FIBONACCI NUMBERS, CHEBYSHEV POLYNOMIALS GENERALIZATIONS AND DIFFERENCE EQUATIONS

R. G. BUSCHMAN, STATE UNIVERSITY OF NEW YORK, BUFFALO, N.Y.

INTRODUCTION

In order to consider Fibonacci numbers, generalized Fibonacci numbers, Chebyshev polynomials, and other related sequences all under one heading we will discuss the sequences generated by the homogeneous linear second order difference equation with constant coefficients,

$$(1) \qquad u_0;\ u_1;\ u_{n+1} = a u_n + b u_{n-1}, \quad \text{for } n \geq 1 .$$

First we note how the special cases arise. If $a = b = 1$, then the generalized Fibonacci numbers, H_n, discussed by Horadam [2] are produced. Further specialization leads to Fibonacci numbers, F_n, for $u_0 = 0$, $u_1 = 1$; to Lucas numbers, L_n, for $u_0 = 2$, $u_1 = 1$. If a and b are polynomials in x, then a sequence of polynomials is generated. In particular, if $a = 2x$ and $b = -1$, then we have Chebyshev polynomials [1:10.11] — of the first kind, $T_n(x)$, for $u_0 = 1$, $u_1 = x$; of the second kind, $U_n(x)$, for $u_0 = 1$, $u_1 = 2x$.

FIBONACCI-CHEBYSHEV RELATIONS

Since the same difference equation can be used to generate these entities, by an appropriate interpretation of a, b, u_0, and u_1, one then expects relationships to exist between some of them. The Fibonacci and Lucas numbers are related to the Chebyshev polynomials by the equations

$$2 i^{-n} T_n(i/2) = L_n; \quad i^{-n} U_n(i/2) = F_{n+1} .$$

The second of these can be obtained, for example, by considering

$$U_0(x) = 1; \quad U_1(x) = 2x; \quad U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x) ,$$

substituting $i/2$ for x, and multiplying by i^{-n-1} so that we have

$$U_0(i/2) = 1; \quad i^{-1}U_1(i/2) = 1; \quad i^{-n-1}U_{n+1}(i/2) = i^{-n}U_n(i/2) + i^{-n+1}U_{n-1}(i/2),$$

which is the same as the Fibonacci sequence,

$$F_1 = 1; \quad F_2 = 1; \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for } n \geq 2 .$$

This close relation leads one to investigate sources for Chebyshev polynomials in order to try to find not too familiar relations involving Fibonacci and Lucas numbers, and vice versa. One such standard source for identities involving Chebyshev polynomials is Erdelyi, et al. [1:10.9, 10.11]. Most of the results which can be obtained were known as early as Lucas [3]; in fact, much of his discussion contains relations with trigonometric identities which lead, of course, to Chebyshev polynomial identities, since

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta .$$

Some examples of such pairs of relations follow.

$$U_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m)!}{m!\,(n-2m)!}(2x)^{n-2m}, \qquad [1:10.11\ (23)].$$

$$F_n = \sum_{m=0}^{[n/2]} \binom{n-m}{m}, \qquad [3:(72)].$$

$$T_n(x) = \frac{n}{2}\sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m} \qquad [1:10.11\ (22)].$$

$$L_n = \sum_{m=0}^{[n/2]} \frac{n}{n-m}\binom{n-m}{m}.$$

Examples of interesting generating functions are given by [1:10.11 (32), (33)] which for $x = i/2$, $z = -iu$ lead to

$$(2) \quad 2^{-\frac{1}{2}}(1-u-u^2)^{-\frac{1}{2}}\{1-u/2+(1-u-u^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} = u^{-1}\sum_{n=0}^{\infty} 2^{-2n}\binom{2n}{n}F_n u^n$$

$$(3) \qquad 2^{-\frac{1}{2}}(1-u-u^2)^{-\frac{1}{2}}\{1-u/2+(1-u-u^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = \sum_{n=0}^{\infty} 2^{-2n-1}\binom{2n}{n} L_n u^n \quad .$$

If the series (2) and (3) are multiplied together as power series, then we have

$$2^{-1}(1-u-u^2)^{-1} = u^{-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 2^{-2n-1}\frac{(2k)!\,(2n-2k)!}{k!\,k!\,(n-k)!\,(n-k)!} L_k F_{n-k}\right)u^n \quad .$$

However, this is a generating function for F_n,

$$2^{-1}(1-u-u^2)^{-1} = u^{-1}\sum_{n=0}^{\infty}(F_n/2)\,u^n \quad ,$$

so that by equating coefficients and rearranging somewhat we obtain

$$\sum_{k=0}^{n}\binom{n}{k}L_k\binom{n}{n-k}F_{n-k}\Big/\binom{2n}{2k} = 2^{2n}F_n\Big/\binom{2n}{n} \quad .$$

Two examples of explicit formulas can be obtained by substituting $\lambda = 1$, $x = i/2$ into the second forms of [1: 10.9 (21), (22)], since $C_n^1(x) = U_n(x)$, and simplifying.

$$F_{2m+1} = (-1)^m \sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{m-k}(-5)^k \quad ;$$

$$F_{2m+2} = (-1)^m \sum_{k=0}^{m}\binom{m+k+1}{m-k}(-5)^k \quad .$$

IDENTITIES FOR THE DIFFERENCE EQUATION

In general, the solution to the linear difference equation can be written

$$(4) \qquad u_n = \{z_2^n(u_1 - z_1u_0) - z_1^n(u_1 - z_2u_0)\}/(z_2 - z_1)$$

provided $z_2 \neq z_1$ are the roots of the characteristic equation $z^2 - az - b = 0$.

(A suitable modification can be made for $z_2 = z_1$ by a passage to the limit — the formulas must be altered appropriately.) An interesting method of arriving at this is given by I. Niven and H. S. Zuckerman [4: pp 90 — 92].(This method can be extended to higher order difference equations and to non-homogeneous equations. Further, it has an analog for differential equations.) If the z's are expressed in terms of a and b and the resulting binomials are expanded, then an alternate form of considerable use is obtained,

$$(5)\qquad u_n = 2^{-n} u_0 \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} (a^2 + 4b)^k + 2^{-n}(2u_1 - au_0) \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} a^{n-1-2k} (a^2 + 4b)^k$$

Here we can define sequences from the sums in (5), for $n \geq 0$. Let $\varphi_0 = 0$, $\varphi_1 = 1$; $\lambda_0 = 2$, $\lambda_1 = a$ so that

$$(6)\qquad \varphi_n = 2^{-n+1} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} a^{n-1-2k} (a^2 + 4b)^k ,$$

$$(7)\qquad \lambda_n = 2^{-n+1} \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} (a^2 + 4b)^k ,$$

which correspond, respectively, to the Fibonacci and Lucas numbers. The general sequence, u_n, can then be written as a linear combination of these; i.e.,

$$(8)\qquad u_n = \frac{1}{2} u_0 \lambda_n + \frac{1}{2}(2u_1 - au_0)\varphi_n .$$

Since also from (4) we can write

$$\varphi_n = (z_2^n - z_1^n) / (z_2 - z_1) ,$$

$$\lambda_n = \{z_2^n (a - 2z_1) - z_1^n (a - 2z_2)\} / (z_2 - z_1) ,$$

and since $z_1 z_2 = -b$, a relation between the φ's and λ's can be obtained,

$$\lambda_n = a\varphi_n + 2b\varphi_{n-1} \quad . \tag{9}$$

This generalizes a known formula, $L_n = F_n + 2F_{n-1}$, relating the Lucas and Fibonacci numbers. The companion expression, $5F_n = L_n + 2L_{n-1}$, becomes

$$(a^2 + 4b)\varphi_n = a\lambda_n + 2b\lambda_{n-1} \quad .$$

These can then be used to express u_n in terms of either the φ's or the λ's;

$$u_n = u_1\varphi_n + bu_0\varphi_{n-1} \quad , \tag{10}$$

$$(a^2 + 4b)u_n = (2bu_0 + au_1)\lambda_n + b(2u_1 - au_0)\lambda_{n-1} \quad .$$

One point of interest is that the list of identities given by Horadam [2] for his generalized Fibonacci numbers, H_n, (u_0, u_1 arbitrary; $a = b = 1$) will yield an analogous list for the general case, with suitable modifications of his formula (1), and with the exception of his formula (16). This latter, "Pythagorean relation," is based upon the identity

$$H_{n+3}^2 - 4H_{n+1}H_{n+2} - H_n^2 = 0$$

for which the analog is

$$u_{n+3}^2 - a(3b + a^2)u_{n+1}u_{n+2} - b^3u_n^2 = (-b)^{n+1}e(b^2 - a) \quad ,$$

where

$$e = u_1^2 - au_1u_0 - bu_0^2 \quad . \tag{11}$$

Unless this extra term is zero; i.e., unless $b^2 = a$ or $u_0z = u_1$, the Pythagorean relation does not generalize. In the set of identities for the general equation the special case φ_n introduced in (6) plays the same role with respect to the u_n as do the Fibonacci numbers with respect to the H_n. For example, (10) provides an extension of Horadam's (7); i.e., if $a = b = 1$ so that $u_n = H_n$ and replacing n by $r + 1$, then (10) becomes

$$H_{r+1} = H_0F_r + H_1F_{r+1} .$$

Two further examples of how one can generalize Horadam's formulas follow. We consider his (8) and (12), several of the others being special cases of these.

$$\text{(8)} \qquad H_{n+r} = H_{n-1}F_r + H_nF_{r+1} ;$$

$$\text{(12)} \qquad H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n-s}eF_sF_{r+s+1} .$$

The general expressions are

$$\text{(12)} \qquad u_{n+r} = bu_{n-1}\varphi_r + u_n\varphi_{r+1} ,$$

$$\text{(13)} \qquad u_nu_{n+r+1} - u_{n-s}u_{n+r+s+1} = (-b)^{n-s}e\varphi_s\varphi_{r+s+1} ,$$

where e is defined by (11); φ_n, by (6).

Proof of (12). We can write, using (10)

$$u_{n+1} = a(u_1\varphi_n + bu_0\varphi_{n-1}) + b(u_1\varphi_{n-1} + bu_0\varphi_{n-2})$$

and then replace $b\varphi_{n-2}$ by $\varphi_n - a\varphi_{n-1}$ and $au_1 + bu_0$ by u_2 to obtain

$$u_{n+1} = u_2\varphi_n + bu_1\varphi_{n-1} .$$

Hence, by induction, the generalization is obtained. The substitution of $r + 1$ for n and $n - 1$ for r with $a = b = 1$ reduces this to the case for H_n's.

Proof of (13). If the appropriate expressions from (4) are substituted into the left side of this equation, and the result is simplified, the right side can then be obtained. Other formulas can sometimes be generalized in the same manner.

The analog to Horadam's (13),

$$b^2u_n^3 + au_{n+1}^3 - (a^2 + b)u_nu_{n+1}^2 = (-b)^ne(au_{n+1} - bu_n) ,$$

is more complicated. It reduces to

$$H_n^3 + H_{n+1}^3 = 2H_nH_{n+1}^2 + (-1)^n eH_{n-1} \quad .$$

We note here the misprint; H_{n-1} was omitted.

GENERALIZED CHEBYSHEV POLYNOMIALS

In (1) let a, b, u_0, u_1 represent polynomials in x. Then u_n becomes a polynomial in x and the various formulas (5) — (13) can be interpreted as formulas involving polynomials. From (8) we note that these polynomials $u_n(x)$ can be expressed in terms of "Fibonacci," $\varphi_n(x)$, and "Lucas," $\lambda_n(x)$, polynomials. The polynomials $\varphi_n(x)$ now play the same special role as the numbers φ_n; for example, formula (12) becomes

$$u_{n+1}(x) = b(x)u_{n-1}(x)\varphi_r(x) + u_n(x)\varphi_{r+1}(x) \quad .$$

The special case $a(x) = 2x$, $b(x) = -1$ leads to the set of polynomials, $H_n(x)$, corresponding to the numbers H_n. We then have analogously from (8),

$$H_n(x) = H_0(x)T_n(x) + (H_1(x) - xH_0(x))U_{n-1}(x) \quad ,$$

where $T_n(x) = \frac{1}{2}\lambda_n(x)$ and $U_{n-1}(x) = \varphi_n(x)$ are again the Chebyshev polynomials. Other identities can be written by inspection from Horadam's list for these "generalized Chebyshev" polynomials.

We note finally that a generating function can be obtained in the usual manner. One assumes a form $g(x,z) = \Sigma u_n(x)z^n$ and obtains a relation by using the difference equation. For the polynomials $u_n(x)$ this is

$$g_n(x,z) = \{u_0(x) + (u_1(x) - a(x)u_0(x))z\}\{1 - a(x)z - b(x)z^2\}^{-1} \quad .$$

Hence the special cases $\lambda_n(x)$ and $\varphi_n(x)$ can be generated from

$$g_\lambda(x,z) = \{2 - a(x)z\}\{1 - a(x)z - b(x)z^2\}^{-1} \quad ,$$

$$g_\varphi(x,z) = z\{1 - a(x)z - b(x)z^2\}^{-1} \quad .$$

REFERENCES

See page 19 for the references to this article.

HISTORY OF THE FIBONACCI QUARTERLY

Toward the close of the year 1962, a small group of mathematicians in Northern California promoted the idea of an organization devoted to the study of Fibonacci numbers and related topics. Almost simultaneously the Fibonacci Quarterly was conceived. Sponsored by a group of charter members who backed the project financially, the publication made its first appearance in the spring of 1963.

At first, subscriptions came in slowly, but with some advertising and favorable notices in various magazines, especially the Scientific American, the tempo increased and, amazingly, continued strong all during the summer. As a result, September 1st saw the Quarterly with six hundred subscribers. By the close of 1963, the total should approximate the one thousand mark.

To the editors, this response has been most heartening inasmuch as it seems to indicate that the mathematical public has looked with favor on this type of magazine and its subject matter—the famed Fibonacci numbers.

CHARACTERISTICS OF THE QUARTERLY

The Fibonacci Quarterly has a number of interesting features. The most obvious is the very specialized nature of its scope. The first natural reactions were: Will people subscribe to such a magazine of limited field? The response to this question is evident in the action of subscribers. Will it be possible to secure enough

(Continued on p. 14)

FIBONACCI NIM

MICHAEL J. WHINIHAN, MEDFORD SENIOR HIGH SCHOOL

The term Nim refers to any mathematical game in which two players remove objects from one or more piles. Fibonacci Nim was invented by Dr. R. E. Gaskell of Oregon State University, and is a variation of One Pile [1]. In One Pile, two players alternately remove at least a, but no more than q objects from a pile of n objects, the winner being the player who removes the last object, where n is a variable integer, and a and q are predetermined integral constants. The strategy is to leave your opponent a situation where $n \equiv 0$ modulo $(a + q)$. This is a "safe position." When $n \equiv i$ modulo $(a + q)$ where $i \not\equiv 0$, the position is "unsafe."

An unsafe [2] position is defined as one in which at least one winning move is possible. A safe position is one in which there are no winning moves possible and every move on this position must make the position unsafe.

In Fibonacci Nim, the determination of safe and unsafe positions is slightly more complex than in One Pile.

RULES OF THE GAME

The rules of Fibonacci Nim are the same as in One Pile with $a = 1$; but q, a constant in One Pile, is a variable in this game. On the first move, q_1 is equal to $n - 1$. After the first move, q_m is equal to twice the number of objects removed by the opponent on the $(m - 1)$th move. Let r_{m-1} be the number of objects removed by a player on the $(m - 1)$th move. Then: $q_m = 2r_{m-1}$. For example, if $n = 16$, on the first move player A may remove up to 15. If he removes 3, player B may remove as many as 6, since $q_2 = 2r_1 = 2 \cdot 3 = 6$. If player B removes 4, then A may remove as many as 8, and so on.

STRATEGY

As in all Nim games, the strategy of Fibonacci Nim calls for the determination of safe positions. The simplest way to determine whether any given situation is safe is to first represent the number of objects left in a Fibonacci number system.*

*See comment No. 1 at the end of this article.

To represent a given number n in the binary system, the binary sequence is used, where $b_n = b_{n-1} + b_{n-1}$, and b_1 is defined as 1. Let the Fibonacci sequence be defined in the following way: $f_n = f_{n-1} + f_{n-2}$, where f_{-1} is defined as 0 and f_0 as 1. f_1 through f_6 are then determined as 1, 2, 3, 5, 8 and 13.

It is generally known that by using either a 1 or a 0 in the nth digit from the left of the decimal point to represent the presence or absence of b_n, any number may be represented. Similarly, using f_1, f_2, etc. in place of b_1, b_2, etc., any number may be represented in a Fibonacci number system, if one remembers to start marking the largest digits first. Thus, 8_{ten} is always represented as 10000_f and never as 1100_f or 1011_f. Notice that using this rule not only makes the representation of any given number unique, it also makes it impossible for two 1's to appear in a number without at least one 0 separating them.*

In the representation of any number $n > 0$ in the Fibonacci number system, there must be at least one 1. Let the 1 that is farthest to the right on the mth move be F_m. If $n = 19_{ten} = 101001_f$, $F = f_1 = 1_{ten}$. If $n = 18_{ten} = 101000_f$, $F = f_4 = 5$. If, on the mth move, $q_m < F_m$, the situation is safe. If $q_m \geq F_m$, the situation is unsafe, and the winning move is to remove exactly F_m objects. For example, if on the first move $n = 10_{ten} = 10010_f$, $q_1 = 9$ and $F_1 = 2$. Since $q_1 > F_1$, the situation is unsafe and the winning move is to remove exactly 2 objects. If player A removes 2 objects, then for player B, $n = 8_{ten} = 10000_f$, $q_2 = 2r_1 = 4$, and $F_2 = 8$. Since $q_2 < F_2$, the situation is safe, and player B will lose unless player A makes a mistake.

PROOF

To prove the strategy correct, it must be proven that unsafe positions can always be made safe and that safe positions can only be made unsafe.

FIRST RESULT

Any unsafe position can be made safe.

By definition, on the mth move, F_m can be removed from an unsafe position. If $F_m = n$, then by removing F_m objects the game is automatically won. If $n > F_m$ then, from the definition of F_m, there is another 1, which is the second 1 from the right. Let the Fibonacci number that this 1 repre-

*See comment No. 2 at the end of this article.

sents be f_k. Let the Fibonacci number that F_m represents be f_i. It has already been shown that between any two 1's in a Fibonacci representation of a number, that there must be at least one 0. It follows that there is at least one Fibonacci number greater than f_i, but less than f_k. Let f_j be the next Fibonacci number after f_i. It may or may not be the immediate predecessor of f_k.

$$f_i < f_j$$

$$2f_i < f_j + f_i$$

$$2f_i < f_k$$

$$q_{m+1} = 2F_m = 2f_i$$

$$q_{m+1} > f_k$$

But $f_k = F_{m+1}$, after F_m has been removed.

$$q_{m+1} < F_{m+1}$$

Thus, by removing F_m objects from an unsafe position on the mth move, the position will be safe on the $(m + 1)$th move.

SECOND RESULT

Any move from a safe position must make it unsafe.

