THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 10 NUMBER 6

CONTENTS

DECEMBER 1972

THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

All subscription correspondence should be addressed to Brother Alfred Brousseau, St. Mary's College, California. All checks ($6.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Two copies of manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State University, San Jose, California. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.

The Quarterly is entered as third-class mail at the St. Mary's College Post Office, Calif., as an official publication of the Fibonacci Association.

A SIMPLE OPTIMAL CONTROL SEQUENCE IN TERMS FIBONACCI NUMBERS

I. McCAUSLAND
University of Toronto, Toronto, Canada

1. INTRODUCTION

It is well known that the Fibonacci numbers are encountered in the optimization of the procedure for searching for the maximum or minimum value of a unimodal function [1-6]. The optimum search procedure can be derived by the method of dynamic programming [3, 4, 5, 6]. In the present note it is shown that the sequence of optimal control inputs, for a simple discrete-time system with a quadratic performance criterion, can be expressed in terms of the Fibonacci numbers.

2. A DISCRETE-TIME SYSTEM

Consider the very simple linear discrete-time system* described by the difference equation

$$x(k+1) = x(k) + u(k) \quad , \tag{1}$$

where $u(k)$ is the control input to the system at discrete time instant k, and $x(k)$ is a state variable of the system at the same instant. Suppose that it is desired to find the sequence of N control inputs $u(1), \cdots, u(N)$ which, starting from an initial system state $x(1)$, gives the minimum possible value to the summation F defined by

$$F = \sum_{k=1}^{N} [x^2(k) + u^2(k)] \; . \tag{2}$$

The final state $x(N)$ may be prescribed or not; assume for the present that the final state is zero.

This problem can easily be solved by dynamic programming [4-6]. The procedure is to start by supposing $N = 1$, use the solution of that simple problem to find the solution for $N = 2$, and proceed to derive a recurrence relationship which gives the solution of the problem for larger values of N. If we define the quantity $S_N(x)$ to be the minimum value of the summation F reached in an N-stage process starting from the initial state x, we obtain the recurrence relationship

$$S_N(x) = \min_u \{x^2 + u^2 + S_{N-1}(x + u)\} \tag{3}$$

*For a discussion of discrete-time systems, see, for example, [7].

The value of $S_1(x)$, for the specified endpoint $x(2) = 0$, can easily be seen to be

$$S_1(x) = 2x^2 \tag{4}$$

for the control input

$$u_1(1) = -x \quad , \tag{5}$$

where the notation $u_1(1)$ means the first (and only) input of the one-stage process, and where the initial state x is understood. In this case there is really no optimization problem, as the specification of the final endpoint leaves no alternative but to choose $u = -x$ as given by (5). Having obtained the solution described by (4) and (5), however, we can proceed to find $S_2(x)$ by substitution in (3) as follows:

$$S_2(x) = \min_u \left\{x^2 + u^2 + 2(x + u)^2\right\} \tag{6}$$

Performing the minimization operation by differentiating the expression in braces with respect to u, we find that the optimum value of u is given by

$$u_2(1) = -\frac{2}{3}x \quad , \tag{7}$$

where the notation $u_2(1)$ represents the first input of the two-stage process. Substituting (7) in (6), we obtain

$$S_2(x) = \frac{5}{3}x^2 \tag{8}$$

Based on Equations (4) and (8), suppose that

$$S_N(x) = K(N)x^2 \quad . \tag{9}$$

$S_N(x)$ can be found by performing the minimizing operation involved in the expression

$$S_N(x) = \min_u \left\{x^2 + u^2 + K(N - 1)(x + u)^2\right\} \quad . \tag{10}$$

This minimization gives the value of u to be

$$u_N(1) = \frac{-K(N - 1)}{K(N - 1) + 1}\, x \quad . \tag{11}$$

Substitution of (11) into (10) leads to the result

$$K(N) = \frac{2K(N - 1) + 1}{K(N - 1) + 1} \quad . \tag{12}$$

We see from (12) that, if $K(N-1)$ is a rational number, $K(N)$ will also be rational. Therefore, because $K(2)$ is rational as shown by Eq. (8), $K(N)$ is rational for all values of N. If $K(N)$ is expressed in the form $A(N)/B(N)$, where A and B are integers with no common factor, the following results can be derived:

$$\frac{A(N)}{B(N)} = \frac{2A(N-1) + B(N-1)}{A(N-1) + B(N-1)} \tag{13}$$

$$A(N) = 2A(N-1) + B(N-1) \tag{14}$$

$$B(N) = A(N-1) + B(N-1). \tag{15}$$

The two first-order difference equations (14) and (15) can be expressed as a second-order difference equation (in either A or B) of the form

$$A(N+1) - 3A(N) + A(N-1) = 0 \quad . \tag{16}$$

Compare Eq. (16) with the following equation for the Fibonacci numbers $F(n)$ for values of n separated by two units instead of one:

$$F(n+2) - 3F(n) + F(n-2) = 0 \ . \tag{17}$$

Equation (17) can easily be obtained from the basic equation for the Fibonacci numbers

$$F(k) = F(k-1) + F(k-2) \tag{18}$$

by taking $k = n,\ n+1,\ n+2$, and manipulating the three equations so obtained. Comparing Eqs. (16) and (17), and using the initial conditions given by Eq. (8), it is found that $K(N)$ can be expressed in the form

$$K(N) = \frac{F(2N+1)}{F(2N)} \quad , \tag{19}$$

where $F(k)$ is the Fibonacci number defined by (18) with initial conditions $F(0) = 0$, $F(1) = 1$. Equation (11) leads to the result

$$u_N(1) = \frac{-F(2N-1)}{F(2N)}\, x \quad . \tag{20}$$

This result shows that the optimal control input is a function of the present state and the number of stages to go to the end of the process.

If the input given by Eq. (20) is applied to the system in initial state $x(1)$, the next state $x(2)$ is given by

$$x(2) = \left[1 - \frac{F(2N-1)}{F(2N)}\right] x(1) \tag{21}$$
$$= \frac{F(2N-2)}{F(2N)}\, x(1) \quad .$$

The next input $u_N(2)$ can be expressed in the form

$$u_N(2) = \frac{-F(2N-3)}{F(2N-2)} \; \frac{F(2N-2)}{F(2N)}\, x(1) \tag{22}$$
$$= \frac{-F(2N-3)}{F(2N)}\, x(1) \quad .$$

The sequence of optimal control inputs $u_N(i)$ can therefore be expressed in the form

$$u_N(i) = \frac{-F(2N-2i+1)}{F(2N)}\, x(1) \tag{23}$$
$$(i = 1, \cdots, N) \; .$$

If the final state is unspecified and therefore allowed to take on any value, the value of the last control input $u_N(N)$ is zero, and the values of $K(N)$ and $u_N(1)$ can be expressed in the forms

$$K(N) = \frac{F(2N)}{F(2N-1)} \tag{24}$$

$$u_N(1) = \frac{-F(2N-2)}{F(2N-1)}\, x \quad . \tag{25}$$

The optimal sequence of control inputs $u_N(i)$ is in this case given by

$$u_N(i) = \frac{-F(2N-2i)}{F(2N-1)}\, x(1) \tag{26}$$
$$(i = 1, \cdots, N) \quad .$$

These results are discussed more fully, and compared with the optimal control input for a continuous-time system, in [6].

REFERENCES

1. R. Bellman, Dynamic Programming, Princeton University Press, 1957, pp. 34-36.
2. L. T. Oliver and D. J. Wilde, "Symmetric Sequential Minimax Search for a Maximum," Fibonacci Quarterly, Vol. 2, No. 3, Oct., 1964, pp. 169-175.
3. D. J. Wilde, Optimum Seeking Methods, Prentice-Hall, 1964.

[Continued on page 608.]

A PROOF OF GOULD'S PASCAL HEXAGON CONJECTURE

A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico
and
V. E. HOGGATT, JR.
San Jose State University, San Jose, California

The binomial coefficients

$$(1) \qquad B_1 = \binom{n-1}{k-1}, \quad B_2 = \binom{n-1}{k}, \quad B_3 = \binom{n}{k+1}, \quad B_4 = \binom{n+1}{k+1}, \quad B_5 = \binom{n+1}{k}, \quad B_6 = \binom{n}{k-1}$$

form a regular hexagon in the Pascal triangle. The identity

$$(2) \qquad B_1B_3B_5 = B_2B_4B_6$$

of Verner E. Hoggatt, Jr., and Walter Hansell [1] has inspired a number of results including Henry W. Gould's remarkable conjecture that

$$(3) \qquad \gcd(B_1, B_3, B_5) = \gcd(B_2, B_4, B_6)$$

for all integers k and n with $0 < k < n$. Gould also had evidence of analogous results including the similar formula for the Fibonomial coefficients

$$(4) \qquad \left\{ {m \atop r} \right\} = F_m F_{m-1} \cdots F_{m-r+1} / F_1 F_2 \cdots F_r ,$$

in which F_n is the n^{th} Fibonacci number. (See [2] and [3].)

In this paper, we prove a generalized Gould hexagon theorem that includes (3) and the analogous property for the Fibonomial coefficients $\left\{ {m \atop r} \right\}$.

Let $a_1, a_2, a_3, \cdots$ be a sequence of nonzero integers such that both

$$(5) \qquad \gcd(a_m, a_n) \mid a_{m+n}$$

and

$$(6) \qquad \gcd(a_m, a_{m+n}) \mid a_n$$

for all m and n in $Z^+ = \{1, 2, 3, \cdots\}$. Let

$$(7)\qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} m \\ r \end{bmatrix} = a_m a_{m-1} \cdots a_{m-r+1} / a_1 a_2 \cdots a_r$$

for m and r in Z^+ with $1 \le r \le m$.

If $a_n = n$, then $\begin{bmatrix} m \\ r \end{bmatrix}$ is the binomial coefficient $\binom{m}{r}$, which is well known to be an integer for m and r in Z^+ with $0 \le r \le m$. If a_n is the Fibonacci number F_n, then $\begin{bmatrix} m \\ r \end{bmatrix}$ is the Fibonomial coefficient $\left\{ {m \atop r} \right\}$ given in (4); these coefficients are also known to be integers. In a later paper we shall show that conditions (5) and (6) imply that each generalized binomial coefficient $\begin{bmatrix} m \\ r \end{bmatrix}$ is an integer. Here we assume this result in our proof of a generalized Gould hexagon property.

Let p be a fixed positive prime. For all nonzero integers a let $E(a)$ denote the greatest integer e such that $p^e | a$.

In terms of this exponent function E, one can translate our hypotheses (5) and (6) into the two following statements:

$$(8)\qquad \min\{E(a_r),\ E(a_s)\} \le E(a_{r+s})$$

$$(9)\qquad \min\{E(a_r),\ E(a_s)\} \le E(a_{|r-s|}) .$$

We now establish the following result:

Lemma 1. For all r and s in Z^+, no one of the integers

$$(10)\qquad E(a_r), \qquad E(a_s), \qquad E(a_{r+s})$$

is smaller than the other two integers in (10), i.e., the minimum integer in (10) occurs at least twice in (10).

Proof. If $E(a_{r+s})$ is a minimum of (10), we see from (8) that at least one of $E(a_r)$ and $E(a_s)$ does not exceed the minimum $E(a_{r+s})$ and hence is also a minimum in (10). If either $E(a_r)$ or $E(a_s)$ is a minimum in (10), then one can use (9) to show similarly that the minimum in (10) occurs at least twice in (10).

Using the definition (7) of the generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$, one can readily establish the proportionality relation

$$(11)\qquad \begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix} : \begin{bmatrix} r+s-1 \\ r \end{bmatrix} : \begin{bmatrix} r+s \\ r \end{bmatrix} = a_r : a_s : a_{r+s} .$$

This proportion and Lemma 1 immediately give us

Lemma 2. The minimum integer in

$$(12)\qquad E\left(\begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix}\right),\ E\left(\begin{bmatrix} r+s-1 \\ r \end{bmatrix}\right),\ E\left(\begin{bmatrix} r+s \\ r \end{bmatrix}\right)$$

occurs at least twice in (12). That is, if u, v, w are the terms of (12) in some order and $u < w$ then $u = v$.

Now let

$$C_0 = \begin{bmatrix} n \\ k \end{bmatrix}, \quad C_1 = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} n-1 \\ k \end{bmatrix}, \quad C_3 = \begin{bmatrix} n \\ k+1 \end{bmatrix},$$
(13)
$$C_4 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}, \quad C_5 = \begin{bmatrix} n+1 \\ k \end{bmatrix}, \quad C_6 = \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

The generalized Hoggatt-Hansell identity

(14) $$C_1C_3C_5 = C_2C_4C_6$$

is established in a straightforward manner. We now turn to the generalized Gould property

(15) $$\gcd(C_1, C_3, C_5) = \gcd(C_2, C_4, C_6) .$$

Let C_i be as in (13) and let $e_i = E(C_i)$ for $0 \le i \le 6$. Then (14) implies

$$e_1 + e_3 + e_5 = e_2 + e_4 + e_6 .$$

The Gould property (15) is equivalent to having (for all primes p)

(17) $$\min(e_1, e_3, e_5) = \min(e_2, e_4, e_6) .$$

If (17) were not true, then either

(18) $$e_i < \min(e_1, e_3, e_5) \qquad \text{for some } i \text{ in } \{2,4,6\}$$

or

(19) $$e_j < \min(e_2, e_4, e_6) \qquad \text{for some } j \text{ in } \{1,3,5\} .$$

We now assume the specific case

(20) $$e_1 < \min(e_2, e_4, e_6)$$

of (19) and show that (20) leads to a contradiction; the other cases of (18) and (19) lead to contradictions similarly.

The special case of (12) in which $r = k$ and $s = n - k$ is

(21) $$e_1, \; e_2, \; e_0 .$$

From (20) we have $e_1 < e_2$. This and Lemma 2 applied to (21) give us $e_1 = e_0$.

Now the inequality $e_1 < e_4$ from (20) and $e_1 = e_0$ tell us that $e_0 < e_4$. The case of (12) with $r = k + 1$ and $s = n - k$ is

$$e_0,\ e_3,\ e_4\ . \tag{22}$$

Using $e_0 < e_4$ and applying Lemma 2 to (22) we find that $e_0 = e_3$.

The inequality $e_1 < e_6$ from (20) and $e_1 = e_0$ lead to $e_0 < e_6$. The case of (12) with $r = k$ and $s = n - k + 1$ is

$$e_6,\ e_0,\ e_5\ . \tag{23}$$

Since $e_0 < e_6$, Lemma 2 applied to (23) gives us $e_0 = e_5$.

Thus we have shown that (20) implies

$$e_0 = e_5 = e_3 = e_1 < \min(e_2, e_4, e_6)\ . \tag{24}$$

But it follows from (24) that $e_1 + e_3 + e_5 < e_2 + e_4 + e_6$ and this contradicts the consequence (16) of the Hoggatt-Hansell identity (14). Hence assumption (20) is false. Similarly, the other cases of (18) and (19) lead to contradictions. Therefore (17) is true and the generalized Gould property (15) is established.

It is now natural for people with Fibonacci interests to ask if properties (5) and (6) are true for sequences $\{a_n\}$ satisfying

$$a_{n+2} = ca_{n+1} - da_n \qquad \text{for} \qquad n = 1, 2, 3, \cdots, \tag{25}$$

with c and d fixed integers. In a later paper, we shall show that if

$$\gcd(c,d) = 1, \qquad a_1 = 1, \qquad \text{and} \qquad a_2 = c \tag{26}$$

then a sequence $\{a_n\}$ satisfying (25) has properties (5) and (6) and hence it gives rise to generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ that are integers with the Gould hexagon property (15).

If one drops the condition $\gcd(c,d) = 1$ in (26), then $\{a_n\}$ need not have properties (5) and (6) and the $\begin{bmatrix} m \\ r \end{bmatrix}$ need not have the Gould hexagon property. An example is the sequence

$$1,\ 2,\ 6,\ 16,\ 44,\ 120,\ 328,\ \cdots$$

with the recursion relation $a_{n+2} = 2a_{n+1} + 2a_n$. For this sequence, the $\begin{bmatrix} m \\ r \end{bmatrix}$ are all integers but the Gould hexagon property (15) is not true when $n = 5$ and $k = 2$.

REFERENCES

1. V. E. Hoggatt, Jr., and Walter Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2 (1971), pp. 120, 133.

[Continued on page 598.]

A CONSTRUCTIVE UNIQUENESS THEOREM ON REPRESENTING INTEGERS

JÁNOS GALAMBOS
Temple University, Philadelphia, Pennsylvania

Let F_n be the n^{th} Fibonacci number, i.e., $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [1] that every integer $N \geq 1$ has a unique representation

(1) $$N = F_{i_1} + F_{i_2} + \cdots + F_{i_\alpha}$$

such that

(2) $$i_1 \geq 1, \qquad i_j - i_{j-1} \geq 2 \quad \text{for} \quad j \geq 2 .$$

Conversely, if for all the integers $N \geq 1$,

(3a) $$N = a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}$$

is unique under (2), then $a_j = F_j$ for all j, i.e., the uniqueness of (1) under (2) characterizes the Fibonacci sequence. Generalizing this theorem, I shall prove in the present note that at most one increasing sequence can represent uniquely all the integers $N \geq 1$ as sums of its elements under a given constraint and I shall give a combinatorial formula for this only possible sequence.

Let $e_1, e_2, \cdots$ be non-negative integers and let C be a property which classifies each finite ordered set $(e_1, e_2, \cdots, e_n)$ into one of the two categories, those which possess C and those which do not. Denote by $C(e)$ the collection of all the sequences satisfying C.

Let $a_1 < a_2 < \cdots$ be positive integers. Assume that every integer $N > 1$ has a unique representation in the form

(3) $$N = \sum e_i a_i , \qquad \{e_i\} \in C(e)$$

and it is further assumed that

(4) $$\text{if} \quad a_n \leq N < a_{n+1} \quad \text{then} \quad e_n \neq 0 .$$

My aim is to prove the following

Theorem. If the property C is expressible independently of $a_1, a_2, \cdots$ then there is at most one sequence $0 < a_1 < a_2 < \cdots$ for which the representation (3) and (4) is unique. In this case, $a_1 = 1$ and for $n > 1$,

(5) $$a_{n+1} = 1 + \sum_{d=1}^{n} k(n,d,C) ,$$

where $k(n,d,C)$ is the number of n-vectors $(e_1, e_2, \cdots, e_n)$ satisfying C and such that exactly d of its coordinates differ from zero.

Before giving its proof, I wish to make some remarks on the theorem itself and on its applications. First of all, I want to emphasize the second part of the theorem, namely, that the sequence a_n is explicitly determined. In several concrete cases when the structure of C(e) is given, the uniqueness of $\{a_i\}$ can be shown by a simple argument but (5) is not obvious even in these cases, and for a general C(e) the usual argument for the uniqueness, too, seems to be very complicated, if it works at all, since several cases should be distinguished. The formula (5) is very useful at obtaining information on the number of non-zero terms in (3) even if no explicit formula for $k(n,d,C)$ is known. As an example, I mention a recent work of A. Oppenheim. Generalizing (1), he considered the following problem (personal communication). Let k_j, $j \geq 1$ be given positive integers and assume that (3a) is unique under the assumption that the first non-zero term in $i_j - i_{j-1} - k_1$, $i_{j+1} - i_j - k_2, \cdots$ is positive for all $j \geq 2$. In our notations it means that C(e) consists of all $(e_1, e_2, \cdots, e_n)$, $n > 2$, where e_j is either zero or one and if the gap between the j^{th} and the $(j+1)^{st}$ one in $(e_1, e_2, \cdots, e_n)$ is m_j, then for all j, $m_j - k_1$, $m_{j+1} - k_2, \cdots$ has the property that the first non-zero term is positive. A. Oppenheim determined the sequences k_j for which such a representation exists (to be published). In our approach we obtain a construction for the corresponding a's though here $k(n,d,C)$ is a complicated expression. However, this combinatorial function has already been investigated in much details since it has close relations to β-expansions, see [3], which has a wide literature. Two special cases of this problem of Oppenheim, namely, when all $k_j = 2$, or more generally, when for all j, $k_j = k$, have been investigated earlier. The case $k_j = 2$ for all j is simply the condition (2), hence the corresponding sequence a_j is the Fibonacci sequence and the formula (5) gives back its relation to the Pascal triangle. When for all j, $k_j = k$, we get the generalized Fibonacci sequence introduced by Daykin [1], the original argument for the validity of (5) being fairly complicated even for this simple case. In my recent paper [2], I obtained (5) for the generalized Fibonacci numbers, and actually that investigation led to the discovery of the short proof of this general theorem, which now follows.

Proof. First of all, note that (3) and (4) imply that there is a one-to-one correspondence between the integers $1 < N < a_{n+1}$ and the set of n-vectors $(e_1, e_2, \cdots, e_n)$ C(e). As a matter of fact, in view of (4), for any $(e_1, e_2, \cdots, e_n)$ belonging to C(e),

$$e_1a_1 + e_2a_2 + \cdots + e_na_n < a_{n+1} \tag{6}$$

namely, if the reversal of the inequality (6) apply, then, putting M for the left-hand side of (6), in view of (4), M would have a representation with an a_j, $j \geq n+1$, taking part, which by the definition of M, contradicts the uniqueness of (3). The converse of the one-to-one correspondence in question is obvious by (4).

From this observation the proof is easily completed. Cancel those terms in (3) for which $e_j = 0$, hence (3) determines a function d(N), the number of non-zero terms in (3). Since

[Continued on page 598.]

GENERATING IDENTITIES FOR FIBONACCI AND LUCAS TRIPLES

RODNEY T. HANSEN
Montana State University, Bozeman, Montana

Using the generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty} \quad \text{and} \quad \{L_{n+m}\}_{n=0}^{\infty},$$

where F_{n+m} denotes the $(n+m)^{th}$ Fibonacci number and L_{n+m} denotes the $(n+m)^{th}$ Lucas number, many basic identities are easily deduced. From certain of these identities and the generating functions, we obtain identities for the triples $F_pF_qF_r$, $F_pF_qL_r$, $F_pL_qL_r$, and $L_pL_qL_r$, where p, q, and r are fixed integers.

To derive the desired generating functions we recall that

$$(0) \qquad F_{n+m} = \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} \quad \text{and} \quad L_{n+m} = \alpha^{n+m} + \beta^{n+m}$$

where

$$\alpha = \frac{1 - \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 + \sqrt{5}}{2} .$$

Note that α and β are the roots of the equation $x^2 - x - 1 = 0$, and hence $\alpha + \beta = 1$ and $\alpha\beta = -1$. The generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty}$$

where m is any fixed integer is found using the given definition of F_{n+m}. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+m}x^n &= \sum_{n=0}^{\infty} \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} x^n \\
&= \frac{1}{\alpha - \beta}\left[\alpha^m \sum_{n=0}^{\infty} \alpha^n x^n - \beta^m \sum_{n=0}^{\infty} \beta^n x^n\right] \\
(1) \qquad &= \frac{1}{\alpha - \beta}\left[\alpha^m \frac{1}{1 - \alpha x} - \beta^m \frac{1}{1 - \beta x}\right] \\
&= \frac{1}{\alpha - \beta}\left[\frac{(\alpha^m - \beta^m) - \alpha\beta(\alpha^{m-1} - \beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)}\right] \\
&= \frac{F_m + F_{m-1}x}{1 - x - x^2} .
\end{aligned}$$

In a similar fashion the generating function of $\{L_{n+m}\}_{n=0}^{\infty}$ is found to be

$$\sum_{n=0}^{\infty} L_{n+m} x^n = \frac{L_m + L_{m-1} x}{1 - x - x^2} \quad . \tag{2}$$

(Any reader who is unfamiliar with the general theory of generating functions will find references [1], [2], [3], and [4] enlightening.)

Before considering important special cases of the above results, two lemmas are given which are proved by appropriate substitution of formulas (0).

Lemma 1. $F_n L_n = F_{2n}$, $n \in Z$, the set of integers.

Lemma 2. $F_n L_{n-1} + F_{n-1} L_n = 2F_{2n-1}$, $n \in Z$.

In utilizing formulas (1) and (2) to generate basic identities, we must first evaluate the formulas at specific values of m. It is sufficient for our purposes to consider the cases $m = -2, -1, 0, 1, 2, 3, 4$.

SPECIAL CASES OF FORMULAS (1) AND (2)

(Let $1 - x - x^2 = \Delta$.)

$$\sum_{n=0}^{\infty} F_{n-2} x^n = \frac{F_{-2} + F_{-1}x}{\Delta} = \frac{-1 + 2x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n-2} x^n = \frac{L_{-2} + L_{-3}x}{\Delta} = \frac{3 - 4x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n-1} x^n = \frac{F_{-1} + F_{-2}x}{\Delta} = \frac{1 - x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n-1} x^n = \frac{L_{-1} + L_{-2}x}{\Delta} = \frac{-1 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_n x^n = \frac{F_0 + F_{-1}x}{\Delta} = \frac{0 + x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_n x^n = \frac{L_0 + L_{-1}x}{\Delta} = \frac{2 - x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{F_1 + F_0x}{\Delta} = \frac{1 + 0x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n+1} x^n = \frac{L_1 + L_0x}{\Delta} = \frac{1 + 2x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \frac{F_2 + F_1x}{\Delta} = \frac{1 + x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n+2} x^n = \frac{L_2 + L_1x}{\Delta} = \frac{3 + x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+3} x^n = \frac{F_3 + F_2x}{\Delta} = \frac{2 + x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n+3} x^n = \frac{L_3 + L_2x}{\Delta} = \frac{4 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+4}\, x^n = \frac{F_4 + F_3 x}{\Delta} = \frac{3 + 2x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n+4}\, x^n = \frac{L_4 + L_3 x}{\Delta} = \frac{7 + 4x}{\Delta} \quad .$$

Using the fact that two series are equal if and only if the corresponding coefficients are equal, we now find several elementary identities.

Since

$$\frac{2 - x}{\Delta} = \frac{1}{\Delta} + \frac{1 - x}{\Delta} ,$$

it follows that

$$\sum_{n=0}^{\infty} L_n\, x^n = \sum_{n=0}^{\infty} F_{n+1}\, x^n + \sum_{n=0}^{\infty} F_{n-1}\, x^n$$

$$= \sum_{n=0}^{\infty} (F_{n+1} + F_{n-1})\, x^n$$

and hence

Lemma 3. $L_n = F_{n+1} + F_{n-1}$, $n \in Z^+ \cup \{0\}$, the set of nonnegative integers.

