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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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SUBSEMIGROUPS OF THE ADDITIVE POSITIVE INTEGERS

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1. INTRODUCTION

Many of the attempts to obtain representations for commutative and/or Archimedean semigroups involve using the additive positive integers or subsemigroups of the additive positive integers. In this regard note references [1], [3], and [4]. The purpose of this paper is to catalogue the results that are known and to present some new results concerning the homomorphic images of such semigroups.

2. PRELIMINARIES

Let I denote the semigroups of additive positive integers. Lower case Roman letters will always denote elements of I. Subsemigroups of I will be denoted by capital Roman letters between J and Q inclusive. Results followed by a bracketed number and page numbers refer to that entry in the references and may be found there. Results not so identified are original and unpublished.

Theorem 1. ([2] pp. 36-48) Let K be a subsemigroup of I, then

i. There is $k \in I$ such that for $n \geq k$, $n \in K$ or

ii. There is $n \in I$, $n > 1$ such that n is a factor of all $k \in K$.

Proof. Suppose there exist $k_1, \cdots, k_m \in K$ such that the collection $(k_1, \cdots, k_m)$ has a greatest common divisor 1. Let K' be the subsemigroup of I generated by $\{k_1, k_2, \cdots, k_m\}$ clearly, $K' \subseteq K$. Let $k = 2k_1 \cdot k_2 \cdot \cdots \cdot k_m$ and for $b > k$, since the g.c.d. of $(k_1, \cdots, k_m)$ is one we may find integers $\alpha_1, \cdots, \alpha_m$ such that $\alpha_1 k_1 + \cdots + \alpha_m k_m = b$. (Note: the α_i are not necessarily positive.) We may now find integers q_i and r_i such that

$$\alpha_i = q_i k_1 \cdots k_{i-1} k_{i+1} \cdots k_m + r_i ,$$

where $0 < r_i \leq k_1 \cdots k_{i-1} \cdots k_m$ $(i = 2, 3, \cdots, m)$. Now let

$c_1 = \alpha_1 + (q_2 + \cdots + q_m)k_2k_3 \cdots k_m, \quad c_i = r_i, \quad (i = 2, 3, \cdots, m).$

We now have

$$b = c_1k_1 + c_2k_2 + \cdots + c_mk_m .$$

We have chosen $c_i \geq 0$ for $i = 2, 3, \cdots, m$. But since

$$c_2k_2 + \cdots + c_mk_m = r_2k_2 + \cdots + r_mk_m \leq k_1k_2 \cdots k_m \leq b,$$

clearly $c_1 \geq 0$. Thus every $b \geq k$ may be expressed as a linear combination of $\{k_1, \cdots, k_m\}$ where only positive integral coefficients are used.

If every finite sub collection of elements of K have g.c.d. greater than one, then clearly all of K have g.c.d. greater than one.]

Corollary 1. ([2] p. 39). Every K is finitely generated.

It is clear that there are essentially two types of subsemigroups of I:

i. Those that contain all integers greater than some fixed positive integer will be called relatively prime subsemigroups of I.

ii. Any other is a fixed integral multiple of a relatively prime subsemigroup.

Theorem 2. Let K, J be subsemigroups of I. Let the mapping $\mathbf{K}$ be a homomorphism from K onto J. Then $\mathbf{K}$ is in fact an isomorphism of K onto J of the type; for $k \in K$. $(k)\mathbf{K} = \gamma k$, where γ is a fixed rational number depending on K and J.

Proof. Since, by Corollary 1, K and J are finitely generated, let $(k_1, \cdots, k_m)$ be a generating set of K. Let $(j_1, \cdots, j_m)$ be the images in J of $(k_1, \cdots, k_m)$ under $\mathbf{K}$. Clearly $(j_1, \cdots, j_m)$. Now generate J.

$$(k_ik_1)\mathbf{K} = k_i(k_1)\mathbf{K} = k_ij_1$$

since $\mathbf{K}$ preserves positive integral multiples, but we also have

$$(k_ik_1)\mathbf{K} = (k_i)\mathbf{K}k_1 = j_ik_1$$

and

$$k_i j_1 = j_i k_1$$

so that

$$j_1 / k_1 k_i . \,]$$

Clearly for a given subsemigroup K not any rational number γ will do. Note that:

$$j_i = \frac{j_1}{k_1} k_i ,$$

but j_i is an integer and, k_1 divides k_i. If the collection $(k_1, \cdots, k_m)$ have greatest common divisor equal to one, then clearly γ is an integer. If the collection $(k_1, \cdots, k_m)$ have greatest common divisor $n \neq 1$, then $(k_1/n, \cdots, k_m/n)$ generates a relatively prime subsemigroup of I, call it K', and K and J are such that

$$K = nK', \qquad L = \gamma nK' ,$$

where γn is an integer. We have now shown:

<u>Corollary 2.</u> Let K and J be subsemigroups of I. For J any homomorphic image of K, K and J are integral multiples of a relatively prime subsemigroup, K', of I.

3. HOMOMORPHISMS

The results of Section 2 make it clear that no subsemigroup of I has a proper homomorphic image contained in I. Let us now examine the proper homomorphic images of subsemigroups of I.

<u>Lemma 1.</u> Let K be a relatively prime subsemigroup of I. Let $\sim$ be a congruence defined on K and satisfying:

$$\ni x, \quad y \in K, \quad x \neq y \quad \text{and} \quad x \sim y .$$

Then, $K/\sim$ is finite.

Proof. Since K is relatively prime there is a least $k \in K$ such that for all $n \geq k$, $n \in K$. Suppose $x < y$ and at $y - x = m$. Now,

$$x + k \sim x + k + im, \qquad i = 1, 2, 3, \cdots$$

since by induction

$$x + k \sim (x + m + k = y + k)$$

and if $x + k \sim x + k + im$, then

$$x + k \sim x + h + (i + 1)m$$

by using the strong form of induction and adding $k + (i)m$ to both sides of: $x \sim x + m$. Clearly then, $x + m + h + 1$ is an upper bound for the order of $K/\sim$.]

Lemma 2. For K, k as in Lemma 1, let n be the least positive integer such that: for x, $y \in K$, $x \sim y$ and $x - y = n$. Then, for any c, $d \in K$, if $c \sim d$, $c < d$, $d - c = m$: we have $d - c = jn$.

Proof. (Let a be the least element of K such that $a \sim a + n$). We may find $k' \in K$ such that $c + k' > a + k$. Thus by Lemma 1, $c + k'$ is in one of the classes determined by

$$a + k, \; a + k + 1, \; \cdots, \; a + k + n - 1.$$

Thus

$$c + k' = a + k + jn + i ,$$

and

$$c + k' + m = a + k + j'n + i' ,$$

but $c + k' + m \sim c + k'$, and $a + k + j'n + i \sim a + k + jn + i'$, but this gives $a + k + i \sim a + k + i'$. Thus, $i = i'$ since n is the least positive integral difference of equivalent elements of K.]

For finite homomorphic images of subsemigroups of I, call n, as defined in Lemma 2, the period of the congruence.

Lemma 3. Let K, k, n, a be as in Lemma 2. Let be a congruence on K such that for $c \sim d$, $d > c$, $d - c \in K$. Then $K/\sim$ has exactly n non-singleton classes.

Proof. Let $d - c = m$. Then by Lemma 2, $m = jn$. We have $jn \in K$ and for p sufficiently large $c + (p)jn > a + k$. Thus, $c + (p)jn \sim a + k + i$ for some i; $0 \leq i \leq n - 1$. But since $jn \in K$, $c + (p)jn \sim c$ for $p = 1, 2, 3, \cdots$. Thus $c \sim a + k + i$ and the non-singleton classes may be represented by $a + k$, $a + k + 1, \cdots, a + k + n - 1$.]

If c is an element of a relatively prime K, where $c \sim a + k + i$ (a,k being as in Lemma 2) then if $\sim$ has period n we have: $c \equiv a + ki$ (mod n). This follows immediately from Lemma 2.

Congruences on a relatively prime K which fail to satisfy the conditions of Lemma 3 may be described as follows. There are the n classes represented by $a + h$, $a + h + 1, \cdots, a + h + n - 1$; there are any number of singleton classes for elements between $a + h$ and the least element of K. There may be finite non-singleton classes of elements between $a + h$ and the least element of K, but from Lemma 3 no two elements in a finite class may differ by an element of K.

4. SUBSEMIGROUPS OF CYCLIC SEMIGROUPS

In this section we treat subsemigroups of finite cyclic semigroups. Let R be the finite cyclic semigroup of index r and period m. Elements of R will be represented by integers; R will be written additively.

Lemma 1. Let T be the subsemigroup of R generated by the elements $t_1, t_2, \cdots, t_k$. If the greatest common divisor of $\{t_1, t_2, \cdots, t_k, m\}$ is one, then T contains the periodic part of R.

Proof. Let t' be the g.c.d. of $\{t_1, t_2, \cdots, t_h\}$. By Theorem 1, Section 2, the subsemigroup of I generated by $\{t_1/t', t_2/t', \cdots, tk/t'\}$ contains all integers greater than some fixed integer k. But for some p all $q \geq p$ are such that $qt' > k$. Now let

$$(k + i)t' - r \equiv_m (k + j)t' - r ,$$

then $(nj - in)t' = n'm$, but t' and m are relatively prime. Thus, m divides $nj - in$.]

The remainder of the subsemigroup of R generated by $\{t_1, t_2, \cdots, t_h\}$ is the intersection in R of the subsemigroup of I generated by the t_i considered as integers. If the g.c.d. of $\{t_1, t_2, \cdots, t_k, m\} = p > 1$, then the subsemigroup generated contains m/p elements of the periodic part of R, and can thus be made isomorphic to a subsemigroup of the type described in Lemma 1 by changing the period of R to m/p.

Finally, let K be the subsemigroup of I generated by $\{t_1, t_2, \cdots, t_k\}$ considered as integers, where $t_1, t_2, \cdots, t_h \in R$ a finite cyclic semigroup of index r and period m, and the g.c.d. of $\{t_1, t_2, \cdots, t_h, m\}$ is one. Let $K' = K \cup N$, where N is all of I greater than r. Clearly K' is a subsemigroup of I. Let $\sim_r$ be the relation:

$$x,\ y \in K',\ x \sim_r y = x = y \quad \text{or} \quad (x, y \geq r \text{ and } x \equiv_m y)\,.$$

The relation $\sim_r$ is a congruence on K'. Now identify the elements of $K'/\sim_r$ with the elements of the subsemigroup of R generated by $\{t_1, \cdots, t_h\}$ in the natural way. We then have:

Theorem 2. The semigroup $K'/\sim_r$ is isomorphic to the subsemigroup of R generated by $\{t_1, t_2, \cdots, t_h\}$.

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PROPERTIES OF TRIBONACCI NUMBERS

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1. INTRODUCTION

Let us define a sequence of Tribonacci numbers

$$(1.1) \qquad \{T_n\}_0^\infty = \{T_n(b,\ c,\ d;\ P,\ Q,\ R)\}_0^\infty$$

by

$$(1.2) \qquad T_n = bT_{n-1} + cT_{n-2} + dT_{n-3}\ ,$$

where n denotes an integer ≥ 3 and T_0, T_1, T_2 are the initial terms P, Q, R respectively. Then it is easy to show that the n^{th} term of this sequence is given by

$$(1.3) \qquad T_n = la^n + mb^n + nr^n\ ,$$

where a, b, r are the roots of $x^3 - bx^2 - cx - d = 0$ and 1, m, n satisfy the following system of equations, viz.,

$$(1.4) \quad 1 + m + n = P, \quad la + mb + ar = Q, \quad la^2 + mb^2 + nr^2 = K.$$

Our aim is to study the properties of this sequence. The 9 special forms which we will refer are as follows:

$$\text{(i)} \qquad \{T_n^{(1)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ 1,\ b)\}_0^\infty\ ,$$

$$\text{(ii)} \qquad \{T_n^{(2)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 1,\ 0,\ c)\}_0^\infty\ ,$$

$$\text{(iii)} \qquad \{T_n^{(3)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ d,\ bd)\}_0^\infty\ ,$$

$$\text{(iv)} \qquad \{T_n^{(4)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ 0,\ 1)\}_0^\infty\ ,$$

(v) $$\{T_n^{(5)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ 1,\ 0)\}_0^\infty\ ,$$

(vi) $$\{T_n^{(6)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 1,\ 0,\ 0)\}_0^\infty\ ,$$

(vii) $$\{T_n^{(7)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 3,\ b,\ b^2 + 2c)\}_0^\infty,$$

(viii) $$\{T_n^{(8)}\}_0^\infty = \{T_n(1,\ 1,\ 1;\ 0,\ 1,\ 0)\}_0^\infty\ ,$$

(ix) $$\{T_n^{(9)}\}_0^\infty = \{T_n(1,\ 1,\ 1;\ 0,\ 0,\ 1)\}_0^\infty\ .$$

2. PROPERTIES OF $\{T_n\}_0^\infty$

First, we recall the following useful relations, viz.,

(2.1)
$$\begin{aligned} 1 &= [\{R - Q(b - a) + Pd/a\}(b - r)] \div D\,, \\ m &= [\{R - Q(b - b) + Pd/b\}(r - n)] \div D\,, \\ n &= [\{R - Q(b - r) + Pd/r\}(a - b)] \div D\,, \end{aligned}$$

where $D = (a - b)(b - r)(a - r)$;

(2.2) $$a = a' + b/3, \quad b = b' + b/3, \qquad r = r' + b/3\ ,$$

where a', b', r' are the roots of the reduced cubic equations $z^3 + 3Hz + G = 0$;

(2.3) $$a' = A^{1/3} + B^{1/3}, \qquad b' = wA^{1/3} + w^2B^{1/3}, \qquad r' = w^2A^{1/3} + wB^{1/3},$$

where $A, B = \{-G \pm \sqrt{(G^2 + 4H^3)}\}/2$;

(2.4) $$D - D' = 3(w - w^2)\sqrt{(G^2 + 4H^3)}\ ,$$

where

$D' = (a' - b')(b' - r')(a' - r')$, $H = -(3c + b^2)/9$ and $G = -(27d + 9bc + 2b^3)/27$.

Some identities will follow.

Identity 1. For q, q_1, q_2, q_3 and u denoting positive integers (where $q > 2$),

$$(2.5)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{q+u} \\ T_{q_1} & T_{q_1-1} & T_{q_1-2} & T_{q_1+u} \\ T_{q_2} & T_{q_2-1} & T_{q_2-2} & T_{q_2+u} \\ T_{q_3} & T_{q_3-1} & T_{q_3-2} & T_{q_3+u} \end{vmatrix} = 0 .$$

Proof. Let

$$(2.6)\qquad T^{(1)}_{n+1}T_q + T^{(2)}_{u+1}T_{q-1} + T^{(3)}_u T_{q-2} - T_{q+u} = 0 .$$

Replace q by q_1, q_2 and q_3 in (2.6). Then we get

$$(2.7)\quad \begin{aligned} &(a)\quad T^{(1)}_{u+1}T_{q_1} + T^{(2)}_{u+1}T_{q_1-1} + T^{(3)}_u T_{q_1-2} - T_{q_1+u} = 0 ,\\ &(b)\quad T^{(1)}_{u+1}T_{q_2} + T^{(2)}_{u+1}T_{q_2-1} + T^{(3)}_u T_{q_2-2} - T_{q_2+u} = 0 ,\\ &(c)\quad T^{(1)}_{u+1}T_{q_3} + T^{(2)}_{u+1}T_{q_3-1} + T^{(3)}_u T_{q_3-2} - T_{q_3+u} = 0 . \end{aligned}$$

Clearly, on eliminating $T^{(1)}_{u+1}$, $T^{(2)}_{u+1}$ and $T^{(3)}_u$ from (2.6), (2.7)a, (2.7)b and (2.7)c, the desired result follows. Note the following particular cases.

$$(2.8)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{q+u} \\ T_{q+1} & T_q & T_{q-1} & T_{q+u+1} \\ T_{q+2} & T_{q+1} & T_q & T_{q+u+2} \\ T_{q+3} & T_{q+2} & T_{q+1} & T_{q+u+3} \end{vmatrix} = 0 ,$$

$$(2.9)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{2q} \\ T_{q+1} & T_q & T_{q-1} & T_{2q+1} \\ T_{q+2} & T_{q-1} & T_q & T_{2q+2} \\ T_{q+3} & T_{q+2} & T_{q+1} & T_{2q+3} \end{vmatrix} = 0\,,$$

$$(2.10)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{3q} \\ T_{q+1} & T_q & T_{q-1} & T_{3q+1} \\ T_{q+2} & T_{q+1} & T_q & T_{3q+2} \\ T_{q+3} & T_{q+2} & T_{q-1} & T_{3q+3} \end{vmatrix} = 0\,.$$

<u>Identity 2.</u> For $q, q_1, q_2, \cdots, q_3$ and u denoting positive integers (where $q + q_1 = r_1$, $q + q_2 = r_2$, $\cdots$, $q + q_9 = r_9$ and $q > 2$),

$$(2.11)\qquad \begin{vmatrix} T_q^2 & T_{q-1}^2 & T_{q-2}^2 & T_{q+u}^2 & T_qT_{q-1} & T_qT_{q-2} & \cdots & T_{q-2}T_{q+u} \\ T_{r_1}^2 & T_{r_1-1}^2 & T_{r_1-2}^2 & T_{r_1+u}^2 & T_{r_1}T_{r_1-1} & T_{r_1}T_{r_1-2} & \cdots & T_{r_1-2}T_{r_1+u} \\ \vdots & & & & & & & \\ T_{r_9}^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

The proof is left to the reader.

<u>Identity 3.</u> For q, q_1, q_2, q_3 and u denoting positive integers (where $q + q_1 = r_1$, $q + q_2 = r_2$, $q + q_3 = r_3$ and $q > 2$) if

$$A_{qq_1} = T_qT_{r_1} + T_{q-1}T_{r_1-1} + T_{q-2}T_{r_1-2} + T_{q+u}T_{r_1+u}\,,$$

then

$$(2.12)\qquad \begin{vmatrix} A_{q0} & A_{qq_1} & A_{qq_2} & A_{qq_3} \\ A_{qq_1} & A_{r_1+q_1} & A_{r_1+q_2} & A_{r_1,q_3} \\ A_{qq_2} & A_{r_2+q_1} & A_{r_2+q_2} & A_{r_2,q_3} \\ A_{qq_3} & A_{r_3+q_1} & A_{r_3+q_2} & A_{r_3,q_3} \end{vmatrix} = 0\,.$$

The proof is left to the reader.

Identity 4. For $q, q_1, q_2, \cdots, q_9$ and u denoting positive integers (where $q + q_1 = r_1$, $q + q_2 = r_2, \cdots, q + q_9 = r_9$ and $q > 2$) if

$$B_{qq_1} = T_q^2 T_{r_1}^2 + T_{q-1}^2 T_{r_1-1}^2 + \cdots + T_q T_{q-1} T_{r_1} T_{r_1-1} + \cdots + T_{q-2} T_{q+u} T_{r_1-2} T_{r_1+u} ,$$

then

$$\text{(2.13)} \qquad \begin{vmatrix} B_{q_0} & B_{qq_1} & B_{qq_2} & B_{qq_3} & \cdots & B_{qq_9} \\ B_{qq_1} & B_{r_1,q_1} & B_{r_1,q_2} & B_{r_1,q_3} & \cdots & B_{r_1,q_9} \\ \vdots & & & & & \\ B_{qq_9} & B_{r_9,q_1} & B_{r_9,q_2} & B_{r_9,q_3} & \cdots & B_{r_9,q_9} \end{vmatrix} = 0 .$$

The proof is left to the reader. We proceed to construct a field closely associated with $\{T_n\}_0^\infty$ which may be called hereafter the "Tribonacci field." The elements of this field are

$$\text{(2.14)} \qquad \frac{X^n}{Y^{n-2}}, \qquad n = 0, 1, 2, \cdots, \infty .$$

For $n \geq 3$, these elements modulo $X^3 - bX^2Y - cXY^2 - dY^3$ are the second-degree polynomials

$$\text{(2.15)} \qquad \frac{X^n}{Y^{n-2}} = T_{n-1}^{(1)} X^2 + T_{n-1}^{(2)} XY + dT_{n-2}^{(1)} Y^2 .$$

In this field, if $P = Y^2$, $Q = XY$ and $R = X^2$, then the above-cited properties hold true.

