+\mathrm{F}_{\mathrm{k}-2}
$$

and Lemma 3.6F we have,

$$
R\left(2 F_{k}\right)=R\left(F_{k+1}+F_{k-2}\right)=R\left(F_{k-2}\right)+R\left(F_{k-1}-1\right)
$$

However, by symmetry and invariance,

$$
R\left(F_{k-1}-1, k-2\right)=R\left(F_{k-2}, k-2\right)=R\left(F_{k-2}\right)
$$

so that

$$
R\left(2 \mathrm{~F}_{\mathrm{k}}\right)=2 \mathrm{R}\left(\mathrm{~F}_{\mathrm{k}-2}\right)
$$

Applying Theorem 1 to $\mathrm{F}_{2 \mathrm{k}-2}$ and $\mathrm{F}_{2 \mathrm{k}-1}$ yields

$$
R\left(2 F_{2 k}\right)=2 R\left(F_{2 k-2}\right)=2 k
$$

and

$$
\mathrm{R}\left(2 \mathrm{~F}_{2 \mathrm{k}+1}\right)=2 \mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}-1}\right)=2 \mathrm{k}
$$

Theorem 3. $R\left(L_{2 k-1}\right)=R\left(L_{2 k}\right)=2 k-1, k \geq 1$. Proof: Since $L_{k} \leq F_{k+2}-1$,
$\mathrm{R}\left(\mathrm{L}_{\mathrm{k}}, \infty\right)=\mathrm{R}\left(\mathrm{L}_{\mathrm{k}}, \mathrm{k}+1\right)=\mathrm{R}\left(\mathrm{F}_{\mathrm{k}+1}+\mathrm{F}_{\mathrm{k}-1}, \mathrm{k}+1\right)$
$=R\left(F_{k-1}, k\right)+R\left(F_{k-2}-1, k\right)$
from Lemma 3.6F.

By symmetry, Lemma 3.2F,

$$
R\left(F_{k-2}-1, k-2\right)=R\left(F_{k-1}, k-2\right)
$$

But, from Lemma 3.5F,

$$
R\left(F_{k-1}, k-1\right)=R\left(F_{k-1}, k-2\right)+R(0, k-2)
$$

and

$$
R\left(F_{k-1}, k-2\right)=R\left(F_{k-1}, k-1\right)-1
$$

from the above equation.
By Lemma 3.3F,

$$
R\left(F_{k-1}, k-1\right)=R\left(F_{k-1}, \infty\right)
$$

Therefore

$$
R\left(L_{k}\right)=2 R\left(F_{k-1}\right)-1
$$

By Theorem 1,

$$
R\left(\mathrm{~F}_{2 \mathrm{k}}\right)=\mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}+1}\right)=\mathrm{k}+1
$$

so that

$$
\begin{aligned}
& R\left(L_{2 k-1}\right)=2 R\left(F_{2 k-2}\right)-1=2 k-1 \\
& R\left(L_{2 k}\right)=2 R\left(F_{2 k-1}\right)-1=2 k-1
\end{aligned}
$$

Lemma 3.7F
(a)

$$
R\left(F_{k+1}^{2}-1\right)=R\left(F_{k-1}^{2}\right)+R\left(F_{k}^{2}-1\right)
$$

$$
\begin{equation*}
R\left(F_{k+1}^{2}-1\right)=R\left(F_{k-1}^{2}-1\right)+R\left(F_{k}^{2}\right) \tag{b}
\end{equation*}
$$

Proof of Lemma 3.7F:
Since

$$
F_{2 n}=F_{n+1}^{2}-F_{n-1}^{2}, \text { then } F_{n+1}^{2}=F_{2 n}+F_{n-1}^{2}
$$

which gives

$$
R\left(\mathrm{~F}_{\mathrm{n}+1}^{2}, 2 \mathrm{n}\right)=R\left(\mathrm{~F}_{2 \mathrm{n}}+\mathrm{F}_{\mathrm{n}-1}^{2}, 2 \mathrm{n}\right)
$$

By addition property, Lemma 3.4F,

$$
R\left(F_{n+1}^{2}, 2 n\right)=R\left(F_{n-1}^{2}, 2 n-1\right)+R\left(F_{2 n+1}-1-F_{n+1}^{2}, 2 n-1\right)
$$

and by symmetry, Lemma 3.2F, and the identity $\mathrm{F}_{2 \mathrm{n}+1}=\mathrm{F}_{\mathrm{n}+1}^{2}+\mathrm{F}_{\mathrm{n}}^{2}$,

$$
R\left(F_{2 n+1}-1-F_{n+1}^{2}, 2 n-1\right)=R\left(F_{n}^{2}-1,2 n-1\right)
$$

Therefore

$$
R\left(F_{n+1}^{2}, 2 n\right)=R\left(F_{n-1}^{2}, 2 n-1\right)+R\left(F_{n}^{2}-1,2 n-1\right)
$$

Similarly,

$$
R\left(F_{n+1}^{2}-1,2 n\right)=R\left(F_{n-1}^{2}-1,2 n-1\right)+R\left(F_{n}^{2}, 2 n-1\right)
$$

Since

$$
F_{n-1}^{2} \leq F_{2 n}-1 ; F_{n}^{2} \leq F_{2 n}-1 ;
$$

and

$$
\mathrm{F}_{\mathrm{n}+1}^{2} \leq \mathrm{F}_{2 \mathrm{n}+1}-1,
$$

then by invariance, Lemma 3.2F,

$$
R\left(F_{n+1}^{2}, 2 n\right)=R\left(F_{n+1}^{2}\right)=R\left(F_{n-1}^{2}\right)+R\left(F_{n}^{2}-1\right)
$$

and

$$
R\left(F_{n+1}^{2}-1,2 n\right)=R\left(F_{n+1}^{2}-1\right)=R\left(F_{n-1}^{2}-1\right)+R\left(F_{n}^{2}\right)
$$

Theorem 4 .
(a)

$$
R\left(F_{2 k-1}^{2}-1\right)=F_{2 k}
$$

(b)

$$
\mathrm{R}\left(\mathrm{~F}_{2 \mathrm{k}-2}^{2}\right)=\mathrm{F}_{2 \mathrm{k}-1}
$$

(c)
(d)

$$
\begin{aligned}
& R\left(F_{2 k}^{2}-1\right)=L_{2 k-1} \\
& R\left(F_{2 k-1}^{2}\right)=L_{2 k-2}
\end{aligned}
$$

Proof: (By induction)

$$
F_{0}=0 ; R\left(F_{0}^{2}\right)=R\left(F_{1}^{2}-1\right)=R\left(F_{2}^{2}-1\right)=1
$$

and

$$
R\left(F_{1}^{2}\right)=2
$$

(a)

$$
R\left(F_{k}^{2}\right)=R\left(F_{k-2}^{2}\right)+R\left(F_{k-1}^{2}-1\right)
$$

(b)

$$
\mathrm{R}\left(\mathrm{~F}_{\mathrm{k}}^{2}-1\right)=\mathrm{R}\left(\mathrm{~F}_{\mathrm{k}-2}^{2}-1\right)+\mathrm{R}\left(\mathrm{~F}_{\mathrm{k}-1}^{2}\right)
$$

by Lemma 3.7F.
Replacing k by 2 k in Lemma 3.7F (a), yields

$$
R\left(F_{2 k}^{2}\right)=R\left(F_{2 k-2}^{2}\right)+R\left(F_{2 k-1}^{2}-1\right)
$$

Thus

$$
R\left(F_{2 k}^{2}\right)=F_{2 k-1}+F_{2 k}=F_{2 k+1}
$$

Replacing k by $2 \mathrm{k}+1$ in Lemma 3.7F (b), yields

$$
\begin{aligned}
R\left(F_{2 k+1}^{2}-1\right) & =R\left(F_{2 k-1}^{2}-1\right)+R\left(F_{2 k}^{2}\right) \\
& =F_{2 k}+F_{2 k+1}
\end{aligned}
$$

Therefore

$$
R\left(\mathrm{~F}_{2 \mathrm{k}+1}^{2}-1\right)=\mathrm{F}_{2 \mathrm{k}+2}
$$

Similarly,

$$
\begin{aligned}
& R\left(F_{2 k+1}^{2}\right)=R\left(F_{2 k-1}^{2}\right)+R\left(F_{2 k}^{2}-1\right) \\
& R\left(F_{2 k+1}^{2}\right)=L_{2 k-2}+L_{2 k-1}=L_{2 k}
\end{aligned}
$$

and

$$
\begin{aligned}
& R\left(F_{2 k+2}^{2}-1\right)=R\left(F_{2 k}^{2}-1\right)+R\left(F_{2 k+1}^{2}\right) \\
& R\left(F_{2 k+2}^{2}-1\right)=L_{2 k-1}+L_{2 k}=L_{2 k+1}
\end{aligned}
$$

Many more fascinating properties of complete sequences will follow in Part II of this paper.

References may be found on page 31.

## ON THE GREATEST PRIMITIVE DIVISORS OF FIBONACCI AND LUCAS NUMBERS WITH PRIME-POWEK SUBSCRIPTS DOV JARDEN, JERUSALEM, ISRAEL

The greatest primitive divisor $F_{n}^{\prime}$ of a Fibonacci number $F_{n}$ is defined as the greatest divisor of $\mathrm{F}_{\mathrm{n}}$ relatively prime to every $\mathrm{F}_{\mathrm{x}}$ with positive $\mathrm{x}<\mathrm{n}$.

Similarly, the greatest primitive divisor $L_{n}^{\prime}$ of a Lucas number $L_{n}$ is defined as the greatest divisor of $L_{n}$ relatively prime to every $L_{x}$ with nonnegative $\mathrm{x}<\mathrm{n}$.

The first 20 values of the sequence $\left(F_{n}^{\prime}\right)$ are:

$$
\begin{aligned}
& F_{1}^{\prime}=1, F_{2}^{\prime}=1, F_{3}^{\prime}=2, F_{4}^{\prime}=3, F_{5}^{\prime}=5, F_{6}^{\prime}=1, F_{7}^{\prime}=13, \\
& F_{8}^{\prime}=7, F_{9}^{\prime}=17, F_{10}^{\prime}=11, F_{11}^{\prime}=89, F_{12}^{\prime}=1, F_{13}^{\prime}=233, F_{14}^{\prime}=29, \\
& F_{15}^{\prime}=61, F_{16}^{\prime}=47, F_{17}^{\prime}=1597, F_{18}^{\prime}=19, F_{19}^{\prime}=4181, F_{20}^{\prime}=41 .
\end{aligned}
$$

As may be seen from these few examples, the growth of the sequence $\left(F_{n}^{\prime}\right)$ is very irregular. However, some special subsequences of $\left(F_{n}^{\prime}\right)$ may occur to be increasing sequences. E.g., the subsequence ( $F_{p}^{\prime}$ ), where $p$ ranges over all the primes, is a strictly increasing sequence (since $F_{p}^{\prime}=F_{p}$ and ( $\mathrm{F}_{\mathrm{n}}$ ) is a strictly increasing sequence beginning with $\mathrm{n}=2$ ).

Similarly, the subsequence $\left(L_{q}^{\prime}\right)$, where $q$ ranges over all the odd primes and over all the powers of 2 beginning with $2^{2}$, is a strictly increasing sequence.

The main object of this note is to prove the following inequalities:

$$
\begin{equation*}
\mathrm{F}_{\mathrm{p}^{\mathrm{X}+1}}^{\prime}>\mathrm{F}_{\mathrm{p}^{\mathrm{x}}}^{\prime}(\mathrm{p}-\mathrm{a} \text { prime, } \mathrm{x}-\mathrm{a} \text { positive integer }) \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
F_{2 p^{x+1}}^{\prime}>F_{2 p^{x}}^{\prime} \quad(p-a \text { prime, } x-a \text { nonnegative integer }) \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
L_{p^{x+1}}^{\prime}>L_{p^{x}}^{\prime} \quad(p-a \text { prime, } x-a \text { nonnegative integer }) \tag{*}
\end{equation*}
$$

In other words: the subsequences $\left(F_{p x}^{\prime}\right)$ and $\left(F_{2 p^{x}}^{\prime}\right)$ of the sequence ( $F_{n}^{\prime}$ ), as well as the subsequence ( $L_{p^{x}}^{\prime}$ ) of the sequence ( $L_{n}^{\prime}$ ), p being a prime and $x=1,2,3, \cdots$, are strictly increasing sequences.

Since (as is well known) the primitive divisors of $F_{2 n}$ and $L_{n}(n \geq 1)$ coincide, we have: $F_{2 n}^{\prime}=L_{n}^{\prime}(n \geq 1)$, and especially: $F_{2^{x+1}}^{\prime}=L_{2^{x}}^{\prime}(x \geq 0)$. Hence, (2) and (2*) are equivalent, and, for $\mathrm{p}=2$, also (1) and ( $2^{*}$ ). Thus it is sufficient to prove (1) for $p \geq 2$ and ( $2^{*}$ ) for $p \neq 2$ 。

We shall even show the stronger inequalities:

$$
\begin{align*}
& F_{p^{x+1}}^{\prime}>F_{p^{x}}(p-a \text { prime, } x-\text { a positive integer })  \tag{3}\\
& L_{p^{x}+1}^{\prime}>L_{p^{x}}(p-a \text { prime }, x-a \text { nonnegative integer })
\end{align*}
$$

Since $F_{n} \geq F_{n}^{\prime}, L_{n} \geq L_{n}^{\prime}$, it is obvious that in order to prove (1) for $p \geq 2$, and $\left(2^{*}\right)$ for $p \neq 2$, it is sufficient to prove (3) for $p \geq 2$ and ( $3^{*}$ ) for $p \neq 2$ 。

The main tools for proving (3) for $\mathrm{p} \geq 2$ and ( $3^{*}$ ) for $\mathrm{p} \neq 2$, are the following inequalities:

$$
\begin{align*}
& F_{n^{x+1}}>F_{n^{x}}^{2}(n \geq 2, x \geq 1)  \tag{4}\\
& F_{5^{x+1}}>5 F_{5^{x}}^{2}(x \geq 1)  \tag{5}\\
& L_{n^{x+1}}>L_{n^{x}}^{2} \quad(n \geq 2, x \geq 0) \tag{*}
\end{align*}
$$

In order to prove (3) for $\mathrm{p} \geq 2$ and ( $3^{*}$ ) for $\mathrm{p} \neq 2$, it is sufficient (as will be shown later) to prove (4) for n a prime $\geq 2$ and (4*) for n an odd prime. However, since (4) and (4*) are interesting by themselves, we shall prove them for any positive integer $\mathrm{n} \geq 2$.

The following formulae are well known.
(6) $\quad \mathrm{F}_{\mathrm{n}}=\frac{1}{\sqrt{5}}\left(\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}\right)$

$$
\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}
$$

$\left(6^{*}\right)$

$$
\mathrm{L}_{\mathrm{n}}=\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}
$$

Since

$$
\alpha=\frac{1+\sqrt{5}}{2} \quad \frac{1+\sqrt{4}}{2}=\frac{3}{2},
$$

we have:
(7)

$$
\alpha=\frac{3}{2}
$$

Since

$$
\begin{aligned}
& \beta=\frac{1-\sqrt{5}}{2}>\frac{1-\sqrt{9}}{2}=-1 \\
& \beta=\frac{1-\sqrt{5}}{2}<\frac{1-\sqrt{4}}{2}-\frac{1}{2}
\end{aligned}
$$

we have

$$
\begin{equation*}
-1<\beta<-\frac{1}{2},|\beta|<1 . \tag{8}
\end{equation*}
$$

Since

$$
\alpha \beta=\frac{1+\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2}=-1,
$$

we have:
(9)

$$
\alpha \beta=-1
$$

For any positive integer $n \geq 3$ we have, by (7):

$$
\begin{align*}
& \alpha^{\mathrm{n}^{\mathrm{X}+1}}=\left(\alpha^{\mathrm{n}^{\mathrm{X}}}\right)^{\mathrm{n}} \geq\left(\alpha^{\mathrm{n}^{\mathrm{x}}}\right)^{3}=\alpha^{\mathrm{n}^{\mathrm{x}}} \alpha^{2 \mathrm{n}^{\mathrm{x}}}>\left(\frac{3}{2}\right)^{2} \alpha^{2 \mathrm{n}^{\mathrm{X}}}>2 \alpha^{2 \mathrm{n}^{\mathrm{X}}}= \\
& \alpha^{2 \mathrm{n}^{\mathrm{x}}}+\alpha^{2 \mathrm{n}^{\mathrm{x}}}>\alpha^{2 \mathrm{n}^{\mathrm{x}}}+\left(\frac{3}{2}\right)^{2}>\alpha^{2 \mathrm{n}^{\mathrm{x}}}+3, \quad \text { whence, } \\
& \alpha^{\mathrm{n}^{\mathrm{x}+1}}>\alpha^{2 \mathrm{n}^{\mathrm{X}}}+3 \quad(\mathrm{n} \geq 3) . \tag{10}
\end{align*}
$$

[Oct.
For odd $\mathrm{n} \geq 3$ we have, by (10), (8), (9):
$\alpha^{\mathrm{n}^{\mathrm{X}+1}}-\beta^{\mathrm{n}^{\mathrm{X}+1}}>\alpha^{2 \mathrm{n}^{\mathrm{X}}}+3=\alpha^{2 \mathrm{n}^{\mathrm{X}}}+2+1>\alpha^{2 \mathrm{n}^{\mathrm{X}}}+2+\beta^{2 \mathrm{n}^{\mathrm{X}}}=\left(\alpha^{\mathrm{nx}}-\beta^{\mathrm{nx}}\right)^{2}$, whence

$$
\begin{equation*}
\alpha^{\mathrm{n}^{\mathrm{X}+1}}-\beta^{\mathrm{n}^{\mathrm{X}+1}}>\left(\alpha^{\mathrm{n}^{\mathrm{X}}}-\beta \mathrm{n}^{\mathrm{x}}\right)^{2}(2 \not X \mathrm{n}, \mathrm{n} \geq 3) . \tag{11}
\end{equation*}
$$

For even $n \geq 3$ we have, by (10), (8), (9):

$$
\begin{aligned}
& \alpha^{\mathrm{n}^{\mathrm{x}+1}}-\beta^{\mathrm{n}^{\mathrm{x}+1}}>\alpha^{2 \mathrm{n}^{\mathrm{x}}}+3-\beta^{\mathrm{n}^{\mathrm{x}+1}}=\alpha^{2 \mathrm{n}^{\mathrm{x}}}-2+\left(5-\beta^{\mathrm{nx}+1}\right. \\
&>\alpha^{2 \mathrm{n}^{\mathrm{x}}}-2+\beta^{2 \mathrm{n}^{\mathrm{x}}}=\left(\alpha^{\mathrm{n}^{\mathrm{x}}}-\beta^{\mathrm{n}^{\mathrm{x}}}\right)^{2} \quad, \text { whence } \\
& \alpha^{\mathrm{n}^{\mathrm{x}+1}}-\beta^{\mathrm{n}^{\mathrm{x}+1}}>\left(\alpha^{\mathrm{n}^{\mathrm{x}}}-\beta^{\mathrm{n}^{\mathrm{x}}}\right)^{2}(2 \mid \mathrm{n}, \mathrm{n} \geq 3)
\end{aligned}
$$

$\overline{\overline{11}})$

Combining ( $\overline{11}$ ) and $(\overline{\overline{11}})$ we have:
(11)

$$
\alpha^{\mathrm{n}^{\mathrm{X}+1}}-\beta^{\mathrm{n}^{\mathrm{X}+1}}>\left(\alpha^{\mathrm{n}^{\mathrm{x}}}-\beta^{\mathrm{n}^{\mathrm{X}}}\right)^{2}(\mathrm{n} \geq 3)
$$