Note from definition (0) that

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right)$$

$$(0') \qquad = \frac{1}{\alpha - \beta}\, \frac{\beta^n - \alpha^n}{(\alpha\beta)^n} = \frac{1}{\alpha - \beta}\, \frac{\beta^n - \alpha^n}{(-1)^n}$$

$$= (-1)^{n+1}\, \frac{\alpha^n - \beta^n}{\alpha - \beta} = (-1)^{n+1}\, F_n$$

and

$$L_{-n} = \alpha^{-n} + \beta^{-n} = (\alpha\beta)^{-n}(\alpha^n + \beta^n)$$

$$(0'') \qquad = (-1)^{-n} L_n = (-1)^n L_n$$

for any positive integer n.

Returning to Lemma 3, we now observe from this lemma and "definitions" (0') and (0") that

$$\begin{aligned} F_{(-n)+1} + F_{(-n)-1} &= F_{-(n-1)} + F_{-(n+1)} \\ &= (-1)^{(n-1)+1} F_{n-1} + (-1)^{(n+1)+1} F_{n+1} \\ &= (-1)^n\, [F_{n-1} + F_{n+1}] \\ &= (-1)^n\, L_n = L_{-n} \quad . \end{aligned}$$

Hence Lemma 3 holds for all integers n.

In a similar manner the additional lemmas are found.

Lemma 4. $5F_n = L_{n+1} + L_{n-1}$, $n \in Z$.

Lemma 5. $2F_{n+1} = F_n + L_n = F_{n+2} + F_{n-1}$, $n \in Z$.

Lemma 6. $2F_{n-1} = F_n - L_n$, $n \in Z$.

Lemma 7. $F_{n+3} = F_{n+1} + L_{n-1}$, $n \in Z$.

Lemma 8. $3F_n = L_{n+1} - F_{n-1}$, $n \in Z$.

Lemma 9. $3L_{n+2} = L_n + L_{n+4}$, $n \in Z$.

Lemma 10. $3F_{n+2} = F_n + F_{n+4}$, $n \in Z$.

Lemma 11. $2F_{n+1} = L_{n+1} - F_{n-2}$, $n \in Z$.

Lemma 12. $L_n + F_n = F_{n+2} + F_{n-1}$, $n \in Z$.

Although these results are of interest in themselves, their principal use is as lemmas to more profound results. The reader is encouraged to consider additional special cases of formulas (0), and then generate additional Fibonacci and Lucas identities.

The next three results are also generated from formulas (1) and (2). These fundamental identities are essential to our development of Fibonacci and Lucas triples.

Theorem 1. $F_nL_m + F_{n-1}L_{m-1} = L_{n+m-1}$, for any $n, m \in Z$.

Proof. Let m be any fixed integer. Then

$$\begin{aligned}\sum_{n=0}^{\infty}(F_nL_m + F_{n-1}L_{m-1})x^n &= L_m\sum_{n=0}^{\infty}F_n x^n + L_{m-1}\sum_{n=0}^{\infty}F_{n-1}x^n \\ &= L_m\frac{x}{\Delta} + L_{m-1}\frac{(1-x)}{\Delta} \\ &= \frac{L_{m-1} + (L_m - L_{m-1})x}{\Delta} \\ &= \frac{L_{m-1} + L_{m-2}x}{\Delta} \\ &= \sum_{n=0}^{\infty} L_{n+m-1}x^n\end{aligned},$$

by formula (2). Results (0') and (0'') complete the proof.

From a development similar to the above proof, we find a companion result to Theorem 1.

Theorem 2. $F_nF_m + F_{n-1}F_{m-1} = F_{n+m-1}$, for any $n, m \in Z$.

Theorem 3. $L_nL_m + L_{n-1}L_{m-1} = L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$, for any $n, m \in Z$.

Proof. Since $L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$ by Lemma 4, we need only consider the first part of the identity. Let m be any fixed integer. Now

$$\begin{aligned}
\sum_{n=0}^{\infty} (L_nL_m + L_{n-1}L_{m-1})x^n &= L_m \sum_{n=0}^{\infty} L_nx^n + L_{m-1} \sum_{n=0}^{\infty} L_{n-1}x^n \\
&= L_m\left(\frac{2 - x}{\Delta}\right) + L_{m-1}\left(\frac{-1 + 3x}{\Delta}\right) \\
&= \frac{[L_m + (L_m - L_{m-1})] + [2L_{m-1} + (L_{m-1} - L_m)]x}{\Delta} \\
&= \frac{[L_m + L_{m-2}] + [L_{m-1} + (L_{m-1} - L_{m-2})]x}{\Delta} \\
&= \frac{L_m + L_{m-1}x}{\Delta} + \frac{L_{m-2} + L_{m-3}x}{\Delta} \\
&= \sum_{n=0}^{\infty} (L_{n+m} + L_{n+m-2})x^n \quad .
\end{aligned}$$

Aided by the partial fractions technique we find the final result needed to generate the specified Fibonacci and Lucas triples. It is the following:

$$\begin{aligned}
\frac{(p + qx)}{\Delta}\frac{(r + tx)}{\Delta} &= \frac{pr + (pt + qr)x + qtx^2}{\Delta^2} \\
&= \frac{-qt}{\Delta} + \frac{(pr + qt) + (pt + qr - qt)x}{\Delta^2} \quad .
\end{aligned} \tag{3}$$

The identities are now found by convoluting series (generating functions) of the forms (1) and (2). We begin by specifying m and s as fixed integers. Now

$$\begin{aligned}
\frac{F_m + F_{m-1}x}{\Delta} \cdot \frac{L_s + L_{s-1}x}{\Delta} &= \sum_{n=0}^{\infty} F_{n+m}x^n \sum_{n=0}^{\infty} L_{n+s}x^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} F_{k+m}L_{n-k+s}x^n \quad ,
\end{aligned} \tag{4}$$

and by Eq. (3) this product also equals

$$\frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{(F_mL_s + F_{m-1}L_{s-1}) + (F_mL_{s-1} + F_{m-1}L_s - F_{m-1}L_{s-1})x}{\Delta^2}$$

$$= \frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{L_{m+s-1} + (F_{m-1}L_s + F_{m-2}L_{s-1})x}{\Delta^2}$$

by Theorem 1 and substitution of F_{m-2} for $F_m - F_{m-1}$.

$$= -F_{m-1}L_{s-1}\frac{1}{\Delta} + \frac{L_{m+s-1} + L_{m+s-2}x}{\Delta}\frac{1}{\Delta}$$

by Theorem 1

$$= -F_{m-1}L_{s-1}\sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} L_{n+m+s-1}x^n \cdot \sum_{n=0}^{\infty} F_{n+1}x^n$$

by definition of generating functions (1) and (2)

$$= \sum_{n=0}^{\infty} [-F_{m-1}L_{s-1}F_{n+1}]x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}x^n$$

(5) $$= \sum_{n=0}^{\infty}[-F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}]x^n .$$

By equating the coefficients of series (4) and (5), the first identity is deduced. It may be expressed as

$$\sum_{k=0}^{n} F_{k+m}L_{n-k+s} = -F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}$$

or

$$F_{m-1}L_{s-1}F_{n+1} = \sum_{k=0}^{n} (F_{k+1}L_{n-k+m+s-1} + F_{k+m}L_{n-k+s}) .$$

Letting $p = m - 1$, $q = n + 1$, and $r = s - 1$, the identity becomes

Theorem 4.

$$F_pF_qL_r = \sum_{k=0}^{q-1} (F_{k+1}L_{p+q+r-k-1} + F_{p+k+1}L_{q+r-k}) ,$$

for any integers p, q, and r.

One notes the need of definitions (0') and (0'') if any of the above integers is negative.

Following the procedure given above, aided by the given lemmas, Theorems 1-3, and definitions, two additional identities are found. The first is a result of the convolution of

$$\frac{F_m + F_{m-1}x}{\Delta}$$

with

$$\frac{F_t + F_{t-1}x}{\Delta},$$

and the second is determined by the convolution of

$$\frac{L_m + L_{m-1}x}{\Delta}$$

with

$$\frac{L_t + L_{t-1}x}{\Delta}.$$

Theorem 5.

$$F_pF_qF_r = \sum_{k=0}^{r-1}(F_{p+q+k+1}F_{r-k} - F_{p+k+1}F_{r+q-k}),$$

for any $p, q, r \in Z$.

Theorem 6.

$$F_pL_qL_r = \sum_{k=0}^{p-1}(5F_{p-k}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k}),$$

for any $p, q, r \in Z$.

Theorem 7.

$$L_pL_qL_r = 5\left[\sum_{k=0}^{p-2}(F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r}\right] - L_{p+q}L_{r+1},$$

for any $p, q, r \in Z$.

Proof. From Lemma 3, we obtain

$$\begin{aligned} L_pL_qL_r &= (F_{p+1} + F_{p-1})L_qL_r \\ &= F_{p+1}L_qL_r + F_{p-1}L_qL_r . \end{aligned}$$

Now from Theorem 6, it follows that

$$L_pL_qL_r = \sum_{k=0}^{p}(5F_{p-k+1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k+1})$$

$$+\sum_{k=0}^{p-2}(5F_{p-k-1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k-1})$$

$$= \sum_{k=0}^{p-2}\left[5F_{q+r+k+1}(F_{p-k+1} + F_{p-k-1}) - L_{q+k+1}(L_{p+r-k+1} + L_{p+r-k-1})\right]$$

$$+ (5F_2F_{p+q+r} - L_{p+q}L_{r+2}) + (5F_1F_{p+q+r+1} - L_{p+q+1}L_{r+1})$$

$$= 5\sum_{k=0}^{p-2}(F_{q+r+k+1}L_{p-k} - F_{p+r+k}L_{q+k+1})$$

$$+ 5F_{p+q+r+2} - (5F_{p+q+r+1} + L_{p+q}L_{r+1})$$

by Lemmas 2 and 4 and Theorem 4

$$= 5\left[\sum_{k=0}^{p-2}(F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r}\right] - L_{p+q}L_{r+1} .$$

Many corollaries to the last three theorems are immediate by making substitution(s) for p, q, and r, respectively, in the given identities. The formulation and derivation of these results we leave to the reader.

REFERENCES

1. Nathan J. Fine, "Generating Functions," Enrichment Mathematics for High School (NCTM Twenty-eighth yearbook), 1963, Washington 6, D. C., pp. 355-367.
2. V. E. Hoggatt and D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI," The Fibonacci Quarterly, Vol. 5, 1967, pp. 445-460.
3. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 1, 1963, pp. 1-16.
4. John Riordan, "Generating Functions for Powers of Fibonacci Numbers," Duke Math. J., Vol. 29, 1962, pp. 5-12.

A NEW GREATEST COMMON DIVISOR PROPERTY OF THE BINOMIAL COEFFICIENTS

H. W. GOULD
West Virginia University, Morgantown, West Virginia

1. INTRODUCTION

The chief object of this paper is to announce the following:

Conjecture. Let k and n be any integers with $0 \le k \le n$, and

$$\binom{n}{k} = n!/k!(n-k)!$$

be the ordinary binomial coefficients. Then

$$\text{(1.1)} \qquad \gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\} .$$

The consideration of this matter was prompted by a result due to Hoggatt and Hansell [3] which is that

$$\text{(1.2)} \qquad \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} .$$

The six coefficients involved form a hexagonal pattern around $\binom{n}{k}$ in the usual Pascal triangle display. See the diagram in [1] where I called (1.2) a Star of David Property. The new conjecture gives a new Star of David property. What is more, I also conjecture that (1.1) holds for Fibonomial coefficients where $n!$ is replaced by

$$[n]! = F_n F_{n-1} \cdots F_2 F_1 , \quad [0]! = 1 ,$$

with

$$F_{n+1} = F_n + F_{n-1}, \qquad F_0 = 0, \qquad F_1 = 1 ,$$

being the ordinary Fibonacci numbers. The manner in which powers of a prime enter as factors of such generalized coefficients suggests that there are many other arrays in which the new arithmetic Star of David property holds. We shall also exhibit some entirely novel pseudo-binomial coefficient arrays where the conjecture holds. It would be of great interest to establish necessary and/or sufficient conditions for the new conjecture. I am certain the conjecture is correct but hesitate to publish a proof as I believe my original proof has a flaw. Computational results will be exhibited here as evidence.

2. EVIDENCE

Table 1 below shows the situation for 21 rows of the Pascal triangle. Shown here is

$$\gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\},$$

for $0 \le k \le n/2$. In every case the value is identical with

$$\gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\} .$$

Spot checks for dozens of other values have failed to turn up any counterexample. In working with numerical examples, it is convenient to draw the Pascal triangle in the usual manner as

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 2^2 & & 2\cdot 3 & & 2^2 & & 1 & & & \\
 & & 1 & & 5 & & 2\cdot 5 & & 2\cdot 5 & & 5 & & 1 & & \\
 & 1 & & 2\cdot 3 & & \boxed{3\cdot 5} & & 2^2\cdot 5 & & 3\cdot 5 & & 2\cdot 3 & & 1 & \\
1 & & 7 & & 3\cdot 7 & & 5\cdot 7 & & 5\cdot 7 & & 3\cdot 7 & & 7 & & 1
\end{array}$$

but in factored form. The way in which the primes appear suggests both (1.1) and (1.2). Because of the recurrence relation governing formation of the binomial coefficients (and the same principle applies to the Fibonomial coefficients) the occurrence of prime factors forms a triangular pattern. Thus, if

$$p^a \,\Big|\, \binom{n}{k} \quad \text{and} \quad p^b \,\Big|\, \binom{n}{k-1} ,$$

then

$$p^c \,\Big|\, \binom{n+1}{k}$$

where $c = \min(a, b)$. But c may be larger!

Let us denote the set of coefficients

$$\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\}$$

by $\lhd$ and the set

$$\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}$$

by $\rhd$, or more generally, we may sometimes use this suggestive notation for the corresponding sets in any general array. If we must be explicit we can write $\lhd_{n,k}$ and $\rhd_{n,k}$, to indicate the values of n and k used. Clearly, if we compute a table of g.c.d. $\lhd$ and the table is symmetrical with an entry in the k spot on row n the same as the entry in the n - k spot, then the property (1.1) holds. This is because of the similar symmetry for the Pascal triangle itself. Table 1, therefore, lists g.c.d. $\lhd$ for $0 \le k \le n/2$ only. The original table was drawn up on a very large sheet of paper and is not easy to reproduce here.

Table 1

n	0	1	2	3	4	···		k	···	[n/2]	
0	1										
1	1										
2	1	1									
3	1	1									
4	1	1	1								
5	1	1	1								
6	1	1	1	5							
7	1	1	1	1							
8	1	1	1	7	7						
9	1	1	1	2	14						
10	1	1	1	3	6	42					
11	1	1	1	5	3	6					
12	1	1	1	11	11	33	66				
13	1	1	1	1	11	11	33				
14	1	1	1	13	13	143	143	429			
15	1	1	1	7	91	91	143	143			
16	1	1	1	5	7	91	13	143	715		
17	1	1	1	4	4	28	52	26	286		
18	1	1	1	17	68	68	68	442	442	4862	
19	1	1	1	3	51	204	204	102	442	442	
20	1	1	1	19	57	969	3876	1938	646	8398	8398

A result like (1.1) using l.c.m. is in general false. The first simple counter-example is

$$\operatorname{lcm} \lhd_{3,1} = \operatorname{lcm}\left\{\binom{2}{1}, \binom{3}{0}, \binom{4}{2}\right\} = \operatorname{lcm}(2, 1, 6) = 6 ,$$

whereas

$$\operatorname{lcm} \triangleright_{3,1} = \operatorname{lcm}\left\{\binom{2}{0}, \binom{3}{2}, \binom{4}{1}\right\} = \operatorname{lcm}(1, 3, 4) = 12 .$$

There are, however, numerous cases where the l. c. m. property does hold.

Except for the first value, it is interesting to note that the sequence of middle numbers in Table 1, i.e., 1, 1, 1, 5, 7, 42, 66, 429, 715, 4862, 8398, ···, are alternately Catalan numbers or one-half Catalan numbers. More precisely: let $n \geq 1$. Then

$$\gcd\left\{\binom{2n-1}{n}, \binom{2n}{n-1}, \binom{2n+1}{n+1}\right\} = \begin{cases} \binom{2n}{n}\dfrac{1}{n+1}, & 2 \nmid n, \\ \dfrac{1}{2}\binom{2n}{n}\dfrac{1}{n+1}, & 2 \mid n. \end{cases} \tag{2.1}$$

We omit the proof.

3. THE FIBONOMIAL CASE

The corresponding result for the Fibonomial coefficients to (1.1) is true because these numbers satisfy a recurrence relation similar to that for the ordinary binomial coefficients. We should remark that the same may be said for the Gaussian or q-binomial coefficients. We omit the details of the proof.

To illustrate the relation (1.1) for Fibonomial coefficients, we give in Table 2 some specimen values. The table starts with $n = 6$, the first row where the g. c. d. > 1 for any k.

Table 2

n	0	1	2	3 ···	k ···	[n/2]
6	1	1	1	5		
7	1	1	1	4		
8	1	1	1	13	52	
9	1	1	1	7	91	
10	1	1	1	17	119	1547
11	1	1	1	55	187	1309

Again one finds a formula for Fibonomial Catalan numbers, but it is not as simple as (2.1).

4. PSEUDO-BINOMIAL COEFFICIENTS

Scrutiny of the discussion above for (1.1) shows that the key to the pattern of prime powers lies in the recurrence relation used. However, we may evidently dispense with the recurrence relation and still have (1.1). To illustrate, we offer the array on the following page of pseudo-binomial coefficients.

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad 1 \qquad 1 \\
1 \qquad 2 \qquad 2 \qquad 1 \\
1 \qquad 3 \qquad 2\cdot3 \qquad 3 \qquad 1 \\
1 \qquad 5 \qquad 3\cdot5 \qquad 3\cdot5 \qquad 5 \qquad 1 \\
1 \qquad 7 \qquad 5\cdot7 \qquad 3\cdot5\cdot7 \qquad 5\cdot7 \qquad 7 \qquad 1 \\
1 \qquad 11 \qquad 7\cdot11 \qquad 5\cdot7\cdot11 \qquad 5\cdot7\cdot11 \qquad 7\cdot11 \qquad 11 \qquad 1 \\
1 \qquad 13 \qquad 11\cdot13 \qquad 7\cdot11\cdot13 \qquad 5\cdot7\cdot11\cdot13 \qquad 7\cdot11\cdot13 \qquad 11\cdot13 \qquad 13 \qquad 1
\end{array}$$

Here we have imposed a perfectly regular pattern of appearance of prime factors. It is easy to see that (1.1) must hold for the pseudo-binomial coefficients $P(n,k)$. A few specimen rows from the g. c. d. triangle are:

$$\begin{array}{c}
5 \\
7 \qquad 7 \\
11 \qquad 7\cdot11 \qquad 11 \\
13 \qquad 11\cdot13 \qquad 11\cdot13 \qquad 13 \\
17 \qquad 13\cdot17 \qquad 11\cdot13\cdot17 \qquad 13\cdot17 \qquad 17
\end{array}$$

where we have tabulated the g. c. d. for $3 \le k \le n - 3$ and $6 \le n \le 10$.

It is also evident that the resulting array itself possesses property (1.1), and this may be seen to repeat forever. The l. c. m. of the two sets of coefficients in (1.1) fail to be equal for the pseudo-binomial coefficients for $k = 0$ $(n \ge 2)$, and for $k = 2$ $(n = 5)$, $k = 3$ $(n = 7)$, $k = 4$ $(n = 9)$, etc. We omit a discussion of the precise behavior of the least common multiples, but it is clearly a matter to be investigated. I have been unable to find an array in which the g. c. d. property and l. c. m. property both hold always. Even l. c. m. arrays are hard to come by.

In contrast to the Pascal triangle and the Fibonomial triangle, the array of pseudo-binomial coefficients does not have the property (1.2) of Hoggatt-Hansell.

Here is still another pseudo-binomial array having the Star of David property (1.1):

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad 2 \qquad 1 \\
1 \qquad 3 \qquad 3 \qquad 1 \\
1 \qquad 7 \qquad 3\cdot7 \qquad 7 \qquad 1 \\
1 \qquad 1 \qquad 7 \qquad 7 \qquad 1 \qquad 1 \\
1 \qquad 2^3 \qquad 2 \qquad 2\cdot7 \qquad 2 \qquad 2^3 \qquad 1 \\
1 \qquad 5^2 \qquad 2\cdot5^2 \qquad 2\cdot5^2 \qquad 2\cdot5^2 \qquad 2\cdot5^2 \qquad 5^2 \qquad 1
\end{array}$$

One may easily extend such a triangle in an infinity of ways.

These are the types of general array suggested by our work, arrays in which the entry of primes occurs in carefully delineated triangles. The most general such triangle has not been written out.

5. MULTINOMIALS

It is, of course, tempting to go further. In [1], [2], [4] will be found methods for finding equal products of any number of binomial and multinomial coefficients in general. Whenever a triangle pattern of prime entry appears, one suspects that interesting g. c. d. and l. c. m. properties will hold in certain cases. Computer calculations would be very useful to make further conjectures, but already I have checked numerous cases and found interesting results. When one realizes that Scharff, Rine, and Gould [2] have found relations such as

$$\binom{n+2}{k-1}\binom{n-3}{k}\binom{n+3}{k+1}\binom{n-2}{k-2}\binom{n+1}{k+2}\binom{n}{k-3}\binom{n-1}{k+3}$$
$$= \binom{n-2}{k-1}\binom{n+3}{k}\binom{n-3}{k+1}\binom{n+1}{k-2}\binom{n+2}{k+2}\binom{n-1}{k-3}\binom{n}{k+3},$$

it becomes clear that there is much more to be investigated. When, for example, are the g. c. d. 's of the above sets of seven coefficients equal? Not in general, as examples are easily shown to the contrary. A computer can easily generate as many tables of this sort as needed. We should remark that the detailed computer print-out in [2] will be deposited in the Fibonacci Bibliographical Center for reference.

In [1] I pointed out that (1.2) generalizes to

$$\binom{n-a}{k}\binom{n}{k-a}\binom{n+n}{k+a} = \binom{n-a}{k-a}\binom{n}{k+a}\binom{n+a}{k}$$

and it is tempting to see if the g. c. d. property holds here. A simple counter-example, $n = 8$, $k = 3$, $a = 2$ suffices to show that the g. c. d. Star of David property does not hold in general here. Again, however, abundant true examples exist.

ADDENDUM

Property (1.1) was first noted by me around December 1971. Since writing the present paper (1.1) was mentioned to Hoggatt (telephone call, August 3, 1972), and I have now heard from him (telephone call August 7) that he and A. P. Hillman [5] have proved conjecture (1.1) as well as for the Fibonomial case and for arrays in general where certain recurrences hold, The method is one due to Hillman based on iteration and the recurrence. Clearly we are at the opening of a new chapter in the discovery of interesting arithmetic properties of arrays of numbers.

[Continued on page 628.]

LINEAR DIFFERENCE EQUATIONS AND GENERALIZED CONTINUANTS
PART I: ALGEBRAIC DEVELOPMENTS

L. R. SHENTON
Computer Center, University of Georgia, Athens, Georgia

1. INTRODUCTION

A continuant determinant (or matrix) has elements in the diagonals through (1,1), (1,2), and (2,1) only, and zeros elsewhere. We can use the notation $K_s(h_1{}^{g_1}_{g_1'})$ for the s^{th} order continuant, where

$$(1) \qquad \begin{vmatrix} h_1 & g_1 & 0 & & \\ g_1' & h_2 & g_2 & & \\ 0 & g_2' & & & \\ & & & & g_s \\ & & & g_s' & h_s \end{vmatrix}_{(s)}$$

As is well known, by expanding this by its last row and column, we find the recurrence relation (omitting the arguments for brevity)

$$(2) \qquad K_s = h_s K_{s-1} - g_s' g_s K_{s-2} \qquad s = 2, 3, \cdots$$

with $K_0 = 1$, $K_1 = h_1$. Note that K_s is unchanged in value if the signs are changed for any subset of the g's along with the corresponding subset of the g''s. Again note that the usual Fibonacci sequence arises from either $g_\lambda = 1$, $g_\lambda' = -1$ (or of course $g_\lambda = -1$, $g_\lambda' = 1$) or $g_\lambda = g_\lambda' = i$ $(= \sqrt{-1})$.

Many elementary properties of recursive schemes such as (2) are well known and in particular Brother Alfred Brousseau [1] has given some of these in the case when the coefficients are constants.

The question arises as to what happens when we add diagonals to (1) through (1,3) and (3,1) and produce a 5-diagonal determinant. We shall call a $(2s + 1)$ diagonal determinant (with elements in the main diagonal and the s super-diagonals, and the s sub-diagonals) a continuant of degree s. The recursions followed by these generalized continuants have been studied by H. D. Ursell [2]. In fact, Ursell gives the following table which refers to the order of the difference equation satisfied by a continuant of degree s:

Order of Recurrence Relation

Degree s	1	2	3	4	5	6
Symmetric Case	2	5	15	49	169	604
Unsymmetric Case	2	6	20	70	252	924

The rate of increase of the difference equation order is very remarkable.

2. THE FIVE DIAGONAL SYMMETRIC CONTINUANT

We use the notation $K_s(h_1, g_1, f_1)$ for a second-degree symmetric continuant with elements $h_1, h_2, \cdots,$ in the principal diagonal, $g_1, g_2, \cdots,$ on the diagonal through (1,2) and (2,1), $f_1, f_2, \cdots,$ on the diagonals through (1,3) and (3,1) and zeros elsewhere. The fifth-order recurrence is then given by (see [3], p. 173, expression (16))

$$
\begin{aligned}
g_{s-2}K_s &= a_sK_{s-1} - b_s(g_{s-1}K_{s-2} - g_{s-2}f_{s-2}K_{s-3}) \\
&\quad - f^2_{s-3}f_{s-2}c_sK_{s-4} + f_{s-2}f^2_{s-3}f^2_{s-4}g_{s-1}K_{s-5}
\end{aligned} \tag{3}
$$

where $s = 3, 4, \cdots,$ with

$$K_{-2} = K_{-1} = 0, \quad K_0 = 1, \quad K_1 = h_1 ,$$

$$K_2 = h_1h_2 - g_1^2 ,$$

where

$$
\begin{aligned}
a_s &= h_sg_{s-2} - f_{s-2}g_{s-1} , \\
b_s &= g_{s-1}g_{s-2} - h_{s-1}f_{s-2} , \\
c_s &= h_{s-2}g_{s-1} - f_{s-2}g_{s-2} .
\end{aligned}
$$

We discuss several special cases.

