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A GENERALIZED FIBONACCI SEQUENCE

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Since the year 1202 when Leonardo Pisano originated the Fibonacci sequence, many interesting results have been obtained [4]. The sequence is usually defined

$$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 .$$

and it is a well-known fact that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} .$$

In this paper we generalize the usual definition of the Fibonacci numbers and the matrix relation, and exhibit some of the many relationships which hold for elements of the generalized sequence.

Consider the following definitions for a generalized sequence.

Definition 1. The k^{th} order Fibonacci sequence is a sequence which satisfies the following conditions:

a. $F_0 = F_1 = \cdots = F_{k-1} = 0$ and $F_{-i} = 0$ for all $i \geq 1$

b. $F_{k-1} = 1$

c. $F_n = \sum_{i=1}^{k} F_{n-i}$ for $n \geq k$.

If we relax the condition as specified in part a of Definition 1, we obtain the following:

Definition 2. A sequence whose members (denoted $\overline{F}_i$) satisfy the following two conditions will be called a generalized k^{th} order Fibonacci sequence.

a. $\overline{F}_i = a_i$ for $0 \leq i \leq k - 1$ where a_i is an arbitrary number,

b. $F_n = \sum_{i=1}^{k} \overline{F}_{n-1}$.

We now define a sequence called the r^{th} auxiliary sequence of order k as a special case of the generalized k^{th} order sequence.

Definition 3. A sequence which satisfies the following three conditions will be called an r^{th} auxiliary sequence of order k, where $1 \le r \le k-2$.

a. $A_i^r = 0$ for $0 \le i \le k-2, \quad i \ne r-1$

b. $A_{r-1}^r = A_{k-1}^r = 1$

c. $A_n^r = \sum_{i=1}^{k} A_{n-i}^r$ for $n \ge k$.

In the following, the superscript of F_i^k will be left off if it is clear from the context that we are concerned with the k^{th} order sequence.

Property 1. $F_j^k = F_{j-1}^{k-1}$ for $1 \le j \le 2(k-1) \quad k > 2$

Property 2. $F_{2k-1}^k = F_{2(k-1)}^{k-1} + 1$ for all $k > 1$

Property 3. $F_{2k}^k = 2^k - 1$ for all $k \ge 2$.

Theorem 1. If A_n^r is an element in the r^{th} auxiliary sequence of order k, and if F_i is an element of the corresponding k^{th} order Fibonacci sequence, then

$$A_n^r = F_n + F_{n-1} + \cdots + F_{n-r+1} + F_{n-r} \quad \text{for } n \ge k.$$

Proof. Let $n = k$, then we will show that

$$A_k^r = F_k + F_{k-1} + \cdots + F_{k-r} . \tag{1}$$

By Definition 1,

$$\sum_{i=0}^{r} F_{k-i} = 2 , \tag{2}$$

and by Definition 2,

$$A_k^r = \sum_{j=1}^{k} A_{k-j}^r \quad . \tag{3}$$

But since r is defined in the range $0 \le r \le k-2$ then there is an element A_{k-j}^r in (3) such that $k - j = r - 1$. From the definition of A_i^r we know

$$A_{r-1}^r = A_{k-1}^r = 1 ,$$

and all the remaining elements are zero. Therefore

$$\sum_{j=1}^{k} A_{j-j}^{r} = 2$$

and from (2) we have the desired result for $n = k$.

Suppose that for $k \le n \le m$ the theorem is true, then for $n = m + 1$, we will also show the theorem is true.

$$A_{m+1}^{r} = \sum_{j=1}^{k} A_{m+1-j}^{r} = A_{m}^{r} + A_{m-1}^{r} + \cdots + A_{m+1-k}^{r} \tag{4}$$

By hypothesis we can rewrite each element of (4) as follows:

$$\begin{aligned}
A_{m}^{r} &= F_{m} + F_{m-1} + \cdots + F_{m-r} \\
A_{m-1}^{r} &= F_{m-1} + F_{m-2} + \cdots + F_{m-r-1} \\
&\vdots \\
A_{m+1-k}^{r} &= F_{m+1-k} + F_{m-k} + \cdots + F_{m+1-k-r}
\end{aligned}$$

and adding the columns we obtain

$$\begin{aligned}
\sum_{i=1}^{k} A_{m-i+1}^{r} &= \sum_{j=1}^{k} F_{m-j+1} + \sum_{j=1}^{k} F_{m-j} + \cdots + \sum_{j=1}^{k} F_{m-j-r+1} \\
&= F_{m+1} + F_{m} + \cdots + F_{m-r+1} \\
&= A_{m+1}^{r}
\end{aligned}$$

which is the desired result.

Lemma 1. The following three identities hold for elements of the auxiliary sequences for $m \ge k$ and $1 \le r \le k - 2$.

A. $$A_{m}^{r} - A_{m-1}^{r-1} = F_{m}$$

B. $$A_{m}^{r} - A_{m}^{r-1} = F_{m-r}$$

C. $$A_{m}^{r} - A_{m-1}^{r} = -F_{m-r-1} + F_{m}$$

Theorem 2. If Q is the $k \times k$ matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
 & \vdots & & & & & \vdots & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{bmatrix},$$

then

$$Q^n = \begin{bmatrix} F_{n-1} & A^1_{n-1} & \cdots & A^r_{n-1} & \cdots & A^{k-2}_{n-1} & F_n \\ F_n & A^1_n & \cdots & A^r_n & \cdots & A^{k-2}_n & F_{n+1} \\ F_{n+1} & A^1_{n+1} & \cdots & A^r_{n+1} & \cdots & A^{k-2}_{n+1} & F_{n+2} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ F_{n+k-4} & A^1_{n+k-4} & \cdots & A^r_{n+k-4} & \cdots & A^{k-2}_{n+k-4} & F_{n+k-3} \\ F_{n+k-3} & A^1_{n+k-3} & \cdots & A^r_{n+k-3} & \cdots & A^{k-2}_{n+k-3} & F_{n+k-2} \\ F_{n+k-2} & A^1_{n+k-2} & \cdots & A^r_{n+k-2} & \cdots & A^{k-2}_{n+k-2} & F_{n+k-1} \end{bmatrix}$$

where n is a positive integer, and the F_n's are elements of the k^{th} order sequence and the A^r_{n+1} are the corresponding terms of the r^{th} auxiliary sequence of that same order.

Proof. This theorem can be proved by induction on n. With $n = 2$, Q^2 is

$$Q^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & A^1_1 & \cdots & A^{k-2} & F_2 \\ F_2 & A^1_2 & \cdots & A^{k-2}_2 & F_3 \\ \vdots & & & & \\ F_{k-2} & A^1_{k-2} & \cdots & A^{k-2}_{k-2} & F_{k-1} \\ F_{k-1} & A^1_{k-1} & \cdots & A^{k-2}_{k-1} & F_k \\ F_k & A^1_k & \cdots & A^{k-2}_k & F_{k+1} \end{bmatrix}$$

Supposing the theorem is true for $1 \leq n \leq m$ we can show it is true for $n = m + 1$.

$$Q^{m+1} = Q \cdot Q^m .$$

But examining the effect of multiplying Q with Q^m, it is obvious that the first $k - 1$ rows of Q cause row i $(2 \leq i \leq k)$ to become row $i - 1$ of Q^{m+1}. The k^{th} row of Q^{m+1} is obtained by summing the columns of Q^m, which using definitions 1 and 3 produces the desired result.

Theorem 3. If $n = k$, then

$$\sum_{i=1}^{n} F_i = -F_{n+k+1} - \frac{1}{k-1} + \sum_{i=1}^{k} iF_{n+i}$$

or

$$\sum_{i=1}^{n} F_i = -\frac{1}{k-1} + \sum_{i=1}^{k} (i-1)F_{n+i} .$$

Proof. The sum of the first n terms of the k^{th} order sequence will appear as an element in a matrix which represents the sum

$$\sum_{i=1}^{n} Q^i \tag{1}$$

in either the $(1,k)$ or the $(2,1)$ position. We rewrite (1) to obtain

$$\sum_{i=1}^{n} Q^i = (Q - I)^{-1}Q(Q^n - I) \tag{2}$$

where I is the $(k \times k)$ identity matrix. The inverse of $Q - I$ shown in (2) can be shown to be

$$\frac{1}{k-1}\begin{bmatrix} -(k-2) & -(k-3) & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & -(k-3) & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & 2 & \cdots & -(k-k) & -(k-(k+1)) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & 2 & \cdots & k-1 & -(k-(k+1)) \end{bmatrix} \tag{3}$$

If we multiply the first row of (3) against the last column of $Q^{n+1} - Q$ we obtain the element in the $(1,k)$ position which represents the desired sum

$$\sum_{i=1}^{n} F_i \; .$$

$$\begin{aligned} \sum_{i=1}^{n} F_i &= \frac{1}{k-1}\sum_{i=1}^{k} -(k - i - 1)(F_{n+i} - F_i) \\ &= \frac{1}{k-1}\left[-(k-1)(F_{n+k+1} - F_{k+1}) + \sum_{i=1}^{k} i(F_{n+i} - F_i)\right] \\ &= F_{n+k+1} + 2 + \frac{1}{k-1}\sum_{i=1}^{k} i\,F_{n+1} - \frac{k}{k-1}\sum_{i=1}^{k} i\,F_i \\ &= -F_{n+k+1} - \frac{1}{k-1} + \sum_{i=1}^{k} i\,F_{n+i} \end{aligned} \tag{4}$$

which is the desired result. The second form can be obtained by substituting in (4) the sum

$$\sum_{i=1}^{k} F_{n+i}$$

for F_{n-k+1}, and combining the two sums.

Using the method given in the proof of Theorem 3, it is obvious that an expression similar to that given in this theorem can be given for the sum

$$\sum_{i=1}^{n} F_{i+j-1} \quad \text{for} \quad 1 \le j - 1 < k \quad \text{and} \quad n \ge k$$

which is

$$\sum_{i=1}^{n} F_{i+j-1} = \frac{1}{k-1}\sum_{i=1}^{k} i\,F_{n+i} + \frac{(zk-1)(k-2)}{k-1} - \sum_{i=j}^{k} F_{n+i}$$

Theorem 4. If $0 \le j \le k - 1$ and if $n \ge k$ then

$$\sum_{i=1}^{n} F_{ki+j} = \frac{1}{k-1}\sum_{i=1}^{k} i\,F_{nk+k-1} - \sum_{i=j+1}^{k} F_{nk+i-1} - \frac{1}{k-1}\,.$$

Proof. Consider the sum

$$\sum_{i=1}^{n} Q^{ki} \tag{1}$$

We can obtain the desired sum

$$\sum_{i=1}^{n} F_{ki+j}$$

in the (k, j) position of the matrix representing the sum given in (1).

$$\begin{aligned} \sum_{i=1}^{n} Q^{ki} &= Q^k\,[I + Q^k + \cdots + Q^{(n-1)k}] \\ &= (Q^k - I)^{-1} Q^k (Q^{nk} - 1) \end{aligned} \tag{2}$$

The characteristic equation of Q is

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1$$

and since Q^{k-1} always has a factor $Q - I$, we can write

$$Q^{k-1} = (Q - I)(Q^{k-1} + Q^{k-2} + \cdots + Q + I) .$$

However

$$Q^{k-1} + Q^{k-2} + \cdots + Q + I = Q^k$$

therefore

$$Q^k - 1 = Q^k(Q - I)$$

and thus

$$(Q^k - I)^{-1}Q^k(Q^{nk} - I) = (Q - I)^{-1}(Q^{nk} - I) \tag{3}$$

and upon multiplying the last column of $Q^{nk} - I$ with the j^{th} row of $(Q - I)^{-1}$ we obtain the desired result.

Theorem 5.

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+k-1} + \sum_{i=1}^{k-2} A^i_{m-1} F_{n+i} ,$$

where m and n are nonnegative integers.

Note. If $k = 2$ then $F_{m+n} = F_{m-1}F_n + F_{n+1}F_m$ which is a well-known result of the usual Fibonacci sequence.

Proof. F_{m+n} occurs in the matrix Q^{m+n} in the $(2,1)$ or $(1,k)$ positions. Since

$$Q^{m+n} = Q^m Q^n ,$$

the required multiplication can be performed to yield the desired result.

Theorem 6. If B is the matrix

$$\begin{bmatrix} \overline{F}_0 & A_0^1 & \cdots & A_0^r & \cdots & A_0^{k-2} & \overline{F}_1 \\ \overline{F}_1 & A_1^1 & \cdots & A_1^r & \cdots & A_1^{k-2} & \overline{F}_2 \\ \overline{F}_2 & A_2^1 & \cdots & A_2^r & \cdots & A_2^{k-2} & \overline{F}_3 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \overline{F}_{k-2} & A_{k-2}^1 & \cdots & A_{k-2}^r & \cdots & A_{k-2}^{k-2} & \overline{F}_{k-1} \\ \overline{F}_{k-1} & A_{k-1}^1 & \cdots & A_{k-1}^r & \cdots & A_{k-2}^{k-2} & \overline{F}_k \end{bmatrix}$$

where the A_i^r, $(1 \le r \le k - 2)$ are the elements of the r^{th} auxiliary sequence of order k, and if Q^{-n} is the matrix

$$\begin{bmatrix} \overline{F}_{n-1} & \cdots & A^r_{n-1} & \cdots & \overline{F}_n \\ \overline{F}_n & \cdots & A^r_n & \cdots & \overline{F}_{n+1} \\ \vdots & & \vdots & & \vdots \\ \overline{F}_{n+k-3} & \cdots & A^r_{n+k-3} & \cdots & \overline{F}_{n+k-2} \\ \overline{F}_{n+k-2} & \cdots & A^r_{n+k-2} & \cdots & \overline{F}_{n+k-1} \end{bmatrix}$$

then $Q^{n-1}B = Q^{-n}$ where Q is the matrix defined in Theorem 2.

Proof. The proof is again done by induction upon n and is similar to that given in Theorem 2.

Using the results of Theorem 6 for the generalized Fibonacci sequence, it is possible to obtain theorems for this sequence corresponding to Theorems 3, 4, 5. These corresponding theorems are stated here without proof.

Theorem 7.

$$\sum_{i=1}^{n} \overline{F}_i = -\overline{F}_{n+k+1} + \overline{F}_{k+1} + \frac{1}{k-1}\sum_{i=1}^{k} i(\overline{F}_{n+1} - \overline{F}_i) \quad \text{for } n \ge k.$$

Theorem 8. If $1 \le j \le k - 1$, and if $n \ge k$, then

$$\sum_{i=1}^{n} \overline{F}_{ik+j} = \frac{1}{k-1}\sum_{i=1}^{k} i(\overline{F}_{nk+i} - \overline{F}_i) + \sum_{i=j}^{k} (\overline{F}_{nk+i} - \overline{F}_i)$$

and with $j = 0$, the expression becomes

$$\sum_{i=1}^{n} \overline{F}_{ki} = \frac{1}{k-1}\left[\sum_{i=1}^{k-1} i(\overline{F}_{nk+i} - \overline{F}_i) - (k-2)(\overline{F}_{(n+1)k} - \overline{F}_k)\right]$$

Theorem 9.

$$\overline{F}_{m+n} = \overline{F}_{m-1}\overline{F}_n + \overline{F}_m\overline{F}_{n+k+1} + \sum_{i=1}^{k-2} A^i_{m-1}\overline{F}_{n+i} \quad \text{for } m,n \ge 1 .$$

There are many known relations involving the elements of the usual or 2^{nd} order Fibonacci sequence. A partial list of these relations appear in [4]. However, when generalizing many of these relations to the terms of either the k^{th} order or the generalized k^{th} order sequence, the relations become quite involved; but in each case a corresponding formula holds in the more general situation.

[Continued on page 354.]

THE FIBONACCI GROUP AND A NEW PROOF THAT $F_{p-(5/p)} \equiv 0 \pmod p$

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It is fairly well known that $F_{p-(5/p)} \equiv 0 \pmod p$, where p is an odd prime; F_p is the p^{th} Fibonacci number, and $(5/p)$ is the Legendre symbol. Three different proofs of this theorem are given in [1], [2], and [3].

My method of proof of this theorem is based on the restricted periods of generalized Fibonacci sequences reduced modulo p and the existence of what I call Fibonacci groups modulo certain primes.

Look at the congruence: $a + ax \equiv ax^2 \pmod p$. This implies $ax^{n-1} + ax^n \equiv ax^{n+1} \pmod p$. Solving for x: $x \equiv (1 \pm \sqrt{5})/2 \pmod p$. Thus we can solve for x iff 5 is a quadratic residue of $p > 2$. If $a \equiv 1 \pmod p$, the recursion relation: $a + ax \equiv ax^2 \pmod p$ will generate the successive terms: $(1, x, x^2, \cdots, x^n, \cdots)$, and we will have a Fibonacci group.

As an example of a Fibonacci group, solve $x \equiv (1 \pm \sqrt{5})/2 \pmod{11}$. We see $x \equiv (1 \pm 4)/2 \pmod{11} \equiv 4$ or $8 \pmod{11}$. If $x \equiv 4 \pmod{11}$, we get the group (1, 4, 5, 9, 3) and if $x \equiv 8 \pmod{11}$, we obtain the group (1, 8, 9, 6, 4, 10, 3, 2, 5, 7). In each case each term is the sum of the preceding two terms (mod 11) and is a constant multiple of the preceding term.

Definitions. Let $\{H_n\}$ be a generalized Fibonacci sequence (hereafter called G. F. S.) reduced modulo p; $H_1 = a$, $H_2 = b$; $H_n \equiv H_{n-1} + H_{n-2} \pmod p$; p an odd prime.

$\{H_n\}$ is periodic modulo p. Let $\mu(a, b, p)$ be the period of the G. F. S. which begins with (a, b) modulo p. That is, $\mu(a, b, p)$ is the least positive integer n such that $H_n \equiv H_0 \equiv H_2 - H_1$ and $H_{n+1} \equiv H_1 \pmod p$.

Also, let $\alpha(a, b, p)$ be the restricted period of $\{H_n\}$ (mod P). Thus, $\alpha(a, b, p)$ is the least positive integer m such that $H_m \equiv sH_0$ and $H_{m+1} \equiv sH_1 \pmod p$ for some s. Let $s(a, b, p) \equiv s \pmod p$; $s(a, b, p)$ will be called the multiplier of $\{H_n\}$ (mod p).

Theorem 1. If the initial pair (a, b) of $\{H_n\} \not\equiv (0, 0)$, $(a, a(1 + \sqrt{5})/2)$, or $(a, a(1 - \sqrt{5})/2) \pmod p$, then $\alpha(a, b, p) = \alpha(1, 1, p)$, $s(a, b, p) = s(1, 1, p)$, and $\mu(a, b, p) = \mu(1, 1, p)$.

Proof. Write out the Fibonacci series reduced modulo p from F_1 to $F_{\mu(1,1,p)}$. There will be $\mu(1, 1, p)$ consecutive pairs in this sequence if we count $(F_{\mu(p)}, F_1) \equiv (0, 1) \pmod p$ as a consecutive pair of terms. If a pair (c, d) does not appear in this sequence, start another G. F. S. with this pair up to $H_{\mu(c,d,p)}$. No pair will be repeated since each pair determines each term that follows and precedes by the recursion relation, and each G. F. S. is periodic modulo p.

One can continue this process until all the p^2 possible pairs are used up. We shall need three lemmas to finish the proof.

Lemma 1. Any linear combination of two G. F. S.'s yields a G. F. S.

Proof. Let $\{G_n\}$, $\{H_n\}$, be two G. F. S's. Then

$$rG_{n-1} + sH_{n-1} + rG_n + sH_n = r(G_{n-1} + G_n) + s(H_{n-1} + H_n) = rG_{n+1} + sH_{n+1} ,$$

and the recursion relation is still satisfied.

Now, we can express any pair of terms (a,b) as $(b - a)$:

$$(F_0, F_1) + a(F_1, F_2) = (b - a)(0,1) + a(1,1) = (b - a, a)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

Lemma 2. For all G. F. S. $\{H_n\}$,

$$(H_{\alpha(1,1,p)+1}, H_{\alpha(1,1,p)+2}) \equiv s(1,1,p)(a,b); \quad a = H_1, \quad b = H_2 ,$$

Proof. Let $\alpha(1, 1, p) = n$. Then

$$(F_n, F_{n+1}) \equiv s(1,1,p)(F_0, F_1)$$

and

$$(F_{n+1}, F_{n+2}) \equiv s(1,1,p)(F_1, F_2) \pmod p,$$

by definition. But

$$\begin{aligned}(H_{n+1}, H_{n+2}) &\equiv (b - a)(F_n, F_{n+1}) + a(F_{n+1}, F_{n+2}) \\ &\equiv (b - a)s(1,1,p)(F_0, F_1) + (a)s(1,1,p)(F_1, F_2) \equiv s(1,1,p)(a,b) \pmod p.\end{aligned}$$

Corollary: $(H_{\mu(1,1,p)+1}, H_{\mu(1,1,p)+2}) \equiv (a,b)$.

This proof is exactly the same as that for Lemma 2. It is interesting to note that this corollary implies that length of a Fibonacci group $\le \mu(1,1,p)$.

Lemma 3. If $b \not\equiv a(1 \pm \sqrt{5})/2$, then $\alpha(a,b,p) = \alpha(1,1,p)$. (Note that if $(a,b) = (F_1, F_2) = (1,1)$, then $b \not\equiv a(1 \pm \sqrt{5})/2 \pmod p$ since this implies that $\sqrt{5} \equiv \pm 1 \pmod p$, which is false for $p \ge 3$.)

Proof. Assume that this assertion is false for some $\{H_n\}$, where $b \not\equiv a(1 \pm \sqrt{5})/2$. Let $\alpha(a,b,p) = n$. By Lemma 2, $n < \alpha(1,1,p)$. Then

$$(H_{n+1}, H_{n+2}) \equiv s(a,b,p)(a,b) \equiv (b - a)(F_n, F_{n+1}) + a(F_{n+1}, F_{n+2}) \pmod p .$$

Let $F_{n+1} = x \cdot F_1 = x$ and $F_{n+2} = y \cdot F_2 = y$; $x \ne y$ since $n < \alpha(1,1,p)$. Then

$$\frac{H_{n+2}}{H_{n+1}} = \frac{s(a,b,p)b}{s(a,b,p)a} \equiv \frac{b}{a} \equiv \frac{(b-a)F_{n+1} + aF_{n+2}}{(b-a)F_n + aF_{n+1}}$$

$$\equiv \frac{(b-a)F_{n+1} + aF_{n+2}}{(b-a(F_{n+2} - F_{n+1}) + aF_{n+1}} \equiv \frac{(b-a)x + ay}{(b-a)(y-x) + ax} \pmod p.$$

Thus,

$$\frac{b}{a} \equiv \frac{bx - ax + ay}{by - ay - bx + 2ax} \pmod p.$$

I claim that neither a nor $(by - ay - bx + 2ax \equiv H_{n+1}) \equiv 0 \pmod p$. If $a \equiv 0$, then $(a,b) \equiv (0,0)$ or $(a,b) \equiv (0,k)$, $k \not\equiv 0 \pmod p$. The pair $(0,0)$ is excluded by hypothesis, and if $(a,b) \equiv (0,k)$, $\{H_n\}$ is a non-zero multiple of the Fibonacci sequence. Since the residues modulo p form a field, there are no divisors of 0 and a multiple of the Fibonacci sequence will have the same restricted period. Therefore $n = \alpha(1,1,p)$ and we have a contradiction. If $H_{n+1} \equiv 0 \pmod p$, the same argument leads to a contradiction.

The congruence

$$\frac{b}{a} \equiv \frac{bx - ax + ay}{by - ay - bx + 2ax} \pmod p$$

leads to the congruence

$$b^2(y-x) - ab(y-x) - a^2(y-x) \equiv 0 \pmod p.$$

Dividing through by the non-zero $(y - x)$ and solving for b, we obtain $b \equiv a(1 \pm \sqrt{5})/2$, a contradiction. Q. E. D.

Corollary. If $b \not\equiv a(1 \pm \sqrt{5})/2$, then $s(a,b,p) = s(1,1,p)$ and $\mu(a,b,p) = \mu(a,b,p)$. This follows from Lemma 2, its corollary and Lemma 3.

With the help of the three lemmas and their corollaries, Theorem 1 is now proved.

We are now ready to prove the main theorem that $F_{p-(5/p)} \equiv 0 \pmod p$. Of the p^2 possible pairs of terms which appear in some G. F. S. reduced modulo p, one pair $(0,0)$ forms the trivial sequence $(0, 0, 0, \cdots)$. We will now look at the $p^2 - 1$ pairs remaining.

If $(5/p) = 1$, then there are two solutions to the congruence: $x \equiv (1 \pm \sqrt{5})/2$, and we can form two Fibonacci groups. By Lagrange's theorem, each group has length $(p-1)/k_i$, $(i = 1, 2)$. If we count the k_i non-zero multiples of each group, there will be $2(p - 1)$ pairs of terms in some non-zero multiple of a Fibonacci group. That leaves $p^2 - 1 - 2(p-1) = (p-1)^2$ pairs remaining.

We will say that two restricted periods belong to the same equivalence class if some pair of consecutive terms of one restricted period is a non-zero multiple of a pair of another restricted period reduced modulo p. In each equivalence class, there are $p - 1$ non-zero multiples of each restricted period. Suppose there are k equivalence classes of restricted periods of length $\alpha(1,1,p)$. Then if $(5/p) = 1$, there will be $(p-1)^2$ pairs in these equivalence classes: $(p-1)(k) \cdot \alpha(1,1,p) = (p-1)^2$, and $\alpha(1,1,p) = (p-1)/k$. Since there are no divisors of 0 (mod p), only terms which are multiples of $(p-1)/k$ will be $\equiv 0 \pmod p$. In particular, $F_{p-1} \equiv 0 \pmod p$.

If $(5/p) = 0$, then $p = 5$ and $\sqrt{5} \equiv 0 \pmod p$. Thus, there is only one root of the congruence: $x \equiv (1 \pm \sqrt{5})/2 - x \equiv 3 \bmod 5$. This leads to the Fibonacci group (1, 3, 4, 2). Excluding the trivial pair (0, 0), there are $p^2 - 1 - (p-1)$ pairs which are not members of multiples of Fibonacci groups (mod p). Then $p(p-1) = (p-1)(k) \cdot \alpha(1,1,p)$, and $\alpha(1,1,p) = p/k$. This implies that $F_p \equiv 0 \pmod p$.

If $(5/p) = -1$, there are no Fibonacci groups (mod p), and $p^2 - 1 = (p-1)(k)\alpha(1,1,p)$. Thus, $\alpha(1,1,p) = (p+1)/k$, and $F_{p+1} \equiv 0 \pmod p$. Q.E.D.

This theorem can easily be generalized. Let us define a Fibonacci-like sequence $\{J_n\}$ as one which satisfies the recursion relation: $J_{n+1} = aJ_n + bJ_{n-1}$; a, b positive integers. In accordance with the notation of Robert P. Backstrom [4], I will call the Fibonacci-like sequence beginning with (1, a) the primary sequence. If $b \not\equiv 0 \pmod p$, then by the recurrence relation $bJ_0 \equiv j_2 - aJ_1 \equiv a - a(1) \equiv 0$, which implies that $J_0 \equiv 0 \pmod p$. Thus, if $b \not\equiv 0 \pmod p$, the primary sequence $\{J_n\}$ will be absolutely periodic and $J_{\alpha(p)}$ will be $\equiv 0 \pmod p$. It should be noted that only in multiples of a primary sequence will all but a finite number of primes (excepting possibly only those primes that divide b) divide some positive term of the sequence.

We can form a Fibonacci-like group analogous to the Fibonacci group by solving the congruence: $bc + acx \equiv cx^2 \pmod p$ for x; $x \equiv (a \pm \sqrt{a^2 + 4b})/2$. As an example of such a group, if $a = 1$, $b = 3$, then a Fibonacci-like group exists iff $(a^2 + 4b/p) = (13/p) = 0$ or 1, if $p = 17$, then a solution of $x \equiv (1 \pm \sqrt{13})/2 \equiv (1 \pm 8)/2 \pmod{17}$ is $x \equiv 13 \bmod 17$, and this gives rise to the Fibonacci-like group (1, 13, 16, 4).

