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PELLIAN REPRESENTATIONS

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1. INTRODUCTION

We define the Pellian numbers by means of

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1} \quad (n \geq 1).$$

By a Pellian representation of the positive integer N we mean a representation of the form

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots , \tag{1.1}$$

where the ϵ_i are non-negative integers. If the ϵ_i are restricted to the values $0, 1$, not all integers N are representable. Indeed we have the sequence of "missing" numbers:

$$4, 9, 10, 11, 16, 21, 22, 23, 24, 25, 26, 27, 28, \cdots .$$

On the other hand we prove that every positive integer N is uniquely representable in the form (1.1) where the ϵ_i satisfy the following conditions:

$$\begin{aligned} &\epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1 \text{ or } 2; \\ &\text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 . \end{aligned} \tag{1.2}$$

It follows that the sequence of "missing" numbers is infinite.

When (1.2) is satisfied we call (1.1) the canonical representation of N. Let A_k denote the set of integers N such that

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k \neq 0 .$$

and let B_k denote the set of integers N such that

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k = 2.$$

As in the previous papers of this series [1, 2, 3, 4], we shall characterize the sets A_k, B_k) in terms of certain arithmetic functions. As we shall see below, the discussion is considerably

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more elaborate than that in the case of Fibonacci representations. The number of functions necessary to describe the sets A_k, B_k is greater than that needed for the corresponding Fibonacci results; moreover some of the relations are more intricate.

To begin with, if N has the canonical representation (1.1) we define

$$e(N) = \epsilon_2 P_1 + \epsilon_3 P_2 + \epsilon_4 P_3 + \cdots \tag{1.3}$$

and

$$p(N) = \epsilon_1 P_2 + \epsilon_2 P_3 + \epsilon_3 P_4 + \cdots . \tag{1.4}$$

Then

$$e\bigl(p(n)\bigr) = n \qquad (n = 1, 2, 3, \cdots)$$

however, for some n,

$$p\bigl(e(n)\bigr) \neq n .$$

Note that the right member of (1.3) need not be canonical.

Next we define the following six functions:

$$\begin{aligned} a(n) &= [\sqrt{2}\, n], \qquad b(n) = [(2 + \sqrt{2})n] \\ d(n) &= [(1 + \sqrt{2})n], \qquad d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n], \\ \delta(n) &= b(n) + d(n), \quad \epsilon(n) = \text{complement of } \delta(n) . \end{aligned}$$

Two (strictly monotone) functions f_1, f_2 from $\underset{\sim}{N}$ to $\underset{\sim}{N}$ are <u>complementary</u> if the sets

$$f_1(\underset{\sim}{N}), \qquad f_2(\underset{\sim}{N})$$

constitute a disjoint partition of $\underset{\sim}{N}$, the set of positive integers. In particular a, b; d, d'; δ, ϵ are complementary pairs of functions.

Of the numerous relations satisfied by these functions we mention in particular the following:

$$\begin{aligned} b(n) &= a(n) + 2n, \qquad d(n) = a(n) + n , \\ ab(n) &= a(n) + b(n), \qquad d'(2n) = b(n) , \\ d(n) &= a\bigl(b(n) - d'(n)\bigr), \quad a^2b(n) = 2b(n) = 1, \\ \epsilon(2n) &= \epsilon(2n - 1) + 1 = d(n), \quad d'\bigl(b(n)\bigr) = \delta(n), \\ a(n+1) &= e(n) + n + 1, \quad b(n+1) = p(n) + n + 3, \\ e\bigl(d(n)\bigr) &= n, \quad e\bigl(b(n)\bigr) = a(n), \quad e\bigl(\delta(n)\bigr) = d(n), \\ p\bigl(d(n)\bigr) &= \delta(n), \qquad p\bigl(\delta(n)\bigr) = d\bigl(\delta(n)\bigr). \end{aligned}$$

The sets A_k, B_k are described by the following formulas:

$$
\begin{aligned}
A_1 &= d(\underset{\sim}{N}) - 1 , \\
A_{2k} &= d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) , \\
A_{2k+1} &= \delta^{k}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) , \\
B_{2k} &= d\delta^{k-1}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) , \\
B_{2k+1} &= \delta^{k}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) .
\end{aligned}
$$

This summarizes the first half of the paper. In the remaining sections of the paper we discuss various other functional relations. For the most part these relations are motivated by the introduction of certain supplementary functions f, f'; g, g' now to be defined. To begin with, we note that the function

$$s(n) = ab(n) - ba(n)$$

takes on only the values $1, 2$; similarly the function

$$t(n) = ad'(n) - d'a(n)$$

takes on only the values $0, 1$. We define f, f' by means of

$$s\big(f(n)\big) = 1, \qquad s\big(f'(n)\big) = 2 ;$$

similarly we define g, g' by means of

$$t\big(g(n)\big) = 0, \qquad t\big(g'(n)\big) = 1 .$$

Thus f, f'; g, g' are complementary pairs.

Alternatively we may define these functions by means of

$$a^2\big(f(n)\big) \equiv 1, \qquad a^2\big(f'(n)\big) \equiv 0 \pmod 2$$

and

$$a\big(g(n)\big) \equiv 1, \qquad a\big(g'(n)\big) \equiv 0 \pmod 2 .$$

In addition, the complementary pair c, c' should also be mentioned:

$$c(n) = b(n) - d'(n) ;$$

as noted above,

$$d(n) = a\big(c(n)\big) .$$

Of the relations satisfied by these functions we note the following:

$$g(n) = a\bigl(f(n)\bigr), \qquad f'(n) = d\bigl(f(n)\bigr)$$

$$b\bigl(f(n)\bigr) - a\bigl(f'(n)\bigr) = 1$$

$$c'(n) = \begin{cases} d(n) & \bigl(n = f'(k)\bigr) \\ d(n) - 1 & \bigl(n = f(k)\bigr) \end{cases}$$

$$c(n) = \begin{cases} d'(n) + 1 & \bigl(n = g(k)\bigr) \\ d'(n) & \bigl(n = g'(k)\bigr) \end{cases}$$

$$a\bigl(c'(n)\bigr) = c'(n) + n - 1 = d'(2n - 1)$$

$$c(n) = \epsilon\bigl(a(n)\bigr) + 1$$

$$e\bigl(c'(n) + 1\bigr) = n .$$

The last section of the paper contains some theorems involving the functions of σ, τ defined as follows by means of (1.1):

$$\sigma(N) \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots \pmod 2$$

$$\tau(N) \equiv k \pmod 2 \qquad (N \in A_k) .$$

In particular we show that

$$b(\underset{\sim}{N}) = \{n \mid \sigma(n) = 0, \quad \tau(n) = 1\}$$

$$g(\underset{\sim}{N}) = \{n \mid \sigma(n) = \tau(n)\}$$

$$= \{n \mid \sigma(n - 1) = 0\} ,$$

$$dg(\underset{\sim}{N}) = \{n | n \in (d), \quad \sigma(n) = 1\}$$

$$dg'(\underset{\sim}{N}) = \{n | n \in (d), \quad \sigma(n) = 0\} .$$

For the convenience of the reader a summary of formulas appears at the end of the paper, as well as several numerical tables.

It should be remarked that most of the theorems in this paper were suggested by numerical data. Thus further numerical data may well suggest additional theorems, particularly in the case of some of the functions defined in the latter part of the paper and not explicitly mentioned in this Introduction.

2. THE CANONICAL REPRESENTATIONS

As above, the Pellian numbers P_n are defined by

(2.1) $$P_0 = 0, \qquad P_1 = 1, \qquad P_n = 2P_{n-1} + P_{n-2} ,$$

so that

$$P_2 = 2, \quad P_3 = 5, \quad P_4 = 12, \quad P_5 = 29, \quad P_6 = 70, \quad \cdots .$$

We consider sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \tag{2.2}$$

of length n, where the ϵ_i satisfy the conditions

$$\begin{cases} \epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1, 2 \quad (i > 1) \\ \text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 . \end{cases} \tag{2.3}$$

It is easily seen by induction on n that the number of sequences (2.2) is precisely P_{n+1}. We prove next that if N is given by

$$N = \epsilon_1 P_1 + \cdots + \epsilon_n P_n ,$$

where the ϵ_i satisfy the conditions (2.3), then $N < P_{n+1}$. For otherwise we would have

$$\begin{aligned} N - \epsilon_n P_n - \epsilon_{n-1}P_{n-1} &\geq P_{n+1} - \epsilon_n P_n - \epsilon_{n-1}P_{n-1} \\ &= (2 - \epsilon_n)P_n + (1 - \epsilon_{n-1})P_{n-1} \geq P_{n-1} \end{aligned} ,$$

which eventually leads to a contradiction. See Keller [7] for similar results.

Theorem 2.1. Every positive integer N can be written uniquely in the form

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \cdots , \tag{2.4}$$

where

$$\begin{cases} \epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1 \text{ or } 2; \\ \text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 \end{cases} \tag{2.5}$$

Proof. In view of the preceding remarks, it is enough to prove that no integer N can have more than one representation (2.4), because if this can be established, the P_{n+1} numbers corresponding to the sequences (2.2) of length n will be precisely

$$0, 1, 2, \cdots, P_{n+1} - 1 .$$

Now suppose N is given by

$$N = \epsilon_1 P_1 + \cdots + \epsilon_n P_n, \qquad \epsilon_n \neq 0 ,$$

where the ϵ_i satisfy (2.5). Then $P_n \leq N < P_{n+1}$, so that n is uniquely determined by N. Now by considering $N - \epsilon_n P_n$ we see that ϵ_n itself is determined uniquely by N. Hence, by induction, the theorem is proved.

In a similar manner we can prove the following theorem.

Theorem 2.2. Every positive integer N can be written uniquely in the form

$$N = \delta_1 P_1 + \delta_2 P_2 + \cdots, \tag{2.6}$$

where

$$\begin{cases} \delta_i = 0,\ 1 \text{ or } 2 & (i = 1, 2, 3, \cdots) \\ \text{if } \delta_1 = \cdots = \delta_{i-1} \neq 0, \quad \delta_i \neq 0, & \text{then } i \text{ is odd.} \end{cases} \tag{2.7}$$

The form (2.4) will be called the first canonical representation for N (or simply the canonical representation); the form (2.6) will be called the second canonical representation.

It will be convenient to abbreviate the formula

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots$$

as follows:

$$N = .\,\epsilon_1\epsilon_2\epsilon_3 \cdots .$$

We shall say that N is a missing number if $\epsilon_i = 2$ for some i. Hence the missing numbers are those which are not the sum of distinct Pell numbers.

Theorem 2.3. The number of missing numbers less than P_{n+1} is equal to $P_{n+1} - 2^n$. Moreover if

$$N = \epsilon_0 + 2\epsilon_1 + \cdots + 2^k\epsilon_k \qquad (\epsilon_i = 0,\ 1)$$

is the binary representation of N, then

$$R_N = \epsilon_0 P_1 + \epsilon_1 P_2 + \cdots + \epsilon_k P_{k+1}$$

is the N^{th} number that can be represented as a sum of distinct Pell numbers.

Proof. The number of sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$$

in which each $\epsilon_i = 0$ or 1 is clearly 2^n. Since the total number of sequences is P_{n+1}, it follows that the number of sequences containing at least one 2 is $P_{n+1} - 2^n$.

For the second half of the theorem it suffices to observe that the proof of Theorem 2.1 shows that R_N is a strictly monotone function of N.

The first few missing numbers are

$$4,\ 9,\ 10,\ 11,\ 16,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28,\ \cdots. \tag{2.8}$$

Let N have the first canonical representation

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots.$$

We define the functions $e(N)$, $p(N)$ by means of

(2.9)
$$e(N) = \epsilon_2 P_1 + \epsilon_3 P_2 + \epsilon_4 P_3 + \cdots$$
and
(2.10)
$$p(N) = \epsilon_1 P_2 + \epsilon_2 P_3 + \epsilon_3 P_4 + \cdots .$$

Theorem 2.4. The functions e and p satisfy the following identities:

(2.11)
$$p(n) = e(n) + 2n$$
(2.12)
$$e\bigl(p(n)\bigr) = n$$
(2.13)
$$e\bigl(p(n) + 1\bigr) = n$$
(2.14)
$$e\ p(n) + 2 \ = n + 1 .$$

Moreover e and p are monotone.

Proof. Let n be given canonically by

$$n = \cdot\,\epsilon_1\epsilon_2\epsilon_3 \cdots .$$
Then by definition
$$p(n) = \cdot\,0\epsilon_1\epsilon_2\epsilon_3 \cdots$$
and
$$e(n) = \cdot\,\epsilon_2\epsilon_3\epsilon_4 \ \cdots .$$

Hence (2.11), (2.12), (2.13) follow at once. If $\epsilon_2 < 2$, $p(n) + 2$ is given canonically by

$$p(n) + 2 = \cdot 0(\epsilon_1 + 1)\epsilon_2\epsilon_3 \cdots$$

and (2.14) follows. Now suppose $\epsilon_2 = 2$. Then $\epsilon_1 = 0$ and

$$p(n) + 2 = (\epsilon_3 + 1)P_4 + \epsilon_4 P_5 + \cdots .$$

As before this is canonical if $\epsilon_4 < 2$ and (2.14) follows. Otherwise we continue until, for some k, $\epsilon_{2k} < 2$, and again (2.14) follows.

To prove the monotonicity of e and p, we again take the canonical representation

$$n = \cdot\,\epsilon_1\epsilon_2\epsilon_3 \cdots .$$
if $\epsilon_1 = 1$, then
$$n - 1 = \cdot\,0\epsilon_2\epsilon_3 \cdots ,$$

so that $e(n - 1) = e(n)$. If $\epsilon_1 = 0$ and $\epsilon_2 \neq 0$, then

$$n - 1 = \cdot 1(\epsilon_2 - 1)\epsilon_3 \cdots$$

and $e(n-1) = e(n) - 1$. If

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k \neq 0, \tag{2.15}$$

then, for k odd,

$$n - 1 = eP_2 + 2P_4 + \cdots + 2P_{k-1} + (\epsilon_k - 1)P_k + \epsilon_{k+1}P_{k+1} + \cdots \tag{2.16}$$

and

$$e(n-1) = 2P_1 + 2P_3 + \cdots + 2P_{k-2} + (\epsilon_n - 1)P_{k-1} + \epsilon_{k+1}P_k + \cdots.$$

This gives $e(n-1) = e(n)$. If in (2.15) k is even, we have

$$n - 1 = P_1 + 2P_3 + \cdots + 2P_{n-1} + (\epsilon_k - 1)P_k + \epsilon_{k+1}P_{k+1} + \cdots \tag{2.17}$$

and

$$e(n-1) = 2P_2 + \cdots + 2P_{k-2} + (\epsilon_k - 1)P_{k-1} + \epsilon_{k+1}P_k + \cdots,$$

which gives $e(n-1) = e(n) - 1$.

This proves that e is monotone and therefore, by (2.12), p is also monotone.

As a corollary we have the following theorem.

Theorem 2.5. For any n, the equation $e(x) = n$ has at most three solutions.

Proof. Assume

$$e(x_1) = e(x_2) = e(x_3) = e(x_4)$$

with

$$x_1 < x_2 < x_3 < x_4.$$

It follows from the definition of p that any n must be of at least one of the three forms $p(j)$, $p(j)+1$ or $p(j)+2$. Take $n = x_2$. Then by Theorem 2.4 we have

$$e(x_1) \neq e(x_4).$$

3. NEWMAN-SKOLEM PAIRS

By a Newman-Skolem pair we shall mean a pair of functions (a,b) defined on the positive integers N and satisfying the conditions

$$a(\underset{\sim}{N}) \cup b(\underset{\sim}{N}) = \underset{\sim}{N}, \tag{3.1}$$

$$a(\underset{\sim}{N}) \cap b(\underset{\sim}{N}) \text{ vacuous}, \tag{3.2}$$

$$a, b \text{ strictly monotone}. \tag{3.3}$$

Hence a and b are complementary functions. The Newman-Skolem pair (a,b) defined uniquely by the condition

$$b(n) = a(n) + n$$

was introduced in [5].

We shall say that (a, b) is <u>ordered</u> if

$$a(n) < b(n) \qquad (n = 1, 2, 3, \cdots) \tag{3.4}$$

and that (a, b) is <u>separated</u> if (a, b) is ordered and

$$b(n + 1) > b(n) + 1 \qquad (n = 1, 2, 3, \cdots) . \tag{3.5}$$

Define

$$d(n) = b(n) - n . \tag{3.6}$$

<u>Theorem 3.1.</u> If (a, b) is separated then

$$ad(n) = b(n) - 1$$

and

$$a(d(n) + 1) = b(n) + 1 . \tag{3.8}$$

<u>Proof.</u> By (3.5) we must have, for some k,

$$b(n) - 1 = a(k), \qquad b(n) + 1 = a(k + 1) .$$

Hence the $k + n$ numbers

$$a(1), a(2), \cdots, a(k); b(1), \cdots, b(n)$$

comprise all the numbers less than or equal to $b(n)$, so that

$$k + n = b(n), \qquad k = b(n) - n = d(n) .$$

This evidently completes the proof of the theorem.

<u>Theorem 3.2.</u> If (a, b) is separated then

$$a(n + 1) = a(n) + 2 \rightleftarrows n \in (d) , \tag{3.9}$$

where (d) denotes the range of the function d.

Proof. Since (a, b) is separated it is clear that, for any n, either $a(n + 1) = a(n) + 1$ or $a(n + 1) = a(n) + 2$. Also we have

$$d(n + 1) = d(n) = b(n + 1) - b(n) - 1 \geq 1 ,$$

so that d is strictly monotone.

Now assume

$$n \neq d(k) \qquad (k = 1, 2, 3, \cdots).$$

Then, for some k,

$$d(k) + 1 \leq n < d(k + 1).$$

If $a(n + 1) = a(n) + 2$ then $a(n) + 1 = b(j)$ for some (j). But

$$b(k) + 2 = a\bigl(d(k) + 1\bigr) + 1 \leq a(n) + 1 < ad(k + 1) + 1 = b(k + 1),$$

so that $a(n) + 1 = b(j)$ is impossible.

Theorem 3.3. If (a, b) is separated and

$$d(n + 1) > d(n) + 1 \qquad (n = 1, 2, 3, \cdots)$$

then

$$a\bigl(d(k) - 1\bigr) = b(k) - 2 \qquad \bigl(d(k) \geq 2\bigr).$$

Proof. Since

$$d(k) - 1 \neq d(j) \qquad (j = 1, 2, 3, \cdots),$$

by Theorem 3.1,

$$b(k) - 1 = ad(k) = a\bigl(d(k) - 1\bigr) + 1.$$

Theorem 3.4. If (a, b) is a Newman-Skolem pair and if, for all n, we have

$$ba(n) < ab(n) < b\bigl(a(n) + 1\bigr),$$

then

$$ab(n) = a(n) + b(n). \tag{3.10}$$

Proof. Using the hypothesis we see that the $a(n) + b(n)$ numbers

$$b(1), b(2), \cdots, ba(n); \quad a(1), a(2), \cdots, ab(n)$$

coincide with the numbers less than or equal to $ab(n)$. Hence (3.10) follows at once.

It is well known that if α, β are positive irrational numbers satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \qquad \alpha < \beta, \tag{3.11}$$

the pair (a, b) defined by

$$a(n) = [\alpha n], \qquad b(n) = [\beta n] \tag{3.12}$$

is a separated Newman-Skolem pair. For the remainder of this paper we define

$$a(n) = [\sqrt{2}\, n]$$

$$b(n) = a(n) + 2n = [(2 + \sqrt{2})n]$$

$$d(n) = b(n) - n = [(1 + \sqrt{2})n]$$

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] .$$

Thus (a, b) and (d', d) are separated Newman-Skolem pairs. Making use of the preceding theorems we get

Theorem 3.5. The functions a, b, d, d' as defined above, satisfy the following relations:

$$ad(n) = b(n) - 1$$

$$a\big(d(n) + 1\big) = b(n) + 1$$

$$a\big(d(n) - 1\big) = b(n) - 2$$

$$d'\big(a(n)\big) = d(n) - 1$$

$$d'\big(a(n) + 1\big) = d(n) + 1$$

$$a(n + 1) = a(n) + 2 \rightleftarrows n \in (d)$$

$$d'(n + 1) = d'(n) + 2 \rightleftarrows n \in (a) .$$

Here we have let (f) denote the range of the function f.

Theorem 3.6. For all positive integers n, we have

$$ab(n) = a(n) + b(n) . \tag{3.13}$$

Proof. Since

$$a(n) < \sqrt{2}\, n < a(n) + 1 ,$$

we see that

$$2a(n) + \sqrt{2}\, a(n) < \sqrt{2}\big(2n + a(n)\big) \le 2\big(a(n) + 1\big) + \sqrt{2}\big(a(n) + 1\big).$$

Hence, taking greatest integers,

$$b\big(a(n)\big) \le ab(n) \le b\big(a(n) + 1\big) .$$

Equality is obviously impossible. Hence, by Theorem 3.4, we get (3.13).

Suppose (d', d) is any separated Newman-Skolem pair and suppose f is any increasing function. Let $d'f = b$ and let a be such that (a, b) is a Newman-Skolem pair. Then since $d'(\underset{\sim}{N})$ $b(\underset{\sim}{N})$, it follows that $d(\underset{\sim}{N})$ $a(\underset{\sim}{N})$. Hence there exists an increasing function c such that

$$d(n) = ac(n) . \tag{3.14}$$

Now, since (d', d) is separated, we have

$$d'(d(n) - n) = d(n) - 1 .$$

Hence, among the numbers

$$1, 2, 3, \cdots, d(n) ,$$

there are exactly j members of $b(\underset{\sim}{N})$, namely

$$d'f(1), \quad d'f(2), \quad \cdots, \quad d'f(j) ,$$

where j is the largest integer such that

$$f(j) \leq d(n) - n .$$

We may write (symbolically)

$$j = \left[\frac{d(n) - n}{f}\right] . \tag{3.15}$$

The remaining $d(n) - j$ members in

$$\{1, 2, 3, \cdots, d(n)\}$$

are members of $a(N)$, so that

$$d(n) = a(d(n) - j),$$

that is

$$c(n) = d(n) - \left[\frac{d(n) - n}{f}\right] . \tag{3.16}$$

Theorem 3.7. For the functions a, b, c, d' previously defined, we have

$$d(n) = a(b(n) - d'(n)) . \tag{3.17}$$

Proof. Since $d'(2n) = b(n)$, the above remarks apply with $f(n) = 2n$. Hence

$$c(n) = d(n) - \left[\frac{d(n) - n}{2}\right] = b(n) - n - \left[\frac{a(n)}{2}\right]$$

But

$$n + \left[\frac{a(n)}{2}\right] = n + \left[\tfrac{1}{2}[2n]\right] = n + \left[\frac{\sqrt{2}\, n}{2}\right]$$

$$= [\tfrac{1}{2}(2 + \sqrt{2})n] = d'(n) ,$$

so that

$$c(n) = b(n) - d'(n) .$$

This evidently completes the proof of the theorem.

4. RELATIONS BETWEEN a, b, d, d' AND e AND p

Theorem 4.1. The functions a, b, c and p are related by the following formulas:

$$a(n + 1) = e(n) + n + 1 \tag{4.1}$$

$$b(n + 1) = p(n) + n + 3 . \tag{4.2}$$

These formulas imply

$$e(n) = [(\sqrt{2} - 1)(n + 1)], \qquad e(0) = 0 \tag{4.3}$$

$$p(n) = [\sqrt{2}(n + 1)] + n - 1, \qquad p(0) = 0 . \tag{4.4}$$

Proof. It is clear by induction that (a, b) is the unique Newman-Skolem pair satisfying

$$b(n) = a(n) + 2n \qquad (n = 1, 2, 3, \cdots) . \tag{4.5}$$

Now let

$$a'(n + 1) = e(n) + n + 1$$

and

$$b'(n + 1) = p(n) + n + 3 .$$

We shall show that (a', b') is a Newman-Skolem pair satisfying

$$b'(n) = a'(n) + 2n . \tag{4.6}$$

This will evidently prove the theorem.

By (2.11) we have

$$p(n) = e(n) + 2n.$$

Hence

$$b'(n + 1) - a'(n + 1) = p(n) - e(n) + 2 = 2n + 2 ,$$

so that (4.6) is satisfied.

Since, by Theorem (2.4),

$$e(p(n)) = e(p(n) + 1) = n, \qquad e(p(n) + 2) = n + 1 ,$$

we get

$$a'(p(n) + 2) = p(n) + n + 2 = b'(n + 1) - 1$$

and

$$a'(p(n) + 3) = p(n) + n + 4 = b'(n + 1) + 1.$$

Hence the ranges of a' and b' are disjoint. Furthermore we see that

$$a'(1), \quad a'(2), \quad \cdots, \quad a'(p(n)) + 2 \; ; \quad b'(1), \quad b'(2), \quad \cdots, \quad b'(n+1)$$

are $p(n) + n + 3$ distinct numbers less than or equal to

$$b'(n+1) = p(n) + n + 3 .$$

Hence all numbers in this range must be included and the theorem is proved.

Theorem 4.2. We have, for all n,

$$e(b(n)) = a(n) , \tag{4.7}$$

$$e(d(n)) = n . \tag{4.8}$$

Proof. By Theorems 3.5 and 4.1 we have

$$b(n) + 1 = a(d(n) + 1) = d(n) + 1 + e(d(n)) .$$

Hence, since $b(n) - d(n) = n$, we get (4.8).

Since $d(n) = [(1 + \sqrt{2})n]$, it follows that

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] .$$

Hence

$$d'(2n) = b(n) .$$

In particular

$$b(n) \notin d(\underset{\sim}{N}) ,$$

so that, by Theorem 3.2,

$$a(b(n) + 1) = ab(n) + 1 .$$

Then

$$b(n) + 1 + e(b(n)) = a(n) + b(n) + 1$$

and therefore

$$e(b(n)) = a(n) .$$

This completes the proof of the theorem.

Further relations between a, b, d, d', e and p will be established in the next section.

5. THE SETS A_k AND B_k

We define the sets A_k and B_k as follows:

$$A_k = \{N \mid \epsilon_1 = \cdots = \epsilon_{k-1} = 0, \quad \epsilon_k \neq 0\} , \tag{5.1}$$

(5.2) $$B_k = \{N \mid \epsilon_1 = \cdots = \epsilon_{k-1} = 0, \quad \epsilon_k = 2\},$$
where
(5.3) $$N = \cdot\, \epsilon_1 \epsilon_2 \epsilon_3 \cdots$$
is the canonical representation of N.

We also define
(5.4) $$\delta(n) = b(n) + d(n) = 2a(n) + 3n$$

and define $\epsilon(n)$ by the requirement

(5.5) (ϵ, δ) is a Newman-Skolem pair.