2.1 $g_1 = g_2 = \cdots = g_{s-1} = 0$. We now have to expand K_s by its last row and column since formula (3) aborts. We find

$$K_s = h_sK_{s-1} - f^2_{s-2}h_{s-1}K_{s-3} + f^2_{s-2}f^2_{s-3}K_{s-4} \qquad (s = 4, 5, \cdots) \tag{4}$$

with

$$
\begin{aligned}
K_0 &= 1 , \\
K_1 &= h_1 , \\
K_2 &= h_1h_2 , \\
K_3 &= h_2(h_1h_3 - f_1^2) .
\end{aligned}
$$

Using (4) we find for the next few cases,

$$
\begin{aligned}
K_4 &= (h_1h_3 - f_1^2)(h_2h_4 - f_2^2) , \\
K_5 &= (h_2h_4 - f_2^2)\big(h_5(h_1h_3 - f_1^2) - h_1f_3^2\big)
\end{aligned}
$$

indicating that K_s is the product of two continuants of degree 1 (three diagonals). This is easily seen from the determinant for K_s by expanding by sub-matrices consisting of elements from odd rows (and columns). For example,

$$K_7 = \begin{vmatrix} h_1 & f_1 & 0 & 0 \\ f_1 & h_3 & f_3 & 0 \\ 0 & f_3 & h_5 & f_5 \\ 0 & 0 & f_5 & h_7 \end{vmatrix} \begin{vmatrix} h_2 & f_2 & 0 \\ f_2 & h_4 & f_4 \\ 0 & f_4 & h_6 \end{vmatrix} \tag{5}$$

and this type of condensation has been given by Muir [4]. We may verify directly from (4) that K_s does in fact factor, and defining first degree continuants

$$K_s^{(2)}(h_1,\ f_1) = \begin{vmatrix} h_1 & f_1 & & \\ f_1 & h_3 & & \\ & & & f_{2s-3} \\ & & f_{2s-3} & h_{2s-1} \end{vmatrix}_{(s)} \tag{6a}$$

$$K_s^{(2)}(h_2,\ f_2) = \begin{vmatrix} h_2 & f_2 & & \\ f_2 & h_4 & & \\ & & & f_{2s-2} \\ & & f_{2s-2} & h_{2s} \end{vmatrix}_{(s)} \tag{6b}$$

it can be demonstrated that

$$\begin{aligned} K_{2s}(h_1,\ 0,\ f_1) &= K_s^{(2)}(h_1,\ f_1)K_s^{(2)}(h_2,\ f_2)\ , \\ K_{2s+1}(h_1,\ 0,\ f_1) &= K_{s+1}^{(2)}(h_1,\ f_1)K_s^{(2)}(h_2,\ f_2)\ . \end{aligned} \tag{7}$$

In particular taking $h_s = 1$, $f_s = i$ in (4) we see that the sequence (K_s) where

$$K_s = K_{s-1} + K_{s-3} + K_{s-4} \qquad (s = 4, 5, \cdots) \tag{8}$$

with $K_0 = 1$, $K_1 = 1$, $K_2 = 1$, $K_3 = 2$, is such that K_{2s-1} is the product of consecutive Fibonacci numbers whereas K_{2s} is the square of a Fibonacci number. For example,

s	4	5	6	7	8	9	10	11	12
K_s	2^2	$2\cdot3$	3^2	$3\cdot5$	5^2	$5\cdot8$	8^2	$8\cdot13$	13^2

It is perhaps not surprising to find the characteristic equation of (8) has zeros $\pm i$, $(1 \pm \sqrt{5})/2$, and indeed

$$K_s = \frac{(2-i)}{10}\, i^s + \frac{(2+i)}{10}\,(-i)^s + \left(\left(\frac{1+\sqrt{5}}{2}\right)^{s+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{s+2}\right)\Big/ 5\ . \tag{9}$$

Again since the characteristic equation has a zero with largest modulus, then

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_s} = \frac{1+\sqrt{5}}{2}\ .$$

2.2 Constant Elements in the Diagonals.

We consider K_s (h, g, f) where h, g, f are either unity in modulus, or zero. The following seem to be the most interesting:

Case	h	g	f
1	0	1	1
2	1	1	-1
3	1	i	-1
4	1	i	1

$(i = \sqrt{-1})$

Case 1

$$K_s = -K_{s-1} - K_{s-2} + K_{s-3} + K_{s-4} + K_{s-5} \qquad s = 3, 4, \cdots$$

with

$$K_{-2} = K_{-1} = 0, \quad K_0 = 1, \quad K_1 = 0, \quad K_2 = -1 .$$

In addition

s	3	4	5	6	7	8	9	10	11	12
K_s	2	0	-2	3	0	-3	4	0	-4	-3

Characteristic Equation

$$(x - 1)(x^2 + x + 1)^2 = 0$$

Roots $x = 1, w, w, w^2, w^2,$ where w is a primitive cube root of unity.

Explicit Formula

$$K_s = \frac{2}{9} + \left(1 + 4w + s(1 + 2w)\right)\left(\frac{w}{9}\right)^{s-1} - \left(1 - 3w + s(1 - w)\right)\left(\frac{w}{9}\right)^{2s-1}$$

from which

$$K_{3s} = s + 1, \quad K_{3s+1} = 0, \quad K_{3s+2} = -s - 1 .$$

Case 2

$$K_s = 2K_{s-1} - 2K_{s-2} - 2K_{s-3} + 2K_{s-4} - K_{s-5}$$

s	K_s	Δ_s $\left(= \sqrt{(K_s^2 - K_{s-1}K_{s+1})}\right)$
0	1	—
1	1	1
2	0	2
3	-4	4
4	-8	6
5	-7	11
6	9	19
7	40	32
8	64	56
9	24	96
10	-135	165
11	-375	285
12	-440	490
13	124	844
14	1584	1454
15	3185	2503

Characteristic Roots

$$x_1 = \left(\sqrt{3}\, e^{i\pi/6} + \sqrt[4]{13}\, e^{i\alpha/2}\right)\Big/ 2 ,$$
$$x_2 = 1/x_1 ,$$
$$x_3 = \bar{x}_1 \qquad \text{(conjugate)} ,$$
$$x_4 = 1/\bar{x}_1 ,$$
$$x_5 = 1 ,$$

where $\tan\alpha = 3\sqrt{3}/5$.

The roots of greatest modulus being complex, "explains" the apparently unpredictable behavior of K_s. On the other hand, notice that $K_s^2 - K_{s-1}K_{s+1}$ is always a perfect square, and in fact Δ_s follows the recurrence

$$\Delta_s = \Delta_{s-1} + \Delta_{s-2} + \Delta_{s-3} - \Delta_{s-4} \qquad (s = 2, \cdots)$$

with

$$\Delta_{-1} = 0, \qquad \Delta_0 = 1, \qquad \Delta_1 = 1 ,$$

and characteristic roots

$$x_1 = -\left(\sqrt{13} + 1 + \sqrt{(2\sqrt{13} - 2)}\right)\Big/ 4 ,$$
$$x_2 = -\left(\sqrt{13} + 1 - \sqrt{(2\sqrt{13} - 2)}\right)\Big/ 4 ,$$
$$x_3 = \left(\sqrt{13} - 1 + i\sqrt{(2\sqrt{13} + 2)}\right)\Big/ 4 ,$$
$$x_4 = \left(\sqrt{13} - 1 - i\sqrt{(2\sqrt{13} + 2)}\right)\Big/ 4 ,$$

in which x_1 has the greatest numerical value, and $|x_3| = |x_4| = 1$. Actually it can be shown that

$$\lim_{s\to\infty} \frac{\Delta_{s+1}}{\Delta_s} = \frac{\sqrt{13} + 1 + \sqrt{2(\sqrt{13} - 1)}}{4} .$$

Case 3

$$K_s = 2K_{s-1} + 2K_{s-4} - K_{s-5} \qquad (s = 4, 5, \cdots)$$

with

s	0	1	2	3	4	5	6	7	8	9	10
K_s	1	1	2	4	10	21	45	96	208	432	933

Characteristic Roots

$$x_1 = 1$$

$$x_{2,3,4,5} = \frac{3 \pm \sqrt{5} \pm \sqrt{(6\sqrt{5} - 2)}}{4}$$

$$\text{Magnitude of largest root} = \left(3 + \sqrt{5} + \sqrt{(6\sqrt{5} - 2)}\right)\Big/ 4$$

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_s} = \frac{3 + \sqrt{5} + \sqrt{(6\sqrt{5} - 2)}}{4}$$
$$= 2.1537$$

Comments (i) K_s is always positive

(ii) $\sqrt{|K_{s+1}K_{s-1} - K_s^2|}$ is an integer.

Case 4

$$K_s = 2K_{s-2} - 2K_{s-3} + K_{s-5}$$

s	0	1	2	3	4	5	6	7	8	9	10
K_s	1	1	2	0	2	-3	5	-8	16	-24	45

Characteristic Roots

$$x_1 = 1 \qquad x_{2,3} = \frac{-(1 + \sqrt{13}) \pm \sqrt{2\sqrt{13} - 2}}{4}$$

$$x_{4,5} = \frac{-(1 - \sqrt{13}) \pm i\sqrt{2\sqrt{13} + 2}}{4}$$

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_0} = -\frac{1}{4}\left\{(1 + \sqrt{13}) + \sqrt{2\sqrt{13} - 2}\right\}$$

$$= -1.7221 \qquad .$$

3. FACTORABLE CONTINUANTS

A number of these have been given by D. E. Rutherford [5], [6]. In particular, Rutherford remarks that the n^{th} Fibonacci number can be expressed as

$$\prod_{r=1}^{n-1}\left(1 - 2i\cos\frac{r\pi}{n}\right) \quad . \tag{10}$$

Moreover, although he does not give the recurrence relation, he quotes the factors of (in our notation) K_s (z, 2a, 1), where

$$K_s \equiv \begin{vmatrix} z, & 2a, & 1, & & & & & & \\ 2a, & z, & 2a, & 1, & & & & & \\ 1, & 2a, & z, & 2a, & 1, & & & & \\ & 1, & 2a, & z, & 2a, & 1, & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & 1, & 2a, & z \end{vmatrix}_{(s)}$$

as

$$\frac{1}{2(\cos 2\alpha - \cos 2\beta)}\left\{\frac{\sin^2(s+2)\alpha}{\sin^2\alpha} - \frac{\sin^2(s+2)\beta}{\sin^2\beta}\right\} \quad ,$$

where

[Continued on page 634.]

TRIANGULAR ARRAYS SUBJECT TO MAC MAHON'S CONDITIONS

L. CARLITZ[1]
Duke University, Durham, North Carolina
and
D. P. ROSELLE[2]
Louisiana State University, Baton Rouge, Louisiana

1. INTRODUCTION

We consider triangular arrays (n_{ij}) $(j = i(1)k,\ i = 1(1)k)$ and (a_{rs}) $(s = 1(1)k + 1 - r,\ r = 1(1)k)$ and let $T(n,k)$ and $C(n,k)$, respectively, denote the number of these arrays in which the entries are non-negative integers subject to the conditions

$$n_{ij} \ge n_{i,j+1}, \qquad n_{ij} \ge n_{i+1,j}, \qquad n_{11} \le n \tag{1.1}$$

$$a_{rs} \ge a_{r,s+1}, \qquad a_{rs} \ge a_{r+1,s}, \qquad a_{11} \le n\ . \tag{1.2}$$

The conditions (1.1) and (1.2) are the same as MacMahon [3] imposed on multi-rowed partitions. Rectangular arrays subject to these conditions have been considered by Carlitz and Riordan [1].

It is easy to evaluate $T(1,k)$ and $C(1,k)$. Indeed, taking row sums, we find that $T(1,k)$ is the number of sequences $j_1, \cdots, j_k$ with $j_i > j_{i+1}$ and $j_1 \le k$. It follows that $T(1,k) = 2^k$. In the same way, we find that $C(1,k)$ is the number of sequences $j_1, \cdots, j_k$ with $k + 1 - i \ge j_i \ge j_{i+1}$. Hence $C(1,k)$ is the familiar Catalan number (c.f. [2])

$$C(1,k) = \frac{1}{k+2}\binom{2k+2}{k+1}\ . \tag{1.3}$$

It will be convenient to have an alternative description of $C(n,k)$ and $T(n,k)$. With each array counted by $T(n,k)$ we associate the $n \times k$ array $M = (m_{ij})$, where m_{ij} is the number of elements in the j^{th} row which are greater than or equal to i. Similarly, with each array counted by $C(n,k)$, associate the $n \times k$ array $B = (b_{ij})$, where b_{ij} is the number of elements in the j^{th} column which are greater than or equal to i. That is, $m_{ij} = \text{card}\{n_{jt} | n_{jt} \ge i\}$ and $b_{ij} = \text{card}\ \{a_{tj} | a_{tj} \ge i\}$. It then follows that the entries of the associated array are subject to the conditions

$$m_{ij} \ge m_{i,j+1}, \qquad m_{ij} \ge m_{i+1,j}, \qquad m_{11} \le k\ , \tag{1.4}$$

[1]Supported in part by National Science Foundation Grant GP17071.
[2]Supported in part by National Science Foundation Grant GP11397.

(1.5) $$b_{ij} \ge b_{i,j+1}, \qquad b_{ij} \ge b_{i+1,j}, \qquad b_{ij} \le k + 1 - j .$$

It also is not difficult to verify that the $n \times k$ arrays subject to (1.4) and (1.5) are equinumerous with those counted by $T(n,k)$ and $C(n,k)$.

Here we prove that

(1.6) $$T(2,k) = \binom{2k+1}{k},$$

(1.7) $$T(3,k) = 2^k \binom{2k+2}{k+1} - 2^k \binom{2k+2}{k}$$

as well as

(1.8) $$C(n,k) = \det \left[\binom{n+k+1-r}{n+r-s} \right] \qquad (r,s = 1, \cdots, k) .$$

It is also shown that

$$\sum_{n=0}^{\infty} C(n,k)x^n = A_k(x) \cdot (1-x)^{\frac{-k(k+1)}{2} - 1}$$

where $A_k(x)$ is a polynomial of degree $\frac{1}{2}k(k-1)$ with integral coefficients and which satisfies the symmetry condition

(1.9) $$x^{\frac{1}{2}k(k-1)} A_k\left(\frac{1}{x}\right) = A_k(x) .$$

2. TRIANGULAR ARRAYS

We consider triangular arrays

(2.1) $$\begin{matrix} n_{11} & n_{12} & \cdots & n_{1k} \\ & n_{22} & \cdots & n_{2k} \\ & & \cdots & \\ & & & n_{kk} \end{matrix}$$

and let $T^*(n,k)$ denote the number of these arrays with non-negative integral coefficients satisfying

(2.2) $$n_{11} = n, \qquad n_{ij} \ge n_{i,j+1}, \qquad n_{ij} \ge n_{i+1,j} ,$$

We also put

$$T(n,k) = \sum_{j=0}^{n} T^*(j,k) .$$

It is immediate that $T(0,k) = T^*(0,k) = 1$ and as observed in Section 1, it is easy to see that $T^*(1,k) = 2^k - 1$. This can also be seen by classifying the arrays according as $n_{11} = 0$ or 1 and noting that this implies the recurrence

$$T^*(1,k) = T^*(1,k - 1) + T(1,k - 1) .$$

A simple verification of the boundary conditions is then all that is necessary to anchor the induction.

Next let $Q(m_{11}, m_{21}, \cdots, m_{n1})$ denote the number of $n \times k$ arrays $M = (m_{ij})$, where the m_{ij} are subject to the conditions (1.4). It is clear from the remarks of Section 1 that

$$T^*(n,k) = \sum Q(s_1, \cdots, s_n) ,$$

where the summation extends over all n-tuples $(s_1, \cdots, s_n)$ for which $k \geq s_1 \geq \cdots \geq s_n \geq 1$. A more useful reformulation of these remarks is the observation that

$$T(n,k) = Q(k + 1, k + 1, \cdots, k + 1) . \tag{2.3}$$

For the case $n = 2$, we find that

$$\begin{aligned} Q(m,r) &= 1 + \sum_{s=1}^{m-1} Q(s) + \sum_{t=1}^{r-1} \sum_{s=t}^{m-1} Q(s,t) \\ &= 2^{m-1} + \sum_{t=1}^{r-1} \sum_{s=t}^{m-1} Q(s,t) , \end{aligned}$$

where we have used (2.3) for the case $n = 1$. A more convenient form of this last equation is

$$Q(m,r + 1) = \sum_{s=r}^{m} Q(s,r) .$$

It is now a simple induction to show that

$$Q(m, r+1) = 2^{m+r-1} - \sum (2^{r-j} - 1)\binom{m+j-1}{j} \qquad (m \geq r+1)$$

which should be compared with [1, Eq. (1.9)]. In particular, we have

$$\begin{aligned} Q(m+1, m+1) &= 2^{2m} - \sum_{j=0}^{m} (2^{m-j} - 1)\binom{m+j}{j} \\ &= \binom{2m+1}{m} \quad . \end{aligned}$$

It now follows from (2.3) that

$$T^*(2,k) = \binom{2k+1}{k} \quad . \tag{2.4}$$

3. THE CASE $n = 3$

The evaluation of $T(3,k)$ is more complicated but leads to a simple result. Let $Q_c(m_{11}, m_{21}, m_{31})$ denote the number of $3 \times c$ arrays (m_{ij}) whose entries are non-increasing down each column and whose positive entries are strictly decreasing along each row. Then, according to the remarks of Section 1, we have

$$T(3,k) = Q_{k+2}(k+1, k+1, k+1) . \tag{3.1}$$

It is not difficult to show (by induction on c) that

$$Q_{c+1}(r,s,t) = \sum_{i \leq j \leq k} D_{c-2i, c-2j-1, c-2k-2} , \tag{3.2}$$

where we put

$$D_{i,j,k} = \begin{vmatrix} \binom{r}{i} & \binom{s}{i+1} & \binom{t}{i+2} \\ \binom{r}{j} & \binom{s}{j+1} & \binom{t}{j+2} \\ \binom{r}{k} & \binom{r}{k+1} & \binom{t}{k+2} \end{vmatrix} \quad .$$

In particular, for $c = m = r = s = t$, it follows from (3.1) that

$$T(3,m-1) = \sum_{i\le j\le k} \begin{vmatrix} \binom{m}{2i} & \binom{m}{2i-1} & \binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \binom{m}{2k+2} & \binom{m}{2k+1} & \binom{m}{2k} \end{vmatrix}$$

$$= \sum_j \begin{vmatrix} \sum_{i\le j}\binom{m}{2i} & \sum_{i\le j}\binom{m}{2i-1} & \sum_{i\le j}\binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \sum_{k\ge j}\binom{m}{2k+2} & \sum_{k\ge j}\binom{m}{2k+1} & \sum_{k\ge j}\binom{m}{2k} \end{vmatrix}$$

$$= \sum_j \begin{vmatrix} \sum_{i\le j}\binom{m}{2i} & \sum_{i\le j}\binom{m}{2i-1} & \sum_{i\le j}\binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \sum_{k}\binom{m}{2k} & \sum_{k}\binom{m}{2k+1} & \sum_{k}\binom{m}{2k} \end{vmatrix}$$

$$= 2^{m-1} \sum_{i\le j} \begin{vmatrix} \binom{m}{2i} & \binom{m}{2i-1} & \binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2^{m-1} \sum_{i<j} \left\{ \binom{m}{2i}\binom{m}{2j} + \binom{m}{2j+1}\binom{m}{2i-2} + \binom{m}{2i-1}\binom{m}{2j-1} \right.$$

$$\left. - \binom{m}{2i-1}\binom{m}{2j+1} - \binom{m}{2i}\binom{m}{2j-1} - \binom{m}{2j}\binom{m}{2i-2} \right\}$$

$$= 2^{m-1}\left\{ \sum_j \binom{m}{j}^2 - \sum_j \binom{m}{j}\binom{m}{j+1} \right\} .$$

This reduces to

$$T(3,k) = 2^k\binom{2k+2}{k+1} - 2^k\binom{2k+2}{k}, \tag{3.3}$$

$$T^*(3,k) = 2^k\binom{2k+2}{k+1} - 2^k\binom{2k+2}{k} - \binom{2k+1}{k}. \tag{3.4}$$

It appears unlikely that this method would lead to a simple result for $T(n,k)$ even though (3.2) can be generalized in an obvious manner.

4. CATALAN DETERMINANTS

We consider triangular arrays

$$\begin{matrix} a_{11} \cdots a_{1,k-1} & a_{1k} \\ a_{21} \cdots a_{2,k-1} & \\ a_{k1} & \end{matrix} \tag{4.1}$$

and let $C(n,k)$ denote the number of these arrays with

$$a_{11} \le n, \qquad a_{ij} \ge a_{i,j+1}, \qquad a_{ij} \ge a_{i+1,j}. \tag{4.2}$$

Then, as observed in Section 1, we have that $C(n,k)$ is also the number of $n \times k$ arrays $B = (b_{ij})$ subject to the conditions (1.5). Also, if we put $C(j_1, \cdots, j_k)$ equal to the number of arrays (4.1) with $a_{1s} = j_s$, then we find that

$$C(j_i, \cdots, j_k) = \sum_{r_{k-1}} \cdots \sum_{r_1} C(r_1, \cdots, r_{k-1}), \tag{4.3}$$

where the i^{th} summand extends over the range $r_{k+1-i} \le r_{k-i} \le j_{k-i}$ and, for convenience, we put $r_k = 0$.

It is an easy induction to show that (4.3) is the same as

$$C(j_1, \cdots, j_k) = \det\left[\binom{j_s + k - r}{k + s - 2r}\right] \qquad (r,s = 1,2,\cdots,k-1).$$

In particular, we find that

$$C(n,k) = \det\left[\binom{n+k+1-r}{n+r-s}\right] \qquad (r,s = 1,\cdots,k). \tag{4.4}$$

Notice that the special case (1.3) follows from (4.4) and the identity

$$\frac{1}{k+2}\binom{2k+2}{k+1} = \sum_{j=0}^{k} (-1)^j \binom{k+1-j}{j+1} \frac{1}{k+1-j}\binom{2k-2j}{k-j} \quad .$$

In the next place if we write (4.4) in the form

$$C(n,k) = \det\left[\binom{n+k+1-r}{k+1-2r+s}\right], \tag{4.5}$$

then we can use this determinant to define $C(n,k)$ for all real numbers n. According to this definition, we find that $C(n,k)$ is a polynomial of degree $\frac{1}{2}k(k+1)$ in n and satisfies the equation

$$C(n,k) = (-1)^{\frac{1}{2}k(k+1)} C(-k-n-1,k) \quad . \tag{4.6}$$

Hence if we put

$$C(n,k) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\binom{n+j}{j} \quad , \tag{4.7}$$

then we have

$$\begin{aligned} C(-k-n-1,k) &= \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\binom{-k-n+j-1}{j} \\ &= \sum_{j=k}^{\frac{1}{2}k(k+1)} (-1)^j a_{kj}\binom{k+n}{j} \quad . \end{aligned}$$

In order to summarize these results in terms of generating functions, we first put $C_k(x) = \sum C(n,k)x^n$ and note that

$$C_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}(1-x)^{-j-1}$$

and

$$\begin{aligned} (-1)^{\frac{1}{2}k(k+1)} C_k(x) &= \sum_{n=0}^{\infty} C(-k-n-1,k)x^n \\ &= \sum_{j=k}^{\frac{1}{2}k(k+1)} (-1)^j a_{kj}\, x^{j-k} (1-x)^{-j-1} \, . \end{aligned}$$

[Continued on page 658.]

2. H. W. Gould, "A New Greatest Common Divisor Property of the Binomial Coefficients," Fibonacci Quarterly, Vol. 10, No. 6 (1972), pp. 579-584.
3. V. E. Hoggatt, Jr., "Fibonacci Numbers and Generalized Binomial Coefficients," Fibonacci Quarterly, Vol. 5, No. 4 (1967), pp. 383-400.

[Continued from page 570.]

for $1 \leq N < a_{n+1}$, $1 \leq d(N) \leq n$, and since the sets $\{d(N) = d\}$ are disjoint, we have that

$$a_{n+1} - 1 = \sum_{d=1}^{n} f(n,d,C) \quad , \tag{7}$$

where $f(n,d,C)$ denotes the number of integers N, such that $1 \leq N < a_{n+1}$ and for which the representation (3) and (4) contains exactly d non-zero terms. By the relation between the n-vectors of $C(e)$ and the interval $1 \leq N < a_{n+1}$, proved in the first paragraph of the proof, $f(n,d,C)$ reduces to the combinatorial function $k(n,d,C)$, hence the formula (5) is proved. Since the property C is, by assumption, independent of the a's, the formula (5), whenever it is defined, determines a single sequence. Note that the whole argument assumed (4), hence that $n \geq 1$. The fact that $a_1 = 1$ follows from applying (3) with $N = 1$, and thus the proof is completed.

To conclude, I wish to remark that if C depends on the a's to be determined, the equation (5) still applies as it can be seen from the argument above; in this case, however, (5) may have more than one solution.

REFERENCES

1. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers," J. London Math. Soc., 35 (1960), pp. 143-160.
2. J. Galambos, "On a Model for a Fair Distribution of Gifts," J. Appl. Prob., 8 (1971), pp. 681-690.
3. W. Parry, "On the β-Expansions of Real Numbers," Acta. Math. Acad. Sci. Hung., II (1960), pp. 401-416.

FIBONACCI NOTE SERVICE

The Fibonacci Quarterly is offering a service in which it will be possible for its readers to secure background notes for articles. This will apply to the following:

(1) Short abstracts of extensive results, derivations, and numerical data.

(2) Brief articles summarizing a large amount of research

[Continued on page 658.]

CONVOLUTION TRIANGLES

VERNER E. HOGGATT, JR.
San Jose State University, San Jose, California
and
MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara, California

If $G(x)$ is the generating function for a sequence, then $(G(x))^{k+1}$ is the column generator for the k^{th} column of the CONVOLUTION TRIANGLE. The original sequence is the zeroth column. We study here the convolution triangle of a class of generalized Fibonacci sequences which are obtained as rising diagonal sums of generalized Pascal triangles induced by the expansions $(1 + x + x^2 + \cdots + x^{r-1})^n$, $n = 0, 1, 2, 3, \cdots$. There are several ways to generate the convolution triangle array for a given generalized Fibonacci sequence. We shall illustrate these with the Fibonacci sequence.

1. GENERATION OF ARRAYS

In [1], it is shown that a rule of formation for the Fibonacci convolution triangle is as follows: to get the n^{th} element in the k^{th} column, add the two elements above it in the same column and the one immediately to the left in the preceding column. One notes in passing that this is equivalent to the following: Start row zero with a row of ones extending to the right. To get an element A in this array, add the two elements directly above A and all those elements in the same two rows and to the left of these.

$$
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots & m & m+1 & \cdots \\
2 & 5 & 9 & 14 & 20 & \cdots & \cdots & \cdots & \cdots \\
3 & 10 & 22 & 40 & 65 & \cdots & \cdots & \cdots & \cdots \\
5 & 20 & 51 & 105 & 190 & \cdots & x & \cdots & \cdots \\
8 & 38 & 111 & 256 & 511 & \cdots & y & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A & &
\end{array} \tag{1.1}
$$

It is easy to prove by mathematical induction that this generates the Fibonacci convolution array.