As before, any arbitrary Fibonacci-like sequence is the linear combination of two primary sequences. If (c, d) are two consecutive terms of a Fibonacci-like sequence and $\{J_n\}$ is a primary sequence, then

$$(c,d) = (d - ac)(J_0, J_1) + c(J_1, J_2) = (d - ac)(0,1) + c(1,a) = (d - ac, c)\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}.$$

Let $a^2 + 4b = k$. If $b \not\equiv 0 \pmod p$, p an odd prime, then by an argument analogous to the one above, we can prove the theorem: $J_{p-(k/p)} \equiv 0 \pmod p$ if $\{J_n\}$ is the primary sequence.

If $a \not\equiv 0 \pmod p$, $b \equiv 0 \pmod p$, then solving the congruence:

$$x \equiv (a \pm \sqrt{a^2 + 4b})/2 \equiv (a \pm \sqrt{a^2})/2 \pmod p,$$

we see that $x \equiv a$ or $0 \pmod p$. Thus, the primary sequence generated by (1, a) will be a Fibonacci-like group and no positive term will be divisible by p.

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THE COEFFICIENTS OF cosh x/cos x

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1. Gandhi [3] defined a set of rational integral coefficients S_{2n} by the generating function

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} \frac{S_{2n} x^{2n}}{(2n)!} \tag{1}$$

The coefficients S_{2n} were the subject of much investigation by Carlitz [1], [2], Gandhi [4], [5], Gandhi and Ajaib Singh [6], Krick [7], Raab [8] and Salie [9]. In the present note we prove that

$$S_{4n+2} \equiv 52 \pmod{100} \quad \text{for} \quad n > 0 . \tag{2}$$

The proof of (2) involves some elementary but interesting results.

2. Gandhi and Ajaib Singh [6] proved that

$$S_{4n+2} = \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} S_{4n+2-4r} + 2^{4n+1} . \tag{3}$$

Assume that (2) is true for any $n > 0$ and we shall prove that it is true for $n + 5$. Since S_6, $S_{10}, \cdots, S_{4n-2} \equiv 52 \pmod{100}$, and $S_2 = 2$, Eq. (3) yields

$$S_{4n+2} \equiv 52 \sum_{r=1}^{n-1} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + \binom{4n+2}{4n} (-1)^{n+1} 2^{2n+1} + 2^{4n+1} \pmod{100}$$

$$\equiv 52 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + \binom{4n+2}{4n} (-1)^{n+1} 2^{2n} [2 - 52] + 2^{4n+1} \pmod{100}.$$

Since $n > 0$, the second term on the right is divisible by 100 and therefore

$$S_{4n+2} \equiv 52 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + 2^{4n+1}$$

$$(4) \qquad \equiv 104 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r-1} + 2^{4n+1}$$

$$\equiv 2 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + 2^{4n+1} \pmod{100}$$

$$\equiv 2A + 2^{4n+1} \pmod{100} \quad ,$$

where

$$A = \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} \quad .$$

We now evaluate the sum for A. Let $\omega = (1 + i)/\sqrt{2}$, then it can be verified that $\omega^4 = -1$ and $\omega^8 = +1$, where $i = \sqrt{-1}$. Now

$$(1 + \omega x)^{4n+2} = \sum_{r=0}^{4n+2} \binom{4n+2}{r} \omega^r x^r \quad ,$$

and

$$(1 - \omega x)^{4n+2} = \sum_{r=0}^{4n+2} \binom{4n+2}{r} (-1)^r \omega^r x^r \quad .$$

Adding these two expansions we get

$$(5) \qquad \frac{(1 + \omega x)^{4n+2} + (1 - \omega x)^{4n+2}}{2} = \sum_{r=0}^{2n+1} \binom{4n+2}{2r} \omega^{2r} x^{2r} \quad .$$

In (5) replace x by $\sqrt{-1}x$ to get

$$(6) \qquad \frac{(1 + \sqrt{-1}\omega x)^{4n+2} + (1 - \sqrt{-1}\omega x)^{4n+2}}{2} = \sum_{r=0}^{2n+1} \binom{4n+2}{2r} (-1)^r \omega^{2r} x^{2r} \; .$$

Adding (5) and (6) and letting $x = \sqrt{2}$ it is easy to see that

(7) $$A = 1 - \tfrac{1}{4}\left[(1+\omega\sqrt{2})^{4n+2} + (1-\omega\sqrt{2})^{4n+2} + (1+\sqrt{-1}\,\omega\sqrt{2})^{4n+2} + (1-\sqrt{-1}\,\omega\sqrt{2})^{4n+2}\right].$$

Since $\omega\sqrt{2} = 1 + i$, Eq. (7) becomes

$$A = 1 - \tfrac{1}{4}\left[(2+i)^{4n+2} + (-1)^{4n+2} + (i)^{4n+2} + (2-i)^{4n+2}\right] = 1 - \tfrac{1}{4}\left[(3+4i)^{2n+1} + (3-4i)^{2n+1} - 2\right].$$

Using this expression for A, Eq. (4) becomes

(8) $$S_{4n+2} \equiv 3 - \tfrac{1}{2}\left[(3+4i)^{2n+1} + (3-4i)^{2n+1}\right] + 2^{4n+1} \pmod{100}.$$

Lemma 1. If α and β are integers, $\alpha \not\equiv 0 \pmod 5$ and if $\alpha^K \equiv \beta \pmod{100}$, then $\alpha^{K+20} \equiv \beta \pmod{100}$. However, if $\alpha = 2$, then K must be greater than 1.

Proof. Trivial.

In view of Lemma 1, for $n > 0$, we have

(9) $$2^{4n+1} \equiv 2^{4(n+5)+1} \pmod{100}.$$

Then we prove that

(10) $$\tfrac{1}{2}\{(3+4i)^{2n+1} + (3-4i)^{2n+1}\} \equiv \tfrac{1}{2}\{(3+4i)^{2n+11} + (3-4i)^{2n+11}\} \pmod{100}.$$

It is easy to see that the above congruence holds for modulus 4 hence we need to prove that

$$(3+4i)^{2n+1} + (3-4i)^{2n+1} \equiv (3+4i)^{2n+11} + (3-4i)^{2n+11} \pmod{25},$$

or

(11) $$(3+4i)^{2n+1}\{(3+4i)^{10} - 1\} + (3-4i)^{2n+1}\{(3-4i)^{10} - 1\} \equiv 0 \pmod{25}.$$

By actual expansion we find that

(12) $$(3+4i)^{10} - 1 \equiv 4(3-4i) \pmod{25}$$

and

$$(3-4i)^{10} - 1 \equiv 4(3+4i) \pmod{25}.$$

Let

(13) $$(3+4i)^{2n+1} = c + id, \quad (3-4i)^{2n+1} = c - id.$$

Expanding we find that

$$c = \sum_{r=0}^{n} \binom{2n+1}{2r} 3^{2n+1-2r}(-1)^r,$$

and

$$d = \sum_{r=0}^{n} \binom{2n+1}{2r+1} 3^{2n+1-(2r+1)}(-1)^r .$$

Lemma 2. $c \not\equiv 0 \pmod 5$ and $d \not\equiv 0 \pmod 5$.

Proof.

$$c \equiv \sum_{r=0}^{n} \binom{2n+1}{2r+1} (-2)^{2n+1-2r}(-1)^r \pmod 5$$

$$\equiv -\sum_{r=0}^{n} \binom{2n+1}{2r} 2^{2n+1-2r}(-1)^r \pmod 5$$

$$\equiv -\frac{(1-2i)^{2n+1} + (1+2i)^{2n+1}}{2i} \pmod 5 .$$

If $c \equiv 0 \pmod 5$ then since $5 = (1+2i)(1-2i)$ and hence $c \equiv 0 \pmod{1+2i}$, which is not true and hence $c \not\equiv 0 \pmod 5$. Similarly it can be proved that $d \not\equiv 0 \pmod 5$. Moreover from (13) we have

$$c^2 + d^2 = (25)^{2n+1} \equiv 0 \pmod{25} . \tag{14}$$

Since $c \not\equiv 0$, $d \not\equiv 0 \pmod 5$ it is easy to see that $(c,d) = 1$ and hence there exist a number a such that

$$c \equiv ad \pmod{25} . \tag{15}$$

Using (11) and (12), Eq. (10) simplifies to

$$3c + 4d \equiv 0 \pmod{25} . \tag{16}$$

Therefore to prove (10), we need to prove (16). Substitute (15) into (14) to get $1 + a^2 \equiv 0 \pmod{25}$ which yields that either (a) $a \equiv 7 \pmod{25}$ or (b) $a \equiv 18 \pmod{25}$. We then prove that condition (a) can only be satisfied and thus will reject condition (b). Assume that (b) is satisfied, i.e., $c \equiv 18d \pmod{25}$ or

$$c \equiv 3d \pmod 5 . \tag{17}$$

We show that (17) is impossible. We have proved that

$$c \equiv -\frac{(1-2i)^{2n+1} + (1+2i)^{2n+1}}{2i} \pmod{5}.$$

Similarly it can be proved that

$$D \equiv \frac{(1+2i)^{2n+1} + (1-2i)^{2n+1}}{2} \pmod{5}.$$

Substituting these expressions in (17) it can be easily proved that (17) will not even hold for modulus $(1+2i)$ or $(1-2i)$. Hence (17) is impossible and condition (b) cannot be satisfied. Therefore condition (a) has to be satisfied and hence $c \equiv 7d \pmod{25}$. Using this congruence we find that (16) is satisfied and hence we have proved the truth of (10). Using these results and (9), from (8) we get $S_{4n+2} \equiv S_{4n+22} \pmod{100}$. But $S_{4n+2} \equiv 52 \pmod{100}$ and therefore $S_{4n+22} \equiv 52 \pmod{100}$ and hence if (2) is true for $n > 0$ it is also true for $n+5$. From Krick's [7] table for S_{2n} up to S_{20} we find that (2) is true for $n = 1, 2, 3, 4$. Also using (3) we verify that $S_{22} \equiv 52 \pmod{100}$. Thus by the usual method of induction (2) has been established.

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[Continued from page 348.]

If $a \equiv 0$, $b \not\equiv 0 \pmod p$, then every term of the primary sequence from the second one on will be $\equiv 0 \pmod p$ and this sequence will satisfy the theorem since

$$J_{p-(k/p)} \equiv J_p \equiv 0 \pmod p .$$

If $a \equiv 0$, $b \not\equiv 0 \pmod p$, then we will get the sequence $(1,0,b,0,b^2,0,b^3,0,\cdots)$ and every second term will be divisible by p. Thus, whether $p - (k/p) = p + 1$ or $p - 1$, the theorem will be satisfied.

I will close the paper by investigating which terms of primary sequence are divisible by the prime 2. If a,b are both odd, we obtain the repetitive sequence $(1,1,0,\cdots)$, and $J_3 = J_{2+1} \equiv 0 \pmod 2$.

If a is odd and b is even, then $\{J_n\}$ is a Fibonacci-like group (mod 2) and we get the sequence $(1, 1, 1, \cdots)$.

If a is even and b odd, we get the sequence $(1,0,1,0,1,0,\cdots)$ and $J_2 = J_{2+0} \equiv 0 \pmod 2$.

If a,b are both even, we obtain the sequence $(1,0,0,0,\cdots)$ and $J_2 \equiv 0 \pmod 2$.

Note. The Fibonacci group was pointed out to me by Stan Perlo, currently a graduate student at the University of Michigan.

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A NEW PRIMALITY CRITERION OF MANN AND SHANKS AND ITS RELATION TO A THEOREM OF HERMITE WITH EXTENSION TO FIBONOMIALS

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1. INTRODUCTION

Henry Mann and Daniel Shanks [4] have found a new necessary and sufficient condition for a number to be a prime. This criterion may be stated in novel terms as a property of a displaced Pascal Arithmetical Triangle as follows. Consider a rectangular array of numbers made by moving each row in the usual Pascal array two places to the right from the previous row. The $n + 1$ binomial coefficients $\binom{n}{k}$, $k = 0, 1, \cdots, n$, are then found in the n^{th} row between columns $2n$ and $3n$ inclusive. We shall say that a given column has the Row Divisibility Property if each entry in the column is divisible by the corresponding row number. Then the new criterion is that the column number is a prime if and only if the column has the Row Divisibility Property.

Column Number

Row Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1																			
1			1	1																
2					1	2	1													
3							1	3	3	1										
4									1	4	6	4	1							
5											1	5	10	10	5	1				
6													1	6	15	20	15	6	1	
7															1	7	21	35	35	21
8																	1	8	28	56
9																			1	9

The Displaced Array

We wish to show here that by relabelling the array it is easy to relate the new criterion with a theorem of Hermite on factors of binomial coefficients. We shall also generalize to a displaced Fibonomial array. The case for arbitrary rectangular arrays of extended coefficients is treated.

2. THEOREMS OF HERMITE

According to Dickson's History [2, p. 272] Hermite stated that

$$\frac{m}{(m,n)} \,\Big|\, \binom{m}{n} \tag{2.1}$$

and

$$\frac{m-n+1}{(m+1,n)} \,\Big|\, \binom{m}{n}\,, \tag{2.2}$$

where (a,b) denotes the greatest common divisor of a and b. Proofs were given by Catalan, Mathews, and Woodall according to Dickson. What is more, there are extensions to multinomial coefficients and Ricci [5] noted that

$$\frac{m!}{a!\,b!\,\cdots\,c!} \equiv 0 \mod \frac{m}{(a,\ b,\ \cdots,\ c)}\,, \quad \text{where } a+b+\cdots+c = m\,. \tag{2.3}$$

These theorems have been used in many ways to derive results in number theory. For example, Eq. (2.1) gives us at once that

$$p \,\Big|\, \binom{p}{k}$$

for each k with $1 \le k \le p-1$ provided that p is a prime. This has been used in various proofs of Wilson's criterion and Fermat's congruence. From (2.2) we obtain at once that

$$n+1 \,\Big|\, \binom{2n}{n}\,,$$

i.e., the numbers generated by

$$\frac{\binom{2n}{n}}{(n+1)}\,:$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, $\cdots$ are all integers. This sequence, the so-called Catalan sequence, is of considerable importance in combinatorics and graph theory, and the reader may consult the historical note of Brown [1] for details.

For the sake of completeness we wish to include proofs of (2.1) and (2.2) to show how easily they follow from the Euclidean algorithm.

By the Euclidean algorithm we know that there exist integers A and B such that

$$(m,n) = d = mA + nB\,.$$

Therefore

$$d\,\frac{(m-1)(m-2)\cdots(m-n+1)}{n!} = \binom{m}{n}A + \binom{m-1}{n-1}B = E, \text{ an integer},$$

so that, upon multiplication by m we have

$$d\binom{m}{n} = mE\ ,$$

from which it is obvious that

$$\frac{m}{d}\ \Bigg|\ \binom{m}{n}$$

as stated in (2.1). This is essentially Hermite's proof [3, 415-416, Letter of 17 April 1889]. Similarly, there exist integers C and D such that

$$(m+1,n) = d = (m+1)C + nD\ .$$

Rearranging,

$$d = (m-n+1)C + (C+D)n\ .$$

Therefore

$$d\,\frac{m(m-1)\cdots(m-n+2)}{n!} = \binom{m}{n}C + \binom{m}{n-1}(C+D) = F, \text{ an integer},$$

whence on multiplying by $m-n+1$ we have

$$d\binom{m}{n} = (m-n+1)F\ ,$$

so that

$$\frac{m-n+1}{d}\ \Bigg|\ \binom{m}{n}$$

as stated in (2.2). Hermite's proof may be applied to other similar theorems.

The usefulness of Hermite's theorems suggested to me that one might get part of the results of Mann and Shanks from them.

3. FIRST RESTATEMENT OF THE CRITERION

Leaving the array arranged as before, it is easy to see that we may restate the condition of Mann and Shanks in the form

$$k = \text{prime} \quad \text{if and only if} \quad n \,\Big|\, \binom{n}{k-2n} \tag{3.1}$$

for all integers n such that $k/3 \le n \le k/2$. If we take all entries in the array other than the binomial coefficients to be zero, we can say "for all n."

By Hermite's theorem (2.1) we have in general

$$\frac{n}{(n, k-2n)} \,\Big|\, \binom{n}{k-2n}$$

for all integers $2n \le k \le 3n$. Equivalently,

$$(n, k-2n)\binom{n}{k-2n} = nE ,$$

for some integer E. Let $k = \text{prime}$. Then, except for $n = k$, $n \nmid k$, whence $n \nmid k - 2n$. But this means that $n \nmid (n, k - 2n)$, so that n must be a factor of

$$\binom{n}{k-2n} .$$

The case $n = k$ offers no difficulty since we are only interested in $k/3 \le n \le k/2$. The converse, that k must be prime if n is a factor of

$$\binom{n}{k-2n}$$

for every n can be handled along the lines of [4];

$$\binom{n}{k-2n} = 1$$

when $2|k$ or $3|k$ so one must consider prime columns that occur for k of form $6j \pm 1$.

4. SECOND RESTATEMENT OF THE CRITERION

Arrange the binomial coefficients in the usual array; $k = 0, 1, \cdots, n$ on the n^{th} row:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

.....

It is easy to see that the criterion may be stated as follows:

$$(4.1)\qquad 2n + 1 = \text{prime if and only if } n - k \,\Big|\, \binom{n-k}{2k+1}$$

for every

$$k = 0, 1, \cdots, \left(\frac{n-1}{3}\right).$$

Since $2n \neq$ prime for $n > 1$ we may ignore the case whether

$$n - k \,\Big|\, \binom{n-k}{2k}.$$

Again, Hermite's theorem (2.1) is the clue for the proof that $n - k$ is a factor when $2n + 1$ is a prime. For (2.1) gives us in general

$$\frac{n-k}{(n-k,\ 2k+1)} \,\Big|\, \binom{n-k}{2k+1}$$

for all integers $0 \le k \le \frac{n-1}{3}$. Equivalently,

$$(n - k,\ 2k + 1)\binom{n-k}{2k+1} = (n - k)E ,$$

for some integer E. Suppose $2n + 1 =$ prime. If $n - k \mid 2k + 1$, then $n - k \mid 2k + 1 + 2(n - k)$, i.e., $n - k \mid 2n + 1$, which is again impossible so that $n - k \nmid (n - k,\ 2k + 1)$, whence we must have

$$n - k \,\Big|\, \binom{n-k}{2k+1} \text{ as desired.}$$

5. QUESTIONS

It would be of interest to see whether (2.2) implies any similar results. We find in general that

$$(5.1)\qquad \frac{3n-k+1}{(n+1,\ k-2n)} \,\Big|\, \binom{n}{k-2n} \quad \text{for } 2n \le k \le 3n ,$$

and

$$(5.2)\qquad \frac{n-3k}{(n-k+1,\ 2k+1)} \,\Big|\, \binom{n-k}{2k+1} \quad \text{for } 0 \le k \le \frac{n-1}{3} .$$

In (5.1) let $n - 3k \mid (n - k + 1,\ 2k + 1)$. Then $n - 3k \mid n - k + 1$ and $n - 3k \mid 2k + 1$. But $n - 3k \mid n - k + 1$ implies also $n - 3k \mid n - k + 1 - (n - 3k)$ or again $n - 3k \mid 2k + 1$.

If then $2k + 1 = \text{prime}$ we again find that $n - 3k$ must divide the binomial coefficient, and this gives us

$$(5.3) \qquad 2k + 1 = \text{prime} \quad \text{implies} \quad n - 3k \,\Big|\, \binom{n - k}{2k + 1} \quad \text{for all} \quad n \neq 5k + 1\,.$$

It is easy to find examples of composite $2k + 1$ such that $n - 3k$ is not a factor of the binomial coefficient; e.g., take $k = 7$ and $n = 24$:

$$3 \not\Big|\, \binom{17}{15} \;.$$

Other possibilities suggest themselves. Letting $n - 3k \mid (n - k + 1,\ 2k + 1)$ gives again $n - 3k \mid n - k + 1$ whence also $n - 3k \mid 3(n - k + 1) - (n - 3k)$ or $n - 3k \mid 2n + 3$. If we then take $2n + 3 = \text{prime}$ we again obtain a useful theorem.

It seems clear from just these samples that the theorems of Hermite can suggest quite a variety of divisibility theorems, some of which may lead to criteria similar to that of Mann and Shanks. Whether any of these have any strikingly simple forms remains to be seen. One possibility suggested by (2.2) is that

$$(n + k, k)\binom{n + k - 1}{k} = nE$$

for some integer E, from which we may argue that under certain hypotheses n divides $\binom{n + k - 1}{k}$.

Result (5.3) is related to a theorem of Catalan [2, p. 265] to the effect that

$$m!\,n! \,\Big|\, (m + n - 1)!$$

provided $(m, n) = 1$.

Finally we wish to recall that some of the results here are related to a problem posed by Erdos (with published solution by F. Herzog) [6] to the effect that for every k there exist infinitely many n such that $(2n)!/n!(n + k)!$ is an integer (the case $k = 1$ yields the Catalan sequence). In fact Erdos claimed a proof that the set of values of n such that this ratio is not an integer has density zero.

6. FIBONOMIAL TRIANGLE

It is tempting to try and extend the primality criterion to arrays other than the binomial. Consider the array of Fibonomial coefficients of Hoggatt [7] displaced in the same manner as the array of Mann and Shanks:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1																			
1			1	1																
1					1	1	1													
2							1	2	2	1										
3									1	3	6	3	1							
5											1	5	15	15	5	1				
8													1	8	40	60	40	8	1	
13															1	13	104	260	260	104
21																	1	21	273	1092
34																			1	34

Here the Fibonomial coefficients are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1}, \qquad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 1 ,$$

where $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$, $F_1 = 1$, being the Fibonacci numbers. In the displaced array, the row numbers are made to be the corresponding Fibonacci numbers. From the above sampling, as well as from extended tables, one is tempted to conjecture that a column number is prime if and only if each Fibonomial coefficient in the column is divisible by the corresponding row Fibonacci number. In other words, it appears that

$$k = \text{prime} \text{ if and only if } F_n \,\Big|\, \left\{ \begin{matrix} n \\ k - 2n \end{matrix} \right\} \text{ for all } k/3 \le n \le k/2 . \tag{6.1}$$

Now as a matter of fact the exact analogue of Hermite's (2.1) is true:

$$\frac{F_m}{(F_m, F_n)} \,\Big|\, \left\{ \begin{matrix} m \\ n \end{matrix} \right\} . \tag{6.2}$$

The proof is an exact replica of Hermite's proof. What is more, Eq. (2.1) holds true for perfectly arbitrary arrays in the sense defined in [8]. That is, take an arbitrary sequence of integers A_n such that $A_0 = 0$ and $A_n \neq 0$ for $n \ge 1$, and define generalized binomial coefficients by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1}, \quad \text{with} \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 1 . \tag{6.3}$$

If all of these turn out to be integers, then we have

(6.4) $$\frac{A_m}{(A_m, A_n)} \,\Big|\, \left\{ {m \atop n} \right\} .$$

However, the corresponding form of (2.2) is in general false. The reason is that the step

$$(A_{m+1}, A_n) = d = CA_{m+1} + DA_n = C(A_{m+1} - A_n) + (D + C)A_n$$

can be applied only if also

$$A_{m+1} - A_n = A_{m-n+1}$$

or something close. Thus it is not at once clear that the Fibonomial Catalan numbers

$$\frac{\left\{ {2n \atop n} \right\}}{F_{n+1}}$$

are integers. The first few of these are in fact 1, 1, 3, 20, 364, 17017, etc.

However, having (2.1) extended to (6.4) is a good result because in order to obtain a theorem such as

(6.5) $$k = \text{prime} \quad \text{implies} \quad A_n \,\Big|\, \left\{ {n \atop k - 2n} \right\} \quad \text{for all } k/3 \le n \le k/2 ,$$

it is only necessary to be able to prove that

(6.6) $$(A_n, A_{k-2n}) = 1 \quad \text{when} \quad k = \text{prime, and} \quad k/3 \le n \le k/2 .$$

For the Fibonacci numbers this is easy because of a known fact that in general

(6.7) $$(F_a, F_b) = F_{(a,b)} .$$

Thus we have $(F_n, F_{k-2n}) = F_{(n,k-2n)}$ so that for k = prime we know as in (3.1) that $(n, k - 2n) = 1$, and since $F_1 = 1$ we have the desired result. The Fibonomial displaced array criterion then parallels the ordinary binomial case studied by Mann and Shanks.

Quite a few standard number-theoretic results have analogues in the Fibonomial case.

7. FIBONOMIAL ANALOGUE OF HERMITE'S SECOND THEOREM

Although we have just indicated that divisibility theorem (2.2) does not hold in general for the generalized binomial coefficients, we now show that it does hold for the Fibonomial coefficients. Thus we now prove that

$$\frac{F_{m-n+1}}{(F_{m+1}, F_n)} \,\Big|\, \left\{ \begin{matrix} m \\ n \end{matrix} \right\} \quad . \tag{7.1}$$

We need to observe that $(a - b, b) = (a, b)$. This follows, e.g., from an easily proved stronger assertion that $(a + c, b) = (a, b)$ when $b|c$. This lemma is used to modify the proof given by Hermite for (2.2), as follows. Let $(F_{m+1}, F_n) = d$. Now, by (6.7) and our lemma,

$$d = (F_{m+1}, F_n) = F_{(m+1,n)} = F_{(m-n+1,n)} = (F_{m-n+1}, F_n) = xF_{m-n+1} + yF_n$$

for some integers x, y. Therefore

$$d\,\frac{F_m F_{m-1} \cdots F_{m-n+2}}{F_n F_{n-1} \cdots F_1} = x \left\{ \begin{matrix} m \\ n \end{matrix} \right\} + y \left\{ \begin{matrix} m \\ n - 1 \end{matrix} \right\} = E, \text{ some integer,}$$

and by multiplication with F_{m-n+1} we obtain

$$d \left\{ \begin{matrix} m \\ n \end{matrix} \right\} = E \cdot F_{m-n+1} ,$$

whence F_{m-n+1}/d is indeed a factor of $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}$.

As a valuable corollary we get the fact that the Fibonomial Catalan numbers are integers; this follows at once from (7.1) by setting $m = 2n$, and noting that $(F_{2n+1}, F_n) = 1$, so that we have

$$F_{n+1} \,\Big|\, \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\} \quad . \tag{7.2}$$

The Fibonomial Catalan sequence 1, 1, 3, 20, 364, 17017, 4194036, $\cdots$ generated by

$$\frac{\left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}}{F_{n+1}}$$

is the exact analogue of the familiar Catalan sequence 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, $\cdots$ generated by

$$\frac{\left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}}{(n + 1)}$$

A brief history of the Catalan sequence is given in [1]. A nice proof of (6.7) is given in [9].

8. SOME CONGRUENCES FOR PRIME FIBONACCI NUMBERS

It is not known whether there exist infinitely many Fibonacci numbers which are primes, but we may easily obtain congruences which such primes satisfy.

Let F_m be a prime number. Then $(F_m, F_n) = 1$ for $1 \le n \le m-1$. Thus from (6.2) we obtain:

$$\text{If } F_m = \text{prime, then } F_n \,\Big|\, \left\{ {m \atop n} \right\} \text{ for } 1 \le n \le m-1 . \tag{8.1}$$

This is a Fibonacci analogue of the fact that $p \,\Big|\, \binom{p}{k}$ for $1 \le k \le p-1$ if p is a prime.

Now, it is known [10] that

$$\sum_{k=0}^{m} (-1)^{k(k+1)/2} \left\{ {m \atop k} \right\} F_{a+1-k}^{m-1} = 0, \qquad m \ge 2 . \tag{8.2}$$

This generalizes such relations as $F_{a+1} - F_a - F_{a-1} = 0$, $F_{a+1}^2 - 2F_a^2 - 2F_{a-1}^2 + F_{a-2}^2 = 0$, etc. See also Hoggatt [7]. Brother Alfred gives a useful table [10] of the Fibonomial coefficients up to $m = 12$. Identity (8.2) may be used with (8.1) to obtain an interesting congruence; for every term in (8.2) is divisible by F_m when F_m is a prime, except the first and last terms. Thus we obtain the congruence:

$$\text{If } F_m = \text{prime, then } F_a^{m-1} \equiv -(-1)^{m(m+1)/2} F_{a-m}^{m-1} \pmod{F_m}, \quad m \ge 2 , \tag{8.3}$$

for all integers a. A special case is of interest. Let $a = m+1$ and we get:

$$\text{If } F_m = \text{prime, then } F_{m+1}^{m-1} \equiv -(-1)^{m(m+1)/2} \pmod{F_m}, \quad m \ge 2 . \tag{8.4}$$

Another result useful for deriving congruences is the identity

$$\sum_{k=0}^{2m+1} \left\{ {2m+1 \atop k} \right\} = \prod_{k=0}^{m} L_{2k}, \quad m \ge 0 , \tag{8.5}$$

where the L's are Lucas numbers, defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$. The identity was noted in Problem H-63 of Jerbic [11].