Theorem 5.1. Let the non-negative integer n have the canonical representation

(5.6) $$n = \cdot\, \epsilon_1 \epsilon_2 \epsilon_3 \cdots .$$
Then
(5.7) $$d(n + 1) - 1 = p(n) + 1 = \cdot 1 \epsilon_1 \epsilon_2 \epsilon_3 \cdots .$$
Hence
(5.8) $$A_1 = d(\underset{\sim}{N}) - 1 .$$

Proof. The theorem follows from the relations

$$b(n + 1) = d(n + 1) + n + 1 = p(n) + n + 3 .$$

Since it is clear that (ϵ, δ) is a separated Newman-Skolem pair, it follows from Theorem 3.1 that
(5.9) $$\epsilon\bigl(2d(n)\bigr) = \delta(n) - 1$$
(5.10) $$\epsilon\bigl(2d(n) + 1\bigr) = \delta(n) + 1 .$$

Since $\delta(n) - n = 2d(n)$, it follows from Theorem 3.3 that

(5.11) $$\epsilon\bigl(2d(n) - 1\bigr) = \delta(n) - 2 .$$
Moreover we have
$$d^2(n) = d(n) + ad(n) = d(n) \neq b(n) - 1 = \delta(n) - 1 ,$$
so that
(5.12) $$e\bigl(\delta(n) - 1\bigr) = d(n) .$$
Also we have
$$3 + d + pd = b(d + 1) = 2(d + 1) + a(d + 1) = 2d + 2 + b + 1 ,$$
so that
(5.13) $$pd(n) = d(n) + b(n) = \delta(n) .$$
Applying e, we get
(5.14) $$e\delta(n) = d(n) .$$

Now using (4.1) and (4.2) we get

$$\begin{aligned} p\delta &= d(\delta + 1) - 2 = (\delta + 1) + a(\delta + 1) - 2 \\ &= (\delta + 1) + (\delta + 1 + e\ ) - 2 = d + 2\delta \\ &= \delta + \delta + e(\delta - 1) = \delta + a\delta = d\delta \quad , \end{aligned}$$

so that

$$p\delta = d\delta\,. \tag{5.15}$$

Theorem 5.2. We have

$$B_2 = d^2(\underset{\sim}{N}) \tag{5.16}$$

$$B_{2k+1} = \delta^k d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) \tag{5.17}$$

$$B_{2k} = d\delta^{k-1} d(\underset{\sim}{N}) \qquad (k = 2, 3, 4, \cdots)\,. \tag{5.18}$$

Proof. It is only necessary to prove (5.16) since (5.17) will then follow by (5.13) and (5.15).

Applying Theorem 5.1 to $d(n + 1) - 1$ we obtain

$$d^2(n + 1) - 1 = \cdot 11\epsilon_1\epsilon_2\epsilon_3 \cdots ,$$

so that

$$d^2(n + 1) = \cdot 02\, \epsilon_1\epsilon_2\epsilon_3 \cdots \quad .$$

This evidently proves (5.16) and therefore the proof of Theorem 5.2 is complete.

Note that if n has the canonical representation

$$n = \cdot \epsilon_1\epsilon_2\epsilon_3 \cdots ,$$

then

$$d(n + 1) - 1 = \cdot 1\epsilon_1\epsilon_2\epsilon_3 \cdots \tag{5.19}$$

is also canonical. Since $\delta(n) = 2d(n) + n$, it follows that

$$\delta(n + 1) - 1 = \cdot 02\, \epsilon_1\epsilon_2\epsilon_3 \cdots \tag{5.20}$$

and

$$d\big(\delta(n + 1)\big) - 1 = \cdot 102\, \epsilon_1\epsilon_2 \cdots \tag{5.21}$$

are both canonical.

Theorem 5.3. We have

$$A_1 = d(\underset{\sim}{N}) - 1 \tag{5.22}$$

$$A_{2k} = d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) \tag{5.23}$$

$$A_{2k+1} = \delta^k\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)\,. \tag{5.24}$$

Proof. We have already proved (5.22). It will therefore suffice to establish

$$A_2 = d\epsilon(\underset{\sim}{N}) . \tag{5.25}$$

Now A_2 consists of all N in the canonical form

$$N = \cdot 0\ \epsilon_2\epsilon_3\epsilon_4 \cdots \qquad (\epsilon_2 \neq 0) .$$

Hence $A_2 - 1$ consists of all N in the canonical form

$$N = \cdot 1\ (\epsilon_2 - 1)\epsilon_3\epsilon_4 \cdots \qquad (\epsilon_3 \neq 2) .$$

Furthermore $d(\underset{\sim}{N}) - 1$ consists of all N in the canonical form

$$N = \cdot 1\ f_2f_3f_4 \cdots$$

and by (5.21), $d\delta(N) - 1$ consists of all N in the canonical form

$$N = \cdot 102\ g_4g_5 \cdots .$$

Therefore since $d(\underset{\sim}{N}) - 1$ is the disjoint union of $d\delta(N) - 1$ and $d\epsilon(N) - 1$, we see that

$$d\epsilon(\underset{\sim}{N}) - 1 = A_2 - 1 ,$$

that is,

$$A_2 = d\epsilon(\underset{\sim}{N}) .$$

This completes the proof of the Theorem.

Theorem 5.4. We have

$$d(\underset{\sim}{N}) = \bigcup_1^\infty A_{2k} \tag{5.26}$$

$$(\underset{\sim}{N}) = \bigcup_1^\infty A_{2k+1} \tag{5.27}$$

$$(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \bigl(d(\underset{\sim}{N}) - 1\bigr) . \tag{5.28}$$

Proof. Since every integer is of the form $\delta^k\epsilon(n)$ for some $k \geq 0$, (5,26) and (5.27) follow from the previous theorem. Since $\epsilon(\underset{\sim}{N})$ is the complement of $\delta(\underset{\sim}{N})$, (5.28) follows from (5.22) and (5.26).

We have seen above that

$$\epsilon(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \bigl(d(\underset{\sim}{N}) - 1\bigr) . \tag{5.29}$$

Hence the numbers in $\epsilon(\underset{\sim}{N})$ are, in order,

$$d(1) - 1, \quad d(1), \quad d(2) - 1, \quad d(2), \quad d(3) - 1, \quad d(3), \quad \cdots .$$

It follows that

(5.30) $$\epsilon(2n) = d(n), \qquad \epsilon(2n - 1) = d(n) - 1 .$$

Applying e, we have

(5.31) $$e(\epsilon(n)) = [n/2] .$$

The following remark concerning the second canonical form is useful. If

$$n = \cdot f_1 f_2 f_3 \cdots \qquad \text{(second canonical)}$$

then

$$d(n) = \cdot 0\, f_1 f_2 f_3 \cdots \qquad \text{(first canonical)}$$

and

$$\delta(n) = \cdot 00\, f_1 f_2 f_3 \cdots \qquad \text{(first and second canonical)} .$$

6. ADDITIONAL RELATIONS INVOLVING a AND b

Theorem 6.1. We have

(6.1) $$a^2 b = 2b - 1 .$$

For the proof we require

Theorem 6.2. The integer n is in (d) if and only if

(6.2) $$\left\{ \frac{n}{1 + \sqrt{2}} \right\} > 2 - \sqrt{2} ,$$

where (α) denotes the fractional part of the real number α.

Proof. Let

$$n = d(k) = [(1 + \sqrt{2})k] ,$$

so that

$$(1 + \sqrt{2})k - 1 < n < (1 + \sqrt{2})k, \qquad k - \frac{1}{1 + \sqrt{2}} < \frac{n}{1 + \sqrt{2}} < k .$$

This is equivalent to

$$\left\{ \frac{n}{1 + \sqrt{2}} \right\} > 1 - \frac{1}{1 + \sqrt{2}} = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2} .$$

Proof of Theorem 6.1. It follows from

$$a(n) = [\sqrt{2}\, n]$$

that

(6.3) $$n - 2 \le a^2(n) \le n - 1 .$$

It therefore suffices to show that

(6.4) $$a^2b(n) \equiv 1 \pmod 2 \qquad (n = 1, 2, 3, \cdots).$$

Assume that there exists an integer k such that

$$a^2b(k) \equiv 0 \pmod 2,$$

that is

$$a\bigl(2d(k)\bigr) \equiv 0 \pmod 2.$$

Then

$$[2\sqrt{2}\, d(k)] = 2j$$

for some integer j. Hence

$$2j < 2\sqrt{2}\, d(k) < 2j + 1,$$

$$j < \sqrt{2}\, d(k) < j + \tfrac{1}{2},$$

so that

(6.5) $$\{\sqrt{2}\, d(k)\} < \tfrac{1}{2}.$$

By Theorem 6.2,

$$\left\{\frac{d(k)}{1 + \sqrt{2}}\right\} > 2 - \sqrt{2},$$

that is

$$\{(\sqrt{2} - 1)d(k)\} > 2 - \sqrt{2}.$$

Hence

$$\{\sqrt{2}\, d(k)\} > 2 - \sqrt{2}.$$

This contradicts (6.5) and so completes the proof of the theorem.

It follows from $ab = a + b$ that

$$b^2 = ab + 2b = a + 3b,$$

$$\begin{aligned} b^3 &= ab + 3b^2 \\ &= a + b + 3(a + 3b) \\ &= 4a + 10b, \end{aligned}$$

$$\begin{aligned} b^4 &= 4(a + b) + 10(a + 3b) \\ &= 14a + 34b. \end{aligned}$$

Put

(6.6) $$b^k = u_k a + v_k b, \quad u_1 = 0, \quad v_1 = 1, \quad u_2 = 1, \quad v_2 = 3.$$

Then

$$\begin{aligned} b^{k+1} &= u_k(a + b) + v_k(a + 3b) \\ &= (u_k + v_k)a + (u_k + 3v_k)b, \end{aligned}$$

so that

(6.7)
$$\begin{cases} u_{k+1} = u_k + v_k \\ v_{k+1} = u_k + 3v_k \end{cases} .$$

It follows that

$$\begin{cases} u_{k+2} - 4u_{k+1} + 2u_k = 0 \\ v_{k+2} - 4v_{k+1} + 2v_k = 0 \end{cases} .$$

Then

$$U(x) = \sum_1^\infty u_k x^k = x^2 + \sum_3^\infty (4u_{k-1} - 2u_{k-2}) x^k$$

$$= x^2 + (4x - 2x^2)U(x) ,$$

so that

$$U(x) = \frac{x^2}{1 - 4x + x^2} .$$

We find that

(6.8)
$$u_k = \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}, \qquad v_k = u_{k+1} - u_k ,$$

where

$$\alpha = 2 + \sqrt{2}, \qquad \beta = 2 - \sqrt{2} .$$

Theorem 6.3. The function b^k is evaluated by means of (6.6) and (6.8). In the next place,

$$ab = a + b ,$$

$$(ab)^2 = a^2b + bab$$
$$= 2a^2b + 2ab$$
$$= 2(2b - 1) + 2(a + b)$$
$$= 2a + 6b - 2 ,$$

$$(ab)^3 = 2a^2b + 6bab - 2$$
$$= 8a^2b + 12ab - 2$$
$$= 8(2b - 1) + 12(a + b) - 2$$
$$= 12a + 28b - 10 ,$$

$$(ab)^4 = 56a + 136b - 50 .$$

Put

(6.9)
$$(ab)^k = u_k a + v_k b - t_k ,$$

$$u_1 = v_1 = 1, \quad t_1 = 0, \quad u_2 = 2, \quad v_2 = 6, \quad t_2 = 2 .$$

Then

$$(ab)^{k+1} = u_k a^2b + v_k bab - t_k$$
$$= (u_k + v_k) a^2b + 2v_k ab - t_k$$

$$= (u_k + v_k)(2b - 1) + 2v_k(a + b) - t_k$$
$$= 2v_k a + (2u_k + 4v_k)b - (u_k + v_k + t_k),$$

so that

$$u_{k+1} = 2v_k$$
$$v_{k+1} = 2u_k + 4v_k = 4v_k + 4v_{k-1}$$
$$t_{k+1} = u_k + v_k + t_k \quad .$$

Let

$$Q_0 = Q_1 = 1, \quad Q_2 = 3, \quad Q_3 = 7, \quad Q_{k+1} = 2Q_k + Q_{k-1}.$$

It is easily verified that

(6.10) $$Q_k = P_{k-1} + P_k$$

k	0	1	2	3	4	5	6
P_k	0	1	2	5	12	29	70
Q_k	1	1	3	7	17	41	99

We find that

(6.11) $$u_k = 2^{k-1}Q_{k-1}, \qquad v_k = 2^{k-1}Q_k$$

(6.12) $$t_k = \frac{1}{7}(2^{k+1}P_{k+1} - 3\cdot 2^k P_k - 2).$$

Theorem 6.4. The function $(ab)^k$ is evaluated by means of (6.9), (6.10), (6.11) and (6.12).

7. THE FUNCTIONS f, f', g, g', c, c'

It follows from

$$a(n) = [\sqrt{2}\,n], \qquad b(n) = [(2 + \sqrt{2})n]$$

that

(7.1) $$ab(n) - ba(n) = 1 \text{ or } 2 \qquad (n = 1, 2, 3, \cdots).$$

We may accordingly define the pair of complementary functions f, f' by means of

(7.2) $$ab(n) - ba(n) = \begin{cases} 1 & (n \in (f)) \\ 2 & (n \in (f')) \end{cases}.$$

An equivalent definition is

(7.3) $$\begin{cases} a^2f(n) \equiv 1 \pmod 2 \\ a^2f'(n) \equiv 0 \pmod 2 \end{cases}.$$

It is also easily verified that

(7.4) $$ad'(n) - d'a(n) = 0 \text{ or } 1 \qquad (n = 1, 2, 3, \cdots).$$

Hence we may define the pair g, g' by means of

$$(7.5)\qquad ad'(n) - d'a(n) = \begin{cases} 0 & (n \in (g)) \\ 1 & (n \in (g')) \end{cases} .$$

It is somewhat more convenient to take as definition

$$(7.6)\qquad \begin{cases} ag(n) \equiv 1 \pmod 2 \\ ag'(n) \equiv 0 \pmod 2 \end{cases} .$$

We shall show that (7.5) and (7.6) are equivalent.

For brevity put

$$(7.7)\qquad s = ab - ba, \qquad t = ad' - d'a .$$

It is easily verified that

$$(7.8)\qquad s(n) = 2n - a^2(n)$$

from which the equivalence of (7.2) and (7.3) is immediate.

It is also immediate from (7.3) and (7.6) that

$$(7.9)\qquad g = af .$$

In the next place

$$t = ad' - d'a = ad' - a - n + 1 ,$$

$$\begin{aligned} ta &= ad'a - a^2 - a + 1 \\ &= a(d - 1) - a^2 - a + 1 \\ &= b - a^2 - a - 1 , \end{aligned}$$

$$ta(n) = 2n - a^2 - 1 ,$$

$$(7.10)\qquad \begin{cases} taf \equiv a^2f + 1 \equiv 0 \pmod 2 , \\ taf' \equiv a^2f' + 1 \equiv 1 \pmod 2 . \end{cases}$$

Also

$$\begin{aligned} tb &= ad'b - db + 1 \\ &= a\delta - ab - b + 1 \\ &= d + \delta - a - 2b + 1 \\ &= b + 2d - a - 2b + 1 \\ (7.11)\qquad &\equiv 1 \pmod 2 . \end{aligned}$$

It follows from (7.10) and (7.11) that

$$(7.12)\qquad t(n) \equiv 0 \pmod 2 \rightleftarrows n \in (g) .$$

This evidently establishes the equivalence of (7.5) and (7.6).

Note that the pair g, g' is not separated.

Theorem 7.1. We have

$$df = f' . \tag{7.13}$$

The proof of this theorem requires a number of preliminary results.

Theorem 7.2

$$bf - 1 = dg . \tag{7.14}$$

Proof.

$$bf - dg - 1 = af + 2f - ag - g - 1$$

$$= 2f - a^2f - 1 = 0 .$$

Theorem 7.3

$$n \in (f) \rightleftarrows \{\sqrt{2}\,n\} < \frac{1}{\sqrt{2}} . \tag{7.15}$$

Proof. By (7.2) or (7.3)

$$n \in (f) \rightleftarrows a^2n = 2n - 1 .$$

Consider

$$\left[\sqrt{2}\,[\sqrt{2}\,n]\right] = 2n - 1, \qquad 2n - 1 < \sqrt{2}\,[\sqrt{2}\,n] < 2n .$$

Put $k = [\sqrt{2}\,n]$, so that

$$\sqrt{2}\,n - 1 < \sqrt{2}\,k < 2n$$

$$\sqrt{2}\,n - \frac{1}{\sqrt{2}} < k < \sqrt{2}\,n$$

$$0 < \sqrt{2}\,n - k < \frac{1}{\sqrt{2}} ,$$

that is

$$\{\sqrt{2}\,n\} < \frac{1}{\sqrt{2}} . \tag{7.16}$$

Hence if $n \in (f)$, Eq. (7.6) is satisfied.

Next let $n \in (f')$, so that $a^2(n) = 2n - 2$. Consider

$$\left[\sqrt{2}\,[\sqrt{2}\,n]\right] = 2n - 2$$

$$2n - 2 < \sqrt{2}\,[\sqrt{2}\,n] < 2n - 1$$

$$2n - 2 < \sqrt{2}\,k < 2n - 1 \qquad (k = [\sqrt{2}\,n])$$

$$\sqrt{2}\,n - \sqrt{2} < k < \sqrt{2}\,n - \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} < \sqrt{2}\,n - k < \sqrt{2} ,$$

that is

$$\{\sqrt{2}\,n\} > \frac{1}{\sqrt{2}} \quad . \tag{7.17}$$

Hence if $n \in (f')$, Eq. (7.17) is satisfied.

Combining (7.16) and (7.17), we get (7.15).

Proof of Theorem 7.1. By Theorem 6.2, $n \in (d)$ if and only if

$$\left\{\frac{n}{1+\sqrt{2}}\right\} > 2 - \sqrt{2} \quad . \tag{7.18}$$

Put

$$(1 + \sqrt{2})f = df + \epsilon;$$

by Theorem 7.3, we have $\epsilon < 1/\sqrt{2}$. Moreover

$$\begin{aligned} f &= \frac{df}{1+\sqrt{2}} + \frac{\epsilon}{1+\sqrt{2}} \\ &= J + \left\{\frac{df}{1+\sqrt{2}}\right\} + \frac{\epsilon}{1+\sqrt{2}} \quad , \end{aligned}$$

where

$$J = \left[\frac{df}{1+\sqrt{2}}\right] .$$

Then

$$\left\{\frac{df}{1+\sqrt{2}}\right\} + \frac{\epsilon}{1+\sqrt{2}} = 1 \quad ,$$

$$\{\sqrt{2}\,df\} + \epsilon(\sqrt{2} - 1) = 1 \quad ,$$

$$\{\sqrt{2}\,df\} > 1 - \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \quad ,$$

so that

$$(df) \subset (f') \quad . \tag{7.19}$$

We shall now show that

$$(f') \subset (df) \quad . \tag{7.20}$$

Let n satisfy $\{\sqrt{2}\,n\} > 1/\sqrt{2}$, so that $n \in (f')$. Then, by (7.18), $n \in (d)$, that is

$$n = d(k) = [(1 + \sqrt{2})k] \quad ,$$

for some integer k. Thus

$$(1 + \sqrt{2})k = n + \{\sqrt{2}\,k\}$$

$$(1 + \sqrt{2})k + (\sqrt{2} - 1)n = \sqrt{2}\,n + \{\sqrt{2}\,k\}$$

$$(1 + \sqrt{2})k + (\sqrt{2} - 1)d(k) = ad(k) + \{\sqrt{2}\,n\} + \{\sqrt{2}\,k\} > b(k) - 1 + \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\sqrt{2}\,k - (2 - \sqrt{2})d(k) + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\sqrt{2}\,k - (2 - \sqrt{2})\big((1 + \sqrt{2})k - \{\sqrt{2}\,k\}\big) + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$(2 - \sqrt{2})\{\sqrt{2}\,k\} + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\frac{\sqrt{2} - 1}{\sqrt{2}} > (\sqrt{2} - 1)\{\sqrt{2}\,k\}$$

$$\frac{1}{\sqrt{2}} > \{\sqrt{2}\,k\} .$$

Therefore $k \in (f)$, $n \in (df)$.

This proves (7.20) and so completes the proof of the theorem.

Theorem 7.4. We have

(7.21) $$bf - af' = 1 .$$

Proof. By (7.14), Eq. (7.21) may be replaced by

(7.22) $$af' = dg = daf ,$$

which by Theorem 7.1 is the same as

(7.23) $$adf = daf .$$

Now

$$ad - da = b - 1 - a^2 - a$$
$$= 2n - 1 - a^2 ,$$
$$adf - daf = 2f - 1 - a^2f = 0 .$$

This proves (7.23) and therefore proves (7.21).

Theorem 7.5. The pair (f, f') is separated.

Proof. By (7.13)

$$f'(n) = df(n) > f(n) ,$$

so that the pair (f, f') is ordered. Since the pair (d', d) is separated, it follows that

$$f'(n + 1) - f'(n) = df(n + 1) - df(n) > 1.$$

Define

(7.24) $$c(n) = b(n) - d'(n) ,$$

so that by (3.17)

$$d = ac. \tag{7.25}$$

Theorem 7.6. We have

$$f' = acf = caf. \tag{7.26}$$

Proof. It suffices to show that

$$acf - caf = 0. \tag{7.27}$$

Now

$$\begin{aligned} ac - ca &= d - ba + d'a \\ &= d - a^2 - 2a + d - 1, \\ acf - caf &= 2df - 2af - a^2f - 1 \\ &= 2f - a^2f - 1 = 0. \end{aligned}$$

Theorem 7.7

$$\begin{cases} n \in (g) \rightleftarrows \left\{\dfrac{n}{\sqrt{2}}\right\} < \dfrac{1}{2} \\ n \in (g') \rightleftarrows \left\{\dfrac{n}{\sqrt{2}}\right\} < \dfrac{1}{2} \end{cases} \tag{7.28}$$

Proof. Let $n \in (g)$, so that $a(n) \equiv 1 \pmod 2$. Then

$$[\sqrt{2}\, n] = 2k - 1$$

$$2k - 1 < \sqrt{2}\, n < 2k$$

$$k - \frac{1}{2} < \frac{n}{\sqrt{2}} < k,$$

so that

$$\left\{\frac{n}{\sqrt{2}}\right\} > \frac{1}{2}.$$

Next let $n \in (g')$ so that $a(n) \equiv 0 \pmod 2$. Then

$$[\sqrt{2}\, n] = 2k$$

$$2k < \sqrt{2}\, n < 2k + 1$$

$$k < \frac{n}{\sqrt{2}} < k + \frac{1}{2},$$

so that

$$\left\{\frac{n}{\sqrt{2}}\right\} < \frac{1}{2}.$$

This completes the proof of the theorem.

Theorem 7.8

$$g' = a(\tfrac{1}{2} ag') + 1. \tag{7.29}$$

Proof. This is equivalent to

$$dg' - 1 = b(\tfrac{1}{2}ag')$$

which in turn is equivalent to

$$d'ag' = b(\tfrac{1}{2}ag') \,. \tag{7.30}$$

Since $d'(2n) = b(n)$, Eq. (7.30) follows at once.

Theorem 7.9

$$\begin{cases} d'(2n) = 2d'(n) + 1 \\ d'(2n) = 2d'(n) \end{cases} \quad \begin{Bmatrix} n \in (g) \\ n \in (g') \end{Bmatrix} \tag{7.31}$$

Theorem 7.10

$$ad'(n) = 2d'(n) - n \,. \tag{7.32}$$

We show first that Theorems 7.9 and 7.10 are equivalent. Since $d'(2n) = b(n)$, (7.31) may be replaced by

$$\begin{cases} bg = 2d'g + 1 \\ bg' = 2d'g' \end{cases} \tag{7.33}$$

while (7.3) may be replaced by

$$\begin{cases} ad'g = 2d'g - g \\ ad'g' = ad'g' - g' \end{cases} . \tag{7.34}$$

Since, by (7.5),

$$ad'g = d'ag, \qquad ad'g' - d'ag' = 1 \,,$$

(7.34) is the same as

$$\begin{cases} d'ag = 2d'g - g \\ d'ag' = 2d'g' - g' - 1 \end{cases} . \tag{7.35}$$

But $d'a = d - 1$, so that (7.35) becomes

$$\begin{cases} dg - 1 = 2d'g - g \\ dg' = 2d'g' - g' \end{cases} \tag{7.36}$$

which is the same as (7.33). This proves the equivalence of (7.31) and (7.32).

We shall now prove (7.32). We have first

$$ad'a = a(d - 1) = b - 2$$

$$2d'a - a = 2(d - 1) - a = b - 2 \,,$$

so that

$$ad'a = 2d'a - a \,. \tag{7.37}$$

Secondly

$$ad'b = a\delta = b + 2d$$

$$2d'b - b = 2\delta - b = b + 2d \,,$$

so that

$$ad'b = 2d'b - b \,. \tag{7.38}$$

Clearly (7.37) and (7.38) imply (7.32).

Theorem 7.11. We have

$$c'(n) + n - 1 = d'(2n - 1) , \tag{7.39}$$

where $c'(n)$ and $c(n)$ are complementary.

Proof. Put

$$\begin{aligned}\bar{c}(n) &= d'(2n-1) - (n-1) \\ &= [\tfrac{1}{2}(2+\sqrt{2})(2n-1)] - (n-1) \\ &= \left[n + \frac{1}{\sqrt{2}}(2n-1)\right] = \left[(1+\sqrt{2})n - \frac{1}{\sqrt{2}}\right] .\end{aligned}$$

It follows from (7.15) that

$$\bar{c}(n) = \begin{cases} d(n) & (n \in (f')) \\ d(n) - 1 & (n \in (f)) \end{cases} . \tag{7.40}$$

In order to prove that $\bar{c}(n) = c'(n)$, it will suffice to show that c and $\bar{c}$ are complementary. Now, by (7.31),

$$c(n) = \begin{cases} d'(n) + 1 & (n \in (g)) \\ d'(n) & (n \in (g')) \end{cases} .$$

Thus

$$(c) = (d'g + 1) \cup (d'g')$$

$$(c) = (df') \cup (df - 1) \quad .$$

Since

$$d'g + 1 = d'af + 1 = df$$

$$df - 1 = d'af = d'g ,$$

it follows that

$$(c) = (df) \cup (d'g')$$

$$(c) = (dg') \cup (d'g) \quad .$$

Therefore

$$\begin{aligned}(c) \cup (\bar{c}) &= (df) \cup (df') \cup (d'g) \cup (d'g') \\ &= (d) \cup (d') = \underset{\sim}{N}\end{aligned}$$

while $(c) \cap (\bar{c})$ is vacuous. This completes the proof of the Theorem.