One might observe that the rising diagonal sums in the array above are the Pell sequence $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$. The rising diagonal sums formed by going up 2 and over 1 are $1, 1, 3, 5, 11, 21, \cdots$, $u_{n+2} = u_{n+1} + 2u_n$. Also, the determinant of the square arrays of order $1, 2, 3, 4, 5$, readily found in the left-hand corner of the array, is in each case equal to one.

By comparison, if Pascal's triangle is written in rectangular form

$$(1.2)\qquad \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

the rule of formation to obtain an element A is to add the one element above A and all elements in that same one row, or, to add the one element above A and the element in the preceding column to the left of A. The rising diagonal sums are the powers of two, and the sums of diagonals formed by going up 2 and over 1 are the Fibonacci numbers 1, 1, 2, 3, 5, 8, $\cdots$, $F_{n+2} = F_{n+1} + F_n$.

When we speak of rising diagonal sums in generalized Pascal's triangles we are thinking of diagonals formed by going up 2 and over 1 in rectangular arrays similar to (1.2) or going up 1 and right 1 in a left-justified array such as

$$(1.3)\qquad \begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

The coefficients of the Fibonacci polynomials

$$f_0(x) = 0, \qquad f_1(x) = 1, \qquad f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

lie along the rising diagonals of Pascal's triangle (1.3) and $f_n(1) = F_n$. It is well known that the generating function for the Fibonacci polynomials is

$$\frac{\lambda}{1 - x\lambda - \lambda^2} = \sum_{n=0}^{\infty} f_n(x)\lambda^n \; .$$

$$\frac{d^m}{dx^m}\left(\frac{\lambda}{1 - x\lambda - \lambda^2}\right) = \frac{\lambda^{m+1}\, m!}{(1 - x\lambda - \lambda^2)^{m+1}} = \sum_{n=0}^{\infty} f_n^{(m)}(x)\lambda^n \; .$$

Since $f_n(x)$ is of degree $n - 1$, setting $x = 1$ yields

$$\frac{\lambda^{m+1}}{(1-\lambda-\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{f_n^{(m)}(1)}{m!}\lambda^n$$

so that since

$$\left(\frac{\lambda}{1-\lambda-\lambda^2}\right)^{m+1} = \sum_{n=0}^{\infty} F_n^{(m)}\lambda^n$$

also generates the m^{th} convolution sequence, equating coefficients shows that

$$\frac{f_n^{(m)}(1)}{m!} = F_n^{(m)} \quad ,$$

where $F_n^{(m)}$ is the n^{th} member of the m^{th} Fibonacci convolution sequence. Thus the Fibonacci polynomials $f_n(x)$ evaluated at $x = 1$ by the Taylor's series expansion have as coefficients elements that lie along diagonals of the Fibonacci convolution triangle (1.1), which are the rows of (1.4):

(1.4)

	$f_n(1)$	$f_n'(1)/1!$	$f_n''(1)/2!$	$f_n'''(1)/2!$
$f_1(x)$	1	0	0	0
$f_2(x)$	1	1	0	0
$f_3(x)$	2	2	1	0
$f_4(x)$	3	5	3	1
$f_5(x)$	5	10	9	4
$f_6(x)$	8	20	22	⋯
⋯	⋯	⋯	⋯	⋯

2. THE JACOBSTHAL POLYNOMIALS

Consider the polynomials $J_1(x) = 1$, $J_2(x) = 1$, and $J_{n+2}(x) = J_{n+1}(x) + xJ_n(x)$. We see, of course, that $J_n(1) = F_n$. The coefficients of the Jacobsthal polynomials also lie on the rising diagonals of Pascal's triangle (1.3) but their order is the reverse of that for the Fibonacci polynomials. The generating function for the Jacobsthal polynomials is

$$\frac{\lambda}{1-\lambda-x\lambda^2} = \sum_{n=0}^{\infty} J_n(x)\lambda^n$$

from which

$$\frac{d^m}{dx^m}\left(\frac{\lambda}{1-\lambda-x\lambda^2}\right) = \frac{\lambda^{2m+1}\, m!}{(1-\lambda-x\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} J_n^{(m)}(x)\lambda^n$$

so that

$$\frac{\lambda^{m+1}}{(1 - \lambda - \lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{J_n^{(m)}(1)}{m!} \lambda^{n-m} .$$

Thus

$$\frac{J_n^{(m)}(1)}{m!} = F_n^{(m)} ,$$

also.

The Jacobsthal polynomial sequence has two polynomials of each degree. The array obtained by listing the polynomials and their derivatives at $x = 1$ appropriately divided by $m!$ also yields the Fibonacci convolution array.

There is another nameless set of polynomials that has interesting related properties, $Q_1(x) = 1$, $Q_2(x) = x$, and $Q_{n+2}(x) = x\big(Q_{n+1}(x) + Q_n(x)\big)$. These polynomials also have their coefficients along the rising diagonals of Pascal's triangle (1.3). Clearly, thus $Q_n(1) = F_n$. The generating function for the $Q_n(x)$ is

$$\frac{\lambda}{1 - x(\lambda + \lambda^2)} = \sum_{n=0}^{\infty} Q_n(x)\lambda^n ;$$

$$\frac{(1 + \lambda)^m \lambda^{m+1}}{(1 - \lambda - \lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{Q_n^{(m)}(1)}{m!} \lambda^n .$$

We will leave the reader to do the analysis of the array obtained from $Q_n^{(m)}(1)/m!$.

3. ROW GENERATING FUNCTIONS

For the Fibonacci convolution array the column generators are

$$\left(\frac{\lambda}{1 - \lambda - \lambda^2}\right)^{k+1} \qquad k = 0, 1, 2, 3, \cdots .$$

Here we are interested in the row generators when the array is written in the form (1.1) which starts with a row of ones. Since an element A of that array is secured by adding the two elements in the column above A and the element in the preceding column directly to the left of A, we now wish to secure the row generating functions based on this same generating scheme. Let $R_n(x)$ be the row generating function; then the recurrence scheme dictates that

$$R_{n+2}(x) = xR_{n+2}(x) + R_{n+1}(x) + R_n(x) .$$

We note that

$$R_0(x) = \frac{1}{1 - x} \qquad \text{and} \qquad R_1(x) = \frac{1}{(1 - x)^2} ,$$

thus since

$$R_{n+2}(x) = \frac{R_{n+1}(x) + R_n(x)}{1 - x}$$

it follows that the general form of

$$R_n(x) = \frac{N_n(x)}{(1 - x)^{n+1}} ,$$

where the numerator polynomials $N_n(x)$ obey the recurrence

$$N_{n+2}(x) = N_{n+1}(x) + (1 - x)N_n(x)$$

with $N_0(x) = 1$ and $N_1(x) = 1$. Surely now we recall the Jacobsthal polynomials discussed earlier and we observe that

$$J_{n+1}(1 - x) = N_n(x) .$$

Expand the polynomials $J_n(x)$ in a Taylor's series about $x = 1$ to yield

$$J_n(x) = J_n(1) + \frac{J_n'(1)}{1!}(x - 1) + \frac{J_n''(1)}{2!}(x - 1)^2 + \cdots$$

$$J_n(1 - x) = J_n(1) - \frac{J_n'(1)}{1!}x + \frac{J_n''(1)}{2!}x^2 - \frac{J_n'''(1)}{3!}x^3 + \cdots$$

Thus we conclude that

$$N_n(x) = J_{n+1}(1 - x)$$

are polynomials whose coefficients lie along the rising diagonals of the Fibonacci convolution triangle (1.4) whose modified column generators are

$$-\left(\frac{-\lambda}{1 - \lambda - \lambda^2}\right)^{m+1} \qquad m = 0, 1, 2, \cdots .$$

4. THE GENERALIZATION OF THE FIBONACCI CONVOLUTION ARRAYS

Consider the arrays whose column generators are

$$\left(\frac{\lambda}{1 - \lambda - \lambda^2 - \cdots - \lambda^r}\right)^{m+1} \qquad m = 0, 1, 2, 3, \cdots .$$

These are convolution arrays for those rising diagonal sum sequences in the generalized Pascal's triangle induced by

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, \cdots$$

and written in a left-justified manner as is (1.3). Such sequences are called the generalized Fibonacci sequences.

We will illustrate the generalization using the Tribonacci sequence

$$T_1 = T_2 = 1, \quad T_3 = 2, \quad \cdots, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad n = 1, 2, 3, \cdots.$$

The Tribonacci convolution triangle written in rectangular form is

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & 5 & 9 & 14 & 20 & \cdots \\ 4 & 12 & 25 & 44 & 70 & \cdots \\ 7 & 26 & 63 & 125 & 220 & \cdots \\ 13 & 56 & \cdots & \cdots & \cdots & \cdots \\ 24 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & & & & & \end{array} \tag{4.1}$$

The rising diagonal sums of the Tribonacci array (4.1) obey the recurrence

$$U_{n+3} = 2U_{n+2} + U_{n+1} + U_n,$$

where

$$U_1 = 1, \quad U_2 = 2, \quad U_3 = 5, \quad U_4 = 13, \quad \cdots$$

These could be called the generalized Pell sequence corresponding to the Trinomial triangle, The diagonals formed by going up 2 and right 1 in the Pascal case were Fibonacci numbers as sums; in the Fibonacci case, going up 3 and right 1 gave Tribonacci numbers; here the diagonals formed by going up 4 and right 1 in (4.1) give Quadranacci numbers, 1, 1, 2, 4, 8, 15, 29, $\cdots$, where $u_{n+4} = u_{n+3} + u_{n+2} + u_{n+1} + u_n$.

The corresponding Jacobsthal polynomials for the trinomial triangle are given by

$$J^\star_{n+3}(x) = J^\star_{n+2}(x) + xJ^\star_{n+1}(x) + x^2J^\star_n(x)$$

with

$$J^\star_1(x), \qquad J^\star_2(x) = 1, \qquad J^\star_3(x) = 1 + x$$

The generating function for the generalized Jacobsthal polynomials is

$$\frac{\lambda}{1 - \lambda - x\lambda - x^2\lambda^2} = \sum_{n=0}^{\infty} J_n^*(x)\lambda^n$$

It is not hard to prove that the row generating functions for the Tribonacci convolution array (4.1) are generally

$$R_n^*(x) = \frac{N_n^*(x)}{(1 - x)^{n+1}} ,$$

where

$$R_0^*(x) = \frac{1}{1 - x}, \qquad R_1^*(x) = \frac{1}{(1 - x)^2}, \qquad R_2^*(x) = \frac{2 - x}{(1 - x)^3}$$

Thus one asserts the polynomials $N_n^*(x)$ obey

$$N_{n+3}^*(x) = N_{n+2}^*(x) + (1 - x)N_{n+1}^*(x) + (1 - x)^2N_n^*(x) ,$$

where

$$N_0^*(x) = 1, \qquad N_1^*(x) = 1, \qquad \text{and} \qquad N_2^*(x) = 2 - x .$$

Further,

$$J_{n+1}^*(1 - x) = N_n^*(x)$$

and

$$N_n^*(x) = J_{n+1}^*(1) - \frac{J_{n+1}^{*\prime}(1)}{1!} x + \frac{J_{n+1}^{*\prime\prime}(1)}{2!} x^2 - \cdots$$

the same as before (this has alternating signs).

There are several polynomial sequences yielding the Tribonacci convolution array when one generates $P_n^{(m)}(1)/m!$, where $P_n(x)$ are the generalized Fibonacci polynomials, or Tribonacci polynomials,

$$P_{n+3}(x) = x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x) ,$$

where

$$P_1(x) = P_2(x) = 1 \qquad \text{and} \qquad P_3(x) = 1 + x .$$

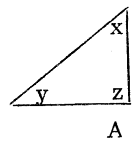
One such example:

$$\frac{\lambda}{1 - x\lambda - \lambda^2 - \lambda^3} = \sum_{n=0}^{\infty} P_n(x)\lambda^n$$

$$\frac{\lambda^{m+1}m!}{(1 - x\lambda - \lambda^2 - \lambda^3)^{m+1}} = \sum_{n=0}^{\infty} P_n^{(m)}(x)\lambda^n$$

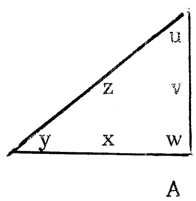
$$\left(\frac{\lambda}{1 - \lambda - \lambda^2 - \lambda^3}\right)^{m+1} = \sum_{n=0}^{\infty} T_n^{(m)} \lambda^n = \sum_{n=0}^{\infty} \frac{P_n^{(m)}(1)}{m!} \lambda^n$$

Recall from Section 1 that the convolution triangle (1.4) for the Fibonacci numbers is generated by adding x, y, and z to get element A as in the diagram below:



$$A = x + y + z\ .$$

Recall this is also the array generated by the numerator polynomials for the row generators of the Fibonacci convolution array. For the Tribonacci convolution array row generators can also be self-generated if element $A = u + v + w + x + y + 2z$ where the elements u, v, w, x, y, and z are found by the diagram



$$A = u + v + w + x + y + 2z$$

beginning the array with a one. Here the coefficients of the numerator polynomials of the row generators of the Tribonacci array lie along the rising diagonals of the triangle array below, which has the Tribonacci numbers in its left column. Of course, one normally asks what are the column generators of this triangle, too.

(4.2)

	1					
	1	1	1			
	2	4	3	2	1	
	4	9	12	11	6	...
R_4^* ——	7	22	37	40	...	...
	13	50	...	...	...	...
	24	...	...	...	...	...
	...	...	...	...	...	...

Here we shall illustrate several row generators:

$$R_0^*(x) = \frac{1}{1 - x}$$

$$R_1^*(x) = \frac{1}{(1 - x)^2}$$

$$R_2^*(x) = \frac{2 - x}{(1 - x)^3}$$

$$R_3^*(x) = \frac{4 - 4x + x^2}{(1 - x)^4}$$

$$R_4^*(x) = \frac{7 - 9x + 3x^2}{(1 - x)^5}$$

The column generators of (4.2) are

$$G_{n+2}(x) = \left(\frac{x}{1 - x - x^2 - x^3}\right)\left((2x + 1)G_{n+1}(x) + G_n(x)\right),$$

$$G_0(x) = \frac{1}{1 - x - x^2 - x^3}, \qquad G_1(x) = \frac{x(1 + 2x)}{(1 - x - x^2 - x^3)^2} .$$

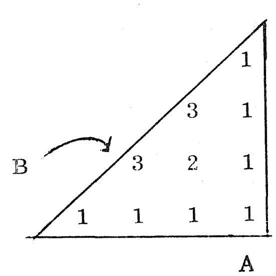
5. THE FULL GENERALIZATION

The generalized Jacobsthal polynomials are $J_1(x) = 1$, $J_2(x) = 1$, $J_3(x) = 1 + x$, $J_4(x) = (1 + x)^2$, $\cdots$, $J_{m+2}(x) = (1 + x)^m$, and

$$J_{n+m+1}(x) = J_{n+m}(x) + xJ_{n+m-1}(x) + x^2J_{n+m-2}(x) + \cdots + x^mJ_n(x)$$

$$= \sum_{j=0}^{m} J_{n+m-j}(x)x^{j-1} .$$

The numerator polynomial triangle is constructed by taking the appropriate size triangle B above A where the multipliers for the elements in B are the elements in the first k rows of Pascal's triangle as illustrated for $k = 4$ below:



The left edge of this triangular array is the Quadranacci sequence:

1						
1	1	1	1			
2	4	6	4	3	2	1
4	12	18	22	22	18	...
8	28	58	88	106	100	...
15	67	...	...	...	...	...
29	154	...	...	...	...	...
...	...	...	...	...	...	...

In each case, the generalized Pascal's array can be generated by adding all the elements in the rectangle with k rows above and to the left of element A (not including elements in the same column as A) to get A. If the rectangle has k rows, then we get the array induced by the expansions $(1 + x + x^2 + \cdots + x^{k-1})^n$, $n = 0, 1, 2, \cdots$. In these rectangular arrays using k rows in formation, if sums are found of elements lying on diagonals formed by going up $(k + 1)$ and right one, the sequence formed obeys the recurrence

$$u_{n+k+1} = u_{n+k} + u_{n+k-1} + \cdots + u_n \,.$$

where $u_1 = u_2 = 1$, $u_n = 2^{n-2}$ for $2 \le n \le k + 1$, generalized Fibonacci sequences, while the rising diagonals yield sums which are generalized Pell sequences obeying the recurrence

$$p_{n+k} = 2p_{n+k-1} + (p_{n+k-2} + p_{n+k-3} + \cdots + p_n)$$

and with the first three members of the sequence the ordinary Pell numbers 1, 2, 5, and the first k members of the sequence the same as the first k members of the sequence found from the rectangular array using $(k - 1)$ rows in its formation.

The convolution triangle for such generalized Fibonacci sequences can be generated by adding all the elements in the rectangle with k rows, including the column above an element A and extending to the extreme left of the array.

In any of these generalized Pascal's arrays or convolution arrays of generalized Fibonacci sequences written in rectangular form, the determinant of any square array found in the upper left-hand corner is always equal to one. The proofs and extensions will appear in later papers [2], [3].

REFERENCES

1. Verner E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 8, No. 2, March, 1970, pp. 158-171.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," Fibonacci Quarterly, in press.
3. Marjorie Bicknell and V. E. Hoggatt, Jr., "Special Determinants Found within Generalized Pascal Triangles," Fibonacci Quarterly, in press.

[Continued from page 564.]

4. G. L. Nemhauser, Introduction to Dynamic Programming, Wiley, 1966.
5. D. J. Wilde and C. S. Beightler, Foundations of Optimization, Prentice-Hall, 1967.
6. I. McCausland, Introduction to Optimal Control, Wiley, 1969.
7. H. Freeman, Discrete-Time Systems, Wiley, 1965.

A COUNTING FUNCTION OF INTEGRAL n-TUPLES

HWA S. HAHN
West Georgia College, Carrollton, Georgia

1. INTRODUCTION

Let P be the set of positive integers and let P^n be the set of n-tuples of positive integers. Many fresnmen books talk about how to count P^2 but rarely exhibit a counting function such as [2]

$$f_2(p_1, p_2) = p_1 + (p_1 + p_2 - 1)(p_1 + p_2 - 2)/2 .$$

E. A. Maier presented a counting function of P^n in this Quarterly [1]. In this note we show another more simple counting function of P^n and also discuss its inverse function and some applications.

2. THEOREM

The following polynomial in n variables

$$(1) \qquad f_n(p_1, p_2, \cdots, p_n) = p_1 + \sum_{k=2}^{n} \binom{s_k - 1}{k} ,$$

where

$$s_k = p_1 + p_2 + \cdots + p_k \quad \text{and} \quad \binom{s_k - 1}{k} = 0$$

for $s_k - 1 < k$, is a counting function of P^n.

<u>Proof.</u> Consider the set, call it the s-layer, of lattice points of positive coordinates $(x_1, x_2, \cdots, x_n)$ satisfying

$$x_1 + x_2 + \cdots x_n = s .$$

This s-layer contains

$$\binom{s - 1}{n - 1}$$

points. For, it is the number of ways of putting $n - 1$ markers in $s - 1$ spaces between 1's in

$$1 + 1 + \cdots + 1 = s .$$

Then the collection of s-layers, call it a pyramid, ranging $n \le s < s_n$, which is the largest pyramid without the given point $(p_1, p_2, \cdots, p_n)$, contains

$$\binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{s_n - 2}{n-1}$$

points. But this sum is simply

$$\binom{s_n - 1}{n}.$$

For,

$$\begin{aligned}\binom{s_n - 1}{n} &= \binom{s_n - 2}{n-1} + \binom{s_n - 2}{n} \\ &= \binom{s_n - 2}{n-1} + \binom{s_n - 3}{n-1} + \binom{s_n - 3}{n} \\ &= \cdots \end{aligned}$$

Next, we count points $(x_1, x_2, \cdots, x_n)$ such that

$$\sum x_i = s_n ,$$

up to $(p_1, p_2, \cdots, p_n)$. Since x_n is determined by $(x_1, x_2, \cdots, x_{n-1})$ and s_n, we need to count only $(n-1)$-tuples from $(1, 1, \cdots, 1)$ to $(p_1, p_2, \cdots, p_{n-1})$. For this we may use the function $f_{n-1}(p_1, p_2, \cdots, p_{n-1})$.

Thus, we obtain

$$f_n(p_1, p_2, \cdots, p_n) = f_{n-1}(p_1, p_2, \cdots, p_{n-1}) + \binom{s_n - 1}{n} .$$

And this recursive formula gives

$$f_n(p_1, p_2, \cdots, p_n) = p_1 + \sum_{k=2}^{n} \binom{s_k - 1}{k} ,$$

(taking $f_1(p_1) = p_1$).

Notes. 1. For $s_0 = 1$,

$$f_n(p_1, p_2, \cdots, p_n) = \sum_{k=0}^{n} \binom{s_k - 1}{k} ,$$

which is a string of pyramids of each dimension from 0 to n.

2. From its counting method f_n is clearly 1 - 1. However, we can also prove as follows. If $(p_1, p_2, \cdots, p_n) \neq (p'_1, p'_2, \cdots, p'_n)$, then there exists m such that $s_m \neq s'_m$ and $s_k = s'_k$ for $k > m$. Say, $s_m < s'_m$ (without loss of generality). Since $1 = s_0 \leq s_1 < \cdots < s_m \leq s'_m - 1$,

$$\sum_{k=0}^{m}\binom{s_k - 1}{k} \le \sum_{k=0}^{m}\binom{s_m - (m - k) - 1}{k} = \sum_{k=0}^{m}\binom{s_m - (m - k) - 1}{s_m - m - 1}$$

$$= \binom{s_m}{s_m - m} = \binom{s_m}{m} < p_1' + \binom{s_m' - 1}{m} \le \sum_{k=0}^{m}\binom{s_k' - 1}{k} .$$

These inequalities imply $f_n(p_1, \cdots, p_n) < f_n(p_1', \cdots, p_n')$. The following section also shows that f_n is onto.

3. THE INVERSE MAPPING $f_n^{-1} : P \to P^n$

The following algorithm produces $s_n, s_{n-1}, \cdots, s_1 (=p_1)$ from a given positive integer p.

First, determine s_n satisfying

$$\binom{s_n - 1}{n} < p \le \binom{s_n}{n} .$$

Then $s_{n-1}, s_{n-2}, \cdots, s_1$ from

$$\binom{s_{n-1} - 1}{n - 1} < p - \binom{s_n - 1}{n} \le \binom{s_{n-1}}{n-1} ,$$

$$\binom{s_{n-2} - 1}{n - 2} < p - \binom{s_n - 1}{n} - \binom{s_{n-1} - 1}{n - 1} \le \binom{s_{n-2}}{n-2} ,$$

$$\cdots$$

$$\binom{s_1 - 1}{1} < p - \sum_{k=0}^{n}\binom{s_k - 1}{k} = \binom{s_1}{1} .$$

Thus

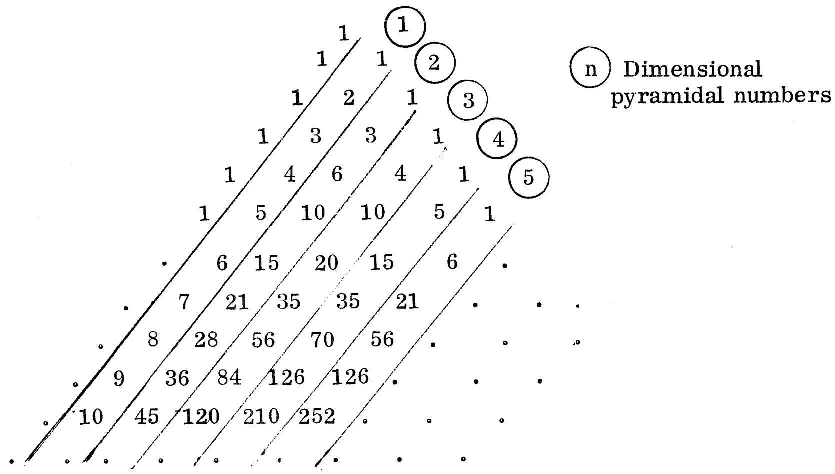
$$f_n - 1(p) = (s_1, s_2 - s_1, \cdots, s_n - s_{n-1}) ,$$

where

$$s_k - s_{k-1} = p_k \quad \text{for} \quad k > 1 \quad \text{and} \quad s_1 = p_1 .$$

4. PYRAMIDAL NUMBERS $\binom{s_n - 1}{n}$ IN PASCAL'S TRIANGLE

In the construction of the inverse image $f_n^{-1}(p)$ it is helpful to use Pascal's triangle, in which $(n + 1)^{st}$ diagonal line is the ordering of all n dimensional pyramids.



For example, to compute $f_3^{-1}(100)$ express 100 as a sum of pyramidal numbers of dimensions 3, 2, and 1 as follows:

$$100 = 84 + 15 + 1 = \binom{10-1}{3} + \binom{7-1}{2} + 1 .$$

Then $s_3 = 10$, $s_2 = 7$, $s_1 = 1$ and thus

$$f_3^{-1}(100) = (1,\ 7-1,\ 10-7) = (1,\ 6,\ 3) .$$

5. COUNTING LATTICE POINTS IN EUCLIDEAN n-SPACE

Take any counting function of Z, the set of integers, for example f_0 defined by

$$f_0(z) = 2\delta z + \frac{1-\delta}{2} ,$$

where

$$\delta = \begin{cases} 1 & \text{for } z > 0 , \\ -1 & \text{for } z \le 0 . \end{cases}$$

Then the ordinal number for $(z_1, z_2, \cdots, z_n)$ is given

$$f_n\big(f_0(z_1),\ f_0(z_2),\ \cdots,\ f_0(z_n)\big) = \sum_{k=0}^{n} \binom{S_k - 1}{k} ,$$

where

$$S_k = \sum_{i=1}^{k} f_0(z_i) .$$

[Continued on page 627.]