Since any move on a safe position on the mth move can never take as many as F_m objects, it follows that $F_{m+1} < F_m$. Let n on the mth move equal $c + F_m = c + f_i$. Let n on the $(m + 1)$th move equal $c + c_1 + F_{m+1} = c + c_1 + f_h$. Suppose $c_1 + f_h$ can be written in the form $f_{i-1} + f_{i-3} + f_{i-5} \cdots f_{h+2} + f_h$. If f_i is written $1000000 \cdots$, i.e., a 1 followed by $i - 1$ 0's, then $c_1 + f_h$ is written $101010 \cdots 101$ followed by enough 0's to make $i - 1$ digits. The last 1, by definition, represents f_h. Let $f_f + f_g$ be the two immediate predecessors of f_h. If f_g is added to $c_1 + f_h$, it is found that:

$$\begin{array}{r} 101010 \cdots\ 101000 \cdots \\ +\ 100 \cdots \\ \hline 1000000 \cdots\ 000000 \cdots \end{array}$$

In other words, $c_1 + f_h + f_g = f_i$.* If $c_1 + f_h$ is less than $f_{i-1} + f_{i-3} + f_{i-5} \cdots + f_{h+2} + f_h$, it follows that $c_1 + f_h + f_g < f_i$. Therefore:

*See comment No. 3 at the end of this article.

$$r_m = f_i - (c_1 + f_h) \geq f_g \tag{1}$$

This means that any move that leaves f_h as F_{m+1} must remove at least f_g objects.

$$f_g \geq f_f$$

$$2f_g \geq f_f + f_g$$

$$2f_g \geq f_h \tag{2}$$

$$q_{m+1} = 2r_m$$

$$q_{m+1} \geq 2f_g \quad \text{(by equation (1))}$$

$$q_{m+1} \geq f_h \quad \text{(by equation (2))}$$

But

$$f_h = F_{m+1} .$$

$q_{m+1} \geq F_{m+1}$, and the position is unsafe. Thus any move on a safe position makes it unsafe.

GENERALIZED FIBONACCI NIM

Suppose $q_m = r_{m-1}$. Then the binary system will determine F_m (or more correctly, B_m). Safe and unsafe positions will be determined in exactly the same way, and the proof parallels the one given above. If the binary sequence is called a Fibonacci sequence of order 1, and the ordinary Fibonacci sequence is called a Fibonacci sequence of order 2, is there a formula for finding a Fibonacci sequence of order n that will satisfy a Fibonacci Nim game where $q_m = n \cdot r_{m-1}$? Dr. Gaskell and the author have worked on this problem independently and have found two different methods of determining an order n Fibonacci sequence. All of the sequences investigated so far take the form of $f_i = f_{i-1} + f_{i-p}$, but as of yet no relationship has been found between the order of the sequence and p.

REFERENCES

1. Richard L. Frey (ed.), The New Complete Hoyle (Philadelphia, David McKay Company, 1947) pp. 705 — 706.
2. Charles L. Bouton, "Nim, A Game with a Complete Mathematical Theory," Annals of Mathematics, 1901 — 1902, pp. 35 — 39.

Comments

1. While the proof makes use only of the fact that every Fibonacci number is at least as large as its immediate predecessor and of the recurrence property, $f_n = f_{n-1} + f_{n-2}$, it should be noted that Lucas numbers cannot be substituted for Fibonacci numbers, because the number 2 cannot be represented in a Lucas number system using only 1's and 0's. One might define L_1 as 2 and L_2 as 1 in order to make a Lucas number system, but this would invalidate the required property that every member of the sequence is at least as large as its predecessor.

2. (Editorial Comment) The uniqueness follows from Zeckendorf's Theorem. If the Fibonacci numbers $u_1, u_2, \cdots$ are defined by $u_1 = 1$, $u_2 = 2$, $u_n = u_{n-1} + u_{n-2}$, $n \geq 3$.

Theorem. For each natural number N there is one and only one system of natural numbers $i_1, i_2, \cdots i_d$ such that

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} \text{ and } i_{\nu+1} \geq i_\nu + 2 \text{ for } 1 \leq \nu < d.$$

3. This is an example of how the Fibonacci number system can be used to prove theorems about Fibonacci numbers. The example shown is a generalized form of the theorems concerning the sum of odd or even Fibonacci numbers. Another simple example is to find the sum of the Fibonacci numbers through f_n. One simply represents all the Fibonacci numbers through an arbitrary n, 5 for example, in the Fibonacci number system: 11111_f. $11111_f = 10101_f + 1010_f$. Since $10101_f = 100000_f - 1_{ten}$ and $1010_f = 10000_f - 1_{ten}$, $11111_f = 110000_f - 2$, or $1000000_f - 2$. In other words, the sum of the Fibonacci numbers through $f_n = f_{n+2} - 2$.

REQUEST

The Fibonacci Bibliographical Research Center desires that any reader finding a Fibonacci reference send a card giving the reference and a brief description of the contents. Please forward all such information to:

Fibonacci Bibliographical Research Center
Mathematics Department,
San Jose State College,
San Jose, California

material? Won't we eventually run out of ideas or become involved in meaningless repetition? So far in the first year, some three hundred and twenty pages of mathematics dealing mainly with Fibonacci numbers has been produced without undue strain. All the evidence points to the fact that there are still many potential writers and untold possibilities of development.

However, to obviate the danger of narrowness, the Fibonacci Quarterly is described officially as "A Journal Devoted to the Study of Integers with Special Properties." While we have not received much material of a non-Fibonacci nature to date, authors are invited to prepare articles along this line and submit them to the editor for publication.

A second characteristic of the Quarterly is that it aims to cater to a wide range of people. The first half of each issue (approximately 50 pages) is devoted to mathematical developments and favors the professional mathematician or those more expert in the field. The second half (approximately 30 pages) is more expository and at a lower level of difficulty. The aim is to provide material that can be read by those who have an acquaintance with mathematics and are interested in it, but who do not claim to be professionals. We suspect, however, that some of the more advanced readers will find this section of value and a welcome relief from the strenuous activity of part one.

(Continued on page 20.)

LEONARDO FIBONACCI

CHARLES KING, SAN JOSE STATE COLLEGE, SAN JOSE, CALIFORNIA

The Fibonacci Quarterly receives its name from Leonardo of Pisa (or Leonardo Pisano), better known as Leonardo Fibonacci (Fibonacci is a contraction of Filius Bonacci, son of Bonacci). Leonardo was born about 1175 in the commercial center of Pisa. This was a time of great interest and importance in the history of Western Civilization. One finds the influence of the crusades stirring and awakening the people of Europe by bringing them in contact with the more advanced intellect of the East. During this time the Universities of Naples, Padua, Paris, Oxford, and Cambridge were established, the Magna Carta signed in England, and the long struggle between the Papacy and the Empire was culminated. Commerce was flourishing in the Mediterranean world and adventurous travelers such as Marco Polo were penetrating far beyond the borders of the known world.

It is in this growing commercial activity that we find the young Leonardo at Bugia on the Northern coast of Africa. Here the merchants of Pisa and other commercial cities of Italy had large warehouses for the storage of their goods. Actually very little is known about the life of this great mathematician. No contemporary historian makes mention of him, and one must look to his writings to find information about him. In the preface of his first and most important work, Liber Abbaci (I), Leonardo tells us that his father, the head of one of the warehouses of Bugia, instructed him to study arithmetic. In Bugia, he received his early education from a Moorish schoolmaster.

Leonardo then traveled about the Mediterranean visiting Egypt, Syria, Greece, Sicily, southern France, and Constantinople. He met with scholars and studied the various systems of arithmetic then in use. Leonardo was persuaded that the Hindu-Arabic system was superior to the methods then adopted in the different countries he had visited and that it was even superior to the Algorithma and the method of Pythagorus. He busied himself with the subject and carried on his own research, intent upon bringing the Hindu-Arabic system to his Italian countrymen. The study and research in mathematics so absorbed him that he seems to have devoted his life to this pursuit and spent little time in commerce which was flourishing at that time and was the favorite occupation of his fellow citizens. Yet most of the applications Leonardo makes in his works are in the field of commerce. In one place, he gives a careful evaluation of the money systems of the countries of his travels.

Leonardo returned to Italy about 1200 and in 1202 wrote Liber Abbaci (I), in which he gave a thorough treatment of arithmetic and algebra, the first that had been written by a Christian. The work is divided into 15 chapters. The chapter contents are given here to indicate the scope of the work: (1) Reading and writing numbers in the Hindu-Arabic system; (2) Multiplication of integers; (3) Addition of integers; (4) Subtraction of integers; (5) Division of integers; (6) Multiplication of integers by fractions; (7) Additional work with fractions; (8) Prices of goods; (9) Barter; (10) Partnership; (11) Alligation; (12) Solutions of problems; (13) Rule of false position; (14) Square and cube roots; (15) Proportions, and Geometry and algebra.

The last and most important chapter is divided into three parts; the first relates to proportions, the second to geometry and the third, to algebra. Each of the three parts begins with definitions and demonstrations credited to the Arabs, then Leonardo considers six questions, three simple and three complex, giving solutions for them.

Leonardo, in 1228, gave a second edition of the Liber Abbaci which he dedicated to Michel Scott, astrologer to the Emperor Frederic II and author of many scientific works. Copies of this edition exist today. Leonardo profusely illustrated and strongly advocated the Hindu-Arabic system in this work. He gave an extensive discussion of the Rule of False Position and the Rule of Three. Leonardo did not use a general method in problem solving; each problem was solved independently of the others. In the solution of a problem he not only considered the problem as it might occur, but considered all of the variations of the question, even those that were not reasonable.

In the Liber Abbaci, Leonardo states and gives the solution to the famous Rabbit Problem [1, Vol. 1, p. 285]. A pair of rabbits are placed in a pen to find out how many offspring will be produced by this pair in one year if each pair of rabbits gives birth to a new pair of rabbits each month starting with the second month of its life; it is assured that deaths do not occur.

Leonardo traces the progress of the rabbits: The first pair has offspring in the first month: thus two pair. The second month there are three pair, the first reproducing in this month. In the third month there are five pair. Continuing in this manner through the twelve months. Leonardo gives the following table:

0
Pairs
1
First Month
2
Second Month
3
Third Month
5
Fourth Month
8
Fifth Month
13
Sixth Month
21
Seventh Month
34
Eighth Month
55
Ninth Month
89
Tenth Month
144
Eleventh Month
233
Twelfth Month
377

It is this sequence of numbers, $1, 2, 3, 5, 8, 13, \cdots,$ that gives rise to the Fibonacci Sequence.

Of the many problems of elementary nature in the *Liber Abbaci*, the following are given as examples.

Seven old women are traveling to Rome and each has seven mules. On each mule there are seven sacks; in each sack there are seven loaves of bread: in each loaf there are seven knives; and each knife has seven sheaths. How many in all are going to Rome?

A man went into an orchard which had seven gates; and there took a certain number of apples. When he left the orchard he gave the first guard half the apples he had and one apple more. To the second he gave half the remaining apples and one apple more. He did the same in the case of each of the remaining five guards, and left the orchard with one apple. How many apples did he gather in the orchard?

A certain man puts one denarius at such a rate that in five years he has two denarii and in every five years thereafter the money doubles. How many denarii would he gain from this one denarius in 100 years?

A certain king sent thirty men into his orchard to plant trees. If they could set out a thousand trees in nine days, in how many days would thirty-six men set out four thousand four hundred trees?

Many readers will recognize these problems.

In 1220, Leonardo wrote *Practica Geometriae*, which he dedicated to Master Dominique, a person of whom there is no record. In this work Leonardo systematized the subject matter of practical geometry with a specialization in measurements of bodies. He included some algebra and trigonometry, square and cube roots, proportions and indeterminate problems. The use of a surveying instrument called the quadrans is included. The work is skillfully done with Euclidean rigor and some originality.

Leonardo's reputation grew and from his writings it can be seen that he had a vast range of knowledge concerning Arabian mathematics and mathematics of antiquity, especially Greek. His treatment shows much originality, completeness and rigor. It is especially noted that his writings did not contain the mysticism of numerology and astrology that were so prevalent in the writing of his day.

Because of Leonardo's great reputation, the Emperor Frederick II, when in Pisa (1225), held a sort of mathematical tournament to test Leonardo's skill. The competitors were informed beforehand of the questions to be asked, some or all of which were composed by Johannes of Palermo [1, Vol. II, p. 227], who was one of Frederick's staff. This is the first case in the history of mathematics that one meets with an instance of these challenges to solve particular problems which were so common in the sixteenth and seventeenth centuries.

The first question propounded was to find a number of which the square when decreased or increased by 5 would remain a square (II)( ). The correct answer given by Leonardo was 41/12. The next question was to find by the methods used in the tenth book of Euclid a line whose length X should satisfy the equation $x^3 + 2x^2 + 10x - 20 = 0$. Leonardo showed by geometry that the problem was impossible, but gave an approximation of the root 1.368808 1075 $\cdots$, which is correct to nine places.

The third question was:

Three men possess a certain sum of money, their shares in the ratio 3:2:1. While making the division, they were surprised by a thief and each took what he could and fled. Later the first man gave up half of what he had, the second gave up one-third, and the third, one-sixth. The money given up was divided equally among them and then each man had the share to which he was entitled. What was the total sum? Leonardo showed that the problem was indeterminate and gave as one solution 47 which is the smallest sum.

The other competitors failed to solve any of these questions. Through the consideration of these problems and others similar to them, Leonardo was led to write his Liber Quadratorum (1225) [No. 1, Vol. II, p. 253] a brilliant and original work containing a well arranged collection of theorems from indeterminate analysis involving equations of the second degree such as $x^2 + 5 = y^2$, $x^2 - 5 = z^2$. This work has marked him as the outstanding mathematician between Diophantus and Fermat in this field.

Two or three works of Leonardo that are known are the Flos [1, Vol. II, p. 227] (blossom or flower), which contains the last two problems of the tournament; the first problem is found in the Liber Quadratorum, and a Letter to Magister Theodoris [1, Vol. II, p. 247], philosopher to Frederick II, relating to indeterminate analysis and to geometry. The last three works show clearly the genius and brilliance of Leonardo as a mathematician and were beyond the abilities of most contemporary scholars.

The works of Leonardo Fibonacci are available in some universities in the United States through B. Boncompagni, Scritte di Leonardo Pisano, Rome, (1857-1862) [1]. The first volume contains the Liber Abbaci and the second volume contains Patricia Geometriae, the Flos, Letter to Magestrum Theodorum, and Liber Quadratorum. A treatment of square numbers composed by Leonardo and addressed to the Emperor Frederick II seems to have been lost.

REFERENCE

1. Boncompagni, Baldassarre, Scritti di Leonardo Pisano; Roma, 1857; 2 vols.

REFERENCES FROM PAGE 7

1. A. Erdélyi, et al., "Higher Transcendental Functions," vol. 2, McGraw-Hill, New York, 1953.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly 68 (1961), pp. 455–459.
3. E. Lucas, "Théorie Des Fonctions Numérique Simplement Périodiques," Amer. J. Math. 1 (1878), pp. 184–240.
4. I. Nivin and H. S. Zuckerman, An Introduction to the Theory of Numbers, Wiley, New York, 1960.

CHARACTERISTICS OF THE QUARTERLY (Cont)

A third feature is the open-ended nature of the publication. Readers are definitely invited to participate in mathematicizing — solving problems, working up articles, continuing discussions that have been left unfinished, etc. Writers do not have to wait endlessly before their articles appear in print. An idea can be propounded in one issue and a development resulting from it may appear in the next. This lends an element of interest and continuity that is badly lacking in most other mathematical publications.

The unified nature of the subject matter means that after a while a steady reader will become conversant with what is being discussed in the magazine. He will be able to understand and appreciate more and more of each succeeding issue and may eventually find himself a publisher.

THE FIBONACCI ASSOCIATION

So much effort has been put into launching the Fibonacci Quarterly that the Fibonacci Association which sponsors this publication has momentarily taken a secondary place. But plans are afoot for a regular type of organization and activity. Already, two Fibonacci conferences have been held at San Jose State College and it is expected that these will become a regular feature of the organization.

The overall picture at the present time is somewhat as follows. Members of the Fibonacci Association would, of course, receive the Fibonacci Quarterly. In addition, they would be

(Continued on page 40)

ON THE PERIODICITY OF THE LAST DIGITS OF THE FIBONACCI NUMBERS

DOV JARDEN, JERUSALEM, ISRAEL

In the FIBONACCI QUARTERLY volume 1, number 2, page 84, Stephen P. Geller announced some empirical data on the periodicity of the last digits of the Fibonacci numbers 1, 1, 2, 3, 5, $\cdots$. Using a table of the first 571 Fibonacci numbers, published by S. L. Basin and V. E. Hoggatt, Jr. in RECREATIONAL MATHEMATICS MAGAZINE issue number 11, October 1962, pp. 19–30, he brought out the fact that the last (units) digit of the sequence is periodic with period 60, and that the last two digits are similarly periodic with period 300. Setting up an IBM 1620 he further found that the last three digits repeat every 1,500 times, the last four every 15,000, the last five every 150,000, and finally after the computer ran for nearly three hours a repetition of the last six digits appeared at the 1,500,000th Fibonacci number. Mr. Geller comments: "There does not yet seem to be any way of guessing the next period, but perhaps a new program for the machine which will permit initialization at any point in the sequence for a test will cut down computer time enough so that more data can be gathered for conjecturing some rule for these repetition periods."

I would like to purse half the money necessary to run a computer that will supply the next periods I know. However, since I know the exact period of any number of last digits, the money of the whole world will not suffice. The next period is 15,000,000. Generally the following theorem holds:

Theorem 1. The last $d \geq 3$ digits of the Fibonacci numbers repeat every $15 \cdot 10^{d-1}$ times.

The proof is based on the following theorems from the theory of Fibonacci numbers.

Notation. $A(n)$ – the period of the Fibonacci sequence relative to n.

$a(n)$ – the least positive subscript of the Fibonacci numbers divisible by n (known as "rank of apparition" of n).

$\{a, b, \cdots\}$ – the least common multiple of a, b, $\cdots$.

Theorem 2. $A(n)$ exists for each whole positive n.

Theorem 3. If $n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ is the canonical decomposition of n into different prime-powers ($p_1, p_2, \cdots, p_k$ being different primes and $d_1, d_2, \cdots, d_k$ being positive integers), then

$$A(n) = \left\{ A\left(p_1^{d_1}\right), A\left(p_2^{d_2}\right), \cdots, A\left(p_k^{d_k}\right) \right\} .$$

Theorem 4. For any odd prime p and whole positive d,

$$A(p^d) = a(p^d), \quad 2a(p^d), \quad \text{or} \quad 4a(p^d)$$

according as

$$a(p^d) \equiv 2, \quad 0, \quad \text{or} \quad \pm 1 \pmod 4 .$$

For $d \geq 3$,

$$A(2^d) = 2a(2^d) .$$

Theorem 5. For $d \geq 3$, $a(2^d) = 3 \cdot 2^{d-2}$.

For any whole positive d, $a(5^d) = 5^d$.

Proof of Theorem 1. Obviously Geller's problem is equivalent with the one of determining the period of the Fibonacci sequence relative to 10^d for any whole positive $d \geq 3$. Now, by the above theorems,

$$\begin{aligned} A(10^d) = A(2^d 5^d) &= \{A(2^d), A(5^d)\} \\ &= \{2a(2^d), 4a(5^d)\} \\ &= \{2 \cdot 3 \cdot 2^{d-2}, 4 \cdot 5^d\} \\ &= 4\{3 \cdot 2^{d-3}, 5^d\} \\ &= 4 \cdot 3 \cdot 2^{d-3} \cdot 5^d \\ &= 15 \cdot 10^{d-1} . \end{aligned}$$

NOTICE TO ALL SUBSCRIBERS!!!

Please notify the Managing Editor AT ONCE of any address change. The Post Office Department, rather than forwarding magazines mailed third class, sends them directly to the dead-letter office. Unless the addressee specifically requests the Fibonacci Quarterly be forwarded at first class rates to the new address, he will not receive it. (This will usually cost about 30 cents for first-class postage.) If possible, please notify us AT LEAST THREE WEEKS PRIOR to publication dates: February 15, April 15, October 15, and December 15.

