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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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FIBONACCI SEARCH WITH ARBITRARY FIRST EVALUATION

CHRISTOPH WITZGALL
Mathematics Research Laboratory, Boeing Scientific Research Laboratory

ABSTRACT

The Fibonacci search technique for maximizing a unimodal function of one real variable is generalized to the case of a given first evaluation. This technique is then employed to determine the optimal sequential search technique for the maximization of a concave function.

1. INTRODUCTION

A real function $f: [a,b] \to R$, where $a < b$ is called

(1.1) $$\text{unimodal} \ ,$$

if there are $\underline{x}$, $\overline{x} \in [a,b]$ such that f is increasing for $x \leq \underline{x}$ and non-increasing for $x \geq \underline{x}$, decreasing for $x \geq \overline{x}$ and nondecreasing for $x \leq \overline{x}$ (Fig. 1).

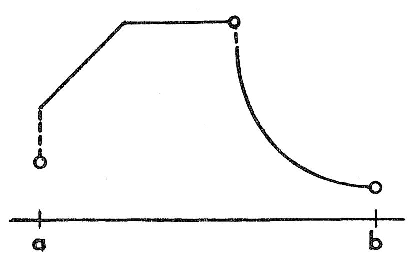

Fig. 1 Example of a Unimodal Function

(1.2) If f is unimodal, then the interval $[\underline{x}, \overline{x}]$ consists of all maxima of f.

Proof. f is constant in $[\underline{x}, \overline{x}]$, since it is by definition nonincreasing for $x \geq \underline{x}$ as well as nondecreasing for $x \leq \overline{x}$. If $x < \underline{x}$, then $f(x) < f(\overline{x})$ as f increases in $[a, x]$. If $x > \overline{x}$, then $f(x) < f(\overline{x})$ as f decreases in $[\overline{x}, b]$.

The definition of unimodality is chosen so as to guarantee that

(1.3) Whenever a unimodal function f has been evaluated for two arguments x_1 and x_2 with $a \leq x_1 < x_2 \leq b$, then some maximum of f must lie in $[x_1, b]$ if $f(x_1) \leq f(x_2)$ and in $[a, x_2]$ if $f(x_1) \geq f(x_2)$

Proof. If $f(x_1) \geq f(x_2)$, then x_1 and x_2 cannot be both in that portion of the interval $[a,b]$ in which the function decreases. In other words, $\bar{x}$ cannot lie to the left of x_1. Thus $\bar{x} \in [x_1, b]$, and $\bar{x}$ is a maximum of f by (1.2). Similarly, if $f(x_1) \leq f(x_2)$, then $\underline{x} \in [a, x_2]$.

As the restriction of a unimodal function to a closed subinterval of $[a,b]$ is again unimodal, this argument can be repeated. Hence, a sequential search based on (1.3) will successively narrow down the interval in which a maximum of f is known to lie. Such an interval is called the

(1.4) Interval of Uncertainty.

Kiefer [3] has asked the question of optimally conducting this search, and answered it by developing his well known Fibonacci search.

The Fibonacci search gives a choice of two arguments for which to make the first evaluation. But what happens if by mistake or for some other reason the first evaluation took place at some argument other than the two optimal ones? How does one optimally proceed from there?

In this paper, we shall therefore ask and answer the question for an optimal sequential search plan with given arbitrary first evaluation. The resulting technique is applied to improving on Fibonacci search for functions known to be concave. The technique may also be of interest in the context of stability of Fibonacci search in the presence of round-off errors as studied by Overholt [6] and Boothroydt [1] (see also Kovalik and Osborne [4]).

2. LENGTH OF UNCERTAINTY

In what follows we assume that $a = 0$ and $b = 1$. Furthermore, we shall permit zero distances between two arguments of evaluation, interpreting each such occurrence as evaluating the (not necessarily unique or finite) derivative of the function f. A more careful analysis would take into account the smallest justifiable distance ϵ between arguments (Kiefer [3], Oliver and Wilde [5]).

By

$$L_k(x), \qquad 0 \le x \le 1,$$

we denote the length to which the interval of uncertainty (1.4) can surely be replaced by k evaluations in addition to a first one at x. Extending a recursive argument due to Johnson [2], we obtain

$$L_k(x) = \min\{M_k(x),\ M_k(1 - x)\}, \tag{2.1}$$

where

$$M_k(x) := \min_{x \le y \le 1} \max\left\{(1 - x)L_{k-1}\left(\frac{1 - y}{1 - x}\right),\ yL_{k-1}\left(\frac{x}{y}\right)\right\}.$$

Proof. Let y denote the first function argument over which we have control. If $x \le y \le 1$, then the two possible intervals of uncertainty are $[0,y]$ and $[x,1]$. The former contains the point of evaluation x. The best upper bound for the length of the interval of uncertainty after the remaining $k - 1$ evaluations is given by

$$yL_{k-1}\left(\frac{x}{y}\right). \tag{2.2}$$

Similarly, y is the evaluation point in $[x,1]$, leading to the best upper bound

$$(1 - x)L_{k-1}\left(\frac{1 - y}{1 - x}\right), \tag{2.3}$$

Whether $[0,y]$ or $[x,1]$ is the first interval of uncertainty depends on the result of the evaluation at y: if $f(y) \le f(x)$, then $[0,y]$, if $f(y) > f(x)$, then $[x,1]$. Hence the maximum $M_k(x)$ of the two expressions (2.2) and (2.3) is the best result achievable if y is selected between x and 1. The expression

$$N_k(x) := \min_{0\le y\le x} \max\left\{xL_{k-1}\left(\frac{y}{x}\right),\ (1-y)L_{k-1}\left(\frac{1-x}{1-y}\right)\right\}$$

analogously describes the best result achievable if y is between 0 and x. Since we control the choice of y, we can choose the smaller one of these two expressions; and this gives

$$L_k(x) = \min\{M_k(x),\ N_k(x)\}$$

Introducing for $0 \le x \le y \le 1$,

$$S_k(x,y) := \max\left\{(1-x)L_{k-1}\left(\frac{1-y}{1-x}\right),\ yL_{k-1}\left(\frac{x}{y}\right)\right\},$$

we have

$$M_k(x) = \min_{x\le y\le 1} S_k(x,y), \qquad N_k(x) = \min_{0\le y\le x} S_k(y,x) .$$

Now for $0 \le x \le y \le 1$,

$$S_k(x,y) = S_k(1-y,\ 1-x) . \tag{2.4}$$

Therefore, $N_k(x) = M_k(1-x)$, and (2.1) is proved.

At the beginning, the interval of uncertainty is the entire interval in which the function is to be examined. A single function evaluation at any point x does not change this situation. Hence

$$L_0(x) = 1 .$$

We then have

$$M_1(x) = \min_{x\le y\le 1} \max\{1-x, y\} = \max\{1-x, x\} = M_1(1-x) .$$

Hence

$$L_1(x) = \max\{1 - x, x\} = \begin{cases} 1 - x & \text{for } 0 \le x \le \frac{1}{2} \\ x & \text{for } \frac{1}{2} \le x \le 1 \end{cases} . \tag{2.5}$$

For $k \ge 2$, we claim (Fig. 2):

$$L_k(x) = \begin{cases} \dfrac{1-x}{F_{k+1}} & \text{for } 0 \le x \le \dfrac{F_k}{F_{k+2}} \\ \dfrac{x}{F_k} & \text{for } \dfrac{F_k}{F_{k+2}} \le x \le \frac{1}{2} \\ \dfrac{1-x}{F_k} & \text{for } \frac{1}{2} \le x \le \dfrac{F_{k+1}}{F_{k+2}} \\ \dfrac{x}{F_{k+1}} & \text{for } \dfrac{F_{k+1}}{F_{k+2}} \le x \le 1 , \end{cases} \tag{2,6}$$

where $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3, \cdots$, $F_k = F_{k-2} + F_{k-1}$ are the Fibonacci numbers.

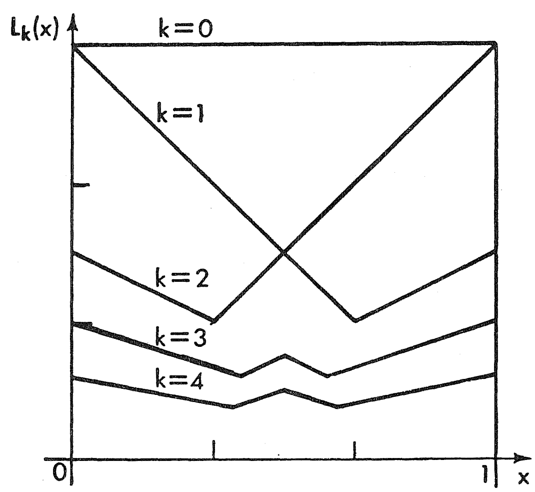

Fig. 2 $L_k(x)$ for $k = 0, \cdots, 4$

Proof. The case $k = 2$ requires special treatment. From (2.5),

$$yL_1\left(\frac{x}{y}\right) = \begin{cases} y - x & \text{for } (x,y) \in A_1 : = \left\{0 \le \frac{x}{y} \le \frac{1}{2}\right\} \\ x & \text{for } (x,y) \in A_2 : = \left\{\frac{1}{2} \le \frac{x}{y} \le 1\right\} \end{cases},$$

$$(1 - x)L_1\left(\frac{1 - y}{1 - x}\right) = \begin{cases} y - x & \text{for } (x,y) \in B_1 : = 0 \le \frac{1 - y}{1 - x} \le \frac{1}{2} \\ 1 - y & \text{for } (x,y) \in B_2 : = \frac{1}{2} \le \frac{1 - y}{1 - x} < 1 \end{cases}.$$

We are now able to determine $S_2(x,y)$ in each of the four regions $A_i \cap B_j$ separately:

$$A_1 \cap B_1 : S_2(x,y) = \max\{y - x,\ y - x\} = y - x .$$

$$A_1 \cap B_2 : S_2(x,y) = \max\{y - x,\ 1 - y\} = 1 - y .$$

$$A_2 \cap B_1 : S_2(x,y) = x \quad \text{by (2.4) and} \quad (1 - y, 1 - x) \in A_1 \cap B_2 .$$

$$A_2 \cap B_2 : S_2(x,y) = \max\{x,\ 1 - y\} = \begin{cases} x & \text{if } y \ge 1 - x \\ 1 - y & \text{if } y \le 1 - x \end{cases}.$$

The sets A_i and B_j are represented in Fig. 3. They are triangles formed by the line segments marked A_i and B_j, respectively, and the corresponding opposite corner of the square. The feathered lines are the minimum lines with respect to constant values of x, i.e., if proceeding vertically the intersection with the feathered lines marks a minimum. The function $M_k(x)$ is defined to be the value of this minimum. Hence

$$M_2(x) = \begin{cases} \frac{1 - x}{2} & \text{if } 0 \le x \le \frac{1}{3} \\ x & \text{if } \frac{1}{3} \le x \le 1 \end{cases}.$$

By (2.1) we then have finally

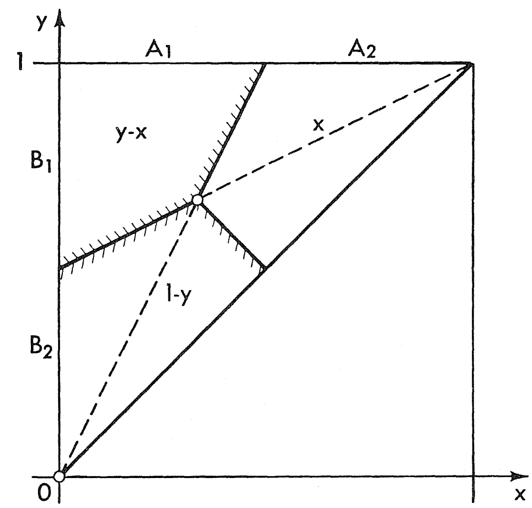

Fig. 3 $S_2(x, y)$

$$L_2(x) = \begin{cases} \frac{1-x}{2} & \text{if } 0 \le x \le \frac{1}{3} \\ x & \text{if } \frac{1}{3} \le x \le \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \le x \le \frac{2}{3} \\ x & \text{if } \frac{2}{3} \le x \le 1 \end{cases},$$

in accordance with (2.6).

The case $k \ge 3$ is now proved by induction over k. We have

$$yL_{k-1}\left(\frac{x}{y}\right) = \begin{cases} \frac{y-x}{F_k} & \text{for } (x, y) \in A_1 : = 0 \le \frac{x}{y} \le \frac{F_{k-1}}{F_{k+1}} \\ \frac{x}{F_{k-1}} & \text{for } (x, y) \in A_2 : = \frac{F_{k-1}}{F_{k+1}} \le \frac{x}{y} \le \frac{1}{2} \\ \frac{y-x}{F_{k-1}} & \text{for } (x, y) \in A_3 : = \frac{1}{2} \le \frac{x}{y} \le \frac{F_k}{F_{k+1}} \\ \frac{x}{F_k} & \text{for } (x, y) \in A_4 : = \frac{F_k}{F_{k+1}} \le \frac{x}{y} \le 1 \end{cases},$$

$$(1 - x)L_{k-1}\left(\frac{1 - y}{1 - x}\right) = \begin{cases} \dfrac{y - x}{F_k} & \text{for } (x, y) \in B_1 : = 0 \le \dfrac{1 - y}{1 - x} \le \dfrac{F_{k-1}}{F_{k+1}} \\[2ex] \dfrac{1 - y}{F_{k-1}} & \text{for } (x, y) \in B_2 : = \dfrac{F_{k-1}}{F_{k+1}} \le \dfrac{1 - y}{1 - x} \le \dfrac{1}{2} \\[2ex] \dfrac{y - x}{F_{k-1}} & \text{for } (x, y) \in B_3 : = \dfrac{1}{2} \le \dfrac{1 - y}{1 - x} \le \dfrac{F_k}{F_{k+1}} \\[2ex] \dfrac{1 - y}{F_k} & \text{for } (x, y) \in B_4 : = \dfrac{F_k}{F_{k+1}} \le \dfrac{1 - y}{1 - x} \le 1 \end{cases},$$

We determine $S_k(x, y)$ in all regions $A_i \cap B_j$ with $i \le j$. For the remaining regions, we use (2.4).

$$A_1 \cap B_1 : S_k(x, y) = \max\left\{\frac{y - x}{F_k}, \frac{y - x}{F_k}\right\} = \frac{y - x}{F_k}$$

$A_1 \cap B_2 : S_k(x, y) = \max\left\{\dfrac{y - x}{F_k}, \dfrac{1 - y}{F_{k-1}}\right\} = \dfrac{1 - y}{F_{k-1}}$ since $(x, y) \in B_2$ gives

$(1 - x)F_{k-1} \le (1 - y)F_{k+1}$, and therefore $(y - x)F_{k-1} = (1 - x)F_{k-1} - (1 - y)F_{k-1} \le (1 - y)F_{k+1} - (1 - y)F_{k-1} = (1 - y)F_k$.

$$A_1 \cap B_3 : S_k(x, y) = \max\left\{\frac{y - x}{F_k}, \frac{y - x}{F_{k-1}}\right\} = \frac{y - x}{F_{k-1}},$$

$A_1 \cap B_4 : S_k(x, y) = \max\left\{\dfrac{y - x}{F_k}, \dfrac{1 - y}{F_k}\right\} = \dfrac{1 - y}{F_k}$ since $(x, y) \in B_4$

gives $1 - x \le 2(1 - y)$ or $y - x \le 1 - y$,

$$A_2 \cap B_2 : S_k(x, y) = \max\left\{\frac{x}{F_{k-1}}, \frac{1 - y}{F_{k-1}}\right\} = \frac{1}{F_{k-1}} \max\{x, 1 - y\},$$

$A_2 \cap B_3 : S_k(x, y) = \max\left\{\dfrac{x}{F_{k-1}}, \dfrac{y - x}{F_{k-1}}\right\} = \dfrac{y - x}{F_{k-1}}$ since $(x, y) \in A_2$

gives $2x \le y$ or $x \le y - x$,

$A_2 \cap B_4 : S_k(x,y) = \max\left\{\frac{x}{F_{k-1}}, \frac{1-y}{F_k}\right\} = \frac{1-y}{F_k}$ since $(x,y) \in A_2$ gives $2x - y \leq 0$, and since $(x,y) \in B_4$ gives $-xF_k + yF_{k+1} \leq F_{k-1}$.

Indeed, multiplying the former inequality by F_k and adding it to the latter gives $xF_k + yF_{k-1} \leq F_{k-1}$.

$A_3 \cap B_3 : S_k(x,y) = \max\left\{\frac{y-x}{F_{k-1}}, \frac{y-x}{F_{k-1}}\right\} = \frac{y-x}{F_{k-1}}$,

$A_3 \cap B_4 : S_k(x,y) = \max\left\{\frac{y-x}{F_{k-1}}, \frac{1-y}{F_k}\right\} = \frac{1-y}{F_k}$ since $(x,y) \in B_4$ gives $(1-x)F_k \leq (1-y)F_{k+1}$, and therefore $(y-x)F_k = (1-x)F_k - (1-y)F_k \leq (1-y)F_{k+1} - (1-y)F_k = (1-y)F_{k-1}$.

$A_4 \cap B_4 : S_k(x,y) = \max\left\{\frac{x}{F_k}, \frac{1-y}{F_k}\right\} = \frac{1}{F_k}\max\{x, 1-y\}$.

The schematic representation of $S_k(x,y)$ then is given by Fig. 4. There are breaks along the line $x = 1 - y$ in areas $A_2 \cap B_2$ and $A_4 \cap B_4$. The feathered lines are again those boundaries of linearity regions at which S_k decreases for fixed x. The abscissae of intersection points of feathered lines are therefore critical. The first one of these critical arguments we denote by v. It is the abscissa of the intersection point of the line

$$\frac{1-y}{1-x} = \frac{F_{k-1}}{F_{k+1}} , \tag{2.7}$$

which separates B_1 from B_2, and the line

$$\frac{x}{y} = \frac{F_{k-1}}{F_{k+1}} , \tag{2.8}$$

which separates A_1 from A_2. Elimination of y yields

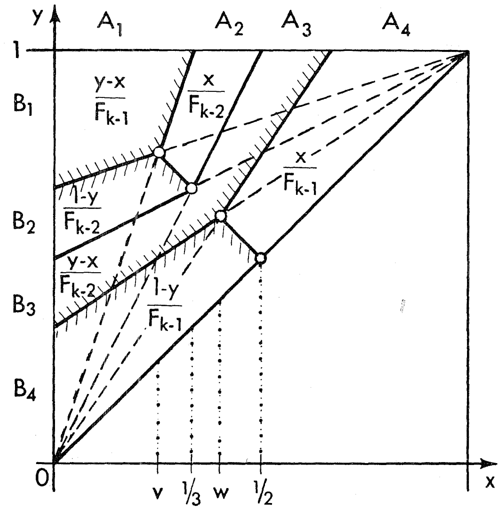

Fig. 4 $S_k(x,y)$ and Critical Arguments

$$v = \frac{F_{k-1}}{F_{k+1} + F_{k-1}} .$$

The next critical argument clearly has the value 1/3. The third one, which we call w, is the intersection of the line

$$\frac{1-y}{1-x} = \frac{F_k}{F_{k+1}} , \tag{2.9}$$

which separates B_3 from B_4, and the line

$$\frac{x}{y} = \frac{F_k}{F_{k+1}} , \tag{2.10}$$

which separates A_3 and A_4. Elimination of y yields

$$w = \frac{F_k}{F_{k+2}} .$$

The last critical argument finally has the value 1/2.

For $0 \leq x \leq v$ the values of $S_k(x, y)$ at the intersection of the vertical through x with the two feathered lines (2.7) and (2.9) are potential minima. The equations of these lines can be rewritten as

$$\frac{1-y}{F_{k-1}} = \frac{1-x}{F_{k+1}} \quad \text{and} \quad \frac{1-y}{F_k} = \frac{1-x}{F_{k+1}} \ .$$

As these terms also represent the value of $S_k(x,y)$, we have

$$M_k(x) = \frac{1-x}{F_{k+1}}$$

for $0 \leq x \leq v$.

For $v < x < 1/3$ locally minimal points are to be found on line (2.9) and in the area where $S_k(x, y)$ assumes the value x/F_{k-1}. Now $x \geq v$ gives $xF_{k+1} \geq (1-x)F_{k-1}$ or

$$\frac{x}{F_{k-1}} \geq \frac{1-x}{F_{k+1}} \ .$$

Thus

$$M_k(x) = \frac{1-x}{F_{k+1}}$$

for $v \leq x \leq 1/3$.

For $1/3 \leq x \leq w$ only the line (2.9) is interesting, and $M_k(x)$ still takes the value

$$\frac{1-x}{F_{k+1}} \ .$$

For $w \leq x \leq 1/2$ and beyond the minimum is assumed within the entire line segments which meets the area in which $S_k(x, y) = x/F_k$.