For $\mathrm{n} \geq 3$ we have, by (6), (11):

$$
\begin{gathered}
\mathrm{F}_{\mathrm{n}^{\mathrm{x}+1}}= \\
=\frac{1}{\sqrt{5}}\left(\alpha^{\mathrm{n}^{\mathrm{X}+1}}-\beta^{\mathrm{n}^{\mathrm{X}+1}}\right)>\frac{1}{\sqrt{5}}\left(\alpha^{\mathrm{n}^{\mathrm{X}}}-\beta^{\left.\mathrm{n}^{\mathrm{x}}\right)^{2}>\left(\frac{1}{\sqrt{5}}\right)^{2}\left(\alpha^{\mathrm{n}^{\mathrm{x}}}-\beta^{\mathrm{n}^{\mathrm{x}}}\right)^{2}} \begin{array}{c}
\left\{\frac{1}{\sqrt{5}}\left(\alpha^{\mathrm{n}^{\mathrm{X}}}-\beta^{\mathrm{n}^{\mathrm{x}}}\right)\right\}^{2}=\mathrm{F}_{\mathrm{n}^{\mathrm{x}}}^{2}, \quad \text { whence, } \\
\mathrm{F}_{\mathrm{n}^{\mathrm{X}+1}}>\mathrm{F}_{\mathrm{n}^{\mathrm{x}}}^{2} \quad(\mathrm{n} \geq 3) .
\end{array} .\right.
\end{gathered}
$$

( $\overline{4}$ )

We have, by (6), (9):

$$
\begin{aligned}
\mathrm{F}_{2^{\mathrm{X}+1}}= & \frac{1}{\sqrt{5}}\left(\alpha^{2^{\mathrm{X}+1}}-\beta^{2^{\mathrm{X}+1}}\right)>\frac{1}{5}\left(\alpha^{2^{\mathrm{X}+1}}-2+\beta^{2^{\mathrm{X}+1}}\right)= \\
& \left\{\frac{1}{\sqrt{5}}\left(\alpha^{2^{\mathrm{X}}}-\beta^{2^{\mathrm{X}}}\right)\right\}^{2}=\mathrm{F}_{2^{\mathrm{X}}}^{2}, \quad \text { whence }
\end{aligned}
$$

$$
\begin{equation*}
\mathrm{F}_{2^{\mathrm{X}+1}}>\mathrm{F}_{2^{\mathrm{X}}}^{2} \tag{4}
\end{equation*}
$$

Combining ( $\overline{4}$ ) and ( $\overline{4}$ ) we have (4).
We have, by (7):*

$$
\begin{gather*}
\alpha^{5^{\mathrm{X}+1}}=\left(\alpha^{5^{\mathrm{X}}}\right)^{5}=\left(\alpha^{5^{\mathrm{X}}}\right)^{3}\left(\alpha^{5^{\mathrm{X}}}\right)^{2}>\left(\frac{3}{2}\right)^{5} \alpha^{2.5^{\mathrm{X}}}=\frac{243}{32} \alpha^{2.5^{\mathrm{X}}}>7 \alpha^{2.5^{\mathrm{X}}}= \\
5 \alpha^{2.5^{\mathrm{X}}}+2 \alpha^{2.5^{\mathrm{X}}}>5 \alpha^{2.5^{\mathrm{X}}}+2\left(\frac{3}{2}\right)^{6}>5 \alpha^{2.5^{\mathrm{X}}}+22 \text {, whence } \\
\alpha^{5^{\mathrm{X}+1}}>5 \alpha^{2.5^{\mathrm{X}}}+22 \tag{12}
\end{gather*}
$$

We have, by (12), (8), (9):

$$
\begin{gathered}
\alpha^{5^{\mathrm{X}+1}}-\beta^{5^{\mathrm{X}+1}}>5 \alpha^{2.5^{\mathrm{X}}}+22-\beta^{5^{\mathrm{X}+1}}>5 \alpha^{2.5^{\mathrm{X}}}+10+\left(12-\beta^{5^{\mathrm{X}+1}}\right) \\
\quad>5 \alpha^{2.5^{\mathrm{X}}}+10+5 \beta^{2.5^{\mathrm{X}}}=5\left(\alpha^{2.5^{\mathrm{X}}}+2+\beta^{2.5^{\mathrm{X}}}\right)=5\left(\alpha^{5^{\mathrm{X}}}-\beta^{5^{\mathrm{X}}}\right)^{2}
\end{gathered}
$$

whence

$$
\begin{equation*}
\alpha^{5^{\mathrm{x}+1}}-\beta^{5^{\mathrm{x}+1}}>5\left(\alpha^{5^{\mathrm{x}}}-\beta^{5^{\mathrm{x}}}\right)^{2} . \tag{13}
\end{equation*}
$$

We have by (6), (13), (8):

$$
\begin{aligned}
\mathrm{F}_{5^{\mathrm{X}+1}} & =\frac{1}{\sqrt{5}}\left(\alpha^{5^{\mathrm{x}+1}}-\beta^{5^{\mathrm{X}+1}}\right)>\frac{1}{\sqrt{5}} 5\left(\alpha^{5^{\mathrm{x}}}-\beta^{5^{\mathrm{x}}}\right)^{2}>5\left\{\frac{1}{\sqrt{5}}\left(\alpha^{5^{\mathrm{x}}}-\beta^{5^{\mathrm{x}}}\right)\right\}^{2} \\
& =5 \mathrm{~F}_{5^{\mathrm{X}}}^{2},
\end{aligned}
$$

whence (5) is valid.
For odd $\mathrm{n} \geq 3$ we have, by $\left(6^{*}\right)$, (10), (8), (9):
$\mathrm{L}_{\mathrm{n}^{\mathrm{X}+1}}=\alpha^{\mathrm{n}^{\mathrm{X}+1}}+\beta^{\mathrm{n}} \mathrm{X}+1>\alpha^{2 \mathrm{n}^{\mathrm{X}}}+3+\beta^{\mathrm{n}^{\mathrm{X}+1}}=\alpha^{2 \mathrm{n}^{\mathrm{X}}}-2+\left(5+\beta^{\mathrm{n}^{\mathrm{X}+1}}\right)=$ $\alpha^{2 n^{\mathrm{X}}}-2+\beta^{2 \mathrm{n}^{\mathrm{X}}}=\left(\alpha^{\mathrm{n}^{\mathrm{X}}}+\beta^{\mathrm{n}^{\mathrm{x}}}\right)^{2}=\mathrm{L}_{\mathrm{n}^{\mathrm{X}}}^{2}$,
whence
*See editorial remark, page 59。

$$
\begin{equation*}
\mathrm{L}_{\mathrm{n}^{\mathrm{x}+1}}>\mathrm{L}_{\mathrm{n}^{\mathrm{x}}}^{2}(2 \nmid \mathrm{n}, \quad \mathrm{n} \geq 3) \tag{4}
\end{equation*}
$$

For even $\mathrm{n} \geq 3$ we have, by $\left(6^{*}\right)$, (10), (8), (9):
$\mathrm{L}_{\mathrm{n}^{\mathrm{x}+1}}=\alpha^{\mathrm{nx}+1}+\beta^{\mathrm{n}^{\mathrm{x}+1}}>\alpha^{2 \mathrm{n}^{\mathrm{x}}}+3=\alpha^{2 \mathrm{n}^{\mathrm{X}}}+2+1>\alpha^{2 \mathrm{n}^{\mathrm{x}}}+2+\beta^{2 \mathrm{n}^{\mathrm{X}}}=$ $\left(\alpha^{n^{x}}+\beta^{n^{\mathrm{X}}}\right)^{2}=L_{n \mathrm{x}}^{2} \quad, \quad$ whence

$$
\begin{equation*}
\mathrm{L}_{\mathrm{n}^{\mathrm{x}+1}}>\mathrm{L}_{\mathrm{n}^{\mathrm{x}}}(2 \mid \mathrm{n}, \quad \mathrm{n} \geq 3) \tag{4}
\end{equation*}
$$

For $n=2$ we have the well-known relation: $L_{2^{x}+1}=L_{2^{x}}^{2}-2$, whence $\left(\overline{\left.\overline{4^{*}}\right)}\right.$

$$
\mathrm{L}_{2^{\mathrm{X}+1}}<\mathrm{L}_{2^{\mathrm{x}}}^{2}
$$

Combining $\left(\overline{4}^{*}\right),\left(\overline{4}^{*}\right)$ and $\left(\overline{\overline{4}}{ }^{*}\right)$ we have $\left(4^{*}\right)$.
Proof of (3), (3*).
For $p \neq 5,\left(p, F_{p^{x}}\right)=1$. Hence, by the law of repetition of primes in ( $\mathrm{F}_{\mathrm{n}}$ ), the greatest imprimitive divisor of $\mathrm{F}_{\mathrm{p}^{\mathrm{x}+1}}$ is $\mathrm{F}_{\mathrm{p}^{\mathrm{x}}}$, whence, by (4):

$$
\mathrm{F}_{\mathrm{p}^{\mathrm{x}+1}}^{\prime}=\mathrm{F}_{\mathrm{p}^{\mathrm{x}+1}} / \mathrm{F}_{\mathrm{p}^{\mathrm{X}}}>\mathrm{F}_{\mathrm{p}^{\mathrm{X}}}
$$

i. e., (3) is valid for $p \neq 5$.

For $p=5$, by the law of repetition of primes in $\left(F_{n}\right)$, the greatest imprimitive divisor of $\mathrm{F}_{5 \mathrm{X}+1}$ is $5 \mathrm{~F}_{5 \mathrm{X}}$, whence, by (5):

$$
\mathrm{F}_{5}^{\mathrm{X}+1}=\mathrm{F}_{5^{\mathrm{X}}+1} / 5 \mathrm{~F}_{5^{\mathrm{X}}}>\mathrm{F}_{5^{\mathrm{X}}},
$$

i. e., $\quad F_{5}^{\prime} \mathrm{X}+1>\mathrm{F}_{5 \mathrm{X}}$, i. e., (3) is valid for $\mathrm{p}=5$.

For $p \neq 2$, by the law of repetition of primes in $\left(L_{n}\right)$, the greatest imprimitive divisor of $L_{p^{x+1}}$ is $L_{p^{x}}$, whence, by (4*): $L_{p^{x}+1}^{1}=L_{p^{x}+1} / L_{p^{x}}$ $>L_{p^{x}}$, i.e., $\left(3^{*}\right)$ is valid for $p \neq 2$.


## A GENERALIZATION OF THE CONNECTION BETWEEN THE FIBONACCI SEQUENCE AND PASCAL'S TRIANGLE JOSEPH A. RAAB, WISCONSIN STATE COLLEGE

Before the main point of this paper can be developed, it is necessary to review some elementary facts about the Fibonacci Sequence and Pascal's triangle.

It is well-known that rectangles exist such that if a full-width square is cut from one end, the remaining part has the same proportions as the original rectangle.


Assuming width to be unity and length x , we have

$$
\frac{1}{x}=\frac{x-1}{1}
$$

or

$$
\begin{equation*}
x^{2}-x-1=0 \tag{1}
\end{equation*}
$$

The greatest root of (1) is the number $\varphi$, called the Golden Ratio, and the rectangle defined is the Golden Rectangle of Greek geometry. Each root of (1) has the property that its reciprocal is itself diminished by 1 , so that

$$
\frac{1}{\varphi}=\varphi-1
$$

Given any two initial integral terms $u_{1}$ and $u_{2}$ not both zero, a Fibonacci Sequence is defined recursively by

$$
\begin{equation*}
u_{n}=u_{n-1}+u_{n-2} \tag{2}
\end{equation*}
$$

It is a well-known property of such sequences that

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\varphi
$$

If $u_{1}=0$ and $u_{2}=1$, we have the Fibonacci sequence.
If a rectangle is defined such that when an integral number $k$ of fullwidth squares are cut from one end, the remaining part has the same proportions as the original rectangle, then

$$
\begin{equation*}
\mathrm{y}^{2}-\mathrm{ky}-1=0 \tag{3}
\end{equation*}
$$

where the width is unity and the length is $y$.


The rectangle defined is a golden-type rectangle. The roots of (3) behave much like $\varphi$, that is, $1 / \mathrm{y}=\mathrm{y}-\mathrm{k}$. The greatest root in absolute value of (3) is the

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}},
$$

where $u_{n}=k u_{n-1}+u_{n-2}$. In fact, it is well-known that under certain conditions Fibonacci-like sequences defined by

$$
\begin{equation*}
u_{n}=a u_{n-1}+b u_{n-2} \tag{4}
\end{equation*}
$$

given initial terms $u_{1}$ and $u_{2}$ not both zero, where $a$ and $b$ are real, have the property that

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\alpha
$$

where $\alpha$ is the greatest root in absolute value of (See [3])

$$
\begin{equation*}
x^{2}-a x-b=0 \tag{5}
\end{equation*}
$$

The condition is that $a$ and $b$ must be such that the roots of (5) are not both distinct, and equal in absolute value.

The above general result can be established in the following way: Consider sequences such that the $n{ }^{\text {th }}$ term $u_{n}$ satisfies

$$
\begin{equation*}
u_{\mathrm{n}}=\mathrm{c} \alpha^{\mathrm{n}}+\mathrm{d} \beta^{\mathrm{n}} \tag{6}
\end{equation*}
$$

By substitution in (4), $\alpha$ and $\beta$ can be determined so that sequences (6) will satisfy (4) and be Fibonacci-like sequences. We find that the coefficients of c and d are $\alpha^{\mathrm{n}-2}\left(\alpha^{2}-\mathrm{a} \alpha-\mathrm{b}\right)$ and $\beta^{\mathrm{n}-2}\left(\beta^{2}-\mathrm{a} \beta-\mathrm{b}\right)$, respectively. Sequences (6), therefore, satisfy (4) if $\alpha$ and $\beta$ are roots of (5).

On the other hand, if $\alpha$ and $\beta$ are roots of (5), then $c \alpha^{\mathrm{n}-2}\left(\alpha^{2}-\mathrm{a} \alpha-\mathrm{b}\right)$ $+\mathrm{d} \beta^{\mathrm{n}-2}\left(\beta^{2}-\mathrm{a} \beta-\mathrm{b}\right)=0$ is satisfied for any choice of c and d . But then we have $\mathrm{c} \alpha^{\mathrm{n}}+\mathrm{d} \beta^{\mathrm{n}}=\mathrm{a}\left(\mathrm{c} \alpha^{\mathrm{n}-1}+\mathrm{d} \beta^{\mathrm{n}-1}\right)+\mathrm{b}\left(\mathrm{c} \alpha^{\mathrm{n}-2}+\mathrm{d} \beta^{\mathrm{n}-2}\right)$. Moreover, if $\alpha \neq \beta$, $c$ and $d$ can be determined given initial terms $u_{1}$ and $u_{2}$. Hence a sequence satisfying (4) satisfies (6) under the conditions stated. If $|\alpha|>|\beta|$, we can use (6) to obtain the

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\lim _{n \rightarrow \infty} \frac{c \alpha+d(\beta / \alpha)^{n} \beta}{c+d(\beta / \alpha)^{n}}=\alpha
$$

The above limit does not exist, of course, if $\alpha=-\beta$. If the roots of (5) are equal, then we can set

$$
\begin{equation*}
u_{\mathrm{n}}=\mathrm{c} \alpha^{\mathrm{n}}+\mathrm{nd} \alpha^{\mathrm{n}} \tag{7}
\end{equation*}
$$

and show by arguments similar to those above that (7) is a Fibonacci sequence if and only if $\alpha$ is the root of (5) and $\mathrm{a} \alpha+2 \beta=0$. But the roots of (5) are equal if and only if $\alpha=a / 2$ and $b=-a^{2} / 4$. Therefore all requirements for (7) being a Fibonacci sequence are met. It is now possible to solve for $c$ and $d$, and to show that for sequences (7),

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\alpha
$$

An interesting observation has been made about the array of numerals known as Pascal's Triangle. If a particular set of parallel diagonals is designated as in Fig. 1, then the sequence resulting from the individual summations of the terms of each diagonal is the Fibonacci sequence. [2]


Figure 1
Therefore, the limit of quotients of sums of terms on these parallel diagonals of the triangle is $\alpha$. We shall now show that some generalizations of this connection can be made.

To begin, we note that the indicated diagonal sums in Fig. 2 are indeed the first few terms (except the first) of (4) if $u_{1}=0$ and $u_{2}=1$.


Other sets of parallel diagonals of Fig. 2 also have interesting properties. It is possible to formalize the definition of the array given as Fig. 2, but it will be more efficacious here to simply refer informally to the figure in the arguments to follow. We will assume only that a and b are real, and that Fig. 2 is a Generalized Pascal's Triangle. The row index shall be $j$, and the term index for each row, $\delta$, each ranging over the non-negative integers. The $j^{\text {th }}$ power of $(a+b)$ is the sum of terms in the $j^{\text {th }}$ row of Fig. 2.

Definition 1. A diagonal sum $\mathrm{x}_{\mathrm{jr}}$ of the generalized Pascal's triangle shall be given by

$$
x_{j r}=\sum_{\delta=0}^{\left[\frac{j}{r+1}\right]}\binom{j-r \delta}{\delta} a^{j-\delta(r+1)} b^{\delta}
$$

Counting from left to right in Fig. 2, the $(\delta+1)$ th term of the diagonal sum is the $(\delta+1)$ th term in the $(j-r \delta)$ th row of the triangle as $\delta$ ranges over the non-negative integers. Hence $x_{j r}$ is a function of $j$ and $r$.

Note that the role of $r$ is simply to determine which terms of the triangle are to be summed. This has the effect of defining a set of parallel diagon-" als for each $r$. For example, if $r=1$, the first term of $x_{61}$ is the first
term of the sixth row of Fig. 2. The second term of $x_{61}$ is the second term of the fifth row of Fig. 2, and so on. If $r=3$, the first term of $x_{63}$ is the first term of the sixth row of Fig. 2, but the second term of $x_{63}$ is the second term of the third row, and so on. When $r=0, x_{j 0}$ is the sum of terms on the $j^{\text {th }}$ row. A sequence $\left\{\mathrm{X}_{\mathrm{jr}}\right\}_{j}$ of diagonal sums is uniquely determined by $r$. Since for $j=0$ the $(j-r \delta)^{\text {th }}$ row is defined for every $r$ only when $\delta=0$, $x_{0 r}=1$ for all $r$. Further, $x_{1 r}=a$ if $r>0$. If $r=2$, the first fewterms of the resulting sequence are:

$$
\left(1, a, a^{2}, a^{3}+b, a^{4}+2 a b, a^{5}+3 a^{2} b, \cdots\right)
$$

Theorem 1. For sequences $\left\{\mathrm{x}_{\mathrm{jr}}\right\}_{\mathrm{j}}$ of sums of terms on parallel diagonals of the generalized Pascal's triangle,

$$
\begin{equation*}
\mathrm{x}_{\mathrm{jr}}=a \mathrm{x}_{(\mathrm{j}-1) r}+\mathrm{bx} \mathrm{j}_{(\mathrm{j}-\mathrm{r}-1) \mathrm{r}} \tag{8}
\end{equation*}
$$

$$
=\sum_{\delta=1}^{\left[\frac{j}{r+1}\right]}\binom{j-r \delta-1}{\delta-1} a^{j-\delta(r+1)_{b} \delta}+\sum_{\delta=0}^{\left[\frac{j-1}{r+1}\right]}\binom{j-r \delta-1}{\delta} a^{j-\delta(r+1)_{b} \delta}
$$

$$
=\sum_{\delta=1}^{\left[\frac{j}{r+1}\right]}\binom{j-r \delta-1}{\delta-1} a^{j-\delta(r+1)} b^{\delta}+a^{j}+\sum_{\delta=1}^{\left[\frac{j-1}{r+1}\right]}\binom{j-r \delta-1}{\delta} a^{j-\delta(r+1) b^{\delta}}
$$

$$
=a^{j}+\sum_{\delta=1}^{\left[\frac{j}{r+1}\right]}\left\{\binom{j-r \delta-1}{\delta-1}+\binom{j-r \delta-1}{\delta}\right\} a^{j-\delta(r+1)_{b} \delta}
$$

but

$$
\binom{\mathrm{j}-\mathrm{r} \delta-1}{\delta-1}=\binom{\mathrm{j}-\mathrm{r} \delta}{\delta} \cdot \frac{\delta}{\mathrm{j}-\mathrm{r} \delta}
$$

and

$$
\binom{j-r \delta}{\delta} \cdot \frac{j-\delta(r+1)}{j-r \delta}=\binom{j-r \delta-1}{\delta}
$$

so

$$
\begin{aligned}
b x_{(j-r-1) r}+a x_{(j-1) r}= & a^{j}+\sum_{\delta=1}^{\left[\frac{j}{r+1}\right]}\left\{\begin{array}{c}
\binom{j-r \delta}{\delta} \cdot \frac{\delta}{j-r \delta} \\
\\
\\
+\binom{j-r \delta}{\delta} \cdot \frac{j-\delta(r+1)}{j-r \delta} \\
\end{array}\right) a^{j-\delta(r+1)_{b} \delta} \\
& =a^{j}+\sum_{\delta=1}^{\left[\frac{j}{r+1}\right]}\binom{j-r \delta}{\delta} a^{j-\delta(r+1)} b^{\delta}=x_{j r} .
\end{aligned}
$$

In view of Theorem 1, any property of sequences defined recursively by

$$
\begin{equation*}
u_{n}=a u_{n-1}+b u_{n-r-1} \tag{9}
\end{equation*}
$$

will be a property of sequences of sums of terms on diagonals of the generalized Pascal's triangle. Further, these diagonal sequences will all be of the special case $u_{1}=0, u_{2}=1, u_{3}=a, \cdots, u_{r+1}=a^{r-1} ;$ since $r+1$ initial terms are required for (9) to generate a sequence. We note that diagonal sum $x_{(n-2) r}$ is $u_{n}$ of (9) given the above initial terms.