Theorem 7.12. We have

$$ac'(n) = c'(n) + n - 1 . \tag{7.41}$$

In view of (7.39), (7.41) is the same as

$$ac'(n) = d'(2n - 1) . \tag{7.42}$$

Proof of (7.41). By (7.40),

$$c'(n) = \begin{cases} d(n) & (n \in (f')) \\ d(n) - 1 & (n \in (f)) \end{cases},$$

so that

$$\begin{cases} c'f' = df' \\ c'f = df - 1 \end{cases}.$$

Thus

$$\begin{cases} ac'f' = adf' = bf' - 1 \\ ac'f = a(df - 1) = bf - 2 \end{cases}.$$

It follows that

$$\begin{cases} ac'f' - c'f' = bf' - 1 - df' = f' - 1 \\ ac'f - c'f = bf - 2 - (df - 1) = f - 1 \end{cases}$$

and therefore

$$ac'(n) - c'(n) = n - 1 .$$

Theorem 7.13. We have

(7.43) $$a^2c'(n) = 2c'(n) - 1 .$$

Proof. By (7.32),

$$ad'(2n - 1) = 2d'(2n - 1) - (2n - 1) .$$

Then by (7.42),

$$a^2c'(n) = ad'(2n - 1) = 2ac'(n) - (2n - 1) .$$

Combining this with (7.41), we get

$$\begin{aligned} a^2c'(n) &= 2(c'(n) + n - 1) - (2n - 1) \\ &= 2c'(n) - 1 \end{aligned} .$$

Theorem 7.14. There exists a strictly monotone function θ such that

(7.44) $$c' = f\theta .$$

Proof. This result is implied by

(7.45) $$f' = cg .$$

To prove (7.45) we take

$$f' = df = acf .$$

Since

$$ac - ca = ab - ba - 1 = s - 1 ,$$

it follows that

$$acf - caf = 0 .$$

Hence

$$f' = caf = cg.$$

Theorem 7.15. There exists a strictly monotone function ψ such that

(7.46) $$f\psi = d' .$$

Proof. This is an immediate consequence of $f' = df$.

Theorem 7.16. There exists a strictly monotone function h such that

$$fh = b \; . \tag{7.47}$$

Proof. Since $f' = df = acf$, it follows that $(f') \subset (a)$ and therefore $(b) \subset (f)$.

Theorem 7.17. We have

$$\psi(2n) = h(n) \; . \tag{7.48}$$

Proof. By (7.46),

$$f\psi(2n) = d'(2n) = b(n)$$

and (7.48) follows at once.

Theorem 7.18. We have

$$c = \epsilon a + 1 \; . \tag{7.49}$$

Proof. We recall that

$$\epsilon(2n) = \epsilon(2n - 1) + 1 = d(n) \; .$$

Also

$$\begin{cases} a(n) \equiv 1 \pmod 2 \rightleftarrows n \in (g) \\ a(n) \equiv 0 \pmod 2 \rightleftarrows n \in (g') \end{cases}$$

1. Let $n = g(k)$. Then

$$\begin{aligned} \epsilon a(n) + 1 &= d\left(\tfrac{1}{2}(a(n) + 1)\right) = d\left(\tfrac{1}{2}(ag(k) + 1)\right) \\ &= d\left(\tfrac{1}{2}(a^2f(k) + 1)\right) = df(k) \qquad , \end{aligned}$$

so that

$$\epsilon ag + 1 = df \; . \tag{7.50}$$

2. Let $n \in (g')$ and put

$$a(n) = [\sqrt{2}\, n] = 2k, \qquad k = \left[\frac{n}{\sqrt{2}}\right] \; .$$

By (7.28)

$$\left\{\frac{n}{\sqrt{2}}\right\} < \frac{1}{2} \; .$$

We have

$$\begin{aligned} \epsilon a(n) + 1 &= d\left(\tfrac{1}{2}a(n)\right) + 1 = d(k) + 1 \\ &= k + [\sqrt{2}\, k] + 1 \\ &= \left[\frac{n}{\sqrt{2}}\right] + \sqrt{2}\left(\frac{n}{\sqrt{2}} - \left\{\frac{n}{\sqrt{2}}\right\}\right) - \left\{n - \sqrt{2}\left\{\frac{n}{\sqrt{2}}\right\}\right\} + 1 \\ &= n + \left[\frac{n}{\sqrt{2}}\right] = \sqrt{2}\left\{\frac{n}{\sqrt{2}}\right\} - \left(1 - \sqrt{2}\left\{\frac{n}{2}\right\}\right) + 1 \\ &= n + \left[\frac{n}{\sqrt{2}}\right] \qquad . \end{aligned}$$

On the other hand

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] = n + \left[\frac{n}{\sqrt{2}}\right] ,$$

so that

(7.51) $$\epsilon a g' + 1 = d'g' .$$

Combining (7.50) and (7.51) we get

$$(\epsilon a + 1) = (df) \cup (d'g') = (c) ;$$

the last equality appeared in the proof of Theorem 7.11.

Theorem 7.19. We have

(7.52) $$e\bigl(c'(n) + 1\bigr) = n .$$

Proof. By (7.40)

$$\begin{cases} c'f(n) = df(n) - 1 \\ c'f'(n) = df'(n) \end{cases} ,$$

so that

$$\begin{cases} c'f(n) + 1 = df(n) \\ c'f'(n) + 1 = df'(n) + 1 \end{cases} .$$

Since

$$df' + 1 = d^2f + 1 = \delta f ,$$

it follows that

$$\begin{cases} c'f(n) + 1 = df(n) \\ c'f'(n) + 1 = \delta f(n) \end{cases} .$$

Therefore

$$\begin{cases} e\bigl(c'f(n) + 1\bigr) = f(n) \\ e\bigl(c'f'(n) + 1\bigr) = df(n) = f'(n) \end{cases} .$$

This evidently proves (7.52).

Remark. $c'(n) + 1 \neq d(n)$.

Theorem 7.20. We have

(7.53) $$\begin{cases} c'f = d'g = d'af \\ c'f' = df' \end{cases} .$$

Proof. We have

(7.54) $$c'(n) = \left[(1 + \sqrt{2})n - \frac{1}{\sqrt{2}}\right]$$

and

$$\{\sqrt{2}\, f\} < \frac{1}{2}, \qquad \{\sqrt{2}\, f'\} > \frac{1}{\sqrt{2}} .$$

Hence

$$\begin{cases} c'f = df - 1 \\ c'f = df' \end{cases} .$$

Since

$$d'g = d'af = df - 1,$$

(7.53) follows at once.

Theorem 7.21. We have

(7.55) $$c'(n) \le d(n) \le c'(n) + 1 \le p(n)$$

and

(7.56) $$e(k) = n \quad \text{if and only if} \quad k \in [d(n),\ p(n) + 1] .$$

The interval $[d(n), p(n) + 1]$ contains exactly three integers if $n \in (d)$ and contains exactly two integers if $n \in (d')$.

Proof. Inequalities (7.55) come from

$$d(n) = [(1 + \sqrt{2})n]$$

together with (4.4) and (7.54). To prove (7.56) we use

$$e\big(d(n)\big) = e\big(p(n) + 1\big) = n$$

and

$$p(n) + 2 = d(n + 1) .$$

The final statement in the theorem follows from

$$d(n + 1) - d(n) = 3 \quad \text{if and only if} \quad n \in (d) .$$

8. THEOREMS INVOLVING σ AND τ

Let

(8.1) $$n = f_1P_1 + f_2P_2 + f_3P_3 + \cdots$$

be the first canonical representation of n. Define $\sigma(n)$ by means of

(8.2) $$\sigma(n) \equiv f_1 + f_2 + f_3 + \cdots \pmod 2 .$$

If

$$f_1 = \cdots = f_{k-1} = 0, \qquad f_k \ne 0 ,$$

put

(8.3) $$\tau(n) \equiv k \pmod 2 .$$

We may assume that $\sigma(n)$, $\tau(n)$ take on the values 0, 1.

It follows from (8.1) that

$$p(n) = \cdot 0\, f_1f_2f_3 \cdots .$$

Since

$$p_k \equiv k \pmod 2$$

it follows that

(8.4) $$n + p(n) \equiv \sigma(n) \pmod 2 .$$

Since

$$b(n+1) = n + p(n) + 3$$

we get

(8.5) $$a(n+1) \equiv b(n+1) \equiv \sigma(n) + 1 \pmod 2 .$$

In the next place, by Theorem 5.4,

(8.6) $$(d) = \{n \mid \tau(n) = 0\}$$

so that

(8.7) $$(d') = \{n \mid \tau(n) = 1\}$$

Since $(b) \subset (d')$ it follows that

(8.8) $$\tau\big(b(n)\big) = 1 \qquad (n = 1, 2, 3, \cdots) .$$

By (8.5)

(8.9) $$\sigma\big(b(n)\big) \equiv a\big(b(n)+1\big) \equiv ab(n) \equiv 0 \pmod 2 .$$

On the other hand, for n such that $a(n) \in (d')$,

$$\sigma\big(a(n)\big) + 1 \equiv a\big(a(n)+1\big) \equiv a^2(n) + 1 .$$

Since $(d') \subset (f)$,

$$a^2(n) = 2n - 1 \equiv 1 \pmod 2$$

and therefore

(8.10) $$\sigma\big(a(n)\big) = 1 \qquad \big(a(n) \in (d')\big) .$$

Combining (8.8), (8.9) and (8.10), we get the following.

Theorem 8.1. The set (b) is characterized by

(8.11) $$(b) = \{n \mid \sigma(n) = 0,\ \ \tau(n) = 1\} .$$

Put

(8.12) $$A_{i,j} = \{n \mid \tau(n) = i,\ \ \sigma(n) = j\} \qquad (i, j = 0, 1)$$

Thus by (8.11)

(8.13) $$(b) = A_{1,0}, \qquad (a) = A_{0,0} \cup A_{0,1} \cup A_{1,1} .$$

Theorem 8.2. We have

(8.14) $$A_{0,0} = (ad'g')$$

(8.15) $$A_{0,1} = (af') = (adf)$$

(8.16) $$A_{1,1} = (ac') = (adf') \cup (ad'g) .$$

Proof.

1. Let $n \in (a) \cap (d')$. By (8.10), $\sigma(n) = 1$; also by (8.7), $\tau(n) = 1$. Therefore

(8.17) $$(a) \cap (d') \subset A_{1,1} .$$

2. Next let $n \in (d)$, so that $\tau(n) = 0$. Since $d = ac$ and $(c) = (df) \cup (d'g')$, we have

$$(d) = (adf) \cup (ad'g') . \tag{8.18}$$

Since $n \in (d)$,

$$\sigma(n) \equiv a(n+1) + 1 \equiv a(n) + 1 .$$

Let $n = a(k)$, $k \in (df)$. Then

$$\sigma\big(a(k)\big) \equiv a^2(k) + 1 \equiv 1 .$$

Hence

$$(adf) \subset A_{0,1} . \tag{8.19}$$

Now let $n = a(k)$, $k \in (d'g')$. Then

$$\sigma\big(a(k)\big) \equiv a^2(k) + 1 \equiv 0 ,$$

so that

$$(ad'f') \subset A_{0,0} . \tag{8.20}$$

Since

$$\begin{aligned}(a) &= \big((a) \cap (d')\big) \cup (ac) \\ &= \big((a) \cap (d')\big) \cup (adf) \cup (ad'g') ,\end{aligned}$$

it follows that the inclusion sign $\subset$ in (8.17), (8.19) and (8.20) may be replaced by equality. This completes the proof of the theorem.

Theorem 8.3. We have

$$\begin{cases} \sigma(n) = \tau(n) & \big(n \in (g)\big) \\ \sigma(n) + \tau(n) = 1 & \big(n \in (g')\big) \end{cases} . \tag{8.21}$$

Proof. Since $g = af$, $(g) \subset (a)$ but $(g) \not\subset (af')$. Consequently, by the last theorem,

$$\begin{cases} (g) = A_{0,0} \cup A_{1,1} \\ (g') = A_{0,1} \cup A_{1,0} \end{cases} \tag{8.22}$$

and (8.21) follows at once.

Theorem 8.4. We have

$$\sigma(n-1) = 0 \rightleftarrows n \in (g) . \tag{8.23}$$

Proof. By (7.6),

$$a(n) \equiv 1 \pmod 2 \rightleftarrows n \in (g) .$$

Since

$$\sigma(n-1) \equiv a(n) + 1 \pmod 2 ,$$

(8.23) follows at once.

Theorem 8.5. We have

$$\begin{cases} (dg) = \{n \mid n \in (d), \quad \sigma(n) = 1\} \\ (dg') = \{n \mid n \in (d), \quad \sigma(n) = 0\} \end{cases} . \tag{8.24}$$

Proof. Since $(d) \subset (a)$ and

$$\tau(n) = 0 \qquad (n \in (d)),$$

it follows from Theorem 8.3 that

$$(d) = A_{0,0} \cup A_{0,1} = (ad'g') \cup (af') .$$

Thus

$$(dg) \cup (dg') = (ad'g') \cup (af') . \tag{8.25}$$

Now assume that

$$n \in (af'), \qquad n \in (dg') = (acg') .$$

It follows that there exists an integer k such that

$$k \in (f'), \qquad k \in (cg') .$$

But

$$f' = df = acf = caf = cg ,$$

so that

$$k \in (cg), \qquad k \in (cg') ,$$

which is impossible.

Next assume that

$$n \in (dg), \qquad n \in (ad'g') .$$

Then there is a k such that

$$k \in (cg), \qquad k \in (d'g') .$$

But

$$cg = caf = acf = df ,$$

so that

$$k \in (dg), \qquad k \in (d'g') ,$$

which is impossible. It therefore follows from (8.25) that

$$(dg) = (af'), \qquad (dg') = (ad'g') .$$

This completes the proof of the theorem.

Theorem 8.6. We have

$$\begin{cases} (\delta g) = \{n \mid n \in (\delta), \quad \sigma(n) = 1\} \\ (\delta g') = \{n \mid n \in (\delta), \quad \sigma(n) = 0\} \end{cases} . \tag{8.26}$$

Proof. Since

$$(\delta) = \bigcup_{1}^{\infty} A_{2k+1}$$

and $e\delta = d$, Theorem 8.6 is an immediate corollary of Theorem 8.5.

SUMMARY OF FORMULAS

1. $p(n) = 2n + e(n)$
2. $e(p(n)) = e(p(n) + 1) = n$
3. $e(p(n) + 2) = n + 1$
4. $a(n + 1) = e(n) + n + 1$
5. $b(n + 1) = p(n) + n + 3$
6. $d(n + 1) = p(n) + 2$
7. $ad(n) = b(n) - 1, \quad a(d(n) + 1) = b(n) + 1, \quad a(d(n) - 1) = b(n) - 2$
8. $ed(n) = n$
9. $eb(n) = a(n)$
10. $d^2(n) = \delta(n) - 1$
11. $e\delta(n) = d(n)$
12. $e^2\delta(n) = n$
13. $e(\delta(n) - 1) = d(n)$
14. $e^2(\delta(n) - 1) = n$
15. $ab(n) = a(n) + b(n) = 2d(n)$
16. $db(n) = bd(n) + 1$
17. $ad - da + 1 = ab - ba$
18. $a\delta(n) = d(n) + \delta(n)$
19. $a(n) = e(b(n) - 1) = ead(n)$
20. $ebd(n) = b(n) - 1$
21. $d'a(n) = d(n) - 1$
22. $d'(a(n) + 1) = d(n) + 1$
23. $\epsilon(2d(n)) = \delta(n) - 1$
24. $\epsilon(2d(n) + 1) = \delta(n) + 1$
25. $e(d(n) - 1) = n - 1$
26. $e(a^2(n) + a(n)) = a(n)$
27. $e(b(n) - 1) = a(n)$
28. $a(d(n) - 1) = b(n) - 2$
29. $\epsilon(2n) = d(n), \quad \epsilon(2n - 1) = d(n) - 1$
30. $e(\epsilon(n)) = [n/2]$

31. $e(n) - e(n-1) = 1 \rightleftarrows n \in (d)$

32. $a(n+1) = a(n) + 2 \rightleftarrows n \in (d)$

33. $d'(n+1) = d'(n) + 2 \rightleftarrows n \in (a)$

34. $d(n) = ac(n), \quad c(n) = b(n) - d'(n)$

35. $A_1 = d(\underset{\sim}{N}) - 1$

36. $A_{2k} = d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

37. $A_{2k+1} = \delta^{k}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

38. $B_{2k} = d\delta^{k-1}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

39. $B_{2k+1} = \delta^{k}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

40. $d(\underset{\sim}{N}) = \bigcup_{1}^{\infty} A_{2k}$

41. $\delta(\underset{\sim}{N}) = \bigcup_{1}^{\infty} A_{2k+1}$

42. $\epsilon(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \left(d(\underset{\sim}{N}) - 1\right)$

43. $a^2b = 2b - 1$

44. $n \in (d) \rightleftarrows \left\{\frac{n}{1+\sqrt{2}}\right\} > 2 - \sqrt{2}$

45. $b^k = u_k a + v_k b,$

where

$$u_k = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, \quad v_k = u_{k+1} - u_k, \quad \alpha = 2 + \sqrt{2}, \quad \beta = 2 - \sqrt{2}.$$

46. $ab^k = u_k n + v_k b - t_k,$

where

$$u_k = 2^{k-1}Q_{k-1}, \quad v_k = 2^{k-1}Q_k, \quad t_k = \frac{1}{7}(2^{k+1}P_{k+1} - 3\cdot 2^k P_k - 2),$$

and

$$Q_k = P_k + P_{k-1}.$$

47. $s = ab - ba$

48. $af(n) = 1, \quad af'(n) = 2$

49. $a^2f(n) \equiv 1, \quad a^2f'(n) \equiv 0 \pmod 2$

50. $t = ad' - d'a$

51. $tg(n) = 0, \quad tg'(n) = 1$

52. $ag(n) \equiv 1, \quad ag'(n) \equiv 0 \pmod 2$

53. $g = af$

54. $df = f'$

55. $df - dg = 1$

56. $n \in (f) \rightleftarrows \{\sqrt{2}\, n\} < \dfrac{1}{\sqrt{2}}$

57. $bf - af' = 1$

58. $f' = acf = caf$

59. $n \in (g) \rightleftarrows \left\{\dfrac{n}{\sqrt{2}}\right\} < \dfrac{1}{2}$

60. $g' = a(\tfrac{1}{2}ag') + 1$

61. $\begin{cases} d'(2n) = 2d'(n) + 1 & (n \in (g)) \\ d'(2n) = 2d'(n) & (n \in (g')) \end{cases}$

62. $ad'(n) = 2d'(n) - n$

63. $ac'(n) = c'(n) + n - 1 = d'(2n - 1)$

64. $c'(n) = \begin{cases} d(n) & (n \in (f')) \\ d(n) + 1 & (n \in (f)) \end{cases}$

65. $\begin{cases} (c) = (df) \cup (d'g') \\ (c') = (df') \cup (d'g') \end{cases}$

66. $a^2c'(n) = 2c'(n) - 1$

67. $c' = f\theta$

68. $d' = f\psi$

69. $fh = b$

70. $\psi(2n) = h(n)$

71. $c = \epsilon a + 1$

72. $e(c'(n) + 1) = n$

73. $\begin{cases} c'f = d'g = d'af \\ c'f = df' \end{cases}$

74. $(b) = \{n \mid \sigma(n) = 0,\ \ \tau(n) = 1\}$

75. $A_{i,j} = \{n \mid \tau(n) = i,\ \ \sigma(n) = j\} \quad (i, j = 0,1)$

76. $A_{0,0} = (ad'g')$

77. $A_{0,1} = (af') = (adf)$

78. $A_{1,1} = (ac') = (adf') \cup (ad'g)$

79. $$\begin{cases} \sigma(n) = \tau(n) & (n \in (g)) \\ \sigma(n) = \tau(n) = 1 & (n \in (g')) \end{cases}$$

80. $\sigma(n - 1) = 0 \rightleftarrows n \in (g)$

81. $$\begin{cases} (dg) = \{n \mid n \in (d), & \sigma(n) = 1\} \\ (dg') = \{n \mid n \in (d), & \sigma(n) = 0\} \end{cases}$$

82. $$\begin{cases} (\delta g) = \{n \mid n \in (\delta), & \sigma(n) = 1\} \\ (\delta g') = \{n \mid n \in (\delta), & \sigma(n) = 0\} \end{cases}$$

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21	22	24	25	26	28
b	3	6	10	13	17	20	23	27	30	34	37	40	44	47	51	54	58	61	64	68
d	2	4	7	9	12	14	16	19	21	24	26	28	31	33	36	38	41	43	45	48
d'	1	3	5	6	8	10	11	13	15	17	18	20	22	23	25	27	29	30	32	34
e	0	1	1	2	2	2	3	3	4	4	4	5	5	6	6	7	7	7	8	8
p	2	5	7	10	12	14	17	19	22	24	26	29	31	34	36	39	41	43	46	48
	1	2	3	4	6	7	8	9	11	12	13	14	15	16	18	19	20	21	23	24
	5	10	17	22	29	34	39	46	51	58	63	68	75	80	87	92	99	104	109	116

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	2	4	5	7	8	9	11	12	14	15	16
ab	4	8	14	18	24	28	32	38	42	48	52	56
ba	3	6	13	17	23	27	30	37	40	47	51	54
s	1	2	1	1	1	1	2	1	2	1	1	2
f	1	3	4	5	6	8	10	11	13	15	16	17
f'	2	7	9	12	14	19	24	26	31	36	38	41

Table 3

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	2	4	5	7	8	9	11	12	14	15	16
d'	1	3	5	6	8	10	11	13	15	17	18	20
ad'	1	4	7	8	11	14	15	18	21	24	25	28
d'a	1	3	6	8	11	13	15	18	20	23	25	27
t	0	1	1	0	0	1	0	0	1	1	0	1
g	1	4	5	7	8	11	14	15	18	21	22	24
g'	2	3	6	9	10	12	13	16	17	19	20	23

Table 4

n	1	2	3	4	5	6	7	8	9	10	11	12
c'	1	4	6	8	11	13	16	18	21	23	25	28
c	2	3	5	7	9	10	12	14	15	17	19	20
θ	1	3	5	6	8	9	11	13	15	17	18	20
d'	1	3	5	6	8	10	11	13	15	17	18	20
ψ	1	2	4	5	6	7	8	9	10	12	13	14
ϵ	1	2	3	4	6	7	8	9	11	12	13	14
h	2	5	7	9	12	14	17	19	22	25	27	29
c'+1	2	5	7	9	12	14	17	19	22	24	26	28

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ENUMERATION OF 3×3 ARRAYS

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1. Let

(1.1) $$A = (a_{ij}) \qquad (i,\ j = 1,\ 2,\ 3)$$

denote an array of non-negative integers. Let $H(r)$ denote the number of arrays (1.1) such that

(1.2) $$\sum_{j=1}^{n} a_{ij} = r = \sum_{j=1}^{n} a_{ji} \qquad (i = 1,\ 2,\ 3)\ .$$

MacMahon [2, p. 161] has proved that

(1.3) $$\sum_{r=0}^{\infty} H(r)x^r = \frac{1 - x^3}{(1 - x)^6} = \frac{1 + x + x^2}{(1 - x)^5}\ .$$

This result has recently been rediscovered by Anand, Demir and Gupta [1].

Let $H(r,t)$ denote the number of arrays (1.1) that satisfy (1.2) and also

(1.4) $$\sum_{i=1}^{3} a_{ii} = t$$

and let $H(r,s,t)$ denote the number of arrays (1.1) that satisfy (1.2), (1.4) and

(1.5) $$\sum_{i=1}^{3} a_{i,4-i} = s\ .$$

MacMahon [2, pp. 162-163] has proved that

(1.6) $$\sum_{r=0}^{\infty} H(r,r)x^r = \frac{1 - x^6}{(1 - x)^3(1 - x^3)^2} = \frac{1 + x^3}{(1 - x)^3(1 - x^3)}$$

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and

$$\sum_{r=0}^{\infty} H(r,r,r)x^r = \frac{(1-x^6)^2}{(1-x^3)^5} = \frac{(1+x^3)^2}{(1-x^3)^3} \tag{1.7}$$

In the present paper we show first that if

$$\overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{\substack{a+b\le r\\ c+d\le r\\ a+c\le r\\ b+d\le r\\ a+b+c+d\ge r}} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d \tag{1.8}$$

then

$$\sum_{r=0}^{\infty} \overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4)x^r$$

$$= \frac{1-\lambda_1\lambda_2\lambda_3\lambda_4x^3}{(1-\lambda_1x)(1-\lambda_2x)(1-\lambda_3x)(1-\lambda_4x)(1-\lambda_1\lambda_4x)(1-\lambda_2\lambda_3x)} \tag{1.9}$$

We show next that (1.9) implies

$$\sum_{r,s,t=0}^{\infty} H(r,s,t)x^r y^s z^t$$

$$= \frac{1-x^3y^3z^3}{(1-xy)^2(1-xz)^2(1-xy^3z)(1-xyz^3)} \tag{1.10}$$

This in turn implies

$$\sum_{r,t=0}^{\infty} H(r,t)\, x^r z^t = \frac{1-x^3z^3}{(1-x)^2(1-xz)^3(1-xz^3)} \tag{1.11}$$

which we show implies (1.6).

In the next place we prove

$$\sum_{r,t=0}^{\infty} H(r,t,t)\, x^r z^t$$

$$= \frac{1+x^2z+4x^3z^3-4x^5z^4-x^6z^6-x^8z^7}{(1-x^2z^4)(1-x^3z^3)(1-x^2z)^3} \tag{1.12}$$

which we show implies (1.7). We also give a combinatorial proof of (1.7).

Finally, if

$$K(s,t) = \begin{cases} H(r,s,t) & (s + t = 2r) \\ 0 & (s + t \text{ odd}) \end{cases}, \tag{1.13}$$

we show that

$$\sum_{s,t=0}^{\infty} K(s,t)\, y^s z^t \tag{1.14}$$

$$= \frac{(1 + y^3z^3)[1 + 4y^3z^3 + y^6z^6 + 4y^2z^2(y^2 + z^2) + yz(y^4 + z^4)]}{(1 - y^5 z)^2(1 - y z^5)^2} .$$

Moreover (1.14) contains (1.7).