Thus, finally

$$M_k(x) = \begin{cases} \dfrac{1 - x}{F_{k+1}} & \text{for} \quad 0 \le x \le \dfrac{F_k}{F_{k+2}} \\[2ex] \dfrac{x}{F_k} & \text{for} \quad \dfrac{F_k}{F_{k+2}} \le x \le 1 \end{cases}, \tag{2.11}$$

and (2.6) follows immediately from (2.1).

Note also that (2.11) implies

$$L_k(x) = \begin{cases} M_k(x) & \text{for} \quad 0 \le x \le \frac{1}{2} \\[1ex] M_k(1 - x) & \text{for} \quad \frac{1}{2} \le x \le 1 \end{cases} \tag{2.12}$$

3. SEARCH STRATEGY

In the previous section, we have determined the optimal length of uncertainty $L_k(x)$, which can be achieved in k evaluations in addition to one evaluation at $x \in [0,1]$. We have yet to describe a search strategy which realizes $L_k(x)$. This amounts to specifying the argument y of the first evaluation in addition to x. In view of (2.12), this reduces to determining y such that $M_k(x) = S_k(x,y)$ for given x between 0 and $1/2$, a task which has been performed already while calculating $M_k(x)$.

If $0 \le x \le v$, then there are two optimal solutions y, since

$$S_k(x,y) = \frac{1 - x}{F_{k+1}}$$

along both feathered lines in Fig. 4. This non-uniqueness is not surprising. Indeed, if $x = 0$, then the evaluation at this argument does not contribute at all towards narrowing the interval of uncertainty, and the optimal continuation is just plain Fibonacci with one evaluation wasted. And in this case there are two optimal arguments, namely the first and second $(k-1)^{st}$ order Fibonacci points

$$\frac{F_{k-1}}{F_{k+1}}, \frac{F_k}{F_{k+1}}.$$

(3.1) If $0 < x < \dfrac{F_{k-1}}{F_{k+1}+F_{k-1}}$, then any of the two $(k-1)^{st}$ order Fibonacci points in the interval $[x,1]$ is an optimal evaluation point

$$y_1 = x + \frac{F_{k-1}}{F_{k+1}}(1-x) = \frac{xF_k + F_{k-1}}{F_{k+1}}$$

$$y_2 = x + \frac{F_k}{F_{k+1}}(1-x) = \frac{xF_{k-1} + F_k}{F_{k+1}} .$$

In both intervals $v \le x \le 1/3$ and $1/3 \le x \le w$, the optimal solution y is unique.

(3.2) If $\dfrac{F_{k-1}}{F_{k+1}+F_{k-1}} \le x \le \dfrac{F_k}{F_{k-1}}$ then the optimal evaluation point y is the first $(k-1)^{st}$ order Fibonacci point of the interval $[x,1]$.

Finally, if $w \le x \le 1/2$, then the optimal solutions fill an entire interval.

(3.3) Let $\dfrac{F_k}{F_{k-1}} \le x \le \frac{1}{2}$. If y_0 is such that x is the second $(k-1)^{st}$ order Fibonacci point in $[0, y_0]$, then all points in $[1-x, y_0]$ are optimal evaluation points.

The following rule will always yield an optimal solution:

(3.4) <u>Theorem</u>. An optimal search strategy after an arbitrary first evaluation at $x_0 \in [a,b]$ is as follows. If $c \le x \le d$ are such that $[c,d]$ constitutes the interval of uncertainty after ℓ additional evaluations, and if x is the argument for which the function has been evaluated already, then:

(i) If x lies between c and the first $(k - \ell)^{th}$ order Fibonacci points in $[c,d]$, then choose y as the first $(k - \ell)^{th}$ order Fibonacci point in $[x,d]$.

(ii) If x lies between the two $(k - \ell)^{th}$ order Fibonacci points of $[c,d]$, then choose y as the symmetric image of x in $[c,d]$, i.e., $y = c + d - x$.

(iii) If x lies between d and the second of the two $(k - \ell)^{th}$ order Fibonacci points in $[c,d]$, then choose y as the second $(k - \ell)^{th}$ order Fibonacci point in $[c,x]$.

We shall refer to any sequential search strategy in keeping with (3.1, 2, 3), in particular the rule described in Theorem (3.4), as

(3.5) Modified Fibonacci Search .

If the interior of the interval of uncertainty does not contain an argument at which the function has been evaluated already, then the selection of the next evaluation by modified Fibonacci search will be the same as in standard Fibonacci search.

4. SPIES

Intervals of uncertainty with nonoptimal evaluation points may be the result of the following situation. Suppose in maximizing a function we avail ourselves of the services of a "spy." This spy operates as follows: every time an interval of uncertainty has been based on the results of prior evaluations, he is consulted, and as a result of this consultation, the interval of uncertainty may sometimes be further reduced (remaining an interval) without additional evaluations. One cannot expect, however, that the remaining evaluation point (if there is any) is in optimal position within the new interval of uncertainty.

In this case, there is a question whether the additional information should be accepted. It is indeed conceivable that reducing the interval of uncertainty and subsequently continuing from a non-optimal evaluation point would in the final analysis lead to a larger interval of uncertainty than ignoring the additional information and doing a straightforward Fibonacci search. That this is not so, is essentially the content of the following.

(4.1) Theorem. The optimal policy in the presence of an unpredictable spy is to heed his advice and to proceed from the interval of uncertainty so achieved by modified Fibonacci search with respect to the remaining evaluation point if there is any.

Proof. Let $[c,d]$ be the interval of uncertainty as determined by the previous step of the search, and let $[\bar{c},\bar{d}]$, $c \le \bar{c} \le \bar{d} \le d$, be the interval of uncertainty after consulting the spy. As the spy is unpredictable, there may be no further information forthcoming. This is the worst case, since even if the spy is providing information, it need not be heeded. Thus all we have to show is that we do not worse by proceeding form $[\bar{c},\bar{d}]$ than from any other interval $[c^*,d^*]$ with $[c,d] \supseteq [c^*,d^*] \supseteq [\bar{c},\bar{d}]$.

Now let x be the evaluation point in $[c,d]$. Then we distinguish two cases, depending on whether $x \in [\bar{c},\bar{d}]$ or not. Suppose $x \in [\bar{c},\bar{d}]$, then $x \in [c^*,d^*]$. Working on the latter interval, the best we can guarantee in remaining steps is reducing the uncertainty to

$$(d^* - c^*)L_\ell\left(\frac{x - c^*}{d^* - c^*}\right) = \begin{cases} \dfrac{d^* - x}{F_{\ell+1}} & \text{for } 0 < \dfrac{x - c^*}{d^* - c^*} < \dfrac{F_\ell}{F_{\ell+2}} \quad (=: I_1) \\[2ex] \dfrac{x - c^*}{F_\ell} & \text{for } \dfrac{F_\ell}{F_{\ell+1}} < \dfrac{x - c^*}{d^* - c^*} < \dfrac{1}{2} \quad (=: I_2) \\[2ex] \dfrac{d^* - x}{F_\ell} & \text{for } \dfrac{1}{2} < \dfrac{x - c^*}{d^* - c^*} < \dfrac{F_{\ell+1}}{F_{\ell+2}} \quad (=: I_3) \\[2ex] \dfrac{x - c^*}{F_{\ell+1}} & \text{for } \dfrac{F_{\ell+1}}{F_{\ell+2}} < \dfrac{x - c^*}{d^* - c^*} < 1 \quad (=: I_4) \end{cases} .$$

For all x such that

$$\frac{x - c^*}{d^* - c^*} \quad \text{and} \quad \frac{x - \bar{c}}{\bar{d} - \bar{c}}$$

are both in one of the four intervals I_i above,

$$(d^* - c^*)L_\ell\left(\frac{x - c^*}{d^* - c^*}\right) \ge (\bar{d} - \bar{c})L_\ell\left(\frac{x - \bar{c}}{\bar{d} - \bar{c}}\right) \tag{4.2}$$

is immediate. Of the remaining twelve cases, we need consider only six, as the others follow by symmetry. Let

$$u^* := d^* - c^* \quad \text{and} \quad \bar{u} := \bar{d} - \bar{c} .$$

$$\frac{x - c^*}{u^*} \in I_1 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_2 : \frac{x - c^*}{u^*} \le \frac{F_\ell}{F_{\ell+2}}$$

implies

$$\frac{d^* - x}{u^*} \ge \frac{F_\ell}{F_{\ell+2}} .$$

Thus

$$\frac{d^* - x}{F_{\ell+1}} \ge \frac{x - c^*}{F_\ell} \ge \frac{x - \bar{c}}{F_\ell} .$$

$$\frac{x - c^*}{u^*} \in I_1 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_3 : x - \bar{c} \ge \frac{\bar{u}}{2}$$

gives $x - \bar{c} \ge \bar{d} - x$. Thus

$$\frac{d^* - x}{F_{\ell+1}} \ge \frac{x - c^*}{F_\ell} \ge \frac{x - \bar{c}}{F_\ell} \ge \frac{\bar{d} - x}{F_\ell}$$

$$\frac{x - c^*}{u^*} \in I_1 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_4 : F_{\ell+1} \ge F_\ell .$$

Thus

$$\frac{d^* - x}{F_{\ell+1}} \ge \frac{x - c^*}{F_\ell} > \frac{x - \bar{c}}{F_\ell} \ge \frac{x - \bar{c}}{F_{\ell+1}} .$$

$$\frac{x - c^*}{u^*} \in I_2 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_3 : x - \bar{c} \geq \frac{\bar{u}}{2}$$

gives $x - \bar{c} \geq \bar{d} - x$. Thus

$$\frac{x - c^*}{F_\ell} > \frac{x - \bar{c}}{F_\ell} > \frac{\bar{d} - x}{F_\ell} \quad .$$

$$\frac{x - c^*}{u^*} \in I_2 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_4 : F_{\ell+1} \geq F_\ell \quad .$$

Thus

$$\frac{x - c^*}{F_\ell} \geq \frac{x - \bar{c}}{F_\ell} \geq \frac{x - \bar{c}}{F_{\ell+1}} \quad .$$

$$\frac{x - c^*}{u^*} \in I_3 \quad \text{and} \quad \frac{x - \bar{c}}{\bar{u}} \in I_4 : \frac{x - c^*}{u^*} \leq \frac{}{F_{\ell+}}$$

implies

$$\frac{d^* - x}{u^*} \leq \frac{F_{\ell-1}}{F_{\ell+1}} \quad .$$

Thus

$$\frac{d^* - x}{F_\ell} \geq \frac{x - c^*}{F_{\ell+1}} \geq \frac{x - \bar{c}}{F_{\ell+1}} \quad .$$

The case in which $x \notin [\bar{c}, \bar{d}]$ remains to be considered. Suppose $x < \bar{c} < \bar{d}$. Since we proceed by standard Fibonacci in any interval of uncertainty not containing x in its interior, starting with $[\bar{c}, \bar{d}]$ is certainly better than starting with $[x, \bar{d}] \subseteq [c, d]$, and we have already seen that $[x, \bar{d}]$ is better than any interval between $[c, d]$ and $[x, \bar{d}]$.

A spy is called

(4.3) almost unpredictable

if for each subinterval $[c^*, d^*]$ of the interval of uncertainty $[c, d]$, which results from the evaluation pattern, the spy has the option of reducing it only to an interval $[\bar{c}, \bar{d}]$ which contains $[c^*, d^*]$. Plainly, we still have

(4.4) Theorem. The optimal policy in the presence of an almost unpredictable spy is to heed his advice and to proceed from the interval of uncertainty so achieved by modified Fibonacci search with respect to the remaining evaluation point if there is any.

5. CONCAVE FUNCTIONS

We shall see that a "spy" is available if the unimodal function to be maximized is known to be concave.

A function $f : [a, b] \to R$ is

(5.1) concave

in $[a, b]$ if

$$f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$$

holds for all $x, y \in [a, b]$, $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. The function is

(5.2) strictly concave

if

$$f(\lambda x + \mu y) > \lambda f(x) + \mu f(y)$$

holds for all x, y, λ, μ which are as above and satisfy in addition $x \neq y$ and $\lambda, \mu > 0$. We state without proof that

(5.3) Every upper semicontinuous concave function on $[a, b]$ is unimodal.

Without the additional hypothesis of upper semicontinuity, (5.3) does not hold as there are concave functions without maximum on $[a, b]$.

Now consider two points

$$P_i := (x_i, f(x_i)) \qquad P_j := (x_j, f(x_j)), \qquad x_i < x_j \ ,$$

of the graph

$$G(f) := \{(x, f(x)) : x \in [a, b]\}$$

and let L_{ij} be the straight line through P_i, P_j. Concavity implies that the graph of f does not lie below L_{ij} in $[x_i, x_j]$ and not above L_{ij} in the remainder of the interval $[a, b]$. Hence if five points of the graph $G(f)$,

$$P_0 := (x_0, f(x_0)), \cdots, P_4 := (x_4, f(x_4))$$

with

$$x_0 < x_1 < x_2 < x_3 < x_4$$

and

$$f(x_2) > f(x_i), \qquad i = 1, 2,$$

are known, then that part of the graph $G(f)$ that lies above $[x_1, x_3]$ is contained in the union of the two triangles Δ_1 and Δ_2 formed by L_{01}, L_{12}, L_{23} and L_{12}, L_{23}, L_{34}, respectively. $f(x_2)$ is a lower bound for the maximum value of f. Therefore

(5.4) a maximum of f must lie in the intersection of $\Delta_1 \Delta_2$ with the horizontal through P_2. (Fig. 5)

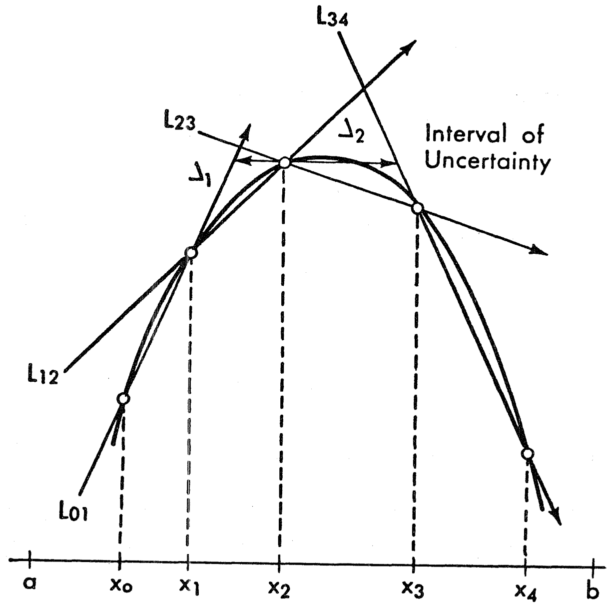

Fig. 5 Bounding a Concave Function by Chords

The information that the function f is concave can thus be used in order to reduce the interval of uncertainty.

In order to complete the description of the proposed search method for concave functions, a few more conventions are necessary. At the ends of the interval $[a, b]$, we pretend that the function has value $-\infty$, and if it has been evaluated there, we pretend that there are two values for the same abscissa, one of the values being infinite. Three evaluations will therefore reduce the interval of uncertainty as indicated in Fig. 6.

We proceed to show that

(5.5) concavity is an almost unpredictable spy (4.3).

Proof. Suppose we have five points

$$a \le x_0 \le x_1 < x_2 < x_3 \le x_4 \le b ,$$

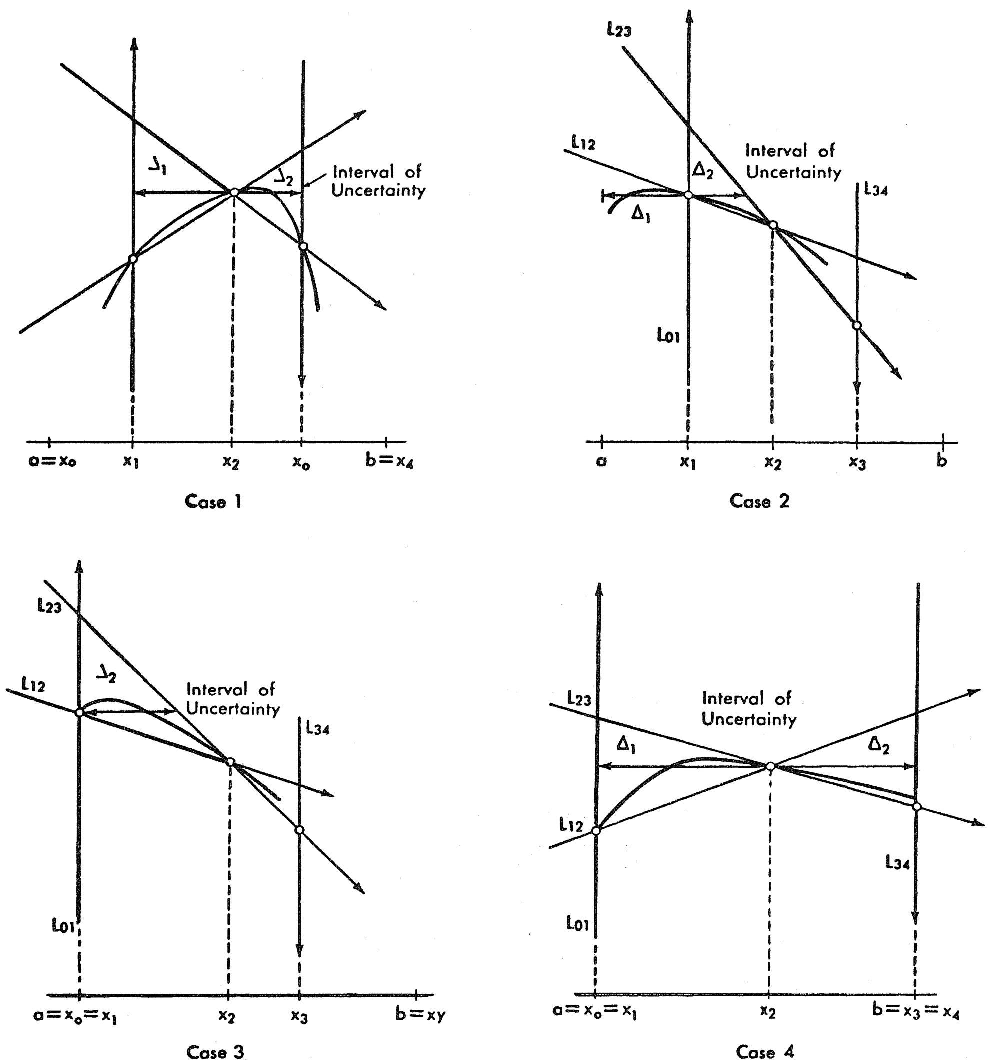

Fig. 6 Three Evaluations

where x_0 and x_1 may both coincide with the left end-point a, and similarly x_3 and x_4 may coincide with the right end-point b. For x_i with $i \neq 0,4$, we have finite function values $f(x_i)$, whereas $f(x_0)$ and $f(x_4)$ are possibly infinite, provided $x_0 = a$ or $x_4 = b$, respectively. We suppose furthermore that

$$f(x_0) < f(x_1) < f(x_2) < f(x_3) < f(x_4) .$$

Let $[c,d]$ be the interval of uncertainty that results in view of concavity. Observe that

$$x_2 \in [c,d] .$$

Now select any x with $c \le x \le x_2$, $x_1 < x$, and assume that $f(x)$ satisfies

$$f(x) = f(x_2) + \delta(x - x_2)$$

for some δ with

$$0 \le \delta \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

Then the new interval of uncertainty taking concavity into account will be of the form $[\bar{c},d]$, where

$$\bar{c} = x + \frac{\delta(x - x_1)(x_2 - x)}{f(x_2) - f(x_1) - \delta(x_2 - x)} > x .$$

The difference $\bar{c} - x$ measures the reduction of uncertainty due to concavity. Now by definition of δ,

$$\bar{c} - x \le \frac{\delta(x - x_1)(x_2 - x)}{f(x_2) - f(x_1) - \delta(x_2 - x_1)} \le \frac{\delta(x_2 - x_1)^2}{f(x_2) - f(x_1) - \delta(x_2 - x_1)}$$

and the last term, independent of x, goes to zero as δ goes to zero. In other words, the contribution of concavity beyond unimodality becomes arbitrarily small as $f(x)$ approaches $f(x_2)$ from below, without assuming it.

The symmetric argument can be carried out for $x_2 < x \le d$ and $x < x_3$. This then will establish concavity as an almost independent spy.

[Continued on page 146.]