As in the proof of

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\varphi
$$

given (2), we shall establish the existence of similar limits for the sequences defined by (9). If we set

$$
\begin{equation*}
\mathrm{u}_{\mathrm{n}}=\mathrm{e}_{0} \alpha_{0}^{\mathrm{n}}+\mathrm{e}_{1} \alpha_{1}^{\mathrm{n}}+\mathrm{e}_{2} \alpha_{2}^{\mathrm{n}}+\cdots+\mathrm{e}_{\mathrm{r}} \alpha_{\mathrm{r}}^{\mathrm{n}} \tag{10}
\end{equation*}
$$

then substituting in (9) the coefficients of the $e_{i}$ are

$$
\alpha_{i}^{n-r-1}\left(\alpha_{i}^{r+1}-\mathrm{a} \alpha_{i}^{\mathrm{r}}-\mathrm{b}\right)(\mathrm{i}=0,1, \cdots, r)
$$

and (9) is satisfied if the $\alpha_{i}$ are the $r+1$ roots of

$$
\begin{equation*}
x^{r+1}-a x^{r}-b=0 \tag{11}
\end{equation*}
$$

Conversely, given that the $\alpha_{i}$ are the roots of (11), it follows that sequences (9) canbe written in the form of (10) if the $e_{i}$ can be determined. One can obtain from the given $(r+1)$ initial terms $(r+1)$ equations $u_{j}=e_{0} \alpha_{0}^{j}+e_{1} \alpha_{1}^{j}$ $+\cdots+\mathrm{e}_{\mathrm{r}} \alpha_{\mathrm{r}}^{\mathrm{j}}(\mathrm{j}=1,2, \cdots, \mathrm{r}+1)$. This set of equations has a non-trivial solution for the $e_{i}$, however, if and only if the $\alpha_{i}$ are distinct. Whether or not the terms of sequences defined by (9) can be written in the form of (10) depends, therefore, on whether or not we can determine conditions for the multiplicity of the roots of (11).

Suppose p is a root of (11) where a and b are both not zero. Then (11) may be written as $(x-p) Q(x)=0$ where
$Q(x)=x^{r}+(p-a) x^{r-1}+(p-a) p x^{r-2}+(p-a) p^{2} x^{r-3}+\cdots+(p-a) p^{r-1}$.

Clearly $p$ is a multiple root of (11) if and only if it is a root of $Q(x)=0$. But then it is easily verified that

$$
p=\frac{a r}{r+1}
$$

Now since $p$ is real, at least all complex roots of (11) are distinct.
DeGua's rule for finding imaginary roots states that when 2 m successive terms of an equation are absent, the equation has 2 m imaginary roots; and when $2 \mathrm{~m}-1$ successive terms are absent, the equation has $2 \mathrm{~m}-2$ or 2 m imaginary roots, according as the two terms between which the deficiency occurs have like or unlike signs. Accordingly, we see that (11) has at most three real roots, since there are $r-1$ successive absent terms and hence at least $r-2$ complex roots. Further, if $f(x)=x^{r+1}-a x^{r}-b$, the two critical numbers of $f$ are zero and $\operatorname{ar} /(r+1)$. Since $f(a r /(r+1))$ is an extremum of $f$, the greatest multiplicity of any real root of (11) is two. [1]

If $b$ is zero but $a$ is not, then the roots of (11) are zero (of multiplicity r ), and a. Other cases are trivial.

If the real roots of (11) are distinct and $\alpha_{0}$ is any root such that $\left|\alpha_{0}\right|$ $>\left|\alpha_{i}\right|(i=1,2, \cdots, r)$, then

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}} & =\lim _{n \rightarrow \infty} \frac{e_{0} \alpha_{0}^{n+1}+e_{1} \alpha_{1}^{n+1}+\cdots+e_{r} \alpha_{r}^{n+1}}{e_{0} \alpha_{0}^{n}+e_{1} \alpha_{1}^{n}+\cdots+e_{r} \alpha_{r}^{n}} \\
& =\lim _{n \rightarrow \infty} \frac{e_{0} \alpha_{0}+e_{1} \alpha_{1}\left(\alpha_{1} / \alpha_{0}\right)^{n}+\cdots+e_{r} \alpha_{r}\left(\alpha_{r} / \alpha_{0}\right)^{n}}{e_{0}+e_{1}\left(\alpha_{1} / \alpha_{0}\right)^{n}+\cdots+e_{r}\left(\alpha_{r} / \alpha_{0}\right)^{n}}
\end{aligned}
$$

Therefore

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\alpha_{0}
$$

It is clear that $\operatorname{ar} /(r+1)$ is a root of (11) if and only if

$$
b=-\frac{a^{r+1} r^{r}}{(r+1)^{r+1}}
$$

Suppose $\alpha_{0}$ and $\alpha_{1}$ are this root. Then we can set

$$
\begin{equation*}
u_{\mathrm{n}}=\mathrm{e}_{0} \alpha_{0}^{\mathrm{n}}+\mathrm{ne}_{1} \alpha_{0}^{\mathrm{n}}+\mathrm{e}_{2} \alpha_{2}^{\mathrm{n}}+\cdots+\mathrm{e}_{\mathrm{r}} \alpha_{\mathrm{r}}^{\mathrm{n}} \tag{12}
\end{equation*}
$$

and use (9) to find the coefficients of the $e_{i}$. The coefficient of $e_{i}$ where $i \neq 1$ is $\alpha_{i}^{n-r-1}\left(\alpha_{i}^{r+1}-\mathrm{a} \alpha_{i}^{r}-b\right)$ and for $e_{i}$ we have

$$
\mathrm{n} \alpha_{0}^{\mathrm{n}-\mathrm{r}-1}\left(\alpha_{0}^{\mathrm{r}+1}-\mathrm{a} \alpha_{0}^{\mathrm{r}}-\mathrm{b}+\frac{\mathrm{a} \alpha_{0}^{\mathrm{r}}}{\mathrm{n}}+\frac{\mathrm{b}(\mathrm{r}+1)}{\mathrm{n}}\right)
$$

It is clear that the required condition is that the $\alpha_{i}$ be the roots of (11) and $\mathrm{a} \alpha_{0}^{\mathrm{r}}+\mathrm{b}(\mathrm{r}+1)=0$. But with $\alpha_{0}$ chosen as above, this is indeed the case. As before, (12) can be used to generate equations which enable us to find the $e_{i}$. Finally

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}
$$

exists and is the greatest root of (11) in absolute value.
Since (9) generates a real sequence given real initial terms, not only is

$$
\lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}
$$

the greatest root of (11) in absolute value, but it must also be real. Hence the greatest root in absolute value of (11) must be real.

If $a, b$, and $r$ in (11) are such that two distinct roots share the greatest absolute value of all roots, then it is easily shown that no limit exists.

Employing simple unit theorems, we can prove that

$$
\lim _{n \rightarrow \infty} \frac{u_{n+s}}{u_{n}}=\alpha_{0}^{s} \text { if } \lim _{n \rightarrow \infty} \frac{u_{n+1}}{u_{n}}=\alpha_{0}
$$

We are now able to state that:

Theorem 2. For all sequences formed by sums of terms on parallel diagonals of the generalized Pascal's triangle, and for all sequences defined by (9) given $r+1$ initial terms,

$$
\lim _{n \rightarrow \infty} \frac{u_{n+s}}{u_{n}}
$$

exists and is the greatest root in absolute value of

$$
x^{\frac{r+1}{s}}-a x^{\frac{r}{s}}-b=0
$$

provided this absolute value is not shared by two distinct roots.

## REFERENCES

1. W. S. Burnside and A. W. Panton, Introduction to the Theory of Binary Algebraic Forms, Dublin University Press, 1918, p. 197.
2. L. E. Dickson, History of the Theory of Numbers, Washington, D. C., Carnegie Institute, 1919-1923.
3. B. W. Jones, The Theory of Numbers, Rinehart and Company, 1955, pp. 77-99.
$\triangle \operatorname{Rec}$
REFERENCES
(Cont. from p. 14)
4. V. E. Hoggatt and C. King, Prob. E1424, American Mathematical Monthly, Vol. 66, 1959, pp. 129-130.
5. H. L. Alder, "The Number System in More General Scales," Mathematical Magazine, June 1962, pp. 147-148.
6. J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, Vol. 68, 1961, pp. 557-560.


## CHARTER MEMBER, MORGAN WARD, PASSES AWAY

Durate, Calif., June 26, 1963 - Dr. Morgan Ward, 61, well-known mathematician on the faculty of the California Institute of Technology, died today of a heart attack at the City of Hope Medical Center.

As a research mathematician, Dr. Ward was noted for his work in algebra and number theory, with particular emphasis on arithmetical sequences. During the past few years he worked with the School Mathematics Study Group set up by the National Science Foundation to reform elementary school mathematics curricula. He was comauthor with Dr. Clarence Hardgrove of a modern elementary mathematics text book which will be published this fall.

Born in New York City, Dr. Ward spent most of his life in Southern California. He received his B. A. at the University of California at Berkeley in 1924 and his Ph. D. in mathematics, summa cum laude, at Caltech in 1928. He joined the Caltech faculty as assistant professor of mathematics in 1929, became associate professor in 1935 and full professor in 1940.

In 1934-1935 he did research work at the Institute for Advanced Study in Princeton, N. J., and from. 1941 to 1944 he served as consultant to the Office of Scientific Research and Development on problems of underwater ballistics and anti-submarine warfare. He was a member of the American Mathematical Society and the American Mathematical Association.

Dr. Ward was an accomplished pianist, a student of poetry, and an expert chess and GO player.

The mathematician, who lived at 1550 San Pasqual Street, Pasadena, is survived by his wife, Sigrid; a daughter, Audrey Ward Gray of China Lake, Calif. ; three sons, Eric, Richard and Samuel of Pasadena; three brothers, Robert Miller, Malcolm Miller and Samuel Ward, and four grandchildren.

# PERIODIC PROPERTIES OF FIBONACCI SUMMATIONS BROTHER U. ALFRED, ST. MARY'S COLLEGE, CALIFORNIA 

## INTRODUCTION

It is well known that if we take the terms of the Fibonacci sequence modulo $m$ that the least positive residues form a periodic sequence. This paper will consider the summation of functions of such residues taken over a period with the further limitations that for most of the results the modulus considered is a prime and the total degree of the product being summed is less than the prime modulus.

## NOTATION

We employ the usual notation $\mathrm{F}_{\mathrm{i}}$ to signify the terms of the Fibonacci sequence: $1,1,2,3,5,8, \ldots$. The letter $p$ represents a prime and $m$ any positive integer.

We shall be considering summations such as:

$$
\sum_{\mathrm{P}} F_{i}^{3} F_{i-3}^{2} F_{i-5}^{4}
$$

where the subscripts of the Fibonacci numbers in the product differ from each other by fixed integers; where the summation is taken over a period for a given modulus $p_{9}$ this being indicated by having $P$ below the summation sign; and where the total degree $n$ of the product being summed is the sum of the exponents of the Fibonacci numbers.

Theorem 1. The summation of the residues of the Fibonacci sequence over a period is congruent to zero modulo m .

Proof: From the basic relation for the Fibonacci sequence

$$
F_{i}=F_{i-1}+F_{i-2}
$$

it follows that

$$
\sum_{P} F_{i}=\sum_{P} F_{i-1}+\sum_{P} F_{i-2}
$$

From the nature of periodicity, it is clear that the summation over a period will always be congruent to the same quantity for a given modulus regardless o: where we start in the sequence. Thus

$$
\sum_{P} F_{i} \equiv \sum_{P} F_{i-1} \equiv \sum_{P} F_{i-2} \quad(\bmod m)
$$

so that

$$
\sum_{P} F_{i} \equiv \underset{P}{2 \Sigma F_{i}} \quad(\bmod m)
$$

which leads to the conclusion that

$$
\sum_{P} F_{i} \equiv 0 \quad(\bmod m)
$$

Theorem 2. The summations

$$
\sum_{\mathrm{P}}^{\Sigma} \mathrm{F}_{\mathrm{i}}^{2} \quad \text { and } \quad \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}} \mathrm{~F}_{\mathrm{i}-1}
$$

are congruent to zero modulo any prime with the possible exception of 2 .
Proof. For convenience we shall replace

$$
\sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{2} \quad \text { by } \quad \text { a } \quad \text { and } \quad \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}} \mathrm{~F}_{\mathrm{i}-1} \text { by } \text { b }
$$

noting once more that the precise subscript of $F$ is inconsequential when computing the residue modulo $p$ over a period. We start as before with the relation $\mathrm{F}_{\mathrm{i}}=\mathrm{F}_{\mathrm{i}-1}+\mathrm{F}_{\mathrm{i}-2}$ and the derived relation $\mathrm{F}_{\mathrm{i}}=2 \mathrm{~F}_{\mathrm{i}-2}+\mathrm{F}_{\mathrm{i}-3}$. By
squaring each of these relations and summing over the period, we obtain
and
or

$$
a \equiv a+2 b+a \quad(\bmod p)
$$

$$
a \equiv 4 a+4 b+a(\bmod p)
$$

$$
\mathrm{a}+2 \mathrm{~b} \equiv 0 \quad(\bmod \mathrm{p})
$$

$$
4 a+4 b \equiv 0 \quad(\bmod p)
$$

Hence we can conclude that $a$ and $b$ must both be congruent to zero modulo $p$ with the possible exception of the case in which the determinant of the coefficients is congruent to zero. But this determinant equals -4 so that the only prime in question would be 2 . We find by direct verification that

$$
\sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{2} \equiv 0(\bmod 2) \text { but that } \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}} \mathrm{~F}_{\mathrm{i}-1} \text { is not. }
$$

Theorem 3. With the possible exception of primes 2 and 3 all summations

$$
\sum_{P} F_{i}^{3}, \sum_{P} F_{i}^{2} F_{i-1}, \text { and } \sum_{P} F_{i} F_{i-1}^{2}
$$

are congruent to zero modulo $p$.
Proof. We employ the same procedure as before after replacing

$$
\sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{3} \text { by a, } \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{2} \mathrm{~F}_{\mathrm{i}-1} \text { by b and } \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}} \mathrm{~F}_{\mathrm{i}-1}^{2} \text { by } \mathrm{c}
$$

Starting with

$$
F_{i}=F_{i-1}+F_{i-2}
$$

and the two derived relations

$$
\begin{aligned}
& F_{i}=2 F_{i-2}+F_{i-3} \\
& F_{i}=3 F_{i-3}+2 F_{i-4}
\end{aligned}
$$

we cube each of them and sum over a period to obtain:

$$
\begin{gathered}
a \equiv a+3 b+3 c+a(\bmod p) \\
a \equiv 8 a+12 b+6 c+a(\bmod p) \\
a \equiv 27 a+54 b+36 c+8 a(\bmod p)
\end{gathered}
$$

or

$$
\begin{gathered}
a+3 b+3 c \equiv 0 \quad(\bmod p) \\
8 a+12 b+6 c \equiv 0 \quad(\bmod p) \\
34 a+54 b+36 c \equiv 0 \quad(\bmod p)
\end{gathered}
$$

The quantities $a, b$, and $c$ are all congruent to zero except possibly when the determinant of the coefficients is congruent to zero modulo $p$. The value of this determinant being $-2^{3} 3^{2}$, the only possible exceptions might be the primes 2 and 3.

## FURTHER DEDUCTION

It should be noted that if $a, b$, and $c$ are congruent to zero modulo $p$, then any expression such as

$$
\sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{2} \mathrm{~F}_{\mathrm{i}-4}
$$

is also congruent to zero modulo $p$. The reason is that $F_{i-4}$ can be expressed as a linear relation in $F_{i}$ and $F_{i-1}$ so that this summation becomes a linear combination of $a, b$, and $c$. Similar considerations apply for any degree whatsoever. Once it is known that all the summations

$$
\sum_{\mathrm{P}}^{\sum F_{i}^{n}}, \quad \sum_{\mathrm{P}} \mathrm{~F}_{\mathrm{i}}^{\mathrm{n}-1} \mathrm{~F}_{\mathrm{i}-1}, \sum_{\mathrm{P}}^{\Sigma F_{i}^{n-2} F_{i-1}^{2}}, \cdots, \sum_{\mathrm{P}}^{\Sigma F_{i}^{2} F_{i-1}^{n-2}, \sum_{P} F_{i} F_{i-1}^{n-1}}
$$

are all congruent to zero modulo $p$, then any summation product of degree $n$ of the type we are considering taken over a period will also be congruent to zero modulo p .

## GENERAL CASE

The pattern established in the above theorems may clearly be extended to higher degrees. To fix ideas, the fifth power summations will be used. As previously, let $\sum_{\mathrm{P}}^{\sum} \mathrm{F}_{\mathrm{i}}^{5}$ be replaced by a, $\sum_{\mathrm{P}} \mathrm{F}_{\mathrm{i}}^{4} \mathrm{~F}_{\mathrm{i}-1}$ by b, $\sum_{\mathrm{P}} \mathrm{F}_{\mathrm{i}}^{3} \mathrm{~F}_{\mathrm{i}-1}^{2}$ by c , $\sum_{P} F_{i}^{2} F_{i-1}^{3}$ by $d$, and $\sum_{P} F_{i} F_{i-1}^{4}$ by $e$.