SOME REMARKS ON CARLITZ' FIBONACCI ARRAY

CHARLES R. WALL, TEXAS CHRISTIAN UNIVERSITY, FT. WORTH, TEXAS

Recently in this journal [Vol. 1, No. 2, pp. 17—27] Carlitz defined a Fibonacci array. Among the properties not included in his discussion are the following summation formulas: (Recall $u_{0,n} = F_n$; $u_{1,n} = F_{n+2}$; $u_{r,n} = u_{r-1,n} + u_{r-2,n}$)

$$\text{(I)} \qquad \sum_{n=0}^{r} u_{r-n,n} = \frac{2}{5}\left[(r+1)L_{r+1} - F_{r+1}\right] ,$$

$$\text{(II)} \qquad \sum_{n=0}^{r} (-1)^n u_{r-n,n} = 0 ,$$

$$\text{(III)} \qquad \sum_{n=0}^{r} \binom{r}{n} u_{r-n,n} = \frac{1}{5}\left[2^{r+1}L_{r+1} - 2\right] ,$$

$$\text{(IV)} \qquad \sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} u_{r-n,n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in J^+ \end{cases} .$$

The similarities between the formulas above and the four below should be noted:

$$\sum_{n=0}^{r} F_n F_{r-n} = \frac{1}{5}\left[rL_r - F_r\right] ,$$

$$\sum_{n=0}^{r} (-1)^{n+1} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd} \\ F_r & \text{if } r \text{ even} \end{cases} ,$$

$$\sum_{n=0}^{r} \binom{r}{n} F_n F_{r-n} = \frac{1}{5}\left[2^r L_r - 2\right] ,$$

$$\sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in J^+ \end{cases} .$$

Because of an overabundance of properties in Carlitz' discussion, we may generalize his array in two ways, taking $H_1 = p$, $H_2 = p + q$,

$$H_{n+1} = H_n + H_{n-1} .$$

We make no attempt to generalize all his results, but consider only the simpler ones. Arabic numerals referring to formulas correspond to those in Carlitz' article.

I. FIRST GENERALIZATION

We define

$$G_{0,n} = H_n , \tag{1'}$$

$$G_{1,n} = H_{n+2} , \tag{2'}$$

as the first two rows of the generalized array G. For $r > 1$ we define $G_{r,n}$ by means of

$$G_{r,n} = G_{r-1,n} + G_{r-2,n} . \tag{3'}$$

It follows that

$$G_{r,n} = pu_{r,n} + qu_{r,n-1}$$

and

$$G_{r,n} = G_{r,n-1} + G_{r,n-2} . \tag{4'}$$

From these properties Table I is easily computed.

Table I
ARRAY G

r \ n	0	1	2	3	4
0	q	p	p + q	2p + q	3p + 2q
1	p + q	2p + q	3p + 2q	5p + 3q	8p + 5q
2	p + 2q	3p + q	4p + 3q	7p + 4q	11p + 7q
3	2p + 3q	5p + 2q	7p + 5q	12p + 7q	19p + 12q
4	3p + 5q	8p + 3q	11p + 8q	19p + 11q	30p + 19q

The symmetry property (5) obviously fails since $G_{0,1} \neq G_{1,0}$. If we put

$$g_r(x) = \sum_{n=0}^{\infty} G_{r,n} x^n \tag{6'}$$

we find that

$$g_0(x) = \frac{q + px - qx}{1 - x - x^2} , \quad g_1(x) = \frac{p + q + px}{1 - x - x^2} . \tag{7'}$$

We also have

$$g_r(x) = g_{r-1}(x) + g_{r-2}(x) , \tag{8'}$$

so that

$$g_r(x) = \frac{H_r + xH_{r+1} + q(F_r - xF_{r+1})}{1 - x - x^2} . \tag{9'}$$

Putting

$$g(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} G_{r,n} x^r y^n$$

we have

$$g(x,y) = \sum_{r=0}^{\infty} \frac{H_r + yH_{r+1} + q(F_r - yF_{r+1})}{1 - y - y^2} x^r$$

so that

$$g(x,y) = \frac{px + py + q - qy + qxy}{(1 - x - x^2)(1 - y - y^2)} . \tag{11'}$$

It appears that

$$\begin{cases} G_{r+1,r-1} - G_{r,r} = (-1)^r (p - q) \\ G_{r-1,r+1} - G_{r,r} = (-1)^r p \end{cases} . \tag{13'}$$

Indeed, following Carlitz' procedure we find that

$$(14') \qquad \begin{cases} G_{r+2,r-2} - G_{r,r} = (-1)^{r+1}(p - 2q) \\ G_{r-2,r+2} - G_{r,r} = (-1)^{r+1}(p + q) \end{cases},$$

$$(15') \qquad \begin{cases} G_{r+3,r-3} - G_{r,r} = (-1)^{r}(4p - 6q) \\ G_{r-3,r+3} - G_{r,r} = (-1)^{r}(4p + 2q) \end{cases},$$

and, in general,

$$(16') \qquad \begin{cases} G_{r+s,r-s} - G_{r,r} = (-1)^{r+s+1}F_s(F_s p - F_{s+1} q) \\ G_{r-s,r+s} - G_{r,r} = (-1)^{r+s+1}F_s H_s \end{cases}.$$

From (16') we note that

$$(5') \qquad G_{r,n} = G_{n,r} + (-1)^n F_{r-n}\, q \quad .$$

We also note that

$$(17') \qquad \sum_{r=0}^{n-1} G_{r,r} = \begin{cases} 2 \cdot F_n H_n & \text{if } n \text{ even} \\ 2 \cdot F_{n+1} H_{n-1} - q & \text{if } n \text{ odd} \end{cases}.$$

Among the elementary properties that do not generalize are (10) and (12); however, the latter failure is the basis for the second generalization. The summation formulas in the introduction generalize as

$$(\mathrm{I}') \quad \sum_{n=0}^{r} G_{r-n,n} = \frac{2}{5}\,[(r + 1)L_{r+1} - F_{r+1}]\,p + \frac{1}{5}\,[2(r + 1)L_r + F_{r+1}]\,q \quad ,$$

$$(\mathrm{II}') \qquad \sum_{n=0}^{r} (-1)^n G_{r-n,n} = qF_r \quad ,$$

$$(\mathrm{III}') \quad \sum_{n=0}^{r} \binom{r}{n} G_{r-n,n} = \frac{1}{5}\,[2^{r+1}L_{r+1} - 2]\,p + \frac{1}{5}\,[2^{r+1}L_r + 3]\,q \quad ,$$

$$\text{(IV')} \qquad \sum_{n=0}^{r} (-1)^n \binom{r}{n} G_{r-n,n} = \begin{cases} q \text{ if } r = 0 \\ 5^{(r-1)/2} \cdot q \text{ if } r \text{ odd} \\ (-2p + q)5^{(r-2)/2} \text{ if } \frac{r}{2} \in J^+ \end{cases} .$$

II. SECOND GENERALIZATION

We define

$$\text{(12'')} \qquad H_{r,n} = H_r H_n + H_{r+n} .$$

It immediately follows that

$$\text{(1'')} \qquad H_{0,n} = H_n (q + 1) ,$$

$$\text{(2'')} \qquad H_{1,n} = pH_n + H_{n+1} ,$$

$$\text{(3'')} \qquad H_{r,n} = H_{r-1,n} + H_{r-2,n} ,$$

$$\text{(4'')} \qquad H_{r,n} = H_{r,n-1} + H_{r,n-2} ,$$

$$\text{(5'')} \qquad H_{r,n} = H_{n,r} .$$

See Table II for array H. We also note that

$$H_{r,n} = p^2 F_r F_n + q^2 F_{r-1} F_{n-1} + pq(F_r F_{n-1} + F_{r-1} F_n)$$
$$+ pF_{r+n} + qF_{r+n-1} .$$

We put

$$\text{(6'')} \qquad h_r(x) = \sum_{n=0}^{\infty} H_{r,n} x^n$$

and see that

$$\text{(7'')} \qquad h_0(x) = \frac{H_{0,0} + xH_{-1,0}}{1 - x - x^2} , \quad h_1(x) = \frac{H_{0,1} + xH_{-1,1}}{1 - x - x^2} .$$

Table II
Array H

$r \backslash n$	0	1	2	3	4
0	$q^2 + q$	$pq + p$	$q^2 + pq + p + q$	$q^2 + 2pq + 2p + q$	$2q^2 + 3pq + 3p + 2q$
1	$pq + p$	$p^2 + p + q$	$p^2 + pq + 2p + q$	$2p^2 + pq + 3p + 2q$	$3p^2 + 2pq + 5p + 3q$
2	$q^2 + pq + p + q$	$p^2 + pq + 2p + q$	$p^2 + q^2 + 2pq + 3p + 2q$	$2p^2 + q^2 + 3pq + 5p + 3q$	$3p^2 + 2q^2 + 5pq + 8p + 5q$
3	$q^2 + 2pq + 2p + q$	$2p^2 + pq + 3p + 2q$	$2p^2 + q^2 + 3pq + 5p + 3q$	$4p^2 + q^2 + 4pq + 8p + 5q$	$6p^2 + 2q^2 + 7pq + 13p + 8q$
4	$2q^2 + 3pq + 3p + 2q$	$3p^2 + 2pq + 5p + 3q$	$3p^2 + 2q^2 + 5pq + 8p + 5q$	$6p^2 + 2q^2 + 7pq + 13p + 8q$	$9p^2 + 4q^2 + 12pq + 21p + 13q$

But by (3'') we have

$$h_r(x) = h_{r-1}(x) + h_{r-2}(x) , \tag{8''}$$

so that

$$h_r(x) = \frac{H_{0,r} + xH_{-1,r}}{1 - x - x^2} . \tag{9''}$$

Putting

$$h(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} H_{r,n} x^r y^n$$

we have

$$h(x,y) = \sum_{r=0}^{\infty} \frac{H_{0,r} + yH_{-1,r}}{1 - y - y^2} x^r$$

(11'')

$$= \frac{q(1+q) + (p-q)(1+q)(x+y) + xy(p^2 - p + 2q + 2pq + q^2)}{(1 - x - x^2)(1 - y - y^2)} .$$

From (12'') we have

$$H_{r+s,r-s} - H_{r,r} = H_{r+s}H_{r-s} - H_r^2 ,$$

so that

(13'') $$H_{r+1,r-1} - H_{r,r} = (-1)^r e ,$$

(14'') $$H_{r+2,r-2} - H_{r,r} = (-1)^{r+1} e ,$$

(15'') $$H_{r+3,r-3} - H_{r,r} = (-1)^r e 4 ,$$

(16'') $$H_{r+s,r-s} - H_{r,r} = (-1)^{r+s+1} e F_s^2 ,$$

where $e = p^2 - pq - q^2$.

The summation formulas previously referred to generalize as

(I'') $$\sum_{n=0}^{r} H_{r-n,n} = (r+1)H_r + qH_r - \frac{e}{5} F_r + \frac{r}{5} [(H_{r+1} + H_{r-1})p + (H_r + H_{r-2})q] ,$$

(II'') $$\sum_{n=0}^{r} (-1)^n H_{r-n,n} = \begin{cases} 0 \text{ if } r \text{ odd} \\ q(F_{r-1} + qF_{r+1} + 2pF_r) \\ + (p - p^2)F_r \text{ if } r \text{ even} \end{cases} ,$$

(III'') $$\sum_{n=0}^{r} \binom{r}{n} H_{r-n,n} = 2^r H_r + \frac{1}{5} [2^r p(H_{r+1} + H_{r-1}) + 2^r q(H_r + H_{r-2}) - 2e] ,$$

(IV'') $$\sum_{n=0}^{r} (-1)^n \binom{r}{n} H_{r-n,n} = \begin{cases} 0 \text{ if } r \text{ odd} \\ q + q^2 \text{ if } r = 0 \\ -2e\, 5^{(r-2)/2} \text{ if } r/2 \in J^+ \end{cases} .$$

RECIPROCALS OF GENERALIZED FIBONACCI NUMBERS

DMITRI THORO, SAN JOSE STATE COLLEGE

One of the oldest procedures for the numerical solution of $f(x) = 0$ is the classical _regula falsi_ method. This "rule of false position" is given by the iteration

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} ,$$

where x_1 and x_2 are the initial estimates. (It may be noted that the _regula falsi_ method is simply inverse linear interpolation.)

For the innocuous equation $x^2 = 0$, this iteration reduces to

$$x_{n+1} = \frac{x_{n-1} x_n}{x_{n-1} + x_n} .$$

If we define the generalized Fibonacci numbers by

$$F_1 = a, \; F_2 = b, \; F_3 = a + b, \; F_4 = a + 2b, \cdots$$
$$F_{n+2} = F_{n+1} + F_n, \cdots$$

it immediately follows that with starting values $x_1 = 1/a$, $x_2 = 1/b$, this application of _regula falsi_ yields the reciprocals of the generalized Fibonacci numbers since

$$\frac{\dfrac{1}{F_{i+1}} \cdot \dfrac{1}{F_{i+2}}}{\dfrac{1}{F_{i+1}} + \dfrac{1}{F_{i+2}}} = \frac{1}{F_{i+1} + F_{i+2}} = \frac{1}{F_{i+3}} .$$

FIBONACCI EXPONENTIALS AND GENERALIZATIONS OF HERMITE POLYNOMIALS

H. W. GOULD WEST VIRGINIA UNIVERSITY MORGANTOWN, W. VA.

Little seems to be known about series of the form

$$\sum_{n=0}^{\infty} A_n x^{F_n} \tag{1}$$

or

$$\sum_{n=0}^{\infty} A_n x^{L_n} \quad , \tag{2}$$

where the exponents are Fibonacci and Lucas numbers, respectively, defined by

$$F_n = \frac{a^n - b^n}{a - b} , \quad L_n = a^n + b^n , \quad a \neq b \quad . \tag{3}$$

It may therefore be of interest to point out that Fibonacci exponentials are intimately related to some generalizations of Hermite polynomials [1]. The existence (or non-existence) of certain generating functions for these generalized Hermite polynomials would possibly shed some light on series of the type (1) and (2).

In the paper [1], a function $H_n^r(x, a, p)$ was introduced by the definition

$$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D_x^n (x^a e^{-px^r}) \quad , \tag{4}$$

which gave the generating function

$$\left(1 - \frac{t}{x}\right)^a e^{p(x^r - (x-t)^r)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^r(x, a, p) \quad . \tag{5}$$

This expansion gives at once in a formal sense

$$x^{F_n} = e^{p(a-b)F_n} = \left(\frac{a}{b}\right)^m \sum_{k=0}^{\infty} \frac{(a - b)^k}{k!} H_k^n(a, m, p) \quad , \tag{6}$$

where p, x satisfy $p(a - b) = \log x$.

Therefore we have

$$\sum_{n=0}^{\infty} A_n t^n x^{F_n} = \left(\frac{a}{b}\right)^m \sum_{k=0}^{\infty} \frac{(a-b)^k}{k!} \sum_{n=0}^{\infty} A_n t^n H_k^n(a,m,p) \quad , \tag{7}$$

from which it is evident that it would be desirable to establish simple generating functions of the sort

$$G_1 = \sum_{n=0}^{\infty} A_n t^n H_k^n(a,m,p) \tag{8}$$

for the generalized Hermite polynomials.

For the Lucas numbers we have

$$x^{L_n} = x^{a^n} \cdot x^{b^n} = e^{pa^n} \cdot e^{pb^n} \quad , \quad \text{with } p = \log x \quad ,$$

and, formally, we have from (5)

$$e^{pa^n} = \sum_{k=0}^{\infty} \frac{a^k}{k!} H_k^n(a,0,p) \quad . \tag{9}$$

Consequently we find

$$x^{L_n} = \sum_{k=0}^{\infty} \frac{b^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{a}{b}\right)^j H_j^n(a,0,p) H_{k-j}^n(b,0,p) \quad . \tag{10}$$

With this approach to a series of the type (2) we should next have to find bilinear generating functions of the form

$$G_2 = \sum_{n=0}^{\infty} A_n t^n H_j^n(a,u,p) H_k^n(b,v,p) \quad , \tag{11}$$

which seem difficult to obtain. Of course this is not the only way to relate the Lucas numbers to the H functions, but it is suggestive of new avenues of research.

One may readily verify (as was found in [1]) that an explicit formula for the H functions is

$$H_n^r(x,a,p) = (-1)^n n! \sum_{k=0}^{n} p^k \frac{x^{rk-n}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{a+rj}{n} \quad . \tag{12}$$

In (7) m is a parameter and we may take m = 0 for our purposes. Thus we find

(13)
$$\sum_{n=0}^{\infty} A_n t^n H_k^n(a,0,p) = (-1)^k k!\, a^{-k} \sum_{s=0}^{k} \frac{p^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \sum_{n=0}^{\infty} A_n \binom{nj}{k} (ta^s)^n ,$$

so that we should have to find some really <u>simple</u> sum for a series of the type

(14)
$$\sum_{n=0}^{\infty} A_n \binom{nj}{k} z^n ,$$

and this also seems difficult. In the case $A_n = 1$ (for all n) it <u>is</u> possible to sum this series as follows.

In general

(15)
$$\sum_{n=0}^{m} f(jn) = \frac{1}{j} \sum_{s=1}^{j} \sum_{n=0}^{jm} \omega_j^{sn} f(n), \quad \text{with } \omega_j = e^{2\pi i/j} .$$

This gives the summation formula

(16)
$$\sum_{n=0}^{\infty} \binom{jn}{k} t^{jn} = \frac{1}{j} \sum_{s=1}^{j} \frac{(t\omega_j^s)^k}{(1 - t\omega_j^s)^{k+1}} , \quad |t| < 1, \; j \geq 1 ,$$

so that in principle we have a (complicated) generating function for (13).

Another direction in which we may go to find generating functions is suggested by the second generalization of Hermite polynomials given in [1]. By definition

(17)
$$g_n^r(x,h) = e^{hD^r} x^n , \quad D = D_x ,$$

and this yields the generating function

(18)
$$e^{tx+ht^r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} g_n^r(x,h) .$$

Thus in a formal sense

$$e^{pa^n} = e^{-az} \sum_{k=0}^{\infty} \frac{a^k}{k!} g_k^n(z,p) \quad . \tag{19}$$

Two such expansions, with parameters a and b, might be multiplied together or perhaps combined with the expansion (9) in order to obtain generating functions involving Fibonacci and Lucas numbers as exponents. It seems clear that what is needed is a collection of interesting and simple generating functions for the generalized Hermite polynomials. It is hoped to offer further results in this direction in a later paper.

REFERENCE

1. H. W. Gould and A. T. Hopper, Operational Formulas Connected With Two Generalizations of Hermite Polynomials, Duke Mathematical Journal, 29 (1962), 51 — 63.

REFERENCES FROM PAGE 39

1. J. A. Jeske, Linear Recurrence Relations — Part I, the Fibonacci Quarterly, Vol. 1, No. 2, pp. 69—74.
2. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 1944.

NOTICE TO ALL SUBSCRIBERS!!!

Please notify the Managing Editor AT ONCE of any address change. The Post Office Department, rather than forwarding magazines mailed third class, sends them directly to the dead-letter office. Unless the addressee specifically requests the Fibonacci Quarterly be forwarded at first class rates to the new address, he will not receive it. (This will usually cost about 30 cents for first-class postage.) If possible, please notify us AT LEAST THREE WEEKS PRIOR to publication dates: February 15, April 15, October 15, and December 15.

