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SOME GENERALIZATIONS SUGGESTED BY GOULD'S SYSTEMATIC TREATMENT OF CERTAIN BINOMIAL IDENTITIES

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In a previous article [1], the writer has presented properties of certain numbers A_n defined by the generating function

$$f(n) = (1 - u)^{-1}(1 + u)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n u^n \quad . \tag{1}$$

In addition, Professor H. W. Gould of West Virginia University, in a recently published paper [2], has indicated several additional identities for the A_n coefficients.

We now introduce the generalized numbers $A_n(x)$ defined by the generating function

$$g(u,x) = (1 - u)^{-1}(1 + u)^{x} = \sum_{n=0}^{\infty} A_n(x)\, u^n \quad , \tag{2}$$

which is valid for all real or complex x; from this, the following relations are evident:

$$f(u) = g(u, -\tfrac{1}{2}) \tag{3}$$

$$A_n = A_n(-\tfrac{1}{2}) \tag{4}$$

$$A_n(x) = \sum_{k=0}^{n} \binom{x}{k} \quad , \tag{5}$$

where the combinatorial coefficients satisfy the basic relations:

$$\binom{x}{k} = \frac{x^{(k)}}{k!} = \frac{x(x - 1) \cdots (x - k + 1)}{k!}; \qquad \binom{x}{0} = 1 \quad . \tag{6}$$

$$\binom{-x}{k} = (-1)^k \binom{x + k - 1}{k} \quad . \tag{7}$$

A useful special result is the identity:

$$\binom{-\frac{1}{2}}{k} = (-\tfrac{1}{4})^k \binom{2k}{k}$$

The purpose of this paper is to present some properties of the coefficients $A_n(x)$. Gould [2] has demonstrated that most of the identities shown in [1] are old results, citing numerous references to substantiate this claim. Likewise, in private communications with the writer, Gould has indicated that the coefficients $A_n(x)$ have been studied extensively by previous mathematicians. However, Gould [2] indicated that one identity proven in [1] appeared to be new in the literature, and restated it in the following form:

$$A_n^2 = \left\{ \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \right\}^2 = \binom{-\frac{1}{2}}{n} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{n-k} \frac{2n+1}{2k+1} \quad . \tag{8}$$

Gould, who as well as being a mathematician of the highest order, is an expert in the field of information retrieval, was impressed by the apparent novelty of the relation in (8), and his closing remarks in [2] stimulated a search for a suitable generalization of (8). This search was initiated by the writer in an effort to find a single-sum expression for the coefficient $A_n^2(x)$. In this respect, he has failed. However, the writer did discover an unexpected generalization of (8) by empirical methods, and this is expressed in the following elegant form:

$$\begin{aligned} A_n(x-\tfrac{1}{2})A_n(-x-\tfrac{1}{2}) &= \left\{ \sum_{k=0}^{n} \binom{x-\frac{1}{2}}{k} \right\} \cdot \left\{ \sum_{k=0}^{n} \binom{-x-\frac{1}{2}}{k} \right\} \\ &= \tfrac{1}{2}\binom{x-\frac{1}{2}}{n} \sum_{k=0}^{n} \binom{-x-\frac{1}{2}}{n-k} \frac{x-\frac{1}{2}-n}{x-\frac{1}{2}-k} + \tfrac{1}{2}\binom{-x-\frac{1}{2}}{n} \sum_{k=0}^{n} \binom{x-\frac{1}{2}}{n-k} \frac{x+\frac{1}{2}+n}{x+\frac{1}{2}+k} \quad . \end{aligned} \tag{9}$$

It is easily seen that when $x = 0$, Eq. (9) reduces to (8). Relation (9) would appear to be a new combinatorial identity.

Before we furnish a proof of (9), we will present a list of various identities involving the $A_n(x)$ coefficients, each identity accompanied with a brief indication of the method used in its derivation. The purpose is to familiarize the reader with some of the known results. The $A_n(x)$'s satisfy the following second-order recursion:

$$(n+1)A_{n+1}(x) - (x+1)A_n(x) + (x-n)A_{n-1}(x) = 0 \, . \tag{10}$$

Recursion (10) is easily verified from the definition of $A_n(x)$ in (5). For $x = -\frac{1}{2}$, it becomes recursion (7) in [1].

$$A_n(x) = (-1)^n \sum_{k=0}^{[n/2]} (-1)^k \binom{x}{k} \binom{-x-1}{n-2k} \quad ; \tag{11}$$

derived by expressing $g(u,x)$ in the form $(1-u)^{-1-x}(1-u^2)^x$, and obtaining the convolute.

For $x = \frac{1}{2}$, (11) becomes formula (11) in [2].

$$A_n(x) = (-1)^n \binom{x}{n} \sum_{k=0}^{n} (-2)^k \binom{n}{k} \frac{x - n}{x - k} \quad ; \tag{12}$$

derived by expressing $g(u,x)$ in the form

$$(1 - u)^{x-1} \left(1 + \frac{2u}{1 - u}\right)^x ,$$

and obtaining the convolute. For $x = -\frac{1}{2}$, this becomes formula (12) in [2], which was previously stated in variant form as formula (22) in [1]. NOTE: Identity (12) has been submitted to Advanced Problems Editor as a proposed problem.

$$A_n(x) = \binom{x}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k}} 2^k \frac{x - n}{x - k} \quad ; \tag{13}$$

derived by obtaining the convolute of the function $g(u,x)$ expressed in the form

$$(1 + u)^{x-1} \left(1 - \frac{2u}{1 + u}\right)^{-1} .$$

For $x = -\frac{1}{2}$, this becomes formula (13) in [2].

$$\begin{aligned} e^{\frac{1}{2}u^2} \int_0^u t^{-2x-1} e^{-t^2} dt &= \sum_{n=0}^{\infty} \frac{u^{2n}}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k-2x}}{k!\,(2k - 2x)} \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n A_n(x) u^{2n-2x}}{(2n - 2x)\binom{x}{n} n!} \quad ; \end{aligned} \tag{14}$$

(using (12) above). For $x = -\frac{1}{2}$, this becomes formula (11) in [1], restated by Gould as formula (14) in [2].

$$2^n = \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{x}{k}} A_k(x) \frac{x - n}{x - k} \tag{15}$$

$$(-2)^n = \binom{x}{n} \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{x}{k}} A_k(x) \frac{x - n}{x - k} \quad . \tag{16}$$

Relations (15) and (16) are obtained from (12) and (13) by inversion. For $x = -\frac{1}{2}$, (15) and (16) become (15) and (16), respectively, in [2], which Gould obtained by the same method.

$$2^{n-1} \left\{ 1 + (-1)^n \binom{x}{n}^{-1} \right\} = \sum_{k=0}^{[n/2]} \frac{\binom{n}{2k}}{\binom{x}{2k}} A_{2k}(x) \frac{x - n}{x - 2k} \tag{17}$$

$$(18)\qquad 2^{n-1}\left\{1 - (-1)^n \binom{x}{n}^{-1}\right\} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{\binom{n}{2k+1}}{\binom{x}{2k+1}} A_{2k+1}(x)\,\frac{x-n}{x-2k-1} \quad .$$

Relations (17) and (18) are obtained by respectively adding and subtracting (15) and (16). When $x = -\frac{1}{2}$, (17) is equivalent to (17) in [2].

$$(19)\qquad (1-u^2)^{-x-1}\int_0^u t^{-2x-1}(1-t^2)^x\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n-2x)\binom{x}{n}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}A_n(x)}{(x-n)\binom{x}{n}} \cdot \frac{u^{2n-2x}}{(2-u^2)^{n+1}} \quad .$$

$$(20)\qquad \int_0^u \frac{t^{-2x-1}}{1+t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n-2x)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}A_n(x)}{(x-n)\binom{x}{n}} \cdot \frac{u^{2n-2x}}{(2+u^2)^{n+1}}$$

$$(21)\qquad A_n(x) = (2n-2x)\binom{x}{n}\int_0^1 t^{-2x-1}(2t^2-1)^n\,dt \quad .$$

A proof of (19) is indicated below. Let the left-hand side of (19) = y, i.e.,

$$y = \sum_{n=0}^{\infty}\binom{-x-1}{n}(-1)^n u^{2n} \sum_{k=0}^{\infty}\binom{x}{k}\frac{(-1)^k u^{2k-2x}}{(2k-2x)} = \sum_{n=0}^{\infty}\theta_n u^{2n-2x} \quad ,$$

or

$$y = \sum_{k=0}^{\infty}\binom{x}{k}\frac{(-1)^k u^{2k-2x}}{(2k-2x)} \sum_{n=k}^{\infty}\binom{-x-1}{n-k}(-1)^{n-k}u^{2n-2k}$$

$$= \sum_{n=0}^{\infty}(-1)^n u^{2n-2x} \sum_{k=0}^{n}\frac{\binom{x}{k}\binom{-x-1}{n-k}}{(2k-2x)} \quad .$$

We will return to the above expression, but first we will direct our efforts toward finding an expression for θ_n, as defined above. If we differentiate the integral expression for y and its series form:

$$y' = (1-u^2)^{-x-1}u^{-2x-1}(1-u^2)^x + 2u(x+1)(1-u^2)^{-1}y = \sum_{n=0}^{\infty}(2n-2x)\,\theta_n u^{2n-2x-1} \quad .$$

$$\therefore\ (1 - u^2)y' = u^{-2x-1} + 2u(x + 1)y\,;$$

converting this differential equation into series form by means of the foregoing relations, we obtain a recursion: $(n + 1 - x)\theta_{n+1} = (n + 1)\theta_n (n = 0, 1, 2, \cdots)$; $1 = -2x\theta_0$ $(x \neq 0)$. By an easy induction on this last recursion, we obtain the expression

$$\theta_n = (-1)^{n+1} \Big/ 2(x - n)\binom{x}{n};$$

this proves the first identity of (19). We may convert such form as follows, by use of (16):

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n-2x}}{(2n - 2x)\binom{x}{n}} = \sum_{n=0}^{\infty} \frac{2^{-n} u^{2n-2x}}{(2n - 2x)\binom{x}{n}} \binom{x}{n} (x - n) \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{x}{k}} \frac{A_k(x)}{(x - k)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k A_k(x)}{(x - k)\binom{x}{k}} \sum_{n=k}^{\infty} -2^{-n-1}\binom{n}{k} u^{2n-2x}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k A_k(x)}{(x - k)\binom{x}{k}} \sum_{n=0}^{\infty} -2^{-n-k-1}\binom{n + k}{k} u^{2n+2k-2x}$$

$$y = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} A_k(x)\, 2^{-k-1}}{(x - k)\binom{x}{k}} u^{2k-2x} (1 - \tfrac{1}{2}u^2)^{-k-1}\ ,$$

which reduces to (19). If we return to the double summation expression for y which we first obtained, we arrive at the interesting identity:

$$\frac{1}{(x - n)\binom{x}{n}} = \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{-x - 1}{n - k}}{x - k}\ . \tag{22}$$

Relation (20) is similarly obtained from (15) being substituted in the first identity of (20), which is readily obtained from the integral expression by direct integration of a geometric series.

Relation (21) is derived from (12), and may be verified by expansion of the integrand in (21), term-by-term integration and comparison with (12).

$$\sum_{n=0}^{\infty} \frac{A_n(x)}{(x - n)\binom{x}{n} n!n!} (\tfrac{1}{2}u)^{2n} = \sum_{n=0}^{\infty} \frac{J_n(u)\, u^n}{(x - n)n!}\ . \tag{23}$$

Relation (23) is obtained by employing the definition of $J_n(u)$, the ordinary Bessel function of n^{th} order, and relation (12).

$$(24)\qquad \int_0^u t^{-2x-1}(1-t^2)^x dt = \sum_{n=0}^{\infty} (-1)^n \binom{x}{n} \frac{u^{2n-2x}}{2n-2x} .$$

Relation (24) is obtained by expansion of the integrand and term-by-term integration of the result.

By the substitution of $x = -\frac{1}{2}$ in (19)-(21), (23) and (24), we obtain Gould's identities (18), (19), (21) (in variant form), (28) and (35) in [2].

By the substitutions $u = iv$, $t = is$ in (19), (20) and (24) (and reconverting back to the dummy variables u and t), we derive the following:

$$(25)\qquad (1+u^2)^{-x-1} \int_0^u t^{-2x-1}(1+t^2)^x dt = \sum_{n=0}^{\infty} \frac{u^{2n-2x}}{(2n-2x)\binom{x}{n}}$$

$$= \sum_{n=0}^{\infty} \frac{A_n(x)}{(n-x)\binom{x}{n}} \frac{u^{2n-2x}}{(2+u^2)^{n+1}}$$

$$(26)\qquad \int_0^u \frac{t^{-2x-1}}{1-t^2} dt = \sum_{n=0}^{\infty} \frac{u^{2n-2x}}{2n-2x} = \sum_{n=0}^{\infty} \frac{A_n(x)}{(n-x)\binom{x}{n}} \frac{u^{2n-2x}}{(2-u^2)^{n+1}}$$

$$(27)\qquad \int_0^u t^{-2x-1}(1+t^2)^x dt = \sum_{n=0}^{\infty} \binom{x}{n} \frac{u^{2n-2x}}{2n-2x} .$$

If, in (19), we make the substitutions

$$\frac{u^2}{2-u^2} = v^2 , \qquad \frac{t^2}{2-t^2} = s^2$$

(and then reconvert to the dummy variables u and t), we obtain:

$$(28)\qquad (1-u^2)^{-x-1} \int_0^u t^{-2x-1} \frac{(1-t^2)^x}{1+t^2} dt = \sum_{n=0}^{\infty} (-1)^n \frac{A_n(x)\, u^{2n-2x}}{(2n-2x)\binom{x}{n}} .$$

The "conjugate" of (28) is the following:

$$(29)\qquad (1+u^2)^{-x-1}\int_0^u t^{-2x-1}\frac{(1+t^2)^x}{1-t^2}\,dt = \sum_{n=0}^{\infty}\frac{A_n(x)\,u^{2n-2x}}{(2n-2x)\binom{x}{n}}\,.$$

Another genus of relations is obtained by considering variations in form of the basic definition of $g(u,x)$ in (2), or related functions. For example, since

$$(1+u)^{x+r}(1+u)^{-x+s} = (1+u)^{r+s}\,,$$

we arrive at

$$(30)\qquad \sum_{k=0}^{n}\binom{x+r}{k}\binom{-x+s}{n-k} = \binom{r+s}{n}\,.$$

This is simply a special case of the Vandermonde convolution theorem; its chief point of interest here is the invariance of (30) with respect to x. Setting $r = 0$ and $s = -1$, as a special case of (30), we have:

$$(31)\qquad \sum_{k=0}^{n}\binom{x}{k}\binom{-x-1}{n-k} = \binom{-1}{n} = (-1)^n\,.$$

By considering the convolution of the expression

$$(1+u)^{x+r}(1+u)^{-x+s}(1-u)^{-1} = (1+u)^{r+s}(1-u)^{-1}\,,$$

we obtain the following identity:

$$(32)\qquad \sum_{k=0}^{n}\binom{x+r}{n-k}A_k(-r+s) = A_n(r+s)\,.$$

Again, the interesting point in (32) is the invariance with respect to x of the right member. When $r = 0$, $s = x$, we obtain the expression in (5) for $A_n(x)$. By setting $x = 0$ and $s = x$ in (32), we obtain the recursion:

$$(33)\qquad A_n(x+r) = \sum_{k=0}^{n}\binom{r}{k}A_{n-k}(x)\,.$$

Another interesting identity displaying invariance on x is obtained by considering the convolution of $(1+u)^{x+r}(1-u)^{-1}\cdot(1+u)^{-x+s}(1-u)^{-1} = (1+u)^{r+s}(1-u)^{-2}$.

$$(34)\qquad \sum_{k=0}^{n} A_k(x + r)\, A_{n-k}(-x + s) = (n + 1)A_n(r + s) - (r + s)A_{n-1}(r + s - 1) .$$

As special cases of (34), for $r = 0$, $s = 0$, we obtain:

$$(35)\qquad \sum_{k=0}^{n} A_k(x)\, A_{n-k}(-x) = n + 1 .$$

For $r = 0$, $s = -1$, (34) yields:

$$(36)\qquad \sum_{k=0}^{n} A_k(x)\, A_{n-k}(-x - 1) = 1 + [\tfrac{1}{2}n] .$$

By considering the sum

$$\sum_{k=0}^{r-1} \frac{(1 + u)^{x+k}}{1 - u} = \frac{(1 + u)^x}{1 - u} \cdot \frac{(1 + u)^r - 1}{u} ,$$

we obtain the following recursion:

$$(37)\qquad \sum_{k=0}^{r-1} A_n(x + k) = \sum_{k=0}^{n} \binom{r}{n - k + 1} A_k(x) .$$

For $r = 2$, we obtain as a special case of (37):

$$(38)\qquad A_n(x + 1) = A_n(x) + A_{n-1}(x) .$$

We may also derive (38) by letting $r = 1$ in (33). As should otherwise be evident, this is the same recursion satisfied by the binomial coefficients, i.e., if $\binom{x}{n}$ is substituted for $A_n(x)$.

The list of identities in (10)-(38) is by no means exhaustive, and indeed it should have by now become evident to the reader that the variety of derivable identities stemming from the basic definition in (2) is virtually unlimited. As previously intimated, Gould [2] has observed that far more general results are available in the existing literature, and it is primarily for this reason that (10)-(38) have been offered with a minimum of explanation. The real purpose of this paper is to give a proof of (9), and the other identities have been presented

solely for the sake of exposition. The proof of (9), which follows, depends on differential equations and the method of equating coefficients. The writer was unable to obtain a more direct proof, and this is left as a project for the interested reader.

We begin by adopting the following definitions, in the interest of simplicity of expression:

$$C_n = A_n(x - \tfrac{1}{2}) ; \quad \overline{C}_n = A_n(-x - \tfrac{1}{2}) \tag{39}$$

$$Q_n = C_n \overline{C}_n \tag{40}$$

$$R_n = Q_n - Q_{n-1} \tag{41}$$

$$J_n = \binom{x - \frac{1}{2}}{n}; \quad \overline{J}_n = \binom{-x - \frac{1}{2}}{n} \tag{42}$$

$$K_n = J_n \overline{J}_n \tag{43}$$

$$q = K_1 = \tfrac{1}{4} - x^2 \quad . \tag{44}$$

Some useful relations are indicated below, which are evident from the definitions given in (39)-(44):

$$C_n = C_{n-1} + J_n ; \quad \overline{C}_n = \overline{C}_{n-1} + \overline{J}_n \tag{45}$$

$$J_n = \left(\frac{x + \frac{1}{2} - n}{n}\right) J_{n-1} ; \quad \overline{J}_n = \left(\frac{-x + \frac{1}{2} - n}{n}\right) \overline{J}_{n-1} \tag{46}$$

$$K_n = \left\{\frac{(\frac{1}{2} - n)^2 - x^2}{n^2}\right\} K_{n-1} = \left\{\frac{q + n(n-1)}{n^2}\right\} K_{n-1} \quad . \tag{47}$$

Our aim is to first obtain a recursion for the coefficients Q_n, then to show that the same recursion, with the same initial conditions, is satisfied by the expression in the right member of (9). The following development makes free use of the relations and definitions in (39)-(47):

$$R_n = C_n \overline{C}_n - C_{n-1} \overline{C}_{n-1} = C_n \overline{C}_n - (C_n - J_n)(\overline{C}_n - \overline{J}_n) \ ,$$

or

$$R_n = J_n \overline{C}_n + \overline{J}_n C_n - K_n \ . \tag{48}$$

If we increase the subscript in (48) by unity, multiply by $(n + 1)$, and apply (45)-(47), we obtain:

$$(n + 1)R_{n+1} = (n + 1)J_{n+1}(\overline{C}_n + \overline{J}_{n+1}) + (n + 1)\overline{J}_{n+1}(C_n + J_{n+1}) - (n + 1)K_{n+1} \ ,$$

or

$$(n + 1)R_{n+1} = (x - \tfrac{1}{2} - n)J_n \overline{C}_n + (-x - \tfrac{1}{2} - n)\overline{J}_n C_n + (n + 1)K_{n+1} \ . \tag{49}$$

If we decrease the subscript in (48) by unity, and again use relations (45)-(47), we obtain:

$$R_{n-1} = \overline{C}_{n-1}J_{n-1} + C_{n-1}\overline{J}_{n-1} - K_{n-1} = (\overline{C}_n - \overline{J}_n)\left(\frac{n}{x + \frac{1}{2} - n}\right)J_n$$

$$+ (C_n - J_n)\left(\frac{n}{-x + \frac{1}{2} - n}\right)\overline{J}_n - \left(\frac{n^2}{q + n(n-1)}\right)K_n \quad .$$

Multiplying the above throughout by

$$\frac{(x + \frac{1}{2} - n)(-x + \frac{1}{2} - n)}{n} = \frac{q + n(n-1)}{n}, \quad \frac{(q + n(n-1))}{n}R_{n-1} = (-x + \frac{1}{2} - n)(\overline{C}_n - \overline{J}_n)J_n$$

$$+ (x + \frac{1}{2} - n)(C_n - J_n)\overline{J}_n - nK_n,$$

or

$$\left(\frac{q}{n} + n - 1\right)R_{n-1} = (-x + \frac{1}{2} - n)J_n\overline{C}_n + (x + \frac{1}{2} - n)\overline{J}_n C_n + (n-1)K_n \quad . \tag{50}$$

If we now multiply (48) throughout by $2n$ and add this result to the sum of (49) and (50), we obtain the following recursion:

$$(n+1)R_{n+1} + 2nR_n + \left(\frac{q}{n} + n - 1\right)R_{n-1} = (n+1)(K_{n+1} - K_n) \quad . \tag{51}$$

If, in (51), we substitute for R_n the expression $Q_n - Q_{n-1}$ from (41), and similarly for the other subscripts, we obtain a third-order recursion involving the Q_n's:

$$(n+1)Q_{n+1} + (n-1)Q_n + \left(\frac{q}{n} - n - 1\right)Q_{n-1} - \left(\frac{q}{n} + n - 1\right)Q_{n-2}$$

$$= (n+1)(K_{n+1} - K_n) \quad . \tag{52}$$

This, then, is the recursion which we now seek to demonstrate is also satisfied by the expression in the right member of (9).

We begin by introducing some additional definitions, again for the sake of brevity:

$$P_n = J_n \sum_{k=0}^{n} \overline{J}_{n-k} \frac{x - \frac{1}{2} - n}{x - \frac{1}{2} - k}; \quad \overline{P}_n = \overline{J}_n \sum_{k=0}^{n} J_{n-k} \frac{x + \frac{1}{2} + n}{x + \frac{1}{2} + k} \tag{53}$$

$$h = w(u,x) = -(1+u)^{-x-\frac{1}{2}} \int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt; \quad \overline{h} = w(u,-x) \quad . \tag{54}$$

The statement of identity (9) may then be condensed to the simple form:

$$Q_n = \tfrac{1}{2}(P_n + \overline{P}_n) \quad . \tag{55}$$

Since

$$(1+u)^{-x-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-x - \frac{1}{2}}{n}u^n \quad ,$$

and

$$-\int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt = \sum_{n=0}^{\infty} \frac{u^{-x+\frac{1}{2}+n}}{x-\frac{1}{2}-n} ,$$

$$h = \sum_{n=0}^{\infty} u^{-x+\frac{1}{2}+n} \sum_{k=0}^{n} \binom{-x-\frac{1}{2}}{n-k} \frac{1}{x-\frac{1}{2}-k} = \sum_{n=0}^{\infty} u^{-x+\frac{1}{2}+n} \sum_{k=0}^{n} \frac{\overline{J}_{n-k}}{x-\frac{1}{2}-k} .$$

Comparing the latter expression with the first definition in (53), we have:

$$h = \sum_{n=0}^{\infty} \frac{P_n u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n}; \quad \text{similarly,} \quad \overline{h} = \sum_{n=0}^{\infty} \frac{\overline{P}_n u^{+x+\frac{1}{2}+n}}{(-x-\frac{1}{2}-n)\overline{J}_n} . \tag{56}$$

By differentiating h, we obtain the expressions:

$$h' = \sum_{n=-1}^{\infty} \frac{-P_{n+1} u^{-x+\frac{1}{2}+n}}{J_{n+1}} = -P_0 u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{-(n+1)P_{n+1} u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} \tag{57}$$

$$\begin{aligned} h'' &= (x+\tfrac{1}{2})P_0 u^{-x-3/2} + \sum_{n=-1}^{\infty} \frac{(n+2)P_{n+2} u^{-x+\frac{1}{2}+n}}{J_{n+1}} \\ &= (x+\tfrac{1}{2})P_0 u^{-x-3/2} + P_1 u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{(n+1)(n+2)P_{n+2} u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} . \end{aligned} \tag{58}$$

On the other hand, if we differentiate h, as defined in (54), we obtain:

$$\begin{aligned} h' &= -(1+u)^{-x-\frac{1}{2}} u^{-x-\frac{1}{2}} (1-u)^{-1} + (x+\tfrac{1}{2})(1+u)^{-x-3/2} \int_0^u \frac{t^{-x-\frac{1}{2}}}{1-t}\,dt \\ &= -(1+u)^{-x-\frac{1}{2}} u^{-x-\frac{1}{2}} (1-u)^{-1} - (x+\tfrac{1}{2})(1+u)^{-1} h , \end{aligned}$$

or:

$$(1-u^2)h' + (x+\tfrac{1}{2})(1-u)h + (1+u)^{-x+\frac{1}{2}} u^{-x-\frac{1}{2}} = 0 . \tag{59}$$

By a second differentiation,

$$\begin{aligned} (1-u^2)h'' - 2uh' + (x+\tfrac{1}{2})(1-u)h' - (x+\tfrac{1}{2})h &= \left\{\frac{x+\frac{1}{2}}{u} + \frac{x-\frac{1}{2}}{1+u}\right\}(1+u)^{-x+\frac{1}{2}} u^{-x-\frac{1}{2}} \\ &= -\left\{\frac{x+\frac{1}{2}}{u} + \frac{x-\frac{1}{2}}{1+u}\right\}\left\{(1-u^2)h' + (x+\tfrac{1}{2})(1-u)h\right\} , \end{aligned}$$

or after simplification:

$$\begin{aligned} (u+u^2-u^3-u^4)h'' + \left\{(x+\tfrac{1}{2}) + (3x+\tfrac{1}{2})u - (x+5/2)u^2 - (3x+5/2)u^3\right\}h' \\ + \left\{(x+\tfrac{1}{2})^2 + (x+\tfrac{1}{2})(x-3/2)u - 2(x+\tfrac{1}{2})^2 u^2\right\}h = 0 . \end{aligned} \tag{60}$$

By means of the series form for h in (56), for h' in (57), and the identity:

$$(1+u)^{-x+\frac{1}{2}}u^{-x-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-x+\frac{1}{2}}{n}u^{-x-\frac{1}{2}+n} = u^{-x-\frac{1}{2}} + \sum_{n=0}^{\infty}\frac{(-x+\frac{1}{2})}{n+1}\overline{J}_n u^{-x+\frac{1}{2}+n} \quad ,$$

we may convert (59) entirely into series form, with certain manipulations based on properties of the binomial coefficients, designed to maintain the exponent of u in the various expressions the same ($-x+\frac{1}{2}+n$, in our development), and to contain the factor J_n in the denominator of each expression, which may subsequently be cancelled. If we do so, we obtain the following:

$$\sum_{n=0}^{\infty}\frac{-(n+1)P_{n+1}u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} + \sum_{n=1}^{\infty}\frac{(x+\frac{1}{2}-n)P_{n-1}u^{-x+\frac{1}{2}+n}}{nJ_n}$$

$$+(x+\tfrac{1}{2})\sum_{n=0}^{\infty}\frac{P_n u^{-x+\frac{1}{2}+n}}{(x-\frac{1}{2}-n)J_n} - (x+\tfrac{1}{2})\sum_{n=1}^{\infty}\frac{P_{n-1}u^{-x+\frac{1}{2}+n}}{nJ_n} + \sum_{n=0}^{\infty}\frac{(-x+\frac{1}{2})}{n+1}\overline{J}_n u^{-x+\frac{1}{2}+n}$$

$$= 0 \ .$$

If we now equate the coefficients of similar powers of u in the above expression and simplify by multiplying throughout by $(x-\frac{1}{2}-n)J_n$, we obtain:

$$P_1 = q+2x\,; \quad -(n+1)P_{n+1} - (x-\tfrac{1}{2}-n)P_{n-1} + (x+\tfrac{1}{2})P_n = (x-\tfrac{1}{2})(x-\tfrac{1}{2}-n)\frac{K_n}{n+1},$$

or

$$(61)\quad (n+1)(P_{n+1}-P_{n-1}) - (x+\tfrac{1}{2})(P_n - P_{n-1}) = \left\{\frac{q}{n+1}+x-\tfrac{1}{2}\right\}K_n \qquad (n=1,2,\cdots)$$

$$P_0 = 1, \quad P_1 = q+2x \ .$$

By a similar, though more complicated manipulation of the series in (56)-(58), we may express (60) in series form, yielding another recursion for the P_n's. The development is omitted here, since it is somewhat lengthy. The interested reader may, with a little elbow grease, verify that the following form of the recursion is first obtained:

$$\frac{-(n+1)^2}{x-\frac{1}{2}-n}P_{n+1} + \left\{\frac{x(n+2)+q+n^2+\frac{1}{2}n}{x-\frac{1}{2}-n}\right\}P_n + \left\{\frac{xn-q-n^2-\frac{1}{2}n}{n}\right\}P_{n-1}$$

$$+\frac{q+n^2-n}{n}P_{n-2} = 0 \ , \qquad (n=2,3,4,\cdots) \ ,$$

with

$$P_1 = q+2x, \quad P_0 = 1, \quad P_2 = (\tfrac{1}{2}q-1)^2 + xq \ .$$

By multiplying the latter expression throughout by $n(x - \frac{1}{2} - n)$, simplifying the result, and shifting the terms around, as the reader may verify, the following form of the recursion is obtained:

$$(62)\quad \begin{aligned} &-n(n+1)^2(P_{n+1} - P_{n-1}) + n(q + n(n+\tfrac{1}{2}))(P_n - P_{n-2}) \\ &+ (\tfrac{1}{2}q - n(n+\tfrac{1}{2}))(P_{n-1} - P_{n-2}) + \{n(n+2)(P_n - P_{n-1}) - (q + n(n-1))(P_{n-1} - P_{n-2})\}x \\ &= 0 \end{aligned}$$

(for $n = 2, 3, \cdots$); with $P_0 = 1$, $P_1 = q + 2x$, $P_2 = (\frac{1}{2}q - 1)^2 + xq$.