2. Proof of (1.9). It follows from (1.8) that

$$\overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4) = S_1(r) - S_2(r) , \tag{2.1}$$

where

$$S_1(r) = \sum_{\substack{a+b\le r\\ c+d\le r\\ a+c\le r\\ b+d\le r}} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d \tag{2.2}$$

and

$$S_2(r) = \sum_{a+b+c+d<r} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d . \tag{2.3}$$

Then by (2.2)

$$S_1(r) = \sum_{b,c\le r} \lambda_2^b\lambda_3^c \sum_{\substack{a\le r-b\\ a\le r-c}} \lambda_1^a \sum_{\substack{d\le r-b\\ d\le r-c}} \lambda_4^d$$

$$= \sum_{b,c\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{\substack{a\le b\\ a\le c}} \lambda_1^a \sum_{\substack{d\le b\\ d\le c}} \lambda_4^d$$

$$= \sum_{b\le c\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{a\le b} \lambda_1^a \sum_{d\le b} \lambda_4^d$$

$$+ \sum_{c\le b\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{a\le c} \lambda_1^a \sum_{d\le c} \lambda_4^d$$

$$- \sum_{b \le r} \lambda_2^{r-b} \lambda_3^{r-b} \sum_{a \le b} \lambda_1^a \sum_{d \le b} \lambda_4^d$$

$$= \sum_{b \le r} \lambda_2^{r-b} \frac{1 - \lambda_3^{r-b+1}}{1 - \lambda_3} \; \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4}$$

$$+ \sum_{c \le r} \lambda_3^{r-c} \; \frac{1 - \lambda_2^{r-c+1}}{1 - \lambda_2} \; \frac{1 - \lambda_1^{c+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{c+1}}{1 - \lambda_4}$$

$$- \sum_{b \le r} \lambda_2^{r-b} \, \lambda_3^{r-b} \; \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} .$$

It follows that

$$\sum_{r=0}^{\infty} S_1(r)\, x^r = \sum_{b=0}^{\infty} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} \; x^b \sum_{r=0}^{\infty} \lambda_2^r \; \frac{1 - \lambda_3^{r+1}}{1 - \lambda_3} \; x^r$$

$$+ \sum_{c=0}^{\infty} \frac{1 - \lambda_1^{c+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{c+1}}{1 - \lambda_4} \; x^c \sum_{r=0}^{\infty} \lambda_3^r \; \frac{1 - \lambda_2^{r+1}}{1 - \lambda_2} \; x^r$$

$$- \sum_{b=0}^{\infty} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \; \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} \; x^b \sum_{r=0}^{\infty} \lambda_2^r \lambda_3^r \, x^r .$$

Carrying out the summations and simplifying, we get

$$\text{(2.4)} \qquad \sum_{r=0}^{\infty} S_1(r) x^r = \frac{(1 - \lambda_1\lambda_4 x^2)(1 - \lambda_2\lambda_3 x^2)}{(1 - x)(1 - \lambda_1 x)(1 - \lambda_2 x)(1 - \lambda_3 x)(1 - \lambda_4 x)(1 - \lambda_1\lambda_4 x)(1 - \lambda_2\lambda_3 x)} .$$

Similarly we find that

$$\text{(2.5)} \qquad \sum_{r=0}^{\infty} S_2(r)\, x^r = \frac{x}{(1 - x)(1 - \lambda_1 x)(1 - \lambda_2 x)(1 - \lambda_3 x)(1 - \lambda_4 x)} .$$

Since, by (2.1)

$$\sum_{r=0}^{\infty} \overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4)x^r = \sum_{r=0}^{\infty} S_1(r)x^r - \sum_{r=0}^{\infty} S_2(r)x^r \ ,$$

it is easily verified that we get (1.9).

3. <u>Proof of (1.10) and (1.6)</u>. Consider the array

$$\boxed{\begin{array}{ccc} a & b & r-a-b \\ c & d & r-c-d \\ r-a-c & r-b-d & k \end{array}} \ . \tag{3.1}$$

If

$$a + b + c + d = k + r \tag{3.2}$$

then clearly all row and column sums of (3.1) equal r. It follows that

$$\overline{H}(r;\ 1,\ 1,\ 1,\ 1) = H(r)\ .$$

Let

$$a + d + k = t \tag{3.3}$$

and

$$2r - 2a - b - c + d = s\ . \tag{3.4}$$

It follows from (3.2) and (3.3) that

$$t + r = 2a + b + c + 2d\ . \tag{3.5}$$

In (1.9) take

$$\lambda_1 = y^{-2}z^2, \qquad \lambda_2 = \lambda_3 = y^{-1}z\ , \qquad \lambda_4 = yz^2 \tag{3.6}$$

and replace x by xy^2z^{-1}. The left member of (1.9) becomes

$$\sum_{r=0}^{\infty} \overline{H}(r;\ y^{-2}z^2,\ y^{-1}z,\ yz^2)(xy^2z^{-1})^r$$

$$= \sum_{r=0}^{\infty} x^r \sum_{\substack{a+b\le r \\ c+d\le r \\ a+c\le r \\ a+b+c+d\ge r}} y^{2r-2a-b-c+d}\, z^{2a+b+c+2d-r}$$

$$= \sum_{r,s,t=0}^{\infty} H(r,s,t)\, x^r y^s z^t\ ,$$

where $H(r,s,t)$ is the number of arrays (3.1) that satisfy (3.3) and (3.4). We have therefore the generating function

$$\sum_{r,s,t=0}^{\infty} H(r,s,t)\, x^r y^s z^t = \frac{1 - x^3 y^3 z^3}{(1 - xy)^2(1 - xz)^2(1 - xy^3z)(1 - xyz^3)} \tag{3.7}$$

To obtain a generating function for $H(r,t)$, the number of arrays (3.1) that satisfy (3.3), we take $y = 1$. Thus

$$\sum_{r,t=0}^{\infty} H(r,t)\, x^r z^t = \frac{1 - x^3 z^3}{(1 - x)^2(1 - xz)^3(1 - xz^3)} \quad . \tag{3.8}$$

We shall now show that (3.8) implies (1.6). Since

$$(1 - x)^{-2}(1 - xz^3)^{-1} = \sum_{a,b=0}^{\infty} (a + 1)x^{a+b} z^{3b} \;,$$

it follows that the terms in which the exponents of x and z are equal contribute

$$\sum_{b=0}^{\infty}(2b + 1)x^{3b} z^{3b} = \frac{1 + x^3z^3}{(1 - x^3 z^3)^2} \quad .$$

Therefore

$$\sum_{r=0}^{\infty} H(r,r)x^r = \frac{1 - x^3}{(1 - x)^3}\,\frac{1 + x^3}{(1 - x^3)^2} = \frac{1 + x^3}{(1 - x)^3(1 - x^3)} \quad .$$

4. Proof of (1.12) and (1.7). Returning to (3.7), we shall now obtain a generating function for $H(r,t,t)$. We have

$$(1 - xy)^{-2}(1 - xz)^{-2}(1 - xy^3z)^{-1}(1 - xyz^3)^{-1}$$

$$= \sum_{a,b,c,d=0}^{\infty} (a + 1)(b + 1)x^{a+b+c+d}\, y^{a+3c+d}\, z^{b+c+3d} \;.$$

For those terms in which y and z have equal exponents

$$a + 3c + d = b + c + 3d \;,$$

so that

$$a + 2c = b + 2d \;.$$

We accordingly get

$$2\sum_{c \le d}\sum_{b}(b - 2c + 2d + 1)(b + 1)x^{2b-c+3d}(yz)^{b+c+3d} - \sum_{a,c}(a + 1)^2x^{2a+2c}(yz)^{a+4c}$$

$$= 2\sum_{b,c,d}(b + 2d + 1)(b + 1)x^{2b+2c+d}(yz)^{b+4c+3d} - \sum_{a,c}(a + 1)^2x^{2a+2c}(yz)^{a+4c}$$

$$= \frac{2}{1 - x^2(yz)^4}\sum_{b,d}(b + 2d + 1)(b + 1)x^{2b+3d}(yz)^{b+3d} - \frac{1}{1 - x^2(yz)^4}\sum_{a}(a + 1)^2x^{2a}(yz)^a.$$

Carrying out the indicated summations, we get

$$\frac{1}{1 - x^2(yz)^4}\left\{\frac{2}{1 - x^3(yz)^3}\,\frac{1 + x^2yz}{(1 - x^2yz)^2} + \frac{4(xyz)^3}{(1 - x^2yz)(1 - x^3)(yz^3)^2} - \frac{1 + x^2yz}{(1 - x^2yz)^3}\right\}$$

which reduces to

$$\frac{1 + x^2yz + 4x^3(yz)^3 - 4x^5(yz)^4 - x^6(yz)^6 - x^8(yz)^7}{(1 - x^2(yz)^4)(1 - x^3(yz)^3)^2(1 - x^2yz)^3}.$$

It follows therefore that

$$\sum_{r,t=0}^{\infty} H(r,t,t)x^r z^t$$

$$(4.1) \qquad = \frac{1 + x^2z + 4x^3z^3 - 4x^5z^4 - x^6z^6 - x^8z^7}{(1 - x^2z^4)(1 - x^3z^3)(1 - x^2z)^3}.$$

To get a generating function for $H(r,r,r)$ we observe that the right member of (4.1) is equal to

$$(4.2) \qquad \frac{(1 + x^2z)(1 + x^3z^3)}{(1 - x^2z^4)(1 - x^2z)^3} + \frac{4x^3z^3}{(1 - x^2z^4)(1 - x^3z^3)(1 - x^2z)^2}.$$

The first fraction

$$= (1 + x^3z^3)\sum_{a,b}(a + 1)^2 x^{2a+2b} z^{a+4b}$$

which will contribute

$$(1 + x^3z^3)\sum_{b}(2b + 1)^2(xz)^{6b} = \frac{(1 + x^3z^3)(1 + 6x^6 + x^{12}z^{12})}{(1 - x^6z^6)^3}.$$

The second fraction in (4.2)

$$= \frac{4x^3z^3}{1 - x^3z^3} \sum_{a,b} (a + 1)\, x^{2a+2b} z^{a+4b}$$

which will contribute

$$\frac{4x^3z^3}{1 - x^3z^3} \sum_{b} (2b + 1)(xz)^{6b} = \frac{4x^3z^3}{1 - x^3z^3} \frac{1 + x^6z^6}{(1 - x^6z^6)^2} .$$

The total contribution is evidently

$$\frac{(1 + x^3z^3)^4}{(1 - x^3z^3)(1 - x^6z^6)^2} = \frac{(1 + x^3z^3)^2}{(1 - x^3z^3)^3} .$$

We have therefore

$$\sum_{r=0}^{\infty} H(r, r, r)x^r = \frac{(1 + x^3)^2}{(1 - x^3)^3} . \tag{4.3}$$

As noted by MacMahon, Eq. (4.3) is equivalent to

$$H(3m, 3m, 3m) = m^2 + (m + 1)^2 . \tag{4.4}$$

We shall now give a combinatorial proof of (4.4). With the notation (3.1) it is clear that $H(r, r, r)$ is equal to the number of solutions of the following system

$$\left\{\begin{array}{l} a + b + c + d = k + r \\ \quad k + a + d = r \\ 2a + b + c - d = r \\ a + b \le r, \quad c + d \le r \\ a + c \le r, \quad b + d \le r \end{array}\right. . \tag{4.5}$$

It follows that $3d = r$. Thus, for $r = 3m$, Eq. (4.5) reduces to

$$\left\{\begin{array}{r} 2a + b + c = 4m \\ a + b \le 3m \\ a + c \le 3m \\ b \le 2m, \quad c \le 2m \end{array}\right. . \tag{4.6}$$

For $0 \le a \le m$, Eq. (4.6) implies

$$b \ge 2m - 2a, \qquad c \ge 2m - 2a .$$

Hence

$$H(3m,3m,3m) = \sum_{a=0}^{m-1} \sum_{2m-2a\le b\le 2m} 1 + \sum_{a=m}^{2m} \sum_{b+c=4m-2a} 1$$

$$= \sum_{a=0}^{m-1} (2a+1) + \sum_{a=m}^{2m} (4m-2a+1)$$

$$= m^2 + \sum_{a=0}^{m} (2a+1)$$

$$= m^2 + (m+1)^2 \quad .$$

5. Proof of (1.14). Returning to (1.10) we replace x by x^2, y by $x^{-1}y$ and z by $x^{-1}z$. If $K(s,t)$ is defined by (1.13) it is clear that

$$\sum_{s,t=0}^{\infty} K(s,t)\, y^s z^t$$

is equal to the sum of the terms in

$$\text{(5.1)} \qquad \frac{1 - y^3z^3}{(1-xy)^2(1-xz)^2(1-x^{-2}y^3z)(1-x^{-2}yz^3)}$$

that are independent of x. Expanding (5.1), this sum is seen to be

$$(1 - y^3z^3) \sum (a+1)(b+1) y^{a+3c+d} z^{b+c+3d} \quad ,$$

where the summation is over all non-negative a, b, c, d such that $a + b = 2c + 2d$. This gives

$$(1 - y^3z^3) \sum_{a,b=o}^{\infty} (2a+1)(2b+1) y^{2a} z^{2b} (yz)^{a+b} \sum_{c+d=a+b} y^{2c} z^{2d}$$

$$+ (1 - y^3z^3) \sum_{a,b=o}^{\infty} (2a+2)(2b+2) y^{2a+1} z^{2b+1} (yz)^{a+b+1} \sum_{c+d=a+b+1} y^{2c} z^{2d} \quad .$$

Carrying out the indicated summations, we get

$$\frac{1 - y^3z^2}{y^2 - z^2}\left\{y^2\,\frac{1 + y^5z}{(1 - y^5z)^2}\,\frac{1 + y^3z^3}{(1 - y^3z^3)^2} - z^2\,\frac{1 + y^3z^3}{(1 - y^3z^3)^2}\,\frac{1 + yz^5}{(1 - yz^5)^2}\right.$$
$$\left. + 4y^4\,\frac{y^2z^2}{(1 - y^5z)^2(1 - y^3z^3)^2} - 4z^4\,\frac{y^2z^2}{(1 - y^3z^3)^2(1 - yz^5)^2}\right\} .$$

A little manipulation gives

$$\frac{(1 + y^3z^3)\,[1 + 4y^3z^3 + y^6z^6 + 4y^2z^2(y^2 + z^2) + yz(y^4 + z^4)]}{(1 - y^5z)^2(1 - yz^5)^2} .$$

This completes the proof of (1.14).

To show that (1.14) contains (1.7), we take

$$(1 - y^5z)^{-2}(1 - yz^5)^{-2} = \sum_{a,b=0}^{\infty} (a + 1)(b + 1)y^{5a+b}\,z^{a+5b} .$$

Since

$$\begin{aligned} &1 + 4y^3z^3 + y^6z^6 + 4y^2z^2(y^2 + z^2) + yz(y^4 + z^4) \\ &\quad = 2(1 + y^3z^3)^2 - (1 - y^5z)(1 - yz^5) + 4y^2z^2(y^2 + z^2) , \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{s=0}^{\infty} H(s,s,s)z^s &= \sum_{s=0}^{\infty} K(s,s)z^s \\ &= 2(1 + z^3)^3 \sum_{a=0}^{\infty} (a + 1)^2 z^{6a} - (1 + z^3) \sum_{a=0}^{\infty} z^{6a} \\ &= 2(1 + z^3)^3\,\frac{1 + z^6}{(1 - z^6)^3} - \frac{1 + z^3}{1 - z^6} \\ &= \frac{2(1 + z^6)}{(1 - z^3)^3} - \frac{1}{1 - z^3} = \frac{(1 + z^3)^2}{(1 - z^3)^3} . \end{aligned}$$

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REPRESENTATIONS FOR A SPECIAL SEQUENCE

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1. INTRODUCTION AND SUMMARY

Consider the sequence defined by

$$(1.1) \qquad u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = u_n + 2u_{n-1} \qquad (n \geq 1).$$

It follows at once from (1.1) that

$$(1.2) \qquad u_n = \frac{1}{3}(2^n - (-1)^n), \quad u_n + u_{n+1} = 2^n .$$

The first few values of u_n are easily computed.

n	1	2	3	4	5	6	7	8	9	10
u_n	1	1	3	5	11	21	43	85	171	341

It is not difficult to show that the sums

$$(1.3) \qquad \sum_{i=2}^{k} \epsilon_i u_i \qquad (k = 2, 3, 4, \cdots),$$

where each $\epsilon_i = 0$ or 1, are distinct. The first few numbers in (1.3) are

$$1, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 17, 19, 20, \cdots .$$

Thus there is a sequence of "missing" numbers beginning with

$$(1.4) \qquad 2, 7, 10, 13, 18, 23, 28, 31, 34, 39, \cdots .$$

In order to identify the sequence (1.4) we first define an array of positive integers R in the following way. The elements of the first row are denoted by $a(n)$, of the second row by $b(n)$, of the third row by $c(n)$. Put

* Supported in part by NSF Grant GP-17031.

$$a(1) = 1, \quad b(1) = 3, \quad c(1) = 2 .$$

Assume that the first $n - 1$ columns of R have been filled. Then $a(n)$ is the smallest integer not already appearing, while

$$b(n) = a(n) + 2n \tag{1.5}$$

and

$$c(n) = b(n) - 1 . \tag{1.6}$$

The sets $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$ constitute a disjoint partition of the positive integers. The following table is readily constructed.

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	4	5	6	9	12	15	16	17	20	21	22
b	3	8	11	14	19	24	29	32	35	40	43	48
c	2	7	10	13	18	23	28	31	34	39	42	47

The table suggests that the numbers $c(n)$ are the "missing" numbers (1.4) and we shall prove that this is indeed the case.

Let A_k Denote the set of numbers

$$\begin{cases} N = u_{k_1} + u_{k_2} + \cdots + u_{k_r} , \\ 2 \le k = k_1 < k_2 < \cdots < k_r \end{cases} \tag{1.7}$$

and $r = 1, 2, 3, \cdots$. We shall show that

$$A_{2k+2} = ab^k a(\underset{\sim}{N}) \cup ab^k c(\underset{\sim}{N}) \qquad (k \ge 0) \tag{1.8}$$

and

$$A_{2k+1} = b^k a(\underset{\sim}{N}) \cup b^k c(\underset{\sim}{N}) \qquad (k \ge 1) , \tag{1.9}$$

where N denotes the set of positive integers.

If N is given by (1.7), we define

$$e(N) = u_{k_r - 1} + u_{k_r - 1} + \cdots + u_{k_r - 1} . \tag{1.10}$$

Then we shall show that

$$e\big(a(n)\big) = n \tag{1.11}$$

and

$$e\big(b(n)\big) = a(n) . \tag{1.12}$$

Clearly the domain of the function $c(n)$ is restricted to $a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$. However, since, as we shall see below, $\big(b(n) - 2\big) \in a(\underset{\sim}{N})$ and

(1.13) $$e(b(n) - 2) = a(n) ,$$

it is natural to define

(1.14) $$e(c(n)) = a(n) .$$

Then $e(n)$ is defined for all n and we show that $e(n)$ is monotone.

The functions a, b, c satisfy various relations. In particular we have

$$\begin{aligned} a^2(n) &= b(n) - 2 = a(n) + 2n - 2 \\ ab(n) &= ba(n) + 2 = 2a(n) + b(n) \\ ac(n) &= ca(n) + 2 = 2a(n) + c(n) \\ cb(n) &= bc(n) + 2 = 2a(n) + 3c(n) + 2 . \end{aligned}$$

Moreover if we define

(1.15) $$d(n) = a(n) + n$$

then we have

$$\begin{aligned} da(n) &= 2d(n) - 2 \\ db(n) &= 4d(n) \\ dc(n) &= 4d(n) - 2 . \end{aligned}$$

It follows from (1.11) and (1.12) that every positive integer N can be written in the form

(1.16) $$N = u_{k_1} + u_{k_2} + \cdots + u_{k_r} ,$$

where now

$$1 \leq k_1 < k_2 < \cdots < k_r .$$

Hence N is a "missing" number if and only if $k_1 = 1$, $k_2 = 2$.

The representation (1.16) is in general not unique. The numbers $a(n)$ are exactly those for which, in the representation (1.7), k_1 is even. Hence in (1.66) if we assume that k_1 is odd, the representation (1.16) is unique. We accordingly call this the canonical representation of N.

Returning to (1.15), we define the complementary function $d'(n)$ so that the sets $\{d(n)\}$, $\{d'(n)\}$ constitute a disjoint partition of the positive integers. We shall show that

(1.17) $$d(n) = 2d'(n) .$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d'	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23
d	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46

.

Let $\delta(n)$ denote the number of $d(k) \leq n$ and let $\delta'(n)$ denote the number of $d'(k) \leq n$. We show that

$$\delta(N) = \left[\frac{N}{2}\right] - \left[\frac{N}{4}\right] + \left[\frac{N}{8}\right] - \cdots$$

$$\delta'(N) = [N] - \left[\frac{N}{2}\right] + \left[\frac{N}{4}\right] - \cdots .$$

Finally, if N has the canonical representation (1.16) we define

$$f(N) = \sum_{i=1}^{r} (-1)^{k_i} . \tag{1.18}$$

It follows that

$$a(N) = 2N + f(N) \tag{1.19}$$

and

$$d(N) = a(N) + N = \sum_{i=1}^{r} 2^{k_i} , \tag{1.20}$$

so that there is a close connection with the binary representation of an integer.

Even though there is no "natural" irrationality associated with the sequence $\{u_n\}$, it is evident from the above summary that many of the results of the previous papers of this series [2, 3, 4, 5, 6] have their counterpart in the present situation.

The material in the final two sections of the paper is not included in the above summary.

2. THE CANONICAL REPRESENTATION

As in the Introduction, we define the sequence $\{u_n\}$ by means of

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = u_n + 2u_{n-1} \qquad (n \geq 1) .$$

We first prove the following.

Theorem 2.1. Every positive integer N can be written uniquely in the form

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots , \tag{2.1}$$

where the $\epsilon_i = 0$ or 1 and

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k = 1 \Rightarrow k \text{ odd} . \tag{2.2}$$

Proof. The theorem can be easily proved by induction on n as follows. Let C_{2n} consist of all sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2n}) \qquad (\epsilon_i = 0 \text{ or } 1)$$

satisfying (2.2). Then the map

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2n}) \longrightarrow \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots + \epsilon_{2n} u_{2n}$$

is $1-1$ and onto from C_{2n} to $[0, \cdots, u_{2n+1} - 1]$. Clearly $C_2 \longrightarrow [0,1]$. Assuming that

$$C_{2n} \longrightarrow [0, \cdots, u_{2n+1} - 1] ,$$

we see that

$$\begin{aligned} C_{2n+2} \longrightarrow & [0, \cdots, u_{2n+1} - 1] \quad [u_{2n+1}, \cdots, 2u_{2n+1} - 1] \\ & \cup [u_{2n+2} + 1, \cdots, u_{2n+1} + u_{2n+2} - 1] \\ & \cup [u_{2n+1} + u_{2n+2}, \cdots, 2u_{2n+1} + u_{2n+2} - 1] \\ & = [0, \cdots, u_{2n+3} - 1] \end{aligned}$$

since

$$2u_{2n+1} - 1 = u_{2n+2} .$$

If (2.2) is satisfied we call (2.1) the <u>canonical representation</u> of N.

In view of the above we have also

<u>Theorem 2.2.</u> If N and M are given canonically by

$$N = \sum \epsilon_i u_i, \qquad M = \sum \delta_i u_i ,$$

then

$$N \le M \quad \text{if and only if} \quad \sum \epsilon_i 2^i \le \sum \delta_i 2^i . \tag{2.3}$$

Let N be given by (2.1) and define

$$\phi(N) = \sum \epsilon_i 2^i . \tag{2.4}$$

Note that since

$$u_n = \frac{1}{3}(2^n - (-1)^n) , \tag{2.5}$$

we have

$$N = \frac{1}{3}(\phi(N) - f(N)) , \tag{2.6}$$

where

$$f(N) = \epsilon(-1)^i \epsilon_i . \tag{2.7}$$

<u>Theorem 2.3.</u> There are exactly N numbers of the form $2^k K$, k, K odd, less than or equal to $\phi(N)$.

<u>Proof.</u> The N numbers of the stated form are simply

$$\phi(1), \phi(2), \cdots, \phi(N) .$$

If N is given canonically by

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots ,$$

we define

(2.8) $$a(n) = \epsilon_1 u_2 + \epsilon_3 u_3 + \cdots .$$

This is of course never canonical. Define

(2.9) $$b(n) = a(N) + 2N = \epsilon_1 u_3 + \epsilon_1 u_4 + \cdots .$$

The representation (2.9) is canonical.

Suppose ϵ_{2k+1} is the first nonzero ϵ_i in the canonical representation of N. Then, since

$$u_1 + u_2 + \cdots + u_{2k+1} = u_{2k+2} ,$$

we see that a(N) is given canonically by

(2.10) $$a(n) = u_1 + u_2 + \cdots + u_{2k+1} + 0 \cdot u_{2k+2} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

Let $c(N) = b(N) - 1$. Then, since

$$u_1 + u_2 + \cdots + u_{2k+2} = u_{2k+3} - 1 ,$$

c(N) is given canonically by

(2.11) $$c(N) = u_1 + u_2 + \cdots + u_{2k+2} + 0 \cdot u_{2k+3} + \epsilon_{2k+2} u_{2k+4} + \cdots .$$

We now state

Theorem 2.4. The three functions a, b, c defined above are strictly monotone and their ranges $a(\underset{\sim}{N})$, $b(\underset{\sim}{N})$, $c(\underset{\sim}{N})$ form a disjoint partition of $\underset{\sim}{N}$.

Proof. We have

(2.12) $$\phi\big(a(N) + 1\big) = 2\phi(N) + 2$$

and

(2.13) $$\phi\big(b(N)\big) = 4\phi(N) .$$

Since ϕ is 1 - 1 and monotone, it follows that a, b, c are monotone. By (2.10), $a(\underset{\sim}{N})$ consists of those N whose canonical representations begin with an odd number of 1's; $b(\underset{\sim}{N})$ of those which begin with 0; and $c(\underset{\sim}{N})$ of those which begin with an even number of 1's. Hence all numbers are accounted for.

It is now clear that the functions a, b, c defined above coincide with the a, b, c defined in the Introduction.

The following two theorems are easy corollaries of the above.

Theorem 2.5. $c(\underset{\sim}{N})$ is the set of integers that cannot be written as a sum of distinct u_i with $i \geq 2$.

Thus the $c(\underset{\sim}{N})$ are the "missing" numbers of the Introduction.

Theorem 2.6. If $K \notin c(\underset{\sim}{N})$, then K can be written uniquely as a sum of distinct u_i with $i \geq 2$.

3. RELATIONS INVOLVING a, b, AND c

We now define

$$d(N) = a(N) + N.$$

Since

$$u_k + u_{k+1} = 2^k,$$

it follows at once from (2.4) and (2.8) that

$$d(N) = \phi(N). \tag{3.1}$$

Hence, by (2.6), we may write

$$2N = a(N) - f(N). \tag{3.2}$$

Let d' denote the monotone function whose range is the complement of the range of d. Since the range of ϕ (that is, of d) consists of the numbers $2^k K$, with k, K both odd, it follows that the range of d' consists of the numbers $2^k K$ with k even and K odd. We have therefore

$$d(N) = 2d'(N). \tag{3.3}$$

Thus (2.12) and (2.13) become

$$d(a + 1) = 2d + 2 \tag{3.4}$$

and

$$db = 4d, \tag{3.5}$$

respectively.