SOME PROPERTIES OF THIRD-ORDER RECURRENCE RELATIONS

A. G. SHANNON*
University of Papua and New Guinea, Boroko, T. P. N. G.
and
A. F. HORADAM
University of New England, Armidale, Australia

1. INTRODUCTION

In this paper, we set out to establish some results about third-order recurrence relations, using a variety of techniques.

Consider a third-order recurrence relation

$$S_n = PS_{n-1} + QS_{n-2} + RS_{n-3} \quad (n \geq 4), \quad S_0 = 0, \tag{1.1}$$

where P, Q, and R are arbitrary integers.

Suppose we get the sequence

$$\{J_n\}, \quad \text{when} \quad S_1 = 0,\ S_2 = 1,\ \text{and}\ S_3 = P, \tag{1.2}$$

and the sequence

$$\{K_n\}, \quad \text{when} \quad S_1 = 1,\ S_2 = 0,\ \text{and}\ S_3 = Q, \tag{1.3}$$

and the sequence

$$\{L_n\}, \quad \text{when} \quad S_1 = 0,\ S_2 = 0,\ \text{and}\ S_3 = R. \tag{1.4}$$

It follows that

$$K_1 = J_2 - J_1, \quad K_2 = J_3 - PJ_2,$$

and for $n \geq 3$,

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*Part of the substance of a thesis submitted in 1968 to the University of New England for the degree of Bachelor of Letters.

(1.5) $$K_n = QJ_{n-1} + RJ_{n-2} ,$$

and

(1.6) $$L_n = RJ_{n-1} .$$

These sequences are generalizations of those discussed by Feinberg [2], [3] and Waddill and Sacks [6].

2. GENERAL TERMS

If the auxiliary equation

$$x^3 - Px^2 - Qx - R = 0$$

has three distinct real roots, suppose that they are given by α, β, γ.

According to the general theory of recurrence relations, J_n can be represented by

(2.1) $$J_n = A\alpha^{n-1} + B\beta^{n-1} + C\gamma^{n-1} ,$$

where

$$A = \frac{\alpha}{(\beta - \alpha)(\gamma - \alpha)}, \quad B = \frac{\beta}{(\gamma - \beta)(\alpha - \beta)} ,$$

and

$$C = \frac{\gamma}{(\alpha - \gamma)(\beta - \gamma)}$$

(A, B and C are determined by J_1, J_2, and J_3.)

The first few terms of $\{J_n\}$ are

$$(J_1) = 0,\ 1,\ P,\ P^2 + Q,\ P^3 + 2PQ + R,\ P^4 + 3P^2Q + 2PR + Q^2 .$$

These terms can be determined by the use of the formula

$$J_{n+2} = \sum_{i=0}^{[n/3]} R^i \sum_{j=0}^{[n/2]} a_{nij} P^{n-3i-2j} Q^j , \tag{2.2}$$

where a_{nij} satisfies the partial difference equation

$$a_{nij} = a_{n-1,i,j} + a_{n-2,i,j-1} + a_{n-3,i-1,j} \tag{2.3}$$

with initial conditions

$$a_{noj} = \binom{n - j}{j}$$

and

$$a_{nio} = \binom{n - 2i}{i} .$$

For example,

$$\begin{aligned} J_5 &= a_{300} P^3 + a_{301} PQ + a_{310} R \\ &= P^3 + 2PQ + R . \end{aligned}$$

Formula (2.2) can be proved by induction. In outline, the proof uses the basic recurrence relation (1.1) and then the partial difference equation (2.3). The result follows because

$$PJ_{n+1} = \sum_{i=0}^{[(n-1)/3]} R^i \sum_{j=0}^{[(n-1)/2]} a_{n-1,i,j} P^{n-3i-2j} Q^j ,$$

$$QJ_n = \sum_{i=0}^{[(n-2)/3]} R^i \sum_{j=1}^{[n/2]} a_{n-2,i,j-1} P^{n-3i-2j} Q^j ,$$

$$RJ_{n-1} = \sum_{i=1}^{[n/3]} R^i \sum_{j=0}^{[(n-3)/2]} a_{n-3,i-1,j} P^{n-3i-2j} Q^j .$$

By using the techniques developed for second-order recurrence relations, it can be shown that

$$(2.4)\qquad (P + Q + R - 1)\sum_{r=1}^{n} J_r = J_{n+3} + (1 - P)J_{n+2} + (1 - P - Q)J_{n+1} - 1.$$

It can also be readily confirmed that the generating function for $\{J_n\}$ is

$$(2.5)\qquad \sum_{n=0}^{\infty} J_n x^n = x^2(1 - Px - Qx^2 - Rx^3)^{-1} .$$

3. THE OPERATOR E

We define an operator E, such that

$$(3.1)\qquad EJ_n = J_{n+1} ,$$

and suppose, as before, that there exist 3 distinct real roots, α, β, γ of the auxiliary equation

$$x^3 - Px^2 - Qx - R = 0 .$$

This can be written as

$$(x - \alpha)(x - \beta)(x - \gamma) = (x^2 - px + q)(x - \gamma) = 0 ,$$

where

$$p = \alpha + \beta = P - \gamma ,$$

and $q = \alpha\beta$.

The recurrence relation

$$J_n = PJ_{n-1} + QJ_{n-2} + RJ_{n-3}$$

can then be expressed as

$$(E^3 - PE^2 - QE - R)J_n = 0 \qquad \text{(replacing } n \text{ by } n + 3)$$

or

$$(E^2 - pE + q)(E - \gamma)J_n = 0 , \tag{3.2}$$

which becomes

$$(E^2 - pE + q)u_n = 0 \tag{3.3}$$

or

$$u_{n+2} - pu_{n+1} + qu_n = 0$$

if we let

$$(E - \gamma)J_n = u_n ,$$

where $\{u_n\}$ is defined by

$$u_{n+2} = pu_{n+1} - qu_n, \quad (n \geq 0), \quad u_0 = 0, \quad u_1 = 1 . \tag{3.4}$$

In other words,

$$u_n = J_{n+1} - \gamma J_n \tag{3.5}$$

and the extensive properties developed for $\{u_n\}$ can be utilized for $\{J_n\}$.

In particular,

$$u_n^2 - u_{n-1} \cdot u_{n+1} = q^{n-1} \tag{3.6}$$

becomes

$$(J_{n+1} - \gamma J_n)^2 - (J_n - \gamma J_{n-1})(J_{n+2} - \gamma J_{n+1}) = q^{n-1} .$$

This gives us

$$(J_{n+1}^2 - J_n J_{n+2}) + \gamma(J_{n+1}J_n - J_{n+2}J_{n-1}) + \gamma^2(J_n^2 - J_{n+1}J_{n-1}) = q^{n-1} . \tag{3.7}$$

Another identity for $\{J_n\}$ analogous to (3.6) is developed below as (4.4).

Since

$$\begin{aligned} J_n &= u_{n-1} + \gamma J_{n-1} \\ &= u_{n-1} + \gamma(u_{n-2} + J_{n-2}) \\ &= u_{n-1} + \gamma u_{n-2} + \gamma^2(u_{n-3} + J_{n-3}) \end{aligned}$$

then

$$J_n = \sum_{r=1}^{n} \gamma^{n-r} u_{r-1} , \tag{3.8}$$

which may be a more useful form of the general term than those expressed in (2.1) and (2.2).

4. USE OF MATRICES

Matrices can be used to develop some of the properties of these sequences. In general, we have

$$\begin{bmatrix} S_5 \\ S_4 \\ S_3 \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_4 \\ S_3 \\ S_2 \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} S_3 \\ S_2 \\ S_1 \end{bmatrix}$$

and so, by finite induction,

$$(4.1) \qquad \begin{bmatrix} S_n \\ S_{n-1} \\ S_{n-2} \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} S_3 \\ S_2 \\ S_1 \end{bmatrix} .$$

Again, since

$$\begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} P^2 + Q & PQ + R & PR \\ P & Q & R \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J_4 & K_4 & RJ_3 \\ J_3 & K_3 & RJ_2 \\ J_2 & K_2 & RJ_1 \end{bmatrix}$$

we can show by induction that

$$(4.2) \qquad \underset{\sim}{S}^n = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{bmatrix} .$$

The corresponding determinants give

$$(4.3) \qquad (\det \underset{\sim}{S})^n = R^n = \begin{vmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{vmatrix}$$

By the repeated use of (1.5), we can show that

$$\begin{vmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{vmatrix} = R^2 \begin{vmatrix} J_{n+1} & J_n & J_{n+1} \\ J_{n+1} & J_{n-1} & J_n \\ J_n & J_{n-2} & J_{n-1} \end{vmatrix}$$

and

$$\begin{vmatrix} J_{n+2} & J_n & J_{n+1} \\ J_{n+1} & J_{n-1} & J_n \\ J_n & J_{n-2} & J_{n-1} \end{vmatrix} = R^{n-2} \tag{4.4}$$

which is analogous to

$$u_n^2 - u_{n-1} \cdot u_{n+1} = q^{n-1} \tag{4.5}$$

for the second-order sequence $\{u_n\}$ defined above, (3.4). In the more general case, we get

$$\underset{\sim}{S}_n = \begin{bmatrix} S_{n+3} & S_{n+1} & S_{n+2} \\ S_{n+2} & S_n & S_{n+1} \\ S_{n+1} & S_{n-1} & S_n \end{bmatrix} = \underset{\sim}{S}^{n-1}\underset{\sim}{S}_1$$

and the corresponding determinants are

$$\begin{vmatrix} S_{n+3} & S_{n+1} & S_{n+2} \\ S_{n+2} & S_n & S_{n+4} \\ S_{n+1} & S_{n-1} & S_n \end{vmatrix} = R^{n-1} \begin{vmatrix} S_4 & S_2 & S_3 \\ S_3 & S_1 & S_2 \\ S_2 & S_0 & S_1 \end{vmatrix} .$$

Matrices can also be used to develop expressions for

$$\sum_{n=0}^{\infty} \frac{J_n}{n!}, \quad \sum_{n=0}^{\infty} \frac{K_n}{n!}, \quad \sum_{n=0}^{\infty} \frac{L_n}{n!} ,$$

by adapting and extending a technique used by Barakat [1] for the Lucas polynomials.

Let

$$\underset{\sim}{X} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with a trace

$$P = a_{11} + a_{22} + a_{33}, \qquad \det \underset{\sim}{X} = R \ ,$$

and

$$Q = \sum_{i,j=1}^{3} a_{ij}a_{ji} - a_{ii}a_{jj}\,, \qquad (i \neq j) \ .$$

For example,

$$\underset{\sim}{X} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

satisfies the conditions.

The characteristic equation of $\underset{\sim}{X}$ is

$$\lambda^3 - P\lambda^2 - Q\lambda - R = 0$$

and so, by the Cayley-Hamilton Theorem [4],

$$\underset{\sim}{X}^3 = P\underset{\sim}{X}^2 + Q\underset{\sim}{X} + R\underset{\sim}{I} \ .$$

Thus

$$\underset{\sim}{X}^4 = P\underset{\sim}{X}^3 + Q\underset{\sim}{X}^2 + R\underset{\sim}{X}$$

$$= (P^2 + Q)\underset{\sim}{X}^2 + (PQ + R)\underset{\sim}{X} + PR\,\underset{\sim}{I}$$

and so on, until

$$\underset{\sim}{X}^n = J_n\underset{\sim}{X}^2 + K_n\underset{\sim}{X} + L_n\underset{\sim}{I}\ . \tag{4.6}$$

Now, the exponential of a matrix $\underset{\sim}{X}$ of order 3 is defined by the infinite series

$$e^{\underset{\sim}{X}} = \underset{\sim}{I} + \frac{1}{1!}\underset{\sim}{X} + \frac{1}{2!}\underset{\sim}{X} + \cdots\quad , \tag{4.7}$$

where $\underset{\sim}{I}$ is the unit matrix of order 3.

Substitution of (4.6) into (4.7) yields

$$e^{\underset{\sim}{X}} = \underset{\sim}{X}^2\sum_{n=0}^{\infty}\frac{J_n}{n!} + \underset{\sim}{X}\sum_{n=0}^{\infty}\frac{K_n}{n!} + \underset{\sim}{I}\sum_{n=0}^{\infty}\frac{L_n}{n!}\quad . \tag{4.8}$$

Sylvester's matrix interpolation formula [5] gives us

$$e^{\underset{\sim}{X}} = \sum_{\lambda_1,\lambda_2,\lambda_3} e^{\lambda_1}\frac{(\underset{\sim}{X} - \lambda_2\underset{\sim}{I})(\underset{\sim}{X} - \lambda_3\underset{\sim}{I})}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}\quad , \tag{4.9}$$

where λ_1, λ_2, λ_3 are the eigenvalues of $\underset{\sim}{X}$.

Simplification of (4.9) yields

$$e^{\underset{\sim}{X}} = \frac{\displaystyle\sum_{\lambda_1,\lambda_2,\lambda_3}\{e^{\lambda_1}(\lambda_3 - \lambda_2)\underset{\sim}{X}^2 + e^{\lambda_1}(\lambda_3^2 - \lambda_2^2)\underset{\sim}{X} + e^{\lambda_1}\lambda_2\lambda_3(\lambda_3 - \lambda_2)\underset{\sim}{I}\}}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)} \tag{4.10}$$

By comparing coefficients of $\underset{\sim}{X}^n$ in (4.8) and (4.10), we get

$$\sum_{n=0}^{\infty} \frac{J_n}{n!} = \frac{\displaystyle\sum_{\lambda_1,\lambda_2,\lambda_3} e^{\lambda_1}(\lambda_3 - \lambda_2)}{\displaystyle\prod_{\lambda_1,\lambda_2,\lambda_3} (\lambda_1 - \lambda_2)} ,$$

$$\sum_{n=0}^{\infty} \frac{K_n}{n!} = \frac{\sum e^{\lambda_1}(\lambda_3^2 - \lambda_2^2)}{\prod(\lambda_1 - \lambda_2)} ,$$

$$\sum_{n=0}^{\infty} \frac{L_n}{n!} = \frac{\sum e^{\lambda_1} \lambda_2\lambda_3(\lambda_3 - \lambda_2)}{\prod(\lambda_1 - \lambda_2)} .$$

The authors hope to develop many other properties of third-order recurrence relations.

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Combining (5.5) with Theorem (4.4) yields

(5.6) Theorem. Using concavity as a spy in a modified Fibonacci search is the optimal strategy for reducing the interval of uncertainty of concave functions.

6. FINAL REMARKS

From the proof of Theorem (5.6), it is apparent that the proposed search strategy for concave function is "min sup" rather than "min max." In other words, the problem is not well set. Indeed, it makes probably more sense for concave functions to decrease the uncertainty in the value of the minimum than in its location.

A similar argument as was used for proving (5.5) can be employed to show that for each $\epsilon > 0$ and each positive integer k there is a concave function for which the reduction of uncertainty by optimal search is improved by less than ϵ over unimodal search. In general, however, the improvement will be drastic, in particular if the function is well rounded, so to speak, and has a maximum in the interior.

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DETERMINANTS AND IDENTITIES INVOLVING FIBONACCI SQUARES

MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara, California

Determinants provide an unusual means of discovering identities involving elements of any Fibonacci sequence. In this paper, a determinant relationship believed to be new provides the derivation of several series of identities for Fibonacci sequences.

1. THE ALTERNATING LAMBDA NUMBER

First is displayed the theorem which provides the foundation for what follows. Only 3 x 3 determinants are discussed here, but the theorem is given in general.

Theorem. Let $A = (a_{ij})$ and $A^* = (a^*_{ij})$ be $n \times n$ matrices such that

$$a^*_{ij} = a_{ij} + (-1)^{i+j}k .$$

Then

$$\det A^* = \det A + k(\det C) ,$$

where $C = (c_{ij})$ is the $(n-1) \times (n-1)$ matrix given by

$$c_{ij} = a_{ij} + a_{i+1,j+1} + a_{i+1,j} + a_{i,j+1} .$$

Proof. Successively replace the k^{th} column by the sum of the $(k-1)^{st}$ and k^{th} columns for $k = n, n-1, \cdots, 2$. Then successively replace the k^{th} row by the sum of the $(k-1)^{st}$ and k^{th} row for $k = n, n-1, \cdots, 2$. The resulting determinant is

$$\begin{vmatrix} a_{11} + k & a_{11} + a_{12} & a_{12} + a_{13} & \cdots \\ a_{21} + a_{11} & c_{11} & c_{12} & \cdots \\ a_{31} + a_{21} & c_{21} & c_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \det A + k(\det C)$$

by noting that the determinant on the left can be expressed as the sum of two determinants by splitting the first column and then reversing the above steps for the determinant which does not contain k in the upper left corner.

Specifically, the theorem says that, for $n = 3$,

$$\begin{vmatrix} a+k & b-k & c+k \\ d-k & e+k & f-k \\ g+k & h-k & i+k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k\begin{vmatrix} a+b+d+e & b+c+e+f \\ d+e+g+h & e+f+h+i \end{vmatrix}$$

Definition. We agree to call $\det C$ of the theorem the alternating lambda number of A, denoted by $\lambda_n(A)$.

The closely related lambda number of a matrix arising with the addition of a constant k to each element of a matrix has been discussed in [1], [2], and [3].

As an illustration of the theorem, evaluate $\det W_n$ for

$$W_n = \begin{bmatrix} L_n^2 & L_{n+1}^2 & L_{n+2}^2 \\ L_{n+1}^2 & L_{n+2}^2 & L_{n+3}^2 \\ L_{n+2}^2 & L_{n+3}^2 & L_{n+4}^2 \end{bmatrix}$$

where each element is the square of a Lucas number L_n, using the usual $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_n + L_{n+1}$. The value of the analogous $\det W_n^*$ where W_n^* is formed from W_n by replacing L_n by the Fibonacci number F_n, defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_n + F_{n+1},$$

has been given in [4] as $2(-1)^{n+1}$. It is not difficult to calculate $\lambda_a(W_n^*)$:

$$\lambda_a(W_n^*) = \begin{vmatrix} F_n^2 + F_{n+2}^2 + 2F_{n+1}^2 & F_{n+1}^2 + F_{n+3}^2 + 2F_{n+2}^2 \\ F_{n+1}^2 + F_{n+3}^2 + 2F_{n+2}^2 & F_{n+2}^2 + F_{n+4}^2 + 2F_{n+3}^2 \end{vmatrix} = \begin{vmatrix} L_{2n+2} & L_{2n+3} \\ L_{2n+3} & L_{2n+4} \end{vmatrix} = 5$$

Since

$$5F_n^2 = L_n^2 + (-1)^{n+1}4 ,$$

$$\det (5W_n^*) = \det W_n + (-1)^{n+1}4\cdot\lambda_a(5W_n^*)$$

$$5^3\cdot 2(-1)^{n+1} = \det W_n + (-1)^{n+1}4\cdot 5^2\cdot 5$$

$$\det W_n = (-1)^n 2\cdot 5^3 .$$

2. DETERMINANTS INVOLVING SQUARES OF ELEMENTS OF ANY FIBONACCI SEQUENCE

Consider the matrix

$$(2.1) \qquad A_n = \begin{bmatrix} H_n^2 & H_{n+1}^2 & H_{n+2}^2 \\ H_{n+1}^2 & H_{n+2}^2 & H_{n+3}^2 \\ H_{n+2}^2 & H_{n+3}^2 & H_{n+4}^2 \end{bmatrix}$$

where each element is the square of a member of a Fibonacci sequence $\{H_n\}$ defined by

$$H_1 = p, \quad H_2 = q, \quad H_{n+2} = H_{n+1} + H_n .$$

Since an identity for such Fibonacci sequences is

$$H_{n+3}^2 = 2H_{n+2}^2 + 2H_{n+1}^2 - H_n^2 ,$$

multiplying each element in columns two and three by (-2) and adding to column one yields the elements $-H_{n+3}^2$, $-H_{n+4}^2$, $-H_{n+5}^2$. Column exchanges show that

$$\det A_n = -\det A_{n+1} ,$$

so increasing the subscript by one in A_n only changes the sign of $\det A_n$, and $|\det A_n|$ is independent of n. It is not difficult (just messy) to evaluate $\det A_n$, then, by picking a value for n, calculating members of $\{H_n\}$ in terms of p and q, and using elementary algebra. This method of calculation for 3 x 3 determinants whose elements are squares of Fibonacci numbers was given by Fuchs and Erbacher in [4].

The results are

$$(2.2)\qquad \begin{aligned} \det A_n &= 2(-1)^n(q^2 - pq - p^2)^3 = 2(-1)^n D_H^3 \\ \lambda_a(A_n) &= 5(q^2 - pq - p^2)^2 = 5D_H^2 \end{aligned},$$

where $|D_H|$ is the characteristic number of the sequence (see [5]). If $\{H_n\} = \{F_n\}$, the Fibonacci sequence, $D_F = -1$ and $\det A_n = 2(-1)^{n+1}$.