We may further simplify (61) and (62), if we introduce another symbol:

$$(63)\quad V_n = P_n - P_{n-1} ,$$

which also yields:

$$V_n + V_{n+1} = P_{n+1} - P_{n-1} .$$

By substituting the appropriate expressions in (61) and (62), we obtain:

$$(64)\quad (n+1)(V_{n+1} - K_{n+1}) = (x - \tfrac{1}{2} - n)(V_n + K_n)$$

(making use of the relation

$$\frac{q}{n+1} K_n = (n+1)K_{n+1} - nK_n ,$$

a variant of (47)), and

$$(65)\quad \begin{aligned} &-n(n+1)^2(V_n + V_{n+1}) + n(q + n(n+\tfrac{1}{2}))(V_{n-1} + V_n) + (\tfrac{1}{2}q - n(n+\tfrac{1}{2}))V_{n-1} \\ &+ n(n+2)\,x\,V_n - (q + n(n-1))\,x\,V_{n-1} = 0 . \end{aligned}$$

Rearranging the terms in (64), we obtain an expression for xV_n:

$$(66)\quad xV_n = -xK_n + (n+\tfrac{1}{2})(V_n + K_n) + (n+1)(V_{n+1} - K_{n+1}) .$$

If we substitute the expression in (66) and the corresponding expression with the subscript reduced by unity in (65), again use (47) in variant form, and simplify, (65) is transformed to the following form:

$$(67)\quad (n+1)V_{n+1} + 2nV_n + \left(\frac{q}{n} + n - 1\right)V_{n-1} = (n+1)(K_{n+1} - K_n) + 2xK_n .$$

The reader may verify the simplification to the above form, using the indicated procedure. If we now replace the V_n's in (67) by the corresponding P_n's, in accordance with (63), we readily obtain the following recursion involving the P_n's:

$$(68)\quad \begin{aligned} &(n+1)P_{n+1} + (n-1)P_n + \left(\frac{q}{n} - n - 1\right)P_{n-1} - \left(\frac{q}{n} + n - 1\right)P_{n-2} \\ &= (n+1)(K_{n+1} - K_n) + 2xK_n . \end{aligned}$$

If we replace x by $-x$ in (68), observing that q and the K_n's are even functions of x, we obtain the "conjugate" of (68):

$$(n+1)\overline{P}_{n+1} + (n-1)\overline{P}_n + \left(\frac{q}{n} - n - 1\right)\overline{P}_{n-1} - \left(\frac{q}{n} + n - 1\right)\overline{P}_{n-2} = (n+1)(K_{n+1} - K_n) - 2xK_n \,. \tag{69}$$

If we add (68) and (69) and divide by 2, we obtain the following recursion involving $\frac{1}{2}(P_n + \overline{P}_n)$, the terms involving x cancelling:

$$(n+1)\tfrac{1}{2}(P_{n+1} + \overline{P}_{n-1}) + (n-1)\tfrac{1}{2}(P_n + \overline{P}_n) + \left(\frac{q}{n} - n - 1\right)\tfrac{1}{2}(P_{n-1} + \overline{P}_{n-1}) - \left(\frac{q}{n} + n - 1\right)\tfrac{1}{2}(P_{n-2} + \overline{P}_{n-2}) = (n+1)(K_{n+1} - K_n) \,. \tag{70}$$

Comparing (52) with (70), we see that $\frac{1}{2}(P_n + \overline{P}_n)$ satisfies the same recursion as Q_n. We need to demonstrate only that $Q_n = \frac{1}{2}(P_n + \overline{P}_n)$ for $n = 0$, 1, and 2, to complete the proof of (55), i.e., (9). We have already demonstrated that

$$P_0 = 1, \quad P_1 = q + 2x, \quad P_2 = (\tfrac{1}{2}q - 1)^2 + qx \,.$$

Therefore,

$$\tfrac{1}{2}(P_0 + \overline{P}_0) = 1\,; \quad \tfrac{1}{2}(P_1 + \overline{P}_1) = \tfrac{1}{2}(q + 2x + q - 2x) = q\,; \quad \tfrac{1}{2}(P_2 + \overline{P}_2) = \tfrac{1}{2}\left\{(\tfrac{1}{2}q - 1)^2 + qx + (\tfrac{1}{2}q - 1)^2 - qx\right\} = (\tfrac{1}{2}q - 1)^2 \,.$$

We may verify that

$$C_0 = 1, \quad C_1 = \tfrac{1}{2} + x, \quad C_2 = \tfrac{1}{2}x^2 + 7/8 = 1 - \tfrac{1}{2}q \,,$$

from (5), substituting $x - \frac{1}{2}$ for x. Then

$$Q_0 = 1, \quad Q_1 = (\tfrac{1}{2} + x)(\tfrac{1}{2} - x) = q, \quad Q_2 = (1 - \tfrac{1}{2}q)^2 \,.$$

This completes the proof of (9).

The limits of convergence of the power series in this paper have been ignored, since we have treated these series as formal generating functions of the coefficients under investigation.

It was initially remarked that this study was originally motivated by a desire to find an expression for $A_n^2(x)$ in single-summation form, and that this effort was unsuccessful. However, certain results were obtained which suggest areas of investigation for the interested reader. A recursion for the $A_n^2(x)$'s may be derived in the following manner. We begin by introducing a new definition:

$$T_n = A_n^2(x) - A_{n-1}^2(x) \,. \tag{71}$$

By using the property

$$A_n(x) - A_{n-1}(x) = \binom{x}{n} \,,$$

and recursion (38), we may obtain an alternate expression for T_n:

$$T_n = \{A_n(x) - A_{n-1}(x)\}\{A_n(x) + A_{n-1}(x)\} ,$$

or

$$T_n = \binom{x}{n} A_n(x+1) . \tag{72}$$

Therefore,

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \left(A_k^2(x) - A_{k-1}^2(x) \right) = \sum_{k=1}^{n} \binom{x}{k} A_k(x+1) ,$$

which yields:

$$A_n^2(x) = \sum_{k=0}^{n} \binom{x}{k} A_k(x+1) . \tag{73}$$

Of course, there is the more obvious identity:

$$A_n^2(x) = A_n(x) \sum_{k=0}^{n} \binom{x}{k} , \tag{74}$$

which is simply (5) multiplied by $A_n(x)$.

Neither (73) nor (74), however, are single-summation expressions, since they involve the coefficient $A_k(x)$ (or $A_n(x)$), which is itself a single-summation expression.

The recursion for the $A_n^2(x)$'s is obtained by substituting

$$\frac{T_n}{\binom{x}{n}}$$

for $A_n(x+1)$ in:

$$(n+1)A_{n+1}(x+1) - (x+2)A_n(x+1) + (x+1-n)A_{n-1}(x+1) = 0 ,$$

which is simply (10) with $x+1$ replacing x. By eliminating the combinatorial terms, we first obtain a second-order recursion involving the T_n's:

$$n(n+1)^2 T_{n+1} - n(x+2)(x-n)T_n + (x-n)(x+1-n)T_{n-1} = 0 . \tag{75}$$

By substituting the expression in (71) for the T_n's in (75), we are led to the required recursion:

$$n(n+1)^2 A_{n+1}^2(x) + \{n(x-n) - (x+1)^2\}\{nA_n^2(x) - (x-n)A_{n-1}^2(x)\} - (x-n)(x+1-n)^2 A_{n-2}^2(x) = 0 \tag{76}$$

It should be observed that if the substitution $x = -\frac{1}{2}$ is made in (76), and the substitution $x = 0$ is made in (52), the same recursion results in either case, namely:

$$(77)\qquad n(n+1)^2A_{n+1}^2 - (n^2+\tfrac{1}{2}n+\tfrac{1}{4})(nA_n^2 + (n+\tfrac{1}{2})A_{n-1}^2) + (n+\tfrac{1}{2})(n-\tfrac{1}{2})^2A_{n-2}^2 = 0\ .$$

(It is not immediately obvious that (52) reduces to (77) for $x = 0$, but if we observe that, in such case, $Q_n = A_n^2$ and $q = \frac{1}{4}$, we may use known relations to show that

$$(n+1)(K_{n+1} - K_n)$$

may be expressed in the form:

$$\frac{1}{4n+2}\left\{(8n^2+14n+6)Q_{n+1} - (4n+3)Q_n - (8n^2+10n+3)Q_{n-1}\right\}\ .$$

Substituting this expression in (52), we obtain a form free of terms involving K_n's which reduces to (77).)

In passing, we leave the reader with one possible form of expression for $A_n^2(x)$, which is suggested below by indicating the first few terms:

$$(78)\qquad A_n^2(x) = \binom{2n}{n}\left\{\binom{x}{2n} + \tfrac{1}{2}(n+2)\binom{x}{2n-1} + \left(\frac{n^3+2n^2+3n-4}{8n-4}\right)\binom{x}{2n-2} + \cdots\right\}\ .$$

It is not difficult to prove (78) by induction, as far as it goes, but the subsequent terms become increasingly obscure, as the difficulty in obtaining them also increases. The writer failed to see any pattern in the terms of (78), but that is not to say that one does not exist.

The writer gratefully acknowledges the impetus provided by Professor Gould for this paper, and his invaluable aid in pointing out the known results.

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A PROCEDURE FOR THE ENUMERATION OF 4 x n LATIN RECTANGLES

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Let M_n^r denote the number of normalized (first row in natural order) $r \times n$ Latin rectangles, $M_n^1 = 1$ for all n. The M_n^2 are the rencontre numbers [1]. Several methods are available for computing the M_n^3 [1, 2, 3]. In this report, a procedure is presented that is effective in finding M_n^r for $r \leq 4$.

CALCULATION OF M_n^2 AND GENERAL FORMULA FOR M_n^r

Consider the diagram (Fig. 1, drawn, as are all the diagrams, for n = 5) consisting of an $n \times n$ square, made up of n^2 cells arranged in n rows and n columns. Label each cell with the numerical indices giving its position in the square, writing them as a one-column matrix with the row index at the top, and mark those cells whose 2 indices are different. Call the marked cells good, the others bad. (For any r, the cells called good will be those with all indices different.) Now "expand" the diagram into "terms" by taking one cell from each row and each column, keeping the row indices in natural order. Each term then corresponds uniquely to a $2 \times n$ rectangular array whose first row is in natural order; a term made up entirely of good cells corresponds to a normalized Latin rectangle. M_n^2 is the number of such "all-good" terms in the expansion. Using the principle of inclusion and exclusion, one can read off M_n^2 from the diagram at sight (cf [4]). The expression so obtained is formula (1), below, with $A_{r,n}^k = n^{(k)}$, where $n^{(k)} \equiv n(n-1)(n-2)\cdots(n-k+1)$.

For $r > 2$, one can prepare an analogous diagram of an r-dimensional hypercube (e. g., Figs. 2 and 4), referred to in this report as the n^r-cube, and obtain the formula, valid for all $r > 0$.

$$M_n^r = \sum_{k=0}^{n} (-1)^k \frac{A_{r,n}^k}{k!} [(n-k)!]^{r-1} . \tag{1}$$

Here $A_{r,n}^k$ denotes the number of k-tuples in the expansion of the n^r-cube. (The word k-tuple, in this report, will always mean an ordered set of k bad cells, no two of which are from the same dimensional level.) $A_{r,n}^0 = 1$ for all r and all n, by definition.

CALCULATION OF M_n^3

The diagram is shown in Fig. 2. The layers are numbered from left to right and the indices are written in the order: layer, row, column. (Column matrices of indices will be written in the form $(a,b,c)'$ for typographical convenience.) There are two types of bad cells: α'-cells, having all 3 indices alike, and β'-cells, having exactly 2 indices alike. The

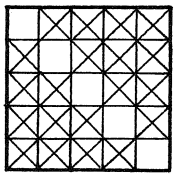

Fig. 1 5^2-Cube. (Good cells outlined and cross-hatched)

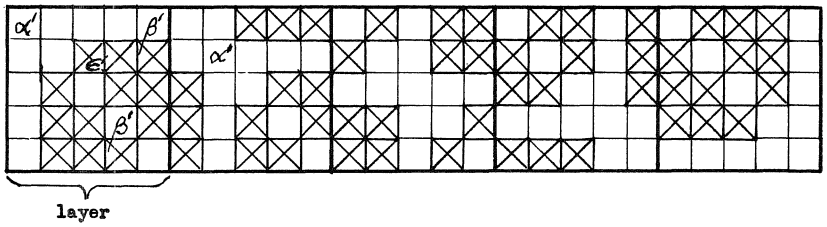

Fig. 2 5^3-Cube (which is also the 5^3 - α'-Cube). (Good cells outlined and cross-hatched.)

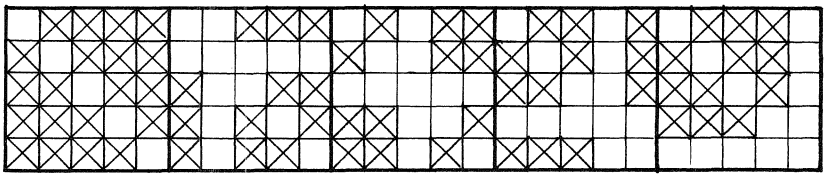

Fig. 3 5^3 - β'-Cube After $(2,1,1)'$. (Good cells outlined and cross-hatched.)

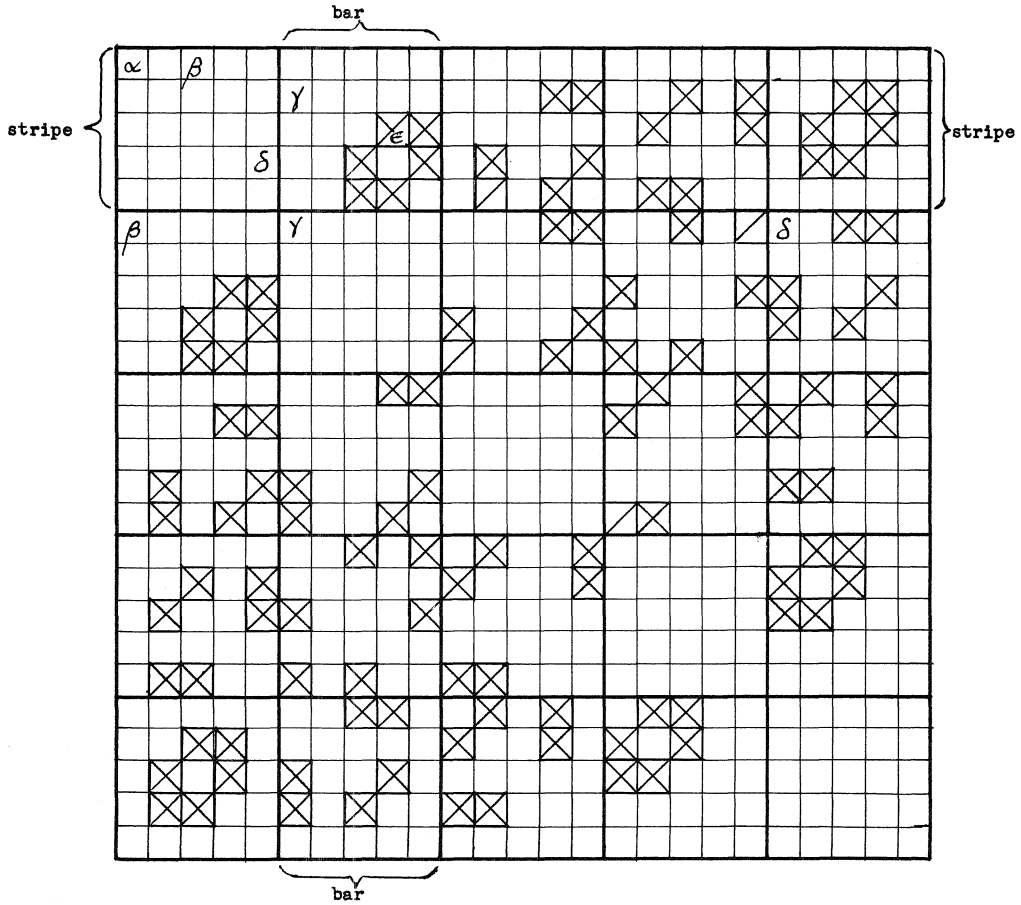

Fig. 4 5^4-Cube (which is also the 5^4 - α-Cube. (Good cells outlined and cross-hatched.)

Table 1

Formulas for the ω_n^k

(Formulas are valid for $k \geq 0$ and for n as indicated. See text for $k = 0$. Except for α_1^1, all ω_n^k are 0 for $k > 0$ and for values of $n > 0$ not covered by the formulas.)

$$\alpha_n^k = n\left\{\alpha_{n-1}^{k-1} + (n-1)\left[4\beta_{n-1}^{k-1} + 3\gamma_{n-1}^{k-1} + 6(n-2)\delta_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$(\alpha_1^1 = 1)$$

$$\beta_n^k = \alpha_n^k - 3k(n-1)(n-2)\delta_{n-1}^{k-1} \qquad (n \geq 1)$$

$$\gamma_n^k = \beta_n^k - k(n-1)(n-2)(2\eta_{n-1}^{k-1} - \delta_{n-1}^{k-1}) \qquad (n \geq 1)$$

$$\delta_n^k = \beta_n^k - k(n-2)\left[2(\theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1} - \delta_{n-1}^{k-1}) + (n-3)(\mu_{n-1}^{k-1} + 2\xi_{n-1}^{k-1} - \epsilon_{n-1}^{k-1})\right] \qquad (n \geq 2)$$

$$(n+1)^{(4)}\epsilon_n^k = (n-k+1)^4\alpha_{n+1}^k - \alpha_{n+1}^{k+1} \qquad (n \geq 3)$$

$$\zeta_n^k = \gamma_n^k - k(n-1)\left[2\beta_{n-1}^{k-1} + (n-2)\delta_{n-1}^{k-1}\right] \qquad (n \geq 1)$$

$$\eta_n^k = \delta_n^k - k\left\{2\gamma_{n-1}^{k-1} + (n-2)\left[4\delta_{n-1}^{k-1} + (n-3)\epsilon_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$\theta_n^k = \delta_n^k - k\left\{2\zeta_{n-1}^{k-1} + (n-2)\left[2\eta_{n-1}^{k-1} + \theta_{n-1}^{k-1} + 2\lambda_{n-1}^{k-1} + \nu_{n-1}^{k-1} + (n-3)(\mu_{n-1}^{k-1} + \xi_{n-1}^{k-1})\right]\right\} \qquad (n \geq 2)$$

$$\lambda_n^k = \delta_n^k - 2k\left\{\zeta_{n-1}^{k-1} + (n-2)\left[\eta_{n-1}^{k-1} + \theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1} + (n-3)\mu_{n-1}^{k-1}\right]\right\} \qquad (n \geq 2)$$

$$\mu_n^k = \epsilon_n^k - 2k\left\{2(\theta_{n-1}^{k-1} + \lambda_{n-1}^{k-1}) + (n-3)\left[3\mu_{n-1}^{k-1} + 2\xi_{n-1}^{k-1} + (n-4)\pi_{n-1}^{k-1}\right]\right\} \qquad (n \geq 3)$$

$$\nu_n^k = \beta_n^k - 3k(n-2)\left[2\theta_{n-1}^{k-1} + (n-3)\mu_{n-1}^{k-1}\right] \qquad (n \geq 2)$$

$$3(n-1)(n-2)\xi_n^k = \alpha_n^k + 2(2n-1)\beta_n^k + 3n\gamma_n^k + (6n-3k)(n-1)\delta_n^k - \left\{6\zeta_n^k + (n-1)\left[6(\eta_n^k + \theta_n^k + \lambda_n^k) + 3(n-2)\mu_n^k + \nu_n^k\right]\right\} \qquad (n \geq 3)$$

$$n^{(4)}\pi_n^k = (n-k+1)^4\beta_{n+1}^k - \beta_{n+1}^{k+1} - n(n-1)\left[3\delta_n^k + (n-2)(\epsilon_n^k + 3\mu_n^k)\right] \qquad (n \geq 4)$$

good cells (not needed in the calculations when $r = 3$) are ϵ'-cells. The numbers of each cell-type in any layer are easily ascertained.

Letting ω' stand for either of α', β', denote as an $n^3 - \omega'$-cube the cube that is obtained when all cells in the layer, row and column occupied by a chosen ω'-cell of the $(n + 1)^3$-cube are removed and the remaining parts of the diagram are allowed to collapse upon themselves (i.e., "ranks are closed"), the good or bad nature of each cell being preserved. Let ${\omega'}_n^k$ denote the number of k-tuples in an $n^3 - \omega'$-cube. It is apparent that the $n^3 - \alpha'$-cube is identical with the n^3-cube and that, for $k > 1$,

$$A_{3,n}^k \equiv {\alpha'}_n^k = n{\alpha'}_{n-1}^{k-1} + 3n(n - 1){\beta'}_{n-1}^{k-1} . \tag{2}$$

$({\alpha'}_n^1 = n^3 - n^{(3)}$ and ${\beta'}_n^1 = {\alpha'}_n^1 - 2(n - 1)$, by direct count in the diagram.)

To find ${\beta'}_n^k$, we can clearly use any β'-cell in the $(n + 1)^3$-cube. Choosing $(2,1,1)'$, we get the $n^3 - \beta'$-cube of Fig. 3. It differs from the $n^3 - \alpha'$-cube only in the first layer, all the differing cells being β'-cells in the $n^3 - \alpha'$-cubes. Accordingly,

$${\beta'}_n^k = {\alpha'}_n^k - 2k(n - 1){\beta'}_{n-1}^{k-1} . \tag{3}$$

The factor k in the second term on the right enters because of the way in which "k-tuple" has been defined.

All $A_{3,n}^k$ can now be calculated, for any given n; when they are substituted in (1), M_n^3 is obtained.

CALCULATION OF M_n^4

The diagram for the n^4-cube is shown in Fig. 4. The dimensional levels are to be written in the order: stripe, bar, row, column; the intersection of stripes a and b is called the field $[a,b]$. The procedure is analogous to that used above for the case $r = 3$. There are now, besides the good or ϵ-cells, four kinds of bad cells: α-cells, with all indices alike; β-cells, with exactly three indices alike; γ-cells, with two distinct pairs of like indices, and δ-cells, with exactly two indices alike.

The numbers of cells of each type are again easily found, and we have at once the formula for $A_{4,n}^k \equiv \alpha_n^k$ given in Table 1. β_n^k is obtained from the n^4-β-cube resulting from choosing $(2,1,1,1)'$ in the $(n + 1)^4$-cube (second formula in Table 1), just as ${\beta'}_n^k$ was obtained in the case $r = 3$. γ_n^k and δ_n^k are not so immediate, but can be found by the method outlined below.

We first analyze the $n^4 - \beta$-cube much as we did the $n^4 - \alpha$-cube. That is to say, we study the types and topographical distributions of the second members of those pairs of cells of the $(n + 1)^4$-cube whose first member is a selected β-cell. The β-cell chosen in the $(n + 1)^4$-cube to obtain the results shown in Fig. 5 is $(2,1,1,1)'$. It is useful to observe that, in a pair of columns of indices, the two numbers in a row may be interchanged without affecting the properties in which we are interested. There prove to be 13 different cell-types, including the five already observed. They are designated by Greek letters, as shown in Fig. 5. (The cells in the first stripe all retain the same designations they had in the $n^4 - \alpha$-cube;

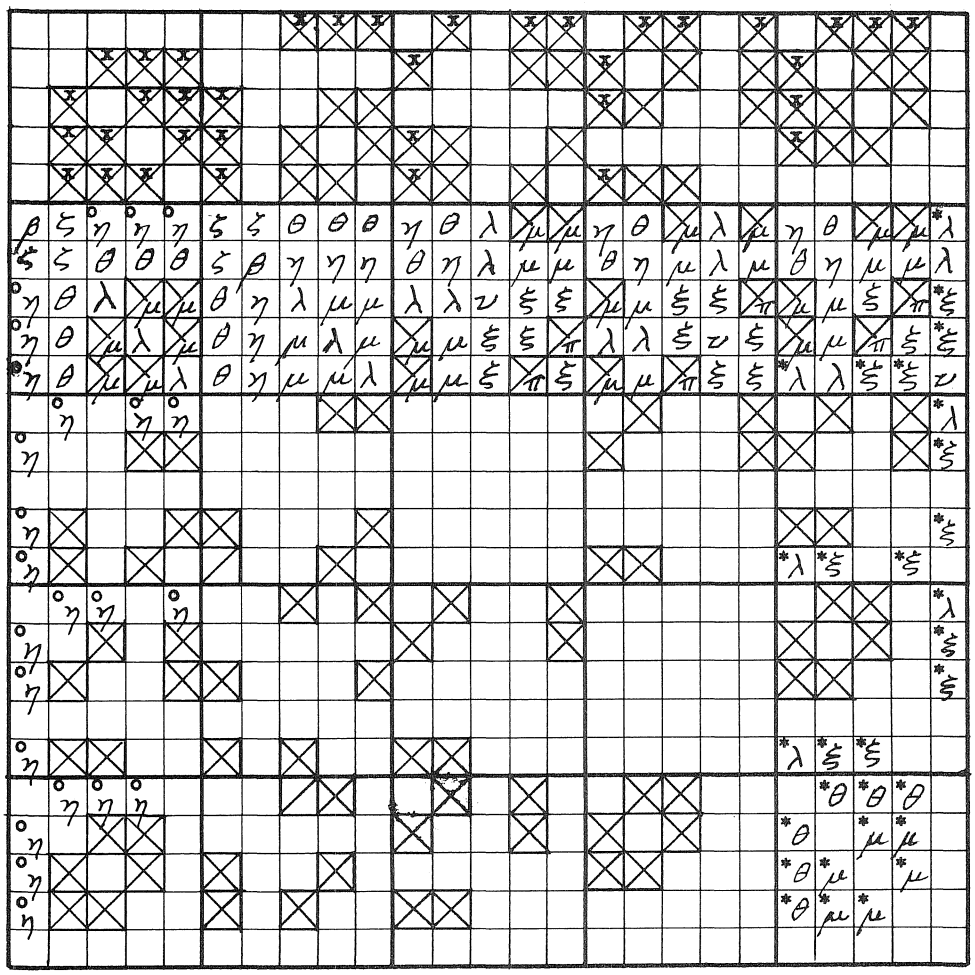

Fig. 5 5^4 - β-Cube after $(2,1,1,1)'$. (Good cells outlined and cross-hatched.) x indicates new good cell; o indicates where good cell is to be inserted to get γ_5^k. * indicates where good cell is to be inserted to get δ_5^k (see text).

each of the lower stripes has the same numbers of cells of types $\beta, \zeta, \cdots, \pi$ as the second stripe, but distributed differently.)

