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*DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES*



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# FIBONACCI REPRESENTATIONS

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## 1. INTRODUCTION

We define the Fibonacci numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

It is known that every positive integer $N$ can be written in the form

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 2 \tag{1.2}$$

and $r$ depends on $N$. We call (1.1) a Fibonacci representation of N. Moreover by the theorem of Zeckendorf, the representation (1.1) is unique provided the $k_j$ satisfy the inequalities

$$k_j - k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, r-1); \qquad k_r \geq 2. \tag{1.3}$$

Such a representation may be called the canonical representation of N.

Now let $A_k$ denote the set of positive integers $\{N\}$ for which $k_r = k$. Then it is clear that the

$$A_k \qquad (k = 2, 3, 4, \cdots)$$

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constitute a partition of the set of positive integers. The chief object of the present paper is to describe the numbers in $A_k$ in terms of the greatest integer function. We shall show that

$$(1.4) \qquad A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots),$$

$$(1.5) \qquad A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots),$$

where

$$(1.6) \qquad a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n], \qquad \alpha = (1 + \sqrt{5})/2$$

and $[x]$ denotes the greatest integer $\leq x$. As is customary, powers and juxtaposition of functions should be interpreted as composition.

Moreover, we shall show that

$$\begin{aligned} A(2t, \overline{2t+2}) &= \{ab^{t-1}a^2(n) \mid n = 1, 2, 3, \cdots\} \\ A(2t, 2t+2) &= \{ab^{t-1}ab(n) \mid n = 1, 2, 3, \cdots\} \\ A(2t+1, \overline{2t+3}) &= \{b^t a^2(n) \mid n = 1, 2, 3, \cdots\} \\ A(2t+1, 2t+3) &= \{b^t ab(n) \mid n = 1, 2, 3, \cdots\}, \end{aligned}$$

where $A(s, s+2)$ denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_{s+2} + F_s ,$$

while $A(s, \overline{s+2})$ denotes the set with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_s \qquad (k_r > s + 2).$$

Using any Fibonacci representation of $N$

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

we define

(1.7) $$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1} .$$

The fact that $e(N)$ is independent of the Fibonacci representation chosen for $N$ was proved in [2].

The following theorems, which will be used in Section 4, were also established in [2].

Theorem 1. For every $N$, $e(N+1) \geq e(N)$ with equality if and only if $N$ is in $A_2$. (See [2], p. 216, Theorem 5 and proof.)

Theorem 2. If $N$ is in $A_2$ then neither $N-1$ nor $N+1$ is in $A_2$. (See [2], p. 217, comments following Theorem 5.)

## 2. THE ARRAY R

As in [3] we form the 3-rowed array $R$ as follows: In the first row we put the positive integers in natural order. We begin the second row with 1. To get an entry of the third row, we add the entries appearing above it in the first and second rows. We get further entries in the second row by choosing the smallest integer which has not appeared so far in the second or third rows.

(2.1)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
| 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 | ... |
| 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | 26 | ... |

Note that $R$ is uniquely determined by the following properties:

(2.2) Every positive integer appears exactly once in row 2 or row 3.

(2.3) Each row is a monotone sequence.

(2.4) The sum of the first two rows is the third row.

Now also consider the 3-rowed array $R'$.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | $\cdots$ |
| a(1) | a(2) | a(3) | a(4) | $\cdots$ |
| b(1) | b(2) | b(3) | b(4) | $\cdots$ |

where $a(n)$, $b(n)$ are defined by (1.6). Since $\alpha + 1 = \alpha^2$, properties (2.3) and (2.4) are obviously satisfied by $R'$. To see that every number appears in $R'$, let $N \geq 2$ be arbitrary. We will show that either $a([N/\alpha])$ or $b([N/\alpha^2])$ is $N - 1$. Suppose not. Then they are both too small; that is,

$$\alpha[N/\alpha] < N - 1 \quad .$$

and

$$\alpha^2[N/\alpha^2] < N - 1 .$$

Dividing the first inequality by $\alpha$, the second by $\alpha^2$, remembering that

$$\frac{1}{\alpha} + \frac{1}{\alpha^2} = 1 ,$$

and adding, we get

$$[N/\alpha] + [N/\alpha^2] < N - 1 .$$

But this is a contradiction since $N/\alpha + N/\alpha^2 = N$.

Now to see that the ranges of $a$ and $b$ are disjoint, suppose for some numbers $N$, $M$ and $P$, we had $a(N) = b(M) = P$. Then

$$\alpha N - 1 < P < \alpha N$$

and

$$\alpha^2 M - 1 < P < \alpha^2 M .$$

Again dividing and adding, we get

$$N + M - 1 < P < N + M ,$$

a contradiction. The fact that no number appears twice in the same row follows simply because both $\alpha$ and $\alpha^2$ are greater than 1. Note that (2.2) was proved using only the fact that $\alpha$ and $\alpha^2$ are irrational and

$$\frac{1}{\alpha} + \frac{1}{\alpha^2} = 1 .$$

The result is not new, of course.

We have established that $R = R'$.

## 3. SOME PROPERTIES OF a(n) AND b(n)

In this section we prove several equalities involving the functions $a(n)$ and $b(n)$. In our proof we use only the properties (2.2), (2.3) and (2.4) of $R(=R')$ from Section 2. Of course, the equalities could, with much more effort, be proved from the definitions (1.6).

$$N + a(N) = b(N) \tag{3.1}$$

$$b(N) = a(a(N)) + 1 \tag{3.2}$$

$$a(N) + b(N) = b(a(N)) + 1 \tag{3.3}$$

$$a(b(N)) = b(a(N)) + 1 \tag{3.4}$$

$$a(N) + b(N) = a(b(N)) \tag{3.5}$$

$$b^2(N) = aba(N) + 2 \tag{3.6}$$

$$ab^2(N) = b^2a(N) + 3 \tag{3.7}$$

$$(3.8) \qquad b^r(N) = ab^{r-1}a(N) + F_{2r-1} \qquad (r = 1, 2, \cdots)$$

$$(3.9) \qquad ab^r(N) = b^r a(N) + F_{2r} \qquad (r = 1, 2, \cdots)$$

$$(3.10) \qquad b^r(1) = F_{2r+1} \qquad (r = 1, 2, \cdots) .$$

Proof. Equation (3.1) is (2.4). For (3.2), note that in $R$, in the third row, to $b(N)$, or the second row to $a(J) = b(N) - 1$, occur all the numbers $1, 2, \cdots, b(N)$. Hence $J + N = b(N)$. Therefore, by (3.1) $J = a(N)$; that is, $a(a(N)) = b(N) - 1$. Equation (3.3) comes from (3.1) and (3.2). To prove (3.4), note that $b(a(N))$ is the $a(N)^{th}$ entry in the third row of $R$, and $a(b(N))$ is the $b(N)^{th}$ entry in the second row. Then the total number of entries is $a(N) + b(N) = b(a(N)) + 1$. Hence $b(a(N))$ cannot be the largest so $a(b(N))$ must be and every integer $\leq b(a(N)) + 1$ must have appeared. Hence $a(b(N)) = b(a(N)) + 1$. Equation (3.5) is obvious from (3.3) and (3.4). Equation (3.6) is obtained by adding (3.2) and (3.4) and using (3.1) and (3.5). Similarly we get (3.7) by adding (3.4) and (3.6). Equations (3.8) and (3.9) arise by induction. If we set $N = 1$ in (3.8) we get

$$b(b^{r-1}(1)) = a(b^{r-1}(1)) + F_{2r-1} ,$$

so, by (3.1),

$$b^{r-1}(1) = F_{2r-1} .$$

## 4. THE SETS $A_k$

We begin with some preliminary theorems.

Theorem 3. If $N$ is in $A_2$, then $N + 1$ is in $A_k$ with $k$ odd.

Proof. By Theorem 2,

$$(4.1) \qquad N + 1 = F_{k_r} + F_{k_{r-1}} + \cdots + F_{k_1} \qquad k_r > 2 .$$

For convenience we let

$$N' = F_{k_{r-1}} + \cdots + F_{k_1} .$$

Then

$$N + 1 = F_{k_r} + N' = F_{k_r-2} + F_{k_r-1} + N'$$
$$= F_{k_r-4} + F_{k_r-3} + F_{k_r-1} + N' .$$

Continuing, we see that $N + 1$ is either

$$F_3 + F_4 + F_6 + \cdots + F_{k_r-1} + N'$$

or

$$F_2 + F_3 + F_5 + \cdots + F_{k_r-1} + N' .$$

If the latter, $N$ would be in $A_3$. Hence

$$N = F_2 + F_4 + \cdots + F_{k_r-1} + N'$$

and $k_r$ is odd.

Theorem 4. If $N$ and $M$ are in $A_2$ and $e(e(N)) = e(e(M))$, then $N = M$.

Proof. Suppose $N \neq M$. If $e(N) = e(M)$ then by Theorem 1, $N$ and $M$ are consecutive integers and by Theorem 2 could not both be in $A_2$. So suppose $e(N) < e(M)$. Then by Theorem 1, $e(N)$ is in $A_2$ and $e(M) = e(N) + 1$. Hence by Theorem 3, $e(M)$ is in $A_{k_r}$ with $k_r$ odd:

$$e(M) = F_{k_r} + F_{k_{r-1}} + \cdots \qquad (k_r \text{ odd}) .$$

Let

$$P = F_{k_r+1} + F_{k_{r-1}+1} + \cdots .$$

Now $e(P) = e(M)$, but $P$ is in $A_{k_r+1}$ so $P \neq M$. Hence, by Theorem 1 we must have $P = M + 1$. Hence $k_r$ is odd, a contradiction. This proves the theorem.

Theorem 5. Let $Q_j$ be the $j^{th}$ largest number in $A_2$. Then

$$e(e(Q_j)) = j .$$

Proof. We can easily see by induction that there are exactly $F_{n-1}$ numbers in $A_2$ whose canonical representations involve only $F_2, F_3, \cdots, F_n$, for let $C_n$ be that set of numbers; i.e., $N \in C_n$ if and only if

$$N = F_2 + \cdots + F_{k_1} \qquad (k_1 \le n) .$$

We want to show that card $(C_n) = F_{n-1}$ and that if $N \in C_n$, $N < F_{n+1}$. This is easily checked for small n. Suppose it is true up to n. Then

$$C_{n+1} = C_n \cup (C_{n-1} + F_{n+1}) .$$

Since this union is disjoint, by the induction hypothesis, the conclusion follows readily.

The point is that $1 + F_{n+1}$ $(n > 3)$ is the $(1 + F_{n-1})^{th}$ number in $A_2$. But

$$e(e(1 + F_{n+1})) = 1 + F_{n-1} ,$$

i.e., the value of $e(e(\cdot))$ on the $(1 + F_{n-1})^{th}$ number of $A_2$ is $1 + F_{n-1}$.

Hence, since $e(e(\cdot))$ is monotone and 1 - 1 on $A_2$ (Theorems 1 and 4), we see that $e(e(\cdot))$ simply counts the members of $A_2$; that is,

$$e(e(Q_j)) = j .$$

Now let $N_i$ be defined by the requirements

$$e(N_i) = i, \qquad e(N_i - 1) \ne i . \tag{4.3}$$

(Set $e(0) = 0$, so that $N_1 = 1$, $N_2 = 3$, etc.)

Theorem 6. For any $N$, $e(a(N)) = N$ and $e(b(N)) = a(n)$. The numbers $(N_1, N_2, \cdots)$ and $(Q_1 + 1, Q_2 + 1, \cdots)$ are the second and third rows of the array $R_1$.

Proof. Note that by Theorem 1, $e((Q_i + 1) - 1) = e(Q_i + 1)$ so that the sets $\{N_i\}$ and $\{Q_i + 1\}$ are disjoint. Furthermore, again by Theorem 1, together they exhaust all positive integers. Now to establish the theorem we only have to show property (2.4) of Section 2 and then that $e(Q_j + 1) = N_j$. Suppose for some $j$ that the latter is false. Then, since

$$e(e(Q_j + 1)) = e(e(Q_j)) = j = e(N_j) ,$$

we must have

$$e(Q_j + 1) = N_j + 1$$

(since $e(N_j - 1) \neq j$, by (4.3)). Furthermore $N_j$ must be in $A_2$. Therefore $e(Q_j + 1) \in A_{k_r}$, $k_r$ odd, so that

$$e(Q_j + 1) = F_{k_r} + \cdots + F_{k_1} \qquad (k_r \text{ odd}) .$$

But then

$$Q_j + 1 = F_{k_r+1} + \cdots + F_{k_1+1} \qquad (k_r + 1 \text{ even}) .$$

Theorem 3 implies that $Q_j$ is not in $A_2$, a contradiction. Hence $e(Q_j + 1) = N_j$.

Now suppose

$$N_j = F_{k_s} + F_{k_{s-1}} + \cdots + F_{k_1}$$

is the canonical representation of $N_j$. Then, since $Q_j + 1$ is not in $A_2$,

$$Q_j + 1 = F_{k_s+1} + F_{k_{s-1}+1} + \cdots + F_{k_1+1} ,$$

so that

(4.4) $$j + N_j = e(N_j) + N_j = Q_j + 1.$$

This proves the theorem.

Theorem 7. We have $A_2 = a^2(\mathbb{N})$ where $\mathbb{N}$ is the set of positive integers. Further,

(4.5) $$A_{2t+1} = b^t a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots)$$

and

(4.6) $$A_{2t} = ab^{t-1}a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) .$$

Proof. We have seen that for any $N$,

$$e(b(N)) = e(a^2(N)) = a(N) .$$

Hence since $b(N) \neq a^2(N)$ and $Q_N + 1 = b(N)$, we get $Q_N = a^2(N)$. This shows that $A_2 = a^2(\mathbb{N})$. Now suppose $N$ is in $A_3$. Then $e(N)$ is in $A_2$ and $e(N) = a^2(M)$ for some $M$. Hence $N$ is either $ba(M)$ or $a^3(M)$. The latter is impossible since $N$ is in $A_3$, not $A_2$. Hence $A_3 = ba(\mathbb{N})$.

Continuing in this way, we complete the proof of the theorem by induction.

## 5. SOME ADDITIONAL PROPERTIES

Since

(5.1) $$\mathbb{N} = a(\mathbb{N}) \cup b(\mathbb{N})$$

it follows from Theorem 7 that

(5.2) $$a(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t}$$

and

$$b(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t+1} . \tag{5.3}$$

Again, by (5.1)

$$a^2(\mathbb{N}) = a^2(\mathbb{N}) \cup a^2b(\mathbb{N}) . \tag{5.4}$$

By (3.2)

$$a^3(n) = ba(n) - 1 .$$

Since, by (4.5),

$$ba(\mathbb{N}) = A_3 , \tag{5.5}$$

it follows that

$$a^3(\mathbb{N}) = A(2, \overline{4}) , \tag{5.6}$$

where the right member denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_2 \qquad (k_r > 4) .$$

Thus by (5.4), we have

$$a^2b(\mathbb{N}) = A(2, 4) , \tag{5.7}$$

where the right member denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_4 + F_2 \qquad (k_r > 5) .$$

Generally if we let $A(s, s+2)$ denote the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_{s+2} + F_s \qquad (k_r > s + 3)$$

and $A(s, \overline{s+2})$ the set with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_s \qquad (k_r > s + 2)$$

then we may state

Theorem 8. For $t \geq 1$ we have

$$ab^{t-1}a^2(\mathbf{N}) = A(2t, 2t + 2), \tag{5.8}$$

$$ab^{t-1}ab(\mathbf{N}) = A(2t, 2t + 2), \tag{5.9}$$

$$b^t a^2(\mathbf{N}) = A(2t + 1, \overline{2t + 3}), \tag{5.10}$$

$$b^t ab(\mathbf{N}) = A(2t + 1, 2t + 3). \tag{5.11}$$

The proof is by induction on $t$. For $t = 1$, Eqs. (5.8) and (5.9) reduce to (5.6) and (5.7), respectively. Next by (5.5)

$$A_3 = ba^2(\mathbf{N}) \cup bab(\mathbf{N}). \tag{5.12}$$

Let $n \in ba^2(\mathbf{N})$; then

$$e(n) \in a^3(\mathbf{N}) = A(2, \overline{4}),$$

that is,

$$e(n) = F_2 + \epsilon F_5 + \cdots ,$$

where $\epsilon = 0$ or $1$. This implies either

$$n = F_2 + \epsilon F_6 + \cdots \qquad \text{or} \qquad F_3 + \epsilon F_6 + \cdots$$

The first possibility contradicts (5.3), so that

$$ba^2(\mathbb{N}) \subset A(a, \overline{5}) \; . \tag{5.13}$$

Now take $n \quad bab(\mathbb{N})$, so that

$$e(n) \in a^2b(\mathbb{N}) = A(2, 4) \; ,$$

$$e(n) = F_2 + F_4 + \epsilon F_6 + \cdots \; .$$

This implies either

$$n = F_2 + F_5 + \epsilon F_7 + \cdots \quad \text{or} \quad F_3 + F_5 + \epsilon F_7 + \cdots \; .$$

The first possibility cannot occur, so that

$$bab(\mathbb{N}) \subset A(3, 5) \; . \tag{5.14}$$

Clearly (5.13) and (5.14) prove (5.10) and (5.11) for $t = 1$.

We now assume that (5.8), $\cdots$, (5.11) hold up to and including the value $t - 1$. Let $n \in ab^{t-1}a^2(\mathbb{N})$, so that

$$e(n) \in b^{t-1}a^2(\mathbb{N}) \; .$$

By the inductive hypothesis this gives

$$e(n) \in A(2t - 1, \overline{2t + 1}) \; ,$$

that is,

$$e(n) = F_{2t-1} + \epsilon F_{2t+2} + \cdots \; .$$

This implies

$$n = F_{2t} + \epsilon F_{2t+3} + \cdots \; ,$$

so that

(5.15) $$ab^{t-1}a^2(\mathbf{N}) \subset A(2t, \overline{2t+2}) .$$

Now take $n \in ab^{t-1}ab(N)$, so that

$$e(n) \in b^{t-1}ab(\mathbf{N}) .$$

Hence by the inductive hypothesis

$$e(n) \in A(2t - 1, 2t + 1) ,$$

that is,

$$e(n) = F_{2t-1} + F_{2t+1} + \epsilon F_{2t+3} + \cdots .$$

This implies

$$n = F_{2t} + F_{2t+2} = \epsilon F_{2t+4} + \cdots ,$$

so that

(5.16) $$ab^{t-1}ab(\mathbf{N}) \subset A(2t, 2t + 2) ,$$

In the next place, take $n \in b^t a^2(\mathbf{N})$, so that

$$e(n) \in ab^{t-1}a^2(\mathbf{N}) .$$

By (5.15) this gives

$$e(n) \in A(2t, \overline{2t+2}) ,$$

that is,

$$e(n) = F_{2t} + \epsilon F_{2t+3} + \cdots .$$

Then either

$$n = F_{2t+1} + \epsilon F_{2t+4} + \cdots$$

or

$$n = F_2 + F_4 + \cdots + F_{2t} + \epsilon F_{2t+4} + \cdots .$$

The second possibility is ruled out, so that

$$ab^{t-1}a^2(\mathbb{N}) \subset A(2t + 1, \overline{2t + 3}) . \tag{5.17}$$

Finally take $n \in b^t ab(\mathbb{N})$, so that

$$e(n) \in ab^{t-1}ab(\mathbb{N}) .$$

Then by (5.16),

$$e(n) \in A(2t, 2t + 2) ,$$

that is,

$$e(n) = F_{2t} + F_{2t+2} + \epsilon F_{2t+4} + \cdots .$$

Then either

$$n = F_{2t+1} + F_{2t+3} + \epsilon F_{2t+5} + \cdots$$

or

$$n = F_2 + F_4 + \cdots + F_{2t} + F_{2t+3} + \epsilon F_{2t+5} + \cdots .$$

Again the second possibility is ruled out, so that

$$b^t ab(\mathbb{N}) \in A(2t + 1, 2t + 3) . \tag{5.18}$$

Combining (5.15), (5.16), (5.17), (5.18), it is clear that we have completed the induction.

