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BERNOULLI NUMBERS AND NON-STANDARD DIFFERENTIABLE STRUCTURES ON (4k - 1) - SPHERES

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ABSTRACT

A number theoretical conjecture of Milnor is presented, examined and the existence of non-standard differentiable structures on $(4k - 1)$-spheres for integers k, $4 \le k \le 265$, is proved.

1. INTRODUCTION

In 1959, J. Milnor [1] proved the following theorem concerning non-standard differentiable structures on $(4k - 1)$-spheres.

<u>Theorem 1.</u> If r is an integer, such that $k/3 < r \le k/2$, then there exists a differentiable manifold M, homeomorphic to S^{4k-1} with $\lambda(M) \equiv s_r s_{k-r} N/s_k \pmod 1$, where $s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!$, all of the prime factors of the integer N are less than $2(k - r)$, B_k is the k^{th} Bernoulli number in the sequence $B_1 = 1/6$, $B_2 = 1/30$, $B_2 = 1/42$, $B_4 = 1/30, \cdots$, and λ is an invariant associated with the manifold M.

Milnor presents an algorithm based on Theorem 1, proves structures exist for $k = 2$, 4, 5, 6, 7, 8, conjectures that Theorem 1 implies the existence of these structures for $k > 3$, and states that he has verified the conjecture for $k < 15$. He points out that for $k = 1$ and $k = 3$ no integers r exist in the interval $(k/3, k/2]$ and that for $k = 1$, two differentiable homeomorphic 3-manifolds are diffeomorphic.

The Milnor algorithm will be described by considering the first seven cases. In each case an actual lower bound will be calculated for the number of said structures; to calculate this bound we consider the denominator of the reduced fraction and drop all prime factors less than $2(k - r)$.

1. $k = r$, $r = 2$.

$$\binom{8}{4}(2^3 - 1)^2 B_2^2 / (2^7 - 1)B_4 = (7^3/3)(1/127), \qquad \text{lb} = 127\,.$$

2. $k = 6$, $r = 3$.

$$\binom{10}{4}(2^3 - 1)(2^5 - 1)B_2 B_3 / (2^9 - 1)B_5 = (11/5)(31/73), \qquad \text{lb} = 73\,.$$

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3. $k = 6, \quad r = 3.$

$$\binom{12}{6}(2^5 - 1)^2B_3^2/(2^{11} - 1)B_6 = (2\cdot5\cdot11\cdot13)(31^2/23\cdot89\cdot691) ,$$

$$1b = 23\cdot89\cdot691 .$$

4. $k = 7, \quad r = 3.$

$$\binom{14}{6}(2^5 - 1)(2^7 - 1)B_3B_4/(2^{13} - 1)B_7 = (11\cdot13/2\cdot5\cdot7)(31\cdot127/8191) ,$$

$$1b = 8191 .$$

5. $k = 8, \quad r = 3.$

$$\binom{16}{6}(2^5 - 1)(2^9 - 1)B_3B_5/(2^{15} - 1)B_8 = (2^2\cdot5^2\cdot13\cdot17/3)(73/151\cdot3617) ,$$

$$1b = 151\cdot3617 .$$

6. $k = 9, \quad r = 4.$

$$\binom{18}{8}(2^7 - 1)(2^9 - 1)B_4B_5/(2^{17} - 1)B_9 = (2\cdot3\cdot7^2\cdot13\cdot17\cdot19)/(73\cdot127/43867\cdot131071) ,$$

$$1b = 43867\cdot131071 .$$

7. $k = 10, \quad r = 4.$

$$\binom{20}{8}(2^7 - 1)(2^{11} - 1)B_4B_6/(2^{19} - 1)B_{10} = (11\cdot17\cdot19/7)(23\cdot89\cdot127/283\cdot617\cdot524287) ,$$

$$1b = 283\cdot617\cdot524287 .$$

8. $k = 10, \quad r = 4.$

$$\binom{20}{10}(2^9 - 1)^2B_5^2/(2^{19} - 1)B_{10} = (2\cdot5^3\cdot7^2\cdot13\cdot17\cdot19/3)(73^2/283\cdot617\cdot524287) ,$$

$$1b = 283\cdot617\cdot524287 .$$

9. $k = 8, \quad r = 4.$

$$\binom{16}{8}(2^7 - 1)^2B_4^2/(2^{15} - 1)B_8 = (3\cdot5\cdot11\cdot13\cdot17/7)(127^2/31\cdot151\cdot3617) ,$$

$$1b = 31\cdot151\cdot3617 .$$

There will be $[k/2] - [k/3]$ integers in the interval $(k/3, k/2]$ and one may choose the largest of the lower bounds. We now restate the positive outcome of the algorithm in the form of the following

Conjecture 1. Let r be an integer, $r \in (k/3, k/2]$, $k > 3$,

$$\binom{2k}{2k}(2^{2r-1} - 1)(2^{2k-2r-1} - 1)B_r B_{k-r}/(2^{2k-1} - 1)B_k = a/b, \qquad (a,b) = 1,$$

then there exists a prime number p, $p > 2(k - r)$, such that p divides b.

This purely number theoretic conjecture implies the existence of more than $2(k - r)$ non-standard differentiable structures for S^{4k-1}, the $(4k - 1)$-dimensional sphere. Conjecture 1 has, aside from its aesthetic number theoretical interest, the additional significance of important topological consequences, and is one more example of the ubiquitous nature of the Bernoulli numbers.

2. REPRESENTATION STRUCTURE OF THE BERNOULLI NUMBERS

Although the Bernoulli numbers have been objects of published mathematical thought for over two centuries, in some respects, embarrassingly little is known about them. We shall present the features of these numbers useful to us in examining Conjecture 1.

As a typical beginning point we write [2]

$$x(e^x - 1) = \sum_{k=0}^{\infty} b_k x^k/k! \tag{1}$$

and since $b_0 = 1$, $b_1 = -1/2$, and $x/(e^x - 1) + x/2$ is an even function, we write

$$b_{2k} = (-1)^{k-1}B_k \quad \text{and} \quad b_{2k+1} = 0, \qquad k \geq 1.$$

We have

$$1 - (1/2)\cot(x/2) = \sum_{k=1}^{\infty} B_k x^{2k}/(2k)! \tag{2}$$

and by the double series theorem [3], we see that

$$B_k = 2(2k)!\,\zeta(2k)/(2\pi)^{2k}, \tag{3}$$

where

$$\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k},$$

the Dirichlet series usually referred to as the even zeta function. An equivalent definition to (1) is the umbral recursion [4].

$$(b + 1)^k - b_k = 0, \qquad b_0 = 1, \tag{4}$$

which reduces to

$$\sum_{r=0}^{k} \binom{k+1}{r} b_r = 0, \qquad b_0 = 1. \tag{5}$$

Equation (1) is the reciprocal of

$$\sum_{k=0}^{\infty} x^k/(k+1)!$$

and an expression for the b_k may be written with symmetric functions of the coefficients of the reciprocal of (1). We may rather write [5], [6]

$$x/(e^x - 1) = \sum_{m=0}^{\infty} (-1)^m \left(\sum_{k=1}^{\infty} x^k/(k+1)! \right)^m \tag{6}$$

so that [7]

$$B_k = (-)^{k-1} \sum_{m=1}^{2k} (-)^m \sum \binom{m}{a_1, \cdots, a_{2k}} \binom{2k}{(1;a_1), \cdots, (2k;a_{2k})} \tag{7}$$

$$x(1/2^{a_1} \cdot 3^{a_2} \cdots (2k+1)^{a_{2k}}$$

where the sum is over the partitions of

$$2k, \quad \sum_{i=1}^{2k} a_i = m, \quad \sum_{i=1}^{2k} i a_i = 2k,$$

$$\binom{m}{a, b, c, \cdots} = m!/a!b!c! \cdots,$$

$$\binom{m}{(a;\alpha), \cdots (d;\beta)} = m!/(a!)^{\alpha} \cdots (d!)^{\beta},$$

and there will be p(2k) terms [8]. A variant of (7) is

$$(-)^{k-1}B_k = -(1/2k+1) + \sum (-)^m \prod_{p<2k} p^{\delta(p,k,a_1,\cdots,a_{2k})} \tag{8}$$

where the product is over all prime numbers less than 2k, the functions $\delta(p, k, a_1, \cdots, a_{2k})$ are all integers and the sum is over all the partitions of 2k but one.

The calculation of Bernoulli numbers has been a lively subject [9], and there exist several tables of these numbers. [The most massive is D. Knuty, MTAC, Unpublished Mathematical Tables File. The caretaker of this file, J. W. Wrench, has informed us that from Knuth's manuscript of 1270D values of $10^{-8k}B_k$ for $k = 1(1)250$ one can obtain the exact values of only the first 159 Bernoulli numbers.] To facilitate the computation of Bernoulli and related numbers, Lehmer generalized a process of Kronecker to produce lacunary recurrences of which the following are typical [10].

$$\sum_{\lambda=0}^{[m/2]} (-)^{\lambda} 2^{m-2\lambda} B_{m-2\lambda} \binom{2m+2}{2\lambda+2} = (-)^{[m/2]}(m+1)/2, \tag{9}$$

$$\sum_{\lambda=0}^{[m/2]} B_{m-2\lambda} \binom{2m+4}{4\lambda+4} \left((-)^{\lambda} 2^{2\lambda+1} + 1\right) = \left((m+2)/2\right)\left((-)^{[m/2]} 2^{m+1} + 1\right), \tag{10}$$

$$\sum_{\lambda=0}^{[m/3]} B_{m-3\lambda} \binom{2m+3}{6\lambda+3} = \begin{cases} -(2m+3)/6, & \text{if } m = 3k-1, \\ (2m+3)/3, & \text{otherwise}, \end{cases} \tag{11}$$

$$\sum_{\lambda=0}^{[m/4]} B_{m-4\lambda} \binom{2m+4}{8\lambda+4} 2^{m+1-2[(m+1)/4]-2\lambda} \mathfrak{M}_{4\lambda+2} = (-)^{[m/2]}(m+2)\mathfrak{M}_{m+2} \tag{12}$$

where

$$\mathfrak{M}_n = -34\mathfrak{M}_{n-4} - \mathfrak{M}_{n-8} \quad \text{and} \quad \mathfrak{M}_n = 2, 0, 3, 10, 14, -12, -99, -338,$$

for n = 0, 1, 2, 3, 4, 5, 6, 7, respectively.

(13)

$$\sum_{\lambda=0}^{[m/6]} B_{m-6\lambda} \binom{2m+6}{12\lambda+6} \left(\mathfrak{B}_{6\lambda+2} + (-)^{\lambda} 2^{6\lambda+2}\right) = \begin{cases} \left((m+3)/3\right)\left(\mathfrak{B}_{m+2} + (-)^{[m/2]} 2^{m+2}\right), \\ \quad \text{if } m \not\equiv 2(3); \end{cases}$$

or
$$\begin{cases} -\big((m+3)/6\big)(\mathfrak{B}_{m+2} + (-)^{[m/2]}2^{m+2} - (-)^{((m+1)/3)}3) , \\ \qquad \text{if } m \equiv 2(3) , \end{cases}$$

where

$$\mathfrak{B}_n = -2702\mathfrak{B}_{n-6} - \mathfrak{B}_{n-12} ,$$

and

$$\mathfrak{B}_n = 1, 5, 26, 97, 265, 362, -1351, -13775, -70226, -262087, -716035, -978122,$$

for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,$ respectively.

The point of creating lacunary recurrences is to avoid dealing with all the B_r, say $r < k$, to calculate B_k. An example of a recursion relation which is not precisely lacunary yet satisfies this last condition is

(14)
$$B_k = (k/2)\binom{2k-2}{k-1} + k\binom{2k}{k}\sum_{r=0}^{[k/2]}(-)^r B_r \binom{k}{2k}\big(1/(2k-2r)\big) + \sum_{0 \le r, s \le [k/2]} B_r B_s$$
$$\times \binom{2k}{2r,\ 2s,\ 2k-2r,\ 2k-2s}\big(1/(2k-2r-2s-1)\big) ,$$

which can be proved [11] by repeated integration of the Fourier series for $(\pi - x)/2$ and then using Parseval's Theorem on the result.

From (2) above, we have the identity

(15)
$$(d/dx)\Big(x\big(1 - (x/2)\cot(x/2)\big)\Big) = x^2/4 + \big(1 - (x/2)\cot(x/2)\big)^2 .$$

Hence, we extract

(16)
$$(2k+1)B_k = \sum_{r=1}^{[k/2]} 2^{g(r)}\binom{2k}{2r} B_r B_{k-r} ,$$

where

$$g(r) = \begin{cases} 1 & \text{if } r < [k/2] \text{ or } r = [k/2],\ k \text{ odd} , \\ 0 & \text{if } r = [k/2],\ k \text{ even} . \end{cases}$$

We observe that this "quasi-convolution" recurrence involves only positive numbers; hence, beginning with

(17)
$$B_1 = 1/2\cdot 3 ,$$

(18)
$$B_2 = 1/2\cdot 3\cdot 5 ,$$

(19) $$B_3 = 1/2\cdot 3\cdot 7\ ,$$

(20) $$B_4 = (1/2\cdot 3^4\cdot 5)(2^2\cdot 5 + 7) = 1/2\cdot 3\cdot 5\ ,$$

(21) $$B_5 = (1/2\cdot 3^3\cdot 11)(2^2\cdot 5 + 7 + 2\cdot 3^2) = (5/2\cdot 3\cdot 11)\ ,$$

(22) $$\begin{aligned} B_6 = {} & (1/2\cdot 3^3\cdot 5\cdot 7\cdot 13)(2^3\cdot 5^2\cdot 7 + 2\cdot 5\cdot 7^2 + 2^2\cdot 5\cdot 7\cdot 11 + 7^2\cdot 11 + 2^2\cdot 3^2\cdot 5\cdot 7 \\ & + 2\cdot 3^2\cdot 5\cdot 11) = 691/(2\cdot 3\cdot 5\cdot 7\cdot 13)\ , \end{aligned}$$

(23) $$\begin{aligned} B_7 = {} & (1/2\cdot 3^5\cdot 5^2)(2^3\cdot 5^2\cdot 7 + 2\cdot 5\cdot 7^2 + 2^2\cdot 3^2\cdot 5\cdot 7 + 2^2\cdot 5\cdot 7\cdot 11 \\ & + 7^2\cdot 11 + 2\cdot 3^2\cdot 5\cdot 11 + 2^2\cdot 5\cdot 7\cdot 13 + 7^2\cdot 13 + 2\cdot 3^2\cdot 7\cdot 13 \\ & + 2^2\cdot 5\cdot 11\cdot 13 + 7\cdot 11\cdot 13) = 7/(2\cdot 3)\ , \end{aligned}$$

(24) $$\begin{aligned} B_8 = {} & (1/2\cdot 3^2\cdot 5\cdot 17)(2^5\cdot 3\cdot 5^2\cdot 7 + 2^3\cdot 3\cdot 5\cdot 7^2 + 2^4\cdot 3^3\cdot 5\cdot 7 \\ & + 2^4\cdot 3\cdot 5\cdot 7\cdot 11 + 2^2\cdot 3\cdot 7^2\cdot 11 + 2^3\cdot 3^3\cdot 5\cdot 11 + 2^4\cdot 3\cdot 5\cdot 7\cdot 13 \\ & + 2^2\cdot 3\cdot 7^2\cdot 13 + 2^3\cdot 3^3\cdot 7\cdot 13 + 2^4\cdot 3\cdot 5\cdot 11\cdot 13 + 2^2\cdot 3\cdot 7\cdot 11\cdot 13 \\ & + 2^5\cdot 3^2\cdot 5^2\cdot 7 + 2^3\cdot 3^2\cdot 5\cdot 7^2 + 2^4\cdot 3^4\cdot 5\cdot 7 + 2^4\cdot 3^2\cdot 5\cdot 7\cdot 11 \\ & + 2^2\cdot 3^2\cdot 7^2\cdot 11 + 2^3\cdot 3^4\cdot 5\cdot 11 + 2^5\cdot 3^2\cdot 5\cdot 13 + 2^3\cdot 3^2\cdot 7\cdot 13 \\ & + 2^4\cdot 3^4\cdot 13 + 2^4\cdot 5^2\cdot 11\cdot 13 + 2^2\cdot 5\cdot 7\cdot 11\cdot 13 + 2^2\cdot 5\cdot 7\cdot 11\cdot 13 \\ & + 7^2\cdot 11\cdot 13) = 3617/(2\cdot 3\cdot 5\cdot 17)\ . \end{aligned}$$

By induction, we express the Bernoulli number B_k by

(25) $$B_k = \prod_{p<2k+2} p^{a(p,k)} \sum_{r=1}^{c(k)} \prod_{p<2k} p^{b(p,r,k)}\ .$$

Where the products are over the primes less than $2k+2$ and $2k$, respectively, $a(p,k)$ is an integer (possibly negative) and $b(p,r,k)$ is a non-negative integer. The number $c(k)$ of terms in the sum clearly possesses the recurrence

(26) $$c(k) = \sum_{r=1}^{[k/2]} c(r)c(k-r)\ ,$$

with initial condition $c(1) = 1$. Kishore [12], [13] has used this technique to develop analogous structure theorems for Rayleigh functions [14], [15].

3. DIVISIBILITY STRUCTURE OF THE BERNOULLI NUMBERS

We first cite the well-known [16], [17]

Theorem 2. (Von Staudt-Clausen). If $B_k = P_k/Q_k$ are the Bernoulli numbers for $k = 1, 2, 3, \cdots$ and $(P_k, Q_k) = 1$, then

$$Q_k = \prod_{p-1 \mid 2k} p, \tag{27}$$

where the product is over all primes whose totients divide $2k$.

This theorem completely characterizes the Bernoulli denominators; hence, questions of divisibility center around the numerators P_k. A sufficient condition on divisors of P_k is given in the following [16, p. 261]

Theorem 3. If $p^\omega \mid 2k$, $p^{\omega+1} \nmid 2k$, $p - 1 \nmid 2k$, then $p^\omega \mid P_k$.

The proof of this theorem follows from a congruence of Voronoi

$$(a^{2k} - 1)P_k \equiv (-)^{k-1} 2k\, a^{2k-1} Q_k \sum_{s=1}^{N-1} s^{2k-1}\,[sa/N] \pmod{N}, \tag{28}$$

where $(a,N) = 1$ and N is any integer greater than one. Clearly if $p^\omega \mid 2k$, $(a^{2k} - 1)P_k \equiv 0 \pmod{p}$ and we may select a to be a primitive root g of p^ω (i.e., if $\omega = 1$, g always exists: if $\omega > 1$ and $g^{p-1} \not\equiv 1 \pmod{p^2}$, take $a = g$; if $g^{p-1} \equiv 1 \pmod{p^2}$, take $a = g + p$).

Equation (28) is a type of congruence used recently [18], [19] to investigate certain divisors of Bernoulli numerators. Specifically, those primes p such that

$$p \nmid P_1 P_2 P_3 \cdots P_{(p-3)/2} \tag{29}$$

are called regular primes and Kummer [20] proved that for these primes, Fermat's inequality, $x^p + y^p \neq z^p$, holds for all nonzero integers x, y and z. We list a number of congruences of the Voronoi type.

$$\sum_{p/6<s<p/4} s^{2k-1} \equiv (2^{p-2k} - 1)(3^{p-2k} - 2^{p-2k} - 1)(-)^k B_k/4k \pmod{p} \tag{30}$$

with [16, p. 268], $p > 3$, $p - 1 \nmid 2k$

$$\sum_{p/6<s<p/5} s^{2k-1} + \sum_{p/3<s<2p/5} s^{2k-1} \equiv (-)^k (6^{p-2k} - 5^{p-2k} - 2^{p-2k} + 1) B_k/4k \pmod{p} \tag{31}$$

with [19, p. 27], $p > 7$, $2k < p - 1$.

(32)
$$\sum_{p/6 < s < p/3} s^{2k-1} \equiv (-)^k (2^{p-2k-1} - 1)(3^{p-2k} - 1)B_k/2k \pmod{p}$$

with [21], $p > 7$, $2k < p - 1$.

(33)
$$\sum_{r=1}^{(p-1)/2} (p - 2r)^{2k} \equiv p2^{2k-1}B_k \pmod{p^3}$$

with [22], $2k \not\equiv 2 \pmod{(p - 1)}$.

(34)
$$b^{a(p-1)}(b^{p-1} - 1)^j \equiv 0 \pmod{p^{j-1}}$$

with [23], p an odd prime, $a > 0$, $j > 0$, $a + j < p - 1$.

From reflections on the divisibility properties of the binomial coefficients, it has been shown [24] that

(35)
$$2B_k \equiv 1 \pmod{2^{r+1}}, \quad \text{for } k > 1,\ 2^r \mid 2k,\ 2^{r+1} \nmid 2k .$$

Also [25],

(36)
$$2B_k \equiv 1 \pmod{4}, \quad k > 1 ,$$

and [26],

(37)
$$B_k \equiv 1 - (1/p) \pmod{p^r}, \quad \text{for } p > 2,\ (p - 1)p^r \mid 2k,\ p^{r+1} \nmid 2k .$$

A more elaborate result [2] is

(38)
$$30B_{2k} \equiv 1 + 600\binom{k-1}{2} \pmod{27000} .$$

The last depends upon special identities such as

$$(e^x - 1)^{-1} - (e^{5x} - 1)^{-1} = \left(\cosh(x/2) + \cosh(3x/2)\right)\cosh(5x/2) .$$

4. APPROACHES TO CONJECTURE 1

Milnor [1, p. 966] asked whether or not

(39) $$8(2k)!/(2^{2k-1} - 1)B_k \not\equiv 0 \pmod 1 .$$

That this is true for $k > 2$ is clear by remarking [27] that $2^{2k-1} - 1$ possesses a primitive divisor q, such that $q \equiv 1 \pmod{2k - 2}$.

In particular, $q > 2k + 1$ and q must occur in the denominator of the fraction in (39). We naturally ask whether or not a prime $q > 2k + 1$ always exists such that

$$q \mid 2^{2k-1} - 1 \quad \text{and} \quad q \nmid 2^{2r-1} - 1, \quad q \nmid 2^{2k-2r-1} - 1, \quad q \nmid B_r, \quad q \nmid B_{k-r} .$$

with $k/3 < r \le k/2$. This suggests

<u>Lemma 1.</u> If $q \mid 2^{2k-1} - 1$ is primitive and regular, then Conjecture 1 is true for k.

We consider $r = k/2$ or $(k - 1)/2$, $k > 3$. Since $q > 2k + 1$ and $q \nmid B_i$ for $i < (q - 1)/2$, $q \nmid B_r^2$, if k is even and $q \nmid B_r B_{k-r}$ if k is odd. Also [28], $q \nmid 2^j - 1$, $j < 2k - 1$. Another natural question is, since Fermat's Last Theorem is true for [29] primes of the form $2^a - 1$, are these numbers and their large factors also regular? Alas,

$$233 \mid B_{42}, \qquad 233 \mid 2^{29} - 1 .$$

As an example of the theorem, $k = 15$, $2k - 1 = 29$; $1103 \mid 2^{29} - 1$, yet 1103 is regular; the nearest irregular primes are 971 and 1061. Also $3391 \mid B_{1116}$, $3391 \mid B_{1267}$ and $3391 \mid 2^{113} - 1$, but $3391 \nmid B_{28}B_{29}$ so that irregular primes may be primitive and still satisfy conjecture 1. Similarly for $263 \mid 2^{131} - 1$ and $263 \mid B_{50}$. These remarks handle cases $k = 57, 66$. The number of primitive primes is infinite. so is the number of irregular primes [30]; Kummer conjectured that the number of regular primes is infinite. Present tables show that known regular primes are more numerous than irregular primes. The intersection of these primitive and regular prime sets, though nonempty, is unknown. It is interesting to note in this connection that

(40) $$2^{2k-1} - 1 = \sum_{r=1}^{k} \binom{2k-1}{2r-1} (2^{2k-2r-1} - 1)(2^{2r} - 1)B_r/r ,$$

which for $2k - 1$ prime is a relation between Mersenne [31] numbers and Bernoulli numbers. We might enjoy having $(2^{4k-1} - 1, B_k) = 1$, for the case of the $(8k - 1)$-sphere; but

$$(2^{27} - 1, B_7) = (2^{111} - 1, B_{28}) = 2^3 - 1,$$

and a similar thing occurs whenever $3 \mid 4k - 1$, $7 \mid 2k$; likewise, if $5 \mid 4k - 1$, $31 \mid 2k$, e.g., $(2^{495} - 1, B_{124}) \ge 31$.

Another approach to (39) is to seek a large (greater than 2k) prime factor of B_k and to apply its existence to Conjecture 1. However, there does not appear to be in the literature any theorem (other than a direct calculation [32] proving the existence of a large prime divisor of B. Equation (25) suggests that if the $b(p, r, k)$ numbers behave appropriately, the sum in (25) would be the source of large factors; for the first few cases the sum has a number of small factors (i.e., equations (17)-(24)). A very general and related problem is whether or not sums of the type

$$\sum_{r=1}^{c(k)} \prod_{p<2k} p^{\eta(p,r,k)} \tag{41}$$

with the function $\eta(p, r, k)$ behaving similarly to the $b(p, r, k)$ possess large factors. It is known 33 that for sums of type (41) where $\eta(p, r, k) \gg b(p, r, k)$ (inequality in a rough distribution sense of the density of primes being greater in one than the other) large factors arise. One must proceed with considerable care because of the copious factors [34] of a sum such as

$$\sum \binom{n}{a_1, \cdots, a_k} \binom{n(k-1)}{n-a_1, \cdots, n-a_k} = \binom{nk}{n, \cdots, n} \quad , \tag{42}$$

where the sum is over the partitions

$$\sum_{i=1}^{k} a_i = n \ .$$

Rather than digging a prime out of P_k, we recognize the obvious

<u>Lemma 2.</u> For m, n arbitrary positive integers, such that $m/n < 1$, then there exists a prime p such that $p | n/(m,n)$ and $p \nmid m/(m,n)$.

We write for integers $r \in (k/3, k/2]$, $k > 3$,

$$\binom{2k}{2r} (2^{2r-1} - 1)(2^{2k\ 2r-1} - 1) B_r B_{k-r} / (2^{2k-1} - 1) B_k \tag{43}$$

$$= \binom{2k}{2r} (Q_k / Q_r Q_{k-r})(2^{2r-1} - 1)(2^{2k-2r-1} - 1) P_r P_{k-r} / (2^{2k-1} - 1) P_k \tag{44}$$

$$= \binom{2k}{2r} {}_{p<2k+2} \; p^{\theta(p,k) - \theta(p,r) - \theta(p,k-r)} (2^{2r-1} - 1)(2^{2k-2r-1} - 1) P_r P_{k-r}$$

$$/ M_k M_k' N_k N_k' \ , \tag{45}$$

where

$$\theta(p,k) = 1 \text{ if } (p-1) | 2k \text{ and zero otherwise} \tag{46}$$

with

$$2^{2k-1} - 1 = M_k M_k' \ , \quad M_k = \prod_{p<2k} p^{\Psi(p,k)} , \quad M_k \text{ largest possible,} \tag{47}$$

and

(48) $$P_k = N_k N'_k, \quad N_k = \prod_{p<2k} p^{\varphi(p,k)}, \quad N_k \text{ largest possible.}$$

Therefore, we have the following

Lemma 3. If

(49) $$M_k N_k < 0.25 \binom{2k}{2r} Q_k / Q_r Q_{k-r}$$

for some integer $r \in (k/3, k/2]$, then Conjecture 1 is true.

From (3),

(50) $$B_r B_{k-r} / B_k = \binom{2k}{2r}^{-1} 2\zeta(2r)\zeta(2k - 2r)/\zeta(2k) < 4 \Big/ \binom{2k}{2r}.$$

In fact, [35], for k even,

(51) $$\zeta^2(k)/\zeta(2k) = \sum_{n=1}^{\infty} 2^{\nu(n)} / n^k ,$$

for $\nu(n)$ equal to the number of distinct prime factors of n.

By hypothesis

(52) $$\begin{aligned} m/n &= (2^{2r-1} - 1)(2^{2k-2r-1} - 1)P_r P_{k-r} / M'_k N'_k \\ &< 4M_k N_k \binom{2k}{2r}^{-1} Q_r Q_{k-r} / Q_k < 1 . \end{aligned}$$

But n has no prime factors less than $2k$ and hence none less than $2(k - r)$ (whether $2k + 1$ is prime or not, n has no factors less than $2k + 2$), so by Lemma 2 there exists some prime greater than $2k$, which provides a non-trivial bound for Conjecture 1. Also, if $2k - 1$ is prime, $M_k = 1$; in general, for say $n = 2k - 1$, an easily refined inequality is $M_k \leq n2^{\varphi(n)+2^{0.09\nu(n)}}$ with φ Euler's totient function.