LINEAR RECURRENCE RELATIONS — PART II

JAMES A. JESKE, SAN JOSE STATE COLLEGE

1. INTRODUCTION

By applying the exponential generating function transformation

$$(1.1) \qquad Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} ,$$

we derived in Part I of this article [1] an explicit formula for the general solution of the homogeneous linear recurrence relation

$$(1.2) \qquad L_k(E)\, y_n \equiv \sum_{j=0}^{k} a_j E^j y_n \equiv \sum_{j=0}^{k} a_j y_{n+j} = 0 ,$$

where the coefficients a_j were constants, and the translation operator E^j was defined by

$$E^j y_n = y_{n+j} \quad (j = 0, 1, \cdots, k) .$$

In the present part of this article, we discuss the non-homogeneous recurrence relations having variable coefficients.

2. EXPLICIT SOLUTION OF A NON-HOMOGENEOUS RECURRENCE RELATION

We consider the linear non-homogeneous recurrence relation

$$(2.1) \qquad \sum_{j=0}^{k} a_j y_{n+j} \equiv L_k(E)\, y_n = b_n$$

with constant coefficients, and where the roots $r_1, r_2, \cdots, r_k$ of the characteristic equation $L_k(r) = 0$ are all distinct. Multiplying both sides of (2.1) by $t^n/n!$ and summing over n from 0 to ∞ yield the transformed equation

$$(2.2) \qquad L_k(D)\, Y = B(t) , \qquad \left(D \equiv \frac{d}{dt} \right)$$

where

(2.3)
$$B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad .$$

Now (2.2) is an ordinary linear differential equation whose general solution is

(2.4)
$$Y(t) = Y_p(t) + \sum_{i=1}^{k} c_i e^{r_i t} \quad ,$$

where, by the method of variation of parameters, the particular solution $Y_p(t)$ can be expressed by

(2.5)
$$Y_p(t) = \sum_{i=1}^{k} \frac{e^{r_i t}}{L_k'(r_i)} \sum_{n=0}^{\infty} \frac{b_n}{n!} \int_0^t s^n e^{-r_i s}\, ds$$

or

(2.6)
$$Y_p(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{i=1}^{k} \frac{r_i^n}{L_r'(r_i)} \sum_{p=0}^{n-1} \frac{b_p}{r_i^{p+1}} \quad .$$

Since $y_n = Y^{(n)}(0)$, we immediately find that

(2.7)
$$y_n = \sum_{i=1}^{k} c_i r_i^n + \sum_{i=1}^{k} \frac{r_i^n}{L_k'(r_i)} \sum_{p=0}^{n-1} \frac{b_p}{r_i^{p+1}}$$

is the general solution of the recurrence relation (2.1). The case where $L_k(r) = 0$ has repeated roots may be treated in a similar way and is left to the reader.

3. LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

A generalization of the recurrence relation (1.2) with constant coefficients is the equation

(3.1)
$$\sum_{j=0}^{k} P_j(n)\, y_{n+j} = 0 \quad ,$$

where $P_j(n)$ are polynomials of degree q_j in the independent discrete variable n. If the exponential generating function (1.1) is applied to (3.1), we obtain the differential equation

(3.2) $$\sum_{j=0}^{k} P_j(\phi)\, Y^{(j)} = 0$$

where ϕ is the operator

(3.3) $$\phi \equiv t\, D \equiv t\frac{d}{dt} \quad ,$$

and where, by definition,

(3.4) $$P_j(n) = \sum_{m=0}^{q_j} \alpha_m\, n^m \quad .$$

Equation (3.2) is an immediate consequence of the following theorem which can easily be established by mathematical induction:

Theorem 3.1. The exponential generating function for the sequence $\{n^m y_{n+j}\}$ is given by

(3.5) $$\phi^m Y^{(j)}(t) = \sum_{n=0}^{\infty} n^m y_{n+j} \frac{t^n}{n!} \quad , \quad (j = 1, 2, \cdots ;\ m = 0, 1, \cdots ,)$$

where ϕ is defined by (3.3).

Since the theory of differential equations is richer in special formulas and techniques than the corresponding formulas and techniques in the theory of recurrence relations, equation (3.2) resulting from the application of the exponential generating function may be more amenable to an explicit solution than the original relation (3.1). We illustrate this fact with the following examples:

4. EXAMPLES WITH VARIABLE COEFFICIENTS

The Bessel functions $J_n(x)$ of order n satisfy the recurrence relation

(4.1) $$x y_{n+2}(x) - 2(n + 1) y_{n+1}(x) + x y_n(x) = 0 \quad ,$$

which is a very special case of (3.1) with $k = 2$, $P_2(n) = x$, $P_1(n) = -2(n + 1)$, $P_0(n) = x$. Equation (3.2) thus yields the differential equation

(4.2) $$(x - 2t) Y'' - 2 Y' + x Y = 0 \quad ,$$

which has the particular solution

$$(4.3)\qquad Y = J_0(\sqrt{x^2 - 2tx}) ,$$

where $J_0(z)$ is Bessel's function of zero order defined by [2]

$$(4.4)\qquad J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m (m!)^2} .$$

Thus, we find

$$Y = J_0(\sqrt{x^2 - 2tx}) = \sum_{m=0}^{\infty} \frac{(-1)^m (x^2 - 2tx)^m}{4^m (m!)^2}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{4^m (m!)^2} \sum_{n=0}^{m} \binom{m}{n} (-1)^n \left(\frac{2t}{x}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2t}{x}\right)^n \sum_{m=n}^{\infty} \frac{(-1)^m}{4^m} \binom{m}{n} \frac{x^{2m}}{(m!)^2} ,$$

or finally

$$(4.5)\qquad Y = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+n}}{2^{2j+n} j!\,(j+n)!} .$$

By definition of the generating function (1.1), we therefore have

$$(4.6)\qquad y_n(x) \equiv J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+n}}{2^{2j+n} j!\,(j+n)!} .$$

As a final example, we consider the second-order recurrence relation

$$(4.7)\qquad y_{n+2}(x) - 2x y_{n+1}(x) + 2(n+1) y_n(x) = 0$$

which is satisfied by the Hermite polynomials $H_n(x)$ of degree n, with initial values

$$(4.8)\qquad y_0(x) = 1, \quad y_1(x) = 2x .$$

The transformed equation of relation (4.6) is the differential equation

$$Y'' - 2(x - t)Y' + 2Y = 0 \tag{4.9}$$

with conditions $Y(0,x) = 1$ and $Y'(0,x) = 2x$. Solution of (4.8) is

$$Y(t,x) = e^{x^2} \cdot e^{-(x-t)^2} = e^{2tx-t^2} , \tag{4.10}$$

and expansion of the right side thus yields

$$Y = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m!\,(n-2m)!} , \tag{4.11}$$

where $[n/2]$ means the integral part $n/2$. From the definition of the exponential generating function (1.1), it is seen that

$$y_n \equiv H_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m n!\,(2x)^{n-2m}}{m!\,(n-2m)!} \tag{4.12}$$

is the explicit solution of the recurrence relation (4.6)

5. REMARKS

The Laguerre polynomials, and in fact most of the important special functions of mathematical physics, satisfy a second-order recurrence relation of the form

$$[A_2(x) + nB_2(x)]y_{n+2}(x) + [A_1(x) + nB_1(x)]y_{n+1}(x) + [A_0(x) + nB_0(x)]y_n(x) = 0 \tag{5.1}$$

whose coefficients are linear in the independent real variable n. Explicit solutions for them, by the method of generating functions, may be obtained as in the above two examples. The method of generating functions can also be easily applied to solve certain partial recurrence relations. In part III of this article we shall show how this may be done and give examples of solutions involving Fibonacci arrays.

REFERENCES

See page 34 for the references to this article.

invited to attend two annual Fibonacci conferences, the present locale being Northern California. However, members who live at a distance would be able to share in this activity by means of a duplicated publication that would appear subsequent to the conference. This same publication would likewise be a news organ in which members of the Association would be able to share their Fibonacci experiences and plans with others in the group.

Finally, there is the matter of library priviliges. Brother Alfred, Managing Editor of the Quarterly, is presently producing a library of photostats of Fibonacci articles. Already, most of the pertinent references in Dickson's History of the Theory of Numbers have been covered and the ultimate objective is to include all available Fibonacci publications. Members of the Association would be allowed to borrow this material without charge. This should provide a great boon to many who are not in a position to examine such works in libraries close at hand. It should also provide an opportunity for research even to those living in somewhat remote places.

The annual fee for membership in the Fibonacci Association (and this includes subscription to the Quarterly) is $5.00.

FURTHER APPEARANCE OF THE FIBONACCI SEQUENCE

A. F. HORADAM, UNIVERSITY OF NEW ENGLAND, ARMIDALE, N. S. W., AUSTRALIA

Besides the widespread use of Fibonacci's sequence in Mathematics generally, and the occurrence of the sequence in such diverse fields as electrical network theory and biology (e.g., in the botanical phenomenon of phyllotaxis and the genealogical tree of the male bee [1]), there are certain non-scientific contexts in which its appearance may be of interest.

Both instances to which I shall refer in a moment involve not only the Fibonacci sequences but the ratio known as the Golden Section which has exercised a powerful influence on men's minds down the ages, and about which there is considerable literature. The idea of the Golden Section, probably of Pythagorean origin, is stated by Euclid (Book 2, proposition 11, according to the standard Heath translation) in the following problem: "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment." A little calculation reveals that for a segment of unit length, the division (Golden Section) occurs at the irrational point distant $X = (\sqrt{5} - 1)/2 = .62$ from the origin, i.e., X is a solution of the equation $x^2 + x - 1 = 0$.

Now

$$X = \lim_{n \to \infty} \left(\frac{F_n}{F_{n+1}} \right) = \lim_{n \to \infty} \left(\frac{H_n}{H_{n+1}} \right)$$

where F_n is the nth term of the ordinary Fibonacci sequence and H_n is the nth term of the generalized Fibonacci sequence [2]. Hence the link between the Golden Section and the Fibonacci sequence.

Psychologists have found by experiment that aesthetically the most pleasing rectangle is the one whose sides are in the ratio $X: 1 - X = (\sqrt{5} - 1): (3 - \sqrt{5}) = 1:X$. Recognizing this aspect of beauty, the ancient Greeks sometimes constructed temples according to these proportions.

A more subtle appreciation of the aethetic qualities of the Golden Section is detailed by Hambidge [3] in his study of Greek vases. After searching inquiries concerning the bases of design in nature and in art, he concludes that the "principle of dynamic symmetry" manifest in shell growth and in leaf distribution in plants was known only to the Egyptians and the Greeks. By meticulous measurements of objects of ancient art, such as Egyptian bas-reliefs and Greek pottery, Hambidge exhibits the constant but hidden occurrence of the Golden Section.

No less meticulous has been the very recent detailed research of Professor G. E. Duckworth [4], of Princeton, into the structural patterns and proportions used by Vergil in the Aeneid. In carefully analyzing the literary architecture of this epic, Duckworth discovers, quite by accident, the basic mathematical symmetry which Vergil consciously used in composing the Aeneid. This is a reminder that ancient poetry was intended to be heard and that, like music, as Duckworth points out, harmony and mathematical proportion appeal to the ear and the imagination.

In his investigations, he gives evidence that

(i) Other poets of Vergil's era, e. g., Catullus, Lucretius, Horace and Lucan, used the Fibonacci sequence in the structure of their poems; and

(ii) Besides the frequent occurrence in the Aeneid of the Golden Section for the ordinary Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, $\cdots$, Vergil also frequently used several Fibonacci sequences, namely, those which in my notation [2] would be labelled H_{12} (the Lucas sequence) H_{13}, H_{14}, H_{15}, H_{23}, H_{34}, H_{45}, H_{67}.

This latter discovery raises a very important point. We are told that Vergil was a serious student of Mathematics. Duckworth produces evidence to show that Vergil, and other poets of his time, were familiar with the Golden Mean and the Fibonacci sequences, a fact which suggests that the Greek and Roman mathematicians knew about the Fibonacci sequence, though there is no record that this is so. [Have we therefore given our Association the correct name?]

Like the work of Hambidge, the minutiae of the painstaking scholarly researches of Duckworth and his fellow-workers reveal a fascinating modern tendency, namely, the successful search for the mathematical expression of beauty and form (and sometimes of chaos). Their discoveries pose a problem

(Continued on p. 46)

ON THE ORDERING OF FIBONACCI SEQUENCES

BROTHER U. ALFRED, ST MARY'S COLLEGE, CALIFORNIA

We may define a Fibonacci sequence by taking any two relatively prime integers and employing the relation

$$f_n = f_{n-1} + f_{n-2}$$

to extend the sequence to subscripts going to plus infinity or subscripts going to minus infinity. Evidently, since any two successive terms of a given sequence define the sequence, there appears to be at first glance an element of confusion in the situation. How may this be obviated and how, once it is removed, is it possible to arrange Fibonacci sequences according to some rational order? Such are the questions that will be answered in this short paper.

It is a remarkable fact that every Fibonacci sequence has two parts: the one going to the right with all the signs the same may be called the monotonic portion; the other going to the left with signs alternating may be denoted the alternating portion. The sequence will be designated positive or negative according as the monotonic portion has all terms plus or minus respectively. However, since a negative sequence may be obtained from a positive sequence by changing the signs of all terms, it will be sufficient in what follows to consider the positive sequence.

Starting, then, with two positive terms a and b with $a < b$, we work back to $a' = b - a$; if this is less than a, we next form $a'' = a - a'$; and so on. Evidently, this process cannot be continued indefinitely and so we finally arrive at a term which is greater than the term which follows it. Once this occurs, the next term to the left is negative and from then on the signs alternate.

It is important to note that in this process we have arrived at a smallest positive term which has the characteristic that it is less than one-half the following positive term. This property of the smallest positive term is unique in the monotonic portion of the sequence. Let us call this smallest non-negative, f_0, and the subsequent term, f_1. We thus have an unambiguous means of representing the Fibonacci sequence by giving these two terms: (f_0, f_1).

If we had started with a positive term and a negative term, as long as terms alternate in sign in going to the right in the sequence, the absolute values decrease in magnitude. Since this cannot go on indefinitely, there must come a term which is of the same sign as the preceding term. From then on the sequence is monotonic. Evidently, the second term in the monotonic portion of the sequence is a minimum for that part of the sequence and so once more it is possible to represent the sequence according to the minimum term and the term that follows it.

Now that a unique representation of each Fibonacci sequence has been achieved, it might appear desirable to have some method of arranging these sequences in order. One means of doing so is by the use of the quantity

$$D = f_1 f_{-1} - f_0^2 = f_1^2 - f_1 f_0 - f_0^2$$

which is characteristic of any given sequence. Intuitively it appears that for any sequence

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n D$$

Suppose this to be true to n. Then

$$f_{n+2} f_n - f_{n+1}^2 = f_{n+1} f_n + f_n^2 - f_{n+1}^2 = f_n^2 - f_{n+1} f_{n-1} = (-1)^{n+1} D$$

so that the formula is seen to hold by mathematical induction.

For any given value of f_1, since

$$D = f_1^2 - f_0 (f_1 + f_0)$$

and since

$$f_0 < f_1/2 \quad ,$$

it follows that

$$D > f_1^2 - f_1 (f_1 + f_1/2)/2$$

or

$$D > f_1^2 / 4$$

Accordingly, by considering successive values of f_1 and the various Fibonacci sequences that may be associated with these values, it is possible to arrive at certain knowledge regarding the Fibonacci sequences that may be associated with allowed values of D. This information is summarized for values of D up to 1000 in the following table.

TABLE OF FIBONACCI SEQUENCES
HAVING A GIVEN VALUE OF D

D	SEQUENCES
1	(0,1)
5	(1,3)
11	(1,4), (2,5)
19	(1,5), (3,7)
29	(1,6), (4,9)
31	(2,7), (2,8)
41	(1,7), (5,11)
55	(1,8), (6,13)
59	(2,9), (5,12)
61	(3,10), (4,11)
71	(1,9), (7,15)
79	(3,11), (5,13)
89	(1,10), (8,17)
95	(2,11), (7,16)
101	(4,13), (5,14)
109	(1,11), (9,19)
121	(3,13), (7,17)
131	(1,12), (10,21)
139	(2,13), (9,20)
145	(3,14), (8,19)
149	(4,15), (7,18)
151	(5,16), (6,17)
155	(1,13), (11, 23)
179	(5,17), (7,19)
181	(1,14), (12,25)
191	(2,15), (11,24)
199	(3,16), (10,23)
205	(4,17), (9,22)
209	(1,15), (5,18), (8,21), (13,27)
211	(6,19), (7,20)
229	(3,17), (11,25)
239	(1,16), (14,29)
241	(5,19), (9,23)
251	(2,17), (13,28)
269	(4,19), (11,26)
271	(1,17), (15,31)
281	(7,22), (8,23)
295	(3,19), (13,29)
401	(7,25), (11,29)
409	(3,22), (16,35)
419	(1,21), (19,39)
421	(4,23), (15,34)
431	(5,24), (14,33)
439	(6,25), (13,32)
445	(7,26), (12,31)
449	(8,27), (11,30)
451	(3,23), (9,28), (10,29), (17,37)
461	(1,22), (20,41)
479	(2,23), (19,40)
491	(7,27), (13,33)
499	(9,29), (11,31)
505	(1,23), (21,43)
509	(4,25), (17,38)
521	(5,26), (16,37)
541	(3,25), (19,41)
545	(8,29), (13,34)
551	(1,24), (10,31), (11,32), (22,45)
569	(5,27), (17,39)
571	(2,25), (21,44)
589	(3,26), (7,29), (15,37), (20,43)
599	(1,25), (23,47)
601	(9,31), (13,35)
605	(4,27), (19,42)
619	(5,28), (18,41)
631	(6,29), (17,40)
641	(7,30), (16,39)
649	(1,26), (8,31), (15,38), (24,49)
655	(9,32), (14,37)
659	(10,33), (13,36)
661	(11,34), (12,35)
671	(2,27)(5,29), (19,43), (23,48)
691	(3,28), (22,47)
695	(7,31), (17,41)
701	(1,27), (25,51)
709	(4,29), (21,46)
719	(11,35), (13,37)
739	(6,31), (19,44)
745	(3,29), (23,49)
751	(7,32), (18,43)
755	(1,28), (26,53)
761	(8,33), (17,42)
769	(9,34), (16,41)
779	(2,29), (11,36), (14,39), (25,52)
781	(5,31), (12,37), (13,38), (21,47)

D	SEQUENCES	D	SEQUENCES
809	(7,33), (19,45)	905	(11,38), (16,43)
811	(1,29), (27,55)	911	(13,40), (14,41)
821	(4,31), (23,50)	919	(3,32), (26,55)
829	(9,35), (17,43)	929	(1,31), (29,59)
839	(5,32), (22,49)	941	(4,33), (25,54)
841	(11,37), (15,41)	955	(9,37), (19,47)
859	(3,31), (25,53)	961	(5,34), (24,53)
869	(1,30), (7,34), (20,47), (28,57)	971	(11,39), (17,45)
881	(8,35), (19,46)	979	(6,35), (13,41), (15,43), (23, 52)
895	(2,31), (27,56)	991	(1,32), (30,61)
899	(5,33), (10,37), (17,44), (23,51)	995	(7,36), (22,51)

By adopting the convention that for several sequences having the same value of D, the ordering will be determined by which has the smaller value of f_0, it becomes possible to give the Fibonacci sequences a precise arrangement. The first few members would be as follows:

$$S_1(0,1),\ S_2(1,3),\ S_3(1,4),\ S_4(2,5),\ S_5(1,5),\ S_6(3,7),\ S_7(1,6),\ S_8(4,9),\ \text{etc.}$$

The above approach in representing Fibonacci sequences and ordering them is all by way of suggestion. There are doubtless other ways of achieving the same objective. It would be very helpful if additional proposals were aired before a final standard is adopted.