Now, to find γ_n^k, note that the n^4 - γ-cube that results from choosing $(2,2,1,1)'$ can be obtained from the n^4 - β-cube (Fig. 5) by removing all the good cells from the field $[1,1]$ and inserting good cells at those places in the first bar indicated with o in the diagram. This leads to the third formula in Table 1.

To get δ_n^k, choose $(2, n+1, 1\ 1)'$ and, in the resulting n^4 - δ-cube jump the first bar over all the others, so that it becomes the n^{th} bar (this will not affect the expansion of the cube). This adjusted n^4 - δ-cube may be obtained from the n^4 - β-cube of Fig. 5 by removing all the good cells in $[1,n]$ and inserting good cells at the places in the n^{th} bar indicated with *. Thus the fourth formula of Table 1 is obtained.

ϵ_n^k is found by considering all (rather than only the bad) cells of the n^4-cube, and proceeding as in the derivation of α_n^k.

Formulas for $\zeta_n^k, \cdots, \pi_n^k$ remain to be derived, in order to make the results for γ_n^k and δ_n^k effective. All but ξ_n^k and π_n^k may be found in the following way, λ_n^k being used as example. Choose an appropriate λ-cell, such as $(2,1,3,3)'$ in the $(n+1)^4 - \beta$-cube, noting that the chosen cell is a δ-cell in the $(n+1)^4 - \alpha$-cube. Now adjust δ_n^k by correcting for the "new" good cells, marked x in Fig. 5 (i.e., cells that are good in the β-cube but bad in the α-cube). The cell pairs that must be examined in this process all prove to be reducible to pairs of the kinds already introduced. There results, finally, the formula for λ_n^k in Table 1. All the rest of the formulas of Table 1 except the last two are derived in the same way.

If the choice of the λ-cell had been inappropriate (e.g., $(2,2,3,3)'$), the procedure would have met an impasse and have failed. This happens for all choices of ξ-cells and π-cells, ξ_n^k can be found, however, by equating the result already known for β_{n+1}^{k+1} with that obtained by expanding the $(n+1)^4 - \beta$-cube in terms of the $\alpha_n^k, \cdots, \zeta_n^k, \cdots, \xi_n^k, \pi_n^k$. The latter expansion is analogous to that used to find α_n^k, above. π_n^k is found by using all, rather than only the bad, cells in the diagram, by analogy with the derivation of ϵ_n^k. There result the last two formulas of Table 1.

As an initial set of values for the recurrences of Table 1, one can use the ω_n^0, whose values are: for $n \geq 3$, all ω_n^0 are 1; for $n = 3$, π_3^0 is 0 and all others are 1; for $n = 2$, $\epsilon_2^0, \mu_2^0, \xi_2^0$ and π_2^0 are 0 and all others are 1; for $n = 1$, $\alpha_1^0, \beta_1^0, \gamma_1^0$ and δ_1^0 are 1 and all the rest are 0. The ω_n^1 can be checked by direct count in the appropriate diagram.

For any given $n > 0$, all the $A_{4,n}^k$ can now be calculated, and M_n^4 can be found by substituting them in (1). Some enumerations of $4 \times n$ Latin rectangles obtained by the method here presented are:

$$M_4^4 = 24$$
$$M_5^5 = 1,344$$
$$M_6^4 = 393,120$$
$$M_7^4 = 155,185,920$$
$$M_8^4 = 143,432,634,240 .$$

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SOME SIMPLE SIEVES

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1. INTRODUCTION

An ancient process for generating the sequence of prime numbers is known by the name of The Sieve of Eratosthenes. This method is presented in many textbooks and is rather widely known. However, it seems to be less widely known that, with some modifications, other interesting sequences can be generated by essentially a sieve process. In particular, we can obtain the sequence of values for some of the common arithmetic functions. First we shall discuss this Sieve of Eratosthenes and some of its modifications, then we will proceed to some "sieves" for generating other sequences.

2. THE SIEVE OF ERATOSTHENES AND MODIFICATIONS

We recall that in order to obtain the sequence of primes by this method, the sequence of integers greater than 1 is first written down. Starting with 2 we then put a slash through each second number beyond 2 in this sequence. This leaves 3 as the first number beyond 2 which is not crossed out, so that we then put a slash through every third number beyond 3. (Note that, for example, 6 now has been crossed out twice.) Since 5 is the first number beyond 3 which is not yet crossed out, we next put a slash through each fifth number beyond 5, and continue in a similar manner. Those numbers remaining (not crossed out) are primes.

In order to place this process in a setting which is more suitable for generalization, we will now modify the process in order to generate the sequence of values of what we shall call the characteristic function for prime numbers, denoted by χ_p. This function has the values $\chi_p(n) = 1$ if n is a prime; $\chi_p(n) = 0$ otherwise. In Table 1 the construction of the successive sequences is illustrated. This table is headed by the sequence of natural numbers in natural order which will thus indicate the position numbers for the elements of the sequences.

Table 1
Successive Sequences for $\chi_p(n)$, N = 33

	1	2	3	4	5	6	7	8	9	1 0	1	2	3	4	5	6	7	8	9	2 0	1	2	3	4	5	6	7	8	9	3 0	1	2	3
$A^{(0)}$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$A^{(1)}$	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$A^{(2)}$	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
$A^{(3)}$	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

The entries in Table 1 are prepared as follows. For the initial sequence, $A^{(0)}$, we enter 0 in the first position and 1 otherwise for this first row of the table and for $n \leq N$. The case $N = 33$ is illustrated. In order to begin the process of sieving, we locate the position of the first non-zero element and denote this position by a_1 (= 2) and then convert the entries in positions ma_1, $m = 2, 3, 4, \cdots$ (every second entry beyond 2) from 1 to 0. The resulting sequence is denoted by $A^{(1)}$. The position number of the first non-zero entry beyond position a_1 in $A^{(1)}$ is denoted by a_2 (= 3) and every entry in position ma_2, $m = 2, 3, 4, \cdots$ (every third entry beyond 3) is converted from 1 to 0, if the entry is not already 0, in order to produce the sequence $A^{(2)}$. In general, in the sequence $A^{(k-1)}$ we locate the position of the first non-zero entry beyond position a_{k-1} and denote its position by a_k. Then every entry in position ma_k, $m = 2, 3, 4, \cdots$ (every a_k-th entry beyond a_k) is converted from 1 to 0, if it is not already 0. This produces the sequence $A^{(k)}$.

It is worth noting that, of course, the process can be terminated at $A^{(k-1)}$ if $a_k > \sqrt{N}$ and the sequence $A^{(k-1)}$ coincides with the sequence χ_p for $n \leq N$; that is, $a_k = p_k$. The reason that this termination is possible is that the smallest number which a_k actually sieves out is a_k^2, since $a_k a_j$ for $a_j < a_k$ has been sieved out at an earlier step.

In this construction the actual sieving out of the number n is indicated by the conversion of an entry 1 to an entry 0 in position n of the sequence. If a 0 has already appeared, this indicates that the number had been sieved out at a previous step; that is, the number actually possessed a smaller prime factor than the number currently being used as the sieving number. The entire process involves (1) the location of a non-zero element, (2) a counting process, and (3) a change of entry. We note that no divisibility checks are used. One further comment. This process does not involve using any of the sequences $A^{(m)}$ for $m < k - 1$, but only the sequence $A^{(k-1)}$ to produce $A^{(k)}$. As a result, those preceding sequences need not be saved. Even though the original process is ancient, in this form it is quite adaptable for digital computers.

For some purposes a simpler sieve which yields slightly different information is of value; this is illustrated in Table 2.

Table 2

First Modification, N = 33

	1	2	3	4	5	6	7	8	9	1 0	1	2	3	4	5	6	7	8	9	2 0	1	2	3	4	5	6	7	8	9	3 0	1	2	3
$A^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$A^{(1)}$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$A^{(2)}$	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
$A^{(3)}$	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

The change of the results is indicated by the appearance of 1 in position 1 and in the appearance of 0 in position p_k for primes $p_k \leq \sqrt{N}$. Hence the result is the sequence of

values of the characteristic function for the set which contains 1 and those primes satisfying the inequality $\sqrt{N} < p_k \leq N$. The process is simpler in the sense that for the general step we sieve out the numbers ma_k for $m = 1, 2, 3, 4, \cdots$; that is, each a_k-th number, counting from the beginning. Here the process must be stopped at the sequence $A^{(k-1)}$ if $a_k > \sqrt{N}$.

These first two processes have the disadvantage in that multiple sieving of elements occurs whenever the position number is composite. The following method will eliminate this problem for the modified sieve, although it is a much more complicated procedure.

Consider the sequence $A^{(0)}$ where $a_k = 1$ for all k. Since the first 1 beyond position 1 which appears is in position a_1 (= 2) we sieve each second element by subtracting 1 from the entries in position ma_k, $m = 1, 2, 3, 4, \cdots$, then we rename the resulting sequence $A^{(1)}$. Next, the first 1 which appears beyond position 1 in $A^{(1)}$ is in position a_2 (= 3) and we use the sequence $A^{(1)}$ itself to generate the sequence of elements which are to be sieved out of $A^{(1)}$ to produce $A^{(2)}$. Subtract the entry in position m of $A^{(1)}$ from the entry in position ma_2 for $m = 1, 2, 3, 4, \cdots$ provided $ma_2 \leq N$. We note that if 2 divides m, then $m = 2m'$ and in position $ma_2 = 2m'a_2$ the entry is 0 so that we subtract 0 from 0. (This replaces the operation of leaving 0 as 0.) In general, we locate the first non-zero entry beyond position 1 in $A^{(k-1)}$ and denote this position by a_k, then we subtract the entry in position m of $A^{(k-1)}$ from the entry in position ma_k for $m = 1, 2, 3, 4, \cdots$ with $ma_k \leq N$. The process is stopped at the sequence $A^{(k-1)}$ if $a_k > \sqrt{N}$ and the final result of this second modification is the same as that of the first modification. The details are illustrated successively in Table 3.

Table 3
Second Modification, N = 33

	1									2									3												
	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
$A^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1		-1	
$A^{(1)}$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		-1			-0			-1			-0			-1			-0			-1			-0			-1			-0			-1	
$A^{(2)}$	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
				-1					-0					-0					-0					-1					-0				
$A^{(3)}$	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0

A useful way of thinking of the process is that $A^{(k-1)}$ is expanded by the factor a_k and subtracted from itself.

Another method closely related to this second modification and which eliminates multiple sieving is discussed by G. S. Arzumanjan [1]. A geometric construction for the second modification is given in [3].

Although these two modifications are not perhaps very important for the problem of generating primes, they have been presented in detail because of their similarity to sieves for other sequences which will be discussed in the next section.

3. SEQUENCES OF VALUES OF CERTAIN ARITHMETIC FUNCTIONS

The sequences which are to be discussed in this section are computed without advance knowledge of a table of primes; that is, the generation of the sequence of primes is contained internally in the process, as needed.

The first function which we will consider is the number of distinct prime factors of n which we denote by $\omega(n)$. We generate the sequence of values of the function ω by a slight alteration of that first modification for the generation of χ_p which was discussed in Sec. 1. In the case of primes we can think of the composites as falling through the sieve and being discarded; this present alteration can be thought of as collecting those things which fall through our sieve in little boxes. In order to indicate how this process goes, let $B^{(0)}$ be the sequence with 0 in each position. To produce $B^{(1)}$ we add 1 to each second entry of $B^{(0)}$. The first 0 which appears beyond position 1 of $B^{(1)}$ is in position p_2 (= 3). We next add 1 to each entry in $B^{(1)}$ in position mp_2, $m = 1, 2, 3, 4, \cdots$ to produce $B^{(2)}$. Continuing in this manner we can state the general step of this iterative process. Locate the first 0 which appears beyond position 1 in $B^{(k-1)}$, this will be in position p_k. Next add 1 to each entry in $B^{(k-1)}$ in position mp_k, $m = 1, 2, 3, 4, \cdots$ for $mp_k < N$. This will produce the sequence $B^{(k)}$. The process can be stopped at sequence $B^{(k-1)}$ if $2p_k > N$, since thereafter only one entry, the entry in position p_k, will be altered. The remaining entries which are 0 and are beyond position 1 now indicate primes satisfying $N/2 < p \leq N$, if $N \geq 4$, and hence the process can be completed by converting the 0 entries to 1 in these positions $N/2 < n \leq N$ of $B^{(k-1)}$. The sequence $B^{(k)}$ coincides with the sequence for ω for $n \leq N$. The results for $N = 33$ are given in Table 4.

Table 4
Sequences for $\omega(n)$, $N = 33$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$B^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
$B^{(1)}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
			1			1			1			1			1			1			1			1			1			1			1
$B^{(2)}$	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1
					1					1				1						1					1					1			
$B^{(3)}$	0	1	1	1	1	2	1	1	1	2	0	2	0	1	2	1	0	2	0	2	1	1	0	2	1	1	1	1	0	3	0	1	1
							1							1							1							1					
$B^{(4)}$	0	1	1	1	1	2	1	1	1	2	1	2	0	2	2	1	0	2	0	2	2	1	0	2	1	1	1	2	0	3	0	1	1
											1											1											1
$B^{(5)}$	0	1	1	1	1	2	1	1	1	2	1	2	1	2	2	1	0	2	0	2	2	2	0	2	1	1	1	2	0	3	0	1	2
													1													1							
$B^{(6)}$	0	1	1	1	1	2	1	1	1	2	1	2	1	2	2	1	0	2	0	2	2	2	0	2	1	2	1	2	0	3	0	1	2
																	↑															Stop	
ω	0	1	1	1	1	2	1	1	1	2	1	2	1	2	2	1	1	2	1	2	2	2	1	2	1	2	1	2	1	3	1	1	2

Since we are adding 1 to the entry at position $n = mp_k$ at the k^{th} step, we count the factor p_k of n exactly once. Hence we have generated the sequence of values for the function ω; that is, $\omega(n)$ appears in position n.

It now becomes an analogous exercise to obtain a sieve process for computing the sequence of values $\tau(n)$, the number of divisors of n. Similarly, the sequence of values $\sigma(n)$, the sum of the divisors of n, can then be computed.

If we next attempt to compute the values $\Omega(n)$, the total number of prime factors of n, the procedure seems to become more complicated. However, one way to proceed is as follows, starting with the sequence $C^{(0)}$ for which all entries are 0. In order to construct $C^{(1)}$ we want to first add 1 to every second entry to count the factor 2^1, then further add 1 to every fourth entry to count the factor 2^2, etc., until $2^k > N$. This subprocess for counting the factors 2 reminds one of the second modification in Sec. 1, if we consider that the first subsequence $C_1^{(1)}$ to be added to $C^{(0)}$ is constructed by entering in position 2m, $m = 1, 2, 3, 4, \cdots$, the value 1, then the second subsequence, $C_2^{(1)}$ which is to be added to $C^{(0)}$ is constructed by entering in position 2m the value from position m of $C_1^{(1)}$, etc. The value 0 is entered otherwise in the subsequences. $C^{(1)}$ is then constructed by adding successively $C_1^{(1)}, C_2^{(1)}, \cdots$ to $C^{(0)}$. From this discussion we can see that if 2^a divides n, then an addition of 1 is carried out in position n for $k = 1, 2, \cdots, a$; thus the process counts the number of prime factors 2 of n. The steps are illustrated in Table 5.

Table 5
Sequences for $\Omega(n)$, N = 33

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$C^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
				1				1				1				1				1				1				1				1	
								1								1				1				1				1				1	
																1																1	
$C^{(1)}$	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	5	0
		1				1			1			1			1			1				1		1			1			1			1
									1									1									1						
$C^{(2)}$	0	1	1	2	0	2	0	3	2	1	0	3	0	1	1	4	0	3	0	2	1	1	0	4	0	1	3	2	0	2	0	5	1
					1					1					1					1					1					1			
$C^{(3)}$	0	1	1	2	1	2	0	3	2	2	0	3	0	1	2	4	0	3	0	3	1	1	0	4	2	1	3	2	0	3	0	5	1
							1							1							1							1					
$C^{(4)}$	0	1	1	2	1	2	1	3	2	2	0	3	0	2	2	4	0	3	0	3	2	1	0	4	2	1	3	3	0	3	0	5	1
											1											1											1
$C^{(5)}$	0	1	1	2	1	2	1	3	2	2	1	3	0	2	2	4	0	3	0	3	2	2	0	4	2	1	3	3	0	3	0	5	2
													1													1							
$C^{(6)}$	0	1	1	2	1	2	1	3	2	2	1	3	1	2	2	4	0	3	0	3	2	2	0	4	2	2	3	3	0	3	0	5	2
																	1																Stop
Ω	0	1	1	2	1	2	1	3	2	2	1	3	1	2	2	4	1	3	1	3	2	2	1	4	2	2	3	3	1	3	1	5	2

To form $C^{(2)}$ from $C^{(1)}$ we locate the first 0 beyond position 1, which is at position p_2 (= 3), and repeat the subprocess, but now using p_2; that is, $C_1^{(2)}$ is the sequence with 1 in position mp_2, $m = 1, 2, 3, 4, \cdots$; $C_2^{(2)}$ is the sequence with 1 in position mp_2^2, $m = 1, 2, 3, 4, \cdots$; etc. Then $C^{(2)}$ is formed by the addition of $C_1^{(2)}, C_2^{(2)}, \cdots$ to $C^{(1)}$. $C_2^{(2)}$ could also be constructed from $C_1^{(2)}$ by entering in position mp_2 the entry from position m of $C_1^{(2)}$ and 0 otherwise.

The general step is somewhat complicated in its description. Consider $C^{(k-1)}$ and locate the first 0 entry beyond position 1; this will be in position p_k by analogy to the sieve for primes. To begin the subprocess, we form $C_1^{(k)}$ by entering 1 in position mp_k, $m = 1, 2, 3, 4, \cdots$, until $mp_k > N$. The subsequence $C_j^{(k)}$ is formed from the subsequence $C_{j-1}^{(k)}$ by entering in position mp_k, $m = 1, 2, 3, \cdots$, the entry from position m of $C_{j-1}^{(k)}$ and 0 otherwise until $mp_k^j > N$. The sequences $C_j^{(k)}$ are successively added to $C^{(k-1)}$ to produce $C^{(k)}$. (It is merely for display purposes that the subsequences are formed separately and then added, the addition process, of course, can be carried out as one progresses.) The process can be stopped at the same point in the computation as for the values of ω and the remaining 0 entries converted to 1, using the same reasoning. It is not difficult to see that we actually have obtained the values $\Omega(n)$.

A slight modification of the entries leads to the sequence $\lambda(n) = (-1)^{\Omega(n)}$, another function of interest in the theory of numbers. After the methods outlined above, this becomes an exercise.

4. THE NUMBER OF REPRESENTATIONS OF A NUMBER AS A PRODUCT OF NUMBERS CONTAINED IN A GIVEN SET

We shall next consider the following problem. Given a subsequence S of natural numbers $1 < a_1 < a_2 < \cdots$, either finite or infinite, we wish to compute the number of distinct representations of a number n as the product of elements of the set S; that is, we wish to compute the number of distinct (except for order) representations of n in the form

$$n = a_1^{b_1} a_1^{b_2} \cdots a_k^{b_k} ,$$

where the a's belong to S and the b's are positive integers. We assume that the set S has been generated separately and we let R(n) denote the number of such representations of n. The sequence of values of R is to be generated by a modified sieve method.

The actual process is somewhat analogous to the procedure of Sec. 3 for computing the sequence Ω. We let $R^{(0)}$ denote the sequence with 1 in position 1 and 0 otherwise. In order to obtain the sequence $R^{(1)}$ we bring down 1 into position 1 and then add the entry from position 1 of $R^{(1)}$ to the entry at position $1a_1$ of $R^{(0)}$ and enter the sum in position a_1 of $R^{(1)}$, then the entry from position 2 of $R^{(1)}$ is added to the entry of position $2a_2$ of $R^{(0)}$ and the sum is entered in position $2a_2$ of $R^{(1)}$, etc., but otherwise the entry at position k of $R^{(1)}$ is taken as the entry at position k of $R^{(0)}$. This set of subprocesses is stopped when $ma_1 > N$. To continue, the entry from position 1 of $R^{(1)}$ is

entered in position 1 of $R^{(2)}$ and the entry from position 1 of $R^{(1)}$ is added to the entry from position $1a_2$ of $R^{(1)}$ and the sum is entered in position a_2 of $R^{(2)}$ and successively in order for $m = 1, 2, 3, \cdots$, the entry at position m of $R^{(2)}$ is added to the entry at position ma_2 of $R^{(1)}$ and the sum entered in position ma_2 of $R^{(2)}$ to produce the sequence $R^{(2)}$. (The other entries are carried from $R^{(1)}$ to $R^{(2)}$, addition only takes place at the positions ma_2.) For the general iterative step, in order to obtain $R^{(k)}$ from $R^{(k-1)}$, we add the entry from position m of $R^{(k)}$ to the entry from position ma_k of $R^{(k-1)}$ and enter the sum in position ma_k of $R^{(k)}$, running successively through $m = 1, 2, 3, \cdots$ and stopping if $ma_k > N$. The process terminates at $R^{(k-1)}$ if $a_k > N$. In Table 6 we have chosen $S = \{2, 3, 4, 5, 12, 30, 72\}$ and we indicate the steps of the computation. Note the iterative process which occurs within the computation for each sequence.

Table 6
R(n) for $S = \{2, 3, 4, 5, 12, 30, 72\}$, $N = 33$

										1										2										3			
	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
$R^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R^{(1)}$	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$R^{(2)}$	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0
$R^{(3)}$	1	1	1	2	0	1	0	2	1	0	0	2	0	0	0	3	0	1	0	0	0	0	0	2	0	0	1	0	0	0	0	3	0
$R^{(4)}$	1	1	1	2	1	1	0	2	1	1	0	2	0	0	1	3	0	1	0	2	0	0	0	2	1	0	1	0	0	1	0	3	0
$R^{(5)}$	1	1	1	2	1	1	0	2	1	1	0	3	0	0	1	3	0	1	0	2	0	0	0	3	1	0	1	0	0	1	0	3	0
R	1	1	1	2	1	1	0	2	1	1	0	3	0	0	1	3	0	1	0	2	0	0	0	3	1	0	1	0	0	2	0	3	0

The iterative procedure from $R^{(k-1)}$ to $R^{(k)}$ can be expressed in terms of the following equations which are to be applied successively for $n = 1, 2, 3, \cdots$ in that order.

$$R^{(k)}(n) = R^{(k-1)}(n), \quad \text{if } n \neq ma_k,$$
$$R^{(k)}(n) = R^{(k-1)}(n) + R^{(k)}(m), \quad \text{if } n = ma_k.$$

For example we have, since $a_4 = 5$,

$$R^{(4)}(20) = R^{(3)}(20) + R^{(4)}(4) = 0 + 2 = 2,$$
$$R^{(4)}(21) = R^{(3)}(21) = 0.$$

Special cases of interest are obtained if $a_k = p_k$, then, of course, $R(n) = 1$ since the representation is unique; if $a_k = k$, then $R(n)$ denotes the number of factorizations of n into integers; if $a_k = k^2$, then $R(n)$ denotes the number of factorizations of n into perfect squares; and if $a_k = p_k^2$, then $R(n)$ is the characteristic function for squares.

5. SOME FURTHER DIRECTIONS

The sequence of lucky numbers has been generated by a sieve technique and some of the properties of this sequence have been ovtained [4, 6]. The question concerning the number of distinct representations of n as a produce of lucky numbers can be approached by means of Sec. 4. A mixed technique of alternately sieving and summing which will generate the sequence of k^{th} powers is due to Moessner [8]; this is discussed and further references are given in a recent paper by C. T. Long [7]. Beginning with V. Brun [2] techniques involving double sieving and other modifications have been used to study the twin prime problem, the Goldbach conjecture, and other problems. An interesting article by David Hawkins [5] on the sieve of Eratosthenes, random sieves, and other matters is well worth consulting.

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TOPOLOGICAL INDEX AND FIBONACCI NUMBERS WITH RELATION TO CHEMISTRY

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INTRODUCTION

This paper deals with the discussion on the graphical aspects of the Fibonacci numbers through the topological index [1] which has been defined by the present author for non-directed graphs.[1]

A graph G consists of points (vertices or atoms) and lines (edges or bounds) [2, 3]. We are concerned with such connected non-directed graphs that have no loop (a line joining to itself) and no multiple lines (double or triple bonds). An adjacency matrix A for graph G with N points is a square matrix for the order N with elements

$$a_{ij} = \begin{cases} 1 & \text{if the points } i \text{ and } j \text{ are neighbors,} \\ 0 & \text{otherwise .} \end{cases} \tag{1}$$

The matrix character is independent of the way of the numbering of the points. A characteristic polynomial or a secular polynomial P(X) is defined as[2]

$$P(X) = \det|A + XE| = \sum_{i=0}^{N} b_i X^{N-i} , \tag{2}$$

where E is a unit matrix of the order N and X is a scalar variable.

Consider a series of path progressions $\{S_N\}$, for which P(X) can be expressed as (see [4])

$$P(X) = \sum_{k=0}^{m} (-1)^k \binom{N-k}{k} X^{N-2k} , \tag{3}$$

where N is the number of points and m is [N/2]. Examples are shown in Table 1 on the following page.

---

[1]Originally this idea came out quite independently from other works especially published in mathematical journals. However, thanks to the communications from the colleagues in this field, several important papers were found to be relevant to this problem. In this paper the relevant papers will be cited as many as possible.

---

[2]Alternative definition

$$P(X) = (-1)^N \det|A - XE| \tag{2'}$$

can be chosen, which, however, makes no difference in the following discussion.