Since for relatively small k, discovery of a large prime divisor of P_k could require more than 10^{38} centuries with our present technology, Lemma 3 presents itself as a most opportune calculational device. Using this lemma we have shown Conjecture 3 to be true for integers $k \in (3, 265]$. The details of this calculation, which appear in the appended tables, materially suggest the truth of the hypothesis of Lemma 3. These calculations make use of congruences of type (28), which gives necessary conditions for all divisors of P_k, conditions which depend upon properties of the sum

$$\sum_{s=1}^{p^{\omega}-1} s^{2k-1}\left[sa/p^{\omega}\right], \qquad (\text{mod } p^{\omega}), \tag{53}$$

for a some primitive root of p (a complication can arise here because p = 3511, which satisfies $2^{p-1} \equiv 1 \pmod{p^2}$, has a Kummer irregularity of 2).

Of (53), the tables present empirical evidence, the most complete to date; the more valuable conceptual information in the form of an upper bound inequality on N_k, for example, would be welcome knowledge at this point.

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CONVERGENCE OF THE COEFFICIENTS IN THE k^{th} POWER OF A POWER SERIES

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1. CONVOLUTED SUM FORMULAS

In this paper we investigate generalized convoluted numbers and sums by using recurring power series

$$\left(1 + \sum_{v=1}^{m} a_v x^v\right)^{-k} = \sum_{n=0}^{\infty} u(n,k,m)\, x^n, \tag{1}$$

where the coefficients a_v and $u(n,k,m)$ are rational integers $k = 1, 2, 3, \cdots$, $u(0,k,m) = 1$ and $m = 1, 2, 3, \cdots$.

By elementary means, it is easy to prove, if

$$(1 - y)^{-k} = \sum_{v=0}^{\infty} b_v^{(k)} y^v \tag{2}$$

then

$$\binom{n + k - 1}{k - 1} = b_n^{(k)},$$

where

$$b_0^{(k)} = 1, \qquad k = 1, 2, 3, \cdots, \qquad n = 0, 1, 2, \cdots,$$

and

$$\binom{n + k - 1}{k - 1} = (n + k - 1)!/n!\,(k - 1)! \;.$$

Elsewhere [1], it has been shown that the following convoluted sum formulas hold:

$$u(n,k,2) = \sum_{j=0} \binom{n + k - 1 - j}{k - 1}\binom{n - j}{j} a_1^{n-2j} a_2^j \tag{3}$$

$$(n = 0, 1, 2, \cdots, \; k = 1, 2, 3, \cdots);$$

and

$$u(n,k,3) = \sum_{r=0}^{r}\sum_{j=0} \left[\binom{k + n - 2r - 2}{k - 1}\binom{n - 2r - 1}{2r + 1 - j}\binom{2r + 1 - j}{j} a_1^S a_2^T a_3^j \right. \tag{4}$$

$$\left. + \binom{k + n - 2r - 1}{k - 1}\binom{n - 2r}{2r - j}\binom{2r - j}{j} a_1^{S+2} a_2^{T-1} a_3^j \right]$$

where $S = n - 4r - 2 + j$, $T = 2r + 1 - 2j$, $n = 0, 1, 2, \cdots$, and $k = 1, 2, 3, \cdots$.

The $u(n,k,2)$ in (3) are called "generalized Fibonacci numbers," the $u(n,k,3)$ in (4) are called "generalized Tribonacci numbers," we shall term the $u(n,k,4)$ as the "generalized Quatronacci numbers," and the general expression $u(n,k,m)$ in (1 for $m = 5, 6, \cdots$) we shall refer to as the "generalized Multinacci numbers."

Now in (2) we let

$$y = \sum_{w=1}^{m} a_w x^w \qquad (m = 2, 3, \cdots)$$

and put

$$(1 - y)^{-k} = \sum_{n=0}^{\infty} u(n,k,m)x^n = \sum_{v=0}^{\infty} b_v^{(k)} y^v , \tag{5}$$

and by comparing the coefficients in (5), it is easy to prove with induction, that

$$\sum_{r_1=0} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_{m-1}=0}^{r_{m-2}} \phi(n,m)F(n,m)\, b_{n-r_1}^{(k)} = u(n,k,m) , \tag{6}$$

where

$$\phi(n,m) = \binom{n - r_1}{r_1 - r_2}\binom{r_1 - r_2}{r_2 - r_3}\cdots\binom{r_{m-3} - r_{m-2}}{r_{m-2} - r_{m-1}}\binom{r_{m-2} - r_{m-1}}{r_{m-1}} ,$$

$$F(n,m) = a_1^{n-2r_1+r_2}\, a_2^{r_1-2r_2+r_3} \cdots a_{m-2}^{r_{m-3}-2r_{m-2}+r_{m-1}}\, a_{m-1}^{r_{m-2}-2r_{m-1}}\, a_m^{r_{m-1}} ,$$

$$b_{n-r_1}^{(k)} = \binom{n + k - r_1 - 1}{1 - 1} ,$$

and $n = 0, 1, 2, \cdots$, $m = 2, 3, 4, \cdots$.

Of course the convoluted sum formula of the generalized Quatronacci number $u(n,k,4)$ is immediate as a special case of (6, with $m = 4$).

2. A GENERAL METHOD TO FIND FORMULAS FOR THE $u(n,k,m)$ AS A FUNCTION OF $u(j,1,m)$ $(n,j = 0, 1, 2, \cdots)$

In [1], it has been shown that the following formulas for the generalized Fibonacci numbers hold:

$$(a_1^2 + 4a_2)ku(n - 1, k + 1, 2) = a_1nu(n, k, 2) + a_2(4k + 2n - 2)u(n - 1, k, 2) , \tag{7}$$

where $u(0,k,2) = 1$, $u(1,k,2) = a_1k$, and $n,k = 1, 2, 3, \cdots$.

Now, using the results in (7) we are able to write the following: where

$A = a_1^2 + 4a_2$, $B(k,n) = 4k + 2n - 2$, $u(0,k,2) = 1$, $u(1,k,2) = a_1k$, $n,k = 1, 2, 3, \cdots$, and

$$u(n,1,2) = u(n-1, 1, 2)a_1 + u(n-2, 1, 2)a_2 ,$$

(where a_1 and a_2 are rational integers) we have

$$u(n-1, 2, 2)A = u(n,1,2)na_1 + u(n-1, 1, 2)B(1,n)a_2 , \tag{8}$$

$$\begin{aligned} u(n-1, 3, 2)A^2 2! &= (a_1a_2nB(1, n+1) + a_1a_2nB(2,n) + a_1^3n(n+1))u(n,1,2) \\ &\quad + (a_2^2B(1,n)B(2,n) + a_1^2a_2n(n+1))u(n-1,1,2), \end{aligned} \tag{8.1}$$

and

$$u(n-1, 4, 2)A^3 3! = M + N , \tag{8.2}$$

where

$$M = \begin{bmatrix} a_1a_2^2nB(1, n+1)B(3,n) + a_1a_2^2nB(2,n)B(3,n) \\ + a_1^3a_2n(n+1)B(3,n) + a_1a_2^2nB(1, n+1)B(2, n+1) \\ + a_1^3a_2n(n+1)(n+2) + a_1^3a_2n(n+1)B(1, n+2) \\ + a_1^3a_2n(n+1)B(2, n+1) + a_1^5n(n+1)(n+2) \end{bmatrix} u(n,1,2) ,$$

and

$$N = \begin{bmatrix} a_2^3B(1,n)B(2,n)B(3,n) + a_1^2a_2^2n(n+1)B(3,n) \\ + a_1^2a_2^2n(n+1)B(1, n+2) + a_1^2a_2^2n(n+1)B(2,n+1) \\ + a_1^4a_2n(n+1)(n+2) \end{bmatrix} u(n-1,1,2) .$$

It should be noted that the method used in [1] to derive the formulas (8), (8.1), and (8.2) may also be used to develop formulas of the $u(n,k,2)$ for values of $k = 5$ and higher.

In this paper we find for the first time a general method to express the $u(n,k,m)$ as a function of the $u(j,1,m)$ $(j = 0, 1, 2, \cdots)$ with $m \geq 2$ $(m = 2, 3, 4, \cdots)$.

Let

$$y = 1 + \sum_{v=1}^{m} a_v x^v , \quad z = \sum_{v=0}^{m-2} d_v x^v, \quad \text{and} \quad w = \sum_{v=0}^{m-1} b_v x^v , \tag{9}$$

where a, d and b are rational integers, $m \geq 2$ $(m = 2, 3, \cdots)$ and

$$M(m) = zy - w(dy/dx) \quad (M(m) \text{ is a rational number}) . \tag{9.1}$$

Now, differentiating the identity $y^{-k} = y^{-k}$, we have

(10) $$-k(dy/dx)/y^{k+1} = d\phi(x)^k/dx \,,$$

where

$$\phi(x)^k = \left(\sum_{n=0}^{\infty} u(n,k,m)x^n\right)^k = y^{-k}, \quad k = 1, 2, 3, \cdots, \text{ and } m \geq 2\,.$$

We then respectively, multiply (9.1) through by k and divide (9.1) through by y^{k+1} and combine the result with (10). This leads to

$$(kM(m) - kzy)/y^{k+1} = (d\phi(x)^k/dx)w \,,$$

and we have

(11) $$xkM(m)/y^{k+1} = xkz/y^k + wx(d\phi(x)^k/dx) \,.$$

Now, comparing coefficients in (11), we conclude that

$$u(n-1), k+1, m)kM(m) =$$

(12) $$k\sum_{v=0}^{m-2} u(n-1-v, k, m)d_v + \sum_{v=0}^{m-1} u(n+v+1-m, k, m)(n+v+1-m)b_{m-v-1} \,.$$

To complete (12), we notice it is necessary to solve (9.1), and this is easily accomplished by collecting the coefficients of x^n. Comparing the coefficients then leads to the following $2m - 1$ equations: (Note: In what follows $B_j = ja_j$, and also for convenience we have replaced a_v with $-a_v$ $(j, v = 1, 2, 3, \cdots, m.)$

(13)

$$d_0 = M(m) + B_1b_0 \,,$$
$$a_1d_0 = d_1 + B_2b_0 + B_1b_1 \,,$$
$$a_2d_0 = -a_1d_1 + d_2 + B_3b_0 + B_2b_1 + B_1b_2 \,,$$

. . . .

$$a_{m-2}d_0 = -a_{m-3}d_1 - a_{m-4}d_2 - \cdots - a_1d_{m-3} + d_{m-2} + B_{m-1}b_0 + B_{m-2}b_1 + \cdots + B_1b_{m-2} \,,$$
$$a_{m-1}d_0 = -a_{m-2}d_1 - \cdots - a_2d_{m-3} - a_1d_{m-2} + B_mb_0 + B_{m-1}b_1 + \cdots + B_1b_{m-1} \,,$$
$$a_md_0 = -a_{m-1}d_1 - \cdots - a_3d_{m-3} - a_2d_{m-2} + B_mb_1 + \cdots + B_2b_{m-1} \,,$$
$$0 = -a_md_1 - \cdots - a_4d_{m-3} - a_3d_{m-2} + B_mb_2 + \cdots + B_3b_{m-1} \,,$$

. . . .

$$0 = -a_md_{m-3} - a_{m-1}d_{m-2} + B_mb_{m-2} + B_{m-1}b_{m-1} \,,$$
$$0 = -a_md_{m-2} + B_mb_{m-1}$$

(dividing through by a_m this last equation becomes $0 = -d_{m-2} + mb_{m-1}$).

Next we consider in (13) the $2m - 1$ equations in the $2m - 1$ unknowns $M(m)$, d_1, d_2, $\cdots$, d_{m-2}, b_0, b_1, $\cdots$, b_{m-1}, where for convenience we write

$$
\begin{array}{llll}
 & S(m,0,0) = M(m)\,; & S(0,1,0) = d_1, & S(0,2,0) = d_2, \cdots, \\
(14.1) & S(0,\ m-2,\ 0) = d_{m-2}\,; & S(0,0,1) = b_1, & S(0,0,2) = b_2, \cdots, \\
 & S(0,0,\ m-1) = b_{m-1}\,; & \text{and} & b_0 = b_0 \quad .
\end{array}
$$

The $2m - 1$ equations in the $2m - 1$ unknowns $S(g)$ (where we consider g to run through all the $2m - 1$ combinations one at a time of the $S(\ )$ (we also include b_0) in (14.1)) can be solved by Cramer's rule to obtain

$$
D(m)S(g) = D(g)\ , \tag{15}
$$

where $D(m)$ and $D(g)$ are the determinants given below:

(15.1)

$$
D(m) = \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & B_1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & B_2 & B_1 & \cdots & 0 & 0 \\
0 & -a_1 & 1 & \cdots & 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & & \cdots & \cdot & \cdot \\
0 & -a_{m-3} & -a_{m-4} & \cdots & -a_1 & 1 & B_{m-1} & B_{m-2} & \cdots & B_1 & 0 \\
0 & -a_{m-2} & -a_{m-3} & \cdots & -a_2 & -a_1 & B_m & B_{m-1} & \cdots & B_2 & B_1 \\
0 & -a_{m-1} & -a_{m-2} & \cdots & -a_3 & -a_2 & 0 & B_m & \cdots & B_3 & B_2 \\
0 & -a_m & -a_{m-1} & \cdots & -a_4 & -a_3 & 0 & 0 & \cdots & B_4 & B_3 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & \cdots & -a_m & -a_{m-1} & 0 & 0 & \cdots & B_m & B_{m-1} \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & m
\end{vmatrix}
$$

(Determinant $D(m)$ = the coefficients of the $S(g)$)

and

(15.2) $D(g)$ is the determinant we get when replacing in (15.1) the appropriate column of the coefficients of any $S(g)$ with the column to the extreme left in (13) (the terms in the column to the extreme left in (13) from top to bottom are: $d_0,\ a_1,\ d_0,\ \cdots,\ a_m,\ d_0,\ 0,\ \cdots,\ 0,\ 0$).

Note. Upon investigation we notice that there is no loss of generality if we put

$$
d_0 = D(m)\ . \tag{15.3}
$$

We shall now use the above method to derive formulas for the generalized Multinacci number.

We first find formulas for the generalized Tribonacci number. We write the generalized Tribonacci power series as follows:

$$(16) \qquad (1 - a_1x - a_2x^2 - a_3x^3)^{-k} = \sum_{n=0}^{\infty} u(n,k,3)x^n ,$$

where $k = 1, 2, 3, \cdots,$ the a are integers and $u(0,k,3) = 1$.

Now combining (16) with (9.1), we write

$$(17) \qquad M(3) = (d_0 + d_1x)(1 - a_1x - a_2x^2 - a_3x^3) + (a_1 + 2a_2 + 3a_3x^3)(b_0 + b_1x + b_2x^2)$$

and combining (17) with (15.1 and 15.3, with $m = 3$), we have

$$(17.1) \qquad d_0 = D(3) = \begin{vmatrix} 1 & 0 & B_1 & 0 & 0 \\ 0 & 1 & B_2 & B_1 & 0 \\ 0 & -a_1 & B_3 & B_2 & B_1 \\ 0 & -a_2 & 0 & B_3 & B_2 \\ 0 & -1 & 0 & 0 & 3 \end{vmatrix}$$

and of course applying the directions in (15.2, with $m = 3$) in combination with the determinant $D(3)$ in (17.1), leads to the following:

$$(17.2) \qquad \begin{aligned} d_0 &= D(3) = 27a_3^2 + 15a_1a_2a_3 - 4a_2^3 \\ d_1 &= 18a_1a_3^2 - 6a_2^2a_3 \\ b_0 &= 4a_1^2a_3 + 3a_2a_3 - a_1a_2^2 \\ b_1 &= 9a_3^2 + 7a_1a_2a_3 - 2a_2^3 \\ b_2 &= 6a_1a_3^2 - 2a_2^2a_3 \\ M(3) &= 27a_3^2 + 18a_1a_2a_3 + 4a_1^3a_3 - 4a_2^3 - a_1^2a_2^2 . \end{aligned}$$

We now combine (16) and (17.2) with (12, with $m = 3$), which leads to

$$(18) \qquad \begin{aligned} &k(27a_3^2 + 18a_1a_2a_3 + 4a_1^3a_3 - 4a_2^3 - a_1^2a_2^2)u(n-1, k+1, 3) \\ &\quad = (4a_1^2a_3 + 3a_2a_3 - a_1a_2^2)nu(n,k,3) \\ &\qquad + ((n-1)(9a_3^2 + 7a_1a_2a_3 - 2a_2^3) + k(27a_3^2 + 15a_1a_2a_3 - 4a_2^3))u(n-1, k, 3) \\ &\qquad + ((n-2)(6a_1a_3^2 - 2a_2^2a_3) + k(18a_1a_3^2 - 6a_2^2a_3))u(n-2, k, 3) . \end{aligned}$$

(18.1) In (18) it is evident that if we put $k = 1$ we can find the $u(n,2,3)$ as a function of the $u(n,1,3)$ and also for $k = 2$ we find $u(n,3,3)$ as a function of the $u(n,2,3)$, so that we have $u(n,3,3)$ as a function of the $u(n,1,2)$. In this way, step by step for $k > 1$ (with induction added), it is easy to see that we can find formulas of the $u(n,k,3)$ as a function of the $u(n,1,3)$.

(19) Using the exact methods which lead to (18) and (18.1), we find formulas for the Quatronacci $(u(n,k,4))$ numbers (with $k > 1$) as a function of the $u(n,1,4)$, and we find formulas for the generalized Multinacci $(u(n,k,m)$ with $m = 5, 6, 7, \cdots$ and $k > 1)$ numbers as a function of the $u(n,1,m)$.

3. THE GENERALIZED MULTINACCI NUMBER EXPRESSED AS A LIMIT

Note. In [1] the generalized Fibonacci number is expressed as the following:

$$(20) \qquad \lim_{n\to\infty} \left(u(n,\ k+1,\ 2)/(n+1)^k u(n,1,2)\right) = \left(1 + a_1(a_1^2 + 4a_2)^{-\frac{1}{2}}\right)^k \Big/ 2^k k!\ ,$$

where

$$k, n = 1, 2, 3, \cdots .$$

In this paper we find asymptotic formulas of the $u(n,k,m)$ (with $k, m \geq 2$) expressed in terms of $u(n,1,m)$, a_v, n, and k.

However, before finding our asymptotic formulas, we make some

(21) SUPPLEMENTARY REMARKS

This author, for the first time, proved the following in 1969 [2]. Define

$$\sum_{w=0}^{f} b_w x^w = F(x) \neq 0$$

(for a finite f),

$$\sum_{w=0}^{t} a_w x^w = \prod_{w=1}^{m} (1 - r_w x)^{d_w} = Q(x)$$

for a finite t and m, where the $d_w \neq 0$ are positive integers, the $r_w \neq 0$ and are distinct and we say $|r_1|$ is the greatest $|r|$ in the $|r_w|$. We then proved the following

Theorem. If

$$F(x)/Q(x) = \sum_{w=0}^{\infty} u_w x^w ,$$

then

$$\lim_{n\to\infty} \left| u_n / u_{n-j} \right|$$

(for a finite $j = 0, 1, 2, \cdots$) converges to $|r_1^j|$, where the $r_w \neq 0$ in $Q(x)$ are distinct with distinct moduli and $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

We are now in a position to discuss the generalized Multinacci number expressed as a limit.

First, we consider when $m = 3$ and we multiply equation (18, with $k = 1$) through by $1/nu(n-1,\ 1,\ 3)$ to get

$$
\begin{aligned}
(22)\qquad M(3)u(n-1,2,3)/nu(n-1,1,3) &= b_0u(n,1,3)/u(n-1,1,3) \\
&+ ((n-1)b_1 + d_0)u(n-1,1,3)/u(n-1,1,3)n \\
&+ ((n-2)b_1 + d_1)u(n-2,1,3)/u(n-1,1,3)n
\end{aligned}
$$

(23) In (21) we have $u(n,1,3)/u(n-1,1,3) = r$ where r is the greatest root in

$$x^3 - a_1x^2 - a_2x - a_3 = 0 ,$$

so that equation (22) may be written as

$$(23.1)\qquad \lim_{n\to\infty} M(3)u(n-1,2,3)/nu(n-1,1,3) = rb_0 + b_1 + b_2/r = \text{(say) } L(3) .$$

Now, we multiply (18, with $k = 2$) through by

$$M(3)/n^2u(n-1,1,3) ,$$

to get

$$
\begin{aligned}
&2(M(3))^2u(n-1,3,3)/n^2u(n-1,1,3) = \\
&\quad + \left[u(n,2,3)M(3)b_0/nu(n-1,1,3)\right]\left[u(n,1,3)/u(n,1,3)\right] \\
&\quad + ((n-1)b_1 + 2d_0)u(n-1,2,3)/n^2u(n-1,1,3) \\
&\quad + \left[((n-2)b_2 + 2d_1)u(n-2,2,3)M(3)/n^2u(n-1,1,3)\right]\left[u(n-1,2,3)/u(n-1,2,3)\right],
\end{aligned}
$$

where combining this result with (23.1), and with $n\to\infty$, leads to

$$
\begin{aligned}
(24)\qquad \lim_{n\to\infty} (2!\,(M(3))^2u(n-1,3,3)/n^2u(n-1,1,3)) &= b_0L(3)r + b_2L(3)/r \\
&= (b_0r + b_1 + b_2/r)L(3) = (L(3))^2 .
\end{aligned}
$$

We continue with the exact method that gave us (24) step by step and with induction, which leads us (for $k = 1, 2, \cdots$) to:

The generalized Tribonacci number expressed as a limit

$$(25)\qquad \lim_{n\to\infty} (k!\,(M(3))^k u(n,k+1,3)/(n+1)^k u(n,1,3)) = (L(3))^k ,$$

where $L(3)$ is defined in (23.1).

Now, with the exact method that was used in finding (25) applied to the equation in (12) and step by step (and with added induction), we prove that:

The generalized Multinacci number expressed as a limit is

$$(26)\qquad \lim_{n\to\infty} (k!\,(M(m))^k u(n,k+1,m)/(n+1)^k u(n,1,m)) = (L(m))^k ,$$

where

$$\lim_{n\to\infty} M(m)u(n,2,m)/(n+1)u(n,1,m) = \sum_{v=0}^{m-1} b_v r^{1-v} = \text{(say)}\ L(m)\ ,$$

r is the greatest root in

$$x^m - \sum_{w=1}^{m} a_w x^{m-w} = 0\ ,$$

the $M(m)$ and the b_v are found by using Cramer's rule as defined in (15) through (15.3), $m = 2, 3, 4, \cdots$, $n = 0, 1, 2, \cdots$, $k = 1, 2, 3, \cdots$, and $u(0,k,m) = 1$.

4. A GENERALIZATION OF THE BINOMIAL FORMULA

Put

$$y = \sum_{w=0}^{m} a_w x^w = \sum_{n=0}^{m} a(n,1,m)x^n\ ,$$

so that

$$y^k = \left(\sum_{w=0}^{m} a_w x^w\right)^k = \sum_{n=0}^{mk} a(n,k,m)\, x^n\ , \tag{27}$$

where $m = 1, 2, 3, \cdots$, $k = 1, 2, 3, \cdots$, and the a_w are arbitrary numbers ($a_0 \neq 0$).

It is evident that $y^{k-1}y = y^k$, and combining this identity with (27) and then comparing the coefficients, leads to

$$a(mk-q,k,m) = \sum_{v=0}^{m} a(v,1,m)a(mk-q-v,k-1,m)\ , \tag{28}$$

where q ranges through the values $q = 0, 1, 2, \cdots, mk-m$, $k = 2, 3, 4, \cdots$, and $m = 1, 2, 3, \cdots$.

Differentiating equation (27) leads to

$$k\left(\sum_{v=0}^{mk-m} a(v,k-1,m)x^v\right)\left(\sum_{v=1}^{m} va(v,1,m)x^v\right) = \sum_{v=1}^{mk} va(v,k,m)x^v\ ,$$

and comparing the coefficients in this result, we have

$$(mk - q)a(mk - q, k, m) = k\sum_{v=1}^{m} va(v,1,m)a(mk - q - v, k - 1, m) , \tag{29}$$

where q ranges through the values $q = 0, 1, 2, \cdots, mk - m$, $k = 2, 3, 4, \cdots$, and $m = 1, 2, 3, \cdots$.

We multiply equation (28) through by $mk - q$ so that the right side of (28) is now an identity with the right side of (29), and arranging the terms in this result leads to

$$\begin{aligned}(mk - q)&a(0,1,m)a(mk - q, k - 1, m) \\ &= \sum_{v=1}^{m} a(v,1,m)a(mk - q - v, k - 1,m)(vk - mk + q) .\end{aligned} \tag{30}$$

Then replacing k with $k + 1$ in (30), we have

$$\begin{aligned}(mk + m - q)&a(0,1,m)a(mk + m - q, k, m) \\ &\sum_{v=1}^{m} a(v,1,m)a(mk + k - q - v, k, m)((v - m)(k + 1) + q) ,\end{aligned} \tag{31}$$

where $m,k = 1, 2, 3, \cdots$, q ranges through the values $q = 0, 1, 2, \cdots, mk$, $mk + k - q = v \geq 0$, and it is evident that

$$a(0,k,m) = (a(0,1,m))^k, \quad \text{and} \quad a(mk,k,m) = (a(m,1,m))^k .$$

As an application of (30) we find a value for $a(1,k,m)$. Let $mk + m - q = 1$, so that

$$a(0,1,m)a(1,k,m) = \sum_{v=1}^{m} a(v,1,m)a(1 - v, k, m)(vk + v - 1) ,$$

then

$$a(0,1,m)a(1,k,m) = ka(0,k,m)a(1,1,m) = k(a(0,1,m))^k a(1,1,m)$$

and we have

$$a(1,k,m) = k(a(0,1,m))^{k-1}a(1,1,m) .$$

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1. J. Arkin and V. E. Hoggatt, Jr., "An Extension of the Fibonacci Numbers — Part 2," Fibonacci Quarterly, Vol. 8, No. 2, March 1970, pp. 199-216.
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ON THE GREATEST COMMON DIVISOR OF SOME BINOMIAL COEFFICIENTS

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Henry W. Gould [1] has raised the conjecture

$$(1)\qquad \gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\},$$

which we shall prove in this note.

It is convenient to express the proof in terms of the p-adic valuation of rationals.

Definition. Let $r = p^{\alpha}(a/b)$ where $(a,p) = (b,p) = 1$ then $|r|_p = p^{-\alpha}$.

We need only two properties of this valuation.

$$(2)\quad \text{Ultrametric inequality.}\qquad |a + b|_p \le \max\{|a|_p, |b|_p\}\quad ;$$

and for all integera $a_1, \cdots, a_n$ we have

$$(3)\qquad |\gcd(a_1, \cdots, a_n)|_p = \max\{|a_1|_p, \cdots, |a_n|_p\}\ .$$

In view of (3) we can rephrase (1) as follows.

Conjecture. For all primes p we have

$$(4)\quad \max\left\{\left|\binom{n-1}{k}\right|_p, \left|\binom{n}{k-1}\right|_p, \left|\binom{n+1}{k+1}\right|_p\right\} = \max\left\{\left|\binom{n-1}{k-1}\right|_p, \left|\binom{n}{k+1}\right|_p, \left|\binom{n+1}{k}\right|_p\right\}.$$

If we divide both sides by

$$\left|\frac{(n-1)(n-2)\cdots(n-k+2)}{(k+1)!}\right|_p$$

we get the equivalent conjecture

$$(5)\qquad \begin{aligned} M_1(n,k) &= \max\{|(n-k)(n-k+1)(k+1)|_p, |nk(k+1)|_p, |(n+1)n(n-k+1)|_p\} \\ &= \max\{|(n-k+1)k(k+1)|_p, |n(n-k)(n-k+1)|_p, |(n+1)n(k+1)|_p\} \\ &= M_2(n,k)\quad . \end{aligned}$$

It thus suffices to prove $M_1 \le M_2$ and $M_2 \le M_1$ by deriving contradictions from the assumptions that one of the terms in M_1 exceeds M_2 or one of the terms in M_2 exceeds M_1. Since $M_2(n,k) = M_1(-k-1, -n-1)$ this involves only three steps.

Step 1. If

$$|(n-k)(n-k+1)(k+1)|_p > M_2$$

then

$$|k|_p < |n-k|_p \le 1, \quad \text{so} \quad |k+1|_p = 1$$

$$|n|_p < |k+1|_p = 1, \quad \text{so} \quad |n+1|_p = 1$$

$$|n|_p = |n(n+1)|_p < |n-k|_p \le \max\{|n|_p, |k|_p\} < |n-k|_p$$

a contradiction.

Step 2. If

$$|nk(k+1)|_p > M_2$$

then

$$|n-k+1|_p < |n|_p \le 1 \quad \text{so} \quad |n-k|_p = 1$$

$$|n-k+1|_p = |(n-k)(n-k+1)|_p < |k(k+1)|_p \le |k|_p$$

$$|n+1|_p < |k|_p = |(n+1)-(n-k+1)|_p \le \max\{|n+1|_p, |n-k+1|_p\}$$
$$< |k|_p$$

a contradiction.

Step 3. If

$$|(n+1)n(n-k+1)|_p > M_2$$

then

$$|k(k+1)|_p < |n(n+1)|_p \le |n+1|_p$$

$$|n-k|_p < |n+1|_p$$

$$|k+1|_p < |n-k+1|_p \quad \text{so} \quad |k|_p = 1 .$$

The first inequality now yields

$$|k+1|_p < |n+1|_p = |(n-k)+(k+1)|_p \le \max\{|n-k|_p, |k+1|_p\}$$
$$< |n+1|_p$$

a contradiction.