If we apply the extended Hermite theorem (8.1) to this, we obtain:

$$\text{If } F_{2m+1} = \text{prime, then } \prod_{k=0}^{m} L_{2k} \equiv 2 \pmod{F_{2m+1}} . \tag{8.6}$$

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A GENERALIZED FIBONACCI NUMERATION

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The sequence of Fibonacci numbers is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ $(n \geq 0)$, and it is well known that

(1) $$F_n = \sum_{s=0}^{m} \binom{n-1-s}{s}, \quad \text{where } m \text{ is the greatest integer } \leq (n-1)/2 .$$

It can be shown, if we allow negative values of the subscript n that

(2) $$F_{-n} = (-1)^{n-1} \cdot F_n .$$

Any sequence satisfying the recurrence relation

(3) $$t_{n+2} = t_n + t_{n+1}$$

is called a "generalized Fibonacci sequence." As soon as the values of any two consecutive terms $t_n = p$ and $t_{n+1} = q$ have been chosen, one can prove by induction that

(4) $$t_{n+s} = p \cdot F_{s-1} + q \cdot F_s \quad \text{and} \quad t_{n-s} = (p \cdot F_{s+1} - q \cdot F_s) \cdot (-1)^s .$$

Note that the subscript n is assumed to run from $-\infty$ to $+\infty$ in the generalized Fibonacci sequences as well as in the sequence of Fibonacci numbers.

a. Whenever t_n and t_{n+1} have the same sign, all terms t_{n+s} $(s \geq 2)$ are positive or negative and their absolute value increases with s.

Let us take, for example, the sequence $t_n = \alpha^n$, where $n \in (-\infty, +\infty)$ and α is a positive real number >1: every term is positive and they increase to infinity.

(Since $t_0 = 1$, $t_1 = \alpha$ and $t_2 = \alpha^2$, α must be the positive root of the "quadratic Fibonacci equation" $\alpha^2 - \alpha - 1 = 0$, that is,

$$\alpha = \frac{1+\sqrt{5}}{2} .$$

Note that the sequence $t_n = \beta^n$, where

$$\beta = \frac{1 - \sqrt{5}}{2}$$

is the negative root, also satisfies the recurrence (3).)

b. If one of the terms t_n and t_{n+1} is positive and the other negative, and if we assume that $|t_n| > |t_{n+1}|$, then $|t_{n+2}| = |t_n| - |t_{n+1}|$ and the following terms **have alternated** signs.

Let us take, for example, the sequence $t_n = \beta^n$, where $n \in (-\infty, +\infty)$ and β is negative and smaller than 1: the terms of this sequence have alternated signs and their absolute value tends to zero when n goes to infinity.

c. In a generalized Fibonacci sequence where positive and negative terms alternate, if there is a term t_{n+1} such that $|t_{n+1}| > |t_n|$, both t_{n+1} and t_{n+2} will have the same sign. So t_{n+1} will start an infinite sequence all of whose terms are either positive or negative, with their absolute values increasing to infinity.

d. In a sequence with alternating positive and negative terms, if there is a term t_{n+1} such that $|t_{n+1}| = |t_n|$, the next term is 0; the terms of this sequence are clearly multiples of the Fibonacci numbers.

Except the sequencex α^n and β^n defined above, the generalized Fibonacci sequences have those two infinite parts: the lower part with alternating terms decreasing in absolute value, followed by the upper part whose terms have the same sign and increase in absolute value.

Let ϵ denote any positive or negative real number. It can be shown that a sequence where $t_0 = \alpha^0$ and $t_1 = \alpha^1 + \epsilon$ has to start with alternating positive and negative terms: for ϵ arbitrarily small, and for n large enough, in

$$t_{-n} = \alpha^{-n} + (-1)^{n-1} \cdot \epsilon \cdot F_n , \quad \left|\epsilon \cdot F_n\right| > \alpha^{-n} .$$

On the other hand, a sequence where $t_0 = \beta^0$ and $t_1 = \beta^1 + \epsilon$ ends with terms increasing in absolute value, all of them being either positive or negative: for ϵ arbitrarily small and for n large enough, in

$$t_n = \beta^n + \epsilon \cdot F_n , \quad \left|\epsilon \cdot F_n\right| > \left|\beta^n\right| .$$

We shall call "primary generalized Fibonacci sequences" those sequences which have at least one term equal to 1. It is no loss of generality to assume $t_0 = 1$. (Among Fibonacci numbers, three (F_{-1}, F_1 and F_2) are equal to 1: any may be the chosen t_0.) For these sequences we may write $t_0 = 1$, $t_1 = q$ and (4) becomes

$$t_s = F_{s-1} + q \cdot F_s \quad \text{and} \quad t_{-s} = (F_{s+1} - q \cdot F_s) \cdot (-1)^s . \tag{5}$$

In this paper, we intend to express the natural numbers 1, 2, 3, $\cdots$ as sums of distinct non-consecutive terms of primary generalized Fibonacci sequences and we shall obtain a coherent system of numeration that could be used in arithmetical operations.

We assumed $t_0 = 1$; all other terms are still undetermined; they might all be positive or negative or they might alternate in sign, and their values might increase or decrease; the recurrent relation (3) will be the only rule. Thus their expression is of an algebraic nature; the value of t_0 only has been fixed. Any other given term may take different values and in general, it is not possible to determine the sum of several given terms, when the sequence they are taken from is unknown. We shall see that the groups of terms belonging to the Generalized Fibonacci Numeration constitute an exception.

The natural numbers will be constructed by successive additions of the unit t_0. More precisely, two rules will be used, one for consecutive terms, namely $t_x + t_{x+1} = t_{x+2}$ and the other for equal terms, namely $2t_x = t_x + (t_{x-1} + t_{x-2}) = t_{x+1} + t_{x-2}$.

Two different notations are possible for a number N.

In the first one, let t_x be the term with the highest subscript and t_{-s} that with lowest subscript used in the expression of N. For each of these terms and for all terms between t_x and t_{-s} (taken from left to right), let us use the digit 1 for the terms involved in the expression of N and the digit 0 for every other term. For convenience of reading, we shall distinguish by punctuation the digit corresponding to t_0 and arrange the other digits in groups of four.

In the second one, the terms involved in the expression of N only are listed as subscripts of the letter t.

For instance, both 10.0100.0.1001.01 and $t_{6,3,-1,-4,-6}$ will represent the number N = 23.

Later on, when arithmetical operations are performed, it may be convenient to avoid the letter t, the expression of N being then shortened to the sequence of subscripts of t.

Applying these rules, we write successively

$$
\begin{aligned}
1 &= t_0 \\
2 &= t_0 + t_0 = t_{1,-2} \\
3 &= t_{1,-2} + t_0 = t_{2,-2} \\
4 &= t_{2,-2} + t_0 = t_{2,0,-2} \\
5 &= t_{2,0,-2} + t_0 = t_2 + (t_1 + t_{-2}) + t_{-2} = t_{3,-1,-4} \\
6 &= t_{3,-1,-4} + t_0 = t_{3,1,-4} \\
7 &= t_{3,1,-4} + t_0 = t_{4,-4} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

These numbers are the groups of terms belonging to the Generalized Fibonacci Numeration (G.F.N.): they represent always the same number and they do not depend on the primary generalized Fibonacci sequence which has been chosen. The other groups of terms do not have this property. One should be able to recognize those particular groups of terms, and so we shall describe them.

We have first to explain how the joined Fibonacci terms taken from various undetermined sequences can yield the same value when added up in a formula of the G. F. N.

The above formulas were constructed by successive additions of the unit t_0 and, according to (4) and (5),

$$t_0 = F_{-1} + q \cdot F_0 = 1 \quad (6)$$

Let us look for the part played by $t_0 = F_{-1} = 1$ and $t_1 = q \cdot F_1 = q$, respectively in the construction of the numbers in the G. F. N.

Formula (6) consists of two parts: the first part, $F_{-1} = 1$, generates the formula for N when the terms t_n are given the value F_{n-1}. The part played by $t_1 = q$ is 0; it is represented in t_0 by $q \cdot F_0$ and in N by a similar expression, when the terms t_n are given the value F_n.

Example a. For $t_n = F_{n-1}$, $t_{6,3,-1,-4,-6} = 23$:

$$\begin{aligned} t_{6,3,-1,-4,-6} &= T_5 + F_2 + F_{-2} + F_{-5} + F_{-7} \\ &= F_7 + 2F_5 \quad \text{according to (2) and} \\ &= F_8 + F_3 = 23 \ . \end{aligned}$$

Example b. For $t_n = F_n$, $t_{6,3,-1,-4,-6} = 0$:

$$\begin{aligned} t_{6,3,-1,-4,-6} &= F_6 + F_3 + F_{-1} + F_{-4} + F_{-6} \\ &= F_3 + F_1 - F_4 \quad \text{according to (2), then by adding } F_0 = 0: \\ &= F_3 + F_1 + F_0 - F_4 = F_4 - F_4 = 0 \ . \end{aligned}$$

We shall now describe the numbers in the G. F. N. The reader is advised to construct the table of the first 50 natural numbers represented by the subscripts of t. (This table can also be found hereafter.) To be clearer, all terms with the same subscript will be written in the same column of the table when they are involved in the expression of several numbers.

Description of the numbers in the G. F. N. Every number is built from one or more independent groups of terms. First we have to describe these groups.

A. The symmetric groups contain

a) the term $t_0 = 1$,

b) the symmetric pairs with even subscripts (e.g., $t_{2,-2}$, $t_{4,-4}$, $\cdots$). These pairs stand for the numbers 3, 7, 18, $\cdots$, that is the Lucas numbers L_{2n}. One or more symmetric group (e.g., $t_{6,0,-6}$, $t_{3,4,-4,-8}$).

When some symmetric pairs and the term t_0 get together without gap, they form the saturated symmetric groups (e.g., $t_{6,4,2,0,-2,-4,-6}$). These saturated groups stand for the numbers 4, 11, 29, $\cdots$, that is the numbers L_{2n+1}.

(Actually, the sum of a symmetric pair and of the corresponding saturated symmetric group gives rise to the next symmetric pair: $t_{6,-6} + t_{6,4,2,0,-2,-4,-6} = t_{8,-8}$.)

B. The asymmetric groups are distinguished by their extreme terms, the upper one with a positive odd subscript $u(t_u)$ and the lower one with negative and even subscript $u + 1$ $(t_{-(u+1)})$. The intermediate terms characterize the varieties of asymmetric groups:

a) In the typical asymmetric group, all intermediate terms have negative odd subscripts following one another without gap from t_{-1} to $t_{-(u-2)}$. These numbers arise by adding the unit t_0 to a saturated symmetric group:

$$t_0 + t_0 = t_{1,-2}\,; \qquad t_{4,2,0,-2,-4} + t_0 = t_{5,-1,-3,-6}\,.$$

b) In the usual asymmetric group, one or more intermediate terms have a positive subscript. These terms replace the symmetric terms with negative odd subscript. By adding the saturated symmetric group $t_{2n,2n-2,\ldots,-2n}$, the intermediate term t_{2n+1} replaces the term with negative symmetric subscript:

$$t_{9,-1,-3,-5,-7,-10} + t_{4,2,0,-2,-4} = t_{9,5,-1,-3,-7,-10}\,.$$

So it is possible to get the asymmetric saturated group (e.g., $t_{9,7,5,3,1,-10}$) where all the intermediate terms have undergone this substitution.

The intermediate terms of the usual asymmetric group are the next ones: $t_{\pm 1}$, $t_{\pm 3}, \ldots, t_{\pm(u-2)}$.

c) In the altered asymmetric group, the presence of an intermediate term with positive even subscript coincides with the suppression of the terms of odd and lower subscript (in absolute value). To change an asymmetric group to such an extent, one has to add the asymmetric saturated group, immediately prior to the new term with even subscript:

$$t_{11,9,-1,-3,-5,-7,-12} + t_{5,3,1,-6} = t_{11,9,6,-7,-12}\,.$$

C. The associated groups. We have seen that symmetric pairs can join with or without t_0 in order to form symmetric groups. Symmetric pairs may also surround the terms of an asymmetric group:

$$t_{6,-6} + t_{3,1,-4} = t_{5,3,1,-4,-6}\,.$$

Usually, nothing of this type occurs with asymmetric groups: the presence of the intermediate terms prevents it. Yet in the altered asymmetric group, the interval between the new term with even subscript and the terms with negative subscripts left over, this interval may be adequate for another group of terms:

$$t_{9,6,-7,-10} + t_{4,0,-4} = t_{9,6,4,0,-4,-7,-10}\,.$$

Estimation of the numbers in the G. F. N. In presence of joined generalized Fibonacci terms, when we have identified a number of the G. F. N., we have to find out its precise value.

A primary generalized Fibonacci sequence may be chosen so as to assign a fixed value to each term as we did with Fibonacci numbers. However, in the next step we had to add or subtract the terms with negative subscript, according to their parity. The difficulty in restoring the formula is still more significant.

Let us try to estimate these formulas by another method involving the terms with non-negative subscript. The precise estimation of a number has to be made by a reckoning process, in spite of the undetermined value of the components.

a) We did assign to t_0 the value 1 and to the symmetric pairs the values 3, 7, 18, $\cdots$, which are those of L_2, L_4, L_6, $\cdots$, in the sequence of Lucas numbers. Let us relate these values to the terms with positive subscript t_2, t_4, t_6, $\cdots$ in these pairs. Then we may disregard the terms with negative subscript and all the symmetric groups will be correctly estimated.

b) Is it likewise possible to estimate the asymmetric groups?

1. Let $t_1 = 1$, $t_3 = 4$, $t_5 = 11$, $\cdots$ in other words, the Lucas numbers L_1, L_3, L_5, $\cdots$; these values were already assigned to the symmetric saturated groups; they are an underestimation for the typical asymmetric groups obtained by adding the unit t_0 to the symmetric saturated groups.

2. An intermediate term with positive subscript is substituted to the one with negative subscript by adding a symmetric saturated — and thus correctly estimated — group. Hence the underestimation of the asymmetric group persists.

3. When an underestimated asymmetric group is altered by adding an asymmetric saturated group that is likewise underestimated, an intermediate term with positive and even subscript appears: this term makes up for the two underestimations of one unit t_0.

As a matter of fact, the values L_1, L_3, $\cdots$, L_{2n-1} have been assigned to the terms with positive subscript t_1, t_3, $\cdots$, t_{2n-1} and the value L_{2n} to the new term with even subscript, t_{2n}. Now, $L_{2n} - (L_1 + L_3 + \cdots + L_{2n-1}) = L_0 = 2$.

Therefore, the altered asymmetric group is correctly estimated and the foreseen estimation of the number N is possible.

i. To the existing terms with positive subscript, we assign the next values: to t_0 the value 1; to the next terms t_1, t_2, t_3, $\cdots$, respectively, the values 1, 3, 4, 7, $\cdots$ of the Lucas numbers.

ii. It remains to add one unit to the sum of these estimations when the number N contains an unaltered asymmetric group. A single one only can exist in the expression for the number N. Therefore, this unit depends on the term with lowest positive subscript: when this subscript is odd, the unit has to be added.

Expression of a natural number by the G. F. N. We now possess all required elements to find such an expression. Let us consider, for instance, the numbers 59 and 87.

$59 = t_8 + t_5 + 1$ — t_5 being the term with lowest positive subscript, the unit is needed to correct the underestimation.

t_8 belongs to the symmetric pair $t_{8,-8}$; t_5 belongs to the typical symmetric group $t_{5,-1,-3,-6}$. Hence the formula: $t_{8,5,-1,-3,-6,-8}$.

$87 = t_9 + 11$ — We cannot use t_5: 5 being odd, one unit more would be needed to correct the underestimation.

$87 = t_9 + t_4 + 4$ — For the same reason, we cannot use t_3 and the use of adjoining terms is not allowed.

$87 = t_9 + t_4 + t_2 + 1$

The last unit has to be t_0.

$87 = t_9 + t_4 + t_2 + t_0$.

t_9 and t_4 belong to the altered asymmetric group $t_{9,4,-5,-7,-10}$: t_2 and t_0 belong to the saturated symmetric group $t_{2,0,-2}$. Hence the formula: $t_{9,4,2,0,-2,-5,-7,-10}$.

Remark. The recurrence relation (3) prevented us from using adjoining terms. An investigation of the numbers in the G. F. N. will show one more peculiarity of this numeration: t_0 does never follow directly a term with odd subscript.

THE FIRST 50 GENERALIZED FIBONACCI NUMBERS. List of Subscripts.

1:	0	21:	6 . 2 . . −2 . −6
2:	1. . −2	22:	6 . 2 .0. −2 . −6
3:	2 . . −2	23:	6 . 3 . .−1 −4. −6
4:	2 .0. −2	24:	6 . 3 1. . −4. −6
5:	3 . .−1 −4	25:	6 . 4 . . −4. −6
6:	3 1. . −4	26:	6 . 4 .0. −4. −6
7:	4 . . −4	27:	6 . 4 1. . −2 −4. −6
8:	4 .0. −4	28:	6 . 4 2 . . −2 −4. −6
9:	4 1. . −2 −4	29:	6 . 4 2 .0. −2 −4. −6
10:	4 2 . . −2 −4	30:	7 . . .−1 −3 .−5 −8
11:	4 2 .0. −2 −4	31:	7 . 1. . −3 .−5 −8
12:	5. . .−1 −3 . −6	32:	7 . 2 . . −3 .−5 −8
13:	5. 1. . −3 . −6	33:	7 . 2 .0. −3 .−5 −8
14:	5. 2 . . −3 . −6	34:	7 . 3 . .−1 .−5 −8
15:	5. 2 .0. −3 . −6	35:	7 . 3 1. . .−5 −8
16:	5. 3 . .−1 . −6	36:	7 . 4 . . .−5 −8
17:	5. 3 1. . . −6	37:	7 . 4 .0. .−5 −8
18:	6 . . . . −6	38:	7 . 4 1. . −2 .−5 −8
19:	6 . .0. . −6	39:	7 . 4 2 . . −2 .−5 −8
20:	6 . 1. . −2 . −6	40:	7 . 4 2 .0. −2 .−5 −8

(Continued on the following page.)

41:	7	5.			.	.	-1	-3	.	-8
42:	7	5.			1.	.		-3	.	-8
43:	7	5.		2	.	.		-3	.	-8
44:	7	5.		2	.	0.		-3	.	-8
45:	7	5.	3		.	.	-1		.	-8

46:	7	5.	3		1.	.		.	-8
47:	8	.			.	.		.	-8
48:	8	.			.	0.		.	-8
49:	8	.			1.	.	-2	.	-8
50:	8	.		2	.	.	-2	.	-8

[Continued from page 364.]

Example: $F_5 = 5$ and $2 \cdot 3 \cdot 7 \cdot 18 \cdot 47 = 35,532 \equiv 2 \pmod 5$.

The congruence is reminiscent of the congruences of Wilson and Fermat.

It is expected that many other interesting and novel consequences follow from the extended Hermite theorems (6.2) and (7.1) giving arithmetic information about Fibonacci, Lucas and other similar numbers.

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DISTRIBUTION OF FIBONACCI NUMBERS MOD 5^k

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It was shown by L. Kuipers and Jau-shyong Shiue [2] that the only moduli for which the Fibonacci sequence $\{F_n\}$, $n = 1, 2, \cdots$, can possibly be uniformly distributed are the powers of 5. In addition, the authors proved the Fibonacci sequence to be uniformly distributed mod 5, and they conjectured that this holds for all other powers of 5 as well. In this note, we settle this conjecture in the affirmative. Thus we show, in particular, that the Fibonacci sequence attains values from each residue class mod 5^k, and each residue class occurs with the same frequency. The weaker property of the existence of a complete residue system mod m in the Fibonacci sequence was investigated earlier by A. P. Shah [3] and G. Bruckner [1]. For definitions and terminology we refer to [2].

Theorem. The Fibonacci sequence $\{F_n\}$, $n = 1, 2, \cdots$, is uniformly distributed mod 5^k for all $k \geq 1$.

Before we start the proof, let us collect some useful preliminaries. It follows from a result of D. D. Wall [5, Theorem 5] that $\{F_n\}$, considered mod 5^k, has period $4 \cdot 5^k$. Therefore it will suffice to show that, among the first $4 \cdot 5^k$ elements of the sequence, we find exactly four elements, or, equivalently, at most four elements from each residue class mod 5^k. It will also be helpful to know that, for $j \geq 1$, the largest exponent e such that 5^e divides $(2j+1)!$ satisfies (see [4]):

$$e = \sum_{i=1}^{\infty} \left[\frac{2j+1}{5^i}\right] < \sum_{i=1}^{\infty} \frac{2j+1}{5^i} = \frac{2j+1}{4} < j \quad . \tag{1}$$

We note the formula

$$\binom{r+s}{t} = \sum_{i=0}^{t} \binom{r}{i}\binom{s}{t-i} \tag{2}$$

with non-negative integers r, s, and t, and $\binom{n}{j} = 0$ for $j > n$, which can be quickly verified by comparing the coefficients of x^t in $(1+x)^{r+s} = (1+x)^r(1+x)^s$.

Proof of the Theorem. We proceed by induction on k. For $k = 1$, the result was already shown in [2]. Now assume that, for some $k \geq 2$ and every integer a, the congruence $F_n \equiv a \pmod{5^{k-1}}$ has exactly four solutions c with $1 \leq c \leq 4 \cdot 5^{k-1}$. If n is a solution of $F_n \equiv a \pmod{5^k}$, $1 \leq n \leq 4 \cdot 5^k$, then $F_n \equiv a \pmod{5^{k-1}}$, hence by periodicity: $n \equiv c \pmod{4 \cdot 5^{k-1}}$ for one of the four c's. We complete the proof by showing that each value of c yields at most one solution n. For suppose we also have $F_m \equiv a \pmod{5^k}$,

$1 \le m \le 4 \cdot 5^k$, $m \equiv c \pmod{4 \cdot 5^{k-1}}$, and WLOG $n \ge m$. Then, in particular, $F_n \equiv F_m \pmod{5^k}$ and $n \equiv m \pmod{4 \cdot 5^{k-1}}$. Using the well-known representation

$$F_n = 2^{1-n} \sum_{j=0}^{\infty} 5^j \binom{n}{2j+1},$$

where $\binom{n}{r} = 0$ for $r > n$, we arrive at

$$\sum_{j=0}^{k-1} 5^j \binom{n}{2j+1} \equiv 2^{n-m} \sum_{j=0}^{k-1} 5^j \binom{m}{2j+1} \pmod{5^k}.$$

Since $2^{4 \cdot 5^{k-1}} \equiv 1 \pmod{5^k}$ by the Euler-Fermat Theorem, we get

$$\sum_{j=0}^{k-1} 5^j \left(\binom{n}{2j+1} - \binom{m}{2j+1}\right) \equiv 0 \pmod{5^k}. \tag{3}$$

We claim that, for $j \ge 1$, the corresponding term in this sum is divisible by 5^k. By (2):

$$5^j \left(\binom{n}{2j+1} - \binom{m}{2j+1}\right) = \sum_{i=1}^{2j+1} 5^j \binom{n-m}{i} \binom{m}{2j+1-i}.$$

We look at $5^j \binom{n-m}{i}$. From (1) we see that cancelling out 5's from $i!$ against 5^j leaves at least one power of 5 in the latter number. Since there is a factor 5^{k-1} in $n - m$, we get the desired divisibility property. Thus, from (3), we are left with the term corresponding to $j = 0$: $n - m \equiv 0 \pmod{5^k}$. Together with $n \equiv m \pmod{4 \cdot 5^{k-1}}$, this implies $n \equiv m \pmod{4 \cdot 5^k}$ or $n = m$.

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A DISTRIBUTION PROPERTY OF THE SEQUENCE OF FIBONACCI NUMBERS

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Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then in order to prove the main theorems of this paper we need the following lemmas (see [2]).

Lemma 1. Every Fibonacci number F_k divides every Fibonacci number F_{nk} for $n = 1, 2, \cdots$.

Lemma 2. $(F_m, F_n) = F_{(m,n)}$ where (x,y) denotes the greatest common divisor of the integers x and y.

Lemma 3. Every positive integer m divides some Fibonacci number whose index does not exceed m^2.

Lemma 4. Let p be an odd prime and $\neq 5$. Then p does not divide F_p.

Proof of Lemma 4. According to [1], p. 394, we have that either F_{p-1} or F_{p+1} is divisible by p. From the well known identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, we derive that $p \nmid F_p$.

Definition 1. The sequence of integers $\{x_n\}$ $(n = 1, 2, \cdots)$ is said to be uniformly distributed mod m where $m \geq 2$ is an integer, provided that

$$\lim_{N \to \infty} \frac{1}{N} \cdot A(N, j, m) = \frac{1}{m} \quad ,$$

for each j, $j = 0, 1, \cdots, m-1$, where $A(N, j, m)$ is the number of x_n, $n = 1, 2, \cdots, N$, that are congruent to $j \pmod{m}$.

Theorem 1. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is uniformly distributed mod 5.

Proof. Let all F_n $(n = 1, 2, \cdots)$ be reduced mod 5. Then we obtain the following sequence of least residues:

1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, , ⋯

Obviously, this sequence is periodic with the period length 20. Now evidently

$$\lim_{N \to \infty} \frac{1}{N} \cdot A(N, j, 5) = \frac{1}{5} \quad \text{for} \quad j = 0, 1, 2, 3, 4 \;,$$

or, $\{F_n\}$ is uniformly distributed mod 5.

Theorem 2. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod 2.

Proof. This follows from the fact that the sequence of least residues of $\{F_n\}$ is 1, 1, 0, 1, 1, 0, ⋯.

Theorem 3. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod p for any prime $p > 2$ and $\neq 5$.

Proof. Let p be a prime > 2 and $\neq 5$. Because of Lemmas 3 and 4 there exists a positive integer $t \neq p$ such that $F_t \equiv 0 \pmod{p}$. We may suppose that t is the smallest positive integer with this property. By Lemma 1, we have $F_{kt} \equiv 0 \pmod{p}$ for $k = 1, 2, \cdots$. Now there does not exist a positive integer q with $kt < q < (k+1)t$ $(k = 1, 2, \cdots)$ such that $F_q \equiv 0 \pmod{p}$, for otherwise there would exist an r $(0 < r < t)$ with $F_r \equiv 0 \pmod{p}$, which can be seen as follows. Let there be a q with the aforementioned property, then by virtue of Lemma 2, we would have

$$(F_{kt}, F_q) = F_{(kt,q)} \equiv 0 \pmod{p}.$$

Now write $q = kt + r$ $(0 < r < t)$ and therefore

$$(kt, q) = (kt, kt + r) = (kt, r) \leq r < t.$$

Because of the above property of t we have that

$$A(N, 0, p) = \left[\frac{N}{t}\right],$$

where $[a]$ denotes the integral part of a, and $A(N, 0, p)$ is related to the Fibonacci sequence (see Definition 1). Let

$$N = \left[\frac{N}{t}\right] t + r$$

with $0 \leq r < t$. Then

$$A(N, 0, p) = \frac{N - r}{t},$$

and therefore

$$\frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} - \frac{r}{Nt},$$

so

$$\lim_{N \to \infty} \frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} \qquad (t \neq p)$$

for any prime $p > 2$ and $\neq 5$. Hence $\{F_n\}$ is not uniformly distributed mod p for any prime $p > 2$ and $\neq 5$.

Theorem 4. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod m for any composite integer $m > 2$ and $m \neq 5^k$ $(k = 3, 4, \cdots)$.

Proof. Suppose that $\{F_n\}$ is uniformly distributed mod m for some composite integer m as indicated in the theorem. According to a theorem of I. Niven [3], Theorem 5.1,
[Continued on page 392.]