From (2.10) we obtain

$$da = 2d - 2 \tag{3.6}$$

and

$$d'a = d - 1. \tag{3.7}$$

Theorem 3.1. We have

$$a^2(N) = b(N) - 2 = a(N) + 2N - 2 \tag{3.8}$$

$$ab(N) = ba(N) + 2 = 2a(N) + b(N) \tag{3.9}$$

$$ac(N) = ca(N) + 2 = 2a(N) + c(N) \tag{3.10}$$

$$cb(N) = bc(N) + 2 = 2a(N) + 3c(N) \tag{3.11}$$

$$da(N) = 2d(N) - 2 \tag{3.12}$$

$$db(N) = 4d(N) \tag{3.13}$$

$$dc(N) = 4d(N) - 2. \tag{3.14}$$

Proof. The first four formulas follow from the definitions. For example if

$$N = u_{2k+1} + \epsilon_{2k+2} u_{2k+2} + \cdots ,$$

then

$$a(N) = 1\cdot u_1 + 1\cdot u_2 + \cdots + 1\cdot u_{2k+1} + \epsilon_{2k+2} u_{2k+3} + \cdots$$

and

$$\begin{aligned} a^2(N) &= 1\cdot u_2 + \cdots + 1\cdot u_{2k+2} + \epsilon_{2k+2} u_{2k+4} + \cdots \\ &= u_{2k+3} - 2 + \epsilon_{2k+2} u_{2k+4} + \cdots \\ &= b(N) - 2 . \end{aligned}$$

Formula (3.12) is the same as (3.6) while (3.13) and (3.14) follow from the formulas for ab and ac.

In view of Theorem 2.6, every

$$N \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$$

can be written uniquely in the form

$$N = \delta_2 u_2 + \delta_3 u_3 + \cdots \tag{3.15}$$

with $\delta_2 = 0, 1$. We define A_k as the set of N for which δ_k is the first nonzero δ_i.

Theorem 3.2. We have

$$A_{2k+2} = ab^k a(\underset{\sim}{N}) \cup ab^k c(\underset{\sim}{N}) \qquad (k \geq 0) \tag{3.16}$$

$$A_{2k+1} = b^k a(\underset{\sim}{N}) \cup b^k c(\underset{\sim}{N}) \qquad (k \geq 1) . \tag{3.17}$$

Proof. By (2.9), (2.10) and (2.11), the union

$$a(\underset{\sim}{N}) \cup c(\underset{\sim}{N})$$

consists of those K for which

$$\epsilon_1 = \epsilon_1(K) = 1 .$$

Hence, applying a, we have

$$A_2 = a^2(\underset{\sim}{N}) \cup ac(\underset{\sim}{N})$$

and, applying b,

$$A_3 = ba(\underset{\sim}{N}) \cup bc(\underset{\sim}{N}) .$$

Continuing in this way, it is clear that we obtain the stated results.

Theorem 3.2 admits of the following refinement.

Theorem 3.3. We have

$$ab^k a(\underset{\sim}{N}) = \{N \in A_{2k+2} \mid N = ab^k a(n) \equiv n \pmod 2\} \tag{3.17}$$

$$ab^k c(\underset{\sim}{N}) = \{N \in A_{2k+2} \mid N = ab^k c(n) \equiv n + 1 \pmod 2\} \tag{3.18}$$

$$b^k a(\underset{\sim}{N}) = \{N \in A_{2k+1} \mid N = b^k a(n) \equiv n \pmod 2\} \tag{3.19}$$

(3.20) $$b^k c(\underset{\sim}{N}) = \{N \in A_{2k+1} \mid N = b^k c(n) \equiv n + 1 \pmod 2\} .$$

Proof. The theorem follows from Theorem 3.2 together with the observation

(3.21) $$a(n) \equiv b(n) \equiv n, \qquad c(n) \equiv n + 1 \pmod 2 .$$

Let

$$N \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N}) ,$$

so that (3.15) is satisfied. We define

(3.21) $$e(N) = \delta_2 u_1 + \delta_3 u_2 + \cdots .$$

Then from the definition of a and b we see that

(3.22) $$e(a(n)) = n$$

and

(3.23) $$e(b(n)) = a(n) .$$

Since

$$a^2(n) = b(n) - 2 < c(n) < b(n) ,$$

we define

(3.24) $$e(c(n)) = a(n) .$$

Thus $e(n)$ is now defined for all n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
e	1	1	1	2	3	4	4	4	5	5	5	6	6	6	7	8	9	9	9	10
n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
e	11	12	12	12	13	14	15	15	15	16	16	16	17	17	17	18	19	20	20	20

Theorem 3.4. The function e is monotone. Indeed $e(n) = e(n - 1)$ if and only if

$$n \in b(\underset{\sim}{N}) \cup c(\underset{\sim}{N}) .$$

Otherwise $(n \in a(\underset{\sim}{N}))$

$$e(n) = e(n - 1) + 1 .$$

Proof. We have already seen that

$$e\, b(n) = e\, c(n) = e\, c(n) - 1 = a(n) .$$

Thus it remains to show that

(3.25) $$e(a(n)) = e(a(n) - 1) + 1 .$$

Let

$$n = u_{2k+1} + \epsilon_{2k+2} u_{2k+2} + \cdots$$

be the canonical representation of n. Then

$$a(n) = u_{2k+2} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

Since

$$u_{2k+2} - 1 = u_2 + u_3 + u_4 + \cdots + u_{2k+1} ,$$

we get

$$a(n) - 1 = u_2 + u_3 + \cdots + u_{2k+1} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

It follows that

$$\begin{aligned} e\bigl(a(n) - 1\bigr) &= u_1 + u_2 + \cdots + u_{2k} + \epsilon_{2k+2} u_{2k+2} + \cdots \\ &= (u_{2k+1} - 1) + \epsilon_{2k+2} u_{2k+2} + \cdots \\ &= n - 1 . \end{aligned}$$

This evidently proves (3.25).

Theorem 3.5. We have

$$\text{(3.26)} \qquad \begin{cases} a(n + 1) = a(n) + 3 & \bigl(n \in a(\mathbb{N})\bigr) \\ a(n + 1) = a(n) + 1 & \bigl(n \in b(\mathbb{N}) \cup c(\mathbb{N})\bigr) . \end{cases}$$

Proof. Formula (3.4) is evidently equivalent to

$$\text{(3.27)} \qquad a\bigl(a(n) + 1\bigr) = b(n) + 1 .$$

By (3.8)

$$a^2(n) = b(n) - 2 = c(n) - 1 ,$$

so that we have the sequence of consecutive integers

$$\text{(3.28)} \qquad a^2(n), \quad c(n), \quad b(n), \quad a\ a(n) + 1 \quad .$$

On the other hand, by (3.9) and (3.10)

$$ab(n) = ac(n) + 1 .$$

Finally, since

$$b(n) + 1 \in a(n) ,$$

we have, by (3.28),

$$\begin{aligned} a\bigl(b(n) + 1\bigr) &= a^2\bigl(a(n) + 1\bigr) = b\ a(n) + 1 - 2 \\ &= a\bigl(a(n) + 1\bigr) + 2a(n) \\ &= 2a(n) + b(n) + 1 \\ &= ab(n) + 1 \qquad . \end{aligned}$$

This completes the proof of the theorem.

If we let $\alpha(n)$ denote the number of $a(k) < n$, it follows at once from Theorem 3.5 that

$$\text{(3.29)} \qquad a(n) = n + 2\alpha(n) \qquad (n \geq 1) .$$

This is equivalent to

(3.30) $$d'(n) = n + \alpha(n) .$$

We shall now show that

(3.31) $$\alpha(n + 1) = e(n) .$$

Let $n \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$. Then

$$n = u_k = \epsilon_{k+1} u_{k+1} + \cdots \qquad (k \geq 2)$$

and

$$e(n) = u_{k-1} + \epsilon_{k+1} u_k + \cdots .$$

Also

$$n + 1 = u_1 + u_k + \epsilon_{k+1} u_{k+1} + \cdots \qquad \text{(canonical)} ,$$

so that

$$a(n + 1) = u_2 + u_{k+1} + \epsilon_{k+1} u_{k+2} + \cdots .$$

It follows that

(3.32) $$a(n + 1) - 2e(n) = n + 1 \qquad (n \notin c(\underset{\sim}{N})) .$$

If $n \in c(\underset{\sim}{N})$ we have $e(n) = e(n + 1)$. Since $n + 1 \in b(\underset{\sim}{N})$, we may use (3.32). Thus

$$2e(n) = 2e(n + 1) = a(n + 2) - (n + 2) = a(n + 1) - (n + 1) ,$$

by (3.26). Hence

$$a(n + 1) - 2e(n) = n + 1$$

for all n. This is evidently equivalent to (3.31).

This proves

Theorem 3.6. The number of $a(k) \leq n$ is equal to $e(n)$. Moreover

(3.33) $$a(n) = n + 2e(n - 1) \qquad (n > 1) .$$

A few special values of $a(n)$ may be noted:

(3.34) $$a(2^{2k-1}) = 2^{2k} \qquad (k \geq 1)$$

(3.35) $$a(2^{2k}) = 2^{2k+1} - 2 \qquad (k \geq 1)$$

(3.36) $$a(2^{2k-1} - 2) = 2^{2k} - 4 \qquad (k > 1)$$

(3.37) $$a(2^{2k} - 2) = 2^{2k+1} - 6 \qquad (k > 2) .$$

4. COMPARISON WITH THE BINARY REPRESENTATION

If N is given in its binary representation

(4.1) $$N = \gamma_0 + \gamma_1 \cdot 2 + \gamma_2 \cdot 2^2 + \cdots ,$$

where $\gamma_1 = 0$ or 1, we define

(4.2) $$\delta(N) = \gamma_0 u_0 + \gamma_1 u_1 + \gamma_2 u_2 + \cdots$$
and
(4.3) $$(N) = \sum_i \gamma_i (-1)^i .$$
Then we have
(4.4) $$\delta(d(N)) = N$$
and
(4.5) $$\chi(d(N)) = f(N) .$$
A simple computation leads to
(4.6) $$\delta(N) = \left[\frac{N}{2}\right] - \left[\frac{N}{4}\right] + \left[\frac{N}{8}\right] - \cdots .$$

Let
(4.7) $$\delta'(N) = N - \left[\frac{N}{2}\right] + \left[\frac{N}{4}\right] - \cdots$$
so that
(4.8) $$\delta(N) + \delta'(N) = N .$$

Theorem 4.1. The number of $d(k) \le n$ is equal to $\delta(N)$. The number of $d'(k) \le n$ is equal to $\delta'(N)$.

Proof. Since δ is monotone, we have $d(k) \le n$ if and only if

$$k = \delta\, d(k) \le \delta(n) .$$

Hence, in view of (4.8), the theorem is proved.

We have seen in Section 3 that if N has the canonical representation

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots$$

then
(4.9) $$a(N) - 2N = f(N) ,$$
where

$$f(N) = \sum_i (-1)^i \epsilon_i .$$

It follows that

(4.10) $$d(N) = a(N) + N = \sum_i \epsilon_i \cdot 2^i .$$

Replacing N by $d(N)$, $d'(N)$ in (4.9), we get

(4.11) $$a(d(N)) - 2d(N) = f(d(N))$$
and
(4.12) $$a(d'(N)) - 2d'(N) = f(d(N)) .$$

Theorem 4.2. The function f(d) takes on every even value (positive, negative or zero) infinitely often. The function f(d') takes on every odd value (positive or negative) infinitely often.

Proof. Consider the number

$$\begin{aligned} N &= u_1 + u_3 + u_5 + \cdots + u_{2k-1} \\ &= \frac{1}{3}(2^1 + 1) + \frac{1}{3}(2^3 + 1) + \cdots + \frac{1}{3}(2^{2k-1} + 1) \\ &= \frac{1}{3}\left(\frac{2}{3}(2^{2k} - 1) + k\right) . \end{aligned}$$

Clearly

(4.13) $$N \equiv 2 \pmod 4$$

if and only if

(4.14) $$k \equiv 0 \pmod 4 .$$

It follows from (4.13) that $N \in d(\underset{\sim}{N})$. Also it is evident that

(4.15) $$f(N) = -k, \qquad k \equiv 0 \pmod 4 .$$

In the next place the number

$$\begin{aligned} N &= u_3 + u_5 + \cdots + u_{2k+1} \\ &= \frac{1}{3}(2^3 + 1) + \frac{1}{3}(2^5 + 1) + \cdots + \frac{1}{3}(2^{2k+1} + 1) \\ &\equiv 3k \pmod 8 . \end{aligned}$$

Hence for $k \equiv 2 \pmod 4$, we have $N \equiv 2 \pmod 4$ and so as above $N \in d(\underset{\sim}{N})$. Also it is evident that (4.15) holds in this case also.

Now consider

$$\begin{aligned} N &= u_1 + u_2 + u_4 + u_6 + \cdots + u_{2k} \\ &= 1 + \frac{1}{3}(2^2 - 1) + \frac{1}{3}(2^4 - 1) + \cdots + \frac{1}{3}(2^{2k} - 1) \\ &\equiv 1 + k \pmod 4 . \end{aligned}$$

Thus for k odd, $N \in d(\underset{\sim}{N})$. Also it is clear that

$$f(N) = k - 1 .$$

This evidently proves the first half of the theorem.

To form the second half of the theorem we first take

$$N = u_1 + u_3 + u_5 + \cdots + u_{2k-1} .$$

Then

$$N \equiv k \pmod 2 .$$

Thus for k odd, $N \in d'(\underset{\sim}{N})$. Moreover

$$f(N) = -k . \tag{4.16}$$

Next for

$$N = u_1 + u_2 + u_4 + u_6 + \cdots + u_{2k} + u_{2k+2}$$

we again have

$$N \equiv k \pmod 2 ,$$

so that $N \in d'(\underset{\sim}{N})$ for k odd. Clearly

$$f(N) = k. \tag{4.17}$$

This completes the proof of the theorem.

As an immediate corollary of Theorem 4.2 we have

Theorem 4.3. The commutator

$$ad(N) - da(N) = fd(N) + 2$$

takes on every even value infinitely often. Also the commutator

$$ad'(N) - d'a(N) = fd'(N) + 1$$

takes on every even value infinitely often.

5. WORDS

By a word function, or briefly, word, is meant a function of the form

$$w = a^{\alpha} b^{\beta} c^{\gamma} a^{\alpha'} b^{\beta'} c^{\gamma'} \cdots , \tag{5.1}$$

where the exponents are arbitrary non-negative integers.

Theorem 5.1. Every word function $w(n)$ can be linearized, that is

$$w(n) = A_w a(n) + B_w n - C_w \qquad (A_w > 0) , \tag{5.2}$$

where A_w, B_w, C_w are independent of n. Moreover the representation (5.2) is unique.

Proof. The representation (5.2) follows from the relations

$$\begin{cases} a^2(n) = a(n) + 2n - 2 \\ ab(n) = 2a(n) + b(n) = 3a(n) + 2n \\ ac(n) = 2a(n) + c(n) = 3a(n) + 2n - 1 . \end{cases} \tag{5.3}$$

If we assume a second representation (5.2) it follows that $a(n)$ is a linear function of n. This evidently contradicts Theorem 3.5.

Theorem 5.2. For any word w, the coefficient B_w in (5.2) is even. Hence the function d is not a word.

Proof. Repeated application of (5.3).

Remark. If we had defined words as the set of functions of the form

$$a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} \cdots , \tag{5.4}$$

then, in view of Theorem 4.3, we would not be able to assert the extended form of Theorem 5.1.

Combining (5.3) with (5.2), we get the following recurrences for the coefficients A_w, B_w, C_w:

$$\begin{cases} A_{wa} = A_w + B_w \\ B_{wa} = 2A_w \\ C_{wa} = 2A_w + C_w \end{cases} \tag{5.5}$$

$$\begin{cases} A_{wb} = 3A_w + B_w \\ B_{wb} = 2A_w + 2B_w \\ C_{wb} = C_w \end{cases} \tag{5.6}$$

$$\begin{cases} A_{wc} = 3A_w + B_w \\ B_{wc} = 2A_w + 2B_w \\ C_{wc} = A_w + B_w + C_w . \end{cases} \tag{5.7}$$

In particular we find that

$$a^k(n) = u_k a(n) + 2u_{k-1} n - (u_{k+1} - 1) , \tag{5.8}$$

$$ab^k(n) = u_{2k+1} a(n) + (u_{2k+1} - 1)n , \tag{5.9}$$

$$ac^k(n) = u_{2k+1} a(n) + (u_{2k+1} - 1)n - \tfrac{1}{3}(4u_{2k} - k) , \tag{5.10}$$

$$b^k(n) = u_{2k} a(n) + (u_{2k} + 1)n , \tag{5.11}$$

$$a^k b^j(n) = u_{k+2j} a(n) + 2u_{k+2j-1} n - (u_{k+1} - 1) , \tag{5.12}$$

$$b^j a^k(n) = u_{k+2j} a(n) + 2u_{k+2j-1} n - (u_{k+2j+1} - u_{2j+1}) , \tag{5.13}$$

$$\begin{aligned} a^k b^j(n) - b^j a^k(n) &= u_{k+2j+1} - u_{k+1} - u_{2j+1} + 1 \\ &= \tfrac{2}{3}(2^k - 1)(2^{2j} - 1) . \end{aligned} \tag{5.14}$$

We shall now evaluate A_w and B_w explicitly. For w as given by (5.1) we define the weight of w by means of

$$p = p(w) = \alpha + 2\beta + 2\gamma + \alpha' + 2\beta' + 2\gamma' + \cdots . \tag{5.15}$$

We shall show that

$$A_w = u_p, \qquad B_w = 2u_{p-1} . \tag{5.16}$$

The proof is by induction on p. For p = 1, (5.16) obviously holds. Assume that (5.16) holds up to and including the value p. By the inductive hypothesis, (5.5), (5.6), (5.7) become

$$\begin{cases} A_{wa} = A_p + B_p = u_p + 2u_{p-1} = u_{p+1} \\ B_{wa} = 2A_p = 2u_p \end{cases} \tag{5.17}$$

$$\begin{cases} A_{wb} = A_{wc} = 3A_p + B_p = 3u_p + 2u_{p-1} = u_{p+2} \\ B_{wp} = B_{wc} = 2A_p + 2B_p = 2u_p + 2u_{p-1} = u_{p+1} . \end{cases} \tag{5.18}$$

This evidently completes the induction.

As for C_w, we have

$$\begin{cases} C_{wa} = 2u_p + C_w \\ C_{wb} = C_w \\ C_{wc} = u_{p+1} + C_w . \end{cases} \tag{5.19}$$

Unlike A_w and B_w, the coefficient C_w is not a function of the weight alone. For example

$$C_{a^2} = 2, \qquad C_b = 0, \qquad C_c = 1 ,$$

$$C_{a^3} = 4, \qquad C_{ab} = 0, \qquad C_{ac} = 1 .$$

Repeated application of (5.19) gives

$$\begin{aligned} C_{a^k} &= 2(u_1 + u_2 + \cdots + u_{k-1}) = u_{k+1} - 1 \\ C_{b^k} &= 0 \\ C_{c^k} &= u_1 + \cdots + u_k = \tfrac{1}{2}(u_{k+2} - 1) , \end{aligned}$$

of which the first two agree with (5.8) and (5.11).

We may state

Theorem 5.3. If w is a word of weight p, then

$$w(n) - u_p a(n) + 2u_{p-1}n - C_w , \tag{5.20}$$

where C_w can be evaluated by means of (5.19). If w, w' are any words of equal weight, then

$$w(n) - w'(n) = C_{w'} - C_w . \tag{5.21}$$

Theorem 5.4. For any word w, the representation

$$w = a^{\alpha} b^{\beta} c^{\gamma} a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$$

is unique.

Proof. The theorem is a consequence of the following observation. If u, v are any words, then it follows from any one of

$$ua = va, \qquad ub = vb, \qquad uc = vc$$

that $u = v$.

Theorem 5.5. The words u, v satisfy $uv = vu$ if and only if there is a word w such that

$$u = w^r, \qquad v = w^s ,$$

where r, s are non-negative integers.

Theorem 5.6. In the notation of Theorem 5.3, $C_w = C'_w$ if and only if $w = w'$.

Remark. It follows from (5.20) that no multiple of $d'(n)$ is a word function.

6. GENERATING FUNCTIONS

Put

$$A(x) = \sum_{n=1}^{\infty} x^{a(n)}, \qquad B(x) = \sum_{n=1}^{\infty} x^{b(n)}, \qquad C(x) = \sum_{n=1}^{\infty} x^{c(n)} \tag{6.1}$$

and

$$D(x) = \sum_{n=1}^{\infty} x^{d(n)}, \qquad D_1(x) = \sum_{n=1}^{\infty} x^{d'(n)} , \tag{6.2}$$

where of course $|x| < 1$. Then clearly

$$A(x) + B(x) + C(x) = \frac{x}{1 - x} \tag{6.3}$$

and

$$D(x) + D_1(x) = \frac{x}{1 - x}. \tag{6.4}$$

Since

$$b(n) = c(n) + 1, \qquad d(n) = 2d'(n) ,$$

(6.3) and (6.4) reduce to

$$A(x) + (1 + x)C(x) = \frac{x}{1 - x} , \tag{6.5}$$

and

$$D_1(x) + D_1(x^2) = \frac{x}{1 - x} , \tag{6.6}$$

respectively.

It follows from (6.6) that

$$D_1(x) = \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{r=1}^{\infty} x^{2^k r}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{2^k r=n} (-1)^k ,$$

so that

$$d'(n) = \sum_{2^k r=n} (-1)^k .$$

This is equivalent to the result previously obtained that

$$d'(\underset{\sim}{N}) = \{2^m M \mid m \text{ even}, \ M \text{ odd}\}.$$

Theorem 6.1. Each of the functions $A(x)$, $B(x)$, $C(x)$, $D(x)$, $D_1(x)$ has the unit circle as a natural boundary.

Proof. It will evidently suffice to prove the theorem for $A(x)$ and $D_1(x)$. We consider first the function $D_1(x)$.

To begin with, $D_1(x)$ has a singularity at $x = 1$. Hence, by (6.6), $D_1(x)$ has a singularity at $x = -1$. Replacing x by x^2, (6.6) becomes

$$D_1(x^2) + D_1(x^4) = \frac{x^2}{1-x^2} .$$

We infer that $D_1(x)$ has singularities at $x = \pm i$. Continuing in this way we show that $D_1(x)$ has singularities at

$$x = e^{2k\pi i/2^n} \qquad (k = 1, 3, 5, \cdots, 2^n - 1; \ n = 1, 2, 3, \cdots) .$$

This proves that $D_1(x)$ cannot be continued analytically across the unit circle.

In the next place if the function

$$f(x) = \sum_{n=1}^{\infty} c_n x^n ,$$

where the $c_n = 0$ or 1, can be continued across the unit circle, then [1, p. 315]

$$f(x) = \frac{P(x)}{1 - x^k},$$

where $P(x)$ is a polynomial and k is some positive integer. Hence

$$c_n = c_{n-k} \qquad (n \geq n_0). \tag{6.7}$$

Now assume that $A(x)$ can be continued across the unit circle. Then by (6.7), there exists an integer k such that

$$a(n) = a(n_1) + k \qquad (n > n_0),$$

where n_1 depends on n. It follows that

$$a(n) = a(n - r) + k \qquad (n > n_0) \tag{6.8}$$

for some fixed r. This implies

$$d(n) = a(n - r) + k + r \qquad (n > u_0). \tag{6.9}$$

However (6.9) contradicts the fact that $D(x) = D_1(x^2)$ cannot be continued across the unit circle.

Theorem 6.2. Let $w(n)$ be an arbitrary word function of positive weight and put

$$F_w(x) = \sum_{n=1}^{\infty} x^{w(n)}. \tag{6.10}$$

Then $F_w(x)$ cannot be continued across the unit circle.

Proof. Assume that $F_w(x)$ does admit of analytic continuation across the unit circle. Then there exist integers r, k such that

$$w(n) = w(n - r) + s \qquad (n > n_0).$$

By (5.2) this becomes

$$A_w a(n) + B_w r = A_w(n - r) + k.$$

This implies

$$A_w d(n) = A_w d(n - r) + (A_w - B_w)r + k. \tag{6.11}$$

Since $A_w > 0$, (6.11) contradicts the fact that $D(x)$ cannot be continued.

Put

$$E(x) = \sum_{n=1}^{\infty} x^{e(n)}. \tag{6.12}$$

Then, by Theorem 3.4,

$$E(x) = \frac{x}{1-x} + 2A(x) . \tag{6.13}$$

Also

$$(1 - x)^{-1}A(x) = \sum_{n=1}^{\infty} e(n)\, x^n . \tag{6.14}$$

In the next place, by (3.8), (3.9), and (3.10),

$$A(x) = \sum_1^{\infty} x^{a^2(n)} + \sum_1^{\infty} x^{ab(n)} + \sum_1^{\infty} x^{ac(n)}$$

$$= x^{-2}B(x) + (1 + x^{-1})F_{ab}(x) .$$

Since

$$A(x) + (1 + x^{-1})B(x) = \frac{x}{1-x} ,$$

it follows that

$$(1 + x)^2 F_{ab}(x) = (1 + x + x^2)A(x) - \frac{x}{1-x} . \tag{6.15}$$

Let w, w' be two words of equal weight. Then by (5.21),

$$x^{C_w}F_w(x) = x^{C_{w'}}F_{w'}(x) . \tag{6.16}$$

Thus it suffices to consider the functions

$$F_{a^k}(x) \qquad (k = 1, 2, 3, \cdots) .$$

We have

$$F_{a^k}(x) = F_{a^{k-1}}(x) + F_{a^k b}(x) + F_{a^k c}(x) .$$

By (5.8)

$$\begin{aligned} a^k b(n) &= u_k ab(n) + 2u_{k-1}b(n) - (u_{k+1} - 1) \\ &= u_k(3a(n) + 2n) + 2u_{k-1}(a(n) + 2n) - (u_{k+1} - 1) \\ &= (3u_k + 2u_{k-1})a(n) + 2(u_k + 2u_{k-1})n - (u_{k+1} - 1) \\ &= u_{k+2}a(n) + 2u_{k+1}n - (u_{k+1} - 1) \\ &= a^{k+2}(n) + 2^{k+1} \end{aligned}$$

$$\begin{aligned} a^k c(n) &= u_k ac(n) + 2u_{k-1}c(n) - (u_{k+1} - 1) \\ &= u_k(3a(n) + 2n - 1) + 2u_{k-1}(a(n) + 2n - 1) - (u_{k+1} - 1) \\ &= u_{k+2}a(n) + 2u_{k+1}n - (2u_{k+1} - 1) \\ &= a^{k+2}(n) + u_{k+2} . \end{aligned}$$

[Continued on page 550.]