We have thus completed the proof of $M_1(n,k) \le M_2(n,k) = M_1(-k-1, -n-1)$ and hence by symmetry the proof of $M_2(n,k) = M_1(-k-1, -n-1) \le M_2(-k-1, -n-1) = M_1(n,k)$.

REFERENCE

1. H. W. Gould, "A New Greatest Common Divisor Property of Binomial Coefficients," Abstract* 72T-A248, Notices AMS 19 (1972), A685.

See the December issue for two pertinent articles.

A TRIANGLE WITH INTEGRAL SIDES AND AREA

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The object of this paper is to discuss the problem [3] of finding all triangles having integral area and consecutive integral sides. The class of all such triangles is determined uniquely by a simple recurrent sequence. We also examine other interesting sequences associated with the triangles. Such triangles have been of interest since the time of Heron of Alexandria and the reader is referred to Dickson's monumental history [9, Vol. 2, Chapter 5] for a detailed account of this and similar problems up to 1920.

The area, K, of a triangle having sides a, b, c must satisfy the formula of Heron

$$K^2 = s(s - a)(s - b)(s - c) ,$$

where

$$s = (a + b + c)/2 .$$

Letting the sides of our triangle be $u - 1$, u, $u + 1$, we have $s = 3u/2$ and the equation

$$K^2 = \frac{3u^2(u^2 - 4)}{16} \quad . \tag{1}$$

Evidently u must be even; for if u were odd then both u^2 and $u^2 - 4$ would be odd and 16 could not divide into the numerator. In order for 3N to be a perfect square it is necessary that N be a multiple of 3. However, u^2 cannot be a multiple of 3 without also being a multiple of 9, and so the only way to account for the factor 3 in the numerator is to impose the Diophantine equation $u^2 - 4 = 3v^2$, or

$$u^2 - 3v^2 = 4 . \tag{2}$$

All solutions to the problem will be determined by solving this equation for u, making certain that we obtain even values of u.

Equation (2) is of the general class $u^2 - Dv^2 = 4$ and a complete solution of this equation may be found in LeVeque [5, Vol. 1, p. 145]. The substance of the solution, as it applies to our work is that if $u_1 + v_1\sqrt{D}$ is the minimal positive solution of $u^2 - Dv^2 = 4$, $D \neq$ square, $D > 0$, then the general solution for positive u, v is given by the symbolic formula

$$u + v\sqrt{D} = 2\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^n , \qquad (n = 0, 1, 2, \cdots)$$

where v and u are found by expanding the right-hand side by the binomial theorem and equating radical and non-radical parts. It is easily seen that the minimal positive solution of (2) is $4 + 2\sqrt{3}$ so that the general solution is given by

$$u + v\sqrt{3} = 2(2 + \sqrt{3})^n = 2\sum_{k=0}^{n}\binom{n}{k} 2^{n-k} (\sqrt{3})^k$$

$$= 2\sum_{k=0}^{[n/2]}\binom{n}{2k} 2^{n-2k}3^k + 2(\sqrt{3})\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1} 2^{n-2k-1}3^k .$$

Thus we have

$$u = 2^{n+1}\sum_{k=0}^{[n/2]}\binom{n}{2k} (3/4)^k .$$

However, it is easy to split up the binomial expansion and obtain the well-known formula

$$\sum_{k=0}^{[n/2]}\binom{n}{2k} x^k = \frac{1}{2}\left\{(1 + \sqrt{x})^n + (1 - \sqrt{x})^n\right\} ,$$

whence we have

$$(3) \qquad u = u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n, \qquad (n = 0, 1, 2, \cdots) .$$

It is of interest to point out that we could also write

$$(4) \qquad u_n = \frac{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}}{2^n} ,$$

but the former relation is easier to use in practice. We also remark that it is easy to prove by induction that u as determined by (3) is indeed even. A shorter derivation of (3) is to note that

$$2u = (u + v\sqrt{3}) + u - v\sqrt{3}) = 2(2 + \sqrt{3})^n + 2(2 - \sqrt{3})^n .$$

Cf. the solution given by E. P. Starke [7].

We also have the recurrence relation

(5) $$u_{n+2} = 4u_{n+1} - u_n, \qquad (u_0 = 2,\ u_1 = 4)$$

since this recurrence is associated with the characteristic equation

$$x^2 = 4x - 1$$

whose roots are $2 + \sqrt{3}$, $2 - \sqrt{3}$. The recurrence relation allows us to compute a short table of values of u, as follows:

n	$u = u_n$
0	2
1	4
2	14
3	52
4	194
5	724
6	2702
7	10084
8	37634
9	140452
10	524174
11	1956244
12	7300802
13	27246964
14	101687054
15	379501252
16	1416317954
17	5285770564
18	19726764302
19	73621286644
20	275758382274

Actually our problem is an old one, rational triangles having always been of interest. A solution of the form (3) was given, for example, by Reinhold Hoppe in 1880 [4]. Also, Cf. solutions in [7], [8].

The first six triangles, together with their areas, are:

1,	2,	3,	0
3,	4,	5,	6
13,	14,	15,	84
51,	52,	53,	1170
193,	194,	195,	16296
723,	724,	725,	228144

The triangle 3, 4, 5 is the only right triangle in the sequence because $(u - 1)^2 + u^2 = (u + 1)^2$ implies $u(u - 4) = 0$ which has only the one non-trivial solution. The triangle 13, 14, 15 has been used widely in the teaching of geometry. In fact the writer first became aware of this example during a course in college where the triangle was used as a standard reference triangle. Such a triangle has rational values for its major constants, as we shall see here, and so makes it possible to have problems with 'nice' answers. For example, in this case the sines of the three angles in the triangle are 4/5, 12/13, and 56/65. The radii of the escribed circle are 21/2, 14, and 12. The altitudes are 168/13, 12, and 168/15. Cf. [7].

It is easy to conjecture that the area $K = K_n$ satisfies the recurrence relation

$$K_{n+2} = 14K_{n+1} - K_n, \qquad (K_0 = 0,\ K_1 = 6)\ . \tag{6}$$

If this were true, we could find an explicit formula for K since the characteristic equation for (6) is $x^2 - 14x + 1 = 0$, whose roots are $7 \pm 4\sqrt{3}$. For suitable constants A, B we should then have

$$K_n = A(7 + 4\sqrt{3})^n + B(7 - 4\sqrt{3})^n.$$

From the initial values, A, B are easily determined and we find that

$$K_n = \frac{\sqrt{3}}{4}\left\{(7 + 4\sqrt{3})^n - (7 - 4\sqrt{3})^n\right\}$$

which simplifies to

$$K_n = \frac{\sqrt{3}}{4}\left\{(2 + \sqrt{3})^{2n} - (2 - \sqrt{3})^{2n}\right\}\ . \tag{7}$$

According to the review in the Fortschritte [4] it was in this form that Hoppe found the area.

Now (7) follows from (6) which we conjectured from tabular values of K. However it is easy to show that K_n given by (7) satisfies (6). Thus we shall prove (6) by proving (7) in a novel way, as follows.

By (1) we have, for any triangle T_n,

$$16K_n^2 = 3u_n^2(u_n^2 - 4)\ ,$$

and it is easy to see that (3) implies

$$u_n^2 = u_{2n} + 2\ , \tag{8}$$

whence

$$16K_n^2 = 3(u_{2n} + 2)(u_{2n} - 2) = 3(u_{2n}^2 - 4) = 3(u_{4n} - 2)\ ,$$

so that we have the formula

(9) $$K_n^2 = \frac{3}{16}(u_{4n} - 2).$$

Thus

(10) $$K_n = \frac{\sqrt{3}}{4}(u_{4n} - 2)^{\frac{1}{2}}.$$

However a short calculation shows that in fact

$$\left\{(2+\sqrt{3})^{2n} - (2-\sqrt{3})^{2n}\right\}^2 = (2+\sqrt{3})^{4n} + (2-\sqrt{3})^{4n} - 2$$
$$= u_{4n} - 2,$$

whence formula (10) gives (7) which we wanted to prove.

We remark that relation (8) is very useful in checking a table of u_n and was used for this purpose here to be certain of the value of u_{20}.

The radius, r, of the inscribed circle of any triangle is given by the formula $K = rs$. In the case at hand this gives

(11) $$r^2 = r_n^2 = \frac{K^2}{a^2} = \frac{u^2 - 4}{12} = \frac{u_n^2 - 4}{12} = \frac{u_{2n} - 2}{12}$$

and it is easy to prove that

(12) $$r_{n+2} = 4r_{n+1} - r_n, \qquad (r_0 = 0,\ r_1 = 1)$$

so that every triangle T_n has an integral inradius. The first few values of r are 0, 1, 4, 15, 56, 209, 780, 2911, 10864, $\cdots$.

Noting that recurrence relation (12) is the same as relation (5) we suspect that there are other intimate relations between u and r. Indeed, the theory of continued fractions provides us an interesting result. Some very handy information on continued fractions is given by Davenport [2] and especially the table on page 105. First of all, our original equation (2) may be transformed as follows. Since u is even, say $u = 2x$, we have $4x^2 - 3y^2 = 4$, whence v is even, say $v = 2y$, and so the equation can be written as

(13) $$x^2 - 3y^2 = 1$$

which suggests that we examine the familiar continued fraction expansion for $\sqrt{3}$. Indeed,

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

and the first few convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \cdots .$$

The interesting point here is that every other numerator is one-half u_n, while every other denominator is precisely r_n. By means of some simple transformations we can bring out the relation more strikingly. In fact the continued fraction

$$C = 1 + \frac{1}{1+}\,\frac{1}{3-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\cdots \tag{14}$$

has successive convergents

$$\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{26}{15}, \frac{97}{56}, \frac{362}{209}, \frac{1351}{780}, \cdots$$

so that each numerator is $\frac{1}{2}u$ and each denominator is r. It can be shown that the continued fraction (14) converges to $\sqrt{3}$. Let us show that $\frac{1}{2}u/r$ also tends to $\sqrt{3}$. We have, by (11)

$$\frac{1}{4}\,\frac{u^2}{r^2} = 3\,\frac{u^2}{u^2-4} = 3\,\frac{1}{1-\dfrac{4}{u^2}} \rightarrow 3 \text{ as } n \rightarrow \infty ,$$

so that we can say that our general T_n has the interesting property that

$$\lim_{n\rightarrow\infty} \frac{u_n}{r_n} = 2\sqrt{3} . \tag{15}$$

It is interesting to recall Heron's formula (iterative) for the square root of 3 :

$$a_{n+1} = \frac{5a_n + 9}{3a_n + 5} .$$

Starting with $a_1 = 5/3$ we find the successive approximations

$$\frac{5}{3}, \frac{26}{15}, \frac{265}{153}, \frac{1351}{780}, \cdots$$

These approximations, especially the value 1351/780, are of historical interest.

One may find formulas for the radii of the escribed circles for the class T_n by recalling that [1, p. 12]

$$rs = (s-a)r_a = (s-b)r_b = (s-c)r_c .$$

Further interesting relations follow from the two formulas

(16) $$r_a + r_b + r_c = r + 4R, \qquad \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c},$$

where R = radius of the circumcirle. Also we recall that $r = (s - a)\tan \frac{1}{2}A$, with other similar formulas.

Thus we have

(17) $$r_u^2 = \frac{3}{4}u^2\left(\frac{u-2}{u+2}\right)$$

(18) $$r_b^2 = \frac{3}{4}(u^2 - 4),$$

whence by (11), $r_b = 3r$,

(19) $$r_c^2 = \frac{3}{4}u^2\left(\frac{u+2}{u-2}\right).$$

The radii of the three escribed circles are easily calculated and the first few values are as follows:

(20) $$r_a : 0,\ 2,\ \frac{21}{2},\ \frac{130}{3},\ \frac{1164}{7},\ \frac{6878}{11},\ \frac{50795}{13},\ \cdots$$

(21) $$r_b : 0,\ 3,\ 12,\ 45,\ 168,\ 627,\ 2340,\ \cdots$$

(22) $$r_c : \infty,\ 6,\ 14,\ \frac{234}{5},\ \frac{679}{4},\ \frac{11946}{19},\ \cdots.$$

Relations (16) become

(23) $$\frac{1}{r_a} + \frac{1}{r_c} = \frac{2}{3r},$$

and

(24) $$r_a + r_c = 4R - 2r = \frac{6r^2 + 2}{r}$$

the last step following because of the fact that we shall find $R = 2r + 1/2r$.

As a simple example of the check afforded by (23), we have $(n = 5)$

$$\frac{19}{11946} + \frac{11}{6878} = \frac{19}{2\cdot3\cdot11\cdot181} + \frac{11}{2\cdot19\cdot181} = \frac{19^2 + 3\cdot11^2}{2\cdot3\cdot11\cdot19\cdot181}$$

$$= \frac{361 + 363}{2\cdot3\cdot11\cdot19\cdot181} = \frac{2}{3\cdot11\cdot19} = \frac{2}{3(209)} = \frac{2}{3r}.$$

One discerns a Pellian equation in this calculation also.

We may combine (23) and (24) to obtain a product formula, which is

$$r_a r_c = 9r^2 + 3 . \tag{25}$$

The equation

$$x^2 - (r_a + r_c)x + r_a r_c = 0$$

has for roots the radii r_a, r_c, and when we substitute into this equation by means of (24) and (25), we have the equation

$$rx^2 - (6r^2 + 2)x + 9r^3 + 3r = 0 .$$

Solving this by the quadratic formula, we obtain the novel formulas

$$r_a = \frac{3r^2 + 1 - \sqrt{3r^2 + 1}}{r} \tag{26}$$

and

$$r_c = \frac{3r^2 + 1 + \sqrt{3r^2 + 1}}{r} \tag{27}$$

which are rather elegant results, especially since $3r^2 + 1$ is a perfect square.

We turn now to the angles of our triangles. From the functional relations

$$2K = ab \sin C = bc \sin A = ca \sin B , \tag{28}$$

we find (by means of (1))

$$\sin^2 A = \frac{3}{4}\,\frac{u^2 - 4}{(u + 1)^2} , \tag{29}$$

$$\sin^2 B = \frac{3}{4}\,\frac{u^2(u^2 - 4)}{(u^2 - 1)^2} , \tag{30}$$

$$\sin^2 C = \frac{3}{4}\,\frac{u^2 - 4}{(u - 1)^2} . \tag{31}$$

Letting $n \to \infty$, each of these tends to $3/4$. This agrees with the fact that in an equilateral triangle the three sines would be each $\sqrt{3}/2$. Of course, our special triangle T_n behaves at ∞ as an equilateral triangle insofar as <u>angular</u> measurements are concerned, but never becomes truly an equilateral triangle because the <u>sides</u> never become equal. We may illustrate this behavior in another way. It is well known that the square of the distance between the circumcenter and incenter in any triangle is $R(R - 2r)$. Since, as we have remarked, it can be shown in our case that $R = 2r + 1/2r$ the number in question has the value $R/2r$. It is also known that $R \geq 2r$ in any case. However, $R - 2r = 1/2r$ which can be made as small as we wish by choosing n sufficiently large. (It follows from (12) that r_n is an increasing sequence.) Thus we have

(32) $$\lim_{n\to\infty} (R_n - 2r_n) = 0$$

and

(33) $$\lim_{n\to\infty} \frac{R_n}{2r_n} = 1 .$$

It follows then that the distance between circumcenter and incenter tends to 1. Only if these two points come together can we speak truly of an equilateral triangle. Of course, in a finite triangle, with R fixed say, then as $2r$ approaches R, $R(R - 2r)$ tends to zero. In our case, however $(R - 2r)^{-1}$ and R increase at the same rate, i.e., $n \to \infty$. The reader will find other peculiarities of T_∞.

Let us agree to write $|P - Q|$ for the distance between points P and Q. Let N = circumcenter; N = orthocenter; I = incenter; G = centroid; M = Nine-point center; A, B, C = vertices. Then we have the following known distance relationships for an arbitrary triangle:

$$|N - H|^2 = 9R^2 - (a^2 + b^2 + c^2) = 9|N - G|^2 = \frac{9}{4}|G - H|^2 ;$$

$$|I = H|^2 = 4R^2 + 2r^2 - \tfrac{1}{2}(a^2 + b^2 + c^2) ;$$

$$|I - N|^2 = R(R - 2r) ;$$

$$|I - A| \cdot |I - B| \cdot |I - C| = 4r^2R ;$$

$$|G - H| = 2|G - N| ;$$

$$|M - N| = |M - H| = \tfrac{1}{2}|N - H| .$$

In our special triangles we also have the following:

(34) $$ab + bc + ca = 3u^2 - 1 = 3u_n^2 - 1 = 3u_{2n} + 5 ,$$

and

(35) $$a^2 + b^2 + c^2 = 3u^2 + 2 = 3u_{2n} + 8 = 36r^2 + 14 .$$

Thus we have

$$|N - H|^2 = 9\left(2r + \frac{1}{2r}\right)^2 - 36r^2 - 14 = 4 + \frac{9}{4r^2}, \tag{36}$$

and since r increases steadily with n we see that for T_∞ the circumcenter and orthocenter will be two units apart.

Moreover,

$$\begin{aligned}|I - H|^2 &= 4(2r + 1/2r)^2 + 2r^2 - \tfrac{1}{2}(36r^2 + 14)\\ &= 1 - 1/r^2,\end{aligned}$$

whence in T_∞ the incenter and orthocenter are also one unit apart.

It is then extremely simple to draw the Euler line for T_∞:

N G I = M H

The Euler line from N to H is two units long and the incenter lies on it and in fact coincides with the Nine-Point center. This then gives some idea of the behavior of T_∞.

In Figure 1 is shown the standard location of the common points in an arbitrary finite triangle. The Nine-point circle has quite a history, having been studied as long ago as 1804. It was first called "le cercle des neuf points" by Terquem in 1842 in Vol. 1 of the journal Nouvelles Annales de Mathématiques. The circle has many properties; it passes through the midpoints of the sides and the feet of the altitudes, it is tangent to the inscribed circle; its residue is $\frac{1}{2}R$; it bisects any line segment drawn from the orthocenter to the circumcircle. Thus it has more than nine points associated with it, and has been called an n-point circle, Terquem's circle, the medioscribed circle, the circumscribed midcircle, Feuerbach's circle, etc. A very interesting history has been given by J. S. MacKay [6]. Coxeter [1, p. 18] quotes Daniel Pedoe: "This circle is the first really exciting one to appear in any course on elementary geometry."

We have now to return to a discussion of the circumradius R. From the formula

$$K = \frac{abc}{4R}$$

we have in our case

$$R = R_n = \frac{u^2 - 1}{6r} = \frac{u_n^2 - 1}{6r} = \frac{u_{2n} + 1}{6r}, \tag{37}$$

or also

$$R_n^2 = \frac{(u^2 - 1)^2}{3(u^2 - 4)}. \tag{38}$$

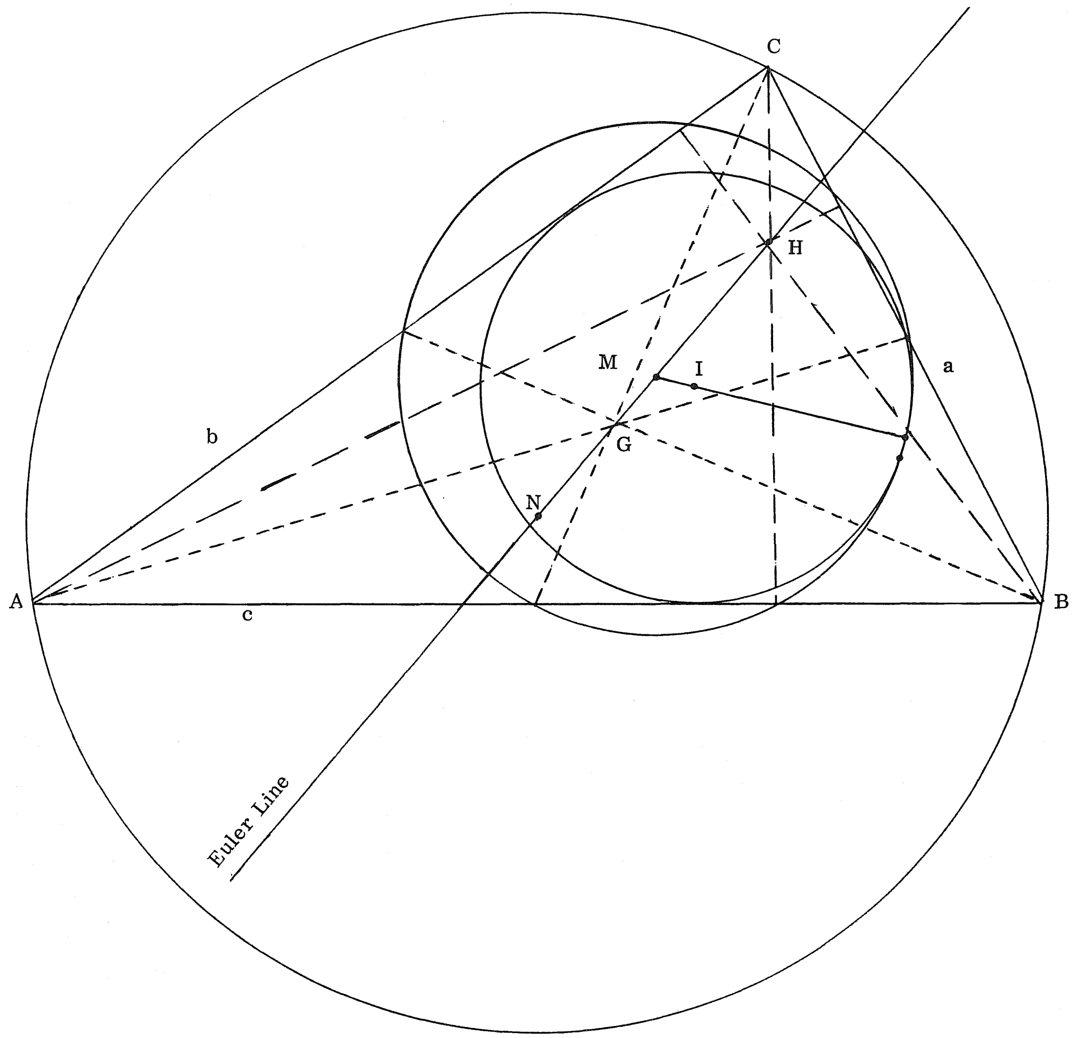

Figure 1

But by (11) we have $u^2 = 12r^2 + 4$, so

$$R^2 = \frac{(12r^2 + 3)^2}{3(12r^2)} = \frac{(4r^2 + 1)^2}{4r^2} \quad ,$$

whence

(39) $$R = 2r + \frac{1}{2r}$$

as we suggested earlier. The first few values of R are

$$\infty, \frac{5}{2}, \frac{65}{8}, \frac{901}{30}, \frac{12545}{112}, \frac{174725}{418}, \cdots$$

or

$$0 + \frac{1}{0}, 2 + \frac{1}{2}, 8 + \frac{1}{8}, 30 + \frac{1}{30}, 112 + \frac{1}{112}, 418 + \frac{1}{418}, 1560 + \frac{1}{1560}, \cdots$$

It is certainly more interesting, for example, in the triangle 13, 14, 15 to think of the circumradius as $8 + 1/8$ than as $65/8$; this together with the inradius being 4. (We apologize for writing $1/0$ but wish to be suggestive.)

The sequence of numbers 1, 5, 65, 901, 12545, $\cdots$ incidentally, has an interesting recurrence. Now we know that these are just $2r$ times R, so let us define a special sequence by

$$g_n = 2rR = 2r_nR_n \,. \tag{40}$$

Then $g_n = (u^2 - 1)/3$, but also

$$g_{n+2} = 14g_{n+1} - g_n - 4, \qquad (g_0 = 1, \quad g_1 = 5)\,. \tag{41}$$

This completes our present discussion of the properties of special number sequences associated with the class of triangles having consecutive integers as sides and having integral areas. The really crucial matter was right at the beginning where it was necessary to set up a criterion for the triangles. It is not enough to guess formula (3) or (5), as we must rule out any other possibility. This we accomplished by setting up the equation (1) and arguing to (2) as a <u>necessary</u> condition. That it is a <u>sufficient</u> condition is clear. Any three consecutive numbers (>1) do generate a real triangle, and sequence (3) turns out to have integral area.

We close by suggesting other possible problems. Let $u \geq 2$ and consider triangles having integral areas and sides $2u - 1$, u, $2u + 1$. Then $s = 5u/2$, and

$$s - a = \frac{1}{2}(u + 2), \qquad s - b = 3u/2, \qquad s - c = \frac{1}{2}(u - 2)\,.$$

Then

$$K^2 = s(s - a)(s - b)(s - c) = \frac{15u^2(u^2 - 4)}{16}\,.$$

Again, u must be even. Thus we have evidently to impose the equation

$$u^2 - 15v^2 = 4\,. \tag{42}$$

The rest of the discussion is similar to what we presented above.

Again, let the sides be consecutive Fibonacci numbers. Then

$$s = \frac{1}{2}(F_{n-1} + F_n + F_{n+1}) = \frac{1}{2}(F_{n+1} + F_{n+1}) = F_{n+1} ,$$

and

$$s - a = F_n , \quad s - b = F_{n-1} , \quad a - c = 0 .$$

Thus $K = 0$. But this is trivial. No triangle is formed; just a degenerate line segment. It would be of interest to modify the values so as to have some really interesting Fibonacci triangle with integral area. We leave this as a problem for any interested reader. Can one, for instance, make anything interesting with sides $F_m - d$, F_m, $F_m + d$ for suitable values of d? What interesting Pellian equations and recurrences might be associated with a tetrahedron?

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A NEW LOOK AT FIBONACCI GENERALIZATION

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1. INTRODUCTION

Our parent topic will be sequences, in the broadest sense. That is to say, we shall be dealing with ordered infinite sets of numbers, mostly or usually positive integers, whose character is determined by (a) some given subsequence of $\underline{s}$ members, and (b) a function linking any given member to its immediate preceding $\underline{s}$ - ad. In this context the case of $\underline{s} = 1$ is trivial, whereas the case of $\underline{s} = 2$ includes many well known examples, in particular those called the Fibonacci and Lucas sequences. Some of the examples of the case $\underline{s} = 3$ have been discussed under the name of Tribonacci sequences.

Here we restrict attention to $\underline{s} = 2$. In characterizing such sequences we use the letters A and B to denote the given pair (and only coprime A and B will be admitted). The determining function will be linear, with parameters N and M. Thus the term following B will be $NA + MB$; the next $NB + M(NA + MB)$; and so on. Similarly, the term preceding A will be $(B - MA)/N$; and the next $A/N - M(B - MA)/N^2$; and so on. Each term is in fact expressible as $aA + bB$, where the coefficients $\underline{a}$ and $\underline{b}$ are polynomials in N and M, and if we work through the algebra the results shown in Table 1 will be reached.

Note that we have not so far mentioned ordinal numbers associated with the terms of the sequence. In thinking of the formal sequence, extending to infinity in both directions, we have to realize that there is an arbitrariness in putting ordinals in one-to-one correspondence with the terms. But it is patently convenient to associate the term A with "first," so that all terms less than A are associated with nonpositive ordinals. Not the least reason for this choice is that the structure of the sequence is such that the expression for terms smaller than A is different from, and more complicated than the expression for terms greater than B (the former involve alternating algebraic signs).

Examining Table 1 we observe that it contains the apices of Pascal Triangles, and it is not difficult to show that, with the proposed ordinal convention, the n^{th} term is

$$(1) \qquad \sum_{i=0}^{\infty} \binom{n-i-2}{i-1} MA + \binom{n-i-2}{i} B \quad N^i M^{n-2i-2} \qquad (n > 2)$$

and

$$(2) \qquad (-1)^{n+1} \sum_{m=0}^{\infty} \binom{-n-i+1}{i} MA - \binom{-n-i}{i} B \quad N^{n+i-1} M^{-n-2i} \qquad (n < 1)\,.$$

Table 1
POLYNOMIALS IN N AND M SPECIFYING THE SEQUENCE
(ONE TERM PER LINE) OF $[aA + bB]$, WHERE $a = f(N, M)$ AND $b = f'(N, M)$

$\underline{a}$ = Coefficient of A	$\underline{b}$ = Coefficient of B
⋮	⋮
$-(N^{-5}M^5 + 4N^{-4}M^3 + 3N^{-3}M)$	$N^{-5}M^4 + 3N^{-4}M^2 + N^{-3}$
$N^{-4}M^4 + 3N^{-3}M^2 + N^{-2}$	$-(N^{-4}M^3 + 2N^{-3}M)$
$-(N^{-3}M^3 + 2N^{-2}M)$	$N^{-3}M^2 + N^{-2}$
$N^{-2}M^2 + N^{-1}$	$-(N^{-2}M)$
$-(N^{-1}M)$	N^{-1}
1	0
0	1
N	M
NM	$M^2 + N$
$NM^2 + N^2$	$M^3 + 2NM$
$NM^3 + 2N^2M$	$M^4 + 3NM^2 + N^2$
$NM^4 + 3N^2M^2 + N^3$	$M^5 + 4NM^3 + 3N^2M$
$NM^5 + 4N^2M^3 + 3N^3M$	$M^6 + 5NM^4 + 6N^2M^2 + N^3$

2. A TWO-PARAMETER SEQUENCE

In what follows, we shall concentrate on an important special case of the "$\underline{s} = 2$" linear sequences, namely, that with $A = M = 1$. The setting of A at unity is actually less of a restriction than at first appears, in that any sequence with $A \neq 1$ can be transformed to the "unity" set by division of every term by A. This new sequence will retain most of the properties of its original form, with the notable exception of number-theoretic properties. The setting of M at unity not only introduces a major simplification into the structure, but, as we shall see later, it ties in with a natural extension of the classic Fibonacci Rabbit Problem.