FURTHER APPEARANCE OF THE FIBONACCI SEQUENCE
(Cont. from p. 42)

for the classicist, no less than for the historian of Mathematics. "Measure and symmetry," observed Socrates, "are beauty and virtue all the world over."

REFERENCES

1. S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly 1(1), 1963, 53 — 56.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math Monthly, 68(5), 1961, 455—459.
3. J. Hambidge, "Dynamics Symmetry," Yale University Press, 1920.
4. G. E. Duckworth, "Structural Patterns and Proportions in Vergil's Aeneid," Univ. of Michigan Press, 1962.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-24 *Proposed by the late Morgan Ward, California Institute of Technology, Pasadena, California.*

Let $\phi_n(x) = x + x^2/2 + \cdots + x^n/n$, and let $k(x) \equiv k_p(x) = (x^{p-1} - 1)/p$, where p is an odd prime greater than 5. (The function $k(x)$ is called the "quotient of Fermat" in the literature.) Let $P = P_p$ be the rank of apparition of p in the sequence $0, 1, 1, 2, 3, 5, \cdots, F_n$, (so $P_{13} = 7$, $P_7 = 8$ and so on). Then

$$F_P \equiv 0 \bmod p^2$$

if and only if

$$\phi_{(p-1)/2}(5/9) \equiv 2k(3/2) \bmod p \quad .$$

H-25 *Proposed by Joseph Erbacker and John A. Fuchs, University of Santa Clara, and F.D. Parker, Suny, Buffalo, N.Y.*

Prove:

$$D_n = \left| a_{ij} \right| = 36, \quad \text{for all } n ,$$

where

$$a_{ij} = F^3_{n+i+j-2} \quad (i,j = 1,2,3) \quad .$$

H-26 *Proposed by Leonard Carlitz, Duke University, Durham, N.C.*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-s}$, then show $R_k^n = (a_{rs})$ such that

$$a_{rs} = \sum_{j=0}^{s-1} \binom{r-1}{j}\binom{k+1-r}{s-1-j} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^{j} \quad .$$

H-27 *Proposed by Harlan L. Umansky, Emerson High School, Union City, N.J.*

Show that

$$F_k^3 = \sum_{j=1}^{k-2} (-1)^{j+1} F_j F_{3k-3j} + (-1)^k F_{k-3} \;, \quad k \geq 4 \;.$$

H-28 *Proposed by H.W. Gould, West Virginia University, Morgantown, W.Va.*

Let $C_j(r,n)$ be the number of numbers, to the base r $(r \geq 2)$ with at most n digits, and the sum of the digits equal to j.

Sum the series:

$$\sum_{j=0}^{\infty} C_j(r,n)\, a^j b^{rn-n-j} \quad .$$

SOLUTIONS

TRINOMIAL COEFFICIENTS

H-9 *Proposed by Olga Taussky, California Institute of Technology, Pasadena, Calif.*

Find the numbers $a_{n,r}$, where $n \geq 0$ and r are integers, for which the relations

$$a_{n,r} + a_{n,r-1} + a_{n,r-2} = a_{n+1,r}$$

and

$$a_{o,r} = \delta_{o,r} = \begin{cases} 0 & r \neq 0 \\ 1 & r = 0 \end{cases}$$

hold.

Solution by the proposer.

It can be shown that $a_{n,r}$ is the coefficient of x^r in the expansion of $(1 + x + x^2)^n$.

This is certainly true for $n = 0$ and it follows for $n > 0$ by using the generating functions

$$\sum_r a_{n,r} x^r \quad .$$

For, multiplying this sum by $(1 + x + x^2)$ and using the recurrence relations it follows that

$$(1 + x + x^2) \sum_r a_{n,r} x^r = \sum_r a_{n+1,r} x^r .$$

This proves the assertion.

SOME FIBONACCI SUMS

H-10 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey*

Show that
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n F_{n+1} F_{n+2}}$$

Solution by Leonard Carlitz, Duke University, Durham, N. C.

$$\sum_2^{\infty} \frac{1}{F_n} - \sum_2^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \frac{F_{n-1}F_{n+1} - F_n^2}{F_{n-1}F_nF_{n+1}} = \sum_2^{\infty} \frac{(-1)^n}{F_{n-1}F_nF_{n+1}} ,$$

$$\sum_2^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \left(\frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \right) = \frac{1}{F_1} + \frac{1}{F_2} = 2 ,$$

$$\sum_1^{\infty} \frac{1}{F_n} = 3 + \sum_2^{\infty} \frac{(-1)^n}{F_{n-1}F_nF_{n+1}} .$$

Also solved by Zvi Dresner.

FIBONACCI AND FOURIER

H-11 *Proposed by John L.Brown,Jr.,Ordnance Research Laboratory, The Pennsylvania State University, University Park, Penna.*

Find the function whose formal Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{F_n \sin nx}{n!} ,$$

where F_n is the nth Fibonacci number.

Solution by Lucile Morton, Santa Clara, Calif.

$$e^{z(\cos x + i \sin x)} = \sum_{n=0}^{\infty} \frac{(\cos x + i \sin x)^n z^n}{n!} = \sum_{n=0}^{\infty} \frac{\cos nx}{n!} z^n + i \sum_{n=0}^{\infty} \frac{\sin nx}{n!} z^n .$$

Therefore

$$e^{z\cos x}\sin(z\sin x) = \sum_{n=0}^{\infty} \frac{\sin nx}{n!} z^n$$

Recalling

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad \text{where } \alpha = (1+\sqrt{5})/2 \text{ and } \beta = (1-\sqrt{5})/2,$$

then

$$f(x) = \frac{1}{\sqrt{5}}\left\{e^{\alpha\cos x}\sin(\alpha\sin x) - e^{\beta\cos x}\sin(\beta\sin x)\right\}$$

and

$$g(x) = e^{\alpha\cos x}\sin(\alpha\sin x) + e^{\beta\cos x}\sin(\beta\sin x) = \sum_{n=0}^{\infty}\frac{\sin nx}{n!}L_n .$$

Also solved by the proposer.

A CURIOUS SEQUENCE

H-12 *Proposed by D.E. Thoro, San Jose State College, San Jose, Calif.*

Find a formula for the nth term in the sequence:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, \cdots .$$

Solution by Malcolm Tallman, Brooklyn, N. Y.

$$N_M \left\{ \begin{array}{l} 1,\ 3,\ 4,\ 6,\ 8,\ 9, 11, 12 \\ 14, 16, 17, 19, 21, 22, 24, 25 \\ 27, 29, 30, 32, 34, 35, 37, 38 \\ 40, 42, 43, 45, 47, 48, 50, 51 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array} \right.$$

Let

$$M = 8m + 1, 2, 3^*, 4, 5, 6, 7, 8$$

$$N_M = 13m + 1, 3, 4^*, 6, 8, 9, 11, 12$$

What is the 19th term? $M = 19 = 8 \times 2 + 3^*$, thus $N_{19} = 13 \times 2 + 4^* = 30$.

Also solved by Maxey Brooke and the proposer.

Editorial Comment: If $T_1 = 1$, $T_2 = 3$, $T_3 = 4$, $T_4 = 6$, $T_5 = 8$, $T_6 = 9$, $T_7 = 11$, $T_8 = 12$, then $T_{8m+k} = 13m + T_k$, $k > 0$, $m = 1, 2, 3, \cdots$.

A MATRIX DERIVED IDENTITY

H-13 *Proposed by H.W. Gould, West Virginia University, Morgantown, W. Va. and Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Show that

$$F_n = \sum_{j=0}^{r}\binom{r}{j} F_{k-1}^{r-j} F_k^j F_{n+j-rk} .$$

See p. 65 of "A Primer for the Fibonacci Numbers—Part III," Oct., 1963, Fibonacci Quarterly.

Also solved by Leonard Carlitz and Merritt Elmore.

IDENTITY FOR FIBONACCI CUBES

H-14 *Proposed by David Zeitlin, Minneapolis, Minnesota, and F.D. Parker, University of Alaska, College, Alaska.*

Prove the Fibonacci identity

$$F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0 \quad .$$

Solution by Maxey Brooke, Sweeny, Texas.

From "Fibonacci Formulas," page 60, April, 1963, Fibonacci Quarterly, one obtains, from paragraph 3,

(1)
$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$$

and the corrected version of Jekuthiel Ginsburg's identity there is

(2)
$$F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = F_{3n}$$

Multiplying equation (1) through by 3 and equating the new left side of (1) to the left side of (2) and simplifying yields

$$F_{n+2}^3 - 3F_{n+1}^3 - 6F_n^3 + 3F_{n-1}^3 + F_{n-2}^3 = 0 \quad .$$

Also solved by J. A. H. Hunter, Zvi Dresner and the proposers.

SOME CHOICE IDENTITIES

H-16 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.*

Define the ordinary Hermite polynomials by $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$.

$$\text{(i)} \quad \sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} = 1 \quad ,$$

Show that

$$\text{(ii)} \quad \sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} F_n = 0$$

$$\text{(iii)} \quad \sum_{n=0}^{\infty} H_n(x/2)\frac{x^n}{n!}L_n = 2e^{-x^2} \quad ,$$

where F_n and L_n are the nth Fibonacci and nth Lucas numbers, respectively.

Solution by Zvi Dresner, Tel-Aviv, Israel.

$$\text{(i)} \quad \sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} = e^{\frac{x_0^2}{4}}\left\{\sum_{n=0}^{\infty}\frac{D^n(e^{-x^2})}{n!}\Bigg|_{x=x_0/2}\left(-\frac{x_0}{2}-\frac{x_0}{2}\right)^n\right\} .$$

The sum in braces on the right is the expansion of e^{-x^2} about the point $+\frac{x_0}{2}$, with $x = -x_0/2$. Hence

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} = e^{+x_0^2/4}\left(e^{-x_0^2/4}\right) = 1 .$$

(ii) In the same way $\left(\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}\right.$, and $\left.F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)\right)$,

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!}F_n = \frac{1}{\sqrt{5}}e^{x_0^2/4}\left\{\sum_{n=0}^{\infty}\frac{D^n e^{-x^2}}{n!}\Bigg|_{x=x_0/2}\left((-x_0\alpha)^n - (-x_0\beta)^n\right)\right\}$$

$$= \frac{1}{\sqrt{5}}e^{x_0^2/4}\left\{e^{-\left(\frac{x_0\sqrt{5}}{2}\right)^2} - e^{-\left(\frac{-x_0\sqrt{5}}{2}\right)^2}\right\} = 0.$$

(iii) Similarly $(L_n = \alpha^n + \beta^n)$

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!}L_n = e^{x_0^2/4}\sum_{n=0}^{\infty}\frac{D^n e^{-x^2}}{n!}\Bigg|_{x=x_0/2}\left[(-x_0\alpha)^n + (-x_0\beta)^n\right]$$

$$= e^{x_0^2/4}\left(2e^{-5x_0^2/4}\right) = 2e^{-x_0^2} .$$

Also solved by L. Carlitz and the proposer.

<u>Correction to Problem H-20 in the October issue</u>

H-20 (Corrected) *Proposed by Verner E. Hoggatt, Jr. and Charles H. King, San Jose State College, San Jose, California.*

If $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, show $D(e^{Q^n}) = e^{L_n}$,

where $D(A)$ is the determinant of matrix A and L_n is the nth Lucas number.

DYING RABBIT PROBLEM REVIVED

BROTHER U. ALFRED, ST. MARY'S COLLEGE, CALIFORNIA

In the first issue of the Fibonacci Quarterly the following problem was proposed [1]. Suppose that in the original Fibonacci rabbit breeding problem, we allow for the dying of rabbits. Those that are bred in February, for example, begin to breed in April and continue breeding monthly through February of the following year. At the end of this month they die. What would be the formula for the number of pairs of rabbits at the end of n months for $n \geq 13$?

Originally, it was thought that the rabbits removed would constitute a sequence which could be readily identified with an expression involving Fibonacci numbers. But after several attempts by a number of people it appeared that it would be difficult to arrive at an answer by straightforward intuition. The following development will indicate why this is so.

First of all we shall set down a table showing how the rabbits propagate over a two-year period. It will be noted that the original table values for the case in which rabbits do not die are positive while a negative term is introduced to show the effect of allowing rabbits to die.

n	Breeding Rabbits	Non-Breeding Rabbits	Bred Rabbits	Dying Rabbits	Rabbits End of Month
1	0	1	0	0	1
2	1	0	1	0	2
3	1	1	1	0	3
4	2	1	2	0	5
5	3	2	3	0	8
6	5	3	5	0	13
7	8	5	8	0	21
8	13	8	13	0	34
9	21	13	21	0	55
10	34	21	34	0	89
11	55	34	55	0	144
12	89	55	89	-1	233 - 1
13	144 - 1	89	144 - 1	0	377 - 2
14	233 - 1	144 - 1	233 - 1	-1	610 - 4
15	377 - 3	233 - 1	377 - 3	-1	987 - 8
16	610 - 5	377 - 3	610 - 5	-2	1597 - 15
17	987 - 10	610 - 5	987 - 10	-3	2584 - 28
18	1597 - 18	987 - 10	1597 - 18	-5	4181 - 51
19	2584 - 33	1597 - 18	2584 - 33	-8	6765 - 92
20	4181 - 59	2584 - 33	4181 - 59	-13	10946 - 164

n	Breeding Rabbits	Non-Breeding Rabbits	Bred Rabbits	Dying Rabbits	Rabbits End of Month
21	6765 - 105	4181 - 59	6765 - 105	-21	17711 - 290
22	10946 - 185	6765 - 105	10946 - 185	-34	28657 - 509
23	17711 - 324	10946 - 185	17711 - 324	-55	46368 - 888
24	28657 - 564	17711 - 324	28657 - 564	-89	75025 - 1541

For the sake of convenience, a table of the negative values is formed with a shift of numbering, the first row in the new table corresponding to n = 14 in the old.

n	a_n	a_{n-1}	a_n	F_n	T_n
1	1	1	1	1	4
2	3	1	3	1	8
3	5	3	5	2	15
4	10	5	10	3	28
5	18	10	18	5	51
6	33	18	33	8	92
7	59	33	59	13	164
8	105	59	105	21	290
9	185	105	185	34	509
10	324	185	324	55	888
11	564	324	564	89	1541

The following relations may be noted apart from those implicit in the column headings.

$$a_{n+1} = a_n + a_{n-1} + F_n$$

$$T_n = 2a_n + a_{n-1} + F_n$$

Using the relation for a_{n+1} we obtain the following succession of relations.

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= a_1 + a_0 + F_1 = 2 + F_1 \\
a_3 &= a_2 + a_1 + F_2 = 3 + F_1 + F_2 \\
a_4 &= a_3 + a_2 + F_3 = 5 + 2F_1 + F_2 + F_3 \\
a_5 &= a_4 + a_3 + F_4 = 8 + 3F_1 + 2F_2 + F_3 + F_4 \\
a_6 &= a_5 + a_4 + F_5 = 13 + 5F_1 + 3F_2 + 2F_3 + F_4 + F_5 \\
a_7 &= a_6 + a_5 + F_6 = 21 + 8F_1 + 5F_2 + 3F_3 + 2F_4 + F_5 + F_6
\end{aligned}$$

It is clear that a formula involving Fibonacci numbers is emerging. For example, a_7 can be written:

$$a_7 = F_8 + F_6F_1 + F_5F_2 + F_4F_3 + F_3F_4 + F_2F_5 + F_1F_6$$

and in general, it could be shown by mathematical induction that:

$$a_n = F_{n+1} + \sum_{k=1}^{n-1} F_k F_{n-k} .$$

The problem then reduces to finding a formula for the summation on the right. Using the roots r and s of the equation $x^2 - x - 1 = 0$ in terms of which:

$$F_n = \frac{r^n - s^n}{\sqrt{5}} \qquad \text{and} \qquad L_n = r^n + s^n$$

where

$$r = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad s = \frac{1 - \sqrt{5}}{2}$$

we have

$$F_k F_{n-k} = \frac{(r^k - s^k)(r^{n-k} - s^{n-k})}{5}$$

$$= \frac{r^n + s^n - r^k s^k (r^{n-2k} + s^{n-2k})}{5}$$

But $r^k s^k = (-1)^k$ since rs, the product of the roots of the given equation, is the constant term -1. Thus

$$F_k F_{n-k} = \frac{L_n + (-1)^{k+1} L_{n-2k}}{5}$$

This is the expression that must be summed from 1 to $n - 1$ over k. However, since the first part $L_n/5$ does not involve k, it is essentially a constant taken $n - 1$ times so that this part of the sum becomes $(n - 1)L_n/5$. The second half is

$$\frac{1}{5} \sum_{k=1}^{n-1} (-1)^{k+1} L_{n-2k} = \frac{1}{5}\left[L_{n-2} - L_{n-4} + L_{n-6} - L_{n-8} + \cdots + (-1)^n L_{-n+2}\right]$$

$$= \frac{1}{5}\left[F_{n-1} + F_{n-3} - F_{n-3} - F_{n-5} + F_{n-5} + F_{n-7} - \cdots + (-1)^n F_{-n+3} + (-1)^n F_{-n+1}\right]$$

It will be noted that all terms cancel out except the first and last and since $(-1)^n F_{-n+1} = F_{n-1}$, the total of the summation is $2F_{n-1}/5$. Thus the value of a_n is given by the expression

$$a_n = F_{n+1} + \frac{(n-1)}{5} L_n + \frac{2F_{n-1}}{5} .$$

Substituting $L_n = F_{n+1} + F_{n-1}$, this can be transformed to

$$a_n = \frac{1}{5}\left[(n + 4) F_{n+1} + (n + 1) F_{n-1}\right]$$

As a check, for $n = 7$, this becomes

$$1/5\,[\,11 \cdot 21 + 8 \cdot 8\,] = 59$$

After suitable transformations one can find a value of T_n equal to

$$1/5\left[(3n + 10)F_{n+1} + (n + 6) F_n\right]$$

Reconverting back to our original notation, the solution of the dying rabbit problem can be expressed as follows $(n \geq 13)$:

$$F_{n+1} - 1/5\left[(3n - 29) F_{n-12} + (n - 7) F_{n-13}\right]$$

REFERENCE

1. Brother U. Alfred, Exploring Fibonacci Numbers, The Fibonacci Quarterly, Vol. 1, No. 1, Feb. 1963, pp. 57 — 63.

PHYLLOTAXIS

SISTER MARY de SALES McNABB, GEORGETOWN VISITATION PREPARATORY SCHOOL

When Nehemial Grew remarked in 1682 that "from the contemplation of plants, men might be invited to Mathematical Enquirys," [5] he might not have been thinking of the amazing relationship between phyllotaxis and Fibonacci numbers, but he could well have been; for the phenomenon of phyllotaxis, literally "leaf arrangement," has long been a subject of special investigation, much speculation, and even heated debate among mathematicians and botanists alike.

By right it is the botanists who deserve the credit for bringing to light the discovery that plants of every type and description seem to have their form elements, that is, their branches, leaves, flowers, or seeds, assembled and arranged according to a certain general pattern; but surely even the old Greek and Egyptian geometers could not have failed to observe the spiral nature of the architecture of plants. Many and varied and even contradictory are the theories on this fascinating phenomenon of phyllotaxis, but it would be beyond the scope of this paper to investigate them here; instead we shall simply try to describe the manifestation of it in the interval-spacing of leaves around a cylindrical stem, in the florets of the sunflower and, finally, in the scales of fir cones and pineapples.