Starting with the relations

$$
\begin{aligned}
& \mathrm{F}_{\mathrm{i}}=\mathrm{F}_{\mathrm{i}-1}+\mathrm{F}_{\mathrm{i}-2} \\
& \mathrm{~F}_{\mathrm{i}}=2 \mathrm{~F}_{\mathrm{i}-2}+\mathrm{F}_{\mathrm{i}-3} \\
& \mathrm{~F}_{\mathrm{i}}=3 \mathrm{~F}_{\mathrm{i}-3}+2 \mathrm{~F}_{\mathrm{i}-4} \\
& \mathrm{~F}_{\mathrm{i}}=5 \mathrm{~F}_{\mathrm{i}-4}+3 \mathrm{~F}_{\mathrm{i}-5} \\
& \mathrm{~F}_{\mathrm{i}}=8 \mathrm{~F}_{\mathrm{i}-5}+5 \mathrm{~F}_{\mathrm{i}-6}
\end{aligned}
$$

we obtain on raising each to the fifth power and summing over a period of the modulus p :

$$
\begin{gathered}
\mathrm{a}+5 \mathrm{~b}+10 \mathrm{c}+10 \mathrm{~d}+5 \mathrm{e} \equiv 0(\bmod \mathrm{p}) \\
2^{5} \mathrm{a}+5 \cdot 2^{4} \mathrm{~b}+10 \cdot 2^{3} \mathrm{c}+10 \cdot 2^{2} \mathrm{~d}+5 \cdot 2 \mathrm{e} \equiv 0(\bmod \mathrm{p}) \\
\left(3^{5}+2^{5}-1\right) \mathrm{a}+5 \cdot 3^{4} 2 \mathrm{~b}+10 \cdot 3^{3} 2^{2} \mathrm{c}+103^{2} 2^{3} \mathrm{~d}+5 \cdot 3 \cdot 2^{4} \mathrm{e} \equiv 0(\bmod \mathrm{p}) \\
\left(5^{5}+3^{5}-1\right) \mathrm{a}+5 \cdot 5^{4} 3 \mathrm{~b}+10 \cdot 5^{3} 3^{2} \mathrm{c}+10 \cdot 5^{2} 3^{3} \mathrm{~d}+5 \cdot 5 \cdot 3^{4} \mathrm{e} \equiv 0(\bmod \mathrm{p}) \\
\left(8^{5}+5^{5}-1\right) \mathrm{a}+5 \cdot 8^{4} 5 \mathrm{~b}+10 \cdot 8^{3} 5^{2} \mathrm{c}+10 \cdot 8^{2} 5^{3} \mathrm{~d}+5 \cdot 8 \cdot 5^{4} \mathrm{e} \equiv 0(\bmod \mathrm{p})
\end{gathered}
$$

Once again, the quantities $a, b, c, d$, and $e$ are all congruent to zero modulo p provided:
(1) The determinant of the coefficients is not identically equal to zero; or
(2) The determinant of the coefficients is not congruent to zero modulo p. Thus precise information on which summations are congruent to zero modulo any given prime is related to knowing the value of the determinant of the coefficients. These determinants have been made the object of extensive study by the author and Terry Brennan who will elaborate the results of their research in a future issue of this publication. For the present, let it suffice to
say that the formulas derived empirically by evaluating these determinants to the nineteenth order have now been theoretically justified.

It will be noted that the binomial coefficients of the fifth degree enter into the equations and that these may all be factored from the determinant. As long as the degree of the summation is less than $p$, these factored binomial coefficients do not affect the issue. Disregarding them, the remaining determinant is as follows.
$\left|\begin{array}{lllll}1 & 1 & 1 & 1 & 1 \\ 2^{5} & 2^{4} & 2^{3} & 2^{2} & 2 \\ 3^{5}+2^{5}-1 & 3^{4} 2 & 3^{3} 2^{2} & 3^{2} 2^{3} & 3 \cdot 2^{4} \\ 5^{5}+3^{5}-1 & 5^{4} 3 & 5^{3} 3^{2} & 5^{2} 3^{3} & 5 \cdot 3^{4} \\ 8^{5}+5^{5}-1 & 8^{4} 5 & 8^{3} 5^{2} & 8^{2} 5^{3} & 8 \cdot 5^{4}\end{array}\right|$

If $n$ be the degree of the summation and the order of the determinant, it is found empirically that:
(1) For $n \equiv 0(\bmod 4)$, the value of the determinant is zero. Thus for summations of degree 4 k , none need be congruent to zero modulo any prime.
(2) For $\mathrm{n} \equiv 2(\bmod 4)$, the value of the determinant is:
(1)

where $L_{i}$ indicates the members of the Lucas sequence which is also of the Fibonacci type but with values $L_{1}=1, L_{2}=3, L_{3}=4$, etc.
(3) For n odd, the value is

$$
\begin{equation*}
\prod_{i=3}^{n} F_{i}^{n-i+1} \prod_{i=1}^{(n+1) / 2} L_{2 i-1} \tag{2}
\end{equation*}
$$

For the convenience of the reader the express value of these determinants up to order 20 are given below, omitting those of order 4 k which are all equal to zero.

## VALUE OF DETERMINANT

n

```
2 2
```

$3 \quad 2^{3}$
$5 \quad 2^{5} 3^{2} 5 \cdot 11$
$6 \quad 2^{12} 3^{5} 5^{2}$
$7 \quad 2^{13} 3^{4} 5^{3} 11 \cdot 13 \cdot 29$
$9 \quad 2^{24} 3^{8} 5^{5} 7^{2} 11 \cdot 13^{3} 17 \cdot 19 \cdot 29$
$10 \quad 2^{30} 3^{12} 5^{7} 7^{5} 11^{3} 13^{4} 17^{2}$
$11 \quad 2^{34} 3^{12} 5^{9} 7^{4} 11^{3} 13^{5} 17^{3} 19 \cdot 29 \cdot 89 \cdot 199$
$13 \quad 2^{52} 3^{20} 5^{13} 7^{6} 11^{5} 13^{7} 17^{5} 19 \cdot 29 \cdot 89^{3} 199 \cdot 233 \cdot 521$
$14 \quad 2^{64} 3^{30} 5^{15} 7^{9} 11^{7} 13^{9} 17^{6} 29^{3} 89^{4} 233^{2}$
$15 \quad 2^{73} 3^{28} 5^{18} 7^{8} 11^{8} 13^{11} 17^{11} 19 \cdot 29^{3} 31 \cdot 61 \cdot 89^{5} 199 \cdot 233^{3} 521$
$17 \quad 2^{93} 3^{38} 5^{24} 7^{12} 11^{10} 13^{15} 17^{9} 19 \cdot 29^{5} 31 \cdot 47^{2} 61^{3} 89^{7} 199 \cdot 233^{5} \cdot 521 \cdot 1597$
- 3571
$18 \quad 2^{111} 3^{49} 5^{27} 7^{16} 11^{11} 13^{17} 17^{11} 19^{3} 29^{7} 47^{5} 61^{4} 89^{8} 233^{6} 1597^{2}$
$19 \quad 2^{119} 3^{48} 5^{30} 7^{16} 11^{12} 13^{19} 17^{13} 19^{3} 29^{7} 31 \cdot 47^{4} 61^{5} 89^{9} 199 \cdot 233^{7} 521 \cdot 1597^{3}$
- $3571 \cdot 4181 \cdot 9349$

## EXAMPLE

For the modulus $\mathrm{p}=19$, it follows from the above determinant values that we might expect to have the sums of powers over a period congruent to zero for $n=1,2,3,5,6,7,10,14$. The actual situation is shown in Table 1 from which it is clear that theory is corroborated.

Table 2 shows the powers at which summations of Fibonacci expressions may cease to be congruent to zero modulo p .

Table 3 shows the comparison of theory and calculation for small primes. A 0 in the table indicates by theory and calculation the summation to degree $n$ modulo the given prime is zero; $x$ means that the summation need not be zero by theory; (x) indicates that theory does not require a sum congruent to zero, but that in reality it is congruent to zero. There is in this no contradiction.

Table 1
RESIDUES OF POWERS OF FIBONACCI NUMBERS MODULO 19
(Captions give n)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 13 | 7 | 14 | 9 | 18 | 17 | 15 | 11 | 3 | 6 | 12 | 5 | 10 | 1 |
| 3 | 9 | 8 | 5 | 15 | 7 | 2 | 6 | 18 | 16 | 10 | 11 | 14 | 4 | 12 | 17 | 13 | 1 |
| 5 | 6 | 11 | 17 | 9 | 7 | 16 | 4 | 1 | 5 | 6 | 11 | 17 | 9 | 7 | 16 | 4 | 1 |
| 8 | 7 | 18 | 11 | 12 | 1 | 8 | 7 | 18 | 11 | 12 | 1 | 8 | 7 | 18 | 11 | 12 | 1 |
| 13 | 17 | 12 | 4 | 14 | 11 | 10 | 16 | 18 | 6 | 2 | 7 | 15 | 5 | 8 | 9 | 3 | 1 |
| 2 | 4 | 8 | 16 | 13 | 7 | 14 | 9 | 18 | 17 | 15 | 11 | 3 | 6 | 12 | 5 | 10 | 1 |
| 15 | 16 | 12 | 9 | 2 | 11 | 13 | 5 | 18 | 4 | 3 | 7 | 10 | 17 | 8 | 6 | 14 | 1 |
| 17 | 4 | 11 | 16 | 6 | 7 | 5 | 9 | 1 | 17 | 4 | 11 | 16 | 6 | 7 | 5 | 9 | 1 |
| 13 | 17 | 12 | 4 | 14 | 11 | 10 | 16 | 18 | 6 | 2 | 7 | 15 | 5 | 8 | 9 | 3 | 1 |
| 11 | 7 | 1 | 11 | 7 | 1 | 11 | 7 | 1 | 11 | 7 | 1 | 11 | 7 | 1 | 11 | 7 | 1 |
| 5 | 6 | 11 | 17 | 9 | 7 | 16 | 4 | 1 | 5 | 6 | 11 | 17 | 9 | 7 | 16 | 4 | 1 |
| 16 | 9 | 11 | 5 | 4 | 7 | 17 | 6 | 1 | 16 | 9 | 11 | 5 | 4 | 7 | 17 | 6 | 1 |
| 2 | 4 | 8 | 16 | 13 | 7 | 14 | 9 | 18 | 17 | 15 | 11 | 3 | 6 | 12 | 5 | 10 | 1 |
| 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 | 18 | 1 |
| 1 | $\frac{1}{18}$ | $\frac{1}{152}$ | $\frac{1}{151}$ | $\frac{1}{152}$ | $\frac{1}{95}$ | $\frac{1}{171}$ | $\frac{1}{111}$ | $\frac{1}{170}$ | $\frac{1}{152}$ | $\frac{1}{127}$ | $\frac{1}{115}$ | $\frac{1}{158}$ | $\frac{1}{95}$ | $\frac{1}{140}$ | $\frac{1}{136}$ | $\frac{1}{126}$ | $\frac{1}{17}$ |

Table 2

| p | n odd | $\mathrm{n}=4 \mathrm{k}+2$ | p | n odd | $\mathrm{n}=4 \mathrm{k}+2$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 2 | 3 | 2 | 43 | 45 | 46 |
| 3 | 5 | 6 | 47 | 17 | 18 |
| 5 | 5 | 6 | 53 | 27 | 30 |
| 7 | 9 | 10 | 59 | 29 | 58 |
| 11 | 5 | 10 | 61 | 15 | 18 |
| 13 | 7 | 10 | 67 | 69 | 70 |
| 17 | 9 | 10 | 71 | 35 | 70 |
| 19 | 9 | 18 | 73 | 37 | 38 |
| 23 | 25 | 26 | 79 | 39 | 78 |
| 29 | 7 | 14 | 83 | 85 | 86 |
| 31 | 15 | 20 | 89 | 11 | 14 |
| 37 | 19 | 22 | 97 | 49 | 50 |
| 41 | 21 |  | 101 | 25 | 50 |

Table 3
ZERO AND NON-ZERO SUMMATIONS
FOR SMALL PRIMES

| n | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | x | x | x | x | x | x | x | x | x |
| 5 |  | 0 | x | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 |  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 |  |  | x | (x) | 0 | 0 | 0 | x | 0 |
| 8 |  |  | x | x | x | x | x | x | x |
| 9 |  |  | x | (x) | (x) | x | 0 | x | 0 |
| 10 |  |  | x | (x) | (x) | 0 | 0 | 0 | 0 |
| 11 |  |  |  | (x) | (x) | x | 0 | x | 0 |
| 12 |  |  |  | x | x | x | x | x | x |
| 13 |  |  |  |  | (x) | x | 0 | x | 0 |
| 14 |  |  |  |  | (x) | 0 | 0 | x | 0 |
| 15 |  |  |  |  | (x) | x | 0 | x | x |
| 16 |  |  |  |  | x | x | x | x | x |
| 17 |  |  |  |  |  | x | 0 | x | x |
| 18 |  |  |  |  |  | x | 0 | x | 0 |
| 19 |  |  |  |  |  |  | 0 | x | x |
| 20 |  |  |  |  |  |  | x | x | X |
| 21 |  |  |  |  |  |  | 0 | x | x |
| 22 |  |  |  |  |  |  | 0 | x | 0 |
| 23 |  |  |  |  |  |  |  | x | x |
| 24 |  |  |  |  |  |  |  | x | x |
| 25 |  |  |  |  |  |  |  | x | x |
| 26 |  |  |  |  |  |  |  | x | 0 |
| 27 |  |  |  |  |  |  |  | x | x |
| 28 |  |  |  |  |  |  |  | x | x |
| 29 |  |  |  |  |  |  |  |  | x |
| 30 |  |  |  |  |  |  |  |  | x |

In addition to the exceptions for $\mathrm{n}=7,9,10,11$ modulo 13 and $\mathrm{n}=9,10$, $11,13,14,15$ modulo 17 , an interesting example was found by Dmitri Thoro using a computer. For modulo 199 (period 22), the power summations should be zero for $1,2,3,5,6,7,9,10,14,18$ but no others need be zero. Actually, an additional summation congruent to zero was found for $\mathrm{n}=156$.

## ADDITIONAL RESEARCH POSSIBILITIES

The following offer additional research possibilities along these lines:
(1) The situation when $\mathrm{n} \geq \mathrm{p}$.
(2) The theory for composite moduli.
(3) Similar summations for other Fibonacci sequences than $\mathrm{F}_{\mathrm{i}}$.
(4) Possibly by means of additional computer data, the study of cases in which summations are congruent to zero when they need not be; patterns and generalizations in these instances.

LETTER TO THE EDITOR

## TWIN PRIMES

Charles Ziegenfus, Madison College, Harrisonburg, Va.

If $p$ and $p+2$ are (twin) primes, then $p+(p+2)$ is divisible by 12 , where $\mathrm{p}>3$.

Two proofs:
If $p>3$, then $p$ must be of the form

$$
6 \mathrm{k}+5 \text { or } 6 \mathrm{k}+1 .
$$

If

$$
p_{\mathrm{n}+1}=\mathrm{p}_{\mathrm{n}}+2
$$

then $p_{n}$ must be of the form $6 k+5$. For otherwise

$$
\mathrm{p}_{\mathrm{n}+1}=(6 \mathrm{k}+1)+2=3(2 \mathrm{k}+1)
$$

and is not prime. Therefore,

$$
\mathrm{p}_{\mathrm{n}}+\mathrm{p}_{\mathrm{n}+1}=(6 \mathrm{k}+5)+(6 \mathrm{k}+5)+2=12(\mathrm{k}+1)
$$

$p_{n}$ must be of the form $3 k, 3 k+1$, or $3 k+2$. Clearly $p_{n}=3 k$ since $\mathrm{p}_{\mathrm{n}}$ is assumed greater than 3.

If $p_{n}=3 k+1$, then $p_{n+1}=3 k+1+2=3(k+1)$ and is not prime. Clearly, $p_{n}+p_{n+1}$ is divisible by 4 .

Now $p_{n}+p_{n+1}=(3 k+3)+(3 k+2)+2=3(2 k+2)$.
So $\mathrm{p}_{\mathrm{n}}+\mathrm{p}_{\mathrm{n}+1}$ is divisible by 12 .
$\triangle x_{0}$

## FIBONACCI NUMBERS AND ZIGZAG HASSE DIAGRAMS*

A. P. HILLMAN, M.T. STROOT, AND R.M. GRASSL, UNIVERSITY OF SANTA CLARA

A Hasse diagram depicts the order relations in a partially ordered set. In this paper Haase diagrams will indicate the inclusion relations between members of a family of subsets of a given universe $U=\left\{e_{1}, \cdots, e_{n}\right\}$ of $n$ elements. Each subset is represented by a vertex and an upward slanting segment is drawn from the vertex for a subset X to the vertex for a subset Y if X is contained in Y. [1]

In a previous paper the senior author described methods for finding the number $f(n)$ of families $\left\{S_{1}, \cdots, S_{r}\right\}$ with each $S_{i}$ a subset of $U$ and with the inclusion relations among the $S_{i}$ pictured by a given Hasse diagram. The formulas $f(n)$ for all diagrams with $r=2,3$, or 4 were listed. The formulas for $r=5$ have also been obtained and will be published subsequently.

We now single out a zigzag diagram for each $r \geq 2$, i.e., the diagrams

$$
\mathrm{I}, \mathrm{~V}, \mathrm{~N}, \mathrm{~W}, \ldots .
$$

More precisely, we consider the problem of determining the number $a_{r}(n)$ of ordered r-tuples ( $S_{1}, \cdots, S_{r}$ ) of subsets $S_{i}$ of $U$ such that $S_{j}$ is contained in $S_{k}$ if and only if $j$ is even and $k=j \pm 1$. Our previous results imply the formulas:

$$
\begin{aligned}
\mathrm{a}_{2}(\mathrm{n})= & 3^{\mathrm{n}}-2^{\mathrm{n}} \\
\mathrm{a}_{3}(\mathrm{n})= & 5^{\mathrm{n}}-2 \cdot 4^{\mathrm{n}}+3^{\mathrm{n}} \\
\mathrm{a}_{4}(\mathrm{n})= & 8^{\mathrm{n}}-3 \cdot 7^{\mathrm{n}}+3 \cdot 6^{\mathrm{n}}-5^{\mathrm{n}} \\
\mathrm{a}_{5}(\mathrm{n})= & 13^{\mathrm{n}}-2 \cdot 12^{\mathrm{n}}-11^{\mathrm{n}}+5 \cdot 10^{\mathrm{n}}-4 \cdot 9^{\mathrm{n}}+8^{\mathrm{n}} \\
\mathrm{a}_{6}(\mathrm{n})= & 21^{\mathrm{n}}-20^{\mathrm{n}}-2 \cdot 19^{\mathrm{n}}-18^{\mathrm{n}}+8 \cdot 17^{\mathrm{n}}-4 \cdot 16^{\mathrm{n}}-2 \cdot 15^{\mathrm{n}}-14^{\mathrm{n}} \\
& \quad+3 \cdot 13^{\mathrm{n}}-12^{\mathrm{n}}
\end{aligned}
$$

[^0]Note that the leading term is the $n^{\text {th }}$ power of the $(r+2)$ nd Fibonacci number. The object of this paper is to prove this for general $r$.

We begin by numbering the $2^{r}$ basic regions of the Venn diagram for $r$ subsets $S_{i}$ of $U$. Express a fixed integer $k$ satisfying $0 \leq k \leq 2^{r}$ in binary form, i.e., let $k=c_{1}+2 c_{2}+2^{2} c_{3}+\cdots+2^{r-1} c_{r}$ where each $c_{i}$ is zero or one. For $i=1, \cdots, r$ let $W_{i}$ be $S_{i}$ if $c_{i}=1$ and let $W_{i}$ be the complement of $S_{i}$ in $U$ if $c_{i}=0$. Now let $E_{k}$ be the intersection of $W_{1}$, $\cdots, W_{r}$. These $E_{k}$ are the sets represented by the basic regions of the Venn diagram.

We next illustrate the process by finding $a_{3}(n)$. In this case the Hasse diagram is a $V$ and we are concerned with ordered triples ( $\mathrm{S}_{1}, \mathrm{~S}_{2}, \mathrm{~S}_{3}$ ) such that $S_{2}$ is contained in $S_{1}$ and in $S_{3}$ and there are no other inclusion relations. The condition that $S_{2}$ is contained in $S_{1}$ forces $E_{2}$ and $E_{6}$ tobe empty. The condition that $S_{2}$ is contained in $S_{3}$ forces $E_{3}$ (and $E_{2}$ ) to be empty. One then sees that there are no other inclusion relations if and only if both $E_{1}$ and $\mathrm{E}_{4}$ are non-empty.

For a given triple $\left(S_{1}, S_{2}, S_{3}\right)$ each of the $n$ objects in the universe is in one and only one of the $E_{k}$. Excluding the empty $E_{2}, E_{3}$, and $E_{6}$, there are $5^{n}$ ways of distributing the n objects among the 5 remaining basic sets $E_{0}, E_{1}, E_{4}, E_{5}$, and $E_{7}$. We subtract the $4^{n}$ ways in which $E_{1}$ turns out to be empty (as well as $\mathrm{E}_{2}, \mathrm{E}_{3}$, and $\mathrm{E}_{6}$ ) and also subtract the $4^{\mathrm{n}}-3^{\mathrm{n}}$ ways in which $E_{4}$, but not $E_{1}$, is empty. The remaining $a_{3}(n)=5^{n}-4^{n}-\left(4^{n}-3^{n}\right)$ ways of distributing the elements of $U$ are all those that meet the conditions associated with the Hasse diagram V.