Let us fix a notation at this point. We shall use $F_{B,N,n}$ to denote the n^{th} member of the sequence whose parameters are B (≥ 0) and N (≥ 1). Thus

$$\left\{ \begin{array}{l} F_{B,N,1} = 1; \quad F_{B,N,2} = B \\ F_{B,N,n} = NF_{B,N,n-2} + F_{B,N,n-1} \end{array} \right. \tag{3}$$

Normally, B and N will be integers. The case of N being any real number $>-1/4$ is worth special consideration; it yields monotonically increasing sequences many of whose properties are shared with those of N integral; but it will not be explored here. Furthermore, we shall not be specifically concerned with $\underline{n}$ negative (although it will occasionally have to be referred to in explication of certain formulas).

The generating function of the sequence is worth noting here. It is the left-hand side of the identity

$$\frac{1 + x(B - 1)/N}{N - x - x^2} = \sum_{n=1}^{\infty} F_{B,N,n} N^{-n} x^{n-1} \quad . \tag{4}$$

This can be verified by multiplying out. And setting B = N = 1 we of course obtain the familiar generating function of the "original" Fibonacci sequence, which is $1/(1 - x - x^2)$.

We shall use $\{B, N\}$ to denote the sequence itself, and it must be pointed out at once that not all $\{B, N\}$ are unique, sequence-wise. Some may differ only in "key," to borrow the musical term, in the sense that a shift in the ordinals (the $\underline{n}$-sequence) will make them identical. For example, the following three sequences can be equalized by such shifts:

n :	-3	-2	-1	0	1	2	3	4
$\{0,1\}$:	5	-3	2	-1	1	0	1	1
$\{1,1\}$:	2	-1	1	0	1	1	2	3
$\{2,1\}$:	-1	1	0	1	1	2	3	5

Explanation is superfluous.

Another type of hidden identity (for the segments with $\underline{n} > 0$) is multiplicative, and is illustrated below:

n :	-1	0	1	2	3	4	5
$\{0,3\}$:	4/9	-1/3	1	0	3	3	12
$\{0,3\}/3$:	4/27	-1/9	1/3	0	1	1	4
$\{1,3\}$:	1/3	0	1	1	4	7	19

Thus $\{0, N\}$, divided throughout by N is identical, over positive $\underline{n}$ (apart from a 2n-keyshift), to $\{1, N\}$.

Using a subscript to denote keyshift, we can summarize the algebra of these sequences as follows:

$$\{0, Y\} = Y\{1, Y\}_{+2} = Y\{Y + 1, Y\}_{+3} \tag{5}$$

which of course includes the special case of Y = 1, illustrated above. Furthermore, if $B|N$, then

(6) $$\{X,Y\} = X\{Y/X + 1, Y\}_{+1}$$

which has a special case $X = Y$, so that

(7) $$\{X,Y\} = X\{2,Y\} = 2X\{Y/2 + 1, Y\}_{+2} \qquad (Y \text{ even})$$

And if $Y = X(X-1)$, the sequence is simply the powers of X, and is infinitely divisible by X — but every quotient is identical to the original dividend, apart from a shift of key. Symbolically,

(8) $$F_{X(X-1),X,n} = X^{n-1} \qquad (X \geq 1) \quad .$$

Finally, if $X > Y + 1$, all $\{X,Y\}$ are unique.

In Figure 1, the distribution pattern of these hidden identities is shown for some of the lower B and N. Each cell is to be regarded as containing a complete sequence $\{B,N\}$ — specifically, $\{X,Y\}$. A blank cell is understood to contain an irreducible sequence (in the sense that it cannot be transformed, by division and/or shift of key, into a smaller-B sequence). Hatched cells contain sequences that are powers of B. Black cells hold all other reducible sequences.

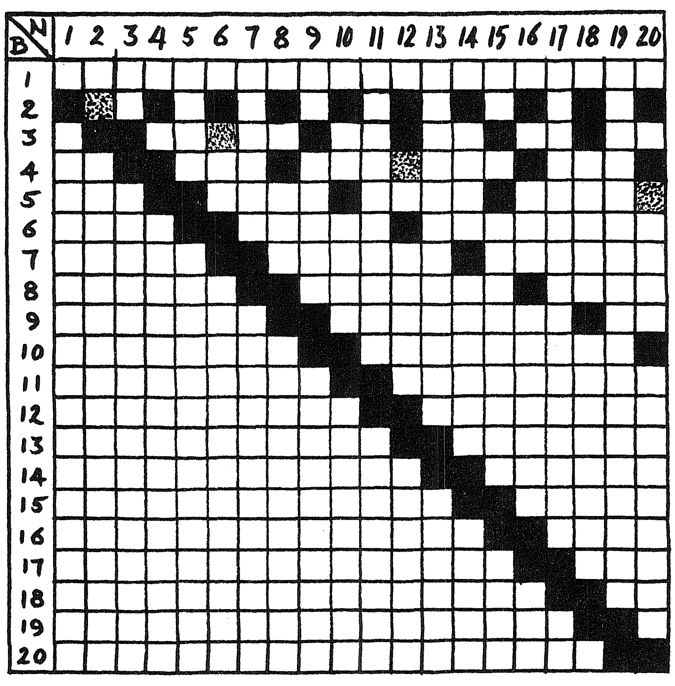

Fig. 1 Distribution of Three Types of $\{B,N\}$: (i) reducible (black); (ii) powers of B (stippled); (ii) irreducibles.

In the Appendix are collected for reference $F_{B,N,n}$ for $n = 1(1)25$, and for certain B ($\not>5$) and N ($\not>10$). Of the possible total of 50 combinations of B and N, only 34 have been tabulated: 14 were omitted because of their being reducibles, and 2 because of their being merely sequences of powers (which in this context are uninteresting). The omissions, in short, are conditioned by Fig. 1.

3. PROLIFIC FIBRABBITS

The sequence $\{1,1\}$ is the original Fibonacci sequence, and $\{3,1\}$ is the Lucas sequence — and we can now see why the Lucas sequence is normally regarded as the one "next" to the Fibonacci sequence; it is because the intervening $\{2,1\}$ is really $\{1,1\}$, with a unit shift of key. We may note in passing that (for any given fixed N, say Y) the identity

$$F_{1,Y,n} + F_{1,Y,n-2} = F_{Y+2,Y,n-1} \tag{9}$$

yields the well known relation between member of the Fibonacci and Lucas sequences when we set Y at unity.

The interesting thing about $\{1,N\}$ is that it furnishes solutions to the Fibonacci Rabbit problem generalized to the situation in which each pair gives birth to N pairs at a time, instead of one. This is perhaps best appreciated by reference to a time-table, as in Table 2.

Table 2

NUMBER OF PAIRS OF IMMORTAL RABBITS ALIVE,
BY MONTH (t) AND GENERATION (g in N^g),
IN A BREEDING REGIME THAT UNFAILINGLY YIELDS N PAIRS PER MONTHLY BIRTH

t = n - 1	N^0	N^1	N^2	N^3	N^4	N^5	N^6	Sum when N = 1	2	3
0	1							1	1	1
1	1							1	1	1
2	1	1						2	3	4
3	1	2						3	5	7
4	1	3	1					5	11	19
5	1	4	3					8	21	40
6	1	5	6	1				13	43	97
7	1	6	10	4				21	85	217
8	1	7	15	10	1			34	171	508
9	1	8	21	20	5			55	341	1159
10	1	9	28	35	15	1		89	683	2683
11	1	10	36	56	35	6		144	1365	6160
12	1	11	45	84	70	21	1	233	2731	14209
13	1	12	55	120	126	56	7	377	5461	32689

We imagine, after Fibonacci, a pair of month-old rabbits mated in an enclosure, and giving birth to N new pairs every month thereafter; and each of the new pairs breeds similarly after a month's maturation. The table can be readily constructed from elementary considerations, with each column representing a generation, beginning at the zeroth — and the construction is in fact the familiar tilted Pascal Triangle. At the beginning of the second month there will be $1 + N$ pairs; at the beginning of the third there will be $1 + 2N$ pairs, and so forth. Clearly, the sums in the end columns will be

$$\sum_{i=0}^{\infty} \binom{n - i - 1}{i} N^i \tag{10}$$

— which is expression (1) with $A = B = M = 1$ and the utilization of Pascal's Rule for the addition of binomial coefficients. In other words, Eq. (10) is $F_{1,N,n}$.

It is possible to sophisticate the treatment by allowance for deaths, the simplest situation being to schedule the death of a mated pair of rabbits immediately after the birth of its m^{th} litter. Hoggatt and Lind [2] have shown how this can be done for the classic case, in which $N = 1$. For $N > 1$ the crude arithmetic of the population growth is straightforward enough, but it does not condense well. The population increment from the g^{th} generation at time $\underline{n}$ $(= t + 1)$ can be written

$$\sum_{i=0}^{\infty} (-1)^i \binom{g-1}{i} \binom{h}{g-1} - \binom{h-m}{g-1} - \binom{h-m-1}{g-1} + \binom{h-2m-1}{g-1} \tag{11}$$,

where

$$h = n - g - im - 2 ,$$

and the summation of (11) over all $\underline{g}$ and all time points to $\underline{n}$ gives the required population size at $\underline{n}$. This is clumsy, but a compact operation is elusive.

Actually, allowance for restricted littering and for mortality does not make a great difference to the population, which, with $N > 1$, soon becomes enormous. For example, $F_{1,5,23} = 3\,912\,125\,981$, and if we limit $\underline{m}$ to 5 (and remove the parents subsequently), the population at the 23^{rd} month will still be 3 759 051 250, which is 96 percent of the former figure (and represents more than one pair of rabbits for every human being on earth).

Incidentally, in considering litters with more than two siblings, we can easily cope with a sex ratio other than 50:50. Suppose, for instance, that litters of five bucks and four does are to be substituted for the classic one buck and one doe (per litter): we carry out the arithmetic for $N = 4$, and then multiply the answer by the factor $(4 + 5)/4$; this will give us the required population (in, of course, rabbits, not pairs of rabbits).

4. τ_N AND THE EXPLICIT FORMULAS

A sequence of the kind we are discussing may intuitively be expected to have a limiting ratio of adjacent terms, and in fact it is well established that such a ratio exists and is independent of B. But it is not independent of N. By extension from the familiar treatment of the case of $\{1,1\}$, we write the auxiliary equation

(12) $$\tau_N^n = N\tau_N^{n-2} + \tau_N^{n-1}$$

and divide it by τ_N^{n-2} to give, after rearrangement,

(13) $$\tau_N^2 - \tau_N - N = 0 .$$

The roots of (13) are $1/2 \pm \sqrt{N + 1/4}$, and we identify the positive root with the required limiting ratio, τ_N. The other root, we note, is $1 - \tau_N$.

So the asymptotic growth rate (per unit interval) of all $\{B,1\}$ (including the original Fibonacci and Lucas sequences) is $1/2 + \sqrt{5}/2 = 1.618034 \cdots$; that of all $\{B,2\}$ is $1/2 + \sqrt{9}/2$; that of all $\{B,3\}$ is $1/2 + \sqrt{13}/2 = 2.302775 \cdots$; and so on. These asymptotes are approached rapidly: turning to the sums at the right foot of Table 2, for example, we shall find that $377/233 = 1.618 \cdots$, that $5461/2731 = 2.000 \cdots$, and that $32689/14209 = 2.301 \cdots$.

The powers of τ_N can be expressed in terms of two F's, thus:

(14) $$\tau_N^n = \frac{F_{1+2X,X,n} + F_{1,N,n}\sqrt{4N+1}}{2}$$

and

(15) $$\tau_N^{-n} = \frac{F_{1+2X,X,n} - F_{1,N,n}\sqrt{4N+1}}{2}\,(-1/N)^n ,$$

where X is the particular value of N and determines B in the first F of the numerator.

The quantity τ_N can be used to derive explicit expressions for any $F_{B,N,n}$ by virtue of the relation

(16) $$F_{B,N,n} = k_1\tau_N^{n-1} + k_2(1 - \tau_N)^{n-1} ,$$

where the k's are constants that can be evaluated from our knowledge of the two parametric members of the sequence

$$F_{B,N,1} = 1 = k_1 + k_2$$

and

$$F_{B,N,2} = B = k_1\tau_N + k_2(1 - \tau_N) ,$$

whence

$$k_1 = (\tau_N + B - 1)/(2\tau_N - 1)$$

$$k_2 = (\tau_N - B)/(2\tau_N - 1) .$$

Therefore,

$$F_{B,N,n} = \frac{(\tau_N + B - 1)\tau_N^{n-1} + (\tau_N - B)(1 - \tau_N)^{n-1}}{2\tau_N - 1} \tag{17}$$

$$= \frac{(\tau_N + B - 1)(\tau_N - 1)\tau_N^n - (\tau_N - B)\tau_N(1 - \tau_N)^n}{N(2\tau_N - 1)}$$

(because $\tau_N(\tau_N - 1) = N$).

It is perhaps worthwhile recasting (17) without τ_N. In so doing we write $\sqrt{N + 1/4} = R$, and obtain

$$F_{B,N,n} = \frac{[N - (B - 1)(1/2 - R)](1/2 + r)^n - [N - (B - 1)(1/2 + R)](1/2 - R)^n}{2NR} . \tag{18}$$

It is here to be noted that, in particular,

$$F_{1,N,n} = \frac{(1/2 + R)^n - (1/2 - R)^n}{2R} , \tag{19}$$

which, with $N = 1$, yields the established explicit formula for a member of the original Fibonacci sequence. And, again,

$$F_{3,N,n} = (1/2 + R)^n + (1/2 - R)^n , \tag{20}$$

which, with $N = 1$, yields the established explicit formula for a member of the Lucas sequence.

5. SOME IDENTITIES

Our topic is rich in interesting identities, and in this section a few of the more important ones will be set out together with their degeneralizations to more familiar forms. We omit proofs, which can be constructed on traditional (and mostly inductional) lines — many exercises and problems can in fact be drawn from the statements.

One of the simplest and most revealing of the identities, an almost obvious consequence of expression (1), is

$$F_{B,N,n} = NF_{1,N,n-2} + BF_{1,N,n-1} . \tag{21}$$

An allied identity is

$$F_{B,N,n} = XF_{1,N,n-1} + F_{B-X,N,n} \tag{22}$$

with the special case in which $X = B - 1$:

$$F_{B,N,n} = (B - 1)F_{1,N,n-1} + F_{1,N,n} . \tag{23}$$

Summations of terms and powers of terms are often neatly expressible. For example:

(24) $$\sum_{i=1}^{n} F_{B,N,i} = (F_{B,N,n+2} - B)/N$$

and its relation to the familiar $\{1,1\}$ is plain to see.

The sum of squares to a given $\underline{n}$ can be eompactly expressed for $N = 1$:

(25) $$\sum_{i=1}^{n} F^2_{B,1,i} = F_{B,1,n}F_{B,1,n+1} - (B - 1)$$

but less so for $B = 1$:

(26) $$\sum_{i=1}^{n} F^2_{1,N,i} = \frac{N^3F^2_{1,N,n-1} + N(N^2 - N - 1)F^2_{1,N,n} - F^2_{1,N,n+1} - (N - 1)}{N(N + 1)(N - 2)}$$

which, with $N = 1$, becomes

$$= (F^2_{1,1,n+1} + F^2_{1,1,n} - F^2_{1,1,n-1})/2 = F_{1,1,n}F_{1,1,n+1} .$$

A central identity, with several useful reductions, is

(27) $$F_{B,N,n}F_{B,N,n+x+y} - F_{B,N,n+x}F_{B,N,n+y} = (-1)^nN^{n-1}F_{1,N,x}F_{1,N,y}(B^2 - B - N) .$$

Setting $y = -x$, and bearing in mind that $F_{1,N,-n} = (-1)^{n-1}N^nF_{1,N,n}$, we can reduce (27) to

(28) $$F^2_{B,N,n} - F_{B,N,n-x}F_{B,N,n+x} = (-1)^{n+x-1}N^{n-x-1}F^2_{1,N,x}(B^2 - B - N) .$$

And setting $x = -y = 1$ gives us

(29) $$F^2_{B,N,n} - F_{B,N,n-1}F_{B,N,n+1} = (-1)^nN^{n-2}(B^2 - B - N) .$$

Lastly, as regards reduction of (27), if we set $x = y = n' - 1$, and $n = 1$, we obtain (after depriming n'):

(30) $$F^2_{B,N,n} - F_{B,N,2n-1} = F^2_{1,N,n-1}(B^2 - B - N)$$

(and this, when $B = N = 1$, becomes the well known two-consecutive-square identity in $\{1,1\}$).

A general "adjacent products" identity is

(31) $$F_{B,N,n+x} = NF_{B,N,n-1}F_{B,N,x} + F_{B,N,n}F_{B,N,x+1} - (B - 1)F_{B,N,n+x-1}$$

which, when $x = n$, can be expressed in several forms:

$$
\begin{aligned}
F_{B,N,2n} &= F_{B,N,n}(NF_{B,N,n-1} + F_{B,N,n+1}) - (B - 1)F_{B,N,2n-1} \\
&= F^2_{B,N,n+1} - N^2F^2_{B,N,n-1} - (B - 1)F_{B,N,2n-1} \qquad (32)\\
&= 2F_{B,N,n}F_{B,N,n+1} - F^2_{B,N,n} - (B - 1)F_{B,N,2n-1}
\end{aligned}
$$

(and from the first of which we readily infer that iff $B = 1$, then $F_{B,N,2n}$ must be composite (being divisible by $F_{B,N,n}$)).

If, in (31), we put $x = 2n$, the result is

$$
(33) \qquad F_{B,N,3n} = NF_{B,N,n-1}F_{B,N,2n} + F_{B,N,n}F_{B,N,2n+1} - (B - 1)F_{B,N,3n-3} \, .
$$

And here are two cubic relations that apply when B is unity:

$$
\begin{aligned}
F_{1,N,3n} &= 3NF_{1,N,n-1}F_{1,N,n}F_{1,N,n+1} + (N + 1)F^3_{1,N,n} \qquad (34)\\
&= F^3_{1,N,n+1} + NF^3_{1,N,n} - N^3F^3_{1,N,n-1}
\end{aligned}
$$

— the former of which, incidentally, tells us that $F_{1,N,0}$ (mod 3) is always composite.

6. SOME MISCELLANEOUS POINTS

1. In Section 2, it is mentioned that real $N < -1/4$ is out of court, so to say. The reason is that the discriminant of the roots of the generalized Fibonacci quadratic is zero at $N = -1/4$, and negative beyond. At $N = -1/4$ we have that $F_{1,N,n} = n/2^{n-1}$, so that

$$
\tau_N = [\lim, n \to \infty](n + 1)/2n = 1/2 \, .
$$

At $N < -1/4$ the terms of the sequence take alternating algebraic signs, and there is no limiting ratio in the usual sense; what happens of course is that τ_N moves onto the gaussian plane.

2. The number-theoretic properties of $\{B, N\}$ need examination. It seems clear that the main theorems of divisibility and primality [3] applicable to $\{1, 1\}$ also apply, mutatis mutandis, to $\{1, N\}$. And squares are rare among the F's in the Appendix (outside of $\{1, 1\}$, in which it is known that only $F_{1,1,12}$ is a square, and beyond $F_{B,N,4}$) I find only $F_{1,4,8} = 441$, and $F_{1,8,6} = 225$. (Note that $X(X - 1), X$, which is a sequence of powers, contains an infinity of squares, but this is an oddity.)

Interesting problems in this area take the form: In how many ways, if at all, can a given natural number be represented as $F_{B,N,n}$?

3. The digits of a Fibonacci number, at a given decimal place, occur in cycles along the ascending sequence. Lagrange, says Coxeter [1], observed that the final digits of $\{1, 1\}$

repeat in cycles of 60. The question naturally arises as to the cycling pattern of other $\{B, N\}$. The answer is in Table 3.

Table 3
CYCLE SIZE OF REPEATED FINAL DIGITS IN $\{B, N\}$ (EXCLUDING $F_{B,N,1}$)

B mod 5 \ N mod 10	0	1	2	3	4	5	6	7	8	9
0, 1, and 2	1	60	4	24	6	3	20	12	24	6
3	1	12	4	24	6	3	4	12	24	6
4	1	60	2	24	6	3	20	6	24	6

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1967, p. 168.
2. V. E. Hoggatt, Jr., and D. A. Lind, "The Dying Rabbit Problem," Fibonacci Quarterly, Vol. 7, No. 4 (1969), pp. 482-487.
3. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.

APPENDIX

VARIOUS $F_{B,N,n}$ TO $n = 25$

The tables appear on the following pages.

CONFERENCE PROGRAM
FIBONACCI ASSOCIATION MEETING

Saturday, October 21, 1972 San Jose State University, Macquarrie Hall

9:15 a.m.	Registration
9:30 - 10:20	SOME QUASI-EXOTIC THEOREMS Dmitri Thoro, Professor of Mathematics, San Jose State University
10:30 - 11:20	GENERALIZED LEO MOSER PROBLEMS Pat Gomez, Student, San Jose State University
11:30 - 12:00	FUN WITH FIBONACCI AT THE CHESS MATCH AND THE BALL PARK Marjorie Bicknell, Mathematics Teacher, A. C. Wilcox High School
1:30 - 2:20	INTERVALS CONTAINING INFINITELY MANY SETS OF ALGEBRAIC INTEGERS — Raphael Robinson, Professor of Mathematics, University of California, Berkeley
2:30 - 3:20	SOME ADDITION THEOREMS IN NUMBER THEORY C. T. Long, Professor of Mathematics, Washington State University, Visiting University of British Columbia
3:30 - 4:10	SOME CONGRUENCES OF THE FIBONACCI NUMBERS MODULO A PRIME, V. E. Hoggatt, Jr., San Jose State University

LINEARLY GENERALIZED FIBONACCI NUMBERS
$F_{B,N,n}$ WITH B = 1

n \ N	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1
3	2	3	4	5	6	7	8	9	10	11
4	3	5	7	9	11	13	15	17	19	21
5	5	11	19	29	41	55	71	89	109	131
6	8	21	40	65	96	133	176	225	280	341
7	13	43	97	181	301	463	673	937	1 261	1 651
8	21	85	217	441	781	1 261	1 905	2 737	3 781	5 061
9	34	171	508	1 165	2 286	4 039	6 616	10 233	15 130	21 571
10	55	341	1 159	2 929	6 191	11 605	19 951	32 129	49 159	72 181
11	89	683	2 683	7 589	17 621	35 839	66 263	113 993	185 329	287 891
12	144	1 365	6 160	19 305	48 576	105 469	205 920	371 025	627 760	1 009 701
13	233	2 731	14 209	49 661	136 681	320 503	669 761	1 282 969	2 295 721	3 888 611
14	377	5 461	32 689	126 881	379 561	953 317	2 111 201	4 251 169	7 945 561	13 985 621
15	610	10 923	75 316	325 525	1 062 966	2 876 335	6 799 528	14 514 921	28 607 050	52 871 731
16	987	21 845	173 383	833 049	2 960 771	8 596 237	21 577 935	48 524 273	100 117 099	192 727 941
17	1 597	43 691	399 331	2 135 149	8 275 601	25 854 247	69 174 631	164 643 641	357 580 549	721 445 251
18	2 584	87 381	919 480	5 467 345	23 079 456	77 431 669	220 220 176	552 837 825	1 258 634 440	2 648 724 661
19	4 181	174 763	2 117 473	14 007 941	64 457 461	232 557 151	704 442 593	1 869 986 953	4 476 859 381	9 863 177 171
20	6 765	349 525	4 875 913	35 877 321	179 854 741	697 147 165	2 245 983 825	6 292 689 553	15 804 569 341	36 350 423 781
21	10 946	699 051	11 228 332	91 909 085	502 142 046	2 092 490 071	7 177 081 976	21 252 585 177	56 096 303 770	134 982 195 491
22	17 711	1 398 101	25 856 071	235 418 369	1 401 415 751	6 275 373 061	22 898 968 751	71 594 101 601	198 337 427 839	498 486 433 301
23	28 657	2 796 203	59 541 067	603 054 709	3 912 125 981	18 830 313 487	73 138 542 583	241 614 783 017	703 204 161 769	1 848 308 388 211
24	46 368	5 592 405	137 109 280	1 544 728 185	10 919 204 736	56 482 551 853	233 431 323 840	814 367 595 825	2 488 241 012 320	6 833 172 721 221
25	75 025	11 184 811	315 732 481	3 956 947 021	30 479 834 641	169 464 432 775	745 401 121 921	2 747 285 859 961	8 817 078 468 241	25 316 256 603 331

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 2

n \ N	3	5	7	9
1	1	1	1	1
2	2	2	2	2
3	5	7	9	11
4	11	17	23	29
5	26	52	86	128
6	59	137	247	389
7	137	397	849	1 541
8	314	1 082	2 578	5 042
9	725	3 067	8 521	18 911
10	1 667	8 477	26 567	64 289
11	3 824	23 812	86 214	234 488
12	8 843	66 197	272 183	813 089
13	20 369	185 257	875 681	2 923 481
14	46 898	516 242	2 780 692	10 241 282
15	108 005	1 442 527	8 910 729	36 552 611
16	248 699	4 023 737	28 377 463	128 724 149
17	572 714	11 236 372	90 752 566	457 697 648
18	1 318 811	31 355 057	289 394 807	1 616 214 989
19	3 036 953	87 536 917	924 662 769	5 735 493 821
20	6 993 386	244 312 202	2 950 426 418	20 281 428 722
21	16 104 245	681 996 787	9 423 065 801	71 900 873 111
22	37 084 403	1 903 557 797	30 076 050 727	254 433 731 609
23	85 397 138	5 313 541 732	96 037 511 334	901 541 589 608
24	196 650 347	14 831 330 717	306 569 866 423	3 191 445 174 089
25	452 841 761	41 399 039 377	978 832 445 761	11 305 319 480 561

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 3

n \ N	1	4	5	7	8	10
1	1	1	1	1	1	1
2	3	3	3	3	3	3
3	4	7	8	10	11	13
4	7	19	23	31	35	43
5	11	47	63	101	123	173
6	18	123	178	318	403	603
7	29	311	493	1 025	1 387	2 333
8	47	803	1 383	3 251	4 611	8 363
9	76	2 047	3 848	10 426	15 707	31 693
10	123	5 259	10 763	33 183	52 595	115 323
11	199	13 447	30 003	106 165	178 251	432 253
12	322	34 483	83 818	338 446	599 011	1 585 483
13	521	88 271	233 833	1 081 601	2 025 119	5 908 013
14	843	226 203	652 923	3 450 723	6 817 107	21 762 843
15	1 364	579 287	1 822 088	11 021 930	23 017 259	80 842 973
16	2 207	1 484 099	5 086 703	35 176 991	77 554 115	298 471 403
17	3 571	3 801 247	14 197 143	112 330 501	261 692 187	1 106 901 133
18	5 778	9 737 643	39 630 658	358 569 438	882 125 107	4 091 615 163
19	9 349	24 942 631	110 616 373	1 144 882 945	2 975 662 603	15 160 626 493
20	15 127	63 893 203	308 769 663	3 654 869 011	10 032 663 459	56 076 778 123
21	24 476	163 663 727	861 851 528	11 669 049 626	33 837 964 283	207 683 043 053
22	39 603	419 236 539	2 405 699 843	37 253 132 703	114 099 271 955	768 450 824 283
23	64 079	1 073 891 447	6 714 957 483	118 936 480 085	384 802 986 219	2 845 281 254 813
24	103 682	2 750 827 603	18 743 456 698	379 708 409 006	1 297 597 161 859	10 529 789 497 643
25	167 761	7 046 403 391	52 318 244 113	1 212 263 769 601	4 376 021 251 611	38 982 602 045 773