PERFECT N-SEQUENCES FOR N, N + 1, AND N + 2

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Frank S. Gillespie and W. R. Utz [1] define a (generalized) perfect n-sequence for m (where $n \geq 2$, $m \geq 2$) to be a sequence of length mn in which each of the integers 1, 2, 3, $\cdots$, m occurs exactly n times and between any two occurrences of the integer x there are x entries. Examples of perfect 2-sequences are numerous: 3 1 2 1 3 2 for $m = 3$ and 4 1 3 1 2 4 3 2 for $m = 4$ are the simplest. However, the author knows of no perfect n-sequence if $n > 2$.

No perfect n-sequence for m exists if $m \leq n$ [1]. (This is a direct corollary of Lemma 1, below.) It will be proved here that no perfect n-sequence for m exists if $m = n$, $m = n + 1$, or $m = n + 2$ (except for the perfect 2-sequences for 3 and 4), extending the result slightly.

In a perfect n-sequence for m, if x is an integer and $1 \leq x \leq m$, then there are n x's in the sequence. The positions in the sequence will be numbered, in order, starting at the left, 1, 2, 3, $\cdots$, mn. Let "$p(x,i)$" mean "the position of the i^{th} occurrence of the integer x". The first occurrence of an integer will have special significance; let $P_x = p(x,1)$.

Example. In the sequence 1 7 1 2 6 4 2 5 3 7 4 6 3 5, $p(6,1) = P_6 = 5$, $p(4,2) = 11$, $P_2 = 4$, etc.

Note that $p(x,i)$ is meaningful if $1 \leq x \leq m$ and $1 \leq i \leq n$, and P_x is meaningful if $1 \leq x \leq m$.

In a perfect n-sequence for m

$$p(x,i) = P_x + (x + 1)(i - 1) \qquad (1 \leq x \leq m;\ 1 \leq i \leq n) \tag{1}$$

which follows from the recursive formula (for $i \geq 2$)

$$p(x,i) = p(x,\ i - 1) + (x + 1)\ . \tag{2}$$

Theorem 1. There is no perfect n-sequence for n.

Proof. Assume such a sequence exists. Then it has n^2 entries. Also

$$p(n,n) = P_n + (n + 1)(n - 1) = P_n + n^2 - 1$$

so that P_n must be 1.

It is impossible that $1 \leq P_{n-1} \leq n$; otherwise $p(n - 1,\ P_{n-1})$ and $p(n,\ P_{n-1})$ are meaningful and

$$p(n - 1, P_{n-1}) = P_{n-1} + nP_{n-1} - n = p(n, P_{n-1})$$

using (1) and $P_n = 1$. But this is impossible since an n and an $n - 1$ cannot occupy the same position.

It is impossible that $n + 1 \le P_{n-1}$; otherwise $p(n - 1, n) \ge n^2 + 1$, but the largest position is n^2.

Now $1 \le n - 1 \le n$ (since $n \ge 2$) so that P_{n-1} is a positive integer, and we have a contradiction.

Theorem 2. There is no perfect n-sequence for $n + 1$, except the perfect 2-sequence for 3.

Proof. Assume such a sequence exists. Then there are $n(n + 1) = n^2 + n$ entries. Also,

$$p(n + 1, n) = P_{n+1} + n^2 + n - 2$$

so that either $P_{n+1} = 1$ or $P_{n+1} = 2$. If $P_{n+1} = 2$, then $p(n + 1, n) = n^2 + n$, the last position, but since a perfect sequence taken in reverse order is still a perfect sequence, this case is symmetrical to the case $P_{n+1} = 1$. Hence only the case $P_{n+1} = 1$ need be considered.

It is impossible that $1 \le P_n \le n$; otherwise $p(n, P_n) = p(n + 1, P_n)$. It is impossible that $n + 2 \le P_n$; otherwise $p(n, n) \ge n^2 + n + 1$. Therefore the only possibility is $P_n = n + 1$. Now we have $P_{n+1} = 1$ and $P_n = n + 1$.

It is impossible that $1 \le P_{n-1} \le n - 1$; otherwise $p(n - 1, P_{n-1} + 1) = p(n, P_{n-1})$. It is impossible that $n + 1 \le P_{n-1} \le 2n$; otherwise $p(n - 1, P_{n-1} - n) = p(n, P_{n-1} - n)$. It is impossible that $2n + 1 \le P_{n-1}$; otherwise $p(n - 1, n) \ge n^2 + n + 1$. Therefore the only possibility is $P_{n-1} = n$.

It is impossible that $1 \le P_{n-2} \le n - 1$; otherwise

$$p(n - 2, P_{n-2} + 1) = p(n - 1, P_{n-2}).$$

It is impossible that $n \le P_{n-2} \le 2n - 1$; otherwise

$$p(n - 2, P_{n-2} - n + 1) = p(n - 1, P_{n-2} - n + 1) .$$

It is impossible that $2n \le P_{n-2} \le 3n - 2$; otherwise

$$p(n - 2, P_{n-2} - 2n + 1) = p(n - 1, P_{n-2} - 2n + 2) .$$

It is impossible that $P_{n-2} = 3n - 1$; otherwise $p(n - 2, n) = p(n, n)$. It is impossible that $3n \le P_{n-2}$; otherwise $p(n - 2, n) \geq n^2 + n + 1$. If $n \neq 2$, then $1 \le n - 2 \le n$ and P_{n-2} is a positive integer, a contradiction. The only possibility is therefore $n = 2$.

From these two theorems some patterns can be seen. They are formulated in the following lemmas.

Lemma 1. In a perfect n-sequence for m, if $1 \le n - r \le m$, then

$$P_{n-r} \le mn - n^2 + nr - r + 1.$$

In particular, in a perfect n-sequence for $n + i$, $P_{n-r} \le nr + in - r + 1$.

Proof. If $P_{n-r} > mn - n^2 + nr - r + 1$, then $p(n - r, n) > mn$, which is impossible since the largest position is mn.

Lemma 2. In a perfect n-sequence for m, if P_x and P_{x+1} are meaningful, then it is impossible that

$$P_{x+1} + (i - 1)x + (2i - 2) \le P_x \le P_{x+1} + (i - 1)x + (i - 2) + n \tag{3}$$

for any integer $i \ge 1$, or that

$$P_{x+1} + (i - 1)x + (i - 1) \le P_x \le P_{x+1} + (i - 1)x + (2i - 3) + n \tag{4}$$

for any integer $i \le 1$.

Proof. Assuming (3) to hold (with $i \ge 1$), we have

$$P_{x+1} + (i - 1)x + (2i - 2) \le P_x \tag{5}$$

$$P_x \le P_{x+1} + (i - 1)x + (i - 2) + n. \tag{6}$$

It follows from (5) and (6), respectively, that

$$P_{x+1} + (i - 1)x + (i - 1) \le P_x \tag{7}$$

$$P_x \le P_{x+1} + (i - 1)x + (2i - 3) + n. \tag{8}$$

From (5) and (8) follows

$$1 \le P_x - P_{x+1} - ix + x - 2i + 3 \le n, \tag{9}$$

and from (7) and (6) follows

$$1 \le P_x - P_{x+1} - ix + x - i + 2 \le n. \tag{10}$$

Finally, we have

$$p(x, P_x - P_{x+1} - ix + x - 2i + 3) = p(x + 1, P_x - P_{x+1} - ix + x - i + 2), \tag{11}$$

which is meaningful by (9) and (10). But (11) is obviously false, hence (3) cannot hold if $i \geq 1$. The proof of the second half is identical.

Corollary to Lemma 2. If P_x and P_{x+1} are meaningful, then either

$$P_{x+1} + (i-1)x + (i-2) + n < P_x < P_{x+1} + ix + 2i$$

for some $i \geq 1$, or

$$P_{x+1} + (i-1)x + (2i-3) + n < P_x < P_{x+1} + ix + i$$

for some $i \leq 0$.

Theorem 3. There is no perfect n-sequence for $n + 2$, except the perfect 2-sequence for 4.

Proof. This sequence has $n^2 + 2n$ entries. By Lemma 1, the only possibilities for P_{n+2} are (case I) $P_{n+2} = 1$, (case II) $P_{n+2} = 2$, and (case III) $P_{n+2} = 3$.

Case I. $P_{n+2} = 1$. By Lemma 1 and the Corollary to Lemma 2, the only possibilities for P_{n+1} are (case IA) $P_{n+1} = n + 1$ and (case IB) $P_{n+1} = n + 2$.

Case IA. $P_{n+1} = n + 1$. By the lemmas, the only possibilities for P_n are 1, $n - 1$, n, $n + 1$, and $2n + 1$. But $P_n = 1$ is impossible; otherwise $p(n,1) = p(n + 2,1)$; $P_n = n + 1$ is impossible; otherwise $p(n,1) = p(n + 1,1)$. Therefore there are three possibilities: (case IA1) $P_n = n - 1$, (case IA2) $P_n = n$, and (case IA3) $P_n = 2n - 1$.

Case IA1. $P_n = n - 1$. The possibilities for P_{n-1} are $n - 2$, $2n - 1$, $3n - 1$, and $3n$. But n even is impossible; otherwise $p(n, n/2) = p(n + 2, n/2)$; so n is odd; $P_{n-1} = n - 2$ is impossible; otherwise $p(n - 1, (n + 1)/2) = p(n + 1, (n - 1)/2)$; $P_{n-1} = 3n - 1$ is impossible; otherwise $p(n - 1, n) = p(n + 1, n)$; $P_{n-1} = 3n$ is impossible; otherwise

$$p(n-1, (n-1)/2) = p(n+1, (n+1)/2) ;$$

Therefore $P_{n-1} = 2n - 1$. The possibilities for P_{n-2} are $n - 1$, $4n - 2$, and $4n - 1$. But $P_{n-2} = n - 1$ is impossible; otherwise $p(n - 2, 1) = p(n, 1)$; $P_{n-2} = 4n - 2$ is impossible; otherwise (noting that $1 \leq (n + 3)/2 \leq n$ since $n \geq 2$ and n is odd)

$$p(n-2, (n-1)/2) = p(n, (n+3)/2) ;$$

$P_{n-2} = 4n - 1$ is impossible; otherwise $p(n - 2, 1) = p(n - 1, 3)$. But $1 \leq n - 2 \leq n$ (since $n \geq 2$ and n odd) so that P_{n-2} is a positive integer, which is a contradiction. Therefore case IA1 is impossible.

This first case indicates the methods used. The others are treated similarly. The other cases are:

[Continued on page 392.]

THE CASE OF THE STRANGE BINOMIAL IDENTITIES OF PROFESSOR MORIARTY

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"My dear fellow," said Sherlock Holmes, as we sat on either side of the fire in his research library at Baker Street, "combinatorial identities are infinitely stranger than anything which the ordinary mortal mind can devise. If we could fly out of that window hand-in-hand, hover over some of the rare geniuses of mathematics, however, and peep in at the queer formulas boiling in their brains, the strange relations and inverse connections, vast chains of implications, we should see some very singular and ineluctable identities. Moreover, they form a beautiful order."

"And yet I am not convinced of it," I answered. "The formulas which appear in the literature are so numerous and diverse that I must quite agree with my old friend John Riordan [17, p. vii] who has often spoken of the protean nature of combinatorial identities. He has said that identities are both inexhaustible and unpredictable; and that the age-old dream of putting order in such a chaos is doomed to failure."

"A certain judicious selection must be made in order to exhibit the order which inheres in this subject," remarked Holmes. "This is wanting in research journals, where stress is placed on novelty and abstraction and the history of the subject is quite often laid entirely aside. A good detective of identities, however, remembers and retrieves numerous facts from the disarray of identities. This requires vast concentration and attention to detail. Depend upon it, behind every identity there is a whole history. As unofficial adviser to everyone who is puzzled by combinatorial identities, I come across many strange and bizarre formulas, none perhaps more strange than those formulas discovered by the infamous Professor Moriarty."

Holmes had now risen from his chair, and was standing before one of the enormous bookcases in his library, a library reputed to be filled with case histories of every binomial identity ever brought to trial. I could tell from his stance that he was about to embark on a story which would be both interesting and educational. He began speaking:

"Professor Moriarty first gave his formulas in the form of a dual pair:

(1) $$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p}$$

and

(2) $$\sum_{k=0}^{n-p} \binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p} 2^{2n-2p} .$$

I am indebted to E. T. Davis [7, p. 71] for calling them to my attention. Davis writes in a footnote that "We shall call these the identities of James Moriarty, since we do not know any other source from which such ingenious formulas could have come. See 'The Final Problem,' The Memoirs of Sherlock Holmes. "Here I have corrected formula (2) in that Davis' version would have $2p + k$ instead of the correct $2p + 2k$, something immediately obvious to an old combinatorial detective! I may say that I have also changed his summation variable from 's' to 'k' because my mind is stamped this way... it simplifies things."

Holmes paused and pulled out another book. He continued:

"The facts of Moriarty's life are well documented. Sabine Baring-Gould [3, pp. 21-23] has given them in a few lines. You see, Moriarty was my own teacher. As you know we developed into arch-enemies. Moriarty's mathematical work is summarized by Baring-Gould as follows:

> "At the age of twenty-one — in 1867 — this remarkable man had written a treatise on the binomial theorem which had a European vogue. On the strength of it — and because of certain connections his West of England family possessed — he won the mathematical chair at one of the smaller English universities. There he produced his magnum opus — a work for which, despite his later infamy, he will be forever famous. He became the author of 'The Dynamics of an Asteroid'."

It is clear from the further remarks given about this monumental work that a genius such as Moriarty could be responsible for formulas such as (1) and (2). Professor Davis, in a letter dated 29 July 1963, told me how elusive he found the proofs of (1) and (2) and could find no explicit reference in the literature. Actually there are many references, as will be clear in the list given below.

Moriarty was a master of disguise, and here we do find Riordan's remark about the protean nature of the identities pertinent. But it needs just a little care to see through the fabric. Replace k by $k - p$ and the two formulas become

$$\sum_{k=p}^{n} \binom{2n+1}{2k+1}\binom{k}{p} = \binom{2n-p}{p} 2^{2n-2p} \tag{3}$$

and

$$\sum_{k=p}^{n} \binom{2n}{2k}\binom{k}{p} = \frac{n}{2n-p}\binom{2n-p}{p} 2^{2n-2p} , \tag{4}$$

and in this form they appeal more to my experienced eye. As a matter of fact two slightly more general such formulas may be found in [12] where formulas (3.120) and (3.121) are as follows:

$$\sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j} \tag{5}$$

and

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} = 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j} \qquad (6)$$

I first came upon (5) and (6) while studying elementary matrix theory. I was trying to determine the n^{th} power of a 2-by-2 matrix. If t_1 and t_2 denote distinct characteristic roots of such a matrix M, then it is easy to prove that

$$M^n = \frac{t_2^n - t_1^n}{t_2 - t_1} M - \frac{t_2^{n-1} - t_1^{n-1}}{t_2 - t_1} |M|I, \quad |M| = t_1t_2, \qquad (7)$$

where $|M| = \det(M)$ and I is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, by successively multiplying M times itself, one is led to conjecture and prove by induction that in fact

$$\begin{aligned} M^n = M &\sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k |M|^k \binom{n-k-1}{k} (t_1 + t_2)^{n-1-2i} \\ &- |M|I \sum_{k=0}^{\left[\frac{n-2}{2}\right]} (-1)^k |M|^k \binom{n-k-2}{k} (t_1 + t_2)^{n-2-2k} \end{aligned} \qquad (8)$$

Upon equation (7) and (8) we obtain a single identity

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} (t_1t_2)^j (t_1 + t_2)^{n-2j} = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1}, \quad t_2 \neq t_1 . \qquad (9)$$

Separating out the even and odd index terms in the binomial expansion it is easy to obtain

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} z^k = \frac{(1+\sqrt{z})^{n+1} - (1-\sqrt{z})^{n+1}}{(1+\sqrt{z}) - (1-\sqrt{z})},$$

and we also have

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1}(x+1)^k = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1}\sum_{j=0}^{k}\binom{k}{j}x^j$$

$$= \sum_{j=0}^{[n/2]} x^j \cdot \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j}.$$

More elegantly, we have proved that

$$\sum_{j=0}^{[n/2]} x^j \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j} = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1}, \tag{10}$$

where $t_2 = 1 + \sqrt{x+1}$, $t_1 = 1 - \sqrt{x+1}$. Since we have in this case $t_1 + t_2 = 2$, $t_1t_2 = -x$, we may apply (10) to (9) when $t_1 + t_2 = 2$, and the result by equating coefficients of powers of x is the identity

$$\sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j},$$

which is precisely formula (5) above. A natural companion piece may be found, and I use what I call an even and odd index argument to do this.

Indeed, write

$$S_n = \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j}.$$

Now

$$\binom{n}{2k} + \binom{n}{2k+1} = \binom{n+1}{2k+1}$$

so that we have

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} + \sum_{k=j}^{[n/2]} \binom{n}{2k+1}\binom{k}{j} = \sum_{k=j}^{[n/2]}\binom{n+1}{2k+1}\binom{k}{j}.$$

Look at the second sum on the left: If n is odd $[n/2] = [(n-1)/2]$; but if n is even $[n/2] = [(n-1)/2] + 1$. It follows readily that we have proved

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} = S_n - S_{n-1} = 2^{n-2j}\binom{n-j}{j} - 2^{n-1-2j}\binom{n-1-j}{j}$$

$$= 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j} \quad ,$$

in other words, we have proved formula (6) above.

The above proof was first obtained by me in the year 1950. It may or may not be original, as it is very difficult to guarantee originality. Formulas (5)-(6) arise naturally in the study of trigonometric identities. In the disguised forms (3) and (4) you will also find them in such studies. Glocksman and Ruderman [10] gave inductive proofs of (5) and (6). They have an interesting footnote calling attention to a pending Sherlock Holmes tale about these strange relations of Professor Moriarty, so that my present remarks are long overdue.

Trigonometric proofs are implicit in Bromwich [5, Chapter 9]. Such proofs are bound up with the well-known expansions

$$\cos nx = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} \frac{n}{n-j}\, 2^{n-2j-1} (\cos x)^{n-2j} \tag{11}$$

and

$$\frac{\sin(n+1)x}{\sin x} = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} 2^{2n-2j}(\cos x)^{n-2j} \quad . \tag{12}$$

See also Deaux [8] who works in reverse, using (6) to prove (11). Formula (1), with n replaced by n/2, was posed by André [1] and the solution given in 1871 by one Moret-Blanc suggests that this may be one of Moriarty's French disguises. Briones [4] was evidently unaware that the summation (6) could be done in closed form, as he carries the sum around consistently in unsummed form. Kaplansky, in a review of a paper of Gonzáles del Valle [11], restates the author's formula in our form somewhat like (1)... but note a few differences. This same related formula of Gonzáles del Valle occurs in [13]. Fred. Schuh [18] again gives something equivalent to (9) and obtains our formula (1) in only a slight variation. Singer [19] rediscovers a special case of the companion to (9) involving the coefficients

$$\frac{n}{n-j}\binom{n-j}{j} \quad ,$$

using formula (6), and cites Netto's famous Combinatorik [16].

The formulas in Netto [16, pp. 246-258] are taken from Father Eagen's valuable treatise [14, pp. 64-68]. It is curious to note that Netto attempts to correct some of Hagen's formulas and introduces further errors where errors sometimes did not appear. The basic error consists in failing to appreciate that $\binom{x}{n}$ is a polynomial of degree n in x, whence some of the inequalities appended by Netto are superfluous. Also, Netto omits one very striking formula (number 17) of Hagen, as being a linear combination of the others, which it is not. Hagen's formula (17) was the motivation of over 25 papers by me since 1956, that formula tracing to 1793 and one H. A. Rothe. A full discussion of the errors in Hagen and Netto must, however, await another time. At any rate, the formulas of Moriarty are in Netto and Hagen.

The connection of our formulas with Fibonacci-type polynomials and numbers should be clear because of formula (9) and the well-known formula

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} = F_{n+1} ,$$

where $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$, $F_1 = 1$, defining the Fibonacci numbers. As further evidence of this connection, Lind [15] has written to me of the following matter. Let $f_1(x) = 1$, $f_2(x) = x$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, denote the Fibonacci polynomials. It is known that

$$f_n(x) = \frac{1}{\sqrt{x^2+4}} \left\{ \left(\frac{x + \sqrt{x^2+4}}{2} \right)^n - \left(\frac{x - \sqrt{x^2+4}}{2} \right)^n \right\} .$$

What is more,

$$f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

so that comparison of coefficients yields

$$\sum_{k=j}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}\binom{k}{j} = 2^{n-2j-1}\binom{n-j-1}{j} ,$$

which, save for a shift of 1 in the index n, is precisely formula (5) above. Our detective work pays off. It shows that Moriarty is implicit in all the literature about Fibonacci polynomials and related generalizations.

As for further references in accessible literature, one should examine ex. 5, pp. 209-210 of Vol. 2 of Chrystal [6]. What is done there is to use an expansion on p. 202 for $t_2^n + t_1^n$

(instead of what we used in (9) above, thereby being a Lucas approach instead of a Fibonacci approach). Chrystal's formula may be written in the form

$$\sum_{s=0}\binom{m}{2r+2s}\binom{r+s}{s} = 2^{m-2r-1}\binom{m-r}{r}\frac{m}{m-r} ,$$

where I have set $n = m - 2r$ in his formulas. The result is, of course, our formula (6) in slight disguise. Both (5) and (6) are in Riordan [17, pp. 87, 243].

Moret-Blanc's solution to Andre's problem [1] finds something equivalent to our (5) as the coefficient of x^{2k+1} in the series expansion of

$$(1+x)^{n+1}\left(1-\frac{1}{x^2}\right)^{-k-1}$$

found in two ways.

Netto's proof of the formula in the form (same as in [11] and [13])

$$\sum_{s=0}\binom{p+s}{s}\binom{2p+m}{2p+2s+1} = 2^{m-1}\binom{m+p-1}{p} \tag{13}$$

is by equating coefficients in the algebraic identity

$$(1-x)^{-2p}\left\{\left(1-\frac{x}{1-x}\right)^2\right\}^{-p} = (1-2x)^{-p} . \tag{14}$$

Our formula (1) of Moriarty then follows by changing s into k and putting $m = 2n - 2p + 1$. A similar argument goes for the companion which he gives and which is of course equivalent to (2) here.

In our sleuthing of these old results we have come across one very old appearance in the literature of a Moriarty type formula in the form (6). The date of this case is 1826, a long time before the historic Moriarty appeared. Perhaps he lifted the results from such an earlier source, being an evil and crafty genius. In any event, Andreas von Ettingshausen, in his surprisingly modern book [9] gives the formula (page 257)

$$\mathfrak{S}_0^r\binom{n}{2r}\binom{r}{r-v} = \frac{n}{n-v}\binom{n-v}{v}2^{n-2v-1} . \tag{15}$$

This is the exact notation he uses. The German 'S' is used in place of Greek sigma for summation, and the indices of summation are arranged differently than modern form, but otherwise it is precisely the same formula. Ettingshausen gives many other formulas which are still today being rediscovered. Incidentally, Ettingshausen's book of 1826 has the first appearance in print of the modern symbol $\binom{n}{r}$ in place of the previously common symbols $\left(\frac{n}{r}\right)$ or $\left[\frac{n}{r}\right]$ used by Euler. Some historians have stated (Cajori notably) that the year was 1827 in another book by Ettingshausen, however the correct item appears to be the 1826 book.

We come now to a modern chapter: INVERSION. A good detective would not earn his pay if he did not make adroit use of inversion of series. Many such inverse series pairs exist, and we can do no better for the time than refer the reader to the excellent discussion by Riordan [17] for information on inverse series pairs of many types. We need one such, a very simple instance.

It is easy to prove by the use of the orthogonality relation

$$\sum_{k=r}^{j} (-1)^{k+j} \binom{k}{r}\binom{j}{k} = \begin{cases} 0, & j \neq r, \\ 1, & j = r, \end{cases}$$

that

$$\sum_{k=r}^{m} (-1)^{k} \binom{k}{r} f(k) = g(r) \tag{16}$$

if and only if

$$\sum_{k=r}^{m} (-1)^{k} \binom{k}{r} g(k) = f(r) .$$

The proof is nothing but inverting order of summation and using the stated orthogonality (cf. Riordan [17], p. 85). Taking

$$f(k) = (-1)^k \binom{n+1}{2k+1} \quad \text{and} \quad g(r) = 2^{n-2r}\binom{n-r}{r}, \quad m = \left[\frac{n}{2}\right],$$

we see that (5) inverts to yield

$$\sum_{k=r}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k}\binom{k}{r} 2^{n-2k} = (-1)^r \binom{n+1}{2r+1} . \tag{17}$$

Similarly, choosing

$$f(k) = (-1)^k \binom{n}{2k}, \quad g(r) = 2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r}, \quad m = \left[\frac{n}{2}\right],$$

we see that (6) inverts to yield

$$\sum_{k=r}^{\left[\frac{n}{2}\right]} (-1)^k \binom{k}{r} 2^{n-2k-1}\binom{n-k}{k}\frac{n}{n-k} = (-1)^r \binom{n}{2r}. \tag{18}$$

I have not noticed any significant appearance of these relations in the literature until now relation (17) has been found by Marcia Ascher [2] who has given an inductive proof. Naturally the skilled combinatorial detective expects a companion formula such as (18)."

Here Holmes rested and then concluded his story by saying that all of these relations are in turn special cases of much more beautiful ones, which must await another time and place. So much for the evil genius of Moriarty.

ADDENDUM

It may be of interest to show that two well-known Fibonacci formulas

$$F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \tag{19}$$

and

$$F_{n+1} = 2^{-n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1} 5^k \tag{20}$$

may be derived, the one from the other, in either of two ways: by use of (5) or by use of (17). We also need the binomial theorem.

We have first, using (5) of the generalized Moriarty,

$$F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} = \sum_{k=0}^{[n/2]} 2^{2k-n} \sum_{j=k}^{[n/2]} \binom{n+1}{2j+1}\binom{j}{k}, \quad \text{by (5)}$$

$$= 2^{-n} \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} \sum_{k=0}^{j} \binom{j}{k} 2^{2k} = 2^{-n} \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} 5^j ,$$

and the steps are reversible, showing (19) and (20) equivalent via (5).

We have next, using (17) of Ascher,

$$F_{n+1} = 2^{-n}\sum_{r=0}^{[n/2]}\binom{n+1}{2r+1}5^r = 2^{-n}\sum_{r=0}^{[n/2]}5^r\sum_{k=r}^{[n/2]}(-1)^{k+r}\binom{n-k}{k}\binom{k}{r}2^{n-2k}$$

$$= \sum_{k=0}^{[n/2]}\binom{n-k}{k}(-1)^k 2^{-2k}\sum_{r=0}^{k}\binom{k}{r}(-5)^r$$

$$= \sum_{k=0}^{[n/2]}\binom{n-k}{k}(-1)^k 2^{-2k}(-4)^k = \sum_{k=0}^{[n/2]}\binom{n-k}{k} ,$$

and again the steps are reversible, showing (19) and (20) equivalent via (17). Similarly it is possible to use the Moriarty and inverse Moriarty relations to show other equivalences.

Thus relations (6) and (18) may be used precisely as above to show that the formulas

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k} = L_n \tag{21}$$

and

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}5^k = 2^{n-1}L_n \tag{22}$$

are equivalent. Here $L_{n+1} = L_n + L_{n-1}$, with $L_0 = 2$, $L_1 = 1$, define the Lucas numbers.

Another similar equivalence which follows from use of (5) or (17) is the pair of combinatorial identities

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-x}{n-k}\binom{n-k}{k}2^{n-2k} = \binom{2n-2x}{n} \tag{23}$$

and

$$\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n+1}{2j+1}\binom{n-x+j}{n} = \binom{2n-2x}{n} \tag{24}$$

valid for all real x. Identity (23) is essentially (3.107) in [12].

Again, the equivalence of the two summations

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n+k}{k}\binom{n-k}{k} 2^{n-2} = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{j}\binom{n+1}{2j+1} \tag{25}$$

follows by use of (5) or by (17), although no closed form is known. By use of (5) it is easy to see that a closed form for

$$\sum_{k=0}^{[n/2]} \binom{n+k}{k}\binom{n-k}{k}$$

might depend on a closed form for

$$\sum_{k=0}^{j} \binom{n+k}{k}\binom{j}{k} 2^{2k} .$$

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[Continued on page 402.]

which says that a sequence of integers, which is uniformly distributed mod m, where m is composite, is also uniformly distributed with respect to any positive divisor of m, we then have that $\{F_n\}$ is uniformly distributed mod p where p is some prime factor of m, >2 and $\neq 5$. This contradicts Theorem 3.