SOME COMBINATORIAL IDENTITIES OF BRUCKMAN
A SYSTEMATIC TREATMENT WITH RELATION TO THE OLDER LITERATURE

H. W. GOULD
West Virginia University, Morgantown, West Virginia

Bruckman [4] has made a study of some properties of numbers A_n defined by the power series expansion

$$f(x) = (1 - x)^{-1}(1 + x)^{-1/2} = \sum_{n=0}^{\infty} A_n x^n . \tag{1}$$

In some cases, for convenience, he uses the modified notation

$$B_n = 2^n n! A_n . \tag{2}$$

By use of the binomial theorem he found that

$$A_n = \sum_{k=0}^{n} (-1)^k \binom{2k}{k} 2^{-2k} . \tag{3}$$

Then by means of an exponential integral he was able to show that

$$A_n = 2^{-2n}(2n + 1)\binom{2n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k}{2k + 1} . \tag{4}$$

The A's satisfy the second-order recurrence relation

$$2n A_n = A_{n-1} + (2n - 1)A_{n-2} , \qquad A_0 = 1, \ A_1 = 1/2 . \tag{5}$$

Using recurrence relations and differential equations, Bruckman obtained the following elegant formula

$$A_n^2 = (2n + 1)\binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} 2^{-2k} \frac{1}{2n + 1 - 2k} . \tag{6}$$

Bruckman proves this interesting formula by showing that

$$\frac{\operatorname{Arctan} x}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} B_n^2 \frac{x^{2n+1}}{(2n+1)!}, \tag{7}$$

while, on the other hand, it is easy to multiply the series for Arctan x and $(1 - x^2)^{-1/2}$ together directly, and the result is (6).

I believe that formula (6) is the most interesting formula given in [4], and it does not appear in any readily accessible source. A direct proof of (6) by squaring (3) is not exactly trivial. The other relations in [4] are not really new, and far more general expansions have been considered in the older literature. However, it is hard to name a single source where all such expansions have been systematically generated. In the work below we shall obtain variant forms and expansions and in passing show that the numbers A_n are special cases of numbers studied by Cauchy [5], Chessin [6, 7], Perna [10], and Graver [9]. Some of the power series expansions are summarized in Adams and Hippisley [1] who also cite other related sums. Since our motive is partly pedagogical, we give considerable detail in some of the proofs below. We end by stating a difficult RESEARCH PROBLEM.

Free use will be made of some elementary identities, such as

$$\binom{-1/2}{k} = (-1)^k \binom{2k}{k} 2^{-2k}, \tag{8}$$

which follow from the polynomial definition of the binomial coefficient

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \qquad \binom{x}{0} = 1.$$

For example, we also have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}, \qquad x = \text{any real number}. \tag{9}$$

We shall use the older notation $((x^n))F(x)$ to denote the coefficient of x^n in the power series expansion of $F(x)$.

We now summarize the main formulas proved and discussed in the present paper:

$$((x^n))(1 - x)^{-1}(1 + x)^{-1/2} = \sum_{k=0}^{n} \binom{-1/2}{k} = \sum_{k=0}^{n} (-1)^k \binom{2k}{k} 2^{-2k} = A_n. \tag{10}$$

This is just Bruckman's first result with a variation by means of (8).

$$((x^n))(1 - x)^{-1/2}(1 - x^2)^{-1/2} = \sum_{k=0}^{[n/2]} (-1)^k \binom{-1/2}{k}\binom{-1/2}{n-2k} = A_n. \tag{11}$$

(12)
$$\left((x^n)\right)(1-x)^{-3/2}\left(1+\frac{2x}{1-x}\right)^{-1/2} = \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{n+1/2}{n-k}2^{-k} = A_n$$
$$= 2^{-2n}(2n+1)\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k}{2k+1} .$$

(13)
$$\left((x^n)\right)(1+x)^{-3/2}\left(1-\frac{2x}{1+x}\right)^{-1}$$
$$= 2^{-2n}\binom{2n}{n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}^{-1}2^{3k}\frac{2n+1}{2k+1} = A_n .$$

(14)
$$e^{x^2/2}\int_0^x e^{-u^2}du = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!\,2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k}{2k+1}$$
$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}2^n n!\,A_n = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}B_n .$$

This is just relation (22) in [4].

(15)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}^{-1}2^{2k}\frac{A_k}{2k+1} = \frac{2^n}{2n+1} .$$

(16)
$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}2^{2k}\frac{A_k}{2k+1} = \frac{2^{3n}}{2n+1}\binom{2n}{n}^{-1} .$$

(17)
$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{4k}{2k}^{-1}2^{4k}\frac{A_{2k}}{4k+1} = \frac{2^{n-1}}{2n+1}\left\{1+2^{2n}\binom{2n}{n}^{-1}\right\} .$$

Relation (17) follows by adding (15) and (16) together so that odd-index terms cancel. Subtracting (15) from (16) yields a similar formula involving A_{2k+1}.

(18)
$$\sum_{n=0}^{\infty}2^{2n+1}\binom{2n}{n}^{-1}\frac{A_n}{2n+1}\cdot\frac{x^{2n+1}}{(2-x^2)^{n+1}} = \frac{\text{Arcsin}\,x}{\sqrt{1-x^2}} .$$

(19) $$\operatorname{Arctan} x = \sum_{n=0}^{\infty} 2^{2n+1}\binom{2n}{n}^{-1} \frac{A_n}{2n+1} \cdot \frac{x^{2n+1}}{(2+x^2)^{n+1}} .$$

(20) $$B_n = \frac{(2n+1)!}{2^n\, n!\, \sqrt{2}} \int_0^{\sqrt{2}} (1-t^2)^n\, dt .$$

This is relation (26) in [4].

(21) $$B_n = \frac{(2n+1)!}{2^n\, n!} \int_0^1 (1-2u^2)^n\, du .$$

This relation follows from (20) by the change of variable $t = u\sqrt{2}$.

(22) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} = \frac{\operatorname{Arctan} x}{\sqrt{1-x^2}} .$$

(23) $$\left\{\sum_{k=0}^{n} \binom{-1/2}{k}\right\}^2 = A_n^2 = \binom{-1/2}{n} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} .$$

This is an equivalent formulation of Bruckman's formula (6) above

(24) $$\sum_{j=0}^{n} \binom{-1/2}{j} \cdot \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k \frac{2n+1}{2k+1} = \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} .$$

This is another equivalent formulation of (6).

(25) $$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{2k+1} = \frac{1}{1-t} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{-xt}{1-t}\right)^k$$

$$= S(x,t) = \frac{1}{1-t} \cdot \frac{\operatorname{Arctan} z}{z} ,$$

where

$$z^2 = xt/(1-t) .$$

For $x = 2$, this may be specialized to involve A_n or B_n, whence

$$S(2,t) = \sum_{n=0}^{\infty} t^n \frac{2^n n!}{(2n+1)!} B_n . \tag{26}$$

$$A_n^2 = (2n+1)^2 \binom{-1/2}{n}^2 \sum_{k=0}^{n} (-1)^k 2^k \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j} \frac{1}{(2j+1)(2k-2j+1)} . \tag{27}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} A_n = \sum_{k=0}^{\infty} x^{k+1} \frac{J_k(x)}{(2k+1)k!} , \tag{28}$$

where $J_k(x)$ is the ordinary Bessel function.

The power series in this paper are treated as formal power series, without regard to regions of convergence. The algebra of such formal power series is developed in Niven's paper [11]. Convergence information for the various series could be developed, but we shall omit this.

The functions expanded in (10), (11), (12), (13) are all identical with Bruckman's definition in (1). The proof of (10) is trivial, being a direct application of the binomial theorem and Cauchy product of series.

Here are details of a proof of (11):

$$\begin{aligned}(1-x)^{-1/2}(1-x^2)^{-1/2} &= \sum_{j=0}^{\infty} (-1)^j \binom{-1/2}{j} x^j \cdot \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+k} \binom{-1/2}{j}\binom{-1/2}{k} x^{j+2k} .\end{aligned}$$

In this, let $j = n - 2k$ to obtain the coefficient of x^n. The result is (11).

In proving (12) we first note that the identity, which is (Z.46) in [8],

$$\frac{2n+1}{2k+1}\binom{2n}{n}\binom{n}{k} 2^{-2n} = \binom{2k}{k}\binom{n+1/2}{n-k} 2^{-2k} , \quad 0 \le k \le n ,$$

can be obtained from the polynomial definition of $\binom{x}{n}$ just as (8) is found. Thus we have only to prove the first form of (12), which can be done as follows:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1/2}{n-k} 2^{-k} (-1)^k x^n$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} 2^{-k}(-1)^k \sum_{n=k}^{\infty} \binom{n + 1/2}{n - k} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k} x^k \sum_{n=0}^{\infty} \binom{n + k + 1/2}{n} x^n ,$$

using the substitution of $n + k$ for n,

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k} x^k \sum_{n=0}^{\infty} (-1)^n \binom{-3/2 - k}{n} x^n ,$$

by use of (9),

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k} x^k (1 - x)^{-3/2-k} ,$$

by the Binomial theorem,

$$= (1 - x)^{-3/2} \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{2x}{1 - x}\right)^k ,$$

using (8),

$$= (1 - x)^{-3/2} \left(1 + \frac{2x}{1 - x}\right)^{-1/2} = (1 - x)^{-1}(1 + x)^{-1/2} .$$

The somewhat similar proof of (13) runs as follows:

$$\sum_{n=0}^{\infty} x^n \binom{2n}{n} 2^{-2n}(2n + 1) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{k}^{-1} 2^{3k} \frac{1}{2k + 1}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} x^n \binom{n + 1/2}{n - k} 2^k (-1)^{n-k} ,$$

using (Z.46) in [8],

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{n+k} \binom{n + k + 1/2}{n} 2^k (-1)^n$$

$$= \sum_{k=0}^{\infty} (2x)^k \sum_{n=0}^{\infty} (-1)^n \binom{-3/2 - k}{n} x^n (-1)^n ,$$

using (9),

$$= \sum_{k=0}^{\infty} (2x)^k (1 + x)^{-3/2-k} = (1 + x)^{-3/2} \sum_{k=0}^{\infty} \left(\frac{2x}{1 + x}\right)^k$$

$$= (1 + x)^{-3/2} \left(1 - \frac{2x}{1 + x}\right)^{-1} = (1 + x)^{-1/2}(1 - x)^{-1} .$$

We have said that the expansions which we consider are special cases of other known expansions. To illustrate this, we note the formula

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{n - k} = \sum_{k=0}^{[n/2]} (-1)^k \binom{x}{k}\binom{y - x}{n - 2k} , \tag{29}$$

valid for all real or complex x and y. This is formula (3.31) in [8]. In this formula, let $x = -1/2$, $y = -1$, and we obtain at once

$$\sum_{k=0}^{[n/2]} (-1)^k \binom{-1/2}{k}\binom{-1/2}{n - 2k} = \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k}\binom{-1}{n - k} ,$$

but by (9) we have

$$\binom{-1}{j} = (-1)^j ,$$

so that we have proved the equivalence of (10) and (11) this way.

Again formula (1.9) in [8] is

$$\sum_{k=0}^{n} \binom{x}{k} y^k = \sum_{k=0}^{n} \binom{n - x}{k} (1 + y)^{n-k}(-y)^k , \tag{30}$$

valid for all real or complex x and y. Letting $x = -1/2$, it is easy to see that we obtain the equivalence of (13) and (10). Here again we need (Z.46) in [8].

Still another way to prove the equivalence of (13) and (10) is to use formula (1.10) from [8]:

$$\sum_{k=0}^{n} \binom{z}{k} x^{n-k} = \sum_{k=0}^{n} \binom{z - k - 1}{n - k} (x + 1)^k . \tag{31}$$

In this, let $z = -1/2$, $x = 1$, and simplify. This time we need the identity

$$\binom{-1/2 - k}{n - k}\binom{2k}{k} = (-1)^{n-k}\binom{2n}{n}\binom{n}{k}2^{2k-2n}, \qquad 0 \le k \le n ,$$

which is easily proved from the polynomial definition of $\binom{x}{n}$.

A direct proof of (14) is as follows:

$$e^{x^2/2}\int_0^x e^{-u^2}\,du = e^{x^2/2}\int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k}}{k!}\,du$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!}\cdot \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^k \frac{x^{2n+2k+1}}{2^n n!\,(2k+1)k!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} (-1)^k \frac{x^{2n+1}}{2^{n-k}(n-k)!\,(2k+1)k!} ,$$

replacing n by $n - k$,

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,2^n}\sum_{k=0}^{n} (-1)^k \binom{n}{k}\frac{2^k}{2k+1} ,$$

as desired to show.

A variant of (14) involving Bessel functions is derived as follows, and is formula (28):

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\, 2^{-2n}\, \frac{(2n+1)!}{n!^2 2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\frac{2^k}{2k+1}$$

$$= x\sum_{k=0}^{\infty} (-1)^k \frac{2^k}{2k+1}\sum_{n=k}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n} \frac{1}{n!\,(n-k)!\,k!}$$

$$= x\sum_{k=0}^{\infty} \frac{2^k}{(2k+1)k!}\left(\frac{x}{2}\right)^{2k}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n} \frac{1}{(n+k)!\,k!}$$

$$= x\sum_{k=0}^{\infty} \frac{2^k}{(2k+1)k!}\left(\frac{x}{2}\right)^{2k}\left(\frac{x}{2}\right)^{-k} J_k(x) = \sum_{k=0}^{\infty} x^{k+1}\frac{J_k(x)}{(2k+1)k!} .$$

Relation (15) is nothing but the inversion of (12), and relation (16) the inversion of (13). What is needed to see this is the well known pair of inverse series:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k) \tag{32}$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) . \tag{33}$$

These in turn depend on nothing deeper than the orthogonality relation

$$\sum_{k=j}^{n} (-1)^{k+j} \binom{n}{k}\binom{k}{j} = \begin{cases} 0, & n \neq j , \\ 1, & n = j , \end{cases}$$

and this is a consequence of the binomial theorem.

To use them, for example, choose $g(k) = 2^k/(2k+1)$ and then by (12),

$$f(n) = 2^n n!\, B_n /(2n + 1)! \quad .$$

Therefore by (33) we find that (12) inverts to yield (15). Relation (13) inverts to give (16) in a similar way.

Adams and Hippisley [1, p. 122, 6.42-(5.)] give the formula

$$\frac{\operatorname{Arcsin} x}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n + 1)!}\, x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \binom{-1/2}{n}^{-1} \tag{34}$$

which may be compared with (18) here. Of course, there is also the well known expansion

$$\operatorname{Arcsin} x = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} \frac{x^{2n+1}}{2n + 1} , \tag{35}$$

which we cite for completeness.

Proof of (18) is obtained in the following way: By (16) we have

$$\frac{2^{2n} n!^2}{(2n + 1)!} = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}^{-1} 2^{2k} \frac{A_k}{2k + 1} ,$$

then, recalling (34), we have

$$\frac{\text{Arcsin } x}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} 2^{-n} x^{2n+1} \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k}^{-1} 2^{2k} \frac{A_k}{2k + 1}$$

$$= x \sum_{k=0}^{\infty} 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k + 1} \sum_{n=k}^{\infty} \binom{n}{k} x^{2n} 2^{-n}$$

$$= x \sum_{k=0}^{\infty} x^{2k} 2^k A_k \binom{2k}{k}^{-1} \frac{1}{2k + 1} \left(1 - \frac{x^2}{2}\right)^{-k-1},$$

which reduces to the desired result.

Relation (19) is proved in a similar way from (15), for we have first of all:

$$\text{Arctan } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = \sum_{n=0}^{\infty} (-1)^n 2^{-n} x^{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k}^{-1} \frac{2^{2k} A_k}{2k + 1}$$

$$= x \sum_{k=0}^{\infty} (-1)^k 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k + 1} \sum_{n=k}^{\infty} (-1)^n x^{2n} 2^{-n} \binom{n}{k}$$

$$= x \sum_{k=0}^{\infty} 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k + 1} 2^{-k} x^{2k} \sum_{n=0}^{\infty} (-1)^n x^{2n} 2^{-n} \binom{n + k}{k}$$

$$= x \sum_{k=0}^{\infty} 2^k \binom{2k}{k}^{-1} \frac{A_k}{2k + 1} x^{2k} \left(1 + \frac{x^2}{2}\right)^{-k-1},$$

which reduces as required.

Proof of (22):

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \sum_{k=0}^{n} \binom{-1/2}{n - k} \frac{2n + 1}{2k + 1}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \sum_{k=0}^{n} \binom{-1/2}{n - k} \frac{1}{2k + 1}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j + 1} \cdot \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k} = \text{Arctan } x \cdot (1 - x^2)^{-1/2}.$$

Relation (23) is found by using (8) in (6), giving

$$A_n^2 = \binom{-1/2}{n} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1},$$

from which the formula follows readily.

Relation (24) is found by writing A_n^2 in (23) as a product of two forms of A_n given by (10) and (12), so that a factor of

$$\binom{-1/2}{n}$$

cancels.

Relation (27) follows from (12) by using the general theorem that

$$\sum_{k=0}^{n} \binom{n}{k} a_k \cdot \sum_{j=0}^{n} \binom{n}{j} b_j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-j} a_j b_{k-j} \tag{36}$$

for arbitrary a_k's and b_j's.

Relations (23), (24), and (27) are offered as small variations on (6).

An alternative proof of (19) can be given by first noting that

$$\text{Arctan } x = \frac{x}{1+x^2} \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n+1)!} \left(\frac{x^2}{1+x^2} \right)^n \tag{37}$$

as noted in [1, p. 122, 6.41-(3.)]. One then expands $2^{2n} n!^2/(2n+1)!$ by (16), and upon reduction and use of the binomial theorem we again find (19).

We note in passing that expansion (37) may be compared with one given by Bromwich [3, p. 199, ex. 17] which is

$$\text{Arctan } x = \sum_{n=0}^{\infty} (-1)^n \binom{3n+1}{n} \frac{t^{2n+1}}{2n+1}, \qquad t = \frac{x}{1+x^2}, \tag{38}$$

which converges, incidentally, for $|t|^2 < 4/27$. Both (37) and (38) are examples of special cases of formulas related to the Lagrange inversion formula.

We now turn to some of the older literature. Cauchy numbers have been defined by the following:

$$N_{-p,\ell,m} = \text{Constant term in expansion of } x^{-p} \left(x + \frac{1}{x} \right)^{\ell} \left(x - \frac{1}{x} \right)^{m}. \tag{39}$$

When $p = \ell + m$, then $N_{-p,\ell,m} = 1$. When $\ell + m - p$ is odd or a negative integer, then $N = 0$. Moreover,

(40) $$N_{-p,\ell,m} = \sum_{k=0}^{n} (-1)^k \binom{\ell}{n-k}\binom{m}{k}, \qquad \text{when } \ell + m - p = 2n.$$

Comparing this with (10), we see that Bruckman's A_n is given by

(41) $$A_n = N_{2n+3/2,-1/2,-1}.$$

Many interesting properties have been found for the Cauchy numbers, and the reader may consult references [5], [6], [7] and [10]. In [10], Perna numbers are defined by

(42) $$A_{m,n,\ell} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{m-2n}{\ell-2k} = N_{2\ell-m,m-n,n}.$$

The late Harry Bateman (1882-1946), a master of special functions (since it was said he knew the properties of over a thousand functions, and he left dozens of card files of such information) worked on manuscripts for about 25 books, living to publish only three of them. In 1961, through the kind generosity of Professor A. Erdélyi, who was then at California Institute of Technology, Bateman's three versions of his manuscript [2] toward a book on binomial coefficients were borrowed for study at West Virginia University. A microfilm of the manuscript is on file now in the West Virginia University Library. The writer has gone through this material and edited it into a single manuscript, adding a few remarks as necessary, correcting obvious mistakes, etc. It is hoped that this version can be made more readily accessible for study by other scholars. Bateman tried to unify some of the material on binomial identities using the Cauchy number definition in one case. Here he summarized many of the properties of these and the related numbers studied by Chessin, Perna, etc. Chessin gave the formula, for example, that

$$J_a(x) = \sum_{n=0}^{\infty} \frac{N_{-a,0,a+2n}}{(a+2n)!}\left(\frac{x}{2}\right)^{a+2n}$$

for the Bessel function.

Such sums of products of two binomial coefficients continue to occur in mathematics. One example is in Graver's combinatorial work [9]. He defines coefficients $P_n(a,b)$ which turn out to be such that

$$(1 - x)^b (1 + x)^{a-b} = \sum_{n=0}^{\infty} P_n(a,b)\, x^n \;. \tag{43}$$

From this, it is easy to see that Bruckman's A_n is given by

$$A_n = P_n(-3/2, -1) \;. \tag{44}$$

Interesting formulas are found by Graver. For example:

$$P_n(a,b) = \sum_{k=0}^{n} (-1)^k \binom{b}{k}\binom{a-k}{n-k} 2^k, \qquad n \le b \le a \;. \tag{45}$$

$$P_n(a,b) = \frac{b!\,(a-b)!}{n!\,(a-n)!}\, P_b(a,n) \;, \tag{46}$$

expressing a symmetry in b and n, and (Graver's actual definition)

$$P_n(a,b) = \sum_{k=0}^{n} (-1)^k \binom{b}{k}\binom{a-b}{n-k} \;. \tag{47}$$

The equality of (45) and (47) is not again a new result, so extensive is the vast literature around the binomial coefficient identities. An expansion of the sort studied by Graver occurs frequently in mathematics, just as the Cauchy numbers have come to attention many times. Graver's numbers relate to Cauchy's numbers by the formula

$$P_n(\ell + m, m) = N_{-p,\ell,m} \qquad \text{with} \quad \ell + m - p = 2n \;. \tag{48}$$

In the older literature, one thing was noted as conspicuously absent; any relation of the form (6) of Bruckman or a suitable extension. Looking at Bruckman's formula in the form (23) it is tempting to generalize and wonder if by chance

$$\left\{\sum_{k=0}^{n}\binom{x}{k}\right\}^2 = \binom{x}{n}\sum_{k=0}^{n}\binom{x}{k}\frac{x-n}{x-n+k} \;,$$

but this turns out to be false. The reader is invited to try and find such a generalization.

Bruckman's formula is an example of a case in which a certain more difficult problem is solvable. The general problem we mean is this:

PROBLEM: Let

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} A_n^2 x^n,$$

for an arbitrary sequence $\{A_n\}$. How are the functions f and g related? In case $A_n = F_n = n^{th}$ Fibonacci number, we not only know the solution for squares but for any power of F_n. Other examples where the function g can be given explicitly when f is known are, e.g.:

$$\sum_{n=0}^{\infty} n x^n = x/(1-x)^2, \qquad \sum_{n=0}^{\infty} n^2 x^n = x(x+1)/(1-x)^3;$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \log(1-x)^{-1}, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t}\,dt;$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!^2} = J_0(2i\sqrt{x});$$

and so on. It is clear that in general there is no really simple relation between f and g, but the writer has not found any result of this type in the literature and tosses it out as a research problem.

Solution of this problem, even with restrictions, would allow us to deal effectively with large classes of difficult problems.

In closing we mention two extensions of relation (22):

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{x}{n-k} \frac{2n+1}{2k+1} = (1-t^2)^x \operatorname{Arctan} t, \tag{49}$$

and

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{x}{n-k} \frac{2n+1}{2k+1} = (1+t^2)^x \cdot \frac{1}{2} \log \frac{1+t}{1-t}. \tag{50}$$

In a later paper we will treat some further properties of such expansions.

REFERENCES

1. E. P. Adams and R. L. Hippisley, "Smithsonian Mathematical Formulas and Tables of Elliptic Functions," Smithsonian Miscellaneous Collections, Vol. 74, No. 1, Publication No. 2672, Washington, D. C., 1922.

2. Harry T. Bateman, Notes on Binomial Coefficients, unpublished manuscript on binomial coefficient identities. About 560 pages. Version edited by H. W. Gould, August, 1961.

3. T. J. I'Anson Bromwich, Introduction to the Theory of Infinite Series, London, Sec. Ed. Revised, 1949.

4. Paul S. Bruckman, "An Interesting Sequence of Numbers Derived from Various Generating Functions," Fibonacci Quarterly, Vol. 10, No. 2 (1972), pp. 169-181.

5. A. L. Cauchy, "Méthode Générale pour la Détermination Numérique des coefficients que Renferme le Développement de la Fonction Perturbatrice," C. R. Acad. Sci. Paris, 11 (1840), 453-475. Oeuvres (1) 5 (1885), 288-322.

6. A. Chessin, "Note on Cauchy's Numbers," Annals of Math., 10 (1896), 1-2.

7. A. Chessin, "On the Relation Between Cauchy's Numbers and Bessel's Functions," Annals of Math., 12 (1899), 170-174.

8. H. W. Gould, Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations, Morgantown, W. Va., 1959.

9. Jack E. Graver, "Remarks on the Parameters of a System of Sets," Annals of New York Acad. Sci., 175 (1970), Article 1, pp. 187-197.

10. A. Perna, "Intorno ad Alcuni Aggregati di Coefficienti Binomiali," Gior. Mat. Battaglini, 41 (1903), 321-335.

11. Ivan Niven, "Formal Power Series," Amer. Math. Monthly, 76 (1969), 871-889.

[Continued from page 612.]

For the inverse mapping $P \to Z^n$ we need

$$f_0^{-1}(p_i) = \frac{(-1)^{\epsilon}(p_i - \epsilon)}{2},$$

where

$$\epsilon = \begin{cases} 0 & \text{for } p_i \text{ even,} \\ 1 & \text{for } p_i \text{ odd} \end{cases} .$$

Then

$$\begin{aligned} f_0^{-1} f_n^{-1}(p) &= f_0^{-1}(p_1, p_2, \cdots, p_n) \\ &= \left(f_0^{-1}(p_1), \cdots, f_0^{-1}(p_n)\right) . \end{aligned}$$

6. POLYNOMIAL COUNTING FUNCTIONS

It is quite easy to see from (1) that there are at least $n!$ polynomial counting functions of P^n (obtained by permuting $p_1, p_2, \cdots, p_n$). But for $n = 3$ besides these six polynomials of degree 3, there are six more polynomials of degree 4 obtained by composition of f_2 such as

$$f_2\left(f_2(p_1, p_2), p_3\right).$$

For $n = 4$ there are 360 polynomials, provided that different compositions yield distinct polynomials.

We are unable to determine the number of counting polynomials of P^n, except the case $n = 1$.

Theorem. The identical function $f_1(p_1) = p_1$ is the only polynomial mapping 1 - 1 from P onto itself.

Proof. Suppose $g(p)$ is a counting polynomial of P. Consider the curve $y = g(x)$. It is clear that after a finite number of ups and downs the curve is monotone increasing (to $+\infty$). Let a be a positive integer such that (1) $g(x)$ is monotone for $x \geq a$ and (2) $g(x) < g(a)$ for $x < a$. Since $g(x)$ is a counting function of P, it has to satisfy

$$g(a) = a, \quad g(a + 1) = a + 1, \cdots .$$

For, if $g(a) < a$, then positive numbers $g(1), g(2), \cdots, g(a)$ cannot all be distinct, and if $g(a) > a$ then the curve must come down beyond a, contrary to (1). Now, by the Fundamental Theorem of Algebra we have $g(x) = x$ for all x.