3. PROPERTIES OF $\{T_n^{(4)}\}_0^\infty$

For this sequence,

$$(3.1) \qquad 1 = (b - r)/D, \quad m = (r - a)/D, \quad n = (a - b)/D$$

and the n^{th} member is given by

$$(3.2) \qquad T_n^{(4)} = \frac{(b - r)a^n + (r - a)b^n + (a - b)r^n}{(a - b)(b - r)(a - r)}$$

or

$$(3.3) \qquad T_n^{(4)} = \frac{(b' - r')(a' + b/3)^n + (r' - a')(b' + b/3)^n + (a' - b')(r' + b/3)^n}{(a' - b')(b' - r')(a' - r')} \quad .$$

We simplify (3.2) and (3.3). Rewrite (3.2) as

$$T_n^{(4)} = \left[r^{n+1} - a^n r - br^n + ba^n - r^{n+1} + r^n a + rb^n - ab^n\right]$$

$$\div \left[(a - b)(b - r)(a - r)\right]$$

$$(3.4) \qquad = \frac{1}{a - b}\left\{\frac{(r^n - a^n)(r - b) - (r - a)(r^n - b^n)}{(r - b)(r - s)}\right\}$$

$$= \frac{1}{a - b}\left\{\frac{r^n - a^n}{r - a} - \frac{r^n - b^n}{r - b}\right\} \quad .$$

This expression may be simplified as

$$(3.5) \qquad T_n^{(4)} = \frac{r^{n-1}}{a - b}\left\{\frac{1 - (ar^{-1})^n}{1 - (a/r)} - \frac{1 - (br^{-1})^n}{1 - (b/r)}\right\}$$

or

$$T_n^{(4)} = \frac{1}{a - b}\left\{r^{n-1}a + r^{n-2}a^2 + r^{n-3}a^3 + \cdots + ra^{n-1} - (r^{n-1}b + r^{n-2}b^2 + r^{n-3}b^3 + \cdots + rb^{n-1})\right\}$$

$$(3.6) \qquad = \frac{1}{a - b}\left\{r^{n-1}(a - b) + r^{n-2}(a^2 - b^2) + r^{n-3}(a^3 - b^3) + \cdots + r(a^{n-1} - b^{n-1})\right\}$$

$$= r^{n-1} + r^{n-2}(a + b) + r^{n-3}(a^2 + ab + b^2) + \cdots + r(a^{n-2} + a^{n-3}b + \cdots + b^{n-2}) .$$

Consider (3.3). Let

$$(3.7)\quad A_1 = \left\{\frac{-G + \sqrt{(G^2 + 4H^3)}}{2}\right\}^{1/3}, \qquad B_1 = \left\{\frac{-G - \sqrt{(G^2 + 4H^3)}}{2}\right\}^{1/3}$$

Clearly,

$$(3.8)\quad a' = A_1 + B_1, \qquad b' = wA_1 + w^2B_1 \qquad \text{and} \qquad b' = w^2A_1 + wB_1 \quad .$$

On substituting for a', b' and r' from (3.8) in (3.3), we have

$$\begin{aligned}
T_n^{(4)} &= [\{(w - w^2)A_1 + (w^2 - w)B_1\}(A_1 + B_1 + b/3)^n + \{(w^2 - 1)A_1 \\
&\quad + (w - 1)B_1\}(wA_1 + w^2B_1 + b/3)^n + \{(1 - w)A_1 + (1 - w^2)B_1\}(w^2A_1 \\
&\quad + wB_1 + b/3)^n] \div D \\
&= \left[\sum_{r=0}^{r=n} {}_nC_r(b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1)\right. \\
&\quad + (wA_1 + w^2B_1)^r[(w^2A_1 + wB_1) - (A_1 + B_1)] + (w^2A_1 + wB_1)^r \\
&\quad \left.\times [(A_1 + B_1) - (wA_1 + w^2B_1)]\}\right] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \\
(3.9)\quad &= \left[\sum_{r=0}^{r=n} {}_nC_r(b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1) + (A_1 + B_1)\right. \\
&\quad \times [(w^2A_1 + wB_1)^r - (wA_1 + w^2B_1)^r] + (w^2A_1 + wB_1)(wA_1 + w^2B_1)^r \\
&\quad \left.- (wA_1 + w^2B_1)(w^2A_1 + wB_1)^r\}\right] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \\
&= \left[\sum_{r=0}^{r=n} {}_nC_r(b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1) + (A_1 + B_1)\right. \\
&\quad \times [(w^2A_1 + wB_1)^r - (wA_1 + w^2B_1)^r] - (A_1^2 - A_1B_1 + B_1^2)[(w^2A_1 \\
&\quad \left.+ wB_1)^{r-1} - (wA_1 + w^2B_1)^{r-1}]\}\right] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \; .
\end{aligned}$$

Since

$$(wA_1 + w^2B_1)^r - (w^2A_1 + wB_1)^r = \frac{\sqrt{3}i}{2^r}\sum_{s=0}^{s=r} {}_rC_s i^{s-1}3^{(s-1)/2} \tag{3.10}$$
$$\times (A_1 + B_1)^{r-s}(A_1 - B_1)^s[1 - (-1)^s]$$

for $i = \sqrt{-1}$, $w = (1 + i\sqrt{3})/2$ and $w^2 = (1 - i\sqrt{3})/3$, (3.9) can be rewritten as

$$T_n^{(4)} = \left[\sum_{r=0}^{r=n} {}_nC_r(b/3)^{n-r}\left\{i\sqrt{3}(A_1 + B_1)^r(A_1 - B_1) - (A_1 + B_1)\right.\right.$$
$$\times\left[\frac{\sqrt{3i}}{2^r}\sum_{s=0}^{s=r} {}_rC_s i^{s-1}3^{(s-1)/2}(A_1 + B_1)^{r-s}(A_1 - B_1)^s\right.$$
$$\left.\times\ (1 - (-1)^s)\right] + (A_1^2 - A_1B_1 + B_1^2)\left[\frac{\sqrt{3i}}{2^{r-1}}\sum_{s=0}^{s=r-1} {}_{r-1}C_s i^{s-1}\right.$$
$$\left.\left.\left.\times\ 3^{(s-1)/2}(A_1 + B_1)^{r-s-1}(A_1 - B_1)^s(1 - (-1)^s)\right]\right\}\right] \tag{3.11}$$
$$\div\ [3i\sqrt{3}(A_1^3 - B_1^3)]$$
$$= \left[\sum_{r=0}^{r=s} {}_nC_r(b/3)^{n-r}\left\{(A_1 + B_1)^r(A_1 - B_1) - (A_1 + B_1)\right.\right.$$
$$\times\left[\frac{1}{2^r}\sum_{r=0}^{r=s} {}_rC_s i^{s-1}3^{(s-1)/2}(A_1 + B_1)^{r-s}(A_1 - B_1)s\right.$$
$$\left.\times\ (1 - (-1)^s)\right] + (A_1^2 - A_1B_1 + B_1^2)\left[\frac{1}{2^{r-1}}\sum_{s=0}^{s=r-1} {}_{r-1}C_s i^{s-1}\right.$$
$$\left.\left.\left.\times\ 3^{(s-1)/2}(A_1 + B_1)^{r-s-1}(A_1 - B_1)^s(1 - (-1)^s)\right]\right\}\right]$$
$$\div\ \left\{3(A_1^3 - B_1^3)\right\}\ .$$

However on combining (2.3) and (3.2),

$$
\begin{aligned}
T_n^{(4)} &= [\{(w - w^2)A^{1/3} + (w^2 - w)B^{1/3}\}(A^{1/3} + B^{1/3} + b/3)^n \\
&\quad + \{(w^2 - 1)A^{1/3} + (w - 1)B^{1/3}\}(wA^{1/3} + w^2B^{1/3} + b/3)^n \\
&\quad + \{(1 - w)A^{1/3} + (1 - w^2)B^{1/3}\}(w^2A^{1/3} + wB^{1/3} + b/3)^n] \qquad (3.12) \\
&\quad \div [3(w - w^2)(A - B)] \\
&= \sum_{\delta=0}^{\delta=n} (b/3^{n-\delta}\, {}_nC_\delta L_\delta \;,
\end{aligned}
$$

where

$$L_{3k} = \left[\sum_{r=o}^{r=k-1} A^{(3k-3r-1)/3} B^{(3r+2)/3} ({}_{3k}C_{3r+1} - {}_{3k}C_{3r+2})\right] \div (B - A),$$

$$L_{3k+1} = \left[\sum_{r=0}^{r=k} A^{(3k+1-3r)/3}\, B^{(3r+1)/3} ({}_{3k+1}C_{3r} - {}_{3k+1}C_{3r+1})\right] \div (B - A)$$

and

$$L_{3k+2} = \left[B^{(3k+3)/2} - A^{(3k+3)/3} + \sum_{r=0}^{r=k-1} A^{(3k-3r)/3} \right.$$
$$\left. B^{(3r+3)/3} ({}_{3k+2}C_{3r+2} - {}_{3k+2}C_{3r+3})\right] \div (B - A) \;.$$

4. PROPERTIES OF $\{T_n^{(5)}\}_0^\infty$

Let

$$
\begin{aligned}
T_n^{(5)} &= \frac{(b - a)(b - r)a^n + (b - b)(r - a)b^n + (b - r)(a - b)r^n}{D} \\
&= [b\{(b - r)a^n + (r - a)b^n + (a - b)r^n\} - \{(b - r)a^{n+1} \qquad (4.1) \\
&\quad + (r - a)b^{n+1} + (a - b)r^{n+1}\}] \div D \\
&= bT_n^{(4)} - T_{n+1}^{(4)} \;.
\end{aligned}
$$

This relation is useful in deriving expressions of $T_n^{(5)}$ similar to those of $T_n^{(4)}$.

5. PROPERTIES OF $\{T_n^{(6)}\}_0^\infty$

Proceeding as in previous section, it is easy to show that

$$T_n^{(6)} = dT_{n-1}^{(4)} . \tag{5.1}$$

Note that this equation connects up expressions of $T_n^{(4)}$ in Section 3.

6. PROPERTIES OF $\{T_n^{(7)}\}_0^\infty$

In this Section, we state without proof the following identities:

$$2(T_{3n}^{(7)} - 3d^n) = T_n^{(7)}\{2T_{2n}^{(7)} - (T_n^{(7)})^2 + T_{2n}^{(7)}\}, \tag{6.1}$$

$$T_{4n}^{(7)} = T_n^{(7)}T_{3n}^{(7)} - T_{2n}^{(7)}[\{(T_n^{(7)})^2 - T_{2n}^{(7)}\}/2] + d^nT_n^{(7)}, \tag{6.2}$$

$$\begin{aligned} T_{4n+4r}^{(7)} - T_{4n}^{(7)} = {} & T_{n+r}^{(7)}T_{3n+3r}^{(7)} - T_n^{(7)}T_{3n}^{(7)} - T_{2n+2r}^{(7)}[\{(T_{n+r}^{(7)})^2 \\ & - 2T_{2n+2r}^{(7)}\}/2] + T_{2n}^{(7)}[\{(T_n^{(7)})^2 - 2T_{2n}^{(7)}\}/2] \\ & + d^n(T_{n+r}^{(7)} - T_n^{(7)}), \end{aligned} \tag{6.3}$$

$$2T_{3n}^{(7)} = 3T_n^{(7)}T_{2n}^{(7)} + 6d^n - (T_n^{(7)})^3 = T_n^{(7)}\{3T_{2n}^{(7)} - (T_n^{(7)})^2\} + 6d^n \tag{6.4}$$

and

$$(T_{n+r}^{(7)})^3 - (T_n^{(7)})^3 = 3\{T_{n+r}^{(7)}T_{2n+2r}^{(7)} - T_n^{(7)}T_{2n}^{(7)}\} - 2\{T_{3n+3r}^{(7)} - T_{3n}^{(7)}\} \tag{6.5}$$

for $d = 1$.

7. PROPERTIES OF $\{T_n^{(8)}\}_0^\infty$

This section will give identities relating to $\{T_n^{(8)}\}_0^\infty$ and $\{T_n^{(9)}\}_0^\infty$. They are:

$$T_n^{(9)} - T_n^{(8)} = T_{n-3}^{(9)} , \tag{7.1}$$

(7.2) $$T_n^{(9)} + T_n^{(8)} = T_{n+1}^{(9)} ,$$

(7.3) $$(T_n^{(9)})^2 - (T_n^{(8)})^2 = T_{n-3}^{(9)} T_{n+1}^{(9)} ,$$

(7.4) $$2(T_n^{(9)})^2 = T_n^{(9)} \{T_{n-3}^{(9)} + T_{n+1}^{(9)}\}$$

and

(7.5) $$4T_n^{(9)} T_n^{(8)} = (T_{n+1}^{(9)})^2 - (T_{n-3}^{(9)})^2 .$$

8. PROPERTIES OF $\{T_n^{(9)}\}_0^\infty$

This section will discuss the congruence properties of $\{T_n^{(9)}\}_0^\infty$ modulo m, a positive integer. We note the following identity:

(8.1) $$T_{q+u}^{(9)} = T_{u+2}^{(9)} T_q^{(9)} + (T_{u+1}^{(9)} + T_u^{(9)}) T_{q-1}^{(9)} + T_{u+1}^{(9)} T_{q-2}^{(9)} .$$

Some theorems concerning $\{T_n^{(9)} \pmod{m}\}_0^\infty$ will follow.

__Theorem a.__ $\{T_n^{(9)} \pmod{m}\}_0^\infty$ is simply periodic.

__Proof.__ For some n and a, let $T_{n-1}^{(9)} \equiv T_{n-1}^{(9)} \pmod{m}$, $T_n^{(9)} \equiv T_a^{(9)} \pmod{m}$ and $T_{n+1}^{(9)} \equiv T_{a+1}^{(9)} \pmod{m}$. From these congruences, we obtain

(8.2) $$T_{n\pm t}^{(9)} \equiv T_{a\pm t}^{(9)} \pmod{m} ,$$

where t denotes an integer ≥ 2. Since m^2 pairs of terms are possible in this series, $\{T_n^{(9)} \pmod{m}\}_0^\infty$ must return to the starting values thus becoming simply periodic.

__Theorem b.__ For a prime factorization of m in the form $m = \Pi p_i^{e_i}$, the period of $\{T_n^{(9)} \pmod{m}\}_0^\infty$ is the lowest common multiple of all the periods of

$$\{T_n^{(9)} \pmod{p_i^{e_i}}\}_0^\infty .$$

__Proof.__ Let k(m) denote the period of $\{T_n^{(9)} \pmod{m}\}_0^\infty$. Then k(m) is of the form

$$c_i k(p_i^{e_i}) ,$$

where c_i denotes a related constant. Therefore,

$$k(m) = \text{l.c.m.}\left[k(p_i^{e_i}) \right] .$$

Theorem c. For some q, if $T_q^{(9)} \equiv 0 \pmod m$ and $T_{q+1}^{(9)} \equiv 0 \pmod m$, then the subscripts for which $T_n^{(9)} \equiv 0 \pmod m$ and $T_{n+1}^{(9)} \equiv 0 \pmod m$ form simple arithmetic progressions.

Proof. Let

$$\begin{aligned} T_{q+q'+1}^{(9)} &\equiv T_{q'+2}^{(9)} T_{q+1}^{(9)} + (T_{q'+1}^{(9)} + T_{q'}^{(9)}) T_q^{(9)} + T_{q'+1}^{(9)} T_{q-1}^{(9)} \\ &\equiv T_{q'+1}^{(9)} T_{q-1}^{(9)} \pmod m . \end{aligned} \tag{8.3}$$

For $q' = q - 1$ and q, this congruence shows that $T_{2q}^{(9)} \equiv 0 \pmod m$ and $T_{2q+1}^{(9)} \equiv 0 \pmod m$. Similarly, we can obtain

$$T_{3q}^{(9)} \equiv 0 \pmod m, \qquad T_{3q+1}^{(9)} \equiv 0 \pmod m ,$$

$$T_{4q}^{(9)} \equiv 0 \pmod m, \qquad T_{4q+1}^{(9)} \equiv 0 \pmod m, \quad \text{etc.}$$

Therefore, it follows that n is of the form xq, $x = 1, 2, \cdots$, so that n and $n + 1$ form simple arithmetic progressions.

Theorem d. Let $H' = 3^2H$, $G' = 3^3G$ and $(G')^2 + 4(H')^3$ be a quadratic residue for primes of the form $3t - 1$. Then $k(p) \mid (p^2 - 1)$.

Proof. Denote p^2 by $3t' + 1$. Then

$$L_{3t'+1} \equiv 3^{3t'}L_{3t'+1} \equiv 3^{3t'}\left[\left\{\sum_{r=0}^{r-t'} A^{(3t'+1-3r)/3} \times B^{(3r+1)/3}({}_{3t'+1}C_{3r-3t'+1}\ C_{3r+1})\right\} \div (B - A)\right] .$$

(8.4)

$$\equiv 3^{3t'}(A^{(3t'+1)/3}B^{1/3} - B^{(3t'+1)/3}A^{1/3}) \div (B - A)$$

$$\equiv 3H'U'_{t'} \pmod p \qquad ,$$

$$L_{3t'+2} \equiv 3^{3t'}L_{3t'+2} \equiv 3^{3t'}\left[\left\{B^{(3t'+3)/3} - A^{(3t'+3)/3} + \sum_{r=0}^{r=t'-1} A^{t--r}B^{r+1}({}_{3t'+2}C_{3r+2-3t'+2}C_{3r+3})\right\} \div (B - A)\right]$$

(8.5)

$$\equiv 3^{3t'}(B^{t'+1} - A^{t'+1}) \div (B - A)$$

$$\equiv U'_{t'+1} \pmod p$$

and

$$L_{3t'+3} \equiv 3^{3t'}\left\{\sum_{r=0}^{r=t'} A^{(3t'+2-3r)/3}B^{(3r+2)/3}({}_{3t'+3}C_{3r+1-3t'+3}\,C_{3r+2})\right\}$$

(8.6)

$$\div (B - A)$$

$$\equiv 3^{3t'}(A^{2/3}B^{(3t'+2)/3} - B^{2/3}A^{(3t'+2)/3}) \div (B - A)$$

$$\equiv (1/3)(H')^2U'_{t'} \pmod p \qquad ,$$

so that

$$T^{(9)}_{3t'+1} \equiv 3H'U'_t \pmod p$$

(8.7)

$$T^{(9)}_{3t'+2} \equiv H'U'_{t'} + U'_{t'+1} \pmod p$$

and

$$T^{(9)}_{3t'+3} \equiv (1/3) + (2/3)U'_{t'+1} + (1/3)(H')^2U'_{t'} + (1/3)H'U'_{t'} \pmod p ,$$

where
$$U'_{n+1} \equiv -G'U'_n + (H')^3U'_{n-1}$$
for $n = 1, 2, \cdots$, $U'_0 = 0$ and $U'_1 = 1$. Since
$$U'_{3t-2} \equiv 0 \pmod p \quad \text{and} \quad U'_{(3t-2)+1} \equiv 1 \pmod p,$$
we get
$$U'_{t'} \equiv 0 \pmod p \quad \text{and} \quad U'_{t'+1} \equiv 1 \pmod p .$$
Therefore

(8.8) $$T^{(9)}_{3t'+1} \equiv 0 \pmod p, \qquad T^{(9)}_{3t'+2} \equiv 1 \pmod p \quad \text{and} \quad T^{(9)}_{3t'+3} \equiv 1 \pmod p.$$

These congruences imply
$$T^{(9)}_{3t'} \equiv 0 \pmod p \quad \text{and} \quad k(p) \mid (p^2 - 1) .$$

<u>Theorem e.</u> For primes of the form $p = 3t - 1$ where $(G')^2 + 4(H')^3$ is a quadratic nonresidue, $k(p) \mid (p^6 - 1)$.

Let $p^6 = 3t'' + 1$. Note that the proof of Theorem 4 holds with t' changed to t'', etc. The proof is left to the reader.

<u>Theorem f.</u> For primes of the form $p = 3t + 1$ where $(G')^2 + 4(H')^3$ is a quadratic nonresidue, $k(p) \mid (p^2 - 1)$.

<u>Proof.</u> Let $p^2 = 3t + 1$. Then
$$T^{(9)}_{3t+1} \equiv 3^{3t}T^{(9)}_{3t+1} \equiv 3^{3t} \sum_{\delta=0}^{\delta=3t+1} (1/3)^{3t+1-\delta}\, {}_{3t+1}C_{\delta}L_{\delta}$$
$$\equiv 3^{3t}L_{3t+1}$$

(8.9) $$\equiv 3^{3t}\left[\left\{\sum_{r=0}^{r=t} A^{(3t+1-3r)/3}B^{(3r+1)/3}\left({}_{3t+1}C_{3r} - {}_{3t+1}C_{3r+1}\right)\right\} \div (B - A)\right]$$
$$\equiv 3^{3t}\left(A^{(3t+1)/3}B^{1/3} - A^{1/3}B^{(3t+1)/3}\right)/(B - A)$$
$$\equiv 3^{3t}A^{1/3}B^{1/3}\left(A^{t} - B^{t}\right)/(B - A)$$
$$\equiv 3H'U'_t \pmod p ,$$

$$T^{(9)}_{3t+2} \equiv 3^{3t}T^{(9)}_{3t+2} \equiv 3^{3t} \sum_{\delta=0}^{\delta=3t+2} (1/3)^{3t+2-\delta}\,{}_{3t+2}C_{\delta}L_{\delta}$$

$$\equiv 3^{3t}\{(1/3)L_{3t+1} + L_{3t+2}\}$$

$$(8.10) \qquad \equiv H'U'_t + 3^{3t}\left[B^{t+1} - A^{t+1} + \sum_{r=0}^{r=t-1} A^{(3t-3r)/3}B^{(3r+3)/2} ({}_{3t+2}C_{3r+2} - {}_{3t+2}C_{3r+3}) \div (B - A)\right]$$

$$\equiv H'U'_t + 3^{3t}(B^{t+1} - A^{t+1})/(B - A)$$

$$\equiv H'U'_t + U'_{t+1} \pmod{p}$$

and

$$T^{(9)}_{3t+3} \equiv 3^{3t}T^{(9)}_{3t+3} \equiv 3^{3t} \sum_{\delta=0}^{\delta=3t+3} (1/3)^{3t+3-\delta}\,{}_{3t+3}C_{\delta}L_{\delta}$$

$$\equiv 3^{3t}\{(1/3)^{3t+3}L_0 + (1/3)^{3t+2}L_1 + (1/3)^{3t+1}L_2 + L_2(1/3)^2L_{3t+1} + (2/3)L_{3t+2} + L_{3t+3}\}$$

$$(8.11) \qquad \equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + 3^{3t}\left[\sum_{r=0}^{r=t} A^{(3t+2-3r)/3} B^{(3r+2)/3}({}_{3t+3}C_{3r+1} - {}_{3t+3}C_{3r+2})\right] \div (B - A)$$

$$\equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + 3^{3t} \{A^{2/3}B^{2/3}(B^t - A^t)/(B - A)\}$$

$$\equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + (1/3)(H')^2U'_t \pmod{p} \quad .$$

For the considered primes, it is easy to show that

$$(8.12) \qquad \begin{array}{ll} U'_{3t+2} \equiv 0 \pmod{p}, & U'_{(3t+2)+1} \equiv \{(-H')^3\} \pmod{p}\ , \\ U'_{2(3t+2)} \equiv 0 \pmod{p}, & U'_{2(3t+2)+1} \equiv \{(-H')^3\}^2 \pmod{p}\ , \\ \cdots \quad \cdots & \cdots \quad \cdots \\ U'_{t(3t+2)} \equiv 0 \pmod{p}, & U'_{t(3t+2)+1} \equiv \{(-H')^3\}^t \pmod{p}\ , \end{array}$$

so that

$$U'_t \equiv 0 \pmod{p} \qquad \text{and} \qquad U'_{t+1} \equiv 2^{3t} \equiv 1 \pmod{p}.$$

Therefore

$$(8.13) \quad T^{(9)}_{3t+1} \equiv 0 \pmod{p}, \quad T^{(9)}_{3t+2} \equiv 1 \pmod{p} \quad \text{and} \quad T^{(9)}_{3t+3} \equiv 1 \pmod{p},$$

when $T^{(9)}_{3t} \equiv 0 \pmod{p}$ and the desired result follows.

Theorem g. For primes of the form $p = 3t + 1$ where $(G')^2 + 4(H')^3$ is a quadratic residue, $k(p) \mid (p^6 - 1)$.

Let $p^6 = 3t' + 1$. Note that the proof of Theorem f holds with t changed to t', etc. The proof is left to the reader.

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NOTE ON THE CHARACTERISTIC NUMBER OF A SEQUENCE OF FIBONACCI SQUARES

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Given a sequence of squares formed from the terms of a general Fibonacci sequence. It is proposed to set up a quadratic expression that will characterize a given sequence of this type.

First let it be noted that since this is equivalent to an expression of the fourth degree in Fibonacci numbers, the characteristic number would be a constant that would not oscillate in sign. To find such an expression we may proceed as follows.