SPECIAL CASES OF FIBONACCI PERIODICITY

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1. INTRODUCTION

This paper will deal with the periodicity of Fibonacci sequences; where the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$; the Lucas sequence

$$\{L_n\}_{n=0}^{\infty}$$

is defined with $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$; and the generalized Fibonacci sequence $\{H_n\}_{n=0}^{\infty}$ has any two starting values with $H_{n+2} = H_{n+1} + H_n$. We will see that in one case, that of modulo 2^n, all generalized Fibonacci sequences will have the same period. In a second case, that of modulo 5^n, different sequences will have different periods. We will also consider the periods modulo 10^n. In each case except that of 10^n, the method of proof will be to show that with sequence $\{A_n\}$, modulus m, and period p, then $A_{n+p} \equiv A_n \pmod m$ and $A_{n+1+p} \equiv A_{n+1} \pmod m$. Identities in the proof may be found in [1].

2. THE FIBONACCI CASE MOD 2^n

Theorem 1. The period of the Fibonacci sequence modulo 2^n is $3 \cdot 2^{n-1}$. We will prove that: (A) $F_{3\cdot 2^{n-1}} \equiv F_0 \pmod{2^n}$ and (B) $F_{3\cdot 2^{n-1}+1} \equiv F_1 \pmod{2^n}$.

A. The proof is by induction.

(1) When n = 1, $F_{3\cdot 2^{\ell-1}} = F_3 = 2 \equiv 0 \pmod{2^\ell}$.

(2) Suppose $F_{3\cdot 2^{k-1}} \equiv 0 \pmod{2^k}$.

(3) Now, $F_{3\cdot 2^k} = F_{3\cdot 2^{k-1}} L_{3\cdot 2^{k-1}}$ from the identity $F_{2n} = F_n L_n$.

(4) We claim $L_{3k} \equiv 0 \pmod 2$.
The proof is by induction.

(5) When k = 1, $L_{3\cdot 1} = 4 \equiv 0 \pmod 2$.

(6) Suppose $L_{3m} \equiv 0 \pmod 2$.

(7) $L_{3(m+1)} = 2L_{3m+1} + L_{3m} \equiv 0 \pmod 2$
and statement (4) is established.

Using (3), with the induction hypothesis (2), and (4), it follows that

(8) $F_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$
and Part A is proved.

B. (9) First, $F_{3\cdot 2^{n-1}+1} = (F_{3\cdot 2^{n-2}+1})^2 + (F_{3\cdot 2^{n-2}})^2$
using the identity $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$. Now, since $F_{3\cdot 2^{n-1}} \equiv 0 \pmod{2^{n-1}}$ from Part A, it follows that

(10) $$(F_{3 \cdot 2^{n-2}})^2 \equiv 0 \pmod{2^n}.$$

(11) Also $$(F_{3 \cdot 2^{n-2}+1})^2 \equiv 1 \pmod{2^n}$$
from the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (10).

Part B follows from these three steps.

3. THE GENERAL FIBONACCI CASE MOD 2^n

Theorem 2. The period of any generalized Fibonacci sequence modulo 2^n is $3 \cdot 2^{n-1}$. We will prove that: (A) $H_{3 \cdot 2^{n-1}+1} \equiv H_1 \pmod{2^n}$ and (B) $H_{3 \cdot 2^{n-1}+2} \equiv H_2 \pmod{2^n}$.

A. We will have to consider three cases.

Case 1: $n = 1$. $H_{3 \cdot 2^{1-1}+1} = H_4 = 2H_2 + H_1 \equiv H_1 \pmod{2^1}$.

Case 2: $n = 2$. $H_{3 \cdot 2^{2-1}+1} = H_7 = 3H_2 + 5H_1 \equiv H_1 \pmod{2^2}$.

Case 3: $n > 2$.

(12) First, $H_{3 \cdot 2^{n-1}+1} = H_{3 \cdot 2^{n-2}+1}F_{3 \cdot 2^{n-2}+1} + H_{3 \cdot 2^{n-2}}F_{3 \cdot 2^{n-2}}$,
from the identity $H_{m+n+1} = H_{m+1}F_{n+1} + H_mF_n$.

(13) We need the fact that $F_{3 \cdot 2^{n-2}} \equiv 0 \pmod{2^n}$ for $n > 2$, which can be proved by induction in the manner of the proof of 1-A.

(14) Next we claim $H_{3 \cdot 2^{n-2}}F_{3 \cdot 2^{n-2}+1} \equiv H_1 \pmod{2^n}$ for $n > 2$.
Since $H_{n+1} = H_1F_{n-1} + H_2F_n$, we can multiply both sides by F_{n+1}

(15) so $$H_{3 \cdot 2^{n-2}+1}F_{3 \cdot 2^{n-2}+1} = H_1F_{3 \cdot 2^{n-2}-1}F_{3 \cdot 2^{n-2}+1} + H_2F_{3 \cdot 2^{n-2}}F_{3 \cdot 2^{n-2}+1}.$$

(16) Now, $$F_{3 \cdot 2^{n-2}-1}F_{3 \cdot 2^{n-2}+1} \equiv 1 \pmod{2^n} \quad n > 2$$
using the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (13).

Our claim in (14) follows from (15), (16), and (13) and Case 3 follows from (12), (13), and (16).

B. (17) First, $H_{3 \cdot 2^{n-1}+2} = H_1F_{3 \cdot 2^{n-1}} + H_2F_{3 \cdot 2^{n-1}+1}$
from the identity $H_{n+2} = H_1F_n + F_2F_{n+1}$.
Since $F_{3 \cdot 2^{n-1}} \equiv 1 \pmod{2^n}$ from 1-A, and $F_{3 \cdot 2^{n-1}+1} \equiv 1 \pmod{2^n}$ from 1-B, Part B follows immediately.

One of the key parts in the proof of Theorem 1 is being able to write $F_{3 \cdot 2^k}$ in terms of $F_{3 \cdot 2^{k-1}}$ as in statement (3). For the next theorem, an analogous result is needed for $F_{5^{n+1}}$ in terms of F_{5^n}.

4. THE FIBONACCI CASE MOD 5^n

We need a simple lemma.

Lemma. $$F_{5^{n+1}} = F_{5^n}(L_{4 \cdot 5^n} - L_{2 \cdot 5^n} + 1), \quad n = 1, 2, \cdots.$$

Proof. We will use the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2} \quad .$$

Note that $\alpha\beta = -1$.

$$F_{5^{n+1}} = \frac{\alpha^{5^{n+1}} - \beta^{5^{n+1}}}{\alpha - \beta} = \frac{\alpha^{5^n \cdot 5} - \beta^{5^n \cdot 5}}{\alpha - \beta}$$

$$= \frac{(\alpha^{5^n} - \beta^{5^n})}{\alpha - \beta}\,(\alpha^{5^n \cdot 4} + \alpha^{5^n \cdot 3}\beta^{5^n} + \alpha^{5^n \cdot 2} + \beta^{5^n \cdot 2} + \alpha^{5^n}\beta^{5^n \cdot 3} + \beta^{5^n \cdot 4})$$

$$= \frac{(\alpha^{5^n} - \beta^{5^n})}{\alpha - \beta}\,[\alpha^{5^n \cdot 4} + \beta^{5^n \cdot 4} + (\alpha\beta)^{5^n}(\alpha^{5^n \cdot 2} + \beta^{5^n \cdot 2}) + (\alpha\beta)^{5^n \cdot 2}]$$

$$= F_{5^n}(L_{5^n \cdot 4} - L_{5^n \cdot 2} + 1) \quad .$$

<u>Theorem 3.</u> The period of the Fibonacci numbers modulo 5^n is $4 \cdot 5^n$.

<u>Proof.</u> We will prove that: (A) $F_{4 \cdot 5^n} \equiv F_0 \pmod{5^n}$ and (B) $F_{4 \cdot 5^n + 1} \equiv F_1 \pmod{5^n}$.

A. (18) Since $F_n \mid F_{kn}$, $\quad F_{4 \cdot 5^n} \equiv F_{5^n} \pmod{5^n}$

(19) Next we claim $\quad F_{5^n} \equiv 0 \pmod{5^n}$.

The proof is by induction.

(20) When $n = 1$, $\quad F_{5^1} \equiv F_5 = 5 \equiv 0 \pmod{5^1}$.

(21) Suppose $\quad F_{5^k} \equiv 0 \pmod{5^k}$.

(22) Now, $F_{5^{k+1}} = F_{5^k}(L_{4 \cdot 5^k} - L_{2 \cdot 5^k} + 1)$ from the Lemma.

(23) $\quad L_{4 \cdot 5^k} \equiv 2 \pmod 5$

from the identity $L_{4n} - 2 = 5F_{2n}^2$,

(24) and $\quad L_{2 \cdot 5^k} \equiv -2 \pmod 5$

from the identity $L_{2(2n+1)} + 2 = 5F_{2n+1}^2$.

Using the induction hypothesis (21), with (22), (23), and (24),

(25) $\quad F_{5^{k+1}} \equiv 0 \pmod{5^{k+1}}$

and Part A follows.

B. (26) First $\quad F_{4 \cdot 5^n + 1} = (F_{2 \cdot 5^n + 1})^2 + (F_{2 \cdot 5^n})^2$

using the identity $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

From (19) it follows that

(27) $\quad (F_{2 \cdot 5^n})^2 \equiv 0 \pmod{5^n}$.

(28) Also $\quad (F_{2 \cdot 5^n + 1})^2 \equiv 1 \pmod{5^n}$

using the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (27).

Consequently Part B is proved.

5. THE LUCAS CASE MOD 5^n

<u>Theorem 4.</u> The period of the Lucas numbers modulo 5^n is $4 \cdot 5^{n-1}$.

<u>Proof.</u> We will prove that: (A) $L_{4 \cdot 5^{n-1}} \equiv L_0 \pmod{5^n}$ and (B) $L_{4 \cdot 5^{n-1} + 1} \equiv L_1 \pmod{5^n}$.

A. (29) First $L_{4\cdot 5^{n-1}} = 5(F_{2\cdot 5^{n-1}})^2 + 2$

from the identity $L_{4n} - 2 = 5F^2_{2n}$.

From (19) it can be shown that

(30) $(F_{2\cdot 5^{n-1}})^2 \equiv 0 \pmod{5^{n-1}}$.

(31) So $5(F_{2\cdot 5^{n-1}})^2 \equiv 0 \pmod{5^n}$

and Part A is proved.

B. (32) First $L_{4\cdot 5^{n+1}+2} = 5(F_{2\cdot 5^{n-1}+1})^2 - 2$

from the identity $L_{4n+2} = 5F^2_{2n+1} - 2$.

(33) In a method similar to that used in showing (28), it can be shown that

$$(F_{2\cdot 5^{n-1}+1})^2 \equiv 1 \pmod{5^n}.$$

(34) Therefore $L_{4\cdot 5^{n-1}+2} \equiv 3 \pmod{5^n}$.

(35) From A and (34), $L_{4\cdot 5^{n-1}+2} - L_{4\cdot 5^{n-1}} \equiv 1 \pmod{5^n}$

(36) $L_{4\cdot 5^{n-1}+1} \equiv 1 \pmod{5^n}$ since $L_{n+2} = L_{n+1} + L_n$.

As shown in [2], the periods of the Fibonacci sequences modulo 10^n will be the least common multiple of the periods mod 2^n and mod 5^n. A summary of the periods is below.

Sequence	mod 2^n $n = 1, 2, \cdots$	mod 5^n $n = 1, 2, \cdots$	mod 10	mod 100	mod 10^n $n = 3, 4, \cdots$
Fibonacci $\{F_n\}$	$3\cdot 2^{n-1}$	$4\cdot 5^n$	60	300	$15\cdot 10^{n-1}$
Lucas $\{L_n\}$	$3\cdot 2^{n-1}$	$4\cdot 5^{n-1}$	12	60	$3\cdot 10^{n-1}$
Generalized Fibonacci $\{H_n\}$	$3\cdot 2^{n-1}$	variable	variable	variable	variable

6. SOME PARTING OBSERVATIONS

We note in passing that we have found some solutions to $n | F_n$ in the statement $F_{5^n} \equiv 0 \bmod 5^n$. To this we add two statements also involving solutions to $L_n \equiv 0 \bmod n$.

<u>Theorem:</u> $L_n \equiv 1 \bmod n$ for n a prime.

<u>Theorem:</u> $L_{2\cdot 3^k} \equiv 0 \bmod 2\cdot 3^k$, $k = 1, 2, 3, \cdots$

<u>Theorem:</u> $F_{2^2\cdot 3^k} \equiv 0 \bmod 2^2\cdot 3^k$, $k = 1, 2, 3, \cdots$

A new paper by Hoggatt and Bicknell will further discuss these ideas.

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[Continued on page 530.]

CONCAVITY PROPERTIES OF CERTAIN SEQUENCES OF NUMBERS

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A set of non-negative real numbers C_k $(k = 1, 2, \cdots, N)$ is said to be <u>unimodal</u> if there exists an integer n such that

$$\begin{aligned} C_k &\leq C_{k+1} \qquad (1 \leq k < n) \\ C_k &\geq C_{k+1} \qquad (n \leq k < N) \;. \end{aligned}$$

A stronger property is <u>logarithmic concavity</u>:

$$C_k^2 \geq C_{k+1} C_{k-1} \qquad (1 < k < N) \,. \tag{1}$$

<u>Strong logarithmic concavity</u> (SLC) means that the inequality in (1) is strict for all k.

In a recent paper, Lieb [1] has proved that the Stirling numbers of the second kind

$$S(N,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^N$$

have the SLC property. The proof makes use of Newton's inequality. If the polynomial

$$Q(x) = \sum_{k=1}^{N} C_k x^k \tag{2}$$

has only real roots, then

$$C_k^2 \geq \frac{k(N - k + 1)}{(k - 1)(N - k)} C_{k+1} C_{k-1} \qquad (1 < k < N) \,.$$

In view of the above, it is of some interest to exhibit sequences $\{C_k\}$ with the SLC property for which the corresponding polynomial does not have the SLC property. Such an example is furnished by

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$$(1 + x + x^2)^n = \sum_{k=0}^{2n} c(n,k)x^k . \tag{3}$$

It follows at once from (3) that $c(n,k)$ satisfies the recurrence

$$c(n + 1,k) = c(n,k - 2) + c(n,k - 1) + c(n,k) . \tag{4}$$

We shall show first that, for $n \geq 2$,

$$c(n,k) < c(n,k + 1) \qquad (0 \leq k < n) , \tag{5}$$

$$c(n,k) > c(n,k + 1) \qquad (n \leq k < 2n) . \tag{6}$$

Since

$$c(n,k) = c(n, 2n - k) , \tag{7}$$

(5) and (6) are equivalent so that it suffices to prove (5). Since

$$(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^2 ,$$

it is clear that (5) holds for $n = 2$. Assume that (5) holds for $2 \leq n \leq m$. Then, for $k < m$,

$$c(m + 1,k + 1) - c(m + 1,k) = c(m,k + 1) - c(m,k - 2) > 0 .$$

For $k = m$ we have

$$\begin{aligned} &c(m + 1,m + 1) - c(m + 1,m) \\ &\qquad = 2c(m,m - 1) + c(m,m) - [c(m,m - 2) + c(m,m - 1) + c(m,m)] \\ &\qquad = c(m,m - 1) - c(m,m - 2) > 0 . \end{aligned}$$

This completes the proof of (5).

We remark that $c(n,n)$ satisfies

$$\sum_{n=0}^{\infty} c(n,n)x^n = (1 - 2x - 3x^2)^{-\frac{1}{2}} .$$

For proof see [2, p. 126, No. 217].

We shall now show that, for $n \geq 2$,

$$c^2(n,k) > c(n,k + 1)c(n,k - 1) \qquad (0 < k < 2n) . \tag{8}$$

This holds for $n = 2$. We assume that (8) holds for $2 \leq n \leq m$.

Note that (8) implies

$$\text{(9)} \qquad c(n,j)c(n,k) > c(n,j-1)c(n,k+1) \qquad (0 < j \leq k < 2n)\,.$$

Indeed, by (8)

$$\frac{c(n,k)}{c(n,k+1)} > \frac{c(n,k-1)}{c(n,k)}\,,$$

which implies

$$\frac{c(n,k)}{c(n,k-1)} \qquad \frac{c(n,j-1)}{c(n,j)}\,.$$

Thus, for $0 < k < 2m$,

$$\begin{vmatrix} c(m+1,k) & c(m+1,k+1) \\ c(m+1,k-1) & c(m+1,k) \end{vmatrix}$$

$$= \begin{vmatrix} c(m,k-2) + c(m,k-1) + c(m,k) & c(m,k-1) + c(m,k) + c(m,k+1) \\ c(m,k-3) + c(m,k-2) + c(m,k-1) & c(m,k-2) + c(m,k-1) + c(m,k) \end{vmatrix}$$

$$= \begin{vmatrix} c(m,k-2) & c(m,k-1) \\ c(m,k-3) & c(m,k-2) \end{vmatrix} + \begin{vmatrix} c(m,k-2) & c(m,k) \\ c(m,k-3) & c(m,k-1) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k-2) & c(m,k+1) \\ c(m,k-3) & c(m,k) \end{vmatrix} + \begin{vmatrix} c(m,k-1) & c(m,k) \\ c(m,k-2) & c(m,k-1) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k-1) & c(m,k+1) \\ c(m,k-2) & c(m,k) \end{vmatrix} + \begin{vmatrix} c(m,k) & c(m,k-1) \\ c(m,k-1) & c(m,k-2) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k) & c(m,k+1) \\ c(m,k-1) & c(m,k) \end{vmatrix}\,.$$

The fourth and sixth determinants cancel while each of the remaining five is positive by (9). Hence

$$c^2(m+1,k) > c(m+1,k-1)c(m+1,k) \qquad (0 < k < 2m)\,.$$

As for the excluded values, we have by (7)

$$c^2(m+1,2m) - c(m+1,2m-1)c(m+1,2m-1) = c^2(m+1,2) - c(m+1,3)c(m+1,1) > 0,$$
$$c^2(m+1,2m+1) - c(m+1,2m)c(m+1,2m+2) = c^2(m+1,1) - c(m+1,2)c(m+1,0) > 0\,.$$

In a similar way we can show that the coefficients of $c_r(n,k)$ defined by

$$(1 + x + \cdots + x^r)^n = \sum_{k=0}^{nr} c_r(n,k)\, x^k$$

have the SLC property for $n \geq 2$.

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A CONJECTURE CONCERNING LUCAS NUMBERS

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Anaya and Crump (now Anaya and Anaya) [1] have proved that

$$[a^k F_n + \tfrac{1}{2}] = F_{n+k} \qquad (n \ge k \ge 1),$$

where $a = \frac{1}{2}(1+\sqrt{5})$ and $[x]$ denotes the greatest integer $\le x$. They remark that it seems reasonable that

$$[a^k L_n + \tfrac{1}{2}] = L_{n+k},$$

when n is somewhat greater than k.

We shall show that

$$(1) \qquad [a^k L_n + \tfrac{1}{2}] = L_{n+k} \qquad (n \ge k+2,\ k \ge 2).$$

Moreover, for $k = 1$,

$$(2) \qquad [a L_n + \tfrac{1}{2}] = L_{n+1} \qquad (n \ge 4).$$

To prove (1), it suffices to show that

$$(3) \qquad \left| a^k L_n - L_{n+k} \right| < \tfrac{1}{2} \qquad (n \ge k+2,\ k \ge 2),$$

that is,

$$(4) \qquad \left| b^n (a^k - b^k) \right| < \tfrac{1}{2} \qquad (n \ge k+2,\ k \ge 2),$$

where we have used

$$L_n = a^n + b^n, \qquad b = \tfrac{1}{2}(1 - \sqrt{5}).$$

Clearly (4) is satisfied if

$$a^{-n}(a^k + a^{-k}) < \tfrac{1}{2} \qquad (n \ge k+2,\ k \ge 2).$$

Thus it is enough to show that

$$a^{-k-2}(a^k + a^{-k}) < \tfrac{1}{2} \qquad (k \ge 2),$$

that is,

$$(5) \qquad a^{-2} + a^{-2k-2} < \tfrac{1}{2} \qquad (k \ge 2).$$

Since

$$a^{-2} + a^{-6} = \frac{3 - \sqrt{5}}{2} - 9 - 4\sqrt{5} = \tfrac{1}{2}(21 - 9\sqrt{5}) < \tfrac{1}{2},$$

[Continued on page 550.]

ADDENDUM TO THE PAPER "FIBONACCI REPRESENTATIONS"

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1. The presentation and investigation of the functions a and b given in the paper cited in the title [1] can be simplified if we consider the following: Every positive integer N has a unique representation in the form

$$N = \delta_2 F_2 + \delta_3 F_3 + \cdots , \tag{1}$$

where δ_i is either 0 or 1 and $\delta_i \delta_{i+1} = 0$. This canonical or Zeckendorf representation may be written more briefly

$$N = \cdot\, \delta_2 \delta_3 \delta_4 \delta_5 \cdots . \tag{2}$$

Let A be the sequence of length 1 consisting of a 0, A = (0), and let B be the sequence of length 2, B = (1,0). Clearly, then, N can be written uniquely as a sequence of A's and B's, and any sequence of A's and B's, infinite on the right, containing only a finite number of B's, represents a non-negative integer. We may regard A and B as functions. For instance A(N) is to be the sequence obtained by adjoining A to the left of the sequence representing N, and similarly for B(N).

Then we see immediately that

$$N + A(N) + 1 = B(N), \qquad (N \geq 0) . \tag{3}$$

Now define

$$\begin{cases} a(N) = A(N-1) + 1 & (N \geq 1) \\ b(N) = B(N-1) + 1 & (N \geq 1) \end{cases} . \tag{4}$$

Then (3) becomes

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$$N + a(N) = b(N), \qquad N \geq 1. \tag{5}$$

Hence properties (2.2), (2.3) and (2.4) of [1] are easily verified, so we have, in fact,

$$\begin{cases} a(N) = [\alpha N] \\ b(N) = [\alpha^2 N], \qquad \alpha = (1 + \sqrt{5})/2 \end{cases} \tag{6}$$

as before, ((1.6) of [1]).

The advantage of introducing A and B appears when we calculate e(a) and e(b). We have

$$\begin{cases} e\bigl(a(N)\bigr) = e\bigl(A(N-1)+1\bigr) = e\bigl(A(N-1)\bigr) + 1 = N \\ e\bigl(b(N)\bigr) = e\bigl(B(N-1)+1\bigr) = 1 + A(N-1) = a(N) \end{cases} \tag{7}$$

The function e is defined by (1.7) in [1]:

$$e(\delta_2 F_2 + \delta_3 F_3 + \cdots) = \delta_2 F_1 + \delta_3 F_2 + \cdots. \tag{8}$$

To obtain (7) we have used the fact that e(N) is independent of the Fibonacci representation chosen for N.

It is also useful to define E(N) by means of

$$e(N) = E(N-1) + 1; \tag{9}$$

this definition may be compared with (4). Let N have the canonical representation (1) and consider

$$N + 1 = 1 + \cdot\, \delta_2\delta_3\delta_4 \cdots. \tag{10}$$

If $\delta_2 = 0$ we may write

$$N + 1 = \cdot 1\, \delta_3\delta_4 \cdots.$$

This representation may not be canonical. However, by (8) we have

$$e(N + 1 = 1 + \cdot\, \delta_3\delta_4\delta_5 \cdots.$$

Hence, by (8) and (9),

$$E(N) = \cdot\, \delta_3 \delta_4 \delta_5 \cdots \tag{11}$$

If $\delta_2 = 1$, then $\delta_3 = 0$ and we get

$$N + 1 = \cdot\, 01\, \delta_4 \delta_5 \cdots .$$

Again this representation may not be canonical but, by (8),

$$e(N + 1) = \cdot\, 1\, \delta_4 \delta_5 \cdots = 1 + \cdot\, \delta_3 \delta_4 \delta_5 \cdots .$$

It follows that

$$E(N) = \cdot\, \delta_3 \delta_4 \delta_5 \cdots .$$

Thus in any case if N has the canonical representation (1), $E(N)$ is determined by (11).

To sum up we state the following.

Theorem. Let N have the canonical representation

$$N = \cdot\, \delta_2 \delta_3 \delta_4 \cdots .$$

Then

$$\begin{aligned} A(N) &= \cdot\, 0\, \delta_2 \delta_3 \delta_4 \cdots \\ B(N) &= \cdot\, 10\, \delta_2 \delta_3 \delta_4 \cdots \\ E(N) &= \cdot\, \delta_3 \delta_4 \delta_5 \cdots \end{aligned} .$$

2. Similar observations may be made for Fibonacci representations of higher order. For instance, if we put

$$A = (0), \qquad B = (10), \qquad C = (110), \tag{12}$$

then the relations between A, B, C and a, b, c of [2] are given by

$$\begin{cases} a(N) = A(N - 1) + 1 \\ b(N) = B(N - 1) + 1 \\ c(N) = C(N - 1) + 1 \end{cases} , \tag{13}$$

where $N \geq 1$.

3. By Theorem 11 of [1]

$$(14) \qquad \begin{cases} N \in (a) \rightleftarrows 0 < \left\{\dfrac{N}{\alpha^2}\right\} < \frac{1}{2}\,, \\ N \in (b) \rightleftarrows \dfrac{1}{\alpha} < \left\{\dfrac{N}{\alpha^2}\right\} < 1\,, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. The possibility $\{N/\alpha^2\} = 1/\alpha$ never occurs.

We should like to point out that (14) can be replaced by the following slightly simpler criterion.

$$(15) \qquad \begin{cases} N \in (a) \rightleftarrows \{\alpha N\} > \dfrac{1}{\alpha^2} \\ N \in (b) \rightleftarrows \{\alpha N\} < \dfrac{1}{\alpha^2} \end{cases} .$$

As above, $\{\alpha N\} = 1/\alpha^2$ is impossible.

To see that (14) and (15) are equivalent, it suffices to observe that

$$\left\{\frac{N}{\alpha^2}\right\} = \{(2 - \alpha)N\} = 1 - \{\alpha N\} .$$

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[Continued from page 525.]