The same method will allow the calculations of the values of several other determinants which follow.

$$(2.3)\qquad \det C_n = \begin{vmatrix} H_n^2 & H_{n+1}^2 & H_{n+2}^2 \\ H_{n+3}^2 & H_{n+4}^2 & H_{n+5}^2 \\ H_{n+6}^2 & H_{n+7}^2 & H_{n+8}^2 \end{vmatrix} = (-1)^n 64\, D_H^3 :$$

$$\lambda_n(C_n) = 160\, D_H^2$$

Continuing since also

$$H_{n+4}H_{n+2} = 2H_{n+3}H_{n+1} + 2H_{n+2}H_n - H_{n+1}H_{n-1},$$

we obtain (2.4) and (2.5):

$$(2.4)\qquad \det R_n = \begin{vmatrix} H_{n+1}H_{n-1} & H_{n+2}H_n & H_{n+3}H_{n+1} \\ H_{n+3}H_{n+1} & H_{n+4}H_{n+2} & H_{n+5}H_{n+3} \\ H_{n+4}H_{n+2} & H_{n+5}H_{n+3} & H_{n+6}H_{n+4} \end{vmatrix} = (-1)^{n+1} 3\, D_H^3 :$$

$$\lambda_a(R_n) = 5\,D_H^2 .$$

$$(2.5)\qquad \det S_n = \begin{vmatrix} H_{n+1}H_{n-1} & H_{n+2}H_n & H_{n+3}H_{n+1} \\ H_{n+4}H_{n+2} & H_{n+5}H_{n+3} & H_{n+6}H_{n+4} \\ H_{n+7}H_{n+5} & H_{n+8}H_{n+6} & H_{n+9}H_{n+7} \end{vmatrix} = (-1)^{n+1} 96\, D_H^3 :$$

$$\lambda_a(S_n) = 160\, D_H^2$$

Since

$$H_n^2 = H_{n+1} H_{n-1} + (-1)^n D_H ,$$

Equations (2.4) and (2.5) can be obtained in a second way with a minimum of effort by using the alternating lambda number theorem. For example, to find (2.5) using (2.3),

$$\begin{aligned} \det C_n &= \det S_n + (-1)^n D_H \lambda_a(C_n) \\ 64(-1)^n D_H^3 &= \det S_n + (-1)^n D_H (160\, D_H^2) \\ \det S_n &= (-1)^{n+1}\, 96\, D_H^3 \end{aligned} .$$

Also, notice that

$$\lambda_a(C_n) = \lambda_a(S_n) .$$

The identity

$$H_{n+6}^2 = 8H_{n+4}^2 - 8H_{n+2}^2 + H_n^2$$

allows one to use the method of Fuchs and Erbacher to find two more values:

$$(2.6) \qquad \det B_n = \begin{vmatrix} H_n^2 & H_{n+2}^2 & H_{n+4}^2 \\ H_{n+2}^2 & H_{n+4}^2 & H_{n+6}^2 \\ H_{n+4}^2 & H_{n+6}^2 & H_{n+8}^2 \end{vmatrix} = (-1)^n 18\, D_H^3 ;$$

$$\lambda_a(B_n) = 9 D_H [(-1)^n 8H_{n+4}^2 + 13 D_H]$$

$$(2.7) \qquad \begin{vmatrix} H_n^2 & H_{n+2}^2 & H_{n+4}^2 \\ H_{n+6}^2 & H_{n+8}^2 & H_{n+10}^2 \\ H_{n+12}^2 & H_{n+14}^2 & H_{n+16}^2 \end{vmatrix} = (-1)^n 2^{11} 3^3 D_H^3 .$$

Compare (2.6) with the Fibonacci result $(18)(-1)^{n+1}$ as given in [6], and notice that D_H^3 is a factor in each determinant value found in this section.

In (2.6) and (2.7) the alternating lambda numbers are not independent of n and hence are not useful in what follows. The alternating lambda number for (2.6) is interesting in that it depends upon the center element of B_n.

3. IDENTITIES FOR MEMBERS OF ANY FIBONACCI SEQUENCE $\{H_n\}$

Before we can continue, we must standardize our sequences. For purposes of forming a Fibonacci sequence, $H_1 = p$ and $H_2 = q$ are arbitrary integers. But surprisingly enough, if enough terms are written, each sequence has a subsequence of terms which alternate in sign as well as a subsequence in which all terms are of the same sign. Since we want a standard way of numbering the terms of these sequences in what follows, when we want the characteristic number

$$D_H = H_2^2 - H_2 H_1 - H_1^2$$

to be positive, then we take H_0 as the first member of the non-alternating subsequence, and H_1 as the second member. When we want $D_H < 0$, we take H_1 as the first or third member of the non-alternating subsequence. Note that $D_H = 5$ for $\{H_n\} = \{L_n\}$, and $D_H = -1$ for $\{H_n\} = \{F_n\}$. Now we are ready to develop several identities which relate two Fibonacci sequences.

The identity

$$L_n^2 + (-1)^{n+1} 4 = 5F_n^2$$

suggests that we seek an identity relating two Fibonacci sequences $\{H_n\}$ and $\{G_n\}$. Returning to (2.1), form matrix A_n with elements from $\{H_n\}$ and matrix A_n^* with elements from $\{G_n\}$. If there exist two integers x and k such that

$$H_n^2 + (-1)^{n+1} x = k G_n^2 \quad ,$$

then the alternating lambda number theorem and (2.2) provide

$$\det A_n + (-1)^{n+1} x\lambda_a(kA_n^*) = \det(kA_n^*)$$

$$2(-1)^n D_H^3 + (-1)^{n+1} x(5k^2 D_G^2) = 2(-1)^n k^3 D_G^3$$

$$x = \frac{(D_H^3 - k^3 D_G^3)(2)}{5k^2 D_G^2}$$

If $-kD_G = D_H$, then $x = 4D_H/5$. Since x must be an integer, D_H must be a multiple of 5. A solution is given by $k = 5$, $D_H = 5(-D_G)$. Since 5 and multiples of 5 do occur as characteristic numbers, we have

$$H_n^2 + (-1)^{n+1} \frac{4}{5} D_H = 5 G_n^2 , \tag{3.1}$$

where $\{H_n\}$ has the positive characteristic number D_H and $\{G_n\}$ has the negative characteristic number $D_G = -D_H/5$.

An example of a solution is given by the pairs of sequences

$$\{H_n\} = \{\cdots, 13, -6, 7, 1, 8, 9, \cdots\}$$

and

$$\{G_n\} = \{\cdots, 5, -1, 3, 2, 5, 7, \cdots\}$$

or their conjugates

$$\{H_n^*\} = \{\cdots, 8, -1, 7, 6, 13, \cdots\}$$

and

$$\{G_n^*\} = \{\cdots, 5, -2, 3, 1, 4, 5, \cdots\} .$$

Since $D_H = 55 > 0$, set $H_1 = 1$ and $H_1^* = 6$, but since $D_0 = -11 < 0$, take $G_1 = 3$ and $G_1^* = 4$. Using $\{H_n\}$ and $\{G_n\}$, notice that

$$H_n^2 + (-1)^{n+1} 44 = 5 G_n^2 . \tag{3.2}$$

Also note that

$$H_n + H_{n+2} = 5G_{n+1}$$

and

$$G_n + G_{n+2} = H_{n+1} .$$

Above, $\{H_n\}$ and $\{G_n\}$ were found by simply referring to a table of characteristic numbers. (See [5] and [7].) To write a pair of sequences $\{H_n\}$ and $\{G_n\}$ to satisfy (3.1), let $p > 0$ be an arbitrary integer. Let z be an integer such that

$$p \equiv 2z \pmod 5 .$$

Then $H_1 = p$ and $H_2 = z$ gives $D_H = 5m$ for some integer m, and

$$G_1 = \frac{2z - p}{5}, \qquad G_2 = \frac{2p + z}{5}$$

gives $\{G_n\}$ with $D_G = -m$. The justification is simple, for if $p \equiv 2z$ (mod 5), then

$$\begin{aligned} D_H &= z^2 - pz - p^2 = (z - p)(z + p) - pz \\ &\equiv (5k - z)(3z) - 2z^2 \equiv 15kz - 5z^2 \equiv 0 \pmod 5 . \end{aligned}$$

The other statements follow by elementary algebra.

Solutions to (3.1) with $D_G = -D_H/5$ for $H_1 = 1, 2, \cdots, 7, \cdots, p, \cdots$ follow. In each case $u, t = 0, 1, 2, \cdots$.

Two more identities relating the two Fibonacci sequences $\{H_n\}$ and $\{G_n\}$ just described follow.

The identity

$$L_n L_{n+2} + (-1)^{n+1} = 5 F_{n-1}^2$$

D_H	$\{H_n\}$ (H_1, H_2)	$\{G_n\}$ (G_1, G_2)
$25t(t-1)+5$	$(1, -2+5t)$	$(2t-1, t)$
$25t^2-5$	$(2, 1+5t)$	$(2t, 1+t)$
$25t(t-1)-5$	$(3, -1+5t)$	$(2t-1, 1+t)$
$25t^2-20$	$(4, 2+5t)$	$(2t, 2+t)$
$25t(t-1)-25$	$(5, 5t)$	$(2t-1, 2+t)$
$25t^2-45$	$(6, 3+5t)$	$(2t, 3+t)$
$25t(t-1)-55$	$(7, 1+5t)$	$(2t-1, 3+t)$
...	...	...
$25t^2-5u^2$	$(2u, u+5t)$	$(2t, u+t)$
$25t(t-1)-5(u^2+u-1)$	$(2u+1, u+5t-2)$	$(2t-1, u+t)$

suggests searching for an identity of the form

$$H_n H_{n+2} + (-1)^{n+1} x = k\, G_{n+1}^2 .$$

The alternating lambda number theorem, (2.2) and (2.4) give

$$\det R_n + (-1)^{n+1} x \lambda_a (kA_n^*) = \det (kA_{n+1}^*)$$
$$3(-1)^{n+2} D_H^3 + (-1)^{n+1} x (5k^2 \cdot D_G^2) = 2(-1)^{n+1} k^3 D_G^3$$
$$x = \frac{2k^3 D_G^3 + 3 D_H^3}{5k^2 \cdot D_G^2}$$

If $kD_G = D_H$, then $x = D_H$, and we have the known identity

(3.3) $$H_n H_{n+2} + (-1)^{n+1} D_H = H_{n+1}^2 .$$

If $kD_G = -D_H$, then $x = D_H/5$. Again let $k = 5$ since D_H must be a multiple of 5, yielding

$$H_n H_{n+2} + (-1)^{n+1} D_H/5 = 5G^2_{n+1} \quad , \tag{3.4}$$

where the characteristic number of $\{G_n\}$ is $-D_H/5$.

A final derivation is suggested by the identity

$$L^2_n + (-1)^n = 5F_{n+1}F_{n-1} \quad .$$

Proceeding as before using (2.2) and (2.4),

$$H^2_n + (-1)^n x = kG_{n+1}G_{n-1}$$
$$\det A_n + (-1)^n x\lambda_a(kR_n) = \det(kR_n)$$
$$2(-1)^n D^3_H + (-1)^n x(5k^2D^2_G) = (-1)^{n+1}3k^3D^3_G$$
$$x = \frac{-3k^3D^3_G - 2D^3_H}{5k^2D^2_G} \quad .$$

If $D_H = -kD_G$, then $x = D_H/5$, and if $k = 5$, we have

$$H^2_n + (-1)^n D_H/5 = 5G_{n+1}G_{n-1} \quad , \tag{3.5}$$

where again $D_G = -D_H/5$. If $D_H = kD_G$, then $x = -D_H$, and taking $k = 1$ gives the known identity

$$H^2_n + (-1)^{n+1} D_H = H_{n+1}H_{n-1} \quad ,$$

which is the same as (3.3).

The possibilities are by no means exhausted by this paper.

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[Continued on page 184.]

A GENERATING FUNCTION FOR PARTLY ORDERED PARTITIONS

L. CARLITZ*
Duke University, Durham, North Carolina

1. In a recent paper [1], Cadogan has discussed the function $\phi_k(n)$ which satisfies the recurrence

$$\phi_k(n) = \phi_k(n-1) + \phi_{k-1}(n-1) \qquad (n > k \geq 1) \tag{1}$$

together with

$$\phi_0(n) = p(n) \tag{2}$$

and

$$\phi_k(k) = 2^{k-1} \qquad (k \geq 1) . \tag{3}$$

As usual $p(n)$ denotes the number of unrestricted partitions of n, so that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1} . \tag{4}$$

The object of the present note is to obtain a generating function for $\phi_k(n)$. Put

$$\Phi_k(x) = \sum_{n=k}^{\infty} \phi_k(n)\, x^n ,$$

$$\Phi(x,y) = \sum_{k=0}^{\infty} \Phi_k(x)\, y^k = \sum_{n,k=0}^{\infty} \phi_k(n)\, x^n y^k .$$

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Then, by (1) and (3), we have

$$\begin{aligned}\Phi_k(x) &= 2^{k-1}x^k + \sum_{n=k+1}^{\infty}\{\phi_k(n-1) + \phi_{k-1}(n-1)\}x^n \\ &= 2^{k-1}x^k + x\sum_{n=k}^{\infty}\phi_k(x^n)x^n + x\sum_{n=k}^{\infty}\phi_{k-1}(n)\,x^n \\ &= 2^{k-1}x^k + x\Phi_k(x) + x\Phi_{k-1}(x) - \phi_{k-1}(k-1)x^k\ ,\end{aligned}$$

so that

(5) $$(1 - x)\Phi_1(x) = x\Phi_0(x)\ ,$$

(6) $$(1 - x)\Phi_k(x) = 2^{k-2}x^k = x\Phi_{k-1}(x) \qquad (k > 1)\ .$$

It follows that

$$\begin{aligned}\Phi(x,y) &= \Phi_0(x) + \Phi_1(x)y + \sum_{k=2}^{\infty}\Phi_k(x)\,y^k \\ &= \Phi_0(x) + \frac{xy}{1-x}\Phi_0(x) + \frac{1}{1-x}\sum_{k=2}^{\infty}\{2^{k-2}x^k + x\Phi_{k-1}(x)\}\,y^k \\ &= \Phi_0(x) + \frac{x^2y^2}{(1-x)(1-xy)} + \frac{xy}{1-x}\Phi(x,y)\ .\end{aligned}$$

We have therefore

(7) $$\begin{aligned}\Phi(x,y) &= \frac{(1-x)\Phi_0(x)}{1-x-xy} + \frac{x^2y^2}{(1-x-xy)(1-2xy)} \\ &= \frac{1-x}{1-x-xy}\prod_{n=1}^{\infty}(1-x^n)^{-1} + \frac{x^2y^2}{(1-x-xy)(1-2xy)}\ .\end{aligned}$$

2. By means of (7) we can obtain an explicit formula for $\Phi_k(x)$. Since

$$\frac{1 - x}{1 - x - xy} = \left(1 - \frac{xy}{1 - x}\right)^{-1} = \sum_{k=0}^{\infty} \frac{x^k y^k}{(1 - x)^k}$$

and

$$\frac{1}{(1 - x - xy)(1 - 2xy)} = \sum_{r=0}^{\infty} \frac{x^r y^r}{(1 - x)^{r+1}} \sum_{s=0}^{\infty} (2xy)^s$$

$$= \sum_{k=0}^{\infty} x^k y^k \sum_{r=0}^{k} \frac{2^{k-r}}{(1 - x)^{r+1}} ,$$

it follows that

$$\Phi_k(x) = \frac{x^k}{(1 - x)^k} \Phi_0(x) + \sum_{r=0}^{k-2} \frac{2^{k-r-2} x^k}{(1 - x)^{r+1}} . \tag{8}$$

Moreover, since

$$\frac{1}{(1 - x)^{r+1}} = \sum_{s=0}^{\infty} \binom{r + s}{r} x^s ,$$

Eq. (8) implies

$$\phi_k(n) = \sum_{r=0}^{n-k} \binom{k + r - 1}{r} p(n - k - r) + \sum_{r=0}^{k-2} 2^{k-r-2} \binom{n - k + r}{r} \tag{9}$$

$$(k \geq 2)$$

For $k = 1$, we have

$$\phi_1(n) = \sum_{r=0}^{n-1} p(n-r) \tag{10}$$

as is evident from (5).

Replacing k by n - k in (9) we get

$$\phi_{n-k}(n) = \sum_{r=0}^{k} \binom{n-k+r-1}{r} p(k-r) + \sum_{r=0}^{n-k-2} 2^{n-k-r-2}\binom{k+r}{r} . \qquad (n \geq k+2) \tag{11}$$

Cadogan [1] has derived the formula

$$\begin{aligned} \phi_{n-k}(n) &= \sum_{r=3}^{k} \binom{n-r-1}{k-r} p(r) + \sum_{r=0}^{n-k-1} \binom{k+r-3}{r} 2^{n-k-r+1} \\ &= \sum_{r=0}^{k-3} \binom{n-k+r-1}{r} p(k-r) + \sum_{r=0}^{n-k-1} \binom{k+r-3}{r} 2^{n-k-r+1} . \qquad (3 \leq k < n,\ n \geq 4) \end{aligned} \tag{12}$$

To show that (11) and (12) are in agreement, it suffices to verify that

$$\begin{aligned} &\sum_{r=0}^{n-k-2} 2^{n-k-r-2}\binom{k+r}{r} \\ &= \sum_{r=0}^{n-k-1} 2^{n-k-r+1}\binom{k+r-3}{r} - \binom{n-1}{k} - \binom{n-2}{k-1} - 2\binom{n-3}{k-2} \\ &= \sum_{r=0}^{n-k-2} 2^{n-k-r+1}\binom{k+r-3}{r} - \binom{n-2}{k} - 2\binom{n-3}{k-1} - 4\binom{n-4}{k-2}. \qquad (n \geq k+2) \end{aligned} \tag{13}$$

Since

$$\sum_{n=k+2}^{\infty} x^{n-k-2} \sum_{r=0}^{n-k-2} 2^{n-k-r-2}\binom{k+r}{r}$$

$$= \sum_{r=0}^{\infty}\binom{k+r}{r}x^r \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{(1-x)^{k+1}(1-2x)}$$

and

$$\sum_{n=k+2}^{\infty} x^{n-k-2}\left\{\sum_{r=0}^{n-k-2} 2^{n-k-r+1}\binom{k+r-3}{r} - \binom{n-2}{k} - 2\binom{n-3}{k-1} - 4\binom{n-4}{k-2}\right\}$$

$$= \frac{8}{(1-x)^{k-2}(1-2x)} - \frac{1}{(1-x)^{k+1}} - \frac{2}{(1-x)^{k}} - \frac{4}{(1-x)^{k-1}}$$

$$= \frac{1}{(1-x)^{k+1}(1-2x)} \quad .$$

it is evident that (13) holds

3. Put

$$\psi_n(y) = \sum_{k=0}^{n} \phi_k(n)\, y^k \quad ,$$

so that

$$\psi_0(y) = 1, \qquad \psi_1(y) = 1 + y, \qquad \psi_2(y) = 2 + 2y + 2y^2$$

Then by (1) and (3), for $n \geq 2$,

$$\psi_n(y) = p(n) + \sum_{k=1}^{n-1} \{\phi_k(n-1) + \phi_{k-1}(n-1)\}\, y^k + 2^{n-1} y^n$$

$$= p(n) + (\psi_{n-1}(y) - p(n-1)) + y(\psi_{n-1}(y) - 2^{n-2} y^{n-1}) + 2^{n-1} y^n \quad .$$

Thus

(14) $$\psi_n(y) = p(n) - p(n-1) + (1+y)\psi_{n-1}(y) + 2^{n-2}y^n \qquad (n \geq 2).$$

For example,

$$\psi_2(y) = 1 + (1+y)^2 + y^2 = 2 + 2y + 2y^2$$

$$\psi_3(y) = 1 + (1+y)(2 + 2y + 2y^2) + 2y^3$$

$$= 3 + 4y + 4y^2 + 4y^3 .$$

It is also evident from (14) that

(15) $$\psi_n(1) = p(n) - p(n-1) + 2^{n-2} + 2\psi_{n-1}(y) \qquad (n \geq 2)$$

and

(16) $$\psi_n(-1) = p(n) - p(n-1) + (-1)^n\, 2^{n-2} \qquad (n \geq 2) .$$

The last two formulas are also implied by (7).

REFERENCE

1. C. C. Cadogan, "On Partly Ordered Partitions of a Positive Integer," Fibonacci Quarterly, Vol. 9, 1971, pp. 329-336.

FIBONACCI PRIMITIVE ROOTS

DANIEL SHANKS
Computation and Mathematics Dept., Naval Ship R & D Center, Washington, D. C.