Let the original sequence be given by $H_n = Ar^n + Bs^n$ where r and s are the roots of the Fibonacci recursion relation. Then the square term

$$G_n = H_n^2 = A^2r^{2n} + 2AB(rs)^n + B^2s^{2n} .$$

We now calculate three expressions.

$$\begin{aligned}
G_n^2 &= A^4r^{4n} + 6A^2B^2 + B^4s^{4n} + 4A^3B(rs)^n r^{2n} + 4AB^3(rs)s^{2n} \\
G_{n-1}G_{n+1} &= A^4r^{4n} + 4A^2B^2 + B^4s^{4n} + 2A^3B[r^{2n-2}(rs)^{n+1} + (rs)^{n-1}r^{2n+2}] \\
&\quad + 2AB^3[(rs)^{n-1}s^{2n+2} + (rs)^{n+1}s^{2n-2}] \\
&\quad + A^2B^2[r^{2n-2}s^{2n+2} + r^{2n+2}s^{2n-2}] \\
G_{n-2}G_{n+2} &= A^4r^{4n} + 4A^2B^2 + B^4s^{4n} + 2AB[r^{2n-4}(rs)^{n+2} + r^{2n+4}(rs)^{n-2}] \\
&\quad + 2AB^3[s^{2n+4}(rs)^{n-2} + s^{2n-4}(rs)^{n+2}] \\
&\quad + A^2B^2[r^{2n-4}s^{2n+4} + r^{2n+4}s^{2n-4}] .
\end{aligned}$$

First let it be noted that the A^2B^2 terms which end the expressions for G_{n-1}, G_{n+1} and $G_{n-2}G_{n+2}$ are $7A^2B^2$ and $47A^2B^2$, respectively. The AB^3 and A^3B terms of $G_{n-1}G_{n+1}$ can be written together as

$$2AB(-1)^{n-1}[A^2r^{2n-2} + B^2s^{2n-2}] + 2AB(-1)^{n-1}[A^2r^{2n+2} + B^2s^{2n+2}] .$$

A similar expression can be obtained for the corresponding terms of G_{n-2}, G_{n+2}. If we let $G_{2n}^* = A^2r^{2n} + B^2s^{2n}$ we have the following relations.

$$G_n^2 = A^4r^{4n} + B^4s^{4n} + 6A^2B^2 + 4AB(-1)^nG_{2n}^*$$

$$G_{n-1}G_{n+1} = A^4r^{4n} + B^4s^{4n} + 11A^2B^2 + 6AB(-1)^{n-1}G_{2n}^*$$
$$G_{n-2}G_{n+2} = A^4r^{4n} + B^4s^{4n} + 51A^2B^2 + 14AB(-1)^n G_{2n}^* .$$

To eliminate all but the terms in A^2B^2 we need three multipliers x, y, z satisfying the relations

$$x + y + z = 0$$
$$-4x + 6y - 14z = 0$$

with the solution $x:y:x = -20:10:10$. Hence the required expression which gives a characteristic number of a quadratic character is

$$2G_n^2 - G_{n-1}G_{n+1} - G_{n-2}G_{n+2} = k .$$

The value of this expression is $K = -50A^2B^2 = -2D^2$ since the characteristic number of the original Fibonacci sequence is given by $D = 5AB$ where D is defined as $H_2^2 - H_1H_3$.

If the initial terms of the sequence of squares are a, b, c, the next two terms are given by the recursion relation $T_{n+1} = 2T_n + 2T_{n-1} - T_{n-2}$. Hence the fourth and fifth terms are $2c + 2b - a$ and $-2a + 3b + 6c$. We form K from these beginning terms of the sequence and find an expression

$$K = 2a^2 - 2b^2 + 2c^2 - 2ab - 2bc - 6ac .$$

a, b, and c are related by the relation $\sqrt{c} = \sqrt{a} + \sqrt{b}$ which becomes

$$a^2 + b^2 + c^2 - 2ab - 2bc - 2ca = 0.$$

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ON SUMS OF FIBONACCI NUMBERS

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For a sequence of integers $S = (s_1, s_2, \cdots)$, we denote by $P(S)$ the set

$$\left\{\sum_{k=1}^{\infty} \epsilon_k s_k : \epsilon_k = 0 \text{ or } 1, \sum_{k=1}^{\infty} \epsilon_k < \infty\right\}.$$

We say that S is <u>complete</u> if all sufficiently large integers belong to $P(S)$. Conditions under which a sequence S is complete have been studied by a number of authors. These sequences have ranged from the slowly growing sequences of Erdős [3] and Folkman [4] ($s_n = 0(n^2)$), the polynomial and near-polynomial sequences of Roth and Szekeres [9], Graham [5] and Burr [1], to the near-exponential sequences of Cassels [2] ($s_n = 0(\exp(n/\log n))$) and the exponential sequences of Lekkerkerker [7] and Graham [6] ($s_n = [t\alpha^n]$). In this note, we investigate sequences in which each term is a Fibonacci number, i.e., an integer F_n defined by the linear recurrence

$$F_{n+2} = F_{n+1} + F_n, \qquad n \geq 0,$$

with $F_0 = 0$, $F_1 = 1$.

For a sequence $M = (m_1, m_2, \cdots)$ of nonnegative integers, let S_M denote the nondecreasing sequence which contains precisely m_k entries equal to F_k. It was noted in [7] that for $M = (1, 1, 1, \cdots)$, S_M is complete but the deletion of any two terms of S_M destroys the completeness. Further, it was shown in [1] that for any fixed a, if $M = (a, a, a, \cdots)$ then some finite set of entries can be deleted from S_M so that the resulting sequence is not complete. This result can be strengthened as follows (where τ denotes $(1 + \sqrt{5})/2$).

Theorem 1. If

$$\sum_{k=1}^{\infty} m_k \tau^{-k} < \infty ,$$

then some finite set of entries of S_M can be deleted so that the resulting sequence is not complete.

Proof. The proof uses the ideas of Cassels [2]. Let $\|x\|$ denote $\min |x - n|$ where n ranges over all integers. It is well known that F_n can be explicitly written as

$$F_n = \frac{1}{\sqrt{5}} (\tau^n - (-\tau)^{-n}) .$$

Thus

$$\begin{aligned}
\sum_{s \in S_M} \|s\tau\| &= \sum_{k=1}^{\infty} m_k \|F_k \tau\| \\
&= \sum_{k=1}^{\infty} m_k \|F_k \tau - F_{k+1}\| \\
&= \frac{1}{\sqrt{5}} \sum_{k=1}^{\infty} m_k \left\| \frac{(\tau^2 + 1)}{\tau} (-\tau)^{-k} \right\| \\
&\le \left| \frac{\tau^2 + 1}{\tau\sqrt{5}} \right| \sum_{k=1}^{\infty} m_k \tau^{-k} < \infty
\end{aligned}$$

by the hypothesis of the theorem. Hence, by deleting a sufficiently large initial segment of S_M, we can form a sequence S_M^* for which

$$\sum_{s \in S_M^*} \|s\tau\| < 1/4 \ .$$

But τ is irrational so that for infinitely many integers m, we have

$$\|m\tau\| > 1/4.$$

The subadditivity of $\|\cdot\|$ shows that such an m cannot belong to $P(S_M^*)$. This proves the theorem.

It follows in particular that if $1 < \theta < \tau$ and $m_k = 0(\theta^k)$ then S_M is not "strongly complete," i.e., the deletion of some finite set of entries from S_M can result in a sequence which is not complete.

In the other direction, however, we have the following result.

Theorem 2. Suppose for some $\epsilon > 0$ and some k_0, $m_k > \epsilon\tau^k$ for $k > k_0$. Then S_M is strongly complete.

Proof. For a fixed integer t, let M' denote the sequence

$$(\underbrace{0, 0, \cdots, 0}_{t}, m_{t+1}, m_{t+2}, \cdots) \ .$$

It is sufficient to show that $S_{M'}$ is complete. We recall the identity

$$F_{n+2k} + F_{n-2k} = L_{2k}F_n \ , \tag{1}$$

where L_r is the sequence of integers defined by $L_{n+2} = L_{n+1} + L_n$, $n \geq 0$, with $L_0 = 2$, $L_1 = 1$. It is easily shown that $F_r \leq \tau^r$ and

$$L_r \geq \frac{1}{2}\tau^r$$

for $r \geq 0$. We can assume without loss of generality that $t > k_0$ and $\epsilon\tau^t > 2$. Choose $\ell > 4/\epsilon$ and $n > t + 2\ell$. We can form sums of pairs $F_{n+2k} + F_{n-2k}$ from $S_{M'}$ to get at least $\epsilon\tau^{n-2k}$ copies of $L_{2k}F_n$ (by (1)) for $0 \leq k \leq \ell$. Since $\epsilon\tau^{n-2\ell} > \epsilon\tau^t > 2$ then these sums can be used to form all the multiples uF_n,

$$1 \le u \le \sum_{k=0}^{\ell} \epsilon\tau^{n-2k} L_{2k} .$$

Since

$$L_r \ge \frac{1}{2}\tau^r ,$$

then we have formed all multiples uF_n,

$$1 \le u \le \frac{\epsilon(\ell + 1)}{2}\tau^n .$$

The same argument can be applied to the terms $F_{n+1\pm 2k}$ (which are distinct from the terms previously considered) to form all multiples vF_{n+1},

$$1 \le v \le \frac{\epsilon(\ell + 1)}{2}\tau^{n+1} .$$

Of course, F_n and F_{n+1} are relatively prime so that the set of integers of the form $xF_n + yF_{n+1}$, x and y nonnegative integers, contains all integers $> F_nF_{n+1} - F_n - F_{n+1}$ (cf. [8]). For any integer

$$N_j = F_nF_{n+1} - F_n - F_{n+1} + j, \quad 1 \le j \le F_{n+2} ,$$

the coefficients x_j and y_j in a representation

$$N_j = x_jF_n + y_jF_{n+1}$$

certainly satisfy $x_j \le F_{n+1}$, $y_j \le F_n$. Thus, $x_j, y_j \le \tau^{n+1} < 2\tau^n$. Since u and v can range up to

$$\frac{\epsilon(\ell + 1)}{2}\tau^n > 2\tau^n$$

then by using the multiples of F_n and F_{n+1} we have just considered, we can represent all the N_j, $1 \le j \le F_{n+2}$, as elements of $P(S_{M'})$. Finally, since we have used at most $\epsilon\tau^{n-2}$ copies of F_{n+i}, $2 \le i$, in this process, we still have available at least $\epsilon(\tau^{n+2} - \tau^{n-2}) > 1$ copies of F_{n+i} to use in forming sums in $P(S_{M'})$. By adding sequentially a single copy of F_{n+i}, $i = 2, 3, 4, \cdots$, to the N_j, it is not difficult to see that all integers $\ge N_1$ belong to $P(S_{M'})$. Thus, $S_{M'}$ is complete and the theorem is proved.

It should be pointed out that the condition

$$\sum_{k=1}^{\infty} m_k \tau^{-k} = \infty$$

is not sufficient for the completeness of S_M as can be seen from the example in which

$$m_k = \begin{cases} [\tau^k] & \text{if } k = 2^n \text{ for some } n \\ 0 & \text{otherwise} \end{cases} .$$

However, the proof of Theorem 2 directly applies to show that if m_ϵ/τ^k is monotone and

$$\sum \frac{m_k}{\tau^k} = \infty$$

then S_M is strongly complete.

It would be of interest to investigate refinements of these questions. Of course, similar results and questions arise for other P - V numbers besides τ but we do not pursue these here.

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A GENERAL Q-MATRIX

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A GENERAL Q-MATRIX

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1. INTRODUCTION

Let F_n be the n^{th} Fibonacci number and let

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix has the interesting property that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

In this paper, we introduce a general type of Q-matrix for the generalized Fibonacci sequence $\{f_{n,r}\}$, and some of the interesting properties of the Q-matrix are then generalized for these sequences. An extension to the general linear recurrent sequence is also given. See [1] for more information on the Q-matrix proper.

2. THE MATRIX Q_r

Recall that the Fibonacci numbers $\{F_n\}$ are defined by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$, $F_1 = 1$. Now let us define the generalized Fibonacci sequences $\{f_{n,r}\}$ for $r \geq 2$ by $f_{n,r} = f_{n-1,r} + \cdots + f_{n-r,r}$, with $f_{0,r} = f_{1,r} = \cdots = f_{r-2,r} = 0$, $f_{r-1,r} = 1$. Note that $r = 2$ gives the Fibonacci numbers.

Now define a matrix Q_r by

$$Q_r = \begin{pmatrix} 1 & 1 & 0 & & \cdots & 0 \\ 1 & 0 & 1 & & \cdots & 0 \\ \vdots & 0 & 0 & 1 & \cdots & 0 \\ \cdot & 0 & & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that Q_r is just the $r - 1$ identity matrix bordered by the first column of 1's and last row of 0's. In order to motivate this definition, note that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} Q_2 = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} .$$

We have thus defined Q_r so that this property holds for the matrix

$$(f_{n+r+1-i-j,r}), \qquad 1 \le i,\ j \le r .$$

Theorem 1.

$$Q_r^n = \begin{pmatrix} f_{n+r-1,r} & f_{n+r-2,r} & \cdots & f_{n,r} \\ \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \sum_{i=0}^{r-2} f_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-2,r} & f_{n+r-3,r} & \cdots & f_{n-1,r} \end{pmatrix}$$

(the general term is

$$q_{jk} = \sum_{i=0}^{r-j} f_{n+r-i-k-1,r} \qquad).$$

Proof. Let r be fixed and use induction on n. This is trivially verified for $n = 1, 2$. Assume true for n, and consider

$$Q_r^{n+1} = Q_r^n Q_r = \begin{pmatrix} f_{n+r-1,r} & f_{n+r-2,r} & \cdots & f_{n,r} \\ \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \sum_{i=0}^{r-2} f_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-2,r} & f_{n+r-3,r} & \cdots & f_{n-1,r} \end{pmatrix} .$$

(equation continued on next page.)

$$\cdot \begin{pmatrix} 1 & 1 & 0 & & \cdots & 0 \\ 1 & 0 & 1 & & \cdots & 0 \\ \vdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & 0 & & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} f_{n+r,r} & f_{n+r-1,r} & & f_{n+1,r} \\ \sum_{i=0}^{r-2} f_{n+r-i-1,r} & \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-1,r} & f_{n+r-2,r} & & f_{n,r} \end{pmatrix} = Q_r^{n+1} ,$$

which completes the proof of the theorem.

We write this matrix in neater form by letting $P_r = (f_{r-i-j+2,r})$, $1 \le i, j \le r$, where $f_{-n,r}$ is found by the recursion relationship. Then

$$P_r Q_r^n = \begin{pmatrix} f_{n+r,r} & \cdots & f_{n+1,r} \\ \vdots & & \vdots \\ f_{n+1,r} & \cdots & f_{n-r+2,r} \end{pmatrix} .$$

An interesting special case of our theorem occurs when $r = 3$, where the numbers $\{f_{n,3}\}$ are the so-called Tribonacci numbers of Mark Feinberg.

3. APPLICATIONS

We now develop some of the interesting properties of the matrices Q_r^n and $P_r Q_r^n$, which in turn are generalizations of interesting properties of the matrix Q^n, which is the special case when $r = 2$.

It is readily calculated that

$$\det (P_r Q_r^n) = (\det P_r)(\det Q_r)^n = (-1)^{(2n+r)(r-1)/2} .$$

For $r = 2$, we have the corresponding Fibonacci identity

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^{n+1} .$$

The traces of Q_r^n and $P_r Q_r^n$ are also readily seen to be

$$\mathrm{Tr}(Q_r^n) = \sum_{j=1}^{r}\left(\sum_{i=0}^{r-j} f_{n+r-i-j-1,r}\right) = \sum_{j=1}^{r} j\, f_{n+r-j-1,r}$$

$$\mathrm{Tr}(P_rQ_r^n) = f_{n+r,r}^{-} + f_{n+r-2,r} + \cdots + f_{n-r+2,r}\,.$$

For $r = 2$, we have

$$\mathrm{Tr}(Q^n) = F_{n-1} + F_{n+1} = L_n\,.$$

The characteristic polynomial of Q_r is $x^r - x^{r-1} - \cdots - x - 1$, which is also the auxiliary polynomial for the sequence $\{f_{n,r}\}$. Since Q_r satisfies its own characteristic equation, $Q_r^r = Q_r^{r-1} + \cdots + Q_r + I$, hence

$$Q_r^{rn} = (Q_r^{r-1} + \cdots + Q_r + I)^n\,.$$

Expanding by the multinomial theorem and equating elements in the upper right-hand corner yields

$$f_{rn,r} = \sum_{\substack{k_1,\cdots,k_r \\ k_1+\cdots+k_r=r}} \frac{n!}{k_1!\cdots k_r!} f_{k_1+2k_2+\cdots+(r-1)k_{r-1},r}$$

For $r = 2$, we recover the familiar

$$F_{2n} = \sum_{k=0}^{n}\binom{n}{k}F_k\,.$$

Now consider the matrix equation $Q_r^{m+n} = Q_r^mQ_r^n$; equating elements in the upper left-hand corner yields

$$f_{m+n+r-1,r} = \sum_{j=1}^{r}\left(\sum_{i=0}^{r-j} f_{m+r-j,r}\, f_{n+r-2-i,r}\right)$$

and for $r = 2$, we have $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$. Note that several other general identities can be obtained in this way.

We now use the matrix Q_r to show that the product of two elements of finite order in a non-abelian group is not necessarily of finite order. This generalizes a counterexample given by Douglas Lind in [2], which results for $r = 2$. Let

$$R_r = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad S_r = \begin{pmatrix} -1 & -1 & & & 0 \\ 1 & & 1 & & \\ \vdots & & 0 & \ddots & \\ \vdots & & & & 1 \\ 1 & & & & 0 \end{pmatrix}$$

be elements of the group of invertible square matrices, then

$$R_r^2 = S_r^{r+1} = I ,$$

so R_r and S_r are of finite order, but $(R_rS_r)^n = Q_r^n \neq I$, for all n, by Theorem 1, so that R_rS_r is not of finite order.

It is of some interest to observe that the matrices Q_r^n give explicit examples of Anosov toral diffeomorphisms. That is, viewed as a linear map on $\mathbb{R}^r$, Q_r^n preserves integer points and is invertible with $\det \neq 1$, hence induces a diffeomorphism on the quotient space $\mathbb{R}^r/\mathbb{Z}^r$. The hyperbolic toral structure follows, since Q_r^n has no eigenvalue of modulus 1, using an argument via the characteristic polynomial as in [3]. Any such Anosov toral diffeomorphism comes from a linear recurrent sequence whose auxiliary equation is given by the polynomial of the diffeomorphism.

4. THE GENERAL LINEAR RECURRENT SEQUENCE

We now show how a Q-type matrix can be determined for the general r^{th} order linear recurrence relation

$$u_{n+r,r} = a_{r-1}u_{n+r-1,r} + \cdots + a_0u_{n,r}$$

with initial values $u_i = b_i$, $i = 0, 1, \cdots, r-1$, where $b_0, b_1, \cdots, b_{r-1}$

are arbitrary constants. This is done in a sequence of successive generalizations.

Define a sequence $\{f^*_{n,r}\}$ by $f_{n+r,r} = f^*_{n+r-1} + \cdots + f^*_{n,r}$, with initial values $f^*_i = b_i$, $0 \le i \le r-1$. (Note that $b_0 = b_1 = \cdots = b_{r-2} = 0$, $b_{r-1} = 1$ give the $\{f_{n,r}\}$ defined previously.) To find a Q-type matrix for the $\{f^*_{n,r}\}$, we need the following identity:

$$f^*_{n,r} = \sum_{i=1}^{r} b_{i-1} \sum_{j=1}^{i} f_{n-j-1,r} ,$$

which is easily proved by induction on n. Now let $B = (b_{r-1} \cdots b_0)$, then

$$BQ_r^n = (f^*_{n+r,r} \cdots f^*_{n+1,r}) ,$$

$$BQ_r^{n-1} = (f^*_{n-r+1}, \cdots, f^*_{n,r}), \cdots, BQ_r^{n-r+1} = (f^*_{n+1,r} \cdots f^*_{n-r+2,r}) ,$$

by our identity. Thus, we have the following Q-type matrix for our sequence $\{f^*_{n,r}\}$:

$$(Q^*_r)^n = \begin{pmatrix} BQ_r^n \\ \vdots \\ BQ_r^{n-r+1} \end{pmatrix} = \begin{pmatrix} f^*_{n+r,r} & \cdots & f^*_{n+1,r} \\ \vdots & & \vdots \\ f^*_{n+1,r} & \cdots & f^*_{n-r+2,r} \end{pmatrix} .$$

Now consider the sequences $\{u^*_{n,r}\}$ defined by

$$u^*_{n+r,r} = a_{r-1} u^*_{n+r-1,r} + a_{r-2} u^*_{n+r-2,r} + \cdots + a_0 u^*_{n,r} ,$$

with initial values $u^*_{n,r} = 0$, $0 \le n \le r-2$, $u^*_{r-1,r} = 1$. As in Theorem 1, we have the following Q-type matrix for the sequence $\{u^*_{n,r}\}$:

$$(R_r^*)^n = \begin{pmatrix} u^*_{n+r-1,r} & u^*_{n+r-2,r} & \cdots & u^*_{n,r} \\ \sum_{i=0}^{r-2} a_{r-2-1}u^*_{n+r-2-1,r} & \sum_{i=0}^{r-2} a_{r-2-i}u^*_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} a_{r-2-i}u^*_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ a_0\, u^*_{n+r-2,r} & a_0\, u^*_{n+r-3,r} & \cdots & a_0\, u^*_{n-1,r} \end{pmatrix}$$

which is proved by induction on n.

We now piece these two partial results together to derive a general Q-type matrix for the general linear recurrent sequence $\{u_{n,r}\}$ defined in the beginning of this section. To do this, we need the following identity:

$$u_{n,r} = \sum_{i=1}^{r} b_{i-1} \sum_{j=1}^{i} a_{i-j}\, u^*_{n-j-1,r} ,$$

which is proved by induction. As before, let $B = (b_{r-1} \cdots b_0)$, then by our identity,

$$B(R_r^*)^n = (u_{n+r,r} \cdots u_{n+1,r}), \cdots, B(R_r^*)^{n-r+1} = (u_{n+1,r} \cdots u_{n-r+2,r}) .$$

Hence, we have the following.

<u>Theorem 2.</u>

$$(R_r)^n = \begin{pmatrix} B(R_r^*)^n \\ \vdots \\ B(R_r^*)^{n-r+1} \end{pmatrix} = \begin{pmatrix} u_{n+r,r} & \cdots & u_{n+1,r} \\ \vdots & & \\ u_{n+1,r} & \cdots & u_{n-r+2,r} \end{pmatrix}$$

Thus, there is a general Q-type matrix for any linear recurrent sequence.