We define a function $\lambda(N)$ by means of $\lambda(1) = 0$ and $\lambda(N) = t$, where $N > 1$ and $t$ is the smallest integer such that

$$e^t(N) = 1. \tag{5.19}$$

Theorem 9. Let

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{5.20}$$

where

$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r - 1); \qquad k_r \geq 2,$$

be the canonical representation of $N$. Then

$$\lambda(N) = \begin{cases} k_r - 2 & (r = 1) \\ k_r - 1 & (r \geq 1) \end{cases}. \tag{5.21}$$

Proof.

1. $r = 1$. Clear.
2. $r = 2$, $N = F_{k_1} + F_{k_2}$.

$$e^{k_2-2}(N) = F_{k_1-k_2+2} + F_2$$

$$e^{k_r k_2 - 2} e^{k_2-2}(N) = F_4 + F_2$$

$$e^{k_1-3}(N) = F_3 + F_2 = F_4$$

$$e^{k_1-2}(N) = F_3$$

$$e^{k_1-1}(N) = F_2 = 1.$$

3. $r > 2$. By induction.

Let $\Lambda_t$ denote the set of positive integers N such that

$$\lambda(N) = t . \tag{5.22}$$

Theorem 10. $\Lambda_t$ consists of the integers N such that

$$F_{t+1} < N \le F_{t+2} . \tag{5.23}$$

Thus

$$|\Lambda_t| = F_t . \tag{5.24}$$

Proof. Let N satisfy (4.22) and assume that N has the canonical representation (5.20). By (5.21) the value $N = F_{t+2}$ satisfies (5.22). For all other values of N, it is clear that $r > 1$. Moreover since

$$F_2 + F_4 + \cdots + F_{2s} = F_{2s+1} - 1,$$
$$F_3 + F_4 + \cdots + F_{2s-1} = F_{2s} - 1 ,$$

it is clear that N must satisfy

$$F_{t+1} \le N < F_{t+2} . \tag{5.25}$$

Conversely all N that satisfy (5.25) are of the form (5.20) with $r > 1$. This evidently completes the proof.

Finally we state

Theorem 11. Let $\{x\} = x - [x]$ denote the fractional part of the real number x. Then

$$N \in a(\mathbf{N}) \rightleftarrows 0 < \left\{\frac{N}{\alpha^2}\right\} < \frac{1}{\alpha} \tag{5.26}$$

$$N \in b(\mathbf{N}) \rightleftarrows \frac{1}{\alpha} < \left\{\frac{N}{\alpha^2}\right\} < 1 . \tag{5.27}$$

Proof. We recall that

$$a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n] .$$

Thus $N = b(n)$ is equivalent to

$$\alpha^2 n = N + \epsilon \qquad (0 < \epsilon < 1) ,$$

so that

$$\frac{N}{\alpha^2} = n - \frac{\epsilon}{\alpha^2} .$$

Thus

$$1 \geq \left\{\frac{N}{\alpha^2}\right\} = 1 - \frac{\epsilon}{\alpha^2} > 1 - \frac{1}{\alpha^2} = \frac{1}{\alpha} .$$

Conversely if

$$\frac{1}{\alpha} < \left\{\frac{N}{\alpha^2}\right\}$$

then

$$\frac{N}{\alpha^2} = m + \epsilon, \qquad \frac{1}{\alpha} < \epsilon < 1 .$$

Thus

$$N = \alpha^2 m + \alpha^2 \epsilon ,$$

so that

$$\alpha^2(m + 1) = N + \alpha^2(1 - \epsilon) .$$

Since

$$\alpha^2(1 - \epsilon) < \alpha^2\left(1 - \frac{1}{\alpha}\right) = \alpha - 1 < 1 ,$$

it follows that $b(m + 1) = N$.

This proves (5.27). The equivalence (5.26) follows from (5.27) since

$$a(\mathbb{N}) \cup b(\mathbb{N}) = \mathbb{N} .$$

## 6. WORD FUNCTIONS

By a word function (or briefly a word) is meant any monomial in the a's and b's. It is convenient to include 1 as a word. Clearly if $u, v$ are any words, then $au \neq bv$. Also if $au = av$ or $bu = bv$ then $u = v$. It follows readily that any word is uniquely represented as a product of "primes" a, b.

We define the weight of a word by means of

(6.1) $$p(1) = 0, \qquad p(a) = 1, \qquad p(b) = 2$$

together with

(6.2) $$p(uv) = p(u) + p(v) ,$$

where $u, v$ are arbitrary words. Thus there is exactly one word of weight 1, two of weight 2, and three of weight 3. Let $N_p$ denote the number of words of weight $p$. If $w$ is any word of weight $p$, then, for $p > 2$, $w = au$ or $bv$, where $u$ is of weight $p - 1$, $v$ of weight $p - 2$. Hence

$$N_p = N_{p-1} + N_{p-2} \qquad (p \geq 2) .$$

It follows that

(6.3) $$N_p = F_{p+1} \qquad (p \geq 0) ,$$

the number of words of weight $p$ is equal to the Fibonacci number $F_{p+1}$.

Consider the equation

$$uv = vu . \tag{6.4}$$

We may assume without loss of generality that $p(u) \geq p(v)$. It then follows from the unique factorization property that $u = vz$, where $z$ is some word. Thus $vzv = v^2z$, so that $zv = vz$. Thus by an easy induction on the total weight of $uv$ we get the following theorem.

Theorem 12. The words $u, v$ satisfy (6.4) if and only if there is a word $w$ such that $u = w^r$, $v = w^s$, where $r, s$ are nonnegative integers.

We show next that any word is "almost" linear. More precisely we prove

Theorem 13. Any word $w$ of weight $p$ is uniquely representable in the form

$$u(n) = F_p a(n) + F_{p-1} n - \lambda_u , \tag{6.5}$$

where $\lambda_u$ is independent of $n$.

Proof. We have

$$b(n) = a(n) + n ,$$

$$a^2(n) = a(n) + n - 1 ,$$

$$ab(n) = 2a(n) + n ,$$

$$ba(n) = 2a(n) + n - 1 .$$

We accordingly assume the truth of (6.5) for words $u$ of weight $<p$. There are two cases to consider. (i) if $u = va$, then $v$ is of weight $p - 1$, so that (6.5) gives

$$v(n) = F_{p-1} a(n) + F_{p-2} n - \lambda_v .$$

Hence

$$u(n) = va(n) = F_{p-1}a^2(n) + F_{p-2}a(n) - \lambda_v$$
$$= F_p a(n) + F_{p-1}n - \lambda_v - F_{p-1} ,$$

(ii) if $u = vb$, $v$ is of weight $p - 2$, so that

$$v(n) = F_{p-2}a(n) + F_{p-3}n - \lambda_v .$$

Then

$$\begin{aligned} u(n) = vb(n) &= F_{p-2}ab(n) + F_{p-3}b(n) - \lambda_v \\ &= F_{p-2}(2a(n) + n) + F_{p-3}(a(n) + n) - \lambda_v \\ &= (2F_{p-2} + F_{p-3})a(n) + (F_{p-2} + F_{p-3})n - \lambda_v \\ &= F_p a(n) + F_{p-1}n - \lambda_v . \end{aligned}$$

This completes the induction.

We now show that the representation (6.5) is unique. Otherwise there exist numbers $r$, $s$, $t$ such that

$$ra(n) + sn = t .$$

Taking $n = 1, 2, 3$ we get

$$\begin{cases} r + s = t \\ 3r + 2s = t \\ 4r + 3s = t \end{cases}$$

and therefore $r = s = t = 0$.

Incidentally, we have proved that $\lambda_u$ satisfies

$$\lambda_{va} = \lambda_v + F_p, \qquad \lambda_{vb} = \lambda_v , \tag{6.6}$$

where $v$ is of weight $p$. Note that

$$\lambda_{vab} = \lambda_{va} = \lambda_v + F_p, \qquad \lambda_{vba} = \lambda_{vb} + F_{p+1} = \lambda_v + F_{p+1} .$$

Note also that (6.5) implies

$$\lambda_u = F_{p+1} - u(1) . \tag{6.7}$$

As an immediate corollary of Theorem 13 we have

Theorem 14. For arbitrary words, $u, v$, we have

$$uv - vu = C , \tag{6.8}$$

where $C$ is independent of $n$.

It may be of interest to mention a few special cases of (6.5):

$$a^k(n) = F_k a(n) + F_{k-1}n - F_{k+1} + 1, \tag{6.9}$$

$$b^k(n) = F_{2k}a(n) + F_{2k-1}n , \tag{6.10}$$

$$b^k(n) = a^{2k}(n) + F_{2k+1} - 1 , \tag{6.11}$$

$$(ab)^k(n) = F_{3k}a(n) + F_{3k-1}n - \tfrac{1}{2}(F_{3k-1} - 1) , \tag{6.12}$$

$$(ba)^k(n) = F_{3k}a(n) + F_{3k-1}n - F_{3k-1} , \tag{6.13}$$

$$(ab)^k(n) - (ba)^k(n) = \tfrac{1}{2}(F_{3k-1} + 1) , \tag{6.14}$$

$$a^k b^j(n) = F_{2j+k}a(n) + F_{2j+k-1}n - F_{k+1} + 1 , \tag{6.15}$$

$$b^j a^k(n) = F_{2j+k}a(n) + F_{2j+k-1}n - F_{2j+k+1} + F_{2j+1} , \tag{6.16}$$

$$a^k b^j(n) - b^j a^k(n) = F_{2j+k+1} - F_{2j+1} - F_{k+1} + 1 . \tag{6.17}$$

## 7. GENERATING FUNCTIONS

Put

$$(7.1) \qquad \phi_j(x) = \sum_{n \in A_j} x^n \qquad (j = 2, 3, 4, \cdots).$$

In view of (4.5) and (4.6), Eq. (5.1) is equivalent to

$$(7.2) \qquad \phi_{2r}(x) = \sum_{n=1}^{\infty} x^{ab^{r-1}a(n)}$$

and

$$(7.3) \qquad \phi_{2r+1}(x) = \sum_{n=1}^{\infty} x^{b^r a(n)} .$$

Also it is clear that

$$(7.4) \qquad \frac{x}{1-x} = \sum_{j=0}^{\infty} \phi_j(x) .$$

It follows from the definition of $A_r$ that

$$(7.5) \qquad \phi_r(x) = x^{F_r} \left\{ 1 + \sum_{j=r+2}^{\infty} \phi_j(x) \right\} \qquad (r = 2, 3, 4, \cdots).$$

This evidently implies

$$(7.6) \qquad x^{-F_r}\phi_r(x) - x^{-F_{r+1}}\phi_{r+1}(x) = \phi_{r+2}(x) \qquad (r = 2, 3, 4, \cdots).$$

In particular, by (7.5),

$$\phi_2(x) = x\left\{1 + \sum_{j=4}^{\infty} \phi_j(x)\right\} .$$

Combining this with (5.4), we get

$$(7.7) \qquad (1 + x)\phi_2(x) + x\phi_3(x) = \frac{x}{1 - x} .$$

It is convenient to define

$$(7.8) \qquad \phi(x) = \sum_{n=1}^{\infty} x^{a(n)} .$$

Since the set $a(\mathbf{N})$ is the union of the sets $a^2(\mathbf{N})$ and $ab(\mathbf{N})$, it follows from (3.4) that

$$(7.9) \qquad \phi(x) = \phi_2(x) + x\phi_3(x) .$$

Therefore by (7.7), we have

$$(7.10) \qquad x\phi_2(x) = \frac{x}{1 - x} - \phi(x)$$

and

$$(7.11) \qquad x^2\phi_3(x) = \frac{x}{1 - x} + (1 + x)\phi(x) .$$

Making use of (7.5), (7.10) and (7.11) we can express all $\phi_j(x)$ in terms of $\phi(x)$. For example, since

$$x^{-1}\phi_2(x) - x^{-2}\phi_3(x) = \phi_4(x) ,$$

we get

$$x^4\phi_4(x) = \frac{x + x^3}{1 - x} - (1 + x + x^2)\phi(x) . \tag{7.12}$$

Generally we have

$$x^{F_{r+1}-1}\phi_r(x) = (-1)^r\left\{\frac{xA_r(x)}{1 - x} - B_r(x)\phi(x)\right\} , \tag{7.13}$$

where $A_r(x)$, $B_r(x)$ are polynomials that satisfy

$$\begin{cases} A_{r+2}(x) = A_{r+1}(x) + x^{F_{r+1}}A_r(x) \\ B_{r+2}(x) = B_{r+1}(x) + x^{F_{r+1}}B_r(x) \end{cases} \tag{7.14}$$

together with the initial conditions

$$\begin{cases} A_2(x) = 1, & A_3(x) = 1, \\ B_2(x) = 1, & B_3(x) = 1 + x . \end{cases}$$

It follows readily that

$$B_r(x) = \frac{1 - x^{F_r}}{1 - x} \tag{7.15}$$

while

$$xA_r(x) = \sum_{j=1}^{F_{r-1}} x^{a(j)} . \tag{7.16}$$

In conclusion we shall show that the function $\phi(x)$ cannot be continued across the unit circle. Indeed by a known theorem [1, p. 315], either $\phi(x)$ is rational or it has the unit circle for a natural boundary. Moreover, it is rational if and only if, for some positive integer $m$, (7.17) $$(1 - x^m)\phi(x) = P(x) ,$$

where $P(x)$ is a polynomial. Clearly the coefficients of $P(x)$ are rational integers. It follows that

(7.18) $$\lim_{x = 1} (1 - x)\phi(x) = C,$$

where $C$ is rational. On the other hand, if we put

$$\phi(x) = \sum_{k=1}^{\infty} c_k x^k ,$$

so that $c_k = 0$ or $1$, it is evident from (7.8) that

$$\sum_{k=1}^{n} c_k \sim \frac{n}{\alpha} .$$

Since this implies

$$\lim_{x = 1} (1 - x)\phi(x) = \frac{1}{\alpha}$$

we have a contradiction with (7.18).

## 8. APPENDIX

In addition to the canonical representation (1.1) we have another representation described in the following

Theorem 15. Every integer $N$ is uniquely represented in the form

(8.1) $$N = F_{k_1} + \cdots + F_{k_r} + F_{2k+1} \qquad (k \geq 0) ,$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r - 1), \qquad k_r - (2k + 1) \geq 2 .$$

Proof. By (5.2),

$$a(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t} . \tag{8.2}$$

Hence, by the first proof of Theorem 6,

$$\mathbb{N} = \bigcup_{t=1}^{\infty} A_{2t-1} .$$

This evidently proves the theorem.

We may refer to (8.1) as the second canonical representation of N.

In view of Theorem 15, we let $\overline{A}_{2k+1}$ denote the set of positive integers $\{N\}$ of the form (8.1). Then the sets

$$\overline{A}_{2k+1} \qquad (k = 0, 1, 2, \cdots)$$

constitute a partition of the positive integers. Clearly

$$\overline{A}_{2k+1} = A_{2k+1} \qquad (k = 1, 2, 3, \cdots) , \tag{8.3}$$

while

$$\overline{A}_1 = \bigcup_{t=1}^{\infty} A_{2t} = a(\mathbb{N}). \tag{8.4}$$

For $N \in \overline{A}_1$, if

$$N = F_{k_1} + \cdots + F_{k_r} + F_1 \tag{8.5}$$
$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r - 1), \qquad k_r > 3 ,$$

then clearly we may replace $F_1$ by $F_2$ and (8.5) reduces to the first canonical representation. In this case, then, $N \in A_2$. However, if $k_r = 3$, the situation is less simple. For example

$$8 = F_6 = F_5 + F_3 + F_1 .$$

Generally, since

$$F_1 + F_3 + F_5 + \cdots + F_{2s-1} = F_{2s} ,$$

it follows that if the number N has the second canonical representation

$$N = F_1 + F_3 + \cdots + F_{2s-1} + F_{k_1} + F_{k_2} + \cdots ,$$

where

$$k_{j+1} - k_j \geq 2 \qquad (j \geq 1), \qquad k_1 \geq 2s + 2 ,$$

then $N \in A_{2s}$ and conversely.

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# LUCAS REPRESENTATIONS

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## 1. INTRODUCTION

We define the Fibonacci and Lucas numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1),$$
$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \qquad (n \geq 1).$$

We recall that every positive integer $N$ can be written uniquely in the form

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, n-1); \qquad k_r \geq 2. \tag{1.2}$$

If $A_k$ denotes the set of positive integers $\{N\}$ for which $k_r = k$, it is clear that the sets

$$\{A_k\} \qquad (k = 2, 3, 4, \cdots) \tag{1.3}$$

constitute a partition of the set of positive integers. We may refer to (1.3) as a Fibonacci partition of the positive integers. It is proved in [2] that the numbers in $A_k$ can be described in terms of the greatest integer function. More precisely, if

$$\alpha = \frac{1}{2}(1 + \sqrt{5})$$

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and we put

$$(1.4) \qquad a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n] ,$$

then we have

$$(1.5) \qquad A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots) ,$$

$$(1.6) \qquad A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots) .$$

As is customary, powers and juxtaposition of functions should be interpreted as composition.

Turning next to representations as sums of Lucas numbers, we show first that every positive integer is uniquely representable either in the form

$$(1.7) \qquad N = L_{k_1} + \cdots + L_{k_r} + L_0 ,$$

where

$$(1.8) \qquad k_j = k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, r - 1); \qquad k_r \geq 3$$

or in the form

$$(1.9) \qquad N = L_{k_1} + \cdots + L_{k_r} ,$$

where now

$$(1.10) \qquad k_j - k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, r - 1); \qquad k_r \geq 1 ;$$

but not in both (1.7) and (1.9).

Let $B_0$ denote the set of positive integers representable in the form (1.7) and let $B_k$ denote the set of positive integers representable in the form (1.9) with $k_r = k$. Then as above the sets

$$(1.11) \qquad B_k \qquad (k = 0, 1, 2, \cdots)$$

constitute a partition of the positive integers which may be called a Lucas partition. In the next section we shall prove the following.

(1.12) $$B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\},$$

(1.13) $$B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\},$$

and

(1.14) $$B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots),$$

(1.15) $$B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots).$$

It is not difficult to show that an integer $N$ is in $B_0$ if and only if it is not representable in the form

(1.16) $$N = L_{k_1} + \cdots + L_{k_r},$$

where

$$k_1 > k_2 > \cdots > k_r \geq 1.$$

Let $\nu(n)$ denote the number of integers $\leq n$ that are not representable in the form (1.17). Hoggatt has conjectured that

(1.17) $$\nu(L_n) = F_{n-1}$$

and that, for fixed $k$,

(1.18) $$\nu(kL_n) = kF_{n-1},$$

if $n$ is sufficiently large. The conjecture (1.17) was proved by Klarner; we shall prove (1.18) in Section 3 below.

## 2. SOME PROPERTIES OF THE LUCAS REPRESENTATION

Let $P_n$ be the set of numbers that can be written in the form (1.7) with $k_1 \le n$, and let $Q_n$ be those that can be written in the form (1.9) with $k_1 \le n$. Then we have

$$
\begin{aligned}
P_3 &= \{2,\ 6\} \\
Q_3 &= \{1,\ 3,\ 4,\ 5\} \\
P_4 &= \{2,\ 6,\ 9\} \\
Q_4 &= \{1,\ 3,\ 4,\ 5,\ 7,\ 8,\ 10\}\ .
\end{aligned} \tag{2.1}
$$

By induction we obtain the following theorem.