Table 1

N	$G(S_N)$	$\binom{N-k}{k} = p(G,k)$					$\sum_{k=0}^{m} p(G,k)$
		k = 0	1	2	3	4	= Z = f_N
1	•	1					1
2	—	1	1				2
3	— —	1	2				3
4	— — —	1	3	1			5
5	— — — —	1	4	3			8
6	— — — — —	1	5	6	1		13
7	— — — — — —	1	6	10	4		21
8	— — — — — — —	1	7	15	10	1	34

On the other hand, from the combinatorial theory we know the following relation under the name of Lucas (see [5, 6]).

$$f_N = \sum_{k=0}^{m} \binom{N-k}{k}, \tag{4}$$

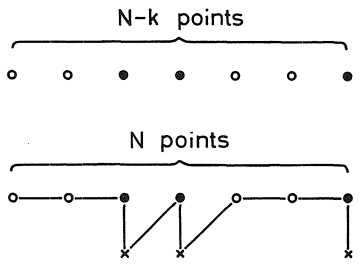
where f_N is the N^{th} Fibonacci number, which is defined as

$$f_0 = f_1 = 1, * \qquad f_N = f_{N-1} + f_{N-2} \quad (N = 2, 3, \cdots) . \tag{5}$$

The sums of the absolute values of the coefficients of the characteristic polynomial for the graph $\{S_N\}$ form the Fibonacci series. This is not new. Turn Table 1 counter-clockwise by 45 degrees, and we get the Pascal's triangle or the pyramid of binomial coefficients, from which the Fibonacci numbers can be obtained by adding the coefficients diagonally (just the reverse of the above procedure!). (See [7].)

Let us consider the physical meaning of the combination $\binom{N-k}{k}$. Consider a group of N - k points which are linearly arranged as in Fig. 1a. Choose an arbitrary set of k points (black circles), place k additional points (crosses) one-by-one below them, and join all the N points together by drawing consecutive N - 1 lines as in Fig. 1b to get a path progression with N points, or N - 1 lines. This means that the value $\binom{N-k}{k}$ is the number of ways in which k disconnected lines (vertical lines in Fig. 1b) are chosen from graph S_N.

*Alternative definition can be used as $f_1 = f_2 = 1$.



(a)

(b)

Fig. 1 Physical Meaning of $\binom{N-k}{k}$

TOPOLOGICAL INDEX [1]

Encouraged by the simple relation above, let us develop a more general theory. Define a non-adjacent number $p(G,k)$ for graph G as the number of ways in which k disconnected lines are chosen from G. A Z-counting polynomial $Q(Y)$ and a topological index Z are defined respectively as

$$Q(Y) = \sum_{k=0}^{m} p(G,k)Y^k \tag{6}$$

and

$$Z = \sum_{k=0}^{m} p(G,k) = Q(1) \quad . \quad \text{(See [1].)} \tag{7}$$

Note that for the series of path progressions $\{S_N\}$ in Table 1, the $p(G,k)$ number is nothing else but $\binom{N-k}{k}$, namely, the absolute value of the coefficients of the X^{N-2k} term in the characteristic polynomial $P(X)$. Thus we get

$$Z_{S_N} = f_N \tag{8}$$

or for brevity

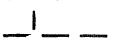
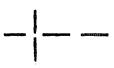
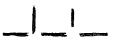
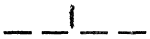
$$S_N = f_N \quad . \tag{8'}$$

Further, for any tree graph with or without branches but with no cycles, the following relation can be proved by term-to-term inspection of the expansion of det $|A + XE|$ into $P(X)$:

$$P(X) = \sum_{k=0}^{m} (-1)^k p(G,k) X^{N-2k} \qquad (G \in \text{Tree}) \quad . \tag{9}$$

Examples are shown in Table 2. More comprehensive tables of $p(G,k)$ and Z numbers have been published for smaller tree [8] and non-tree [9] graphs. For non-tree graphs, Eq. (9) no longer holds but $P(X)$ can be expressed as the sums of the contributions like the right-

Table 2

G	p(G,k) k = 0	1	2	3	Z
	1	3			4
	1	4			5
	1	5			6
	1	4	2		7
	1	5	3		9
	1	5	4		10
	1	5	5		11
	1	5	5	1	12

hand side of Eq. (9) of subgraphs of G. (See [10, 11].) As well as the characteristic polynomial [12, 13] the topological index does not uniquely determine the topology of a graph. However, it is generally observed that the Z value gets smaller with branching and larger with cyclization. Thus for a group of graphs with the same number of points, Z roughly represents the topological nature of the graph. For evaluating the Z values of larger and complicated graphs, the following composition principles are useful. They can be proved by the aid of Q(Y). (See [1, 10].)

COMPOSITION PRINCIPLES (CP)

Composition Principle 1 (CP1). (See [1].) Consider a graph G in Fig. 2a and choose from it a line ℓ, (1) Delete line ℓ and we get subgraphs L and M. (2) Delete all the lines in L and M that were incident to ℓ and we get subgraphs A, B, $\cdots$, F. Then the topological index Z for G can be obtained as

$$G = L \times M + A \times B \times C \times D \times E \times F . \tag{10}$$

For applying this principle there is no restriction in the number of subgraphs incident to the chosen line ℓ, since the Z values of a point graph (S_1) and a vacant graph (S_0) are both unity,

$$S_0 = S_1 = 1 . \tag{11}$$

Application of CP1 to the terminal line of graph S_N gives the recursion formula

$$S_N = S_{N-1} + S_{N-2} . \tag{12}$$

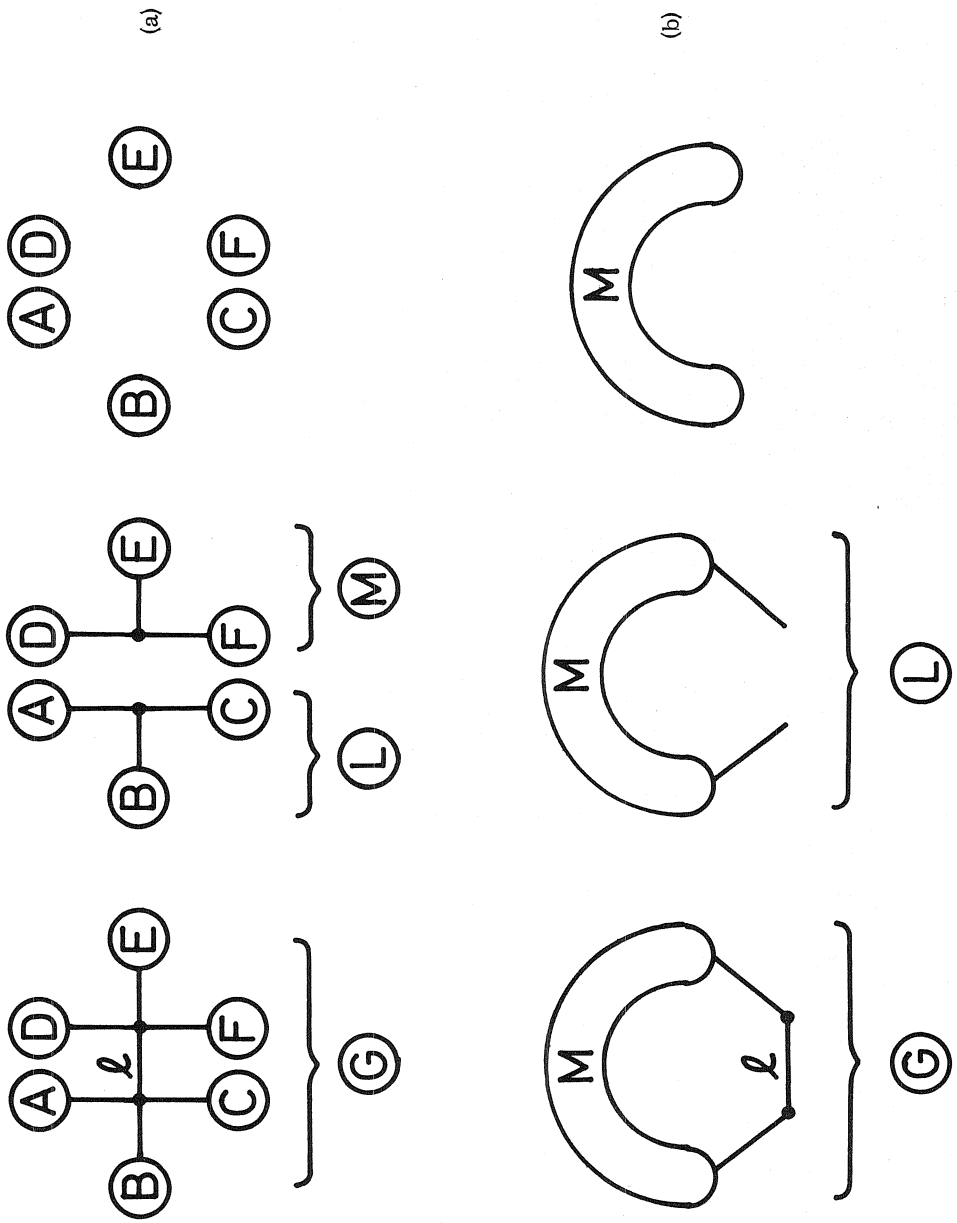


Fig. 2 Composition Principle 1

Comparison of Eqs. (11) and (12) with Eq. (5) yields Eq. (8).

For graph S_N with even N (= 2n) we get the relation

$$S_{2n} = S_n^2 + S_{n-1}^2 \tag{13}$$

by choosing the central line as ℓ. This is the graphical equivalent of the relation for the Fibonacci numbers [5-7]

$$f_{2n} = f_n^2 + f_{n-1}^2 \quad . \tag{14}$$

Similarly, we get

$$S_{2n+1} = S_n(S_{n+1} + S_{n-1}) \tag{15}$$

or

$$f_{2n+1} = f_n(f_{n+1} + f_{n-1}) \quad . \tag{16}$$

Corollary to CP1. If the line to be deleted is a member of a cycle, the deletion gives only one subgraph L as in Fig. 2b. In this case, we have

$$G = L + M \quad . \tag{17}$$

By use of this corollary the Z values for the series of N-membered cycles (N-gon, abbreviated as C_N) are obtained as in Table 3. It is apparent from Eq. (17) that

$$C_N = S_N + S_{N-2} \tag{18}$$

and the series of these Z values form what are known as the Lucas sequences $\{g_N\}$; namely, (see [5])

$$\begin{aligned} C_N &= g_N \\ g_1 &= 1, \quad g_2 = 3 \\ g_N &= g_{N-1} + g_{N-2} \quad . \end{aligned} \tag{19}$$

Then Eq. (18) is equivalent to the relation

$$g_N = f_N + f_{N-2} \quad . \tag{20}$$

From the correspondence relation of Eq. (19), a monogon and a digon may be defined, respectively, as a point graph C_1 (= S_1) and a graph C_2 with two points joined by two lines (see Table 3).*

*By extending this definition a topological index for a graph with multiple bonds can be defined.

Table 3

N	$G(C_N)$	p(G,k) k = 0	1	2	3	4	Z
1		1					1
2		1	2				3
3		1	3				4
4		1	4	2			7
5		1	5	5			11
6		1	6	9	2		18
7		1	7	14	7		29
8		1	8	20	16	2	47

Composition Principle 2 (CP2). (See [8].) Consider a graph G in Fig. 3 and choose from it a point p. The number of the lines incident to point p should be at least two but not necessarily be six as in this example. (1) Divide them into two groups. In this case, we chose the division as (a,b,c) and (d,e,f). (2) Delete a group of lines a, b and c in G, and we get subgraphs A, B, C and M. (3) Delete another group of lines d, e and f in G, and we get subgraphs D, E, F and L. (4) Delete both of the groups of lines a, b, $\cdots$, f in G, and we get subgraphs A, B, $\cdots$, F. With these subgraphs we have

$$G = A \times B \times C \times M + D \times E \times F \times L - A \times B \times C \times D \times E \times F . \tag{21}$$

Composition Principle 3 (CP3). Further consider a graph G in Fig. 4a in which two subgraphs A and B are joined by path progression S_3, i.e., three consecutive lines. (1) Delete a line from S_3 and rejoin the two resultant subgraphs to get L. (2) Delete one more line from S_2 in L and rejoin the subgraphs to get M. The Z value for G is given by (see Fig. 4)

$$G = L + M . \tag{22}$$

This is also applied to the case in which A and B are joined with two paths to form a cycle to give the relation (19) (Fig. 4b).

RECURSIVE SEQUENCES

A recursive sequence $\{a_N\}$ is defined as

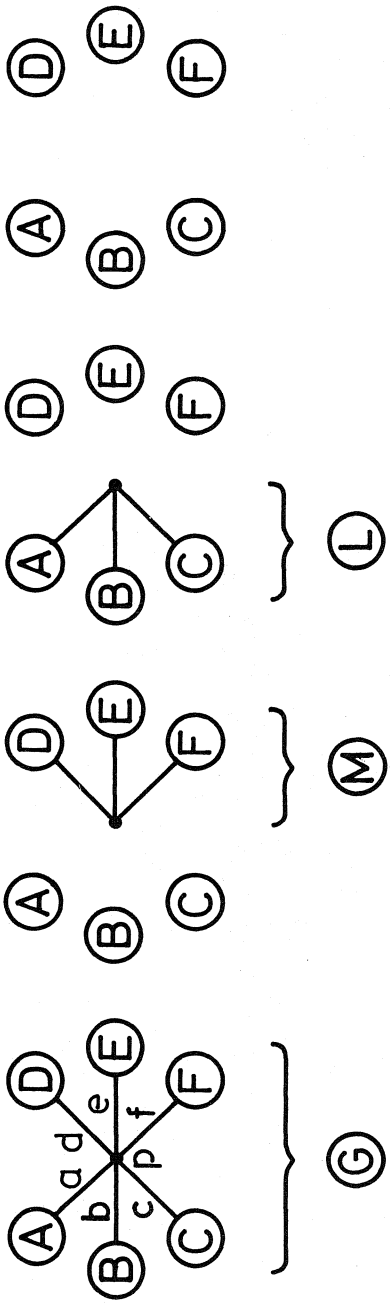


Fig. 3 Composition Principle 2

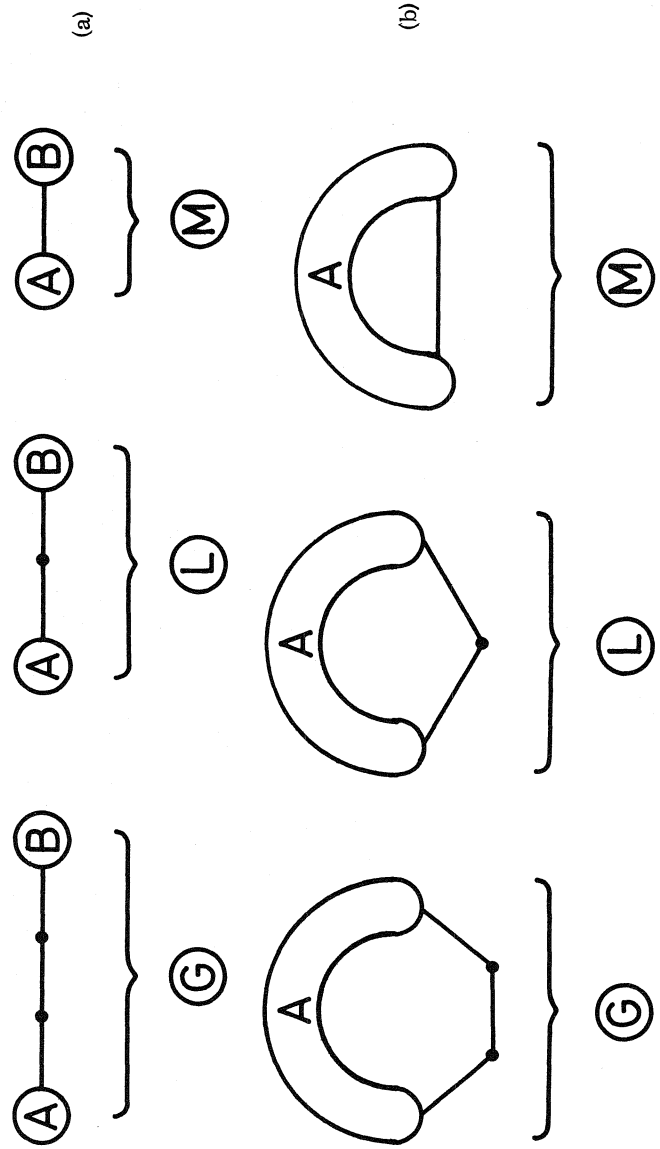


Fig. 4 Composition Principle 3

$$a_N = \sum_{i=1}^{k} C_i a_{N-i} \quad (23)$$

Both of the Fibonacci and Lucas sequences are the special cases with $C_1 = C_2 = 1$, $C_i = 0$ $(i > 2)$ but with different initial conditions. One can find a number of graphical series whose topological indices form recursive sequences as in Fig. 5. They can be proved by the composition principles. More interesting graphical sequences might be discovered through the topological index.

The most important point in this discussion is that a number of relations in the recursive sequences can be inspected and proved by applying the composition principles to the graphical equivalent of the sequences.

APPLICATION TO CHEMISTRY

Let us confine ourselves to a class of chemical compounds, saturated hydrocarbons, whose topological structure is expressed as a structural formula. An example is shown in Fig. 6 for 2-methylbutane (a). Since carbon (C) and hydrogen (H) atoms, respectively, have tetra- and mono-valencies, for describing the whole structure only the carbon atom skeleton (b) is sufficient, which is equivalent to graph (c). The series of graphs in Table 1 are read in chemical language as methane, ethane, propane, butane, etc. They form a family of normal paraffins. Thus the topological indices of normal paraffins are shown to form the Fibonacci sequences. Table 3 indicates that the topological indices of cycloparaffins (cyclopropane ...) form the Lucas sequences.

As was discussed earlier the topological index does not uniquely determine the topology of the molecular structure. For example, normal butane (the 4^{th} entry in Table 1) and neopentane (the 2^{nd} entry in Table 2) both have $Z = 5$. However, it was shown that the topological index can be used as a rough sorting device for coding the complicated structures of chemical compounds [14].

It was also shown that the topological index of a saturated hydrocarbon is correlated well with some of the thermodynamic quantities such as boiling point through its entropy, which is a measure of the degree of freedom in internal rotations of a flexible molecule [15].

Characteristic polynomials appear in the application of quantum mechanics to the study of the electronic structure of molecules. The simplest method is the Hückel molecular orbital method, in which the problem is reduced to obtaining the solution of a secular equation $P(X) = 0$ (see [10, 16, 17]).

SYNOPSIS

Define a topological index Z as the sum of the non-adjacent number, $p(G,k)$, which is the number of ways in which such k disconnected lines are chosen from graph G. The Z values for the path progressions $\{S_N\}$ form the Fibonacci sequences, while those for the

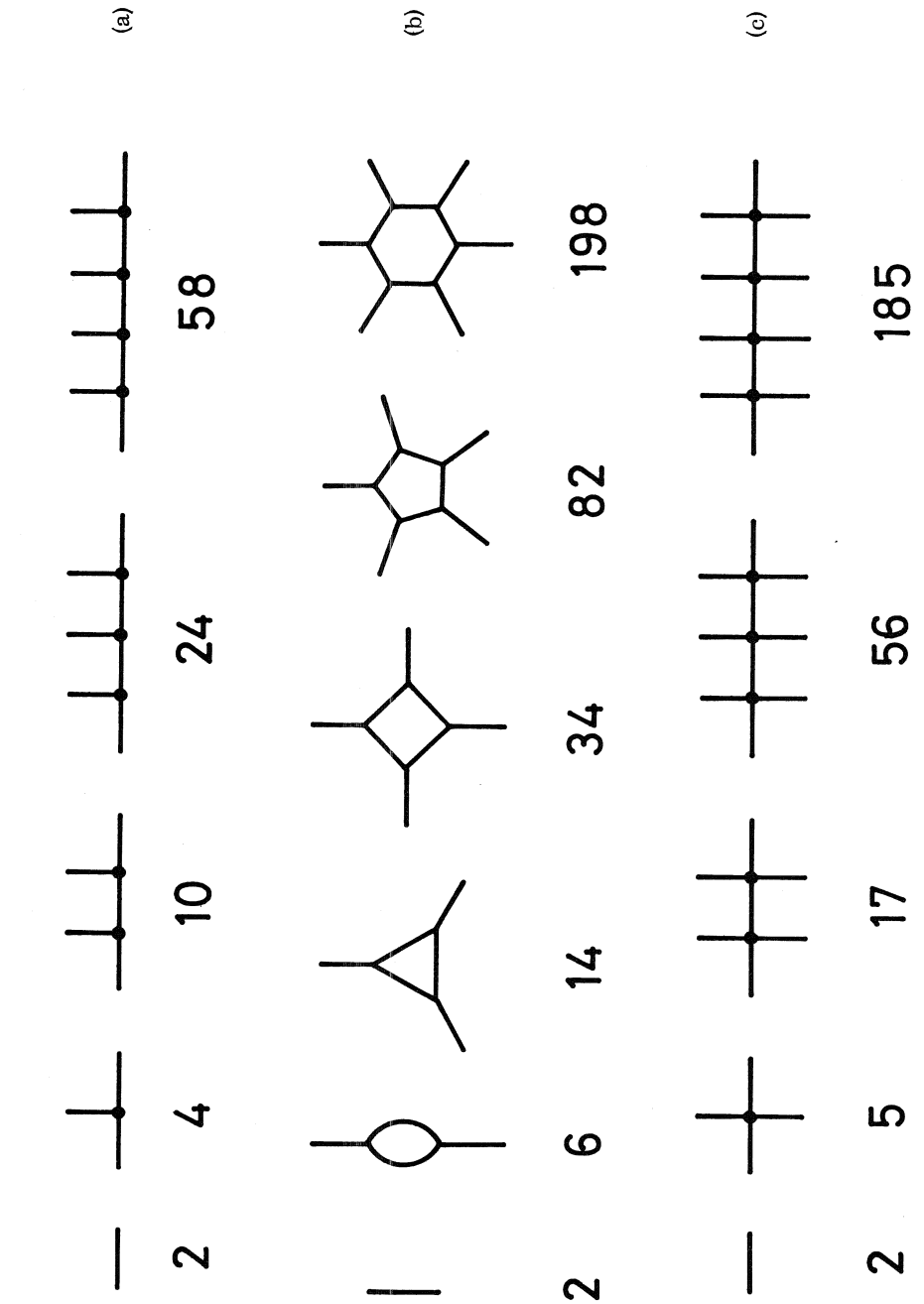


Fig. 5 Examples of Graphical Equivalent of Recursive Sequences $a_N = 2a_{N-1} + a_{N-2}$ for (a) and (b) and $a_N = 3a_{N-1} + a_{N-2}$ for (c).

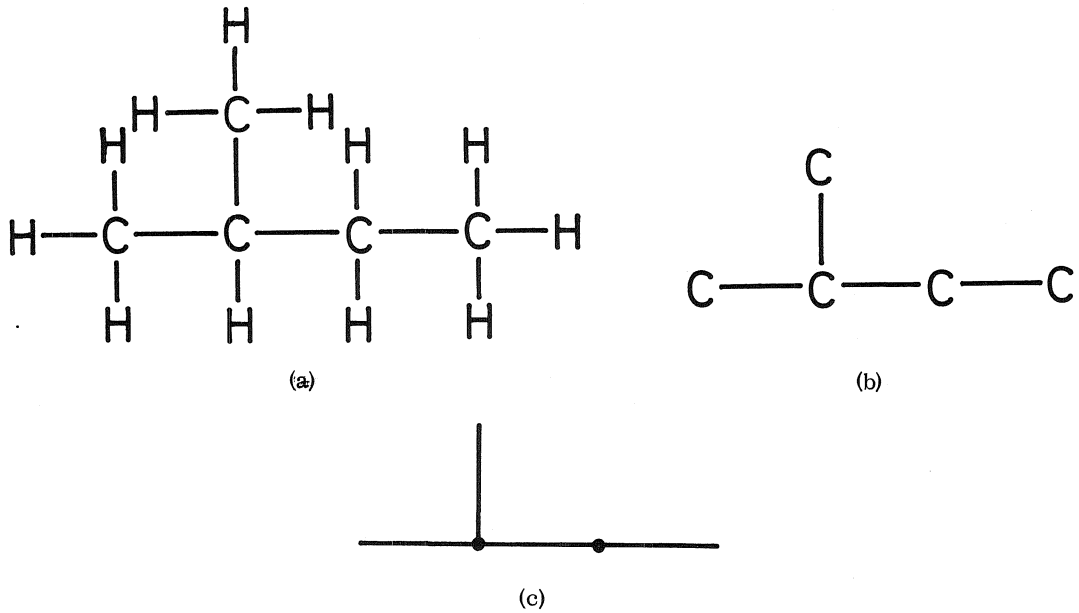


Fig. 6 Structure and Graph of 2-methylbutane

series of cycles $\{C_N\}$ the Lucas sequences. Many relations for them can be proved by the aid of the composition principles for Z. Application of Z to chemistry is discussed.

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BOOK REVIEW: I CHING GAMES

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I Ching Games of Duke Tan of Chou and C. C. T'ung, by H. Y. Li and Sibley S. Morrill, The Cadleon Press, P. O. Box 24, San Francisco, California 94101: 1971. 138 pages plus game pieces. $5.95.

The I Ching Games, whose names translate as "The Wisdom Plan" and "The Beneficial to Wisdom Plan," are considered among the most important ever written, since they are thought to improve the player's ability to learn while advancing his psychological development. The first game is also called the Tangram, being a seven-piece dissection of a square into a smaller square, five isosceles right triangles, and a parallelogram, which can be reassembled into an infinite variety of recognizable pictures. The 15-game is a dissection of a square to also include the circle, and the problems become jigsaw puzzles of a thousand delights.

The authors, as well as hoping to re-introduce the Tangram game and introduce the 15-game to the West for the first time, give a history of the games and describe their relationship to the I Ching, the ancient Chinese Book of Change, the oldest book now known.

THE NUMBER OF SDR'S IN CERTAIN REGULAR SYSTEMS

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ABSTRACT

Let $(a_1, \cdots, a_k) = \overline{a}$ denote a vector of numbers, and let $C(\overline{a}, n)$ denote the $n \times n$ cyclic matrix having $(a_1, \cdots, a_k, 0, \cdots, 0)$ as its first row. It is shown that the sequences $(\det C(\overline{a}, n) : n = k, k+1, \cdots)$ and $(\operatorname{per} C(\overline{a}, n) : n = k, k+1, \cdots)$ satisfy linear homogeneous difference equations with constant coefficients. The permanent, per C, of a matrix C is defined like the determinant except that one forgets about $(-1)^{\operatorname{sign} \pi}$ where π is a permutation.

INTRODUCTION

While she was a student at Lowell High School, Beverly Ross [2] generalized an exercise given by Marshall Hall, Jr. [1], and found an elegant solution. Hall's exercise was posed in the context of systems of distinct representatives, or SDR's for short. Let $\overline{A} = (A_1, \cdots, A_m)$ denote an m-tuple of sets, then an m-tuple $(a_1, \cdots, a_m)$ with $a_i \in A_i$ for $i = 1, \cdots, m$ is an SDR of $\overline{A}$ if the elements $a_1, \cdots, a_m$ are all distinct. Hall's exercise is the case $m = 7$ of the following problem posed and solved by Ross: Let $A_i = \{i, i+1, i+2\}$ denote a 3-set of consecutive residue classes modulo m for $i = 1, \cdots, m$. The number of SDR's of $(A_i : i = 1, \cdots, m)$ is $2 + L_m$ where L_m is the m^{th} term of the Lucas sequence $1, 3, 4, 7, 11, \cdots$ defined by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n = 3, 4, \cdots$. For example, it follows from this result that the solution to Hall's exercise is $2 + L_7 = 31$.