Question. Are

$$x_1 + \binom{s_2 - 1}{2} \qquad \text{and} \qquad x_2 + \binom{s_2 - 1}{2}$$

the only two polynomials mapping 1 - 1 from P^2 onto P?

REFERENCES

1. E. A. Maier, "One-One Correspondence Between the set N of Positive Integers and the Sets N^n and $\bigcup_{n \in N} N^n$," Fibonacci Quarterly, Oct. 1970, pp. 365-371.
2. P. W. Zehna and R. L. Johnson, Elements of Set Theory, Allyn and Bacon, Inc., Boston, 1962, p. 108.

[Continued from p. 584.]

REFERENCES

1. H. W. Gould, "Equal Products of Generalized Binomial Coefficients," Fibonacci Quarterly, Vol. 9, No. 4 (1971), pp. 337-346.
2. H. W. Gould, D. C. Rine, and W. L. Scharff, "Algorithm and Computer Program for the Determination of Equal Products of Generalized Binomial Coefficients," to be published.
3. V. E. Hoggatt, Jr., and Walter Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2 (1971), pp. 120, 133.
4. V. E. Hoggatt, Jr., and G. L. Alexanderson, "A Property of Multinomial Coefficients," Fibonacci Quarterly, Vol. 9, No. 4 (1971), pp. 351-356, 420-421.
5. V. E. Hoggatt, Jr., and A. P. Hillman, "Proof of Gould's Conjecture on Greatest Common Divisors," Fibonacci Quarterly, Vol. 10, No. 6 (1972), pp. 565-568.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-198 Proposed by E. M. Cohn, National Aeronautics and Space Administration, Washington, D.C.

There is an infinite sequence of square values for triangular numbers,[1]

$$k^2 = m(m + 1)/2 .$$

Find simple expressions for k and m in terms of Pell numbers, P_n. ($P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$.

H-199 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

A certain country's coinage consists of an infinite number of types of coins: $\cdots, C_{-2}, C_{-1}, C_0, C_1, \cdots$. The value V_n of the coin C_n is related to the others as follows: for all n,

$$V_n = V_{n-3} + V_{n-2} + V_{n-1} .$$

Show that any (finite) pocketful of coins is equal in value to a pocketful containing at most one coin of each type.

H-200 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Let $M(n)$ be the number of primes (distinct) which divide the binomial coefficient,[2]

$$C_k^n \equiv \binom{n}{k} .$$

[1]A. V. Sylvester, Am. Math. Monthly, 69 (1962), p. 168; quoted in C. W. Trigg, Mathematical Quickies (1967), p. 164.

[2]Divide at least one C_k^n where $0 \le k \le n$.

Clearly, for $1 \le n \le 15$, we have $M(1) = 0$, $M(2) = M(3) = 1$, $M(4) = M(5) = 2$, $M(6) = M(7) = M(8) = M(9) = 3$, $M(10) = 4$, $M(11) = M(12) = M(14) = 5$, $M(13) = M(15) = 6$, etc. Show that

$$\{M(n)\}_{n=1}^{\infty}$$

has an upper bound and find an asymptotic formula for $M(n)$.

H-201 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Copy 1, 1, 3, 8, $\cdots$, F_{2n-2} $(n \ge 1)$ down in staggered columns as in display C:

$$\begin{array}{llllll} & 1 & & & & \\ & 1 & 1 & & & \\ C & 3 & 1 & 1 & & \\ & 8 & 3 & 1 & 1 & \\ & 21 & 8 & 3 & 1 & 1 \end{array}$$

i) Show that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$

ii) Show that, if the columns are multiplied by 1, 2, 3, $\cdots$ sequentially to the right, then the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

iii) Show that the rising diagonal sums ($\nearrow$) are F^2_{n+1} $(n = 0, 1, 2, \cdots)$.

H-202 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \frac{F_k F_{k-1} \cdots F_{k-j+1}}{F_1 F_2 \cdots F_j}, \qquad \left\{ \begin{matrix} k \\ 0 \end{matrix} \right\} = 1 .$$

Show that

$$(\star) \qquad \begin{cases} \displaystyle\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j+1)} \left\{ \begin{matrix} 2k \\ j+k \end{matrix} \right\} = \prod_{j=1}^{k} L_{2j-1} \\ \displaystyle\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j-1)} \left\{ \begin{matrix} 2k \\ j+k \end{matrix} \right\} = (-1)^k \prod_{j=1}^{k} L_{2j-1} \end{cases},$$

$$(\star\star) \qquad \begin{cases} \displaystyle\sum_{j=o}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} L_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}k} F_1 F_3 \cdots F_{2k-1} & (k \text{ even}) \\ \displaystyle\sum_{j=o}^{2k} (-1)^j \left\{ \begin{matrix} 2k \\ j \end{matrix} \right\} F_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}(k-1)} F_1 F_3 \cdots F_{2k-1} & (k \text{ odd}) . \end{cases}$$

H-203 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

A. Let there be n $(n \geq 1)$ edge-connected squares. How many configurations are there which have each row starting one square to the right of the row above?

B. For the above configurations, how many have each row starting k $(k \geq 0)$ squares to the right of the row above?

H-204 Proposed by Dwarka Nivas, Berhampur, Orissa, India.

Given that 65537 $(= 256^2 + 1)$ is prime, find the remainder when it divides

i) $$\binom{32768}{16384}$$

and

ii) $$16384!$$

SOLUTIONS

NOBODY IS EVEN PERFECT

H-188 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Prove that there are no even perfect Fibonacci numbers.

Solution by S. L. Padwa, Applied Mathematics Department, Brookhaven National Laboratory.

As is well known, all even perfect numbers are of the form $2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime.

In particular, all even perfect numbers > 28 are multiples of 16.

Now the only Fibonacci numbers which are multiples of 16 are also multiples of 9; namely, F_{12k}.

Thus no Fibonacci number which is a multiple of 16 is of the required form for perfect numbers since they are all multiples of 9, while even perfect numbers cannot have an odd composite factor.

Also solved by the Proposer.

SOME SUMS

H-191 Proposed by David Zeitlin, Minneapolis, Minnesota

Prove the following identities:

(a) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 L_{2k} = L_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!}\, 5^{n-k}$$

(b) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 L_{2k} = F_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!} 5^{n+1-k}$$

(c) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 F_{2k} = F_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(d) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 F_{2k} = L_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)}{(k!)^3(2n+1-2k)!} 5^{n-k} ,$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Solution by the Proposer.

From the solution to H-180, we recall that

(1) $$\sum_{k=0}^{p} \binom{p}{k}^3 x^k = \sum_{2k \le p} \frac{(p+k)!}{(k!)^3(p-2k)!} x^k (x+1)^{p-2k} .$$

Since $\alpha^2 + 1 = \alpha^2 - \alpha\beta = \alpha(\alpha - \beta)$ and $\beta^2 + 1 = \beta(\beta - \alpha)$

$$\left(\alpha = \frac{1+\sqrt{5}}{2} , \quad \beta = \frac{1-\sqrt{5}}{2}\right) ,$$

we obtain from (1) for $x = \alpha^2$ and $x = \beta^2$,

(2) $$\sum_{k=0}^{p} \binom{p}{k}^3 \alpha^{2k} = \sum_{2k \le p} \frac{(p+k)!}{(k!)^3(p-2k)!} \alpha^{2k} [\alpha(\alpha - \beta)]^{p-2k}$$

and

(3) $$\sum_{k=0}^{p} \binom{p}{k}^3 \beta^{2k} = \sum_{2k \le p} \frac{(p+k)!}{(k!)^3(p-2k)!} \beta^{2k} [-\beta(\alpha - \beta)]^{p-2k} ,$$

respectively. So, by addition of (2), (3), we get

$$\sum_{k=0}^{p} \binom{p}{k}^3 L_{2k} = [\alpha^p + (-1)^p \beta^p] \sum_{2k \le p} \frac{(p+k)!}{(k!)^3(p-2k)!} 5^{\frac{p-2k}{2}} ,$$

since $(\alpha - \beta)^2 = 5$ and $\alpha^{2k} + \beta^{2k} = L_{2k}$. For $p = 2n$, we obtain (a); for $p = 2n+1$, we obtain (b). By subtraction of (3) from (2), we get

$$\sum_{k=0}^{p} \binom{p}{k}^3 (\alpha^{2k} - \beta^{2k}) = \alpha^p - (-1)^p \beta^p \sum_{2k \le p} \frac{(p+k)!}{(k!)^3 (p-2k)!} 5^{\frac{p-2k}{2}} ,$$

since $(\alpha^{2k} - \beta^{2k})/(\alpha - \beta) = F_{2k}$.

For $p = 2n$, we obtain (c); for $p = 2n+1$, we obtain (d).

SECOND DEGREE FOR DIOPHANTUS

H-194 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Solve the Diophantine equations:

(i) $$x^2 + y^2 \pm 5 = 3xy$$

(ii) $$x^2 + y^2 \pm e = 3xy ,$$

where

$$e = p^2 - pq - q^2 ;$$

p, q positive integers.

Solution by the Proposer.

Rewrite (ii) as

(1) $$(x + y)^2 - 5xy = \pm e .$$

Let $H_0 = q$, $H_1 = p$, and $H_{n+2} = H_{n+1} + H_n$, $n \ge 0$ be the generalized Fibonacci sequence. Then we have the following identities viz.

(2) $$(H_{2r-1} + H_{2r+1})^2 - 5H_{2r-1}H_{2r+1} = (-1)^{2r-1}e$$

and

(3) $$(H_{2r} + H_{2r+2})^2 - 5H_{2r}H_{2r+2} = (-1)^{2r}e ,$$

from which the solution of (ii) easily follows. (i) is a particular case, where $e = -5$.

EDITORIAL NOTES

Correction to H-185.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \binom{2k}{k} (1 - x)^{n-k} {}_2F_1[-k, n+k+1; k+1; x] ,$$

where ${}_2F_1[a,b;\ c;\ x]$ denotes the hypergeometric function.

Comment on H-193.

The proposer has pointed out that the stated condition does hold for the following examples.

Examples:

$$5 + 1 + 1 = 7 = 2^3 - 1, \qquad 5^3 + 1^3 + 1^3 = 127 = 2^7 - 1,$$
$$19 + 11 + 1 = 31 = 2^5 - 1, \qquad 19^3 + 11^3 + 1^3 = 8191 = 2^{13} - 1,$$
$$79 + 29 + 19 = 127 = 2^7 - 1, \qquad 79^3 + 29^3 + 19^3 = 524287 = 2^{19} - 1.$$

The validity of the statement would be a pleasant surprise.

Late Acknowledgements

H-183 P. Lindstrom, D. Klarner, S. Smith, D. Priest, and L. Carlitz.

Notice: The editor would be happy to override the "two months after publication" clause for solutions of problems prior to H-180, for which no solutions have been published. The next issue will contain a complete list of unresolved problems. Please send your solutions!

[Continued from page 590.]

$$\begin{aligned} 2\alpha &= \theta - \psi, & 2\beta &= \theta + \psi, \\ x &= 2\cos\theta, & y &= 2\cos\psi, \\ z &= xy + 2, & a &= \frac{1}{2}(x + y). \end{aligned}$$

We shall consider the asymmetric five diagonal determinant on another occasion.

REFERENCES

1. Brother Alfred Brousseau, "Lesson Eight — Asymptotic Ratios in Recurrence Relations," Fibonacci Quarterly, Vol. 8, No. 3, 1970, p. 311.
2. H. D. Ursell, "Simultaneous Recurrence Relations with Variable Coefficients," Proc. Edinb. Math. Soc., Vol. 9, Pt. IV, p. 183, 1958.
3. R. L. Shenton, "A Determinantal Expansion for a Class of Definite Integral," Proc. Edinb. Math. Soc., Vol. 10, Part IV, p. 167, 1957.
4. Thomas Muir, "Note on the Condensation of a Special Continuant," Proc. Edinb. Math. Soc., Vol. II, pp. 16-18, 1884.
5. D. E. Rutherford, "Some Continuant Determinants Arising in Physics and Chemistry," Proc. Roy. Soc. Edinb., Vol. LXII, p. 229, 1947.
6. D. E. Rutherford, "Some Continuant Determinants Arising in Physics and Chemistry," Proc. Roy. Soc. Edinb., Vol. LXIII, p. 232, 1952.

A PRIMER FOR THE FIBONACCI NUMBERS
PART X: ON THE REPRESENTATION OF INTEGERS

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

The representation of integers is a topic that has been implicit in our mathematics education from our earliest years due to the fact that we employ a positional system of notation. A number such as 35864 in base ten assumes the existence of a sequence 1, 10, 100, 1000, 10000, $\cdots$, running from right to left. The digits multiplied by the members of the sequence taken in order give the indicated integer. In this case, the representation means

$$3\cdot 10000 + 5\cdot 1000 + 8\cdot 100 + 6\cdot 10 + 4 .$$

Another way of thinking of these multipliers is this: they are the number of times various members of the sequence are being used.

It is instructive to see that such a sequence used as a base for representing integers arises naturally. Suppose we allow multipliers 0, 1, or 2. We wish to have a sequence that will enable us to represent all the positive integers and furthermore we want this sequence with the multipliers to do this uniquely; that is, for each integer there is one and only one representation by means of the sequence and the multipliers. Clearly, the first member of the sequence will have to be 1; otherwise, we could never represent the first integer 1. With this, we can represent 0, 1, or 2. Hence, the next integer we need is 3. The following table shows how at each step we are able to represent additional integers and likewise what is the next integer that is needed.

Sequence	Representations added
1	0, 1, 2
3	3, 4, 5, 6, 7, 8
9	9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, $\cdots$, 26
27	27, 28, 29, $\cdots$, 79, 80
81	81, 82, 83, $\cdots$, 241, 242

Note that, as far as we have gone, the representation is unique. Assume that we have unique representation when the sequence goes to 3^n and that this representation extends to $3^{n+1} - 1$. Adding 3^{n+1} to the sequence enables us to go from 3^{n+1} to $2\cdot 3^{n+1} - 1$ in a unique manner, but this sum is $3^{n+2} - 1$. Thus, the base three representation of integers using the sequence 1, 3, 9, 27, 81, $\cdots$ arises naturally in the case of allowed multipliers 0, 1, 2, and the requirements of complete and unique representation.

Perhaps the most interesting case of representation is that in which the allowed multipliers are 0, 1. We build up the sequence that goes with these multipliers giving complete and unique representation.

Sequence	Representations added
1	1
2	2, 3
4	4, 5, 6, 7
8	8, 9, 10, 11, 12, 13, 14, 15
16	16, 17, 18, ⋯, 30, 31
32	32, 33, 34, ⋯, 62, 63

Thus far the representation is unique. If we have unique and complete representation when the largest term of the sequence is 2^n and the representation extends to $2^{n+1} - 1$, then on adding 2^{n+1} to the sequence, we extend complete and unique representation to $2^{n+1} + 2^{n-1} - 1 = 2^{n+2} - 1$.

Another way of thinking of representation when the multipliers are 0 and 1 is this: We have a sequence where integers are represented by distinct members of the sequence. Thus the base two integer 110111010 says that the number in question is the sum of 2^8, 2^7, 2^5, 2^4, 2^3, and 2. The powers of two along with 1 enable us to represent all integers uniquely by combining different powers of two.

INCOMPLETE AND NON-UNIQUE SEQUENCES

Let us return to the representation with multipliers 0, 1, and 2. Clearly, if instead of taking 1, 3, 9, 27, 81, ⋯, we take some larger numbers such as 1, 3, 10, 28, 82, 244, ⋯, it will not be possible to represent all integers.

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
10	10-18, 20-28
28	28-36, 38-46, 48-56, 56-64, 66-74, 76-84
82	82-90, 92-100, etc.

Below 100, the numbers that cannot be represented are 9, 19, 37, 47, 65, 75, and 91. On the other hand, 28, 56, 82, 83, and 84 have two representations.

Suppose that instead of making the numbers of the sequence slightly larger we make them a bit smaller. Let us take the sequence 1, 3, 8, 26, 80, 242, ⋯, as before:

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
8	8-16, 16-24
26	26-34, 34-42, 42-50, 52-60, 60-68, 68-76
80	80-88, 88-96, 96-104, 106-114, 114-122, 122-130,
	132-140, 140-148, 148-156, 160, etc.

Up to 160, the missing integers are 25, 51, 77, 78, 79, 105, 131, 157, 158, and 159. Duplicated integers are 8, 16, 34, 42, 60, 68, 88, 96, 114, 122, 140, and 148.

The sequence 1, 3, 8, 23, 68, 203, $\cdots$, gives complete but not unique representation.

Sequence	Representations added
1	1, 2
3	3-8
8	8-16, 16-24
23	23-31, 31-39, 39-47, 46-54, 54-62, 62-70
68	68-76, 76-84, 84-92, 91-99, 99-107, 107-115,
	114-122, 122-130, 130-138, 136-144, 144-152, etc.

Up to 140 there is complete representation but duplicate representation for the following: 8, 16, 23, 24, 31, 39, 46, 47, 54, 62, 68, 69, 70, 76, 84, 91, 92, 99, 107, 114, 115, 122, 130, 136, 137, and 138.

FIBONACCI REPRESENTATIONS

Let us now consider the case in which the multipliers are 0, 1 and the basic sequence is the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, $\cdots$. That this sequence gives complete representation is not difficult to prove. In fact, the representation is still complete if we eliminate the first 1 and use the sequence 1, 2, 3, 5, 8, 13, $\cdots$. In the table following, note that the representation at each stage gives complete representation up to and including $F_{n+2} - 2$. Assume this to be so up to a certain F_n. Then upon adjoining F_{n+1} to the sequence the representation will be complete to $F_{n+1} + F_{n+2} - 2$, which is much beyond F_{n+2}, the next term to be added. Thus the representation is complete, but it is evidently not unique.

Sequence	Representations added
1	1
2	2, 3
3	3, 4, 5, 6
5	5-8, 8-11
8	8-11, 11-14, 13-16, 16-19
13	13-16, 16-19, 18-21, 21-24, 21-24, 24-27, 26-29, 29-32

AN INTERESTING THEOREM

To get a new perspective on representation by this Fibonacci sequence we write down the representations of the integers in their various possible forms. (Read 10110 as 8 + 3 + 2 or $1 \cdot F_6 + 0 \cdot F_5 + 1 \cdot F_4 + 1 \cdot F_3 + 0 \cdot F_2$.)

Integer	Representations	Integer	Representations
1	1	11	10100, 10011, 1111
2	10	12	10101
3	11, 100	13	11000, 10110, 100000
4	101	14	100001, 11001, 10111
5	110, 1000	15	100010, 11010
6	111, 1001	16	100100, 100011, 11100, 11011
7	1010	17	100101, 11101
8	1100, 1011, 10000	18	101000, 100110, 11110
9	10001, 1101	19	101001, 100111, 11111
10	10010, 1110	20	101010

Now the Fibonacci sequence has the property that the sum of two consecutive members of the sequence gives the next member of the sequence. Accordingly, one might argue, it is superfluous to have two successive members of the sequence in a representation since they can be combined to give the next member. If this is done, we arrive at representations in which there are no two consecutive ones in the representation. Looking over the list of integers that we have represented thus far, it appears that there is just one such representation for each integer in this form.

Suppose we go at this from another direction. We are building up a sequence that will represent the integers uniquely with multipliers 0 and 1. However, we stipulate that no two consecutive members of the sequence may be found in any representation. We form a table as before.

Sequence	Representations added
1	1
2	2
3	3, 4
5	5, 6, 7
8	8, 9, 10, 11, 12
13	13, 14, 15, 16, 17, 18, 19, 20

To this point the representation is unique and the sequence that is emerging is the Fibonacci sequence 1, 2, 3, 5, 8, 13, $\cdots$. Assume that up to F_n there is unique representation to $F_{n+1} - 1$. On adding F_{n+1} to the sequence, we cannot use F_n in conjunction with it but only terms up to F_{n-1}. But by supposition these may represent all integers up to $F_n - 1$ in a unique way. Hence with F_{n+1} we can represent uniquely all integers from F_{n+1} to $F_{n+1} + F_n - 1 = F_{n+2} - 1$. Hence the uniqueness and completeness of this type of representation are established, which is known as Zeckendorf's Theorem.

MORE ZEROES IN THE REPRESENTATION

A natural question to ask is: Would it be possible to require that there be at least two zeroes between 1's in the representation and obtain unique representation? We can build up the sequence as before taking into account this requirement.

Sequence	Representations added
1	1
2	2
3	3
4	4, 5
6	6, 7, 8
9	9, 10, 11, 12
13	13, 14, 15, 16, 17, 18
19	19, 20, 21, 22, 23, 24, 25, 26, 27
28	28-40

Up to this point, the representation is complete and unique. We have a sequence, but it would be difficult to operate with it unless we knew the way it builds up according to some recursion relation. The relation appears as

$$T_{n+1} = T_n + T_{n-2} .$$

Now assume that up to T_n we have unique representation to $T_{n+1} - 1$, where T_{n+1} is given by the recursion relation in terms of previous members of the sequence. Then on adding T_{n+1} to the sequence we may not use T_n or T_{n-1} in conjunction with it but only terms up to T_{n-2}. But these give unique and complete representation to $T_{n-1} - 1$. Hence upon adding T_{n+1} to the sequence we have extended unique and complete representation from T_{n+1} to $T_{n+1} + T_{n-1} - 1 = T_{n+2} - 1$. Thus, the uniqueness and completeness are established in general.

The sequences required for unique and complete representation when three, four, or more zeroes are required between 1's in the representation can be built up in the same way. Some are listed on the following page.

Zeroes	Sequence derived	Recursion relation
3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250, ⋯	$T_{n+1} = T_n + T_{n-3}$
4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, 140, 185, ⋯	$T_{n+1} = T_n + T_{n-4}$
5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, 34, 43, 55, 71, 92, 119, ⋯	$T_{n+1} = T_n + T_{n-5}$
6	1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 17, 22, 28, 35, 43, 53, 66, 83, 105, 133, ⋯	$T_{n+1} = T_n + T_{n-6}$

For k zeroes, the sequence is $1, 2, 3, 4, \cdots, k, k+1, k+2$, which enables us to get $k+3$; then $k+4$ which gives $k+5$, $k+6$; and so on. Up to this point the representation is unique and complete; the recursion relation beginning with $k+2$ is $T_{n+1} = T_n + T_{n-k}$. Assume that the sequence up to T_n gives unique and complete representation to $T_{n+1} - 1$. Then upon adding T_{n+1} the highest term we can use in conjunction with it is $T_{n+1-k-1} = T_{n-k}$ which gives unique representation to $T_{n-k+1} - 1$ by hypothesis. Hence upon adding T_{n+1} we have unique representation from T_{n+1} to $T_{n+1} + T_{n-k+1} - 1 = T_{n+2} - 1$.

MULTIPLIERS 0, 1, 2

We know that we obtain unique and complete representation using multipliers 0, 1, 2 when we have the geometric progression 1, 3, 9, 27, ⋯. Can we find a unique and complete representation if we demand that there be a zero between any two non-zero digits in the representation? Let us build this up as before.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 5, 6, 8, 9, 10
7	7, 8, 9, 10, 13, 14, 15, 16, 17, 20
11	11-14, 17, 15-17, 19-25, 28, 26-28, 30-32
18	18-21, 24, 22-24, 26-28, 25-28, 31-35, 38, 36-39, 42, 40-42, 44-46, 43-46, 49-53, 56

It appears that the sequence is the Lucas numbers. The representation is not unique. But a Lucas number L_n allows complete representation to the next Lucas number L_{n+1} (and beyond) without any additional Lucas numbers being represented. Assume that this is the case up to a certain n. Upon adding L_{n+1} we may not use L_n. Going back to L_{n-1} and preceding terms we can represent all integers up to $L_n - 1$ without being able to represent any Lucas numbers $L_n, L_{n+1}, \cdots$. Thus adding L_{n+1} allows the representation of numbers L_{n+1} to $L_{n+1} + L_n - 1 = L_{n+2} - 1$, but does not give L_{n+2} since this would require L_n. If we use $2L_{n+1}$ we would need L_n to get L_{n+3}, but since we do not have L_n it is not possible to arrive at this Lucas number. To dispose of L_{n+4} and higher Lucas numbers, we have to set a bound on the highest number at which we may arrive. Starting with L_{n-1} and working backward, the highest sum we can have is twice the sum of alternate terms beginning with L_{n-1}. If $n - 1$ is odd, this sum is $2(L_n - 2)$, and if $n - 1$ is even, this sum is $2(L_n - 1)$. In either case, the sum is less than $2L_n$. Hence an upper bound for terms when L_{n+1} is added to the sequence is $2L_{n+1} + 2L_n = 2L_{n+2}$. But $L_{n+4} = 2L_{n+2} + L_{n+1}$ which is greater than $2L_{n+2}$. Hence it is not possible to arrive at L_{n+4} or higher Lucas numbers.

This result was very encouraging and led to an investigation of cases with multipliers 0, 1, 2, 3; then 0, 1, 2, 3, 4; etc., where we still require one zero between non-zero digits. The first few terms looked interesting.

Multipliers 0, 1, 2, 3:	1, 4, 5, 9, 14, ···
Multipliers 0, 1, 2, 3, 4:	1, 5, 6, 11, 17, ···
Multipliers 0, 1, 2, 3, 4, 5:	1, 6, 7, 13, 20, ···

Unfortunately, in the sequence 1, 4, 5, 9, 14, ···, if we continue with the terms 23, 27, 60, we find that 60 is already represented by 14 and lower terms. In the sequence 1, 5, 6, 11, 17, 28, ···, the 28 is represented by earlier terms. We have run into a DRY HOLE.

Next, keeping the multipliers 0, 1, 2, the case of two zeroes between non-zero digits was investigated. This led to the sequence 1, 3, 4, 5, 9, 13, 22, 31, 53, 75, 128, 181, ···, where there are two apparent laws of formation, one for odd-numbered terms, and a second for even-numbered terms,

$$T_{2n+1} = T_{2n} + T_{2n-1}\,, \tag{1}$$

$$T_{2n+2} = T_{2n+1} + T_{2n-1}\,. \tag{2}$$

There are equivalent representations of these relations. By (1) and (2),

$$T_{2n+1} = (T_{2n-1} + T_{2n-3}) + T_{2n-1} = 2T_{2n-1} + T_{2n-3}\,, \tag{3}$$

$$T_{2n+2} = (T_{2n} + T_{2n-1}) + T_{2n-1} = T_{2n} + 2T_{2n-1}\,. \tag{4}$$

Since by (1) $T_{2n-1} = T_{2n+1} - T_{2n}$, we have from (4) $T_{2n+2} = 2T_{2n+1} - T_{2n}$, or,

$$2T_{2n+1} = T_{2n+2} + T_{2n}\,. \tag{5}$$

Therefore, by using (5) to express $2T_{2n-1}$ in (4),

$$T_{2n+2} = 2T_{2n} + T_{2n-2}\,. \tag{6}$$

Hence, combining (3) and (6), there is one recursion relation for the entire sequence,

$$T_{n+1} = 2T_{n-1} + T_{n-3}\,. \tag{7}$$

The manner in which the sequence builds up is shown by the following table.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 8
5	5, 6, 7, 10, 11, 12
9	9-12, 15, 18-21, 24
13	13-16, 19, 17, 21, 26-29, 32, 30, 34
22	22-25, 28, 26, 30, 27-29, 32-34, 44-47, 50, 48, 52, 49-51, 54-56

To show that the sequence will continue to be built up in this way we note the following as a basis for our induction:

(1) Adding a term T_k covers all representations up to $T_{k+1} - 1$.