1. INTRODUCTION

A prime p possesses a Fibonacci Primitive Root g if g is a primitive root of p and if it satisfies

$$g^2 \equiv g + 1 \pmod{p} . \tag{1}$$

It is obvious that if (1) holds then so do

$$g^3 \equiv g^2 + g \pmod{p} , \tag{2}$$

$$g^4 \equiv g^3 + g^2 \pmod{p} , \tag{3}$$

etc.

For example, $g = 8$ is one of the four primitive roots of $p = 11$ (the others being 2, 6, 7), and $g = 8$ (only) satisfies (1). Thus, its powers 8^n (mod 11) are

$$1, 8, 9, 6, 4, 10, \cdots \pmod{11}$$

and may be computed not only by

$$9 \equiv 8^2, \quad 6 \equiv 9 \cdot 8, \quad 4 \equiv 9 \cdot 8, \cdots \pmod{11} ,$$

but also, more simply, by

$$9 \equiv 8 + 1, \quad 6 \equiv 9 + 8, \quad 4 \equiv 6 + 9, \cdots \pmod{11} .$$

Thus the name: Fibonacci Primitive Root.

The brief Table 1 shows every $p < 200$ that has an F. P. R., and every such g satisfying $0 < g < p$ that it possesses. By incomplete induction (a

TABLE 1

p	g	p	g
5	3	71	63
11	8	79	30
19	15	109	11, 99
31	13	131	120
41	7, 35	149	41, 109
59	34	179	105
61	18, 44	191	89

fine old expression seldom used these days), we observe the following properties, all of which are easily proved in the next section.

A. Except for the singular $p = 5$, all p having an F.P.R. are $\equiv \pm 1 \pmod{10}$.

B. But not all $p \equiv \pm 1 \pmod{10}$ have an F.P.R., since, e.g., $p = 29$ and 101 do not.

C. Except for the singular $p = 5$, the number of g in $0 < g < p$, if any, is 1 or 2 according as $p \equiv -1$ or $+1 \pmod 4$.

D. In the latter case, the two g satisfy

$$g_1 + g_2 = p + 1 . \tag{4}$$

2. ELEMENTARY PROPERTIES

The solutions of (1) are

$$g \equiv (1 \pm \sqrt{5})2^{-1} \pmod p \tag{5}$$

and therefore exist if, and only if, $p = 5$, $g = 3$, or $p = 10k \pm 1$, since only these p have 5 as a quadratic residue. This proves A. For $p = 29$, the two solutions of (1) are $g = 6$ and 24, but since these are also quadratic residues of 29, they cannot be primitive roots, thus proving B. The product of the two solutions (5) is given by

(6) $$g_1 g_2 \equiv -1 \pmod{p} .$$

Thus, if $p \equiv -1 \pmod 4$, one g is a quadratic residue and one g is not. There can, therefore, then be at most one F. P. R. On the other hand, for $p \equiv +1 \pmod 4$, consider

$$g_2 \equiv -g_1^{-1} .$$

If g_1 is primitive, and g_2 is of order m, then

$$g_1^m \equiv (-1)^m .$$

Therefore, m is even, and so g_2 is primitive also. Thus, g_1 and g_2 are both primitive, or neither is. This completes C. Finally,

(7) $$g_1 + g_2 \equiv 1 \pmod{p}$$

and (4) follows from $0 < g < p$.

3. THE ASYMPTOTIC DENSITY

Let $F(x)$ be the number of primes $p \le x$ having an F. P. R. (We do not distinguish in this count whether p has one or two.) Then with $\pi(x)$ being the total number of primes $\le x$, we

Conjecture: As $x \to \infty$,

(8) $$\frac{F(x)}{(x)} \sim \frac{27A}{38} = 0.2657054465 \cdots ,$$

where

(9) $$A = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136 \cdots$$

is Artin's constant.

Artin originally conjectured, cf. [1], [2, page 81] that if $\nu_a(x)$ is the number of $p \le x$ having a as a primitive root, and if

$$a \ne b^n \qquad (n > 1) ,$$

then

$$\frac{\nu_a(x)}{\pi(x)} \sim A . \tag{10}$$

Subsequently, [3] it was found that the heuristic argument was faulty for $a = 5, -3$, and infinitely many other a but it was still considered reasonable for $a = 2, 3, 6, 7, 10$, etc. Both heuristically and empirically, Eq. (10) seems correct for these a, and Hooley [4] recently proved that (10) is then true provided one assumes a sufficient number of Riemann Hypotheses.

The heuristic argument for (8) is similar to that which leads to (10), but we must modify two of the factors in (9). Consider the primes in the eight residue classes

$$20k + 1, 3, 7, 9, 11, 13, 17, 19 .$$

Those in $20k + 3, 7, 13, 17$ cannot have an F.P.R. For those in $20k + 11, 19$ the factor

$$1 - \frac{1}{2(2 - 1)}$$

in (9) must be deleted. This represented the probability that a is not a quadratic residue and therefore could be a primitive root. But for $20k + 11, 19$, one of g_1 and g_2 must always be a quadratic nonresidue as we have shown with (6). The factor

$$1 - \frac{1}{5(5 - 1)}$$

in (9) represented the probability that a is not a quintic residue and therefore could be a primitive root. For $20k + 9$, 19 p has no quintic residues since these p are not $\equiv 1 \pmod 5$, and so this factor is deleted. For $20k + 1$, 11, p is always $\equiv 1 \pmod 5$, and the factor must be changed to

$$1 - \frac{1}{5}.$$

Therefore, the expected density of p in these eight residue classes having an F. P. R. is the following:

$20k + 1$	$16A/19$	$20k + 11$	$32A/19$
$20k + 3$	0	$20k + 13$	0
$20k + 7$	0	$20k + 17$	0
$20k + 9$	$20A/19$	$20k + 19$	$40A/19$

As $x \to \infty$, the eight classes of primes are equinumerous, and so (8) follows from this table by averaging these densities. On the other hand, it is known that the number of primes in

$$20k + 1, \qquad 20k + 9$$

will generally lag somewhat behind the other six classes since 1 and 9 are quadratic residues of 20, cf. [5]. We therefore expect that the convergence of $F(x)/\pi(x)$ to $27A/38$ will be mostly from above.

The empirical facts are given in Table 2.

TABLE 2

x	F(x)	(x)	$F(x)/\pi(x)$
500	31	95	0.3263
1000	46	168	0.2738
1500	66	239	0.2762
2000	81	303	0.2673
2500	97	367	0.2643

This seems thoroughly satisfactory.

It seems likely that one could transcribe Hooley's theory [4] to the present variant, and thereby prove (8), assuming a sufficient number of Riemann Hypotheses. But the theory in [4] is by no means simple, and this transcription has not been attempted so far.

4. SEVERAL REFERENCES

In closing, we indicate three references related to the concept developed here. The idea for a Fibonacci Primitive Root was suggested by Exercise 158 in [2, page 206]. It is shown there that if g is <u>any</u> primitive root of <u>any</u> prime p, the sequence of first differences

$$g^{n+1} - g^n \pmod{p} \tag{11}$$

is the same as the sequence

$$g^{n-d} \pmod{p} \tag{12}$$

for some fixed displacement d. If, now, one has the first d powers of g:

$$1,\ g,\ g^2,\ \cdots,\ g^d,$$

one can obtain all further powers <u>additively</u> from (11). Our construction here forces $d = 1$ and therefore allows this additive computation <u>ab initio</u>.

In [6], W. Schooling gives a curious method of computing logarithms based on the fact that all powers of

$$\varphi = (1 + \sqrt{5})/2$$

can be computed additively:

$$\varphi^2 = \varphi + 1,$$

$$\varphi^3 = \varphi^2 + \varphi,$$

[Continued on page 181.]

AN INTERESTING SEQUENCE OF NUMBERS DERIVED FROM VARIOUS GENERATING FUNCTIONS

PAUL S. BRUCKMAN
San Rafael, California

The following development, to the best of the author's knowledge, is new. At any rate, it is original and very interesting. We begin by defining the function

$$(1) \qquad f(x) = 1/(1 - x)\sqrt{1 + x} \quad .$$

This may be thought of as the generating function of a power series in x, whose coefficients we are to determine. Thus, we seek the values of the coefficients A_k, where

$$(2) \qquad f(x) = \sum_{k=0}^{\infty} A_k x^k \quad .$$

That this representation is valid may be seen by observing that $f(x)$ is expressible as the product of the two functions $(1 - x)^{-1}$ and $(1 + x)^{-\frac{1}{2}}$, each of which is of the same form as (2). In fact,

$$(3) \qquad (1 - x)^{-1} = \sum_{k=0}^{\infty} x^k , \qquad \text{and} \qquad (1 + x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k}\left(-\frac{1}{4}\right)^k x^k \quad .$$

Therefore, it follows that

$$(4) \qquad A_k = \sum_{i=0}^{k} \binom{2i}{i}\left(-\frac{1}{4}\right)^i \quad .$$

From the foregoing expression for A_k, it is evident that

$$A_k = A_{k-1} + \binom{2k}{k}\left(-\frac{1}{4}\right)^k, \qquad A_0 = 1 . \tag{5}$$

Recursion (5) may be expressed in the form

$$A_k = A_{k-1} - \frac{2k-1}{2k} \cdot \binom{2k-2}{k-1}\left(-\frac{1}{4}\right)^{k-1} . \tag{6}$$

If, in recursion (6), we multiply throughout by $(2k)/2k - 1$, and if, in recursion (5), we replace the subscript k by $k - 1$, we may add the two results, thereby eliminating the factorial term. Upon simplification, this process yields the following recursion, which involves three successive values of A_k:

$$2kA_k = A_{k-1} + (2k - 1)A_{k-2} . \tag{7}$$

This is valid for $k = 2, 3, 4, \cdots$, and if we affix the values $A_0 = 1$ and $A_1 = \frac{1}{2}$, we have fully characterized the coefficients A_k.

We shall now define the sequence of numbers B_k, such that for each non-negative integer k,

$$B_k = 2^k \cdot k! \cdot A_k . \tag{8}$$

Substituting this definition in recursion (7),

$$\frac{2k \cdot B_k}{2^k \cdot k!} = \frac{B_{k-1}}{2^{k-1}(k-1)!} + \frac{(2k-1)B_{k-2}}{2^{k-2}(k-2)!} .$$

If we multiply this result throughout by $2^{k-1} \cdot (k - 1)!$, we obtain:

$$B_k = B_{k-1} + (2k - 1)(2k - 2)B_{k-2} . \tag{9}$$

Recursion (9), plus the initial conditions $B_0 = B_1 = 1$, completely characterize the coefficients B_k. Furthermore, from (9), it is evident that all the B_k's are integers. Upon application of (9), for the first few values of k, we obtain the following values:

$$B_0 \equiv B_1 = 1, \quad B_2 = 7, \quad B_3 = 27,$$

$$B_4 = 321, \quad B_5 = 2,265, \quad B_6 = 37,575, \quad B_7 = 390,915,$$

etc. We may summarize the results thus far derived in the following form:

$$f(2x) = 1/(1 - 2x)\sqrt{1 + 2x} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}, \tag{10}$$

where

$$B_k = 2^k \cdot k! \sum_{i=0}^{k} \binom{2i}{i} \left(-\frac{1}{4}\right)^i .$$

What struck the author as interesting was the fact that the sequence of numbers B_k appears in other power series, derived from generating functions of totally different form from (10).

Specifically, we will demonstrate that

$$g(x) \equiv e^{x^2/2} \int_0^x e^{-u^2}\, du = \sum_{k=0}^{\infty} B_k \frac{x^{2k+1}}{(2k+1)!}, \tag{11}$$

and

$$h(x) = \tan^{-1} x \Big/ \sqrt{1 - x^2} = \sum_{k=0}^{\infty} (B_k)^2 \frac{x^{2k+1}}{(2k+1)!} . \tag{12}$$

Let $y = g(x)$. If we differentiate y, as defined in (11),

$$y' = e^{x^2/2} \cdot e^{-x^2} + x e^{x^2/2} \int_0^x e^{-u^2}\, du = e^{-x^2/2} + xy .$$

Differentiating again, we obtain

$$y'' = -x\,e^{-x^2/2} + xy' + y = -x\,e^{-x^2/2} + x\,e^{-x^2/2} + x^2y + y = (1 + x^2)y.$$

Next, we observe that $g(x)$ is an odd function of x. This is demonstrated by replacing x with $-x$ and the dummy variable u with $-u$ in (11), which yields $g(-x) = -g(x)$.

Therefore, $g(x)$ may be expressed in the form

$$\sum_{k=0}^{\infty} r_k\, x^{2k+1} .$$

Negative powers of x are excluded, for otherwise $g(x)$ would be discontinuous at $x = 0$, along with the first and higher order derivatives. However, it is readily seen that $g(0) = 0$, $g'(0) = 1$, and $g'(0) = 0$.

We will use these conditions to develop a recursion involving the coefficients r_k. If we differentiate the series expression for $g(x)$,

$$\text{(13)}\quad g'(x) = \sum_{k=0}^{\infty} (2k + 1)\, r_k\, x^{2k} ; \qquad g''(x) = \sum_{k=1}^{\infty} 2k(2k + 1) r_k\, x^{2k-1} .$$

We use the differential equation $y'' = (1 + x^2)y$ derived above, which becomes transformed to the following relationship:

$$\text{(14)}\quad \sum_{k=0}^{\infty} (2k + 2)(2k + 3) r_{k+1}\, x^{2k+1} = \sum_{k=0}^{\infty} r_k\, x^{2k+1} + \sum_{k=1}^{\infty} r_{k-1}\, x^{2k+1} .$$

If we equate the coefficients of similar powers of x, we obtain:

$$\text{(15)}\quad r_0 = 6\, r_1; \qquad (2k + 2)(2k + 3) r_{k+1} = r_k + r_{k-1}, \text{ if } k = 1, 2, 3, \cdots .$$

Using the condition $g'(0) = 1$, we see that $r_0 = 1$, and therefore,

$$r_1 = \frac{1}{6} .$$

We now define the sequence of numbers R_k such that, for every non-negative integer k, $R_k = (2k+1)!\,r_k$. Substituting this definition in recursion (15), and multiplying throughout by $(2k+1)!$, we obtain:

$$R_{k+1} = R_k + 2k(2k+1)R_{k-1}\ ; \qquad \text{also,} \qquad R_0 = R_1 = 1\ . \tag{16}$$

But if we replace k by $k-1$ in (16), we obtain precisely the same recursion as (9). Since the initial values of R_k are identical to those of B_k, we conclude that $R_k = B_k$ for all values of k, and the validity of (11) is established.

The proof of (12) is similar, though somewhat more complicated. We begin by squaring both sides of (9), and solving for $B_{k-1}B_{k-2}$:

$$B_{k-1}B_{k-2} = \frac{B_k^2 - B_{k-1}^2 - (2k-1)^2(2k-2)^2 B_{k-2}^2}{2(2k-1)(2k-2)}\ . \tag{17}$$

Next, we may multiply (9) throughout by B_{k-1}, obtaining

$$B_k B_{k-1} = B_{k-1}^2 + (2k-1)(2k-2)B_{k-1}B_{k-2}\ . \tag{18}$$

If, in (18), we substitute the expression derived in (17) for $B_{k-1}B_{k-2}$, and the corresponding expression for $B_k B_{k-1}$ obtained by increasing the subscript from $k-1$ to k, we arrive at a recursion which involves only the squares of successive B_k's. Upon simplification, this becomes

$$\begin{aligned} B_{k+1}^2 &= (4k^2+2k+1)(B_k^2 + 2k(2k+1)B_{k-1}^2) \\ &\qquad - (2k-2)^2(2k-1)^2 2k(2k+1)B_{k-2}^2\ . \end{aligned} \tag{19}$$

Next, we observe that $h(x)$ is an odd function of x, continuous at $x = 0$. Therefore, as before, $h(x)$ may be expressed in the form

$$\sum_{k=0}^{\infty} s_k x^{2k+1}$$

As before, we will develop a recursion involving the s_k's. If we let $z = h(x)$, as defined in (12), we differentiate as follows:

$$z' = \frac{(1 - x^2)^{\frac{1}{2}} \cdot (1 + x^2)^{-1} + x \tan^{-1}x \cdot (1 - x^2)^{-\frac{1}{2}}}{1 - x^2} = \frac{(1 - x^2)^{-\frac{1}{2}}}{1 + x^2} + \frac{xz}{1 - x^2} .$$

Differentiating again,

$$z'' = \frac{x(1 + x^2)(1 - x^2)^{-3/2} - 2x(1 - x^2)^{-\frac{1}{2}}}{(1 + x^2)^2} + \frac{(1 - x^2)(xz' + z) + 2x^2 z}{(1 - x^2)^2} .$$

From the first differentiation,

$$(1 - x^2)^{-\frac{1}{2}} = (1 + x^2)\left(z' - \frac{xz}{1 - x^2}\right) .$$

Substituting this result in the second differentiation, we eliminate all irrational functions of x, and upon simplifying the result:

$$(20) \qquad (1 + x^2)(1 - x^2)^2 z'' + 4x^3(x^2 - 1) z' + (2x^4 - 3x^2 - 1) z = 0 .$$

In the series expression for $h(x)$, there will be no loss in generality if we make the substitution $s_k = S_k + (2k + 1)!$. Then

$$z = \sum_{k=0}^{\infty} S_k \frac{x^{2k+1}}{(2k + 1)!}, \qquad z' = \sum_{k=0}^{\infty} S_k \frac{x^{2k}}{(2k)!}, \qquad z'' = \sum_{k=0}^{\infty} S_{k+1} \frac{x^{2k+1}}{(2k + 1)!} .$$

Each term in differential equation (20) may be expressed in series form by means of the latter expressions. Using the method of equating coefficients (the development is omitted here, in the interest of brevity), we arrive at the following recursion:

$$(21) \qquad S_{k+1} = (4k^2 + 2k + 1)S_k + 2k(2k + 1)(4k^2 + 2k + 1)S_{k-1} - 2k(2k + 1)(2k - 1)^2(2k - 2)^2 S_{k-2}$$

valid for $k = 0, 1, 2, 3, \cdots$. But this recursion is of the same form as (19), and becomes identical to it if $S_k = B_k^2$ for all non-negative values of k. It remains to show that such is the case for the initial values, where $k = 0$ and 1. We observe that $h(0) = 0$, and from the first-order differential equation, $h'(0) = 1$. But we see from the series expression for z' that $h'(0) = S_0 = 1$. From (21), we readily obtain the values $S_1 = 1$, $S_2 = 49$, $S_3 = 729$, etc. This establishes the truth of (12).

We have overlooked the question of convergence in the manipulation of the foregoing infinite series. A more rigorous treatment would only have served to detract interest from the remarkable properties of these series which link them together. It may be demonstrated, however, that f(x) and h(x) are convergent within the interval (-1,1), excluding the end points; g(x) converges for all real values of x.

The purpose of this paper was to demonstrate the validity of (10), (11) and (12). Now that this has been accomplished, it would be desirable to deduce some properties for the coefficients B_k. The remaining portion is devoted to the derivation of several such properties and relationships.

We begin by noting that g(x) and h(x) are expressible as the products of two functions, as is the case with f(x). By application of Maclaurin's formula,

$$e^{x^2/2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}; \qquad \int_0^x e^{-u^2}\,du = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} .$$

Multiplying these two series term-by-term, we obtain:

$$g(x) = \sum_{k=0}^{\infty} c_k x^{2k+1} ,$$

where

$$c_k = \sum_{i=0}^{k} \frac{(-1)^i}{2^{k-i}(k-i)!\,i!\,(2i+1)} .$$

But, as we have already shown, $c_k = B_k + (2k+1)!$. Therefore, we are led to an alternate expression for B_k:

$$B_k = \frac{(2k+1)!}{2^k \cdot k!} \sum_{k=0}^{k} \binom{k}{i} \frac{(-2)^i}{(2i+1)} . \tag{22}$$

In a similar fashion, we may derive an expression for B_k^2 by using the component functions of $h(x)$:

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} ;$$

$$(1 - x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k} (x/2)^{2k} .$$

Therefore,

$$h(x) = \sum_{k=0}^{\infty} d_k x^{2k+1} ,$$

where

$$d_k = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{2k - 2i + 1} \frac{\binom{2i}{i}}{2^{2i}} ,$$

But, since $d_k = B_k^2 + (2k+1)!$, we are led to the expression:

$$B_k^2 = (-1)^k (2k+1)! \sum_{i=0}^{k} \binom{2i}{i} \frac{\left(-\frac{1}{4}\right)^i}{2k - 2i + 1} . \tag{23}$$

We may also express each B_k in the form of a definite integral as follows:

First, we define the polynomial $P_k(x)$ by the following summation:

$$P_k(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{x^{2i+1}}{2i+1} \quad . \tag{24}$$

If we differentiate,

$$P'_k(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^{2i} \quad .$$

But the latter expression is equivalent to the binomial expansion for $(1 - x^2)^k$. Noting that $P_k(0) = 0$, we may integrate and obtain:

$$P_k(x) = \int_0^x (1 - u^2)^k \, du \tag{25}$$

Next, we observe that

$$P_k(\sqrt{2}) = \sqrt{2} \sum_{i=0}^{k} \binom{k}{i} \frac{(-2)^i}{2i+1} \; .$$

Comparing this with the expression for B_k in (22), we obtain:

$$B_k = \frac{(2k+1)!}{2^{k+\frac{1}{2}} k!} \int_0^{\sqrt{2}} (1 - u^2)^k \, du \quad . \tag{26}$$

Next, we prove the following property:

(27) B_k is divisible by $\dfrac{(2m)!}{2^m m!}$, where m is the greatest integer in $\frac{1}{2}(k+1)$.