Theorem 1. Every positive integer can be uniquely represented in either the form (1.7) or the form (1.9), but not both. Moreover,

$$P_n \cup Q_n = \{1,\ 2,\ \cdots,\ L_{n+1} - 1\} \tag{2.2}$$

$$\mathrm{card}\,(P_n) = F_n \tag{2.3}$$

$$\mathrm{card}\,(Q_n) = F_{n+2} - 1\ . \tag{2.4}$$

Proof. We will prove (2.2)—(2.4) and also

$$P_n \cap Q_n = \phi \tag{2.5}$$

by induction. Hence let us assume (2.2)–(2.5) up to and including the value n. Now by definition

$$
\begin{aligned}
P_{n+1} &= P_n \cup (P_{n-1} + L_{n+1}) \\
Q_{n+1} &= Q_n \cup (Q_{n-1} + L_{n+1}) \cup \{L_{n+1}\}
\end{aligned}
$$

and these unions are disjoint; if for instance, $N \in P_{n-1} + L_{n+1}$, then $N > L_{n+1}$ and by (2.2) $N \notin P_n$, etc. Hence

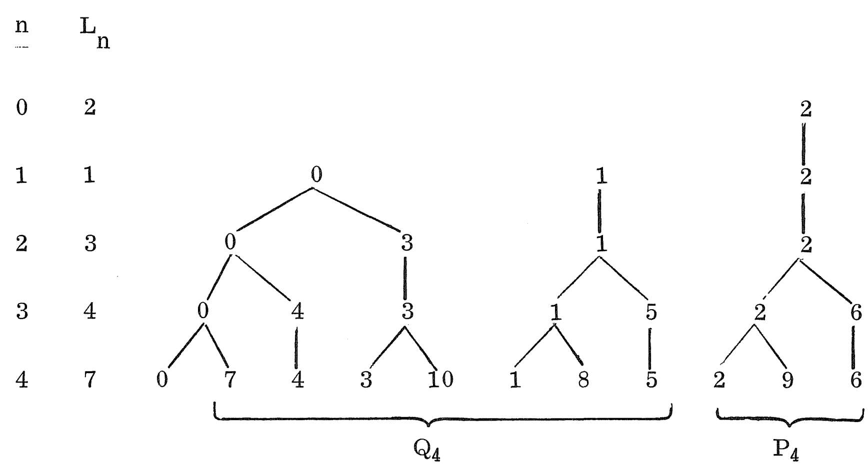
$$\text{card}\,(P_{n+1}) = \text{card}\,(P_n) + \text{card}\,(P_{n-1}) = F_{n+1}$$

and

$$\text{card}\,(Q_{n+1}) = F_{n+2} - 1 + F_{n+1} - 1 + 1 = F_{n+3} - 1\,.$$

The other properties are easily checked.

The following tree may aid the reader.




We turn next to the relations (1.12)–(1.15). We make use of the function e defined in [1]. The properties we need are the following (see [1] and [2]):

(i) If

$$N = F_{k_1} + \cdots + F_{k_r}, \qquad k_r \geq 2, \quad k_1 > k_2 > \cdots > k_r\,,$$

then

$$e(n) = F_{k_1-1} + \cdots + F_{k_r-1}\,.$$

(ii) For every N,

$$e(a(N)) = N$$

and

$$e(b(N)) = a(N).$$

Theorem 2. The following relations hold.

(2.6) $$B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\}$$

(2.7) $$B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\}$$

(2.8) $$B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots)$$

(2.9) $$B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots).$$

Proof. Let $N$ be an arbitrary positive integer. By (1.5), we have $a^2(N) \in A_2$. Hence

(2.10) $$a^2(N) = F_2 + \epsilon_4 F_4 + \cdots ,$$

where $\epsilon_i$ may assume the values 0 or 1. Applying $e$ twice, we get

(2.11) $$N = F_1 + \epsilon_4 F_2 + \cdots .$$

Adding (2.10) and (2.11), we get

(2.12) $$a^2(N) + N = 2 + \epsilon_4 L_3 + \cdots \in B_0 .$$

On the other hand, suppose

(2.13) $$M = L_0 + \epsilon_3 L_3 + \epsilon_4 L_4 + \cdots$$

is in $B_0$. Let

$$K = F_2 + \epsilon_3 F_4 + \epsilon_4 F_5 + \cdots .$$

Since $K \in A_2$, by (1.5) K must be of the form $a^2(M)$ for some M. Also $M = e^2(a^2(M))$. Hence

$$a^2(M) = F_2 + \epsilon_3 F_4 + \epsilon_4 F_5 + \cdots$$
$$M = F_1 + \epsilon_3 F_2 + \epsilon_4 F_3 + \cdots$$

and

$$N = M + a^2(M) .$$

This proves (2.6). Equation (2.7) is clear from the definition. To prove (2.8), let N be arbitrary. Then

$$b^t a(N) \in A_{2t+1} ,$$

by (1.6), so

$$b^t a(N) = F_{2t+1} + \epsilon_{2t+3} F_{2t+3} + \cdots .$$

Applying e twice and adding we get

$$b^t a(N) + b^{t-1}(N) = L_{2t} + \epsilon_{2t+2} L_{2t+2} + \cdots \in B_{2t} .$$

Conversely, suppose $N \in B_{2t}$, so that

$$N = L_{2t} + \epsilon_{2t+2} L_{2t+2} + \cdots .$$

Put

$$M = F_{2t+1} + \epsilon_{2t+2} F_{2t+3} + \cdots .$$

Then, by (1.6), $M = b^t a(K)$ for some K. Moreover, since

$$e^2(M) = b^{t-1} a(K) ,$$

we have

$$N = b^t a(K) + b^{t-1} a(K) .$$

This proves (2.8), and the proof of (2.9) is similar.

## 3. PROOF OF HOGGATT'S CONJECTURES

<u>Theorem 3.</u> An integer $N$ is in $B_0$ if and only if it is not representable in the form

$$N = L_{j_1} + \cdots + L_{j_s} , \tag{3.1}$$

where

$$j_1 > j_2 > \cdots > j_s > 1 . \tag{3.2}$$

<u>Proof.</u> If

$$j_t - j_{t+1} \geq 2 \qquad (t = 1, \cdots, s - 1) , \tag{3.3}$$

then Theorem 3 is an immediate consequence of Theorem 1. Let $u$ be the least positive integer such that

$$j_u - j_{u+1} = 1 .$$

In (3.1), replace

$$L_{j_u} + L_{j_{u+1}} \qquad \text{by} \qquad L_{j_u+1}$$

and then repeat the process. Since

$$L_1 + L_2 + \cdots + L_k = L_{k+2} - 3,$$

we ultimately reach a representation of the form (3.1) that satisfies (3.3). This evidently proves the theorem.

Let $\nu(n)$ denote the number of positive integers $N \leq n$ that are not representable in the form (3.1), so that by the theorem just proved, $\nu(n)$ is also the number of integers $\leq n$ in $B_0$.

Theorem 4. We have

$$\nu(n) = \left[\frac{n+2}{\alpha^2+1}\right] \quad . \tag{3.4}$$

Proof. By Theorem 2,

$$\begin{aligned} B_0 &= \{aa(k) + k \mid k = 1, 2, 3, \cdots\} \\ &= \{b(k) + k - 1 \mid k = 1, 2, 3, \cdots\} . \end{aligned}$$

Thus $\nu(n)$ is the largest integer $k$ such that

$$b(k) + k \leq n + 1 .$$

Since $b(k) = [\alpha^2 k]$, $\nu(n)$ is the largest $k$ such that

$$[(\alpha^2 + 1)k] \leq n + 1 ,$$

that is, the largest $k$ such that

$$(\alpha^2 + 1)k < n + 2 .$$

Thus (3.4) follows at once.

Theorem 5. We have

$$\nu(L_n) = F_{n-1} \qquad (n \geq 1) . \tag{3.5}$$

Proof. Since

$$L_n = \alpha^n + \beta^n \qquad (\alpha\beta = -1) ,$$

it follows that

$$\frac{L_n + 2}{\alpha^2 + 1} = \frac{\alpha^n + \beta^n + 2}{\alpha^2 + 1} = \frac{\alpha^{n-1} - 2\beta - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{-2\beta + \beta^{n-1} - \beta^{n+1}}{\alpha - \beta}$$

$$= F_{n-1} + \frac{2 + \beta^{n-1}}{\alpha^2 + 1} \quad .$$

It is easily verified that

$$0 < \frac{2 + \beta^{n-1}}{\alpha^2 + 1} < 1 \qquad (n \geq 1) \ .$$

Theorem 6. Let k be a fixed positive integer. Then

$$\nu(kL_n) = kF_{n-1} \tag{3.6}$$

for n sufficiently large.

Proof. We have

$$\frac{kL_n + 2}{\alpha^2 + 1} = \frac{k(\alpha^n + \beta^n) + 2}{\alpha^2 + 1} = \frac{k(\alpha^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= k\,\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{k(\beta^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

For n sufficiently large it is clear that

$$0 < \frac{k(\beta^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta} < 1$$

so that

$$\left[\frac{kL_n + 2}{\alpha^2 + 1}\right] = kF_{n-1}\,.$$

This completes the proof of the theorem.

Theorem 7. We have

$$\nu(5F_n) = L_{n-1} \qquad (n > 1) \tag{3.7}$$

and

$$\nu(5kF_n) = kL_{n-1} \tag{3.8}$$

for sufficiently large $n$.

Proof. To prove (3.7), note that

$$\frac{5F_n + 2}{\alpha^2 + 1} = \frac{(\alpha - \beta)(\alpha^n - \beta^n) + 2}{\alpha^2 + 1} = \frac{(\alpha - \beta)(\alpha^{n-1} + \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= \alpha^{n-1} + \beta^{n-1} - \beta^{n-1}(1 - \beta^2) - \frac{2\beta}{\alpha - \beta}$$

$$= L_{n-1} + \beta^n - \frac{2\beta}{\alpha - \beta}\,.$$

Since

$$0 < \beta^n - \frac{2\beta}{\alpha - \beta} < 1 \qquad (n \geq 1)\,,$$

(3.7) follows.

Next to prove (3.8) we take

$$\frac{5kF_n + 2}{\alpha^2 + 1} = \frac{k(\alpha - \beta)(\alpha^n - \beta^n) + 2}{\alpha^2 + 1} = \frac{k(\alpha - \beta)(\alpha^{n-1} + \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= k(\alpha^{n-1} + \beta^{n+1}) - \frac{2\beta}{\alpha - \beta} = k(\alpha^{n-1} + \beta^{n-1}) + k\beta^n - \frac{2\beta}{\alpha - \beta}\,.$$

Since

$$0 < k\beta^n - \frac{2\beta}{\alpha - \beta} < 1$$

for $n$ sufficiently large, Eq. (3.8) follows at once.

The last two theorems were also conjectured by Hoggatt.

## 4. GENERATING FUNCTIONS

Put

$$\psi_j(0) = \sum_{n \in B_j} x^n \qquad (j = 0, 1, 2, \cdots). \tag{4.1}$$

In view of Theorem 2, Eq. (4.1) is equivalent to

$$\psi_0(x) = \sum_{n=1}^{\infty} x^{a^3(n)+n}, \tag{4.2}$$

$$\psi_1(x) = \sum_{n=1}^{\infty} x^{a^2(n)+n-1}, \tag{4.3}$$

$$\psi_{2t+1}(x) = \sum_{n=1}^{\infty} x^{ab^{t-1}a(n)+ab^t a(n)} \qquad (t \geq 1), \tag{4.4}$$

$$\psi_{2t}(x) = \sum_{n=1}^{\infty} x^{b^{t-1}a(n)+b^t a(n)} \qquad (t \geq 1). \tag{4.5}$$

Clearly

$$\psi_0(x) = x\psi_1(x) \quad . \tag{4.6}$$

Also it is evident that

$$\frac{x}{1-x} = \sum_{j=0}^{\infty} \psi_j(x) \quad , \tag{4.7}$$

so that, by (4.6),

$$\frac{x}{1-x} = (1+x)\psi_1(x) + \sum_{j=2}^{\infty} \psi_j(x) \quad . \tag{4.8}$$

In the next place it follows from the definition of $A_r$ that

$$\psi_r(x) = x^{L_r}\left\{1 + \sum_{j=r+2}^{\infty} \psi_j(x)\right\} \qquad (r \geq 1) \; . \tag{4.9}$$

This implies

$$x^{-L_r}\psi_r(x) - x^{-L_{r+1}}\psi_{r+1}(x) = \psi_{r+2}(x) \qquad (r \geq 1) \quad . \tag{4.10}$$

In particular, by (4.9),

$$\psi_1(x) = x\left\{1 + \sum_{j=3}^{\infty} \psi_j(x)\right\} \quad .$$

Combining this with (4.8), we get

$$\frac{x}{1-x} = (1 + x + x^2)\psi_1(x) + x\psi_2(x) \quad . \tag{4.11}$$

By means of (4.10) and (4.11) we can express all $\psi_j(x)$, $j > 1$, in terms of $\psi_1(x)$. The first few formulas are

$$x\psi_2(x) = \frac{x}{1-x} - (1 + x + x^2)\psi_1(x)$$

$$x^4\psi_3(x) = -\frac{x}{1-x} + (1 + x + x^2 + x^3)\psi_1(x)$$

$$x^8\psi_4(x) = \frac{x + x^5}{1-x} + \frac{1 - x^7}{1-x}\psi_1(x) \quad .$$

Generally we have

$$x^{L_{r+1}-3}\psi_r(x) = (-1)^r\left\{\frac{xA_r(x)}{1-x}\right\} - B_r(x)\psi_1(x) \tag{4.12}$$

where

$$\begin{cases} A_{r+2}(x) = A_{r+1}(x) + x^{L_{r+1}}A_r(x) \\ B_{r+2}(x) = B_{r+1}(x) + x^{L_{r+1}}B_r(x) \end{cases} \tag{4.13}$$

together with the initial conditions

$$\begin{cases} A_2(x) = 1, & A_3(x) = 1 \\ B_2(x) = 1 + x + x^2, & B_3(x) = 1 + x + x^2 + x^3 \end{cases}$$

It follows that

$$B_r(x) = \frac{1 - x^{L_r}}{1-x} \tag{4.14}$$

while
[Continued on page 70.]

# FIBONACCI REPRESENTATIONS OF HIGHER ORDER

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## 1. INTRODUCTION

Let

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1) .$$

It is well known that every positive integer N can be uniquely represented in the form

$$N = F_{k_1} + F_{k_2} + F_{k_3} + \cdots , \tag{1.1}$$

where

$$k_1 \geq 2, \quad k_{i+1} - k_i \geq 2 \qquad (i = 1, 2, 3, \cdots ) ; \tag{1.2}$$

Equation (1.1) is called the canonical representation of N. Let $A_k$ denote the set of positive integers N with $k_1 = k$ in (1.1). It was proved in [2] that

$$A_{2t} = ab^{t-1}a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) , \tag{1.3}$$

$$A_{2t+1} = b^t a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) , \tag{1.4}$$

where $\mathbb{N}$ denotes the set of positive integers and the functions a(n), b(n) are defined by means of

$$a(n) = [\alpha n], \quad b(n) = [\alpha^2 n], \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}) , \tag{1.5}$$

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and $[x]$ denotes the greatest integer $\leq x$. In the paper cited, considerable use is made of the function $e(N)$ defined by

$$(1.6) \qquad e(N) = F_{k_1-1} + F_{k_2-1} + F_{k_3-1} + \cdots ;$$

This function was introduced in an earlier paper [1].

It is natural to try to extend the results of [2] to Fibonacci numbers of higher order. For a number of reasons we limit ourselves in the present paper to the numbers defined by

$$(1.7) \qquad G_0 = 0, \quad G_1 = G_2 = 1, \quad G_{n+1} = G_n + G_{n-1} + G_{n-2} \qquad (n \geq 2).$$

To begin with, we have the unique canonical representation

$$(1.8) \qquad N = \epsilon_2 G_2 + \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots ,$$

where each $\epsilon_i$ is either 0 or 1 and now

$$(1.9) \qquad \epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0 \qquad (i = 2, 3, 4, \cdots).$$

Corresponding to the function $e(N)$ defined by (1.6) we introduce the function

$$(1.10) \qquad f(N) = \epsilon_2 G_1 + \epsilon_3 G_2 + \epsilon_4 G_3 + \cdots .$$

Moreover if

$$(1.11) \qquad N = \epsilon_2' G_2 + \epsilon_3' G_3 + \epsilon_4' G_4 + \cdots ,$$

where each $\epsilon_i'$ is either 0 or 1, is any representation of N, then

$$f(N) = \epsilon_2' G_1 + \epsilon_3' G_2 + \epsilon_4' G_4 + \cdots .$$

Let $C_k$ denote the set of positive integers $\{N\}$ for which $\epsilon_k$ is the first nonzero $\epsilon_i$ in (1.8). We obtain results analogous to (1.3) and (1.4), namely

$$C_{3k+2} = ac^k a(\mathbf{N}) \cup ac^k b(\mathbf{N}) \qquad (k \geq 0), \tag{1.12}$$

$$C_{3k+3} = bc^k a(\mathbf{N}) \cup bc^k b(\mathbf{N}) \qquad (k \geq 0), \tag{1.13}$$

$$C_{3k+4} = c^{k+1} a(\mathbf{N}) \cup c^{k+1} b(\mathbf{N}) \qquad (k \geq 0). \tag{1.14}$$

The functions a, b, c are defined in Section 3 below; we have been unable to find explicit formulas analogous to (1.5). We show, however, that the functions can be characterized in the following way. They are strictly monotone functions whose ranges constitute a disjoint partition of the positive integers; moreover

$$b(n) = a^2(n) + 1, \qquad c(n) = a(n) + b(n) + n. \tag{1.15}$$

In addition to the canonical representation (1.8), we find it convenient to introduce a second canonical representation

$$N = G_{3k+1} + \epsilon_{3k+2} G_{3k+2} + \cdots, \tag{1.16}$$

where $k \geq 0$ and as before

$$\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0 \qquad (i \geq 3k+1).$$

Moreover, making use of the representation (1.16),

$$\begin{cases} a(N) = G_{3k+2} + \epsilon_{3k+2} G_{3k+3} + \cdots \\ b(N) = G_{3k+3} + \epsilon_{3k+2} G_{3k+4} + \cdots \\ c(N) = G_{3k+4} + \epsilon_{3k+2} G_{3k+5} + \cdots \end{cases} \tag{1.17}$$

It is because of these formulas for a(N), b(N), c(N) that (1.16) is particularly useful.

## 2. PRELIMINARIES

Let $Q_n$ be the set of non-negative N's which can be written <u>canonically</u> in the form

$$N = \epsilon_2 G_2 + \epsilon_3 G_3 + \cdots + \epsilon_n G_n . \tag{2.1}$$

Then we have

$$\begin{aligned} Q_2 &= \{0, 1\}, \qquad Q_3 = \{0, 1, 2, 3\} , \\ Q_4 &= \{0, 1, 2, 3, 4, 5, 6 , \cdots \} \end{aligned} \tag{2.2}$$

We can see easily by induction that $Q_n$ is a disjoint union:

$$Q_n + (Q_{n-3} + G_{n-1} + G_n) \cup (Q_{n-2} + G_n) \cup Q_{n-1} \tag{2.3}$$

and that

$$Q_n = \{0, 1, 2, \cdots, G_{n+1} - 1\} . \tag{2.4}$$

These remarks imply the following theorem.

Theorem 1. Any positive integer N can be uniquely represented in the canonical form (2.1).