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A CHARACTERIZATON OF THE FIBONACCI NUMBERS SUGGESTED BY A PROBLEM ARISING IN CANCER RESEARCH

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1. INTRODUCTION

Cancerous growths consist of multiplying cells which invade the surrounding normal tissue. The mechanisms whereby the cancer cells penetrate among the normal cells are little understood and we have been investigating several mathematical models with a view to analyzing the movements of cells. In one such model it was necessary to enumerate the number of distinct ways a system of n cells could transform itself if adjacent cells are permitted to exchange position at most one time. It is shown below that this is simply the Fibonacci number F_{n+1}. From this characterization a very simple argument leads to a general identity for the F_n. No special knowledge of biology is required to follow the proofs and the words "person," "jumping bean," etc., could be substituted for "cell."

2. CHARACTERIZATION OF THE FIBONACCI NUMBERS

Consider a line of n cells

(1) $$A_1A_2A_3 \cdots A_{n-1}A_n$$

and suppose during a unit of time a cell either exchanges position with one of its adjacent neighbors or remains fixed. It is assumed that each cell performs at most one exchange during this time. Let G_n be the number of possible distinct arrangements of the cells after the unit of time. Then $G_n = F_{n+1}$.

For $n > 2$ the number of distinct arrangements of (1) after unit time (G_n) is equal to the number in which A_n did not exchange (G_{n-1}) plus the number in which A_n exchanged with A_{n-1} (G_{n-2}), i.e.,

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$$G_n = G_{n-1} + G_{n-2} . \tag{2}$$

This is the well known recurrence relation for the Fibonacci numbers and since $G_1 = 1 = F_2$, $G_2 = 2 = F_3$ it follows from (2) by induction on n that $G_n = F_{n+1}$. We define for later use: $G_0 = F_1 = 1$, $G_{-1} = F_0 = 0$.

3. A GENERAL IDENTITY

In order to avoid as much as possible the typographical difficulties of subscripted subscripts, the notation for the Fibonacci numbers and the G_n will be modified as follows in this section:

$$G(n) = G_n, \qquad F(n) = F_n .$$

Let $N^2 = 2$ and $M_1, M_2, \cdots, M_N$ all positive integers. Let S_{N-1} be the set of 2^{N-1} (N-1)-tuples $(k_1, k_2, \cdots, k_{N-1})$ where $k_j = 0$ or 1, $j = 1, 2, \cdots, N-1$. Then the identity is

$$F\left(\sum_{j=1}^{N} M_j + 1\right) = \sum_{S_{N-1}} F(M_1 - k_1 + 1) \cdot F(M_N - k_{N-1} + 1) \prod_{j=2}^{N-1} F(M_j - k_{j-1} - k_j + 1), \tag{3}$$

where the sum is taken over all the (N-1)-tuples in S_{N-1}. (For $N = 2$ the product is defined to be 1.)

For $N = 2$, Eq. (3) reduces to the well known identity

$$F(M_1 + M_2 + 1) = F(M_1 + 1)F(M_2 + 1) + F(M_1)F(M_2) . \tag{4}$$

It is, of course, possible to prove (3) directly from (4) by induction on N. However, the proof based on the characterization of Section 2 is extremely simple, and at the same time may suggest new geometric approaches for the analysis of multi-dimensional generalizations of the Fibonacci numbers.

Consider a line of

$$n = \sum_{j=1}^{N} M_j$$

cells which, as in (1) are performing the type of exchange described in Section 2. The number of distinct arrangements of this line after unit time is

$$G\left(\sum_{j=1}^{N} M_j\right) .$$

Now partition this group of cells before the exchanges occur in groups of M_1, $M_2, \cdots, M_N$ cells.

$$A_1^1 A_2^1 \cdots A_{M_1}^1 A_1^2 A_2^2 \cdots A_{M_2}^2 A_1^3 A_2^3 \cdots A_{M_3}^3 \cdots A_1^N A_2^N \cdots A_{M_N}^N . \tag{5}$$

If during the unit of time $A_{M_1}^1$ and A_1^2 exchange with each other then there are only $G(M_1 - 1)$ possible distinct arrangements for the first group of cells. Set $k_1 = 1$ if these two cells do exchange and $k_1 = 0$ if they do not exchange. Then there are $G(M_1 - k_1)$ possible distinct arrangements for the first group of cells. Similarly, define $k_2 = 1$ if $A_{M_2}^2$ and A_1^3 exchange, $k_2 = 0$ otherwise. Then there are $G(M_2 - k_1 - k_2)$ possible distinct arrangements for the second group of cells. Thus for each of the four possible values of the pair (k_1, k_2) the number of distinct rearrangements of the first two groups of cells considered as a whole is

$$G(M_1 - k_1) + G(M_2 - k_1 - k_2) ,$$

and the total number of distinct rearrangements for the two groups combined is

$$\sum_{k_1=0}^{1} \sum_{k_2=0}^{1} G(M_1 - k_1) + G(M_2 - k_1 - k_2) . \tag{6}$$

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LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS

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1. LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS

Since its founding, this quarterly has essentially devoted its effort towards the study of recursive relations described by certain difference equations. The solutions of many of these difference equations can be expressed in closed form, not seldom referred to as Binet forms.

A previous article [2, p. 41] offered a closed form solution for the linear homogeneous difference equation

$$\sum_{j=0}^{N} A_j y(t + j) = 0 , \tag{1.1}$$

where

$$y(t) = a_n ; \quad n \le t < n + 1 ; \quad n = 0, 1, 2, \cdots$$

with the characteristic equation

$$\sum_{j=0}^{N} A_j z^j = 0 \tag{1.2}$$

expressed as

$$\prod_{j=0}^{N} (z - r_j) = 0 \tag{1.3}$$

with distinct roots r_j. The method of solution involved the use of Laplace Transforms. It was noted after the appearance of that article that many linear homogeneous difference equations actually encountered in practice do not have distinct roots to the characteristic equation (1.2). In other words, Eq. (1.2) is often of the form

$$\prod_{i=1}^{M}(z - r_i)^{m_i} = 0 \qquad \left(N = \sum_{i=1}^{M} m_i\right) , \tag{1.4}$$

where m_i is the multiplicity of the root r_i.

With respect to Laplace Transforms, the problem of handling multiple roots lies in the inversion of the transform Y(s). It has been suggested that the definition of a "Maclaurin Series" could be regarded as a transform pair

$$\begin{aligned} G(w) &= \sum_{t=0}^{\infty} [y(t)] \frac{w^t}{t!} \\ y(t) &= D_w^t [G(w)] \Big|_{w=0} \end{aligned} \tag{1.5}$$

which has the property that the transform of $y(t + j)$ is $G^{(j)}(w)$. Since the solution of linear homogeneous differential equations is already well known when involving multiple roots [1, p. 46], it was a straightforward procedure to establish the form for the complementary problem for difference equations.

The Laplace Transform of Eq. (1.1) given in [2, p. 44] is

$$Y(s) = \left\{\frac{e^s - 1}{s}\right\} \frac{\sum_{j=1}^{N} A_j \sum_{k=0}^{j-1} a_k e^{s(j-k-1)}}{\sum_{j=0}^{N} A_j e^{sj}} , \tag{1.6}$$

and can be broken up into parts using the following theorem.

Theorem 1. (The Heaviside Theorem) If

$$Q(z) = \sum_{i=1}^{M} (z - r_i)^{m_i} ,$$

then

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^{M} \sum_{j=1}^{m_i} \frac{C_{ij}}{(z - r_i)^j} ,$$

where

$$C_{ij} = \lim_{z \to r_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{P(z)}{Q(z)} (z - r_i)^j \right\} .$$

The reader can verify the formula for C_{ij} by creating the expression being operated on, and carry out the differentiation and limit. The essence of this theorem, however, is that the transform $Y(s)$ can be expressed in the form

$$Y(s) = \frac{e^s - 1}{s} \sum_{i=1}^{M} \sum_{j=1}^{m_i} \frac{C_{ij}}{(c^s - r_i)^j} . \tag{1.7}$$

The inverse of each of these terms is given by the next theorem.

Theorem 2.

$$L\left\{ \binom{n}{j - 1} r^{n-j+1} \right\} = \frac{e^s - 1}{s} \frac{1}{(e^s - r)^j} ,$$

where

$$\binom{n}{j - 1} = 0 \quad \text{when} \quad n < j - 1$$

(r represents an arbitrary root) .

Proof. Since

$$L\left\{\binom{n}{j-1}r^{n-j+1}\right\} = \int_0^\infty f(t)c^{-st}dt$$

$$= \sum_{n=0}^{\infty}\int_n^{n+1}\binom{n}{j-1}r^{n-j+1}e^{-st}dt$$

$$= \left(\frac{1-e^{-s}}{s}\right)\sum_{n=j-1}^{\infty}\binom{n}{j-1}r^{n-j+1}c^{-sn},$$

we need only show that

$$\sum_{n=j-1}^{\infty}\binom{n}{j-1}r^{n-j+1}e^{-sn} = \frac{e^s}{(e^s-r)^j},$$

by induction. The formula is true for $j = 1$ since

$$\sum_{n=0}^{\infty}(re^{-s})^n = \frac{1}{1-re^{-s}} = \frac{c^s}{(e^s-r)}.$$

Assume now that it holds for $j = k$, that is

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}r^{n-k+1}e^{-sn} = \frac{e^s}{(e^s-r)^k}.$$

Differentiating once, term-by-term, with respect to r yields

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}(n-k+1)r^{n-k}e^{-sn} = \frac{ke^s}{(e^s-r)^{k+1}}$$

or

$$\sum_{n=k}^{\infty} \binom{n}{k} r^{n-k} e^{-sn} = \frac{e^s}{(e^s - r)^{k+1}} ,$$

and thus implies the truth of the formula for the $(k+1)^{st}$ case. As a result of this theorem, the more general solution for the linear homogeneous difference equation (1.1) is given by

$$(1.8) \qquad y(t) = \sum_{i=1}^{M} \sum_{j=1}^{m_i} C_{ij} \binom{n}{j-1} r_i^{n-j+1} ,$$

where the C_{ij} are given (by Theorem 1) as

$$C_{ij} = \lim_{z \to m_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{\sum_{k=1}^{N} A_k \sum_{\ell=0}^{k-1} a_\ell z^{k-\ell-1}}{\prod_{k=1}^{M} (z - r_k)^{m_k}} (z - r_i)^j \right\}$$

or, by re-ordering the double summation according to z,

$$(1.9) \qquad C_{ij} = \lim_{z \to m_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{\sum_{k=0}^{N-1} \sum_{\ell=k+1}^{N} A_\ell a_{\ell-k-1} z^k}{\prod_{k=1}^{M} (z - r_k)^{m_k}} (a - r_i)^j \right\} .$$

2. CONVOLUTION OF FIBONACCI SEQUENCES

The following problem related to the previous discussion was brought to my attention by Prof. V. E. Hoggatt, Jr. Initially, we are given that a convolution of a Fibonacci sequence is described by

$$H_{n+2} - H_{n+1} - H_n = F_n , \tag{2.1}$$

where F_n is the famous n^{th} Fibonacci number. The problem is to find a closed form (Binet form) for H_n. Since F_n satisfies the relationship

$$F_{n+2} - F_{n+1} - F_n = 0 , \tag{2.2}$$

Eq. (2.1) can be made homogeneous by substitution; that is, Eq. (2.2) can be re-written as

$$(H_{n+4} - H_{n+3} - H_{n+2}) - (H_{n+3} - H_{n+2} - H_{n+1}) - (H_{n+2} - H_{n+1} - H_n) = 0$$

or, collecting terms,

$$H_{n+4} - 2H_{n+3} - H_{n+2} + 2H_{n+1} + H_n = 0 . \tag{2.3}$$

Since $F_0 = 0$ and $F_1 = 1$, the starting values depending on H_0 and H_1 are

$$\begin{aligned} H_0 &= H_0 \\ H_1 &= H_1 \\ H_2 &= H_0 + H_1 \\ H_3 &= 1 + H_0 + 2H_1 . \end{aligned}$$

The characteristic equation of the difference relation (2.3) is

$$z^4 - 2z^3 - z^2 + 2z + 1 = 0$$

or

$$(z - \alpha)^2(z - \beta)^2 = 0 ,$$

where α is the well known golden ratio and β is the conjugate,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

[Continued on page 292.]

GENERALIZED FIBONACCI NUMBERS IN PASCAL'S PYRAMID

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1. INTRODUCTION

It is well known that the Fibonacci numbers are the rising diagonals of Pascal's triangles. Harris and Styles [2] generalized the Fibonacci numbers to other diagonals. Hoggatt and Bicknell further generalized these to other Pascal triangles in [3]. Mueller in [5] discusses sums taken over planar sections of Pascal's pyramid. Here we further extend the results in [5] to many relations with the Fibonacci numbers.

In [1] many nice derivations were obtained using generating functions for the columns of Pascal's binomial triangle. Further results will be forthcoming in [6]. The earliest results were laid out by Hochster in [7].

2. COLUMN GENERATORS

The simple Pascal pyramid has column generators, when it is double left-justified, which are

$$G_{m,n} = \frac{x^{m+n}\binom{m+n}{n}}{(1-x)^{m+n+1}} .$$

These columns can be shifted up and down with parameters p and q. The parameter p determines the alignment of the left-most slice of columns and the parameter q determines the alignment of the slices relative to that left-most slice. Now the modified simple column generators are

$$G^*_{m,n} = \frac{x^{pm+qn}\binom{m+n}{n}}{(1-x)^{m+n+1}} .$$

We desire to get the generating function of the planar section sum sequence. Each such planar section now has summands which are all multiplied by the same power of x. For instance,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^{m+n}\binom{m+n}{n}}{(1-x)^{m+n+1}} = \sum_{n=0}^{\infty}\frac{x^n}{(1-x)^{n+1}}\left\{\sum_{m=0}^{\infty}\frac{x^m}{(1-x)^m}\binom{m+n}{n}\right\} .$$

But

$$\sum_{m=0}^{\infty}\binom{m}{n}z^m = \frac{z^n}{(1-z)^{n+1}} ,$$

so that

$$\sum_{m=0}^{\infty}\binom{m+n}{n}z^m = \frac{1}{(1-z)^{n+1}} .$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^{m+n}}{(1-x)^{m+n+1}}\binom{m+n}{n} = \sum_{n=0}^{\infty}\frac{x^n}{(1-x)^{n+1}}\cdot\frac{1}{\left(1-\frac{x}{1-x}\right)^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{(1-2x)^{n+1}} = \frac{1}{1-3x} = \sum_{n=0}^{\infty}3^n x^n$$

which was to be expected as each planar section contains the numbers in the expansion $(1 + 1 + 1)^n$.

We next let p and q be utilized.

$$G = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}G^*_{m,n} = \frac{1}{1 - x - x^p - x^q} .$$

Here clearly when $p = 2$ and $q = 3$ we get

$$G = \frac{1}{1 - x - x^2 - x^3} = \sum_{n=0}^{\infty} T_{n+1}\, x^n ,$$

the generating function for the Tribonacci numbers,

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 1, \quad \text{and} \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n .$$

If, on the other hand, we set $p = 1$ and $q = 2$, then

$$G = \frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1}\, x^n ,$$

the generating function for the Pell numbers, $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. One can get even more out of this.

Let $p = t + 1$ and $q = 2t + 1$; then,

$$G = \frac{1}{1 - x - x^{t+1} - x^{2t+1}} = \sum_{n=0}^{\infty} u(n;\, t, 1) x^n$$

the generating function for the generalized Fibonacci numbers of Harris and Styles [2] applied to the trinomial triangle whose coefficients are induced by the expansions

$$(1 + x + x^2)^n, \qquad n = 0, 1, 2, \cdots .$$

See also Hoggatt and Bicknell [3].

Consider

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} \frac{x^{mp+qn}\binom{m+n}{n}}{(1 - x)^{m+n+1}}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\left[\frac{x^p}{1 - x}\right]^m \binom{m+n}{n}\right)\frac{x^{qn}}{(1 - x)^{n+1}} .$$

Let us now take every r^{th} slice in the general p, q case

$$\sum_{n=0}^{\infty} \frac{x^{qnr}}{(1 - x - x^p)^{rn+1}} = \frac{1}{1 - x - x^p} \cdot \frac{1}{1 - \dfrac{x^{qr}}{(1 - x - x^p)^r}}$$

$$\frac{(1 - x - x^p)^{r-1}}{(1 - x - x^p)^r - x^{rq}} = \frac{(1 - x - x^p)^{r-1}}{(1 - x - x^p)^r - x^{r+q'}} = \sum_{n=0}^{\infty} U(n; q', r)x^n ,$$

where $q' = r(q - 1)$, which is the generating function for the generalized Fibonacci numbers of Harris and Styles $U(n; q', r)$ as applied to the CONVOLUTION triangle of the number sequence $u(n; p - 1, 1)$ which are themselves generalized Fibonacci numbers of Harris and Styles in the binomial triangle. (See "Convolution Triangles for Generalized Fibonacci Numbers" [4].)

3. THE GENERAL CASE

In [5] Pascal's pyramid in standard position has as its elements in a horizontal plane the expansions of $(a + b + c)^n$, $n = 0, 1, 2, 3, \cdots$ with each planar section laid out as an equilateral lattice. In our configuration it is a right isosceles lattice.

The general column generator is

$$G^*_{m,n} = \frac{x^{pm+qn} b^m c^n \binom{m+n}{n}}{(1 - ax)^{m+n+1}}$$

and it is not difficult to derive that

$$G = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G^*_{m,n} = \frac{1}{1 - ax - bx^p - cx^q} .$$

Thus by selecting the five parameters one can get many other known generating functions.

Example 1. $a = 2$, $b = 2$, $c = -1$, $p = 2$, $q = 3$,

$$G = \frac{1}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+1}F_{n+2}x^n .$$

Example 2. $a = 1$, $b + c = 1$, $p = q = 2$, then

$$G = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n .$$

One notes that the condition $b + c = 1$ allows an infinitude of choices of integers b and c.

Example 3. Let

$$a = 3(1 - x^2), \quad b = 6, \quad c = -1, \quad p = 2, \quad \text{and} \quad q = 4,$$

then

$$G = \frac{1}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{m=0}^{\infty} \left(\begin{matrix} m+3 \\ 3 \end{matrix}\right) x^m ,$$

where $\left(\begin{matrix} m \\ n \end{matrix}\right)$ are the Fibonomial coefficients. See H-78 and [8], or it can be written as

$$G = \sum_{m=0}^{\infty} \left(\frac{F_{m+1}F_{m+2}F_{m+3}}{1 \cdot 1 \cdot 2}\right) x^m .$$

The possibilities seem endless.

4. FURTHER RESULTS

Consider

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{x^{pm} b^m \binom{m+n}{n}}{(1-ax)^m} \right\} \frac{c^n x^{nq}}{(1-ax)^{n+1}} \quad .$$

Now let's take every r^{th} slice:

$$G = \sum_{n=0}^{\infty} \frac{(cx^q)^{rn}}{(1 - ax - bx^p)^{rn+1}} = \frac{(1 - ax - bx^p)^{r-1}}{(1 - ax - bx^p)^r - c^r x^{r+q'}} \,,$$

where $q' = q(r-1)$. If $c = 1$, $a = 2$, $b = -1$, $p' = r + q'$, and $p = 2$, then

$$G = \frac{(1-x)^{2r-2}}{(1-x)^{2r} - x^{2r+p'}} \quad .$$

Recall from [1] and [3] that

$$H = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;\, p, q)x^n$$

for the generalized Fibonacci numbers in Pascal's triangle so that G is the generating function for $H/(1-x)$ or

$$G = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} u(k;\, p',\, 2r) \right\} x^n \quad .$$

Another example: If $a = 1 + x$, $b = 1$, $p = 3$, $c = 1$, then

$$G = \frac{(1 - x - x^2 - x^3)^{r-1}}{(1 - x - x^2 - x^3)^r - x^{r+q'}} \quad ,$$

[Continued on page 293.]

MODULO ONE UNIFORM DISTRIBUTION OF CERTAIN FIBONACCI-RELATED SEQUENCES

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Let $\{x_j\}_1^\infty$ be a sequence of real numbers with corresponding fractional parts $\{\beta_j\}_1^\infty$, where $\beta_j = x_j - [x_j]$ and the bracket denotes the greatest integer function. For each $n \geq 1$, we define the function F_n on $[0,1]$ so that $F_n(x)$ is the number of those terms among $\beta_1, \cdots, \beta_n$ which lie in the interval $[0,x)$, divided by n. Then $\{x_j\}_1^\infty$ is said to be uniformly distributed modulo one iff $\lim_{n\to\infty} F_n(x) = x$ for all $x \in [0,1]$.

In other words, each interval of the form $[0,x)$ with $x \in [0,1]$, contains asymptotically a proportion of the β_n's equal to the length of the interval, and clearly the same will be true for any sub-interval (α,β) of $[0,1]$. The classical Weyl criterion [1, p. 76] states that $\{x_j\}_1^\infty$ is uniformly distributed mod 1 iff

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i\nu x_j} = 0 \quad \text{for all } \nu \geq 1 .$$

An example of a sequence which is uniformly distributed mod 1 is $\{n\xi\}_{n=0}^\infty$, where ξ is an arbitrary irrational number. (See [1, p. 81] for a proof using Weyl's criterion.)

The purpose of this paper is to show that the sequence $\{\ln F_n\}_1^\infty$ and $\{\ln L_n\}_1^\infty$ are uniformly distributed mod 1. More generally, we show that if $\{V_n\}_1^\infty$ satisfies the Fibonacci recurrence

$$V_{n+2} = V_{n+1} + V_n$$

for $n \geq 1$ with $V_1 = K_1 > 0$ and $V_2 = K_2 > 0$ as initial values, then $\{\ln V_n\}_1^\infty$ is uniformly distributed mod 1. Toward this end, the following two lemmas are helpful.

Lemma 1. If $\{x_j\}_1^\infty$ is uniformly distributed mod 1 and $\{y_j\}_1^\infty$ is a sequence such that

$$\lim_{j \to \infty} (x_j - y_j) = 0 ,$$

then $\{y_j\}_1^\infty$ is uniformly distributed mod 1.

Proof. From the hypothesis and the continuity of the exponential function, it follows that

$$\lim_{j \to \infty} \left(e^{2\pi i \nu x_j} - e^{2\pi i \nu y_j} \right) = 0 .$$

But it is well known [2, Theorem B, p. 202], that if $\{\gamma_n\}$ is a sequence of real numbers converging to a finite limit L, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_1^n \gamma_j = L .$$

Taking

$$\gamma_j = e^{2\pi i \nu x_j} - e^{2\pi i \nu y_j} ,$$

we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_1^n \left(e^{2\pi i \nu x_j} - e^{2\pi i \nu y_j} \right) = 0 .$$

Since

$$\lim_{n \to \infty} \frac{1}{n} \sum_1^n e^{2\pi i \nu x_j} = 0$$

by Weyl's criterion, we also have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} e^{2\pi i \nu y_j} = 0$$

and the sufficiency of Weyl's criterion proves the sequence $\{y_j\}_1^\infty$ to be uniformly distributed mod 1.