If we multiply (5) throughout by $2^k k!$ and apply relation (8), we obtain the recursion

$$(28)\quad B_k = 2k B_{k-1} + (-1)^k \frac{(2k)!}{2^k k!} = 2k B_{k-1} + (-1)^k (1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)) .$$

Recursion (28) may be expressed in the following alternative forms, depending on whether k is even or odd:

$$(28a)\qquad B_{2m} = 4m B_{2m-1} + 1\cdot 3\cdot 5\cdot \cdots \cdot (4m-1)$$

$$(28b)\qquad B_{2m+1} = (4m+2) B_{2m} - 1\cdot 3\cdot 5\cdot \cdots \cdot (4m+1) .$$

We may now prove (27) by induction. Let us first assume that (27) is true for $k = 2m$, i.e., B_{2m} is divisible by $1\cdot 3\cdot 5\cdot \cdots \cdot (2m-1)$. Then, by (28b), B_{2m+1} is divisible by $1\cdot 3\cdot 5\cdot \cdots \cdot (2m+1)$. But this is equivalent to the assertion of (27), where $k = 2m+1$. Now, if we replace m by $m+1$ in (28a), we see that B_{2m+2} is also divisible by $1\cdot 3\cdot 5\cdot \cdots \cdot (2m+1)$. This, in turn, is equivalent to the assertion of (27), where $k = 2m+2$. This establishes the inductive chain. Since (27) is true for $k = 0$, it is therefore true for all values of k.

The readers are invited to discover any other properties of the sequence B_k which they feel might be of interest. It is the belief of the author that a deeper analysis of this series of numbers, though perhaps not of any lasting value, might be a source of recreation for those who derive pleasure from such studies.

APPENDIX

DERIVATION OF EQUATION (21)

In addition to the series expressions for the derivatives of $h(x)$, we will need the following expressions:

$$x^2z = \sum_{k=1}^{\infty} S_{k-1}(2k + 1)^{(2)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^4z = \sum_{k=2}^{\infty} S_{k-2}(2k + 1)^{(4)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^3z' = \sum_{k=1}^{\infty} S_{k-1}(2k + 1)^{(3)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^5z' = \sum_{k=2}^{\infty} S_{k-2}(2k + 1)^{(5)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^2z'' = \sum_{k=1}^{\infty} S_{k}(2k + 1)^{(2)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^4z'' = \sum_{k=2}^{\infty} S_{k-1}(2k + 1)^{(4)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^6z'' = \sum_{k=3}^{\infty} S_{k-2}(2k + 1)^{(6)} \frac{x^{2k+1}}{(2k + 1)!} .$$

In the foregoing, the symbol $(2k + 1)^{(r)}$ represents

$$(2k + 1)(2k)(2k - 1)(2k - 2) \cdots (2k + 1 - (r - 1)) = \frac{(2k + 1)!}{(2k + 1 - r)!} .$$

Equation (20) may be expressed in the following manner:

$$(1 - x^2 - x^4 - x^6)z'' + (4x^5 - 4x^3)z' + (2x^4 - 3x^2 - 1)z = 0 .$$

Substituting the previous expressions in the latter equation, we obtain:

$$\sum_{k=0}^{\infty} S_{k+1} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} S_k (2k+1)^{(2)} \frac{x^{2k+1}}{(2k+1)!}$$

$$- \sum_{k=2}^{\infty} S_{k-1}(2k+1)^{(4)} \frac{x^{2k+1}}{(2k+1)!} + \sum_{k=3}^{\infty} S_{k-2}(2k+1)^{(6)} \frac{x^{2k+1}}{(2k+1)!}$$

$$+ \sum_{k=2}^{\infty} 4S_{k-2}(2k+1)^{(5)} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} 4S_{k-1}(2k+1)^{(3)} \frac{x^{2k+1}}{(2k+1)!}$$

$$+ \sum_{k=2}^{\infty} 2S_{k-2}(2k+1)^{(4)} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} 3S_{k-1}(2k+1)^{(2)} \frac{x^{2k+1}}{(2k+1)!}$$

$$- \sum_{k=0}^{\infty} S_k \frac{x^{2k+1}}{(2k+1)!} = 0 .$$

If we equate like coefficients, we obtain the following recursions:

$$S_1 - S_0 = 0; \qquad S_2 - 6S_1 - 24S_0 - 18S_0 - S_1 = 0;$$

$$S_3 - 20S_2 - 120S_1 + 480S_0 - 240S_1 + 240S_0 - 60S_1 - S_2 = 0 ;$$

if $k = 3, 4, 5, \cdots$,

$$S_{k+1} - (2k(2k+1)+1)S_k - 2k(2k+1)Q_k S_{k-1}$$
$$+ (2k+1)^{(4)}((2k-3)(2k-4) + 4(2k-3) + 2)S_{k-2} = 0 ,$$

where

$$Q_k = (2k-1)(2k-2) + 4(2k-1) + 3 .$$

Upon simplification, these results become:

$$(21)\quad S_{k+1} = (4k^2 + 2k + 1)S_k + 2k(2k + 1)(4k^2 + 2k + 1)S_{k-1} - 2k(2k + 1)(2k - 1)^2(2k - 2)^2 S_{k-2} ,$$

balid for $k = 0, 1, 2, \cdots$.

[Continued from page 168.]

FIBONACCI PRIMITIVE ROOTS

etc. Of course, that is (abstractly) the same thing we are doing in (2), (3).

In [7], Emma Lehmer examines the quadratic character of

$$\theta = (1 + \sqrt{5})/2 \qquad (\text{mod } p) .$$

If θ is a quadratic residue of p, but not a higher power residue, then all quadratic residues can be generated by addition. In our construction, θ is a primitive root and generates the quadratic nonresidues also.

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TABLE OF INDICES WITH A FIBONACCI RELATION

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

In preparing tables of residues for indices of primitive roots the following situation was noted for the modulus 109. The primitive root, 11, has residues as shown corresponding to indices as given on the borders of the table. Thus the residue of 11 to the index 82 is 36.

RESIDUES OF POWERS OF 11 MODULO 109

	0	1	2	3	4	5	6	7	8	9
0		11	12	23	35	58	93	42	26	68
1	94	53	38	91	20	2	22	24	46	70
2	7	77	84	52	27	79	106	76	73	40
3	4	44	48	92	31	14	45	59	104	54
4	49	103	43	37	80	8	88	96	75	62
5	28	90	9	99	108	98	97	86	74	51
6	16	67	83	41	15	56	71	18	89	107
7	87	85	63	39	102	32	25	57	82	30
8	3	33	36	69	105	65	61	17	78	95
9	64	50	5	55	60	6	66	72	29	101
10	21	13	34	47	81	19	100	10	1	

It is noteworthy from the early entries of the table that each succeeding entry is the sum of the two that precede it. This relation can be verified for the entire table if the sums are taken modulo 109. Clearly this is an unusual situation for a table of this kind. The questions that come to mind are: Is this something very extraordinary? Under what conditions does a table of this type have this Fibonacci property?

Since the entries in the table are residues of successive powers of some quantity x, the conditions that must be fulfilled are two: (1) x must satisfy the relation

$$x^{n+1} \equiv x^n + x^{n-1} \pmod{p}$$

or what is equivalent presuming that $(x,p) = 1$ as must be the case for a primitive root,

$$x^2 \equiv x + 1 \pmod{p}$$

(2) x must be a primitive root modulo p.

The first condition leads to the congruence

$$(2x - 1)^2 \equiv 5 \pmod{p}$$

so that a necessary condition is that 5 be a quadratic residue of p. This means that p is a prime of the form $10n \pm 1$. The solutions of this quadratic congruence for primes of this type fulfill the first requirement. It is necessary, however, to determine whether they are primitive roots.

The results of this investigation for primes of the required form up to 300 are shown in the table below.

PRIME	SOLUTIONS	PRIMITIVE ROOTS
11	4, 8	8
19	5, 15	15
29	6, 24	
31	19, 13	13
41	7, 35	7, 35
59	34, 26	34
61	44, 18	44, 18
71	9, 63	63
79	50, 25	
89	10, 80	
101	23, 79	
109	11, 99	11, 99
131	12, 120	120
139	76, 64	

149	104, 41	41
151	28, 124	
179	105, 75	105
181	13, 169	
191	103, 79	
199	138, 62	
211	33, 179	
229	148, 82	
239	16, 224	224
241	52, 190	52, 190
251	134, 118	134
269	198, 72	198, 72
271	17, 225	255
281	38, 244	

The conclusion would seem to be that this phenomenon is not particularly uncommon and that there is a straightforward method of determining additional instances of this type.

[Continued from page 156.]

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7. New book of number theory tables, to be published by the Fibonacci Association.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-189 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{r,s=0}^{\infty} \frac{(2r+3s)!}{r!\,s!\,(r+2s)!} \frac{(a-by)^r b^s y^{r+2s}}{(1+y)^{2r+3s+1}} = \frac{1}{1-ay-by^2}$$

H-190 Proposed by H. H. Ferns, Victoria, British Columbia.

Prove the following

$$2^r F_n \equiv n \pmod 5$$

$$2^r L_n \equiv 1 \pmod 5\ ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, and r is the least residue of $n - 1 \pmod 4$.

H-191 Proposed by David Zeitlin, Minneapolis, Minnesota.

Prove the following identities:

(a) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 L_{2k} = L_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(b) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 L_{2k} = F_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!} 5^{n+1-k}$$

(c) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 F_{2k} = F_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(d) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 F_{2k} = L_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!} 5^{n-k},$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

SOLUTIONS

KEEPING THE Q's ON CUE

H-176 Proposed by C. C. Yalavigi, Government College, Mercara, India.

In the "Collected Papers of Srinivas Ramanujan," edited by G. H. Hardy, P. V. Sheshu Aiyer, and B. M. Wilson, Cambridge University Press, 1927, on p. 326, Q. 427 reads as follows:

Show that (corrected)

$$\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \frac{3}{e^{6\pi}-1} + \cdots = \frac{1}{24} - \frac{1}{8\pi} .$$

Provide a proof.

Solution by Clyde A. Bridger, Springfield, Illinois.

A typical term on the left-hand side can be written as

$$\frac{m\,e^{-2m\pi}}{1 - e^{-2m\pi}} = \frac{m\,q^{2m}}{1 - q^{2m}} .$$

This suggests a logarithmic derivative of a product. A suitable well-known product is

(1) $$Q_0 = \prod_{m=1}^{\infty} (1 - q^{2m}) .$$

(See Harris Hancock, Theory of Elliptic Functions, p. 396, Dover, 1958) where (loc cit p. 107)

(2) $$q = \exp(-\pi K'/K) ,$$

in which K and K' have the same relation to elliptic functions as 2π has to trigonometric functions. For example, for the sine-amplitude function, we have

$$\mathrm{sn}(u + 4K + 2iK') = \mathrm{sn}\, u$$

and for the sine function,

$$\sin(x + 2\pi) = \sin x .$$

Define K itself as the complete elliptic integral of the first kind

(3) $$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$$

with modulus $\underline{k}$. Let K', L, and L' be complete elliptic integrals of the first kind with moduli k', ℓ, ℓ', respectively.

The problem now is to find something that contains Q_0 and K. On page 400 (Hancock) appears

$$(kk')^{\frac{1}{12}} = 2^{\frac{1}{6}} q^{\frac{1}{24}} Q_1 Q_3 , \qquad Q_1 Q_2 Q_3 = 1 ,$$

and

$$q^{\frac{1}{8}} \frac{Q_0}{Q_2} = \sqrt{\frac{K}{\pi} \sqrt{kk'}} \quad .$$

Then

$$q^{\frac{1}{12}} Q_0 = 2^{\frac{1}{6}} (kk') \sqrt{\frac{K}{\pi}} \tag{4}$$

is the starting equation.

Suppose that the four elliptic integrals are connected by

$$\frac{nK'}{K} = \frac{L'}{L} , \tag{5}$$

with $k^2 + k'^2 = 1$ and $\ell^2 + \ell'^2 = 1$. (Arthur Cayley, An Elementary Treatise on Elliptic Functions, p. 45, Dover, 1961.)

Then

$$q^n = e^{-\frac{\pi L'}{L}} \tag{2'}$$

and

$$q^{\frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm}) = 2^{\frac{1}{6}} (\ell\ell')^{\frac{1}{6}} \sqrt{\frac{L}{\pi}} \ . \tag{4'}$$

If we divide Eq. (4) by Eq. (4') and let $n \quad 1$, we should get $1 = 1$. Of the conditions to do this, putting

$$\ell = k' \text{ and } \ell' = k \tag{6}$$

gives a suitable form in $\underline{n}$ only. We find from Eq. (3) that

$$L = K' \quad \text{and} \quad L' = K. \tag{7}$$

Then Eq. (5) becomes

$$K/K' = \sqrt{n}. \tag{5'}$$

Equation (2) becomes

$$q = e^{-\pi/\sqrt{n}} \tag{2''}$$

and Eq. (2') becomes

$$q^n = e^{-\pi/\sqrt{n}} .$$

We can now write the quotient of Eq. (4) by Eq. (4') as

$$\begin{aligned} &e^{-\pi/12\sqrt{n}}(1 - e^{-2\pi/\sqrt{n}})(1 - e^{-4\pi/\sqrt{n}})(1 - e^{-6\pi/\sqrt{n}}) \cdots \\ &\qquad = n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1 - e^{-2\pi\sqrt{n}})(1 - e^{-4\pi\sqrt{n}})(1 - e^{-6\pi\sqrt{n}}) \cdots \end{aligned} \tag{8}$$

Both are infinite products. We now differentiate this logarithmically with respect to $\underline{n}$ to have

$$\begin{aligned} &\frac{\pi}{24 n\sqrt{n}}\left\{1 - 24\left[\frac{e^{-2\pi/\sqrt{n}}}{1 - e^{-2\pi/\sqrt{n}}} + \frac{2e^{-4\pi/\sqrt{n}}}{1 - e^{-4\pi/\sqrt{n}}} + \cdots\right]\right\} \\ &\quad = \frac{1}{4n} - \frac{\pi}{24\sqrt{n}}\left\{1 - 24\left[\frac{e^{-2\pi\sqrt{n}}}{1 - e^{-2\pi\sqrt{n}}} + \frac{e^{-4\pi\sqrt{n}}}{1 - e^{-4\pi\sqrt{n}}} + \cdots\right]\right\}. \end{aligned} \tag{9}$$

This reduces readily to

$$1 - 24 \sum_{m=1} m/(e^{2m\pi/\sqrt{n}} - 1) +$$

$$+ n\left[1 - 24 \sum_{m=1} m/(e^{2m\pi\sqrt{n}} - 1)\right] = \frac{6\sqrt{n}}{\pi} \quad . \tag{9'}$$

Now let $n \to 1$. We find the correct solution to be

$$\frac{1}{e^{2\pi} - 1} + \frac{2}{e^{4\pi} - 1} + \frac{3}{e^{6\pi} - 1} + \cdots = \frac{1}{24} - \frac{1}{8\pi} .$$

We have followed Ramanujan's development and have filled in a number of gaps because his procedure is quite esoteric.

Also solved by the Proposer, who used the reference cited in the problem to pick it up at (9').

PARTITION

H-177 Proposed by L. Carlitz, Duke University, Durham, North Carolina. (corrected)

Let $R(N)$ denote the number of solutions of

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \qquad (r = 1, 2, 3, \cdots),$$

where

$$k_1 \geq k_2 \geq \cdots \geq k_r \geq 1 .$$

Show that

(1) $\quad R(F_{2n}F_{2m}) = R(F_{2n+i}F_{2m}) = (n - m)F_{2m} + F_{2m-1} \qquad (n \geq m)$,

(2) $\quad R(F_{2n}F_{2m+1}) = (n - m)F_{2m+1} \qquad (n > m)$,

(3) $$R(F_{2n+1}F_{2m+1}) = (n - m)F_{2m+1} \qquad (n > m),$$

(4) $$R(F^2_{2n+1}) = R(F^2_{2n}) = F_{2n-1} \qquad (n \geq 1).$$

Solution by the Proposer. (See reference below.)

The Proposer has proved that if

$$N = F_{2k} + F_{2k+4} + F_{2k+8} + \cdots + F_{2k+4r-4} \qquad (k \geq 1),$$

then

(*) $$R(N) = kF_{2r} - F_{2r-1} .$$

Also the same result holds for

$$N = F_{2k+1} + F_{2k+5} + \cdots + F_{2k+4r-3} \qquad (k \geq 1).$$

1. Since

$$F_{2n}F_{2m} = F_{2n-2m+2} + F_{2n-2m+6} + \cdots + F_{2n+2m-2} \qquad (n \geq m),$$

it follows from (*) that

$$\begin{aligned} R(F_{2n}F_{2m}) &= (n - m + 1)F_{2m} - F_{2m-2} \\ &= (n - m)F_{2m} + F_{2m-1} \qquad (n \geq m). \end{aligned}$$

Since

$$F_{2n+1}F_{2m} = F_{2n-2m+3} + F_{2n-2m+7} + \cdots + F_{2n+2m-1} \qquad (n \geq m),$$

it follows that

$$R(F_{2n+1}F_{2m}) = R(F_{2n}F_{2m}) .$$

L. Carlitz, "Fibonacci Representations," Fibonacci Quarterly, Vol. 6, pp. 193-220.

2. It is proved in Theorem 1 of the paper cited above that if

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

where

$$k_1 > k_2 > \cdots \geq k_r \geq 2 ,$$

then

$$(**) \qquad R(N) = R(F_{k_1-k_r+1} + \cdots + F_{k_{r-1}-k_r+1}) + \left(\left[\tfrac{1}{2}k_r\right] - 1\right) R(F_{k_1-k_r+2} + \cdots + F_{k_{r-1}-k_r+2}) ,$$

and in particular if k_r is odd, then

$$(***) \qquad R(N) = R(F_{k_1-1} + \cdots + F_{k_r-1}) .$$

Since

$$F_{2n}F_{2m+1} = (F_{2n+2m-1} + F_{2n+2m-3} + \cdots + F_{2n-2m+3}) + F_{2n-2m} \quad (n > m) ,$$

it follows from (**) and (***) that

$$\begin{aligned} R(F_{2n}F_{2m+1}) &= R(F_{4m} + F_{4m-4} + \cdots + F_4) + (n - m - 1)\, R(F_{4m+1} + \cdots + F_5) \\ &= (n - m) R(F_{4m} + F_{4m-4} + \cdots + F_4) \\ &= (n - m)(2F_{2m} - F_{2m-2}) = (n - m)F_{2m+1} \qquad (n > m). \end{aligned}$$

3. Since

$$F_{2n+1}F_{2m+1} = (F_{2n+2m} + F_{2n+2m-4} + \cdots + F_{2n-2m+4}) + F_{2n-2m+1} \quad (n \geq m),$$

it follows from (***) and (**) that

$$\begin{aligned} R(F_{2n+1}F_{2m+1}) &= R(F_{2n+2m-1} + F_{2n+2m-5} + \cdots + F_{2n-2m+3}) + F_{2n-2m}) \\ &= R(F_{4m} + F_{4m-4} + \cdots + F_4) + (n - m - 1)R(F_{4m+1} + \\ &\qquad + \cdots + F_5) \\ &= (n - m)R(F_{4m} + F_{4m-4} + \cdots + F_4) \\ &= (n - m)F_{2m+1} \qquad (n \geq m)\,. \end{aligned}$$

4. Since

$$F^2_{2n+1} = (F_{4n} + F_{4n-4} + \cdots + F_4) + F_2$$

we get

$$\begin{aligned} R(F^2_{2n+1}) &= R(F_{4n-1} + F_{4n-5} + \cdots + F_3) \\ &= R(F_{4n-2} + F_{4n-6} + \cdots + F_2) \\ &= F_{2n} - F_{2n-2} = F_{2n-1} \qquad (n \geq 1)\,. \end{aligned}$$

Similarly, since

$$F^2_{2n} = F_{4n-2} + F_{4n-6} + \cdots + F_2\,,$$

we have

$$R(F^2_{2n}) = F_{2n-1} \qquad (n \geq 1)\,.$$

WHAT'S THE DIFFERENCE?