Conjecture: The Fibonacci Sequence $\{F_n\}$ is uniformly distributed mod 5^k $(k = 3, 4, \cdots)$.

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[Continued from page 380.]

PERFECT N-SEQUENCES FOR N, N + 1, AND N + 2

IA1	$P_{n+2} = 1$	$P_{n+1} = n + 1$	$P_n = n - 1$	
IA2a	1	1	n	$P_{n-1} = 2n$
IA2b	1	n + 1	n	3n
IA3a	1	n + 1	2n + 1	n
IA3b	1	n + 1	2n + 1	2n
IB1	1	n + 2	n	
IB2	1	n + 2	n + 1	
I1A1	2	1	n + 1	
I1A2	2	1	n + 2	
(I1B	2	n + 2	symmetrical to case IIA)	
(III	3	symmetrical to case I).		

Each of these cases is impossible except IA3a and its mirror image in case III which give only the perfect 2-sequence for 4.

Applying these methods to higher cases would either disprove them or produce examples. The length of such an application would be prohibitive, however.

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REPRERESENTATIONS OF AUTOMORPHIC NUMBERS

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An n-place automorphic number $x > 1$ is a natural number with n digits such that the last n digits of x^2 are equal to x (see, for example [1]). In number-theoretic notation, this definition can be expressed quite compactly as $x - x^2 \equiv 0 \pmod{10^n}$. An example of a 3-place automorphic number is 625. A recent report [2] indicates that automorphic numbers with 100,000 digits have been computed.

It is a simple matter to prove that automorphic numbers with any number of digits exist. Further, if x is an automorph of n digits, then it follows that $y = 10^n + 1 - x$ is also. In other words, n-place automorphic numbers occur in pairs. (This statement is not quite accurate. For example, the "two" 4-place automorphs are 9376 and 0625. If we accept the convention that a leading zero is distinctive, then 0625 may be considered a 4-place automorph different from the 3-place automorph 625).

The purpose of this paper is to present the following representations for automorphic numbers:

Theorem. If x is an n-place automorphic number, then $y \pmod{10^{tn}}$, defined by

$$y = x^t \sum_{k=0}^{t-1} (-1)^k \binom{t+k-1}{k} \binom{2t-1}{t+k} x^k \tag{1}$$

is a tn-place automorphic number, $t = 1, 2, 3, \cdots$. Moreover,

$$y = \frac{t}{2} \binom{2t}{t} \int_0^x (u - u^2)^{t-1}\, du \,. \tag{2}$$

Remarks. a. In the case $t = 1$, the theorem gives the trivial identity $y = x$.

b. These representations, for the case $t = 2$, are presented in [3, page 257] and in [2].

c. Apparently (due to multiple-precision requirements on digital computers) these representation formulas do not give any special advantage to their user in computing automorphic numbers with large numbers of digits. Even other alternatives for doing the necessary arithmetic with large integers, e.g., modular arithmetic, seem also to present major problems in appying these formulas.

d. The following definition and binomial coefficient identities are used in the proof of the theorem.

$$\binom{r}{k} \equiv \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots(1)}, & \text{integer } k \ge 0, \\ 0, \quad \text{integer } k < 0; \end{cases}$$

(3) $$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \quad \text{integer } k \neq 0 ;$$

(4) $$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k}, \quad \text{integer } n ;$$

(5) $$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}, \quad \text{integers } m, k ;$$

(6) $$\sum_{k=0}^{n} (-1)^k \binom{r}{k} = (-1)^n \binom{r-1}{n}, \quad \text{integer } n \ge 0 .$$

Proof of Theorem. First, by divisibility properties of primes, a necessary and sufficient condition that $x > 1$ is an n-place automorphic number is that either

(7) $$\begin{gathered} x \equiv 0 \pmod{2^n} \quad \text{and} \quad x \equiv 1 \pmod{5^n} \\ \text{or} \\ x \equiv 1 \pmod{2^n} \quad \text{and} \quad x \equiv 0 \pmod{5^n} . \end{gathered}$$

Hence $x = qp^n + r$, where $p = 2$ or 5 and $r = 0$ or 1. By the binomial expansion formula,

$$x^k = \sum_{m=0}^{k} \binom{k}{m} (qp^n)^{k-m} r^m, \quad \text{where } k = 1, 2, \cdots .$$

Suppose now that it is possible to find integers a_k, independent of x, such that

(8) $$y = \sum_{k=0}^{s} a_k x^k$$

is an tn-place automorph for any n-place automorph x. Then

(9) $$y \equiv r \pmod{p^{tn}}, \quad p = 2 \text{ or } 5, \quad r = 0 \text{ or } 1 .$$

By replacing x^k with its binomial expansion and interchanging orders of summation,

$$y \equiv \sum_{j=0}^{t-1} A_j (qp^n)^j \equiv r \pmod{p^{tn}} ,$$

where

$$A_j \equiv \sum_{k=j}^{s} \binom{k}{j} a_k r^{k-j} .$$

Due to our assumption that y is automorphic for any automorph x, it follows that

$$A_j = \delta_0^j r, \qquad j = 0, 1, 2, \cdots, t - 1 .$$

Hence, for $r = 0$, it follows that

$$\text{(10)} \qquad a_k = 0, \qquad k = 0, 1, 2, \cdots, t - 1$$

Further, for $r = 1$, the remaining $s - t + 1$ coefficients are related in t linear equations. If we choose $s = 2t - 1$, then the necessary conditions on the remaining t coefficients in the representation y are the t linear equations

$$\text{(11)} \qquad \sum_{k=t}^{2t-1} \binom{k}{j} a_k = \delta_0^j , \qquad j = 0, 1, 2, \cdots, t - 1 .$$

We now verify that this system has a solution, indeed

$$\text{(12)} \qquad a_k = (-1)^{k-t} \binom{k - 1}{t - 1} \binom{2t - 1}{k} , \qquad k = t, \cdots, 2t - 1$$

defines a set of solutions of the linear system (11). Having proven this result, then $y \equiv r \pmod{p^{tn}}$, i.e., $y \pmod{10^{tn}}$ is a tn-place automorph.

First, consider the cases $j = 1, \cdots, t - 1$. Let

$$S \equiv \sum_{k=t}^{2t-1} (-1)^{k-t} \binom{k}{j} \binom{k - 1}{t - 1} \binom{2t - 1}{k} .$$

So, by using binomial identities (4) and (5)

$$S = \binom{2t - 1}{j} \sum_{k=t}^{2t-1} (-1)^{k-t} \binom{2t - 1 - j}{k - j} \sum_{i=0}^{t-1} \binom{j - 1}{i} \binom{k - j}{t - 1 - i} .$$

By again using binomial coefficient identity (5), we get

$$S = \binom{2t-1}{j}\sum_{i=0}^{j-1}\binom{j-1}{i}\binom{2t-1-j}{t-1-i} T ,$$

where

$$T \equiv \sum_{k=t}^{2t-1}(-1)^{k-t}\binom{t-j+i}{k-t-j+1+i}$$

$$\equiv (-1)^{j-i-1}\left[\sum_{k=t+j-i-1}^{2t-1}(-1)^{k-t-j+i+1}\binom{t-j+i}{k-t-j+i+1}\right.$$

$$\left. - \sum_{k=t+j-i-1}^{t-1}(-1)^{k-t-j+i+1}\binom{t-j+1}{k-t-j+i+1}\right] .$$

with the first sum in the brackets equal to zero (by a special case of identity (6)). Then, by binomial coefficient identity (6) again,

$$T = \binom{t-j+i-1}{i-j} .$$

Hence $T = 0$, since $i - j < 0$. Therefore $S = 0$ for $j = 1, 2, \cdots, t-1$.

In case $j = 0$,

$$S = (2t-1)\binom{2t-2}{t-1}\sum_{k=t}^{2t-1}\frac{(-1)^k}{k}\binom{t-1}{k-t}$$

by first applying identity (3), then (5). Hence replacing $k - t$ by t gives

$$S = (2t-1)\binom{2t-2}{t-1}\sum_{k=0}^{t-1}\frac{(-1)^k}{t+k}\binom{t-1}{k} = 1 ,$$

since for the Beta function B,

$$B(t,t) \equiv \int_0^1 v^{t-1}(1-v)^{t-1}dv = \frac{1}{t\binom{2t-1}{t-1}} ,$$

and expanding $(1-v)^{t-1}$ and integrating each term yields

[Continued on page 402.]

A PRODUCT IDENTITY FOR SEQUENCES DEFINED BY $W_{n+2} = dW_{n+1} - cW_n$

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1. INTRODUCTION

Let W_0, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define

(1.1) $$W_{n+2} = dW_{n+1} - cW_n, \qquad d^2 - 4c \neq 0, \qquad (n = 0, 1, \cdots),$$

(1.2) $$Z_n = (a^n - b^n)/(a - b) \qquad (n = 0, 1, \cdots),$$

(1.3) $$V_n = a^n + b^n \qquad (n = 0, 1, \cdots),$$

where $a \neq b$ are the roots of $y^2 - dy + c = 0$. We shall define

(1.4) $$W_{-n} = (W_0V_n - W_n)/c^n \qquad (n = 0, 1, \cdots).$$

If $W_0 = 0$ and $W_1 = 1$, then $W_n \equiv Z_n$, $n = 0, 1, \cdots$; and if $W_0 = 2$ and $W_1 = d$, then $W_n \equiv V_n$, $n = 0, 1, \cdots$. The phrase, Lucas functions (of n) is often applied to Z_n and V_n.

It should be noted that

(1.5) $$W_n = W_0Z_{n+1} + (W_1 - dW_0)Z_n \qquad (n = 0, 1, \cdots);$$

and we shall refer to Z_n, $n = 0, 1, \cdots$, as the fundamental solution of (1.1). Let W_n^* be a second, general solution of (1.1) with initial values W_0^* and W_1^*. Since W_n^* also satisfies (1.5), we now see that the product sequence, $W_nW_n^*$, can be represented as a linear combination of Z_{n+1}^2, Z_mZ_{n+1}, and Z_n^2. We observe that

(1.6) $$W_nW_n = C_1a^{2n} + C_2b^{2n} + C_3c^n \qquad (n = 0, 1, \cdots),$$

where C_i, $i = 1, 2, 3$, are arbitrary constants, is the general solution of a third-order linear difference equation whose characteristic equation is

(1.7) $$(x - c)(x^2 - V_2x + c^2) = 0.$$

If the initial conditions of W_n and W_n^* are chosen such that $C_3 \equiv 0$, then $W_nW_n^*$ is also a solution of a second-order linear difference equation, and its representation is of interest.

2. STATEMENT OF RESULTS

Theorem 1. Let W_n and W_n^*, $n = 0, 1, \cdots$, be solutions of (1.1). Then (see (1.6))

$$W_2W_2^* - V_2W_1W_1^* + c^2W_0W_0^* = 0 \tag{2.1}$$

is a necessary and sufficient condition that $C_3 \equiv 0$. If $C_3 \equiv 0$, then

$$W_nW_n^* = ((W_1W_1^* - (d^2 - c)W_0W_0^*)/d)Z_{2n} + W_0W_0^*Z_{2n+1}\ ; \tag{2.2}$$

and if $P_n \equiv W_nW_n^*$, then

$$P_{n+2} - V_2P_{n+1} + c^2P_n = 0 \qquad (n = 0, 1, \cdots)\ , \tag{2.3}$$

and

$$\frac{P_0 + (P_1 - V_2P_0)x}{1 - V_2x + c^2x^2} = \sum_{n=0}^{\infty} P_nx^n, \qquad (V_2 = d^2 - 2c)\ . \tag{2.4}$$

Corollary 1. If $d = -c = 1$, then $W_n \equiv H_n$, where H_n is the generalized Fibonacci number. Since $V_2 = 3$ and $Z_n \equiv F_n$, the ordinary Fibonacci number, we obtain from (2.2)

$$\begin{aligned} H_nH_n^* &= (H_1H_1^* - 2H_0H_0^*)F_{2n} + H_0H_0^*F_{2n+1} \\ &= H_1H_1^*F_{2n} - H_0H_0^*F_{2n-2} \end{aligned}\ , \tag{2.5}$$

(since $F_{2n+1} = 2F_{2n} - F_{2n-2}$), where (see (2.1))

$$H_2H_2^* - 3H_1H_1^* + H_0H_0^* = 0\ . \tag{2.6}$$

If $H_n^* = H_{n-1} + H_{n+1} \equiv G_n$, $n = 0, 1, \cdots$, then (2.6) is satisfied and thus (2.5) gives

$$H_nG_n = H_1G_1F_{2n} - H_0G_0F_{2n-2} \qquad (n = 0, 1, \cdots)\ ; \tag{2.7}$$

and from (2.4), we obtain

$$\frac{H_0G_0 + (H_1G_1 - 3H_0G_0)x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} H_nG_nx^n\ . \tag{2.8}$$

Remarks. Our special result (2.7) solves completely the problem posed by Brother U. Alfred [1], where (2,9), for example, must stand for (H_0, H_1), and not, as incorrectly indicated (H_1, H_2). If $H_n \equiv F_n$, then $G_n \equiv L_n$, and (2.7) reduces to the well-known identity, $F_nL_n = F_{2n}$; and (2.8) gives

$$\frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n .$$

3. PROOF OF THEOREM 1

For $n = 0, 1,$ and 2, Eq. (1.6) gives a linear system of three equations for the three unknowns C_1, C_2, and C_3. We readily find that $C_3 = N/D$, where $D = cd(a - b)^3 \neq 0$ is the determinant of the system

$$W_0W_0^* = C_1 + C_2 + C_3 \tag{3.1}$$

$$W_1W_1^* = a^2C_1 + b^2C_2 + cC_3 \tag{3.2}$$

$$W_2W_2^* = a^4C_1 + b^4C_2 + c^2C_3 \tag{3.3}$$

and

$$N = \begin{vmatrix} 1 & 1 & W_0W_0^* \\ a^2 & b^2 & W_1W_1^* \\ a^4 & b^4 & W_2W_2^* \end{vmatrix} .$$

If we set $N = 0$, we obtain the necessary condition (2.1) for $C_3 = 0$.

For the sufficiency proof, we assume that (2.1) is true. If we multiply both sides of (3.1) by c^2 and both sides of (3.2) by $-V_2$, then the addition of the resulting equations to (3.3) gives, using (2.1),

$$0 = (c^2 - a^2V_2 + a^4)C_1 + (c^2 - b^2V_2 + b^4)C_2 + (c^2 - cV_2 + c^2)C_3 . \tag{3.4}$$

Since $c = ab$ and $V_2 = a^2 + b^2$, we obtain from (3.4)

$$0 = -ab(a - b)^2C_3 .$$

Since $a \neq b \neq 0$, we must have $C_3 = 0$.

If $C_3 \equiv 0$, then (see (1.6))

$$P_n \equiv W_nW_n^* = C_1a^{2n} + C_2b^{2n}, \qquad n = 0, 1, \cdots .$$

Since $P_0 = C_1 + C_2$, we obtain, respectively, noting (1.2),

$$P_n = C_2(b - a)Z_{2n} + P_0a^{2n} \quad (n = 0, 1, \cdots), \tag{3.5}$$

$$P_n = C_1(a - b)Z_{2n} + P_0b^{2n} \quad (n = 0, 1, \cdots). \tag{3.6}$$

Evaluating C_2 in (3.5) (for $n = 1$) and C_1 in (3.6) (for $n = 1$), we obtain, respectively, after simplification,

$$P_n = [(P_1 - a^2P_0)/d]Z_{2n} + P_0a^{2n} \quad (n = 0, 1, \cdots), \tag{3.7}$$

$$P_n = [(P_1 - b^2P_0)/d]Z_{2n} + P_0b^{2n} \quad (n = 0, 1, \cdots). \tag{3.8}$$

Addition of (3.7) and (3.8) gives

$$2P_n = [(2P_1 - V_2P_0)/d]Z_{2n} + P_0V_{2n} \quad (n = 0, 1, \cdots). \tag{3.9}$$

Since (see (1.5)) $V_{2n} = 2Z_{2n+1} - dZ_{2n}$, we obtain from (3.9)

$$2dP_n = (2P_1 - P_0(V_2 + d^2))Z_{2n} + 2dP_0Z_{2n+1}. \tag{3.10}$$

Noting that $V_2 + d^2 = 2d^2 - 2c$, we obtain from (3.10),

$$P_n = [(P_1 - P_0(d^2 - c))/d]Z_{2n} + P_0Z_{2n+1}. \tag{3.11}$$

Since $P_n \equiv W_nW_n^*$, Eq. (3.11) reduces to (2.2).

If we set $(E^2 - V_2E + c^2)W_nW_n^* = Q_n$, where $E^mA_n = A_{n+m}$, then (1.7) becomes

$$(E - c)Q_n = 0. \tag{3.12}$$

The solution to (3.12) is

$$Q_n = Kc^n \quad (K, \text{ a constant}). \tag{3.13}$$

But $K = Q_0$, and so (3.13) reads

$$W_{n+2}W_{n+2}^* - V_2W_{n+1}W_{n+1}^* + c^2W_nW_n^* = Q_0c^n, \tag{3.14}$$

where

$$Q_0 = W_2W_2^* - V_2W_1W_1^* + c^2W_0W_0^*.$$

If (2.1) is true, then $Q_0 = 0$, and $P_n \equiv W_nW_n^*$ satisfies (2.3); and (2.4) follows readily from (2.3).

4. COMMENTS

If $W_n^* = W_{n-1} - (1/c)W_{n+1}$ in Theorem 1, then (2.1) is satisfied. For example, if $W_{n+2} = 2W_{n+1} + W_n$, then $\{Z_n\}_0^\infty = \{0, 1, 2, 5, 12, \cdots\}$, where Z_n is Pell's sequence. If we choose

$$\{W_n\}_0^\infty = \{2, 3, 8, 19, \cdots\}$$

and set

$$W_n^* = W_{n-1} + W_{n+1},$$

then

$$\{W_n\}_0^\infty = \{2, 10, 22, \cdots\};$$

and since $d = 2$ and $c = -1$, we obtain from (2.2) in Theorem 1

$$W_nW_n^* = 5Z_{2n} + 4Z_{2n+1} \qquad (n = 0, 1, \cdots), \tag{4.1}$$

where Z_n is Pell's sequence.

Using results of the author [2, p. 242], it seems reasonable that the conclusions of Theorem 1 may be extended (properly interpreted) to p products of solutions of (1.1), where $p = 2, 4, 6, \cdots$. For example, if $P_n = W_nW_n^*W_n^{**}W_n^{***}$, where W_n, W_n^*, W_n^{**}, and W_n^{***} are independent solutions of (1.1), then P_n satisfies a fifth-order linear difference equation (see [2, (2.2), p. 242] whose characteristic equation is

$$(x - c^2)\prod_{j=0}^{1}(x^2 - c^jV_{4-2j}x + c^4) = 0. \tag{4.2}$$

Since

$$P_n = C_1a^{4n} + C_2(a^3b)^n + C_3c^{2n} + C_4(ab^3)^n + C_5b^{4n},$$

we believe that $C_3 \equiv 0$ if and only if

$$\left[\prod_{j=0}^{1}(E^2 - c^jV_{4-2j}E + c^4)\right]P_0 = 0. \tag{4.3}$$

However, the representation of P_n under (4.3) is another matter.

For the case $d^2 = 4c$, $d \neq 0$, it appears that (2.2) of Theorem 1 holds under (2.1). Moreover, if $2W_1 = dW_0$, then (2.1) holds for any arbitrary sequence W_n^*. Since $a = b$, we have $Z_n = na^{n-1}$, $n = 0, 1, \cdots$, in (2.2).

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$$B(t,t) = \sum_{k=0}^{t-1} \binom{t-1}{k} \frac{(-1)^k}{t+k} .$$

Hence $y \pmod{10^{tn}}$, defined by (8), with coefficients given by (10) and (12), is an automorphic number of tn places. By replacing $k - t$ by k, we get the representation (1). Further, by using identity (5),

$$y = t\binom{2t-1}{t} x^t \sum_{k=0}^{t-1} \frac{(-x)^k}{t+k} \binom{t-1}{k},$$

where

$$\frac{1}{x^t} \int_0^x u^{t-1}(1-u)^{t-1}du = \int_0^1 v^{t-1}(1-xv)^{t-1}dv$$

$$= \sum_{k=0}^{t-1} \binom{t-1}{k} \frac{(-x)^k}{t+k} ,$$

by expanding $(1 - xv)^{t-1}$ and integrating term-by-term. This result yields the representation (2).

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PELL NUMBER TRIPLES

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Horadam [1] has shown that Pythagorean triples are Fibonacci-number triples. It has now been found that Pythagorean triples are Pell-number triples as well.

The Diophantine solution for Pythagorean triples (x, y, z) is $x = 2pq$, $y = p^2 - q^2$, and $z = p^2 + q^2$, where $p > q$. For primitive solutions $(p, q) = 1$, p and q are of different parity, x or $y \equiv 0 \pmod 3$ [2], $x \equiv 0 \pmod 4$, and all prime factors of z are congruent to 1 modulo 4. Since $x \neq y$, regardless of primitivity, let

$$y - x = p^2 - q^2 - 2pq = \pm c \tag{1}$$

which is readily transformed into

$$p - q = \sqrt{2q^2 \pm c} \tag{2}$$

and

$$p + q = \sqrt{2p^2 \pm c} \quad . \tag{3}$$

It may be noted in passing that all values of c for primitive triples are of the form $12d \pm 1$ and $12d \pm 5$, where $d = 0, 1, 2, 3, \cdots$. However, less than fifty percent of numbers of this form are possible values of c, because this representation by means of three Pell numbers includes all odd numbers not divisible by 3.

Two characteristic identities of the Pell-number sequence,

$$P_{n+2} = 2P_{n+1} + P_n \qquad (P_0 = 0,\ P_1 = 1) \tag{4}$$

and

$$(P_{n+1} + P_n)^2 - 2P_{n+1}^2 = (-1)^{n+1} \tag{5}$$

were used [3] to prove that Pell numbers generate all values for (x, y, z) when $c = 1$. Multiplication of (5) by a^2 shows that Pell numbers also generate all values for (x, y, z) when $c = a^2$, regardless of primitivity. Thus, when $c = 1$, $q_n = P_n$ and $p_n = P_{n+1}$; $c = 4$, $q_n = 2P_n$; $c = 9$, $q_n = 3P_n$, etc. Similarly, Pell numbers generate all (x, y, z) when $c = 2a^2$, obviously nonprimitive. When $c = 2$, $q_n = P_{n+1} + P_n$ and $p_n = P_{n+2} + P_{n+1}$; $c = 8$, $q_n = 2(P_{n+1} + P_n)$; $c = 18$, $q_n = 3(P_{n+1} + P_n)$, etc.

All other Pythagorean triples are represented by generalized Pell numbers, similar to Horadam's generalized Fibonacci numbers [4], in such a way that a pair of equations is associated with each value of c.

(6a) $$q_{2n+1} \text{ or } q_{2n+2} = aP_{n+1} - bP_n$$

and

(6b) $$q_{2n+2} \text{ or } q_{2n+1} = bP_{n+1} + aP_n$$

where $a > b$. The value of p, associated with a given value of q, is obtained by replacing n by $(n + 1)$. It will be noted that the odd and even values form two distinct sequences.

Upon combining (2) and (3) with (6), we obtain

(7a) $$p_{2n+1} - q_{2n+1} = (a - b)P_{n+1} + (a + b)P_n$$

(7b) $$p_{2n+2} - q_{2n+2} = (a + b)P_{n+1} + (b - a)P_n$$

and

(8a) $$p_{2n+1} + q_{2n+1} = (3a - b)P_{n+1} + (a - b)P_n$$

(8b) $$p_{2n+2} + q_{2n+2} = (a + 3b)P_{n+1} + (a + b)P_n ,$$

where the subscripts for p and q may be interchanged between (7a) and (7b) as well as between (8a) and (8b) as needed.

Since Pell numbers proper, and generalized Pell numbers for primitive solutions, are alternately of different parity, and with $p \pm q$ odd for primitive solutions, $a \pm b$ must be odd in view of (7) and (8). All other possible values of $a \pm b$ also occur and give rise to non-primitive triples. Thus, all (x, y, z) can be generated, and no impossible values occur. Once obtained, all values are easily verified, and any oversight of a permissible value of c becomes obvious by the absence of an expected pair (a,b). But there appears to be no systematic, analytical method of determining a priori either possible values of c or their associated pair or pairs of (a,b), except for $c = a^2$ and $c = 2a^2$, where $b = 0$.

Following is a table of the first 33 values for c, a, and $b > 0$. Values of c giving rise to primitive solutions are underlined.

c	a	b	c	a	b	c	a	b
<u>7</u>	2	1	56	6	2	<u>97</u>	7	6
14	3	1	62	7	1	98**	9	1
<u>17</u>	3	2	63	6	3	<u>103</u>	8	3
<u>23</u>	4	1	68	6	4	112	8	4
28	4	2	<u>71</u>	6	5	<u>113</u>	9	2
<u>31</u>	4	3	<u>73</u>	7	2	<u>119</u>***	8	5
34	5	1	<u>79</u>	8	1	<u>119</u>***	10	1
<u>41</u>	5	2	82	7	3	124	8	6
46	5	3	<u>89</u>	7	4	126	9	3
<u>47</u>	6	1	92	8	2	<u>127</u>	8	7
<u>49</u>*	5	4	94	7	5	136	10	2

*This also has the solution $7P_n$.
**This also has the solution $7(P_{n+1} + P_n)$
*** This is the first value with two pairs of solutions.

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FIBONACCI NUMBERS OBTAINED FROM PASCAL'S TRIANGLE WITH GENERALIZATIONS

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1. INTRODUCTION

Consider the following array of numbers obtained from the first k lines of Pascal's Triangle.

$$
\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & \cdots & 0 \\
\hdashline
1 & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & 1 \\
\hline
k & \binom{k}{2} & \binom{k}{3} & \cdots & 1 \\
2k-1 & 2\binom{k}{2} & 2\binom{k}{3} & \cdots & 2 \\
4k-3 & 4\binom{k}{2}-1 & 4\binom{k}{3} & \cdots & 4 \\
\hdashline
\end{array}
$$

If we let the element in the $i+1^{\text{th}}$ column and n^{th} row be $F_{i,n}$, then $F_{i,n} = \binom{n}{i}$ $(n, i = 0, 1, 2, \cdots, k-1)$ and

$$F_{i,n} = \sum_{j=1}^{k} F_{i,n-j} \qquad (i = 0, 1, 2, \cdots, k-1; \quad |n| = k, k+1, \cdots).$$

If $k = 2$, $F_{0,n} = f_{n+1}$, $F_{1,n} = f_n$, where f_n is the n^{th} Fibonacci number; and if $k = 3$, $F_{1,n} = L_{n+1}$ and $F_{2,n} = K_{n-1}$, where L_n and K_n are the general Fibonacci numbers of Waddill and Sacks [8]. Also, $F_{k-1,h} = f_{h,k}$ where the $f_{n,k}$ are the k-generalized Fibonacci numbers of Miles [5], and $F_{0,n} = U_{k,n}$ of Ferguson [2]. Both the numbers $f_{n,k}$ and $U_{k,n}$ are of use in polyphase merge sorting techniques (see, for example, Gilstad [3] and Reynolds [6]).

The purpose of this paper is to investigate some of the properties of a more general set of functions which include the functions $F_{i,n}$ $(i = 0, 1, 2, \cdots, k-1)$ and several others as special cases.