Lemma 2. If α is an algebraic number, then $\ln \alpha$ is irrational.

Proof. Assume, to the contrary, that $\ln \alpha = p/q$, where p and q are non-zero integers. Then $e^{p/q} = \alpha$, so that $e^p = \alpha^q$. But α^q is algebraic, since the algebraic numbers are closed under multiplication [1, p. 84]. Thus e^p is algebraic, in turn implying e is algebraic. But e is known to be transcendental [1, p. 25] so that a contradiction is obtained.

Theorem. Let $\{V_n\}_1^\infty$ be a sequence generated by the recursion formula

$$V_{n+2} = V_{n+1} + V_n$$

for $n \geq 1$ given that $V_1 = K_1 > 0$ and $V_2 = K_2 > 0$. Then the sequence $\{\ln V_n\}_1^\infty$ is uniformly distributed modulo one.

Proof. The recursion (difference equation) has general solution

$$V_n = C_1\alpha^n + C_2\beta^n ,$$

where α, β are the roots of the equation $x^2 - x - 1 = 0$ and C_1, C_2 are constants determined by the initial conditions. Thus

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} ,$$

while $C_1\alpha + C_2\beta = K_1$ and $C_1\alpha^2 + C_2\beta^2 = K_2$. Now,

$$|V_n - C_1\alpha^n| = |C_2\beta^n|$$

for $n \geq 1$, so that, noting $|\beta| < 1$, we have

$$\lim_{n \to \infty} |V_n - C_1\alpha^n| = 0 .$$

Moreover, from the fact that $\{V_n\}_1^\infty$ is an increasing positive sequence,

$$\left|1 - \frac{C_1\alpha^n}{V_n}\right| = \left|\frac{V_n - C_1\alpha^n}{V_n}\right| \leq \frac{1}{K_1}\left|V_n - C_1\alpha^n\right| ,$$

so that

$$\lim_{n \to \infty} \frac{C_1\alpha^n}{V_n} = 1 .$$

Thus,

$$\lim_{n \to \infty} \ln\left(\frac{C_1\alpha^n}{V_n}\right) = 0 ,$$

or equivalently,

$$\lim_{n \to \infty} [\ln (C_1\alpha^n) - \ln V_n] = 0 . \tag{2}$$

Since α is algebraic $(\alpha^2 = \alpha + 1)$, it follows from Lemma 2 that $\ln \alpha$ is irrational and consequently [1, p. 84] that

$$\{n \ln \alpha\}_1^\infty = \{\ln (\alpha^n)\}_1^\infty$$

is uniformly distributed mod 1. Then

$$\{\ln (C_1\alpha^n)\}_1^\infty$$

[Continued on page 294.]

NOTE ON SOME SUMMATION FORMULAS

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In a recent paper [1], the writer has proved the following multiple summation formula:

$$(1) \qquad \sum_{s_1, s_2, \cdots, =0}^{\infty} \frac{(k + 2s_1 + 3s_2 + \cdots)!}{s_1!\, s_2! \cdots (k + s_1 + 2s_2 + \cdots)!} \cdot \frac{u_1^{s_1} u_2^{s_2} \cdots}{(1 + u_1 + u_2 + u_3 + \cdots)^{2s_1+3s_2+\cdots}} = \frac{(1 + u_1 + u_2 + \cdots)^{k+1}}{1 - u_1 - 2u_2 - 3u_3 - \cdots}$$

$$(k = 0, 1, 2, \cdots),$$

where the series

$$(2) \qquad u_1 + u_2 + u_3 + \cdots ,$$

is absolutely convergent but otherwise arbitrary.

In the present note we should like to point out that (1) admits of the following extension:

$$(3) \qquad \sum_{s_0, s_1, s_2, \cdots =0}^{\infty} \frac{(k + s_0 + 2s_1 + 3s_2 + \cdots)!}{s_0!\, s_1!\, s_2! \cdots (k + s_1 + 2s_2 + \cdots)!} \cdot \frac{u_0^{s_0} u_1^{s_1} u_2^{s_2} \cdots}{(1 + u_0 + u_1 + u_2 + \cdots)^{s_0+2s_1+\cdots}}$$

$$= \frac{(1 + u_0 + u_1 + u_2 + \cdots)^{k+1}}{1 - u_1 - 2u_2 - 3u_3 - \cdots} \qquad (k = 0, 1, 2, \cdots),$$

where again the series (2) is absolutely convergent.

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Proof of (3). By (1),

$$\frac{(1+u_0+u_1+u_2+\cdots)^{k+1}}{1-u_1-2u_2-3u_3-\cdots} = \frac{(1+u_1+u_2+\cdots)^{k+1}}{1-u_1-2u_2-3u_3-\cdots}\left(\frac{1+u_0+u_1+u_2+\cdots}{1+u_1+u_2+\cdots}\right)^{k+2}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!\,s_2!\cdots(k+s_1+2s_2+\cdots)!}\,\frac{u_1^{s_1}u_2^{s_2}\cdots}{(1+u_1+u_2+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot\left(\frac{1+u_0+u_1+\cdots}{1+u_1+u_2+\cdots}\right)^{k+1}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!}\,\frac{u_1^{s_1}u_2^{s_2}\cdots}{(1+u_0+u_1+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot\left(1-\frac{u_0}{1+u_0+u_1+\cdots}\right)^{-k-2s_1-3s_2-\cdots-1}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!}\,\frac{u_1^{s_1}u_2^{s_2}}{(1+u_0+u_1+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot\sum_{s_0=0}^{\infty}\binom{k+s_0+2s_1+3s_2+\cdots}{s_0}\frac{u_0^{s_0}}{(1+u_0+u_1+\cdots)^{s_0}}$$

$$= \sum_{s_0,s_1,s_2,\cdots=0}^{\infty} \frac{(k+s_0+2s_1+3s_2+\cdots)!}{s_0!s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!}\,\frac{u_0^{s_0}u_1^{s_1}u_2^{s_2}\cdots}{(1+u_0+u_1+\cdots)^{s_0+2s_1+3s_2+\cdots}}.$$

This evidently proves (3).

Exactly as in [1], we can show that (3) holds for arbitrary k, provided we replace the coefficient

$$\frac{(k+s_0+2s_1+3s_2+\cdots)!}{s_0!s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!}$$

[Continued on page 291.]

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-192 Proposed by Ronald Alter, University of Kentucky, Lexington, Kentucky.

If

$$c_n = \sum_{j=0}^{3n+1} \binom{6n+3}{2j+1}(-11)^j ,$$

prove that

$$c_n = 2^{6n+3}N, \qquad (N \text{ odd}, \; n \geq 0).$$

H-193 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Prove or disprove: If

$$x + y + z = 2^{2n+1} - 1 \qquad \text{and} \qquad x^3 + y^3 + z^3 = 2^{6n+1} - 1,$$

then $6n + 1$ and $2^{6n+1} - 1$ are primes.

H-194 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Solve the Diophantine equations:

(i) $$x^2 + y^2 \pm 5 = 3xy$$

(ii) $$x^2 + y^2 \pm e = 3xy\,,$$

where

$$e = p^2 - pq - q^2\,,$$

p, q positive integers.

SOLUTIONS

BINET GAINS IDENTITY

H-180 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{n} \binom{n}{k}^3 F_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} F_{(2n-3k)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 L_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} L_{(2n-3k)}\,,$$

where F_k and L_k denote the k^{th} Fibonacci and Lucas numbers, respectively.

Solution by David Zeitlin, Minneapolis, Minnesota.

A more general result is that

$$\sum_{k=0}^{n} \binom{n}{k}^3 b^{n-k} a^k W_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} b^k a^k W_{2n-3k}\,, \tag{1}$$

where $W_{n+2} = aW_{n+1} + bW_n$, $n = 0, 1, \cdots$. For $a = b = 1$, we obtain the desired results with $W_k = F_k$ and $W_k = L_k$.

Proof. From a well-known result*, we note that

$$(2)\qquad \sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{2k\le n} \frac{(n+k)!}{(k!)^3(n-2k)!}\, x^k (x+1)^{n-2k} .$$

Set $x = (ay)/b$ in (2) to obtain:

$$(3)\qquad \sum_{k=0}^{n} \binom{n}{k}^3 b^{n-k} a^k y^k = \sum_{2k\le n} \frac{(n+k)!}{(k!)^3(n-2k)!}\, b^k a^k y^k (ay+b)^{n-2k} .$$

Let α, β be the roots of $y^2 = ay + b$. Noting that $W_n = C_1\alpha^n + C_2\beta^n$, we obtain (1) by addition of (3) for $y = \alpha$ and $y = \beta$.

Remarks. If $a = 2x$, $b = -1$, then with $W_k = T_k(x)$, the Chebyshev polynomial of the first kind, we obtain from (1)

$$(4)\qquad \sum_{k=0}^{n} \binom{n}{k}^3 (-1)^{n-k}(2x)^k T_k(x) = \sum_{2k\le n} \frac{(n+k)!}{(k!)^3(n-2k)!}\, (-2x)^k T_{2n-3k}(x) .$$

For $a = 2$, $b = 1$, one may choose $W_k = P_k$, the Pell sequence.

Let $V_0 = 2$, $V_1 = a$, and $V_{k+2} = aV_{k+1} + bV_k$. Then, from (1), we obtain the general result

$$(5)\qquad \begin{aligned} &\sum_{k=0}^{n} \binom{n}{k}^3 V_k^k \left\{(-1)^{m+1} b^m\right\}^{n-k} W_{mk+p} \\ &\qquad = \sum_{2k\le n} \frac{(n+k)!}{(k!)^3(n-2k)!} \left((-1)^{m+1} b^m V_m\right)^k W_{m(2n-3k)+p} \end{aligned},$$

for $m, p = 0, 1, \cdots$.

It should be noted that (1) is valid for equal roots, i.e., $\alpha = \beta$.

---

*J. Riordan, Combinatorial Identities, p. 41.

Also solved by F. D. Parker, A. G. Shannon, and the Proposer.

SUM-ER TIME

H-181 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$\sum_{m,n=0}^{\infty} (am + cn)^{m}(bm + dn)^{n}\, \frac{u^{m}v^{n}}{m!\,n!} = \frac{1}{(1 - ax)(1 - dy) - bcxy},$$

where

$$u = xe^{-(ax+by)}, \qquad v = ye^{-(cx+dy)}.$$

Solution by the Proposer.

$$\sum_{m,n=0}^{\infty} (am + cn)^{m}(bm + dn)^{n}\, \frac{u^{m}v^{n}}{m!\,n!}$$

$$= \sum_{m,n=0}^{\infty} (am + cn)^{m}(bm + dn)^{n}\, \frac{x^{m}y^{n}}{m!\,n!}\, e^{-(am+cn)x-(bm+dn)y}$$

$$= \sum_{m,n=0}^{\infty} (am + cn)^{m}(bm + dn)^{n}\, \frac{x^{m}y^{n}}{m!\,n!} \sum_{j=0}^{\infty} (-1)^{j}\, \frac{(am+cn)^{j}}{j!}\, x^{j} \sum_{k=0}^{\infty} (-1)^{k}\, \frac{(bm+dn)^{k}}{k!}\, y$$

$$= \sum_{m,n=0}^{\infty} \frac{x^{m}y^{n}}{m!\,n!} \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j}\binom{n}{k} {a(m-j)+c(n-k)}^{m}\, {b(m-j)+d(n-k)}^{n}$$

$$= \sum_{m,n=0}^{\infty} \frac{x^{m}y^{n}}{m!\,n!} \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} (aj + ck)^{m}(bj + dk)^{n}.$$

But

$$S_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj + ck)^m(bj + dk)^n$$

$$(*) \qquad = \sum_{r=0}^{m}\sum_{n=0}^{n} \binom{m}{r}\binom{n}{a} a^{m-r}o^r b^{n-s}d^s \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} j^{m+n-r-s}$$

$$\cdot \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} d^{r+s} \quad .$$

Since

$$\sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} j^t = \begin{cases} 0 & (t < m) \\ m! & (t = m) \end{cases} ,$$

we need only consider those terms in (*) such that

$$\begin{cases} m + n - r - s \geq m \\ \qquad\qquad r + s \geq n \end{cases} ,$$

that is, $r + s = n$.

We therefore get

$$S_{m,n} = m!\,n! \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r}d^{n-r}(bc)^r ,$$

so that

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!\, n!}$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r} d^{n-r} (bc)^r$$

$$= \sum_{r=0}^{\infty} (bcxy)^r \sum_{m,n=r}^{\infty} \binom{m+r}{r}\binom{n+r}{r} (ax)^m (dy)^n$$

$$= \sum_{r=0}^{\infty} (bcxy)^r (1 - ax)^{-r-1} (1 - dy)^{-r-1}$$

$$= (1 - ax)^{-1}(1 - dy)^{-1}\left\{1 - \frac{bcxy}{(1 - ax)(1 - dy)}\right\}^{-1}$$

$$= \left\{(1 - ax)(1 - dy) - bcxy\right\}^{-1}$$

ARRAY OF HOPE

H-183 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the display indicated below.

1						
1	1					
2	2	1				
5	4	3	1			
13	9	7	4	1		
34	22	16	11	5	1	
89	56	38	27	16	6	1
...						

Pascal Rule of Formation Except for Prescribed Left Edge.

(i) Find an expression for the row sums.

(ii) Find a generating function for the row sums.

(iii) Find a generating function for the rising diagonal sums.

Solution by the Proposer.

i) An inspection of the array reveals that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$

ii) If the columns are multiplied by $1, 2, 3, \cdots$ sequentially to the right, then the row sums have the generating function,

$$\frac{(1 - x)}{(1 - 3x + x^2)} \cdot \frac{(1 - x)}{(1 - 2x)} \quad .$$

Thus the row sums are the convolution of the two sequences:

a) $A_1 = 1, \quad A_n = F_{2n+1} \quad (n \geq 1)$ and

b) $B_1 = 1, \quad B_n = 2^{n-1} \quad (n \geq 1)$.

iii) The rising diagonal sums, E_n, are the convolution of the two sequences:

c) $C_n = F_{n-1}$ and

d) $D_n = F_{2n-1} \quad (n = 0, 1, 2, \cdots)$.

Hence

$$\frac{(1 - x)^3}{(1 - x - x^2)(1 - 3x + x^2)} = \sum_{n=0}^{\infty} E_n x^n \quad .$$

FIBO-CYCLE

H-184 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the cycle α_n $(n = 1, 2, \cdots)$ as follows:

i) $$\alpha_n = (1, 2, 3, 4, \cdots, F_n) \quad ,$$

where F_n denotes the n^{th} Fibonacci number. Now construct a sequence of permutations

$$\left\{\alpha_n^{F_i}\right\}_{i=1}^{\infty}, \qquad (n = 1, 2, \cdots)$$

(ii) $$\alpha_n^{F_{i+2}} = \alpha_n^{F_i} \cdot x_n^{F_{i+1}} \qquad (i \geq 1) \quad .$$

Finally, define a sequence

$$\{u_n\}_{n=1}^{\infty}$$

as follows: u_n is the period of

$$\left\{\alpha_n^{F_i}\right\}_{i=1}^{\infty}$$

i.e., u_n is the smallest positive integer such that

(iii) $$\alpha_n^{F_{i+u_n}} = \alpha_n^{F_i} \qquad (i \geq N) .$$

a) Find a closed form expression for u_n.

b) If possible, show $N = 1$ is the minimum positive integer for which iii) holds for all $n = 1, 2, \cdots$.

Solution by the Proposer.

Since α_n is of order F_n, it follows that the exponents of α_n may be replaced by residues mod F_n and u_n is thus the period of the Fibonacci sequence mod F_n. Therefore $u_1 = u_2 = 1$, $u_3 = 3$. Consider the First n residue classes of the Fibonacci seuqence, mod F_n $(n \geq 4)$; $1, 1, 2, 3, \cdots, F_{n-1}, 0$. The $(n+1)^{st}$ residue class is $F_{n-1} = 1 + (F_{n-1} - 1)$ and $(2n-1)^{st}$ class is

$$F_{n-1} + F_{n-1}(F_{n-1} - 1) = F_{n-1}^2 .$$

However,

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geq 2)$$

and

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n \geq 2)$$

implies

$$F_{n-1}^2 \equiv (-1)^n \pmod{F_n} .$$

If n is even $(n \geq 4)$, we have $F_{n-1}^2 \equiv 1 \pmod{F_n}$ and $u_n = 2n$. If n is odd $(n > 4)$, $F_{n-1}^2 \equiv -1 \pmod{F_n}$ and $u_n = 4n$.

From the above, it is obvious that $N = 1$ is the smallest positive integer for which (iii) holds for all $n = 1, 2, \cdots$. It is interesting to note that

$$\{u_n | n = 1, 2, \cdots\} \cap \{F_n | n = 1, 2, \cdots\} = \{F_1, F_4, F_6, F_9, F_{12}, \cdots\} .$$

[Continued from page 282.]

NOTE ON SOME SUMMATION FORMULAS

by

$$\frac{\prod_{i=1}^{s_0+s_1+s_2+\cdots} (k + s_1 + 2s_2 + 3s_3 + \cdots + i)}{s_0!s_1!s_2! \cdots} .$$

REFERENCE

1. L. Carlitz, "Some Summation Formulas," Fibonacci Quarterly, Vol. 9 (1971), pp. 28-34.

[Continued from page 270.]
The solution is then given by Eq. (1.8) as

$$H_n = C_{11}\alpha^n + C_{12}n\alpha^{n-1} + C_{21}\beta^n + C_{22}n\beta^{n-1} \tag{2.5}$$

with the C_{ij} given by Eq. (1.9). In practice, however, the C_{ij} are most easily found by solving the set of simultaneous equations derived by applying the initial values, H_0, H_1, H_2, H_3, for $n = 0, 1, 2, 3$. The solution yields:

$$C_{11} = \frac{3-\alpha}{5}H_0 + \frac{2\alpha - 1}{5}H_1 + \frac{2}{25}(1 - 2\alpha)$$

$$C_{12} = 1/5$$

$$C_{21} = \frac{2+\alpha}{5}H_0 + \frac{1 - 2\alpha}{5}H_1 + \frac{2}{25}(2\alpha - 1)$$

$$C_{22} = 1/5 \quad .$$

REFERENCES

1. Gustav Doetsch, Guide to the Applications of the Laplace and Z Transforms, Van Nostrand Reinhold Company, New York, 1971.
2. Robert M. Giuli, "Binet Forms by Laplace Transform," Fibonacci Quarterly, Vol. 9, No. 1, p. 41.

[Continued from page 264.]

(If $M_2 = 1$, i.e., there is only one cell in the second group, then it cannot exchange with both $A^1_{M_1}$ and A^3_1. The rearrangements corresponding to this case are eliminated in (6) since it occurs when $k_1 = k_2 = 1$ and $G(-1) = 0$.)

The remainder of the proof follows the same procedure. Define $k_j = 1$ if $A^j_{M_j}$ and A^{j+1}_1 exchange, $k_j = 0$ otherwise, $j = 3, \cdots, N-1$. For each of 2^{N-1} possible values of $(k_1, k_2, \cdots, k_{N-1})$ the number of distinct arrangements of the N groups combined is

$$G(M_1 - k_1) + G(M_N - k_{N-1}) \cdot \prod_{j=2}^{N-1} G(M_j - k_{j-1} - k_j) \,. \tag{7}$$

[Continued on page 293.]

[Continued from page 276.]
(where $q' = r(q'-1)$), which are the numbers $u(n; q, r)$ in the Tribonacci convolution triangle! See [4].

REFERENCES

1. V. E. Hoggatt, Jr., "A New Angle on Pascal's Pyramid," Fibonacci Quarterly, Vol. 6 (1968), pp. 221-234.
2. V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 2 (1964), pp. 277-289.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 7 (1969), pp. 341-358.
4. V. E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 8 (1970), pp. 158-171.
5. Stephen Mueller, "Recursions Associated with Pascal's Pyramid," Pi Mu Epsilon Journal, Vol. 4, No. 10, Spring 1969, pp. 417-422.
6. Stanley Carlson and V. E. Hoggatt, Jr., "More Angles on Pascal's Triangle," Fibonacci Quarterly, to appear.
7. Melvin Hochster, "Fibonacci-Type Series and Pascal's Triangle," Particle, Vol. IV (1962), pp. 14-28.
8. V. E. Hoggatt, Jr., "Fibonacci Numbers and Generalized Binomial Coefficients," Fibonacci Quarterly, Vol. 5 (1967), pp. 383-400.

[Continued from page 292.]

The total number of distinct arrangements of the N groups combined is obtained by summing the expression in (7) over all possible values of $(k_1, k_2, \cdots, k_{N-1})$, i.e., over the set S_{N-1}. But the total number of distinct arrangements is also equal to

$$G\left(\sum_{j=1}^{N} M_j\right) .$$

The identity in (3) then follows from $G(n) = F(n+1)$.

[Continued from page 280.]

is also uniformly distributed mod 1 and the mod 1 uniform distribution of $\ln V_n$ then follows from (2) in conjunction with Lemma 1. q.e.d.

Corollary. The sequences $\{\ln F_n\}_1^\infty$ and $\{\ln L_n\}_1^\infty$ are uniformly distributed mod 1. Here

$$\{F_n\} = \{1, 1, 2, 3, \cdots\} \quad \text{and} \quad \{L_n\} = \{2, 1, 3, 4, \cdots\}$$

are the Fibonacci and Lucas sequences, respectively.

Proof. Both sequences satisfy the recursion, $V_{n+2} = V_{n+1} + V_n$ for $n \geq 1$ with $(K_1, K_2) = (1,1)$ for the Fibonacci sequence and $(K_1, K_2) = (2,1)$ for the Lucas sequence, so that the result follows directly from the theorem.

REFERENCES

1. I. Niven, "Irrational Numbers," Carus Mathematical Monograph Number 11, The Math. Assn. of America, John Wiley and Sons, Inc., New York,
2. P. R. Halmos, Measure Theory, D. Van Nostrand Co., Inc., New York, New York, 1950.

Renewal notices, normally sent out to subscribers in November or December, are now sent by bulk mail. This means that if your address has changed the notice will not be forwarded to you. If you have a change of address, please notify:

Brother Alfred Brousseau
St. Mary's College
St. Mary's College, Calif.