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SEQUENCES, PATHS, BALLOT NUMBERS

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1. INTRODUCTION

Consider the sequences of positive integers $(a_1, a_2, \cdots, a_n)$ that satisfy the following conditions:

$$1 = a_1 \le a_2 \le \cdots \le a_n \tag{1.1}$$

and

$$a_i \le i \qquad (1 \le i \le n). \tag{1.2}$$

The number of such sequences with $a_n = k$, where k is fixed, $1 \le k \le n$, will be denoted by $f(n,k)$. Thus the total number of sequences satisfying (1.1) and (1.2) is equal to

$$\sum_{k=1}^{n} f(n,k) \quad . \tag{1.3}$$

The numbers $f(n,k)$ were called two-element lattice permutations by MacMahon [6, p. 167]. Two-element lattice permutations have n elements of one kind, k of a second kind with $k \le n$, and are such that if a_r is the number of the first kind in the first r and b_r is the corresponding number of the second kind, then $a_r \ge b_r$ for every r. Another way of putting it is that the elements of the first kind are thought of as votes for candidate A, those of the second kind as votes for candidate B; the lattice permutation is then an election return with final vote (n,k) which is such that all partial returns correctly predict the winner. As still another interpretation, let each element of the first kind be represented as a unit horizontal line and each of the second kind as a unit vertical line, then the permutation represents a path from $(1,1)$ to (n,k) which does not cross the line $y = k$. The illustration at the top of the following page shows an admissible path from $(1,1)$ to $(7,6)$.

In the present paper we discuss some of the basic properties of $f(n,k)$ and related functions. We also discuss briefly some extensions, in particular the q-analog [3]

$$f(n,k,q) = \sum q^{a_1+a_2+\cdots+a_n} , \tag{1.4}$$

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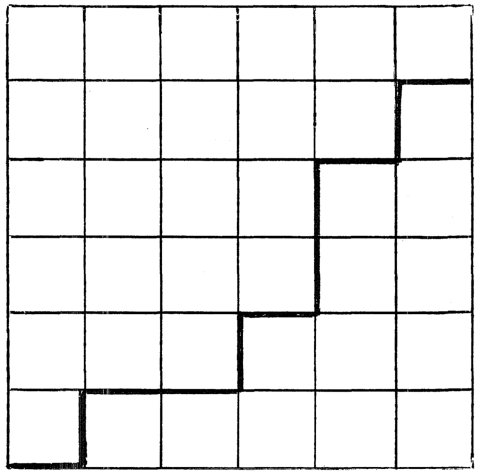

where the summation is over all $a_1, a_2, \cdots, a_n$ such that

$$1 = a_1 \le a_2 \le \cdots \le a_n = k$$

and

$$a_i \le i \qquad (1 \le i \le n),$$

furnishes a useful generalization of $f(n,k)$. Many of the properties of $f(n,k)$ carry over to the general case.

The list of references at the end of the paper is by no means complete. A comprehensive bibliography of Catalan numbers is included in [1].

2. It follows at once from the definition that

$$f(n,k) = \sum_{j=1}^{k} f(n-1,j) \qquad (1 \le k \le n;\ n > 1), \tag{2.1}$$

where it is understood that $f(n-1, n) = 0$. From (2.1) we get

$$f(n,k) = f(n, k-1) + f(n-1, k), \tag{2.2}$$

where again $1 \le k \le n$, $n > 1$.

Making use of either (2.1) or (2.2) we can easily compute the table shown at the top of the following page.

It is evident from (2.1) that the total number of sequences satisfying (1.1) and (1.2) is equal to

$$f(n+1, n+1) = f(n+1, n) \quad. \tag{2.3}$$

$f(n,k)$:

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	2				
4	1	3	5	5			
5	1	4	9	14	14		
6	1	5	14	28	42	42	
7	1	6	20	48	90	132	132

We now define

$$b(n,k) = \sum_{j=1}^{k} 2^{k-j} f(n,j) \quad . \tag{2.4}$$

Using (2.2) we get

$$\begin{aligned} b(n,k) &= \sum_{j=1}^{k} 2^{k-j} \{f(n,\ j-1) + f(n-1,\ j)\} \\ &= \sum_{j=1}^{k-1} 2^{k-j-1} f(n,j) + \sum_{j=1}^{k} 2^{k-j} (f(n-1,\ j) \quad , \end{aligned}$$

so that

$$b(n,k) = b(n,\ k-1) + b(n-1,\ k) \qquad (1 \le k < n) \ . \tag{2.5}$$

However, for $k = n$, we get

$$b(n,n) = b(n,\ n-1) + 2b(n-1,\ n-1) \ . \tag{2.6}$$

It follows from (2.5) that

$$b(n,k) = \sum_{j=1}^{k} b(n-1,\ j) \qquad (1 \le k \le n) \ ; \tag{2.7}$$

however

$$b(n,n) = \sum_{j=1}^{n-1} b(n-1,\ j) + 2b(n-1,\ n-1) \ . \tag{2.8}$$

The table shown at the top of the following page is easily computed using either (2.4) or (2.5) and (2.6).

Examination of the table suggests the following formula.

$$b(n,k) = \binom{n+k-1}{k-1} \qquad (1 \le k \le n) \ . \tag{2.9}$$

It is clear from (2.4) that

b(n, k) :

n \ k	1	2	3	4	5	6
1	1					
2	1	3				
3	1	4	10			
4	1	5	15	35		
5	1	6	21	56	126	
6	1	7	28	84	210	462

(2.10) $$b(n,1) = 1 \qquad (n = 1, 2, 3, \cdots),$$

in agreement with (2.9). Assume that (2.9) holds for $n = 1, 2, \cdots, m$ and $1 \le k \le n$. Then by (2.7), for $k \le m$,

$$b(m+1, k) = \sum_{j=1}^{k} b(m,j) = \sum_{j=1}^{k} \binom{m+j-1}{j-1} = \sum_{j=1}^{k} \binom{m+j-1}{m}$$
$$= \binom{m+k}{m+1} = \binom{m+k}{k-1}.$$

On the other hand, by (2.8),

$$b(m+1, m+1) = \sum_{j=1}^{m} b(m,j) + 2b(m,m)$$
$$= \sum_{j=1}^{m} \binom{m+j-1}{j-1} + 2\binom{2m-1}{m-1}$$
$$= \binom{2m}{m-1} + \binom{2m}{m} = \binom{2m+1}{m}.$$

This evidently completes the proof of (2.9).

Returning to (2.4), it is evident that

(2.11) $$f(n,k) = b(n,k) - 2b(n, k-1) \qquad (1 \le k \le n).$$

Therefore, by (2.9),

(2.12) $$f(n,k) = \binom{n+k-1}{k-1} - 2\binom{n+k-2}{k-2} = \frac{n-k+1}{n}\binom{n+k-2}{n-1} \qquad (1 \le k \le n).$$

In particular, for $k = n$,

(2.13) $$f(n,n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Thus, by (2.3), the number of sequences that satisfy (1.1) and (1.2) is equal to the Catalan number

$$c(n) = \frac{1}{n+1}\binom{2n}{n} . \tag{2.14}$$

Making use of (2.12), it is easy to verify that

$$\begin{vmatrix} f(n, k) & f(n, k+1) \\ f(n+1, k) & f(n+1, k+1) \end{vmatrix} > 0 \tag{2.15}$$

and

$$f^2(n,k) > f(n, k-1)f(n, k+1) . \tag{2.16}$$

3. Put

$$C(x) = \sum_{n=0}^{\infty} c(n)\, x^n . \tag{3.1}$$

Since

$$\begin{aligned}(1 - 4x)^{\frac{1}{2}} &= \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} (4x)^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n \\ &= 1 - 2x\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n ,\end{aligned}$$

it follows from (2.14) and (3.1) that

$$C(x) = \frac{1}{2x}(1 - \sqrt{1 - 4x}) . \tag{3.2}$$

Thus

$$xC^2(x) = C(x) - 1 , \tag{3.3}$$

which is equivalent to

$$c(n+1) = \sum_{k=0}^{n} c(k)c(n-k) . \tag{3.4}$$

In a letter to the writer, Dr. Jürg Rätz had inquired about the possibility of proving (3.4) without the use of generating functions. This can be done in the following way. Put

$$\frac{(x+\frac{1}{2})_n}{(x)_{n+1}} = \sum_{k=0}^{n} \frac{A_k}{x+k} ,$$

where

$$(a)_n = a(a+1)\cdots(a+n-1)\,.$$

Then

$$A_k\left[\frac{(x)_{n+1}}{x+k}\right]_{x=-k} = \binom{\frac{1}{2}-k}{}$$

and a little manipulation leads to

$$2^{2n}A_k = \binom{2k}{k}\binom{2n-2k}{n-k}\,.$$

Thus we have proved the identity

$$2^{2n}\,\frac{(x+\frac{1}{2})_n}{(x)_{n+1}} = \sum_{k=0}^{n}\frac{1}{x+k}\binom{2k}{k}\binom{2n-2k}{n-k}\,. \tag{3.5}$$

It is easily verified that

$$\frac{2^{2n}(x+\frac{1}{2})_n}{(x)_{n+1}} = \frac{(2x)_{2n}}{(x)_n(x+1)_n}\,,$$

so that (3.5) becomes

$$\sum_{k=0}^{n}\frac{1}{x+k}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{(2x)_{2n}}{(x)_n\cdot(x+1)_n}\,.$$

In particular, for $x = 1$, we have

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{(2n+1)!}{n!\,(n+1)!} = \binom{2n+1}{n}\,.$$

It follows that

$$\binom{2n+2}{n+1} = 2\binom{2n+1}{n} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} + \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$$

$$= \sum_{k=0}^{n}\frac{n+2}{(k+1)(n-k+1)}\binom{2k}{k}\binom{2n-2k}{n-k} = (n+2)\sum_{k=0}^{n}c_k\,c_{n-k}\,.$$

This evidently proves (3.4).

4. We now define

$$F(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f(n,k)\, x^n y^k . \tag{4.1}$$

Then, by (2.1),

$$\begin{aligned}
F(x,y) &= xy + x \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} f(n+1,k)\, x^n y^k \\
&= xy + x \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} \sum_{j=1}^{k} f(n,j)\, x^n y^k \\
&= xy + x \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^j \sum_{k=j}^{n+1} y^{k-j} \\
&= xy + \frac{x}{1-y} \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^j (1 - y^{n-j+2}) \\
&= xy + \frac{x}{1-y} F(x,y) - \frac{xy^2}{1-y} \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^n .
\end{aligned}$$

Since, by (2.13) and (2.14),

$$\sum_{j=1}^{n} f(n,j) = f(n+1,\, n+1) = c(n) ,$$

we get

$$(1 - x - y)F(x,y) = xy(1-y) - xy^2 \sum_{n=1}^{\infty} c(n)(xy)^n$$

and therefore

$$(1 - x - y)F(x,y) = xy - xy^2 C(xy) . \tag{4.2}$$

Now put

$$F_n(x) = \sum_{k=1}^{\infty} f(n+k,\, k) x^{n+k} \qquad (n \geq 0) . \tag{4.3}$$

Then

$$F(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} f(n+k,\, k) x^{n+k} y^k = \sum_{n=0}^{\infty} y^{-n} \sum_{k=1}^{\infty} f(n+k,k)(xy)^{n+k} ,$$

so that

$$F(x,y) = \sum_{n=o}^{\infty} y^{-n} F_n(xy) \quad . \tag{4.4}$$

It follows from (4.2) and (4.4), with x replaced by xy^{-1}, that

$$(1 - xy^{-1} - y) \sum_{n=0}^{\infty} y^{-n} F_n(x) = x - xy\, C(x) ,$$

or preferably

$$(1 - y + xy^2) \sum_{n=0}^{\infty} y^n F_n(x) = x\, C(x) - xy . \tag{4.5}$$

Comparison of coefficients yields

$$\begin{aligned} &F_0(x) = x\, C(x) - F_1(x) + F_0(x) = x , \\ &F_n(x) - F_{n-1}(x) + xF_{n-2}(x) = 0 \qquad (n \geq 2) . \end{aligned} \tag{4.6}$$

Thus

$$F_1(x) = F_0(x) - x = x\, C(x) - x = x^2 C^2(x) ,$$

by (3.3). Next

$$F_2(x) = F_1(x) - xF_0(x) = x^2 C(x)\bigl(C(x) - 1\bigr) = x^3 C^3(x) .$$

Generally we have

$$F_n(x) = x^{n+1} C^{n+1}(x) \quad , \tag{4.7}$$

as is easily proved by induction, using (4.6).

Clearly (4.7) implies

$$F_n(x) = xC(x)F_{n-1}(x) \qquad (n \geq 1) . \tag{4.8}$$

Since

$$xC(x) = \sum_{n=1}^{\infty} f(n,n)x^n \quad ,$$

it follows from (4.8) that

$$f(n + k, k) = \sum_{i=1}^{k} f(j,j)f(n + k - j, k - j + 1) \qquad (n \geq 1) . \tag{4.9}$$

When $n = 1$, (4.9) reduces to (3.4).

If we define

$$f_n(x) = \sum_{k=1}^{n} f(n,k)x^k \quad , \tag{4.10}$$

we have

$$(1 - x)f_n(x) = \sum_{k=1}^{n} \{f(n,k) - f(n, k - 1)\} x^k - f(n,n)x^{n+1}$$

$$= \sum_{k=1}^{n-1} f(n - 1,k)x^k - f(n,n)x^{n+1} ,$$

so that

$$(4.11) \qquad (1 - x)f_n(x) = f_{n-1}(x) - f(n,n)x^{n+1} .$$

By iteration of (4.11),

$$(1 - x)^2 f_n(x) = f_{n-2}(x) - f(n - 1, n - 1)x^n - f(n,n)x^n(1 - x),$$

$$(1 - x)^3 f_n(x) = f_{n-3}(x) - f(n - 2, n - 2)x^{n-1} - f(n - 1, n - 1)x^n(1 - x) - f(n,n)x^{n+1}(1 - x)^2$$

and generally

$$(4.12) \qquad (1 - x)^k f_n(x) = f_{n-k}(x) - \sum_{j=o}^{k-1} f(n - j, n - j)x^{n-j+1}(1 - x)^{k-j-1} .$$

In particular, for $k = n - 1$, Eq. (4.12) becomes

$$(1 - x)^{n-1} f_n(x) = x - \sum_{j=o}^{n-2} f(n - j, n - j)x^{n-j+1}(1 - x)^{n-j-2} ,$$

so that

$$(4.13) \qquad (1 - x)^n f_n(x) = x - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{j-1} .$$

For example, for $n = 3$,

$$(1 - x)^3(x + 2x^2 + 2x^3) = x - x^2 - x^3 - x^4 + 4x^5 - 2x^6$$

$$= x - x^2 - x^3(1 - x) - 2x^4(1 - x)^2 .$$

If we put

$$(4.14) \qquad G_n(x) = x - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{j-1} ,$$

(4.13) becomes

$$(4.15) \qquad (1 - x)^n f_n(x) = G_n(x) .$$

We shall show that (4.15) characterizes the $f(n,k)$ in the following sense. Let

$$(a_1, a_2, a_3, \cdots)$$

be a sequence of numbers such that

$$A_n(x) = x - \sum_{j=1}^{n} a_j x^{j+1}(1-x)^{j-1} \tag{4.16}$$

is divisible by $(1-x)^n$ for $n = 1, 2, 3, \cdots$. Define $g_n(x)$ by means of

$$A_n(x) = (1-x)^n g_n(x) , \tag{4.17}$$

so that $g_n(x)$ is a polynomial of degree n. We shall show that

$$g_n(x) = f_n(x) \qquad (n = 1, 2, 3, \cdots) . \tag{4.18}$$

For $n = 1$, it follows from (4.16) and (4.17) that $a_1 = 1$, $g_1(x) = x$. For $n = 2$ we have

$$x - x^2 - a_2 x^3(1-x) = (1-x)^2 g_2(x) ,$$

so that $a_2 = 1$, $g_2(x) = 1 + x$. For $n = 3$ we have

$$x - x^2 - x^3(1-x) - a_3 x^4(1-x)^2 = (1-x)^3 g_3(x) ,$$

which gives

$$a_3 = 2, \qquad g_3(x) = 1 + 2x + 2x^2 .$$

It follows from (4.16) that

$$A_{n-1}(x) - A_n(x) = a_n x^{n+1} (1-x)^{n-1} ,$$

so that, by (4.17),

$$(1-x)^{n-1} g_{n-1}(x) - (1-x)^n g_n(x) = a_n x^{n+1}(1-x)^{n-1} .$$

Thus

$$g_{n-1}(x) - (1-x)g_n(x) = a_n x^{n+1} . \tag{4.19}$$

On the other hand, by (4.11),

$$f_{n-1}(x) - (1 - x)f_n(x) = f(n,n)x^{n+1} \quad (4.20)$$

Now assume that (4.18) holds for $n = 1, 2, \cdots, m - 1$. Then by (4.19) and (4.20) we have

$$(1 - x)\left[f_m(x) - g_m(x)\right] = \left[a_m - f(m,m)\right]x^{m+1} .$$

This implies

$$a_m = f(m,m), \qquad f_m(x) = g_m(x) ,$$

thus completing the proof of (4.18).

In the next place, we have, by (4.13),

$$\begin{aligned} f_n(x) &= x(1 - x)^{-n} - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{-n+j-1} \\ &= x\sum_{k=o}^{\infty}\binom{n + k - 1}{k}x^k - \sum_{j=1}^{n} f(j,j)x^{j+1}\sum_{k=o}^{\infty}\binom{n - j + k}{k}x^k \\ &= x\sum_{k=o}^{\infty}\binom{n + k - 1}{k}x^k - \sum_{k=o}^{\infty} x^{k+1}\sum_{\substack{j=1 \\ j\le k}}^{n}\binom{n - 2j + k}{k - j} f(j,j) . \end{aligned}$$

Equating coefficients, we get

$$f(n, k + 1) = \binom{n + k - 1}{k} - \sum_{j=1}^{k}\binom{n - 2j + k}{k - j} f(j,j) \qquad (0 \le k \le n) . \quad (4.21)$$

and

$$\sum_{j=1}^{n}\binom{n - 2j + k}{k - j} f(j,j) = \binom{n + k - 1}{k} \qquad (k \ge n) . \quad (4.22)$$

In particular, for $k = n$, (4.22) becomes

$$\sum_{j=1}^{n}\binom{2n - 2j}{n - j} f(j,j) = \binom{2n - 1}{n - 1} = \frac{1}{2}\binom{2n}{n} \qquad (n \ge 1) . \quad (4.23)$$

Then

$$\frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}x^n = \sum_{j=1}^{\infty} f(j,j)x^j \sum_{n=0}^{\infty}\binom{2n}{n}x^n .$$

Since

$$(1 - 4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n ,$$

it follows that

$$\sum_{j=1}^{\infty} f(j,j)\, x^j = \tfrac{1}{2}(1 - 4x)^{\frac{1}{2}} \left\{(1 - 4x)^{-\frac{1}{2}} - 1\right\}$$

$$= \tfrac{1}{2}\left\{1 - (1 - 4x)^{\frac{1}{2}}\right\} .$$

Thus we have obtained another proof of (3.2).

We can also prove (3.2) — or the equivalent formula (3.4) — directly from the definition of $f(n,k)$. Consider the sequence $(a_1, a_2, \cdots, a_{n+1})$ with

$$1 = a_1 \le a_2 \le \cdots \le a_{n+1} = n; \qquad a_i \le i \qquad (i = 1, 2, \cdots, n) .$$

Let k be the largest integer such that $a_k = k$. Clearly $k \le n$ and $a_{k+1} = k$. Now break the given sequence into two subsequences

$$(a_1, \cdots, a_k), \qquad (a_{k+1}, \cdots, a_{n+1}) .$$

Put

$$b_j = a_{k+j} - k + 1 \qquad (j = 1, 2, \cdots, n - k + 1) .$$

Then

$$\begin{cases} 1 = b_1 \le b_2 \le \cdots \le b_{n-k+1} = n - k + 1 ; \\ b_j \le j \qquad (j = 1, 2, \cdots, n - k + 1) . \end{cases}$$

It follows that

$$f(n + 1, n) = \sum_{k=1}^{n} f(k,k) f(n - k + 1, n - k + 1) . \tag{4.24}$$

Since

$$c(n) = f(n + 1, n) = f(n + 1, n + 1) ,$$

(4.24) reduces to

$$c(n) = \sum_{k=1}^{n} c(k) c(n - k) .$$

More generally consider the sequence $(a_1, a_2, \cdots, a_{n+1})$ with

$$\begin{cases} 1 = a_1 \le a_2 \le \cdots \le a_{n+1} = m \le n ; \\ a_i \le i \qquad (i = 1, 2, \cdots, n) . \end{cases}$$

As above let k be the greatest integer such that $a_k = k$. Then $k \le m \le n$ and $a_{k+1} = k$. Break the given sequence into two pieces:

$$(a_1, \cdots, a_k), \qquad (a_{k+1}, \cdots, a_{n+1})$$

and put

$$b_j = a_{k+j} - k + 1 \qquad (j = 1, 2, \cdots, n - k + 1) .$$

Then

$$\begin{cases} 1 = b_1 \le b_2 \le \cdots \le b_{n-k+1} = m - k + 1 ; \\ b_j \le j \qquad (j = 1, 2, \cdots, n - k + 1) . \end{cases}$$

It follows that

$$f(n + 1, m) = \sum_{k=1}^{m} f(k, k) f(n - k + 1, m - k + 1) . \tag{4.25}$$

Replacing $n + 1$ by $r + m$, (4.25) becomes

$$f(r + m, m) = \sum_{k=1}^{m} f(k, k) f(r + m - k, m - k + 1) . \tag{4.26}$$

This furnishes another proof of (4.9).

5. We now consider the following generalization of $f(n,k)$:

$$f(n, k, q) = \sum q^{a_1 + a_2 + \cdots + a_n} , \tag{5.1}$$

where the summation is over all $a_1, a_2, \cdots, a_n$ that satisfy

$$1 = a_1 \le a_2 \le \cdots \le a_n = k \tag{5.2}$$

and

$$a_i \le i \qquad (i = 1, 2, \cdots, n) . \tag{5.3}$$

It is clear that $f(n, k, 1) = f(n, k)$.

It follows at once from the definition that

$$f(n, k, q) = q^k \sum_{j=1}^{k} f(n - 1, j, q) , \tag{5.4}$$

where $f(n - 1, n, q) = 0$. From (5.4) we get

(5.5) $$f(n,k,q) = qf(n, k-1, q) + q^k f(n-1, k, q) .$$

Making use of either (5.4) or (5.5) we can compute the following table.

n \ k	1	2	3	4	5
1	q				
2	q^2	q^3			
3	q^3	$q^4 + q^5$	$q^5 + q^6$		
4	q^4	$q^5 + q^6 + q^7$	$q^6 + q^7 + 2q^8 + q^9$	$q^7 + q^8 + 2q^9 + q^{10}$	
5	q^5	$q^6 + q^7 + q^8 + q^9$	$q^7 + q^8 + 2q^9 + 2q^{10} + 2q^{11} + q^{12}$	$q^8 + q^9 + 2q^{10} + 3q^{11} + 3q^{12} + 3q^{13} + q^{14}$	$q^9 + q^{10} + 2q^{11} + 3q^{12} + 3q^{13} + 3q^{14} + q^{15}$

It is evident that

(5.6) $$f(n,1,q) = q^n \qquad (n = 1, 2, 3, \cdots)$$

and

(5.7) $$f(n,n,q) = qf(n, n-1, q) .$$

Also, by (5.4), the sum

(5.8) $$f(n,q) = \sum \lambda^{a_1+a_2+\cdots+a_n} ,$$

where now

(5.9) $$1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n; \qquad a_i \leq i ,$$

satisfies

(5.10) $$f(n,q) = q^{-n} f(n+1, n, q) = q^{-n-1} f(n+1, n+1, q) .$$

It is also easily verified that

(5.11) $$f(n,2,q) = q^{n+1} \frac{1 - q^{n-1}}{1-q} .$$

6. It is convenient to consider the polynomial $a_n(x,q)$ defined by means of

(6.1) $$(a)_{n+1} a_n(x,q) = 1 - x \sum_{k=0}^{n} a(k,k,q)(qx)^k (x)_k ,$$

where

(6.2) $$(x)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x), \qquad (x)_0 = 1 .$$

The definition (6.1) may be compared with (4.13).

For $n = 0$, (6.1) becomes

$$(1 - x)a_0(x,q) = 1 - xa(0,0,q) ,$$

so that

$$a(0,0,q) = 1, \qquad a_0(x,q) = 1 .$$

For $n = 1$ we have

$$(1 - x)(1 - qx)a_1(x,q) = 1 - xa(0,0,q) - qx^2(1 - x)a(1,1,q) ,$$

which implies

$$a(1,1,q) = q, \qquad a_1(x,q) = 1 + qx .$$

For $n = 2$ we have

$$(1 - x)(1 - qx)(1 - q^2x)a_2(x,q) = 1 - x - q^2x^2(1 - x) - q^2x^3(1 - x)(1 - qx)a(2,2,q).$$

This yields

$$a(2,2,q) = 1 + q, \qquad a_2(x,q) = 1 + (q + q^2)x + (q^3 + q^4)x^2 .$$

We now show that (6.1) uniquely determines $a(n,n,q)$ and $a_n(x,q)$. Clearly (6.1) implies

$$(x)_{n+1}\, a_n(x,q) = (x)_n\, a_{n-1}(x,q) - xa(n,n,q)(qx)^n(x)_n ,$$

or

$$(1 - q^nx)a_n(x,q) = a_{n-1}(x,q) - q^n\, a(n,n,q)x^{n+1} . \tag{6.3}$$

For $x = q^{-n}$ this becomes

$$a(n,n,q) = q^n\, a_{n-1}(q^{-n},\, q) . \tag{6.4}$$

Substitution from (6.4) in (6.2) shows that $a_n(x,q)$ is uniquely determined and is of degree n in x. We may accordingly put

$$a_n(x,q) = \sum_{k=0}^{n} a(n,k,q)x^n , \tag{6.5}$$

thus incidentally justifying the notation $a(n,n,q)$ in (6.1).

It now follows from (6.2) that

$$\begin{cases} a(n,k,q) = q^n a(n,\ k - 1,\ q) + a(n - 1,\ k,\ q) \\ a(n,n,q) = q^n a(n,\ n - 1,\ q) \end{cases} . \tag{6.6}$$

Iteration of the first of (6.6) leads to

$$a(n,k,q) = \sum_{j=0}^{k} q^{jn} a(n - 1,\ k - j,\ q) . \tag{6.7}$$

If we put

$$a(n,k,q) = q^{\frac{1}{2}k(k+1)}b(n,k,q) , \tag{6.8}$$

(6.6) becomes

$$\begin{cases} b(n,k,q) = b(n-1,k,q) + q^{n-k}b(n,k-1,q) \\ b(n,n,q) = b(n,n-1,q) \end{cases} \tag{6.9}$$

The following table is easily computed.