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 4

n \ N	1	2	5	6	7	9	10
1	1	1	1	1	1	1	1
2	4	4	4	4	4	4	4
3	5	6	9	10	11	13	14
4	9	14	29	34	39	49	54
5	14	26	74	94	116	166	194
6	23	54	219	298	389	607	734
7	37	106	589	862	1 201	2 101	2 674
8	60	214	1 684	2 650	3 924	7 564	10 014
9	97	426	4 629	7 822	12 331	26 473	36 754
10	157	854	13 049	23 722	39 799	94 549	136 894
11	254	1 706	36 194	70 654	126 116	332 806	504 434
12	411	3 414	101 439	212 986	404 709	1 183 747	1 873 374
13	665	6 826	282 409	636 910	1 287 521	4 179 001	6 917 714
14	1 076	13 654	789 604	1 914 826	4 120 484	14 832 724	25 651 454
15	1 741	27 306	2 201 649	5 736 286	13 133 131	52 443 733	94 828 594
16	2 817	54 614	6 149 669	17 225 242	41 976 519	185 938 249	351 343 134
17	4 558	109 226	17 157 914	51 642 958	133 908 436	657 931 846	1 299 629 074
18	7 375	218 454	47 906 259	154 994 410	427 744 079	2 331 376 087	4 813 060 414
19	11 933	436 906	133 695 829	464 852 158	1 365 103 121	8 252 762 701	17 809 351 154
20	19 308	873 814	373 227 124	1 394 818 618	4 359 311 604	29 235 147 484	65 939 955 294
21	31 241	1 747 626	1 041 706 269	4 183 931 566	13 915 033 451	103 510 011 793	244 033 466 834
22	50 549	3 495 254	2 907 841 889	12 552 843 274	44 430 214 679	366 626 339 149	903 433 019 774
23	81 790	6 990 506	8 116 373 234	37 656 432 670	141 835 448 836	1 298 216 445 286	3 343 767 688 114
24	132 339	13 981 014	22 655 582 679	112 973 492 314	452 846 951 589	4 597 853 497 627	12 378 097 885 584
25	214 129	27 962 026	63 237 448 849	338 912 088 334	1 445 695 093 441	16 281 801 505 201	45 815 774 766 994

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 5

n \ N	1	2	3	6	7	8	9
1	1	1	1	1	1	1	1
2	5	5	5	5	5	5	5
3	6	7	8	11	12	13	14
4	11	17	23	41	47	53	59
5	17	31	47	107	131	157	185
6	28	65	116	353	460	581	716
7	45	127	257	995	1 377	1 837	2 381
8	73	257	605	3 113	4 597	6 485	8 825
9	118	511	1 376	9 083	14 236	21 181	30 254
10	191	1 025	3 191	27 761	46 415	73 061	109 769
11	309	2 047	7 319	82 259	146 067	242 509	381 965
12	500	4 097	16 892	248 825	470 972	826 997	1 369 076
13	809	8 191	38 849	742 379	1 493 441	2 767 069	4 806 761
14	1 309	16 385	89 525	2 235 329	4 790 245	9 383 045	17 128 445
15	2 118	32 767	206 072	6 689 603	15 244 332	31 519 597	60 389 294
16	3 427	65 537	474 647	20 101 577	48 776 047	106 583 957	214 545 299
17	5 545	131 071	1 092 863	60 239 195	155 486 371	358 740 733	758 048 945
18	8 972	262 145	2 516 804	180 848 657	496 918 700	1 211 412 389	2 688 956 636
19	14 517	524 287	5 795 393	542 283 827	1 585 323 297	4 081 338 253	9 511 397 141
20	23 489	1 048 577	13 345 805	1 627 375 769	5 063 754 197	13 772 637 365	33 712 006 865
21	38 006	2 097 151	30 731 984	4 881 078 731	16 161 017 276	46 423 343 389	119 314 581 134
22	61 495	4 194 305	70 769 399	14 645 333 345	51 607 296 655	156 604 442 309	422 722 642 919
23	99 501	8 388 607	162 965 351	43 931 805 731	164 734 417 587	527 991 189 421	1 496 553 873 125
24	160 996	16 777 217	375 273 548	131 803 805 801	525 985 494 172	1 780 826 727 893	5 301 057 659 396
25	260 497	33 554 431	864 169 601	395 394 640 187	1 679 126 417 281	6 004 756 243 261	18 770 042 517 521

ON THE LENGTH OF THE EUCLIDEAN ALGORITHM

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Throughout this article let a and b be integers, $a > b > 0$. The Euclidean algorithm generates finite sequences of nonnegative integers,

$$\{q_j\}_{j=1}^{n} \qquad \text{and} \qquad \{r_j\}_{j=1}^{n}$$

such that

$$\begin{aligned}
a &= q_1 b + r_1, & 0 &< r_1 < b\ ,\\
b &= q_2 r_1 + r_2, & 0 &< r_2 < r_1\ ,\\
r_1 &= q_3 r_2 + r_3, & 0 &< r_3 < r_2\ ,\\
&\cdots\\
r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1}, & 0 &< r_{n-1} < r_{n-2}\\
r_{n-2} &= q_n r_{n-1} + r_n, & r_n &= 0 \qquad .
\end{aligned} \tag{1}$$

The integers r_{n-1} is the greatest common divisor of a and b and $q_n \geq 2$.

Define $\ell(a,b)$ to be the number of divisions n in the algorithm (1). Some basic properties of $\ell(a,b)$ are

(i) $$\ell(a,a) = 1\ ;$$

(ii) $$\ell(ac,bc) = \ell(a,b), \qquad c > 0\ ;$$

(iii) $$\ell(a+b,b) = \ell(a,b)\ ;$$

(iv) $$\ell(a+b,a) = 1 + \ell(a,b)\ .$$

Each of these properties is proved directly from the definition (1). Property (ii) permits us to assume a and b are relatively prime.

This paper is concerned with maximizing $\ell(a,b)$ when the integers a and b are drawn from certain subclasses of positive integers. There are some classical results in this direction such as the theorem of Lamé [3, p. 43] which states that $\ell(a,b)$ is never greater than five times the number of digits in b. We begin with a known result, the proof of which is instrumental for the justification of the main theorem of the paper.

<u>Theorem 1.</u> Let $\{F_j\}$ be the Fibonacci sequence generated by

(2) $$F_{j+2} = F_{j+1} + F_j, \quad F_{-1} = 0, \quad F_0 = 1 \quad (j = -1, 0, 1, 2, \cdots)\ .$$

Editorial note: This is not our standard Fibonacci sequence.

If $a < F_{m+1}$ or $b < F_m$ for some integer $m > 0$, then $\ell(a,b) < \ell(F_{m+1}, F_m) = m$.

Proof. From (1) the rational number a/b has a continued fraction expansion

$$\text{(3)} \qquad \frac{a}{b} = q_1 + \frac{1}{q_2} + \frac{1}{q_3} + \cdots + \frac{1}{q_n}, \qquad 0 < q_j \quad (1 \le j < n), \qquad q_n \ge 2\,.$$

The k^{th} numerator A_k and the k^{th} denominator B_k of this continued fraction are determined from the equations

$$\text{(4)} \qquad A_k = q_k A_{k-1} + A_{k-2}, \qquad B_k = q_k B_{k-1} + B_{k-2} \qquad (k = 1, 2, \cdots, n)\,,$$

where

$$A_0 = 1, \ B_0 = 0, \ A_1 = q_1, \quad B_1 = 1 \quad [2, \text{p. } 3]\,.$$

Since $q_k > 0$ for each index $k \le n$, it follows from (4) that

$$A_k > A_{k-1}, \qquad B_k > B_{k-1} \qquad (k = 2, 3, \cdots, n)\,.$$

Moreover, by (1) and (4) we have $a \ge A_n$, $b \ge B_n$.

Suppose a and b are integers for which $n = \ell(a,b) \ge m$. Since $q_k \ge 1$ $(1 \le k \le n)$, we have $A_0 = F_0$, $A_1 \ge F_1$, $A_2 \ge F_1 + F_0 = F_2$, and, in general,

$$A_k \ge A_{k-1} + A_{k-2} \ge F_{k-1} + F_{k-2} = F_k \qquad (1 < k < n)\,.$$

Finally, since $q_n \ge 2$, we have by (2)

$$A_n \ge 2A_{n-1} + A_n \ge 2F_{n-1} + F_{n-2} = F_{n-1} + F_n = F_{n+1}\,.$$

Similarly, $B_k \ge F_{k-1}$ $(1 \le k < n)$ and $B_n \ge F_n$. Furthermore, $A_n = F_{n+1}$ if and only if $q_k = 1$ $(1 \le k < n)$, $q_n = 2$ and $B_n = F_n$ if and only if $q_k = 1$ $(1 < k < n)$, $q_n = 2$. Since $a \ge A_n \ge F_{n+1} \ge F_{m+1}$ and $b \ge B_n \ge F_n \ge F_m$, we have the contrapositive of the first part of the implication in the statement of the theorem proved. The fact that $\ell(F_{m+1}, F_m) = m$ is a consequence of the statements concerning equality of A_m and B_m with F_{m+1} and F_m, respectively [1].

The ordered pairs of integers (a,b) can be partially ordered by defining $(a,b)\alpha(a',b')$ if $a \le a'$ and $b \le b'$. Relative to this partial order, the theorem states, in particular, that (F_{m+1}, F_m) is the "first" pair for which $\ell(a,b) = m$, i.e., if $(a', b')\alpha(F_{m+1}, F_m)$, then $\ell(a',b') < \ell(F_{m+1}, F_m)$ unless $a' = F_{m+1}$ and $b' = F_m$.

The proofs of our next results are dependent on the following known lemma.

Lemma 1. $\qquad F_{p+q} = F_p F_q + F_{p-1} F_{q-1} \quad (p, q = 1, 2, \cdots)\,.$

Proof. Set $S_{p,q} = F_p F_q + F_{p-1} F_{q-1}$. Then by (2)

$$S_{p,q} = (F_{p-1} + F_{p-2})F_q + F_{p-1}F_{q-1} = F_{p-1}F_{q+1} + F_{p-2}F_q = S_{p-1,q+1}\,.$$

Repeated application of this identity yields

$$S_{p,q} = S_{1,q+p-1} = F_1F_{p+q-1} + F_0F_{p+q-2} = F_{p+q} .$$

Corollary (Lamé). If m is the number of digits in the integer b, then $\ell(a,b) \le 5m$.

Proof. We first show $F_{5n+1} > 10^n$ by induction. For $n = 1$, $F_6 = 13 > 10$. If the inequality is valid for an integer n, then by Lemma 1

$$F_{5n+6} = F_{5n+1}F_5 + F_{5n}F_4 > 8 \cdot 10^n + \frac{5}{2}10^n = \frac{21}{2}10^n > 10^{n+1}$$

since

$$F_{5n} > \frac{1}{2}F_{5n+1} .$$

Thus, the inequality is valid for all integers.

Now if b has m digits, then $b < 10^m$ and, hence, $b < F_{5m+1}$. By Theorem 1 it follows that $\ell(a,b) < 5m + 1$ and Lamé theorem is proved.

It is interesting to observe that equality is possible in Lamé theorem if $b < 10^3$. If b has four digits, then $b < F_{20} = 10946$ and, by Theorem 1, $\ell(a,b) < \ell(F_{21}, F_{20}) = 20$. More generally, equality cannot hold in the Corollary for $m > 3$. Indeed, by Lemma 1 and the argument used in the proof of the corollary, we have $F_p > 10^k$ implies $F_{p+5} > 10^{k+1}$. Since $F_{20} > 10^4$, it follows that $F_{5m} > 10^m$ for $m \ge 4$. If $b < 10^m$ $(m \ge 4)$, then

$$\ell(a,b) < \ell(F_{5m+1}, F_{5m}) = 5m .$$

The next problem considered in this article pertains to the number of distinct pairs (a,b) such that

$$(F_{m+1}, F_m)\alpha(a,b)\alpha(F_{m+2}, F_{m+1})$$

and $\ell(a,b) = m$. We prove there are $m + 1$ such pairs and obtain formulas for the integers a and b that comprise the pairs. It is convenient to establish these results from a sequence of lemmas.

Lemma 2. Let the Euclidean algorithm for a and b, a and b are relatively prime, be (1) where for some integer m $(1 < m < n)$ - $q_m = 2$ and $q_k = 1$ $(k \ne m,\ 1 \le k < n)$, $q_n = 2$. Then

$$a = F_{n+1} + F_{n-m+1}F_{m-1}$$

and

$$b = F_n + F_{n-m+1}F_{m-2} .$$

Moreover, $(a,b)\alpha(F_{n+2}, F_{n+1})$.

Proof. From the proof of Theorem 1, we have that the k^{th} numerator and denominator of the continued fraction expansion for a/b when $\ell(a,b) = n$ satisfy, for $k < m$, the conditions $A_k = F_k$, $B_k = F_{k-1}$. From this fact and (4), we have

$$A_m = 2F_{m-1} + F_{m-2} = F_m + F_{m-1} = F_m + F_0F_{m-1},$$
$$B_m = 2F_{m-2} + F_{m-3} = F_{m-1} + F_{m-2} = F_{m-1} + F_0F_{m-2},$$
$$A_{m+1} = (F_m + F_{m-1}) + F_{m-1} = F_{m+1} + F_1F_{m-1},$$
$$B_{m+1} = (F_{m-1} + F_{m-2}) + F_{m-2} = F_m + F_1F_{m-2}.$$

Thus, by induction, we obtain

$$A_{n-1} = F_{n-1} + F_{m-1}F_{n-m-1},$$
$$B_{n-1} = F_{n-2} + F_{m-2}F_{n-m-1}.$$

Finally, by (4) and these formulas,

$$A_n = 2F_{n-1} + F_{n-2} + (2F_{n-m+1} + F_{n-m-2})F_{m-1} = F_{n+1} + F_{n-m+1}F_{m-1}$$

and, similarly, $B_n = F_n + F_{n-m+1}F_{m-2}$. Therefore, $a = A_n$ and $b = B_n$ and the first part of the lemma is proved.

Next, by Lemma 1, it follows that

$$F_{n+1} < A_n = F_{n+1} + F_{n-m+1}F_{m-1} = F_{n+1} + F_n - F_{n-m}F_{m-2} < F_{n+2}$$

and, similarly, $F_n < B_n < F_{n+1}$.

This lemma gives us $n - 2$ pairs $(m = 2, 3, \cdots, n-1)$ of integers (a,b) such that

$$F_{n+1} < a < F_{n+2}, \quad F_n < b < F_{n+1},$$

and $\ell(a,b) = n$. Since $\ell(F_{n+1}, F_n)$ and

$$\ell(F_{n+2}, F_n) = \ell(F_{n+1} + F_n, F_n) = \ell(F_{n+1}, F_n) = n,$$

there are so far n pairs in the range

$$(F_{n+1}, F_n)\alpha(a,b)\alpha(F_{n+2}, F_{n+1})$$

for which $\ell(a,b) = n$. The fact that there exists only one additional such pair is proved by the next two lemmas.

Lemma 3. Let $q_k = 1$ $(k = 1, 2, \cdots, n-1)$, $q_n = 3$ in the Euclidean algorithm (1) for the relatively prime integers a and b. Then

$$a = F_{n+1} + F_{n-1}, \qquad b = F_n + F_{n-2},$$

and

$$(F_{n+1}, F_n)\alpha(a, b)\alpha(F_{n+2}, F_{n+1}).$$

If $q_k \geq 1$ $(k = 1, 2, \cdots, n-1)$, $q_n > 3$, then the corresponding integers a and b obey the inequalities $a > F_{n+2}$ and $b > F_{n+1}$.

Proof. From the proof of Theorem 1, we have $A_{n-1} = F_{n-1}$ and $B_{n-1} = F_{n-2}$ when $q_k = 1$ $(1 \leq k < n)$. If $q_n = 3$, then by (4),

$$A_n = 3F_{n-1} + F_{n-2} = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}$$

and, similarly, $B_n = F_n + F_{n-2}$. Since $F_{n-2} < F_{n-1} < F_n$, we have

$$a = A_n < F_{n+1} + F_n = F_{n+2}$$

and

$$b = B_n < F_n + F_{n-1} = F_{n+1}.$$

Next, if $q_k \geq 1$ $(1 \leq k < n)$ and $q_n \geq 4$, we have $A_{n-1} \geq F_{n-1}$ and $B_{n-1} \geq F_{n-2}$. By (4)

$$\begin{aligned} a = A_n &\geq 4A_{n-1} + A_{n-2} \geq 4F_{n-1} + F_{n-2} \\ &= F_{n+1} + 2F_{n-1} > F_{n+1} + F_n = F_{n+2}. \end{aligned}$$

Similarly, $b = B_n > F_{n+1}$.

Lemma 4. Let the Euclidean algorithm for the integers a and b be (1), where $q_k \geq 2$ for at least three indices k or where $q_p \geq 2$, $q_m \geq 3$ for $1 \leq p, m \leq n$, $p \neq m$. Then $a > F_{n+2}$.

Proof. Let $q_k \geq 2$ for $k = m, p$ $(1 \leq m < p < n)$. Then, paralleling the proof of Lemma 2, we obtain

$$a \geq A_n \geq F_{n+1} + F_{n-m+1}F_{m-1} + F_{n-p+1}F_{p-1}. \tag{5}$$

Now the last expression is greater than F_{n+2} provided

$$F_{n-m+1}F_{m-1} + F_{n-p+1}F_{p-1} > F_n. \tag{6}$$

Since

$$F_{n-s+1}F_{s-1} > \frac{1}{2}F_n$$

for $1 \le s \le n$ by Lemma 1, the inequality (6) is valid. We conclude from (5) that

$$a \ge A_n > F_{n+1} + F_n = F_{n+2}.$$

If for some index m, $1 \le m < n$, we have $q_m \ge 3$, then $A_k \ge F_k$ for $k = 1, 2, \cdots, m-1$ and by (4)

$$A_m \ge 3F_{m-1} + F_{m-2} = F_{m+1} + F_{m-1} > F_{m+1},$$
$$A_{m+1} \ge (F_{m+1} + F_{m-1}) + F_{m-1} > F_{m+1} + F_m = F_{m+2}.$$

By induction, $A_k > F_{k+1}$ for $m \le k < n$. Now

$$A_n \ge 2A_{n-1} + A_{n-2} > 2F_n + F_{n-1} = F_{n+2}$$

so $a > F_{n+2}$.

The final case to consider is when $q_m = 2$ for some index m, $1 \le m < n$ and $q_n \ge 3$. As in the proof of Lemma 2, it is easily shown that

$$A_k \ge F_k + F_{m-1} F_{k-m} \qquad (k = m, m+1, \cdots, n-1).$$

Thus,

$$A_n \ge 3A_{n-1} + A_{n-2} \ge 3F_{n-1} + F_{n-2} + (3F_{n-m-1} + F_{n-m-2})F_{m-1}$$
$$\ge F_{n+1} + F_{n-1} + (F_{n-m+1} + F_{n-m-1})F_{m-1} > F_{n+2},$$

provided

$$F_{n-m+1} F_{m-1} + F_{n-m-1} F_{m-1} > F_{n-2}.$$

This is the case since, by Lemma 1,

$$F_{n-s+1} F_{s-1} > \frac{1}{2} F_n$$

for $1 \le s \le n$ and, hence,

$$F_{n-m+1} F_{m-1} + F_{n-m-1} F_{m-1} > \frac{1}{2} (F_n + F_{n-2}) > F_{n-2}.$$

Therefore, $a > F_{n+2}$ in all cases considered in this Lemma.

Collecting the results in the last three lemmas, we have proved the following:

Theorem 2. Let $\mathfrak{A}$ be the set of ordered pairs (a,b) such that $(a,b)\alpha(F_{n+2}, F_{n+1})$. There are exactly $n+1$ pairs in $\mathfrak{A}$ such that $\ell(a,b) = n$. These pairs are obtained from the formulas

$$a = F_{n+1} + F_{n-m+1} F_{m-1}, \qquad b = F_n + F_{n-m+1} F_{m-2}$$

($m = 0, 1, 2, \cdots, n$), where $F_{-2} = F_{-1} = 0$ and F_j for each $j \geq 0$ is the j^{th} Fibonacci number (2).

The results in Theorem 2 were suggested to the authors by considering a number of special cases on an IBM 360/65 computer.

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2. O. Perron, Die Lehre von den Kettenbruchen, Vol. 1, Teubner, Stuttgart, 1954.
3. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, 1939.

LETTERS TO THE EDITOR

Dear Editor:

In the paper (*) by W. A. Al-Salam and A. Verma, "Fibonacci Numbers and Eulerian Polynomials," Fibonacci Quarterly, February 1971, pp. 18-22, an error occurs in (9), which is readily corrected. I will generalize their (4) by defining a general polynomial operator M by

(I) $$Mf(x) = Af(x + c_1) + Bf(x + c_2), \qquad c_1 \neq c_2 ,$$

where $f(x)$ is a polynomial and A, B, c_1, and c_2 are given numbers. With $D = d/dx$, we note that $M = Ae^{c_1D} + Be^{c_2D}$ so that

$$Mf(x) = A\sum_{n=0}^{\infty} \frac{c_1^n}{n!} D^nf(x) + B\sum_{n=0}^{\infty} \frac{c_2^n}{n!} D^nf(x) ,$$

or

(II) $$Af(x + c_1) + Bf(x + c_2) = \sum_{n=0}^{\infty} \frac{W_n}{n!} D^nf(x) ,$$

where $W_n = Ac_1^n + Bc_2^n$ is the solution of $W_{n+2} = PW_{n+1} - QW_n$ and $c_1 \neq c_2$ are the roots of $x^2 = Px - Q$. In (*), Eq. (4) is a special case of (I) with $A = \mu$ and $B = 1 - \mu$. There are two cases of (II) to consider:

Case 1. $A + B \neq 0$. If $A = B$, we obtain from (II)

(III) $$f(x + c_1) + f(x + c_2) = \sum_{n=0}^{\infty} \frac{V_n}{n!} D^nf(x) ,$$

where $V_0 = 2$, $V_1 = P$, and $V_{n+2} = PV_{n+1} - QV_n$. If c_1 and c_2 are roots of $x^2 = x + 1$,

[Continued on page 71.]

ON SUMMATIONS AND EXPANSIONS OF FIBONACCI NUMBERS

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INTRODUCTION

One of the early delights a neophyte in the study of Fibonacci numbers experiences may be an encounter with some elementary summation properties such as

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 .$$

As soon as his curiosity is aroused, he may wish to investigate summations which "skip" a constant number of Fibonacci numbers, for instance the problem of obtaining a formula for the sum of the first n Fibonacci numbers of odd position index.

But — as has often been observed — mathematicians are like lovers: give them the little finger, and they will want the whole hand. Can one find a relationship which spells out the sum of any finite Fibonacci sequence whose subindices follow the pattern of an arithmetic progression?

A SUMMATION THEOREM (Theorem 1)

Seeking a pattern for the sum of a number of equally spaced Fibonacci numbers means a concern with

$$\sum_{i=0}^{n} F_{n_i}, \qquad (n_i = k\,i+r) ,$$

r is a non-negative integer, whereas k is a natural number.

Let us use the Binet formula

$$F_n = \frac{a^n - b^n}{\sqrt{5}} \quad \text{with} \quad a = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{5}}{2} .$$

We also note that $ab = -1$. The n^{th} Lucas number, L_n, is $L_n = a^n + b^n$. Then

$$\sum_{i=0}^{n} F_{n_i}$$

becomes:

$$\frac{1}{\sqrt{5}}\sum_{i=0}^{n}(a^{ik+r} - b^{ik+r}) = \frac{1}{\sqrt{5}}\left[a^r\,\frac{a^{(n+1)k} - 1}{a^k - 1} - b^r\,\frac{b^{(n+1)k} - 1}{b^k - 1}\right]$$

$$= \frac{[a^{(n+1)k+r} - a^r](b^k - 1) - [b^{(n+1)k+r} - b^r](a^k - 1)}{\sqrt{5}\,[(-1)^k + 1 - L_k]} .$$

Performing the indicated operations and again employing the Binet formula, we are ready to give the sum of n Fibonacci numbers beginning with F_r. The sequence continues equally spaced such that $(k - 1)$ Fibonacci numbers are left out from any one term to the next.

Theorem 1.

$$\sum_{i=1}^{n} F_{k(i-1)+r} = \frac{(-1)^k F_{(n-1)k+r} + (-1)^{\min(k,r)+t} F_{|r-k|} - F_{nk+r} + F_r}{(-1)^k + 1 - L_k} ,$$

where k is any natural number and r any non-negative integer. The number t is defined by:

$$t = \begin{cases} 0, & \text{when } r < k \\ 1, & \text{when } r > k \end{cases} .$$

Since $F_{|r-k|}$ vanishes for $r = k$, t need not be defined in this case.

Attention should be drawn to the fact that we may restrict r to the condition $0 \leq r < k$ by the

Reduction Formula: (2)

If $r \equiv \bar{r} \pmod{k}$, i.e.: $r = ak + \bar{r}$ where a is a natural number and $0 \leq \bar{r} < k$, then

$$\sum_{i=1}^{n} F_{(i-1)k+r} = \sum_{i=1}^{n} F_{(a+i-1)k+\bar{r}}$$

$$= \sum_{i=1}^{n+a} F_{(i-1)k+\bar{r}} - \sum_{i=1}^{a} F_{(i-1)k+\bar{r}} .$$

While the restriction on $\bar{r}$ is useful for reduction purposes, it is not a necessary condition for relationship (2).

Special Cases of Theorem 1.

We notice that the result of our summation involves an expression which combines no more than four terms. Thus, this relationship would be quite helpful whenever n is "fairly large." For $r = 0$, the special case

(3)
$$\sum_{i=1}^{n} F_{ki} = \frac{(-1)^k F_{nk} + F_k - F_{(n+1)k}}{(-1)^k + 1 - L_k}$$

may merit attention.

It is evident that Theorem 1 embraces the basic elementary summation formulas of this kind. Obviously, $k = 1$, $r = 0$ yields:

$$\sum_{i=1}^{n} F_i = F_n + F_{n+1} - 1 = F_{n+2} - 1 ,$$

which is the formula we previously quoted for the sum of the first n Fibonacci numbers.

However, it is aesthetically satisfying that the summation formulas for the first n Fibonacci numbers of odd indices and those of even indices also become special cases of our pattern. Thus, by letting $k = 2$ and $r = 0$, we get

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1 ,$$

whereas $r = 1$ yields:

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n} .$$

If one relationship combining the two cases were required, Theorem 1 — for $k = 2$ and $r = 0$ or 1 — becomes:

$$\sum_{i=1}^{n} F_{2(i-1)+r} = F_{2n+2-2} - (-1)^r F_{2-r} - F_r ,$$

or, more simply:

(4)
$$\sum_{i=1}^{n} F_{2(i-1)+r} = F_{2n+r-1} + r - 1 .$$

It may be instructive to check other cases of small "skipping numbers" k. Owing to reduction formula (2), the condition $r < k$ does not limit the generality of the results.

For $k = 3$ we obtain

$$\sum_{i=1}^{n} F_{3(i-1)+r} = \frac{2F_{3n+r-1} - (-1)^r F_{3-r} - F_r}{4} ,$$

which may also be stated as

$$\sum_{i=1}^{n} F_{3(i-1)+r} = \frac{F_{3n+r-1} - |r - 1|}{2} \tag{5}$$

and, for $k = 4$, we have

$$\sum_{i=1}^{n} F_{4(i-1)+r} = \frac{2F_{4n+r-3} + F_{4n+r-2} - (-1)^r F_{4-r} - F_r}{5}$$

or, alternatively:

$$\sum_{i=1}^{n} F_{4(i-1)+r} = F_{2n-2}F_{2n+r} + \left[\frac{r+1}{2}\right] . \tag{6}$$

These equivalences, relationships (5) and (6), may easily be verified by straight substitution of the few r-values to which we are restricted. All of these formulas can, however, readily be established either by using the Binet formula, or else, employing mathematical induction. They were stated here merely as a matter of interest since none of them seem too obvious.

Two further observations may be mentioned.

We might wish to impose the condition $r = k$ on Theorem 1. Then

$$\sum_{i=1}^{n} F_{ik} = \frac{(-1)^k F_{nk} - F_{(n+1)k} + F_k}{(-1)^k + 1 - L_k} . \tag{7}$$

Clearly, the summation formula for the first n Fibonacci numbers of even subindex is a special case of this.

It may also be of interest to note that on the basis of Theorem 1, L_k divides into all sums of our kind, provided k is odd, i.e., the number of Fibonacci numbers "skipped over" in our summation is even. If this number were odd, $(2 - L_k)$ would be a divisor of our sum.

AN EXPANSION THEOREM (Theorem 2)

But hasn't Jacobi advised us: "Man muss immer umkehren" (one must always turn around)? Thus — having obtained summation results as expressions involving Fibonacci numbers — we may now experiment with an inversion and pose the problem: **Can a Fibonacci** number be expanded into a series reminiscent of an expansion for the n^{th} power of a binomial?

Partly analogous to Theorem 1, and primarily for the sake of developing some notions, we symbolize our Fibonacci numbers F_n as F_{km+r}, where all letters represent non-negative integers.

The proposed expansion reads:

Theorem 2.

$$F_n = \sum_{i=0}^{k-1} \binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i F_{m+r-k+i+1}, \qquad (n = km + r).$$

In our proof, we use mathematical induction on n. Symbolizing Theorem 2 by R(n), we readily verify R(n) for the first few natural numbers. Now we need to show that the correctness of R(s - 1) and of R(s) implies correctness of R(s + 1), where s represents any natural number. This means that we investigate whether

$$\binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i [F_{m+r-k+i} + F_{m+r-k+i+1}]$$

equals

$$\binom{k-1}{i} F_m^{k-1-i} F_{m+1}^i F_{m+r-k+i+2}.$$

However, the iterative definition of Fibonacci numbers assures the correctness of this equality and, hence, completes the proof.