SOME NEW NARCISSISTIC NUMBERS

JOSEPH S. MADACHY
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A Narcissistic number is one which can be represented as some function of its digits. For example,

$$153 = 1^3 + 5^3 + 3^3, \qquad 145 = 1! + 4! + 5!, \quad \text{and} \quad 2427 = 2^1 + 4^2 + 2^3 + 7^4$$

are narcissistic numbers. One special class of these numbers, represented by the first example above, are called Digital Invariants. These are integers which are equal to the sum of the n^{th} powers of the digits of the integers. Extensive studies of digital invariants have been in progress during the past two years. Robert L. Patton, Sr., Robert L. Patton, Jr., and the author have completed the search for all digital invariants for n^{th} powers up to $n = 15$ and will publish the results in the near future.

This short note reports on various narcissistic numbers other than digital invariants. An abbreviated form for these numbers is used in Table 1.

$$abc\cdots \quad \text{means} \quad 10^p a + 10^{p-1} b + 10^{p-2} c + \cdots + 10p + q\,,$$

where $a, b, c, \cdots, q$ are the digits of the integer and the number of digits is $p + 1$. That is,

$$349 = 10^2 \cdot 3 + 10 \cdot 4 + 9\,.$$

The general form is shown in the Table along with the known solutions, their discoverers, and some notes. Trivial solutions, 0 and 1, are not included.

The search for solutions to the first form

$$(abc \cdots = a^n + b^{n+1} + c^{n+2} + \cdots)$$

*Mound Laboratory is operated by Monsanto Research Corporation for the Atomic Energy Commission under Contract No. AT-33-1-GEN-53.

shown in Table 1 is far from complete. If $n = 1$, a complete search would entail checking all integers less than 23 digits in length (more precisely, integers less than about 1.108×10^{21}). There are comparable, though larger, searches if $n > 1$. A WANG 700 Programmable Calculator took about five hours to find the list shown in Table 1.

The search for the second solution to the form

$$abc\cdots = a^a + b^b + c^c + \cdots$$

took about one hour on an IBM 360/50 Computer. The factorial and subfactorial forms were searched to check for the possibility of missed solutions. In less than 15 minutes on the IBM 360/50 Computer the solutions shown were confirmed to be the only ones.

A secondary search was made in isolated cases for recurring forms. For example:

$$\begin{aligned} 169: \quad 1! + 6! + 9! &= 36301 \\ 3! + 6! + 3! + 0! + 1! &= 1454 \\ 1! + 4! + 5! + 4! &= 169 \end{aligned}$$

or, briefly, digital factorial $169 \to 36301 \to 1454 \to 169$ (3 cycles). Similarly, digital factorial $871 \to 45361 \to 871$ (2 cycles),
$872 \to 45362 \to 872$ (2 cycles).

No other recurring forms for digital factorials were found, but the cycle search was limited to five or less. There are undoubtedly many others with a greater number of cycles.

A few recurring forms for the digital exponent form

$$(abc\cdots = a^a + b^b + c^c + \cdots)$$

were found by sheer trial and error on a WANG 700 Calculator. The initial integer in the following examples is the smallest member of the cycle series.

Digital exponent $288 \to 33554436 \to \cdots \to 140023 \to 288$ (58 cycles).

Digital exponent 3439 ⟶ 387420799 ⟶ ··· ⟶ 53423 ⟶ 3439 (52 cycles).
Digital exponent 50119 ⟶ 387423618 ⟶ ··· ⟶ 33601354 ⟶ 50119 (25 cycles).

Searching for interesting integers is obviously endless! I hope some readers will warm up their pencils, calculators, or computers and search further into the Table and report any new additions — including forms not shown here. (Notes and discoverers are shown on the following page.)

Table 1
NARCISSISTIC NUMBERS

Form	Solutions	Discoverer	Notes
$abc\cdots = a^n + b^{n+1} + c^{n+2} + \cdots$	$43 = 4^2 + 3^3$	3	
	$63 = 6^2 + 3^3$	3	
	$89 = 8^1 + 9^2$	8	
	$135 = 1^1 + 3^2 + 5^3$	2	
	$175 = 1^1 + 7^2 + 5^3$	3	
	$518 = 5^1 + 1^2 + 8^3$	3	
	$598 = 5^1 + 9^2 + 8^3$	2	
	$1306 = 1^1 + 3^2 + 0^3 + 6^4$	8	
	$1676 = 1^1 + 6^2 + 7^3 + 6^4$	8	B
	$2427 = 2^1 + 4^2 + 2^3 + 7^4$	8	
	$6714 = 6^3 + 7^4 + 1^5 + 4^6$	5	
	$47016 = 4^2 + 7^3 + 0^4 + 1^5 + 6^6$	5	
	$63760 = 6^3 + 3^4 + 7^5 + 6^6 + 0^7$	5	
	$63761 = 6^3 + 3^4 + 7^5 + 6^6 + 1^7$	5	
	$542186 = 5^2 + 4^3 + 2^4 + 1^5 + 8^6 + 6^7$	5	
$abc = a^n + b^{n-1} + c^{n-2} + \cdots$	$24 = 2^3 + 4^2$	7	
	$332 = 3^5 + 3^4 + 2^3$	7	
	$1676 = 1^5 + 6^4 + 7^3 + 6^2$	7	B
$abc = a^a + b^b + c^c + \cdots$	$3435 = 3^3 + 4^4 + 3^3 + 5^5$	8	
	$438579088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 0^0 + 8^8 + 8^8$	6	A
$abc = a! + b! + c! + \cdots$	$2 = 2!$		
	$145 = 1! + 4! + 5!$	8	
	$40585 = 4! + 0! + 5! + 8! + 5!$	4	
$abc = !a + !b + !c + \cdots$	$148349 = 1! + !4 + !8 + !3 + !4 + !9$	1	C

Notes for this table are found on the following page.

A. Since 0^0 is indeterminant, two assumed values were tested: $0^0 = 0$ and $0^0 = 1$. There are no solutions using $0^0 = 1$, so the solution (438579088) shown assumes $0^0 = 0$.

B. 1676 is most interesting: appearing in two places in this table!

C. !n is the subfactorial n and is given by the formula:

$$!n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\left(\frac{1}{n!}\right)\right]$$

so that $!0 = 0$, $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, and so on. The number shown in the Table is the only non-trivial solution for this form.

DISCOVERERS

1. Ron S. Dougherty, in a letter to the author dated April 28, 1965. published in Mathematics on Vacation by J. S. Madachy (Scribner's Sons, 1966), page 167.
2. Dale Kozniuk, included in "Curious Number Relationships," Recreational Mathematics Magazine, No. 10, August 1962, page 42.
3. J. A. H. Hunter, "Number Curiosities," Recreational Mathematics Magazine, No. 13, February 1963, page 28.
4. Leigh Janes, discovered in 1964 and published in Mathematics on Vacation (see [1] above) without proper credit, inadvertently.
5. Joseph S. Madachy, discovered 1970 on WANG 700 Programmable Calculator.
6. Joseph S. Madachy, discovered 1970 on IBM 360/50 Computer.
7. Joseph S. Madachy, discovered 1971 on Hewlett-Packard 9100B Programmable Calculator.
8. Unknown.

FIBONACCI NUMBERS AND WATER POLLUTION CONTROL

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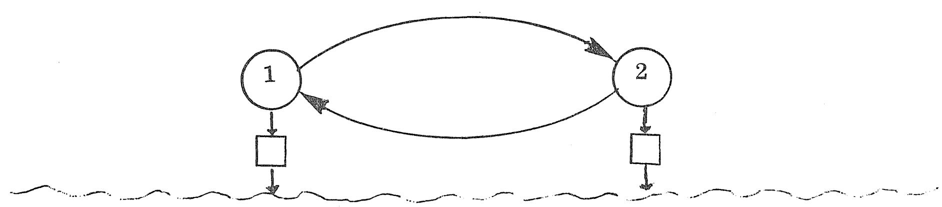
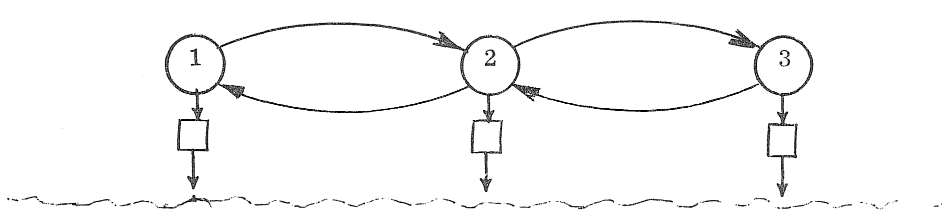
Consider a number of cities along a major water course which discharge presently their wastes untreated to the stream. To control the polution of the waters they have the obligation to build treatment plants. The major question is, where should one build these plants to minimize the cost of pollution control? Construction as well as operation costs of treatment plants exhibit large economies of scale, and therefore it is generally economically advantageous to build one or more central treatment plants. Given one possible location for a treatment plant for each city, and the possibility to transport the waste waters from any city to another one, the problem arises of how many possible solutions there are. Due to the economies of scale it is known that it would not be economical to "split" the waste flow of one city, that is, transport part of the waste upstream and part of it downstream.

Consider two cities only:



The number of possible solutions is $A(2) = 3$; namely, a treatment plant at each city, one treatment plant at city 1, and finally, one treatment plant at city 2.

Consider now 3 cities:



The interconnecting sewers between the cities are the only decision variables. Let a zero indicate no transport between cities, a 1 for upstream transport of wastes, and a 2 for downstream transport of wastes. For n cities there are (n - 1) connecting sewers between the cities, each of which may take on 3 values. So the total number of solutions would be 3^{n-1} were it not for the economic requirement that a city may not simultaneously transport wastes upstream <u>and</u> downstream. For three cities, the total number of solutions may be represented as follows:

00   01   02   10   11   12   20   21*   22

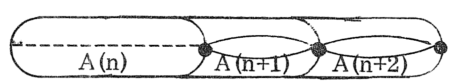
The 8^{th} solution indicated by an asterisk is ruled out, since we do not allow transport of waste water from city 2 simultaneously to 1 and 3. And thus the total number of economical solutions will be $A(3) = 8$.

Consider now n cities:

Let $A(n + 2)$ stand for the number of solutions for $n + 2$ cities, $A(n + 1)$ for the number of solutions to $n + 1$ cities, and $A(n)$ for the solutions for n cities.

Then, the following recursive relation can be established:

$$A(n + 2) = 3A(n + 1) - A(n) .$$



This relation may be deduced by the following reasoning. Given the value of $A(n + 1)$, the adding of one city increases the number of solutions to

$$3 \cdot A(n + 1)$$

since the new added sewer may assume the values of 0, 1, or 2. However, of this total number there are some which are not economical, namely, all those which end in a 2 1 sequence. But the number of those is exactly $A(n)$.

[Continued on page 302.]

A NUMBER GAME

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Preliminary condition for participation in the game: elementary knowledge of arithmetic.

1. First of all, give all participants of the game the same task, as follows:

Build up a sequence of numbers with nine terms in which the first and the second term may be any arbitrary cipher and their sum should build up the third term of the sequence. Every following term of the sequence is the sum of the two preceding terms (for instance; starting with the numbers of 3 and 4 we have the sequence 3, 4, 7, 11, 18, $\cdots$).

2. Now put the individual task for every participant of the game:

Divide the eighth or the ninth term of the sequence by the ninth or the eighth term, respectively (limit the result to two decimals behind the point!). Then multiply the received quotient by a small integer, for instance: by 2, 3, or 5, etc.

The final result of the computation can immediately be told to every participant of the game as soon as he has finished his computation. The participant is required only to state what ratio, i.e.,

$$\frac{\text{8th term}}{\text{9th term}} \text{ or } \frac{\text{9th term}}{\text{8th term}}$$

was used and by what integer it was multiplied. Since for any figures of the first two terms of the sequence the ratio

$$\frac{\text{8th term}}{\text{9th term}}$$

equals 0.62 (roundly) and

$$\frac{\text{9th term}}{\text{8th term}}$$

equals 1.62 (both quotients are an approximation to the "golden ratio"-value), it follows that the final result of the computation can easily be guessed. Thus for instance in the case

$$\frac{\text{8th term x 3}}{\text{9th term}}$$

the answer should be 0.62 x 3 = 1.86 and in the case

$$\frac{\text{9th term x 2}}{\text{8th term}}$$

the answer is 1.62 x 2 = 3.24.

If the properties of the recurrent sequences are unknown or too little known to the participants of the game, the guessing of the final results of their computations will have a startling effect.

[Continued from page 300.]

FIBONACCI NUMBERS AND WATER POLLUTION CONTROL

Upon generating the number of solutions for varying n the similarity of the series to the Fibonacci number series was noted.

n	1	2	3	4	5	6
A(n)	1	3	8	21	55	144

And thus we concluded that the total number of economical solutions for n cities is

$$A(n) = F_{2n},$$

where F_k stands for the k^{th} Fibonacci number. This still does not indicate which of the F_{2n} solutions is the most economical one, but places an upper bound on the total number of economical solutions to be investigated.

FIBONACCI NUMBERS AND GEOMETRY

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The Fibonacci relations we are going to develop represent a special case of algebra. If we are able to relate them to geometry we should take a quick look at the way algebra and geometry can be tied together.

One use of geometry is to serve as an illustration of an algebraic relation. Thus

$$(a + b)^2 = a^2 + 2ab + b^2$$

is exemplified by Figure 1.

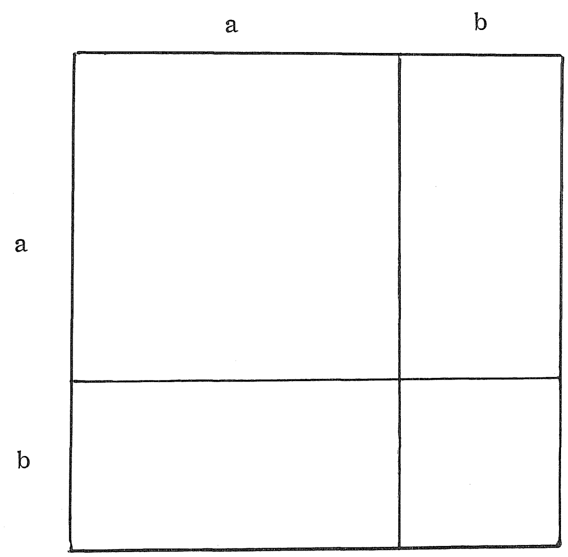


Figure 1

A second use of geometry is to provide a PROOF of an algebraic relation. As we ordinarily conceive the Pythagorean Theorem (though this was not the original thought of the Greeks) we tend to think of it as an algebraic relation on the sides of the triangle, namely,

$$c^2 = a^2 + b^2 .$$

One proof by geometry of this algebraic relation is shown in Figure 2.

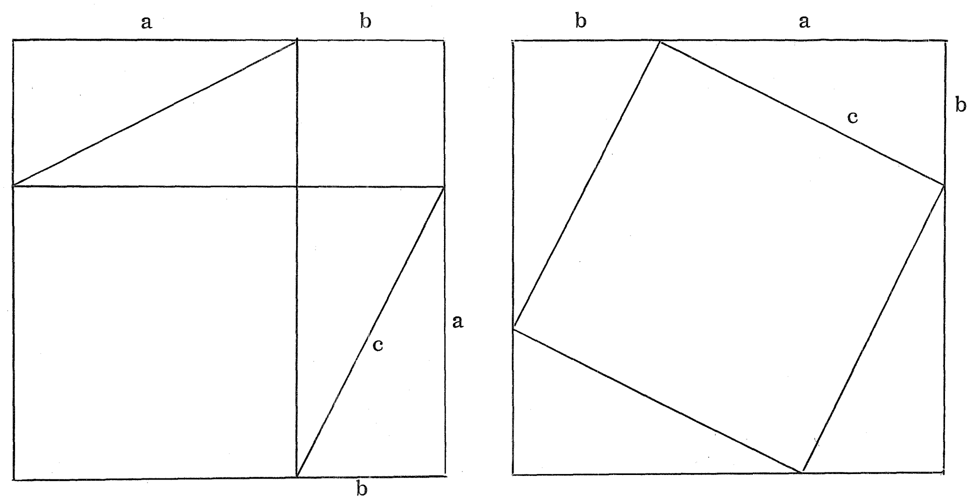


Figure 2

In summary, geometric figures may illustrate algebraic relations or they may serve as proofs of these relations. In our development, the main emphasis will be on proof though obviously illustration occurs simultaneously as well.

SUM OF FIBONACCI SQUARES

In the standard treatment of the Fibonacci sequence, geometry enters mainly at one point: summing the squares of the first n Fibonacci numbers. Algebraically, it can be shown by intuition and proved by induction that the sum of the squares of the first n Fibonacci numbers is

$$F_n F_{n+1} .$$

But there is a geometric pattern which ILLUSTRATES this fact beautifully as shown in Figure 3.

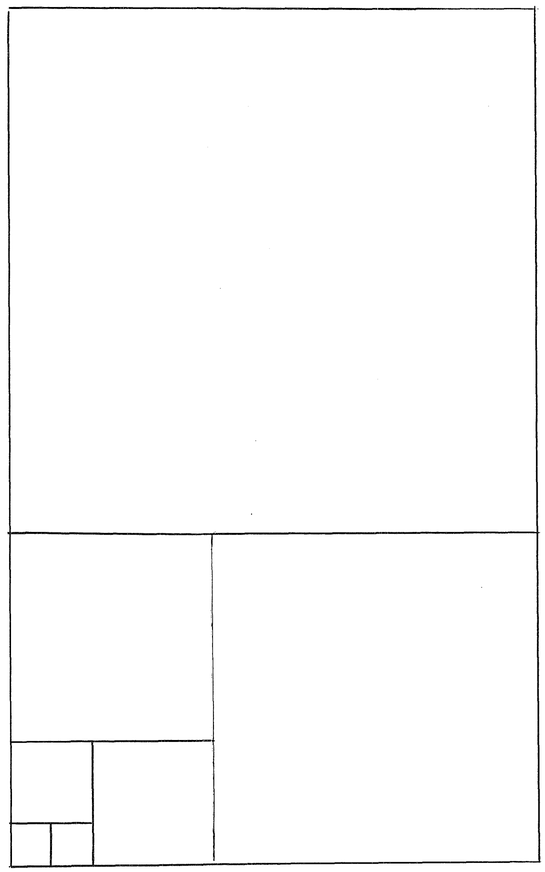
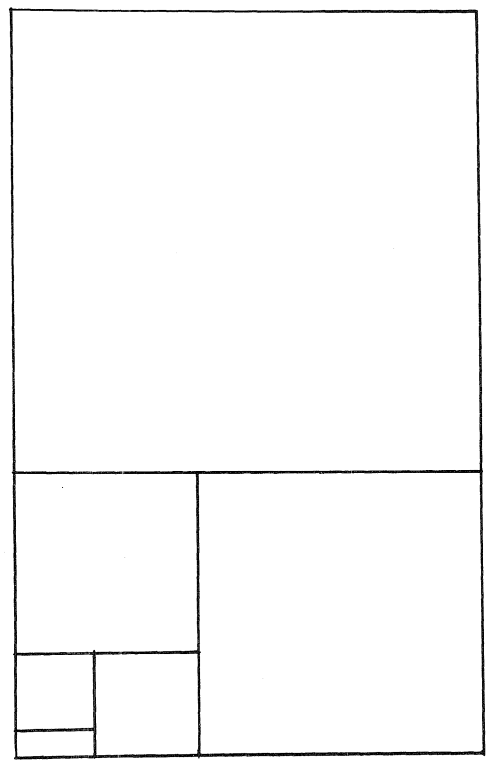


Figure 3

The figure is built up as follows. We put down two unit squares which are the squares of F_1 and F_2. Now we have a rectangle of dimensions 1 by 2. On top of this can be placed a square of side 2 (F_3) which gives a 2 by 3 rectangle. Then to the right can be set a square of side 3 (F_4) which produces a rectangle of sides 3 by 5. On top of this can be placed a square of side 5 (F_5) which gives a 5 by 8 (F_5F_6) rectangle, and so on.

This is where geometry begins and ends in the usual treatment of Fibonacci sequences. For if one tries to produce a similar pattern for the sum of the squares of any other Fibonacci sequence, there is an impasse. To meet this road block the following detour was conceived.

Suppose we are trying to find the sum of the squares of the first n Lucas numbers. Instead of starting with a square, we put down a rectangle whose sides are 1 and 3, the first and second Lucas numbers. (Figure 4 illustrates the general procedure.) Then on the side of length 3 it is possible to place



$$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1(T_2 - T_1)$$

Figure 4

a square of side 3: this gives a 3 by 4 rectangle. Against this can be set a square of side 4 thus producing a 4 by 7 rectangle. On this a square of side 7 is laid giving a 7 by 11 rectangle. Thus the same process that operated for the Fibonacci numbers is now operating for the Lucas numbers. The only difference is that we began with a 1 by 3 rectangle instead of a 1 by 1 square. Hence, if we subtract 2 from the sum we should have the sum of the squares of the first n Lucas numbers. The formula for this sum is thus:

(5) $$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 .$$

Using a direct geometric approach it has been possible to arrive at this algebraic formula with a minimum of effort. By way of comparison it may be noted that the intuitional algebraic route usually leads to difficulties for students.

Still more striking is the fact that by using the same type of procedure it is possible to determine the sum of the squares of the first n terms of ANY Fibonacci sequence. We start again by drawing a rectangle of sides T_1 and T_2 (see Fig. 4). On the side T_2 we place a square of side T_2 to give a rectangle of sides T_2 and T_3. Against the T_3 side we set a square of side T_3 to produce a rectangle of sides T_3 and T_4. The operation used in the Fibonacci and Lucas sequences is evidently working again in this general case, the sum being $T_n T_{n+1}$ if we end with the n^{th} term squared. But instead of having the squre of T_1 as the first term, we used instead T_1T_2. Thus it is necessary to subtract

$$T_1T_2 - T_1^2$$

from the sum to arrive at the sum of the squares of the first n terms of the sequence. The formula that results is:

(6) $$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1(T_2 - T_1) = T_n T_{n+1} - T_1 T_0 .$$

ILLUSTRATIVE FORMULAS

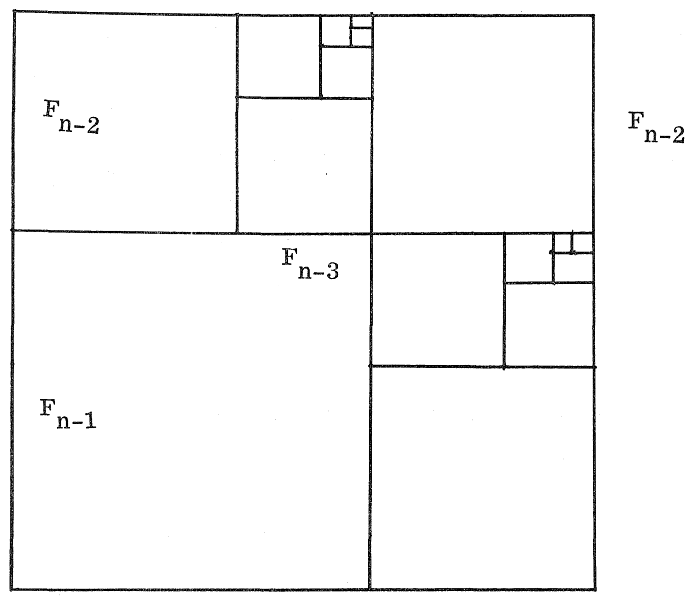
The design in Figure 1 for $(a + b)^2 = a^2 + 2ab + b^2$ can be used to illustrate Fibonacci relations that result from this algebraic identity. For example, Formulas (2), (3), and (4) could be employed for this purpose. Thus

$$L_n^2 = F_{n+1}^2 + 2F_{n+1}F_{n-1} + F_{n-1}^2 .$$

This evidently leads to nothing new but the algebraic relations can be exemplified in this way as special cases of a general algebraic relation which is depicted by geometry.