Theorem 2. If N is given (not necessarily canonically) by

$$N = \epsilon'_2 G_2 + \epsilon'_3 G_3 + \cdots ,$$

then

$$f(N) = \epsilon'_2 G_1 + \epsilon'_3 G_2 + \cdots .$$

Proof. Given any representation $\epsilon' = (\epsilon'_2, \epsilon'_3, \cdots)$ of N we obtain another representation $s(\epsilon')$ of N by choosing, in $\epsilon$, the block of the form (1, 1, 1, 0) that is farthest right and replacing it by the block (0, 0, 0, 1). If there is no such block, $\epsilon'$ is canonical and we set $s(\epsilon') = \epsilon'$. It is clear that sufficiently many applications of s will yield the canonical representation of N, but it is also clear that

$$\epsilon'_2 G_1 + \epsilon'_3 G_2 + \cdots = s(\epsilon')_2 G_1 + s(\epsilon')_3 G_2 + \cdots , \tag{2.5}$$

establishing the theorem.

Theorem 3. We have $f(N + 1) \geq f(N)$, with equality if and only if $N \in C_2$.

Proof. If $N \notin C_2$ then

$$N = \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots$$

and

$$N + 1 = G_2 + \epsilon_3 G_3 + \cdots .$$

Hence

$$f(N + 1) = G_1 + \epsilon_3 G_2 + \cdots = f(N) + 1 .$$

If $N \in C_2$ then either

(a) $$N = G_2 + G_3 + \epsilon_5 G_5 + \cdots$$

or

(b) $$N = G_2 + \epsilon_4 G_4 + \cdots .$$

In case (a)

$$N + 1 = G_4 + \epsilon_5 G_5 + \cdots$$

and

$$f(N + 1) = G_3 + \epsilon_5 G_4 + \cdots = f(N) .$$

In case (b)

$$N + 1 = G_3 + \epsilon_4 G_4 + \cdots$$

and

$$f(N+1) = G_2 + \epsilon_4 G_3 + \cdots = f(N) .$$

This completes the proof.

Theorem 4. We have $N - 1 \notin C_2$ if and only if $N \in C_k$, where $k \equiv 2 \pmod 3$.

Proof. If $N \in C_2$, there is nothing to prove, so suppose $N \in C_k$, $k > 2$; let $N$ have the canonical representation

$$N = G_k + \epsilon_{k+1} G_{k+1} + \cdots .$$

Then we have

$$(2.6) \qquad G_k = \begin{cases} G_0 + G_1 + G_2 + (G_4 + G_5) + \cdots + G_{k-2} + G_{k-1} & k \equiv 0 \pmod 3 \\ G_1 + G_2 + G_3 + (G_5 + G_6) + \cdots + (G_{k-2} + G_{k-1}) & k \equiv 1 \pmod 3 \\ G_2 + G_3 + G_4 + (G_6 + G_7) + \cdots + (G_{k-2} + G_{k-1}) & k \equiv 2 \pmod 3). \end{cases}$$

Thus we see that only in the case $k \equiv 2 \pmod 3$ we have $G_k - 1 \notin C_2$.

Theorem 5. The following identities hold for $k > 2$.

$$(2.7) \qquad f(G_k - 1) = \begin{cases} G_{k-1} & k \equiv 0 \pmod 3 \\ G_{k-1} & k \equiv 1 \pmod 3 \\ G_{k-1} - 1 & k \equiv 2 \pmod 3 \end{cases}$$

Proof. Making use of (2.6), we readily get (2.7).

## 3. THE FUNCTIONS a, b, AND c

In this section we define three strictly monotone functions on the positive integers, which we display as an array:

(3.1) R :

| 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|
| a(1) | a(2) | a(3) | a(4) | a(5) | ... |
| b(1) | b(2) | b(3) | b(4) | b(5) | ... |
| c(1) | c(2) | c(3) | c(4) | c(5) | ... |

We begin by setting $a(1) = 1$, $b(1) = 2$, $c(1) = 4$, $a(2) = 3$, and fill the rest of the array by induction. Suppose that columns 1 to n have been filled, and also that $a(n+1)$ is known. Then we fill row a to column $a(n+1)$ in increasing order with the first integers that have not appeared so far in the array. Then we let $b(n+1)$ be the next integer that has not appeared, and we set

$$c(n + 1) = n + 1 + a(n + 1) + b(n + 1) .$$

Thus we get

(3.2) R :

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| a | 1 | 3 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 18 |
| b | 2 | 6 | 9 | 13 | 15 | 19 | 22 | 26 | 30 | |
| c | 4 | 11 | 17 | 24 | 28 | 35 | 41 | 48 | 55 | |

It is clear from the definition of R that the ranges $a(N)$, $b(N)$, and $c(N)$ are disjoint and exhaust the positive integers. We will now establish several relations between a, b and c.

Theorem 6. For every positive integer N, the following identities hold:

$$c(N) = a(N) + b(N) + N , \tag{3.3}$$

$$b(N) = a^2(N) + 1 , \tag{3.4}$$

$$ab(N) = ba(N) + 1 , \tag{3.5}$$

$$c(N) = ab(N) + 1 = ba(N) + 2 . \tag{3.6}$$

Proof. 1. ((3.3)) This is the definition of $c(N)$.

2. ((3.4)) Let N be the first integer for which (3.4) fails. Then we must have, for some $K < N$,

$$c(K) = a^2(N) + 1 \qquad \text{and} \qquad b(N) - a^2(N) + 2\,.$$

Hence the array has the form

| K | N | a(N) |
|---|---|---|
| . | . | $a^2(N)$ |
| . | $a^2(N) + 2$ | |
| $a^2(N) + 1$ | | |

Now $K + N + a(N)$ numbers have been entered. Since they must be the numbers $1, 2, \cdots, a^2(N) + 2$, we get

$$K + N + a(N) = a^2(N) + 2\,. \tag{3.7}$$

But

$$K + a(K) + b(K) = c(K) = a^2(N) + 1\,.$$

Therefore

$$a(K) + b(K) + 1 = N + a(N)\,. \tag{3.8}$$

Now if we had $a(K) < N$, we would have $a^2(K) < a(N)$, but from (3.8) we would have $b(K) + 1 > a(N)$. However $b(K) = a^2(K) + 1$ since (3.4) holds for $K < N$. This is a contradiction, since $a(N) < b(K)$. In a similar way we contradict the supposition $a(K) > N$. Hence $a(K) = N$ and we have

$$K + N + a(N) = K + a(K) + a^2(K) = K + a(K) + b(K) - 1 = c(K) - 1 = a^2(N),$$

contradicting (3.7).

3. ((3.5) and (3.6)). Consider the array:

$$
\begin{array}{cccc}
N & a(N) & a^2(N) & b(N) \\
\hline
\cdot & a^2(N) & ba(N) - 1 & ab(N) \\
\cdot & ba(N) & & \\
c(N) & & &
\end{array}
\tag{3.9}
$$

Assume $ba(N) > c(N)$. Then no number $\leq c(N)$ can be missing from the enclosed portion (since it's too late to enter it in any row). Hence in the enclosed portion we have at least the numbers $1, 2, \cdots, c(N)$, $ba(N) - 1$ and $ba(N)$. However these are only $N + a(N) + b(N) - 1 = c(N) - 1$ entries, a contradiction. Hence $ba(N) < c(N)$ and one number $M < c(N)$ is missing from the enclosed portion. Then we must have $M = ab(N)$. Now $M$ is exceeded only by $c(N)$, so we must have $ab(N) = c(N) - 1$, $ba(N) = c(N) - 2$ proving (3.5) and (3.6).

We conclude this section with a characterization of the array R.

Theorem 7. Let $a_1$, $b_1$ and $c_1$ be strictly monotone functions whose ranges form a disjoint partition of the positive integers. Suppose further that they satisfy (3.3) and (3.4). Then $a_1 = a$, $b_1 = b$ and $c_1 = c$.

Proof. Clearly

$$b(N) = a^2(N) + 1 > a^2(N) \geq a(N) .$$

Hence we must have $a(1) = 1$ and $b(1) = a^2(1) + 1 = 2$. Then $c(1) = 4$, and further, since $b(N) > a(N)$, $a(2) = 3$.

Now by induction on the columns of the array formed by the functions $a_1$, $b_1$ and $c_1$, we see that it is the array R.

## 4. RELATIONS INVOLVING f

Since every number appears in the range of f and f is monotone, the following definition makes sense. For every N, we let $A(N)$ be defined as follows:

$$f(A(N)) = N; \qquad f(A(N) - 1) = N - 1 . \tag{4.1}$$

We define $B(N)$ by

$$B(N) = A(A(N)) + 1 \tag{4.2}$$

and $C(N)$ by

$$C(N) = N + A(N) + B(N) . \tag{4.3}$$

Theorem 8. $A(A(\mathbf{N}) \subseteq C_2$ .

Proof. Suppose for some $N$, $A(A(N))$ is not in $C_2$. Put (canonical representation)

$$A(A(N)) = G_k + \epsilon_{k+1} G_{k+1} + \cdots \qquad (k > 2) .$$

Then applying $f$ we get

$$\begin{aligned} A(N) &= G_{k-1} + \epsilon_{k+1} G_k + \cdots \\ N &= G_{k-2} + \epsilon_{k+1} G_{k-1} + \cdots \quad . \end{aligned} \tag{4.4}$$

By the definition of $A$ and Theorem 3, $A(A(N)) - 1 \notin C_2$, so by Theorem 4, $k \equiv 2 \pmod 3$. But neither is $A(N) - 1$ in $C_2$. Hence $k - 1 \equiv 2 \pmod 3$. This is a contradiction and proves the theorem.

Theorem 9. $C_2 = A(A(\mathbf{N})) \cup A(B(\mathbf{N}))$.

Proof. Suppose $A(B(N)) \notin C_2$. Put (canonical representation)

$$A(B(N)) = G_k + \epsilon_{k+1} G_{k+1} + \cdots \qquad (k > 2) .$$

As in the previous theorem, we must have $k \equiv 2 \pmod 3$ so that $k \geq 5$ and

$$A(A(N)) + 1 = B(N) = G_{k-1} + \epsilon_{k+1} G_k + \cdots .$$

Hence

$$A(A(N)) = G_{k-1} - 1 + \epsilon_{k+1} G_k + \cdots$$

and, from Theorem 5,

$$A(N) = G_{k-2} + \epsilon_{k+1} G_{k-1} + \cdots .$$

Now again since $A(N) - 1 \notin C_2$, we get $k - 2 \equiv 2 \pmod 3$, a contradiction.

Theorem 10. Let $K$ be arbitrary and suppose $K - 1$ is given canonically by

$$K - 1 = \epsilon_2 G_2 + \epsilon_3 G_3 + \cdots . \tag{4.5}$$

Then

$$A(K) = G_1 + \epsilon_2 G_3 + \epsilon_3 G_4 + \cdots \tag{4.6}$$

$$A(A(K) = G_2 + \epsilon_2 G_4 + \epsilon_3 G_5 + \cdots \tag{4.7}$$

$$C(K) = G_4 + \epsilon_2 G_5 + \epsilon_3 G_6 + \cdots . \tag{4.8}$$

Proof. From the previous theorem, the number

$$P = G_2 + \epsilon_2 G_4 + \epsilon_3 G_5 + \cdots$$

is either of the form $A(A(L))$ or $A(B(L))$. Hence

$$f(P) = G_1 + \epsilon_2 G_3 + \epsilon_3 G_4$$

is either of the form $A(L)$ or $B(L)$. But $f(P) - 1 \notin C_2$, so $f(P)$ cannot have the form $B(L)$. Hence $P$ is, in fact, $A(A(K))$ and all of the relations follow, the third using (4.2) and (4.3).

Theorem 11. $A = a$, $B = b$, $C = c$ and for any integer $N$,

$$\begin{cases} f(a(N)) = N \\ f(b(N)) = a(N) \\ f(c(N)) = b(N) \end{cases} . \tag{4.9}$$

Proof. We prove the first part of the theorem by verifying the conditions of Theorem 7. The second part will be established incidentally in the course of the proof. Only the requirement that $A(N)$, $B(N)$ and $C(N)$ be disjoint and exhaustive is not clear.

Now $A(A(N)) \in C_2$ by Theorem 8, so by Theorem 3,

$$f(B(N) - 1) = f(A(A(N))) = f(B(N)) .$$

Hence $B(N) \notin A(N)$ (and $f(B(N)) = A(N)$). Now

$$C(N) = N + A(N) + A(A(N)) + 1 .$$

Let (canonical representation)

$$A(A(N)) = G_2 + \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots .$$

Then

$$A(N) - 1 = \epsilon_3 G_2 + \epsilon_4 G_4 + \cdots ,$$

and, since $A(N) - 1 \notin C_2$, it follows that $\epsilon_3 = 0$. Applying $f$ we get

$$N - 1 = \epsilon_3 G_1 + \epsilon_4 G_2 + \cdots ,$$

so that

$$\begin{aligned} C(N) &= 3 + F_2 + \epsilon_3 G_4 + \epsilon_4 G_5 + \cdots \\ &= G_4 + \epsilon_4 G_5 + \epsilon_5 G_6 + \cdots . \end{aligned}$$

This is not necessarily the canonical representation of $C(N)$ but

$$f(C(N)) = A(A(N)) + 1 \qquad ( = B(N))$$

and

$$f(C(N) - 1) = A(A(N)) + 1 .$$

Hence $C(N) \not\subset A(\mathbf{N})$. Now suppose $C(N) = A(A(M)) + 1$ for some $M$. Then

$$B(N) = A(A(N)) + 1 = f(C(n)) = A(M) ,$$

a contradiction. Hence we have shown that $A(\mathbf{N})$, $B(\mathbf{N})$ and $C(\mathbf{N})$ are disjoint.

Now suppose $N \notin A(\mathbf{N}) \cup B(\mathbf{N})$. Let (canonical representation)

$$N = G_k + \epsilon_{k+1} G_{k+1} + \cdots . \tag{4.10}$$

By Theorem 9 this is equivalent to assuming $A(N) \notin C_2$, that is,

$$A(N) = G_{k+1} + \epsilon_{k+1} G_{k+2} + \cdots , \tag{4.11}$$

and since, always, $A(N) - 1 \notin C_2$, we have $k + 1 \equiv 2 \pmod 3$, that is, $k \equiv 1 \pmod 3$.

First let us consider the case $k = 4$. Then, if we put

$$K - 1 = \epsilon_5 G_2 + \epsilon_6 G_3 + \cdots , \tag{4.12}$$

we get, by Theorem 10,

$$c(K) = N .$$

Now suppose $k > 4$; $k = 3t + 1$, $t > 1$. Then let $s = t - 1$ and set

$$\begin{aligned} K &= G_{3s+1} + \epsilon_{3t+2} G_{3s+2} + \cdots \\ &= G_{-2} + (G_{-1} + G_0) + (G_2 + G_3) + \cdots + (G_{3s-1} + G_{3s}) + \epsilon_{3t-2} G_{3s+2} \cdot \end{aligned}$$

Now, applying Theorem 10 to $K - (G_{-2} + (G_{-1} + G_0))$, we get

$$\begin{aligned} &C(K - (G_{-2} + G_{-1} + G_0) + 1) \\ &\quad = G_4 + (G_5 + G_6) + \cdots + (G_{3t-1} + G_{3t}) + \epsilon_{3t+2} G_{3t+2} + \cdots \\ &\quad = N. \end{aligned}$$

This proves the Theorem.

## 5. THE SECOND CANONICAL REPRESENTATION

Theorem 12. Every positive integer N can be written in a unique way in the form

$$N = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \tag{5.1}$$

where $s \geq 0$ and, as before, $\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0$. Moreover,

$$a(N) = G_{3s+2} + \epsilon_{3s+2} G_{3s+3} + \cdots , \tag{5.2}$$

$$b(N) = G_{3s+3} + \epsilon_{3s+2} G_{3s+4} + \cdots , \tag{5.3}$$

and

$$c(N) = G_{3s+4} + \epsilon_{3s+2} G_{3s+5} + \cdots . \tag{5.4}$$

Proof. We saw in the proof of the previous theorem that an integer M is of the form c(K) if and only if it is given canonically by

$$M = G_k + \epsilon_{k+1} G_{k+1} + \cdots , \qquad k \equiv 1 \pmod 3.$$

Hence for some $s \geq 0$, c(N) is given canonically by

$$c(N) = G_{3s+4} + \epsilon_{3s+2} G_{3s+5} + \cdots .$$

Apply f repeatedly to get the existence of the representation and formulas (5.2) and (5.3). Now if we assume that N can be written in two different ways in the form (5.1), we should obtain two different canonical representations of c(N). Hence the theorem is proved.

We may call (5.1) the second canonical representation.

In view of the representation (5.1), it is natural to let $C_{3s+1}$ denote the set of integers representable in the form (5.1), for a fixed s. Then clearly

$$\overline{C}_{3s+1} = C_{3s+1} \qquad (s \geq 1) \tag{5.5}$$

while

$$\overline{C}_1 = \bigcup_{k=0}^{\infty} (C_{3k+2} \cup C_{3k+3}) \tag{5.6}$$

Making use of the last theorem, we obtain several formulas relating a, b and c. The details are similar in all cases so we will prove only two of the formulas.

Theorem 13. The following formulas hold.

$$\begin{cases} a^2 = b - 1 \\ ab = c - 1 \\ ac = a + b + c \\ ba = c - 2 \\ b^2 = a + b + c - 1 \\ bc = a + 2b + 2c \\ ca = a + b + c - 3 \\ cb = a + 2b + 2c - 2 \\ c^2 = 2a + 3b + 4c \end{cases} \tag{5.7}$$

Proof. To prove, for instance, that

$$bc = a + 2b + 2c ,$$

we suppose that

$$c(N) = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \qquad (s \geq 1) . \tag{5.8}$$

Then by Theorem 12

$$bc(N) = G_{3s+3} + \epsilon_{3s+2} G_{3s+4} + \cdots \qquad (s \geq 1) .$$

But, by applying f to c(N) we see that

$$b(N) = G_{3s} + \epsilon_{3s+2} G_{3s+1} + \cdots \qquad (s \geq 1)$$

and

$$a(N) = G_{3s-1} + \epsilon_{3s+2} G_{3s} + \cdots \qquad (s \geq 1) .$$

Now the result follows if we observe that

$$G_{n-1} + 2G_n + 2G_{n-1} = G_{n+3} .$$

Similarly, to prove that

$$b^2 = a + b + c - 1 ,$$

suppose that

$$c(N) = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \qquad (s \geq 1) .$$

Then

$$b(N) = G_{3s} + \epsilon_{3s+2} G_{3s+1} + \cdots \qquad (s \geq 1)$$

and

$$a(N) = G_{2s-1} + \epsilon_{3s+2} G_{3s} + \cdots \qquad (s \geq 1) .$$

Now we write b(N) in the second canonical form:

$$b(N) = (G_1 + G_2) + \cdots + (G_{3s-2} + G_{3s-1}) + \epsilon_{3s+2} G_{3s+1} + \cdots \quad .$$

Then

$$b^2(n) = (G_3 + G_4) + \cdots + (G_{3s} + G_{3s+1}) + \epsilon_{3s+2} G_{3s+3} + \cdots .$$

Hence

$$b^2(N) + 1 = b^2(N) + G_2 = a(N) + b(N) + c(N) .$$

A word function (or simply word) $u$ is a monomial in $a, b, c$:

$$u = a^{i_1} b^{j_1} c^{k_1} \cdots a^{i_r} b^{j_r} c^{k_r} , \tag{5.9}$$

where the exponents are arbitrary nonnegative integers. Since $au = bv$, for example, is impossible, and $au = av$ implies $u = v$, it follows that the representation (5.9) is unique. In other words factorization into prime elements $a, b, c$ is unique. We define the weight of a word by means of

$$p(a) = 1, \qquad p(b) = 2, \qquad p(c) = 3$$

together with

$$p(uv) = p(u) + p(v) ,$$

where $u, v$ are arbitrary words. Let $N_p$ denote the number of words of weight $p$. If $u$ is any such word then either

$$u = au_1, \qquad u = bu_2 \qquad \text{or} \qquad u = cu_3 ,$$

where

$$p(u_1) = p - 1, \qquad p(u_2) = p - 2, \qquad p(u_3) = p - 3 .$$

Hence

$$N_p = N_{p-1} + N_{p-2} + N_{p-3} \qquad (p \geq 3) .$$

Moreover

$$N_0 = N_1 = 1, \qquad N_2 = 2 .$$

It follows that

$$N_p = G_{p+1} . \tag{5.10}$$

Theorem 14. The words $u, v$ satisfy

$$uv = vu \tag{5.11}$$

if and only if there is a word $w$ such that

$$u = w^r, \qquad v = w^s ,$$

where $r$ and $s$ are nonnegative integers.