2. NOTATION AND DEFINITIONS

Let $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a fixed set of k integers such that

$$F(x) = x^k - \alpha_1 x^{k-1} + \alpha_2 x^{k-2} + \cdots + (-1)^k \alpha_k$$

has distinct zeros $\rho_0, \rho_1, \cdots, \rho_{k-1}$. Let $a_0, a_1, \cdots, a_{k-1}$ be any k integers and define

$$\phi_i = \sum_{j=0}^{k-1} a_j \rho_i^j \qquad (i = 0, 1, 2, \cdots, k-1).$$

Finally, let

$$D = \begin{bmatrix} 1 & \rho_0 & \rho_0^2 & \cdots & \rho_0^{k-1} \\ 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^{k-1} \\ \hdashline 1 & \rho_{k-1} & \rho_{k-1}^2 & \cdots & \rho_{k-1}^{k-1} \end{bmatrix}$$

$$\Delta = |D| .$$

We shall concern ourselves with the functions

$$(2.1) \qquad A_{i,n} = \frac{1}{\Delta} \begin{vmatrix} 1 & \rho_0 & \cdots & \rho_0^{i-1} & \phi_0^n & \rho_0^{i+1} & \cdots & \rho_0^{k-1} \\ 1 & \rho_1 & \cdots & \rho_1^{k-1} & \phi_1^n & \rho_1^{i+1} & \cdots & \rho_1^{k-1} \\ \hdashline 1 & \rho_{k-1} & \cdots & \rho_{k-1}^{i-1} & \phi_{k-1}^n & \rho_{k-1}^{i+1} & \cdots & \rho_{k-1}^{k-1} \end{vmatrix}$$

$$(i = 0, 1, 2, \cdots, k-1).$$

It is clear that

$$(2.2) \qquad A_{i,n} = \frac{1}{\Delta} \sum_{j=0}^{k-1} c_{ij} \phi_j^n \qquad (i = 0, 1, 2, \cdots, k-1),$$

where c_{ij} is the cofactor of ρ_i^j in D.

If $a_1 = 1$, $a_i = 0$ $(i = 0, 2, 3, \cdots, k-1)$, we have $\phi_i = \rho_i$; and, in this case, we define $A_{i,n}$ to be $z_{i,n}$. These functions, which are quite useful in the determination of the properties of $A_{i,n}$, have been dealt with in some detail by authors such as Bell [1], Ward [9] and Selmer [7]. When $\alpha_i = (-1)^{i+1}$ $(i = 1, 2, \cdots, k)$, $\{z_{k-1,n}\}$ is the general Fibonacci sequence discussed by Miles [5] and Williams [10].

Since matrix methods are advantageous in the treatment of the $A_{i,n}$ functions, we introduce the following:

$$C = \frac{1}{\Delta}\begin{bmatrix} c_{00} & c_{10} & \cdots & c_{k-1\,0} \\ c_{01} & c_{11} & \cdots & c_{k-1\,1} \\ \hdashline c_{0\,k-1} & c_{1\,k-1} & \cdots & c_{k-1\,k-1} \end{bmatrix},$$

$$C_i = \operatorname{diag}(c_{i0}, c_{i1}, \cdots, c_{i\,k-1}),$$

$$Z_i = \begin{bmatrix} z_{i,0} & z_{i,1} & \cdots & z_{i,k-1} \\ z_{i,1} & z_{i,2} & \cdots & z_{i,k} \\ \hdashline z_{i,k-1} & z_{i,k} & \cdots & z_{i,2k-2} \end{bmatrix}$$

$$P_{n,r} = \begin{bmatrix} \phi_0^n & \phi_1^n & \cdots & \phi_{k-1}^n \\ \phi_0^{n+r} & \phi_1^{n+r} & \cdots & \phi_{k-1}^{n+r} \\ \hdashline \phi_0^{n+(k-1)r} & \phi_1^{n+(k-1)r} & \cdots & \phi_{k-1}^{n+(k-1)r} \end{bmatrix}$$

$$B_{n,r} = \begin{bmatrix} A_{0,n} & A_{1,n} & \cdots & A_{k-1,n} \\ A_{0,n+r} & A_{1,n+r} & \cdots & A_{k-1,n+r} \\ \hdashline A_{0,n+(k-1)r} & A_{1,n+(k-1)r} & \cdots & A_{k-1,n+(k-1)r} \end{bmatrix}$$

$$B_{i,n,r} = \begin{bmatrix} A_{i,n} & & A_{i,n+(k-1)r} \\ A_{i,n+r} & A_{i,n+2r} & A_{i,n+kr} \\ \hdashline A_{i,n+(k-1)r} & A_{i,n+kr} & A_{i,n+(2k-2)r} \end{bmatrix}$$

3. SPECIAL CASES

The $A_{i,n}$ functions include a number of interesting functions as special cases. We have already mentioned the $z_{i,n}$ functions in the previous section and in this section we describe several other special cases. We first show the relation of the function $F_{i,n}$ to $A_{i,n}$.

Let H_j $(j = 1, 2, \ldots, k)$ be the j^{th} elementary symmetric function of $\phi_0, \phi_1, \cdots, \phi_{k-1}$, then

$$A_{i,n} = \sum_{j=1}^{k} (-1)^{j+1} H_j A_{i,n-j} \tag{3.1}$$

If

$$\alpha_i = (-1)^i \left[\binom{n}{i} - \binom{n}{i-1}\right] \qquad (i = 1, 2, \cdots, k)$$

and $a_0 = a_1 = 1$, $a_i = 0$ $(i = 2, 3, \cdots, k-1)$, we have $\phi_i = 1 + \rho_i$ and $H_j = (-1)^{j+1}$. Hence,

$$A_{i,n} = \sum_{j=1}^{k} A_{i,n-j}$$

and

$$A_{i,n} = \binom{n}{i} \qquad (0 \leq i, \quad n \leq k-1);$$

thus, in this case,

$$A_{i,n} = F_{i,n} .$$

When $k - 2$, $\alpha_1 = 1 - 2a$, $\alpha_2 = a^2 - a - 1$, $a_0 = a$, $a_1 = 1$, we have $H_1 = 1$, $H_2 = -1$, and $A_{0,n} = af_n + f_{n-1}$, $A_{i,n} = f_n$. If $\alpha_1 = 0$, $\alpha_2 = -5$, $a_0 = a_1 = 1$, we have

$$A_{0,n} = 2^{n-1}\ell_n \quad \text{and} \quad A_{1,n} = 2^{n-1}f_n ,$$

where ℓ_n is the n^{th} Lucas number. If (x_1, y_1) is the fundamental solution of the Pell equation

$$x^2 - dy^2 = 1 , \tag{3.2}$$

$$\alpha_1 = 0, \quad \alpha_2 = -d, \quad a_0 = x_1, \quad a_1 = y_1 ,$$

Then $A_{0,n} = x_n$ and $A_{1,n} = y_n$, where (x_n, y_n) is the n^{th} solution of (3.2).

When $k = 3$, we also have some interesting cases. For example, if $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and $a_0 = -a_1 = a_2 = 1$, we have $H_1 = -H_2 = H_3 = 1$ and $A_{0,n} = U_{3,n}$, $A_{1,n} = -L_{n-1}$, $A_{2,n} = f_{3,n+1} = K_n$. If (x_1, y_1, z_1) is a fundamental solution of the diophantine equation (Mathews [4])

$$x^3 + dy^3 + d^2z^3 - 3dxyz = 1 , \tag{3.3}$$

and $\alpha_0 = \alpha_2 = 0$, $\alpha_3 = d$, $a_0 = x_1$, $a_1 = y_1$, $a_2 = z_1$, then all the solutions of (3.3) are given by

$$(A_{0,n},\ A_{1,n},\ A_{2,n}) \qquad (|n| = 0, 1, 2, \cdots).$$

4. IDENTITIES

We now obtain several of the important relations satisfied by the $A_{i,n}$ functions. It will be seen that each of these relations is a generalization of a corresponding identity satisfied by the Fibonacci numbers. The most important properties of the Fibonacci numbers are the identities which connect the numbers f_{n+m}, f_{n-m} and f_{nm} to other Fibonacci numbers. For the sake of convenience, we shall call these relations the addition, subtraction, and multiplication formulas.

By (2.1),

$$B_{i,n+m,r} = P_{n,r}C_iP'_{m,r},$$

where we denote by B' the transpose of the matrix B. Since

$$CD = DC = I,$$

$$B_{i,n+m,r} = P_{n,r}C'D'C_iDCP'_{m,r} = B_{n,r}Z_iB'_{m,r};$$

hence,

$$\begin{aligned} A_{i,m+n} &= (A_{0,n}A_{1,n}A_{2,n}\cdots A_{k,n})Z_i(A_{0,m}A_{1,m}\cdots A_{k,m})' \\ &= \sum_{h=0}^{k-1}\sum_{j=0}^{k-1} z_{i,h+j}A_{h,n}A_{j,m} \quad . \end{aligned} \tag{4.1}$$

This is the addition formula for $A_{i,n}$.

By the definition of $A_{i,n}$ it follows that

$$\phi_i^m = \sum_{j=0}^{k-1} A_{j,m}\rho_i^j ; \tag{4.2}$$

thus,

$$\begin{aligned} \rho_i^h\phi_i^m &= \sum_{j=0}^{k-1} \rho_i^{j+h} A_{j,m} \\ &= \sum_{j=0}^{k-1} \rho_i^h \left(\sum_{j=0}^{k-1} z_{h,j+k} A_{j,m}\right) \quad . \end{aligned}$$

Now if $H = H_k = \phi_0\phi_1\phi_1 \cdots \phi_{k-1}$,

$$H^m A_{i,n-m} = \frac{1}{\Delta}\begin{vmatrix} \phi_0^m & \rho_0\phi_0^m & \cdots & \rho_0^{i-1}\phi_0^m & \phi_0^n & \rho_0^{i+1}\phi_0^m & \cdots & \rho_0^{k-1}\phi_0^m \\ \phi_1^m & \rho_1\phi_1^m & \cdots & \rho_1^{i-1}\phi_1^m & \phi_1^n & \rho_1^{i-1}\phi_1^m & \cdots & \rho_1^{k-1}\phi_1^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{k-1}^m & \rho_{k-1}\phi_{k-1}^m & \cdots & \rho_{k-1}^{i-1}\phi_{k-1}^m & \phi_{k-1}^m & \rho_{k-1}^{i+1}\phi_{k-1}^m & \cdots & \rho_{k-1}^{k-1}\phi_{k-1}^m \end{vmatrix}$$

By (4.2)

$$(4.3) \qquad H^m A_{i,n-m} =$$

$$\begin{vmatrix} \sum_{j=0}^{k-1} z_{0,j}A_{j,m} & \cdots \sum z_{0,i+j-1}A_{j,m} & A_{0,m} & \sum z_{0,i+j+1}A_{j,m} & \cdots \sum z_{0,i+k-1}A_{j,m} \\ \sum z_{i,j}A_{j,m} & \cdots \sum z_{1,i+j-1}A_{j,m} & A_{1,m} & \sum z_{1,i+j+1}A_{j,m} & \cdots \sum z_{1,i+k-1}A_{j,m} \\ \sum z_{k-1,j}A_{j,m} & \cdots \sum z_{k-1,i+j-1}A_{j,m} & A_{k-1,m} & \sum z_{k-1,i+j+1}A_{j,m} & \cdots \sum z_{k-1,i+k-1}A_{j,m} \end{vmatrix}$$

this is the subtraction formula for $A_{i,n}$.

Since

$$A_{i,nm} = \frac{1}{\Delta}\sum_{j=0}^{k-1} c_{ij}\phi_j^{nm} \quad ,$$

$$A_{i,nm} = \sum_{j=0}^{k} (\Delta^{-1}c_{ij})\left(\sum_{h=0}^{k-1} A_{h,m}\rho_j^h\right)^m \quad .$$

From (4.2), we get the multiplication formula

$$A_{i,nm} = \sum \frac{m!}{i_1!i_2! \cdots i_k!}\prod_{j=1}^{k} A_{j,m}^{i_j} z_{i,s} \quad ,$$

where the sum is taken over all non-negative integers $i_1, i_2, \cdots, i_k$ such that $\sum i_k = m$; and $\sum(j-1)i_j$.

We may easily evaluate the determinants of $B_{n,r}$ and $B_{i,n,r}$ by first introducing the matrix.

$$Q_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \phi_0^n & \phi_1^n & \cdots & \phi_{k-1}^n \\ \cdots & \cdots & \cdots & \cdots \\ \phi_0^{(k-1)n} & \phi_1^{(k-1)n} & \cdots & \phi_{k-1}^{(k-1)n} \end{vmatrix}$$

Now

$$Q_n C = \begin{vmatrix} A_{0,0} & A_{1,0} & \cdots & A_{k-1,0} \\ A_{0,n} & A_{1,n} & \cdots & A_{k-1,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{0,(k-1)n} & A_{1,(k-1)n} & \cdots & A_{k-1,(k-1)n} \end{vmatrix}$$

hence

$$|Q_n| = \Delta \begin{vmatrix} A_{1,n} & \cdots & A_{k,n} \\ A_{1,2n} & \cdots & A_{k,2n} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)n} & \cdots & A_{k,(k-1)n} \end{vmatrix}$$

Since

$$|P_{n,r}| = H^n |Q_r| ,$$

we have

$$|B_{n,r}| = \frac{H^n}{\Delta} \begin{vmatrix} A_{1,r} & \cdots & A_{n-1,r} \\ A_{1,2r} & \cdots & A_{n-1,2r} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)r} & \cdots & A_{n-1,(k-1)r} \end{vmatrix} \tag{4.5}$$

and

$$|B_{i,n,r}| = H^n |Z_i| \begin{vmatrix} A_{1,r} & \cdots & A_{n-1,r} \\ A_{1,2r} & \cdots & A_{n-1,2r} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)r} & \cdots & A_{n-1,(k-1)r} \end{vmatrix}^2 . \tag{4.6}$$

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...ANNOUNCEMENT...

The Editor (who always parks in a Fibonacci-numbered parking space) noted the following in the latest publication of the Fibonacci Association, A Primer for the Fibonacci Numbers: There are 13 authors, each of whom wrote a Fibonacci number of articles. Each co-author has a Fibonacci number of articles with a given co-author. There are 11 articles with one author, and 13 articles have co-authors. Of the twenty-four articles, 13 are Primer articles, and 11 are not.

The Primer, co-edited by Marjorie Bicknell and V. E. Hoggatt, Jr., is a compilation of elementary articles which have appeared over the years. These articles were selected and edited to give the reader a comprehensive introduction to the study of Fibonacci sequences and related topics. The 175-page Primer will be available in the Fall of 1972 at a cost of $5.00.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-195 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California

Consider the array indicated below:

1	1						
1	2						
2	4	1	1				
5	9	3	4				
13	22	7	11	1	1		
34	56	16	27	5	6		
89	145	38	65	16	22	1	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

(i) Show that the row sums are F_{2n}, $n \geq 2$.

(ii) Show that the rising diagonal sums are the convolution of

$$\left\{F_{2n-1}\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{u(n;\ 2,2)\right\}_{n=0}^{\infty} \quad ,$$

the generalized numbers of Harris and Styles.

H-196 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

(a) Let A_0 be the set of integral parts of the positive integral multiples of τ, where

$$\tau = \frac{1+\sqrt{5}}{2} \quad ,$$

and let A_{m+1}, $m = 0, 1, 2, \cdots$, be the set of integral parts of the numbers $n\tau^2$ for $n \in A_m$. Prove that the collection of Z^+ of all positive integers is the disjoint union of the A_j.

(b) Generalize the proposition in (a).

H-197 Proposed by Lawrence Somer, University of Illinois, Urbana, Illinois.

Let $\{u_n^{(t)}\}_{n=1}^{\infty}$ be the t-Fibonacci sequences with positive entries satisfying the recursion relationship:

$$u_n^{(t)} = \sum_{i=1}^{t} u_{n-i} \quad .$$

Find

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \frac{u_{n+1}^{(t)}}{u_n^{(t)}} \quad .$$

SOLUTIONS

HYPER-TENSION

H-185 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \binom{2k}{k} (1 - v)^{n-k} {}_2F_1[-k;\ n + k + 1;\ k + 1;\ x] \ ,$$

where ${}_2F_1[a, b;\ c;\ x]$ denotes the hypergeometric function.

Solution by the Proposer.

We start with the identity

$$\sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!\,s!\,(r + 2s)!} \frac{(y - z)^r z^s}{(1 + y)^{2r+3s+1}} = \frac{1}{1 - y - z} \quad .$$

Now put $y = u + v$, $z = v$, so that

$$(*) \qquad \sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!\,s!\,(r + 2s)!} \frac{u^r v^s}{(1 + u + v)^{2r+3s+1}} = \frac{1}{1 - u - 2v} \quad .$$

The right-hand side of (*) is equal to

$$\sum_{n=0}^{\infty} (u + 2v)^n ,$$

while the left-hand side

$$= \sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} u^r v^s \sum_{k=0}^{\infty} (-1)^k \binom{2r + 3s + k}{k} (u + v)^k .$$

It follows that

$$(u + 2v)^n = \sum_{k=0}^{n} (u + v)^k \sum_{r+s=n-k} (-1)^k \binom{2r + 3s + k}{k} \frac{(2r + 3s)!}{r!s!(r + 2s)!} u^r v^s$$

$$= \sum_{r+s \le n} (-1)^{n-r-s} \frac{(r + 2s + n)!}{r!s!(r + 2s)!(n - r - s)!} u^r v^s (u + v)^{n-r-s}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \frac{(u + v)^{n-k}}{(n - k)!} \sum_{s=0}^{k} \frac{(s + n + k)!}{s!(k - s)!(k + s)!} u^{k-s} v^s .$$

Taking $u = 1$, $v = -x$, we get

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n + k)!}{k!k!(n - k)!} (1 - v)^{n-k} {}_2F_1[-k, n + k + 1; k + 1; x] .$$

A CONGRUENCE IN ITS PRIME

H-186 Proposed by James Desmond, Florida State University, Tallahassee, Florida.

The generalized Fibonacci sequence is defined by the recurrence relation

$$U_{n-1} + U_n = U_{n+1} ,$$

where n is an integer and U_0 and U_1 are arbitrary fixed integers.

For a prime p and integers n, r, s and t, show that

$$U_{np+r} \equiv U_{sp+t} \pmod{p} ,$$

if $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$, and that

$$U_{np+r} \equiv (-1)^{r+t} U_{sp+t} \pmod p$$

if $p \equiv \pm 2 \pmod 5$ and $n - r = s - t$.

Solution by the Proposer.

We have from Hoggatt and Ruggles, "A Primer for the Fibonacci Sequence — Part III," Fibonacci Quarterly, Vol. 1, No. 3, 1963, p. 65, and by Fermat's theorem, that

$$F_{np+r} = \sum_{i=0}^{p} \binom{p}{i} F_{i+r} F_n^i F_{n-1}^{p-i} \equiv F_r F_{n-1}^p + F_{p+r} F_n^p \equiv F_r F_{n-1} + F_{p+r} F_n \pmod p$$

for all n and r. From I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2, 1963, p. 79, we have that

$$F_{i+j} = F_{i+1} F_j + F_i F_{j-1}$$

for all i and j. Therefore,

$$F_{np+r} \equiv F_r F_{n-1} + F_{r+1} F_p F_n + F_r F_{p-1} F_n \pmod p$$

for all n and r. We have from Hardy and Wright, Theory of Numbers, Oxford University Press, London, 1954, p. 150, that

$$F_{p-1} \equiv 0 \pmod p \quad \text{and} \quad F_p \equiv 1 \pmod p$$

if $p \equiv \pm 1 \pmod 5$, and that

$$F_{p+1} \equiv 0 \pmod p \quad \text{and} \quad F_p \equiv 1 \pmod p$$

if $p \equiv \pm 2 \pmod 5$. Let $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$. Then

$$F_{np+r} \equiv F_r F_{n-1} + F_{r+1} F_n \equiv F_{r+n} \pmod p$$

for all n and r. Therefore

$$F_{sp+t} \equiv F_{s+t} \equiv F_{n+r} \equiv F_{np+r} \pmod p.$$

It is easily verified by induction that

$$U_n = U_1F_n + U_0F_{n-1}$$

for all n. Therefore

$$U_{np+r} \equiv U_1 F_{np+r} + U_0 F_{np+r-1} \equiv U_1F_{sp+t} + U_0F_{sp+t-1} \equiv U_{sp+t} \pmod{p}.$$

Now, let $p \equiv +2 \pmod 5$ and $n - r = s - t$. From page 77 of the reference to Ruggles, we have

$$F_{i+j} - F_iL_j = (-1)^{j+1}F_{i-j}$$

for all i and j. Therefore

$$\begin{aligned} F_{np+r} &\equiv F_r F_{n-1} - F_{r+1} F_n + F_r F_n \equiv F_r F_{n-1} + F_{r+1} F_n - 2F_{r+1} F_n + F_rF_n \\ &\equiv F_{r+n} - L_r F_n \equiv (-1)^{r+1} F_{n-r} \pmod{p} \end{aligned}$$

for all n and r. Thus

$$(-1)^{r+t} F_{sp+t} \equiv (-1)^{r+t}(-1)^{t+1} F_{s-t} \equiv (-1)^{r+1} F_{n-r} \equiv F_{np+r} \pmod{p}.$$

Hence

$$\begin{aligned} U_{np+r} &\equiv U_1 F_{np+r} + U_0 F_{np+r-1} \equiv U_1(-1)^{r+t}F_{sp+t} + U_0(-1)^{r-1+t-1} F_{sp+t-1} \\ &\equiv (-1)^{r+t} (U_1F_{sp+t} + U_0 F_{sp+t-1}) \equiv (-1)^{r+t}U_{sp+t} \pmod{p}. \end{aligned}$$

FIBONACCI IS A SQUARE

H-187 Proposed by Ira Gessel, Harvard University, Cambridge, Massachusetts.

Problem: Show that a positive integer n is a Fibonacci number if and only if either $5n^2 + 4$ or $5n^2 - 4$ is a square.

Solution by the Proposer.

Let $F_0 = 0$, $F_1 = 1$, $F_{r+1} = F_r + F_{r-1}$ be the Fibonacci series and $L_0 = 2$, $L_1 = 1$, $L_{r+1} = L_r + L_{r-1}$ be the Lucas series. It is well known that

$$(1) \qquad (-1)^r + F_r^2 = F_{r+1} F_{r-1}$$

$$(2) \qquad L_r = F_{r+1} + F_{r-1} \quad .$$

Subtracting four times the first from the square of the second equation, we have

$$L_r^2 - 4(-1)^r - 4F_r^2 = (F_{r+1} - F_{r-1})^2 = F_r^2 \ ,$$

whence

$$5F_r^2 + 4(-1)^r = L_r^2 \quad .$$

Thus if n is a Fibonacci number, either $5n^2 + 4$ or $5n^2 - 4$ is a square.

I have two proofs of the converse.

First Proof. We use the theorem (Hardy and Wright, An Introduction to the Theory of Numbers, p. 153) that if p and q are integers, x is a real number, and $|(p/q) - x| < 1/2q^2$, then p/q is a convergent to the continued fraction for x, and that (Hardy and Wright, p. 148) the convergents to the continued fraction for $(1 + \sqrt{5})/2$ in lowest terms are F_{r+1}/F_r.

Assume that $5n^2 \pm 4 - m^2$. Then since m and n have the same parity, $k = (m + n)/2$ is an integer. Then substituting $m = 2k - n$ in $5n^2 \pm 4 = m^2$, we get $k^2 - kn - n^2 = \pm 1$, so that k and n are relatively prime and

$$\pm 1/n^2 = (k/n)^2 - (k/n) - 1 = [(k/n) - (\sqrt{5} + 1)/2][(k/n) + (\sqrt{5} - 1)/2] .$$

Thus

$$|(k/n) - (\sqrt{5} + 1)/2| = 1/n^2 \; |(k/n) + (\sqrt{5} - 1)/2| .$$

Since 1 is a Fibonacci number, we may assume $n \geq 2$. Then

$$(2k - n)^2 = m^2 \geq 5n^2 - 4 = 4n^2 + (n^2 - 4) \geq 4n^2 ,$$

so $2k - n \geq 2n$, whence $k/n \geq 3/2$. Thus $(k/n) + (\sqrt{5} - 1)/2 > 2$, so by the two theorems quoted above, $k/n = F_{r+1}/F_r$ for some r, and since both fractions are reduced, $n = F_r$.

Second Proof. Assume $5n^2 \pm 4 = m^2$. Then $m^2 - 5n^2 = \pm 4$, so

$$\frac{m + \sqrt{5}n}{2} \cdot \frac{m - \sqrt{5}n}{2} = \pm 1 ,$$

and since m and n have the same parity,

$$\frac{m + \sqrt{5}n}{2} \quad \text{and} \quad \frac{m - \sqrt{5}n}{2}$$

are integers in $Q(\sqrt{5})$, where Q is the rationals, and since their product is ± 1, they are units. It is well known (Hardy and Wright, p. 221) that the only integral units of $Q(\sqrt{5})$ are of the form $\pm x^{\pm r}$, where $x = (1 + \sqrt{5})/2$.

Then we have

$$(m + \sqrt{5}n)/2 = x^r = \frac{1}{2}\left[(x^r + y^r) + \frac{x^r - y^r}{\sqrt{5}} \cdot \sqrt{5}\right],$$

where $y = -1/x$. Now $x^r + y^r = L_r$ and

$$(x^r - y^r)/\sqrt{5} = F_r$$

(Hardy and Wright, p. 148). Thus

$$\frac{1}{2}(m + \sqrt{5}n) = \frac{1}{2}(L_r + \sqrt{5}F_r),$$

so $n = F_r$.

SUM SERIES

H-189 Proposed by L. Carlitz, Duke University, Durham, North Carolina (Corrected).

Show that

$$\sum_{r,s=0}^{\infty} \frac{(2r+3s)!}{r!s!(r+2s)!} \frac{(a-by)^r b^s y^{r+2s}}{(1+ay)^{2r+3s+1}} = \frac{1}{1-ay-by^2}$$

Solution by the Proposer.

Put

$$\frac{1}{1-ax-bx^2} = \sum_{m=0}^{\infty} G_m x^m ,$$

so that

$$\frac{1}{(1-ax-bx^2)(1-y)} = \sum_{m,n=0}^{\infty} G_m x^m y^n .$$

Replacing y by $x^{-1}y$ this becomes

$$\frac{1}{(1-ax-bx^2)(1-x^{-1}y)} = \sum_{m,n=0}^{\infty} G_m x^{m-n} y^n .$$

Hence that part of the expansion of

$$\frac{1}{(1-ax-bx^2)(1-x^{-1}y)}$$

that is independent of x is equal to

$$(*) \qquad \frac{1}{1 - ay - by^2}$$

On the other hand, since

$$(1 - ax - bx^2)(1 - x^{-1}y) = (1 + ay) - x(a - by) - bx^2 - x^{-1}y,$$

we have

$$(1 - ax - bx^2)^{-1}(1 - x^{-1}y)^{-1} = \sum_{k=0}^{\infty} \frac{[x(a - by) + bx^2 + x^{-1}y]^k}{(1 + ay)^{k+1}}$$

$$= \sum_{r,s,t=0}^{\infty} \frac{(r + s + t)!}{r!s!t!} \frac{(a - by)^r b^s y^t}{(1 + ay)^{r+s+t+1}} x^{r+2s-t} .$$

The part of this sum that is independent of x is obtained by taking $t = r + 2s$. We get

$$\sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} \frac{(a - by)^r b^s y^{r+2s}}{(1 + ay)^{2r+3s+1}} .$$

Since this is equal to (*), we have proved the stated identity.

IT'S A MOD WORLD

H-190 Proposed by H. H. Ferns, Victoria, British Columbia.

Prove the following

$$2^r F_n \equiv n \pmod 5$$
$$2^r L_n \equiv 1 \pmod 5 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, and r is the least residue of $n - 1 \pmod 4$.

Solution by the Proposer.

In an unpublished paper by the proposer, it is shown that

$$2^{n-1} F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k .$$

Hence

$$2^{n-1}F_n = n + \sum_{k=1}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k .$$

Thus

$$2^{n-1}F_n \equiv n \pmod{5} .$$

Let $n - 1 = 4m + r$, where $0 \le r < 4$. Then

$$2^{n-1}F_n = 2^{4m+r}F_n = 2^{4m}2^rF_n \equiv n \pmod{5} .$$

But

$$2^{4m} = (2^4)^m \equiv 1 \pmod{5} .$$

Hence

$$2^rF_n \equiv n \pmod{5} .$$

To prove

$$2^rL_n \equiv 1 \pmod{5} ,$$

use

$$2^{n-1}L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} 5^k$$

(which is derived in the same paper) and proceed as above.

JUST SO MANY TWO'S

H-192 Proposed by Ronald Alter, University of Kentucky, Lexington, Kentucky.

If

$$c_n = \sum_{j=0}^{3n+1} \binom{6n+3}{2j+1} (-11)^j ,$$

prove that

$$c_n = 2^{6n+3} \cdot N, \qquad (N \text{ odd}, \; n \ge 0) .$$

Solution by the Proposer.