For a general $r$ the inclusion relations of the zigzag diagram force $g(r)$ of the $2^{r}$ basic sets $E_{k}$ to be empty. The technique illustrated above canbe used to show that these are the $E_{k}$ such that the $r$-tuple $\left(c_{1}, \ldots, c_{r}\right)$ of binary coefficients for $k$ has an even-subscripted $c_{i}=1$ with an adjacent $c_{i \pm 1}=0$. The remaining r-tuples will be called allowable; there are $h(r)=$ $2^{r}-g(r)$ such $r$-tuples. We wish to show that $h(r)$ is the Fibonacci number $\mathrm{F}_{\mathrm{r}+2}$. It will then be clear that the leading term in $\mathrm{a}_{\mathrm{r}}(\mathrm{n})$ is $\left(\mathrm{F}_{\mathrm{r}+2}\right)^{\mathrm{n}}$ and that
the other terms result from subtracting numbers of ways of distributing the elements of $U$ among fewer $E_{k}$ than the allowable ones.

For $r=3$ the allowable triples are

$$
\begin{equation*}
(0,0,0), \quad(1,0,0)_{2} \quad(0,0,1), \quad(1,0,1), \quad(1,1,1), \tag{1}
\end{equation*}
$$

i.e., those for $E_{0}, E_{1}, E_{4}, E_{5}$, and $E_{7}$. The allowable quadruples for $r=4$ can be made by attaching a zero in the fourth place to the 2 triples in (1) that have a zero in the third place and by attaching either a zero or a one in the fourth place to each of the remaining 3 triples in (1). There are thus 3 allowable quadruples with a one in the fourth place, $2+3=5$ of them with a zero in the fourth place, and a total of $h(4)=8=F_{6}$ such quadruples. Similarly the number of quintuples of our desired form with a zero in the fifth place is 5 , the number with a one is $3+5=8$, and the total number of such quintuples is $h(5)=13=F_{7}$. Using mathematical induction, one now easily shows that $h(r)=F_{r+2}$.

## REFERENCES

1. G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publications, vol. 25, Rev. Ed., 1961.
2. A. P. Hillman, On the Number of Realizations of a Hasse Diagram by Finite Sets, Proceedings of the Amer. Math. Soc., vol. 6, No. 4, pp. 542-548, 1955.

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## adVanceld problems and solutions

EDI TED BY VERNER E. HOGGATT, JR。, SAN JOSE STATE COLLEGE

H-19 Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas.

In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle,
$B$, divide side, $A C$, into segments of length $F_{n}, F_{n+1}, F_{n+3}$. Find:
(i) $\lim _{\mathrm{n} \rightarrow \infty} \theta$
(ii) $\lim \varphi$ $n \rightarrow \infty$


H-20 Proposed by Verner E. Hoggatt, Jr., and Charles H. King, San Jose State College, San Jose, California.
If $\quad Q=\left(\begin{array}{ll}1 & 1 \\ 0 & 0\end{array}\right)$, show $\quad D\left(e^{Q^{n}}\right)=e^{L_{n}} \quad$,
where $D(A)$ is the determinant of matrix $A$ and $L_{n}$ is the $n^{\text {th }}$ Lucas number.
H-21 Proposed by Francis D. Parker, University of Alaska, College, Alaska

Find the probability, as $n$ approaches infinity, that the $n^{\text {th }}$ Fibonacci number, $F(n)$, is divisible by another Fibonacci number ( $\neq \mathrm{F}_{1}$ or $\mathrm{F}_{2}$ ).

H-22 Proposed by Verner E. Hoggatt, Jr.

$$
\text { If } P(x)=\prod_{i=1}^{\infty}\left(1+x^{F_{i}}\right)=\sum_{n=0}^{\infty} R(n) x^{n},
$$

then show
(i)

$$
\begin{gathered}
R\left(F_{2 n}-1\right)=n \\
R(N)>n \text { if } N>F_{2 n}-1 .
\end{gathered}
$$

(ii)
(See first paper of this issue.)
H-23 Proposed by Malcolm H. Tallman, Brooklyn, New York

1, 3, 21, and 55 are Fibonacci numbers. Also, they are triangular numbers. What is the next higher number that is common to both series?

## SOLUTIONS

H-3 Show $\mathrm{F}_{2 \mathrm{n}-2}<\mathrm{F}_{\mathrm{n}}^{2}<\mathrm{F}_{2 \mathrm{n}-1}$, $\mathrm{n} \geq 3$;

$$
\mathrm{F}_{2 \mathrm{n}-1}<\mathrm{L}_{\mathrm{n}-1}^{2}<\mathrm{F}_{2 \mathrm{n}}, \quad \mathrm{n} \geq 4
$$

where $F_{n}$ and $L_{n}$ are the $n^{\text {th }}$ Fibonacci and Lucas numbers, respectively.
Solution by Francis D. Parker, University of Alaska.
The identities $F^{2}(n)=F(2 n-2)+F^{2}(n-2)$
and $F^{2}(n)=F(2 n-1)-F^{2}(n-1)$
are valid for $n \geq 3$ and can be proved from the explicit formulas for $F(n)$. From these it follows that $F(2 n-2)<F^{2}(n)<F(2 n-1), n \geq 3$. Again, from the explicit formulas for $L(n)$ and $F(n)$ it is possible to prove the identities $L^{2}(\mathrm{n}-1)=\mathrm{F}(2 \mathrm{n}-1)+\mathrm{F}(2 \mathrm{n}-3)+2(-1)^{\mathrm{n}+1}$ and $\mathrm{L}^{2}(\mathrm{n}-1)=\mathrm{F}(2 \mathrm{n})-\mathrm{F}(2 \mathrm{n}-4)$ $+2(-1)^{n+1}$. From these it follows that $F(2 n-1)<L^{2}(n-1)<F(2 n),(n \geq 4)$. This problem was also solved by Dov Jarden, Jerusalem, Israel.

H-4 Prove the identity:

$$
\mathrm{F}_{\mathrm{r}+1} \mathrm{~F}_{\mathrm{S}+1} \mathrm{~F}_{\mathrm{t}+1}+\mathrm{F}_{\mathrm{r}} \mathrm{~F}_{\mathrm{S}} \mathrm{~F}_{\mathrm{t}}-\mathrm{F}_{\mathrm{r}-1} \mathrm{~F}_{\mathrm{S}-1} \mathrm{~F}_{\mathrm{t}-1}=\mathrm{F}_{\mathrm{r}+\mathrm{s}+\mathrm{t}}
$$

Are there any restrictions on the integral subscripts?

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania
We shall prove the assertion under the subscript restrictions, $r \geq-1$, $\mathrm{s} \geq-1, \mathrm{t} \geq-1$, where $\mathrm{F}_{-2}=-1, \mathrm{~F}_{-1}=1$ and $\mathrm{F}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}-1}+\mathrm{F}_{\mathrm{n}-2}$ for $\mathrm{n} \leq 0$. The proof is by an induction on $n$, where $n=r+s+t$. To show that the result holds for $\mathrm{n}=1$ and $\mathrm{n}=2$, a symmetry argument shows that it suffices to verify the result for the nine triples $(\mathrm{r}, \mathrm{s}, \mathrm{t})=(1,0,0),(-1,1,1),(3,-1,-1)$, $(-1,2,0),(2,0,0),(1,1,0),(-1,3,0),(2,-1,1)$ and $(4,-1,1)$.

Now assume as an inductionhypothesis that the result has been proved for all $n$ satisfying $n \leq k$, where $k \geq 2$. Then consider any triple ( $r, s, t$ ) such that $r+s+t=k+1$. Assume without loss of generality that $r=\max (r, s, t)$. Then $r \geq 1$ and

$$
\begin{aligned}
\Delta_{\mathrm{k}+1} \equiv & \mathrm{~F}_{\mathrm{r}+1} \mathrm{~F}_{\mathrm{s}+1} \mathrm{~F}_{\mathrm{t}+1}+\mathrm{F}_{\mathrm{r}} \mathrm{~F}_{\mathrm{s}} \mathrm{~F}_{\mathrm{t}}-\mathrm{F}_{\mathrm{r}-1} \mathrm{~F}_{\mathrm{s}-1} \mathrm{~F}_{\mathrm{t}-1} \\
= & \left(\mathrm{F}_{\mathrm{r}} \mathrm{~F}_{\mathrm{s}+1} \mathrm{~F}_{\mathrm{t}+1}+\mathrm{F}_{\mathrm{r}-1} \mathrm{~F}_{\mathrm{S}+1} \mathrm{~F}_{\mathrm{t}+1}\right) \\
& +\left(\mathrm{F}_{\mathrm{r}-1} \mathrm{~F}_{\mathrm{s}} \mathrm{~F}_{\mathrm{t}}+\mathrm{F}_{\mathrm{r}-2} \mathrm{~F}_{\mathrm{s}} \mathrm{~F}_{\mathrm{t}}\right) \\
& -\left(\mathrm{F}_{\mathrm{r}-2} \mathrm{~F}_{\mathrm{s}-1} \mathrm{~F}_{\mathrm{t}-1}+\mathrm{F}_{\mathrm{r}-3} \mathrm{~F}_{\mathrm{s}-1} \mathrm{~F}_{\mathrm{t}-1}\right)
\end{aligned}
$$

But

$$
F_{r} F_{s+1}+F_{t+1}+F_{r-1} F_{s} F_{t}-F_{r-2} F_{s-1} F_{t-1}=F_{r-1+s+t}
$$

by the induction hypothesis applied to the triple ( $\mathrm{r}-1, \mathrm{~s}, \mathrm{t}$ ), which has the sum $\mathrm{r}-1+\mathrm{s}+\mathrm{t}=\mathrm{k}$. Similarly

$$
\mathrm{F}_{\mathrm{r}-1} \mathrm{~F}_{\mathrm{s}+1} \mathrm{~F}_{\mathrm{t}+1}+\mathrm{F}_{\mathrm{r}-2} \mathrm{~F}_{\mathrm{S}} \mathrm{~F}_{\mathrm{t}}-\mathrm{F}_{\mathrm{r}-3} \mathrm{~F}_{\mathrm{s}-1} \mathrm{~F}_{\mathrm{t}-1}=\mathrm{F}_{\mathrm{r}-2+\mathrm{s}+\mathrm{t}}
$$

by the induction hypothesis applied to the triple ( $r-2, s, t$ ), which has the sum $\mathrm{r}-2+\mathrm{s}+\mathrm{t}=\mathrm{k}-1$. Thus

$$
\Delta_{\mathrm{k}+1}=\mathrm{F}_{\mathrm{r}-1+\mathrm{s}+\mathrm{t}}+\mathrm{F}_{\mathrm{r}-2+\mathrm{s}+\mathrm{t}}=\mathrm{F}_{\mathrm{r}+\mathrm{s}+\mathrm{t}}
$$

as required and the result follows by induction.

H-5 (i) If

$$
\left[m F_{n}\right]=\frac{\left(F_{m} F_{m-1} \cdots F_{1}\right)}{\left(F_{n} F_{n-1} \cdots F_{1}\right)\left(F_{m-n} F_{m-n-1} \cdots F_{1}\right)}
$$

then

$$
2\left[\mathrm{~m}_{\mathrm{n}}\right]=\left[m-1 \mathrm{~F}_{\mathrm{n}}\right] \mathrm{L}_{\mathrm{n}}+\left[m-1 F_{\mathrm{n}-1}\right] L_{m-n}
$$

where $F_{n}$ and $L_{n}$ are the $n^{\text {th }}$ Fibonacci and $n^{\text {th }}$ Lucas numbers, respectively.
(ii) Show that this generalized binomial coefficient $\left[\mathrm{m}^{\mathrm{F}} \mathrm{n}_{\mathrm{n}}\right.$ is always an integer.

Solution by J. L. Brown, Jr.
(i) The identity $L_{n} F_{m-n}+L_{m-n} F_{n}=2 F_{m}$ for $m \geq n \geq 0$ is easily verified by induction. Multiplying both sides of the identity by $\left[\mathrm{m}_{\mathrm{n}}\right] \quad \mathrm{F}_{\mathrm{m}}$ then gives the required relation $L_{n}\left[m-1 F_{n}\right]+L_{m-n}\left[m-1 F_{n-1}\right]=2\left[m_{n}\right]$.

From the expression for $F_{m}$, it follows that

$$
\mathrm{F}_{\mathrm{m}}=\alpha^{\mathrm{m}-\mathrm{n}} \mathrm{~F}_{\mathrm{n}}+\beta^{\mathrm{n}} \mathrm{~F}_{\mathrm{m}-\mathrm{n}} \text { for } \mathrm{m} \geq \mathrm{n}
$$

Then
(*) $\quad\left[m^{*} F_{n}\right]=\frac{\left[m^{F_{n}}\right]}{F_{m}} \cdot F_{m}=\alpha^{m-n}\left[m-1 F_{n-1}\right]+\beta^{n}\left[m-1 F_{n}\right]$.
but $\left[m^{F} n_{n}\right]=\left[m^{F}{ }_{n-m}\right]$. If we replace $n$ by $m-n$ on the right-hand side of $\left(^{*}\right)$, then we have
$\left(^{* *}\right) \quad\left[m^{F_{n}}\right]=\alpha^{n}\left[m-1 F_{m-n-1}\right]+\beta^{m-n}\left[m-1 F_{m-n}\right]$.
However, $\left[m-1 F_{m-n-1}\right]=\left[m-1 F_{n}\right]$ and $\left[m-1 F_{m-n}\right]=\left[m-1 F_{n-1}\right]$, so that adding $\left({ }^{*}\right)$ and $\left({ }^{* *}\right)$ yields

$$
\begin{aligned}
2\left[\mathrm{~m}^{\mathrm{F}} \mathrm{n}\right] & =\left(\alpha^{\mathrm{m}-\mathrm{n}}+\beta^{\mathrm{m}-\mathrm{n}}\right)\left[\mathrm{m}-1 \mathrm{~F}_{\mathrm{n}-1}\right]+\left(\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}\right)\left[\mathrm{m}-1 \mathrm{~F}_{\mathrm{n}}\right] \\
& =\mathrm{L}_{\mathrm{m}-\mathrm{n}}\left[\mathrm{~m}_{\mathrm{m}} \mathrm{~F}_{\mathrm{n}-1}\right]+\mathrm{L}_{\mathrm{n}}\left[\mathrm{~m}-1 \mathrm{~F}_{\mathrm{n}}\right] \text { as required. }
\end{aligned}
$$

(ii) A proof of the second part which makes use of relation (*) can be found on p. 45 of D. Jarden's "Recurring Sequences."

H-6 Determine the last three digits, in base seven, of the millionth Fibonacci number. (Series: $\mathrm{F}_{1}=1, \mathrm{~F}_{2}=1, \mathrm{~F}_{3}=2$, etc.)

Solution by Brother U. Alfred, St. Mary's College, Calif.
The last three digits base seven may be determined if we find the residue modulo seven cubed of the millionth Fibonacci number.

Seven has a period of 16 and $7^{3}$ has a period of $7^{2} \times 16=784$.
In $1,000,000$, there are a number of complete periods and a partial period of 400 .

For a period of the form $2^{m}(2 \lambda+1)$ where $m \geq 2$, there is a zero at the half-period of 392. Also, for a prime or a power of a prime, the adjacent terms are congruent to -1 modulo the power of the prime. Hence we know that we have the following series of values:

| n | Residue (modu |
| :---: | :---: |
| 392 | 0 |
| 393 | 342 |
| 394 | 342 |
| 395 | 341 |
| 397 | 340 |
| 397 | 338 |
| 398 | 335 |
| 399 | 330 |
| 400 | 322 |

This expressed to base seven is $(440)_{7}$, so that these are the last three digits of the millionth Fibonacci number expressed in base 7.
$H-7$ If $F_{n}$ is the $n^{\text {th }}$ Fibonacci number find $\lim _{n \rightarrow \infty} \sqrt[{n_{\sqrt{F}}}]{n}=L$ and show that

$$
\sqrt[2 n]{\sqrt{5} F_{2 n}}<L<\sqrt[2 n+1]{\sqrt{5} F_{2 n+1}} \text { for } n \geq 2 .
$$

Solution by John L. Brown, Jr.
Let $\mathrm{a}=(1+\sqrt{5}) / 2$. Then, it is well-known (see, e.g., pp. 22-23 of "Fibonacci Numbers" by N. N. Vorobév) that

$$
\left|F_{n}-\frac{a^{n}}{\sqrt{5}}\right|<\frac{1}{2} \text { for all } n \geq 1
$$

Therefore $F_{n}=\frac{a^{n}}{\sqrt{5}}+\theta_{n}$, where $\left|\theta_{n}\right|<\frac{1}{2}$ and $n_{n}=\sqrt[n]{\frac{a^{n}}{\sqrt{5}}+\theta_{n}}$. But for $\mathrm{n} \geq 1$,

$$
\begin{aligned}
\frac{a}{\sqrt[n]{2} \sqrt[2 n]{5}} & =\sqrt[n]{\frac{a^{n}}{2 \sqrt{5}}}<\sqrt[n]{\frac{2 a^{n}-\sqrt{5}}{2 \sqrt{5}}}=\sqrt[n]{\frac{a^{n}}{\sqrt{5}}-\frac{1}{2}}<\sqrt[n]{\frac{a^{n}}{\sqrt{5}}+\theta_{n}} \\
& <\sqrt[n]{\frac{a^{n}}{\sqrt{5}}+\frac{1}{2}}=\sqrt[n]{\frac{2 a^{n}+1}{\sqrt{5}}}<\sqrt[n]{\frac{3 a^{n}}{2 \sqrt{5}}}=\frac{a}{\sqrt[n]{\frac{2 \sqrt{5}}{3}}}
\end{aligned}
$$

Taking $\lim _{\mathrm{n} \rightarrow \infty}$, we find

$$
\mathrm{L}=\lim _{\mathrm{n} \rightarrow \infty} \sqrt[n]{\frac{\mathrm{a}^{\mathrm{n}}}{\sqrt{5}}+\theta_{\mathrm{n}}}=a
$$

Thus $\mathrm{L}=\mathrm{a}=\frac{1+\sqrt{5}}{2}$.
Now, let $b=\frac{1-\sqrt{5}}{2}$ so that $b<0$. Then, since $\sqrt{5} F_{n}=a^{n}-b^{n}$ for $\mathrm{n} \geq 1$, we have

$$
\sqrt[2 n]{\sqrt{5} F_{2 n}}=\sqrt[2 n]{a^{2 n}-b^{2 n}}<a<\sqrt[2 n+1]{a^{2 n+1}-b^{2 n+1}}=\sqrt[2 n+1]{\sqrt{5} F_{2 n+1}}
$$

thus the desired inequality follows for all $\mathrm{n} \geq 1$ on noting that $\mathrm{L}=\mathrm{a}$.
Also solved by Donna Seaman.
H-8 Prove

$$
\left|\begin{array}{lll}
F_{n}^{2} & F_{n+1}^{2} & F_{n+1}^{2} \\
F_{n+1}^{2} & F_{n+2}^{2} & F_{n+3}^{2} \\
F_{n+2}^{2} & F_{n+3}^{2} & F_{n+4}^{2}
\end{array}\right|=2(-1)^{n+1}
$$

where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number.
Solution by John Allen Fuchs and Joseph Erbacher, University of Santa Clara, Santa Clara, California
The squares of the Fibonacci numbers satisfy the linear homogeneous recursion relationship $\mathrm{F}_{\mathrm{n}+3}^{2}=2 \mathrm{~F}_{\mathrm{n}+2}^{2}+2 \mathrm{~F}_{\mathrm{n}+1}^{2}-\mathrm{F}_{\mathrm{n}}^{2}$. (See H. W. Gould, Generating Functions for Products of Powers of Fibonacci Numbers, this Quarterly, Vol. 1, No. 2, p. 2,)

We may use this recursion formula to substitute for the last row of the given determinant, $D_{n}$, and then apply standard row operations to get

$$
\begin{aligned}
& D_{n}=\left|\begin{array}{lll}
F_{n}^{2} & F_{n+1}^{2} & F_{n+2}^{2} \\
F_{n+1}^{2} & F_{n+2}^{2} & F_{n+3}^{2} \\
2 F_{n+1}^{2}+2 F_{n}^{2}-F_{n-1}^{2} & 2 F_{n+2}^{2}+2 F_{n+1}^{2}-F_{n}^{2} & 2 F_{n+3}^{2}+2 F_{n+2}^{2}-F_{n+1}^{2}
\end{array}\right| \\
&=\left|\begin{array}{lll}
F_{n}^{2} & F_{n+1}^{2} & F_{n+2}^{2} \\
F_{n+1}^{2} & F_{n+2}^{2} & F_{n+3}^{2} \\
-F_{n-1}^{2} & -F_{n}^{2} & -F_{n+1}^{2}
\end{array}\right|=-D_{n-1} .
\end{aligned}
$$

It follows immediately by induction that $D_{n}=(-1)^{n-1} D_{1}$. Since $D_{1}=2, D_{n}$ $=2(-1)^{\mathrm{n}-1}=2(-1)^{\mathrm{n}+1}$.