Proof. The proof is by induction on $p(u) + p(v)$. We may assume that both $u$, $v$ have positive weight. Also we may assume that $p(u) \geq p(v)$. It then follows from (5.11) and unique factorization that $u = vz$, where $z$ is a word. Thus (5.11) reduces to

$$zv = vz . \tag{5.12}$$

Since $p(zv) < p(uv)$, the inductive hypothesis gives

$$z = w^r, \qquad v = w^s ,$$

so that $u = w^{r+s}$.

For the next theorem we require, in addition to the weight of $u$, the degree of $u$, $d(u)$, defined by

$$d(u) = i_1 + j_1 + k_1 + \cdots + i_r + j_r + k_r , \tag{5.13}$$

where $u$ is given by (5.9). We also define the integers $H_n$ by means of

(5.14) $H_0 = 0, \quad H_1 = 1, \quad H_2 = 2, \quad H_{n+1} = H_n + H_{n-1} + H_{n-2} \quad (n \geq 2).$

It is easy to show that

$$H_n = G_n + G_{n-1}\,. \tag{5.15}$$

Theorem 15. Let u be a word of weight p. Then

$$u(n) = G_{p-3}a(n) + H_{p-3}b(n) + G_{p-2}c(n) = \lambda_u\,, \tag{5.16}$$

where $\lambda_u$ is independent of n but depends on u.

To have the theorem hold for all $p \geq 1$ we extend the definition of $G_n$, $H_n$ for negative values of n. In particular, we have the following table of values

| n | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| G | -1 | 1 | 0 | 0 | 1 | 1 | 2 |
| H | -1 | 0 | 1 | 0 | 1 | 2 | 3 |

It is now easily verified that the theorem holds for the words a, b and c. We assume that (5.16) holds for words of degree k. Let u be an arbitrary word of degree k + 1. There are three cases according as u = va, vb or vc. Assume v has weight p.

(i) For u = va, we have, by the inductive hypothesis and Theorem 13,

$$\begin{aligned} u = va &= G_{p-3}a^2 + H_{p-3}ba + g_{p-2}ca - \lambda_v \\ &= G_{p-3}(b-1) + H_{p-3}(c-2) + G_{p-2}(a+b+c-3) - \lambda_v \\ &= G_{p-2}a + (G_{p-2} + G_{p-3})b + (H_{p-3} + G_{p-2})c - (G_{p-3} + 2H_{p-3} + 3G_{p-2} + \lambda_v) \\ &= G_{p-2}a + H_{p-2}b + G_{p-1}c - (H_p + \lambda_v)\,. \end{aligned}$$

(ii) For u = vb, we have

$$
\begin{aligned}
u = vb &= G_{p-3}ab + H_{p-3}b^2 + G_{p-2}cb - \lambda_v \\
&= G_{p-3}(c - 1) + H_{p-3}(a + b + c - 1) + G_{p-2}(a + 2b + 2c - 2) - \lambda_v \\
&= (G_{p-2} + H_{p-3})a + (2G_{p-2} + H_{p-3})b + (2G_{p-2} + G_{p-3} + H_{p-3})c \\
&\qquad - (2G_{p-2} + G_{p-3} + H_{p-3} + \lambda_v) \\
&= G_{p-1}a + H_{p-1}b + G_pc - (G_p + \lambda_v) .
\end{aligned}
$$

(ii) For $u = vc$, we have

$$
\begin{aligned}
u = vc &= G_{p-3}ac + H_{p-3}bc + G_{p-2}c^2 - \lambda_v \\
&= G_{p-3}(a + b + c) + H_{p-3}(a + 2b + 2c) + G_{p-2}(2a + 3b + 4c) = \lambda_v \\
&= (2G_{p-2} + G_{p-3} + H_{p-3})a + (3G_{p-2} + G_{p-3} + 2H_{p-3})b \\
&\qquad + (4G_{p-2} + G_{p-3} + 2H_{p-3})c = \lambda_v \\
&= G_pa + H_pb + G_{p+1}c - \lambda_v .
\end{aligned}
$$

This completes the proof. Incidentally, we have proved the following relations:

$$
(5.17) \qquad \begin{cases} \lambda_{va} = H_p + \lambda_v \\ \lambda_{vb} = G_p + \lambda_v \\ \lambda_{vc} = \lambda_v \end{cases} ,
$$

where $v$ is of weight $p$.

As an immediate corollary of the last theorem, we state:

Theorem 16. Let $u$ and $v$ be arbitrary words. Then there is an integer $C$ such that

$$
(5.18) \qquad uv - vu = C .
$$

## 6. AN ESTIMATE OF $a(n)$

Let $\alpha$ be the real root of $x^3 - x^2 - x - 1 = 0$ and let $\beta$ and $\gamma$ be the complex roots, $\beta = re^{i\theta}$, $\gamma = re^{-i\theta}$. Then we have

(6.1) $$G_{n+1} - \alpha G_n = \frac{\gamma^{n+1} - \beta^{n+1}}{\gamma - \beta}$$

which we can verify by taking $n$ equal to $-1$, $0$ and $1$ and noting that both sides of (6.1) satisfy the recurrence

$$u_{n+3} = u_{n+2} + u_{n+1} + u_n \cdot$$

If $N$ is given in the second canonical representation by

(6.2) $$N = G_{3k+1} + \epsilon_{3k+2} G_{3k+2} + \cdots \qquad (k \geq 0) ,$$

we have

(6.3) $$\begin{cases} a(N) = G_{3k+2} + \epsilon_{3k+2} G_{3k+3} + \cdots \\ b(N) = G_{3k+3} + \epsilon_{3k+2} G_{3k+4} + \cdots \\ c(N) = G_{3k+4} + \epsilon_{3k+2} G_{3k+5} + \cdots \ . \end{cases}$$

Now $\alpha = 1.8, \cdots$, so that $|\beta| = |\gamma| = \sqrt{\beta\gamma} = \sqrt{1/\alpha} < 1$. Then, using (6.1), (6.2), and (6.3) we get the following.

Theorem 17. The three sequences

$$a(N) - [\alpha N], \qquad b(N) - [\alpha^2 N], \qquad c(N) - [\alpha^3 N]$$

are all bounded.

Next we prove

Theorem 18. The difference $a(N) - [\alpha N]$ is positive infinitely often, negative infinitely often and $0$ infinitely often.

Proof. If $\theta$ were a rational multiple of $2\pi$ we should have, for some $m$,

$$\gamma^{m+1} = \beta^{m+1} = r^{m+1}$$

and, by (6.1), $G_{m+1} = \alpha G_m$. But $\alpha$ is irrational so this is impossible. Hence for infinitely many $k$ we must have

(6.4)
$$G_{3k+2} - \alpha G_{3k+1} = r^{3k+1} \frac{\sin\{(3k + 2)\theta\}}{\sin \theta} > 0 ,$$

that is,

$$a(G_{3k+1}) - [\alpha G_{3k+1}] > 0 ,$$

for infinitely many k.

To get the second part of the theorem we must find an infinite number of integers N for which

$$a(N) - \alpha N < -1 .$$

Let N have the form

$$N = G_1 + G_3 + G_k ,$$

where k is very large. Then

$$\begin{aligned} a(N) - \alpha N &= G_2 - \alpha G_1 + G_4 - \alpha G_3 + G_{k+1} - \beta G_k \\ &= 1 - \alpha + 3 - 2\alpha + G_{k+1} - \alpha G_k \approx -1.4 . \end{aligned}$$

This proves the theorem.

Finally to prove that the difference vanishes infinitely often, it suffices to show that (compare (6.4))

$$-1 \leq r^{3k+1} \frac{\sin(3k + 2)}{\sin \theta} < 0$$

for infinitely many values of n. This is clear since $0 < r < 1$ and $\theta$ is an irrational multiple of $2\pi$.

## 7. GENERATING FUNCTIONS

Put

$$\phi_k(x) = \sum_{n \in C_k} x^n \qquad (k = 2, 3, 4, \cdots) \tag{7.1}$$

and

$$\overline{\phi}_{3k+1}(x) = \sum_{n \in \overline{C}_{3k+1}} x^n \qquad (k = 0, 1, 2, \cdots). \tag{7.2}$$

In view of (5.5) and (5.6), we have

$$\overline{\phi}_{3k+1}(x) = \phi_{3k+1}(x) \qquad (k = 1, 2, 3, \cdots) \tag{7.3}$$

and

$$\overline{\phi}_1(x) = \sum_{k=0}^{\infty} \phi_{3k+2}(x) + \sum_{k=0}^{\infty} \phi_{3k+3}(x) . \tag{7.4}$$

It is evident that

$$\frac{x}{1-x} = \sum_{k=2}^{\infty} \phi_k(x) . \tag{7.5}$$

Also it follows from the definition of $C_k$ that

$$\begin{aligned} \phi_k(x) &= x^{G_k}\left\{1 + \sum_{j=k+2}^{\infty} \phi_j(x)\right\} \\ &\quad + x^{G_k + G_{k+1}}\left\{1 + \sum_{j=k+3}^{\infty} \phi_j(x)\right\} . \end{aligned} \tag{7.6}$$

From (1.1) we get the recurrence

$$
\begin{aligned}
(7.7)\qquad & (1 + x^{G_{k+1}})(\phi_{k+1}(x) + x^{G_{k+1}+G_{k+2}}\phi_{k+3}(x)) \\
& = x^{G_{k+1}-G_k}(1 + x^{G_{k+2}})(\phi_k(x) - x^{G_k}\phi_{k+2}(x)) .
\end{aligned}
$$

It is also convenient to define

$$
(7.8)\qquad A(x) = \sum_{n=1}^{\infty} x^{a(n)}, \qquad B(x) = \sum_{n=1}^{\infty} x^{b(n)}, \qquad C(x) = \sum_{n=1}^{\infty} x^{c(n)} ,
$$

so that

$$
(7.9)\qquad A(x) + B(x) + C(x) = \frac{x}{1 - x} .
$$

Moreover

$$
(7.10)\qquad A(x) = \sum_{k=0}^{\infty} \phi_{3k+2}(x) ,
$$

$$
(7.11)\qquad B(x) = \sum_{k=0}^{\infty} \phi_{3k+3}(x) ,
$$

$$
(7.12)\qquad C(x) = \sum_{k=0}^{\infty} \phi_{3k+4}(x) = \frac{x}{1 - x} - \bar{\phi}_1(x) .
$$

Now by (7.1) and (1.12)

$$
\phi_2(x) = \sum_{n=1}^{\infty} x^{a^2(n)} + \sum_{n=1}^{\infty} x^{ab(n)} .
$$

Since

$$a^2(N) = b(n) - 1, \qquad ab(n) = c(n) - 1 ,$$

It follows that

(7.13) $$x\phi_2(x) = B(x) + C(x) .$$

In the next place, by (1.13)

$$\begin{aligned}\phi_3(x) &= \sum_{n=1}^{\infty} x^{ba(n)} + \sum_{n=1}^{\infty} x^{b^2(n)} \\ &= x^{-2}C(x) + x^{-1}\sum_{n=1}^{\infty} x^{a(n)+b(n)+c(n)} .\end{aligned}$$

Since

$$\begin{aligned}A(x) &= \sum_{n=1}^{\infty} x^{a^2(n)} + \sum_{n=1}^{\infty} x^{ab(n)} + \sum_{n=1}^{\infty} x^{ac(n)} \\ &= x^{-1}A(x) + x^{-1}B(x) + \sum_{n=1}^{\infty} x^{a(n)+b(n)+c(n)} ,\end{aligned}$$

it follows that

(7.14) $$x^2\phi_3(x) = xA(x) - B(x) .$$

By (1.1)

$$\begin{aligned}\phi_2(x) &= x\left\{1 + \sum_{j=4}^{\infty}\phi_j(x)\right\} + x^3\left\{1 + \sum_{j=5}^{\infty}\phi_j(x)\right\}\\ &= x + x^3 + x\phi_4(x) + (x + x^3)\sum_{j=5}^{\infty}\phi_j(x)\\ &= x + x^3 + x\phi_4(x) + (x + x^3)\left\{\frac{x}{1 - x} - \phi_2(x) - \phi_3(x) - \phi_4(x)\right\}.\end{aligned}$$

Combining this with (7.13) and (7.14), we get

$$x^4\phi_4(x) = x^2n(x) - C(x)\ . \tag{7.15}$$

In a similar manner we get

$$x^8\phi_5(x) = -xA(x) + B(x) + (1 + x^4)C(x)\ , \tag{7.16}$$

$$x^{15}\phi_6(x) = (x + x^8)A(x) - (1 + x^2 + x^7)B(x) - x^7C(x)\ . \tag{7.17}$$

Generally it can be shown that

$$x^{S_{k-1}}\phi_k(x) = p_1(x)A(x) + p_2(x)B(x) + p_3(x)C(x)\ , \tag{7.18}$$

where

$$S_k = G_1 + G_2 + \cdots + G_k$$

and $p_1(x)$, $p_2(x)$, $p_3(x)$ are polynomials with integral coefficients.

In the next place, exactly as in [1, Sec. 7], we can show that $A(x)$, $B(x)$ and $C(x)$ cannot be continued analytically across the unit circle. For the proof it suffices to use

$$\sum_{a(k)\le n} 1 \sim \frac{n}{\alpha}, \qquad \sum_{b(k)\le n} 1 \sim \frac{n}{\alpha^2}, \qquad \sum_{c(k)\le n} 1 \sim \frac{n}{\alpha^3}\ . \tag{7.19}$$

which follow from Theorem 17. Indeed, we can show in this way that none of the functions can be continued across the unit circle. Moreover if we put

$$\phi_k(x) = \phi_k^a(x) + \phi_k^{(b)}(x) ,$$

where (compare (1.12), (1.13), (1.14))

$$\phi_{3k+2}^a(x) = \sum_{n=1}^{\infty} x^{ac^k a(n)}, \quad \phi_{3k+3}^a(x) = \sum_{n=1}^{\infty} x^{bc^k a(n)},$$

$$\phi_{3k+4}(x) = \sum_{n=1}^{\infty} x^{c^{k+1} a(n)}, \quad \phi_{3k+2}^b(x) = \sum_{n=1}^{\infty} x^{ac^k b(n)}, \qquad \text{and}$$

$$\phi_{3k+3}^b(x) = \sum_{n=1}^{\infty} x^{bc^k b(n)}, \quad \phi_{3k+4}^b(x) = \sum_{n=1}^{\infty} x^{c^{k+1} b(n)} ,$$

then neither $\phi_k^a(x)$ nor $\phi_k^b(x)$ can be continued across the unit circle.

We can also show that $A(x)$, $B(x)$, $C(x)$ do not satisfy any relation of the form

$$f_1(x)A(x) + f_2(x)B(x) + f_3(x)C(x) = 0 , \tag{7.20}$$

where $f_1(x)$, $f_2(x)$, $f_3(x)$ are polynomials. In the first place we may assume without loss of generality that the coefficients of $f_1(x)$ are rational (for proof compare [3, p. 141, No. 151]) and that

$$(f_1(x), \quad f_2(x), \quad f_3(x)) = 1 . \tag{7.21}$$

Since (7.19) implies

$$\lim_{x=1} (1 - x)A(x) = \frac{1}{\alpha} \qquad \lim_{x=1} (1 - x)B(x) = \frac{1}{\alpha^2}, \qquad \lim_{x=1} (1 - x)C(x) = \frac{1}{\alpha^3},$$

[Continued on page 94.]

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$$A_r(x) = \sum_{n=1}^{F_{r-1}} x^{a^2(n)+n-2} \qquad (4.15)$$

Note that by (4.15) and (4.3), we have

$$\lim_{r\to\infty} A_r(x) = x^{-1}\psi_1(x)$$

in agreement with (4.12) and (4.14).

Exactly as in [1] it can be shown that the function $\psi_1(x)$ has the unit circle for a natural boundary. In view of (4.12) the same is true of each of the functions $\psi_k(x)$.

It would be of interest to know whether there is any simple relation connecting $\psi_1(x)$ with

$$\phi(x) = \sum_{n=1}^{\infty} x^{a(n)} .$$

In particular, do there exist polynomials $P(x)$, $Q(x)$, $R(x)$ such that

$$P(x)\phi(x) + Q(x)\psi_1(x) = R(x) \ ? \qquad (4.16)$$

## 5. FURTHER RESULTS

In [3] the following are given:

$\nu(kL_n) = kF_{n-1}$, for $n$ sufficiently large;
$\nu(5kF_n) = kL_{n-1}$, for $n$ sufficiently large;

$\nu(L_{2n}^2) = F_{4n-1}$, $(n \ge 1)$; $\nu(L_{2n-1}^2) = F_{4n-3} - 1$, $(n \ge 1)$;
$\nu(F_{2n}) = F_nF_{n-1}$, $(n \ge 2)$; $\nu(L_nL_{n-1}) = F_{2n-2}$, $(n \ge 2)$;
$\nu(L_{2n+1}L_{2n-1}) = F_{4n-1} - 1$, $(n \ge 1)$; $\nu(L_{2n+2}L_{2n}) = F_{4n+1} + 1$, $(n \ge 1)$;
$\nu(5F_n) = L_{n-1}$, $(n \ge 2)$; $\nu(5F_n^2) = F_nL_{n-1}$, $(n \ge 3)$;
$\nu(5F_nF_{n+1}) = F_{2n}$, $(n \ge 1)$; $\nu(5F_{2n}F_{2n-2}) = F_{4n-3} - 1$, $(n \ge 1)$.

[Continued on page 112.]

# FIBONACCI REPRESENTATIONS OF HIGHER ORDER - II

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## 1. INTRODUCTION

Let $N \geq 2$ be a fixed integer. We wish to discuss various properties of sequences $\{v_n\}$ $(n = 0, \pm 1, \pm 2, \cdots)$ of complex numbers satisfying the recurrence

(1.1) $$v_{n+N} = v_{n+N-1} + \cdots + v_{n+1} + v_n \qquad (n = 0, \pm 1, \pm 2, \cdots) .$$

We let $\mathbb{V}$ be the set of sequences satisfying (1.1) and we let $\mathbb{D}$ be the set of all sequences $\delta_n$ $(n = 0, \pm 1, \pm 2, \cdots)$ which are non-zero on only a finite number of coordinates. For $\delta \in \mathbb{D}$ and $v \in \mathbb{V}$ we define

$$\delta(v) = \sum \delta_n v_n \quad .$$

We will call $\delta \in \mathbb{D}$ canonical if

(1.2) $$\delta_i \neq 0 \Rightarrow \delta_i = 1 \qquad (i = 0, \pm 1, \cdots)$$

and

(1.3) $$\delta_i \delta_{i+1} \cdots \delta_{i+N-1} = 0 \qquad (i = 0, \pm 1, \cdots) .$$

We will say $\epsilon$ and $\epsilon' \in \mathbb{D}$ are equivalent $(\epsilon \equiv \epsilon')$ if $\epsilon(v) = \epsilon'(v)$ for all $v \in R$.