Before we proceed to consider the actual arrangement of the form elements, however, it is interesting to note the relationship between the number of petals of many well-known flowers and the Fibonacci numbers. Two-petaled flowers are not common but enchanter's nightshade is one such example. Several members of the iris and the lily families have three petals, while five-petaled flowers, including the common buttercup, some delphiniums, larkspurs and columbines, are the most common of all. Other varieties of delphiniums have eight petals, as does the lesser celandine, and in the daisy family, squalid and field senecio likewise have eight petals in the outer ring of ray florets. Thirteen-petaled flowers are quite common and include the globe flower and some double delphiniums as well as ragwart, corn marigold, mayweed, and several of the chamomiles. Many garden and wild flowers, including some heleniums and asters, chicory, doronicum, and some hawk-bits, have twenty-one petals, while thirty-four is the most common number in the daisy family and is characteristic of the field daiseys, ox-eye daisies, some heleniums, gaillardias, plantains, pyrethrums, and a number of hawk-bits and hawkweeds. Some field daisies have fifty-five petals, and Michaelmas daisies often have either fifty-five or eighty-nine petals. It is difficult to trace this relationship much further, but it must be remembered that this number pattern is not necessarily followed by every plant of a species but simply seems to be characteristic of the species as a whole.

Fibonacci numbers occur in other types of patterns too. The milkwort will commonly be found to have two large sepals, three smaller sepals, five petals and eight stamens, and Frank Land 4 reports that he found a clump of alstroemerias in his garden in which one plant had two flowers growing on each of three stalks and that, where the three stalks grew out from the top of the main stem, a whorl of five leaves grew out radially; while another plant had three flowers on each of five stems with a whorl of eight leaves at the base of the flower stalks.

The Fibonacci number pattern, however, which has received the most attention is that associated with the spiral arrangement of the form elements of the plants. In its simplest manifestation it may be observed in plants and trees which have their leaves or buds or branches arranged at intervals around a cylindrical stem. If we should take a twig or a branch of a tree, for instance, and choose a certain bud, then by revolving the hand spirally around the branch until we came to a bud directly above the first one counted, we would find that the number of buds per revolution as well as the number of revolutions itself are both Fibonacci numbers, consecutive or alternate ones depending on the direction of revolution, and different for various plants and trees. If the number of revolutions is m and the number of leaves or buds is n, then the leaf or bud arrangement is commonly called an m/n spiral or m/n phyllotaxis. Hence in some trees, such as the elm and basswood, where the leaves along a twig seem to occur directly opposite one another, we speak of 1/2 phyllotaxis, whereas in the beech and the hazel, where the leaves are separated by one-third of a revolution, we speak of 1/3 phyllotaxis. Likewise, the oak, the apricot, and the cherry tree exhibit 2/5 phyllotaxis, the poplar and the pear 3/8, while that of the willow and the almond is 5/13. Much investigation along these lines seems to indicate that, at least as far as leaves and blossoms are concerned, each species is characterized by its own particular phyllotaxis ratio, and that almost always, except where damage or abnormal growth has modified the arrangement, the ratios encountered are ratios of consecutive or alternate terms of the Fibonacci sequence.

When the form elements of certain plants are assembled in the form of a disk rather than along a cylindrical stem, we have a slightly different form of phyllotaxis. It is best exemplified in the head of a sunflower, which consists of a number of tightly packed florets, in reality the seeds of the flower. Very clearly the seeds can be seen to be distributed over the head in two distinct sets of spirals which radiate from the center of the head to the outermost edge in both clockwise and counterclockwise directions. These spirals, logarithmic in character, are of the same nature as those mentioned earlier in plants with cylindrical stems, but in those instances, the adjacent leaves being generally rather far apart along the stem, it is more difficult for the eye to detect the regular spiral arrangement. Here in the close-packed arrangement of the head of the sunflower, we can see the phenomenon in almost two-dimensional form. As was the case with the cylindrical-stemmed plants, the number pattern exhibited by the double set of spirals is intimately bound up with Fibonacci numbers. The normal sunflower head, which is about five or six inches in diameter, will generally have thirty-four spirals winding in one direction and fifty-five in the other. Smaller sunflower heads will commonly exhibit twenty-one spirals in one direction and thirty-four in the other or a combination of thirteen and twenty-one. Abnormally large heads have been developed with a combination of fifty-five and eighty-nine spirals and even a gigantic one at Oxford with eighty-nine spirals in one direction and a hundred and forty-four in the other. In each instance the combination of clockwise and counterclockwise spirals consists of successive terms of the Fibonacci sequence.

One other interesting manifestation of phyllotaxis and its relation to the Fibonacci numbers is observed in the scales of fir cones and pineapples. These scales are really modified leaves crowded together on relatively short stems, and so, in a sense, we have a combination of the other two forms of the phenomenon; namely, a short conical or cylindrical stem and a close-packed arrangement which easily enables us to observe that the scales are arranged in ascending spirals or helical whorls called parastichies. In the fir cone, as in the sunflower head, two sets of spirals are obvious, and hence in many cones, such as those of the Norway spruce or the American larch, five rows of scales may be seen to be winding steeply up the cone in one direction while three rows wind less steeply the other way; in the common larch we usually find eight rows winding in one direction and five in the other, and frequently the two arrangements cross each other on different parts of the cone. In the pineapple, on the other hand, three distinct groups of parastichies may be observed; five rows winding slowly up the pineapple in one direction, eight rows ascending more steeply in the opposite direction and, finally, thirteen rows winding upwards very steeply in the first direction. The fact that pineapple scales are of irregular hexagonal shapes accounts for the three sets of whorls, for three distinct sets of scales can consequently be contiguous and, hence, constitute a different formation. Moreover, Fibonacci numbers manifest themselves in still another way in connection with the scales of the pineapple. If the scales should be numbered successively around the fruit from the bottom to the top, the numbering being based on the corresponding lateral distances of the scales along the axis of the pineapple, we would find that each of the three observable groups of parastichies winds through numbers which constitute arithmetic sequences with common differences of 5, 8 and 13, the same three successive Fibonacci numbers observed above. Thus a spiral of the first group would ascend through the numbers 0, 5, 10, $\cdots$; one of the second group through the numbers 0, 8, 16, $\cdots$; and, finally, a spiral of the third group would wind steeply up the pineapple through the numbers 0, 13, 26, $\cdots$.

In all these many and varied ways, then, in the number of petals possessed by different species of flowering plants, in the interspacing of leaves or buds around a cylindrical stem, in the double spirals of the close-packed florets of sunflowers, and in the ascending spirals or parastichies of the fir cone and the pineapple, we have encountered number patterns which again and again involve particular terms of the Fibonacci sequence. These Fibonacci number patterns or combinations occur so continually in the varied manifestations of phyllotaxis that we often hear of the "law" of phyllotaxis. However, it must be admitted that not all four-petaled flowers are so rare as the four-leaf clover's reputed to be and that other combinations also occur, notably in those species exhibiting symmetrical arrangements. Moreover, in the cases of fir cones and some large sunflowers, where the spiral number pattern can be verified more carefully, deviations, sometimes even large ones, from the Fibonacci pattern have been found. If this is at all disturbing to the modern botanist, it is not at all so to the Fibonacci devotee, for whom the whole phenomenon, if not a "law," is at least, in the words of H. S. M. Coxeter [1], a fascinatingly prevalent tendency!

[References for this article are found on page 71.]

THE "GOLDEN RATIO" AND THE FIBONACCI NUMBERS IN THE WORLD OF ATOMS

J. WLODARSKI

In the world of atoms there are four fundamental asymmetries. They appear

- In the structure of atomic nuclei of protons and neutrons,
- In the distribution of fission fragments by mass number resulting from the bombardment of most heavy nuclei by thermal neutrons,
- In the distribution of numbers of isotopes of even stable elements,
- In the distribution of emitted particles in two opposite directions at "weak" nuclear interactions.

It turns out that the numerical values of all these asymmetries are equal approximately the "golden ratio" ("g. r.") and the numbers forming these numerical values are sometimes Fibonacci or "near"-Fibonacci numbers as follows:

1. The number of protons Z in the lightest stable nucleus as a rule is equal the number of neutrons N. When the atomic number Z increases, the proton-neutron ratio in the nucleus Z/N decreases to about 0.6.

A practical stable nucleus, found in nature may possess a maximum of 92 protons and 146 neutrons (nucleus ${}_{92}U^{238}$). The ratio of both these numbers Z/N is equal to 0.630 and differs from the "g. r."-value (if we limit the "g.r."-value to three decimals behind the point) by 0.012 only.

2. It is known that symmetrical fission of most heavy nuclei by slow neutrons is very rare. For example, in the case of ${}_{92}U^{235} + {}_0n^1$ the atomic mass of fission-fragments $A = 118$ happens in only about 0.01 of all cases. The most common event in this case is a splitting into two fragments with mass numbers in the range 89 – 99 and 144 – 134 respectively. The mass numbers 89 and 144 appearing in the nuclear reaction

$$ {}_{92}\overset{235}{U}_{143} + {}_0\overset{1}{n}_1 \rightarrow {}_{36}\overset{89}{Kr}_{53} + {}_{56}\overset{144}{Ba}_{88} + 3({}_0n_1^1) $$

belong to two neighboring terms in the Fibonacci sequence. The ratio of $89/144 = 0.618056\cdots$ yields one of the best approximations to the "g. r. "-value found in nature. The same ratio of 89/144 in the world of plants yields, for example, the distribution of sees-spirals on the disk of the sunflower.

It is interesting that the number of protons and neutrons of fission-fragments in above nuclear reaction is also one of the Fibonacci or "near"-Fibonacci numbers as the following table shows.

Table 1

Nucleon-numbers	Compound Nucleus	Fission Fragments		Terms of Fibonacci Sequence		
number of protons	92	36	56	89	34	55
number of neutrons	144	53	88	144	55	89
mass-number	236	89	144	233	89	144

Remarks:

- We have to consider the amount of 3 (on the average) emitting neutrons
- A variety of other pairs of nuclei, as Kr - Ba pair, may be produced in the above nuclear reaction, but this pair is one of the most abundant.

3. When the atomic number Z of elements in the periodic table increases, the number of isotopes of even stable elements also increases little by little and reaches a maximum (10 isotopes) at Z = 50. Behind Z = 50 the number of isotopes of even elements gradually decreases with the advancing atomic number. With that the whole row of stable elements in the periodic table is divided in the ratio 32 : 50 = 0.640. The last value differs from the "g. r."-value by 0.022 and
4. The recent (1957) discovery of parity non-conservation at "weak" interactions showed that:
   a. The β-decay of polarized neutrons is a process with an asymmetrical feature. The result of an experiment decisive for violation of the parity principle at the "weak" interactions was as follows: [1]

(1) $$\frac{\text{Intensity of } \beta\text{-emission parallel to neutron spin}}{\text{Intensity of } \beta\text{-emission antiparallel to neutron spin}} = 0.62 \mp 0.10$$

This ratio lies in the range of the "g. r."-value.

b. Various types of the hyperon-decay are also processes with an asymmetrical feature. A series of experiments was performed at some laboratories in order to investigate the distribution of emitted particles. Thereby it was found that upward emission prevailed over downward emission.

In the following table, some data of these experiments (colums 2, 3) and out of that computed values (columns 4, 5) are given:

Table 2

Experiment No.	The Emitted Particles — Number in the direction: down	up	Ratio of d/up (roughly)	Divergence of the Ratio of d/up From "g. r."-value by (roughly)
1 [2]	138	215	0.642	0.024
2 [3]	81	129	0.628	0.010
3 [4]	105	158	0.665	0.047

Finally it would appear that a Nobel Prize-winning English chemist and physicist F. W. Aston [5] probably was the first who showed the appearance of the Fibonacci numbers in the world of atoms. He observed that all the atoms with atomic number Z in the range 1 to 30 have the gaps representing the mass numbers of atoms which either entirely are non-existent in nature or too rare to be found. If one takes the recurring series 2, 3, 5, 8, 13, 21, 34, 55, ⋯ then the first 7 of these terms correspond to the missing mass numbers, but the relation breaks down at Mn^{55} and again at Y^{89}.

REFERENCES

1. Phys. Rev. 107, 1731 (1957)
2. Phys. Rev. 108, 1102 (1957)
3. Phys. Rev. 108, Footnote 1103 (1957)
4. Phys. Rev. 108, 1353 (1957)
5. F. W. Aston, Isotopes (second Edition, 1924)

SUBSCRIPTION FOR THE FIBONACCI QUARTERLY

The annual subscription price for the Fibonacci Quarterly is \$4.00 per year regardless of where the subscriber may live. Renewals and new subscriptions for 1964 are now being taken.

We realize, however, that most people like to have a complete set of a magazine and so we are going to make every effort to see that Volume 1 (1962) is available to those who want it. Again, the price for the entire volume (four issues) is \$4.00. (See inside front cover.)

SUSTAINING MEMBERS

The production of a mathematics magazine of any type on the basis of subscriptions alone is not easy of accomplishment. The financial problems involved are considerable. Thus, a supplementary income is a desideratum. The original charter members provided the impetus for the Quarterly by their contributions. Such help will be furnished on a continuing basis by means of SUSTAINING MEMBERS who will make an annual donation of \$10.00. In addition to receiving all the privileges of membership, these sustaining members will be given recognition in each issue of the Fibonacci Quarterly.

EDITORIAL ASSOCIATES

Apart from financial assistance, the Fibonacci Quarterly requires a contribution of ideas and talent for its continuance and development. Those who are active in writing Fibonacci articles for the Quarterly or other publications, as

(Continued on p. 80)

A PRIMER FOR THE FIBONACCI NUMBERS — PART IV

V. E. HOGGATT, JR. AND I. D. RUGGLES, SAN JOSE STATE COLLEGE

1. INTRODUCTION

In the primer, Part III, it was noted that if $V = (x,y)$ is a two-dimensional vector and A is a 2 by 2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $V' = AV$ is a two-dimensional vector, $V' = (x',y') = (ax + by, cx + dy)$. Here, V and consequently V', are expressed as column vectors. The matrix A is said to transform, or map, the vector V onto the vector V'. The matrix A is called the mapping matrix or transformation matrix.

2. SOME MAPPING MATRICES

The zero matrix, $Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, maps every vector V onto the zero vector $\phi = (0,0)$.

The identity matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ maps every vector V onto itself; that is, $IV = V$.

The matrix $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ maps vectors $V = (k,-k)$, (k any real number), onto the zero vector ϕ. Such a mapping as determined by B is called a many-to-one mapping.

If the only vector mapped onto ϕ is the vector ϕ itself, the mapping is a one-to-one mapping. A matrix A determines a one-to-one mapping of two-dimensional vectors onto two-dimensional vectors if, and only if, $\det A \neq 0$. If $\det A \neq 0$, for each vector U, there exists a vector V such that $AV = U$. Note, however, that for matrix B above, $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}$. There is no vector V such that $BV = (0,1)$.

3. GEOMETRIC INTERPRETATIONS OF 2x2 MATRICES AND 2-DIMENSIONAL VECTORS

As in Primer III, the vector $V = (x,y)$ is interpreted as a point in a rectangular coordinate system. Thus the geometric concepts of length, direction, slope and angle are associated with the vector V.

A non-zero scalar multiple of the identity matrix, kI, maps the vector $U = (a,b)$ onto the vector $V = (ka,kb)$. The length of V, $|V|$, is equal to $|k|\,|U|$. There is no change in slope but if $k < 0$ the sense or direction is reversed.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ maps a vector onto the reflection vector with respect to the line through the origin with slope one. Note that different vectors may be rotated through different angles!

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ preserves the first component of a vector while annihilating the second component. Every vector $U = (x,y)$ is mapped into a vector on the x-axis.

The matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates all vectors through the same angle θ (theta), in a counterclockwise direction if theta is a positive angle. There is no change in length. This seems to contradict the notion of a matrix having vectors whose slopes are not changed but in this case the characteristic values are complex; thus, there are no real characteristic vectors.

4. THE CHARACTERISTIC VECTORS OF THE Q-MATRIX

The Q matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ does not generally preserve the length of a vector $U = (x,y)$. Also, different vectors are in general rotated through different angles.

The characteristic equation of the Q matrix is

$$\lambda^2 - \lambda - 1 = 0$$

with roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \quad ,$$

which are the characteristic roots, or eigenvalues, for Q.

To solve for a pair of corresponding characteristic vectors consider

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}, \; x^2 + y^2 \neq 0 \quad .$$

Then

$$(1-\lambda)x + y = 0 \quad .$$

Thus, a pair of characteristic vectors are

$$X_1 = (\lambda_1 x, x) \quad , \; |X_1| \neq 0 \quad ,$$

with slope

$$m_1 = \frac{\sqrt{5}-1}{2} \quad \text{and} \quad X_2 = (\lambda_2 x, x), \; |X_2| \neq 0 \quad ,$$

with slope

$$m_2 = -\left(\frac{\sqrt{5}+1}{2}\right) \quad .$$

What happens when the matrix Q^2 is applied to the characteristic vectors X_1 and X_2 of matrix Q? Since

$$Q^2X_1 = Q(QX_1) = Q(\lambda X_1) = \lambda QX_1 = \lambda^2 X_1 \quad ,$$

clearly X_1 is a characteristic vector of the matrix Q^2 as well as a characteristic vector of matrix Q. The characteristic roots of Q^2 are the squares of the characteristic roots of matrix Q. In general if λ_1 and λ_2 are the characteristic roots of Q then λ_1^n and λ_2^n are the characteristic roots of Q^n. But the characteristic equation for Q^n is

$$\lambda^2 - (F_{n+1} + F_{n-1})\lambda + (F_{n+1}F_{n-1} - F_n^2) = 0 \quad .$$

Recalling that $L_n = F_{n+1} + F_{n-1}$, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, and $L_n^2 = 5F_n^2 + 4(-1)^n$, it follows that, since $\lambda_1 = \alpha = (1 + \sqrt{5})/2$ and $\lambda_2 = \beta = (1 - \sqrt{5})/2$,

$$\alpha^n = \lambda_1^n = (L_n + \sqrt{5}F_n)/2 \text{ and } \beta^n = \lambda_2^n = (L_n - \sqrt{5}F_n)/2 \quad .$$

5. FIBONACCI AND LUCAS VECTORS AND THE Q MATRIX

Let $U_n = (F_{n+1}, F_n)$ and $V_n = (L_{n+1}, L_n)$ be denoted as Fibonacci and Lucas vectors, respectively. We note

$$|U_n|^2 = F_{n+1}^2 + F_n^2 = F_{2n+1} \text{ and } |V_n|^2 = L_{n+1}^2 + L_n^2 = (5F_{n+1}^2 + (-1)^{n+1}4 + 5F_n^2 + (-1)^n 4) = 5(F_{n+1}^2 + F_n^2) = 5F_{2n+1}.$$

It is well known that the slopes of the vectors U_n and V_n (the ratios F_n/F_{n+1} and L_n/L_{n+1}) approach the slope, $(\sqrt{5} - 1)/2$, of the characteristic vector, X_1.

Since $Q^mQ^n = Q^{m+n}$, it is easy to verify that

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}$$

by equating elements in the upper left in the above matrix equation. In a similar manner it follows that

$$F_{m+1}F_{n+2} + F_mF_{n+1} = F_{m+n+2}$$
$$F_{m+1}F_n + F_mF_{n-1} = F_{m+n}$$

Adding these two equations and using $L_{n+1} = F_{n+2} + F_n$ it follows that

$$F_{m+1}L_{n+1} + F_mL_n = L_{m+n+1} \quad .$$

From the above identities it is easy to verify that

$$Q^{n+1}V_0 = QV_n = V_{n+1} ,$$
$$Q^{n+1}U_0 = QU_n = U_{n+1} ,$$
$$Q^nV_m = V_{m+n+1} ,$$
$$Q^nU_m = U_{m+n+1} .$$

6. A SPECIAL MATRIX

Let $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, then from

$$L_{n+1} = F_{n+1} + 2F_n , \quad L_n = 2F_{n+1} - F_n ,$$
$$5F_{n+1} = L_{n+1} + 2L_n , \quad 5F_n = 2L_{n+1} - L_n ,$$

it follows that

$$PU_n = (F_{n+1} + 2F_n,\ 2F_{n+1} - F_n) = V_n$$
$$PV_n = (L_{n+1} + 2L_n,\ 2L_{n+1} - L_n) = 5U_n$$

Also

$$PQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}$$

$$P^2Q^n = 5Q^n$$

$$D\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = D(P)\,D(Q^n) = 5(-1)^{n+1}$$

We now discuss two geometric properties of matrix P. Let $U = (x,y)$, $|U|^2 = x^2 + y^2 \neq 0$.

$$PU = (x + 2y,\ 2x - y) \quad |PU|^2 = 5(x^2 + y^2) = 5|U|^2$$

Thus matrix P magnifies each vector length by $\sqrt{5}$.