In this note we give a new proof of Ross' theorem, and indicate a generalization.

ROSS' THEOREM

We shall require a simple result which appears in Ryser [3]; namely, the number of SDR's of an m-tuple $\overline{B} = (B_1, \cdots, B_m)$ of sets $B_1, \cdots, B_m$ is equal to the permanent of the incidence matrix of $\overline{B}$. Since this fact is an immediate consequence of definitions, we give them here. Let m and n denote natural numbers with $m \leq n$, and let $B_1, \cdots, B_m$ denote subsets of $\{1, \cdots, n\}$. The incidence matrix $[b(i,j)]$ of $\overline{B} = (B_1, \cdots, B_m)$ is defined by

$$b(i,j) = \begin{cases} 1, & \text{if } j \in B_i, \\ 0, & \text{if } j \notin B_i, \end{cases}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. The permanent of an $m \times n$ matrix $[r(i,j)]$ is defined to be

$$\text{per } [r(i,j)] = \sum_{\pi} r(i,\pi 1)\, r(2,\pi 2) \cdots r(m,\pi m) ,$$

where the index of summation extends over all one-to-one mappings π sending $\{1, \cdots, m\}$ into $\{1, \cdots, n\}$.

The incidence matrix C_m of the m-tuple $\overline{A} = (A_1, \cdots, A_m)$ of sets $A_1, \cdots, A_m$ considered by Ross is an $m \times m$ cyclic matrix having as its first row $(1,1,1,0,\cdots,0)$; that is, the first row has its first three components equal to 1 and the rest of its components equal to 0. For example, the incidence matrix for Hall's exercise is

$$C_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Ross' Theorem is equivalent to showing that $\text{per } C_m = 2 + L_m$. To do this, we define three sequences of matrices:

$$D_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad D_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \cdots ;$$

$$E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \cdots ;$$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \cdots .$$

Let $\text{per } C_m = c_m$, $\text{per } D_m = d_m$, $\text{per } E_m = e_m$, and $\text{per } F_m = f_m$. We use the following properties of the permanent function. First, the permanent of a 0-1 matrix is equal to

the sum of the permanents of the minors of the 1's in a row or in a column of the matrix. Second, the permanent of a matrix is unchanged by permuting the rows or by permuting the columns of the matrix. Third, the permanent of a matrix having a row or column of 0's is equal to 0. Fourth, the permanent of a square matrix is equal to the permanent of the transpose of the matrix. Expanding per C_m in terms of the minors of the 1's in the first row of C_m, we find

$$c_m = 2d_{m-1} + e_{m-1} \quad (m = 4, 5, \cdots) . \tag{1}$$

Expanding per D_m in terms of the minors of the 1's in the first column of D_m, we find

$$d_m = e_{m-1} + f_{m-1} \quad (m = 4, 5, \cdots) . \tag{2}$$

It is easy to show that

$$e_m = e_{m-1} + e_{m-2} \quad (m = 4, 5, \cdots) , \tag{3}$$

$$f_m = f_{m-1} = \cdots = f_3 = 1 . \tag{4}$$

Using the system (1)-(4) it is easy to show by induction that $e_m = F_{m+1}$, where F_m denotes the m^{th} term of the Fibonacci sequence $(1,1,2,3,\cdots)$, $d_m = 1 + F_m$, and $c_m = 2 + 2F_{m-1} + F_m = 2 + F_{m-1} + F_{m+1} = 2 + L_m$ for $m = 3, 4, \cdots$.

A GENERALIZATION

Let $\bar{a} = (a_1, \cdots, a_k)$ denote a k-tuple of numbers and let T denote a $k \times (k-1)$ matrix having all of its entries in the set $\{0, a_1, \cdots, a_k\}$. For each $n \geq k$ define an $n \times n$ matrix $C(T,n)$ as follows:

$$C(T,n) = \begin{array}{|c|c|} \hline T_1 & \begin{matrix} a_k & & & 0 \\ \vdots & & & \\ a_1 & a_k & & \\ & & \ddots & \\ & & & a_k \\ & & & \vdots \\ & & & a_1 \end{matrix} \\ \hline 0 & \\ \hline T_2 & \\ \hline \end{array}$$

The first $k-1$ columns of $C(T,n)$ have the upper triangular half T_1 of T in the upper right corner, and the lower triangular half T_2 of T in the lower left corner. All other entries in the first $k-1$ columns of $C(T,n)$ are 0. The remaining $n-k+1$ columns of

$C(T,n)$ consist of $n - k + 1$ cyclic shifts of the column $(a_k, \cdots, a_2, a_1, 0, \cdots, 0)$.

Given a $k \times (k - 1)$ matrix T having all of its entries in $\{0, a_1, \cdots, a_k\}$ and having $(t_1, \cdots, t_{k-1})$ as its top row, we expand per $C(T,n)$ by the minors of elements in the top row of $C(T,n)$. It turns out that these minors always have the form $C(T_i, n - 1)$ where T_i is a $k \times (k - 1)$ matrix having all its entries in $\{0, a_1, \cdots, a_k\}$. Thus, there exist $k \times (k - 1)$ matrices $T, \cdots, T$ having all their entries in $\{0, a_1, \cdots, a_k\}$ such that

$$\text{per } C(T,n) = \sum_{i=1}^{k} t_i \text{ per } C(T_i, n - 1) , \tag{1}$$

where $t_k = a_k$. (If we are dealing with determinants, $(-1)^i$ must be put into the summand.)

We have an equation like (1) for each matrix T; hence, we have a finite system of equations if we let T range over all possible $k \times (k - 1)$ matrices with their entries in $\{0, a_1, \cdots, a_k\}$. The existence of this system of difference equations implies the existence of a difference equation satisfied by the sequence $(\text{per } C(T,n) : n = k, k + 1, \cdots)$ for each fixed matrix T. (This is also true for the sequence $(\det C(T,n) : n = k, k + 1, \cdots)$.) A consequence of the foregoing is the result proved by Ross, but evidently much more is true.

Let $r_1, \cdots, r_n$ denote natural numbers with $1 = r_1 < \cdots < r_n = k$, and for each natural number $m \geq k$ define the collection $\overline{A}_m = \{A_1, \cdots, A_m\}$ of sets A_i of residue classes modulo m where

$$A_i = \{r_1 + i, \cdots, r_n + i\} .$$

Let $a(m)$ denote the number of SDR's of $\overline{A}_m$, then the sequence $(a(m) : m = k, k + 1, \cdots)$ satisfies a linear homogeneous difference equation with constant coefficients. The proof of this fact follows the proof of Ross' Theorem given in the preceding section.

Note that our existence theorem has a constructive proof, but we do not have an explicit expression for a difference equation satisfied by the sequence $(\text{per } C(T,n) : n = k, k + 1, \cdots)$. This gives rise to a host of interesting research problems. For example, give a difference equation satisfied by the sequence $(\text{per } C(k,n) : n = k, k + 1, \cdots)$ where $C(k,n)$ is the cyclic $n \times n$ matrix having as its first row $(1, \cdots, 1, 0, \cdots, 0)$ consisting of k 1's followed by $n - k$ 0's.

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ROOTS OF FIBONACCI POLYNOMIALS

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Usually the roots of polynomial equations of degree n become more difficult to find exactly as n increases, and for $n \geq 5$, no general formula can be applied. But, for certain classes of polynomials, the roots can be derived by using hyperbolic trigonometric functions. Here, we solve for the roots of Fibonacci and Lucas polynomials of degree n.

The Fibonacci polynomials $F_n(x)$, defined by

$$F_1(x) = 1, \quad F_2(x) = x, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x),$$

and the Lucas polynomials $L_n(x)$,

$$L_1(x) = x, \quad L_2(x) = x^2 + 2, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x),$$

have the auxiliary equation

$$Y^2 = xY + 1$$

which arises from the recurrence relation, and which has roots

$$\alpha = \frac{x + \sqrt{x^2 + 4}}{2}, \quad \beta = \frac{x - \sqrt{x^2 + 4}}{2} . \tag{1}$$

It can be shown by mathematical induction that

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n(x) = \alpha^n + \beta^n . \tag{2}$$

The first few Fibonacci and Lucas polynomials are given in Table 1. Observe that, when $x = 1$, $F_n(x) = F_n$ and $L_n(x) = L_n$, the n^{th} Fibonacci and Lucas numbers, respectively, See [1] for an introductory article on Fibonacci polynomials.

Now, we develop formulae for finding the roots of any Fibonacci or Lucas polynomial equation using hyperbolic functions defined by

$$\sinh z = (e^z - e^{-z})/2, \quad \cosh z = (e^z + e^{-z})/2$$

Table 1
Fibonacci and Lucas Polynomials

n	F_n	$L_n(x)$
1	1	x
2	x	$x^2 + 2$
3	$x^2 + 1$	$x^3 + 3x$
4	$x^3 + 2x$	$x^4 + 4x^2 + 2$
5	$x^4 + 3x^2 + 1$	$x^5 + 5x^3 + 5x$
6	$x^5 + 4x^3 + 3x$	$x^6 + 6x^4 + 9x^2 + 2$
7	$x^6 + 5x^4 + 6x^2 + 1$	$x^7 + 7x^5 + 14x^3 + 7x$
8	$x^7 + 6x^5 + 10x^3 + 4x$	$x^8 + 8x^6 + 20x^4 + 16x^2 + 2$
9	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$	$x^9 + 9x^7 + 25x^5 + 30x^3 + 9x$

which satisfy, among many other identities,

$$\cosh^2 z - \sinh^2 z = 1$$

$$\cosh iy = \cos y, \qquad \sinh iy = i \sin y .$$

If we let $x = 2 \sinh z$, then $\sqrt{x^2 + 4} = 2 \cosh z$, and from (1), $\alpha = \cosh z + \sinh z = e^z$ while $\beta = \sinh z - \cosh z = -e^{-z}$. Then,

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{e^{zn} - (-1)^n e^{-nz}}{e^z + e^{-z}}$$

$$L_n(x) = \alpha^n + \beta^n = e^{nz} + (-1)^n e^{-nz} .$$

Thus

$$F_{2n}(x) = \frac{\sinh 2nz}{\cosh z}, \qquad F_{2n+1}(x) = \frac{\cosh (2n + 1)z}{\cosh z}$$

(3)

$$L_{2n(x)} = 2 \cosh 2nz, \qquad L_{2n+1}(x) = 2 \sinh (2n + 1)z .$$

Now, clearly the polynomial equation equals zero when the corresponding hyperbolic function vanishes. For $z = x + iy$ (see [2], p. 55)

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y .$$

Thus, since for real x, $\sinh x = 0$ if and only if $x = 0$, this implies that the zeroes of $\sinh z$ are those of $\sinh iy = i \sin y$, and the zeroes of $\cosh z$ are the zeroes of $\cosh iy = \cos y$. Thus, we can easily find the z's necessary and sufficient for $F_n(x)$ and $L_n(x)$ to be zero.

Example. $F_{2n}(x) = 0$ implies that $\sinh 2nz = 0$, $\cosh z \neq 0$, so that $\sin 2ny = 0$, $\cos y \neq 0$, so $2ny = k\pi$ and $z = iy$. Thus, $x = \pm 2i \sin k\pi/2n$, $k = 0, 1, 2, \cdots, n - 1$. Specifically, the zeroes of $F_6(x)$ are given by $x = \pm 2i \sin k\pi/6$, $k = 0, 1, 2$, so that $x = 0, \pm i, \pm i\sqrt{3}$. As a check, since $F_6(x) = x(x^2 + 1)(x^2 + 3)$, we can see that the formula is working.

$F_{2n+1}(x) = 0$ only if $\cosh (2n + 1)z = 0$, $\cosh z \neq 0$, or when $\cosh (2n + 1)iy = \cos (2n + 1)y = 0$, $\cos y \neq 0$. Then, $(2n + 1)y = (2k + 1)\pi/2$, so that

$$z = iy = \frac{i(2k + 1)\pi}{(2n + 1)2} ,$$

so that

$$x = \pm 2i \sin\left(\frac{2k + 1}{2n + 1}\right) \cdot \frac{\pi}{2} , \qquad k = 0, 1, \cdots, n - 1 .$$

To summarize, taking $x = 2 \sinh z$ leads to the following solutions:

$$F_{2n}(x) = 0: \quad x = \pm 2i \sin \frac{k\pi}{2n} , \qquad k = 0, 1, \cdots, n - 1$$

$$F_{2n+1}(x) = 0: \quad x = \pm 2i \sin\left(\frac{2k + 1}{2n + 1}\right) \cdot \frac{\pi}{2} , \qquad k = 0, 1, \cdots, n - 1$$

$$L_{2n}(x) = 0: \quad x = \pm 2i \sin\left(\frac{2k + 1}{2n}\right) \cdot \frac{\pi}{2} , \qquad k = 0, 1, \cdots, n - 1$$

$$L_{2n+1}(x) = 0: \quad x = \pm 2i \sin \frac{k\pi}{2n + 1} , \qquad k = 0, 1, \cdots, n - 1 .$$

Compare with Webb and Parberry [3].

Suppose that, on the other hand, we start over again with $x = 2i \cosh z$ so that $\sqrt{x^2 + 4} = 2i \sinh z$, and $\alpha = ie^z$, $\beta = ie^{-z}$. Then, by (2),

$$F_n(x) = i^{n-1}\left(\frac{e^{zn} - e^{-zn}}{e^z - e^{-z}}\right) = i^{(n-1)} \frac{\sinh nz}{\sinh z}$$

(4)

$$L_n(x) = e^{nz} + e^{-nz} = 2 \cdot i^n \cosh nz .$$

Now this looks better. For the Fibonacci polynomials, $F_n(x) = 0$ when $\sinh nz = 0$, $\sinh z \neq 0$. Since $\sinh nz = 0$ if and only if $\sin ny = 0$ or when $z = iy$, we must have $ny = \pm k\pi$ so that $z = \pm ik\pi/n$. Since $i \cosh iy = i \cos y$, $x = 2i \cosh z = 2i \cos k\pi/n$, $k = 1, 2, \cdots, n - 1$.

Now, for the Lucas polynomials, $L_n(x) = \cosh nz = 0$ if and only if $\cos ny = 0$, or when ny is an odd multiple of $\pi/2$, and again $z = iy$, so that $x = 2i \cosh z$ becomes $x = 2i \cos (2k + 1)\pi/2n$, $k = 0, 1, \cdots, n - 1$.

To summarize, taking $x = 2 \cosh z$ leads to the following solutions:

$$F_n(x) = 0: \quad x = 2i \cos \frac{k\pi}{n} \qquad k = 1, 2, \cdots, n - 1$$

$$L_n(x) = 0: \quad x = 2i \cos \frac{(2k + 1)\pi}{2n}, \quad k = 0, 1, \cdots, n - 1$$

Actually, there is another way, using $F_{2n}(x) = F_n(x)L_n(x)$. Now, if we can solve $F_m(x) = 0$, then the roots of $L_n(x)$ are those roots of $F_{2n}(x)$ which are not roots of $F_n(x)$. Please note how this agrees with our results:

$$F_{2n}(x) = 0 \quad x = 2i \cos \frac{k\pi}{2n}, \qquad k = 1, 2, \cdots, 2n - 1$$

$$F_n(x) = 0 \quad x = 2i \cos \frac{2j\pi}{2n}, \qquad j = 1, 2, \cdots, n - 1$$

$$L_n(x) = 0 \quad x = 2i \cos \frac{(2j + 1)\pi}{2n}, \quad j = 0, 1, \cdots, n - 1 .$$

Thus the roots separate each other.

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EXPONENTIAL GENERATING FUNCTIONS FOR FIBONACCI IDENTITIES

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1. INTRODUCTION

Generating functions provide a starting point for an apprentice Fibonacci enthusiast who would like to do some research. In the Fibonacci Primer: Part VI, Hoggatt and Lind [1] discuss ordinary generating functions for identities relating Fibonacci and Lucas numbers. Also, Gould [2] has worked with generalized generating functions. Here, we use exponential generating functions to establish some Fibonacci and Lucas identities.

2. THE EXPONENTIAL FUNCTION AND EXPONENTIAL GENERATING FUNCTIONS

The exponential function e^x appears in studying radioactive decay, bacterial growth, compound interest, and probability theory. The transcendental constant $e \doteq 2.718$ is the base for natural logarithms. However, the particular property of e^x that interests us is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad . \tag{1}$$

Then

$$e^{\alpha t} = 1 + \frac{\alpha t}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} + \cdots$$

and algebra shows that

$$e^{\alpha t} - e^{\beta t} = (1 - 1) + \frac{(\alpha - \beta)t}{1!} + \frac{(\alpha^2 - \beta^2)t^2}{2!} + \frac{(\alpha^3 - \beta^3)t^3}{3!} + \cdots \quad . \tag{2}$$

To relate (2) to Fibonacci numbers, if F_n is the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, and if $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, then it is well known that

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \, . \tag{3}$$

Thus, dividing Eq. (2) by $(\alpha - \beta)$ gives

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \frac{F_1 t}{1!} + \frac{F_2 t^2}{2!} + \frac{F_3 t^3}{3!} + \frac{F_4 t^4}{4!} + \cdots = \sum_{n=1}^{\infty} F_n \frac{t^n}{n!} \quad .$$

since $F_0 = 0$, we can add the term $F_0 \frac{t^0}{0!}$ and write the following exponential generating function for Fibonacci numbers:

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \quad . \tag{4}$$

An elementary companion to the Fibonacci exponential generating function generates Lucas number coefficients. The Lucas numbers are defined by $L_1 = 1$, $L_2 = 3$, $L_n + L_{n-1} = L_{n+1}$, and have the property that

$$L_n = \alpha^n + \beta^n \quad . \tag{5}$$

If the power series for $e^{\alpha t}$ and $e^{\beta t}$ are calculated and then added term-by-term, the result is

$$e^{\alpha t} + e^{\beta t} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!} \quad . \tag{6}$$

For a novel use for these elementary generating functions, the reader is directed to [3] for a proof that the determinant of e^{Q^n} is e^{L_n}, where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

3. PROPERTIES OF INFINITE SERIES

We list without proof some properties of infinite series necessary to our development of exponential generating functions.

Given

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad ,$$

it follows that

$$A(t)\,B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!} \quad ,$$

$$A(t)\,B(-t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!} \quad . \tag{7}$$

Thus, if $B(t) = e^t$, then $b_n = 1$ for all n, and

$$A(t)\,e^t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right) \frac{t^n}{n!} \quad .$$

To help the reader with the double summation notation, let

$$A(t) = \sum_{n=0}^{\infty} n \frac{t^n}{n!} \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Then

$$\begin{aligned}
A(t)\,B(t) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} k \right) \frac{t^n}{n!} \\
&= \left(\sum_{k=0}^{0} \binom{0}{k} k \right) \frac{t^0}{0!} + \left(\sum_{k=0}^{1} \binom{1}{k} k \right) \frac{t^1}{1!} + \left(\sum_{k=0}^{2} \binom{2}{k} k \right) \frac{t^2}{2!} + \cdots \\
&= \binom{0}{0} 0 \frac{t^0}{0!} + \left(\binom{0}{0} 0 + \binom{1}{1} 1 \right) \frac{t^1}{1!} + \left(\binom{2}{0} 0 + \binom{2}{1} 1 + \binom{2}{2} 2 \right) \frac{t^2}{2!} + \cdots \\
&= 0 + \frac{t}{1!} + \frac{4t^2}{2!} + \cdots + te^{2t} = \sum_{n=0}^{\infty} \frac{t(2t)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)2^n t^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(n2^{n-1})t^n}{n!}
\end{aligned}$$

where $\binom{n}{k}$ is the binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

4. EXPONENTIAL GENERATING FUNCTIONS FOR FIBONACCI IDENTITIES

Generating function (4) and algebraic properties of α and β, the roots of $x^2 - x - 1 = 0$, give us an easy way to generate Fibonacci identities. Useful algebraic properties of $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ include:

$$\begin{array}{lll}
\alpha\beta = -1 & \alpha^2 = \alpha + 1 & F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \\
\alpha - \beta = \sqrt{5} & \alpha^m = \alpha F_m + F_{m-1} & L_n = \alpha^n + \beta^n \\
\alpha + \beta = 1 & &
\end{array}.$$

Take $B(t) = e^t$ and $A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$. (See Eqs. (1) and (4).) Then their series product $A(t)$ and $B(t)$ gives

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k\right)\frac{t^n}{n!} = \frac{e^{(\alpha+1)t} - e^{(\beta+1)t}}{\alpha-\beta} = \frac{e^{\alpha^2 t} - e^{\beta^2 t}}{\alpha-\beta}$$

(8)

$$= \sum_{n=0}^{\infty} F_{2n}\frac{t^n}{n!} \quad .$$

On the left, we used series property (7). On the right, we multiplied $A(t)\,B(t)$ and used algebraic properties of α and β, and then combined our knowledge of Eqs. (1) through (4). Lastly, equating coefficients of $t^n/n!$ gives us the identity

$$\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n} \quad .$$

If we follow the same steps with $B(t) = e^{-t}$ and $A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$, then

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_k\right)\frac{t^n}{n!} = \frac{e^{(\alpha-1)t} - e^{(\beta-1)t}}{\alpha-\beta}$$

(9)

$$= \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha-\beta} = \sum_{n=0}^{\infty}(-1)^{n+1}F_n\frac{t^n}{n!} \quad .$$

The identity resulting from (9) is

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_k = (-1)^{n+1}F_n \quad .$$

The technique, then, is this: Take $B(t)$ and $A(t)$ as simple functions in terms of powers of e. Follow algebra as outlined in Eqs. (1) through (7), and equate coefficients of $t^n/n!$ The reader is invited to use $B(t) = e^{-t}$ and $A(t) = (e^{\alpha^2 t} - e^{\beta^2 t})/(\alpha - \beta)$ to derive

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_{2k} = F_n \quad .$$

For an identity relating Fibonacci and Lucas numbers, let

$$A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta) \,, \qquad B(t) = e^{\alpha t} + e^{\beta t} \quad .$$

Since B(t) is the generating function for Lucas number coefficients (see Eq. (6)), computing the series product A(t) B(t) gives

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k L_{n-k}\right)\frac{t^n}{n!} = \frac{e^{2\alpha t} - e^{2\beta t}}{\alpha - \beta} = \sum_{n=0}^{\infty} 2^n F_n \frac{t^n}{n!} , \tag{10}$$

yielding

$$\sum_{k=0}^{n}\binom{n}{k}F_k L_{n-k} = 2^n F_n .$$

Similarly, let $A(t) = B(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$, leading to

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k}\right)\frac{t^n}{n!} = \left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right)^2 = \frac{1}{5}(e^{2\alpha t} + e^{2\beta t} - 2e^{t})$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}(2^n L_n - 2)\frac{t^n}{n!} , \tag{11}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k} = \frac{1}{5}(2^n L_n - 2) .$$

The reader should use $A(t) = B(t) = e^{\alpha t} + e^{\beta t}$ to derive

$$\sum_{k=0}^{n}\binom{n}{k}L_k L_{n-k} = 2^n L_n + 2 . \tag{12}$$

To generalize, try combinations using $e^{\alpha^m t}$ and $e^{\beta^m t}$, such as

$$A(t) = (e^{\alpha^m t} - e^{\beta^m t})/(\alpha - \beta), \qquad B(t) = e^{\alpha^m t} + e^{\beta^m t} ,$$

which generalize Eq. (10) as follows:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk} L_{mn-mk}\right)\frac{t^n}{n!} = \frac{e^{2\alpha^m t} - e^{2\beta^m t}}{\alpha - \beta} = \sum_{n=0}^{\infty} 2^n F_{mn}\frac{t^n}{n!} . \tag{10'}$$

By taking $A(t) = B(t) = (e^{\alpha^m t} - e^{\beta^m t})/(\alpha - \beta)$, Eq. (11) becomes

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk}F_{mn-mk}\right)\frac{t^n}{n!} = \left(\frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}\right)^2$$

$$= \frac{1}{5}\left(e^{2\alpha^m t} + e^{2\beta^m t} - 2e^{(\alpha^m+\beta^m)t}\right) \tag{11'}$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left(2^n L_{mn} - 2L_m^n\right)\frac{t^n}{n!} \quad .$$

The generalization of (12) found by $A(t) = B(t) = e^{\alpha^m t} + e^{\beta^m t}$ is

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}L_{mk}L_{mn-mk}\right)\frac{t^n}{n!} = \left(e^{\alpha^m t} + e^{\beta^m t}\right)^2$$

$$= e^{2\alpha^m t} + e^{2\beta^m t} + 2e^{(\alpha^m+\beta^m)t} \tag{12'}$$

$$= \sum_{n=0}^{\infty}\left(2^n L_{mn} + 2L_m^n\right)\frac{t^n}{n!}$$

The reader should now experiment with other simple functions involving powers of e. A suggestion is to use some combinations which lead to hyperbolic sines or cosines, which are defined in terms of e.

5. GENERATING FUNCTIONS FOR MORE GENERALIZED IDENTITIES

To get identities of the type

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+r} = F_{2n+r}$$

note that the r^{th} derivative with respect to t of $A(t)$ is

$$D_t^r A(t) = \sum_{n=0}^{\infty} a_{n+r}\frac{t^n}{n!}$$

so that if $A(t) = (e^{\alpha t} + e^{\beta t})/(\alpha - \beta)$, $B(t) = e^t$,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k+r}\right)\frac{t^n}{n!} = e^t D_t^r\left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right) = \frac{\alpha^r e^{(\alpha+1)t} - \beta^r e^{(\beta+1)t}}{\alpha - \beta}$$

$$= \frac{\alpha^r e^{\alpha^2 t} - \beta^r e^{\beta^2 t}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_{2n+r}\frac{t^n}{n!} \tag{13}$$

all of which suggests a whole family of identities; e.g., for

$$A(t) = (e^{\alpha^{4m}t} - e^{\beta^{4m}t})/(\alpha - \beta), \qquad B(t) = e^{t},$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{4mk+r}\right)\frac{t^n}{n!} = \frac{\alpha^{4rm}e^{(\alpha^{4m}+1)t} - \beta^{4rm}e^{(\beta^{4m}+1)t}}{\alpha - \beta}$$

$$= \frac{\alpha^{4rm}e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})t} - \beta^{4rm}e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})t}}{\alpha - \beta} \tag{14}$$

$$= \sum_{n=0}^{\infty} L_{2m}^{n} F_{2mn+4mr}\frac{t^n}{n!} .$$

From the other direction one can get identities of the type

$$\sum_{n=0}^{\infty} F_{mn}\frac{t^n}{n!} = \frac{e^{\alpha^{m}t} - e^{\beta^{m}t}}{\alpha - \beta} = \frac{e^{(\alpha F_m + F_{m-1})t} - e^{(\beta F_m + F_{m-1})t}}{\alpha - \beta}$$

$$= e^{F_{m-1}t}\left(\frac{e^{\alpha F_m t} - e^{\beta F_m t}}{\alpha - \beta}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m-1}^{n-k}F_m^k F_k\right)\frac{t^n}{n!} . \tag{15}$$

Taking the r^{th} derivative of Eq. (15) leads to

$$\sum_{n=0}^{\infty} F_{mn+rm}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m-1}^{n-k}F_m^k F_{k+rm}\right)\frac{t^n}{n!} . \tag{16}$$

Replace rm by s in Eq. (16) and compare with Vinson's result [4, p. 38].
See also H. Leonard [5].