We shall also have occasion to use the translation operator $T$ on sequences from $\mathbb{D}$ or $\mathbb{V}$ defined by

(1.4) $$(Tv)_n = v_{n+1} \qquad (v \in \mathbb{D} \text{ or } \mathbb{V}) .$$

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The main theorem of the present paper is the following.

Theorem A. Let $\epsilon \in \mathbf{D}$ have integral coordinates. Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element of $\mathbf{D}$.

We use this theorem first to generalize a result of Klarner's [4] for Fibonacci numbers to $N^{th}$ order Fibonacci numbers $P = \{P_n\}$ defined by

(i) $$P \in \mathbf{W}$$

(ii) $$P_{-(N-2)} = \cdots = P_0 = 0, \qquad P_1 = 1 .$$

The generalization is as follows:

Theorem B. Let $K_1, K_2, \cdots, K_N$ be positive integers. Then there is a unique canonical $\delta \in \mathbf{D}$ such that

$$K_i = \delta(T^i P) \qquad (i = 1, 2, \cdots, N) . \tag{1.5}$$

If $\gamma$ is a root of

$$x^N - x^{N-1} - \cdots - x - 1 = 0 \tag{1.6}$$

we let $\underline{\gamma}$ be the sequence in $\mathbf{W}$ defined by

$$(\underline{\gamma})_n = \gamma^n . \tag{1.7}$$

We let $\alpha$ be the largest positive root of (1.6). Note that $\alpha > 1$.

As a corollary to the main theorem we get

Theorem C. A positive real number $x$ is of the form $\delta(\underline{\alpha})$ for some canonical $\delta \in \mathbf{D}$ if and only if, for some positive $k$ and some integers $Q_1$, $Q_2, \cdots, Q_N$ we have

$$\alpha^k x = Q_1 + Q_2\alpha + \cdots + Q_N\alpha^{N-1} . \tag{1.8}$$

In Section 4, we assume that $N = 3$ and verify some conjectures of Hoggatt concerning certain functions introduced and discussed in [1], [2] and

[3]. The authors believe that the results obtained in Section 4 for the case $N = 3$ are strongly indicative of those that might hold for larger values of N.

## 2. PROPERTIES OF CANONICAL ELEMENTS

Theorem 1. Suppose $\delta$ and $\epsilon \in \mathbf{D}$ are canonical. Then either $\delta - \epsilon$ or $\epsilon - \delta$ is equivalent to $\gamma \in \mathbf{D}$.

Proof. The non-zero coordinates of $\eta = \delta - \epsilon$ are 1's and -1's. Suppose the first non-zero coordinate of $\eta$ (starting from the left) is -1, and let $\eta_k = 1$ be the first 1. Now change $\eta_k$ to 0 and add 1 to each of $\eta_{k-1}, \eta_{k-2}, \cdots, \eta_{k-N}$. The resulting sequence is equivalent to $\eta$, and since $\delta$ and $\epsilon$ are canonical, it can be seen that not all of $\eta_{k-1} + 1, \cdots, \eta_{k-N} + 1$ are 0. Performing this "change" repeatedly, we finally come to a sequence $\eta'$ equivalent to $\eta$ all of whose non-zero coordinates are either 1 or -1. This of course implies that either $\eta$ or $-\eta$ is equivalent to a canonical element of $\mathbf{D}$.

Theorem 2. Let $\epsilon \in \mathbf{D}$ have integral coordinates. Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element of $\mathbf{D}$. If the coordinates of $\epsilon$ are non-negative then $\epsilon$ is equivalent to a canonical element of $\mathbf{D}$.

Proof. We set $\epsilon = \epsilon^+ - \epsilon^-$. The previous theorem shows that the first statement of the present theorem follows from the second; so we assume $\epsilon = \epsilon^+$.

Now a simple induction shows that it is enough to prove the following statement: If $\epsilon$ is canonical, then $\epsilon + \chi_i$ is equivalent to a canonical element, where $\chi_i$ is defined by

$$\chi_i(V) = v_i \qquad v \in \mathbf{V} . \tag{2.1}$$

Note that $\epsilon + \chi_i = \epsilon - \chi_{i-1} - \cdots - \chi_{i-N+1} + \chi_{i+1} \equiv \gamma_1 + \chi_{i+1}$ where, by Theorem 1 either $\gamma_1$ or $-\gamma_1$ is canonical. If $-\gamma_1$ is canonical, then again by Theorem 1, $\gamma_1 + \chi_{i+1}$ is equivalent to a canonical element. Hence we may suppose $\gamma_1$ is canonical. Then we get

$$\epsilon + \chi_i \equiv \gamma_1 + \chi_{i+1} \equiv \gamma_2 + \chi_{i+2} \equiv$$

with $\gamma_1, \gamma_2, \cdots$ canonical. But this is impossible for, if so, we would have

$$[\epsilon + \chi_i](\underline{\alpha}) \geq \chi_{i+n}(\underline{\alpha}) = \alpha^{i+n} \qquad (n = 1, 2, \cdots) \,. \tag{2.2}$$

This completes the proof.

Let $P \in \mathbb{V}$ be the sequence defined by the initial conditions

$$P_{-(N-2)} = \cdots = P_0 = 0; \qquad P_1 = 1 \,. \tag{2.3}$$

Theorem 3. Let $K$ be a positive integer. Then there is a unique canonical $\delta \in \mathbb{D}$ such that, for all $n$,

$$P_n K = \sum_i \delta_i P_{i+n} \,. \tag{2.4}$$

Proof. Let $\epsilon \in \mathbb{D}$ be the sequence

$$\epsilon_n = \begin{cases} K & n = 0 \\ 0 & \text{otherwise} \end{cases} \,. \tag{2.5}$$

Then by Theorem 2 there is a unique canonical $\delta \in \mathbb{D}$ satisfying

$$\epsilon(v) = \delta(v), \qquad v \in \mathbb{V} \,. \tag{2.6}$$

Letting $v$ be translates of $P$ we get (2.4) immediately since $\epsilon(v) = v_0 K$ for any $v \in \mathbb{V}$.

The uniqueness of $\delta$ will follow if we can show that any $\gamma \in \mathbb{V}$ is determined by its value on translates of $P$. We state this as a separate theorem.

Theorem 4. $\mathbb{V}$ is N-dimensional as a complex vector space. It is spanned by $P, TP, \cdots, T^{N-1}P$. Moreover, the $N \times N$ matrix

$$\Delta_i = \{(T^j P)_n\} \qquad \begin{matrix} (j = 0, 1, \cdots, N-1) \\ (n = 0, i+1, \cdots, i+N-1) \end{matrix}$$

has determinant

$$|\Delta_i| = \left((-1)^{N+1}\right)^{i+1} .$$

Proof. The fact that V is N-dimensional is well-known, so the calculation of the determinant will complete the proof: we have

$$\Delta_i = \begin{pmatrix} P_i & P_{i+1} & \cdots & P_{i+n-1} \\ P_{i+1} & P_{i+2} & \cdots & P_{i+n} \\ \vdots & & & \\ P_{i+n-1} & & \cdots & P_{i+2n-2} \end{pmatrix}$$

Adding the last N - 1 columns to the first, using the recurrence and interchanging columns we get

$$|\Delta_i| = (-1)^{N+1} |\Delta_{i+1}| . \tag{2.7}$$

But

$$\Delta_{-(N-2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 1 & \cdots & & & \\ 1 & 1 & \cdots & & & \end{pmatrix}$$

so that

$$|\Delta_{-(N-2)}| = (-1)^{N+1} .$$

Hence

$$|\Delta_i| = \left((-1)^{N+1}\right)^{i+1} .$$

Theorem 5. Let $v \in \mathbf{V}$. Then

(2.8) $$v = v_0TP + (v_1 - v_0)P + (v_2 - v_1 - v_0)T^{-1} + \cdots + (v_{N-1} + \cdots - v_1 - v_0)T^{-(N-2)}P \ .$$

Proof. Let $0 \le j \le N - 1$. The $j^{th}$ coordinate of the right side is

$$\begin{aligned} v_0P_{j+1} + (v_1 + v_0)P_j + \cdots + (v_{N-1} - \cdots - v_1 - v_0)P_{j-(N-2)} \\ = v_0(P_{j+1} - P_j - \cdots - P_{j-(N-2)}) \\ + v_1(P_j - P_{j-1} - \cdots - P_{j-(N-2)}) \\ + v_k(P_{j+1-k} - \cdots - P_{j-(N-2)}) \\ \vdots \\ + v_{N-2}(P_{j-(N-2)+1} - P_{j-(N-2)}) \\ + v_{N-1}(P_{j-(N-2)}) \ . \end{aligned}$$

The coefficient of $v_k$ is non-zero only when $j + 1 - k = 1$, i.e., only when $k = j$. In this case it is 1.

We can generalize a theorem proved by Klarner for the Fibonacci numbers as follows.

Theorem 6. Let $K_1, K_2, K_3, \cdots, K_N$ be positive integers. Then there is a unique canonical $\delta$ such that

(2.10) $$K_i = \delta(T^iP) \qquad (i = 1, 2, \cdots, N) \ .$$

Proof. It will be enough to find a canonical $\delta$ satisfying

(2.11) $$K_i = \delta\left(T^{i-(N-1)}P\right) \qquad (i = 1, 2, \cdots, N)$$

because then a translate of $\delta$ will satisfy (2.10). Let $\gamma$ be one of the N roots of $x^N - x^{N-1} - \cdots - x - 1 = 0$, and let

(2.12) $$v = \underline{\gamma} \ .$$

Then by the previous theorem, if $\delta$ exists and satisfies (2.11) it must also satisfy

$$\begin{aligned}\delta(\underline{\gamma}) &= K_N + (\gamma - 1)K_{N-1} + \cdots + (\gamma^{N-1} - \gamma^{N-2} - \cdots - \gamma - 1)K_1 \\ &= \frac{1 + \gamma + \cdots + \gamma^{N-1}}{\gamma^N} K_N + \frac{1 + \gamma + \cdots + \gamma^{N-2}}{\gamma^{N-1}} + \cdots + \frac{1}{\gamma} K_1 \qquad (2.13)\\ &= K_N\gamma^{-N} + (K_N + K_{N-1})\gamma^{-(N-1)} + \cdots + (K_N + \cdots + K_1)\gamma^{-1} .\end{aligned}$$

Hence we should define $\delta$ to be the unique canonical form in $\mathbb{D}$ equivalent to $\beta \in \mathbb{D}$ where $\beta$ is given by

$$\beta_i = \begin{Bmatrix} K_N + \cdots + K_i & (-N \le i \le -1) \\ 0 & (\text{otherwise}) \end{Bmatrix} \tag{2.14}$$

Now

$$\begin{aligned}\beta\left(T^{i-(N-1)}P\right) &= \sum_{j=1}^{N} (K_N + \cdots + K_j)P_{-j+i-(N-1)} \qquad (2.15)\\ &= \sum_{t=1}^{N} K_t\left(\sum_{j=1}^{t} P_{-j+i-(N-1)}\right) = K_i .\end{aligned}$$

## 3. FURTHER APPLICATIONS OF THE MAIN THEOREM

We recall that $\alpha$ is the largest positive root of

$$x^N - x^{N-1} - \cdots - x - 1 = 0$$

and

$$\underline{\alpha} = (\ldots, \alpha^{-1}, 1, \alpha, \cdots) .$$

Theorem 7. Let $K$ be any positive integer. Then there exists a unique canonical $\delta \in \mathbb{D}$ such that

$$K = \delta(\underline{\alpha}) .$$

Moreover,

$$K = \delta(P) .$$

Proof. Choose $\delta$ as in Theorem 3. Then

$$\delta(\underline{\alpha}) = \epsilon(\underline{\alpha}) = K .$$

Theorem 8. A positive real number $x$ is of the form $\delta(\underline{\alpha})$ for some canonical $\delta \in \mathbb{D}$ if and only if, for some positive $k$ and some integers $Q_1$, $Q_2, \cdots, Q_N$ we have

$$\alpha^k x = Q_1 + Q_2 \alpha + \cdots + Q_N \alpha^{N-1} . \tag{3.1}$$

Proof. Suppose first that $x$ is of the form $\delta(\underline{\alpha})$:

$$x = \sum_{j=-k} \epsilon_j \alpha^j . \tag{3.2}$$

Then

$$\alpha^k x = \sum_{j=0} \epsilon_j \alpha^{j+k}$$

and powers of $\alpha$ higher than $\alpha^{N-1}$ can be successively reduced to lower powers eventually giving (3.1).

Now suppose (3.1) holds. Let $\epsilon \in \mathbb{D}$ be defined by

$$\epsilon_n = \begin{cases} Q_{n+k+1} & -k \le n \le N-k-1 \\ 0 & \text{otherwise} \end{cases} . \tag{3.3}$$

Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element $\delta \in \mathbb{D}$. But

$$\epsilon(\alpha) = x > 0 .$$

Hence we must have $\epsilon \equiv \delta$.

4 .

For the notation used in the remainder of the paper we refer the reader to [3].

Let $\nu_k(M)$ denote the number of numbers $n \in C_k$ such that $n \leq M$.

Theorem 9. If $M \notin C_2$ then

$$\nu_2(M) = M - f(M) . \tag{4.1}$$

More generally, if

$$M \notin C_2 \cup C_3 \cup \cdots \cup C_r$$

then

$$\nu_r(M) = f^{r-2}(M) - f^{r-1}(M) \qquad (r = 2, 3, 4, \cdots) . \tag{4.2}$$

Proof. Let

$$K_r = \{K | K \notin C_2 \cup C_3 \cup \cdots \cup C_r\} , \qquad r \geq 2$$

and let $K_1 = \mathbb{N}$. Then clearly $f^{r-1}$ is $1 - 1$, onto and monotone from $K_r$ to $\mathbb{N}$. In particular,

$$\operatorname{card} \{K | K \in K_r, \ K \leq M\} = f^{r-1}(M) \qquad (r = 1, 2, \cdots) . \tag{4.3}$$

Hence

$$\nu_r(M) = \operatorname{card} \{K | K \in C_r; \ K < M\} = \operatorname{card} \{K | K \in K_{r-1}, \ K \leq M\}$$
$$- \operatorname{card} \{K | K \in F_r, \ K \leq M\} = f^{r-2}(M) - f^{r-1}(M).$$

The following theorem is an immediate corollary.

Theorem 10. We have

$$\nu_2(G_n) = G_n - G_{n-1} = G_{n-2} + G_{n-3} \qquad (n \ge 3) . \tag{4.4}$$

More generally

$$\nu_r(G_n) = G_{n-r+2} - G_{n-r+1} = G_{n-r} + G_{n-r-1} \qquad (n \ge r + 1) . \tag{4.5}$$

Theorem 11. Let $k$ and $r$ be fixed integers, $k \ge 1$, $r \ge 2$. Then

$$\nu_r(kG_n) = k(G_{n-r} + G_{n-r-1}) \tag{4.6}$$

for $n$ sufficiently large.

Proof. Using Theorem 3, we let $\delta \in \mathbf{D}$ be canonical such that

$$kG_n = \sum \delta_i G_{i+n}, \qquad (n = 0, 1, 2, \cdots) . \tag{4.7}$$

Hence for $n$ sufficiently large we will have

$$kG_n \notin C_2 \cup \cdots \cup C_r ,$$

so

$$\begin{aligned} \nu_r(kG_n) &= f^{r-2}(kG_n) - f^{r-1}(kG_n) \\ &= \sum \delta_i G_{i+n-(r-2)} - \sum \delta_i G_{i+n-(r-1)} \\ &= kG_{n-(r-2)} - kG_{n-(r-1)} \\ &= k(G_{n-r} + G_{n-r-1}) . \end{aligned} \tag{4.8}$$

The last three theorems were conjectured by Hoggatt.

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# SOME GENERAL RESULTS ON REPRESENTATIONS

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DEDICATED TO THE MEMORY OF FRANCIS DE KOVEN

## 1. INTRODUCTION

Let $P = \{P_1, P_2, P_3, \cdots\}$ be any sequence of distinct positive integers, then

$$(*) \qquad \prod_{i=1}^{\infty}\left(1 + X^{P_i}\right) = \lim_{m\to\infty} \prod_{i=1}^{m}\left(1 + X^{P_i}\right) = \sum_{n=0}^{\infty} R(n)X^n \ ,$$

where $R(n)$ is the number of representations of the integer $n$ as the sum of distinct elements of $P$. If $P_i = 2^{i-1}$ $(i = 1, 2, \cdots)$, then $R(n) = 1$ for all $n \geq 0$. Brown [1] has shown that if $P_1 = 1$ and

$$P_{n+1} \leq 1 + \sum_{i=1}^{n} P_i \ ,$$

then $R(n) \geq 1$ for all $n \geq 0$. Here we discuss some consequences of the condition

$$(**) \qquad P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i \ .$$

Let $P_1 = 1$, if equality holds for each $n \geq 1$, then $P_i = 2^{i-1}$, $i \geq 1$. If for some $n$, the inequality holds, then $R(m) = 0$ for some $m > 0$, which we call an integer which is non-representable by $P$.

## 2. SOME GENERAL RESULTS

The condition (**) guarantees that $P_i \neq P_j$ for $i \neq j$. Further we may prove

Theorem 1. Every positive integer N which has a representation by the sum of distinct elements of P, then that representation is unique.

Proof. Clearly each $P_i$ is its own unique representation since the sequence is strictly increasing and $P_{n+1} > P_1 + P_2 + P_3 + \cdots + P_n$. Suppose N had two different representations

$$N = \sum_{i=1}^{k} \alpha_i P_i = \sum_{i=1}^{m} \beta_i P_i ,$$

where $\alpha_i$ and $\beta_i = 0$ or $1$ independently, with $\alpha_k = \beta_m = 1$. If $m = k$, then delete $P_m = P_k$ from each side and continue to do so step-by-step until the highest order term on the left is different from the highest order term on the right. Now assume $P_k > P_m$. This is an immediate contradiction since $P_k > P_1 + P_2 + \cdots + P_m + \cdots + P_{k-1}$, thus both representations cannot represent N. This evidently proves Theorem 1.

## 3. THE NON-REPRESENTABLE INTEGERS

In certain cases, the integers which cannot be represented by sequence P can be described by a suitable closed form. See [3] and [4], however, that is not the general situation.

Definition. Let M(n) be the number of positive integers less than n which cannot be represented by the sequence P.

Theorem 2. If

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i ,$$

then

$$M(P_{n+1}) = P_{n+1} - 2^n .$$

Proof. All the sums of the $2^n$ subsets of $\{P_1, P_2, P_3, \cdots, P_n\}$ distinct by Theorem 1. These sums are less than $P_{n+1} > P_1 + P_2 + \cdots$

$+ P_n$, thus

$$M(P_{n+1}) = (P_{n+1} - 1) - (2^n - 1) = P_{n+1} - 2^n$$

since $P_{n+1} - 1$ is the number of positive integers $< P_{n+1}$ and the empty subset yields the non-positive sum zero. In fact it is simple to prove further.

Theorem 3. $M(P_1 + P_2 + \cdots + P_n) = M(P_1) + \cdots + M(P_n)$.