Also solved by Marjorie Bicknell and Dov Jarden.

Continued from p. 80, "Elementary Problems and Solutions"
Then

$$
\begin{aligned}
F_{k+2} p^{k+1}+F_{k+1} p^{k+2} & =\left(F_{k+1}+F_{k}\right) p^{k+1}+\left(F_{k}+F_{k-1}\right) p^{k+2} \\
& =p\left(F_{k+1} p^{k}+F_{k} p^{k+1}\right)+p^{2}\left(F_{k} p^{k+1}+F_{k-1} p^{k}\right)
\end{aligned}
$$

But

$$
p\left(F_{k+1} p^{k}+F_{k} p^{k+1}\right)+p^{2}\left(F_{k} p^{k-1}+F_{k-1} p^{k}\right) \equiv p+p^{2}\left(\bmod p^{2}+p-1\right)
$$

Since $F_{k+1} p^{k}+F_{k} p^{k+1}$ and $F_{k} p^{k-1}+F_{k-1} p^{k}$ are both congruent to $1\left(\bmod p^{2}\right.$ $+\mathrm{p}-1)$ by the induction hypothesis and $\mathrm{p}+\mathrm{p}^{2} \equiv 1\left(\bmod \mathrm{p}^{2}+\mathrm{p}-1\right)$, the desired result follows by induction on $n$.

Also solved by Marjorie R. Bicknell and Donna J. Seaman.

## BEGINNERS' CORNER

EDITED BY DMITRI THORO, SAN JOSE STATE COLLEGE,

THE GOLDEN RATIO: COMPUTATIONAL CONSIDERATIONS

## 1. INTRODUCTION

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel" - so wrote Kepler (1571-1630) [1].

The famous golden section involves the division of a given line segment into mean and extreme ratio, i.e., into two parts $a$ and $b$, such that $a / b=$ $b /(a+b)$, $a<b$. Setting $x=b / a$ we have $x^{2}-x-1=0$. Let us designate the positive root of this equation by $\phi$ (the golden ratio). Thus

$$
\begin{equation*}
\phi^{2}-\phi-1=0 \tag{1}
\end{equation*}
$$

Since the roots of (1) are $\phi=(1+\sqrt{5}) / 2$ and $-1 / \phi=(1-\sqrt{5}) / 2$ we may write Binet's formula [2], [3], [4] for the $\mathrm{n}^{\text {th }}$ Fibonacci number in the form

$$
\begin{equation*}
\mathrm{F}_{\mathrm{n}}=\frac{\phi^{\mathrm{n}}-(-\phi)^{-\mathrm{n}}}{\sqrt{5}} \tag{2}
\end{equation*}
$$

## 2. POWERS OF THE GOLDEN RATIO

Returning to (1), let us "solve for $\phi^{2}$ " by writing

$$
\begin{equation*}
\phi^{2}=\phi+1 . \tag{3}
\end{equation*}
$$

Multiplying both members by $\phi$, we get $\phi^{3}=\phi^{2}+\phi=(\phi+1)+\phi=2 \phi+1$. Now $\phi^{3}=2 \phi+1$ yields $\phi^{4}=2 \phi^{2}+\phi=2(\phi+1)+\phi=3 \phi+2$. Similarly,

$$
\phi^{5}=3 \phi^{2}+2 \phi=3(\phi+1)+2 \phi=5 \phi+3
$$

This pattern suggests

$$
\begin{equation*}
\phi^{\mathrm{n}}=\mathrm{F}_{\mathrm{n}} \phi+\mathrm{F}_{\mathrm{n}-1}, \mathrm{n}=1,2,3, \cdots \tag{4}
\end{equation*}
$$

To prove (4) by mathematical induction [5], [6], we note that it is true for $n$ $=1$ and $\mathrm{n}=2$ (since $\mathrm{F}_{0}=0$ by definition). Assume $\phi^{\mathrm{k}}=\mathrm{F}_{\mathrm{k}} \phi+\mathrm{F}_{\mathrm{k}-1}$. Then $\phi^{\mathrm{k}+1}=\mathrm{F}_{\mathrm{k}} \phi^{2}+\mathrm{F}_{\mathrm{k}-1} \phi=\mathrm{F}_{\mathrm{k}}(\phi+1)+\mathrm{F}_{\mathrm{k}-1} \phi=\left(\mathrm{F}_{\mathrm{k}}+\mathrm{F}_{\mathrm{k}-1}\right) \phi+\mathrm{F}_{\mathrm{k}}=$ $\mathrm{F}_{\mathrm{k}+1} \phi+\mathrm{F}_{\mathrm{k}}$, which completes the proof. The computational advantage of (4) over expansion of

$$
\left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}}
$$

by the binomial theorem is striking.
Dividing both members of (3) by $\phi$, we obtain

$$
\begin{equation*}
\frac{1}{\phi}=\phi-1 \tag{5}
\end{equation*}
$$

Thus $1 / \phi^{2}=1-1 / \phi=1-(\phi-1)=-(\phi-2)$. Using this result and (5), $1 / \phi^{3}$ $=2 / \phi-1=2(\phi-1)-1=2 \phi-3$. Similarly, $1 / \phi^{4}=2-3 / \phi=2-3 \phi+3=$ $-(3 \phi-5)$. Via induction, the reader may provide a painless proof of

$$
\begin{equation*}
\phi^{-\mathrm{n}}=(-1)^{\mathrm{n}+1}\left(\mathrm{~F}_{\mathrm{n}} \phi-\mathrm{F}_{\mathrm{n}+1}\right), \mathrm{n}=1,2,3, \cdots \tag{6}
\end{equation*}
$$

## 3. A LIMIT OF FIBONACCI RATIOS

If we "solve" $x^{2}-x-1=0$ for $x$ by writing $x=1+1 / x$ and then consider the related recursion relation

$$
\begin{equation*}
x_{1}=1, \quad x_{n+1}=1+\frac{1}{x_{n}} \tag{7}
\end{equation*}
$$

Fibonacci numbers start popping out! We immediately deduce $\mathrm{x}_{2}=1+1 / \mathrm{x}_{1}$ $=1+1 / 1=2, x_{3}=1+1 / x_{2}=1+1 / 2=3 / 2, x_{4}=5 / 3, x_{5}=8 / 5$, etc. This suggests that $x_{n}=F_{n+1} / F_{n}$.

Now suppose the sequence $x_{1}, x_{2}, x_{3}, \cdots$ has a limit, say $L$, as $n \rightarrow$ $\infty$. Then

$$
\lim _{n \rightarrow \infty} x_{n+1}=\lim _{n \rightarrow \infty} x_{n}=L
$$

whence (7) yields $L=1+1 / L$ or $L=\phi$ since the $x_{i}$ are positive. Indeed, there are many ways of proving Kepler's observation that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{F_{n+1}}{F_{n}}=\phi \tag{8}
\end{equation*}
$$

E.g., from (2)
$\frac{\mathrm{F}_{\mathrm{n}+1}}{\mathrm{~F}_{\mathrm{n}}}=\left[\phi^{\mathrm{n}+1}-(-\phi)^{-\mathrm{n}-1}\right] /\left[\phi^{\mathrm{n}}-(-\phi)^{-\mathrm{n}}\right]=\frac{\phi-\frac{1}{(-\phi)^{\mathrm{n}+1} \phi^{\mathrm{n}}}}{1-\frac{1}{(-\phi)^{\mathrm{n} \phi^{\mathrm{n}}}} \rightarrow \phi}$
since $\phi=(1+\sqrt{5}) / 2>1$
implies that the fractions involving $\phi^{n}$ approach 0 as $n \rightarrow \infty$.

## 4. AN APPROXIMATE ERROR ANALYSIS

Just how accurate are the above approximations to the golden ratio? Let us denote the exact error at the $\mathrm{n}^{\text {th }}$ iteration by

$$
\begin{equation*}
e_{n} \equiv x_{n}-\phi \tag{9}
\end{equation*}
$$

The trick is to express $e_{n+1}$ in terms of $e_{n}$ and then to make use of the identity

$$
\begin{equation*}
\frac{1}{1+w}=1-w+w^{2}-w^{3}+w^{4}-\cdots \quad, \quad w<1 . \tag{10}
\end{equation*}
$$

(The latter may be discovered by dividing 1 by $1+w ; ~ c f . ~[7])$.
Thus

$$
\begin{aligned}
e_{n+1} & \equiv x_{n+1}-\phi=1+\frac{1}{x_{n}}-\phi \\
& =1-\phi+\frac{1}{e_{n}+\phi}=1-\phi+\frac{1}{\phi}\left[\frac{1}{1+\left(e_{n} / \phi\right)}\right] \\
& =1-\phi+\frac{1}{\phi}\left[1-\left(e_{n} / \phi\right)+\left(e_{n} / \phi\right)^{2}-\left(e_{n} / \phi\right)^{3}+\cdots\right] \\
& =-\frac{e_{n}}{\phi^{2}}+\frac{e_{n}^{2}}{\phi^{3}}-\frac{e_{n}^{3}}{\phi^{4}}+\cdots \quad \text { since } \frac{1}{\phi}=\phi-1 \text { by }(5) .
\end{aligned}
$$

However, the terms involving the higher powers of $e_{n}$ are quite small in comparison with the first term. Thus, following the customary practice of neglecting high order terms, we will approximate the error at the ( $n+1$ )st step by $\epsilon_{\mathrm{n}+1}=-\epsilon_{\mathrm{n}} \phi^{-2}$. Finally, we may note that $\epsilon_{2}=-\epsilon_{1} \phi^{-2}, \epsilon_{3}=-\epsilon_{2} \phi^{-2}=\epsilon_{1} \phi^{-4}$, $\epsilon_{4}=-\epsilon_{1} \phi^{-6}$, and, in general,

$$
\begin{equation*}
\epsilon_{\mathrm{n}}=(-1)^{\mathrm{n}+1} \epsilon_{1} \phi^{-2(\mathrm{n}-1)} \tag{11}
\end{equation*}
$$

## 5. COMPUTATION OF $\phi$ VIA MATRICES

We recall (cf. [8]) that if the matrix

$$
\mathrm{M}=\left(\begin{array}{ll}
\mathrm{a} & \mathrm{~b} \\
\mathrm{c} & \mathrm{~d}
\end{array}\right)
$$

and the column vector

$$
\mathrm{v}=\binom{\mathrm{r}}{\mathrm{~s}}
$$

then the product Mv is defined to be the column vector

$$
\binom{a r+b s}{c r+d s}
$$

Let us investigate the recursion relation

$$
\begin{equation*}
\mathrm{v}_{\mathrm{n}+1}=A \mathrm{v}_{\mathrm{n}}, \quad \mathrm{n}=1,2,3, \cdots \tag{12}
\end{equation*}
$$

where $A$ is a given matrix and $v_{1}$ a given vector. (For convenience we will always take $v_{1}$ to be the first column of $A$.)
(a) If A is the Q matrix [9], [10] $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$, then $\mathrm{v}_{1}=\binom{1}{1}$ and $\mathrm{v}_{2}=$ $A v_{1}=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)\binom{1}{1}=\binom{2}{1}, v_{3}=A v_{2}=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)\binom{2}{1}=\binom{3}{2}, v_{4}=\binom{5}{3}, v_{5}=$ $\binom{8}{5}, \cdots, v_{n}=\binom{F_{n+1}}{F_{n}}, \cdots$. Thus if $v_{i}=\binom{r_{i}}{s_{i}}$, then for $A=Q$ the ratio $r_{i} / s_{i}$ is precisely the approximation to $\phi$ obtained from (7).
(b) Let $\mathrm{A}=\left(\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right)$. Then
$\mathrm{v}_{2}=\left(\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right)\binom{2}{1}=\binom{5}{3}, \quad \mathrm{v}_{3}=\left(\begin{array}{l}2 \\ 1 \\ 1\end{array} 1\right)\binom{5}{3}=\binom{13}{8}, \quad \mathrm{v}_{4}=\binom{34}{21}, \ldots$
This time $v_{n}=\binom{F_{2 n+1}}{F_{2 n}}$. Note that here the ratio obtained from, say, $v_{3}$ is exactly that obtained from $\mathrm{V}_{6}$ when A is taken to be the Q matrix.
(c) For $A=\left(\begin{array}{cc}0 & 1 \\ 1 & -1\end{array}\right)$, the successive approximations suggested by (12) turn out to be

$$
\begin{equation*}
\frac{1}{-1}, \frac{-1}{2}, \frac{2}{-3}, \frac{-3}{5}, \frac{5}{-8}, \cdots \tag{13}
\end{equation*}
$$

From the discussion in (a) above it is easy to deduce that the limit of the sequence (13) is $-\frac{1}{\bar{Q}}$ : the negative root of (1)!

BEGINNERS' CORNER
[Oct.
Similarly pleasant results may be obtained from (infinitely many) other A's. Several possibilities are suggested in the following exercises. The mathematical basis for this approach will be explored in a future issue.

## 6. EXERCISES

E1. Show that the definition of the golden section leads to the equation $x^{2}-x-1=0$.

E2. Use mathematical induction to prove (6).
E3. How shouldyou define $\mathrm{F}_{-\mathrm{k}}(\mathrm{k}>0)$ in order that (4) would hold for negative values of $n$ ?

E4. Verify (10) by long division. Find an additional check by starting with the right-hand member.

E5. Give an induction proof of (11).
E6. Show that when $x_{1}=1$, the estimated error given by (11) becomes

$$
\epsilon_{\mathrm{n}}=(-1)^{\mathrm{n}} \phi^{1-2 \mathrm{n}}
$$

Hint: Use (5).
E7. Using the results of E 6 (with $\phi=1.618$ ) compute an estimate of $F_{11} / F_{10}-\phi$. Compare this approximate error to the actual error (given $\phi=$ 1.61803). Thus although $\epsilon_{\mathrm{n}}$ is a function of $\phi$ itself, it can be usedin approximating $\phi$ to a desired number of decimal places.

E8. A comparison of the three values of A exhibited above reveals that in each case A has the form

$$
\left(\begin{array}{cc}
w & 1 \\
1 & w-1
\end{array}\right)
$$

It turns out that $w$ need not be an integer. Experiment with different values of w. Hint: consider the cases
(a)
(b)

$$
\begin{gathered}
\mathrm{w}>\phi \\
\frac{1}{2}<\mathrm{w}<\phi
\end{gathered}
$$

(c)
$-\frac{1}{\phi}<\mathrm{w}<\frac{1}{2}$
(d)
$\mathrm{w}<-\frac{1}{\phi}$

E9. What happens, in the preceding exercise, when $w=1 / 2$ ?
E10. Explain why the first two illustrations of (12) are essentially "computationally equivalent." Hint: count the minimum number of arithmetic operations required in each case.

## REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley and Sons, Inc., New York, 1961, p. 160.
2. The Fibonacci Quarterly, 2 (1963), 66-67.
3. $\qquad$ , 2 (1963), 73.
4. $\qquad$ , 2 (1963), 75.
5. $\qquad$ , 1 (1963), 61.
6. $\qquad$ , 1 (1963), 67 .
7. $\qquad$ , 2 (1963), 65.
8. $\qquad$ , 2 (1963), 61.
9. $\qquad$ , 2 (1963), 47.
10. $\qquad$ , 2 (1963), 61-62.


## EDITORIAL REMARK FROM PAGE 19:

In the European notation, 763,26 means what 763.26 does to us, and $2.5^{\mathrm{X}}$ means $2\left(5^{\mathrm{x}}\right)=2 \cdot 5^{\mathrm{X}}$.

## 

HAVE YOU SEEN?
Nathan J. Fine, Generating Functions, Enrichment Mathematics for High School, Twenty-eighth Yearbook National Council of Teachers of Mathematics, Washington, D. C. , 1963, pp. 355-367. This is an excellent and inspiring article.

## EXPLORING FIBONACCI POLYGONS <br> EDITED BY BROTHER U. ALFRED, ST. MARY'S COLLEGE, CALIFORNIA

We shall define a Fibonacci polygon as any closed plane figure bounded by straight lines all of whose lengths correspond to Fibonacci numbers of the series: $1,1,2,3,5,8,13, \cdots$. Specifically, we shall investigate one subset of this group of figures, namely, those for which all sides are unequal.

The question for study is: Under what circumstances may a polygon be formed from line segments all of whose lengths correspond to Fibonacci numbers? Three situations may be envisaged:
(1) The greatest length is greater than the sum of all the other lengths in which case no polygon can be formed;
(2) The greatestlength is equal to the sum of all the other lengths. Again, no polygon can beformed, but this case is interesting as itrepresents the division point between polygons and non-polygons.
(3) The greatest length is less than the sum of all the other lengths in which case a polygon can be formed.

Research could begin by studying specific polygons beginning with the triangle and working upward. We might ask such questions as the following:
(1) Is it possible to have a Fibonacci triangle with all sides unequal?
(2) Is a Fibonacci quadrilateral possible? Under what circumstances?
(3) What is the limiting situation between polygons and non-polygons for the pentagon?
(4) Is there some situation in which we can be sure that a polygon can always be formed if the number of sides is greater than a given quantity?
(5) Is there some situation in which we can be certain that a Fibonacci polygon can never be formed?

This study leads to some interesting results. It is not difficult but it is rewarding in mathematical insights into the properties of Fibonacci numbers.

Readers are encouraged to send their discoveries to the editor of this section by December 15, 1963, so that it may be possible to give due recognition to all contributors in the issue of February, 1964.

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A PRIMER FOR THE FIBONAOCI SEQUENCE - PART III VERNER E. HOGGATt, JR. and I.D. RUGGLES, SAN JOSE STATE COLLEGE

## 1. INTRODUCTION

The algebra of vectors and matrices will be further pursued to derive some more Fibonacci identities.

## 2. THE ALGEBRA OF (TWO-DIMENSIONAL) VECTORS

The two-dimensional vector, V , is an ordered pair of elements, called scalars, of a field: (The real numbers, for example, form a field.)

$$
V=(a, b)
$$

The zero vector, $\phi$, is a vector whose elements are each zero (i.e., $a=0$ and $b=0$ ).

Two vectors, $\mathrm{U}=(\mathrm{a}, \mathrm{b})$ and $\mathrm{V}=(\mathrm{c}, \mathrm{d})$, are equal if and only if $\mathrm{a}=$ $c$ and $b=d_{3}$ that is (iff) their corresponding elements are equal.

The vector $W$, which is the product of a scalar, $k$, and a vector, $U$ $=(a, b)$, is

$$
\mathrm{W}=\mathrm{kU}=(\mathrm{ka}, \mathrm{~kb})=\mathrm{Uk}
$$

We see that if $k=1$, then $k U=U$. We shall define the additive inverse of $U,-U$, by $-U=(-1) U$.

The vector $W$, which is the vector sum of two vectors $U=(a, b)$ and $\mathrm{V}=(\mathrm{c}, \mathrm{d})$ is

$$
W=U+V=(a, b)+(c, d)=(a+c, b+d)
$$

The vector $W=U-V$ is

$$
\mathrm{W}=\mathrm{U}-\mathrm{V}=\mathrm{U}+(-\mathrm{V})
$$

which defines subtraction.
The only binary multiplicative operation between two vectors, $U=(a, b)$ and $V=(c, d)$, considered here is the scalar or inner product, $U \cdot V$,

$$
\mathrm{U} \cdot \mathrm{~V}=(\mathrm{a}, \mathrm{~b}) \cdot(\mathrm{c}, \mathrm{~d})=\mathrm{ac}+\mathrm{bd}
$$

which is a scalar.