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NOTES ON BINOMIAL COEFFICIENTS: IV - PROOF OF A CONJECTURE OF GOULD ON THE GCD'S OF TWO TRIPLES OF BINOMIAL COEFFICIENTS

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Let n and k be integers, $n \geq 2$, and $1 \leq k \leq n - 1$. Hoggatt has recently noted that

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

Gould [1] conjectures that

$$\mathrm{GCD}\left(\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right) = \mathrm{GCD}\left(\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right).$$

In this note, I shall prove this conjecture and obtain the corollary that these GCD's are equal to

$$\mathrm{GCD}\left(\binom{n-1}{k-2}, \binom{n-1}{k-1}, \binom{n-1}{k}, \binom{n-1}{k+1}\right).$$

Before proceeding, let us note that the six binomial coefficients involved form a hexagon about $\binom{n}{k}$ in the Pascal triangle. The two groups of three involved are the two equilateral triangles of this hexagon.

Theorem. For $n \geq 2$, and $1 \leq k \leq n - 1$, we have that

$$\mathrm{GCD}\left(\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right) = \mathrm{GCD}\left(\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right).$$

Proof. Let the two GCD's be G_1 and G_2, respectively. We write out the involved section of the Pascal triangle as:

$$\begin{array}{ccccccc} a & & b & & c & & d \\ & a+b & & b+c & & c+d & \\ & & a+2b+c & & b+2c+d & & \end{array},$$

where $b + c = \binom{n}{k}$, etc. Then $G_1 = \mathrm{GCD}(b, c + d, a + 2b + c)$ and $G_2 = \mathrm{GCD}(c, a + b, b + 2c + d)$. (If $k = 1$ (or $k = n - 1$) then $a = 0$ (or $d = 0$). The following argument still holds in these cases, but one can see that $G_1 = G_2 = 1 = \mathrm{GCD}(a, b, c, d)$ directly.)

We shall show that $p^e \mid G_1$ if and only if $p^e \mid G_2$, for any prime power p^e.

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Case 1. If $p^e \mid b$ and $p^e \mid c$, then $p^e \mid G_1$ iff $p^e \mid (a,d)$ iff $p^e \mid G_2$.

Case 2. If $p^e \nmid b$ and $p^e \nmid c$, then $p^e \nmid G_1$ and $p^e \nmid G_2$.

Case 3. If $p^e \mid b$ and $p^e \nmid c$, then $p^e \nmid G_2$. Suppose that $p^e \mid G_1$. Then we have $p^e \mid c + d$ and $p^e \mid a + c$, whence $p^e \nmid a$ and $p^e \nmid d$. We claim that the four conditions $p^e \nmid a$, $p^e \mid b$, $p^e \nmid c$ and $p^e \mid c + d$ are inconsistent. For this we require a lemma.

Lemma. For $0 \le k < n$,

$$p^e \Big| \binom{n}{k} \quad \text{and} \quad p^e \nmid \binom{n}{k+1}$$

implies $p \mid k + 1$.

Proof. Let $n = \Sigma a_i p^i$ and $b = \Sigma b_i p^i$ be the p-ary expansions of n and k. A result of Glaisher [2, Corollary 6.1] asserts that

$$p^{\alpha} \Big\| \binom{n}{k}$$

if and only if α is the number of borrows in the p-ary subtraction $n - k$. Consider now b_0 and a_0. If $0 \le b_0 < a_0$ or $a_0 < b_0 < p - 1$, then $n - k$ and $n - (k + 1)$ have the same number of borrows. If $b_0 = a_0 < p - 1$, then $n - (k + 1)$ has more borrows than $n - k$. Hence $b_0 = p - 1$ is the only case consistent with

$$p^e \Big| \binom{n}{k} \quad \text{and} \quad p^e \nmid \binom{n}{k+1}.$$

Corollary. For $0 \le k < n$,

$$p^e \nmid \binom{n}{k} \quad \text{and} \quad p^e \Big| \binom{n}{k+1}$$

implies $p \mid n - k$.

Proof. Use $\binom{n}{k} = \binom{n}{n-k}$ and the Lemma.

Returning to the Theorem, we have

$$p^e \nmid \binom{n-1}{k-2} = a \quad \text{and} \quad p^e \Big| \binom{n-1}{k-1} = b,$$

hence $p \mid n - k + 1$, and we have

$$p^e \Big| \binom{n-1}{k-1} = b \quad \text{and} \quad p^e \nmid \binom{n-1}{k} = c,$$

hence $p \mid k$. Thus $p \mid n + 1$. Now $c + d = \binom{n}{k+1}$. Let $n = \Sigma a_i p^i$ and $k + 1 = \Sigma b_i p^i$ be the p-ary expansions. From $p \mid n + 1$, we have $a_0 = p - 1$ and from $p \mid k$, we have $b_0 = 1$. Hence $n - (k + 1)$ has the same number of borrows as $(n - 1) - k$. From Glaisher's result and

$$p^e \nmid \binom{n-1}{k} = c,$$

we deduce that $p^e \nmid \binom{n}{k+1} = c + d$, which demonstrates the claimed inconsistency. Thus, in Case 3, $p^e \nmid G_1$ and $p^e \nmid G_2$.

Case 4. If $p^e \nmid b$ and $p^e \mid c$, then the symmetry of the binomial coefficients converts this to Case 3 and this completes the theorem.

Corollary. $G_1 = G_2 = \mathrm{GCD}(a,b,c,d)$.

Proof. We have $G_1 = G_2$ from the Theorem and so we have $G_1 \mid b$, $G_1 \mid c$, $G_1 \mid c + d$, $G_1 \mid d$, $G_1 \mid a$ and $G_1 \mid \text{GCD}(a,b,c,d)$. Conversely, GCD(a,b,c,d) clearly divides G_1.

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LETTERS TO THE EDITORS

Dear Editors:

On page 165 of Professor Coxeter's Introduction to Geometry (New York, 1961), we read: "In 1202, Leonardo of Pisa, nicknamed Fibonacci ("son of good nature"), came across his celebrated sequence ···."

This translation of Leonardo's nickname differs, of course, from the one I've seen in the Quarterly.

Who can solve the historic mystery for us?

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Dear Editors:

Thank you for the reprints I have just received. Sorry to bother you again, but somehow the main sentence from "An Old Fibonacci Formula and Stopping Rules," (Vol. 10, No. 6) was omitted. The formula is

$$\sum_{0}^{\infty} \frac{F(n)}{2^{n+1}} = 1$$

and it is based on Wald's proof that the defined stopping rule is a real stopping rule (the process terminates after a final number of steps with probability 1).

R. Peleg
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POLYNOMIALS ARISING FROM REFLECTIONS ACROSS MULTIPLE PLATES

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1. INTRODUCTION

It is known that reflections of light rays within two glass plates can be expressed in terms of the Fibonacci numbers as mentioned by Moser [1]. Here, we will explore what happens when the number of glass plates is increased. As will be seen, a new set of sequences and polynomials arises.

Assume that one starts with a single light ray and that the surfaces of the glass plates are half-mirrors, such that they both transmit and reflect light. The initial reflection, as a light ray enters the stack of plates, is ignored. Let P(n,k) be the number of possible distinct light paths, where n is the number of reflections and k the number of plates. Figure 1 illustrates the particular case of two glass plates for n = 0, 1, 2, and 3, where we already know that the possible light paths result in the Fibonacci numbers. The dots on the upper surface in this figure indicate the start of a light ray for a distinct possible path for each particular number of reflections.

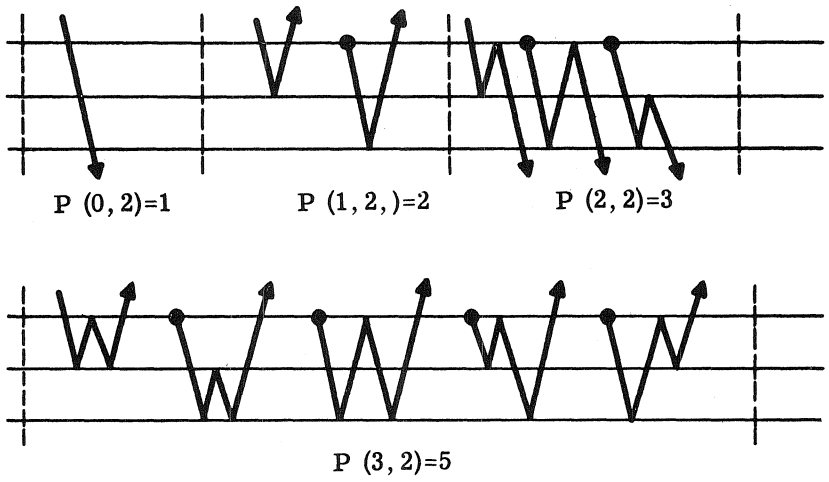

Figure 1

We will now derive a matrix equation which relates the number of distinct reflected paths to the number of reflections and to the number of glass plates and examine a sequence of polynomials arising from the characteristic equations of these matrices.

2. THE $k \times k$ MATRIX Q_k

Consider the bundle of distinct paths along which each light ray has been reflected exactly n times in a collection of k glass plates, as shown in Fig. 2. Let $Q(n,i)$ be the number of rays added by reflection to the bundle at the surface i, $1 \le i \le k$, at which point the rays make the n^{th} reflection. Since the number of rays emerging from the stack of k plates after exactly n reflections is identical to the number of possible distinct light paths for n reflections,

(1) $$P(n,k) \equiv Q(n,k) \quad \text{for all } n \text{ and } k .$$

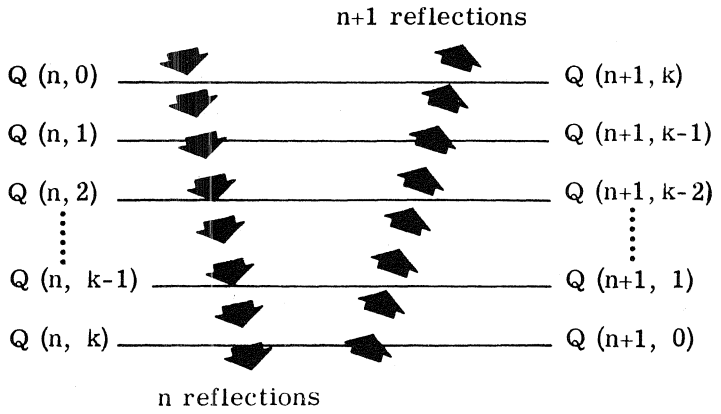

Figure 2

From Figure 2, the following set of equations is then obtained:

$$\begin{aligned}
Q(n+1,k) &= Q(n,k) + Q(n,k-1) + \cdots + Q(n,2) + Q(n,1) \\
Q(n+1,k-1) &= Q(n,k) + Q(n,k-1) + \cdots + Q(n,2) \\
&\cdots \qquad\qquad \cdots \\
Q(n+1,2) &= Q(n,k) + Q(n,k-1) \\
Q(n+1,1) &= Q(n,k) \qquad .
\end{aligned}$$

We can write this set of equations as a matrix equation,

(2) $$\begin{Vmatrix} Q(n+1,k) \\ Q(n+1,k-1) \\ \cdots \\ Q(n+1,2) \\ Q(n+1,1) \end{Vmatrix} = \begin{Vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} Q(n,k) \\ Q(n,k-1) \\ \cdots \\ Q(n,2) \\ Q(n,1) \end{Vmatrix}$$

and define Q_k as the square matrix of order k which arises with its elements above and on the minor diagonal all ones and with all zeros below the minor diagonal,

$$(3) \qquad Q_k = \begin{Vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{Vmatrix}_{k \times k}$$

Next, we find a recursion relation for the characteristic polynomials for the matrices Q_k. We let

$$(4) \qquad D_k(y) = \det (Q_k - yI_k) \ ,$$

where Q_k is given by (3) and I_k is the identity matrix of order k. We display Eq. (4) as

$$(5) \qquad D_k(y) = \begin{Vmatrix} 1-y & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-y & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1-y & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & -y & 0 \\ 1 & 0 & 0 & \cdots & 0 & -y \end{Vmatrix}_{k \times k}$$

The determinant on the right side of (5) is now modified by subtracting row 2 from row 1, after which column 2 is subtracted, resulting in (6):

$$(6) \qquad \begin{Vmatrix} -2y & y & 0 & \cdots & 0 & 1 \\ y & 1-y & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1-y & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & -y & 0 \\ 1 & 0 & 0 & \cdots & 0 & -y \end{Vmatrix}_{k \times k}$$

This determinant (6) is then expanded by the elements in the first column, giving

$$(7) \qquad D_k(y) = (-2y)A_1 - yA_2 + (-1)^{k+1}A_3 \ ,$$

where A_1, A_2, and A_3 are cofactors still to be evaluated. When A_1 and A_3 are expanded by the elements in their last row and column, respectively, they become $A_1 = -yD_{k-2}(y)$ and $A_3 = (-1)^k D_{k-2}(y)$. If the determinant A_2 is expanded according to the elements of its first row, the resulting determinant according to the elements of its last row, and finally this new determinant according to the elements of its last row, one finds $A_2 = y^3 D_{k-4}(y)$. The above expressions for A_1, A_2, and A_3 are then substituted into Eq. (7), which yields the result

$$D_k(y) = (2y^2 - 1)D_{k-2}(y) - y^4 D_{k-4}(y) \quad , \tag{8}$$

the desired recursion formulas for the polynomials $D_k(y)$ for all $k \geq 1$.

3. THE COEFFICIENTS OF $D_n(y)$ AND RECURSION FORMULAS FOR $P(n,k)$

The polynomials $D_n(y)$ can be expanded in power series in y as

$$D_n(y) = \sum_{i=0}^{n} A_{n,i}\, y^{n-i} \quad , \tag{9}$$

where $A_{n,i}$ are constants. By substituting these power series into the recursion formula (8) and equating coefficients of like powers of y, the following recursion relation among the coefficients are obtained:

$$A_{n,i} = 2A_{n-2,i} - A_{n-2,i-2} - A_{n-4,i} \quad , \qquad 0 \leq i \leq n \quad , \tag{10}$$

where we take $A_{n,i} = 0$ whenever $i < 0$ or $n < i$. For $n = 2, 4, 6$ and $n = 1, 3, 5$, one obtains from the recursion formula (8),

$$\begin{aligned}
D_2(y) &= y^2 - y - 1 \\
D_4(y) &= y^4 - 2y^3 - 3y^2 + y + 1 \\
D_6(y) &= y^6 - 3y^5 - 6y^4 + 4y^3 + 5y^2 - y - 1 \\
D_1(y) &= -y + 1 \\
D_3(y) &= -y^3 + 2y^2 + y - 1 \\
D_5(y) &= -y^5 + 3y^4 + 3y^3 - 4y^2 - y + 1 \quad .
\end{aligned}$$

The coefficients $A_{n,i}$ can now be evaluated by using the above polynomials and recursion relations, resulting in the set of specific formulas in addition to those of (10):

$$\begin{aligned}
A_{2n,0} &= 1 & A_{2n+1,0} &= -1 \\
A_{2n,1} &= -n & A_{2n+1,1} &= n + 1 \\
A_{2n,2n} &= A_{2n,2n-1} = (-1)^n & A_{2n+1,2n} &= -A_{2n+1,2n+1} = (-1)^{n+1} \\
A_{2n,2n-2} &= (-1)^{n+1}(2n - 1) & A_{2n+1,2n-1} &= (-1)^{n+1}2n \\
A_{2n,2n-3} &= (-1)^{n+1}(2n - 2) & A_{2n+1,2n-2} &= (-1)^n(2n - 1) \quad .
\end{aligned}$$

These sets of formulas for the coefficients will then permit one to write the polynomials $D_n(y)$ as power series in y, which is very useful in obtaining recursion formulas for $P(n,k)$.

If we let $D_n(y) = 0$ in the power series expansion (9), the resulting equation implies that

$$\sum_{i=0}^{n} A_{k,i} P(n - i, k) = 0 \tag{11}$$

for all k. Then, Eq. (11) is the recursion relation for the numbers $P(n,k)$, and for $k \leq 5$, we can write

$$P(n + 1, 1) = P(n,1)$$

$$P(n + 2, 2) = P(n + 1, 2) + P(n,2)$$

$$P(n + 3, 3) = 2P(n + 2, 3) + P(n + 1, 3) - P(n,3)$$

$$P(n + 4, 4) = 2P(n + 3, 4) + 3P(n + 2, 4) - P(n + 1, 4) - P(n,4)$$

$$P(n + 5, 5) = 3P(n + 4, 5) + 3P(n + 3, 5) - 4P(n + 2, 5) - P(n + 1, 5) + P(n,5) .$$

4. A GENERATING FUNCTION FOR THE POLYNOMIALS $D_n(y)$

Theorem 1. A generating function for $D_{2n}(y)$ is

$$[1 - y(y + 1)t][1 - (2y^2 - 1)t + y^4 t^2]^{-1} = \sum_{n=0}^{\infty} D_{2n}(y)t^n . \tag{12}$$

Proof. In Eq. (12), multiply both sides by the denominator on the left side to obtain

$$\begin{aligned} 1 - y(y + 1)t &= [1 - (2y^2 - 1)t + y^4 t^2] \cdot \sum_{n=0}^{\infty} D_{2n}(y)t^n \\ &= D_0(y) + [D_2(y) - (2y^2 - 1)D_0(y)]t \\ &\quad + \sum_{n=0}^{\infty} [D_{2n+4}(y) - (2y^2 - 1)D_{2n+2}(y) + y^4 D_{2n}(y)]t^{n+2} . \end{aligned} \tag{13}$$

Equating like powers of t on the left and right sides of Eq. (13), one obtains

$$D_0(y) = 1$$

$$D_2(y) = (2y^2 - 1)D_0(y) - y(y + 1) = y^2 - y - 1$$

$$D_{2n+4}(y) = (2y^2 - 1)D_{2n+2}(y) - y^4 D_{2n}(y)$$

Since these last three equations are in full agreement with the recursion formulas derived earlier, one concludes that the theorem has been proved.

Theorem 2. A generating function for $D_{2n+1}(y)$ is

$$[y + 1 - y^3t][1 - (2y^2 - 1)t + y^4t^2]^{-1} = \sum_{n=0}^{\infty} D_{2n+1}(y)t^n \quad . \tag{14}$$

The proof of this theorem readily follows if one uses the same procedure as in proving the preceding theorem.

The generating functions of Eqs. (12) and (14) can be used to obtain closed form solutions for $D_{2n}(y)$ and $D_{2n+1}(y)$ with the aid of the following equations from Rainville [2]:

$$(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \tag{15}$$

$$(a)_n = \prod_{k=1}^{n} (a + k - 1) = a(a + 1)(a + 2) \cdots (a + n - 1), \quad n \geq 1 ,$$
$$(a)_0 = 1, \quad a \neq 0 \quad . \tag{16}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{n - k}{k} \quad . \tag{17}$$

By applying Eqs. (15) and (17) to the denominator of Eq. (12),

$$[1 - (2y^2 - 1)t + y^4t^2]^{-1} = \sum_{n=0}^{\infty} [(1)_n / n!] [(2y^2 - 1)t - y^4t^2]^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2y^2 - 1)^{n-k} y^{4k} t^{n+k} \tag{18}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n - k}{k} (2y^2 - 1)^{n-2k} y^{4k} t^n \quad .$$

Hence,

$$[1 - y(y+1)t][1 - (2y^2 - 1)t + y^4t^2]^{-1}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y-1)^{n-2k} y^{4k} t^n + 1$$

$$- y(y+1) \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2-1)^{n-2k-1} y^{4k} t^n \quad . \tag{19}$$

Now

$$\sum_{n=0}^{\infty} D_{2n}(y)t^n = \sum_{n=1}^{\infty} D_{2n}(y)t^n + 1 \quad . \tag{20}$$

Therefore, by equating coefficients of like powers of t in Eq. (12), a closed-form solution for $D_{2n}(y)$ is extracted:

$$D_{2n}(y) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y^2-1)^{n-2k} y^{4k}$$

$$- y(y+1) \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2-1)^{n-2k-1} y^{4k} \quad . \tag{21}$$

The closed-form solution for $D_{2n+1}(y)$ follows readily from the above derivation,

$$D_{2n+1}(y) = (y+1) \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2y^2-1)^{n-2k} y^{4k}$$

$$- y^3 \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2y^2-1)^{n-2k-1} y^{4k} \quad . \tag{22}$$

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REMARK ON A PAPER BY DUNCAN AND BROWN ON THE SEQUENCE OF LOGARITHMS OF CERTAIN RECURSIVE SEQUENCES

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In the present paper, it is shown that the main theorem in [1], see p. 484, can be established by using one of J. G. van der Corput's difference theorems [2]. Moreover, by using a theorem of C. L VandenEynden [3] we show the property that the sequence of the integral parts of the logarithms of the recursive sequence under consideration is also uniformly distributed modulo m for any integer $m \geq 2$.

Lemma 1. Let (x_n), $n = 1, 2, \cdots$, be a sequence of real numbers. If

$$\lim_{n \to \infty} (x_{n+1} - x_n) = \alpha ,$$

α irrational, then (x_n) is u.d. mod 1 ([2], p. 378).

Lemma 2. Let (x_n), $n = 1, 2, \cdots$, be a sequence of real numbers. Assume that the sequence (x_n/m), $n = 1, 2, \cdots$, is u.d. mod 1 for all integers $m \geq 2$. Then the sequence of the integral parts $([x_n])$, $n = 1, 2, \cdots$, is u.d. mod m [3].

For the notion of uniform distribution modulo m we refer to [4].

Theorem. Let (V_n), $n = 1, 2, \cdots$, be a sequence generated by the recursion relation

$$V_{n+k} = a_{k-1}V_{n+k-1} + \cdots + a_1V_{n+1} + a_0V_n, \qquad n \geq 1 , \tag{1}$$

where $a_0, a_1, \cdots, a_{k-1}$ are non-negative rational coefficients with $a_0 \neq 0$, k is a fixed integer, and

$$V_1 = \gamma_1, \quad V_2 = \gamma_2, \quad \cdots, \quad V_k = \gamma_k \tag{2}$$

are given positive values for the initial terms. It is assumed that the polynomial

$$x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$$

has k distinct real roots $\beta_1, \beta_2, \cdots, \beta_k$ satisfying $0 < |\beta_k| < \cdots < |\beta_k|$ and such that none of the roots has magnitude equal to 1. Then:

1. The sequence $(\log V_n)$, $n = 1, 2, \cdots$, is u.d. mod 1 [1].
2. The sequence $([\log V_n])$, $n = 1, 2, \cdots$, is u.d.

Proof. By (1) and (2), we have that

$$V_n = \sum_{j=1}^{k} \alpha_j \beta_j^n \qquad (n \geq 1) ,$$

where the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_k$ are uniquely determined by assumption (2). Let p be the largest value of j for which $a_j \neq 0$. We have $p \geq 1$. Hence

$$V_n = \sum_{j=1}^{p} \alpha_j \beta_j^p \quad .$$

Now

$$\frac{V_{n+1}}{V_n} = \frac{\alpha_1 \beta_1^{n+1} + \cdots + \alpha_p \beta_p^{n+1}}{\alpha_1 \beta_1^n + \cdots + \alpha_p \beta_p^n} \to \beta_p \qquad \text{as} \qquad n \to \infty \quad ,$$

since $\beta_1^n \,/\, \beta_p^n \to 0$ (as $n \to \infty$), $i = 1, 2, \cdots, p-1$, because of the conditions on the absolute values of the β_j. (From the conditions follows that $\beta_p > 0$.) Hence we have that

$$\log V_{n+1} - \log V_n \to \log \beta_p, \qquad \text{as} \qquad n \to \infty \quad .$$

The number β_p is algebraic and therefore $\log \beta_p$ is an irrational number (see [1]). Hence Lemma 1 applies and we obtain that the sequence $(\log V_n)$ is u.d. mod 1. This proves Duncan and Brown's result.

In order to show the second part of the theorem we observe that for every integer $m \geq 2$

$$\frac{\log V_{n+1}}{m} - \frac{\log V_n}{m} \to \frac{\beta_p}{m}, \qquad \text{as} \qquad n \to \infty \quad ,$$

hence the sequence $((\log V_n)/m)$, $n = 1, 2, \cdots$ is u.d. mod 1, and according to Lemma 2 we obtain that the sequence of the integral parts $([\log V_n])$ is u.d. mod m for every integer $m \geq 2$.

Remark. By restricting the order of the recurrence we may relax the conditions on the coefficients a_j and the initial values of V_n. The values of elements of (V_n) can be negative in that case, and so we obtain a result regarding the logarithms of the absolute value of V_n.

Let (V_n), $n = 1, 2, \cdots$, be a sequence generated by the recurrence

$$V_{n+2} = a_1 V_{n+1} + a_0 V_n, \qquad n \geq 1 ,$$

where $V_1 = \gamma_1$, $V_2 = \gamma_2$. We assume that γ_1, γ_2, a_0 and a_1 are rational numbers, where γ_1 and γ_2 are $\neq 0$, and a_0 and a_1 not both 0. Moreover, it is assumed that the polynomial $x^2 - a_1 x - a_0$ has distinct real roots, β_1 and β_2, one of which has an absolute value

different from 1. Then the sequence $(\log|V_n|)$ is u.d. mod 1, and the sequence of integral parts $([\log|V_n|])$ is u.d.

Proof. We have

$$V_n = \frac{(\gamma_2 - \gamma_1\beta_2)\beta_1^{n-1} - (\gamma_2 - \gamma_1\beta_1)\beta_2^{n-1}}{\beta_1 - \beta_2} ,$$

where

$$\beta_1 = \tfrac{1}{2}(a_1 + \sqrt{a_1^2 + 4a_0}), \qquad \beta_2 = \tfrac{1}{2}(a_1 - \sqrt{a_1^2 + 4a_0}) .$$

Now

$$\log|V_{n+1}| - \log|V_n| = \log\left|\frac{(\gamma_2 - \gamma_1\beta_2)\beta_1^n - (\gamma_2 - \gamma_1\beta_1)\beta_2^n}{(\gamma_2 - \gamma_1\beta_2)\beta_1^{n-1} - (\gamma_2 - \gamma_1\beta_1)\beta_2^{n-1}}\right| .$$

We may suppose that $|\beta_1| \neq 1$, $|\beta_2/\beta_1| < 1$.

Since $\log|V_{n+1}| - \log|V_n| \to \log|\beta_1|$ as $n \to \infty$, and as $|\beta_1|$ is algebraic when β_1 is algebraic, we may complete the proof in the same way as done above.

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ERRATA

Please make the following changes in the article, "A Triangle with Integral Sides and Area," by H. W. Gould, appearing in Vol. 11, No. 1, pp. 27-39.

Page 28, line 3 from bottom:	For $+u - v\sqrt{3})$	read	$+(u - v\sqrt{3})$.
Page 31, Eq. (11):	For $\frac{K^2}{a^2}$	read	$\frac{K^2}{s^2}$.
Page 31, line 6 from bottom:	For $4x^2 - 3y^2$	read	$4x^2 - 3v^2$.
Page 33, Eq. (17):	For r_u^2	read	r_a^2 .
Page 33, Eq. (22):	For r_c: , 6, 14	read	r_c:∞, 6, 14 .
Page 35, Line 13:	For i.e.	read	as .
Page 35, Line 16:	For N = orthocenter	read	H = orthocenter.
Page 35, line 9 from bottom:	For $\|I = H\|^2$	read	$\|I - H\|^2$.
Page 36, line 12 from bottom:	For residue	read	radius .
Page 39, Ref. 4.	Underline Jahrbuch uber die.		
Page 39, Ref. 4.	Closed quotes should follow sind rather than Dreieck.		