LARGE SQUARE IN ONE CORNER

We shall deal with a number of geometric patterns which can be employed in a variety of ways in many cases. In the first type we place in one corner of a given figure the largest possible Fibonacci (or Lucas) square that will fit into it. Take, for example, a square whose side is F_n. (See Fig. 5.)



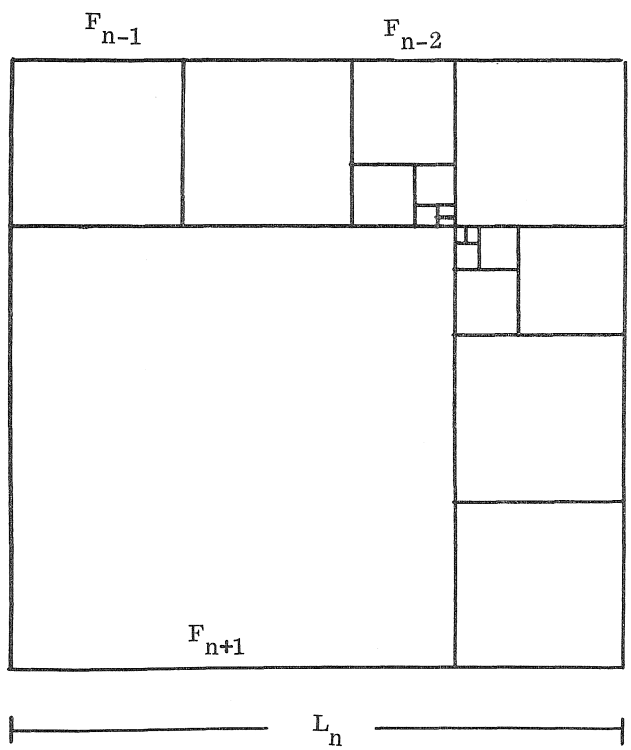
$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2$$

Figure 5

This being the sum of F_{n-1} and F_{n-2}, a square of side F_{n-1} can be put into one corner and its sides extended. In the opposite corner is a square of side F_{n-2}. From the two rectangles can be taken squares of side F_{n-2} leaving two smaller rectangles of dimensions F_{n-2} and F_{n-3}. But by what was found in the early part of this discussion, such a rectangle can be represented as the sum of the first $n - 3$ Fibonacci squares. We thus arrive at the formula:

$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2 \quad . \tag{7}$$

As a second example, take a square of side $L_n = F_{n+1} + F_{n-1}$. (See Fig. 6.)



$$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2$$

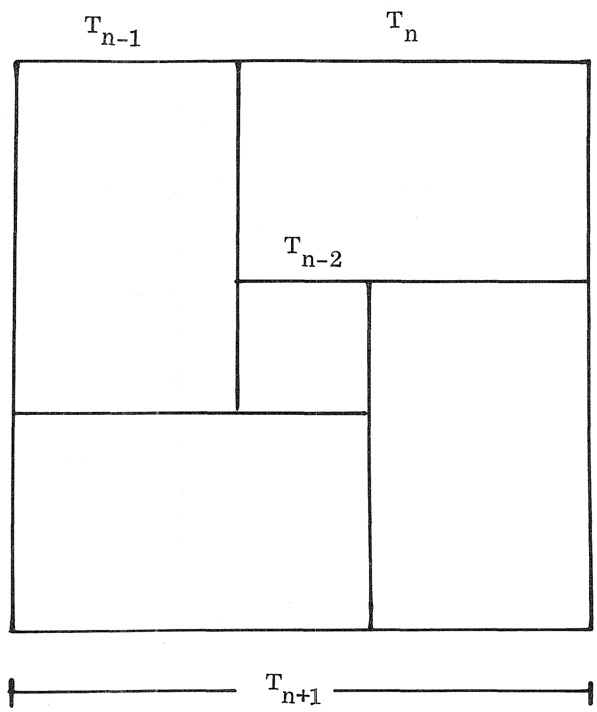
Figure 6

In one corner is a square of side F_{n+1} and in the opposite a square of side F_{n-1}. The rectangles have dimensions F_{n+1} and F_{n-1}. But F_{n+1} equals $2F_{n-1} + F_{n-2}$ by (3), so that each rectangle contains two squares of side F_{n-1} and a rectangle of sides F_{n-1} and F_{n-2}. Thus the following formula results:

$$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2 . \tag{8}$$

CYCLIC RECTANGLES

A second type of design leading to Fibonacci relations is one that may be called cyclic rectangles. Take a square of side T_{n+1}, a general Fibonacci number. Put in one corner a rectangle of sides T_n and T_{n-1} (Fig. 7). The



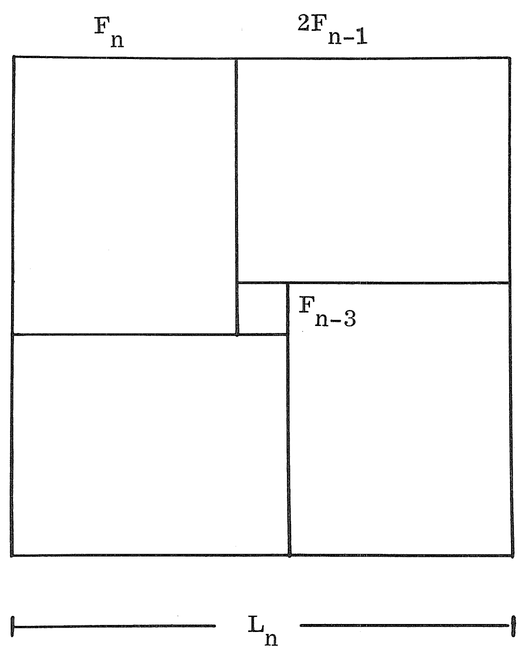
$$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2$$

Figure 7

process can be continued until there are four such rectangles in a sort of whorl with a square in the center. This square has side $T_n - T_{n-1}$ or T_{n-2}. Accordingly the general relation for all Fibonacci sequences results:

$$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2. \tag{9}$$

As another example of this type of configuration consider a square of side L_n and put in each corner a rectangle of dimensions $2F_{n-1}$ by F_n. (See Fig. 8.)



$$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2$$

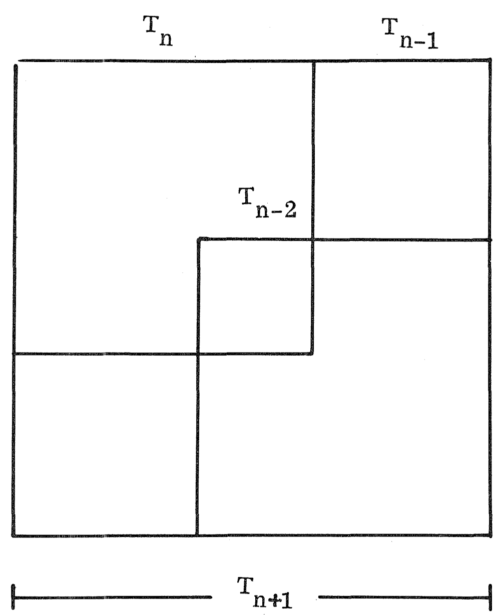
Figure 8

Again, there is a square in the center with side $2F_{n-1} - F_n$ or $F_{n-1} - F_{n-2} = F_{n-3}$. Hence:

$$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2\ . \tag{10}$$

OVERLAPPING SQUARES IN TWO OPPOSITE CORNERS

Construct a square whose side is T_{n+1} which equals $T_n + T_{n-1}$. In two opposite corners place squares of side T_n (Fig. 9). Since T_n is greater



$$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2$$

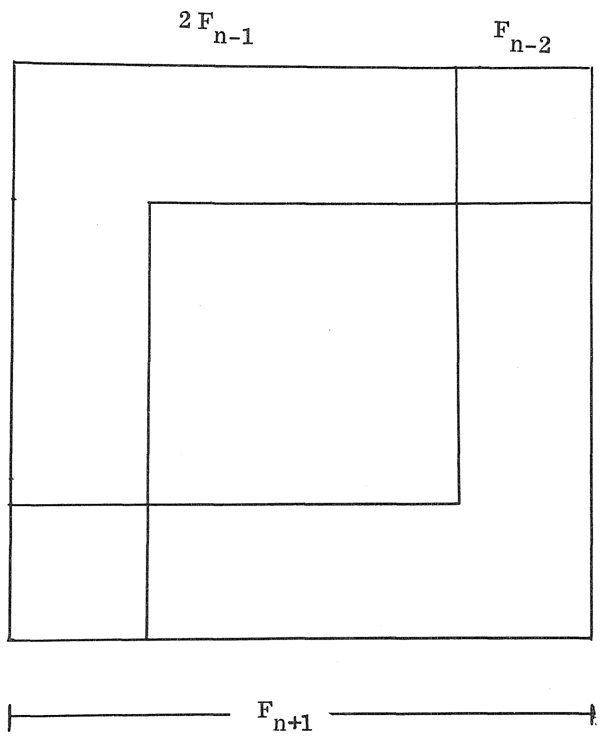
Figure 9

than half of T_{n+1} it follows that these squares must overlap in a square. The side of this square is $T_n - T_{n-1} = T_{n-2}$. The entire square is composed of two squares of side T_n and two squares of side T_{n-1}. But since the area of the central square of side T_{n-2} has been counted twice, it must be subtracted once to give the proper result. Thus:

$$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2 , \tag{11}$$

a result applying to all Fibonacci sequences.

Example 2. Take a square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$. In opposite corners, place squares of side $2F_{n-1}$. (See Fig. 10). Then the overlap square



$$F^2_{n+1} = 8F^2_{n-1} + 2F^2_{n-2} - L^2_{n-2}$$

Figure 10

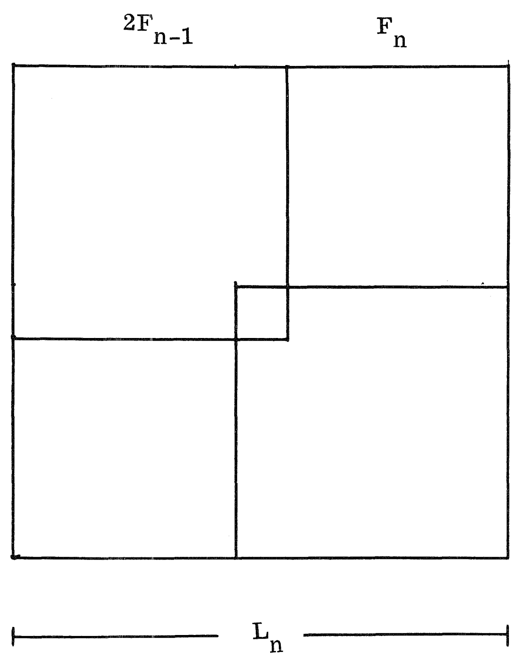
in the center has side $2F_{n-1} - F_{n-2} = F_{n-1} + F_{n-3} = L_{n-2}$. Thus:

(12)
$$F^2_{n+1} = 8F^2_{n-1} + 2F^2_{n-2} - L^2_{n-2} \quad .$$

Third example. A square of side $L_n = F_{n+1} + F_{n-1}$ has a central overlapping square of side $F_{n+1} - F_{n-1} = F_n$. Accordingly:

(13)
$$L^2_n = 2F^2_{n+1} + 2F^2_{n-1} - F^2_n \quad .$$

Final example. In a square of side $L_n = 2F_{n-1} + F_n$, place in two opposite corners squares of side $2F_{n-1}$. The overlap square in the center has side $2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. (See Fig. 11.)



$$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2$$

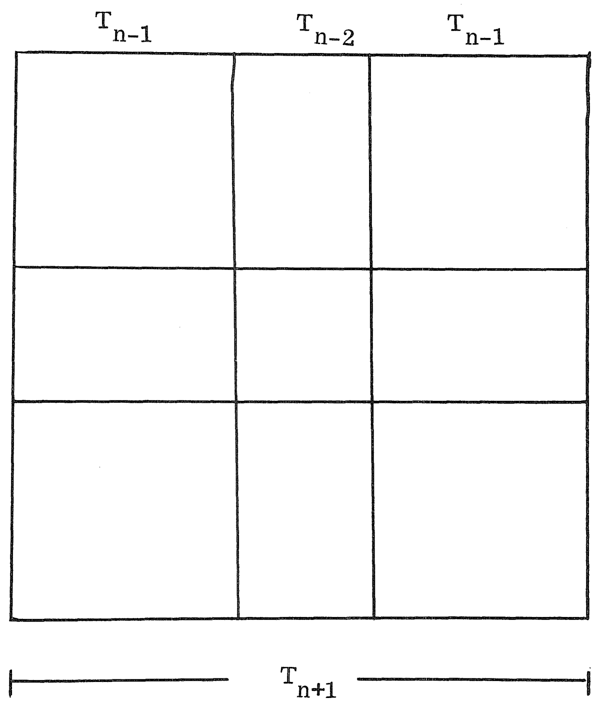
Figure 11

Hence:

$$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2 \; . \tag{14}$$

NON-OVERLAPPING SQUARES IN FOUR CORNERS

Consider the relation $T_{n+1} = 2T_{n-1} + T_{n-2}$. Each side of the square can be divided into segments T_{n-1}, T_{n-2}, T_{n-1} in that order (Fig. 12).



$$T_{n+1}^2 = 4T_{n-1}^2 + 4T_{n-1}T_{n-2} + T_{n-2}^2$$

Figure 12

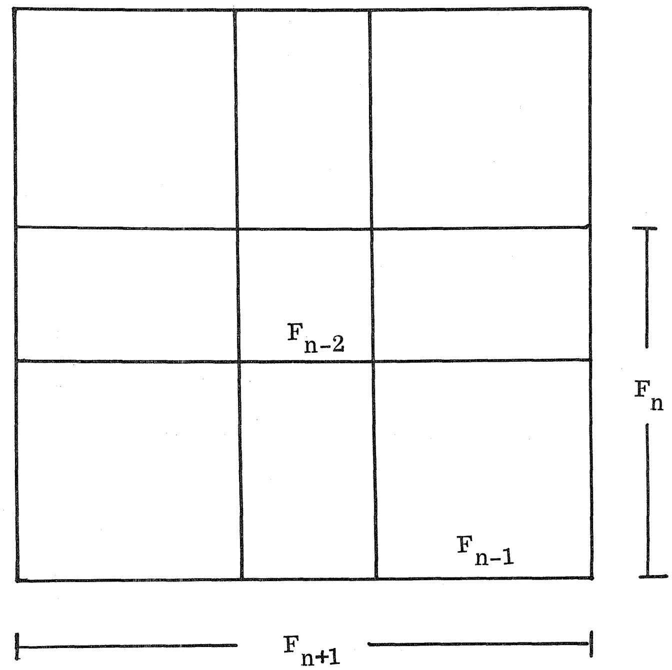
There are now four squares of side T_{n-1} in the corners, a square of side T_{n-2} in the center and four rectangles of dimensions T_{n-1} and T_{n-2}. From

(15) $$T_{n+1}^2 = 4T_{n-1}^2 + 4T_{n-1}T_{n-2} + T_{n-2}^2$$

which applies to ALL Fibonacci sequences.

OVERLAPPING SQUARES IN FOUR CORNERS

We start with $F_{n+1} = F_n + F_{n-1}$ and put four squares of side F_n in the corners (Fig. 13). Clearly there is a great deal of overlapping. The



$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$$

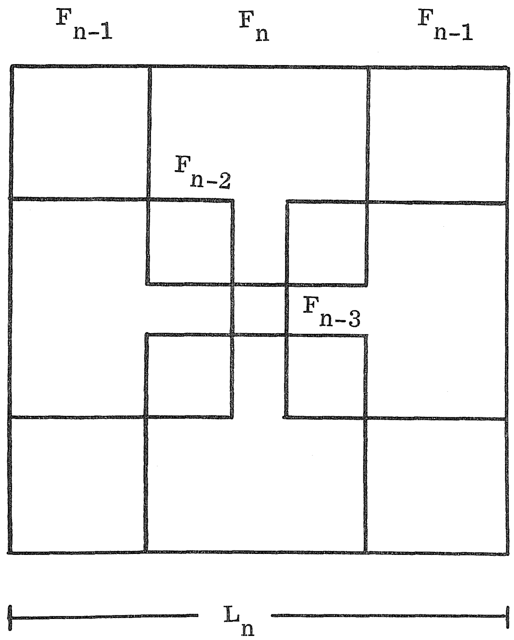
Figure 13

square at the center of side $F_n - F_{n-1} = F_{n-2}$ is covered four times; the four rectangles are each found in two of the corner squares so that this rectangle must be subtracted out four times. The central square being covered four times must be subtracted out three times. As a result the following formula is obtained:

$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2 \tag{16}$$

OVERLAPPING SQUARES PROJECTING FROM THE SIDES

We start with the relation $L_n = F_n + 2F_{n-1}$ and divide the side into segments F_{n-1}, F_n, F_{n-1} in that order (Fig. 14). On the F_n segments build squares which evidently overlap as shown. The overlap squares in the corners of these four squares have a side $F_n - F_{n-1} = F_{n-2}$ while the central



$$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2$$

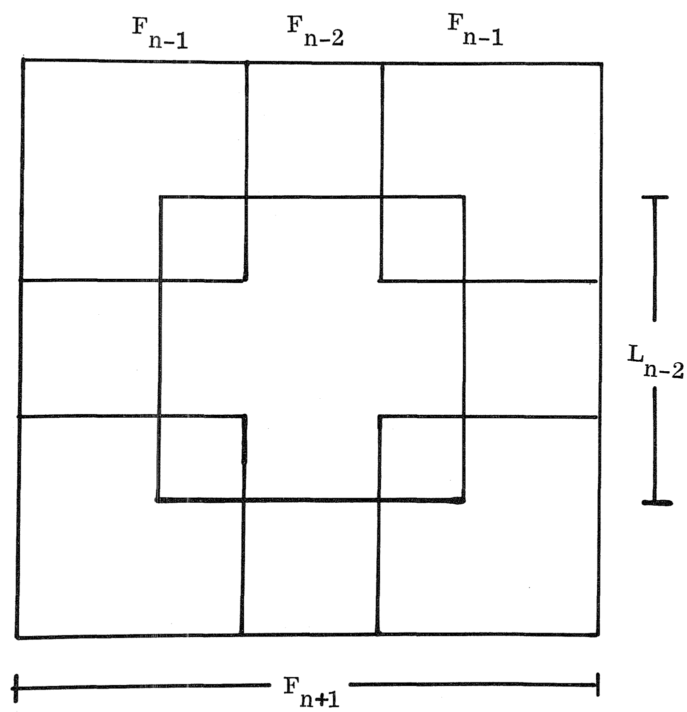
Figure 14

square has a side $L_n - 2F_n = F_{n+1} + F_{n-1} - 2F_n = 2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. Taking the overlapping areas into account gives the relation:

(17). $$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2 \, .$$

FOUR CORNER SQUARES AND A CENTRAL SQUARE

A square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$ has its sides divided into segments F_{n-1}, F_{n-2}, F_{n-1} in that order (Fig. 15). In each corner, a square of side F_{n-1} is constructed. Then a centrally located square of side L_{n-2} is constructed. It may be wondered where the idea for doing this came from. Since



$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-2}^2 + L_{n-2}^2 - 4F_{n-3}^2$$

Figure 15

$$L_{n-2} = F_{n-1} + F_{n-3} = F_{n-2} + 2F_{n-3},$$

it follows that such a square would project into the corner squares in the amount of F_{n-3}, thus giving three squares of this dimension. Taking overlap into account leads to the formula:

$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-2}^2 + L_{n-2}^2 - 4F_{n-3}^2 \tag{18}$$

CONCLUSION

In this all too brief session we have explored some of the relations of Fibonacci numbers and geometry. It is clear that there is a field for developing

[Continued on page 323.]

PROPORTIONS AND THE COMPOSER*

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Music is a combinatorial art. It is a combinatorial art operating in time.

Music is not, technically, a creative art in the sense that sculpture is. No resource such as a solid mass of tone exists from which a composer can carve out a musical composition in the way that a sculptor executes a statue from a block of stone. The piano keyboard, for instance, embraces 88 notes. Were these 88 notes struck all at once the result would be a sort of tonal "fence" consisting of 88 layers. Any or all of these notes can be utilized in whatever vertical or horizontal combinations meet a composer's specific artistic requirements. Thus, creativity in music is achieved through the ingenious combining of pre-established sounds within a limited spectrum of complex tonal effects.

Music differs from painting and sculpture in that it operates in time rather than in space. In this respect it is more closely allied to poetry. Poetry, likewise, is a combinatorial art. Its raw material is words instead of musical sounds. But a basic difference does exist. Words are encumbered by meaning which restricts their combinatorial sequences. A musical sound is by itself entirely devoid of meaning. From this point of view no combinatorial restrictive factor exists.

It is in the combinatorial structure of music that proportions become artistically pregnant.

If a composer is to be credited with as highly developed a sense of discipline as a painter or sculptor, it must be assumed that his initial concept towards a new composition is a pre-determined time span. Such a time span can be defined in terms of minutes, measures or notes. Should the projected composition be incidental music for a film the time span will have been established for him, leaving no available options. Essentially the same problem confronts a painter executing a mural within a given area. And the identical problem must be solved by the writer preparing a script for a radio or television drama with a fixed format.

Let us formulate a specific compositional problem; namely, a piano solo in a fairly quick tempo with a performance time of three minutes, twenty seconds. (So far nothing is being said about the content, mood, or style of the projected composition.)

First Decision: Tempo

If the tempo is determined by a metronome setting of 96 for a quarter-note, 96 quarter-notes per minute will make 320 quarter-notes in the composition. With four quarter-notes per measure, this tempo decision will result in a composition 90 measures long. Herewith is established a definite commitment as to the outside dimensions of the projected compositional exercise.