If $\tan\alpha = y/x$, we say $\alpha = \mathrm{Tan}^{-1}\, y/x$, read "$\alpha$ is an angle whose tangent is y/x." Let $\tan\alpha = y/x$ and $\tan\beta = (2x - y)/(x + 2y)$. From $\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$ we may now see what effect P has on the slope of vector $U = (x,y)$.

Now (recalling $x^2 + y^2 \neq 0$ says x and y are not both zero at the same time.)

$$\tan(\alpha + \beta) = \tan\left(\mathrm{Tan}^{-1}\frac{y}{x} + \mathrm{Tan}^{-1}\frac{2x - y}{x + 2y}\right) = \frac{2(x^2 + y^2)}{x^2 + y^2} .$$

Thus, since $x^2 + y^2 \neq 0$, then

$$\tan(\alpha + \beta) = 2 .$$

What does this mean? Consider two vectors A and B, the first inclined at an angle α with the positive x-axis and the second inclined at an angle β with the positive x-axis and the angles are measured positively in the counterclockwise direction. The angle bisector, ψ, of the angle between vectors A and B is such that $\alpha - \psi = \psi - \beta$ whether or not α is greater than β or the other way around. Solving for ψ yields

$$\psi = (\alpha + \beta)/2.$$

Thus ψ is the arithmetic average of α and β. Also we note that $\alpha + \beta = 2\psi$. The tangent of double the angle is given by

$$\tan 2\psi = (2\tan\psi)/(1 - \tan^2\psi) .$$

Let

$$\tan\psi = \frac{\sqrt{5} - 1}{2} ,$$

then it is an easy exercise in algebra to find $\tan 2\psi = 2$, but $\tan(\alpha + \beta) = 2$, therefore we would like to conclude that the angle bisector between vectors U and PU is precisely one whose slope is $(\sqrt{5} - 1)/2$, but this is the slope of X_1, the characteristic vector of Q. Can you show that X_1 is also a characteristic vector of P?

We have shown

Theorem 1. The matrix $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ maps a vector $U = (x,y)$ into a vector PU such that

(1) $$|P(U)| = \sqrt{5}\,|U|$$

and

(2) The angle bisector of the angle between the vector U and the vector PU is X_1, a characteristic vector of Q and P. Thus Matrix P reflects vector U across vector X_1.

Theorem 2. The vectors U_n and V_n are equally inclined to the vector X_1 whose slope is $(\sqrt{5} - 1)/2$.

Corollary. The vectors V_n are mapped into vectors $\sqrt{5}\, U_n$ by P and the vectors U_n are mapped into V_n by P.

7. SOME INTERESTING ANGLES

An interesting theorem is

Theorem 3.

$$\mathrm{Tan}\left\{\mathrm{Tan}^{-1} L_n/L_{n+1} - \mathrm{Tan}^{-1} \frac{L_{n+1}}{L_{n+2}}\right\} = \frac{(-1)^n}{F_{2n+2}}$$

$$\mathrm{Tan}\left\{\mathrm{Tan}^{-1} F_n/F_{n+1} - \mathrm{Tan}^{-1} F_{n+1}/F_{n+2}\right\} = \frac{(-1)^{n+1}}{F_{2n+2}}$$

Theorem 4.

$$\mathrm{Tan}^{-1} \frac{F_n}{F_{n+1}} = \sum_{m=1}^{n} (-1)^{m+1} \mathrm{Tan}^{-1} \frac{1}{F_{2m}} \quad .$$

We proceed by mathematical induction. For $n = 1$, it is easy to verify $\mathrm{Tan}^{-1} 1 = \mathrm{Tan}^{-1}(1/F_2)$.

Assume true for $n = k$, that is

$$\mathrm{Tan}^{-1} \frac{F_k}{F_{k+1}} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} \frac{1}{F_{2m}}$$

But, by Theorem 3,

$$\mathrm{Tan}^{-1} \frac{F_{k+1}}{F_{k+2}} = \mathrm{Tan}^{-1} \frac{F_k}{F_{k+1}} + \mathrm{Tan}^{-1} \frac{(-1)^k}{F_{2k+2}}$$

Thus, if

$$\mathrm{Tan}^{-1} \frac{F_k}{F_{k+1}} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} \frac{1}{F_{2m}}$$

then

$$\mathrm{Tan}^{-1} \frac{F_{k+1}}{F_{k+2}} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} \frac{1}{F_{2m}} + \mathrm{Tan}^{-1} \frac{(-1)^k}{F_{2k+2}}$$

$$= \sum_{m=1}^{k+1} (-1)^{m+1} \mathrm{Tan}^{-1} \frac{1}{F_{2m}}$$

because $\mathrm{Tan}^{-1}(-X) = -\mathrm{Tan}^{-1}X$ and $(-1)^k = (-1)^{k+2}$ and the proof is complete.

8. AN EXTENDED RESULT

Theorem 5. The series

$$A = \sum_{m=1}^{\infty} (-1)^{m+1} \operatorname{Tan}^{-1} \frac{1}{F_{2m}}$$

converges and $A = \operatorname{Tan}^{-1} (\sqrt{5} - 1)/2$.

Proof: Since the series is an alternating series, and, since $\operatorname{Tan}^{-1} X$ is a continuous increasing function, then

$$\operatorname{Tan}^{-1} \frac{1}{F_{2n}} > \operatorname{Tan}^{-1} \frac{1}{F_{2n+2}} \text{ and } \operatorname{Tan}^{-1} 0 = 0 .$$

The angle A must lie between the partial sums S_N and S_{N+1} for every $N > 2$ by the error bound in the alternating series, but $S_N = \operatorname{Tan}^{-1} (F_N / F_{N+1})$. Thus the angles of U_N and U_{N+1} lie on opposite sides of A. By the continuity of $\operatorname{Tan}^{-1} X$ then

$$\lim_{n\to\infty} \operatorname{Tan}^{-1} (F_n / F_{n+1}) = A = \operatorname{Tan}^{-1} (\sqrt{5} - 1)/2 .$$

Comment: The same result can be obtained simply from

$$\operatorname{Tan} \left\{ \operatorname{Tan}^{-1} \frac{F_n}{F_{n+1}} - \frac{\sqrt{5} - 1}{2} \right\} = (-1)^{n+1} \left(\frac{\sqrt{5} - 1}{2} \right)^{2n+1} .$$

Which slope gives a better numerical approximation to $\frac{\sqrt{5} - 1}{2}$, F_n / F_{n+1} or L_n / L_{n+1}? Hmmm?

REFERENCES FROM PAGE 60

1. Coxeter, H. S. M., Introduction to Geometry, New York: John Wiley and Sons, 1961.
2. Hambidge, J. Practical Applications of Dynamic Symmetry, New Haven: Yale Press, 1932.
3. Hunter, J. A. H. and Madachy, J. S., Mathematical Diversions, New Jersey: Van Nostrand Co., 1963.
4. Land, F., The Language of Mathematics, London: John Murray, 1960.
5. Thompson, D'Arcy W., On Growth and Form, New York: McMillan Co., 1944.

EXPLORING THE FIBONACCI REPRESENTATION OF INTEGERS

BROTHER U. ALFRED, ST. MARY'S COLLEGE, CALIFORNIA

Every integer may be represented as the sum of Fibonacci numbers or as a single such number. What is being considered in this investigation is the smallest number of different Fibonacci numbers required in the representation of an integer. For example, 125 is the sum of $89 + 34 + 2$. This seems to be the smallest number of Fibonacci numbers required to represent 125.

The following question is being proposed: Is it possible to set up an upper limit to this minimum number of Fibonacci numbers required to represent any integer? Possibly, there are many approaches to a solution, but one particular line of development will be indicated here.

We need first of all some notation. A well known symbol is the square bracket [] which means "the greatest integer in." Thus

$$[6.3] = 6; \qquad [5] = 5; \qquad [17/3] = 5 \quad .$$

Along with this we are going to introduce a similar notation to mean "the greatest Fibonacci number in." Thus

$$[63]^* = 55; \qquad [189/4]^* = 34; \qquad [13]^* = 13 \quad .$$

One way to solve the proposed question may be indicated by the following partially stated theorem:

<u>Theorem.</u> The maximum number of different Fibonacci numbers required to represent an integer N for which $[N]^* = F_n$ is given by [] . The answer in the bracket is some function of n. Explorers who find this result are encouraged to report their solution. In addition, there is a line of proofs that could be formulated to show that the theorem holds in general.

The above investigation will be reported in the April, 1964, issue of the Fibonacci Quarterly.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY S. L. BASIN, SYLVANIA ELECTRONIC SYSTEMS, MT. VIEW, CALIF.

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

B-24 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

It is evident that the determinant

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$

has a value of zero. Prove that if the same quantity k is added to each element of the above determinant, the value becomes $(-1)^{n-1}k$.

B-25 *Proposed by Brother U. Alfred.*

Find an expression for the general term(s) of the sequence $T_0 = 1, T_1 = a, T_2 = a, \cdots$ where

$$T_{2n} = \frac{T_{2n-1}}{T_{2n-2}} \quad \text{and} \quad T_{2n+1} = T_{2n}T_{2n-1} \quad .$$

B-26 *Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Given polynomials $b_n(x)$ and $B_n(x)$ defined by

$$b_0(x) = 1, \quad B_0(x) = 1$$

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x) \quad (n \geq 1)$$

$$B_n(x) = (x + 1)\, B_{n+1}(x) + b_{n-1}(x) \quad (n \geq 1)$$

show that

$$b_n(x) = P_{2n}(x)$$

and

$$B_n(x) = P_{2n+1}(x)$$

where

$$P_m(x) = \sum_{j=0}^{\left[\frac{m}{2}\right]} \binom{n-j}{j} x^{\left[\frac{m}{2}\right]-j}$$

$\left[\frac{m}{2}\right]$ being the greatest integer less than or equal to $\frac{m}{2}$.

B-27 *Proposed by D.C. Cross, Birmingham, England.*

Let $x = \cos\phi$, $[z]$ is the greatest integer contained in z.

$$\begin{aligned}
\cos\phi &= x \\
\cos 2\phi &= 2x^2 - 1 \\
\cos 3\phi &= 4x^3 - 3x \\
\cos 4\phi &= 8x^4 - 8x^2 + 1 \\
\cos 5\phi &= 16x^5 - 20x^3 + 5x \\
\cos 6\phi &= 32x^6 - 48x^4 + 18x^2 - 1 \\
\cos n\phi &= P_n(x) = \sum_{j=1}^{N} A_{jn} X^{n+2-2j} \quad (N = [(n+1)/2] \text{ is}
\end{aligned}$$

greatest integer function.)

Show

(i) $A_{1n} = 2^n$

(ii) $A_{j+1,n+1} = 2A_{j+1,n} - A_{j,n} \quad (j = 1, 2, \cdots N-1)$

(iii) $P_{n+2}(x) = 2x P_{n+1}(x) - P_n(x)$

(iv) If $A_n = \sum_{j=1}^{N} |A_{jn}|$, then $A_{n+2} = 2A_{n+1} + A_n$.

Note: $(A_1 = 1,\ A_2 = 3,\ 7 = A_3 = 2A_2 + A_1 = 2 \cdot 3 + 1)$.

B-28 *Proposed by Brother U. Alfred.*

Using the nine Fibonacci numbers F_2 to F_{10} (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

B-29 *Proposed by A.P. Boblett, U.S. Naval Ordnance Laboratory, Corona, California.*

Define a general Fibonacci sequence such that

$$F_1 = a; \quad F_2 = b; \quad F_n = F_{n-2} + F_{n-1} \quad n \geq 3$$
$$F_n = F_{n+2} - F_{n+1} \quad n \leq 0$$

Also define a characteristic number, C, for this sequence, where $C = (a + b)(a - b) + ab$.

Prove:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n C, \quad \text{for all } n \quad .$$

SOLUTIONS

Solutions to Problems B6 and B9 through B15, Vol. 1, No. 2, April, 1963

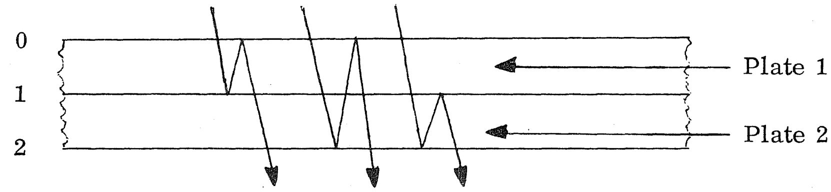
SOME REFLECTIONS

B-6 *Proposed by Leo Moser, University of Alberta, Edmonton, Alberta.*

Light rays fall upon a stack of two parallel plates of glass, one ray goes through without reflection, two rays (one from each interval interface opposing the ray) will be reflected once but in different ways, three will be reflected twice but in different ways. Show that the number of distinct paths, which are reflected exactly n times, is F_{n+2}.

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania

All rays which experience exactly n reflections will emerge from the same face, either top or bottom of the stack; furthermore, if those having n - 1 reflections emerge from the top face, then those having n reflections will emerge from the bottom face. Let us assume, without loss of generality that the rays having exactly n reflections will emerge from the bottom face as shown below for the case of two reflections.



Let α_n be the number of distinct paths which have exactly n reflections. If we consider any emergent ray which has had n reflections ($n \geq 2$), then it must have had its last, or n^{th} reflection from either face 0 or interface 1. The number of distinct paths having the n^{th} reflection at face 0 is equal to the number of distinct paths reaching face 0 after n - 1 reflections, or α_{n-1}. Similarly, the paths whose n^{th} reflection is at interface 1 must have had the (n - 1)th reflection at face 2, and the number of distinct paths is then equal to the number of distinct paths reaching face 2 after (n - 2) reflections, or α_{n-2}. Since the two possibilities are mutually exclusive and exhaustive, we have $\alpha_n = \alpha_{n-1} + \alpha_{n-2}$ for $n \geq 2$. The initial conditions, $\alpha_0 = 1$, $\alpha_1 = 2$ establish that $\alpha_n = F_{n+2}$ for $n \geq 0$.

FIBONACCI SUMS

B-9 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, N.J.*

Prove

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2 \quad ,$$

where F_n is the n^{th} Fibonacci number.

B-9 Solution by Francis D. Parker, University of Alaska.

Since

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$$

then

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\left[\frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right] = \left[\frac{1}{1\cdot 1} - \frac{1}{1\cdot 2}\right] + \left[\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right]$$

$$+ \left[\frac{1}{2\cdot 3} - \frac{1}{3\cdot 5}\right] + \cdots = 1$$

Similarly,

$$\frac{F_n}{F_{n-1}F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \quad \text{and}$$

and

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{5}\right] + \left[\frac{1}{3} - \frac{1}{8}\right] + \cdots = \frac{1}{F_1} + \frac{1}{F_2} = 2$$

Editorial Comment: The above solution to problem B-9 is a good example of a principle found in many other problems in number theory, namely in forming a sum, it is often helpful to judiciously group the terms in a certain fashion. An example of this may be found in proving the following theorem concerning the divisor function $\tau(n)$. Prove $\tau(n)$ is odd if and only if n is a square.

LUCAS-FIBONACCI IDENTITY

B-10 *Proposed by Stephen Fisk, San Francisco, California.*

Prove the "de Moivre-type" identity,

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = \frac{L_{np} + \sqrt{5}\,F_{np}}{2}$$

where L_n denotes the nth Lucas number and F_n denotes the nth Fibonacci number.

B-10 Solution by Charles Wall, Ft. Worth, Texas.

Since

$$\frac{L_n + \sqrt{5}\,F_n}{2} = \frac{\alpha^n + \beta^n + \alpha^n - \beta^n}{2} ,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} ,$$

we have

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = \alpha^{np} = \frac{\alpha^{np} + \beta^{np} + \alpha^{np} - \beta^{np}}{2} = \frac{L_{np} + \sqrt{5}\,F_{np}}{2} .$$

B-11 *Proposed by S.L.Basin, Sylvania Electronic Defense Laboratory,*

Show that the hypergeometric function

$$G(x,n) = \sum_{k=0}^{n-1} \frac{2^k (n+k)!\,(x-1)^k}{(n-k-1)!\,(2k+1)!}$$

generates the sequence

$$G\left(\frac{3}{2}, n\right) = F_{2n}, \qquad n = 1,2,3,\cdots .$$

B-11 Solution by S. L. Basin, Sylvania Electronic Systems, Mountain View, California and San Jose State College

$$\sum_{k=0}^{n-1} \frac{2^k (n+k)!\,(x-1)^k}{(n-k-1)!\,(2k+1)!} = U_{n-1}(x)$$

where $U_{n-1}(x)$ are the Chebyshev polynomials of the second kind and

$$U_{n-1}(x) = \frac{1}{2\sqrt{x^2 - 1}} \left\{(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n\right\}$$

$$U_{n-1}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{5}} \left\{ \left(\frac{3 + \sqrt{5}}{2}\right)^n - \left(\frac{3 - \sqrt{5}}{2}\right)^n \right\}$$

Observing that

$$\left(\frac{3 + \sqrt{5}}{2}\right) = \left(\frac{1 + \sqrt{5}}{2}\right)^2 \text{ and } \left(\frac{3 - \sqrt{5}}{2}\right) = \left(\frac{1 - \sqrt{5}}{2}\right)^2 ,$$

we have

$$U_{n-1}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{2n} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n}\right\} = F_{2n} .$$

Comment: Setting $x = 3/2$, the summation becomes

$$\sum_{k=0}^{n-1} \frac{(n + k)!}{(2k + 1)!\ (n - k - 1)!} = \sum_{k=0}^{n-1} \binom{n + k}{2k + 1} = F_{2n} \quad \left(\begin{array}{c}\text{Rising diagonals}\\ \text{of Pascal's}\\ \text{triangle}\end{array}\right)$$

See Fig. 1, page 24, October, 1963, Fibonacci Quarterly.

A LUCAS DETERMINANT

B-12 *Proposed by Paul F. Byrd, San Jose State College, San Jose, Calif.*

Show that

$$L_{n+1} = \begin{vmatrix} 3 & i & 0 & 0 & \cdots & 0 & 0 \\ i & 1 & i & 0 & \cdots & 0 & 0 \\ 0 & i & 1 & i & \cdots & 0 & 0 \\ 0 & 0 & i & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & i \\ 0 & 0 & 0 & 0 & \cdots & i & 1 \end{vmatrix}_n \qquad n \ge 1$$

where L_n is the nth Lucas number given by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, and $i = \sqrt{-1}$.

B-12 Solution by Marjorie Bicknell, San Jose State College, San Jose, Calif.

Let D_n denote the determinant of order n. Expanding the determinant by its nth row we have, $D_n = D_{n-1} + D_{n-2}$ with $D_1 = 3$, $D_2 = 4$ so that $D_n = L_{n+1}$.