REPRESENTATIONS AS PRODUCTS OR AS SUMS

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1. INTRODUCTION

In a previous paper [1] the idea of a "sieve" was extended in order to give a method for the computation of the sequences of values for certain functions which occur in the theory of numbers. Some of the important functions are generated as the sequences of coefficients of suitable Dirichlet series or of suitable power series; see, for example, G. H. Hardy and E. M. Wright [2, Chapters 17 and 19]. We will consider the similar problems of the number of representations of an integer as a product with factors chosen from a given set of positive integers and the problem of the number of representations of an integer as a sum with terms chosen from a given set of positive integers. Although quite analogous, the two problems are rarely mentioned together.

To be specific, we consider a subsequence $S = \{a_n\}$, of positive integers, finite or infinite, which satisfies the conditions $a_1 < a_2 < a_3 < \cdots$ with $1 < a_1$ for the case of products and $0 < a_1$ for the case of sums. Our problems can then be stated as (1) compute the number, $R(n)$, of distinct representations of a number n as a product of the form

$$n = a_j^{b_j} a_k^{b_k} \cdots ,$$

with $a_i \in S$, $b_i > 0$, and (2) compute the number, $P(n)$, of distinct representations of a number n as a sum of the form $n = b_j a_j + b_k a_k + \cdots$, $a_i \in S$, $b_i > 0$. Analogous problems are obtained by the restriction $b_i = 1$; these are (1') compute the number, $R'(n)$, of distinct representations of a number n as a product of the form $n = a_j a_k \cdots$, $a_i \in S$, that is, of distinct factors, and (2') compute the number, $P'(n)$, of distinct representations of a number n as a sum of the form $n = a_j + a_k + \cdots$, $a_i \in S$, that is, of distinct terms. The generating functions for the sequences R, P, R', and P' are given by

$$\prod (1 - a_i^{-s})^{-1} = \sum R(n) n^{-s} ,$$

$$\prod (1 - x^{a_i})^{-1} = \sum P(n) x^n ,$$

$$\prod (1 + a_i^{-s}) = \sum R'(n) n^{-s} ,$$

$$\prod (1 + x^{a_i}) = \sum P'(n) x^n .$$

The reciprocals of these generating functions are also of interest. For example, those for R and P lead to generalizations of the Möbius function ($\mu(n)$ is obtained for $a_i = p_i$) and of the Euler identity (whis is obtained for $a_i = i$), respectively,

$$\prod (1 - a_i^{-s}) = \sum M(n)n^{-s} ,$$

$$\prod (1 - x^{a_i}) = \sum K(n)x^n .$$

2. REPRESENTATIONS AS A PRODUCT

From the generating function we can develop a "sieve" technique ("sieve" used in the sense given in [1]) for computing the values of the sequence R. The generating function is rewritten in the form

$$\prod(1 - a_n^{-s})^{-1} = \prod(1 + a_n^{-s} + a_n^{-2s} + \cdots) ,$$

where the products extend over $a_n \in S$. We need to know the sequence $\{a_n\}$ for $1 \le n \le N$ in advance as the input, if we are to compute $R(n)$ for $1 \le n \le N$. Table 1 illustrates this process for the input sequence of Fibonacci numbers 2, 3, 5, 8, $\cdots$.

Table 1
Product Representations, $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	1 0	1	2	3	4	5	6	7	8	9	2 0	1	2	3	4	5	6	7	8	9	3 0	1	2
$R^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R^{(1)}$	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$R^{(2)}$	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1
$R^{(3)}$	1	1	1	1	1	1	0	1	1	1	0	1	0	0	1	1	0	1	0	1	0	0	0	1	1	0	1	0	0	1	0	1
$R^{(4)}$	1	1	1	1	1	1	0	2	1	1	0	1	0	0	1	2	0	1	0	1	0	0	0	2	1	0	1	0	0	1	0	2
$\cdots$	$\cdots$																															
R	1	1	1	1	1	1	0	2	1	1	0	1	1	0	1	2	0	1	0	1	1	0	0	2	1	1	1	0	0	1	0	2

To begin, let $R^{(0)}$ denote the sequence with 1 in position 1 and 0 in position n for $2 \le n \le N$. In order to obtain the sequence $R^{(1)}$ we bring down 1 into position 1 and then add the entry from position 1 of $R^{(1)}$ to the entry at position $1a_1$ of $R^{(0)}$ and enter the sum in position a_1 of $R^{(1)}$, then the entry from position 2 of $R^{(1)}$ is added to the entry in position $2a_1$ of $R^{(0)}$ and the sum is entered in position $2a_1$ of $R^{(1)}$, etc. At position $n \neq ma_1$ of $R^{(1)}$ we simply use the entry from position n of $R^{(0)}$. This set of subprocesses using a_1 is stopped when $ma_1 > N$. To continue, the entry from position 1 of $R^{(1)}$ is entered in position 1 of $R^{(2)}$ and in general the entry in position n of $R^{(1)}$ is computed successively for $1 < n \le N$ by either entering in position n of $R^{(2)}$ the entry from position

n of $R^{(1)}$ if $n \neq ma_2$, but if $n = ma_2$ by adding the entry at position m of $R^{(2)}$ to the entry at position ma_2 of $R^{(1)}$ and entering the sum in position n of $R^{(2)}$. For the general iterative step going from $R^{(k-1)}$ to $R^{(k)}$, since $R^{(k)}(n)$ sometimes depends on previous entries in $R^{(k)}$, we use these formulas sequentially for $n = 1, 2, 3, \cdots$,

$$R^{(k)}(n) = R^{(k-1)}(n) \quad \text{if} \quad a_k \nmid n ,$$

$$R^{(k)}(n) = R^{(k-1)}(n) + R^{(k)}(n/a_k) \quad \text{if} \quad a_k \mid n .$$

We stop each subprocess when $ma_k > N$ and we stop the entire process when $a_k > N$; the result is the sequence of values of R for $1 \leq n \leq N$. It should be noted that the sequence $R^{(k-1)}$ can be destroyed entry-by-entry as $R^{(k)}$ is generated.

The reasoning behind the workings of the process is as follows. Suppose that we have already generated the sequence $R^{(k-1)}$. We want to multiply that series generated by the function

$$\prod_{n=1}^{k-1} (1 - a_n^{-s})^{-1}$$

by the series

$$(1 - a_k^{-s})^{-1} = 1 + a_k^{-s} + a_k^{-2s} + \cdots .$$

This is equivalent to the generation of $R^{(k)}$ from $R^{(k-1)}$. Actually, this multiplication is equivalent to successively expanding the scale for $R^{(k-1)}$ by a factor a_k and adding the expanded result to the sequence $R^{(k-1)}$ itself. This is illustrated in Table 2 in which $R^{(k-1)}(n)$ is denoted by c_n.

Table 2
Product Pattern

$$\begin{array}{llllllllll}
c_1 c_2 \cdots & c_m \cdots & c_{a_k} \cdots & c_{2a_k} \cdots & c_{ma_k} \cdots & c_{a_k^2} \cdots & c_{ma_k^2} \cdots \\
 & c_1 & c_2 \cdots & c_m \cdots & c_{a_k} \cdots & c_{ma_k} \cdots & \\
 & & & & c_1 \cdots & c_m \cdots & \\
 & & & & & & \cdots
\end{array}$$

From the diagram in Table 2 we note that rows, beginning with the second, are repeats of the first row, but with the scale expanded successively by the factor a_k. If we consider a column which is to be summed, we note that the quantity to be added to c_{na_k} is merely the sum appearing in the column headed by c_n, hence we obtain the iteration equations.

In the intermediate steps the number $R^{(k)}(n)$ has an interpretation as the number of representations of n as a product using only elements of the finite subsequence $a_1, a_2, \cdots, a_k$.

3. REPRESENTATIONS AS A SUM (PARTITIONS)

Here we have a quite closely analogous case to that of the previous section. The generating function is rewritten

$$\Pi(1 - x^{a_n})^{-1} = \Pi(1 + x^{a_n} + x^{2a_n} + \cdots) .$$

In Table 3 the operations are diagrammed analogous to Table 2 except that here we notice that the rows are shifted by an amount a_k and then successively added to the $(k - 1)$-sequence.

Table 3
Sum Pattern

$$\begin{array}{llllllll}
c_0\ c_1\ c_2 \cdots & c_{a_k}\ c_{a_k+1} \cdots & c_{2a_k} \cdots & c_{3a_k} \cdots & c_{ma_k} & \cdots \\
c_0 \quad c_1 \quad \cdots & c_{a_k} \cdots & c_{2a_k} \cdots & c_{(m-1)a_k} & \cdots \\
c_0 \quad \cdots & c_{a_k} \cdots & c_{(m-2)a_k} & \cdots \\
c_0 \quad \cdots & & & \\
\cdots & & & \\
c_0 & & &
\end{array}$$

(Here the sequence is indexed from 0.) Reading the columns as in Table 2 we have the interation formulas

$$P^{(k)}(n) = P^{(k-1)}(n) \qquad \text{if } a_k \nmid n ,$$

$$P^{(k)}(n) = P^{(k-1)}(n) + P^{(k)}(n - a_k) \qquad \text{if } a_k \leq n .$$

The initial sequence $P^{(0)}$ corresponds to $R^{(0)}$; a 1 appears in position 0 and 0 appears otherwise. The process is stopped similarly. It is of interest to note that $a_k \leq n$ here replaces $a_k \mid n$ and that $n - a_k$ replaces n/a_k. In Table 4, the case of the Fibonacci sequence 1, 2, 3, 5, 8, $\cdots$ is illustrated.

Table 4
Sum Representations, $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$P^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$P^{(2)}$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11
$P^{(3)}$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48
$P^{(4)}$	1	1	2	3	4	6	8	10	13	16	20	24	29	34	40	47	54	62	71	80	91	102
$\cdots$	$\cdots$																					
P	1	1	2	3	4	6	8	10	14	17	22	27	33	41	49	59	71	83	99	115	134	157

In the intermediate steps the number $P^{(k)}(n)$ denotes the number of partitions of n involving only those numbers $a_1, a_2, \cdots, a_k$.

4. REPRESENTATIONS WITH DISTINCT ELEMENTS

If it is required that the representations involve no repeated elements, that is, "squarefree" products or "pairfree" sums, the generating functions are simpler and hence the computational process is also simpler. By reasoning somewhat analogous to the previous two sections we obtain the recurrence formulas

$$R'^{(k)}(n) = R'^{(k-1)}(n) \quad \text{if } a_k \nmid n,$$
$$R'^{(k)}(n) = R'^{(k-1)}(n) + R'^{(k-1)}(n/a_k) \quad \text{if } a_k \mid n,$$

where we start with $R'^{(0)}(1) = 1$ and $R'^{(0)}(n) = 0$ for $n > 1$. The case of distinct terms of a sum is quite analogous to this. Starting with $P'^{(0)}(0) = 1$, $P'^{(0)}(n) = 0$ for $n > 0$ we obtain the recurrences

$$P'^{(k)}(n) = P'^{(k-1)}(n) \quad \text{if } a_k \nleq n,$$
$$P'^{(k)}(n) = P'^{(k-1)}(n) + P'^{(k-1)}(n - a_k) \quad \text{if } a_k \leq n.$$

The only alteration required to change the formulas of the previous sections to these is the change in the upper index on the second term. Tables 5 and 6 display the computations of R' and P' for the Fibonacci sequence.

Table 5
Squarefree Products, $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	1 0	1	2	3	4	5	6	7	8	9	2 0	1	2	3	4	5	6	7	8	9	3 0	1	2
$R'^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(1)}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(2)}$	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$R'^{(3)}$	1	1	1	0	1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
...		...																														
R	1	1	1	0	1	1	0	1	0	1	0	0	1	0	1	1	0	0	0	0	1	0	0	1	0	1	0	0	0	1	0	0

Table 6
Pairfree Sums, $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	1 0	1	2	3	4	5	6	7	8	9	2 0	1	2	3	4	5	6	7	8	9	3 0	1
$P'^{(0)}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P'^{(1)}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(Continued on following page.)

Table 6 (Continued

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$P_1^{(2)}$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_1^{(3)}$	1	1	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_1^{(4)}$	1	1	1	2	1	2	2	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_1^{(5)}$	1	1	1	2	1	2	2	1	3	2	2	3	1	2	2	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
...	...																															
P	1	1	1	2	1	2	2	1	3	2	2	3	1	3	3	2	4	2	3	3	1	4	3	3	5	2	4	4	2	5	3	3

5. OTHER FORMS

As was remarked in the first section, the reciprocals of the generating functions are also of interest. Tables 7 and 8 illustrate these analogs for the Fibonacci sequence.

Table 7
Product "Reciprocal," $S = \{2, 3, 5, 8, 13, \cdots\}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$M^{(0)}$	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(1)}$	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(2)}$	+1	-1	-1	0	0	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$M^{(3)}$	+1	-1	-1	0	-1	+1	0	0	0	+1	0	0	0	0	+1	0	0	0	0	0	0
...	...																				
M	+1	-1	-1	0	-1	+1	0	-1	0	+1	0	0	-1	0	+1	+1	0	0	0	0	-1

Table 8
Sum "Reciprocal," $S = \{1, 2, 3, 5, 8, 13, \cdots\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$K^{(0)}$	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(1)}$	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(2)}$	+1	-1	-1	+1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(3)}$	+1	-1	-1	0	+1	+1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$K^{(4)}$	+1	-1	-1	0	+1	0	0	+1	0	-1	-1	+1	0	0	0	0	0	0	0	0	0	0
...	...																					
K	+1	-1	-1	0	+1	0	0	+1	-1	0	0	+1	-1	-1	+1	0	0	0	+1	-1	-1	0

Note that the only alterations required in the techniques used involves the changes of signs; this is equivalent to switching the primed and unprimed formulas and subtracting the second terms instead of adding these terms.

All of the cases which have been considered are special cases of the general formulas

$$\prod(1 + f(a_k)a_k^{-s} + f(a_k^2)a^{-2s} + \cdots) = \sum F(n)n^{-s},$$

$$\prod(1 + g(a_k)x^{a_k} + g(a_k^2)x^{2a_k} + \cdots) = \sum G(n)x^n.$$

Other cases can certainly be derived and similar lines of reasoning can be carried out for the simpler cases. More complicated cases can also be worked out, if the process is generalized somewhat, but they become messy.

REFERENCES

1. R. B. Buschman, "Some Simple Sieves," Fibonacci Quarterly, Vol. 11, No. 3, pp. 247-254.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 3rd Ed., Oxford, Clarendon Press, 1954.

ERRATA

Please make the following changes in the article, "A New Look at Fibonacci Generalizations," by N. T. Gridgeman, appearing in Vol. 11, No. 1, pp. 40-55.

Page 40, Eqs. (1) and (2). Please insert an opening bracket immediately following the summation sign, and a closing bracket immediately following "B" in both cases. In Eq. (2), please change the lower limit of the summation to read: "i=0" instead of "m=0."

Page 41, Table 1. Please add continue signs, i.e., $\vdots \quad \vdots$, at the end of the table.

Page 42, line 14 from bottom: Please correct spelling from "superflulous" to "superfluous."

Page 42, line 7 from bottom: Please insert a space between "over" and "positive."

Page 44, line 10: Please change "member" to read "members."

Page 45, Eq. (11): Please add an opening bracket immediately before $\binom{h}{g-1}$ and a closing bracket at the end of the line.

Page 46, line 10: Please change the first fraction to read " $\sqrt{9}/2 = 2$;"

Page 47, Eq. (18): Please correct the numerator to read:

$$[N - (B - 1)(1/2 - R)](1/2 + R)^n - [N - (B - 1)(1/2 + R)](1/2 - R)^n$$

[Continued on page 306.]

MULTIPLE REFLECTIONS

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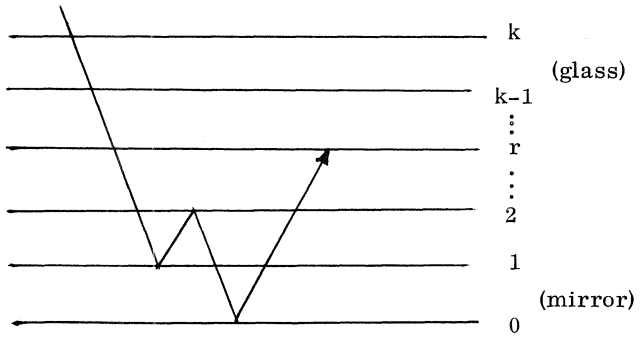
Let us consider a set of k parallel plane prisms and a mirror arranged as follows:



Let $f_r(n)$ be the number of paths which start at the top of the plate and reach plate r after n upward reflections. Further, let $A = (a_{ij})$ be a $k \times k$ enumerating matrix such that

$$f_r(n) = \sum_{j=1}^{k} a_{rj} f_j(n - 1) \quad . \tag{1.1}$$

If $F(n)$ denotes the one column matrix

$$F(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \\ \cdots \\ f_k(n) \end{bmatrix} \tag{1.2}$$

then (1.1) can be written

$$F(n) = AF(n - 1) \quad . \tag{1.3}$$

Hence, by iteration we have

$$F(n) = A^n F(0) \quad . \tag{1.4}$$

Thus (1.4) provides an explicit solution for $F(n)$ in terms of $F(0)$. This form is not suitable to compute the asymptotic behavior of $F(n)$ for large values of n. We now derive a second explicit form by means of which the asymptotic behavior is easily calculated.

The characteristic equation of the matrix A is

$$|\lambda I - A| = \lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_0 = 0 \ . \tag{1.5}$$

Let us assume that the roots of (1.5) are $\lambda_1, \lambda_2, \cdots, \lambda_k$ and that these roots are distinct. The case of multiple roots can also be treated easily by the method we shall use.

Since every matrix A satisfies its own characteristic equation we also have

$$A^k + c_{k-1}A^{k-1} + \cdots + c_0 I = 0 \ . \tag{1.6}$$

Multiplying by A^{n-k}, we have

$$A^n + c_{k-1}A^{n-1} + \cdots + c_0 A^{n-k} = 0. \tag{1.7}$$

From (1.4) and (1.7) we immediately have

$$F(n) + c_{k-1}F(n-1) + \cdots + c_0 F(n-k) = 0 \ , \tag{1.8}$$

and

$$f_r(n) + c_{k-1}f_r(n-1) + \cdots + c_0 f_r(n-k) = 0 \ . \tag{1.9}$$

However, Eq. (1.9) depends for its solution on the equation

$$\lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_0 = 0 \ . \tag{1.10}$$

Hence, the general solution for (1.9) is

$$f_r(n) = \sum_{j=1}^{k} B_{rj}\lambda_j^n \ , \tag{1.11}$$

where the constants B_{rj} do not depend on n. Since k is considered fixed we may consider the matrices $1, A, \cdots, A^{k-1}$ as having been computed. Hence from (1.4) and the boundary conditions we may consider $f_r(0), f_r(1), \cdots, f_r(k-1)$ as being known. **Hence** from (1.11) we will have k equations that determine $B_{r1}, B_{r2}, \cdots, B_{rk}$. Explicit expressions for these constants can be given. From (1.10) we can easily see the asymptotic behavior of $f_r(n)$. Let us write λ_j in the form $\lambda_r = r_j \exp(i\theta_j)$. Further let us assume $r_1 = r_2 = \cdots = r_p > r_{p+1} > r_{p+2} > \cdots > r_k$. Clearly,

$$f_r(n) \sim r_1^n \sum_{j=1}^{k} B_{ij} \exp(in\theta_j) \ . \tag{1.12}$$

If $p = 1$ then

$$f_r(n) \sim \lambda_1^n B_{r1} \ . \tag{1.13}$$

Example. If the matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & k \end{bmatrix} \tag{1.14}$$

the characteristic determinant

$$D_k(\lambda) = |\lambda I - A| \tag{1.15}$$

can be shown to satisfy the recurrence relation

$$D_k(\lambda) = (1 - 2\lambda)D_{k-1}(\lambda) - \lambda^2 D_{k-2}(\lambda), \quad D_0(\lambda) = 1, \quad D_1(\lambda) = 1 - \lambda . \tag{1.16}$$

The solution of (1.16) is easily obtained to be

$$\begin{aligned} D_k(\lambda) &= H\left(\frac{(1 - 2\lambda) + \sqrt{(1 - 2\lambda)^2 - 4\lambda^2}}{2}\right)^k \\ &\quad + K\left(\frac{(1 - 2\lambda) - \sqrt{(1 - 2\lambda)^2 - 4\lambda^2}}{2}\right)^k \\ &= HR_1 + KR_2 , \end{aligned} \tag{1.17}$$

where H, K are constants depending on λ but not on k. Filling the boundary conditions, we find

$$D_k(\lambda) = (R_1 + R_2)/2 + (R_1 - R_2)/(2\sqrt{(1 - 2\lambda)^2 - 4\lambda^2}) , \tag{1.18}$$

where R_1 and R_2 are defined in (1.17).

In order to find the roots of $D_k(\lambda) = 0$ we make the substitution

$$\lambda = \frac{1}{2(1 + \cos\theta)} = \frac{1}{4}\sec^2\frac{\theta}{2} . \tag{1.19}$$

Hence (1.19) becomes

$$\cos k\theta + \frac{1 + \cos\theta}{\sin\theta}\sin k\theta = 0 ,$$

$$\frac{\sin (k + \frac{1}{2})\theta}{\sin\theta}\cos\tfrac{1}{2}\theta = 0 .$$

Obviously, the roots of (1.19) are

$$\theta = \frac{s\pi}{k + \frac{1}{2}}, \quad s = 1, 2, \cdots , \tag{1.20}$$

where $s = 0$ must be excluded because of the denominator. Hence the roots of $D_k(\lambda) = 0$ are given by

$$\lambda = \tfrac{1}{4}\sec^2\left(\frac{s\pi}{2k+1}\right), \qquad s = 1, 2, \cdots . \tag{1.21}$$

Obviously only k are distinct, and arranged in order of magnitude we have

$$\lambda_1 = \tfrac{1}{4}\sec^2\left(\frac{k\pi}{2k+1}\right), \quad \lambda_2 = \tfrac{1}{4}\sec^2\left(\frac{(k-1)\pi}{2k+1}\right), \cdots, \lambda_k = \tfrac{1}{4}\sec^2\left(\frac{\pi}{2k+1}\right). \tag{1.22}$$

Thus

$$f_r(n) \sim \left(\tfrac{1}{2}\sec\left(\frac{\pi}{2k+1}\right)\right)^n B_{r1} . \tag{1.23}$$

TWO NUMERICAL CASES

Case 1. $k = 2$.

$$D_2(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1 = 0 ,$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{3-\sqrt{5}}{2} ,$$

$$f_1(0) = 0, \quad f_2(0) = 1, \quad f_1(1) = 1, \quad f_2(1) = 2 ,$$

$$f_1(n) = B_{11}\lambda_1^n + B_{12}\lambda_2^n ,$$

$$\begin{cases} 0 = B_{11} + B_{12} \\ 1 = B_{11}\lambda_1 + B_{12}\lambda_2 \end{cases}$$

$$\therefore \quad B_{11} = \frac{1}{\lambda_1 - \lambda_2} \quad \text{and} \quad B_{12} = \frac{1}{\lambda_1 - \lambda_2} .$$

$$\therefore \quad f_1(n) = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\lambda_1^n(1 - \lambda_2^{2n}) \sim \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n$$

Similarly,

$$f_2(n) = B_{21}\lambda_1^n + B_{22}\lambda_2^n$$

$$1 = B_{21} + B_{22}$$

$$2 = B_{21}\lambda_1 + B_{22}\lambda_2$$

$$\therefore \quad B_{21} = \frac{2-\lambda_2}{\lambda_1 - \lambda_2} = \frac{1+\sqrt{5}}{2\sqrt{5}}, \qquad B_{22} = \frac{2-\lambda_1}{\lambda_2 - \lambda_1} = -\frac{1-\sqrt{5}}{2\sqrt{5}} .$$

$$\therefore f_2(n) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}[\alpha^{2n+1} - \beta^{2n+1}] = F_{2n+1}$$

which gives the complete solution.

Case 2. k = 3.

$$\lambda_1 = \tfrac{1}{4}\sec^2\left(\frac{3\pi}{7}\right), \quad \lambda_2 = \tfrac{1}{4}\sec^2\left(\frac{2\pi}{7}\right), \quad \lambda_3 = \tfrac{1}{4}\sec^2\left(\frac{\pi}{7}\right)$$

$$\begin{array}{lll} f_1(0) = 0 & f_2(0) = 0 & f_3(0) = 1 \\ f_1(1) = 1 & f_2(1) = 2 & f_3(1) = 3 \\ f_1(2) = 6 & f_2(2) = 11 & f_3(2) = 14 \end{array} .$$

Thus

$$\begin{aligned} f_3(n) &= B_{31}\lambda_1^n + B_{32}\lambda_2^n + B_{33}\lambda_3^n \\ 1 &= B_{31} + B_{32} + B_{33} \\ 3 &= B_{31}\lambda_1 + B_{32}\lambda_2 + B_{33}\lambda_3 \\ 14 &= B_{31}\lambda_1^2 + B_{32}\lambda_2^2 + B_{33}\lambda_3^2 \end{aligned} .$$

Solving simultaneously,

$$B_{31} = \frac{\lambda_2\lambda_3 - 3(\lambda_2 + \lambda_3) + 14}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} .$$

Calculating λ_1, λ_2, λ_3 and substituting above gives $B_{31} \doteq 0.537$, so that

$$f_3(n) \sim 0.537\left(\tfrac{1}{2}\sec\left(\frac{3\pi}{7}\right)\right)^{2n} .$$

[Continued from page 301.]

Page 49, Eq. (33): Please change the last number on the line from "3" to "1."

Page 49, Line following Eq. (34): Please raise "(mod 3) " to the main line of type.

Page 49, line 6 from bottom: Please insert brackets around X(X - 1), X.

Page 53, line 2 from bottom: In the third column from the left, please change the number to read: " 2 750 837 603 ."

RECURSION - TYPE FORMULAE FOR PARTITIONS INTO DISTINCT PARTS

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A recursion formula for $p(n)$, the number of partitions of n, is given by the Euler identity

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - - + + \cdots$$

(1)

$$= \sum_{i \neq 0} (-1)^{i+1} p\left(n - \tfrac{1}{2}(3i^2 + i)\right),$$

where the sum extends over all integers i, except $i = 0$, for which the arguments of the partition function are nonnegative (see [1]).

This paper presents a recursion type formula for $q(n)$, the number of partitions of n into distinct parts, in terms of $p(k)$ for certain $k \leq n$. In addition a recursion type formula is presented for $q(a,m,n)$, the number of partitions of n into distinct parts congruent to $\pm a \pmod m$, in terms of $p(k)$ for certain $k \leq n$.

Theorem 1. If $n \geq 0$, and $q(n)$ is the number of partitions of n into distinct parts, then

$$q(n) = \sum_{i=-\infty}^{\infty} (-1)^i p(n - (3i^2 + i)), \tag{2}$$

where the sum extends over all integers i for which the arguments of the partition function are nonnegative.

Proof. We have

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i=0}^{\infty} (1 + x^i) = \prod_{j=0}^{\infty} (1 - x^j)^{-1}$$

$$\cdot \prod_{i=0}^{\infty} \{(1 - x^i)(1 + x^i)\} = \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \prod_{i=0}^{\infty} (1 - (x^2)^i)$$

$$= \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \left(\sum_{i=-\infty}^{\infty} (-1)^i (x^2)^{\frac{3i^2+i}{2}}\right) = \left(\sum_{j=0}^{\infty} p(j)x^j\right) \cdot \left(\sum_{i=-\infty}^{\infty} (-1)^i x^{3i^2+i}\right),$$

and the result follows by equating coefficients of x^n on both sides of this equation.