## 3. A GEOMETRIC INTERPRETATION OF A TWO--DIMENSIONAL VECTOR

One interpretation of the vector, $U=(a, b)$, is a directed line segment from the origin $(0,0)$ to the point $(a, b)$ in a rectangular coordinate system. Every vector, except the zerovector, $\phi$, will have the direction from the origin $(0,0)$ to the point $(a, b)$ and a magnitude or length, $U=\sqrt{a^{2}+b^{2}}$. The zero vector, $\phi$, has a zero magnitude and no defined direction.

The inner or scalar product of two vectors, $U=(a, b)$ and $V=(c, d)$ can be shown to equal

$$
U \cdot V=|U||V| \cos \theta
$$

where $\theta$ is the angle between the two vectors.

## 4. TWO-BY-TWO MATRICES AND TWO-DIMENSIONAL VECTORS

If $U=(a, b)$ is written as (a b), then $U$ is a $1 \times 2$ matrix which we shall call a row-vector. If $U=(a, b)$ is written $\binom{a}{b}$, then $U$ is a $2 \times 1$ matrix, which we shall call a column-vector.

The matrix

$$
A=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

for example, can be considered as two row vectors $R_{1}=(a b)$ and $R_{2}=(c d)$ in special position or, as two column vectors, $C_{1}=\binom{a}{c}$ and $C_{2}=\binom{b}{d}$ in special position.

The product $W$ of a matrix $A$ and a column-vector $X=\binom{x}{y}$ is a column-vector, $X^{\prime}$,

$$
X^{\prime}=A X=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{x}{y}=\binom{a x+b y}{c x+d y}=\binom{x^{\prime}}{y^{\prime}} .
$$

Thus matrix $A$, operating upon the vector, $X$, yields another vector, $X$ '. The zero vector, $\phi=\binom{0}{0}$, is transformedinto the zero vector again. In general, the direction and magnitude of vector, $X$, are different from those of vector X'. (See "Beginners' Corner" this issue.)

## 5. THE INVERSE OF A TWO-BY-TWO MATRIX

If the determinant, $d(A)$, of a two-by-two matrix, $A$, is non-zero, then there exists a matrix, $A^{-1}$, the inverse of matrix $A$, such that

$$
\mathrm{A}^{-1} \mathrm{~A}=\mathrm{AA}^{-1}=\mathrm{I}
$$

From the equation $A X=X^{\prime}$ or pair of equations

$$
a x+b y=x^{\prime} \quad c x+d y=y^{\prime}
$$

one can solve for the variables $x$ and $y$ in terms of $a, b, c, d$; and $x^{9}, y^{\prime}$ provided $D(A)=a d-b c \neq 0$. Suppose this has been done so that (let $D=D(A)$ * 0 )

$$
\begin{aligned}
& \frac{d}{D} x^{\prime}-\frac{b}{D} y^{p}=x \\
& \frac{-c}{D} x^{p}+\frac{a}{D} y^{p}=y
\end{aligned}
$$

Thus the matrix $B$, such that $\mathrm{BX}^{\prime}=\mathrm{X}$ is given by

$$
B=\left(\begin{array}{cc}
\frac{d}{D} & \frac{-b}{D} \\
\frac{-c}{D} & \frac{a}{D}
\end{array}\right), \quad D \neq 0
$$

It is easy to verify that $B A=A B=I$. Thus $B$ is $A^{-1}$, the inverse matrix to matrix $A$. The inverse of the $Q$ matrix is $Q^{-1}=\left(\begin{array}{rr}0 & 1 \\ 1 & -1\end{array}\right)$.
6. FIBONACCI IDENTITY USING THE Q MATRIX

Suppose we prove, recalling $Q=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ and

$$
Q^{n}=\left(\begin{array}{ll}
F_{n+1} & F_{n} \\
F_{n} & F_{n-1}
\end{array}\right) \text {, that } F_{1}+F_{2}+\cdots+F_{n}=F_{n+2}-1
$$

It is easy to establish by induction that

$$
\left(\mathrm{I}+\mathrm{Q}+\mathrm{Q}^{2}+\cdots+\mathrm{Q}^{\mathrm{n}}\right)(\mathrm{Q}-\mathrm{I})=\mathrm{Q}^{\mathrm{n}+1}-\mathrm{I}
$$

If $Q-I$ has an inverse $(Q-I)^{-1}$, then multiplying on each side
yields

$$
\mathrm{I}+\mathrm{Q}+\mathrm{Q}^{2}+\cdots+\mathrm{Q}^{\mathrm{n}}=\left(\mathrm{Q}^{\mathrm{n}+1}-\mathrm{I}\right)(\mathrm{Q}-\mathrm{I})^{-1}
$$

It is easy to verify that $Q=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ satisfies the matrix equation $Q^{2}=Q+I$. Thus $(\mathrm{Q}-\mathrm{I}) \mathrm{Q}=\mathrm{Q}^{2}-\mathrm{Q}=\mathrm{I}$ and $(\mathrm{Q}-\mathrm{I})^{-1}=\mathrm{Q}$. Therefore

$$
\mathrm{Q}+\cdots+\mathrm{Q}^{\mathrm{n}}=\mathrm{Q}^{\mathrm{n}+2}-(\mathrm{Q}+\mathrm{I})=\mathrm{Q}^{\mathrm{n}+2}-\mathrm{Q}^{2} .
$$

Equating elements in the upper right (in the above matrix equation) yields

$$
\mathrm{F}_{1}+\mathrm{F}_{2}+\cdots+\mathrm{F}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}+2}-\mathrm{F}_{2}=\mathrm{F}_{\mathrm{n}+2}-1
$$

## 7. THE CHARACTERISTIC POLYNOMIAL OF MATRIX A

In Section 4, we discussed the transformation $A X=X^{\prime}$. Generally the direction and magnitude of vector, $X$, are different from those of vector, $X^{\prime}$. If we ask which vectors $X$ have their directions unchanged, we are led to the equation

$$
A X=\lambda X, \quad(\lambda, \text { a scalar })
$$

This can be rewritten $(A-\lambda I) X=0$. Since we want $|X| \neq 0$, the only possible solution occurs when $D(A-\lambda I)=0$. This last equation is called the characteristic equation of matrix $A$. The values of $\lambda$ are called characteristic values of eigenvalues and the associated vectors are the characteristic vectors of matrix $A$. The characteristic polynomial of $A$ is $D(A-\lambda I)$.

The characteristic polynomial for the $Q$ matrix is $\lambda^{2}-\lambda-1=0$. The Hamilton-Cayley theorem states a matrix satisfies its own characteristic equation, so that for the $Q$ matrix

$$
Q^{2}-Q-I=0
$$

## 8. SOME MORE IDENTITIES

Let $Q=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$, which satisfies $Q^{2}=Q+I$, thus (remembering $Q^{0}=I$ )

$$
\begin{equation*}
Q^{2 n}=\left(Q^{2}\right)^{n}=(Q+I)^{n}=\sum_{i=0}^{n}\binom{n}{i} Q^{i} \tag{1}
\end{equation*}
$$

Equating elements in the upper right yields

$$
F_{2 n}=\sum_{i=0}^{n}\left(\begin{array}{ll}
n \\
i_{i} & F_{i}
\end{array}\right.
$$

(Compare with problems $\mathrm{H}-18$ and $\mathrm{B}-4$ ).
From (1)

$$
Q^{p} Q^{2 n}=\sum_{i=0}^{n}\binom{n}{i} Q^{i+p}
$$

which gives

$$
F_{2 n+p}=\sum_{i=0}^{n}\binom{n}{i} F_{i+p} \quad(n \geq 0, \text { integral } p)
$$

From part $\Pi_{2} Q^{n}=F_{n} Q+F_{n-1} I$,

$$
Q^{m n+p}=\sum_{i=0}^{m}\binom{m}{i} Q^{i+p} F_{n}^{i} F_{n-1}^{m-i}
$$

Equating elements in upper right of the above matrix equation gives

$$
F_{m n+p}=\sum_{i=0}^{m}\binom{m}{i} F_{i+p} F_{n}^{i} F_{n-1}^{m-i}
$$

with $\mathrm{m} \geq 0$, any integral p and n .
(See the result p. 38, line 12, issue 2, and H-13).

## 

HAVE YOU SEEN??
Melvin Hochster, "Fibonacci-Type Series and Pascal's Triangle," Particle, Vol. IV, No. 1, Summer 1962, pp. 14-28. (Written while author was a sophomore at Harvard University, but the work was done while he was a senior student at Stuyvesant High School, New York, New York.)

Particle is a quarterly by and for science students with editorial and publishing offices located at 2531 Ridge Road, Berkeley 9, California. The present editor is Steve Kahn.
A. Hamilton Bolton, The Elliott Wave Principle-A Critical Appraisal, BoltonTremblay and Company, Montreal 2, Canada, Chap. IX, pp. 61-67.

This is an interesting application of the Fibonacci Sequence to business cycles, and will merit some interest.

# TRIANGLE INSCRIBED IN RECTANGLE 

## J. A. H. HUNTER

Arising from a problem proposed recently by Ben Cohen in a letter to myself, yet another example of the famous Golden Section has been revealed. The problem was:

Within a given rectangle, inscribe a triangle such that the remainder of the rectangle will comprise three triangles of equal area.


Referring to the figure above, we have:

$$
\mathrm{xw}=\mathrm{yz}, \text { and } \mathrm{x}=\mathrm{yw} /(\mathrm{w}+\mathrm{z})
$$

whence

$$
z^{2}+z w-w^{2}=0, \text { so } 2 z=w(\sqrt{5}-1)
$$

Then,

$$
2 \mathrm{x}=\mathrm{y}(\sqrt{5}-1)
$$

So, as a necessary condition to meet the requirements, we have:

$$
\frac{y}{x}=\frac{w}{z}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}
$$

the Golden Section.

## 

## FIBONACCI SUMMATIONS

KEN SILER, ST. MARY'S COLLEGE, CALIFORNIA

In the first issue of the FIBONACCI QUARTERLY, several problems regarding summation of terms of the Fibonacci series were proposed [1]. They can be solved without too much difficulty by means of intuition followed by mathematical induction. The results for the series suggested in the article "Exploring Fibonacci Numbers" are as follows:

$$
\begin{array}{rlrl}
\sum_{k=1}^{n} F_{2 k} & =F_{2 n+1}-1 & \sum_{k=1}^{n} F_{4 k-3} & =F_{2 n-1} F_{2 n} \\
\sum_{k=1}^{n} F_{4 k-2} & =F_{2 n}^{2} & \sum_{k=1}^{n} F_{4 k-1} & =F_{2 n} F_{2 n+1} \\
\sum_{k=1}^{n} F_{4 k} & =F_{2 n+1}^{2}-1 & 2 \sum_{k=1}^{n} F_{3 k-2} & =F_{3 n} \\
2 \sum_{k=1}^{n} F_{3 k-1} & =F_{3 n+1}-1 & 2 \sum_{k=1}^{n} F_{3 k}=F_{3 n+2}-1
\end{array}
$$

The attempt to extend this work by intuition to such summations as

$$
\sum_{k=1}^{n} F_{5 k-4}
$$

leads to difficulties. One is led therefore to adopt a more mathematical approach in solving the general case of all Fibonacci series summations with subscripts in arithmetic progression, namely,

$$
\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{~F}_{\mathrm{ak}-\mathrm{b}}
$$

where $a$ and $b$ are positive integers and $b<a$.

We recall that Fibonacci numbers can be given in terms of the roots of the equation $x^{2}-x-1=0$ [2]. If these roots are

$$
\mathrm{r}=\frac{1+\sqrt{5}}{2} \text { and } \mathrm{s}=\frac{1-\sqrt{5}}{2}
$$

then

$$
\mathrm{F}_{\mathrm{n}}=\frac{\mathrm{r}^{\mathrm{n}}-\mathrm{s}^{\mathrm{n}}}{\sqrt{5}} \text { and } \mathrm{L}_{\mathrm{n}}=\mathrm{r}^{\mathrm{n}}+\mathrm{s}^{\mathrm{n}}
$$

where $F_{n}$ is the $n^{\text {th }}$ term of the Fibonacci sequence $1,1,2,3,5, \cdots$ and $L_{n}$ is the $\mathrm{n}^{\text {th }}$ term of the Lucas sequence $1,3,4,7,11,18, \cdots$. In these terms

$$
\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{~F}_{\mathrm{ak}-\mathrm{b}}=\frac{1}{\sqrt{5}}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{r}^{\mathrm{ak}-\mathrm{b}}-\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{~s}^{\mathrm{ak}-\mathrm{b}}\right]
$$

One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions.

$$
\sum_{k=1}^{n} r^{a k-b}=r^{a-b}\left[1+r^{a}+r^{2 a}+r^{3 a}+\cdots+r^{(n-1) a}\right]=r^{a-b}\left(\frac{r^{a n}-1}{r^{a}-1}\right)
$$

There is an entirely similar formula for the " $s$ " summation. Substituting into the original formula and combining fractions, one obtains

$$
\left.\begin{array}{r}
\sum_{k=1}^{n} F_{a k-b}=\frac{1}{\sqrt{5}}\left[\frac{s^{a} r^{a n+a-b}-r^{a} s^{a n+a-b}-r^{a n+a-b}+s^{a n+a-b}}{r^{a} s^{a}-r^{a}-s^{a}+1}\right. \\
\left.+\frac{s^{a} r^{a-b}+r^{a} s^{a-b}+r^{a-b}-s^{a-b}}{r^{a} s^{a}-r^{a}-s^{a}+1}\right]
\end{array}\right]
$$

Various simplifications result using the definitions of $F_{n}$ and $L_{n}$ in terms of $r$ and $s$ together with the relation $r s=-1$, the product of the roots in the equation $x^{2}-x-1=0$ being the constant term -1 . For example,

$$
s^{a} r^{a n+a-b}-r^{a} s^{a n+a-b}=(r s)^{a}\left(r^{a n-b}-s^{a n-b}\right)=(-1)^{a} \sqrt{5} F_{a n-b} .
$$

The denominator can be transformed into $(-1)^{\mathrm{a}}-\mathrm{L}_{\mathrm{a}}+1$. Using these relations the reader may verify without too much difficulty that the final formula is

$$
\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{~F}_{\mathrm{ak}-\mathrm{b}}=\frac{(-1)^{\mathrm{a}} \mathrm{~F}_{\mathrm{an}-\mathrm{b}}-\mathrm{F}_{\mathrm{a}(\mathrm{n}+1)-\mathrm{b}}+(-1)^{\mathrm{a}-\mathrm{b}} \mathrm{~F}_{\mathrm{b}}+\mathrm{F}_{\mathrm{a}-\mathrm{b}}}{(-1)^{\mathrm{a}}+1-\mathrm{L}_{\mathrm{a}}}
$$

With this formula particular cases can be handled with little effort. For example, let $\mathrm{a}=7, \mathrm{~b}=3$ and $\mathrm{n}=6$. Then

$$
\begin{aligned}
\sum_{\mathrm{k}=1}^{6} \mathrm{~F}_{7 \mathrm{k}-3} & =\frac{(-1)^{7} \mathrm{~F}_{39}-\mathrm{F}_{46}+(-1)^{4} \mathrm{~F}_{3}+\mathrm{F}_{4}}{(-1)^{7}+1-\mathrm{L}_{7}} \\
& =\frac{-63245986-1836311903+2+3}{-29} \\
& =65501996
\end{aligned}
$$

This result may be checked by actually summing the series:

$$
\begin{aligned}
\mathrm{F}_{4}+\mathrm{F}_{11}+\mathrm{F}_{18}+\mathrm{F}_{25}+\mathrm{F}_{32}+\mathrm{F}_{39} \text { or } 3+89+2584+75025 & +2178309 \\
& +63245986
\end{aligned}
$$

the result being 65501996 .

## REFERENCES

1. Brother U. Alfred, Exploring Fibonacci Numbers, Fibonacci Quarterly, Vol. 1, No. 1, February 1963, pp. 57-64.
2. I. Dale Ruggles, Some Fibonacci Results Using Fibonacci-Type Sequences, Fibonacci Quarterly, Vol. 1, No. 2, April 1963, pp. 75-80.

## Editorial Comment

Mark Feinberg is a fourteen-year-old student in the ninth grade of the Susquehanna Township Junior High School and recently became the Pennsylvania State Grand Champion in the Junior Academy of Science. This paper is based on his winning project and is editorially uncut. Mark Feinberg, in this editor's opinion, will go far in his chosen field of endeavor. Congratulations from the editorial staff of the Fibonacci Quarterly Journal, Mark?


Figure 1


Figure 3

## FIBONACCI-TRIBONACCI* <br> MARK FEINBERG

For this Junior High School Science Fair project two variations of the Fibonacci series were worked out.
"TRIBONACCI"
Just as in the Fibonacci series where each number is the sum of the preceding two, or $p_{n+1}=p_{n}+p_{n-1}$, the first variation is a series in which each number is the sum of the preceding three, or $q_{n+1}=q_{n}+q_{n-1}+q_{n-2}$; hence the series is called "Tribonacci." Its first few numbers are

$$
1,1,2,4,7,13,24,44,81,149,274,504 \cdots
$$

Like the Fibonacci series, the Tribonacci series is convergent. Where the Fibonacci fractions $p_{n} /\left(p_{n+1}\right.$ and $p_{n+1} / p_{n}$ converge on $.6180339 \cdots$ and $1.6180339 \ldots$, the Tribonacci fraction of any number of the series divided by the preceding one $\left(q_{n} / q_{n+1}\right)$ approaches $.54368901 \cdots$. While the Fibonacci convergents are termed "Phi" $(\phi)$, the Tribonacci convergents might be called "Tri-Phi" ( $\phi_{3}$ )

Series-repeating characteristics are shown in the famed Fibonacci Golden Rectangle.* A rectangle can be made of the Tribonacci series which also has series-repeating characteristics but since they are less obvious this rectangle might be called the "Silver Rectangle." Its length $\left(q_{n+1}\right)$ and its width ( $q_{n}$ ) make it proportionately longer than the Golden Rectangle.

By removing the squares $q_{n}$ by $q_{n}$ and $q_{n-1}$ by $q_{n-1}$, two new rectangles in the proportion of the original appear (shaded areas). One is $q_{n-1}$ by $q_{n-2}$; but the other is composed of numbers not found in the Tribonacci series. This rectangle is $\left(q_{n+1}-q_{n}\right)$ by $\left(q_{n}-q_{n-1}\right)$ and is formed of numbers from an intermediate series obtained by subtracting each Tribonacci number from the one after it.
*See editorial remarks, page 70. Figure 1 appears on page 70.