In the sequence

$$b_k = b_{k-1} - 3b_{k-2}, \qquad (k \ge 1, \; b_1 = b_2 = 1) ,$$

it is easy to show that 2 is the highest power of 2 that divides b_k if and only if $k \equiv 3 \pmod{6}$. Also, by deriving the appropriate Binet formula, it follows that

$$b_k = \frac{1}{\sqrt{-11}}\left\{\left(\frac{1+\sqrt{-11}}{2}\right)^k - \left(\frac{1-\sqrt{-11}}{2}\right)^k\right\}, \qquad k \geq 1.$$

Thus

$$b_k = \frac{1}{2^{k-1}}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{k}{2j+1}(-11)^j, \qquad k \geq 1.$$

The desired result follows by observing

$$b_{6n+3} = \frac{1}{2^{6n+2}}\, c_n \quad .$$

Editorial Note: Please submit solutions for any of the problem proposals. We need fresh blood!

A GOLDEN SECTION SEARCH PROBLEM

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After tiring of using numerous quadratic functions as objective functions for examples in my mathematical programming course, I posed the following problem for myself: Design a unimodal function over the (0,1) interval which is concave, has a maximum in the interior of (0,1), and is not a quadratic function. The purpose was to demonstrate numerically the golden section search.*

My first thoughts were to add two functions which are concave over the (0,1) interval with the property that one goes to $-\infty$ at 0 and the other goes to $-\infty$ at 1. My two initial choices were $\log x$ and $1/(x - 1)$. The golden section search starts at the two points $x_1 = 1 - (1/\phi)$ and $x_2 = 1/\phi$ where $\phi = (1 + \sqrt{5})/2$. After searching with 8 points, I noticed that the interval of uncertainty still contained the first search point so I thought it about time to find the location of the maximum analytically. I was dumfounded to discover that if I continued indefinitely with the search my interval of uncertainty would still contain the initial search point.

*Douglas J. Wilde, Optimum Seeking Methods, Prentice Hall, Inc. (1964).

INTRODUCTION TO PATTON POLYGONS

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This paper introduces an extraordinarily elementary topic which is accessible to any patient high school student with little or no sophisticated number theory. The ideas covered are presented in a straight-forward fashion, with many proofs and extensions left for the reader to work through. Deeper connexions with additive sequences and number theory are left to those with interest to pursue matters in the standard references on Fibonacci numbers. In the following (*) designates assertions which must be proved or developed by the reader. Drawing all the figures carefully is certainly essential to an understanding of what is going on.

1. Choose a coordinate system (which is to say, use some convenient graph paper) and draw any parallelogram $0A_0A_2A_1$ where 0 is the origin, and the letters are taken around the figure.

2. Find the unique point A_3 so that $0A_1A_3A_2$ is a parallelogram (Fig. 1).

3. In general, find the point A_{n+1} so that $0A_{n-1}A_{n+1}A_n$ is a parallelogram.

*4. Consider the situation if $n = -1, -2, -3, \cdots$ in (3) and study Fig. 2.

5. If we have been successful so far, we now have a set of points $\{A_n\}$ where n is any integer, positive or negative; we may consider these points as forming an infinite polygon $\cdots A_{-2}A_{-1}A_0A_1A_2A_3\cdots$ (Fig. 3).

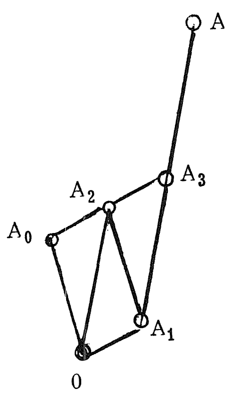

Figure 1

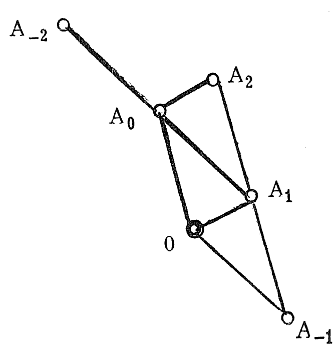

Figure 2

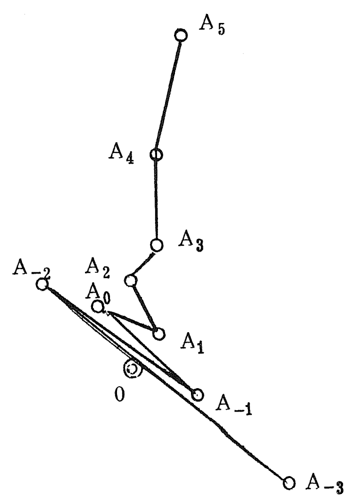

Figure 3

6. This curious polygon has many properties which are somewhat surprising. Evidently, A_{n+2} is the midpoint of $\overline{A_n A_{n+3}}$ for any integer n. This can be easily shown since $A_0A_2 = 0A_1 = A_2A_3$ as opposite sides in the first two parallelograms. This process may be continued along the polygon.

7. But there is a more interesting and related result. In Fig. 1, area $0A_1A_2 = \frac{1}{2}$ area $0A_1A_2A_0$ = area $A_0A_1A_2$, and $A_0A_2 = A_2A_3$, so that area $A_0A_1A_2$ = area $A_1A_2A_3$. Continuing along the polygon we find that area $A_1A_2A_3$ = area $A_2A_3A_4$. In general then, area $0A_0A_1$ = area $A_nA_{n+1}A_{n+2}$. In a sense, the polygon is an infinite stack of triangles with the same area.

8. Vectors are now introduced to make calculations a bit simpler. Let $\overrightarrow{0A_n}$ be represented by the vector v_n. We may apply the "Parallelogram Law" for vector addition to $0A_0A_1A_2$ so that we have $\overrightarrow{0A_0} + \overrightarrow{0A_1} = \overrightarrow{0A_2}$, or $v_0 + v_1 = v_2$. In general, we have that $v_{n+2} = v_{n+1} + v_n$, since by (3), $0A_nA_{n+2}A_{n+1}$ is a parallelogram.

9. The entire polygon is based on $0A_0A_2A_1$, so in some way, the vectors v_0 and v_1 are fundamental. In fact,

$$
\begin{aligned}
v_2 &= v_1 + v_0 \\
v_3 &= v_2 + v_1 = 2v_1 + v_0 \\
v_4 &= v_3 + v_2 = 3v_1 + 2v_0 \\
v_5 &= v_4 + v_3 = 5v_1 + 3v_0 \\
v_6 &= v_5 + v_4 = 8V_1 + 5v_0 \ .
\end{aligned}
$$

And we recognize our old friend the Fibonacci sequence where $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. In short, we are able to write: $v_n = F_nv_1 + F_{n-1}v_0$.

*10. In the negative direction along the polygon, check that $v_{-n} = F_{-n}v_1 + F_{-n-1}v_0$. We already know one of the properties of the Fibonacci sequence is that

$$F_{-n} = (-1)^{n+1}F_n \ ,$$

and so we have $v_{-n} = (-1)^{n+1}(F_nv_1 - F_{n+1}v_0)$.

11. Using the coordinate system we set up in (1), we may assign coordinates (f_n, g_n) to the point A_n; and, of course, the vector v_n will have the same coordinates. Then, since vectors are added coordinate-wise, we have:

$$f_n = F_nf_1 + F_{n-1}f_0 \qquad \text{and} \qquad g_n = F_ng_1 + F_{n-1}g_0 \ ,$$

for any integer n.

*12. Since our polygons seem to be deeply involved with the Fibonacci sequence, we need a short detour to pick up some well known properties of this sequence. Let

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}) \ ,$$

so that $\varphi^2 = \varphi + 1$. Then F_{n+1}/F_n is an increasing sequence of rational numbers bounded by φ if n is odd, and a decreasing sequence bounded by φ if n is even. As n becomes large, F_{n+1}/F_n can be shown to approach φ as limit. As a result of all this we can write that:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi ;$$

and that $F_n - \varphi F_{n-1} > 0$ and $\varphi F_n - F_{n+1} > 0$ if and only if n is odd.

13. Returning to the polygon, consider the <u>slope</u> of $\overline{0A_n}$ for large positive n, where A_n is the point (f_n, g_n):

$$\frac{g_n - 0}{f_n - 0} = \frac{F_n g_1 + F_{n-1} g_0}{F_n f_1 + F_{n-1} f_0} = \frac{\frac{F_n}{F_{n-1}} g_1 + g_0}{\frac{F_n}{F_{n-1}} f_1 + f_0} .$$

As n becomes very large, F_n/F_{n-1} approaches φ and so the slope approaches the value

$$M = \frac{\varphi g_1 + g_0}{\varphi f_1 + f_0} .$$

14. For the slope of $\overline{0A_{-n}}$, we find, using (10), that:

$$\frac{g_{-n} - 0}{f_{-n} - 0} = \frac{(-1)^{n+1}(F_n g_1 - F_{n+1} g_0)}{(-1)(F_n g_1 - F_{n+1} f_0)} = \frac{g_1 - \frac{F_{n+1}}{F_n} g_0}{f_1 - \frac{F_{n+1}}{F_n} f_0} .$$

Again, as n becomes large, the slope approaches

$$N = \frac{g_1 - \varphi g_0}{f_1 - \varphi f_0} .$$

15. Another way of thinking about (13) and (14) is to call the lines $y = Mx$ and $y = Nx$ the <u>asymptotes</u> of the polygon (Fig. 4), where M and N are given in (13) and (14). For large n, the polygon runs along the asymptote $y = Mx$ in the positive direction, and along $y = Nx$ in the negative direction

*16. It is easy to show that the asymptotes are distinct lines through the origin 0. Merely show that $M \neq N$ if $0A_0A_1$ form a triangle.

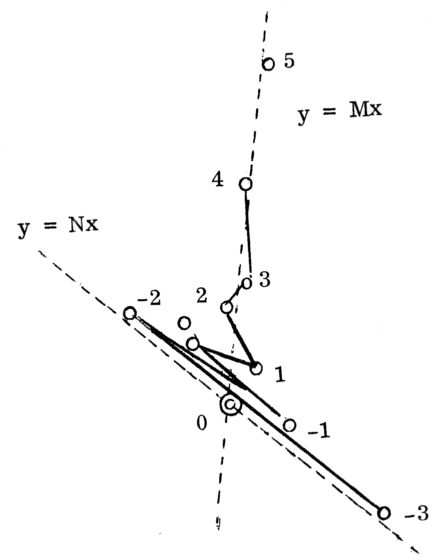

Figure 4

17. Figure 4 suggests a more intriguing relationship between the polygon and its asymptotes. In order to get at this, let

$$d = f_0 g_1 - f_1 g_0 = \begin{vmatrix} f_0 & g_0 \\ f_1 & g_1 \end{vmatrix} .$$

Check to see that we may choose A_0 and A_1 so that $d \neq 0$, $\varphi f_1 + f_0 \neq 0$ and $f_1 - \varphi f_0 \neq 0$. A calculation shows that for positive n:

$$g_n - Mf_n = \frac{d(F_n - \varphi F_{n-1})}{\varphi f_1 + f_0} .$$

Since $d/(\varphi f_1 + f_0)$ is a constant, the sign of $g_n - Mf_n$ depends on $F_n - \varphi F_{n-1}$ which is positive if n is odd and negative if n is even (see (12)). Hence $g_n - Mf_n$ is alternately greater and less than 0, which is equivalent to saying that g_n is alternately greater and less than Mf_n. Hence, the vertices $A_n = (f_n, g_n)$ lie alternately above and below the line $y = Mx$.

*18. A similar analysis for $g_n - Nf_n$, $g_{-n} - Mf_{-n}$ and $g_{-n} - Nf_{-n}$ yields this result: the polygon (its vertices, at any rate) lies on alternate sides of the asymptote $y = Mx$, and entirely on one side of $y = Nx$. This explains the "T"-shape of the polygon (Fig. 4). The asymptotes divide the plane into 4 regions: one containing the even-numbered vertices, another the odd ones, and the last two regions are empty.

19. We know from (7) that the absolute value of the area of triangle $A_n A_{n+1} A_{n+2}$ equals area $0A_0A_1$. More precisely, from analytic geometry, the area $0A_0A_1$ is given by the determinant:

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \end{vmatrix} ,$$

which gives after expansion: $\frac{1}{2}(f_0g_1 - f_1g_0) = \frac{1}{2}d$, as in (17). Using determinants to find area, we must recall that lettering a triangle in the opposite sense changes the sign of its area. Hence we get:

$$d = \begin{vmatrix} 1 & 0 & 0 \\ 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \end{vmatrix} = -\begin{vmatrix} 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \\ 1 & f_2 & g_2 \end{vmatrix} ,$$

which is twice the area $A_0A_1A_2$, and, in general:

$$d = (-1)^{n+1}\begin{vmatrix} 1 & f_n & g_n \\ 1 & f_{n+1} & g_{n+1} \\ 1 & f_{n+2} & g_{n+2} \end{vmatrix} .$$

This in turn may be simplified to:

$$d = \begin{vmatrix} f_0 & g_0 \\ f_1 & g_1 \end{vmatrix} = (-1)^n \begin{vmatrix} f_n & g_n \\ f_{n+1} & g_{n+1} \end{vmatrix}$$

for any n. This is a rather simple and unexpected result.

*20. A little more digging around can give us even more curious results. For example, confine attention to the even-numbered vertices. These form an "hyperbola"-shaped polygon with the obvious asymptotes (Fig. 5). It can be shown without much trouble that

$$\frac{1}{2}d = \text{area } 0A_{2n}A_{2n+2} = \text{area } A_{2n}A_{2n+2}A_{2n+4}$$

in absolute value. Notice also that $F_{n+4} = 3F_{n+2} - F_n$.

*21. Check the situation for the odd-numbered vertices.

22. What happens if we demand the asymptotes be perpendicular? Borrowing a result from analytic geometry again, we see that $MN = -1$ in that case. This can be simplified to:

$$\frac{g_1^2 - g_0g_2}{f_1^2 - f_0f_2} = -1 .$$

A simple way (not the only way, of course) for this to happen is for $g_n = f_{n-1}$. This gives us the polygon with vertices (f_n, f_{n-1}) and the asymptotes are $y = (1/\varphi)x$ and $y = -\varphi x$ (which are clearly perpendicular).

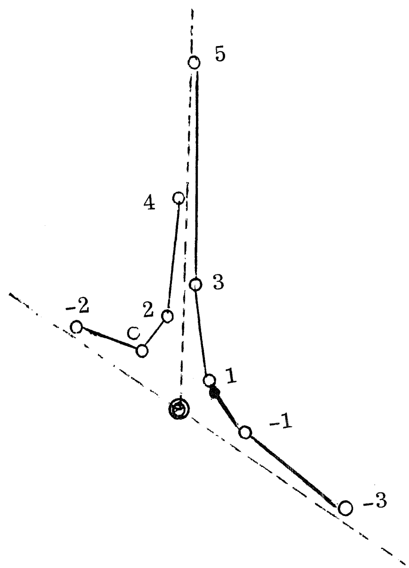

Figure 5

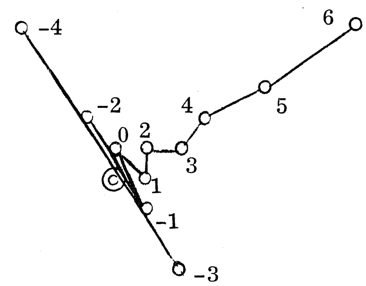

Figure 6

23. All polygons of the form (f_n, f_{n-1}) have the same asymptotes and so must be of the same general shape. The simplest one is (F_n, F_{n-1}) so that $A_0 = (0,1)$, $A_1 = (1,0)$ and $A_2 = (1,1)$ as in Fig. 6. Thus the polygon is based on the unit square, and so

$$d = F_0F_0 - F_1F_{-1} = -1.$$

Also, the result in (19) becomes:

$$\begin{vmatrix} F_n & F_{n-1} \\ F_{n+1} & F_n \end{vmatrix} = (-1)^{n+1} .$$

24. Investigate all eight polygons based on unit squares at the origin. For example, in addition to polygon (23), we also have (F_{n-1}, F_n). What are the asymptotes, etc. ?

25. This material reveals a great many properties of the Fibonacci-type sequences in a very geometric and graphic fashion. One obvious and several not-so-obvious generalizations are immediately available. But these will be the subject of another article.

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ALGORITHM FOR ANALYZING A LINEAR RECURSION SEQUENCE

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The basic idea of the algorithm to be presented in this paper may be illustrated by the simple example of a third-order linear recursion sequence: 1, 4, 8, 21, 67, 199, 568, 1641, 4782, 13904, 40353, 117161, ⋯ with a recursion relation of the form

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} .$$

The problem is to find a, b, c. The obvious procedure is to set up a set of linear equations

$$\begin{aligned}
L_1&: \quad c + 4b + 8a = 21 \\
L_2&: \quad 4c + 8b + 21a = 67 \\
L_3&: \quad 8c + 21b + 67a = 199 \\
L_4&: \quad 21c + 67b + 199a = 568 .
\end{aligned}$$

Only three equations are needed to find a, b, and c; the fourth is introduced since it brings out the fact that it must be a linear combination of the first three, the multipliers being in fact the quantities a, b, c for the given sequence; that is,

$$L_4 = aL_3 + bL_2 + cL_1$$

Hence we can ascertain that the sequence is of the third order by fact that these four relations are linearly dependent and no smaller number has this property. Thereafter, the first three equations can be used to find the quantities a, b, c.

The algorithm that does both these jobs simultaneously is Gaussian elimination. Stripped of the excess baggage we start with a matrix of quantities:

(1a)	1	4	8	21
(2a)	4	8	21	67
(3a)	8	21	67	199
(4a)	21	67	199	568

Multiply the first set of quantities by -4 and the second by 1 and add to get (2b); multiply the first set by -8 and the third by 1 to get (3b); multiply the first by -21 and the fourth by 1 to get (4b).

(2b)	0	-8	-11	-17
(3b)	0	-11	3	31
(4b)	0	-17	31	127

Now use -8 in (2b) as the pivot value and eliminate -11 and -17 in (3b) and (4b).

(3c)	0	0	-145	-435
(4c)	0	0	-435	-1305

Finally by another elimination

(4d)	0	0	0	0

This shows that there is a third-order linear recursion relation among the quantities we have used in these equations.

To find the constants $a, b, c,$ we have equivalently from (4c):

$$-435a = -1305 \quad \text{so that} \quad a = 3 .$$

Then from (3b) $-11b + 3c = 31$ which gives $b = -2$. Finally from (1a), $c + 4b + 8a = 21$ we have $c = 5$. The recursion relation in question is:

$$T_{n+1} = 3T_n - 2T_{n-1} + 5T_{n-2} .$$

It can now be ascertained whether the remaining terms are governed by this recursion relation.

In general, given a linear recursion relation for which we do not know the order or the coefficients, we can proceed by Gaussian elimination until we find a row of zeros. If this is the n^{th} row, then the order of the linear recursion relation is $n - 1$. The coefficients can then be found by back-substitution as was done in the illustrative example.

For example, the sequence $\cdots 527, 110, 23, 5, 2, 5, 23, 110, 527, \cdots$ was obtained as a fourth-order sequence. Is it a proper fourth-order sequence or does it have a lower-order factor which governs the sequence? We proceed to make our analysis.

(1a)	527	110	23	5	2
(2a)	110	23	5	2	5
(3a)	23	5	2	5	23
(4a)	5	2	5	23	110
(5a)	2	5	23	110	527

(2b)	0	21	105	504	2415
(3b)	0	105	525	2520	12075
(4b)	0	504	2520	12096	57960
(5b)	0	2415	12075	57960	277725
(3c)	0	0	0	0	0

The sequence is governed by a second-order recursion relation:

$$T_{n+1} = 5T_n - T_{n-1} .$$

As a more ambitious example, consider the sequence:

77, -20, 1, 0, -8, -2, 5, -2, 1, 9, 1, -2, 5, -2, -8, 0, 1, -20, 77, -425, ··· ,

which is supposed to be of the sixth order.

(1a)	1	0	-8	-2	5	-2	1
(2a)	0	-8	-2	5	-2	1	9
(3a)	-8	-2	5	-2	1	9	1
(4a)	-2	5	-2	1	9	1	-2
(5a)	5	-2	1	9	1	-2	5
(6a)	-2	1	9	1	-2	5	-2
(7a)	1	9	1	-2	5	-2	-8
(2b)	0	-8	-2	5	-2	1	9
(3b)	0	-2	-59	-18	41	-7	9
(4b)	0	5	-18	-3	19	-3	0
(5b)	0	-2	41	19	-24	8	0
(6b)	0	1	-7	-3	8	1	0
(7b)	0	9	9	0	0	0	-9
(3c)	0	0	468	154	-332	58	-54
(4c)	0	0	154	-1	-142	19	-45
(5c)	0	0	-332	-142	188	-62	18
(6c)	0	0	58	19	-62	-9	-9
(7c)	0	0	-54	-45	18	-9	-9
(4d)	0	0	0	-3023	-1916	-5	-1593
(5d)	0	0	0	-1916	-2780	-1220	-1188
(6d)	0	0	0	-5	-1220	-947	-135
(7d)	0	0	0	-1593	-1188	-135	-891

(5e)	0	0	0	0	4732884	3678480	539136
(6e)	0	0	0	0	3678480	2862756	400140
(7e)	0	0	0	0	539136	400140	155844

(6f)	0	0	0	0	0	17876957904	-89384789520
(7f)	0	0	0	0	0	-89384789520	446923947600

(7g)	0	0	0	0	0	0	0

Back substitution gives for the coefficients -5, 4, -2, 4, -5, 1, so that the recursion relation is $T_{n+1} = -5T_n + 4T_{n-1} - 2T_{n-2} + 4T_{n-3} - 5T_{n-4} - T_{n-5}$.

A complication can arise when there is a zero in a position at which the pivot should be found. The general procedure here is to take as pivot in a given column the first candidate among the sets of coefficients that might serve as a possible pivot in this column.

The following example will illustrate the manner of proceeding. Let there be a sequence of unknown order: 1, 2, 4, 8, 11, 7, -11, -47, -94, -123, -76, 123, $\cdots$. We set up quantities to cover up to the fifth order as an initial guess.

(1a)	1	2	4	8	11	7	-11
(2a)	2	4	8	11	7	-11	-47
(3a)	4	8	11	7	-11	-47	-94
(4a)	8	11	7	-11	-47	-94	-123
(5a)	11	7	-11	-47	-94	-123	-76
(6a)	7	-11	-47	-94	-123	-76	123

(2b)	0	0	0	-5	-15	-25	-25
(3b)	0	0	-5	-25	-55	-75	-50
(4b)	0	-5	-25	-75	-135	-150	-35
(5b)	0	-15	-55	-135	-215	-200	45
(6b)	0	-25	-75	-150	-200	-125	200

The first pivot element in the second column occurs in (4b). So we carry down (2b) and (3b) and pivot on (4b).

(2c)	0	0	0	-5	-15	-25	-25
(3c)	0	0	-5	-25	-55	-75	-50
(5c)	0	0	-100	-450	-950	-1250	-750
(6c)	0	0	-250	-1125	-2375	-3125	-1875

The first pivot element in the third column is found in (3c).
[Continued on page 438.]

FUN WITH FIBONACCI AT THE CHESS MATCH

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As first noted by Hoggatt [1], the results of the world championship chess match held in Iceland this summer between Boris Spassky, U. S. S. R., and Bobby Fischer, U. S. A., were heavily influenced by Fibonacci and Lucas numbers (underlined in what follows). Fischer started as a 5 to 2 favorite with the British bookmakers, and Jimmy "The Greek" Snyder in Las Vegas, Nevada, gave Fischer 6 to 5 which was changed to 8 to 5 as reported in Time, August 21. After the first and second games, Fischer won all games whose numbers are Fibonacci numbers, while all games numbered by Lucas numbers (apart from 3) were either draws or wins for Spassky. They played 7 consecutive draws (games 14 to 20 inclusive); there were 8 consecutive games where Fischer was 3 games ahead of Spassky, and Fischer was 4 games ahead of Spassky when he won the match. The 21 games ended with 11 draws, 7 wins for Fischer and 3 for Spassky, with a prize split of 5/8 for Fischer and 3/8 for Spassky, or, a gold ratio of 5 to 3.

There were 8 occasions when both scores were positive integers. After 5 of the games (1, 2, 3, 8, 13), both scores were Fibonacci numbers and consecutive Fibonacci numbers in 4 of those cases; in 2 cases (3 and 7), both scores were Lucas numbers; in 8 cases (1, 2, 3, 7, 8, 12, 13, 19 — where there are 4 Lucas-numbered and 5 Fibonacci-numbered games listed), the scores were each Fibonacci or Lucas numbers. In 7 cases (1, 2, 3, 7, 8, 12, 19) there were one Lucas and one Fibonacci number.

Even when the scores were non-integral, all was well with Fibonacci. It is well-known that there are several ways to round off numbers ending with 1/2. If one score is rounded up and the other rounded down, in 8 cases (4, 5, 6, 9, 10, 11, 18, 21) the scores will both be either a Fibonacci or a Lucas number; in 3 cases (4, 5, 21) the scores are both distinct Fibonacci numbers, being consecutive Fibonacci numbers in games 5 and 21 and making Fibonacci numbers 2 ways for the scores from game 4; in 5 cases (4, 6, 10, 11, 18) both scores will be Lucas numbers with positive subscripts, being consecutive Lucas numbers in games 4, 11, and 18; in 3 cases (4, 5, 6) both scores are both Fibonacci and Lucas numbers for both ways to round up and down. Notice that, in 3 cases (4, 5, 21), the scores are both distinct Fibonacci numbers while in 4 other cases (6, 10, 11, 18) the scores are both Lucas numbers. There were 5 games (4, 6, 10, 11, 18) after which if both scores were rounded up, they would both be either (distinct) Fibonacci or Lucas numbers, and again 3 cases (4, 5, 6) where both scores are both Fibonacci and Lucas numbers. Similarly, there are 5 cases (4, 6, 9, 14, 20) in which if both scores were rounded down, both numbers resulting are either (distinct) Fibonacci or Lucas numbers. Game 5 yields tied Fibonacci and/or Lucas numbers upon rounding both scores either up or down. Further, the non-integral entries, upon dropping the fraction 1/2, yield 13 entries which are Fibonacci numbers, and 11 which are Lucas numbers, made up of 7 distinct numbers; further, 8 are both Fibonacci and Lucas numbers, while 8 are separately Fibonacci or Lucas numbers. Games 4, 5, and 6 make two scores which are each both Fibonacci and Lucas numbers for all 3 ways to round off the scores, and a Fibonacci or Lucas number of game numbers are underlined in each case listed in this paragraph. There were 11 games where the scores were non-integral. Writing all non-integral scores as improper fractions yields 8 Fibonacci only numerators and 7 Lucas only numerators and 1 which is both.

The ratio of the two scores equalled the ratio of two positive Fibonacci numbers after 7 of the games (3, 4, 5, 7, 13, 15, 19), while after 3 of the games (5, 7, 8) the ratio of the scores was equal to the ratio of two Lucas numbers with positive subscripts. After the 8 games 1, 2, 3, 5, 6, 8, 13 and 21, the ratios of the scores are the ratios of two Fibonacci numbers, and after the 7 games 3, 4, 6, 7, 10, 11, and 18 the scores are two Lucas numbers if Spassky's score is rounded down and Fischer's rounded up. Further, in each list of games cited, a Fibonacci or Lucas number of games is underlined.

Note that game 5 fits all 5 criteria given for non-integral scores. Lastly, there is exactly one game, game 16, whose scores fit into none of the preceding patterns, again a Fibonacci count. The scores of this remarkable match follow.

SCORES IN WORLD CHAMPIONSHIP CHESS MATCH

(Scoring: Win, 1 point; Draw, 1/2 point)

Winner	Game	Spassky	Fischer	Winner	Game	Spassky	Fischer
S	1	1	0	D	12	5	7
S	2	2	0	F	13	5	8
F	3	2	1	D	14	5-1/2	8-1/2
D	4	2-1/2	1-1/2	D	15	6	9
F	5	2-1/2	2-1/2	D	16	6-1/2	9-1/2
F	6	2-1/2	3-1/2	D	17	7	10
D	7	3	4	D	18	7-1/2	10-1/2
F	8	3	5	D	19	8	11
D	9	3-1/2	5-1/2	D	20	8-1/2	11-1/2
F	10	3-1/2	6-1/2	F	21	8-1/2	12-1/2
S	11	4-1/2	6-1/2				

Making a different count, notice that, out of 42 scores occurring, if fractions are discarded, Lucas or Fibonacci numbers occur 34 times while there are only 8 occurrences of non-Fibonacci, non-Lucas numbers, and, further, each score occurs a Fibonacci or Lucas number of times. If both scores are rounded up, each score occurs a Lucas or Fibonacci number of times.