Proof. $M(P_{n+1}) = P_{n+1} - 2^n$. Since $P_1 + P_2 + \cdots + P_n < P_{n+1}$, then all the integers between

$$\sum_{i=1}^{n} P_i$$

and $P_{n+1}$ are non-representable. Thus

$$M(P_1 + P_2 + P_3 + \cdots + P_n) = (P_{n+1} - 2^n) - \left(P_{n+1} - \left(\sum_{i=1}^{n} P_i\right) - 1\right)$$

$$= P_1 + P_2 + P_3 + \cdots + P_n - (2^n - 1)$$

$$= P_1 + P_2 + P_3 + \cdots + P_n - (1 + 2^1 + 2^2 + \cdots + 2^{n-1})$$

$$= (P_1 - 2^0) + (P_2 - 2^1) + (P_3 - 2^2) + \cdots + (P_n - 2^{n-1})$$

$$= \sum_{i=1}^{n} M(P_i),$$

which concludes the proof of Theorem 3.

## 4. M(N) FOR REPRESENTABLE N

The main result in this section is the statement and proof of

Theorem 4. If

$$N = \sum_{i=1}^{k} \alpha_i P_i ,$$

then

$$M(N) = N - \sum_{i=1}^{k} \alpha_i 2^{i-1} ,$$

where each $\alpha_i = 1$ or $0$.

Proof. Let

$$N = \sum_{i=1}^{k} \alpha_1 P_i ,$$

then $P_k \le N < P_{k+1}$. Thus

$$M(N) = (P_k - 2^{k-1}) + M(N - P_k) ,$$

by virtue

$$\prod_{i=1}^{k-1}\left(1 + X^{P_i}\right) = \sum_{n=0}^{q} R(n)X^n, \qquad q = \sum_{i=1}^{k-1} P_i .$$

In forming these polynomials, the representations using only $P_1$, $P_2$, $\cdots$, $P_{k-1}$ are enumerated by the $R(n)$ for $n = 0$ to $n = P_1 + P_2 + \cdots + P_{k-1}$. The polynomial

$$\prod_{i=1}^{k-1}\left(1 + X^{P_i}\right),$$

which has degree $n = q$, has zeros behind this $N$. Thus, when the factor

$$\left(1 + X^{P_k}\right)$$

is multiplied in, the $R(n)$ between $n > P_k$ and $n = P_1 + P_2 + \cdots + P_k$ are precisely those from $n = 0$ to $n = P_1 + P_2 + \cdots + P_{k-1}$ followed by zero up to $P_k - 1$. Thus if we proceed by induction on the number of summands, we see the theorem is true for $N = P_k$. Assume for all N having a representation with precisely $k - 1$ summands is such that

$$N = \sum_{j=1}^{k-1} P_{i_j} ,$$

and

$$M(N) = \sum_{j=1}^{k-1}\left(P_{i_j} - 2^{i_j - 1}\right) = N - \sum_{j=1}^{k-1} 2^{i_j - 1} ,$$

then if

$$N = \sum_{j=1}^{k} P_{i_j}$$

then

$$\begin{aligned} M(N) &= \left(P_{i_k} - 2^{i_k - 1}\right) + M\left(N - P_{i_k}\right) \\ &= P_{i_k} - 2^{i_k - 1} + \sum_{j=1}^{k-1}\left(P_{i_j} - 2^{i_j - 1}\right) \\ &= \sum_{i=1}^{k}\left(P_{i_j} - 2^{i_j - 1}\right) = N - \sum_{i=1}^{k} 2^{i_j - 1} . \end{aligned}$$

which evidently proves the theorem by mathematical induction. This completes the proof of Theorem 4.

## 5. SOME GENERAL REMARKS

The foregoing theorems are applicable to a large class of sequences. The restriction

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i$$

in particular, fits $u_0 = 0$ and $u_1 = 1$, while

$$u_{n+2} = ku_{n+1} + u_n \qquad n \geq 0, \quad k \geq 2 .$$

The Pell sequence is the special case when $k = 2$.

<u>Theorem 5.</u> If $P_1 = 1$, $P_2 = k$, and $P_{n+2} = kP_{n+1} + P_n$ $n \geq 1$, then

$$P_{m+1} \geq 1 + \sum_{i=1}^{m} P_i .$$

It is true that, if $S_n = P_1 + P_2 + \cdots + P_n$, then

$$P_{n+2} + P_{n+1} - P_2 - P_1 + S_n = k(P_{n+1} - P_1 + S_n) + S_n .$$

From $P_{n+2} - kP_{n+1} = P_n$ and $P_2 - kP_1 = 0$, we assert

$$P_{n+1} = kS_n - P_n + P_1 = 1 + S_n + (k - 2)P_n + kS_{n-1} .$$

Since $k \geq 2$, the proof would be complete by induction provided it holds for $n = 1$, which one sees as follows:

$$P_2 = k \geq 1 + \sum_{i=1}^{1} P_1 = 2 .$$

This completes the proof of Theorem 5.

Another large family of sequences is given by $P_0 = 1$, $P_1 = 1$ and $P_{n+2} = P_{n+1} + kP_n$ for $n \geq 0$, $k \geq 2$. It is not difficult to establish

Theorem 6. If $P_1 = 1$, $P_2 = k + 1$, and, for $n \geq 0$,

$$P_{n+2} = P_{n+1} + kP_n ,$$

then

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i .$$

Proof. We proceed by induction. $P_1 = 1$ and $P_2 = k + 1$, thus $P_2 \geq 1 + 1$ for $k \geq 2$. Now assume

$$P_m \geq 1 + \sum_{i=1}^{m-1} P_i$$

for $m = 2, 3, \cdots, n$, then

$$\begin{aligned} P_{n+1} &= P_n + kP_{n-1} = P_n + P_{n-1} + (k-1)P_{n-1} \\ &\geq P_n + P_{n-1} + \left(1 + \sum_{i=1}^{n-2} P_i\right) + (k-2)P_{n-1} \\ &\geq 1 + \sum_{i=1}^{n} P_i + (k-2)P_{n-1} . \end{aligned}$$

Clearly

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i$$

for $k \geq 2$, $n \geq 1$. This concludes the proof of Theorem 6.

We add a couple of more sequences to show we haven't captured them all. Let $P_n = F_{2n}$. ($F_n$ is the $n^{th}$ Fibonacci number.) Then, since

$$F_2 + F_4 + \cdots + F_{2n} + 1 = F_{2n+1} < F_{2n+2}$$

so that here, too,

$$P_{n+1} \geq 1 + \sum_{i=1}^{n} P_i .$$

So does $P_n = F_{2n-1}$, $n \geq 1$.

## 6. A FINAL CONJECTURE

Conjecture. Let $H_1$ and $H_2$ be distinct positive integers, sequence H, generated by $H_{n+2} = H_{n+1} + H_n$ $n \geq 1$, then condition (*) yields $R(n)$ such that $R(H_n)$ is independent of the choice of $H_1$ and $H_2$.

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# GENERALIZED ZECKENDORF THEOREM

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## 1. INTRODUCTION

The Zeckendorf theorem states that every positive integer can be uniquely represented as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in any given sum.

D. E. Daykin [1] proved the converse of the Zeckendorf theorem. Keller [2] generalized the Zeckendorf theorem and proved a restricted converse for monotone increasing integer sequences. Hence we generalize the Zeckendorf theorem in a different way and also get a restricted converse. This leaves two open questions as to validity of the unrestricted converse theorems.

## 2. THE GENERALIZED ZECKENDORF THEOREM

Theorem 1. Let $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = kU_{n+1} + U_n$ ($n \ge 0$, $k \ge 1$), then every positive integer $N$, has a unique representation in the form

$$N = \epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n ,$$

where

$$\begin{aligned} &\epsilon_1 = 0, 1, 2, 3, \cdots, \text{ or } k - 1 \\ &\left.\begin{aligned} &\epsilon_1 = 0, 1, 2, 3, \cdots, \text{ or } k \\ &\text{If } \epsilon_i = k, \text{ then } \epsilon_{i-1} = 0 \end{aligned}\right\} i \ge 2 \end{aligned} .$$

First we prove two useful lemmas.

Lemma 1. (i) $U_{2n} = k\,(U_{2n-1} + \cdots + U_3 + U_1)$
(ii) $U_{2n+1} = k\,(U_{2n} + \cdots + U_2) + 1$ .

Proof of the Lemma. (The proof will proceed by induction.)

$$U_1 = 1, \quad U_2 = k, \quad \text{and} \quad U_3 = k^2 + 1$$

from recurrence.

(i) $U_{2n+2} = kU_{2n+1} + U_{2n}$

$= k\{kU_{2n} + kU_{2n-2} + \cdots + kU_2 + 1\} + \{kU_{2n-1} + kU_{2n-2} + \cdots + kU_3 + kU_1\}$

$= k\{(kU_{2n} + U_{2n-1}) + (kU_{2n-2} + U_{2n-2}) + \cdots + (kU_2 + U_1) + 1\}$

$= k\{U_{2n+1} + U_{2n-1} + \cdots + U_3 + 1\}$

$= k\{U_{2n+1} + U_{2n-1} + \cdots + U_3 + U_1\}$, since $U_1 = 1$. End of proof of (i).

(ii) $U_{2n+3} = k\,U_{2n+2} + U_{2n+1}$

$= k\{kU_{2n+1} + \cdots + kU_3 + kU_1\} + k\{U_{2n} + \cdots + U_2)\} + 1$

$= k\{(kU_{2n+1} + U_{2n}) + (kU_{2n-1} + U_{2n-2}) + \cdots + (kU_3 + U_2)\} + 1 + k^2U_1$

$= k\{U_{2n+2} + U_{2n} + \cdots + U_4 + kU_1\} + 1$

$= k\{U_{2n+2} + U_{2n} + \cdots + U_4 + U_2\} + 1$, since $U_1$ and $U_2 = k$.

Lemma 2.

$$\begin{cases} U_{2n} - 1 = k(U_{2n-1} + \cdots + U_3) + (k-1)U_1 \\ U_{2n+1} - 1 = k(U_{2n} + U_{2n-2} + \cdots + U_2) \end{cases}.$$

Proof of Lemma 2. Both parts follow easily from Lemma 1. We need to know the maximum admissible sum using $U_1, U_2, \cdots, U_m$, subject to the coefficient constraints of Theorem 1.

$$\begin{aligned} U_{2n} - 1 &= k(U_{2n-1} + U_{2n-3} + \cdots + U_1) - 1 \\ &= k(U_{2n-1} + U_{2n-3} + \cdots + U_3) + (k-1)U_1 \end{aligned}.$$

Thus the maximum admissible sum using

$$U_1, \quad U_2, \quad U_3, \quad \cdots, \quad U_{2n-1}$$

is $U_{2n} - 1$. Now,

$$U_{2n+1} - 1 = k(U_{2n} + U_{2n-2} + \cdots + U_4 + U_2) .$$

Thus the maximum admissible sum using

$$U_1, \quad U_2, \quad U_3, \quad \cdots, \quad U_{2n}$$

is $U_{2n+1} - 1$, since $U_2$ has coefficient $k$, $U_1$ can have only coefficient zero.

Proof of the Theorem. The proof will proceed by induction. Verification for $s = 1$, $m < U_2 = k$ implies $n = n \cdot U_1$. Assume every integer $n < U_{s+1}$ has a unique admissible representation using only $U_1, U_2, U_3, \cdots U_s$. The maximum such representation has sum $U_{s+1} - 1$ by Lemma 2. Thus $U_{s+1}$ is its own unique representation. For the representations for numbers

$$jU_{s+1} \le n' < (j + 1)U_{s+1} \qquad 1 \le j \le k - 2$$

we simply add $j\,U_{s+1}$ to the representations for $1 \le n \le U_{s+1}$ to get a unique representation. The coefficient of $U_s$ can be $k$ since the coefficient of $U_{s+1} < k$. In the interval

$$k\,U_{s+1} < n'' < U_{s+2} ,$$

the representations cannot contain $U_s$ thus the greatest admissible representation uses $U_1, U_2, \cdots, U_{s-1}$ whose maximal admissible sum is $U_s - 1$. Thus we add to $kU_{s+1}$ a unique representation for $n \le U_s - 1$. Thus we have now covered the interval $U_{s+1} < n < U_{s+2}$ and furthermore each such constructed representation is UNIQUE. The proof of the Theorem is complete by mathematical induction. END OF PROOF.

## 3. THE RESTRICTED CONVERSE TO THE GENERALIZED ZECKENDORF THEOREM

Definition: For fixed integer $K \ge 1$, a sequence $\{V_n\}_1^\infty$ of positive integers will be called a Zeckendorf K-basis (or briefly a K-basis) if every positive integer $n$ has a unique representation in the form

$$n = \sum_{i=1}^{m} \epsilon_i V_i \; , \tag{1}$$

where the coefficients $\epsilon_i$ satisfy constraints

$$\begin{cases} \epsilon_1 = 0, 1, \cdots, K - 1 & \\ \epsilon_i = 0, 1, \cdots, K & \text{for} \quad i \geq 2 \\ \epsilon_{i-1} = 0 \text{ if } \epsilon_i = K & \text{for} \quad i \geq 2 \end{cases} \quad . \tag{2}$$

A representation in form (1) with coefficients satisfying (2) will be called admissible.

Lemma 3. If $\{V_n\}_1^\infty$ is a K-basis with $K \geq 2$, then $V_j \neq V_n$ for $j \neq n$, $1 \leq j$, $n < \infty$.

Proof. Obvious from uniqueness requirement. (For $K = 1$, $V_1 = V_2$, but $V_1$ has a zero coefficient in any admissible representation.)

Lemma 4. If $\{V_n\}_1^\infty$ is a non-decreasing K-basis, then $V_n$ for $n \geq 2$ is characterized as the smallest positive integer not representable in admissible form using only $V_1, V_2, \cdots, V_{n-1}$.

Proof. Let $N_n$ = smallest positive integer not capable of being represented in admissible form using only $V_1, V_2, \cdots, V_{n-1}$. If $N_n > V_n$, then $V_n$ would have two admissible representations, thereby contradicting uniqueness. On the other hand, if $N_n < V_n$, then $N_n$ itself would have no admissible representation (recalling $\{V_n\}$ is non-decreasing).

Theorem 2. Let $\{V_n\}_1^\infty$ be a non-decreasing K-basis with $K \geq 1$. Then defining $V_0 = 0$, we have

$$V_{n+2} = KV_{n+1} + V_n \quad \text{for} \quad n \geq 0, \; K \geq 1 \; . \tag{3}$$

Proof. Since $K = 1$ corresponds to Zeckendorf's theorem, we may confine our attention for $K \geq 2$. Then $\{V_n\}_1^\infty$ is strictly increasing by Lemma 3. Clearly $V_1 = 1$, and Lemma 4 in conjunction with the coefficient constraints (2) implies $V_2 = K$ [since $\epsilon_1 V_1$ can represent only the integers $1, 2, \cdots, K - 1$].

For fixed $K \geq 2$, let $\{U_n\}_1^\infty$ be the sequence defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = KU_{n+1} + U_n$ for $n \geq 0$. Then $V_0 = U_0$, $V_1 = U_1$, $V_2 = U_2$. Now, assume as an induction hypothesis that $V_i = U_i$ for $i = 1, 2, \cdots, n$, where $n \geq 2$. We wish to show $V_{n+1} = U_{n+1}$. Contained in the proof of the generalized Zeckendorf theorem is the fact that the smallest integer not representable by an admissible combination of $U_1, U_2, \cdots, U_n$ is $U_{n+1}$. Since $U_i = V_i$ for $i = 1, \cdots, n$, Lemma 2 implies $V_{n+1} = U_{n+1}$ and the theorem is established.

I wish to thank John L. Brown, Jr., for the details of the restricted converse theorem.

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we get

$$\frac{1}{\alpha} f_1(1) + \frac{1}{\alpha^2} f_2(1) + \frac{1}{\alpha^3} f_3(1) = 0 .$$

This evidently implies

$$f_1(1) = f_2(1) = f_3(1) = 0 ,$$

which contradicts (7.21).

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# GENERALIZATIONS OF ZECKENDORF'S THEOREM

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The Fibonacci numbers $F_n$ are defined by the recurrence relation

$$F_1 = F_2 = 1 ,$$
$$F_n = F_{n-1} + F_{n-2} \qquad (n > 2) .$$

Every natural number has a representation as a sum of distinct Fibonacci numbers, but such representations are not in general unique. When constraints are added to make such representations unique, the result is Zeckendorf's theorem [1], [5]. Statements of Zeckendorf's theorem and its converse follow. (Alpha is an integer.)

Theorem. (Zeckendorf). Every natural number $N$ has a unique representation in the form

$$N = \sum_2^n \alpha_k F_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+1} = 1$, then $\alpha_k = 0$.

Theorem. (Converse of Zeckendorf's Theorem) ([1], [3]). Let

$$\{x_n\}_1^\infty$$

be a monotone sequence of distinct natural numbers such that every natural number $N$ has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+1} = 1$, then $\alpha_k = 0$. Then

$$\{x_n\}_1^\infty = \{F_n\}_2^\infty .$$

There are generalizations of Zeckendorf's theorem for every monotone sequence $\{x_n\}_1^\infty$ of distinct natural numbers for which $x_1 = 1$. The following theorem is the first of many such generalizations.

Theorem 1. Let the numbers $x_n$ be defined by the recurrence relation

$$x_1 = 1, \qquad x_2 = a,$$

$$x_n = m_1 x_{n-1} + m_2 x_{n-2} \qquad (n > 2) ,$$

where $m_1 > 0$, $m_2 > 0$, and $a > 1$. Then every natural number $N$ has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $\alpha_k \geq 0$ and if $\alpha_{k+p+1} \neq m_1$, $\alpha_{k+i} = m_1$ for $1 \leq i \leq p$.

i) and $p$ is odd, then $\alpha_k < m_2$ ;
ii) $p$ is even, and $k > 1$, then $\alpha_k \leq m_1$ ;
iii) $p$ is even, and $k = 1$, then $\alpha_1 < a$.

The special case $m_1 = m_2 = 1$, $a = 2$ is Zeckendorf's theorem, and the case $m_2 = 1$, $a = m_1$ is a generalization proved by Hoggatt. (See p. 89)

Proof. We prove the existence of a representation by induction on $N$. For $N < x_2$, we have $N = Nx_1$. Take $N \geq x_2$ and assume representability for $1, 2, \cdots, N - 1$. Since $\{x_n\}_1^\infty$ is a monotone sequence of distinct natural numbers, any natural number lies between some pair of successive elements of $\{x_n\}_1^\infty$. More explicitly, there is a unique $n \geq 2$ such that $x_n \leq N < x_{n+1}$. First let $N < m_1 x_n$. There are unique integers $m$ and $r$ such that

$$N = mx_n + r ,$$

where $0 < m < m_1$ and $0 \le r < x_n$. If $r = 0$, then $N = mx_n$, whereas if $r > 0$, then the induction hypothesis shows that $r$ is representable. Thus $N$ is representable. Now let $N \ge m_1x_n$. Since

$$x_{n+1} = m_1x_n + m_2x_{n-1}$$

for $n \ge 2$, there are unique integers $m$ and $r$ such that

$$N = m_1x_n + m\,x_{n-1} + r\,,$$

where $0 \le m < m_2$ and $0 \le r < x_{n-1}$. If $r = 0$, then

$$N = m_1x_n + mx_{n-1}\,,$$

whereas if $r > 0$, then $r$ is representable. Thus $N$ is representable. Now use the induction principle.