H-178 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$a_{m,n} = \binom{m+n}{m}^2$$

Show that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r}\sum_{h=0}^{s} c_{j,k}a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq s) ,$$

where the $c_{j,k}$ and r,s are all independent of m,n.

Show also that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r}\sum_{k=0}^{n} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq 0) ,$$

where the $c_{j,k}$ and r are independent of m,n.

Solution by the Proposer.

1. Assume that

$$(1) \qquad \sum_{j=0}^{r}\sum_{k=0}^{s} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq s) ,$$

where $c_{j,k}$ and r,s are independent of m,n.

$$F(x,y) = \sum_{m,n=0}^{\infty} a_{m,n}\, x^m y^n .$$

Then we have

$$(2) \qquad F(x,y) = \left\{(1 - x - y)^2 - 4xy\right\}^{-\frac{1}{2}} .$$

Indeed,

$$\left\{(1 - x - y)^2 - 4xy\right\}^{-\frac{1}{2}} = (1 - x - y)^{-1}\left\{1 - \frac{4xy}{(1 - x - y)^2}\right\}^{-\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(xy)^k}{(1 - x - y)^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} (xy)^k \sum_{n=0}^{\infty} \binom{2k + n}{n} (x + y)^n$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} (xy)^k \sum_{m,n=0}^{\infty} \binom{2k + m + n}{m + n} \binom{m + n}{m} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{k=0}^{\min(m,n)} \frac{(m + n)!}{k!\,k!\,(m - k)!\,(n - k)!} .$$

The inner sum is equal to

$$\binom{m + n}{m} \sum_{k}^{\infty} \binom{m}{k}\binom{n}{k} = \binom{m + n}{m}^2 ,$$

which proves (2).)

Now

$$\sum_{j=0}^{r} \sum_{k=0}^{s} c_{j,k} x^j y^k F(x,y) = \sum_{j=0}^{r} \sum_{k=0}^{s} c_{j,k}\, x^j\, y^k \sum_{m=0}^{r} \sum_{n=0}^{s} a_{m,n}\, x^m\, y^n$$

$$= \sum_{m,n=0}^{\infty} b_{m,n}\, x^m\, y^n ,$$

where

$$b_{m,n} = \sum_{j,k} c_{j,k}\, a_{m-j,n-k} \quad .$$

By (1), we have

$$b_{m,n} = 0 \qquad (m \geq r,\ n \geq r)\ ,$$

so that

$$(3) \qquad \sum_{j=0}^{r}\sum_{k=0}^{s} c_{j,k}\, x^j y^k F(x,y)$$

$$= \sum_{m=0}^{r-1}\sum_{n=0}^{\infty} b_{m,n} x^m y^n - \sum_{m=0}^{\infty}\sum_{n=0}^{s-1} b_{m,n} x^m y^n - \sum_{m=0}^{r-1}\sum_{n=0}^{s-1} b_{m,n} x^m y^n\ .$$

For fixed m, $a_{m,n}$ is a polynomial in n, hence $b_{m,n}$ is also a polynomial in n. Similarly, for fixed n, $b_{m,n}$ is a polynomial in m. Consequently, each of the sums

$$\sum_{m=0}^{r-1}\sum_{n=0}^{\infty} b_{m,n}\, x^m y^n, \qquad \sum_{m=0}^{\infty}\sum_{n=0}^{s-1} b_{m,n}\, x^m y^n$$

is a rational function of x,y. Hence, by (3), $F(x,y)$ is a rational function of x,y. This contradicts (2).

2. Assume that

$$(4) \qquad \sum_{j=0}^{r}\sum_{k=0}^{n} c_{j,k}\, a_{m-n,n-k} = 0 \qquad (m \geq r,\ n \geq 0)\ .$$

Then as in 1, we have

[Continued on page 202.]

FIBONACCI MAGIC CARDS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

According to the well-known theorem of Zeckendorf, if adjacent members of the Fibonacci sequence (1, 2, 3, 5, 8, 13, ···) are not allowed in the same representation, then each positive integer can be expressed uniquely as the sum of one or more Fibonacci numbers. On the basis of this unique representation theorem, each integer is associated with just certain Fibonacci numbers. For example: 35 = 34 + 1; 51 = 34 + 13 + 3 + 1.

Accordingly, if one places on a set of cards those integers which have a given Fibonacci number as a component, one creates a set of magic cards with the following property. Let someone select all the cards in the set which contain a certain integer. Knowing the particular Fibonacci number associated with each card, it is then possible to add these numbers together and thus be able to say what the selected integer was.

The following sets of integers provide the numbers for each card, the smallest number on the card being the Fibonacci number which is a component of each of the integers on the card. One could possibly conceal the trick more effectively by a random distribution of the numbers on each card.

Card 1

1, 4, 6, 9, 12, 14, 17, 19, 22, 25, 27, 30, 33, 35, 38, 40, 43, 46, 48, 51, 53, 56, 59, 61, 64, 67, 69, 72, 74, 77, 80, 82, 85, 88, 90, 93, 95, 98

Card 2

2, 7, 10, 15, 20, 23, 28, 31, 36, 41, 44, 49, 54, 57, 62, 65, 70, 75, 78, 83, 86, 91, 96, 99

Card 3

3, 4, 11, 12, 16, 17, 24, 25, 32, 33, 37, 38, 45, 46, 50, 51, 58, 59, 66, 67, 71, 72, 79, 80, 87, 88, 92, 93, 100

Card 4

5, 6, 7, 18, 19, 20, 26, 27, 28, 39, 40, 41, 52, 53, 54, 60, 61, 62, 73, 74, 75, 81, 82, 83, 94, 95, 96

Card 5

8, 9, 10, 11, 12, 29, 30, 31, 32, 33, 42, 43, 44, 45, 46, 63, 64, 65, 66, 67, 84, 85, 86, 87, 88, 97, 98, 99, 100

Card 6

13, 14, 15, 16, 17, 18, 19, 20, 47, 48, 49, 50, 51, 52, 53, 54, 68, 69, 70, 71, 72, 73, 74, 75

Card 7

21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88

Card 8

34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54

Card 9

55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88

Card 10

89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

CORRECTIONS TO A FIBONACCI CROSTIC

H. 104 should be 102

J. needs two 144

In diagram O 81 should be G 81

F 93 should be E 93

THE LAMBERT FUNCTION

WRAY G. BRADY
Slippery Rock State College, Slippery Rock, Pennsylvania

The sum of certain reciprocal Fibonacci series can be summed in terms of the so-called Lambert series or Lambert function:

$$L(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} T_n(z)^n \ ,$$

where T_n is the number of divisors of N*. For example, let

$$\beta = \frac{1 - \sqrt{5}}{2} \quad ,$$

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k}} = \sqrt{5}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - L\left(\frac{7 - 3\sqrt{5}}{2}\right)\right] = \sqrt{5}\,[L(\beta^2) - L(\beta^4)]$$

or to generalize:

$$\sum_{k=1}^{\infty} \frac{1}{F_{2km}} = \sqrt{5}\,[L(2m\beta) - L(4m\beta)] \ ,$$

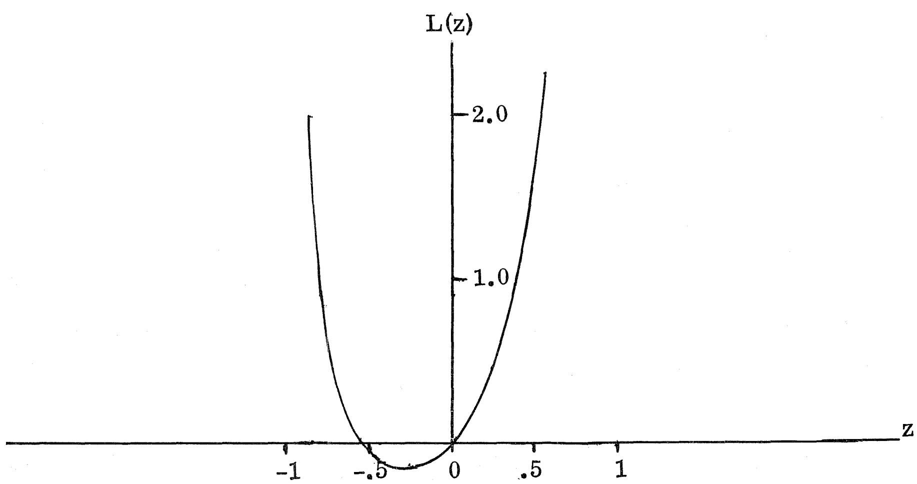
for an integer m, such that $m > 0$.

In this note, we tabulate the Lambert function for selected real values of z. The results are given in the table below. The calculations were made by machine evaluation. The graph of the approximation polynomial to L(z) is shown on the following page.

---

*Konrad Knopp, Theory and Application of Infinite Series, Harper, New York.

z	L_z	$L_{(-z)}$
.95	19.7372	4.7378
.90	14.4885	3.1728
.85	10.6987	2.0953
.80	7.9593	1.3565
.75	5.9724	.8513
.70	4.5224	.5066
.65	3.4550	.2720
.60	2.6605	.1130
.55	2.0615	.0062
.50	1.6035	-.0645
.45	1.2482	-.1096
.40	.9687	-.1363
.35	.7464	-.1493
.30	.5667	-.1518
.25	.4211	-.1456
.20	.3017	-.1316
.15	.2035	-.1103
.10	.1223	-.0817
.05	.0553	-.0452
.00	.0000	



FIBONACCI ONCE AGAIN

J. A. H. HUNTER
88 Bernard Avenue, Apt. 1004, Toronto 180, Canada

Many popular-type math teasers are based on the concept that may be expressed symbolically as:

$$(\underline{X})(\underline{Y}) = \underline{Y}^2 - \underline{X}^2 .$$

Examples are:

$$34\ 68 = 68^2 - 34^2$$

$$216\ 513 = 513^2 - 216^2 .$$

The true algebraical representation, of course, is:

$$10^{\underline{n}}\underline{X} + \underline{Y} = \underline{Y}^2 - \underline{X}^2$$

$\underline{Y}$ having $\underline{n}$ digits including any initial zero. For example, with $n = 6$, we have:

$$2230\ 047276 = 47276^2 - 2230^2 .$$

Working recently on such examples, it seemed interesting to determine the limiting minimal value of the ratio Y:X, that is of Y/X. This proved quite simple, the derivation being as follows:

For very large values of $\underline{n}$ we may take the maximum value of $\underline{Y}$ as being $10^{\underline{n}}$.

Hence we have

$$10^{\underline{n}}\,\underline{X} + 10^{n} = 10^{\underline{2n}} - \underline{X}^2 .$$

Solving for X,

$$2X = -10^n + \sqrt{10^{2n} + 4 \cdot 10^{2n} - 4 \cdot 10^n}$$
$$= -10^n + \sqrt{5 \cdot 10^{2n} - 4 \cdot 10^n} \quad .$$

Again for very large values we may ignore $4 \cdot 10^n$ in the expression under the square-root sign, so having, as $n \to \infty$,

$$2X \to -10^n + 10^n\sqrt{5} \quad ,$$

i.e.,

$$X \to \frac{10^n(\sqrt{5} - 1)}{2} \quad .$$

Hence

$$X/Y \to (\sqrt{5} - 1)/2, \qquad Y/X \to (\sqrt{5} + 1)/2 \quad .$$

Fibonacci again!

It may be noted that with $n = 6$, the greatest value of Y (giving the minimal $X:Y$ ratio) gives

$$569466\ 945388 = 945388^2 - 569466^2 \ .$$

And for this we have $Y/X = 1.6601 \cdots$.

[Continued from page 196.]

$$\sum_{j=0}^{r} \sum_{k=0}^{\infty} c_{j,k}\, x^j y^k F(x,y) = \sum_{m=0}^{r-1} x^m \sum_{j=0}^{m} \sum_{k=0}^{\infty} c_{j,k}\, y^k \sum_{n=0}^{\infty} a_{m-j,n}\, y^n \ .$$

It follows that $F(x,y)$ is rational in x, again contradicting (2).

Remark. We note that $a_{m,n}$ does satisfy recurrences of the type

[Continued on page 217.]

A NOTE ON PYTHAGOGEAN TRIPLETS

HARLAN L. UMANSKY
Emerson High School, Union City, New Jersey

A Pythagorean triplet is defined as a, b, c, in which $a^2 + b^2 = c^2$. It is well known that, where u and v are any two integers, $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$.

Triplets like 9, 40, 41, and 133, 156, 205, are of particular interest because $a + b$ is also a square. Not all Pythagorean triplets possess this property; for example, 3, 4, 5, and 20, 21, 29.

I have found that, x and y being any two integers, Pythagorean triplets possessing this property can be generated where $u = x^2 + (x + y)^2$ and $v = 2y(x + y)$. Then

I. $$a = u^2 - v^2 = 4x^4 + 8x^3y + 4x^2y^2 - 4xy^3 - 3y^4$$

II. $$b = 2uv = 8x^3y + 16x^2y^2 + 12xy^3 + 4y^4$$

III. $$c = u^2 + v^2 = 4x^4 + 8x^3y + 12x^2y^2 + 12xy^3 + 5y^4$$

IV. $$a + b = (2x^2 + 4xy + y^2)^2$$

V. $$b + c = (2x^2 + 4xy + 3y^2)^2 .$$

In triplets like 3, 4, 5, and 5, 12, 13, where $u = v + 1$, there is the further property that $a^2 = b + c$. Of the triplets in the series in which $a^2 = b + c$, only certain triplets possess the property that $a + b$ is also a square. The first six such triplets are listed below:

u	v	a	b	c
5	4	9	40	41
29	28	57	1,624	1,625
169	168	337	56,784	56,785

985	984	1,969	1,938,480	1,938,481
5,741	5,740	11,481	65,906,680	65,906,681
33,461	33,460	66,921	2,239,210,120	2,239,210,121

The series of u's (5, 29, 169, 985, $\cdots$) is a recurrent series which is defined as

$$u_n = 6u_{n-1} - u_{n-2} ,$$

where $u_0 = 1$ and $u_1 = 5$.

Since the generator

$$u = x^2 + (x + y)^2 ,$$

it can be expressed as the sum of two squares:

$$\begin{aligned}
u_1 &= 1^2 + 2^2 = 5 \\
u_2 &= 2^2 + 5^2 = 29 \\
u_3 &= 5^2 + 12^2 = 169 \\
u_4 &= 12^2 + 29^2 = 985 \\
u_5 &= 29^2 + 70^2 = 5741 \\
u_6 &= 70^2 + 169^2 = 33,461 \\
&\vdots
\end{aligned}$$

As expressed in this manner, the series of u's forms the recurrent series

$$\begin{aligned}
u_1 &= 1^2 + 2^2 = 5 \\
u_2 &= 2^2 + (1 + 2\cdot 2)^2 = 29 \\
u_3 &= 5^2 + (2 + 2\cdot 5)^2 = 169 \\
u_4 &= 12^2 + (5 + 2\cdot 12)^2 = 985 \\
u_5 &= 29^2 + (12 + 2\cdot 29)^2 = 5741 \\
u_6 &= 70^2 + (29 + 2\cdot 70)^2 = 33,461 \\
&\vdots
\end{aligned}$$

Pythagorean triplets possessing the properties that (1) $a^2 = b + c$ and that (2) $a + b$ is a square can be derived in another way.

For a triplet to possess the first property, the necessary and sufficient condition is that $u = v + 1$:

$$(u^2 - v^2)^2 = 2uv + u^2 + v^2$$

$$(u^2 - v^2)^2 = (u + v)$$

$$u^2 - v^2 = u + v$$

$$(u - v)(u + v) = u + v$$

$$u - v = 1$$

$$u = v + 1 .$$

We already know that for a triplet to possess property (2),

$$u = x^2 + (x + y)^2$$

and

$$v = 2y(x + y) .$$

Since $u = v + 1$, set

$$x^2 + (x + y)^2 = 2y(x + y) + 1 .$$

Then

$$x = \pm\sqrt{\frac{y^2 + 1}{2}}$$

(symbolized by 1) and

$$y = \pm\sqrt{2x^2 - 1}$$

(symbolized by k).

Substituting

$$x = \pm\sqrt{\frac{y^2 + 1}{2}}$$

in Eqs. I, II, III, IV, and V, we find that

$$a = 4y^2 + 4yl + 1$$

$$b = 12y^4 + 16y^3l + 8y^2 + 4yl$$

$$c = b + 1$$

$$a + b = (2y^2 + 4yl + 1)^2$$

$$b + c = (4y^2 + 4yl + 1)^2$$

Now

$$\pm\sqrt{\frac{y^2 + 1}{2}}$$

is integral for 1, 7, 41, 239, $\cdots$. This is a recurrent series which is defined as

$$r_n = 6r_{n-1} - r_{n-2} ,$$

where $r_1 = 1$ and $r_2 = 7$. Substituting alternately the positive and negative values of

$$\pm\sqrt{\frac{y^2 + 1}{2}}$$

in a, b, c, we obtain the desired triplets.

Substituting $y = \pm\sqrt{2x^2 - 1}$ in Eqs. I, II, III, IV, and V, we find that
[Continued on page 212.]

A GENERALIZED GREATEST INTEGER FUNCTION THEOREM

ROBERT ANAYA and JANICE CRUMP
San Jose State College, San Jose, California

Theorem:

$$\left[a^k F_n + \frac{1}{2}\right] = F_{n+k}, \qquad n \geq k, \quad k \geq 1,$$

where

$$a = \frac{1 + \sqrt{5}}{2}$$

and $[x]$ is the greatest integer contained in x.

Proof. For $k = 1$,

$$\left[aF_n + \frac{1}{2}\right] = F_{n+1} .$$

See [1, Thm. III]. The Binet form for the Fibonacci numbers is

$$F_n = \frac{a^n - b^n}{\sqrt{5}} ,$$

where

$$a = \frac{1 + \sqrt{5}}{2} \text{ and } b = \frac{1 - \sqrt{5}}{2} .$$

Thus

$$a^k F_n = \frac{a^{n+k} - b^n a^k}{\sqrt{5}} = \frac{a^{n+k} - b^n a^k - b^{n+k} + b^{n+k}}{\sqrt{5}}$$

$$= \frac{a^{n+k} - b^{n+k}}{\sqrt{5}} + \frac{b^{n+k} - b^n a^k}{\sqrt{5}}$$

$$= F_{n+k} - b^n \left(\frac{a^k - b^k}{\sqrt{5}}\right) = F_{n+k} - b^n F_k .$$

See [2]. Therefore,

$$a^kF_n + \frac{1}{2} = F_{n+k} + \left(\frac{1}{2} - b^nF_k\right) .$$

The next step is to prove that $\left|b^nF_k\right| < \frac{1}{2}$, $n \geq k$, $k \geq 2$. Since $n \geq k$, let $n = k$ for a fixed k. When $n = k$, $\left|b^nF_k\right|$ will have its largest value. As $n \to \infty$, $\left|b^n\right| \to 0$ monotonically. When k is even:

$$\left|b^kF_k\right| = \left|\frac{b^k(a^k - b^k)}{\sqrt{5}}\right| = \left|\frac{(ba)^k - b^{2k}}{\sqrt{5}}\right| = \left|\frac{1 - b^{2k}}{\sqrt{5}}\right| ,$$

since $ab = -1$. The sequence

$$\left|\frac{1 - b^{2k}}{\sqrt{5}}\right|$$

is monotone increasing, and also

$$\lim_{k \to \infty} \left|\frac{1 - b^{2k}}{\sqrt{5}}\right| = \left|\frac{1}{\sqrt{5}}\right| = \frac{1}{\sqrt{5}} < \frac{1}{2} .$$

Thus,

$$0 \leq \left|b^nF_k\right| < \frac{1}{2}$$

for even k. Now for odd k, we have

$$\left|b^kF_k\right| = \left|\frac{b^k(a^k - b^k)}{\sqrt{5}}\right| = \left|\frac{(ab)^k - b^{2k}}{\sqrt{5}}\right| = \left|\frac{-1 - b^{2k}}{\sqrt{5}}\right|$$

since $ab = -1$. Here we are considering $k = 3, 5, 7, \cdots$. When $k = 3$,

$$b^{2k} = b^6 \approx 0.055726 ;$$

and as k increases, b^{2k} gets smaller rapidly and

$$\left|\frac{-1 - b^{2k}}{\sqrt{5}}\right|$$

becomes smaller. Therefore, if

$$\left|\frac{-1 - b^{2k}}{\sqrt{5}}\right| < \frac{1}{2}$$

for $k = 3$, then it is less than 1/2 for any odd k greater than 3. Thus:

$$\left|\frac{-1 - b^{2k}}{\sqrt{5}}\right| = \left|\frac{1 + b^{2k}}{\sqrt{5}}\right| .$$

If

$$\left|\frac{-1 - b^{2k}}{\sqrt{5}}\right| < \frac{1}{2} ,$$

then

$$\left|1 + b^{2k}\right| < \frac{\sqrt{5}}{2} \quad \text{or} \quad \frac{-\sqrt{5} - 2}{2} < b^{2k} < \frac{\sqrt{5} - 2}{2} .$$

Since $\sqrt{5}$ is approximately 2.2361, the upper bound is approximately 0.1181, and since

$$b^{2k} = b^6 = 0.055726 ,$$

then certainly

$$0 < b^{2k} < \frac{\sqrt{5} - 2}{2} .$$

Therefore:

$$\left| b^k F_k \right| < \frac{1}{2}$$

for all odd k, and, moreover,

$$\left| b^n F_k \right| < \frac{1}{2}$$

for all $k \geq 2$ and $n \geq k$. Finally, since we know that

$$\left| b^n F_k \right| < \frac{1}{2} ,$$

we have

$$-\frac{1}{2} < b^n F_k < \frac{1}{2} .$$

Multiplying by -1 and adding 1/2, we have

$$0 < \frac{1}{2} - b^n F_k < 1 .$$

Since

$$\frac{1}{2} - b^n F_k > 0 ,$$

(i) $$a^k F_n + \frac{1}{2} = F_{n+k} + \left(\frac{1}{2} - b^n F_k \right)$$

implies that

$$\left(a^k F_n + \frac{1}{2} \right) > F_{n+k} .$$

Also, since

$$\left(\frac{1}{2} - b^n F_k\right) < 1 ,$$

(ii) $$F_{n+k} + \left(\frac{1}{2} - b^n F_k\right) < F_{n+k} + 1 \quad \text{and} \quad a^k F_n + \frac{1}{2} < F_{n+k} + 1.$$

Therefore, combining (i) and (ii), we obtain

$$F_{n+k} < a^k F_n + \frac{1}{2} < F_{n+k} + 1$$

or

$$\left[a^k F_n + \frac{1}{2}\right] = F_{n+k} .$$

REFERENCES

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969, pp. 34-35.
2. V. E. Hoggatt, Jr., John W. Phillips, and H. T. Leonard, Jr., "Twenty-Four Master Identities," The Fibonacci Quarterly, Vol. 9, Feb., 1971, pp. 2-5.