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	$1 + q$	$1 + q$		
3	1	$1 + q + q^2$	$1 + 2q + q^2 + q^3$	$1 + 2q + q^2 + q^3$	
4	1	$1 + q + q^2 + q^3$	$1 + 2q + 2q^2 + 2q^3 + q^4 + q^5$	$1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$	$1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$

Comparison with the table for $f(n,k,q)$ suggests that

$$f(n+1, k+1, q) = q^{(k+1)(n+1)-\frac{1}{2}k(k+1)}b(n,k,q^{-1}) . \tag{6.10}$$

To prove (6.10), substitute from (6.9) in (5.5). We get

$$q^{(k+1)(n+1)-\frac{1}{2}k(k+1)}b(n,k,q^{-1}) = q^{k(n+1)-\frac{1}{2}k(k-1)+1}b(n, k-1, q^{-1}) + q^{(k+1)n-\frac{1}{2}k(k+1)+k+1}b(n-1, k, q^{-1}) ,$$

that is

$$b(n,k,q^{-1}) = q^{k-n}b(n, k-1, q^{-1}) + b(n-1, k, q^{-1}) .$$

Replacing q by q^{-1}, this becomes

$$b(n,k,q) = q^{n-k}b(n, k-1, q) + b(n-1, k, q) ,$$

which is identical with (6.9). This evidently proves (6.10).

7. It follows from (6.9) that $b(n,k,q)$ is a polynomial of degree k in q^n. Put

$$b(n,k,q) = \frac{1}{(q)_k}\sum_{s=0}^{k} c(k,s)q^{ns} , \tag{7.1}$$

where $c(k,s) = c(k,s,q)$ is independent of n. Using (6.9) we get

$$c(k,s) = -q^{s-k}\,\frac{1-q^k}{1-q^s}\,c(k-1,\ s-1)\,.$$

This yields

$$c(k,s) = (-1)^s q^{s(s-k)}\begin{bmatrix} k \\ s \end{bmatrix} c(k-s)\,, \tag{7.2}$$

where $c(k-s) = c(k-s,0)$ and

$$\begin{bmatrix} k \\ s \end{bmatrix} = \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}\,.$$

Thus (7.1) becomes

$$b(n,k,q) = \frac{1}{(q)_k}\sum_{s=0}^{k}(-1)^s\, q^{s(s-k)}\begin{bmatrix} k \\ s \end{bmatrix} c(k-s)q^{ns}\,. \tag{7.3}$$

By (6.9) and (7.3) we have

$$b(j,\ k+1,\ q) - b(j-1,\ k+1,\ q) = \frac{q^{-k-1}}{(q)_k}\sum_{s=0}^{k}(-1)^s q^{s(s-k)}\begin{bmatrix} k \\ s \end{bmatrix} c(k-s)q^{j(s+1)} \qquad (j > k).$$

Summing over j gives

$$\begin{aligned} b(n,\ k+1,\ q) &= \frac{q^{-k-1}}{(q)_k}\sum_{s=0}^{k}(-1)^s\, q^{s(s-k)}\begin{bmatrix} k \\ s \end{bmatrix} c(k-s)\ \frac{q^{(k+1)(s+1)} - q^{(n+1)(s+1)}}{1-q^{s+1}} \\ &= \frac{1}{(q)_{k+1}}\sum_{s=0}^{k+1}(-1)^s\, q^{s(s-k-1)}\begin{bmatrix} k+1 \\ s \end{bmatrix} c(k-s+1)q^{ns} \\ &= \frac{1}{(q)_{k+1}}\sum_{s=1}^{k+1}(-1)^s\, q^{s(s-1)}\begin{bmatrix} k+1 \\ s \end{bmatrix} c(k-s+1)\,. \end{aligned}$$

Comparison with (7.3) yields

$$c_{k+1} = -\sum_{s=1}^{k+1}(-1)^s\, q^{s(s-1)}\begin{bmatrix} k+1 \\ s \end{bmatrix} c(k-s+1)\,,$$

that is,

$$\sum_{s=0}^{k}(-1)^s\, q^{s(s-1)}\begin{bmatrix} k \\ s \end{bmatrix} c(k-s) = 0 \qquad (k > 0)\,. \tag{7.4}$$

The $c(k)$ are uniquely determined by (7.4) and $c(0) = 1$. In particular

$$c(1) = 1, \quad c(2) = 1 + q - q^2, \quad c(3) = 1 + 2q + 2q^2 + q^3 - q^6 .$$

To get a generating function for $c(k)$, put

$$f(t) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)} t^k/(q)_k . \tag{7.5}$$

Then, by (7.4),

$$\sum_{n=o}^{\infty} c(n)t^n/(q)_n = \frac{1}{f(t)} . \tag{7.6}$$

In the next place put

$$\frac{f(xt)}{f(t)} = \sum_{n=0}^{\infty} \psi_n(x) \frac{t^n}{(q)_n} , \tag{7.7}$$

where

$$\psi_n(n) = \sum_{s=o}^{n} (-1)^s q^{s(s-1)} \begin{bmatrix} n \\ s \end{bmatrix} c(n - s)x^s . \tag{7.8}$$

In particular, by (7.3),

$$b(n + k - 1, k, q) = \frac{1}{(q)_k} \psi_k(q^n) , \qquad b(k,k,q) = \frac{1}{(q)_k} \psi_k(q) \tag{7.9}$$

Therefore, by (7.7),

$$\sum_{k=0}^{\infty} b(n + k - 1, k, q)\, t^k = \frac{f(q^n t)}{f(t)} \qquad (n > 0) . \tag{7.10}$$

While $f(t)$ resembles the familiar series

$$\sum_{n=o}^{\infty} t^n/q^n = \prod_{n=o}^{\infty} (1 - q^n t)^{-1}$$

and

$$\sum_{n=o}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} t^n/(q)_n = \prod_{n=o}^{\infty} (1 - q^n t) ,$$

its properties seem to be less simple.

It follows from (7.5) that

$$f(t) - f(qt) + tf(q^2t) = 0 . \tag{7.11}$$

Repeated application of (7.11) leads to

$$q^{\frac{1}{2}n(n-1)}t^n f(q^{n+1}t) = -A_n(t)f(t) + B_n(t)f(qt) \qquad (n > 0) , \tag{7.12}$$

where $A_{n+1}(t) = B_n(qt)$ and

$$B_n(t) = \sum_{2s \leq n} (-1)^s q^{s(s-1)} \begin{bmatrix} n - s \\ s \end{bmatrix} t^s . \tag{7.13}$$

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Brother Alfred Brousseau
St. Mary's College
St. Mary's College, Calif.

We have therefore

(6.17)
$$F_{a^k}(x) = F_{a^{k+1}}(x) + (x^{2^{k+1}} + x^{u_{k+2}})F_{a^{k+2}}(x) .$$

Now

$$B(x) = x^2 F_{a^2}(x) ,$$

so that $F_{a^2}(x)$ is rationally related to $A(x) = F_a(x)$. Then by (6.17) the same is true of $F_{a^2}(x)$ and so on.

We may state

Theorem 6.3. For arbitrary w, the function $F_w(x)$ is rationally related to $A(x)$, that is, there exist polynomials $P_w(x)$, $Q_w(x)$, $R_w(x)$ such that

$$P_w(x)F_w(x) = Q_w(x)A(x) + R_w(x) .$$

It seems plausible that $A(x)$ and $D_1(x)$ are not rationally related but we have been unable to prove this.

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it is clear that we have proved (5).

As for (2), we have

$$aL_n - L_{n+1} = b^n(a - b) = b^n\sqrt{5} .$$

For $n \ge 4$

$$\left|b^n\sqrt{5}\right| \le b^4\sqrt{5} = \tfrac{1}{2}(7 - 3\sqrt{5})\sqrt{5} < \tfrac{1}{2} .$$

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SOME THEOREMS ON COMPLETENESS

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1. INTRODUCTION

The notion of completeness was introduced in [1].

Definition. A sequence of positive integers, A, is "complete" if and only if every positive integer, N, is the sum of a subsequence of A. The theorem of Brown [2] gives a necessary and sufficient condition for completeness.

Theorem 1. A sequence of monotonic increasing positive integers, A, is "complete" if and only if:

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} \le 1 + \sum_{k=1}^{n} a_k \;.$$

Corollary. As an easy consequence of Theorem 1, the sequence $a_n = 2^{n-1}$, $n = 1, 2, 3, \cdots$ is complete, since $2^{n+1} = 1 + (2^n + \cdots + 2 + 1)$, a well known result.

Theorem 2. The Fibonacci Sequence is complete.

Proof. The identity

$$F_{n+2} - 1 = \sum_{k=1}^{n} F_k$$

gives us

$$F_{n+1} \le 1 + \sum_{k=1}^{n} F_k = F_{n+2} \;,$$

since

$$F_{n+2} = F_{n+1} + F_n \;.$$

2. ANOTHER SUFFICIENT CONDITION

Theorem 3. If (i) $a_1 = 1$, (ii) $a_{n+1} \ge a_n$, (iii) $a_{n+1} \le 2a_n$, then sequence A is complete.

Proof.

$$\begin{aligned} a_{n+1} &\le a_n + a_n \\ &\le a_n + a_{n-1} + a_{n-1} \\ &\le a_n + a_{n-1} + \cdots + a_1 + a_1 \end{aligned}$$

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by repeated use of conditions (ii) and (iii), thus

$$a_{n+1} \leq 1 + \sum_{k=1}^{n} a_k$$

since (i) gives $a_1 = 1$.

Corollary. The Fibonacci sequence is complete. $F_1 = 1$, $F_{n+1} \leq 2F_n$ and $F_{n+1} \geq F_n$.

Theorem 4. The sequence $\{1, p_n\}$ is complete, where p_n is the n^{th} prime.

Proof. By Bertrand's postulate there is a prime in $[n, 2n]$ for $n \geq 1$. Now $p_n < p_{n+1} \leq 2p_n$. Thus by Theorem 3, Theorem 4 is proved.

Theorem 5. The sequence $\{1, p_n\}$ is complete even when an arbitrary prime ≥ 7 is removed.

Proof. By Sierpiñski's Theorem VII in [3], we have for $n > 5$ there exists at least two primes between n and $2n$.

Thus

$$p_n < p_{n+1} < p_{n+2} < 2p_n \quad .$$

Clearly, if some p_{n+1} is deleted, then Theorem 3 is still valid. Thus Theorem 5.

Theorem 6. The sequence $\{1, p_n\}$ remains complete even if for $n > 5$ we remove an infinite subsequence of primes no two of which are consecutive.

3. COMPLETENESS OF FIBONACCI POWERS

Theorem 7. The sequence of 2^{m-1} copies of F_k^m is complete.

Proof.

$$\frac{F_{n+1}}{F_n} \leq 2 \quad \text{for} \quad n \geq 3$$

and

$$\left(\frac{F_{n+1}}{F_n}\right)^4 \leq 2^3 \quad \text{for} \quad n \geq 3 \; .$$

Thus

$$\left(\frac{F_{n+1}}{F_n}\right)^m \leq 2^{m-1} \quad \text{for} \quad m \geq 4, \; n \geq 3 \; .$$

Now:

$$F_{n+1}^m \leq 2^{m-1} F_n^m \leq 1 + 2^{m-1} \sum_{k=1}^{n} F_k^m \quad .$$

For $m = 1$, the theorem is true by Theorem 2. For $m = 2$, we have

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

shows that one copy is not enough.

Let $a_{2n} = a_{2n+1} = F_n^2$, then clearly

$$a_{2n+1} \le 1 + \sum_{k=1}^{2n} a_k ,$$

since

$$a_{2n+1} = a_{2n}$$

$$\sum_{k=1}^{2n} a_k = 2(F_1^2 + F_2^2 + \cdots + F_n^2) = 2F_n F_{n+1} .$$

Thus

$$\begin{aligned} a_{2n+2} &= F_{n+1}^2 \\ &\le 1 + 2F_n F_{n+1} , \end{aligned}$$

since

$$F_{n+1} \le 2F_n .$$

Therefore by Theorem 1 it is complete. For $m = 3$, four copies of F_n^3 is complete from [5]. Theorem 7 is proved. See Brown [4].

In [5] is the following Theorem which we cite without proof:

Theorem 8. If any a_n, $n \le 6$, is deleted from the two copies of the Fibonacci Squares, the sequence remains complete, while if $n \ge 7$, the sequence becomes incomplete.

In [5] the following theorem is given:

Theorem 9. If four copies of F_n^3 forms a sequence, then the sequence remains complete if F_k^3 is removed for k odd and becomes incomplete if F_k^3 is removed for k even.

The following conjecture was given by O'Connell in [5]:

Theorem 10. If $m \ge 4$, the 2^{m-1} copies of F_n^m remains complete even if a F_k^m is removed.

Proof. Since $F_{n+1}^m \le 2^{m-1} F_n^m$ for $n \ge 3$; $m \ge 4$, then

$$F_{n+k+1}^m \le 2^{m-1} F_{n+k}^m \le 1 + 2^{m-1} \sum_{s=1}^{n+k} F_s^m - F_n^m .$$

From Theorem 8, the sequence is complete up to terms using $2^{m-1} F_n^m$ clearly if we delete one F_k^m the first possible difficulty appears at $k = 1$ above. Clearly this causes no trouble for $k \ge 0$. The result follows and the proof of Theorem 10 is finished.

Theorem 11. If $m = 4k$, then the sequence of 2^{m-1} copies of F_n^k remains complete even if 2^{k-1} of the F_n^m are deleted.

Proof.

$$\left(\frac{F_{n+1}}{F_n}\right) \le 2 \quad \text{for} \quad n \ge 3$$

then

$$\left(\frac{F_{n+1}}{F_n}\right)^{4k} \le 2^{3k}$$

$$\begin{aligned} F_{n+1}^{4k} &\le 2^{3k} F_n^{4k} = F_n^{4k} + (2^{3k} - 1)F_n^{4k} \\ &\le 2^{3k} F_{n-1}^{4k} + 2^{k-1}(2^{3k} - 1)F_n^{4k} \\ &\le 2^{4k-1} \sum_{i=1}^{n-1} F_i^{4k} + (2^{4k-1} - 2^{k-1})F_n^{4k} \\ &\le 1 + 2^{4k-1} \sum_{i=1}^{n} F_i^{4k} - 2^{k-1} F_n^{4k} \quad . \end{aligned}$$

then let $m = 4k$;

$$F_{n+1}^m \le 1 + 2^{m-1} \sum_{i=1}^{n} F_i^m - 2^{k-1} F_n^m \quad .$$

Thus 2^{k-1} copies of F_n^m can be deleted without loss of completeness. Further:

<u>Theorem 12.</u>

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m$$

can be deleted without loss of completeness, and where $\alpha_i \le 2^{k-1}$

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m \le 2^{k-1} F_{s_k}^m \quad .$$

<u>Proof.</u> As a consequence of Theorem 11, we have

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m \le 2^{k-1} F_{s_k}^m \qquad \alpha_i \le 2^{k-1} \quad .$$

Thus:

$$F_{n+1}^m \le 1 + 2^{m-1} \sum_{i=1}^{n} F_i^m - \sum_{i=1}^{k} \alpha_i F_{s_i}^m$$

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FIBONACCI AND LUCAS TRIANGLES

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1. INTRODUCTION

We first define four sequences of polynomials and lay out two fundamental identities. Let

(a) $$f_0(x) = 0, \quad f_1(x) = 1, \quad \text{and} \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x).$$

These are the Fibonacci polynomials, and $f_n(1) = F_n$. Let

(b) $$L_0(x) = 2, \quad L_1(x) = x, \quad \text{and} \quad L_{n+2}(x) = xL_{n+1}(x) + L_n(x),$$

which are the Lucas polynomials. It is easy to show that

$$L_n(x) = f_{n+1}(x) + f_{n-1}(x), \quad L_n^2(x) - (x^2+4)f_n^2(x) = (-1)^n 4, \quad \text{and} \quad L_n(1) = L_n.$$

These are two well known polynomial sequences which have been much discussed in these pages. Both enjoy Binet forms. Let $\lambda^2 - x\lambda - 1 = 0$ have roots

$$\lambda_1 = \frac{x + \sqrt{x^2+4}}{2} \quad \text{and} \quad \lambda_2 = \frac{x - \sqrt{x^2+4}}{2}.$$

Then

$$f_n(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad L_n(x) = \lambda_1^n + \lambda_2^n,$$

$$\lambda_1^n = \frac{L_n(x) + \sqrt{x^2+4}\, f_n(x)}{2} \quad \text{and} \quad \lambda_2^n = \frac{L_n(x) - \sqrt{x^2+4}\, f_n(x)}{2}.$$

Next we introduce two polynomial sequences closely related to the Chebysheff polynomials of the first and second kind which were introduced in [2]. Let

$$g_0(x) = 0, \quad g_1(x) = 1, \quad \text{and} \quad g_{n+2}(x) = xg_{n+1}(x) - g_n(x),$$
$$h_0(x) = 2, \quad h_1(x) = x, \quad \text{and} \quad h_{n+2}(x) = xh_{n+1}(x) - h_n(x).$$

It is easy to establish

$$h_n^2(x) - (x^2 - 4)g_n^2(x) = 4 \quad \text{and} \quad h_n(x) = g_{n+1}(x) + g_{n-1}(x) ,$$

and if $\lambda^2 - x\lambda + 1 = 0$ has roots

$$\lambda_1^*(x) = \frac{x + \sqrt{x^2 - 4}}{2} \quad \text{and} \quad \lambda_2^*(x) = \frac{x - \sqrt{x^2 - 4}}{2} ,$$

then, for $x \neq \pm 2$,

$$g_n(x) = \frac{\lambda_1^{*n}(x) - \lambda_2^{*n}(x)}{\lambda_1^*(x) - \lambda_2^*(x)} \quad \text{and} \quad h_n(x) = \lambda_1^{*n}(x) + \lambda_2^{*n}(x) ,$$

while $g_n(2) = n$ and $g_n(-2) = -n$, $n = 0, 1, 2, \cdots$. As with Fibonacci polynomials, $g_n(x)$ have their coefficients lying along the rising diagonals of Pascal's triangle.

2. SUBSTITUTIONS INTO POLYNOMIAL SEQUENCES

Consider

$$\lambda_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}$$

for x replaced by $L_{2n+1}(x)$. From $L_n^2(x) - (x^2 + 4)f_n^2(x) = 4(-1)^n$ we see that

$$\lambda_1(L_{2n+1}(x)) = \frac{L_{2n+1}(x) + \sqrt{x^2 + 4}\, f_{2n+1}(x)}{2}$$

which from

$$\lambda_1^n(x) = [L_n(x) + \sqrt{x^2 + 4}\, f_n(x)]/2$$

becomes

$$\lambda_1(L_{2n+1}(x)) = \lambda_1^{2n+1}(x) .$$

Similarly,

$$\lambda_2(L_{2n+1}(x)) = \lambda_2^{2n+1}(x) .$$

Now let us look at

$$f_m(L_{2n+1}(x)) = \frac{\lambda_1^m(L_{2n+1}(x)) - \lambda_2^m(L_{2n+1}(x))}{\lambda_1(L_{2n+1}(x)) - \lambda_2(L_{2n+1}(x))}$$

$$= \frac{\lambda_1^{m(2n+1)}(x) - \lambda_2^{m(2n+1)}(x)}{\lambda_1^{2n+1}(x) - \lambda_2^{2n+1}(x)} = \frac{f_{m(2n+1)}(x)}{f_{2n+1}(x)}$$

by dividing numerator and denominator by $\lambda_1(x) - \lambda_2(x)$ and using the Binet form for the Fibonacci polynomials. We note that since the coefficients of both polynomial sequences $f_n(x)$

and $L_n(x)$ are integers, then $f_m(L_{2n+1}(x))$ is a polynomial and $f_{m(2n+1)}(x)/f_{2n+1}(x)$ is a polynomial. Letting $x = 1$ shows that $F_{2n+1} \mid F_{m(2n+1)}$.

If instead we use $L_{2n}^2(x) - 4 = (x^2 + 4)f_{2n}^2(x)$, then

$$\lambda_1^*(x) = \frac{x + \sqrt{x^2 - 4}}{2}, \qquad \lambda_2^*(x) = \frac{x - \sqrt{x^2 - 4}}{2}$$

becomes

$$\lambda_1^*(L_{2n}(x)) = \frac{L_{2n}(x) + \sqrt{x^2 + 4}\, f_{2n}(x)}{2} = \lambda_1^{2n}(x)$$

$$\lambda_2^*(L_{2n}(x)) = \frac{L_{2n}(x) - \sqrt{x^2 + 4}\, f_{2n}(x)}{2} = \lambda_2^{2n}(x)$$

so that

$$g_m(x) = \frac{\lambda_1^{*m}(x) - \lambda_2^{*m}(x)}{\lambda_1^*(x) - \lambda_2^*(x)}$$

becomes, when x is replaced by $L_{2n}(x)$,

$$g_m(L_{2n}(x)) = \frac{\lambda_1^{*m}(L_{2n}(x)) - \lambda_2^{*m}(L_{2n}(x))}{\lambda_1^*(L_{2n}(x)) - \lambda_2^*(L_{2n}(x))}$$

$$= \frac{\lambda_1^{2mn}(x) - \lambda_2^{2mn}(x)}{\lambda_1^{2n} - \lambda_2^{2n}} = \frac{f_{2mn}(x)}{f_{2n}(x)}$$

as before using the Binet form for the Fibonacci polynomials. Again $g_m(L_{2n}(x))$ is a polynomial when $x = 1$, $F_{2n} \mid F_{2mn}$.

Summarizing,

(A) $$f_{(2n+1)m}(x) = f_{2n+1}(x)\, f_m(L_{2n+1}(x))$$

(B) $$f_{2nm}(x) = f_{2n}(x)\, g_m(L_{2n}(x)) .$$

Using the explicit formulas for the polynomials $f_n(x)$ and $g_n(x)$, we have

$$f_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n+1-k}{k} x^{n-2k}, \qquad g_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n+1-k}{k} (-1)^k x^{n-2k}$$

Then, we can combine (A) and (B) into one formula,

$$f_{nk}(x) = f_k(x) \sum_{j=0}^{[n/2]} \binom{n-1-j}{j} (-1)^{(k+1)(j+1)} L_k^{n-2j-1}(x) .$$

This justifies the formula reported by Brother Alfred in [3], Table 41, when $x = 3$, but, of course, holds for other x as well.

3. THE LUCAS TRIANGLE

The polynomial sequences $L_n(x)$ and $h_n(x)$ for each n have the same coefficients except those of $h_n(x)$ alternate in sign. If we call the coefficient array for the Lucas polynomials the Lucas Triangle, then we can get a result similar to that above as reported for Table 42 in [3]. See also [1], [4], [5], and [6]. First,

$$\lambda_1(L_{2n+1}(x)) = \lambda_1^{2n+1}(x) \qquad \text{and} \qquad \lambda_2(L_{2n+1}(x)) = \lambda_2^{2n+1}(x)$$

so that

$$L_m(L_{2n+1}(x)) = \lambda_1^{(2n+1)m}(x) + \lambda_2^{(2n+1)m}(x) = L_{m(2n+1)}(x) .$$

Next

$$\lambda_1^*(L_{2n}(x)) = \lambda_1^{2n}(x) \qquad \text{and} \qquad \lambda_2^*(L_{2n}(x)) = \lambda_2^{2n}(x)$$

so that

$$h_n(L_{2n}(x)) = \lambda_1^{2mn}(x) + \lambda_2^{2mn}(x) = L_{2mn}(x) .$$

This evidently establishes the counterpart for the Lucas polynomials.

We note in passing that $L_0(x) = 2$, $L_1(x) = x$, and from

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \qquad L_2(x) = x^2 + 2 .$$

Thus the $L_{2n+1}(x)$ are divisible by x. This also holds for $h_{2n+1}(x)$.

Thus,

$$L_{2m+1}(L_{2n+1}(x)) = L_{(2m+1)(2n+1)}(x)$$
$$h_{2m+1}(L_{2n}(x)) = L_{(2m+1)(2n)}(x)$$

implies that $L_p(x) \mid L_{(2m+1)p}(x)$. Similarly, $f_p(x) \mid f_{mp}(x)$. Setting $x = 1$ establishes $L_p \mid L_{(2m+1)p}$ and $F_p \mid F_{mp}$ for $m \geq 0$.

4. SOME OTHER RESULTS

Suppose

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x); \qquad f_0(x) = 0, \quad f_1(x) = 1 .$$

Next let $x = \alpha$, where $\alpha^2 = \alpha + 1$; then

$$f_n(\alpha) = \alpha P_n + Q_n .$$

Here we seek recurrences for the sequences P_n and Q_n. Thus

$$\alpha P_{n+2} + Q_{n+2} = \alpha(\alpha P_{n+1} + Q_{n+1}) + (\alpha P_n + Q_n)$$

and

$$P_{n+2} = P_{n+1} + P_n + Q_{n+1}$$
$$Q_{n+2} = P_{n+1} + Q_n$$
$$P_{n+1} = P_n + P_{n-1} + Q_n$$
$$P_{n+3} = P_{n+2} + P_{n+1} + Q_{n+2} .$$

Subtracting

$$P_{n+3} - P_{n+1} = P_{n+2} - P_n + P_{n+1} - P_{n-1} + Q_{n+2} - Q_n .$$

Thus

$$P_{n+3} = P_{n+2} + 3P_{n+1} - P_n - P_{n-1}$$

since

$$Q_{n+2} - Q_n = P_{n+1} ,$$

whose auxiliary polynomial is

$$x^4 = x^3 + 3x^2 - x - 1 .$$

This agrres with the results in [8].

Now, let $\lambda^2 = x\lambda + 1$. Then

$$f_n(\lambda) = \lambda P_n + Q_n ,$$

when P_n and Q_n are polynomials in λ.

$$\begin{aligned}\lambda P_{n+2} + Q_{n+2} &= \lambda(\lambda P_{n+1} + Q_{n+1}) + \lambda P_n + Q_n \\ &= x\lambda P_{n+1} + P_{n+1} + \lambda Q_{n+1} + \lambda P_n + Q_n .\end{aligned}$$

Thus

$$P_{n+2} = xP_{n+1} + P_n + Q_{n+1} ,$$
$$Q_{n+2} = P_{n+1} + Q_n ,$$

so that, using the same techniques as before, we find

$$P_{n+3} = xP_{n+2} + P_{n+1} + Q_{n+2}$$
$$P_{n+1} = xP_n + P_{n-1} + Q_n$$
$$P_{n+3} - P_{n+1} = xP_{n+2} + P_{n+1} - xP_n - P_{n-1} + (Q_{n+2} - Q_n)$$

yielding

$$P_{n+3} = xP_{n+2} + 3P_{n+1} - xP_n - P_{n-1}$$

which agrees with Eq. (8), particularly result (iii), in [8].

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SOME THEOREMS ON COMPLETENESS

holds true and Theorem 12 is completed.

Corollary. The hypothesis of Theorem 3 is not a necessary condition. From Theorem 7, clearly $F_{n+1}^m \le 2F_n^m$ for $n \ge 3$, $m \ge 4$, and that the sequence 2^{m-1} copies of F_n^m is complete.

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