As an illustration, we might wish to expand F_{11} by letting $m = 3$ and $r = 2$. We assert that

$$F_{11} = \sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{3+i},$$

which is easily verified.

Special Cases of Theorem 2.

Some special cases might be pointed to. Considering Fibonacci numbers with even subindex, Theorem 2 reduces to:

$$F_n = \sum_{i=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{i} 2^i F_{3-(n/2)+i}. \tag{8}$$

But those of odd subindex may be expanded on the basis of

$$F_n = \sum_{i=0}^{\frac{n-3}{2}} \binom{\frac{n-3}{2}}{i} 2^i F_{(9-n)/2+i}. \tag{9}$$

A Corollary of Theorem 2.

We propose a corollary of expansion formula 2 (Theorem 2) which gives a prescribed number of terms for the expansion. Let the symbol a stand for that number. In our condition $n = km + r$ we stipulate that $m = 1$ and $k = a$, and we obtain:

Corollary of Theorem 2.

$$F_n = \sum_{i=0}^{a-1} \binom{a-1}{i} F_{n+2(1-a)+i} , \tag{10}$$

where

$$2 \le a \le \frac{n+1}{2} .$$

Special Cases of the Corollary:

The following two special cases seem worth mentioning. We desire to let a be the largest possible number.

Case 1:

If n is even, $a = n/2$ is chosen. Then

$$F_n = \sum_{i=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{i} F_{i+2} \tag{11}$$

and there are $n/2$ terms in the expansion.

Case 2:

If n is odd, $a = \frac{n+1}{2}$,

$$F_n = \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} F_{i+1} \tag{12}$$

and the expansion has $\frac{n+1}{2}$ terms.

To illustrate, let us expand F_{21} into a five-term series. Then $n = 21$. Using relationship (10) and letting $a = 5$, we have:

$$F_{21} = \sum_{i=0}^{4} \binom{4}{i} F_{13+i} ,$$

which is correct. For the maximum number of terms in the expansion we would designate a as being 11 and use (12). Then

$$F_{21} = \sum_{i=0}^{10} \binom{10}{i} F_{i+1} ,$$

a relationship which can also be easily verified.

BACK TO ANOTHER SUMMATION THEOREM (Theorem 3)

Once again, we might "invert." Our summation theorem (Theorem 1) gave us an expansion involving Fibonacci numbers as the result of the addition. Now let us give a summation which results in one Fibonacci number. This problem may possibly use Theorem 2 to the best advantage.

Starting with a summation involving Fibonacci numbers of prescribed indices, can we predict the resulting Fibonacci number? Again recalling Jacobi's advice, we reverse the expansion of a given Fibonacci number to a sum. Now designate a sum which leads to a predictable Fibonacci number. Symbolize m by u, and $u + r - (n - r)/u + 1$ by v. Then $r = v - 1 - u + k$ and Theorem 2 becomes:

Theorem 3.

$$\sum_{i=0}^{k-1} \binom{k-1}{i} F_u^{k-1-i} F_{u+1}^i F_{v+i} = F_{(k-1)(u+1)+v}$$

for any arbitrarily chosen natural numbers u and v. The number k may be any integer greater than or equal to 2.

To illustrate this summation idea, we try a summation involving F_4 and F_7. Here we let $u = 4$, and $v = 7$, and get:

$$\sum_{i=0}^{k-1} \binom{k-1}{i} 3^{k-1-i} 5^i F_{7+i} .$$

We predict F_{5k+2} as our result which is correct.

Pre-assigning the Fibonacci Number Resulting from Summation Theorem 3:

Formula 3 is a method for a summation which uses prescribed Fibonacci numbers and predicts a Fibonacci number as the result. What about assigning the resulting Fibonacci number without prescribing Fibonacci numbers involved in the summation?

This summation, not necessarily unique, can be had by considering two cases.

Case 1. The Fibonacci number to be attained has odd subindex n. We choose $u = v = 1$, and have

$$\sum_{i=0}^{k-1} \binom{k-1}{i} F_{i+2} = F_{2k-1} \quad . \tag{13}$$

Case 2. We wish to obtain a Fibonacci number of even subindex. For this purpose we let u and v take on the values 1 and 2, respectively. Here:

$$\sum_{i=0}^{k-2} \binom{k-1}{i} F_{i+2} = F_{2k}. \tag{14}$$

Obviously, the number of terms in these summations will be $(n+1)/2$ for odd subindices n, and $n/2$ for even ones. We realize, however, that our choices for u and v have forfeited the ability to discern the powers of F_u and F_v which characterize the terms of Theorem 3.

Pre-Assigning the Fibonacci Number Resulting from Summation Theorem 3 as well as the Number of Terms in the Summation, and Retaining Generality.

Finally, we prescribe the resulting Fibonacci number F_n as well as k, the number of terms in the summation. Moreover, to avoid the difficulty encountered above, exclude the somewhat trivial cases which involve $F_1 = F_2 = 1$ among the summation terms. We impose the condition: $u, v \geq 3$. Furthermore, the iterative definition of Fibonacci numbers:

$$\sum_{i=0}^{1} F_{n+i} = F_{n+2}$$

inherently provides a summation of two terms resulting in a Fibonacci number (even though the summation is not of our general type). Therefore, impose the condition: $k \geq 3$. Then, for all $n \geq 4k - 1$; i.e., for all $n \geq 11$, we can do justice to our data by assigning appropriate values to u and v such that

$$n = (k-1)(u+1) + v \tag{15}$$

is satisfied. Again, no claim is made for uniqueness.

For example, to obtain F_{11} through a summation of three terms, the following choice proves successful:

$$\sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{3+i} = F_{11}.$$

For a summation of three terms for F_{15}, we can already write:

$$\sum_{i=0}^{2} \binom{2}{i} F_3^{2-i} F_4^i F_{7+i} = \sum_{i=0}^{2} \binom{2}{i} F_4^{2-i} F_5^i F_{5+i} = \sum_{i=0}^{2} \binom{2}{i} F_5^{2-i} F_6^i F_{2+i} = F_{15}.$$

Lack of Uniqueness — Predicting the Number of Different Summations

Can you foretell the number of different summation representations of our type, each having k terms, and leading to the same Fibonacci number F_n? Using relationship (15), our prediction becomes:

If set T is defined by

$$T = \left\{ t : 4 \le t \le \frac{n-3}{k-1} \right\},$$

then the numerosity of T, that is, the number

$$\left[\frac{n-3}{k-1}\right] - 3 \tag{16}$$

predicts the possible number of different summations of our type, each having k terms and leading to the Fibonacci number F_n.

To illustrate, there will be 52 ten-term summations of our kind leading to F_{500}. We would have:

$$\sum_{i=0}^{9} \binom{9}{i} F_{54}^{9-i} F_{55}^{i} F_{5+i} = \sum_{i=0}^{9} \binom{9}{i} F_{53}^{9-i} F_{54}^{i} F_{14+i} = \sum_{i=0}^{9} \binom{9}{i} F_{52}^{9-i} F_{53}^{i} F_{23+i}$$

$$= \cdots = \sum_{i=0}^{9} \binom{9}{i} F_{3}^{9-i} F_{4}^{i} F_{464+i} = F_{500}.$$

[Continued from page 62.]

then $V_n = L_n$, the Lucas sequence, and so (III) now gives the correct expression for (9) in (*).

Case 2. $A + B = 0$. We now obtain from (II)

$$\frac{f(x + c_1) - f(x + c_2)}{c_1 - c_2} = \sum_{n=0}^{\infty} \frac{U_n}{n!} D^n f(x), \tag{IV}$$

where $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = PU_{n+1} - QU_n$. Thus for $P = 1$, $Q = -1$, $U_n = F_n$; and for $P = 2$, $Q = -1$, $U_n = P_n$, the Pell sequence. For $m = 1, 2, \cdots$, we obtain from (IV)

$$\frac{f(x + c_1^m) - f(x + c_2^m)}{c_1 - c_2} = \sum_{n=0}^{\infty} \frac{V_{mn}}{n!} D^n f(x). \tag{V}$$

Remarks. The same ideas in (*) show that the generating function of the moments of the inverse operator

[Continued on page 84.]

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-205 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Evaluate the determinants of n^{th} order

$$D_n = \begin{vmatrix} z & 1 & & & & \\ -1 & qz & 1 & & & \\ & -1 & q^2z & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & q^{n-2}z & 1 \\ & & & & -1 & q^{n-1}z \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} z & 1 & & & \\ -1 & z & q & & \\ & -1 & z & q^2 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -1 & z & q^{n-2} \\ & & & -1 & z \end{vmatrix} .$$

H-206 Proposed by P. Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity:

$$1/(1 - x^n) = \frac{1}{n}\sum_{k=0}^{n-1} 1/(1 - x \cdot e^{2k\pi i/n}) .$$

H-207 Proposed by C. Bridger, Springfield, Illinois.

Define $G_k(x)$ by the relation

$$\frac{1}{1 - (x^2 + 1)s^2 - xs^3} = \sum_{n=0}^{\infty} G_k(x)s^k ,$$

where x is independent of s.

1. Find a recursion formula connecting the $G_k(x)$.

2. Put $x = 1$ and find $G_k(1)$ in terms of Fibonacci numbers.
3. Also with $x = 1$, show that the sum of any four consecutive G-numbers is a Lucas number.

H-208 Proposed by P. Erdos, Budapest, Hungary.

Assume

$$\frac{n!}{a_1!\,a_2!\,\cdots\,a_k!} \qquad (a_i \geq 2, \quad 1 \leq i \leq k)$$

is an integer. Show that the

$$\max \sum_{i=1}^{k} a_i < \frac{5}{2} n ,$$

where the maximum is to be taken with respect to all choices of the a_i's and k.

H-209 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$u_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} ,$$

where $\alpha = \beta = \alpha\beta = z$. Determine the coefficients $C(n,k)$ such that

$$z^n = \sum_{k=1}^{n} C(n,k)u_{n-k+1} \qquad (n \geq 1).$$

H-210 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that a positive integer n is a Lucas number if and only if $5n^2 + 20$ or $5n^2 - 20$ is a square.

H-211 Proposed by S. Krishman, Orissa, India.

A. Show that $\binom{2n}{n}$ is of the form $2n^3k + 2$ when n is prime and $n > 3$.

B. Show that $\binom{2n-2}{n-1}$ is of the form $n^3k - 2n - n$, when n is prime.

$\binom{m}{j}$ represents the binomial coefficient, $\dfrac{m!}{j!\,(m-j)!}$.

H-212 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let n be a positive integer. Consider n edge-connected squares. How many configurations are there if each row starts k squares to the right of the row above? (k denotes a non-negative integer.)

H-213 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

A. Let A_n be the left adjusted Pascal triangle, with n rows and columns and 0's above the main diagonal. Thus

$$A_n = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 \cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{n \times n}$$

Find $A_n \cdot A_n^T$ where A_n^T represents the transpose of matrix, A_n.

B. Let

$$C_n = \begin{pmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & 0 \cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{n \times n}$$

where the i^{th} column of C_n is the i^{th} row of Pascal's triangle adjusted to the main diagonal and the other entries are 0's. Find $C_n \cdot A_n^T$.

H-214 Proposed by E. Karst, University of Arizona, Tucson, Arizona.

Let $x = y^2 + z^2$ be the first prime in a sequence of 10 primes in A.P. and

$$x + 2^2 \cdot 3^4 = (y + 2 \cdot 3^2 \cdot 7)^2 + (z - 2^5 \cdot 3^2)^2$$

the first prime in another sequence of 10 primes in A.P. where both sequences have the same common difference. The second member after the 10^{th} prime in the first sequence is divisible by 17 and has a factor which is the square of a 3-digit prime; the second member before the first prime in the second sequence is also divisible by 17, and its first three digits are a permutation of the last three digits which form a perfect square. The common difference consists of prime factors, each of them smaller than 17. Find x, y, and z.

SOLUTIONS

AN OLD FRIEND REVISITED

H-118 Proposed by G. Ledin, Jr., San Francisco, California.

Solve the difference equation

$$C_{n+2} = F_{n+2}C_{n+1} + C_n \qquad (n \geq 1)$$

with $C_1 = a$, $C_2 = b$, and F_n, the n^{th} Fibonacci number.

Solution by Clyde A. Bridger, Springfield, Illinois.

Write the following series of equations, beginning with $n = 1$,

$$\begin{aligned}
C_3 &= F_3 C_2 + a \\
C_4 &= F_4 C_3 + C_2 \\
C_5 &= F_5 C_4 + C_3 \\
&\vdots \\
C_{n+1} &= F_{n+1} C_n + C_{n-1} \\
C_{n+2} &= F_{n+2} C_{n+1} + C_n
\end{aligned}$$

We see at once that

$$C_3 = F_3 b + a = \begin{vmatrix} b & a \\ -1 & F_3 \end{vmatrix}$$

$$C_4 = F_4(F_3 b + a) + b = \begin{vmatrix} b & a & 0 \\ -1 & F_3 & 1 \\ 0 & -1 & F_4 \end{vmatrix}$$

etc. So the solution in determinant form is

$$C_{n+2} = \begin{vmatrix} b & a & 0 & 0 & \cdots & 0 & 0 \\ -1 & F_3 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & F_4 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & F_5 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot & \cdots & & \cdot \\ \cdot & & \cdot & \cdot & \cdots & & \cdot \\ 0 & 0 & 0 & 0 & \cdots & F_{n+1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & F_{n+2} \end{vmatrix}$$

as may be verified by expanding in terms of the minors of the last row.

The ratio of two adjacent solutions of the difference equation can be developed into a continued fraction. Write, using the above sets of equations,

$$\begin{aligned}
\frac{C_3}{C_2} &= F_3 + \frac{a}{b} \\
\frac{C_4}{C_3} &= F_4 + \frac{1}{C_3/C_2} = F_4 + \cfrac{1}{F_3 + \cfrac{a}{b}} \\
&\vdots \\
\frac{C_{n+2}}{C_{n+1}} &= F_{n+2} + \cfrac{1}{F_{n+1} + \cfrac{1}{F_n + \cfrac{\cdot}{\cdot \; \cdot \; \cfrac{1}{F_3 + \cfrac{a}{b}}}}} \quad .
\end{aligned}$$

Also solved by R. Whitney.

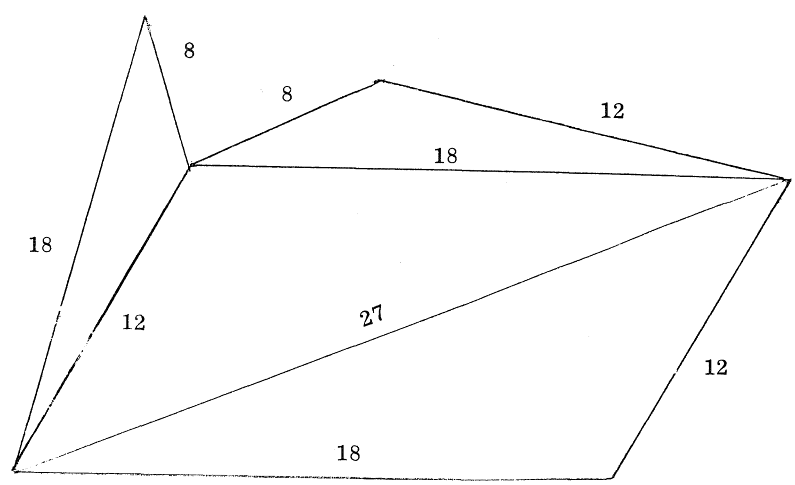
ANOTHER OLD TIMER

H-108 Proposed by H. E. Huntley, Hutton, Somerset, U.K.

Find the sides of a tetrahedron, the faces of which are all scalene triangles similar to each other, and having sides of integral lengths.

Solution by the Proposer.

The interesting article, "Mystery Puzzle and Phi," by Marvin H. Holt (Fibonacci Quarterly, Vol. 3, No. 2, p. 135) contains a solution. See H. E. Huntley's The Divine Proportion, Dover, New York, N. Y., 1970, pp. 108-109, Section entitled "The Tetrahedron Problem."



SHADES OF THE PAST

H-86 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, Calif. (Corrected)

Let p, q be integers such that $p + q \geq 1$, $q \geq 0$; show that if $x^p(x - 1)^q - 1 = 0$ has roots $r_1, r_2, \cdots, r_{p+q}$ and $(x - 1)^{p+q} - x^p = 0$ has roots $s_1, s_2, \cdots, s_{p+q}$ then $s_i^q = r_i^{q+p}$ for $i = 1, 2, \cdots, p + q$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Presumably the problem should read:

Show that if $x^p(x - 1)^q - 1 = 0$ has roots $r_1, r_2, \cdots, r_{p+q}$ and $(y - 1)^{p+q} - y^p = 0$ has roots $s_1, s_2, \cdots, s_{p+q}$, then the roots can be so numbered that

$$r_i^{p+q} = s_i^q \qquad (i = 1, 2, \cdots, p + q) .$$

Proof. Consider the transformation

$$x - 1 = \frac{1}{y - 1} .$$

This implies

$$y = \frac{x}{x - 1} .$$

Hence, if x satisfies $x^p(x - 1)^q = 1$, we get

$$y^q = \frac{x^q}{(x - 1)^q} = \frac{x^{p+q}}{x^p(x - 1)^q} = x^{p+q} .$$

This evidently yields the stated result.

PARTIAL SOLUTION

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with this string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b. P_n, where P_n is the n^{th} prime.

c. $n!$

d. F_n, where F_n is the n^{th} Fibonacci number.

Partial Solution by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

b. The article "Initial Digits for the Sequence of Primes," by R. E. Whitney (Amer. Math. Monthly, Vol. 79, No. 2, 1972, pp. 150-152) established a positive relative logarithmic density for the set of primes with initial digit sequence $\{a_n, a_{n-1}, \cdots, a_1\}$ in the set of primes. Thus P_n is left-normal. On the other hand, no member of P_n ends in "4," so P_n is not right-normal.

I believe that the left-normality of F_n can also be established with a density argument.

Editorial Note

The following list represents those problems for which no solutions have been submitted. Let's fight problem pollution!

H-76, H-84, H-87, H-90, H-91, H-84, H-100, H-110, H-113, H-114, H-115, H-116,

H-122, H-125 (partial), H-130, H-146, H-148, H-152, H-170, H-174, H-179, H-182.

This list represents problems less than or equal to H-185.

NUMBERS COMMON TO TWO POLYGONAL SEQUENCES

DIANNE SMITH LUCAS
China Lake, California

The polygonal sequence (or sequences of polygonal numbers) of order r (where r is an integer, $r \geq 3$) may be defined recursively by

$$(1) \qquad (r,i) = 2(r, i - 1) - (r, i - 2) + r - 2$$

with $(r,0) = 0$, $(r,1) = 1$.

It is possible to obtain a direct formula for (r,i) from (1). A particularly simple way of doing this is via the Gregory interpolation formula. (For an interesting discussion of this formula and its derivation, see [3].) The result is

$$(2) \qquad (r,i) = i + (r - 2)i(i - 1)/2 = [(r - 2)i^2 - (r - 4)i]/2 .$$

It is comforting to note that the "square" numbers — the polygonal numbers of order 4 — actually are the squares of the integers.

Using either (1) or (2), we can take a look at the first few, say, triangular numbers $(r = 3)$

$$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, \cdots .$$

One observation we can make is that three of these numbers are also squares — namely 0, 1, and 36. We can pose the following question: Are there any more of these "triangular-square" numbers? Are there indeed infinitely many of them? What can be said about the numbers common to <u>any</u> pair of polygonal sequences?

We shall begin by answering the last of these questions, and then return to the other two. Suppose that s is an integer common to the polygonal sequences of orders r_1 and r_2 (say $r_1 < r_2$). Then there exist integers p and q such that

$$s = [(r_1 - 2)p^2 - (r_1 - 4)p]/2 = [(r_2 - 2)q^2 - (r_2 - 4)q]/2 ,$$

so that

$$(3) \qquad (r_1 - 2)p^2 - (r_1 - 4)p = (r_2 - 2)q^2 - (r_2 - 4)q ,$$

This paper is based on work done when the author was an undergraduate research participant at Washington State University under NSF Grant GE-6463.

and in fact, since both sides of the equation (3) are always even, every pair of non-negative integers p, q which satisfy (3) determine such an integer s.

As a quadratic in p, this has integral solutions, so — since all coefficients are integers — the discriminant

$$(r_1 - 4)^2 + 4(r_1 - 2)(r_2 - 2)q^2 - 4(r_1 - 2)(r_2 - 4)q$$

must be a perfect square, say x^2, so that

$$x^2 = 4(r_1 - 2)(r_2 - 2)q^2 - 4(r_1 - 2)(r_2 - 4)q + (r_1 - 4)^2 .$$

As a quadratic in q, this also has integral solutions, and the discriminant — and hence $1/16^{\text{th}}$ of the discriminant — must again be a perfect squre, say y^2, so that

$$(4) \qquad y^2 - (r_1 - 2)(r_2 - 2)x^2 = (r_1 - 2)^2(r_2 - 4)^2 - (r_1 - 2)(r_2 - 2)(r_1 - 4)^2 ,$$

where p and q are given by

$$(5) \qquad p = \frac{(r_1 - 4) + x}{2(r_1 - 2)} \qquad q = \frac{(r_1 - 2)(r_2 - 4) + y}{2(r_1 - 2)(r_2 - 2)}$$

Although it can be shown, by solving (5) for x and y and substituting into (4), that every solution of (4) gives a solution of (3), it should be noted that some of the integer solutions of (4) may not give integer values for p and q. Nevertheless, (4) and (5) give us all possible candidates for integer solutions of (3).

Now (4) is in the form of Pell's equation, $y^2 - dx^2 = N$, which has a finite number of integral solutions in x and y if d is a perfect square while N does not vanish. For then the left side can be factored into $(y - ax)(y + ax)$, where a is an integer; and N has only finitely many integral divisors.

So we already have a partial answer to our question. If $(r_1 - 2)(r_2 - 2)$ is a perfect square and the quantity on the right side of (4) is non-zero, we have only finitely many candidates for integers common to the two sequences of orders r_1 and r_2.

On the other hand, if $(r_1 - 2)(r_2 - 2)$ is a perfect square and the right side of (4) is zero, then (4) reduces to a linear equation in x and y:

$$y = \pm\sqrt{(r_1 - 2)(r_2 - 2)}\, x \quad .$$

Since the coefficient of x is an integer, this has infinitely many integral solutions.

An analysis of the right side of (4) reveals that, with $r_1 \neq r_2$, this quantity vanishes only when one of r_1 and r_2 is 3 and the other is 6. In that case, (4) becomes $y^2 - 4x^2 = 0$, or $y = \pm 2x$; and equations (5), with y replaced by $\pm 2x$, become $p = (x - 1)/2$; $q = (1 \pm x)/4$.

At this point it is not too hard to see that for infinitely many integers x, the above equations yield non-negative integral values for both p and q. Therefore, there are infinitely many hexagonal-triangular numbers. In this case, however, we have taken the long way around; for it can be shown directly, using (3), that indeed every hexagonal number is also a triangular number.

It remains for us to investigate what happens when $(r_1 - 2)(r_2 - 2)$ is not a perfect square (and here the right side of (4) is necessarily non-zero). If this is the case, then there are infinitely many positive integral solutions to (4) if there is one such solution [2, p. 146], But in fact we can always exhibit at least one solution — namely $x_1 = r_1$, $y_1 = r_2(r_1 - 2)$ — corresponding to $p = q = 1$. We still have the job, however, of showing that infinitely many of these solutions of (4) give us integer solutions of (3).

Consider the related equation

$$u^2 - (r_1 - 2)(r_2 - 2)v^2 = 1 . \tag{6}$$

With $(r_1 - 2)(r_2 - 2)$ not a perfect square, this has infinitely many integral solutions, generated by

$$u_n + v_n\sqrt{(r_1 - 2)(r_2 - 2)} = (u_1 + v_1\sqrt{(r_1 - 2)(r_2 - 2)})^n ,$$

where u_1, v_1 is the smallest positive solution [2, p. 142]. We obtain u_1, v_1 by inspection. In particular, u_2, v_2, given by

$$u_2 + v_2\sqrt{(r_1 - 2)(r_2 - 2)} = (u_1 + v_1\sqrt{(r_1 - 2)(r_2 - 2)})^2 ,$$

is a solution of (6), and by expanding the right side and comparing coefficients, we get

$$\begin{aligned} u_2 &= u_1^2 + (r_1 - 2)(r_2 - 2)v_1^2 \\ v_2 &= 2u_1v_1 \end{aligned} \tag{7}$$

Now infinitely many (but not necessarily all) of the positive solutions of (4) are given by

$$y_{n+1} + x_{n+1}\sqrt{(r_1 - 2)(r_2 - 2)} = (u_i + v_i\sqrt{(r_1 - 2)(r_2 - 2)})(y_n + x_n\sqrt{(r_1 - 2)(r_2 - 2)}) \tag{8}$$

where u_i, v_i is any positive solution of (6) [2, p. 146], say u_2, v_2. Again comparing coefficients, we get

$$\begin{aligned} y_{n+1} &= u_2y_n + (r_1 - 2)(r_2 - 2)v_2x_n , \\ x_{n+1} &= v_2y_n + u_2x_n , \end{aligned} \tag{9}$$

with the side conditions $x_1 = r_1$, $y_1 = r_2(r_1 - 2)$.

Consider the first of equations (9). This can, by adding a suitable quantity to each side, be changed to

$$y_{n+1} + (r_1 - 2)(r_2 - 4) + (r_1 - 2)(r_2 - 4)(u_2 - 1) = u_2(y_n + (r_1 - 2)(r_2 - 4)) + (r_1 - 2)(r_2 - 2)v_2x_n ,$$

and using (6) and (7), we get

$$y_{n+1} + (r_1 - 2)(r_2 - 4) = u_2(y_n + (r_1 - 2)(r_2 - 4)) + 2(r_1 - 2)(r_2 - 2)u_1v_1x_n - 2(r_1 - 2)^2(r_2 - 2)(r_2 - 4)v_1^2 . \tag{10}$$

Recalling that $y_1 = r_2(r_1 - 2)$, clearly

$$y_1 \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)} ;$$

and letting $n = k$ in (10), we see that if

$$y_k \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)}$$

for some integer k, then each term on the right of (10) is divisible by $2(r_1 - 2)(r_2 - 2)$. Hence the left side of (10) is divisible by this same quantity, and

$$y_{k+1} \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)} .$$

By mathematical induction, and with reference to the second of equations (5), all of the y_n's given by (9) produce positive integral values for q.

Similarly, the second of equations (9) can be transformed into

$$x_{n+1} + (r_1 - 4) + (u_2 - 1)(r_1 - 4) + v_2(r_1 - 2)(r_2 - 4) = v_2(y_n + (r_1 - 2)(r_2 - 4)) + u_2(x_n + (r_1 - 4)),$$

and again using (6) and (7), we get

$$x_{n+1} + (r_1 - 4) = v_2(y_n + (r_1 - 2)(r_2 - 4)) + u_2(x_n + r_1 - 4)) - 2v_1^2(r_1 - 2)(r_2 - 2)(r_1 - 4) - 2u_1v_1(r_1 - 2)(r_2 - 4) . \tag{11}$$

Since

$$y_n \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)}$$

for all n, certainly

$$y_n \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)} .$$

We have that

$$x_1 \equiv -(r_1 - 4) \pmod{2(r_1 - 2)} ,$$

since $x_1 = r_1$; and it can be seen from (11) that if

$$x_k \equiv -(r_1 - 4) \pmod{2(r_1 - 2)}$$

for some integer k, then

$$x_{k+1} \equiv -(r_1 - 4) \pmod{2(r_1 - 2)} .$$

That is, $2(r_1 - 2)$ divides $x_n + (r_1 - 4)$ for every positive integer n.

To summarize, for $(r_1 - 2)(r_2 - 2)$ not a perfect square, we have exhibited (in (9)) infinitely many — but not necessarily all — of the solutions to the Pell-type equation (4); and all of these give positive integral solutions p, q of (3). These, in turn, give integers s which are common to the two polygonal sequences of orders r_1 and r_2.

In view of the above, we can now state the following theorem:

Theorem. Given two distinct integers r_1 and r_2, with $3 \leq r_1 < r_2$, each defining the order of a polygonal sequence, there are infinitely many integers common to both sequences if and only if one of the following is true:

i. $r_1 = 3$ and $r_2 = 6$, or

ii. $(r_1 - 2)(r_2 - 2)$ is not a perfect squre.

In practice, given particular integers r_1 and r_2, we can get all of the solutions of (4) by using at most finitely many equations of the form (8), with a different x_1, y_1 for each one. Some of these equations can be eliminated or modified to leave out those solutions which give non-integer values for either p or q. We may then obtain equations generating all pairs p, q for which $(r_1, p) = (r_2, q)$; and, if desired, finitely many equations generating the numbers s common to the two sequences. The procedure for finding all solutions of (4) is arduous and depends erratically on the actual values of r_1 and r_2. For the general machinery, see G. Chrystal [1, pp. 478-486].