REMARK

With the aid of an ingenious programmer, Galen Jarvinen, it seems reasonable that

$$\left[a^k L_n + \frac{1}{2}\right] = L_{n+k} ,$$

and in general that

$$\left[a^k H_n + \frac{1}{2}\right] = H_{n+k} ,$$

with n somewhat greater than k.

$$a = 8x^2 + 4xk - 3$$

$$b = 48x^4 + 32x^3k - 32x^2 - 12xk + 4$$

$$c = b + 1$$

$$a + b = (4x^2 + 4xk - 1)^2$$

$$b + c = (8x^2 + 4xk - 3)^2 \quad .$$

Now $\pm\sqrt{2x^2 - 1}$ in integral for 1, 5, 29, 169, $\cdots$, a recurrent series that has already been defined. Substituting alternately the positive and negative values of $\pm\sqrt{2x^2 - 1}$ in a, b, c, we obtain the desired triplets.

Several minor but interesting relationships may be noted in conclusion. Since

$$u = x^2 + (x + y)^2 \quad ,$$

it follows that

$$u = x^2 + (x + k)^2 = 4x^2 + 2xk - 1$$

$$u = l^2 + (l + y)^2 = 2y^2 + 2yl + 1 \quad ,$$

and, since $v = u - 1$,

$$a + b = 2u^2 - 1 \; ,$$

and

$$u = \sqrt{\frac{1}{2}(a + b + 1)} \quad .$$

BACK-TO-BACK: SOME INTERESTING RELATIONSHIPS BETWEEN REPRESENTATIONS OF INTEGERS IN VARIOUS BASES

J. A. H. HUNTER
Toronto, Ontario, Canada
and
JOSEPH S. MADACHY
Mound Laboratory, Miamisburg, Ohio

A back-to-back relationship between integer representations is one in which the representation of an integer in one base is the reverse of its representation in some other base. Finding such integers and bases is elementary, but the concept does not appear to have received any attention in the literature. A double back-to-back relationship goes one step further: the base indices (written in scale 10 notation) are also the reverses of each other. Examples of single and double back-to-back relationships are:

$$73_{10} = 37_{22}$$

$$169_{82} = 961_{28}$$

Table 1 gives all solutions for integers that have 2, 3, or 4 digits in base-10 notation. The reader may feel tempted to find examples with 5 or more digits. Table 2 lists some of the known double back-to-back examples, leaving a wide open field for the computing-minded enthusiast.

For single back-to-backs we concentrated on finding reverses for base-10 cases. Without that restriction there would be an unlimited number of examples, such as:

$$74_{13} = 47_{22}$$

$$35_{26} = 53_{16}$$

If A, B, C, $\cdots$, represent the digits of an integer N, in base b notation, we seek relationships of the form:

---

*Mound Laboratory is operated by Monsanto Research Corporation for the Atomic Energy Commission under Contract No. AT-33-1-GEN-53.

(1) $$N = (A)(B)(C) \cdots (M)_{10} = (M) \cdots (C)(B)(A)_b ,$$

or solutions to the equation

(2) $$A \cdot 10^{d-1} + B \cdot 10^{d-2} + C \cdot 10^{d-3} + \cdots + M = M \cdot b^{d-1} + \cdots + C \cdot b^2 + B \cdot b + A ,$$

where d represents the number of digits in N. For 2-digit cases we have:

$$(A)(B)_{10} = (B)(A)_b$$

or

(3) $$10A + B = bB + A$$

The solution of (3) is obviously a simple matter. Somewhat more tedious, the 3-digit cases entail integral solutions of

(4) $$100A + 10B + C = b^2C + bB + A .$$

Both the 2-digit and 3-digit cases were found by hand. The lists were checked and confirmed as complete with a Hewlett-Packard 9100A programable calculator — this taking barely two minutes. The same calculator discovered all the 4-digit cases in less than 90 minutes.

The problem of solving Eq. (2) may appear formidable, but there are limits which reduce the amount of numerical work. For a 3-digit case the largest base to be considered is 31. This is so because with $b = 32$, we must have a 4-digit case since $32^2 = 1024$. Similarly the maximum bases for 2, 4, 5, and 6 digits would be 82, 21, 17, and 15, respectively.

Finding solutions for double back-to-backs is more complicated since both the representations and the bases must be in reverse relationship. If $a, b, c, \cdots,$ represent the digits of the bases written in base-10 notation, we have

Table 1
SINGLE BACK-TO-BACKS

2-Digit

$13_{10} = 31_4$	$51_{10} = 15_{46}$	$82_{10} = 28_{37}$
$21_{10} = 12_{19}$	$53_{10} = 35_{16}$	$83_{10} = 38_{25}$
$23_{10} = 32_7$	$61_{10} = 16_{55}$	$84_{10} = 48_{19}$
$31_{10} = 13_{28}$	$62_{10} = 26_{28}$	$86_{10} = 68_{13}$
$41_{10} = 14_{37}$	$63_{10} = 36_{19}$	$91_{10} = 19_{82}$
$42_{10} = 24_{19}$	$71_{10} = 17_{64}$	$93_{10} = 39_{28}$
$43_{10} = 34_{13}$	$73_{10} = 37_{22}$	
$46_{10} = 64_7$	$81_{10} = 18_{73}$	

3-Digit

$190_{10} = 091_{21}$	$774_{10} = 477_{13}$
$371_{10} = 173_{16}$	$834_{10} = 438_{14}$
$441_{10} = 144_{19}$	$882_{10} = 288_{19}$
$445_{10} = 544_9$	$912_{10} = 219_{21}$
$511_{10} = 115_{22}$	$961_{10} = 169_{28}$
$551_{10} = 155_{21}$	

4-Digit

$0801_{10} = 1080_9$	$3290_{10} = 0923_{19}$
$1090_{10} = 0901_{11}$	$5141_{10} = 1415_{16}$
$1540_{10} = 0451_{19}$	$7721_{10} = 1277_{19}$
$2116_{10} = 6112_7$	$9471_{10} = 1749_{19}$

$$(5)\qquad (A)(B)(C)\cdots(M)_{(a)(b)(c)\cdots(m)} = (M)\cdots(C)(B)(A)_{(m)\cdots(c)(b)(a)} \quad .$$

In order to keep computation within reasonable limits, examples were sought with bases of only two or three digits. A 3-digit integer representation with a 2-digit (in scale-10) base would involve the equation

(6) $$A[(a)(b)]^2 + B[(a)(b)] + C$$

$$= C[(b)(a)]^2 + B[(b)(a)] + A \ .$$

For example, if $A = 1$, $B = 6$, $C = 9$, $a = 8$, $b = 2$, we have:

$$1[82]^2 + 6[82] + 9 = 9[28]^2 + 6[28] + 1 = 7225 \ ;$$

that is,

$$169_{82} = 961_{28} \quad .$$

In Table 2 are listed examples of double back-to-backs. All those in the second part of Table **2** were found by us without calculator aid.

Variations on this type of recreation are endless. Some of the simpler ones could provide classroom enrichment material without entailing too much time on computation. This type of number search could also add zest to the current emphases on modular arithmetic in the so-called "new mathematics."

Table 2
SOME DOUBLE BACK-TO-BACKS

$$051_{91} = 150_{19}$$

$$144_{73} = 441_{37}$$

$$169_{82} = 961_{28}$$

$$508_{43} = 805_{34}$$

If terms in parentheses are considered as single "digits" in the given base we may have examples such as:

$$(1)(12)(7)_{31} = (7)(12)(1)_{13}$$

$$(1)(10)(10)_{41} = (10)(10)(1)_{14}$$

$$(6)(10)(15)_{74} = (15)(10)(6)_{47}$$

$$(10)(0)(16)_{43} = (16)(0)(10)_{34}$$

$$(12)(20)(30)_{74} = (30)(20)(12)_{47}$$

$$(17)(10)(33)_{64} = (33)(10)(17)_{46}$$

$$(18)(30)(45)_{74} = (45)(30)(18)_{47}$$

$$(19)(25)(37)_{64} = (37)(25)(19)_{46}$$

$$(21)(40)(41)_{64} = (41)(40)(21)_{46}$$

$$(6)(149)(17)_{251} = (17)(149)(6)_{152}$$

$$(19)(44)(52)_{251} = (52)(44)(19)_{152}$$

$$(38)(88)(104)_{251} = (104)(88)(38)_{152}$$

$$(47)(13)(91)_{352} = (91)(13)(47)_{253}$$

$$(94)(26)(182)_{352} = (182)(26)(94)_{253}$$

[Continued from page 202.]

$$\sum_{j=0}^{m} \sum_{k=0}^{n} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m + n > 0) .$$

However this is true of arbitrary $a_{m,n}$ with $a_{00} \neq 0$. We may define $c_{j,k}$ by means of

$$\left(\sum_{m,n=0}^{\infty} a_{mn}\, x^m y^n \right)^{-1} = \sum_{j,k=0}^{\infty} c_{j,k}\, x^j y^k \quad .$$

Late Acknowledgements. David Klarner solved H-168 and H. Krishna solved H-173.

Commentary on H-169. The theorem is false. Let $a = F_{2n+2}$, $b = c = F_{2n+1}$, $d = F_{2n}$. Thus from $F_{m+1}F_{m-1} - F_m^2 = (-1)^m$, we have $ad - bc = -1$, while $ab + cd = (F_{2n+2}F_{2n+1} + F_{2n}F_{2n+1}) = F_{2n+1}L_{2n+1} = F_{4n+2}$. However, let $N = F_{2n} \neq F_{4n+2}$, so that $F_{2n}^2 + 1 = F_{2n+1}F_{2n-1}$ and $N^2 + 1$ is composite. CONTRADICTION.

The Editors, V. E. Hoggatt, Jr., and R. E. Whitney

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

The Fibonacci Numbers F_n and the Lucas Numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS

B-226 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Find the smallest number in the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$ that is not the sum of the squares of three integers.

B-227 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Let $H_0, H_1, H_2, \cdots$ be a generalized Fibonacci sequence satisfying $H_{n+2} = H_{n+1} + H_n$ (and any initial conditions $H_0 = q$ and $H_1 = p$). Prove that

$$F_1H_3 + F_2H_6 + F_3H_9 + \cdots + F_nH_{3n} = F_nF_{n+1}H_{2n+1}\,.$$

B-228 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Extending the definition of the F_n to negative subscripts using

$$F_{-n} = (-1)^{n-1}F_n ,$$

prove that for all integers k, m, and n

$$(-1)^k F_n F_{m-k} + (-1)^m F_k F_{n-m} + (-1)^n F_m F_{k-n} = 0 .$$

B-229 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Using the recursion formulas to extend the definition of F_n and L_n to all integers n, prove that for all integers k, m, and n

$$(-1)^k L_n F_{m-k} + (-1)^m L_k F_{n-m} + (-1)^n L_m F_{k-n} = 0 .$$

B-230 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $\{C_n\}$ satisfy

$$C_{n+4} - 2C_{n+3} - C_{n+2} + 2C_{n+1} + C_n = 0$$

and let

$$G_n = C_{n+2} - C_{n+1} - C_n .$$

Prove that $\{G_n\}$ satisfies $G_{n+2} = G_{n+1} + G_n$.

B-231 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

A GFS (generalized Fibonacci sequence) $H_0, H_1, H_2, \cdots$ satisfies the same recursion formula

$$H_{n+2} = H_{n+1} + H_n$$

as the Fibonacci sequence but may have any initial values. It is known that

$$H_n H_{n+2} - H^2_{n+1} = (-1)^n c ,$$

where the constant c is characteristic of the sequence. Let $\{H_n\}$ and $\{K_n\}$ be GFS and let

$$C_n = H_0K_n + H_1K_{n-1} + H_2K_{n-2} + \cdots + H_nK_0 .$$

Show that

$$C_{n+2} = C_{n+1} + C_n + G_n ,$$

where $\{G_n\}$ is a GFS whose characteristic is the product of those of $\{H_n\}$ and $\{K_n\}$.

SOLUTIONS

GENERALIZED FIBONACCI IDENTITY

B-208 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$$

Prove both of the following and generalize:

(a) $$F^2_{n+2} = 3F^2_{n+1} - F^3_n = 2(-1)^n$$

(b) $$L^2_{n+2} = 3L^2_{n+1} - L^2_n = 10(-1)^n .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

In the paper by David Zeitlin, "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$," this Quarterly, Vol. 3, No. 4, 1965, pp. 241-255, it is shown on page 251, Eq. (4.5) that

(1) $$H^2_{n+2} - 3H^2_{n+1} + H^2_n = 2(-1)^{n+1}(H_1^2 - H_1H_0 - H_0^2) ,$$

where

$$H_{n+2} = H_{n+1} + H_n, \qquad n = 0, 1, \qquad .$$

Thus, (1) gives (a) for $H_n \equiv F_n$ and (b) for $H_n \equiv L_n$.

Also solved by Richard Blazej, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, C. B. A. Peck, A. G. Shannon, and the Proposer.

FURTHER GENERALIZATION

B-209 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California

Do the analogue of B-208 for the Pell sequence defined by

$$P_0 = 0,\quad P_1 = 1,\quad P_{n+2} = 2P_{n+1} + P_n,\quad \text{and}\quad Q_n = P_n + P_{n-1}\,.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

In the paper quoted in B-208, there is given Eq. (3.1) on p. 245 which states that

$$(1)\qquad W_{n+2}^2 - (d^2 - 2c)W_{n+1}^2 + c^2W_n^2 = 2c^{n+1}(W_1^2 - dW_0W_1 + cW_0^2)\ ,$$

where

$$W_{n+2} = dW_{n+1} - cW_n\,.$$

Thus, for $d = 2$, $c = -1$, and $W_n \equiv P_n$, (1) gives

$$(2)\qquad P_{n+2}^2 - 6P_{n+1}^2 + P_n^2 = 2(-1)^{n+1}\ .$$

Since

$$Q_{n+2} = 2Q_{n+1} + Q_n\ ,$$

we obtain from (1) for $d = 2$, $c = -1$, and $W_n \equiv Q_n$, $Q_0 = 1$, $Q_1 = 1$,

$$(3)\qquad Q_{n+2}^2 - 6Q_{n+1}^2 + Q_n^2 = 4(-1)^n\ .$$

Also solved by Herta T. Freitag, Ralph Garfield, A. G. Shannon, and the Proposer.

SUMMING OF FIBONACCI RECIPROCALS

B-210 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $S > 803/240$, where

$$S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots .$$

Solution by Peter A. Lindstrom, Genesee Community College, Batavia, New York.

Consider the finite sum S_n, where

$$S_n = (1/F_1) + (1/F_2) + \cdots + (1/F_n) .$$

Then one finds that

$$240\,S_{13} = 240 + 240 + 120 + 80 + 48 + 30 + 18\frac{6}{13} + 11\frac{9}{21} + 7\frac{2}{34}$$

$$+ 4\frac{20}{55} + 2\frac{62}{89} + 1\frac{96}{144} + 1\frac{7}{233} .$$

and hence $240\,S_{13} > 803$. Then $S > S_{13} > 803/240$.

Also solved by R. Garfield, C. B. A. Peck, and the Proposer.

FIBONACCI WITH A GEOMETRIC PROGRESSION

B-211 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California. (Corrected)

Let F_n be the n^{th} term in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$. Solve the recurrence

$$D_{n+1} = 2D_n + F_{2n+1}$$

subject to the initial condition $D_1 = 1$.

Composite of solutions by Herta T. Freitag, Hollins, Virginia, and R. Garfield, College of Insurance, New York, New York.

The condition $D_2 = 3$ is unnecessary and is indeed false since the recurrence gives $D_2 = 2D_1 + F_3 = 2\cdot 1 + 2 = 4$.

By writing a few terms in the D_n sequence it is easy to show that

$$D_{n+1} = 2^n D_1 + 2^{n-1}F_3 + 2^{n-2}F_5 + \cdots + 2F_{2n-1} + F_{2n+1}\,.$$

Using the Binet formula and summing geometric progressions, we find that

$$D_n = F_{2n+2} - 2^n\,.$$

It is easier to prove this by mathematical induction than to check the details.

Also solved by the Proposer.

A QUESTION WITH MANY ANSWERS

B-212 Proposed by Tomas Djerverson, Albrook College, Tigertown on the Rio.

Give examples of interesting functions f and g such that

$$f(m,n) = g(m+n) - g(m) - g(n)\,.$$

(One example is $f(m,n) = mn$ and

$$g(n) = \binom{n}{2} = n(n-1)/2\,.\ )$$

<u>EPS Editor's Note.</u> We tabulate some of the submitted answers as follows:

Solver	$f(m,n)$	$g(m)$
Proposer	mn	$\binom{m}{2} = m(m-1)/2$
Herta T. Freitag	mn	$m(m+c)/2$, c constant
Herta T. Freitag	$g(m)g(n)$	$r^m - 1$, r constant
John W. Milsom	$2mn$	m^2
John W. Milsom	$3mn(m+n)$	m^3
Phil Mana	$\log\binom{m+n}{m}$	$\log(m!)$

UNFRIENDLY SUBSETS ON A LINE OR CIRCLE

B-213 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Given n points on a straight line, find the number of subsets (including the empty set) of the n points in which consecutive points are not allowed. Also find the corresponding number when the points are on a circle.

Solution by Theodore J. Cullen, Cal Poly, Pomona, California.

Let T_n be the solution for the line. It is easily seen that $F_1 = 2$ and $T_2 = 3$. For $n \geq 3$, let p be an extreme point, i.e., p has only one neighbor. Then the subsets can be divided into two types, those with p absent and those with p present. Clearly there are T_{n-1} of the first type and T_{n-2} of the second type, so that

$$T_n = T_{n-1} + T_{n-2}\,.$$

Therefore $T_n = F_{n+2}$ for $n \geq 1$, where $F_1 = F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 3$, the Fibonacci numbers.

Let V_n be the solution for the circle. One can check that $V_1 = 2$, $V_2 = 3$, $V_3 = 4$. For $n \geq 4$ let p be any fixed point, and again consider subsets with p absent and then p present. The numbers of these are T_{n-1} and T_{n-3}, respectively, so that

$$V_n = T_{n-1} + T_{n-3} = F_{n+1} + F_{n-1} = L_n\,,$$

the n^{th} Lucas number.

Also solved by Sister Marion Beiter, Herta T. Freitag, and the Proposer.