By carrying the rectangle out farther new numbers found in neither the original Tribonacci series nor the intermediate series appear. These are of a second intermediate series and are obtained by subtracting each number of the first intermediate series from the succeeding one. New numbers of new intermediate series also appear by further carrying out the rectangle. These other series are formed by triangulating in the same way as the first two intermediate series. All these intermediate series are convergent upon the "Tri-Phi" values and each number in each of these series is the sum of the preceding three. Figure 2 shows the first two intermediate series.

$$
\begin{aligned}
& 1,1,1,3,5,4,17,31 \cdots \\
& 0,1,2,3,6,11,20,37,68 \cdots \\
& 1, \quad 1, \quad 2,4, \quad 7,13,24,44,81,149 \cdots
\end{aligned}
$$

Figure 2
The two Fibonacci convergents fit the quadratic equation $x=1+1 / x$. The Tribonacci convergent of any number in the series divided by the preceding one $\left(q_{n+1} / q_{n}\right)$ fits the cubic equation $y=1+1 / y+1 / y^{2}$. It is derived thus:

The formula giving any number in the series is

$$
q_{n+1}=q_{n}+q_{n-1}+q_{n-2}
$$

Dividing by $q_{n-1}$ :

$$
\frac{q_{n+1}}{q_{n-1}}=\frac{q_{n}}{q_{n-1}}+1+\frac{q_{n-2}}{q_{n-1}}
$$

Let

$$
\frac{q_{n+1}}{q_{n}}=t_{n} ; \frac{q_{n}}{q_{n-1}}=t_{n-1} ; \frac{q_{n-1}}{q_{n-2}}=t_{n-2}
$$

Then since

$$
\frac{q_{n+1}}{q_{n-1}}=\frac{q_{n+1}}{q_{n-1}} \cdot \frac{q_{n}}{q_{n}}=\frac{q_{n+1}}{q_{n}} \cdot \frac{q_{n}}{q_{n-1}}=t_{n} \cdot t_{n-1}
$$

Therefore

$$
t_{n} \cdot t_{n-1}=\frac{q_{n}}{q_{n-1}}+1+\frac{q_{n-2}}{q_{n-1}}
$$

Substituting for the rest of the formula:

$$
t_{n} \cdot t_{n-1}=t_{n-1}+1+\frac{1}{t_{n-2}}
$$

Dividing by $t_{n-1}$ :

$$
t_{n}=1+\frac{1}{t_{n-1}}+\frac{1}{t_{n-1} \cdot t_{n-2}}
$$

All the $t_{n}$ terms converge upon one value (y). Therefore " $y$ " can be substituted for all $t_{n}$ terms. So

$$
\mathrm{y}=1+\frac{1}{\mathrm{y}}+\frac{1}{\mathrm{y}^{2}}
$$

The convergent approached by any number of the series divided by the succeeding one $\left(q_{n} / q_{n+1}\right)$ fits the cubic equation $1 / y=1+y+y^{2}$ and is derived through a similar process.

Charting the Fibonacci convergent $.6180339 \ldots$ on polar coordinate paper is known to produce the famed spiral found all over nature. By charting the Tribonacci convergent $.54368901 \cdots$ a slightly tighter spiral is produced.

It is not known whether the Tribonacci series has any natural applications. A well-known Fibonacci application is of a hypothetical tree. If each limb were to sprout another limb one year and rest the next, the number of limbs per year would total $1,2,3,5,8 \cdots$ in Fibonacci sequence. However if each limb on the tree were to sprout for two years and rest for a year, the number of limbs per year would total $1,2,4,7,13 \cdots$ in Tribonacci sequence. See Figure 3, page 70.

Could such a tree as that on the right be called a "Tree-bonacci?"

## "TETRANACCI"

The second variation of the Fibonacci sequence is a series in which each number is the sum of the preceding four numbers or $r_{n+1}=r_{n}+r_{n-1}+r_{n-2}$
$+r_{n-3}$. Therefore this series is called "Tetranacci." Its first few numbers are:

$$
1,1,2,4,8,15,29,56,108,208,401,773 \ldots
$$

Like the Fibonacci and Tribonacci series, the Tetranacci series is convergent. The fraction $r_{n+1} / r_{n}$ converges upon $1.9275619 \cdots$ and fits the fourth power equation $z=1+1 / z+1 / z^{2}+1 / z^{3}$.

The fraction $r_{n} / r_{n+1}$ converges upon $.51879006 \cdots$ which fits the equation $1 / z=1+z+z^{2}+z^{3}$.

The derivation of these formulas follows the same algebraic process as that given above and will be gladly furnished upon request.

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## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY S.L. BASIN, SYLVANIA ELECTRONIC SYSTEMS, MT。VIEW, CALIF。

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

```
B-17 Proposed by Charles R. Wall, Ft. Worth, Texas
```

If $m$ is an integer, prove that

$$
F_{n+4 m+2}-F_{n}=L_{2 m+1} F_{n+2 m+1}
$$

where $F_{p}$ and $L_{p}$ are the $p^{\text {th }}$ Fibonacci and Lucas numbers, respectively. B-18 Proposed by J. L. Brown, Jr.. Pennsylvania State University.

Show that

$$
\mathrm{F}_{\mathrm{n}}=2^{\mathrm{n}-1} \sum_{\mathrm{k}=0}^{\mathrm{n}-1}(-1)^{\mathrm{k}} \cos ^{\mathrm{n}-\mathrm{k}-1} \frac{\pi}{5} \sin ^{\mathrm{k}} \frac{\pi}{10} \text {, for } \mathrm{n} \geq 0
$$

B-19 Proposed by L. Carlitz, Duke University, Durham, N.C.
Show that

$$
\sum_{n=1}^{\infty} \frac{1}{F_{n} F_{n+2}^{2} F_{n+3}}+\sum_{n=1}^{\infty} \frac{1}{F_{n} F_{n+1}^{2} F_{n+3}}=\frac{1}{2}
$$

B-20 Proposed by Louis G. Brökling, Redwood City, Calif.
Generalize the well-known identities,

$$
\begin{align*}
& \mathrm{F}_{1}+\mathrm{F}_{2}+\mathrm{F}_{3}+\cdots+\mathrm{F}_{\mathrm{n}}=\mathrm{F}_{\mathrm{n}+2}-1  \tag{i}\\
& \mathrm{~L}_{1}+\mathrm{L}_{2}+\mathrm{L}_{3}+\cdots+\mathrm{L}_{\mathrm{n}}=\mathrm{L}_{\mathrm{n}+2}-3
\end{align*}
$$

B-21 Proposed by L. Carlitz, Duke University, Durham, N.C.
If

$$
u_{\mathrm{n}}=\frac{1}{2}\left[(\mathrm{x}+1)^{2^{\mathrm{n}}}+(\mathrm{x}-1)^{2^{n}}\right]
$$

show that

$$
u_{n+1}=u_{n}^{2}+2^{2 n} u_{0}^{2} u_{1}^{2} \cdots u_{n-1}^{2}
$$

B-22 Proposed by Brother U. Alfred. St. Mary's College, Calif.
Prove the Fibonacci identity

$$
\mathrm{F}_{2 \mathrm{k}^{2}} \mathrm{~F}_{2 \mathrm{k}^{\mathbf{t}}}=\mathrm{F}_{\mathrm{k}+\mathrm{k}^{\prime}}^{2}-\mathrm{F}_{\mathrm{k}-\mathrm{k}^{\prime}}^{2}
$$

and find the analogous Lucas identity.
B-23 Proposed by S.L. Basin, Sylvania Electronic Systems, Mt. View, Calif.
Prove the identities
(i)
$F_{n+1}=\prod_{i=1}^{n}\left(1+\frac{F_{i-1}}{F_{i}}\right)$
(ii)

$$
\frac{F_{n+1}}{F_{n}}=1+\sum_{i=1}^{n} \frac{(-1)^{i}}{F_{i} F_{i-1}}
$$

(iii)

$$
\frac{1+\sqrt{5}}{2}=1+\sum_{i=1}^{\infty} \frac{(-1)^{i}}{\mathrm{~F}_{\mathrm{i}} \mathrm{~F}_{\mathrm{i}-1}}
$$

SOLUTIONS TO PROBLEMS IN VOL. 1, FEBRUARY, 1963
B-1 Show that the sum of twenty consecutive Fibonacci numbers is divisible by $\mathrm{F}_{10}$, i. e. $, \sum_{i=1}^{20} \mathrm{~F}_{\mathrm{n}+1} \equiv 0\left(\bmod \mathrm{~F}_{10}\right), \quad \mathrm{n} \geq 0$.
Solution by Marjorie R. Bicknell, San Jose State College, San Jose, Calif.
The proof is by induction. When $\mathrm{n}=0$

$$
\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{i}}=\mathrm{F}_{22}-1=\mathrm{F}_{10}\left(\mathrm{~F}_{13}+\mathrm{F}_{11}\right)
$$

Assume that the proposition is true for all $n \leq k$, i.e.,

$$
\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{k}+\mathrm{i}} \equiv 0\left(\bmod \mathrm{~F}_{10}\right)
$$

and

$$
\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{k}-1+\mathrm{i}} \equiv 0 \quad\left(\bmod \mathrm{~F}_{10}\right)
$$

Now addition of the congruences yields
$\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{k}+\mathrm{i}}+\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{k}-1+\mathrm{i}}=\sum_{\mathrm{i}=1}^{20}\left(\mathrm{~F}_{\mathrm{k}+\mathrm{i}}+\mathrm{F}_{\mathrm{k}-1+\mathrm{i}}\right)=\sum_{\mathrm{i}=1}^{20} \mathrm{~F}_{\mathrm{k}+1+\mathrm{i}} \equiv 0\left(\bmod \mathrm{~F}_{10}\right)$
Also solved by J. L. Brown, Jr., Dermott A. Breault, and the proposer.
B-2 Show that $u_{n+1}+u_{n+2}+\cdots+u_{n+10}=11 u_{n+7}$ holds for generalized Fibonacci numbers such that $u_{n+2}=u_{n+1}+u_{n}$, where $u_{1}=p$ and $u_{2}=q$.
Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania
It is easily shown, by induction, that $u_{n}$ may be written in terms of the Fibonacci numbers, $F_{m}$, as $u_{n}=p F_{n-2}+q F_{n-1}$ for $n \geq 1$. Using this result, we have

$$
\sum_{\mathrm{k}=1}^{10} \mathrm{u}_{\mathrm{n}+\mathrm{k}}=\mathrm{p} \sum_{\mathrm{k}=1}^{10} \mathrm{~F}_{\mathrm{n}+\mathrm{k}-2}+\mathrm{q} \sum_{\mathrm{k}=1}^{10} \mathrm{~F}_{\mathrm{n}+\mathrm{k}-1}
$$

and

$$
11 u_{n+7}=11\left(p F_{n+5}+q F_{n+6}\right)
$$

The result follows if

$$
\begin{equation*}
\sum_{\mathrm{k}=1}^{10} \mathrm{~F}_{\mathrm{n}+\mathrm{k}}=11 \mathrm{~F}_{\mathrm{n}+7} \text { for } \mathrm{n} \geq 0 \tag{1}
\end{equation*}
$$

however,

$$
\begin{aligned}
\sum_{\mathrm{k}=1}^{10} \mathrm{~F}_{\mathrm{n}+\mathrm{k}}=\sum_{\mathrm{i}=1}^{\mathrm{n}+10} \mathrm{~F}_{\mathrm{i}}-\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~F}_{\mathrm{i}} & =\left(\mathrm{F}_{\mathrm{n}+12}-1\right)+\left(\mathrm{F}_{\mathrm{n}+2}-1\right) \\
& =\mathrm{F}_{\mathrm{n}+12}-\mathrm{F}_{\mathrm{n}+2}, \mathrm{n} \geq 0
\end{aligned}
$$

Equation (1) is therefore equivalent to

$$
\begin{equation*}
\mathrm{F}_{\mathrm{n}+12}-\mathrm{F}_{\mathrm{n}+2}=11 \mathrm{~F}_{\mathrm{n}+7}, \mathrm{n} \geq 0 \tag{2}
\end{equation*}
$$

By direct calculation, $\mathrm{F}_{12}-\mathrm{F}_{2}=11 \mathrm{~F}_{7}$ and $\mathrm{F}_{13}-\mathrm{F}_{3}=11 \mathrm{~F}_{8}$; now adding these identities we have, $\mathrm{F}_{14}-\mathrm{F}_{4}=11 \mathrm{~F}_{9}$. Proceeding in this fashion, (2) is verified by induction.

Also solved by Dermott A. Breault, Marjorie R. Bicknell and Edward Balizer.
B-3 Show that $\quad F_{n+24} \equiv F_{n}(\bmod 9) \quad$.
Solution by Marjorie R. Bicknell, San Jose State College, San Jose, Calif.
Proof is by mathematical induction. When $n=0$,

$$
F_{24}=F_{12}\left(F_{13}+F_{11}\right)=144\left(F_{13}+F_{11}\right) \equiv F_{0}(\bmod 9)
$$

Assuming that the proposition holds for all integers $n \leq k$,

$$
\mathrm{F}_{\mathrm{k}-1+24} \equiv \mathrm{~F}_{\mathrm{k}-1}(\bmod 9)
$$

and

$$
\mathrm{F}_{\mathrm{k}+24} \equiv \mathrm{~F}_{\mathrm{k}} \quad(\bmod 9)
$$

Adding the congruences, we have

$$
\mathrm{F}_{\mathrm{k}+1+24} \equiv \mathrm{~F}_{\mathrm{k}+1}(\bmod 9)
$$

and the proof is complete by mathematical induction.
Solution by Dermott A. Breault, Sylvania ARL, Waltham, Mass.
Using the identity $\mathrm{F}_{\mathrm{n}+\mathrm{p}+1}=\mathrm{F}_{\mathrm{n}+1} \mathrm{~F}_{\mathrm{p}+1}+\mathrm{F}_{\mathrm{n}} \mathrm{F}_{\mathrm{p}}$ write

$$
\mathrm{F}_{24+\mathrm{n}}=\mathrm{F}_{23+\mathrm{n}+1}=\mathrm{F}_{24} \mathrm{~F}_{\mathrm{n}+1}+\mathrm{F}_{23} \mathrm{~F}_{\mathrm{n}}
$$

Therefore,

$$
\mathrm{F}_{\mathrm{n}+24}-\mathrm{F}_{\mathrm{n}}=\mathrm{F}_{24} \mathrm{~F}_{\mathrm{n}+1}+\left(\mathrm{F}_{23}-1\right) \mathrm{F}_{\mathrm{n}} \text {, but }
$$

$$
\mathrm{F}_{24}=9(5153) \quad \text { and } \quad \mathrm{F}_{23}-1=9(3184),
$$

hence

$$
\mathrm{F}_{\mathrm{n}+24}-\mathrm{F}_{\mathrm{n}} \equiv 0(\bmod 9)
$$

Also solved by J. L. Brown, Jr.

B-4 Show that

$$
\sum_{\mathrm{i}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{~F}_{\mathrm{i}}=\mathrm{F}_{2 \mathrm{n}}
$$

Generalize.
Solution by Joseph Erbacher, University of Santa Clara, Santa Clara, Calif., and J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

Using the Binet formula,

$$
F_{2 k n}=\frac{\left(a^{2}\right)^{k n}-\left(b^{2}\right)^{k n}}{a-b}
$$

where

$$
\mathrm{a}^{2}=1+\mathrm{a}, \quad \mathrm{~b}^{2}=1+\mathrm{b}, \quad \mathrm{a}=\frac{1+\sqrt{5}}{2}, \quad \mathrm{~b}=\frac{1-\sqrt{5}}{2}
$$

we have
$F_{2 k n}=\frac{(1+a)^{k n}-(1+b)^{k n}}{a-b}=\frac{1}{a-b}\left[\sum_{i=0}^{\operatorname{kn}}\binom{n}{i} a^{i}-\sum_{i=0}^{k n}\binom{n}{i} b^{i}\right]=\sum_{i=0}^{k n}\binom{n}{i} \frac{a^{i}-b^{i}}{a-b}$

Therefore,

$$
=\sum_{\mathrm{i}=0}^{\mathrm{kn}}\binom{\mathrm{n}}{\mathrm{i}} \mathrm{~F}_{\mathrm{i}}
$$

$$
F_{2 k n}=\sum_{i=0}^{k n}\binom{n}{i} F_{i}
$$

Also solved by the proposers.
B-5 Show that with order taken into account, in getting past an integral number N dollars, using only one-dollar and two-dollar bills, that the number of different ways is $\mathrm{F}_{\mathrm{N}+1}$.
Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.
Let $\alpha_{N}$ for $N \geq 1$ be the number of different ways of being paid $N$ dollars in one and two dollar bills, taking order into account. Consider the case where $N \geq 2$. Since a one-dollar bill is received as the last bill if andonly if N-1 dollars have been received previously and a two-dollar bill is received as the last bill if and only if N-2 dollars have been received previously, the two possibilities being mutually exclusive, we have $\alpha_{N}=\alpha_{N-1}+\alpha_{N-2}$ for $N \geq 2$. But $\alpha_{1}=1, \alpha_{2}=2$, therefore $\alpha_{N}=F_{N+1}$ for $N \geq 1$.

Is the expansion valid at $x=14$, i. e., does $\sum_{i=0}^{\infty} F_{i}^{2} 4^{i}=\frac{12}{25}$
Solution by Wm. E. Briggs, University of Colorado, Boulder, Colo.

$$
\begin{aligned}
& \text { Write }\left(1-2 x-2 x^{2}+x^{3}\right)^{-1}=\sum_{n=0}^{\infty} a_{n} x^{n} \text { where } \\
& a_{k}=2 a_{k-1}+2 a_{k-2}-a_{k-3}, k>2 \text {, and } a_{0}=1, a_{1}=2 \text {, } a=6 .
\end{aligned}
$$

Therefore,

$$
\frac{x(1-x)}{1-2 x-2 x^{2}+x^{3}}=x+\sum_{n=2}^{\infty}\left(a_{n-1}-a_{n-2}\right) x^{n} .
$$

It follows that the coefficient of $\mathrm{x}^{\mathrm{k}}$ is $\mathrm{F}_{\mathrm{k}}^{2}$ for $\mathrm{k}=1,2,3,4$; assume this is true for all $k \leq n$. From above, the coefficient of $x^{n+1}$ is

$$
a_{n}-a_{n-1}=2\left(a_{n-1}-a_{n-2}\right)+2\left(a_{n-2}-a_{n-3}\right)-\left(a_{n-3}-a_{n-4}\right) .
$$

Therefore, $\mathrm{a}_{\mathrm{n}}=\mathrm{a}_{\mathrm{n}-1}=2 \mathrm{~F}_{\mathrm{n}}^{2}+2 \mathrm{~F}_{\mathrm{n}-1}^{2}-\mathrm{F}_{\mathrm{n}-2}^{2}$; however,

$$
\mathrm{F}_{\mathrm{n}+1}^{2}+\mathrm{F}_{\mathrm{n}-2}^{2}=\left(\mathrm{F}_{\mathrm{n}}+\mathrm{F}_{\mathrm{n}-1}\right)^{2}+\left(\mathrm{F}_{\mathrm{n}}-\mathrm{F}_{\mathrm{n}-1}\right)^{2}=2 \mathrm{~F}_{\mathrm{n}}^{2}+2 \mathrm{~F}_{\mathrm{n}-1}^{2}
$$

so that the coefficient of $x^{n+1}$ is $F_{n-1}^{2}$. The zero of $1-2 x-2 x^{2}+x^{3}$ with smallest modulus is $r=(1 / 2)(3-\sqrt{5})$ which is the radius of convergence of the power series. Since $r>|1 / 4|$, the series converges for $x=1 / 4$ to the value $12 / 25$.
Also solved by J. L. Brown, Jr.
B-8 Show that
(ii)

$$
\begin{align*}
\mathrm{F}_{\mathrm{n}+1} 2^{\mathrm{n}}+\mathrm{F}_{\mathrm{n}} 2^{\mathrm{n}+1} & \equiv 1(\bmod 5)  \tag{i}\\
\mathrm{F}_{\mathrm{n}+1} 3^{\mathrm{n}}+\mathrm{F}_{\mathrm{n}} 3^{\mathrm{n}+1} & \equiv 1(\bmod 11) \\
\mathrm{F}_{\mathrm{n}+1} 5^{\mathrm{n}}+\mathrm{F}_{\mathrm{n}} 5^{\mathrm{n}+1} & \equiv 1(\bmod 29)
\end{align*}
$$

(iii)

Generalize.
Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.
The general result,

$$
\mathrm{F}_{\mathrm{n}+1} \mathrm{p}^{\mathrm{n}}+\mathrm{F}_{\mathrm{n}} \mathrm{p}^{\mathrm{n}+1} \equiv 1\left(\bmod \mathrm{p}^{2}+\mathrm{p}-1\right)
$$

where p is a prime and $\mathrm{n} \leq 0$ is proved by mathematical induction.
The proposition is clearly true for $\mathrm{n}=0$ and $\mathrm{n}=1$, with the usual definition $\mathrm{F}_{0}=0$. Suppose the proposition is true for all $\mathrm{n} \leq \mathrm{k}$ where $\mathrm{k} \geqq 1$. (Continued on p. 52)


[^0]:    *This work was partially supported by the Undergraduate Research Participation Program of the National Science Foundation through G-21681.