(2) Adding another term of the sequence does not give additional terms of the representing sequence.

(3) The largest term that can be represented by adding T_k is less than T_{k+3}.

Now, if the above is true to T_n, add the term T_{n+1}. We can use only terms to T_{n-2} and smaller in the sequence in conjunction with T_{n+1}. Such terms can represent values up to $T_{n-1} - 1$. Hence adding T_{n+1} enables us to represent values from T_{n+1} to $T_{n+1} + T_{n-1} - 1$, which gives $T_{n+2} - 1$ if $n + 1$ is odd. If $n + 1$ is even, $T_{n+1} + T_{n-1} - 1 = 2T_{n+2} - 1$. Hence all representations up to $T_{n+2} - 1$ are covered.

On adding T_{n+1} to the sequence we do not obtain any other sequence terms. For $T_{n+2} = T_{n+1} + T_{n-1}$ and $T_{n+2} = 2T_{n+1} + T_{n-1}$ if $n + 1$ is odd, and T_{n-1} is not available in conjunction with T_{n+1}. Similarly, if $n + 1$ is even, $T_{n+2} = T_{n+1} + T_n$ and $T_{n+3} = 2T_{n+1} + T_{n-1}$ where neither T_n nor T_{n-1} is available. Finally, T_{n+4} is larger than any term that can be formed using T_{n+1} and smaller terms.

CONCLUSION

A great deal of work has been done on representations of integers in recent years. Much of this has appeared in the Fibonacci Quarterly which has published some two dozen articles totalling approximately 300 pages by such mathematicians as Carlitz, Brown, Hoggatt, Ferns, Klarner, Daykin, and others. The number of byways that may be investigated is great. It could be the project of a lifetime.

FIBONACCI DRAINAGE PATTERNS

W. E. SHARP
University of South Carolina, Columbia, South Carolina

1. INTRODUCTION

Some 25 years ago, an engineer, Robert E. Horton, developed the notion of stream order or class [1] as a measure of the position of a stream in the hierarchy of tributaries as observed in natural river basins. Using a map of a specified scale, he designated first-class streams as those which have no tributaries, second-class streams as those which have tributaries only of the first class, and third-class streams as those which have only first- and second-class tributaries, and so on (Fig. 1). Although Horton in his original analysis next renumbered the streams to show the headward extent of the main thread of the river, it has since been shown [6] that for most purposes this is an unnecessary complication. If the renumbering procedure is omitted then the basic property of stream class numbers is that if two streams of the same class i combine [5], the resulting stream is of class i + 1, that is

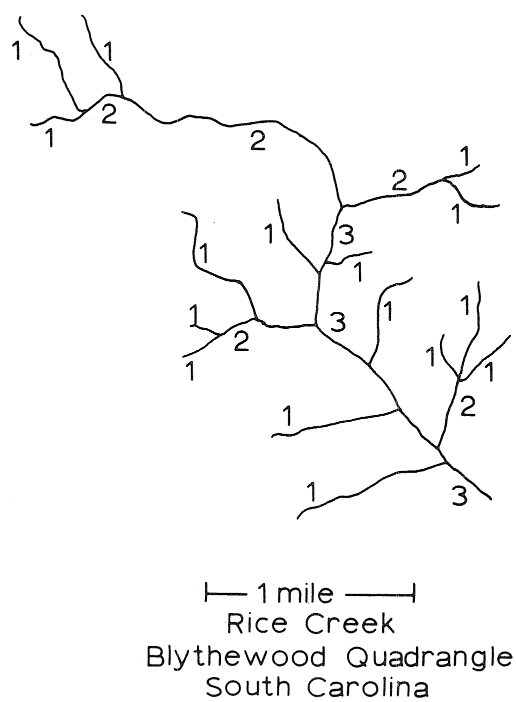


Fig. 1 A Third-Class Natural Stream Illustrating the Ordering Procedure

$$i * i = i + 1 ,$$

where an asterisk (*) signifies the junction or combination of two streams. If, however, the streams are of different class, then the lower class stream is lost in the one of higher class and the combination is expressed by [5]:

$$i * j = j * i = j \qquad (j > i) .$$

Using this system of stream classification, Horton [1] noticed that in natural river basins when the logarithm of the number of streams of each class was plotted versus the stream class (Fig. 2), the graph formed a straight line. The constant slope of the line, Horton called the bifurcation ratio. Measured values of this ratio for natural streams range between 2 and 5 but seem to average about 3.5 [2, p. 138]. Although many geomorphologists have been greatly intrigued by this natural relationship between numbers of streams and their class, it is still not clearly understood why the relationship holds so well.

Because the bifurcation ratio is given by

$$r_b = \frac{N_i}{N_i + 1}$$

Horton then summarized the result of his observations for a basin using the relation

$$N_i = r_b^{m-i} ,$$

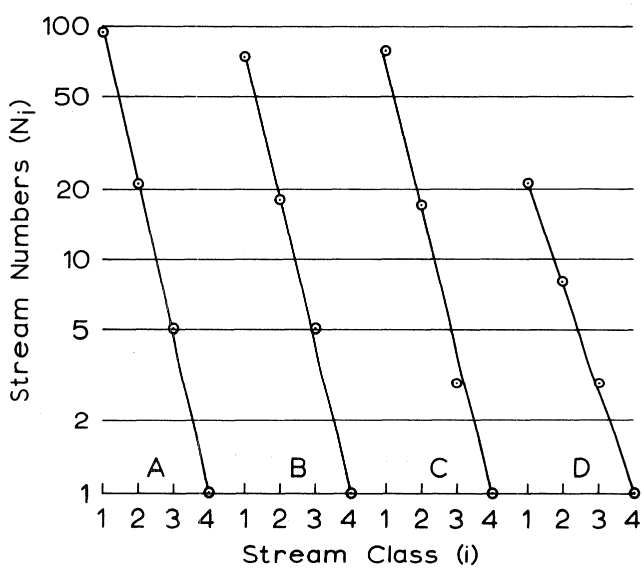


Fig. 2 A Graph Showing the Relationships Between the Stream Class and the Number of Streams in each Class. Examples are: (A) Hightower Creek, Georgia [9, p. 19]; (B) Tar Hollow, Ohio [4, p. 1036]; (C) Green Lick, Pa. [4, p. 1036]; (D) Fibonacci Pattern (8th Order)

where m is the class of the main stream in the basin and N_i is the number of streams of each class i. This equation is analogous to the simple population growth equation [3, p. 129] when the initial population is taken equal to one. The bifurcation ratio then corresponds to the net reproductive rate, m - i corresponds to the number of generations and N_i corresponds to the population size after m - i generations (Fig. 3).

Norton's equation by itself implies that drainage nets only have junctions of the type where streams of class i meet at a single place to form streams of class i + 1 (Figs. 3 and 4). That is, only junctions of the type

$$i * i = i + 1$$
$$i * i * i = i + 1$$
$$i * i * i * i = i + 1$$

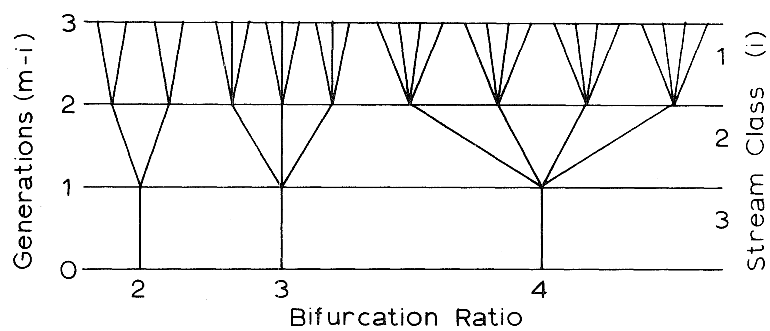


Fig. 3. Branching Systems for Various Bifurcation Ratios which Obey the Simple Population Growth Equation

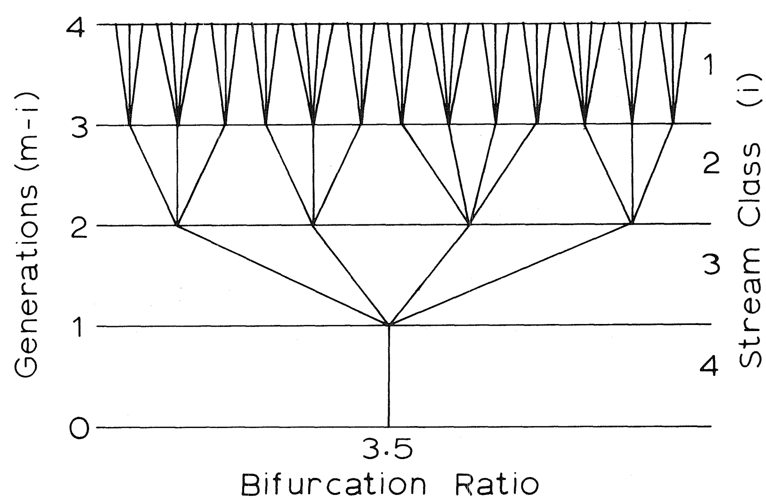


Fig. 4. A Simple Branching System having an Average Bifurcation Ratio of 3.5

etc., exist with no junctions of the type

$$i * j = j \qquad (j > i) .$$

This would suggest that natural streams should appear as shown in Fig. 4. However, this branching pattern has little resemblance to natural drainage patterns (Fig. 1) because there are no junctions of the type where a low-class stream becomes lost in a higher-class stream.

If we next examine the drainage patterns of randomly selected small second- and third-class streams, one will soon notice that a significant percentage of these patterns resemble the branching pattern obtained by constructing a Fibonacci tree [10, p. 47]. This resemblance can be illustrated by comparing an unnamed portion of Rice Creek and the upper reaches of Crane Creek (Figs. 5 and 6) Blythewood Quadrangle, South Carolina with the 5^{th} and 6^{th} order Fibonacci trees (Fig. 7), respectively.

2. SIMPLE PROPERTIES OF FIBONACCI DRAINAGE PATTERNS

If a Fibonacci tree of any order is treated as if it were a drainage pattern [7], we can apply Horton's numbering procedure to the branches of the tree and call these branches streams (Fig. 7). Inspection of any such Fibonacci tree (Fig. 7) shows that the total number

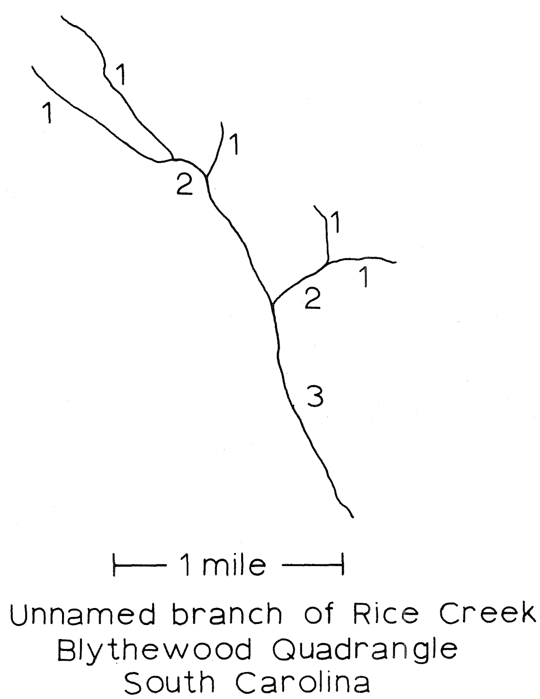


Fig. 5. A Third-Class Natural Stream Which can be Represented as a Fifth-Order Fibonacci Tree

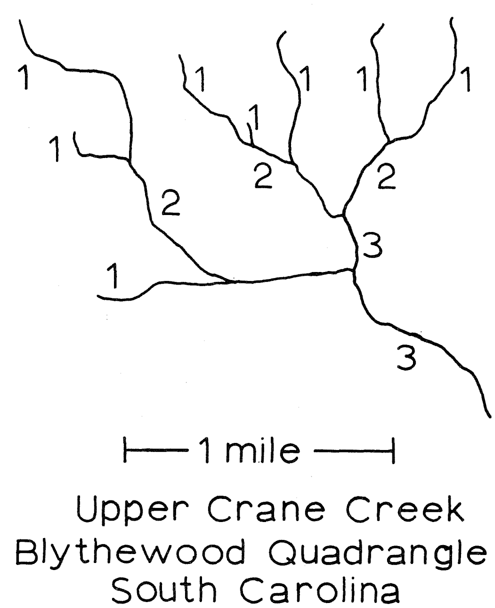


Fig. 6. A Third-Class Natural Stream Which can be Represented as a Sixth-Order Fibonacci Tree

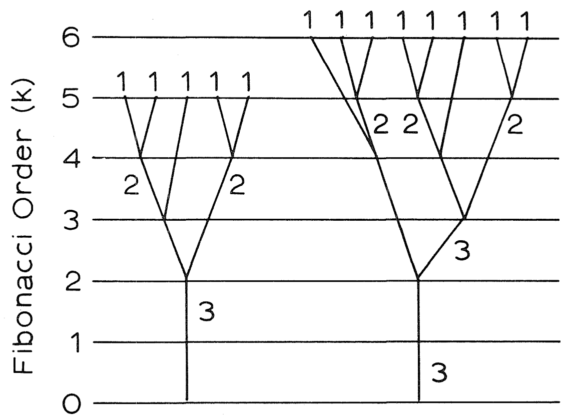


Fig. 7. Fibonacci Trees of Fifth and Sixth Order

of pendant vertices (first-class streams) at the end of k Fibonacci orders is given by F_k, the Fibonacci number having index k where F_k is evaluated using the recursion formula:

$$F_k = F_{k-1} + F_{k-2}$$

and the initial conditions:

$$F_{-k} = 0, \qquad F_0 = 0, \qquad F_1 = 1$$

so that in a Fibonacci stream basin

$$N_1 = F_k .$$

Similarly, inspection of the Fibonacci tree (Fig. 7) having a maximum Fibonacci order of k shows that the total number of streams of class i is given by $N_i = F_n$ where $n = k - 2(i - 1)$ and the total number of streams of i^{th} class lost in a stream of class $i + 1$ is given by $L_i = F_\ell$ where $\ell = k - 2i - 1$. The stream of highest class (m) in a tree of Fibonacci order k is given by

$$m = \left[\frac{k+1}{2}\right]$$

where [ ] signifies the integral value. The bifurcation ratio is given by

$$\frac{N_i}{N_{i+1}} = \frac{F_{k-2i+2}}{F_{i-2i}} = 1 + \frac{F_{k-2i+1}}{F_{k-2i}}$$

and for large values of $(k - 2i)$, this ratio converges to the constant $1 + \tau$ where τ is the famous golden ratio. Comparison of the Fibonacci bifurcation ratio of 2.618 (Fig. 2) with the average ratio of 3.5 shows the Fibonacci ratio to be significantly smaller than the ratio seen in natural streams.

In Fibonacci streams, the disappearance of low-class streams into higher-class streams has the following fixed restrictions:

1. When $i * j = j$, then $0 \le j - 1 \le 1$.
2. No more than one stream of class i can be lost in any given stream of class $i + 1$.

3. A GENERALIZED FIBONACCI DRAINAGE PATTERN

The restriction imposed on Fibonacci patterns that no more than one i^{th} class stream can be lost in any given $i + 1$ class stream can be partially relaxed by considering a simple form of a generalized Fibonacci tree [8, p. 922]. In the usual construction of a Fibonacci tree [10, p. 47] it is assumed that the trunk and each limb has a maturing time of one period and a gestation time of one period. This growth pattern can be modified by changing the maturing period from one to some other integral period (Fig. 10).

If p is equal to one (gestation period) plus the maturing period (any integer greater than one) and k is the total number of elapsed periods (order), then ${}^pF_k$ is a generalized Fibonacci number which can be evaluated from the recursion formula:

$${}^pF_k = {}^pF_{k-1} + {}^pF_{k-p}$$

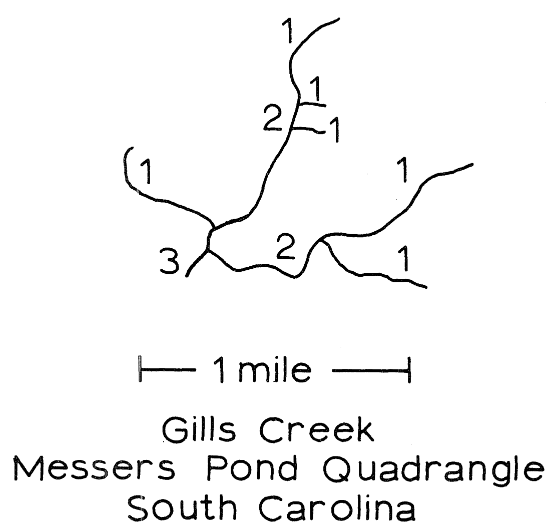


Fig. 8. A Third-Class Natural Stream Which Can Be Represented as a Seventh-Order Modified Fibonacci Tree with Two-Period Maturation

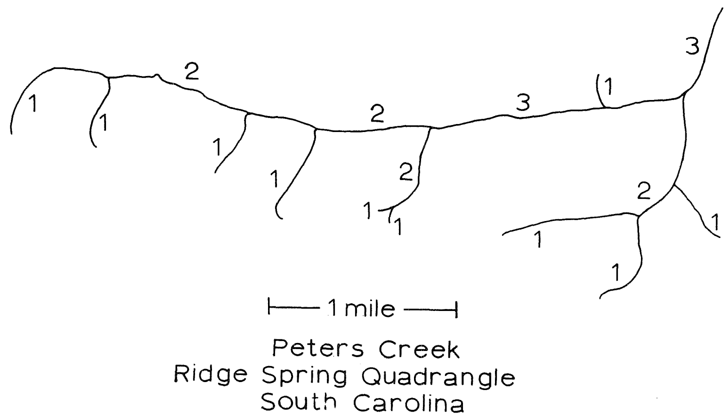


Fig. 9. A Third-Class Natural Stream Which can be Represented as an Eighth-Order Modified Fibonacci Tree with Two-Period Maturation

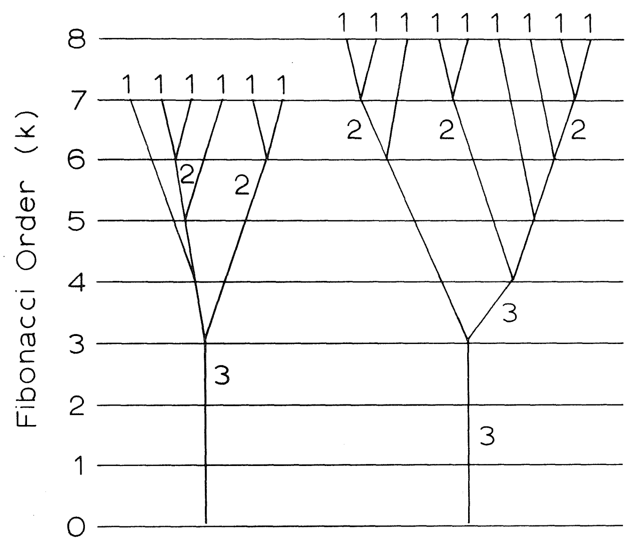


Fig. 10. Modified Fibonacci Trees of Seventh and Eighth Order with Two-Period Maturation

and the initial conditions:

$$p_{F_{-k}} = 0, \qquad p_{F_0} = 0, \qquad p_{F_1} = 1 .$$

In a manner similar to regular Fibonacci patterns this group of generalized Fibonacci patterns has a total number of pendant vertices (first-class streams) at the end of k orders given by

$$N_1 = {}^{p}F_k \ .$$

The total number of streams of class i is then given by

$$N_i = {}^{p}F_n \quad ,$$

where

$$n = k - p(i - 1)$$

the total number of streams i lost in class $i + 1$ is given by

$$L_i = {}^{p}F_{\ell} - {}^{p}F_{k-pi} \ ,$$

where

$$\ell = k - 1 - p(i - 1) \ ,$$

and the bifurcation ratio is given by

$$r_b = \frac{N_i}{N_{i+1}} = \frac{{}^{p}F_{k-p(i-1)}}{{}^{p}F_{k-pi}} = 1 + \frac{{}^{p}F_{k-1-p(i+1)}}{{}^{p}F_{k-pi}} \ ,$$

which will converge to a constant for large values of $(k - pi)$.

This group of generalized Fibonacci streams has the following fixed restrictions governing the loss of low-class streams into higher-class streams:

1. when $i * j = j$, then $0 \leq j - i \leq 1$,
2. No more than $(p - 1)$ streams of class i can be lost in any given stream of class $i + 1$.

Natural streams which resemble this type of generalized Fibonacci pattern are illustrated by comparing the upper reaches of Gills Creek (Fig. 8) Messers Pond Quadrangle, and Peters Creek (Fig. 9) Ridge Spring Quadrangle, South Carolina with 7^{th} and 8^{th} order generalized Fibonacci trees (Fig. 10) having two-period maturation $(p = 3)$.

4. CONCLUSION

Because natural second-class streams can only have first-class tributaries and third-class streams can only have first- and second-class tributaries, the very restrictive junction rule $(|j - i| \leq 1)$ for Fibonacci patterns is commonly satisfied. This produces a superficial resemblance between these small natural patterns and Fibonacci patterns. In fourth- and higher-class basins the opportunities for violation of the Fibonacci junction rule are suddenly increased and the resemblance rapidly fades. Yet even in basins of the highest class, whenever the branching among two or three adjacent classes is emphasized, a Fibonacci pattern can often be discerned.

[Continued on page 655.]

MORE ABOUT MAGIC SQUARES CONSISTING OF DIFFERENT PRIMES

EDGAR KARST
University of Arizona, Tucson, Arizona

Let a magic square of order n be surrounded by numbers such that square plus numbers form another magic square of order $n + 2$ and similar magic squares of order $n + 4$, $n + 6$, and so on; then the center square may be called a nucleus and the surrounding numbers a frame.

In a letter of August 8, 1971, V. A. Golubev concocts and gives permission to publish the following magic square of order 11 consisting of primes of the form $30x + 17$ and including similar magic squares of order 3, 5, 7, and 9.

GOLUBEV'S PRIME MAGIC SQUARE

73547	52757	52457	74567	51287	75767	49787	49727	24527	119087	72977
80177	59447	54767	71987	54167	72647	53597	50147	84407	68687	46457
80897	73127	67217	60527	60257	58427	59387	70937	66467	53507	45737
81077	53117	75437	64877	60497	54347	71147	65717	51197	73517	45557
81647	52727	55967	60017	64577	61637	63737	66617	70667	73907	44987
44927	74507	69737	72707	62477	63317	64157	53927	56897	52127	81707
44417	51257	57737	58067	62897	64997	62057	68567	68897	75377	82217
43787	101537	56957	60917	66137	72287	55487	61757	69677	25097	82847
84437	46187	60167	66107	66377	68207	67247	55697	59417	80447	42197
27917	57947	71867	54647	72467	53987	73037	76487	42227	67187	98717
53657	73877	74177	52067	75347	50867	76847	76907	102107	7547	53087

The nucleus of order 3 contains the elements 61637, 62057, $\cdots$, 64997 which are the nine primes in A. P. given in the appendix of [3]. A pair of opposite primes in each frame adds up to $126634 = 2 \cdot 63317$. Important for constructing the frames is the fact that the sums of two opposite sides without the corners must be the same. Hence, the frame of order 5 has

$$60497 + 54347 + 71147 + 66137 + 72287 + 55487 = 66617 + 53927 + 68567 + 60017 +$$
$$+ 72707 + 58067 = 379902 = 2 \cdot 3 \cdot 63317 ,$$

the frame of order 7 has

$$\begin{aligned} &60527 + 60257 + 58427 + 59387 + 70937 + 66107 + 66377 + 68207 + 67247 + 55697 \\ &\quad = 51197 + 70667 + 56897 + 68897 + 69677 + 75437 + 55967 + 69737 + 57737 + 56957 \\ &\quad = 633170 = 2 \cdot 5 \cdot 63317 , \end{aligned}$$

and so on. This comprises the simpler part of the construction. For the corners, the two pairs of diagonally opposite primes must each not only add up to $126634 = 2 \cdot 63317$, but the sum of the elements of each of the two diagonals must also agree with the magic constant already obtained by summing up n members in a vertical or horizontal way. This is the more difficult part of the construction. Is someone able to attach a frame of order 13 to Golubev's beautiful magic square of primes $30x + 17$?

If we have prime magic squares of odd order, it is not necessary that the nucleus consists of primes in A. P. such that

$$p_1 + d = p_2, \quad p_2 + d = p_3, \quad \cdots, \quad p_8 + d = p_9 .$$

In fact, the 3^{rd} and 6^{th} d in those equations may be replaced by any number $y = 6m$ such that the elements still remain primes. For example,

$$-17 + 6 = -11, \quad -11 + 6 = -5, \quad -5 + 12 = 7, \quad 7 + 6 = 13, \quad 13 + 6 = 19,$$

$$19 + 12 = 31, \quad 31 + 6 = 37, \quad 37 + 6 = 43 \quad \text{with} \quad d = 6 \quad \text{and} \quad y = 12.$$

Choosing now the standard magic square of order 3

8	1	6
3	5	7
4	9	2

and putting the right side of those equations, starting with -17, in that order into it, we obtain

37	-17	19
-5	13	31
7	43	-11

yielding a prime magic square with magic constant 39. For the frames we may not request that their primes are of a special form. Of course, all means of construction should be the same as in Golubev's prime magic square. Has such a magic square of primes, say of order 13, ever been constructed? Yes, one can find it in [5], and it may be republished here as a good example of magic squares of primes with no restrictions attached to their construction. It says there: "This tremendous prime magic square was sent to Francis L. Miksa of Aurora, Illinois, from an inmate in prison who, obviously, must remain nameless." The nucleus of order 3 consists of triples of primes in A. P. with $d = 6$ and $y = 3558$. Each opposite prime pair in any frame adds up to $10874 = 2 \cdot 5437$, the magic constant of order 3 is $16311 = 3 \cdot 5437$, of order 5 is $27185 = 5 \cdot 5437, \cdots,$ of order 13 is $70681 = 13 \cdot 5437$. It is constructed in the same way as Golubev's magic square, but while there the difference between the largest prime, 119087, and the smallest prime, 7547, is $111540 = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13^2$, in the prisoner's magic square it is 9967 and 907 with $9060 = 2^2 \cdot 3 \cdot 5 \cdot 151$.