Also solved by William A. Beyer, Los Alamos, New Mexico

FIBONACCI CONTINUANT

B-13 *Proposed by S.L. Basin.*

Determinants of order n which are of the form,

$$K_n(b,c,a) = \begin{vmatrix} c & a & 0 & 0 & 0 & \cdots \\ b & c & a & 0 & 0 & \cdots \\ 0 & b & c & a & 0 & \cdots \\ 0 & 0 & b & c & a & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

are known as CONTINUANTS

Prove that

$$K_n(b,c,a) = \frac{(c + \sqrt{c^2 - 4ab})^{n+1} - (c - \sqrt{c^2 - 4ab})^{n+1}}{2^{n+1}\sqrt{c^2 - 4ab}}$$

and show, for special values of a, b, and c, that $K_n(b,c,a) = F_{n+1}$.

B-13 Solution by Marjorie Bicknell, San Jose State College, San Jose, Calif.

Expanding $K_n(b,c,a)$ by the nth row we obtain,

(1) $$K_n(b,c,a) = cK_{n-1}(b,c,a) - ab\,K_{n-2}(b,c,a) \quad .$$

If u and v are the roots of the quadratic equation $x^2 - cx + ab = 0$, then

(2) $$u = \frac{1}{2}(c + \sqrt{c^2 - 4ab}), \quad v = \frac{1}{2}(c - \sqrt{c^2 - 4ab})$$

Now $K_n(b,c,a) = (u^{n+1} - v^{n+1})/(u - v)$ by induction and $K_n(b,c,a) = F_{n+1}$ for values of a, b, and c which yield the quadratic $x^2 - x - 1$, i.e., $a = c = 1$, and $b = -1$; $a = -1$ and $b = c = 1$; $a = b = i = \sqrt{-1}$ and $c = 1$.

A LITTLE SURPRISE

B-14 *Proposed by Maxey Brooke, Sweeny, Texas and C.R. Wall, Ft. Worth, Tex*

Show that

$$\sum_{n=1}^{\infty} \frac{F_n}{10^n} = \frac{10}{89} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n}{10^n} = \frac{10}{109} \quad .$$

B-14 Solution by Charles Wall, Ft. Worth, Texas

Since
$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

then

$$\sum_{n=1}^{\infty} F_n(.1)^n = \frac{.10}{1 - .10 - .01} = \frac{.10}{.89} = \frac{10}{89}$$

and

$$\sum_{n=1}^{\infty} [-F_n(-.1)^n] = \frac{-(-.10)}{1 + .10 - .01} = \frac{.10}{1.09} = \frac{10}{109} .$$

Also solved by Dermott A. Breault, Sylvania, ARL, Waltham, Mass.

FIBONACCI SEQUENCE PERIODS

B-15 *Proposed by R.B.Wallace, Beverly Hills, Calif., and Stephen Geller, University of Alaska, College, Alaska.*

If p_k is the smallest positive integer such that

$$F_{n+p_k} \equiv F_n \bmod (10^k)$$

for all positive n, then p_k is called the period of the Fibonacci sequence relative to 10^k. Show that p_k exists for each k, and find a specific formula for p_k as a function of k.

Editorial Comment: This problem is discussed in this issue in a paper by Dov Jarden which is a reply to Stephen Geller's letter to the editor, p. 84, April, 1963, Fibonacci Quarterly.

EDITORIAL ASSOCIATES (Cont.)

well as those who have the intention of doing so, will receive recognition as Editorial Associates. The Editor should be contacted by anyone who wishes to be associated with the Fibonacci Quarterly in this manner.

THE FIBONACCI QUARTERLY

Official Publication of

THE FIBONACCI ASSOCIATION

INDEX TO VOLUME 1

Numbers 1, 2, 3 and 4

CONTENTS

THE FIBONACCI QUARTERLY

1963 FIBONACCI BIBLIOGRAPHY

Brother U. Alfred, Additional Factors of the Fibonacci and Lucas Sequences, FQ, Feb., 1963, pp. 34-42

Dying Rabbit Problem Revived, FQ, Dec., 1963, pp. 53-56

Exploring Fibonacci Numbers (Dying rabbit problem), FQ, Feb., 1963, pp. 57-63

Exploring Recurrent Sequences, FQ, April, 1963, pp. 81-83

Exploring Fibonacci Polygons, FQ, Oct., 1963, p. 60

Exploring the Fibonacci Representation of Integers, FQ, Dec., 1963, p. 72

On the Form of Primitive Factors of Fibonacci Numbers, FQ, Feb., 1963, pp. 43-45

On the Ordering of the Fibonacci Sequences, FQ, Dec., 1963, pp. 43-46

Periodic Properties of Fibonacci Summations, FQ, Oct., 1963, pp. 33-42

Problems Proposed

- H-6, FQ, Feb., 1963, p. 47
- H-8, FQ, Feb., 1963, p. 48
- H-17, FQ, April, 1963, p. 55
- B-22, FQ, Oct., 1963, p. 76
- B-24, FQ, Dec., 1963, p. 73
- B-25, FQ, Dec., 1963, p. 73
- B-28, FQ, Dec., 1963, p. 74-75

Problem solution: H-6, FQ, Oct., 1963, p. 50
Research Project: Fibonacci Nim, FQ, Feb., 1963, p. 63

A. Asimov: "T-Formation", The Magazine of Fantasy and Science Fiction, August, 1963, pp. 45-56. A popularized science article about huge integers which define the Fibonacci numbers and display F_{571} explicitly.

S. L. Basin and V. E. Hoggatt, Jr., A Primer on the Fibonacci Sequence, Part I, FQ, Feb., 1963, pp. 65-72
Same authors: A Primer on the Fibonacci Sequence — Part II, FQ, April, 1963, pp. 61-68

S. L. Basin, Elementary Problems and Solutions, FQ, Feb., 1963, pp. 73-75
Elementary Problems and Solutions, FQ, April, 1963, pp. 85-87
Elementary Problems and Solutions, FQ, Oct., 1963, pp. 75-80
Elementary Problems and Solutions, FQ, Dec., 1963, pp. 73-80
Generalized Fibonacci Sequences and Squared Rectangles, American Mathematical Monthly, April, 1963, pp. 372-379
Problems Proposed
B-4, FQ, Feb., 1963, p. 74
B-11, FQ, April, 1963, p. 85
B-13, FQ, April, 1963, p. 86
B-23, FQ, Oct., 1963, p. 76
B-26, Dec., 1963, pp. 73-74
Problem solution: B-11, FQ, Dec., 1963, p. 77-78

S. L. Basin and V. E. Hoggatt, Jr., Representations by Complete Sequences — Part I (Fibonacci), FQ, Oct., 1963, pp. 1-14

S. L. Basin, The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory, Mathematics Magazine, March, 1963, pp. 84-97

S. L. Basin, The Fibonacci Sequence as it Appears in Nature, FQ, Feb., 1963, pp. 53-56

Marjorie Bicknell and Verner E. Hoggatt, Jr., Fibonacci Matrices and Lambda Functions, FQ, April, 1963, pp. 47-52

Marjorie Bicknell, Problem proposed: B-16, FQ, April, 1963, p. 87
Problem Solutions
B-1, FQ, Oct., 1963, pp. 76-77
B-3, FQ, Oct., 1963, p. 78
B-12, FQ, Dec., 1963, p. 78
B-13, FQ, Dec., 1963, p. 79

A. P. Boblett, Problem Proposed, B-29, FQ, Dec., 1963, p. 75

Terry Brennan, Problem Proposed, H-5, FQ, Feb., 1963, p. 47

Wm. E. Briggs, Problem Solution, B-7, FQ, Oct., 1963, p. 80

Louis G. Brokling, Problem Proposed, B-20, FQ, Oct., 1963, p. 75

Maxey Brooke, Fibonacci Formulas, FQ, April, 1963, p. 60
Problem Proposed, B-14, FQ, April, 1963, p. 86
Problem Solution, H-14, FQ, Dec., 1963, p. 51

J. L. Brown, Jr., A Generalization of Semi-Completeness for Integer Sequences, FQ, Feb., 1963, pp. 3-15
Problems Proposed
H-11, FQ, April, 1963, p. 53
B-18, FQ, Oct., 1963, p. 75
Problem Solutions
H-4, FQ, Oct., 1963, pp. 47-48
H-5, FQ, Oct., 1963, pp. 49-50
H-7, FQ, Oct., 1963, pp. 50-51
B-2, FQ, Oct., 1963, pp. 77-78
B-4, FQ, Oct., 1963, p. 79
B-5, FQ, Oct., 1963, p. 79
B-8, FQ, Oct., 1963, p. 80
B-6, FQ, Dec., 1963, pp. 75-76

R. G. Buschman, Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations, FQ, Dec., 1963, pp. 1-7
Problem Proposed, H-18, FQ, April, 1963, p. 55

Paul F. Byrd, Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers, FQ, Feb., 1963, pp. 16-28

Problem Proposed, B-12, FQ, April, 1963, p. 86

L. Carlitz, A Fibonacci Array, FQ, April, 1963, pp. 17-27

Generating Functions for Powers of Certain Sequences, of Numbers, Duke Mathematics Journal, 29 (1962), 521-37. Mathematical Reviews, May, 1963, #3896

Problems Proposed

H-2, FQ, Feb., 1963, p. 46

B-19, FQ, Oct., 1963, p. 75

B-21, FQ, Oct., 1963, p. 76

H-26, FQ, Dec., 1963, p. 47-48

L. Carlitz, Problem Solution, H-10, FQ, Dec., 1963, p. 49

D. C. Cross, Problem Proposed, B-27, FQ, Dec., 1963 p. 74

Zvi Dresner, Problem Solution, H-16, FQ, Dec., 1963, pp. 51-52

Editor(s)

Generating Fibonacci Series on a Desk Calculator, FQ, April, 1963, pp. 56-88

Fibonacci Century Mark Reached, FQ, Feb., 1963, p. 45 (credit to John Brillhart)

Charter Member, Morgan Ward, Passes Away, FQ, Oct., 1963, p. 32

Joseph Ehrbacher, Problem Proposed, H-25, FQ, Dec., 1963, p. 47

Problem Solutions

H-8, FQ, Oct., 1963, pp. 51-52

B-4, FQ, Oct., 1963, p. 79

Mark Feinberg, Fibonacci Tribonacci, FQ, Oct., 1963, pp. 71-74

Stephen Fisk, Problem Proposed, B-10, FQ, April, 1963, p. 85

John A. Fuchs, Proposed Problem, H-25, FQ, Dec., 1963, p. 47

Problem Solution, H-8, FQ, Oct., 1963, pp. 51-52

M. Gardner, mention of Fibonacci Quarterly, Scientific American, June, 1963, p. 152

Stephen P. Geller, A Computer Investigation of a Property of the Fibonacci Sequence, FQ, April, 1963, p. 84

Ben K. Gold, Review of the Fibonacci Quarterly, Mathematics Magazine, Sept.-Oct., 1963, pp. 260-261

H. W. Gould, Fibonacci Exponentials and Generalizations of Hermite Polynomials, FQ, Dec., 1963, pp. 31-34

Generating Functions for Products of Powers of Fibonacci Numbers, FQ, April, 1963, pp. 1-16

Operational Recurrences Involving Fibonacci Numbers, FQ, Feb., 1963, pp. 30-33

Proposed Problems

H-1, FQ, Feb., 1963, p. 46

B-7, FQ, Feb., 1963, p. 74

H-13, FQ, April, 1963, p. 54

H-16, FQ, April, 1963, p. 54

H-28, FQ, Dec., 1963, p. 48

R. L. Graham, Problems Proposed, H-10, FQ, April, 1963, p. 53

B-9, FQ, April, 1963, p. 85

A. P. Hillman, M. T. Stroot, and R. M. Grassl, Fibonacci Numbers and Zigzag Hasse Diagrams, Fibonacci Quarterly, Oct., 1963, pp. 43-45

Verner E. Hoggatt, Jr., Advanced Problems and Solutions, FQ, Feb., 1963, pp. 46-48

Advanced Problems and Solutions, FQ, April, 1963, pp. 53-55

Advanced Problems and Solutions, FQ, Oct., 1963, pp. 46-52

Advanced Problems and Solutions, FQ, Dec., 1963, pp. 47-52

V. E. Hoggatt, Jr., and S. L. Basin, A Primer on the Fibonacci Sequence, Part I, FQ, Feb., 1963, pp. 65-72

Same authors, A Primer on the Fibonacci Sequence, Part II, FQ, April, 1963, pp. 61-68

Verner E. Hoggatt, Jr., and I. D. Ruggles, A Primer for the Fibonacci Sequence, Part III, FQ, Oct., 1963, pp. 61-65

Same authors, A Primer for the Fibonacci Numbers — Part IV, FQ, Dec., 1963, pp. 65-71

Verner E. Hoggatt, Jr., and Marjorie Bicknell, Fibonacci Matrices and Lambda Functions, FQ, April, 1963, pp. 47-52

Verner E. Hoggatt, Jr., Problems Proposed
H-3, FQ, Feb., 1963, p. 46
H-7, FQ, Feb., 1963, p. 47
B-2, FQ, Feb., 1963, p. 73
H-13, FQ, April, 1963, p. 54
H-20, FQ, Oct., 1963, p. 46 and Dec., 1963, p. 52
H-22, FQ, Oct., 1963, p. 47

Verner E. Hoggatt, Jr., and S. L. Basin, Representations by Complete Sequences — Part I (Fibonacci), FQ, Oct., 1963, pp. 1-14

A. F, Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, reviewed in Mathematical Reviews, Oct., 1963, #3663
Complex Fibonacci Numbers and Fibonacci Quaternions, American Mathematical Monthly, March, 1963, pp. 289-291
Further Appearance of the Fibonacci Sequence, FQ, Dec., 1963, pp. 41-42, 46

J. A. H. Hunter, Triangle Inscribed in Rectangle, FQ, Oct., 1963, p. 66

J. E. Householder, Problem Proposed, B-3, FQ, Feb., 1963, p. 73

Vladimir Ivanoff, Problem Proposed, B-4, FQ, Feb., 1963, p. 74

Dov Jarden, On the Greatest Primitiv Divisors of Fibonacci and Lucas Numbers with Prime-Power Subscripts, FQ, Oct., 1963, pp. 15-20
On the Periodicity of the Last Digits of the Fibonacci Numbers, FQ, Dec., 1963, pp. 21-22

James A. Jeske, Linear Recurrence Relations — Part I, FQ, April, 1963, pp. 69-74
Linear Recurrence Relations — Part II, FQ, Dec., 1963, p. 35-39

Charles King, Leonardo Fibonacci, FQ, Dec., 1963, pp. 15-19
Problem Proposed, H-20, FQ, Oct., 1963, p. 46 and Dec. 1963, p. 52

J. A. Maxwell, Problem Proposed, B-8, FQ, Feb., 1963, p. 75

Sister Mary de Sales McNabb, Phyllotaxis, FQ, Dec., 1963, pp. 57-60

Lucile Morton, Problem Solution, H-11, FQ, Dec., 1963, pp. 49-50

L. Moser, Problems Proposed, H-2, FQ, Feb., 1963, p. 46
B-5, FQ, Feb., 1963, p. 74
B-6, FQ, Feb., 1963, p. 74

F. D. Parker, Problems Proposed, H-14, FQ, April, 1963, p. 54
H-21, FQ, Oct., 1963, p. 46
H-24, FQ, Dec., 1963, p. 47

F. D. Parker, Problem Solutions, H-3, FQ, Oct., 1963, p. 47
B-9, FQ, Dec., 1963, pp. 76-77

Joseph A. Raab, A Generalization of the Connection between the Fibonacci Sequence and Pascal's Triangle, FQ, Oct., 1963, pp. 21-31

D. W. Robinson, The Fibonacci Matrix Modulo m, FQ, April, 1963, pp. 29-36

I. D. Ruggles and Verner E. Hoggatt, Jr., A Primer for the Fibonacci Sequence — Part III, FQ, Oct., 1963, pp.61-65

Same authors, A Primer for the Fibonacci Numbers — Part IV, FQ, Dec., 1963, pp. 65-71

I. D. Ruggles, Problems Proposed, H-4, FQ, Feb., 1963, p. 47
B-1, FQ, Feb., 1963, p. 73
Some Fibonacci Results Using Fibonacci-type Sequences, FQ, April, 1963, pp. 75-80

Ken Siler, Fibonacci Summations, FQ, Oct., 1963, pp. 67-69

A. P. Hillman, M. T. Stroot, and R. M. Grassl, Fibonacci Numbers and Zigzag Hasse Diagrams, FQ, Oct., 1963, pp. 43-45

Malcolm H. Tallman, Problems Proposed, H-15, April, 1963, p. 54
H-23, FQ, Oct., 1963, p. 47
Problem Solution, H-12, FQ, Dec., 1963, p. 50

Olga Taussky, Problem Proposed, H-9, FQ, Dec., 1963, p. 48
Problem Solution, H-9, FQ, Dec., 1963, p. 48

D. E. Thoro, Beginners' Corner, The Golden Ratio: Computational Considerations, FQ, Oct., 1963, pp. 53-59
Reciprocals of Generalized Fibonacci Numbers, FQ, Dec., 1963, p. 30
Problem Proposed, H-12, FQ, April, 1963, p. 54
Regula Falsi and the Fibonacci Numbers, American Mathematical Monthly, Oct., 1963, p. 869

D. E. Thoro, Beginners' Corner, Divisibility I, Feb., 1963, pp. 49-52, 64

-----Beginners' Corner, Divisibility II, April 1963, pp. 57-60

Charles W. Trigg, Review of the Fibonacci Quarterly Journal, Recreational Mathematics Magazine, Feb., 1963, pp. 23-24

Harlan L. Umansky, Proposed Problem, H-27, FQ, Dec., 1963, p. 48

John Vinson, The Relation of the Period Modulo to the Rank of Apparition of m in the Fibonacci Sequence, FQ, April, 1963, pp. 37-45

N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Co., 1961. Mathematical Reviews, Oct., 1963, #3664.

Charles R. Wall, Problems Proposed, H-19, FQ, Oct., 1963, p. 46
B-17, FQ, Oct., 1963, p. 75
Problem Solutions, B-10, FQ, Dec., 1963, p. 77
B-14, FQ, Dec., 1963, pp. 79-80
Some Remarks on Carlitz' Fibonacci Array, FQ, Dec., 1963, p. 47

R. B. Wallace, Problem Proposed, B-15, FQ, April, 1963, p. 87

Morgan Ward, Proposed Problem, H-24, FQ, Dec., 1963, p. 47

Morgan Ward, The Prime Divisors of Fibonacci Numbers, Pacific Journal of Mathematics, 11 (1961), pp. 379-386. Mathematical Reviews, March, 1963, #2029

Michael J. Whinihan, Fibonacci Nim, FQ, Dec., 1963, pp. 9-13

Wlodarski, J., The "Golden Ratio" and the Fibonacci Numbers in the World of Atoms, FQ, Dec., 1963, pp. 61-63

Marvin Wunderlich, On the Non-Existence of Fibonacci Squares, Mathematics of Computation, Oct., 1963, pp. 455-457

Same, Mathematical Reviews, April, 1963, #3579

M. Wyman, Problem Proposed, B-6, FQ, Feb., 1963, p. 74

David Zeitlin, On Identities for Fibonacci Numbers, American Mathematical Monthly, Nov., 1963, pp. 987-991

Problem Proposed, H-14, FQ, April, 1963, p. 54

Charles Ziegenfus, Twin Primes, FQ, Oct., 1963, p. 42

1963 FIBONACCI INDEX

Volume 1, Number 1, February 1963

Volume 1, Number 2, April 1963

Volume 1, Number 3, October 1963

Volume 1, Number 4, December 1963