Now we can easily answer our questions about triangular squares. Letting $r_1 = 3$ and $r_2 = 4$, $(r_1 - 2)(r_2 - 2)$ becomes 2, which is not a perfect square. There are, then, infinitely many triangular squares. As a matter of fact, this result has been known for some time. To exhibit these numbers, we note that since the coefficient of q in (3) becomes 0, we can get a formula like (4) by applying the quadratic formula only once. The result is

$$x^2 - 8q^2 = 1$$

or

$$x^2 - 2y^2 = 1 , \tag{12}$$

where $p = (x - 1)/2$ and $q = y/2$. Conveniently enough, (12) is already in the form of (6); and since $x_1 = 3$, $y_1 = 2$ is the smallest positive solution, all non-negative solutions of (12) are given by

$$x_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n \qquad (n = 0, 1, 2, \cdots) . \tag{13}$$

Certainly the "next" solution is given by

$$x_{n+1} + y_{n+1}\sqrt{2} = (x_n + y_n\sqrt{2})(3 + 2\sqrt{2}) ,$$

and by comparing coefficients we get

$$\begin{aligned} x_{n+1} &= 3x_n + 4y_n , \\ y_{n+1} &= 2x_n + 3y_n , \end{aligned} \tag{14}$$

with (from (13)) $x_0 = 1$, $y_0 = 0$.

It follows by induction from (14) that all values of y_n are even non-negative integers, and all x_n's are odd positive integers. Therefore, for any solution p, q of (3) — in non-negative integers and with $r_1 = 3$, $r_2 = 4$ — there exists an n $(n = 0, 1, 2, \cdots)$ such that

$$\begin{aligned} p = p_n &= (x_n - 1)/2 \\ q = q_n &= y_n/2 \end{aligned} \tag{15}$$

where x_n, y_n are given by (14). Furthermore, p_n, q_n given by (15) forms a non-negative integral solution for any n, since the x_n's are always odd and all of the y_n's are even.

All triangular square numbers, then, are given by

$$s_n = (p_n^2 + p_n)/2 = q_n^2 . \tag{16}$$

Solving (14) with $x_0 = 1$, $y_0 = 0$, we get

$$\begin{aligned} x_n &= [(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]/2 \\ y_n &= [(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]/2\sqrt{2} , \end{aligned}$$

and combining these with (15) and (16), we obtain

$$s_n = \frac{(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2}{32} ,$$

where s_n is the n^{th} triangular-square number.

Likewise, we can compute a formula for the n^{th} triangular-pentagonal number. The result is

$$s_n = \frac{(2 - \sqrt{3})(97 + 56\sqrt{3})^n + (2 + \sqrt{3})(97 - 56\sqrt{3})^n - 4}{48} .$$

This agrees with a result recently published by W. Sierpiński [4].

I am thankful to Dr. D. W. Bushaw, whose suggestions and encouragement made the writing of this paper possible.

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[Continued from page 71.]

$$M^{-1} = \sum_{k=0}^{\infty} \frac{m_k^*}{k!} D^k$$

is given by

$$\text{(VII)} \qquad \sum_{k=0}^{\infty} \frac{m_k^*}{k!} t^k = 1/(Ae^{c_1 t} + Be^{c_2 t}) .$$

We now note that for Case 2, where $A + B = 0$, Eq. (VII) does not exist for $t = 0$, and hence there is no inverse operator M^{-1}. Thus, a sufficient condition for M^{-1} (see (I)) to exist is that $A + B \neq 0$, i.e., Case 1. For $A + B \neq 0$, one readily finds that

$$\text{(VIII)} \qquad (A + B)m_k^* = (c_2 - c_1)^k H_k\left(\frac{c_1}{c_1 - c_2} \middle| -A/B\right) ,$$

where $H_k(x|\lambda)$ is the Eulerian polynomial cited in (*).

Many more identities can be quoted. Indeed, for $m, n = 0, 1, \cdots,$ one has

[Continued on page 112.]

A PRIMER FOR THE FIBONACCI NUMBERS
PART XI: MULTISECTION GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

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1. INTRODUCTION

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Often one desires generating functions which multisect the sequence $\{a_n\}$,

$$G_i(x) = \sum_{j=0}^{\infty} a_{i+mj} x^j, \qquad (i = 0, 1, 2, \cdots, m-1).$$

For the bisection generating functions the task is easy. Let

$$H_1(x^2) = \frac{f(x) + f(-x)}{2},$$

$$H_2(x^2) = \frac{f(x) - f(-x)}{2x};$$

then clearly $H_1(x^2)$ and $H_2(x^2)$ contain only even powers of x so that

$$H_1(x) = \sum_{n=0}^{\infty} a_{2n} x^n \qquad \text{and} \qquad H_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$$

are what we are looking for.

Let us illustrate this for the Fibonacci sequence. Here

$$f(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n;$$

then

$$H_1(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n}\, x^n$$

and

$$H_2(x) = \frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1}\, x^n \quad .$$

Exercise: Find the bisection generating functions for the Lucas sequence.

Let us find the general multisecting generating functions for the Fibonacci sequence, using the method of H. W. Gould [1]. The Fibonacci sequence enjoys the Binet Form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha = \frac{1 + \sqrt{5}}{2}, \qquad \beta = \frac{1 - \sqrt{5}}{2} \quad .$$

Let $f(x) = 1/(1 - x)$; then

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{mn+j}\, x^n &= \frac{\alpha^j f(\alpha^m x) - \beta^j f(\beta^m x)}{\alpha - \beta} \\
&= \frac{1}{\alpha - \beta}\left(\frac{\alpha^j}{1 - \alpha^m x} - \frac{\beta^j}{1 - \beta^m x}\right) \\
&= \frac{\dfrac{\alpha^j - \beta^j}{\alpha - \beta} + (\alpha\beta)^j\, \dfrac{\alpha^{m-j} - \beta^{m-j}}{\alpha - \beta}\, x}{1 - (\alpha^m + \beta^m)x + (\alpha\beta)^m x^2} \\
&= \frac{F_j + (-1)^j F_{m-j}\, x}{1 - L_m x + (-1)^m x^2}, \qquad (j = 0, 1, 2, \cdots, m - 1),
\end{aligned}$$

since $\alpha\beta = -1$, $\alpha^m + \beta^m = L_m$, and

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad .$$

Exercise: Find the general multisecting generating function for the Lucas sequence.

The same technique can be used on any sequence having a Binet Form. The general problem of multisecting a general sequence rapidly becomes very complicated according to Riordan [2], even in the classical case.

2. COLUMN GENERATORS OF PASCAL'S TRIANGLE

The column generators of Pascal's left-justified triangle [3], [4], [5], are

$$G_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n, \qquad k = 0, 1, 2, \cdots .$$

We now seek generating functions which will m-sect these,

$$G_i(m, k; x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{n+k+1}, \qquad (i = 0, 1, \cdots, m-1).$$

We first cite an obvious little lemma.

Lemma 1.

$$\binom{n}{k} = \sum_{j=1}^{m} \binom{n-j}{k-1} + \binom{n-m}{k} .$$

Definition. Let $G_{i,k}(x)$, $i = 0, 1, 2, \cdots, m-1$, be the m generating functions

$$G_{i,k}(x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{i+mn+k} .$$

Lemma 2.

$$G_{i,k+1}(x) = \frac{xG_{i,k}(x) + x^2 G_{i-1,k}(x) + \cdots + x^m G_{i-m+1,k}(x)}{1 - x^m} .$$

The proof follows easily from Lemma 1.

Let

$$(1 + x + x^2 + \cdots + x^{m-1})^n = \sum_{j=0}^{n(m-1)} \binom{n}{j}_m x^j$$

define the row elements of the m-nomial triangle. Further, let

$$f_i(m, k; x) = \sum_{j=0} \binom{k}{i+jm}_m x^j, \qquad i = 0, 1, \cdots, m-1,$$

where j is such that $i + jm \le k(m-1)$. These are multisecting polynomials for the rows of the m-nomial triangle. Now, we can state an interesting theorem:

Theorem. For $i = 0, 1, 2, \cdots, m-1$,

$$G_i(m, k; x) = \frac{x^{k+i} f_i(m, k; x)}{(1 - x)^{k+1}} .$$

Proof. Recall first that the m-nomial coefficients obey

$$\binom{n}{r}_m = \binom{n-1}{r}_m + \binom{n-1}{r-1}_m + \cdots + \binom{n-1}{r-m+1}_m$$

where the lower arguments are non-negative and less than or equal to $n(m - 1)$.

Clearly, for $k = 0$, from the definition just before Lemma 2,

$$G_{i,0}(x) = \frac{x^i}{1 - x^m}, \qquad i = 0, 1, 2, \cdots, m - 1 .$$

Assume now that

$$G_{i,k}(x) = \frac{x^{k+i} f_i(m, k; x^m)}{(1 - x^m)^{k+1}}$$

for $i = 0, 1, 2, 3, \cdots, (m - 1)$. From Lemma 2,

$$G_{i,k+1}(x) = \frac{xG_{i-1,k}(x) + \cdots + x^m G_{i-m+1,k}(x)}{1 - x^m} .$$

Thus,

$$G_{i,k+1}(x) = \frac{\sum_{s=0}^{m-1}\left(\sum_{j=0}\binom{k}{i-s+jm}_m\right) x^{k+(i-s)+s+jm+1}}{(1 - x^m)^{k+2}}$$

$$= \frac{\sum_{j=0}\left(\sum_{s=0}^{m-1}\binom{k}{i-s+jm}_m\right) x^{k+1+i+jm}}{(1 - x^m)^{k+2}}$$

$$= \frac{x^{k+1+i}\sum_{j=0}\binom{k+1}{i+jm}_m x^{jm}}{(1 - x^m)^{k+2}}$$

$$= \frac{x^{k+1+i} f_i(m, k; x^m)}{(1 - x^m)^{k+2}} .$$

This completes the induction.

The x^{k+1+i} merely position the column generators. Here the non-zero entries are separated by $m - 1$ zeros. To get rid of the zeros, let

$$G_i(m,k;\ x) = \frac{x^{k+i}\, f_i(m,k;\ x)}{(1 - x)^{k+1}}$$

for $i = 0, 1, 2, \cdots, m - 1$. This concludes the proof of the theorem.

If we write this in the form

$$G_i(m,k;\ x) = \sum_{j=0}^{\infty} \binom{i + jm + k}{k} x^{j+k+1} = \frac{\sum_{j=0} \binom{k}{i + jm}_m x^{k+i+j}}{(1 - x)^{k+1}}$$

it emphasizes the relation of the multisection of the k^{th} column of Pascal's triangle and the multisection of the k^{th} row of the m-nomial triangle.

3. A NEAT GENERATING FUNCTION

Lemma 3

$$\binom{n}{k} = \sum_{j=0}^{r} \binom{r}{j}\binom{n - r}{k - j}$$

This is easy to prove by starting with

$$\text{(A)}\qquad \begin{aligned} \binom{n}{k} &= \binom{n - 1}{k} + \binom{n - 1}{k - 1} \\ &= \binom{n - 2}{k} + \binom{n - 2}{k - 1} + \binom{n - 2}{k - 1} + \binom{n - 2}{k - 2} \\ &= 1 \cdot \binom{n - 2}{k} + 2 \cdot \binom{n - 2}{k - 1} + 1 \cdot \binom{n - 2}{k - 2}\ . \end{aligned}$$

Apply (A) to each term on the right repeatedly.

Now let $H_i(m,k;\ x)$ m-sect the k^{th} column of Pascal's triangle $(i = 0, 1, 2, \cdots, m - 1)$; then, using Lemma 3, it follows that

Lemma 4

$$H_i(m,k;\ x) = \frac{x}{1 - x} \sum_{j=1}^{m} \binom{m}{j} H_i(m,k - j;\ x)\ .$$

The results using the method of Polya for small m and i seem to indicate the following [3].

Theorem. The generating functions for the rising diagonal sums of the rows of Pascal's triangle $i + jm$ (all other rows are deleted) are given by

$$H_i(x) = \frac{(1+x)^i}{1 - x(1+x)^m}, \qquad i = 0, 1, \cdots, m-1 .$$

Exercise: Show that

$$\sum_{i=0}^{m-1} x^i H_i(x^m) = \frac{1}{1 - x(1 + x^m)} .$$

This is a necessary condition which now makes the theorem plausible. These are the generalized Fibonacci numbers obtained as rising diagonal sums from Pascal's triangle, beginning in the left-most column and going over 1 and up m 3 . The theorem is proved by careful examination of its meaning with regards to Pascal's triangle as follows:

$$\frac{(1+x)^i}{1 - x(1+x)^m} = \sum_{n=0}^{\infty} x^n (1+x)^{mn+i} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{m(n-j)+i}{j} x^n ,$$

Recall that $\binom{n}{k} = 0$ if $0 \le n \le k$.

ILLUSTRATION

$n = 0$	$x^0(1+x)^{0+1}$	=	$1 + x$
$n = 1$	$x^1(1+x)^{2+1}$	=	$x + 3x^2 + 3x^3 + x^4$
$n = 2$	$x^2(1+x)^{4+1}$	=	$x^2 + 5x^3 + 10x^4 + 10x^5 + 5x^6 + x^7$
$n = 3$	$x^3(1+x)^{6+1}$	=	$x^3 + 7x^4 + 21x^5 + \cdots$
$\cdots$	$\cdots$		$\cdots\cdots\cdots$
Sum:			$1 + 2x + 4x^2 + 9x^3 + 19x^4 + \cdots$

Here, $m = 2$ and $i = 1$. Now, write a left-justified Pascal's triangle. Form the sequence of sums of elements found by beginning in the left-most column and proceeding right one and up two throughout the array: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, $\cdots$. Notice that the coefficients of successive powers of x give every other term in that sequence.

The general problem of finding generating functions which multisect the column generators of Pascal's triangle has been solved by Nilson [6], although interpretation of the numerator polynomial coefficients has not been achieved as in our last few theorems.

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A CURIOUS PROPERTY OF UNIT FRACTIONS OF THE FORM $1/d$ WHERE $(d, 10) = 1$

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INTRODUCTION

One of the rewards of teaching is seeing your students discover for themselves a profound mathematical result. Over twelve years of teaching I have had more than my share of such observances. Perhaps the most rewarding came about as a "spin off" of a problem dealing with the nature of a repeating decimal. A student at San Carlos High School, Frank Stroshane, made the original discovery described in this article while trying to find out why with some fractions its period has a "nines-complement" split, that is, its period can be split into two halves that have a nines complement relationship. For example: $1/7 = .\overline{142857}$ has 1 and 8; 4 and 5; and 2 and 7. Frank, typical of talented students, found a different "gem." He could not prove his result but it was clear to me and the others with whom he shared it that it was unquestionably true. It is this observation and its subsequent justification that represents the main thrust of the article.

The property alluded to is:

<u>Theorem</u>. The period of a fraction of the form $1/d$ where $(d,10) = 1$ can be completely determined without dividing.

For example, to find the decimal expansion of $1/7$ we "know" (this knowledge will be proven later) that the last digit in the period must be 7. Now, multiply this terminal digit by our "magic" number 5 (this too will be explained later). Continue the process of multiplying the previous digit by 5 (allow for carries) until the digits of the period repeat. The full process follows:

a.	$1/7$ has 7 for its last digit and its period	7
b.	Multiply the 7 by 5 giving	$\overset{3}{5}7$ where 3 is the carry
c.	Multiply the 5 by 5 and add the previous carry of 3 giving	$\overset{2}{8}57$ where 2 is the carry
d.	Repeating the process gives	$\overset{4}{2}857$
e.	Again giving	$\overset{1}{4}2857$

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[1]Provided the proof of the algorithm described in the article.
[2]A teacher at San Carlos High School. Presented the problem that led, after several years, to the proof. He also compiled this article.
[3]A student at San Carlos High School. Discovered the magic number and provided most of the lemmas and their proofs leading to Brother Brousseau's proof.

f. Again giving $\begin{matrix} 2 \\ 142857 \end{matrix}$

g. Again giving $\begin{matrix} 0 \\ 7142857 \end{matrix}$

which indicates the period is repeating.

Therefore $1/7 = .\overline{142857}$

Before launching into a statement of the algorithm employed and its proof, some preliminaries need to be established. We have assumed that all fractions of the form $1/d$ where $(d,10) = 1$ have a decimal expansion which repeats, furthermore they begin their period immediately.* First a Lemma about the final digit in the repeated block.

Lemma. If $1/d = .\overline{a_1a_2 \cdots a_k}$ where $(d,10) = 1$, then

$$d \cdot a_k = 9 \mod 10$$

or

$$d \cdot a_k \text{ ends in a } 9 .$$

Proof. Since

$$1/d = .\overline{a_1 a_2 \cdots a_k}$$

then

$$\frac{10^k}{d} = a_1 a_2 \cdots a_k \cdot \overline{a_1 a_2 \cdots a_k}$$

subtracting

$$\frac{10^k - 1}{d} = a_1 a_2 \cdots a_k$$

or

$$\frac{10^k - 1}{d} = a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} 10^1 + a_k 10^0$$

$$10^k - 1 = d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} \cdot 10^1 + a_k \cdot 10^0)$$

but

$$10^k - 1 \equiv 9 \pmod{10}$$

$$d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} 10) + d \cdot a_k \equiv 9 \pmod{10}$$

or

$$d \cdot a_k \equiv 9 \pmod{10}$$

since

$$d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} \cdot 10) \equiv 0 \pmod{10}$$

or

$$d \cdot a_k = 10N + 9$$

where N is some integer; that is, $d \cdot a_k$ ends in a 9.

This shows that in expanding any unit fraction of the type described, the product of the denominator and the last digit must end in a nine. Hence $1/7$ has for the last digit in its

*See The Enjoyment of Mathematics, Rademacher and Toeplitz, pp. 149-152.

period a 7; 1/11 has for the last digit in its period a 9; and 1/23 has for the last digit in its period a 3.

FINDING THE "MAGIC NUMBER"

An algorithm for determining the magic number is as follows:

1. Find the terminal digit in the period (see Lemma).
2. Multiply by d.
3. Add 1.
4. Drop final digit in sum. (It will always be zero.)
5. This number will be the "magic number."

Briefly, if m is the "magic number,"

$$m = \frac{d \cdot a_k + 1}{10} ,$$

where d is the denominator of the given unit fraction, a_k is the terminal digit in the period of 1/d, and k is length of period. Therefore, using the above algorithm, the "magic number" for the following unit fractions are:

a. For 1/7 the magic number is 5, since

$$5 = \frac{7(7) + 1}{10} .$$

b. For 1/11 the magic number is 10, since

$$10 = \frac{9(11) + 1}{10} .$$

c. For 1/27 the magic number is 19, since

$$19 = \frac{7(27) + 1}{10} .$$

d. For 1/43 the magic number is 13, since

$$13 = \frac{3(43) + 1}{10} ,$$

etc.

PROOF OF ALGORITHM

On inspection one can see this algorithm is equivalent to finding the quantity which on being multiplied by 10 and divided by the denominator gives a remainder of 1. That is,

$$10m \equiv 1 \pmod{d}.$$

If we visualize the process of division in complete detail, m is the remainder in the division process just prior to the remainder 1 which initiates a new cycle.

How does one go about justifying such an algorithm? First, it may be pointed out that the length of the period of such a decimal is found by the smallest value of k for which

$$10^k = 1 \pmod{d},$$

where d is an odd integer.* Thus for 7

$$10^1 \equiv 3 \pmod{7}; \quad 10^2 \equiv 2 \pmod{7}; \quad 10^3 \equiv 6 \pmod{7};$$
$$10^4 \equiv 4 \pmod{7}; \quad 10^5 \equiv 5 \pmod{7}; \quad 10^6 \equiv 1 \pmod{7}.$$

Note also that these quantities are the successive remainders in the division process. The magic number is given by $10^5 \equiv 5 \pmod{7}$. In other words, the magic number m is the least positive residue for which $10^{k-1} \equiv m \pmod{d}$. It is also the last remainder in the division process that precedes a remainder of 1 which is the first remainder. That is

$$10^{k-1} = r_k \pmod{d},$$

where r_k is the last remainder where the length of period is k.

To understand the ensuing analysis, let us parallel division by 7 and the corresponding notation that will be employed.

$.142857$	$.a_1 a_2 a_3 a_4 a_5 a_6$
$7\,)\,\overline{1.000000}$	$d\,)\,\overline{1.0\,0\,0\,0\,0\,0}$
$\underline{7}$	$\underline{n_1}$
30	$r_2 0$
$\underline{28}$	$\underline{n_2}$
20	$r_3 0$
$\underline{14}$	$\underline{n_3}$
60	$r_4 0$
$\underline{56}$	$\underline{n_4}$
40	$r_5 0$
$\underline{35}$	$\underline{n_5}$
50	$r_6 0$
$\underline{49}$	$\underline{n_6}$
1	$1 = r$

*The proof of this statement can be found in The Enjoyment of Mathematics.

In the above illustration, $n_2 = a_2 \cdot d$, while $r_3 0$ is the remainder with a zero attached. From the nature of the division operation we have the following equations:

$$\begin{aligned} 10\, r_1 &= a_1 \cdot d + r_2 \\ 10\, r_2 &= a_2 \cdot d + r_3 \\ &\cdots \\ 10\, r_{k-1} &= a \cdot d + r_k \\ 10\, r_k &= a_k \cdot d + 1 \quad . \end{aligned}$$

Taking

$$r_1 = 1; \qquad 10 \cdot r_1 = a_1 \cdot d + r_2$$

implies

$$10 = a_1 \cdot d + r_2 \quad \text{or} \quad 10 - a_1 \cdot d = r_2$$

and

$$10 \cdot r_2 = a_2 \cdot d + r_3$$

leads to

$$10 \cdot (10 - a_1 \cdot d) = a_2 \cdot d + r_3$$

or

$$10^2 - r_3 = (10a_1 + a_2)\, d$$

or equivalently

$$10^2 \equiv r_3 \pmod{d}$$

and

$$10 \cdot r_3 = a_3 \cdot d + r_4$$

leads to

$$10(10^2 - 10\, a_1 \cdot d - a_2 \cdot d) = a_3 \cdot d + r_4$$

or

$$10^3 - r_4 = (10^2\, a_1 + 10\, a_2 + a_3)d$$

or

$$10^3 \equiv r_4 \pmod{d}$$

and in general

$$10^{\ell} \equiv r_{\ell+1} \pmod{d} \quad .$$

Now since $r_k \equiv 10^{k-1} \pmod{d}$ where r_k is the last remainder in the division process for the unit fraction which has a decimal expansion with a period of length k it follows (recalling $10^k \equiv 1 \pmod{d}$),

$$r_k^2 \equiv (10^{k-1})^2 \equiv 10^{2k-2} \equiv 10^{k-2} \equiv r_{k-1} \pmod{d}$$

or equivalently

$$r_{k-1} \equiv 10^{k-2} \pmod{d}$$

so that

$$r_k^2 = d \cdot b_k + r_{k-1} \, ,$$

where b_k is an integer. Therefore

$$r_k \cdot r_{k-1} \equiv 10^{k-1} 10^{k-2} \equiv 10^{2k-2} \equiv 10^{k-3} \equiv r_{k-2} \pmod{d} \, .$$

In general,

$$r_k \cdot r_{k-\lambda} = 10^{k-1}10^{k-\lambda-1} = 10^{2k-\lambda-2} = 10^{k-\lambda-2} = r_{k-\lambda-1} \pmod{d}.$$

Hence

$$r_k^2 = d \cdot b_k + r_{k-1}$$
$$r_k \cdot r_{k-1} = d \cdot b_{k-1} + r_{k-2}$$
$$r_k \cdot r_{k-2} = d \cdot b_{k-2} + r_{k-3}$$
$$\cdot \quad \cdot \quad \cdot$$
$$r_k \cdot r_2 = d \cdot b_2 + 1$$

where the b_i's are integers.

From the first set of relations,

$$a_k d = 10\, r_k - 1$$
$$r_k a_k d = 10\, r_k^2 - r_k = 10\, d \cdot b_k + 10\, r_{k-1} - r_k$$
$$= 10\, d \cdot b_k + a_{k-1} d$$

therefore

$$r_k a_k = 10\, b_k + a_{k-1}.$$

This shows that the product of a magic number r_k by the last digit in the period a gives the penultimate digit in the period, viz, a_{k-1}. Continuing in like manner:

$$a_{k-1} d = 10\, r_{k-1} - r_k$$
$$r_k a_{k-1} d = 10\, r_k r_{k-1} - r_k^2$$
$$= 10\, d \cdot b_{k-1} + 10\, r_{k-2} - d \cdot b_k - r_{k-1}$$
$$= 10\, d \cdot b_{k-1} + a_{k-2} d - d b_k$$

since $10\, r_{k-2} - r_{k-1} = a_{k-2} d$ or simplifying,

$$r_k a_{k-1} = 10\, b_{k-1} + a_{k-2} - b_k$$
$$r_k a_{k-1} + b_k = 10\, b_{k-1} + a_{k-2}.$$

This shows that multiplying r_k by a_{k-1}, the next to last digit in the period and adding b_k from previous operation gives a_{k-2} as the last digit. In general,

$$a_{k-\lambda} d = 10\, r_{k-\lambda} - r_{k-\lambda+1}$$
$$r_k a_{k-\lambda} d = 10\, r_k \cdot r_{k-\lambda} - r_k \cdot r_{k-\lambda+1}$$
$$= 10\, d \cdot b_{k-\lambda} + 10\, r_{k-\lambda-1} - d b_{k-\lambda+1} - r_{k-\lambda}$$
$$= 10\, d b_{k-\lambda} - d b_{k-\lambda+1} + d \cdot a_{k-\lambda-1},$$

since $10\, r_{k-\lambda-1} - r_{k-\lambda} = d \cdot a_{k-\lambda-1}$ or

$$r_k \cdot a_{k-\lambda} + b_{k-\lambda+1} = 10\, b_{k-\lambda} + a_{k-\lambda-1}.$$

This shows that the process continues at each step of the operation and completes the proof.

FINAL COMMENTS

It is not difficult to expand the remarks concerning unit fractions developed in this article to all fractions of the form c/d where $0 < c < d$ and $(d, 10) = 1$. Also the fact that the remainders in the division process are all relatively prime to the division is useful in determining the length of the period of a given fraction. A proof of this result concludes the article.

Theorem. All of the remainders in the division process associated with $1/d$ where $(d, 10) = 1$ are relatively prime to d.

Proof. Since $r_1 = 1$ then $(r, d) = 1$.

$$10 r_1 = a_1 \cdot d + r_2 \qquad (0 \le r_2 < d) .$$

It must be that $(r_2, d) = 1$ since if

$$(r_2, d) = t_1 ; \qquad (t_1 \neq 1)$$

then

$$r_2 = pt_1 \quad \text{and} \quad d = kt_1 .$$

$$10 = a_1(kt_1) + pt_1 = (a_1 \cdot k + p)t_1 .$$

Therefore, t_1 must divide 10 but $(d, 10) = 1$ and $d = kt_1$ hence a contradiction and $(r_2, d) = 1$. Continuing,

$$10 r_2 = a_2 \cdot d + r_3 \qquad (0 \le r_3 < d) .$$

Again it must be that $(r_3, d) = 1$ since if

$$(r_3, d) = t_2 \qquad (t_2 \neq 1)$$

then

$$r_3 = pt_2 \quad \text{and} \quad d = pt_2$$

$$10 r_1 = a_2 \cdot k \cdot t_2 + pt_2 = (a_2 \cdot k + p)t_2$$

but $(t_2, r_1) = 1$ since $(d, r_1) = 1$ hence t_2 must divide 10 but $(d, 10) = 1$ thus $(r_2, d) = 1$. Since the argument continues in like manner, the theorem is proved.

EDITORIAL COMMENT
Marjorie Bicknell

Puzzles intimately related to the results of the paper, "A Curious Property of Unit Fractions of the Form $1/d$ Where $(d, 1) = 1$," have the following form:

Find a number whose left-most digit is k which gives a number $1/m$ as large when k is shifted to the far right-end of the number.

The solution to such puzzles can be obtained by multiplying k by the "magic multiplier" m to produce the original number, which is the repeating block of the period of c/d, where m is the "magic number" for d, and $1 < c < d$.

For example, find a number whose left-most digit is 6 which gives a number 1/4 as large when 6 is shifted to the far right end of the number. Multiplying 6 by the "magic multiplier" 4 as explained in the paper above gives a solution of 615384, which is four times as great as 153846. Notice that

$$4 = \frac{d \cdot a_k + 1}{10}$$

gives the solutions, in positive integers,

$$d \cdot a_k = 39 = 39 \cdot 1 = 13 \cdot 3 \;,$$

where $d = 13$ or $d = 39$ give the same solution as follows. 1/13 ends in 3, 2/13 ends in 6,

$$4 \times \frac{2}{13} = \frac{8}{15}$$

has the original number of the puzzle as its period.

As a second example, re-read the puzzle using $k = 4$ and $m = 2$. Multiplying 4 using the "magic multiplier" 2 yields

421052631578947368

which is twice as large as

210526315789473684 .