Is someone able to attach a frame of order 15 to the prisoner's remarkable magic square?

Somewhat differently behave the prime magic squares of even order. The greatest attraction is here the prime magic square of order 12 by J. N. Muncey of Jessup, Iowa, which is the smallest possible magic square of consecutive odd primes, starting with 1, ending with 827, and reproduced in [2]. It speaks for the attitude of mathematical journals

THE PRISONER'S PRIME MAGIC SQUARE

1153	8923	1093	9127	1327	9277	1063	9133	9661	1693	991	8887	8353
9967	8161	3253	2857	6823	2143	4447	8821	8713	8317	3001	3271	907
1831	8167	4093	7561	3631	3457	7573	3907	7411	3967	7333	2707	9043
9907	7687	7237	6367	4597	4723	6577	4513	4831	6451	3637	3187	967
1723	7753	2347	4603	5527	4993	5641	6073	4951	6271	8527	3121	9151
9421	2293	6763	4663	4657	9007	1861	5443	6217	6211	4111	8581	1453
2011	2683	6871	6547	5227	1873	5437	9001	5647	4327	4003	8191	8863
9403	8761	3877	4783	5851	5431	9013	1867	5023	6091	6997	2113	1471
1531	2137	7177	6673	5923	5881	5233	4801	5347	4201	3697	8737	9343
9643	2251	7027	4423	6277	6151	4297	6361	6043	4507	3847	8623	1231
1783	2311	3541	3313	7243	7417	3301	6967	3463	6907	6781	8563	9091
9787	7603	7621	8017	4051	8731	6427	2053	2161	2557	7873	2713	1087
2521	1951	9781	1747	9547	1597	9811	1741	1213	9181	9883	1987	9721

shortly before the outbreak of World War I that they would rather publish abstract mathematics than such a genuine gem of mathematical thinking. Hence, one doesn't wonder that Muncey's magic square of consecutive primes finally appeared in a philosophical journal [The Monist, 23 (1913), 623-630]. We see at a glance that this prime magic square is of a different kind. Neither has it a nucleus of order 4 nor does it include similar magic squares of order 6, 8, and 10. Its magic constant is $4514 = 2 \cdot 37 \cdot 61$.

Another gem is the magic square of order 4 consisting of 16 primes in A.P. by S. C. Root of Brookline, Massachusetts. It is published in [4]. Its magic constant is

$$15637321864 = 2^3 \cdot 43 \cdot 45457331 ,$$

the common difference is

$$223092870 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 .$$

It is not known whether there exists a sequence of 16 primes in A.P. with a smaller common difference d. Theoretically, it should be possible to find such a sequence with $d = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

If we have prime magic squares of an even order, the nucleus has not to consist of primes in A.P. Assuming again,

$$p_1 + d = p_2, \quad p_2 + d = p_3, \quad \cdots, \quad p_{15} + d = p_{16}$$

we shall see that the 4^{th}, the 8^{th}, and the 12^{th} d can be replaced by 6m, but these all different, say u, v, and w. Each $(2m - 1)^{th}$ d may be 30 and each $2(2m - 1)^{th}$ d may be 12. In this way we obtain the pair of prime magic squares due to the late Leo Moser of the University of Alberta which are published in [5]. Moser uses not only primes, but twin

MUNCEY'S CONSECUTIVE PRIME MAGIC SQUARE

1	823	821	809	811	797	19	29	313	31	23	37
89	83	211	79	641	631	619	709	617	53	43	739
97	227	103	107	193	557	719	727	607	139	757	281
223	653	499	197	109	113	563	479	173	761	587	157
367	379	521	383	241	467	257	263	269	167	601	599
349	359	353	647	389	331	317	311	409	307	293	449
503	523	233	337	547	397	421	17	401	271	431	433
229	491	373	487	461	251	443	463	137	439	457	283
509	199	73	541	347	191	181	569	577	571	163	593
661	101	643	239	691	701	127	131	179	613	277	151
659	673	677	683	71	67	61	47	59	743	733	41
827	3	7	5	13	11	787	769	773	419	149	751

ROOT'S MAGIC SQUARE OF PRIMES IN A. P.

2236133941	5359434121	5136341251	2905412551
4690155511	3351598291	3574691161	4020876901
3797784031	4243969771	4467062641	3128505421
4913248381	2682319681	2459226811	5582526991

MOSER'S TWIN MAGIC SQUARES OF PRIMES IN A.P.

29	1061	179	227
269	137	1019	71
1049	101	239	107
149	197	59	1091

31	1063	181	229
271	139	1021	73
1051	103	241	109
151	199	61	1093

primes. We see that $u = 6$, $v = 18$, and $w = 750$. The magic constant of the left square is $1496 = 2^3 \cdot 11 \cdot 17$, the magic constant of the right square is $1504 = 2^5 \cdot 47$. The author remembers that Leo Moser had always a little self-fabricated poem on hand which served as a kind of donkey bridge to his brain twisters: does someone recall the poem for the twin prime magic squares?

We have attempted to give a glimpse into the more recent investigations on prime magic squares and to somewhat analyze the regular ones of them. Of course, a detailed treatise on their construction would not be permissible here, but can be found in the almost classic collection of [1].

REFERENCES

1. W. S. Andrews, Magic Squares and Cubes, Dover Publications, New York, 1960.

2. Martin Gardner, "The Remarkable Lore of the Prime Numbers," Scientific American, March 1964, pp. 122-126.
3. Edgar Karst, "Magic Squares Consisting of Primes in A. P.," The Fibonacci Quarterly, April 1970, pp. 317-323.
4. Edgar Karst, "12 to 16 Primes in Arithmetical Progression," Journal of Recreational Mathematics, Oct. 1969, pp. 214-215.
5. J. S. Madachy, "Strictly for Squares," Recreational Mathematics Magazine, Oct. 1961, pp. 24-29.

[Continued from page 650.]

The determination of the branching characteristics of natural streams of class five and higher is an extremely difficult and tedious task. Thus any hypothesis proposed for stream patterns of high class is very difficult to test. If it could be shown that a Fibonacci or one of the generalized Fibonacci patterns could serve as a first approximation to natural patterns, then any hypothesis proposed could quickly and easily be explored to very high orders and the results used to plan tests that could be applied to natural patterns.

REFERENCES

1. R. E. Horton, "Erosional Development of Streams and Their Drainage Basins," Bull. Geol. Soc. Am. 56 (1945), pp. 275-370.
2. L. B. Leopold, M. G. Wolman, and J. P. Miller, Fluvial Processes in Geomorphology, W. H. Freeman, San Francisco, 1964.
3. R. B. MacArthur and J. H. Connell, The Biology of Populations, John Wiley and Sons, New York, 1966.
4. M. E. Morisawa, "Quantitative Geomorphology of Some Watersheds in the Appalachian Plateau," Bull. Geol. Soc. Am. 73 (1957), pp. 1025-1046.
5. A. E. Scheidegger, "The Algebra of Stream-Order Numbers," U. S. Geol. Survey Prof. Paper, 525B (1965), pp. 187-189.
6. A. E. Scheidegger, "Horton's Law of Stream Numbers," Water Resources Res., 4 (1968), pp. 655-658.
7. W. E. Sharp, "An Analysis of the Laws of Stream Order for Fibonacci Drainage Patterns," Water Resources Res. 7 (1971), pp. 1548-1557.
8. W. E. Sharp, "Stream Order as a Measure of Sample Source Uncertainty," Water Resources Res. 6 (1970), pp. 919-926.
9. R. L. Shreve, "Statistical Law of Stream Numbers," Jour. Geol. 74 (1966), pp. 17-37.
10. H. Steinhaus, Mathematical Snapshots, Oxford Univ. Press, New York, 1969.

ERRATA

FOR

A NEW PRIMALITY CRITERION OF MANN AND SHANKS AND ITS RELATION TO A THEOREM OF HERMITE WITH EXTENSION TO FIBONOMIALS[1]

On page 356, Eq. (2.3): for "$\mathrm{mod}\ \dfrac{m}{(a, b, \cdots, c)}$" read "$\left(\mathrm{mod}\ \dfrac{m}{(a, b, \cdots, c)}\right)$" .

On page 359, line 4: for "$\left(\dfrac{n-1}{3}\right)$" read "$\left[\dfrac{n-1}{3}\right]$" .

The sixth and eighth lines from the bottom of page 360 should read "Erdős," NOT "Erdos."

In Reference 2, page 372, change "Institute" to "Institution."

These are typographical errors arising in the process of printing. H. W. Gould

FOR

THE CASE OF THE STRANGE BINOMIAL IDENTITIES OF PROFESSOR MORIARTY[2]

Page 382, line 1: For "E. T. Davis" read "H. T. Davis".

Page 382, line 4: For "Holmes. "Here I . . ." read "Holmes." Here I. . ." .

Page 383, Eq. (8): For "n - 1 - 2i" read "n - 1 - 2k". (manuscript error)

Page 383, 4th line from bottom: for "equation" read "equating".

Page 385, line 7: for "Glocksman" read "Glicksman" .

Page 385, Eq. (12): for "2n - 2j" read "n - 2j".

Page 385, 3rd line from bottom: for "Eagan" read "Hagen".

Page 387, line 5: Before "87" insert "36," (new reference cited by Riordan).

Page 387, Eq. (14): for $\left(1 - \dfrac{x}{1-x}\right)^2$ read $1 - \left(\dfrac{x}{1-x}\right)^2$.

Page 387, line 6: for "Andre" read "André" .

Page 391, Eq. (25): for the exponent "n - 2" read "n - 2k" .

Page 391, Ref. 5: for "Bromwhich" read "Bromwich" .

Page 391, Ref. 10: for "Glocksman" read "Glicksman" .

Page 391, Ref. 11: for "Gonsáles" read "Gonzáles" ,
and for "numerous" read "numeros" .

Page 402, Ref. 16: for "Leipsig" read "Leipzig" .

Page 402, Ref. 18: for "Niew" read "Nieuw" .

Page 402, Ref. 20: for "letters" read "lettere" .

Except as indicated, these are typographical errors arising in the printing. H. W. Gould

---

[1] Appearing in Vol. 10, No. 4, Fibonacci Quarterly, pp. 355-364; 372.
[2] Appearing in Vol. 10, No. 4, Fibonacci Quarterly, pp. 381-391; 402.

A NOTE ON THE NUMBER OF FIBONACCI SEQUENCES

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

In an article entitled "On the Ordering of Fibonacci Sequences" [1], the author pointed out that if we consider Fibonacci sequences with relatively prime successive terms and a series of positive terms extending to the right, there is (apart from the case of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, $\cdots$), one point in the sequence and only one where a positive term is less than half the next positive term. Such being the case, it is convenient to identify a Fibonacci sequence by these two numbers, as this gives a unique means of specifying a sequence.

The present note is concerned with this question: If the two identifying numbers of a Fibonacci sequence as presently defined are less than or equal to a positive integer m, how many Fibonacci sequences does this give?

Theorem. If the starting numbers of a Fibonacci sequence are $\leq m$ $(m \geq 2)$, the number of Fibonacci sequences that can be formed is:

$$1/2 \sum_{k=1}^{m} \phi(k)$$

where $\phi(m)$ is Euler's totient function.

Proof. The following table indicates the situation for small values of m and serves as the basis of the subsequent mathematical induction

m	$\phi(m)$	$\Sigma\phi(k)$	$\frac{1}{2}\Sigma\phi(k)$	Sequences
1	1			
2	1	2	1	(1,1)
3	2	4	2	(1,3)
4	2	6	3	(1,4)
5	4	10	5	(1,5), (2,5)
6	2	12	6	(1,6)
7	6	18	9	(1,7), (2,7), (3,7)

Within the limits of this table, it is clear that the total number of sequences that may be formed for any given m is $\frac{1}{2}\Sigma\phi(k)$.

Assume that this is true to some given m. If we enlarge the domain by including $m + 1$, the new sequences added will be those involving this quantity as well as those quantities less than half of $m + 1$ and relatively prime to it. But the number of such quantities is $\frac{1}{2}\phi(m + 1)$. Thus it follows that if the formula is true for m, it is true for $m + 1$ and the theorem is proved in general.

REFERENCE

1. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Dec. 1963, pp. 43-46.

[Continued from page 597.]

That is, we have shown that

$$C_k(x) = A_k(x) \cdot (1 - x)^{-\frac{1}{2}k(k+1)-1} , \tag{4.8}$$

where $A_k(x)$ is a polynomial in x of degree $\frac{1}{2}k(k - 1)$ given by either of

$$A_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}(1 - x)^{\frac{1}{2}k(k+1)-j} \tag{4.9}$$

or

$$A_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\, x^{j-k}(x - 1)^{\frac{1}{2}k(k+1)-j} . \tag{4.10}$$

Notice that the symmetry property (1.9) follows by comparing (4.9) and (4.10). The first few values of $A_k(x)$ are $A_1(x) = 1$, $A_2(x) = 1 + x$, $A_3(x) = 1 + 7x + 7x^2 + x^3$.

REFERENCES

1. L. Carlitz and John Riordan, "Enumeration of Certain Two-Line Arrays," Duke Math. J., Vol. 32 (1965), pp. 529-539.
2. L. Carlitz and R. A. Scoville, Problem E2054, MAA Monthly, Vol. 75 (1968), p. 77.
3. P. A. MacMahon, Combinatory Analysis, Vol. 1, Cambridge, 1915.

[Continued from page 598.]

(3) Articles of standard size for which additional background material may be obtained.

Articles in the Quarterly for which this note service is available will indicate the fact, together with the number of pages in question. Requests for these notes should be made to

Brother Alfred Brousseau
St. Mary's College
Moraga, California 94574

The notes will be Xeroxed.

The price for this service is four cents a page (including postage, materials and labor).

FIBONACCI NUMBERS IN PHYSICS

BASIL DAVIS
Student, Indian Statistical Institute, Calcutta, India

Since mathematics has great application in Physics, it would be surprising if the Fibonacci numbers, which have a wide application in unexpected branches of science, play no part in Physics. However, Fibonacci numbers do occur in Physics, though the importance of their occurrence is not certain.

ELECTRO-STATICS

Consider the following problem in electrostatics: A charge of $+e$ and two charges $-e$ are to be arranged along a straight line such that the potential energy of the whole system is equal to zero.

The potential energy of a system of static charges is the work done in bringing the charges from infinity to those points. The potential energy of two charges may be taken as the product of the charges divided by the distance between them. In the problem, let the charges $+e$, $-e$, and $-e$ occupy points A, B, and C, respectively. Let $AB = x$, $BC = y$.

$$\begin{array}{ccccc} A & x & B & y & C \\ +e & & -e & & -e \end{array}$$

Potential energy due to $+e$ at A and $-e$ at

$$B = \frac{(+e) \cdot (-e)}{x} = \frac{-e^2}{x}$$

Potential energy due to $+e$ at A and $-e$ at

$$C = \frac{(+e) \cdot (-e)}{x + y} = \frac{-e^2}{x + y}$$

Potential energy due to $-e$ at B and $-e$ at

$$C = \frac{(-e) \cdot (-e)}{y} = \frac{+e^2}{y}$$

For the potential energy of the system to be zero,

$$\frac{-e^2}{x} + \frac{-e^2}{x + y} + \frac{e^2}{y} = 0$$

or

$$-(x + y)y - xy + x(x + y) = 0$$

Therefore

$$x^2 - xy - y^2 = 0$$

or

$$\frac{x^2}{y^2} - \frac{x}{y} - 1 = 0 .$$

Hence, x/y, the golden ratio, $= 1.618 \cdots$.

Thus we find that for the potential energy to be zero, x/y must be the golden ratio.

Now we consider charges in dynamic equilibrium.

THE ATOM

The atom consists of a positively charged nucleus surrounded by electrons orbiting around it in fixed orbits. Bohr, Schroedinger, Pauli and others contributed to the building up of the shell model of the atom that has successfully explained the physical and chemical properties of matter.

Certain gases, called noble or rare gases, are exceptionally stable chemically. On looking at the periodic table, one finds an interesting relationship between the atomic numbers of these inert gases. With the exception of Helium, the atomic numbers of the gases roughly correspond to the Fibonacci numbers. Also, if the atomic numbers are divided by 18 and the results expressed to the nearest integers, the Fibonacci numbers from 0 to 5 are attained.

Gas	Symbol	Atomic No. = Z	F Numbers	Z/18 to Nearest Integer
Helium	He	2	$F_6 = 8$	$0 = F_0$
Neon	Ne	10	$F_7 = 13$	$1 = F_1$
Argon	Ar	18	$F_8 = 21$	$1 = F_2$
Krypton	Kr	36	$F_9 = 34$	$2 = F_3$
Xenon	Xe	54	$F_{10} = 55$	$3 = F_4$
Radon	Rn	86	$F_{11} = 89$	$5 = F_5$

Thus, there is a double correlation between the atomic numbers of stable atoms and the Fibonacci Series.

THE NUCLEUS

The structure of the nucleus remained a mystery for several years. It was known that the nucleus consisted of two kinds of particles — the protons and the neutrons. A proton has a charge equal and opposite to that of an electron, while the neutron is neutral. The atomic number Z = number of protons. The neutron number N = number of neutrons. The mass number $A = N + Z$. Exactly how the particles were arranged in the nucleus was unknown. Various models were put forward, but none was satisfactory. None of them could explain a

[Continued on page 662.]

AN OLD FIBONACCI FORMULA AND STOPPING RULES

REUVEN PELEG
Jerusalem, Israel

A fair coin is tossed, a head giving a return of +1, a tail of -1. Let the sum of these returns for a sequence of m throws be designated S_m. We define a stopping rule for the sequence: The sequence of throws will end if S_m is outside the closed interval -2 to +1.

At the end of m throws, if all possible variations are considered, there will be a certain number of 1's, 0's, -1's and -2's which will be designated n(1), n(0), n(-1), and n(-2), respectively. The number of sequences that terminate at m because of the stopping rules will be denoted $\phi(m)$.

Let us consider the first few steps. At the end of the first throw, there are two possible values +1, -1, and no terminations. Hence n(1) = 1, n(-1) = 1, $\phi(1) = 0$.

At the end of two throws, the possible values are +2, 0, 0, -2, the first being a termination. Hence n(0) = 2, n(-2) = 1, $\phi(2) = 1$. Continuing with the non-terminating sequences, we have values -1, +1, -1, +1, -3, -1 at the end of three throws. Hence $\phi(3) = 1$, n(1) = 2, n(-1) = 3.

The following table summarizes a few additional steps.

	m								
	1	2	3	4	5	6	2m - 1	2m	2m + 1
$\phi(m)$	0	1	1	2	3	5	F_{2m-2}	F_{2m-1}	F_{2m}
n(1)	1	0	2	0	5	0	F_{2m-1}	0	F_{2m+1}
n(0)	0	2	0	5	0	13	0	F_{2m+1}	0
n(-1)	1	0	3	0	8	0	F_{2m}	0	F_{2m+2}
n(-2)	0	1	0	3	0	8	0	F_{2m}	0

The general pattern is shown under the columns 2m - 1, 2m, 2m + 1. Now assume that we have the pattern in column 2m - 1. The 1's get out of bounds at 2 giving $\phi(2m) = F_{2m-1}$. The 1's and -1's combine to give $F_{2m-1} + F_{2m} = F_{2m+1}$ zeros. The -1's go to -2 giving F_{2m}. Starting at 2m, the -2's go out of bounds giving $\phi(2m+1) = F_{2m}$. The 0's and -2's combine to give $F_{2m} + F_{2m+1} = F_{2m+2}$ for -1. The 0's also go to 1 putting F_{2m+1} in that place. Thus the process is seen to continue indefinitely.

REFERENCE

A. Wald, "On Cumulative Sums of Random Variables," Annals of Mathematical Statistics, Vol. 15 (1944), p. 281.

puzzling phenomenon: nuclei having certain values for their N or Z numbers are considerably more stable than others. These values are 2, 8, 14, 20, 28, 50, 82, and 126. These numbers were called "magic numbers," since their origin was a mystery.

Let us divide the magic numbers by 10 and express the results to the nearest integers.

X	2	8	14	20	28	50	82	126
X/10	0.2	0.8	1.4	2	2.8	5	8.2	12.6
Nearest Integer	0	1	1	2	3	5	8	13

We get the Fibonacci numbers from 0 to 13 ··· !

We saw that the shell structure of the atom showed that the atomic numbers of stable atoms should be related to the Fibonacci series. The phenomenon of the magic numbers thus indicates that the nucleus also might have a shell structure. The first successful model of the nucleus, the shell model, was put forward by Maria Goeppert Mayer, Hans Jensen, and Eugene Wigner. Calculations based on the shell model successfully explained the phenomenon of magic numbers.

Thus the Fibonacci numbers seem to be associated with the stability of systems in dynamic equilibrium. Perhaps the Fibonacci sequence might help solve a number of problems in Physics.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

NOTATION: $F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n;\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let Q be the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the sum of a finite number of matrices chosen from the sequence $Q, Q^2, Q^3, \cdots$. Prove that $b = c$ and $a = b + d$.

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term F_n with $n > 0$ in the sequence $F_0, F_1, F_2, \cdots$ is expressible as $x^2 + y^2$ or $x^2 - y^2$ with x and y terms of the sequence with distinct subscripts.

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational:

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}\ , \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}\ .$$

B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that m and n are integers with $0 < n < m$ and $F_n \mid L_m$, prove that n is 1, 2, 3, or 4.

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $h = 5^k$. Prove that $h \mid F_h$.

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $g = 2 \cdot 3^k$. Prove that $g \mid L_g$.

SOLVERS INADVERTENTLY OMITTED FROM PREVIOUS ISSUES

J. L. Brown, Jr: B-219
Herta T. Freitag: B-202, B-203, B-206, B-207.
D. V. Jaiswal: B-214, B-215, B-216, B-217, B-218, B-219.
Graham Lord: B-202, B-203, B-204, B-205

SOLUTIONS

TWIN PRIMES SLIGHTLY DISGUISED

B-220 Proposed by Guy A. R. Guillotte, Montreal, P. Q., Canada.

Let p_m be the m^{th} prime. Prove that p_m and p_{m+1} are twin primes (i.e., $p_{m+1} = p_m + 2$) if and only if

$$\sum_{n=1}^{m} (p_{n+1} - p_n) = p_m .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

The sum telescopes to $p_{m+1} - p_1 = p_{m+1} - 2$.

Also solved by Wray G. Brady, Paul S. Bruckman, Warren Cheves, R. Garfield, Herta T. Freitag, Peter A. Lindstrom, Graham Lord, John W. Milsom, Richard W. Sielaff, and the Proposer.

SIMPLE SUBSTITUTION IN A CONVERGENT SERIES

B-221 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\sum_{n=1}^{\infty} (1/F_nL_n) = \sum_{n=1}^{\infty} (1/F_{2n}) .$$

Solution by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

It follows from the identity $F_{2n} = F_n L_n$ that the series are identical. Since

$$\lim_{n \to \infty} (F_{2n}/r^{2n}) = 1$$

where

$$r = (1 + \sqrt{5})/2 > 1,$$

the series converge.

Also solved by Paul S. Bruckman, Herta T. Freitag, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Richard W. Sielaff, Gregory Wulczyn, and the Proposer.

A NONHOMOGENEOUS RECURSION

B-222 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Find a formula for K_n where $K_1 = 1$ and

$$K_{n+1} = (K_1 + K_2 + \cdots + K_n) + F_{2n+1}.$$

Solution by C. B. A. Peck, State College, Pennsylvania.

Reducing the subscript $n + 1$ to n in the given recursion, we have $K_n = (k_1 + K_2 + \cdots + K_{n-1}) + F_{2n-1}$. Subtracting from corresponding sides of the original gives us

$$K_{n+1} - K_n = K_n + F_{2n+1} - F_{2n-1} = K_n + F_{2n}$$

or

$$K_{n+1} = 2K_n + F_{2n}.$$

Then $K_1 = 1$, $K_2 = 2 \cdot 1 + 1$, $K_3 = 2^2 \cdot 1 + 2 \cdot 1 + 3$, and generally

$$K_n = 2^{n-1} + (2^{n-2}F_2 + 2^{n-3}F_4 + \cdots + 2F_{2n-4} + F_{2n-2}).$$

Using a result of Herta T. Freitag (Fibonacci Quarterly, Vol. 8, No. 5, p. 344), we have

$$K_n = F_{2n+1} - 2^{n-1}.$$

Also solved by Paul S. Bruckman, L. Carlitz, Herta T. Freitag, Graham Lord, David Zeitlin, Gregory Wulczyn, and the Proposer.

FORMIDABLE ARITHMETIC

B-223 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Find a solution of

$$x^y + (x + 3)^y - (x + 4)^y = u^v + (u + 3)^v - (u + 4)^v$$

in the form

$$x = F_m, \quad y = F_n, \quad u = L_r, \quad \text{and} \quad v = L_s .$$

Solution by the Proposer.

A solution is

$$x = 13 = F_7, \quad y = 5 = F_5, \quad u = 7 = L_4, \quad \text{and} \quad v = 3 = L_2 .$$

QUADRATIC NONRESIDUES

B-224 Proposed by Lawrence Somer, Champaign, Illinois

Let m be a fixed positive integer. Prove that no term in the sequence $F_1, F_3, F_5, F_7, \cdots$ is divisible by $4m - 1$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since

$$5F_{2n+1}^2 - 4 = L_{2n+1}^2 ,$$

it would follow from

$$F_{2n+1} \equiv 0 \pmod{4m - 1}$$

that

$$L_{2n+1}^2 \equiv -4 \pmod{4m - 1} .$$

This implies the solvability of the congruence

$$x^2 \equiv -1 \pmod{4m - 1}$$

which is impossible.

Also solved by Paul S. Bruckman, Graham Lord, and the Proposer.

STILL UNCHARACTERIZED SEQUENCES

B-225 Proposed by John Ivie, Berkeley, California.

Let $a_0, \cdots, a_{j-1}$ be constants and let $\{f_n\}$ be a sequence of integers satisfying

$$f_{n+j} = a_{j-1}F_{n+j-1} + a_{j-2}f_{n+j-2} + \cdots + a_o f_n ; \qquad n = 0, 1, 2, \cdots .$$

Find a necessary and sufficient condition for $\{f_n\}$ to have the property that every integer m is an exact divisor of some f_k.

EDITORIAL NOTE: A necessary and sufficient condition that m divides some f_k is that m divides some f_k for $1 \leq k \leq m^2 + 1$, for every m. o

VOLUME INDEX