Second Decision: Principal Division of Form Resulting from First Decision

For the beginner in the use of proportions a time span embracing 80 one-measure units presents an extremely elementary problem: merely partition it into two sections of 30 and 50 measures, respectively, thereby achieving a simple 3:5 proportion. Likewise, it can be split into 50 and 30 measures, thereby reversing the proportion. Thus, two simple form plans become available for artistic exploitation:

30 measures + 50 measures

or

50 measures + 30 measures .

Both of these forms can, of course, operate at once. One might determine the harmonic plan and the other the shape of the melody. A variety of harmonic and melodic applications of a proportion and its retrograde operating concurrently will inevitably occur to an enterprising and inventive composer.

Third Decision: Subdivision of Overall Form Plan Resulting from the Second Decision

At this level, the opportunities for ingenuity in the formal utilization of proportions is greatly increased. Several summation series can function concurrently in sophisticated time exploitation. For example, let us consider the extremely simple 30 + 50 measure form division developed under the second Decision. The opening 30-measure section can be subdivided into

18 + 12 measures in the Fibonacci proportion of 3:2. On the other hand, the 50-measure section can be subdivided into 19 + 31 measures in the series.

$$2 : 5 : 7 : \underline{12} : \underline{19} : \underline{31} : 50 : 81, \text{ etc.}$$

Thus, we now have a form divided as follows:

30 measures	:	50 measures
18 + 12		19 + 31 .

Now the 12 measures of the opening 30-measure section relates in the same series with the 19- and 31-measure divisions of the closing 50-measure section. Thus, the form is evolving into more complex relationships:

30 measures	(3 : 5)	50 measures
(3 : 2)		
18 + 12		
12	+	19 + 31 .

In this light, the 12 measures at the end of the opening 30-measure section becomes a kind of "pivot" relating the two different summation series.

Fourth Decision: Treating the subdivisions resulting from the third Decision

Added formal sophistication can be achieved, and thereby greater diversity or complexity — whichever may be desired, by subdividing the sections shown above at the Third Decision into smaller units either within the series already in operation or by introducing a new series. For instance, the initial 18 measures of the opening 30-measure section of the fundamentally binary form lends itself to a 7 + 11 or 11 + 7 Lucas division, and since 7 is also operative in the series mentioned under the Third Decision, it would be quite easy to visualize how a systematically recurring 7-measure phrase could almost automatically become a characteristic feature in the design of the entire composition.

The above is an extremely elementary problem. And the formal solution is equally elementary. Any composer with a bit of imagination and structural ingenuity can think of many ways to divide and subsequently subdivide a time span of any given number of units.

* * * * * * * *

The second area in which proportions are useful is in the musical content of a given form as distributed in its various sections and subdivisions. Listed herewith are a few of the distribution possibilities:

(1) kinds of harmonies
(2) duration of harmonies
(3) dissonance effects
(4) rests and textures
(5) registers and ranges
(6) tonalities.

To illustrate, under the Third Decision there is evolved a 12-measure subdivision that serves as a "pivot" span that is common to two different summation series. If this is treated in terms of the series given under the Third Decision, it will be seen that it can be readily fragmented into 5 + 7. Some possibilities for exploiting this diminutive time span are

(1) 5 major triads + 7 minor triads, or vice versa
(2) 5 triads + 7 chords of the 7^{th}, or vice versa
(3) 5 measures containing two chords + 7 measures containing one chord, or vice versa
(4) 5 measures having no discords + 7 measures containing discords, or vice versa, etc.

This list of proportion possibilities can be extended as long as the composer has within his technique sufficient contrasting resources to originate additional complementary relationships.

But, the above listings do not imply merely a one-dimensional division. Suppose a composer decides on five major and seven minor triads, utilizing two contiguous numbers in the series quoted under the Third Decision. Now comes the problem of selecting the horizontal arrangement of the five major and seven minor triads within the pre-determined twelve measures. Since this choice can be made only from the available number of placements, the process is one of selection rather than creativity. Herewith comes into play an intriguing aspect of the combinatorial art: namely, the systematic choice of effect placements within a time span. The Chorale harmonizations of Bach demonstrate rare genius in this respect.

The third area for proportion utilization is in the vertical arrangement of chordal and dissonance effects. Since the compositional exercise under consideration is a piano piece, it is idiomatic to maintain a large number of notes in motion for effective performance results.

Suppose, then, that in the first section of the piece that three notes were assigned to the left hand and five to the right, which arrangement can be inverted for contrast. In the second section, greater activity may be devised for increased interest. This is obtainable by increasing the number of notes to, let us say, five and eight still in the Fibonacci series or to another series such as Lucas' four and seven. To heighten the organization further, the notes assigned to each hand could be proportionately divided between concords and discords.

These are mere clues to a kind of organizational thinking that is available to composers. It would, of course, be impractical to maintain such rigid internal organization throughout an entire composition, although there are movements in Bach where this does actually occur. It is more likely that such a plan would constitute a norm from which the composer can deviate. either systematically according to intentions or whimsically and freely. Above all, a systematic substructure must leave the composer unfettered and free. Any technique must be a help to the composer, never an obstacle to be conquered. Thus, it is quite possible that the proportion scheme from which a composition has its arising may never be definitely identified through the conventional academic processes.

[Continued from page 318.]

FIBONACCI NUMBERS AND GEOMETRY

geometrical ingenuity and thereby arriving simply and intuitively at algebraic relations involving Fibonacci numbers, Lucas numbers and general Fibonacci numbers. It appears that there is a considerable wealth of enrichment and discovery material in the general area of Fibonacci numbers as related to geometry.

Reports of other types of geometric designs that lead to the discovery of Fibonacci formulas would be welcome by the Editor of the Elementary Section of the Quarterly.

A NUMBER PROBLEM

M. S. KLAMKIN
Scientific Research Staff, Ford Motor Company, Dearborn, Michigan

In a recent note (this Journal, Vol. 4 (1971), p. 195), Wlodarski gives two solutions for the problem of determining the smallest number ending in 6 such that the number formed by moving the 6 to the front of the number is equivalent to multiplying the given number by 6. Here we give a more compact solution and answer.

If the given number is represented by

$$N = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + \cdots + a_{n-1} \cdot 10 + 6 ,$$

then

$$I = 6[10^{n+1} - 1]/59 .$$

By Fermat's theorem, $a^{p-1} \equiv 1 \pmod{p}$, and thus

$$I = 6[10^{58} - 1]/59 . \tag{1}$$

Since it can be shown that $10^{29} \equiv -1 \pmod{59}$, it follows that the number in (1) is the least one with the desired property.

There is no need to assume the number ends in 6. For if the number ended in 7, then

$$I = 7(10^{58} - 1)/59$$

would satisfy the deleted conditions but would be larger.

A similar example and solution for the case 6 is replaced by 9 had been given by the author previously.[1]

[1]M. S. Klamkin, "On the Teaching of Mathematics so as to be Useful," Educ. Studies in Math., Vol. 1 (1968), p. 140.

A LUCAS NUMBER COUNTING PROBLEM

BEVERLY ROSS*
San Francisco, California

Marshall Hall, Jr., [1] proposes the problem: Given

$$S_1, S_2, \cdots, S_n, \qquad S_i = (i, i + 1, i + 2)$$

(reduced mod 7, representing 0 as 7), show that there are 31 different sets, formed by choosing exactly one element from each original set and including each number from 1 to 7 exactly once.

The problem of how many new sets can be formed from this type of group of sets can be generalized in terms of the Fibonacci and Lucas numbers.

Given sets

$$S_1, S_2, \cdots, S_n, \qquad S_i = (i, i + 1, i + 2)$$

(reduced mod n, representing 0 as n), the number of new sets (for all $n \geq 4$) formed by choosing one element from each original set, including each number from 1 to n exactly once is $L_n + 2$, where L_n is the n^{th} Lucas number,

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}.$$

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

$$L_n = F_{n-1} + F_{n+1}.$$

One more Fibonacci identity is needed:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1}.$$

The number of sets shall be counted by arranging the sets in ascending order (base $n + 1$) to avoid missing any possible sets. A series in a group of sets

*Student at Lowell High School when this was written.

which begin with the same number and are not determined by the first two numbers. The base set of a series is the set with all elements after the first arranged in ascending order; e.g., $\{2, 3, 4, 5, 6, 7, 8, 9, 1\}$.

The set beginning with 1, 2, is obviously determined.

The first base set is $\{1, 3, 4, 5, \cdots, n, 2\}$.

The 2 at the end of the set can't be chosen from any other set, so the first change which can be made must be the interchange of the n and the n - 1. There can be no other sets between them because there are no numbers less than n - 1 in the last two original sets; e.g., 1, 3, 4, 5, 6, 7, 8, 2 is changed to: 1, 3, 4, 5, 6, 8, 7, 2.

The interchange of n - 1 and n - 2 would create one new set.

The interchange of n - 2 and n - 3 would create two new sets (isomorphic to the first two of the series, but with the n - 2 and n - 3 reversed).

For the set $\{2, 3, 4, 5, 6, 7, 8, 1\}$, the first 3 interchanges create the sets

$$
\begin{gathered}
\{2, 3, 4, 5, 6, 7, 8, 1\} \\
\{2, 3, 4, 5, 6, 7, 1, 8\} \\
\{2, 3, 4, 5, 6, 8, 7, 1\} \\
\{2, 3, 4, 5, 7, 6, 8, 1\} \\
\{2, 3, 4, 5, 7, 6, 1, 8\}
\end{gathered}
$$

(The last 2 sets are similar to the first 2 except for the interchange of 7 and 6.)

The new sets created by the 5^{th} interchange are:

$$
\begin{gathered}
\{2, 3, 5, 4, 6, 7, 8, 1\} \\
\{2, 3, 5, 4, 6, 7, 1, 8\} \\
\{2, 3, 5, 4, 6, 8, 7, 1\} \\
\{2, 3, 5, 4, 7, 6, 8, 1\} \\
\{2, 3, 5, 4, 7,,6, 1, 8\}
\end{gathered}
$$

The sets are similar to those created by the first 3 interchanges but with the 5 and 4 interchanged.

An interchange involves only two elements. All elements after those two are left unchanged. Therefore the i^{th} interchange creates as many new sets as all the interchanges before $i - 1$ did.

The $i - 1$ interchange creates as many new sets as all interchanges before $i - 2$. The number of new sets before the $i - 2$ interchange plus the number of sets created by the $i - 2$ interchange equals the number of sets created by the $i - 1$ interchange. Therefore the number of sets created by the i interchange is equal to the number of sets created by the $i - 1$ interchange plus the number of sets created by the $i - 2$ interchange.

There are $n - 3$ interchanges in the first series.

There are $n - 2$ interchanges in the second series because the position of the 1 is not determined.

The set beginning with 3, 4, is determined.

There are $n - 3$ interchanges in the third series because the interchange of 2 and 4 would produce a determined set.

The number of sets in the first series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1} .$$

The number of sets in the second series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-2} = F_n .$$

The number of sets in the third series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1} .$$

The number of determined sets is 2.

The sum is:

$$\begin{aligned} &F_{n-1} + F_n + F_{n-1} + 2 \\ &\quad = F_{n+1} + F_{n-1} + 2 \\ &\quad = L_n + 2 \qquad . \end{aligned}$$

REFERENCES

1. Marshall Hall, Combinatorial Theory, Blaisdell Publishing Company, Waltham, Mass., 1967 (Problem 1, p. 53).

CORRIGENDA
FOR
ON PARTLY ORDERED PARTITIONS OF A POSITIVE INTEGER
Appearing in the Fibonacci Quarterly, May, 1971

Lines 3 and 4 of Proof of Theorem 1, page 330, should read:

"Then each $V_j^!$ $(j \neq 1)$ which has the same components as $V_i^!$ (in a different order) will give the same partition of n as V_i after rearrangement, hence, $\cdots$."

The last line of page 330 should read: $(1 \leq r \leq n - 1)$.

The third line of the Proof of Theorem 3, page 331, should read:

$$U = (u_1, u_2, \cdots, u_r) \in [U] \quad .$$

On page 331, the second line of expression for $\phi_k(n)$ should read:

$$" = \left(1 + \sum_{r=1}^{n-2} \phi_{k-1}(r)\right) + \phi_{k-1}(n - 1) \quad ,$$
$$= \ldots \qquad "$$

In Table 1, page 332, values for ϕ_1 are:

$$\phi_1 \cdot 1, 2, 4, 7, 12, 19, 30, \cdots \quad .$$

The first two lines of the Proof for Lemma 1, page 333, should read:

"Proof.

$$\sum_{r=0}^{n-j-1} \binom{j - 3 + r}{r} = \cdots$$
$$= \binom{n - 3}{n - j - 1} - 1 + 1$$
$$= \cdots \qquad ."$$

C. C. Cadogan

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed postcards.

DEFINITIONS

$F_0 = 0,\ \ F_1 = 1,\ \ F_{n+2} = F_{n+1} + F_n;\ \ L_0 = 2,\ \ L_1 = 1,\ \ L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-232 Proposed by Guy A. R. Guillotte, Quebec, Canada.

In the following multiplication alphametic, the five letters, F, Q, I, N, and E represent distinct digits. The dashes denote not necessarily distinct digits. What are the digits of FINE FQ ?

```
  FQ
  FQ
 ----
  --
 ---
 ----
FINE
```

B-233 Proposed by Harlan L. Umansky, Emerson High School, Union City, N. J.

Show that the roots of

$$F_{n-1}x^2 - F_n x - F_{n+1} = 0$$

are $x = -1$ and $x = F_{n+1}/F_{n-1}$. Generalize to show a similar result for all sequences formed in the same manner as the Fibonacci sequence.

B-234 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California

Prove that

$$L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 .$$

B-235 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the largest positive integer n such that F_n is smaller than the sum of the cubes of the digits of F_n.

B-236 Proposed by Paul S. Bruckman, San Rafael, California.

Let P_n denote the probability that, in n throws of a coin, two consecutive heads will not appear. Prove that

$$P_n = 2^{-n}F_{n+2} .$$

B-237 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let (m,n) denote the greatest common divisor of the integers m and n.

(i) Given $(a,b) = 1$, prove that $(a^2 + b^2,\ a^2 + 2ab)$ is 1 or 5.

(ii) Prove the converse of Part (i).

APOLOGIES FOR SOME OMISSIONS

Following are some of the solvers whose names were inadvertently omitted from the lists of solvers of previous problems:

B-197 David Zeitlin

B-202 Herta T. Freitag, N. J. Kuenzi and Robert W. Prielipp

B-203 Herta T. Freitag and Robert W. Prielipp

B-206 Herta T. Freitag

B-207 Herta T. Freitag

SOLUTIONS

LUCKY 11 MODULO UNLUCKY 13

B-214 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let n be a random positive integer. What is the probability that L_n has a remainder of 11 on division by 13? [Hint: Look at the remainders for $n = 1, 2, 3, 4, 5, 6, \cdots$.]

Composite of solutions by Paul S. Bruckman, San Rafael, California, and Phil Mana, Albuquerque, New Mexico.

Let R_n be the remainder in the division of L_n by 13. Then

$$R_{n+2} \equiv R_{n+1} + R_n \pmod{13} .$$

Calculating the first 30 values of R_n, one finds that $R_{29} = 1 = R_1$ and $R_{30} = 3 = R_2$. It then follows from the recursion formula that $R_{n+28} = R_n$. The only n's with $R_n = 11$ and $1 \le n \le 28$ are $n = 5, 9$, and 14. Hence, in each cycle of 28 terms, the remainder 11 occurs exactly 3 times. Therefore, the required probability is 3/28.

Also solved by Debby Hesse and the Proposer.

QUOTIENT OF POLYNOMIALS

B-215 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that for all positive integers n the quadratic $q(x) = x^2 - x - 1$ is an exact divisor of the polynomial

$$P_n(x) = x^{2n} - L_n x^n + (-1)^n$$

and establish the nature of $p_n(x)/q(x)$. [Hint: Evaluate $p_n(x)/q(x)$ for $n = 1, 2, 3, 4, 5$.]

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let α, β denote the roots of $x^2 - x - 1$. Since $L_n = \alpha^n + \beta^n$, it is clear that

$$Q_n(x) = \frac{x^{2n} - L_n x^n + (-1)}{x^2 - x - 1} = \frac{(x^n - \alpha^n)(x^n - \beta^n)}{(x - \alpha)(x - \beta)}$$

is a polynomial.

To find the coefficients of $Q_n(x)$ we put

$$Q_n(x) = \sum_{r=0}^{n-1} \alpha^r x^{n-r-1} \sum_{s=0}^{n-1} \beta^s x^{n-s-1}$$

$$= \sum_{k=0}^{2n-2} \left(x^{2n-k-2} \sum_{\substack{r+s=k \\ r<n,\, s<n}} \alpha^r \beta^s \right) = \sum_{k=0}^{2n-2} x^{2n-k-2} c_k ,$$

say. Then for $k \leq n - 1$,

$$c_k = \sum_{r+s=k} \alpha^r \beta^s = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} = F_{k+1} .$$

For $k \geq n$, we have

$$c_k = \sum_{r=k-n+1}^{n-1} \alpha^r \beta^{k-r} = (\alpha\beta)^{k-n+1} \sum_{j=0}^{2n-k-2} \alpha^j \beta^{2n-k-j-2}$$

$$= (-1)^{k-n+1} F_{2n-k-1} .$$

Also solved by Paul S. Bruckman, Ralph Garfield, G. A. R. Guillotte, Herta T. Freitag, David Zeitlin, and the Proposer.

A NONHOMOGENEOUS RECURSION

B-216 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Solve the recurrence $D_{n+1} = D_n + L_{2n} - 1$ for D_n, subject to the initial condition $D_1 = 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Since $D_0 = 0$ and

$$\sum_{k=0}^{n-1} L_{2k} = 1 + L_{2n-1} ,$$

we have, with n replaced by k in the recurrence,

$$D_n = \sum_{k=0}^{n-1} (D_{k+1} - D_k) = \sum_{k=0}^{n-1} (-1) + \sum_{k=0}^{n-1} L_{2k}$$

$$= -n + 1 + L_{2n-1} .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, G. A. R. Guillotte, and the Proposer.

MODIFIED PASCAL TRIANGLE

B-217 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array of numbers $A(n,k)$ $(n = 0, 1, 2, \cdots, 0 \le k \le n)$ is defined by the recurrence

$$A(n+1,k) = A(n, k-1) + (n+k+1)A(n,k) \qquad (1 \le k \le n)$$

together with the boundary conditions

$$A(n,0) = n! , \quad A(n,n) = 1 .$$

Find an explicit formula for $A(n,k)$.

Solution by Paul S. Bruckman.

Let $A(n,k) = (n!/k!)B(n,k)$. Substituting this expression in the given recursion, we obtain

$$[(n+1)!/k!]B(n+1,k) = [n!/(k-1)!]B(n,k-1) + [(n+k+1)n!/k!]B(n,k) .$$

Multiplying throughout by $k!/n!$ gives us

$$(n+1)B(n+1,k) = kB(n,k-1) + (n+k+1)B(n,k)$$

or

(1) $$(n+1)[B(n+1,k) - B(n,k)] = k[B(n,k-1) + B(n,k)] .$$

Next we demonstrate that recursion (1) is satisfied by

$$B(n,k) = \binom{n}{k} .$$

Since

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \binom{m-1}{r-1} = r\binom{m}{r},$$

$$(n+1)\left[\binom{n+1}{k} - \binom{n}{k}\right] = (n+1)\binom{n}{k-1} = k\binom{n+1}{k}.$$

Also,

$$k\left[\binom{n}{k-1} + \binom{n}{k}\right] = k\binom{n+1}{k} .$$

Therefore,

$$B(n,k) = \binom{n}{k}$$

satisfies (1). The boundary conditions for this $B(n,k)$ are $B(n,0) = 1 = B(n,n)$, which lead to the desired boundary conditions for

$$A(n,k) = (n!/k!)B(n,k) = \frac{n!}{k!}\binom{n}{k} = (n-k)!\binom{n}{k}^2 = \frac{(n!)^2}{(n-k)!(k!)^2} .$$

Also solved by David Zeitlin and the Proposer.

ARCTAN OF A SUM EQUALS SUM OF ARCTANS

B-218 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and show that

$$\text{Arctan} \sum_{n=1}^{\infty} [1/(aF_{n+1} + F_n)] = \sum_{n=1}^{\infty} \text{Arctan}\,(1/F_{2n+1}) .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since $aF_{n+1} + F_n = a^{n+1}$ and

$$\sum_{n=1}^{\infty} \frac{1}{a^{n+1}} = \frac{1}{a(a-1)} = 1 ,$$

the stated result may be replaced by

$$(*) \qquad \frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan \frac{1}{F_{2n+1}} .$$

Now

$$\arctan \frac{1}{F_{2n}} - \arctan \frac{1}{F_{2n+1}} = \arctan \frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+1}}}{1 + \frac{1}{F_{2n}F_{2n+1}}}$$

$$= \arctan \frac{F_{2n-1}}{F_{2n}F_{2n+1} + 1} = \arctan \frac{1}{F_{2n+2}} ,$$

using the well-known identity $F_{2n-1}F_{2n+2} - F_{2n}F_{2n+1} = 1$. Hence

$$\arctan \frac{1}{F_{2n+1}} = \arctan \frac{1}{F_{2n}} - \arctan \frac{1}{F_{2n+2}} .$$

Take $n = k, \; k+1, \; k+2, \cdots$ and add the resulting equations. We get

$$\sum_{n=k}^{\infty} \arctan \frac{1}{F_{2n+1}} = \arctan \frac{1}{F_{2k}} .$$

In particular, for $k = 1$, this reduces to (*).

Also solved by Paul S. Bruckman, David Zeitlin, and the Proposer.

HILBERT MATRIX

B-219 Proposed by Tomas Djerverson, Albrook College, Tigertown, New Mexico.

Let k be a fixed positive integer and let $a_0, a_1, \cdots, a_k$ be fixed real numbers such that, for all positive integers n.

$$\frac{a_0}{n} + \frac{a_1}{n+1} + \cdots + \frac{a_k}{n+k} = 0 .$$

Prove that $a_0 = a_1 = \cdots = a_k = 0$.

Solution by David Zeitlin, Minneapolis, Minnesota.

For $n = 1, 2, \cdots, k+1$, we have a homogeneous system of $(k+1)$ linear equations in the $k+1$ unknowns: $a_0, a_1, \cdots, a_k$. The coefficient matrix is the well-known Hilbert matrix, which is non-singular. Thus, the determinant of the system is non-zero; and so, by Cramer's rule, $a_0 = a_1 = a_2 = \cdots = a_k = 0$.

Also solved by Paul S. Bruckman and the Proposer.