To prove the uniqueness of this representation, it is sufficient to prove that $x_n$ is greater than the maximum admissible sum of numbers less than $x_n$ according to constraints (i)-(iii). We prove this by induction on $n$. For $n = 1$, this is obviously true. Take $n > 1$ and assume that the sufficient condition is true for $1, 2, \cdots, n - 1$. From

$$\sum_2^n \{m_1x_{2i-2} + (m_2 - 1)x_{2i-3}\} = \sum_2^n x_{2i-1} - \sum_1^{n-1} x_{2i-1} = x_{2n-1} - 1,$$

$$\sum_2^n \{m_1x_{2i-1} + (m_2 - 1)x_{2i-2}\} = \sum_2^n x_{2i} - \sum_1^{n-1} x_{2i} = x_{2n} - a\,,$$

we obtain the identities

$$x_{2n-1} = \sum_{2}^{n} \{m_1 x_{2i-2} + (m_2 - 1)x_{2i-3}\} + 1 ,$$

(1)

$$x_{2n} = \sum_{2}^{n} \{m_1 x_{2i-1} + (m_2 - 1)x_{2i-2}\} + (a - 1)x_1 + 1 .$$

The induction hypothesis together with (1) shows that $x_n$ is greater than the maximum admissible sum of numbers less than $x_n$. Now use the induction principle.

Theorem 1 can be extended to the case where the numbers $x_n$ are defined by the recurrence relation

$$x_1 = 1, \qquad x_n = a_n (2 \le n \le q) ,$$

$$x_n = \sum_{1}^{q} m_k x_{n-k} \qquad (n > q) ,$$

where $m_1 > 0$, $m_k \ge 0$ for $1 < k < q$, $m_q > 0$, and $1 < a_n < a_{n+1}$ for $1 < n < q$. Every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $\alpha_k \ge 0$ and other constraints similar to those in Theorem 1 are added. For example, if $\alpha_{n-k+1} = m_k$ for $1 \le k < p < q$, then $\alpha_{n-p+1} \le m_p$. If $p = q$, then $\alpha_{n-q+1} < m_q$. These constraints must be modified to fit the initial conditions $a_n$. The proof of this extension follows that of Theorem 1 and uses the identity

$$x_{qn-r} = \sum_1^{q-1} m_k \sum_2^n x_{qi-r-k} + (m_q - 1)\sum_1^{n-1} x_{qi-r} + \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] x_{q-r-1}$$

$$+ \left[\frac{a_{q-r} - \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] a_{q-r-1} - 1}{a_{q-r-2}}\right] x_{q-r-2} + \cdots + \left(a_{q-r} - \left[\frac{a_{q-r} - 1}{a_{q-r-1}}\right] a_{q-r-1} - \cdots - 1\right)$$

$$\cdot\, x_1 + 1 \qquad (0 \le r < q) .$$

Statements of two special cases and the proof of the second one follow.

Theorem. (Daykin [3]). Let the numbers $x_n$ be defined by the recurrence relation

$$x_n = n (1 \le n \le q) ,$$

$$x_n = x_{n-1} + x_{n-q} \qquad (n > q) .$$

Then every natural number N has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k \le 1$ and if $\alpha_{k+q-1} = 1$, then $\alpha_{k+i} = 0$ for $0 \le i < q - 1$.

Theorem 2. Let the numbers $x_n$ be defined by the recurrence relation

$$x_n = (m + 1)^{n-1} (1 \le n \le q ),$$

$$x_n = m \sum_1^q x_{n-k} \qquad (n > q) .$$

Then every natural number N has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k \le m$ and if $\alpha_{k+i} = m$ for $1 \le i < q$, then $\alpha_k < m$.

Proof. Following the proof of Theorem 1, we prove the existence of a representation by induction on $N$. For $N < x_q$, we have

$$N = \sum_{1}^{q-1} \alpha_k x_k ,$$

where $0 \le \alpha_k \le m$. Take $N \ge x_q$ and assume representability for 1, 2, $\cdots$, $N - 1$. There is a unique $n \ge q$ such that $x_n \le N < x_{n+1}$. Since

$$x_{n+1} = m \sum_{0}^{q-1} x_{n-k}$$

for $n \ge q$, there are unique integers $p, m'$, and $r$ such that

$$N = m \sum_{0}^{p-1} x_{n-k} + m'x_{n-p} + r ,$$

where $0 \le p < q$, $0 \le m' < m$, and $0 \le r < x_{n-p}$. If $r = 0$, then

$$N = m \sum_{0}^{p-1} x_{n-k} + m'x_{n-p} ,$$

whereas if $r > 0$, then $r$ is representable. Thus $N$ is representable. Now use the induction principle.

To prove the uniqueness of this representation, we prove that $x_n$ is greater than the maximum admissible sum of numbers less than $x_n$ according to the constraints by induction on $n$. For $1 \le n \le q$, we have

$$m \sum_{1}^{n-1} x_k = m \sum_{1}^{n-1} (m + 1)^{k-1} = (m + 1)^{n-1} - 1 < (m + 1)^{n-1} = x_n .$$

Take $n > q$ and assume that the sufficient condition is true for $n - q$. Then

$$x_n = m \sum_1^{q-1} x_{n-k} + (m - 1)x_{n-q} + x_{n-q} .$$

The induction hypothesis shows that $x_n$ is greater than the maximum admissible sum of numbers less than $x_n$. Now use the induction principle.

Zeckendorf's theorem can be further generalized to cases where the numbers $x_n$ are defined by recurrence relations with negative coefficients.

Theorem 3. Let the numbers $x_n$ be defined by the recurrence relation

$$x_1 = 1, \qquad x_2 = a,$$
$$x_n = m_1x_{n-1} - m_2x_{n-2} \qquad (n > 2) ,$$

where $0 < m_2 < m_1$ and $a > m_2$. Then every natural number $N$ has a unique representation in the form

$$N = \sum_1^n \alpha_k x_k ,$$

where $0 \le \alpha_k < m_1$ for $k > 1$, $0 \le \alpha_1 < a$, and if $\alpha_{k+p+1} = m_1 - 1$,

$$\alpha_{k+i} = m_1 - m_2 - 1$$

for $1 \le i \le p$, and

(i) $k > 1$, then $\alpha_k < m_1 - m_2$ ;
(ii) $k = 1$, then $\alpha_1 < a - m_2$ .

The proof, which will not be given, follows that of Theorem 1 and uses the identity

$$x_n = (m_1 - 1)x_{n-1} + (m_1 - m_2 - 1)\sum_{2}^{n-2} x_i + (a - m_2 - 1)x_1 + 1.$$

The converse of Zeckendorf's theorem can be generalized to include as special cases the converses of the generalizations of Zeckendorf's theorem given so far.

Theorem 4. Let $\{x_n\}_1^\infty$ be a monotone sequence of distinct natural numbers such that every natural number N has a unique representation in the form

$$N = \sum_{1}^{n} \alpha_k x_k ,$$

where $\alpha_k \geq 0$ and other constraints on $\{\alpha_k\}_1^n$ are added such that the representation of $x_n$ is itself. Then $\{x_n\}_1^\infty$ is the only such sequence.

Proof. Assume the sequences $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$ both satisfy the hypotheses, where

$$\{x_n\}_1^N = \{y_n\}_1^N$$

and $y_{n+1} \leq x_{N+1}$. Then $y_{N+1}$ has a unique representation as a sum of numbers $x_n$, each of which in turn has a unique representation as a sum of numbers $y_n$, where $n \leq N$. On the other hand, $y_{N+1}$ obviously represents itself and, thus, $y_{N+1}$ has two representations in terms of numbers $y_n$. This contradicts the uniqueness of representation, and we conclude that

$$\{x_n\}_1^\infty = \{y_n\}_1^\infty .$$

Theorem 4 does not include the converse of the following generalization of Zeckendorf's theorem.

Theorem (Brown [2]). Every natural number N has a unique representation in the form of

[Continued on page 111.]

# REPRESENTATIONS OF INTEGERS AS SUMS OF FIBONACCI SQUARES

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## 1. COMPLETENESS

If elements of a sequence can be selected, with each element being selected at most once, such that their sum is a given integer, then this integer is said to have a representation with respect to the sequence. A sequence of positive integers is complete if and only if every positive integer has at least one representation with respect to the sequence.

Consider the sequence of Fibonacci squares:

$$1,\ 1,\ 4,\ 9,\ 25,\ 64,\ \cdots\ .$$

Clearly this sequence is not complete as there are no representations for 3, 7, 8, 12, and infinitely many other integers. Let us now consider using two copies of the sequence of Fibonacci squares. Consider the sequence

$$1,\ 1,\ 1,\ 1,\ 4,\ 4,\ 9,\ 9,\ 25,\ 25,\ 64,\ 64,\ \cdots\ .$$

A few simple calculations will lead one to suspect that we now have a complete sequence. This can be proved using the following theorem given by Brown [1].

Theorem 1. Let $\{a_k\}$ be a non-decreasing sequence of positive integers with $a_1 = 1$. If

$$a_{n+1} \le 1 + \sum_{k=1}^{n} a_k \quad ,$$

then the sequence $\{a_k\}$ is complete.

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Let us now define our sequence so that the notation will be similar to that used in Theorem 1. Let

$$a_{2k-1} = F_k^2, \qquad a_{2k} = F_k^2 .$$

Then we have

$$\sum_{k=1}^{2m} a_k = 2\sum_{k=1}^{m} F_k^2 = 2F_m F_{m+1}$$

$$\sum_{k=1}^{2m-1} a_k = \sum_{k=1}^{2m-2} a_k + F_m^2 = 2F_{m-1}F_m + F_m^2 = F_{2m} .$$

<u>Theorem 2.</u> The sequence of two of each of the Fibonacci squares is complete.

<u>Proof.</u> Let $n$ be even with $n = 2m$. Then we must show that

$$a_{2m+1} \le 1 + \sum_{k=1}^{2m} a_k$$

or that

$$F_{m+1}^2 \le 1 + 2F_m F_{m+1} .$$

For $m \ge 1$,

$$F_{m-1} \le F_m$$

$$F_{m-1} + F_m \le 2F_m$$

$$F_{m+1} \le 2F_m$$

$$F_{m+1}^2 \le 2F_m F_{m+1}$$
$$F_{m+1}^2 \le 1 + 2F_m F_{m+1} \ .$$

The case for $n$ odd is handled in a similar manner to complete the proof.

Theorem 3. Exactly one of the first six elements of the sequence $\{a_k\}$ can be deleted without loss of completeness.

This theorem is proved by showing that if one $F_n^2$ is deleted, with $n \ge 4$, then there is no representation for the integer $F_{n+1}^2 - 1$. Further, one shows that if any two of the first six elements are deleted, then completeness is again lost. The proof is completed by showing that if any one of the first six elements is deleted, it is still possible to find enough representations to establish a foundation for mathematical induction.

## 2. BASIC LEMMAS

Let

$$P(x) = \prod_{j=1}^{\infty} (1 + x^{a_j}) = \sum_{k=0}^{\infty} R(k)\, x^k \ ,$$

$$P_{2n}(x) = \prod_{j=1}^{2n} (1 + x^{a_j}) = \sum_{k=0}^{2F_nF_{n+1}} R_{2n}(k)\, x^k \ ,$$

$$P_{2n-1}(x) = \prod_{j=1}^{2n-1} (1 + x^{a_j}) = \sum_{k=0}^{F_{2n}} R_{2n-1}(k)\, x^k \ ,$$

where $a_j$ is an element from our sequence. Then $R(k)$ is the number of representations of $k$ as a sum of Fibonacci squares. Paralleling the method used by Klarner [2] we can prove the following lemmas.

Lemma 1.

(a) $\qquad R_{2n}(k) = R_{2n}(2F_nF_{n+1} - k), \qquad 0 \le k \le 2F_nF_{n+1}$

(b) $$R_{2n-1}(k) = R_{2n-1}(F_{2n} - k), \qquad 0 \le k \le F_{2n}.$$

Lemma 2.

(a) $$R_{2n+1}(k) = R_{2n}(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n+1}(k) = R_{2n}(k) + R_{2n}(k - F_{n+1}^2), \qquad F_{n+1}^2 \le k \le 2F_nF_{n+1}$$

(c) $$R_{2n+1}(k) = R_{2n}(k - F_{n+1}^2), \qquad 2F_nF_{n+1} + 1 \le k \le F_{2n+2}.$$

Lemma 3.

(a) $$R_{2n}(k) = R_{2n-1}(k), \qquad 0 \le k \le F_n^2 - 1$$

(b) $$R_{2n}(k) = R_{2n-1}(k) + R_{2n-1}(k - F_n^2), \qquad F_n^2 \le k \le F_{2n}$$

(c) $$R_{2n}(k) = R_{2n-1}(k - F_n^2), \qquad F_{2n} + 1 \le k \le 2F_nF_{n+1}.$$

Lemma 4.

(a) $$R_{2n}(k) = R(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n}(k) = R(2F_nF_{n+1} - k), \qquad F_{2n} + 1 \le k \le 2F_nF_{n+1}.$$

Lemma 5.

(a) $$R_{2n+1}(k) = R(k), \qquad 0 \le k \le F_{n+1}^2 - 1$$

(b) $$R_{2n+1}(k) = R(2F_nF_{n+1} - k) + R(k - F_{n+1}^2), \qquad n \ge 2,$$
$$F_{n+1}^2 \le k \le 2F_nF_{n+1}$$

(c) $$R_{2n+1}(k) = R(F_{2n+2} - k), \qquad 2F_nF_{n+1} + 1 \le k \le F_{2n+2}.$$

Lemma 6.

$$R(F_nF_{n+1} - k) = R(F_nF_{n+1} + k), \qquad n \geq 2, \qquad 0 \leq k \leq F_{n-1}F_{n+1} - 1 .$$

<u>Lemma 7.</u> For $n \geq 3$,

(a) $R(k) = 2R(k - F_n^2) + R(2F_nF_{n-1} - k), \qquad F_n^2 \leq k \leq 2F_nF_{n-1}$

(b) $R(k) = 2R(k - F_n^2), \qquad 2F_nF_{n-1} + 1 \leq k \leq 2F_n^2 - 1$

(c) $R(k) = R(2F_nF_{n+1} - k), \qquad 2F_n^2 \leq k \leq F_{n+1}^2 - 1 .$

Lemma 7 can now be used to prove many representation theorems suggested by a table of values for $R(k)$, with $0 \leq k \leq 25{,}000$.

## 3. REPRESENTATION THEOREMS

<u>Theorem 4.</u>

$$R(F_nF_{n+1}) = 2R(F_{n-1}F_n), \qquad n \geq 3 .$$

<u>Proof.</u> For $n \geq 3$,

$$2F_nF_{n-1} + 1 \leq F_nF_{n+1} \leq 2F_n^2 - 1 .$$

Using Lemma 7(b), we have

$$\begin{aligned} R(F_nF_{n+1}) &= 2R(F_nF_{n+1} - F_n^2) \\ &= 2R(F_n[F_{n+1} - F_n]) \\ &= 2R(F_nF_{n-1}) . \end{aligned}$$

<u>Theorem 5.</u>

$$R(F_nF_{n+1}) = 3 \cdot 2^{n-1} .$$

<u>Proof.</u> From the table of values we have

$$n = 1: \quad R(F_1 F_2) = 3 = 3 \cdot 2^0$$

$$n = 2: \quad R(F_2 F_3) = 6 = 3 \cdot 2^1$$

$$n = 3: \quad R(F_3 F_4) = 12 = 3 \cdot 2^2$$

which gives us a basis for induction. Now assume the statement holds for $n = k$. Then

$$R(F_k F_{k+1}) = 3 \cdot 2^{k-1} .$$

By Theorem 4,

$$R(F_{k+1} F_{k+2}) = 2R(F_k F_{k+1}) = 2 \cdot 3 \cdot 2^{k-1} = 3 \cdot 2^k .$$

We have shown that if the statement is true for $n = k$, then it is also true for $n = k + 1$. Therefore, by induction, the proof is complete.

In a thesis on this subject forty-four theorems such as theorems four and give were proved and another nine were suggested.

## 4. MAXIMUM AND MINIMUM VALUES OF R(k)

Since by Theorem 5,

$$R(F_n F_{n+1}) = 3 \cdot 2^{n-1} ,$$

we see that $R(k)$ increases without bound. However, maximum and minimum values of $R(k)$ can be found in each interval

$$F_n^2 \le k \le F_{n+1}^2 - 1 .$$

Theorem 6. For

$$F_n^2 \le k \le F_{n+1}^2 - 1 ,$$

the maximum value of $R(k)$ is $R(F_n F_{n+1})$.

This theorem is proved by induction.

Theorem 7. For

$$F_n^2 \le k \le F_{n+1}^2 - 1 ,$$

the minimum value of $R(k)$ is $R(k) = 3$, where

$$k = 2\left[1 + \sum_{i=1}^{n} F_{2i}^2\right], \qquad F_{2n}^2 \le k \le F_{2n+1}^2 - 1 ,$$

$$k = 2\sum_{i=1}^{n} F_{2i+1}^2 , \qquad F_{2n+1}^2 \le k \le F_{2n+2}^2 - 1 \quad .$$

By inspecting the table we see that three is the minimum value of $R(k)$ for all $k$ included in the table. Lemma 7 assures us that no later values of $R(k)$ will be less than three. Induction is used to show that $R(k) = 3$ as specified above.

## 5. SIMPLE REPRESENTATIONS

A simple representation is a representation in which each Fibonacci square is used at most once. Since $F_1^2 = F_2^2 = 1$ we will allow two ones to be included in a simple representation. By distinct simple representations we mean representations whose elements are not identical.

$$R_1 = F_1^2 + \sum_{i=1}^{n} F_{k_i}^2$$

and

$$R_2 = F_2^2 + \sum_{i=1}^{n} F_{k_i}^2 \qquad (k_i \ge 3)$$

are taken to be the same simple representation since when the representations are actually written out we cannot distinguish between $F_1^2$ and $F_2^2$.

Theorem 8. An integer has at most one simple representation.

Proof. Let

$$I = F_{j_1}^2 + F_{j_2}^2 + \cdots + F_{j_n}^2$$

be a simple representation for I.

$$\sum_{i=1}^{j_n-1} F_i^2 = F_{j_n-1}F_{j_n} < F_{j_n}^2 < 1 .$$

Hence $F_{j_n}^2$ must be used in a simple representation for I. Similar arguments show that each $F_{j_i}^2$ must also be used.

Theorem 9. Every representation of $F_nF_{n+1}$ is simple.

Proof. Recall that

$$1 + 1 + 4 + 9 + \cdots + F_n^2 = F_nF_{n+1} .$$

Using our sequence there are $\binom{4}{2}$ ways to select the two ones and two ways to select each succeeding summand. Therefore, the number of simple representations is

$$\binom{4}{2} \cdot 2^{n-2} = 6 \cdot 2^{n-1} = 3 \cdot 2^{n-1} .$$

From Theorem 5, we have

$$R(F_nF_{n+1}) = 3 \cdot 2^{n-1} .$$

Note that we have $3 \cdot 2^{n-1}$ simple representations and a total of $3 \cdot 2^{n-1}$ representations for $F_nF_{n+1}$. Hence, every representation for $F_nF_{n+1}$ is simple.

As a result of Theorem 9 we have the following theorem, which may be called a Non-Four-Square Theorem.

Theorem 10. There does not exist a finite number $n$ such that every positive integer can be represented as a sum of at most $n$ Fibonacci squares.

## 6. VALUES OF $m$ SUCH THAT $R(k) \neq m$

Using Lemma 7 and mathematical induction, it is possible to prove

$$R(k) \neq 5, \qquad R(k) \neq 7, \qquad R(k) \neq 13$$

for any positive integer $k$. It is suggested that there are an infinite number of integers $m$ such that $R(k) \neq m$ for any positive integer $k$.

Further expansion of these ideas is contained in [3].

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$$N = \sum_{2}^{n} \alpha_k F_k ,$$

where $0 \leq \alpha_k \leq 1$ and if $\alpha_{k+1} = 0$, then $\alpha_k = 1$.

Zeckendorf's theorem provides the representation of $N$ in terms of the minimum number of distinct Fibonacci numbers, and Brown's theorem provides the representation of $N$ in terms of the maximum number of distinct Fibonacci numbers.

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