[Continued on page 438.]

POLYHEDRA, PENTAGRAMS, AND PLATO

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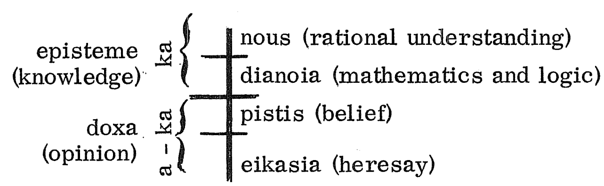
1. INTRODUCTION

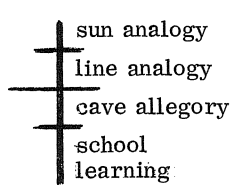
The Divided Line

Plato believed that mathematics and logic were a necessary step in the pursuit of the Good, but that an extra step of enlightenment was necessary to achieve it. This is illustrated by the discussion of the divided line in the Republic. Plato divides a line into "unequal segments," [1] by first dividing the line (length a) into two unequal segments (ka and a - ka), the shorter one near the top. He then divides each segment by the same proportion k, and he labels the segments as shown in Fig. 1a. The divided line is one of four explanations of the Good offered in the Republic (Fig. 1b), where the line is itself an example of dianoia. Now, the problem with the line is that, following orders, one cannot construct the second and third segments unequal [2]. I leave the proof to the reader.

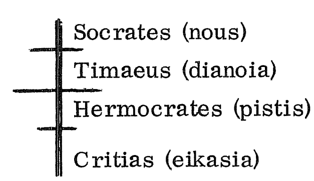
Platonists conclude that mathematics, symbolized by the line, while useful for describing the Good, contains inconsistencies reconciled only by a higher perception. In [2] is an excellent detailed explanation of the line.



(a) Steps in the Attainment of the Good



(b) Ways to Understand the Good



(c) Characters in the Timaeus

Fig. 1 The Divided Line

The irreconcilability of mathematics and the highest good was a particularly sore area for Plato. Irrational numbers were a case in point. Plato felt that, even though they were embodied in many beautiful objects (the Golden Ratio appeared in many buildings of his day), irrational numbers were without reason and impure.

2. THE DODECAHEDRON IN THE TIMAEUS

In this dialogue, Plato describes the celestial orbs as consisting of the five regular polyhedra, each of whose faces can be decomposed into the basic triangles which constitute matter [3]. He divides them up as shown in Fig. 2. The Pythagoreans divided the pentagonal faces of the dodecahedron into 30 elementary scalene triangles [4], as shown in Fig. 3a.

POLYHEDRON	FACE	ELEMENTARY TRIANGLE
pyramid	(4) triangles	
octahedron	(8) triangles	(6) $30°$-$60°$-$90°$
icosahedron	(20) triangles	
cube	(6) squares	(4) $45°$-$45°$-$90°$
dodecahedron	(12) pentagons	Plato does not divide these.

Fig. 2 The Celestial Orbs and their Constituent Triangles, Squares, and Pentagons

(a) Decomposed into elementary triangles (b) Represented as a pentagram

Fig. 3 The Pentagonal Face of a Dodecahedron [4]

This decomposition provides the outline of the famous pentagram (Fig. 3b), the Pythagorean symbol of recognition, meaning "Health" [5]. The heavy outline in Fig. 3a marks a $72°$-$72°$-$36°$ isosceles triangle, the ratio of whose sides is the Golden Ratio, which is irrational [6].

The first four polyhedra describe the Sun, the Moon, and planets [7], and comprise collectively the Circle of the Different; but the dodecahedron, the Circle of the Same, is the celestial sphere itself. The twelve faces of the dodecahedron are the twelve signs of the Zodiac [8]. Where the other orbs rotate at various intervals, the dodecahedron rotates exactly once each day (actually the rotation of the earth). Plato gives the dodecahedron special compliments. Because of its diurnal regularity, it has Sameness and Supremacy and is Self-Moving, quite a nice Platonic praise. Most importantly, the dodecahedron is rational. He says:

> Now whenever discourse that is alike true... is about that which is sensible, and the circle of the Different, moving aright, carries its message throughout all its soul--then there arise judgments and beliefs that are sure and true. But whenever discourse is concerned with the rational, and the circle of the Same, running smoothly, declares it, the result must be rational understanding and knowledge [9].

Plato contradicts himself. At the root of the dodecahedron is the Golden Ratio, which is irrational and Platonically imperfect; yet Plato describes the dodecahedron as rational and perfect.

The easiest explanation of this contradiction is that it is a Platonic aberration. But I think that Plato knew it all along, and that it is an attempt to show a flaw in Timaeus' argument.

3. INTERPRETATION

Plato is giving mathematicians a subtle lesson on the limits of their perceptions. It is not Plato speaking in this dialogue, it is Timaeus--in fact, there are four speakers: Critias, Hermocrates, Timaeus, and Socrates. This is an important Platonic clue. The characters might be ranked as shown in Fig. 1c. Critias begins with a story about Atlantis which he heard from the old Critias, who heard it from Solon, who heard it from the priest of an ancient Egyptian province (an example of heresay). Hermocrates is the one who introduces the story to Plato (belief). Timaeus is the scientist, describing the universe with natural laws and mathematics as he sees them (mathematics). Plato (rational understanding) never gets the last word.

Timaeus' discussion is a model of <u>dianoia.</u> We Timaeuses might describe the world in our mathematical terms and point to the beauty of our models, but according to Plato, our models have built-in contradictions. Like the Line, the <u>Timaeus</u> is a mathematical description of nature; and like the Line, it must contain hidden contradictions and imperfections. Timaeus' mathematical construct, the dodecahedron, is superficially beautiful and rational, but it contains hidden the Golden Ratio and its imperfect, irrational $\sqrt{5}$.

Now most of us probably do not see anything wrong with $\sqrt{5}$; after all, it's much neater than e or i. And I personally think mathematics is quite beautiful. But Plato believed that mathematics cannot simultaneously retain its simplicity and achieve beauty, that mathematics alone is insufficient to achieve the Good, and that the Golden Ratio is the paradigm of mathematics' aesthetic inadequacy, as shown by the dodecahedron.

We Fibonacci lovers can at least savor the knowledge that the great Greek genius spent so much time thinking about one of our favorite numbers.

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2. Robert S. Brumbaugh, <u>Plato's Mathematical Imagination</u>, Bloomington: Indiana University Press, 1954, p. 98.
3. Plato, <u>Timaeus</u>, Francis M. Cornford (trans.), Indianapolis: Bobbs-Merrill Co., 1959, 54d-55c.
4. Thomas L. Heath, <u>The Thirteen Books of Euclid's Elements</u>, New York: Dover Publications, Inc., 1956, Vol. 2, p. 98.
5. <u>Ibid.</u>, pp 99. This is the answer to word "K" in Marjorie Bicknell, "A Fibonacci Crostic," <u>Fibonacci Quarterly</u>, Vol. 9, No. 5, Dec. 1971, pp. 538-540. The pentagram is the emblem of the Fibonacci Association (see cover).
6. According to legend, Hippasus was struck down at sea for discovering the dodecahedron and thus introducing irrational numbers to the world.
7. Kepler published a model of these in <u>Mysterium cosmographicum</u> in 1595.
8. Plato, <u>Timaeus</u>, <u>op. cit.</u>, 55c.
9. Ibid, 37b-c. The underlines are mine.

(2d)	0	0	0	-5	-15	-25	-25
(5d)	0	0	0	-250	-750	-1250	-1250
(6d)	0	0	0	-625	-1875	-3125	-3125 .

The first pivot element in the fourth column is in (2d).

$$(5e) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,.$$

Back substituting in (2d), (3c), (4b) and (1a), we find the coefficients in the recursion relation to be $a = 3$, $b = -4$, $c = 2$, $d = -1$, so that the recursion relation of the fourth order is:

$$T_{n+1} = 3T_n - 4T_{n-1} + 2T_{n-2} - T_{n-3} \,.$$

[Continued from page 434.]

Turning to the players and the match itself, Fischer, who is now 29, won the U. S. Junior Open Championship at age 13 and became an international grandmaster at 15, while Spassky was 18, three years older than Fischer was, when designated an international grandmaster. Larry Evans, American grandmaster, in Time, September 11, 1972, analyzes the match as having three parts: games 1-5, opening, 5 games; games 6-13, middle, 8 games; games 14-21, ending, 8 games. Fischer's "poisoned pawn" bobble came on the 29^{th} move of the first game, after Fischer had arrived 7 minutes late, while the 11^{th} move of the third game was the key move in his first win. Finally, observe that the match was played in the 7^{th} and 8^{th} months at longitude 21°W.

In conclusion, returning to the opening sentence of this paper, notice that every proper noun has a Fibonacci number of letters in the word (except for Fibonacci itself).

The odds were really in our favor since 11 out of the first 21 integers are Fibonacci or Lucas numbers and of these 4 are even integers and 7 are odd integers. Again, of these 21, 7 are Lucas, 7 are Fibonacci and 3 are both.

The factual information in this article was gleaned for the most part from Time Magazine and the San Jose Mercury-News.

REFERENCE

1. V. E. Hoggatt, Jr., letters, September 5, 8, 11, and 13, 1972.

MAKING GOLDEN CUTS WITH A SHOEMAKER'S KNIFE

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1. The problem of finding the "golden cut" (or section) of a line segment was known to the early Greeks and is solved in Euclid II, 11 [1]: let segment AB be divided into two segments by the point G such that $AB/AG : AG/GB$. Then G is the golden cut of AB, and AG/GB, the golden ratio. The "shoemaker's knife" (or arbelos) was the name given by Archimedes [2] to the following figure: let K be any point on segment CD and let semicircles be drawn on the same side of CD, with CD, CK, and KD as diameters. The figure whose boundary consists of these semicircles is called a shoemaker's knife (see Fig. 1).

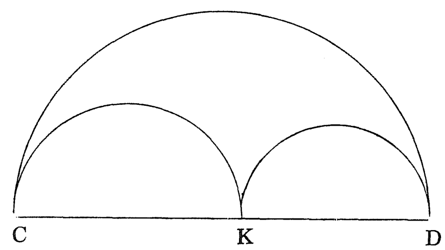

Figure 1

In this note, we will show how, given a golden cut G in a segment AB, we can, with the aid of an arbelos, generate a golden cut on any segment with length smaller than AB, in a swift and straightforward fashion. This in itself should justify bestowing the title of "golden" on the arbelos; however, we will also give a justification which conforms more with historical criteria.

2. Let segment AB be given and let G denote the golden cut of AB. Let C be any interior point of segment AB. The problem is to find the golden cut of segment AC. Construct a semicircle with diameter AB. Locate point D on the semicircle such that the distance AD is equal to AC. Draw chord AD, and drop a perpendicular from G to AD. Denote the foot of the perpendicular by G'. (See Fig. 2.)

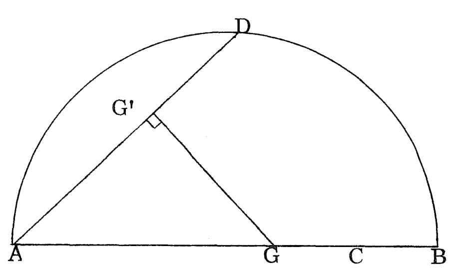

Figure 2

Claim. G' is the golden cut of chord AD.

Proof. Draw chord BD. Since angle ABD is a right angle, it follows that the triangles $AG'G$ and ADB are similar, from which it follows that $AG'/G'D = AG/GB$. Hence, G' is the golden cut of AD, which was to be shown. Finally, by locating point G'' on segment AB such that $AG'' = AG'$, we have that G'' is the golden cut of segment AC.

Since angle $AG'G$ is a right angle, it follows that the points A, G', G determine a semicircle. Thus, we have the following corollary: let AD be any chord in the semicircle with diameter AB. Let G' denote the golden cut of AD. Then the locus of all such points G' is a semicircle with diameter AG (G, the golden cut of AB).

It is easy to see that both the construction and above corollary carry over with obvious modifications if we reference everything at point B, rather than point A. Thus, if BD is any chord in the semicircle, and H its golden cut, then the locus of all such points H is a semicircle with GB as diameter.

The figure consisting of the semicircles with diameters AB, AG, and GB is, of course, an arbelos. Now we are ready to reverse the above procedure and deal with the main problem: viz, to utilize the shoemaker's knife to effect golden cuts. Let AB be a given line segment and G its golden cut. Draw semicircles with diameters AB, AG, and GB, respectively. Let D be any point on the semicircle with diameter AB, and draw chord AD intersecting the semicircle on diameter AG at G'. Then G' is the golden cut of AD. (See Fig. 3.)

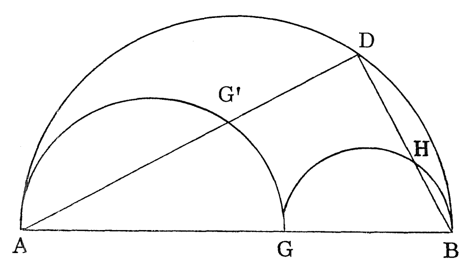

Figure 3

By drawing chord BD intersecting the semicircle on diameter GB at H, we see also that H is the golden cut of BD. The argument for both these statements is the same as that given above.

In light of the latter argument, it is reasonable that the arbelos in Fig. 3 should be termed "golden." To see that this terminology is in fact also historically justified, observe that the arbelos can be viewed as a continuous deformation of a right triangle, where the hypotenuse corresponds to the largest semicircle and the legs to the smaller semicircles. Historically [1], a right triangle is called golden if the ratio of its legs is the golden ratio. In light of the above observation, it would be in keeping to term the arbelos "golden" if the ratio of its "legs" (i.e., its minor semicircles) were the golden ratio. A simple computation for our arbelos reveals that this is in fact the case; the length of the semicircle on AG is evi-

[Continued on page 444.]

YE OLDE FIBONACCI CURIOSITY SHOPPE

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In the good old days when Jekuthiel Ginsburg was Editor of Scripta Mathematica, there were many brief items of interesting mathematics, some with proof, some without, contributed by a wide variety of people. Some of these were labeled curiosities; others without being tagged as such were evidently in the same category. A fair amount of this material dealt with Fibonacci sequences. We offer a few for-instances translated into symbolism more familiar to readers of the Fibonacci Quarterly.

Charles W. Raine [1] noted that if four consecutive Fibonacci numbers are taken, then the product of the extreme terms can be used as one leg of a Pythagorean triangle, twice the product of the mean terms as the other, to give a hypotenuse which is a Fibonacci number whose index is the sum of the indices of the terms in one of the sides. For example, if

$$F_6 = 8, \quad F_7 = 13, \quad F_8 = 21, \quad F_9 = 34$$
$$a = 8 \times 34 = 272; \qquad b = 2 \times 13 \times 21 = 546;$$
$$c = \sqrt{272^2 + 546^2} = 610 = F_{15}.$$

Harlan L. Umansky [2] following up on Raine's idea extended the result to a generalized Fibonacci sequence. The sides of the Pythagorean triangle in this case would be given by:

$$a = T_kT_{k+3}; \qquad b = 2T_{k+1}T_{k+2}; \qquad c = b + T_k^2 \quad \text{or} \quad c = T_{k+1}^2 + T_{k+2}^2 .$$

For example, using the sequence 1, 4, 5, 9, 14, 23, $\cdots$ and taking the four values 5, 9, 14, 23, $a = 5 \times 23 = 115$; $b = 2 \times 9 \times 13 = 252$.

$$c = \sqrt{115^2 + 252^2} = 277,$$

while $$b + T_k^2 = 252 + 5^2 = 277 \quad \text{and} \quad 9^2 + 14^2 = 277.$$

Gershon Blank [3] pointed out that

$$(F_n + F_{n+6})F_k + (F_{n+2} + F_{n+4})F_{k+1} = L_{n+3}L_{k+1} .$$

For example, if $n = 5$, $k = 4$, $(5 + 89) \times 3 + (13 + 34) \times 5 = 517$, while $47 \times 11 = 517$.

A note signed G. (evidently J. Ginsburg himself) [4] quoted the cubic relation given by Dickson

$$F_{n+1}^2 + F_n^3 - F_{n-1}^3 = F_{3n}$$

and offered a second

$$F^3_{n+2} - 3F^3_n + F^3_{n-2} = 3F_{3n} .$$

Thus for $n = 5$,

$$13^3 - 3 \times 5^3 + 2^3 = 1830 = 3F_{15} .$$

Fenton Stancliff [5] (A Curious Property of a_{11}) showed the following arrangement for finding the value of $1/89 = 0.011235955056\cdots$

```
1/89 = 0.0112358
              13
               21
                34
                 55
                  89
                  144
                   233
                    377
       ------------------
       0.011235955056107
```

P. Schub [6] (A Minor Fibonacci Curiosity) offered the relation:

$$5F^2_n + 4(-1)^n = L^2_n .$$

G. Candido [7] produced a fourth-power relation:

$$2(F^4_n + F^4_{n+1} + F^4_{n+2}) = (F^2_n + F^2_{n+1} + F^2_{n+2})^2$$

When $n = 5$, this becomes

$$2(5^4 + 8^4 + 13^4) = 66564$$

while

$$(5^2 + 8^2 + 13^2)^2 = 66564 .$$

Royal V. Heath [8] noted that the sum of ten consecutive Fibonacci numbers is divisible by 11 with a cofactor the seventh of the ten quantities. The sum of fourteen consecutive Fibonacci numbers is divisible by 29 with the cofactor the ninth; etc.

Harlan L. Umansky [9] (Curiosa, Zero Determinants) offered the following. For any series a, d, a + d, a + 2d, 2a + 3d, etc., if N^2 consecutive terms $(N \geq 2)$ are taken and placed consecutively in the columns of a determinant, the value of the determinant is zero. Thus:

$$\begin{vmatrix} 1 & 11 & 76 & 521 \\ 3 & 18 & 123 & 843 \\ 4 & 29 & 199 & 1364 \\ 7 & 47 & 322 & 2207 \end{vmatrix} = 0$$

These examples are sufficient to give the general flavor of the contributions of those days. I believe there must be people today likewise who would be happy to express themselves in this fashion once more. Recently, in a private communication, William H. Huff stated that he had discovered the following. Add up any number of consecutive Fibonacci numbers; then add the second to the result; the final answer is always a Fibonacci number. The same seems to hold for any Fibonacci sequence starting with two values a, b.

Another curiosity is the fact that the sum of the squares of consecutive Fibonacci numbers is always divisible by $F_{10} = 15$. What about other sums of squares either of the Fibonacci sequence proper or other Fibonacci sequences? Are there similar results for sums of cubes, fourth powers, etc.?

This article is an invitation to our readers to engage in the type of activity that used to be featured by Scripta Mathematica. While it is probably true that one man's "curiosity" is another man's formula, article, or thesis, we shall content ourselves by defining a "curiosity" as follows: A fact or relation that seems interesting and calculated to arouse intellectual curiosity which is offered without proof for the consideration of the readers of the Fibonacci Quarterly. If you want something to appear in this department, be sure to label it FIBONACCI CURIOSITY.

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1. Charles W. Raine, "Pythagorean Triangles from the Fibonacci Series 1, 1, 2, 3, 5, 8," Scripta Mathematica, 14, 1948, p. 164.
2. Harlan L. Umansky, 433. "Pythagorean Triangles from Recurrent Series," Scripta Mathematica, xxii, No. 1, March 1956, p. 88.
3. Gershon Blank, "Another Fibonacci Curiosity (401)," Scripta Mathematica, xxi, No. 1, March 1955, p. 30.
4. G., 343, "A Relationship Between Cubes of Fibonacci Numbers," Scripta Mathematica, Dec. 1953, p. 242.
5. Fenton Stancliff, 323. "A Curious Property of a_{11}," Scripta Mathematica, xix, No. 2-3, June-Sept. 1953, p. 126.
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7. G. Candido, "A Relationship Between the Fourth Powers of the Terms of the Fibonacci Series," Scripta Mathematica, xvii, No. 3-4, Sept.-Dec. 1951, p. 230.
8. Royal V. Heath, 229. "Another Fibonacci Curiosity," Scripta Mathematica, xvi, No. 1-2, March-June 1950, p. 128.
9. H. L. Umansky, "Curiosa. Zero Determinants," Scripta Mathematica, xxii, No. 1, March 1956, p. 88.

AN INTERESTING BOOK

A MATHEMATICAL MODEL OF LIFE AND LIVING by Li Kung Shaw, Second Edition, 1972. Paperback, good quality printing and binding, 94 pages. (Libreria Inglesa, P. O. Box 94 (Suc. 25), Buenos Aires, Argentina, \$3.00 postpaid.)

The Generalized Equation of the Golden Ratio

$$S + h = \sqrt{S^2 + S} \tag{1}$$

which arises in Li Kung Shaw's mathematical model of human life induces a new development in that historically famous topic of mathematics and magic, the Golden Section Ratio. Here, S represents a man's service hours and h his work hours. The equation (1) arises as the condition for the happiness function

$$F = h\left(\frac{1 - h}{S + h}\right)$$

to be maximized. The mathematical model begins with five basic assumptions and has derived values which are in conformity with natural human behavior.

Li Kung Shaw graduated from Chiao Tung University in Shanghai, China, in 1937, where he had studied physics. He was Chief of Air Transportation of the Chinese Civil Aeronautics Administration in Nanking, following service in the Air Forces as an aeronautical engineer. After three years of refugee life, he immigrated to Argentina where he practices operations research.

---Marjorie Bicknell

[Continued from page 440.]

MAKING GOLDEN CUTS WITH A SHOEMAKER'S KNIFE

dently $(\pi/2)AG$, while the length of the semicircle on GB is $(\pi/2)GB$. The ratio of these two lengths is then seen to be AG/GB, the golden ratio.

REFERENCES

1. H. E. Huntley, The Divine Proportion, Dover, New York, 1970.
2. Levi S. Shively, An Introduction to Modern Geometry, Wiley, New York, 1949.

THE FIBONACCI ASSOCIATION

PROGRAM OF THE NINTH ANNIVERSARY MEETINGS
FIFTH ANNUAL SPRING CONFERENCE AND BANQUET

Saturday, 22 April, and Sunday, 23 April, 1972

HARVEY SCIENCE CENTER — UNIVERSITY OF SAN FRANCISCO

Sponsored by the Institute of Chemical Biology

INTRODUCTORY TOPICS

Elementary Session: FIBONACCI OPEN HOUSE
Saturday, April 22, 1972

8:30 — Registration

9:00 — WELCOME
George Ledin, Jr., University of San Francisco

9:15 — FIBONACCI REPRESENTATIONS
Brother Alfred Brousseau, St. Mary's College, Moraga, California

10:00 — THE GOLDEN SECTION AND THE GREEK CRISIS
G. D. Chakerian, University of California, Davis, California

10:45 — Question and Answer Period

11:00 — Short talks to small groups to be presented by high school students.
Topics and speakers to be announced

11:45 — A GENERALIZATION OF BEVERLY ROSS' PROBLEM
Brian Peterson, student, San Jose State University, San Jose, California

ADVANCED TOPICS

Research Session
Sunday, April 23, 1972

10:00 — Registration

10:30 — REDUNDANCY: WHY TWO RABBITS ARE BETTER THAN ONE
Loran P. Meissner, Dept. of Computer Science, University of California, Berkeley

11:20 — GENERAL LINEAR FIBONACCI AND LUCAS IDENTITIES
Rodney Hansen, Montana State University, Bozeman, Montana

12:10 — ALGORITHM FOR FINDING PRIME FACTORS
Capt. N. A. Draim, Ventura, California

2:15 — SOME RESULTS IN GRAPH THEORY
Frank Harary, University of Michigan, Ann Arbor, Michigan

3:15 — THE ARITHMETIC PROPERTIES OF CERTAIN RECURSIVELY DEFINED SEQUENCES — David Klarner, Dept. of Computer Science, Stanford University

4:15 — SOME COMBINATORIAL PROBLEMS ON PERMUTATIONS
Leonard Carlitz, Duke University, Durham, North Carolina

5:30 — Cocktails
6:00 — Dinner-Banquet

FIBONACCI MAKES THE SPORTS PAGE

The following appeared in an article clipped from the sports page (p. 51) of the San Francisco Chronicle, Wednesday, August 2, 1972, by A. P. Hillman.

"5 Homers in Sweep: COLBERT'S 13 RBIs

"Nate Colbert set a major league record when he drove in 13 runs on five home runs, including a grand slammer, and a single in leading the San Diego Padres to a doubleheader sweep over Atlanta 9-0 and 11-7."

The article continues, stating that Colbert hit two homers in the first game and three in the second, driving in five runs in the opener and eight in the nightcap.

"In other games, Cleon Jones' single in the 18th gave the New York Mets a 3-2 win over Philadelphia in the opener, but Steve Carlton's 11th straight win gave the Phils the nightcap, 4-1; Matty Alou's three hits sparked St. Louis to a 7-4 win over Pittsburgh; Joe Morgan's two-run homer led Cincinnati to a 3-1 victory over Houston and Montreal edged the Chicago Cubs, 3-2, in ten innings.

"Colbert homered with two mates aboard in the first inning of the opener, singled home a run in the third and added a solo homer in the seventh.

"In the nightcap, Colbert blasted a grand slam homer off Pat Jarvis in the second inning and hit a two-run shot off Jim Hardin in the seventh. His final homer, another two-run shot, came in the ninth."

The reader is left to find his own Fibonacci and Lucas number counts. If the Padres had scored 8 in the first game against Atlanta, and if Montreal had taken 11 innings, and Colbert had homered in the 8th inning instead of the ninth, then every number mentioned including the date would be either a Fibonacci or a Lucas number. Isn't that an amazing coincidence? (And the article should have appeared on page 55.)

FIBONACCI NOTE SERVICE

The Fibonacci Quarterly is offering a service in which it will be possible for its readers to secure background notes for articles. This will apply to the following:

(1) Short abstracts of extensive results, derivations, and numerical data.

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(3) Articles of standard size for which additional background material may be obtained.

Articles in the Quarterly for which this note service is available will indicate the fact, together with the number of pages in question. Requests for these notes should be made to:

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The notes will be Xeroxed.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within five months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-238 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Can you guess WHO IS SHE? This is an easy simple addition and SHE is divisible by 29.

$$\begin{array}{r} \text{WHO} \\ \text{IS} \\ \hline \text{SHE} \end{array}$$

B-239 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let $p > 0$, $q > 0$, $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = pu_n + qu_{n-1}$ $(n \ge 1)$. Put

$$\left\{ {n \atop k} \right\} = u_n u_{n-1} \cdots u_{n-k+1} / u_1 u_2 \cdots u_k, \qquad \left\{ {n \atop 0} \right\} = 1.$$

Show that

$$(*) \qquad \left\{ {n \atop k} \right\}^2 - p^2 \left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} > 0 \qquad (0 \le k \le n)\ .$$

B-240 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California.

Prove that, for all positive integers n, $3F_{n+2}F_{n+3}$ is an exact divisor of

$$7F_{n+2}^3 - F_{n+1}^3 - F_n^3 \ .$$

B-241 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

If $2F_{2n-1}F_{2n+1} - 1$ and $2F_{2n}^2 + 1$ are both prime numbers, then prove that

$$F_{2n}^2 + F_{2n-1}F_{2n+1}$$

is also a prime number.

B-242 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n}{k-1} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k < n$).

B-243 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n+1}{k} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k \le n$).

ERATTA FOR

A CHARACTERIZATON OF THE FIBONACCI NUMBERS SUGGESTED BY A PROBLEM ARISING IN CANCER RESEARCH

Please make the following changes in "A Characterization of the Fibonacci Numbers Suggested by a Problem Arising in Cancer Research" by Leslie E. Blumenson, appearing on pp. 262-264, Fibonacci Quarterly, April 1972.

Page 263, line 11: For "$N^2 = 2$," read "$N = 2$";

Page 264, fourth line from bottom of page: For "+" read "•";

Page 264, Eq. (6): For "+" read "•";

Page 292, Eq. (7): For "+" read "•" .