Here 2 in the "magic number" formula produces

$$2 = (d \cdot a_k + 1)/10$$

so that

$$d \cdot a_k = 19 = 19 \cdot 1 .$$

1/19 ends in 1, 4/19 ends in 4, $2 \cdot 4/19 = 8/19$ which has the original number as its period. (14/19 also ends in 4 but $2 \cdot 14 > 19$.)

One can also find $1/m$, $(m,10) \neq 1$ by methods of this paper. $1/6 = (1/2)(1/3)$. Find $1/3 = .3333\cdots$ without dividing. Then $(.5) \times (.3333\cdots)$, remembering that the multiplication on the right begins with 1 to carry, makes $.1666\cdots$.

THE AUTOBIOGRAPHY OF LEONARDO PISANO

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For the mathematical historian interested in biographical details, Leonardo Pisano, better known to readers of this journal as Fibonacci, was a frustratingly modest genius. In his extant writings he tells us next to nothing of himself. In only one place, the second paragraph of the 1228 edition of his revised Liber Abbaci (Book of Calculation), first published in 1202, does he convey to us information about his earlier life; and even then the information, given merely as an incidental backdrop for his explanation of his purpose in writing the Liber, is very scanty and lamentably lacking in the precision which he displays in his mathematical elucidations. This second paragraph had, in the 1202 edition, been placed at the very beginning of the book; but in the revised, second edition of 1228, Leonardo wrote a dedication to the celebrated court astrologer of Frederick II, Michael Scott, who had requested a copy of the work, and thus this dedication became the work's first paragraph, with the "autobiographical paragraph" following immediately after it. Today's mathematicians are familiar with only this second, revised edition, since it is the one which Baldassare Boncompagni printed as Volume 1 of his two-volume Scritti di Leonardo Pisano (Rome, 1857-1862). Although Boncompagni knew of the existence of six manuscripts containing this autobiography, he based his edition — the first, and still the only complete printed edition which we possess — on only one manuscript, the handsome but frequently badly faded Conventi Soppressi C. I. 2616, dated to the early fourteenth century. This manuscript is now housed in the Biblioteca Nazionale Centrale in Florence; for convenience, I shall hereafter refer to it as Boncompagni's manuscript.

His failure to collate his manuscripts and his reliance upon a manuscript often difficult correctly to read led Boncompagni into an astonishing number of errors, both of transcription and of punctuation. The brief autobiographical second paragraph is unfortunately not immune from either type of error; yet this section forms the basis for most of the statements about Leonardo's early life which are found in current histories of mathematics, encyclopedias, and special articles. Unfortunately, there has also been a considerable amount of embroidering upon Leonardo's spare Latinity by many of those who have employed Boncompagni's text — which is to say all scholars who during the past eleven decades have written on Leonardo's life. It is not my intention here to refute point-by-point the many extravagant statements found about Leonardo in this more than century-old literature. Instead, I wish to present the second paragraph anew, basing my text on a collation of the six manuscripts which contain it. Following the text I shall offer a translation, along with some footnotes, keyed both to the Latin text and to the translation. Let me state at once that not all the problems in this paragraph are hereby forever resolved. I hope only that some misconceptions about

Fibonacci can be laid aside and that we can more accurately assess what his Latinity does allow us to assert.

Cum genitor meus a patria publicus scriba[1] in duana bugee pro pisanis mercatoribus ad eam confluentibus constitutus preesset, me in pueritia mea ad se venire faciens, inspecta utilitate et commoditate futura, ibi me studio abbaci per aliquot dies[2] stare voluit et doceri. Vbi ex mirabili magisterio in arte[3] per novem figuras indorum introductus, scientia artis in tantum mihi pre ceteris placuit, et intellexi ad illam[4] quod quicquid studebatur ex ea[5] apud egyptum, syriam, greciam, siliciam, et provinciam cum suis variis modis, ad que loca negotiationis causa[7] postea[6] peragravi per multum studium et disputationis didici conflictum[8]. Sed hoc totum etiam, et algorismum atque artem pictagore[9] quasi errorem computavi respectu modi indorum. Quare, amplectens strictius ipsum modum indorum et attentius studens in eo, ex proprio sensu quedam addens et quedem etiam ex subtilitatibus euclidis geometrice artis apponens, summam huius libri, quam intelligibilius potui, in quindecim capitulis distinctam componere laboravi, fere omnia que inserui certa probatione ostendens, ut extra perfecto pre ceteris modo hanc scientiam[10] appetentes instruantur, et gens latina[11] de cetero, sicut hactenus, absque illa minime inveniatur. Si quid forte minus aut plus iusto vel necessario intermisi, mihi deprecor indulgeatur, cum nemo sit qui vitio careat et in omnibus undique sit circumspectus.[12]

...

After my father's appointment by his homeland as state official[1] in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and, in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days[2]. There, following my introduction, as a consequence of marvelous instruction in the art[3], to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it[4] I realized that all its aspects[5] were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter[6], while on business[7], I pursued my study in depth and learned the give-and-take of disputation[8]. But all this even, and the algorism, as well as the art of Pythagoras[9] I considered as almost a mistake in respect to the method of the Hindus. Therefore, embracing more stringently that method of the Hindus, and taking stricter pains in its study, while adding certain things from my own understanding and inserting also certain things from the niceties of Euclid's geometric art, I have striven to compose this book in its entirety as understandably as I could, dividing it into fifteen chapters. Almost everything which I have introduced I have displayed with exact proof, in order that those further seeking this knowledge, with its pre-eminent method[10], might be instructed, and further, in order that the Latin[11] people might not be discovered to be without it, as they have been up to now. If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things.[12]

1. This unsatisfactory translation is the most that should be advanced for publicus scriba, I feel. Its vagueness matches the vagueness of the Latin. We simply do not know the precise nature of the position held by Leonardo's father. He was appointed (constitutus) by Pisa to this post, which certainly involved duties at Bugia (present-day Bugie in Algeria) in connection with the Pisan duana, a word which we perhaps translate too easily as customs-house. The text as it stands offers no basis for much of the standard lore found in biographies of Leonardo regarding his father as "secretary," "merchant," "agent," "business man," "head of a factory," "warehouse head," etc.

2. Note that Leonardo says specifically that his father wanted him to be instructed for some days in the study of calculation. The phrase per aliquot dies, which looks like a rendering of the Italian per qualche giorno, is vague indeed, but it would be generous to consider it to imply more than a fortnight. Further, this was the period of time Leonardo's father wanted him to study the "abacus." How much time he actually spent at Bugia in his study Leonardo does not tell us. Finally, it should be noted that Leonardo uses the word abbacus for "calculation." By the twelfth century, in the latter part of which Leonardo was born, the older meaning of abacus as a calculation board had grown to include the operations which the abacus performed, namely calculation in general.

3. Just who gave Leonardo this "marvelous instruction" is not stated. It has been frequently assumed that his instructor was Moorish, but there is no hint of this in the text.

4. My translation is the best I have been able to do with ad illam, which I strongly suspect is corrupt, though all the manuscripts have it. As it stands, illa must refer to either scientia, the knowledge of the Hindu system, or to ars, the art of its exposition; but ad illam as a shorthand way of saying ad illam cognoscendam or discendam ("for learning it") is very harsh, and the loss of the gerundive early in the manuscript tradition is a strong probability.

5. The difficult quicquid studebatur ex ea, coming immediately after the strange ad illam, compels us to refer ea and illa to the same thing; the phrase can be tortured into sense by taking "whatever was studied of it" to mean "all there was of it was studied," and hence "all its aspects were studied," as the present translation renders it. It is somewhat mystifying that Leonardo mentions these particular five regions as containing all aspects of the Hindu lore, when we know that he also spent time in Constantinople. Did his grecia embrace the Byzantine capital?

6. The word for "thereafter," postea, gives us no indication of the amount of time which elapsed between Leonardo's boyhood experiences in Bugia and his travels around the Mediterranean. It is very probable that he returned to Pisa and went abroad again several years later, after reaching maturity. It should not be forgotten that he was still a lad (in pueritia mea, as he says) when he came to Bugia.

7. This rendering, "while on business," is based on an examination of the six autobiographical manuscripts. Boncompagni's manuscript reads ad que loca negotiationis tam postea peragraui per multum studium et disputationis didici conflictum. With this reading, tam must modify postea, and negotiationis is genitive with loca: "... to which places of business

so much later I wandered, through [= in the course of?] considerable study," etc. (italics mine). This is an extremely forced rendering. *Tam postea* is bad Latin for *tanto postea*; I cannot believe Leonardo wrote it, especially since all the other manuscripts give *causa* instead of *tam*. In the ligature employed by the scribes copying Leonardo's manuscripts in the twelfth to the fourteenth century, $\widehat{r2}$, *tam*, and $\widehat{ca}$, *causa*, are easily confused. The phrase *ad que loca negotiationis causa posta* is, I think, Leonardo's succinct way of saying "Later, while on business at these places."

8. *Peragravi per multum studium* I have rendered as "I pursued my study in depth." The phrase possibly means that in Egypt, Syria, and the other lands he has just mentioned, Leonardo utilized the opportunities which his business trips provided to investigate the Hindu number system more thoroughly. The final phrase *et disputationis didici conflictum*, also cryptic, seems a reference to the medieval practice of discussion and debate on set topics. Leonardo, it may be surmised, sought out local scholars on his business trips and mastered not only the theoretical material of the Hindu number system, but also the method of expounding it in scholarly debate.

9. The Latin here, from *sed* to *pictagore*, is a mare's nest of difficulty which has not been adequately investigated by those who have read it. Almost certainly, to judge by the variety of readings which the manuscripts exhibit at this point, there is deep -possibly incurable — textual corruption, and my translation must rely in part on emendation. There are three principal areas of difficulty.

(1) Does *hoc totum*, "all this," refer to the disputationis conflictum at the end of the preceding sentence? Or does it have as appositive *algorismum* two words later? I doubt the latter alternative. *Hoc totum, algorismum*, "all this, algorismus," would almost certainly be a reference to al-Khwarizmi, the great ninth-century Arab mathematician, whose very name was corrupted to "algorism" and referred to the practice of calculating with Hindu numbers. Would Leonardo say that he regarded algorism as *quasi errorem* when compared to the methods of the Hindus? (I propose a tentative answer in the next note.) Again, Leonardo has not previously discussed *hoc totum, algorismum*; the *hoc* should refer to something under discussion. One is practically forced back to the preceding *disputationis conflictum*, the method of argumentation itself, which Leonardo would then be contrasting with the theoretical basis of the system of Hindu numerals. This is a poor contrast at best, and I am not happy with it.

(2) The words *etiam et* are in five of the autobiographical manuscripts but are strange. If the reading is correct, *etiam* should probably be taken with *hoc totum* (= "all this, even"), and *et algorismum* should mean "and the algorismus." Once again, would Leonardo regard this algorism as "almost a mistake" when compared with the Hindu system? If the text is kept as is, I can only believe that Leonardo intends some contrast between the Hindu system as transcribed through the Arabs and the "original" system developed in its pure form by the Hindus. Had he seen some earlier work of the Hindus in his travels which made the Arab adaptation seem inferior? Kurt Vogel in his article on Fibonacci in the *Dictionary of Scientific Biography* (Vol. IV, pp. 603-613), speculates,

p. 605, that Fibonacci might mean the later *algorismus linealis,* reconing with lines, but this seems unlikely. When *algorismus* is mentioned by itself, without qualifying adjective, it would have for Leonardo's readers but one reference, and that is to the Hindu system of calculation.

(3) The final phrase, *atque artem pictagore*, is the last of the three things which Leonardo regards as "almost a mistake" when compared to the Hindu system. Boncompagni's text reads *atque arcus pictagore*, a phrase which has considerably exercised the ingenuity of scholars. What, they have asked, are Pythagoras' arcs? The answer, I suspect, is "a scribal concoction." My reasons for so believing and my justification for the proposed emendation are as follows.

A. The literature on Pythagoras, so far as I have ascertained, contains no allusion to any such phrase, and since Leonardo here considers ______ *pictagore* important enough to be classified alongside the *algorismus*, discussed above, it is logical to assume that he is making a reference to some large category of Pythagorean mathematics which parallels the *algorismus*. A reference to the "arcs of Pythagoras" is too esotoric and restricted, even if Leonardo (and presumably, his readers) knew something about Pythagoras which we today do not.

B. Of the six autobiographical manuscripts, only Boncompagni's clearly reads *arcus*, written in ligature ārc by the scribe. One other manuscript, the Biblioteca Laurenziana No. 783, written at least a century later than Boncompagni's, reads arc, which could stand for *arcus*, though in extensive checking elsewhere I have found the long *us* ending for fourth declension nouns such as *arcus* and *gradus* written out by the scribe. The other four manuscripts all omit the word *arcus*; three have *atque pictagore,* one (obviously guilty of a slip) *adque pictagore*. It should be noted that two of these are roughly contemporary with Boncompagni's manuscript and that the latter has no special claim to paleographic superiority.

C. To balance *algorismum*, a noun is needed between *atque* ("and also") and *pictagore* ("of Pythagoras"). In the four "noun-less" manuscripts, which on other grounds appear to belong to a common tradition, it seems obvious that for some reason the word after *atque* dropped out early. Could this word have been *arcus*? In manuscript, the two words would have appeared as atq3 ārc ; I find it difficult to believe that some early scribe would have carelessly omitted a relatively uncommon word like *arcus*. He might, however, have been guilty of haplography if he had found atq3 artē, *atque artem,* since both words are common (the word *ars* appears thrice in this paragraph) and in manuscript more closely resemble each other than do *atque arcus*.

D. The scribe of Boncompagni's manuscript, moreover, has already shown himself to be guilty of confusing *c* and *t* when he read câ as *tam* instead of *causa*. Hence it is possible that, finding something like artē, he read ārc, hence *arcus*.

E. Certainly *artem pictagore*, "the art of Pythagoras," makes excellent sense in context, balancing as it does the earlier mention of the *ars* of the Hindus and the immediately following mention of the *ars* of Euclid. It also serves as a satisfactory balance to algorismum, if the interpretation of the word which I have given above is accepted.

F. I propose, then, artem instead of the arcus of Boncompagni's text, as a more reasonable, though — I freely admit — by no means certain reading. Arcus, however, should be given a decent burial, since both logically and paleographically it is unworthy of serious consideration.

10. The Latin here, from ut through scientiam, is rather murky, and the manuscripts admit considerable variation. However, three of the autobiographical manuscripts have Boncompagni's reading, and I have kept it, though other interpretations of the text than the one my translation implies are possible.

11. Leonardo's name for the Italians.

12. To me, this last sentence might well serve as a motto for scholars who write books. Leonardo's humility graces his genius.

[Continued from page 90.]

A PRIMER FOR THE FIBONACCI NUMBERS

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

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DEFINITIONS. The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS PROPOSED IN THIS ISSUE

B-250 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

$$\begin{array}{r} \text{DO} \\ \text{YOU} \\ \underline{\text{LIKE}} \\ \text{SUSY} \end{array}$$

In this alphametic, each letter stands for a particular but different digit, nine digits being shown here. What do you make of the perfect square sum SUSY?

B-251 Proposed by Paul S. Bruckman, San Rafael, California

A and B play a match consisting of a sequence of games in which there are no ties. The odds in favor of A winning any one game is m. The match is won by A if the number of games won by A minus the number won by B equals 2n before it equals -n. Find m in terms of n given that the match is a fair one, i.e., the probability is 1/2 that A will win the match.

B-252 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k}{i!j!k!} = \frac{1}{n!} .$$

B-253 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k L_{j+2k}}{i!j!k!} = 0 = \sum_{i+j+k=n} \frac{(-1)^k F_{j+2k}}{i!j!k!} .$$

B-254 Proposed by Clyde A. Bridger, Springfield, Illinois.

Let $A^n = a^n + b^n + c^n$ and $B^n = d^n + e^n + f^n$ where a, b, and c are the roots of $x^3 - 2x - 1$ and d, e, and f are the roots of $x^3 - 2x^2 + 1$. Find recursion formulas for the A_n and for the B_n. Also express B_n in terms of A_n.

B-255 Proposed by L. Carlitz and Richard Scoville, Duke University, Durham, North Carolina.

Show that

$$\sum_{2k \le n} k\binom{n-k}{k} = \sum_{k=0}^{n} F_k F_{n-k} = [(n-1)F_{n+1} + (n+1)F_{n-1}]/5.$$

SOLUTIONS

FIBONACCI SUM OF FOUR SQUARES

B-226 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Find the smallest number in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$ that is not the sum of the squares of three integers.

Solution by Paul S. Bruckman, San Rafael, California.

It is a well-known result in number theory (see, for example, The Higher Arithmetic, by H. Davenport, p. 127, Harper Torchbooks, 1960) that any number of the form $4^u(8v + 7)$ is not representable as the sum of three squares, whereas all other numbers are representable. The first few numbers in this sequence are as follows:

$$7, 15, 23, 28, 31, 39, 47, 55, \cdots.$$

The smallest number of this set which is also a Fibonacci number is 55, which is therefore the solution to the problem.

Also solved by Ralph Fecke, J. A. H. Hunter, Peter A. Lindstrom, C. B. A. Peck, Stephen Rayport, and the Proposer.

GENERALIZATION OF RECKE'S FORMULA

B-227 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Let $H_0, H_1, H_2, \cdots$ be a generalized Fibonacci sequence satisfying $H_{n+2} = H_{n+1} + H_n$ (and any initial conditions $H_0 = q$ and $H_1 = p$). Prove that

$$F_1H_3 + F_2H_6 + F_3H_9 + \cdots + F_nH_{3n} = F_nF_{n+1}H_{2n+1}.$$

Solution by John W. Milsom, Butler County Community College, Butler, Pennsylvania.

This is a generalization of Problem B-153 in which it was established that

$$F_1F_3 + F_2F_6 + F_3F_9 + \cdots + F_nF_{3n} = F_nF_{n+1}F_{2n+1}.$$

An induction proof follows.

$$\sum_{i=1}^{n} F_iH_{3i} = F_nF_{n+1}H_{2n+1}$$

for $n = 1$. Assume that for some positive integer k that

$$\sum_{i=1}^{k} F_iH_{3i} = F_kF_{k+1}H_{2k+1}.$$

The difference between

$$\sum_{i=1}^{k+1} F_i H_{3i}$$

and

$$\sum_{i=1}^{k} F_i H_{3i}$$

is $F_{k+1} H_{3k+3}$. If it can be shown that

$$F_{k+1}F_{k+2}H_{2k+3} - F_kF_{k+1}H_{2k+1} = F_{k+1}H_{3k+3},$$

then it will follow that

$$\sum_{i=1}^{k+1} F_i H_{3i} = F_{k+1} F_{k+2} H_{2k+3} .$$

$$\begin{aligned} F_{k+1} F_{k+2} H_{2k+3} - F_k F_{k+1} H_{2k+1} &= F_{k+1}(F_{k+2} H_{2k+3} - F_k H_{2k+1}) \\ &= F_{k+1} [(F_{k+1} + F_k)(H_{2k+1} + H_{2k+2}) - F_k H_{2k+1}] \\ &= F_{k+1}(F_{k+1} H_{2k+3} + F_k H_{2k+2}) \\ &= F_{k+1} H_{3k+3} . \end{aligned}$$

This last statement follows from the known statement of equality

$$H_{n+r} = F_{r-1} H_n + F_r H_{n+1}$$

with $n = k + 1$ and $r = 2k + 2$. Thus it can be said for all positive integral values of n that

$$F_1 H_3 + F_2 H_6 + F_3 H_9 + \cdots + F_n H_{3n} = F_n F_{n+1} H_{2n+1} .$$

Also solved by Paul S. Bruckman, A. Carroll, Herta T. Freitag, Ralph Garfield, Pierre J. Malraison, Jr., C. B. A. Peck, A. Sivasubramanian, David Zeitlin, and the Proposer.

A CYCLICALLY SYMMETRIC FORMULA

B-228 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Extending the definition of the F_n to negative subscripts using $F_{-n} = (-1)^{n-1} F_n$, prove that for all integers k, m, and n

$$(-1)^k F_n F_{m-k} + (-1)^m F_k F_{n-m} + (-1)^n F_m F_{k-n} = 0 .$$

Solution by Paul S. Bruckman, San Rafael, California

Using the Binet definitions of the Fibonacci and Lucas numbers,

$$F_n = (a^n - b^n)/\sqrt{5}, \qquad L_n = a^n + b^n,$$

where

$$a = \tfrac{1}{2}(1 + \sqrt{5}), \qquad b = \tfrac{1}{2}(1 - \sqrt{5}) ;$$

$$\begin{aligned} (-1)^k F_n F_{m-k} &= (-1)^k (a^n - b^n)(a^{m-k} - b^{m-k}) \div 5 \\ &= (-1)^k (a^{m+n-k} - b^{n-m+k}(ab)^{m-k} - a^{n-m+k}(ab)^{m-k} + b^{m+n-k})/5 \\ &= \tfrac{1}{5}(-1)^k L_{m+n-k} - \tfrac{1}{5}(-1)^m L_{n-m+k} , \end{aligned}$$

since $ab = -1$. Similarly,

$$(-1)^m F_k F_{n-m} = \frac{1}{5}(-1)^m L_{n+k-m} - \frac{1}{5}(-1)^n L_{k-n+m}$$

and

$$(-1)^n F_m F_{k-n} = \frac{1}{5}(-1)^n L_{m+k-n} - \frac{1}{5}(-1)^k L_{m-k+n} .$$

Adding these three expressions, the term on the R. H. S. vanish, yielding the desired result.

Also solved by Herta T. Freitag, R. Garfield, C. B. A. Peck, David Zeitlin, and the Proposer.

AN ANALOGUE OF B-228 GENERALIZED

B-229 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Using the recursion formulas to extend the definition of F_n and L_n to all integers n, prove that for all integers k, m, and n

$$(-1)^k L_n F_{m-k} + (-1)^m L_k F_{n-m} + (-1)^n L_m F_{k-n} = 0 .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

To solve B-228 and B-229 simultaneously, we let $\{H_n\}$ satisfy $H_{n+2} = H_{n+1} + H_n$. Then it is well known that

$$(1) \qquad (-1)^a H_i F_j = H_{a+i} F_{a+j} - H_{a+i+j} F_a .$$

In (1) we let $(a, i, j) = (k, n, m - k)$, $(m, k, n - m)$, and $(n, m, k - n)$ and add the results to obtain

$$(-1)^k H_n F_{m-k} + (-1)^m H_k F_{n-m} + (-1)^n H_m F_{k-n} = 0 ,$$

which contains B-228 and B-229 as special cases.

Also solved by Paul S. Bruckman, Herta T. Freitag, R. Garfield, C. B. A. Peck, and the Proposer.

A SIMPLE RESULT, GENERALIZED

B-230 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $\{C_n\}$ satisfy

$$C_{n+4} - 2C_{n+3} - C_{n+2} + 2C_{n+1} + C_n = 0$$

and let

$$G_n = C_{n+2} - C_{n+1} - C_n.$$

Prove that $\{G_n\}$ satisfies $G_{n+2} = G_{n+1} + G_n$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Theorem 1. Let A and B be real constants, and let

$$W_{n+4} = AW_{n+3} + BW_{n+2} + (3 - B - 2A)W_{n+1} + (2 - A - B)W_n$$

for $n = 0, 1, \cdots$. Let

$$Q_{n+2} = W_{n+2} + (1 - A)W_{n+1} + (2 - A - B)W_n .$$

Then

$$Q_{n+2} = Q_{n+1} + Q_n, \qquad n = 0, 1, \cdots .$$

Theorem 1 is proved easily and gives the desired result for $A = 2$ and $B = 1$. We also have

Theorem 2. Let A be a real constant and let

$$W_{n+3} = AW_{n+2} + (2 - A)W_{n+1} + (1 - A)W_n$$

for $n = 0, 1, \cdots$. Let

$$Q_n = W_{n+1} + (1 - A)W_n .$$

Then

$$Q_{n+2} = Q_{n+1} + Q_n, \qquad n = 0, 1, \cdots .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, R. Garfield, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Richard W. Sielaff, A. Sivasubramanian, and the Proposer.

GENERALIZED FIBONACCI SEQUENCES

B-231 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

A GFS (generalized Fibonacci sequence) H_0, H_1, H_2, $\cdots$ satisfies the same recursion formula $H_{n+2} = H_{n+1} + H_n$ as the Fibonacci sequence but may have any intial values. It is known that

$$H_n H_{n+2} - H_{n+1}^2 = (-1)^n c ,$$

where the constant c is characteristic of the sequence. Let $\{H_n\}$ and $\{K_n\}$ be GFS and let

$$C_n = H_0K_n + H_1K_{n-1} + H_2K_{n-2} + \cdots + H_nK_0 .$$

Show that

$$C_{n+2} = C_{n+1} + C_n + G_n ,$$

where $\{G_n\}$ is a GFS whose characteristic is the product of those of $\{H_n\}$ and $\{K_n\}$.

Solution by Paul S. Bruckman, San Rafael, California.

Let $G_n = C_{n+2} - C_{n+1} - C_n$. By the definition of C_n, we obtain:

$$G_n = \sum_{i=0}^{n+2} H_iK_{n+2-i} - \sum_{i=0}^{n+1} H_iK_{n+1-i} - \sum_{i=0}^{n} H_iK_{n-i}$$

$$= H_{n+2}K_0 + H_{n+1}K_1 - H_{n+1}K_0 + \sum_{i=0}^{n} H_i\,(K_{n+2-i} - K_{n+1-i} - K_{n-i})$$

$$= H_{n+2}K_0 + H_{n+1}K_1 - H_{n+1}K_0$$

(since the terms in the summation vanish)

$$= (H_{n+1} + H_n)K_0 + H_{n+1}K_1 - H_{n+1}K_0 = H_{n+1}K_1 + H_nK_0 .$$

Substituting the latter expression for G_n in the following, we obtain:

$$\begin{aligned} G_{n+1}G_{n-1} - G_n^2 &= (H_{n+2}K_1 + H_{n+1}K_0)(H_nK_1 + H_{n-1}K_0) - (H_{n+1}K_1 + H_nK_0)^2 \\ &= H_{n+2}H_nK_1^2 + H_nH_{n+1}K_0K_1 + H_{n+2}H_{n-1}K_0K_1 + H_{n+1}H_{n-1}K_0^2 \\ &\quad - H_{n+1}^2K_1^2 - 2H_nH_{n+1}K_0K_1 - H_n^2K_0^2 \\ &= K_1^2(H_{n+2}H_n - H_{n+1}^2) + K_0K_1(H_nH_{n+1} + H_{n+2}H_{n-1} - 2H_nH_{n+1}) \\ &\quad + K_0^2(H_{n+1}H_{n-1} - H_n^2) \quad . \end{aligned}$$

The coefficient of K_1^2 in the above expression, by hypothesis, is equal to $(-1)^nc$. The coefficient of K_0K_1 may be expressed as:

$$\begin{aligned} H_{n+2}H_{n-1} - H_nH_{n+1} &= (H_{n+1} + H_n)H_{n-1} - H_n(H_n + H_{n-1}) \\ &= H_{n+1}H_{n-1} - H_n^2 = (-1)^{n-1}c = -(-1)^nc \ . \end{aligned}$$

The coefficient of K_0^2 is also equal to $-(-1)^nc$. Therefore,

$$\begin{aligned} G_{n+1}G_{n-1} - G_n^2 &= (-1)^nc(K_1^2 - K_0K_1 - K_0^2) = (-1)^nc\ K_1^2 - K_0(K_1 + K_0) \\ &= (-1)^nc(K_1^2 - K_0K_2) = (-1)^{n-1}cd \quad , \end{aligned}$$

where d is the characteristic of the sequence $\{K_n\}$. It remains now to prove that $\{G_n\}$ is a GFS. Using the expression $G_n = H_{n+1}K_1 + H_nK_0$, derived above, we see that

$$G_{n+2} - G_{n+1} - G_n = (H_{n+3} - H_{n+2} - H_{n+1})K_1 + (H_{n+2} - H_{n+1} - H_n)K_0 = 0 .$$

Also solved by R. Garfield, C. B. A. Peck, and the Proposer.

[Continued from page 84.]

$$\text{(IX)} \qquad \sum_{k=0}^{p} \binom{p}{k} c_1^{r(p-k)} c_2^{rk} f(x + c_1^{m(p-k)} c_2^{mk}) = \sum_{n=0}^{\infty} \frac{V_{mn+r}^p}{n!} D^n f(x) ,$$

$$\text{(X)} \qquad \sum_{k=0}^{p} \left[(-1)^k \binom{p}{k} c_1^{r(p-k)} c_2^{rk} f(x + c_1^{m(p-k)} c_2^{mk}) \right] \Big/ (c_1 - c_2)^p$$

$$= \sum_{n=0}^{\infty} \frac{U_{mn+r}^p}{n!} D^n f(x) .$$

David Zeitlin
Minneapolis, Minnesota

Dear Editor:

I recently noted problem H-146 in Vol. 6, No. 6 (December 1968), p. 352, by J. A. H. Hunter of Toronto. (I am a slow reader.) I don't know whether you have printed a solution as yet; in any case, the answer is in a paper by Wilhelm Ljunggren, Vid.-Akad. Avhandlinger I, NR. 5 (Oslo 1942).

Indeed, $P_7 = 169$ is the only non-trivial square Pell number.

Ernst M. Cohn
Washington, D.C.

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