Corollary. If $n \geq 0$, then

$$q(n) = p(n) + \sum_{i=1}^{\infty} (-1)^i \{p(n - (3i^2 - i)) + p(n - (3i^2 + i))\}$$

$$= p(n) - p(n - 2) - p(n - 4) + p(n - 10) + p(n - 14) - p(n - 24)$$

$$- p(n - 30) + + - - \cdots .$$

Proof. This follows from Eq. (2) by rearranging the right-hand side.

Theorem 2. If $m \geq 3$, $1 \leq a < m/2$, $n \geq 0$ and $q(a,m,n)$ is the number of partitions of n into distinct parts congruent to $\pm a \pmod{m}$, then

$$q(a,m,n) = \sum_{m \mid (n+aj)} p\left(\frac{n + aj}{m} - \frac{j^2 + j}{2}\right) . \tag{3}$$

Proof. We have

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \prod_{i=0}^{\infty} \{(1 + x^{im+a})(1 + x^{im+m-a})\}$$

$$= \prod_{i=0}^{\infty} (1 - x^{im+m})^{-1} \cdot \prod_{i=0}^{\infty} \{(1 - x^{im+m})(1 + x^{im+a})(1 + x^{im+m-a})\}$$

$$= \left(\sum_{i=0}^{\infty} p(i)x^{im}\right) \cdot \prod_{r=1}^{\infty} \{(1 - x^{rm})(1 + x^{rm+a-m})(1 + x^{rm-a})\} .$$

By Jacobi's identity,

$$\prod_{r=1}^{\infty} \{(1 - q^{2r})(1 + zq^{2r-1})(1 + z^{-1}q^{2r-1})\} = \sum_{j=-\infty}^{\infty} z^j q^{j^2} .$$

with

$$q = x^{\frac{m}{2}} \quad \text{and} \quad z = x^{a-\frac{m}{2}} ,$$

we find

$$\prod_{r=1}^{\infty}\{(1 - x^{rm})(1 + x^{rm+a-m})(1 + x^{rm-a})\} = \sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj} .$$

Therefore,

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \left(\sum_{i=0}^{\infty} p(i)x^{im}\right)\left(\sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj}\right).$$

Since $p(i) = 0$ for $i < 0$, we have

$$\sum_{n=0}^{\infty} q(a,m,n)x^n = \left(\sum_{i=-\infty}^{\infty} p(i)x^{im}\right)\left(\sum_{j=-\infty}^{\infty} x^{m\left(\frac{j^2+j}{2}\right)-aj}\right).$$

Thus,

$$q(a,m,n) = \sum p\left(\frac{n - \left(m\left(\frac{j^2+j}{2}\right) - aj\right)}{m}\right),$$

where the sum extends over all integral values of j for which

$$\frac{n - \left(m\left(\frac{j^2+j}{2}\right) - aj\right)}{m}$$

is an integer. Clearly, this is an integer if and only if $m \mid (n + aj)$. Therefore,

$$q(a,m,n) = \sum_{m \mid (n+aj)} p\left(\frac{n+aj}{m} - \frac{j^2+j}{2}\right),$$

as required.

Corollary. Let $m \geq 3$, $1 \leq a < m/2$, and $n \geq 0$. Let

$$a_1 = \frac{a}{(a,m)} \quad \text{and} \quad m_1 = \frac{m}{(a,m)} .$$

If $(a,m) \nmid n$, then $q(a,m,n) = 0$. If $(a,m) \mid n$ and j_0 is some solution of the congruence

$$a_1 j \equiv -\frac{n}{(a,m)} \pmod{m_1},$$

then

$$q(a,m,n) = \sum_{k=-\infty}^{\infty} p\left(\left(\frac{n + aj_0}{m} - \frac{j_0^2 + j_0}{2}\right) - \left(\frac{m_1^2}{2}k^2 + \left(j_0m_1 + \frac{m_1}{2} - a_1\right)k\right)\right) .$$

Proof. If $(a,m) \nmid n$, then there are no values of j for which $m \mid (n + aj)$. Therefore, the sum in Theorem 2 is empty and $q(a,m,n) = 0$.

Suppose $(a,m) \mid n$ and

$$a_1j_0 \equiv -\frac{n}{(a,m)} \pmod{m_1} .$$

Then for any integer j, $m \mid (n + aj)$, if and only if $j \equiv j_0 \pmod{m_1}$. By Theorem 2,

$$q(a,m,n) = \sum_{j\equiv j_0 \,(\mathrm{mod}\, m_1)}^{\infty} p\left(\frac{n + aj}{m} - \frac{j^2 + j}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty} p\left(\frac{n + a(j_0 + km_1)}{m} - \frac{(j_0 + km_1)^2 + (j_0 + km_1)}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty} p\left(\left(\frac{n + aj_0}{m} - \frac{j_0^2 + j_0}{2}\right) - \left(\frac{m_1^2}{2}k^2 + \left(j_0m_1 + \frac{m_1}{2} - \frac{am_1}{m}\right)k\right)\right) .$$

But

$$\frac{am_1}{m} = \frac{a}{m}\,\frac{m}{(a,m)} = \frac{a}{(a,m)} = a_1 ,$$

so the proof is complete.

Example. Let $m = 3$ and $a = 1$. Then $(a,m) = 1$, $a_1 = 1$, and $m_1 = 3$. The congruence for j_0 is $j_0 \equiv -n \pmod 3$.

If $n \equiv 0 \pmod 3$, let $j_0 = 0$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 + \frac{1}{2}k\right)\right) = p\left(\frac{n}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 - \frac{1}{2}k\right)\right)\right.$$

$$\left. + p\left(\frac{n}{3} - \left(\frac{9}{2}k^2 + \frac{1}{2}k\right)\right)\right\} = p\left(\frac{n}{3}\right) + p\left(\frac{n}{3} - 4\right) + p\left(\frac{n}{3} - 5\right)$$

$$+ p\left(\frac{n}{3} - 17\right) + p\left(\frac{n}{3} - 19\right) + p\left(\frac{n}{3} - 39\right) + p\left(\frac{n}{3} - 42\right) + \cdots .$$

If $n \equiv 1 \pmod 3$, let $j_0 = -1$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 - \frac{5}{2}k\right)\right)$$

$$= p\left(\frac{n-1}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 - \frac{5}{2}k\right)\right) + p\left(\frac{n-1}{3} - \left(\frac{9}{2}k^2 + \frac{5}{2}k\right)\right)\right\}$$

$$= p\left(\frac{n-1}{3}\right) + p\left(\frac{n-1}{3} - 2\right) + p\left(\frac{n-1}{3} - 7\right) + p\left(\frac{n-1}{3} - 13\right)$$

$$+ p\left(\frac{n-1}{3} - 23\right) + p\left(\frac{n-1}{3} - 33\right) + p\left(\frac{n-1}{3} - 48\right) + \cdots .$$

If $n \equiv 2 \pmod 3$, let $j_0 = 1$. Then

$$q(1,3,n) = \sum_{k=-\infty}^{\infty} p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 + \frac{7}{2}k\right)\right)$$

$$= p\left(\frac{n-2}{3}\right) + \sum_{k=1}^{\infty}\left\{p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 - \frac{7}{2}k\right)\right) + p\left(\frac{n-2}{3} - \left(\frac{9}{2}k^2 + \frac{7}{2}k\right)\right)\right\}$$

$$= p\left(\frac{n-2}{3}\right) + p\left(\frac{n-2}{3} - 1\right) + p\left(\frac{n-2}{3} - 8\right) + p\left(\frac{n-2}{3} - 11\right)$$

$$+ p\left(\frac{n-2}{3} - 25\right) + p\left(\frac{n-2}{3} - 30\right) + p\left(\frac{n-2}{3} - 51\right) + \cdots .$$

REFERENCE

1. Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, 3rd ed., John Wiley and Sons, Inc., New York, 1972, pp. 226-227.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-221 *Proposed by L. Carlitz, Duke University, Durham, North Carolina*

Let $p = 2m + 1$ be an odd prime, $p \neq 5$. Show that if m is even then

$$\begin{cases} F_m \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = +1\right) \\ F_{m+1} \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = -1\right); \end{cases}$$

if m is odd, then

$$\begin{cases} L_m \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = +1\right) \\ L_{m+1} \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}$$

where $\left(\frac{5}{p}\right)$ is the Legendre symbol.

H-222 *Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.*

A natural number, n, is called semiperfect, if there is a collection of distinct proper divisors of n whose sum is n. A number, n, is called abundant if $\sigma(n) > 2n$, where $\sigma(n)$ represents the sum of the distinct divisors of n (not necessarily proper). Finally a number, n, is called weird* if it is abundant and not semiperfect.

Are any Fibonacci or Lucas numbers weird? (All known weird numbers are even.)

*Elementary Problem E2308, American Mathematical Monthly, 79 (1972), p. 774.

H-223 *Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.*

Let S be a set of k elements. Find the number of sequences $(A_1, A_2, \cdots, A_n)$ where each A_i is a subset of S, and where $A_1 \subseteq A_2$, $A_2 \supseteq A_3$, $A_3 \subseteq A_4$, $A_4 \supseteq A_5$, etc.

H-224 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let

$$A = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \\ 3 & 10 & 22 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n\times n}$$

denote the Fibonacci convolution determinant, and

$$B = \begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 5 & 9 & 14 & \cdots & \\ 10 & 22 & \cdots & & \\ \cdots & & & & \end{vmatrix}_{n\times n},$$

where the first row and column of A have been deleted. Show

(i) $A = 1$ and

(ii) $B = n + 1$.

H-225 Proposed by Guy A. R. Guillotte, Quebec, Canada.

Let p denote an odd prime and $x^p + y^p = z^p$ for positive integers, x, y, and z. Show that for a large value of p,

$$p \doteq \frac{z}{z - x} + \frac{z}{z - y} .$$

Also show that

$$p < \frac{z}{z - x} + \frac{z}{z - y} .$$

H-226 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

(i) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \le a_i \le k \quad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \neq a_{i-1}$ for $i = 2, \cdots, n$.

(ii) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \le a_i \le k \quad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \neq a_{i-1}$ for $i = 2, \cdots, n$; moreover $a_i = 0$ for exactly r values of i.

SOLUTIONS

PROBABLY?

H-179 Proposed by D. Singmaster, Bedford College, University of London, England.

Let k numbers $p_1, p_2, \cdots, p_k$ be given. Set $a_n = 0$ for $n < 0$; $a_0 = 1$ and define a_n by the recursion

$$a_n = \sum_{i=1}^{n} p_i a_{n-i} \qquad n > 0 .$$

1. Find simple necessary and sufficient conditions on the p_i for $\lim_{n\to\infty} a_n$ to exist and be a) finite and non zero, b) zero, c) infinite.

2. Are the conditions: $p_i \geq 0$ for $i = 1, 2, \cdots, p_1 > 0$ and

$$\sum_{i=1}^{n} p_i = 1$$

sufficient for $\lim_{n\to\infty} a_n$ to exist, be finite and be nonzero?

Comment by the Proposer

This problem arises in the following probabilistic situation. We have a k-sided die (or other random device) such that the probability of i occurring is p_i. We have a game board consisting of a sequence of squares indexed 0, 1, 2, $\cdots$. Beginning at square 0, we use the die to determine the n number of squares moved, as in Monopoly or other board games. Then a_n gives the probability of landing on square n. Since the average (expected) move is

$$E = \sum_{i=1}^{n} i \cdot p_i ,$$

one would hope that $a_n \to 1/E$. If all $p_i = 1/k$, this can be seen.

The restriction $p_1 > 0$ (or some more complicated restriction) is necessary to avoid situations such as $p_1 = 0$, $p_2 = 1$ which gives $a_{2n} = 1$, $a_{2n+1} = 0$ for all n.

Editorial Comment

Since p_i would not be defined for $i > k$, an infinite set of numbers $\{p_1, p_2, \cdots\}$ would have to be given.

AN INEQUALITY IN A CERTAIN DIOPHANTINE EQUATION

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The Diophantine Equation

$$x_1^p + \cdots + x_n^p = y^p \quad , \tag{1}$$

where p is an odd prime number >1, $n \geq 2$ and $1 \leq x_1 \leq \cdots \leq x_n$, is known to possess general solutions for $n = 2$, $p = 2$; $n = 3$, $p = 3$; for other values of n and p, no general solutions are known, although computer searches for solutions of such equations can easily be carried through by assigning a value to y and n and then allowing the corresponding x's to take all values from $x_1 = \cdots = x_n = 1$ to $x_1 = \cdots = x_n = y - 1$; in each case, different primes p are tested to see whether Eq. (1) is satisfied. The labor involved, however, is drastically reduced by realizing that for a given n and y possible primes p which can satisfy Eq. (1), have an upper bound, above which no solutions are possible. This statement is a consequence of examining properties of the function ψ which is defined by

$$\psi = (x_1 + \cdots + x_n)/y \quad , \tag{2}$$

where y is given by Eq. (1), subject to the restrictions stated above. The relevant property of ψ is given by the following:

Theorem. The function ψ is bounded above by $n^{1-\frac{1}{p}}$ and below by $(1 + 2p/y)$ or 1 depending on whether the solution to Eq. (1) are integers or not, respectively.

Proof of Theorem. From elementary calculus

$$d\psi = \sum_{i=1}^{m} \frac{\partial \psi}{\partial x_i} \, dx_i$$

and ψ has a turning point when

$$\frac{\partial \psi}{\partial x_i} = 0 \qquad (1 \leq i \leq n) \quad .$$

The conditions

$$\frac{\partial \psi}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x_2} = 0 \qquad (i \neq 1)$$

result in the equations

$$x_1^p + \cdots + x_i^p + \cdots + x_n^p - x_1^{p-1}(x_1 + \cdots + x_i + \cdots + x_n) = 0 \tag{3}$$

$$x_1^p + \cdots + x_i^p + \cdots + x_n^p - x_i^{p-1}(x_1 + \cdots + x_i + \cdots + x_n) = 0 \quad . \tag{4}$$

Subtracting Eq. (3) from Eq. (4) gives the condition for a turning point as

$$x_1 = x_2 = \cdots = x_n \quad . \tag{5}$$

From which we deduce that $\max(\psi) = n^{1-\frac{1}{p}}$. The lower bound of x depends on whether the x's are restricted to integers. Thus if the x's are non-integral, then we note that $y^p = x_1^p + \cdots + x_n^p < (x_1 + \cdots + x_n)^p$ so that $1 < \psi$. If the x's are integers only, we use the little Fermat theorem $x_i^p \equiv x_i \pmod{p}$. But since $(x_i^p - x_i)$ is even and p is odd by hypothesis, it follows that $x_i^p \equiv x_i \pmod{2p}$ and hence using Eq. (1) we deduce that $x_1 + \cdots + x_m \equiv y \pmod{2p}$ from which it follows immediately that $y + 2p \leq x_1 + \cdots + x_n$. The case of $p = 2$ deserves special attention. Using the same reasoning as above, we obtain the inequality:

$$y + 2 \leq x_1 + \cdots + x_n \leq \sqrt{n}\, y \quad . \tag{6}$$

Moreover, it is easy to derive solutions for the equation

$$\sum_{i=1}^{m} x_i^2 = y^2$$

for any n by using the well known general solution for $n = 2$ — i.e., the identity $(2ab)^2 + (a^2 - b^2)^2 = (a^2 + b^2)^2$. Thus putting $a = n$, and $b = n + 1$, we obtain:

$$(2n + 1)^2 + (2n(n + 1))^2 + (2n(n + 1) + 1)^2 \quad .$$

Now putting $n = m\ (m + 1)$ and using Eq. (6) gives:

$$(2m + 1)^2 + (2U)^2 + (2U(U + 1))^2 = (2U(U + 1) + 1)^2 \quad , \tag{7}$$

with $U = m\,(m + 1)$. It is easy to use induction to show that this method gives an identity in m. We may write $m = a/b$ and multiply throughout by b^{2n} to obtain an identity in a and b.

A PRIMER FOR THE FIBONACCI NUMBERS: PART XII

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ON REPRESENTATIONS OF INTEGERS USING FIBONACCI NUMBERS

In how many ways may a given positive integer p be written as the sum of distinct Fibonacci numbers, order of the summands not being considered? The Fibonacci numbers are $1, 1, 2, 3, 5, \cdots, F_n, \cdots$, where $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For example, $10 = 8 + 2 = 2 + 3 + 5$ is valid, but $10 = 5 + 5 = 1 + 1 + 8$ would not be valid. The original question is an example of a representation problem, which we do not intend to answer fully here. We will explore representations using the least possible number or the greatest possible number of Fibonacci numbers.

1. THE ZECKENDORF THEOREM

First we prove by mathematical induction a lemma which has immediate application.

Lemma: The number of subsets of the set of the first n integers, subject to the constraint that no two consecutive integers appear in the same subset, is F_{n+2}, $n \geq 0$.

Proof. The theorem holds for $n = 0$, for when we have a set of no integers the only subset is ϕ, the empty set. We thus have one subset and $F_{0+2} = F_2 = 1$.

For $n = 1$, 2 subsets: $\{1\}, \phi$; $F_{1+2} = F_3 = 2$

$n = 2$, 3 subsets: $\{1\}, \{2\}, \phi$; $F_{2+2} = F_4 = 3$

$n = 3$, 5 subsets: $\{1,3\}, \{3\}, \{2\}, \{1\}, \phi$; $F_{3+2} = F_5 = 5$.

Assume that the lemma holds for $n \leq k$. Then notice that the subsets formed from the first $(k + 1)$ integers are of two kinds — those containing $(k + 1)$ as an element and those which do not contain $(k + 1)$ as an element. All subsets which contain $(k + 1)$ cannot contain element k and can be formed by adding $(k + 1)$ to each subset, made up of the $(k - 1)$ integers, which satisfies the constraint. By the inductive hypothesis there are F_{k+2} subsets satisfying the constraint and using only the first k integers, and there are F_{k+1} subsets satisfying the constraints and using the first $(k - 1)$ integers. Thus there are precisely

$$F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}$$

subsets satisfying the constraint and using the first $(k + 1)$ integers. The proof is complete by mathematical induction.

Now, for the application. The number of ways in which n boxes can be filled with zeros or ones (every box containing exactly one of those numbers) such that no two "ones" appear in

adjacent boxes is F_{n+2}. (To apply the lemma simply number the n boxes.) Since we do not wish to use all zeros (ϕ, the empty set in the lemma) the number of logically useable arrangements is $F_{n+2} - 1$. Now, to use the distinctness of the Fibonacci numbers in our representations, we must omit the initial $F_1 = 1$, so that to the n boxes we assign in order the Fibonacci numbers $F_2, F_3, \cdots, F_{n+1}$. This gives us a binary form for the Fibonacci positional notation. The interpretation to give the "zero" or "one" designation is whether or not one uses that particular Fibonacci number in the given representation. If a one appears in the box allocated to F_k, then F_k is used in this particular representation. Notice that since no two adjacent boxes can each contain a "one," no two consecutive Fibonacci numbers may occur in the same representation.

Since the following are easily established identities,

$$F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1 ,$$
$$F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2} - 1 ,$$

using the Fibonacci positional notation the largest number representable under the constraint with our n boxes is $F_{n+2} - 1$. Also the number F_{n+1} is in the n^{th} box, so we must be able to represent at most $F_{n+2} - 1$ distinct numbers with $F_2, F_3, \cdots, F_{n+1}$ subject to the constraint that no two adjacent Fibonacci numbers are used. Since there are $F_{n+2} - 1$ different ways to distribute ones and zeros in our n boxes, there are $F_{n+2} - 1$ different representations which could represent possibly $F_{n+2} - 1$ different integers. That each integer p has a unique representation is the Zeckendorf Theorem [1]:

Theorem. Each positive integer p has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in the representation.

We shall defer the proof of this until a later section. Now, a minimal representation of an integer p uses the least possible number of Fibonacci numbers in the sum. If both F_k and F_{k-1} appeared in a representation, they could both be replaced by F_{k+1}, thereby reducing the number of Fibonacci numbers used. It follows that a representation that uses no two consecutive Fibonacci numbers is a minimal representation and a Zeckendorf representation.

2. ENUMERATING POLYNOMIALS

Next, we use enumerating polynomials to establish the existence of at least one minimal representation for each integer.

An enumerating polynomial counts the number of Fibonacci numbers necessary in the representation of each integer p in a given interval $F_m \leq p < F_{m+1}$ in the following way. Associated with this interval is a polynomial $P_{m-1}(x)$. A term ax^j belongs to $P_{m-1}(x)$ if in the interval $F_m \leq p < F_{m+1}$, there are a integers p whose minimal representation requires j Fibonacci numbers. For example, consider the interval $F_6 = 8 \leq p < 13 = F_7$. Here, we can easily determine the minimal representations

$$8 = 8$$
$$9 = 1 + 8$$
$$10 = 2 + 8$$
$$11 = 3 + 8$$
$$12 = 1 + 3 + 8 .$$

Thus, $P_5(x) = x^3 + 2x^2 + x$ because one integer required 3 Fibonacci numbers, 3 integers required 2 Fibonacci numbers, and one integer required one Fibonacci number in its minimal representation. We note in passing that all the minimal representations in this interval contain 8 but not 5. We now list the first nine enumerating polynomials.

	$F_m \le p < F_{m+1}$	$P_{m-1}(x)$
$m = 1$	$1 \le p < 1$	$0 = P_0(x)$
$m = 2$	$1 \le p < 2$	$x = P_1(x)$
$m = 3$	$2 \le p < 3$	$x = P_2(x)$
$m = 4$	$3 \le p < 5$	$x^2 + x = P_3(x)$
$m = 5$	$5 \le p < 8$	$2x^2 + x = P_4(x)$
$m = 6$	$8 \le p < 13$	$x^3 + 3x^2 + x = P_5(x)$
$m = 7$	$13 \le p < 21$	$3x^3 + 4x^2 + x = P_6(x)$
$m = 8$	$21 \le p < 34$	$x^4 + 6x^3 + 5x^2 + x = P_7(x)$
$m = 9$	$34 \le p < 55$	$4x^4 + 10x^3 + 6x^2 + x = P_8(x)$

We shall now proceed by mathematical induction to derive a recurrence relation for the enumerating polynomials $P_m(x)$. It is evident from the definitions that an enumerating polynomial for $F_m \le p < F_{m+2}$ is the sum of the enumerating polynomials for $F_m \le p < F_{m+1}$ and $F_{m+1} \le p < F_{m+2}$. Also it will be proved that the minimal representation of any integer p in the interval $F_m \le p < F_{m+1}$ contains F_m but not F_{m-1}. If we added F_{m+2} to each such minimal representation of p in $F_m \le p < F_{m+1}$ we would get a minimal representation of an integer in the interval

$$L_{m+1} = F_m + F_{m+2} \le p < F_{m+1} + F_{m+2} = F_{m+3} .$$

Clearly the enumerating polynomial for this interval is $xP_{m-1}(x)$ since each integer p in this interval has one more Fibonacci number in its minimal representation than did the corresponding integer p in the interval $F_m \le p < F_{m+1}$.

Next, the integers p in the interval $F_{m+2} \le p < F_{m+3}$ require an F_{m+2} in this minimal representation while all the numbers in the interval $F_{m+1} \le p < F_{m+2}$ have F_{m+1} in their minimal representation. In each of these minimal representations remove the F_{m+1}

and put in an F_{m+2}. The resulting integer will have a minimal representation with the same number of Fibonacci numbers as was required before. In other words, the enumerating polynomial $P_{m-1}(x)$ is also the enumerating polynomial for

$$F_{m+2} - F_{m+1} + F_{m+1} \leq p' < F_{m+2} - F_{m+1} + F_{m+2} = L_{m+1} .$$

Now, the intervals $F_{m+2} \leq p' < L_{m+1}$ and $L_{m+1} \leq p' < F_{m+3}$ are not overlapping and exhaust the interval $F_{m+2} \leq p < F_{m+3}$. Thus, the enumerating polynomial for this interval is

$$P_{m+1}(x) = P_m(x) + xP_{m-1}(x), \qquad P_0(x) = 0, \qquad P_1(x) = x ,$$

which is the required recurrence relation.

Now, to show by mathematical induction that the minimal representation of any integer p in the interval $F_m \leq p < F_{m+1}$ contains F_m but not F_{m-1}, re-examine the preceding steps. Each minimal representation in the interval $F_{m+2} \leq p < F_{m+3}$ contains F_{m+2} explicitly since we added F_{m+2} to a representation from the interval $F_m \leq p < F_{m+1}$ and by the inductive hypothesis those representations did not contain F_{m+1} but all contained F_m. Next, for the representations from $F_{m+1} \leq p < F_{m+2}$, all of which used F_{m+1} explicitly by inductive assumption, we removed the F_{m+1} and replaced it by F_{m+2} so that each representation in $F_{m+2} \leq p < F_{m+3}$ contains F_{m+2} but not F_{m+1}. Thus, if the integers p in the previous two intervals, namely, $F_m \leq p < F_{m+1}$ and $F_{m+1} \leq p < F_{m+2}$, had Zeckendorf representations, then the representations of the integers p in the interval $F_{m+2} \leq p < F_{m+3}$ are also Zeckendorf representations.

Now, notice that $P_m(1)$ is the sum of the coefficients of $P_m(x)$, or the count of the numbers for which a minimal representation exists in the interval $F_{m+1} \leq p < F_{m+2}$. But, $P_m(1) = F_m$ because $P_1(1) = P_2(1) = 1$ and $P_{m+1}(1) = P_m(1) + 1 \cdot P_{m-1}(1)$, so that the two sequences have the same beginning values and the same recursion formula. The number of integers in the interval $F_{m+1} \leq p < F_{m+2}$ is $F_{m+2} - F_{m+1} = F_m$, so that every integer is represented. Thus, at least one minimal representation exists for each integer, and we have established Zeckendorf's theorem, that each integer has a unique minimal representation in Fibonacci numbers. Notice that this means that it is possible to express any integer as a sum of distinct Fibonacci numbers. Also, notice that the coefficients of $P_m(x)$ are the summands along the diagonals of Pascal's triangle summing to F_m with increasing powers as one proceeds up the diagonals beginning with x.

3. THE DUAL ZECKENDORF THEOREM

Suppose that, instead of a minimal representation, we wished to write a maximal representation, or, to use as many distinct Fibonacci numbers as possible in a sum to represent an integer. Then, we want no two consecutive Fibonacci numbers to be missing in the representation. Returning to our n non-empty boxes, for this case we wish to fill the boxes with zeros and ones with no two consecutive zeros. Here we consider n ones interposed

by at most one zero. Thus, we have boxes to zero or not to zero. These zeros can occur between the left-most one and the next on the right, between any adjacent pair of ones, and on the right of the last one if necessary. Thus, there are precisely 2^n possibilities, or, 2^n maximal representations can be written using n Fibonacci numbers from among 1, 2, 3, 5, $\cdots$, F_{2n+1}.

Now, associate with integers p in the interval $F_n - 1 \leq p < F_{n+1} - 1$ an enumerating maximal polynomial $P^*_{n-1}(x)$ which has a term ax^j if a of the integers p require j Fibonacci numbers in their maximal representation. For example, in the interval $F_6 - 1 = 7 \leq p < 12 = F_7 - 1$, the maximal representations are

$$\begin{aligned} 7 &= 5 + 2 \\ 8 &= 5 + 2 + 1 \\ 9 &= 5 + 3 + 1 \\ 10 &= 5 + 3 + 2 + 1 \\ 11 &= 5 + 3 + 2 + 1 \ . \end{aligned}$$

Thus, $P_5(x) = x^4 + 3x^3 + x^2$ because one integer requires 4 Fibonacci numbers, 3 integers require 3 Fibonacci numbers, and one integer requires 2 Fibonacci numbers in its maximal representation. Notice that all maximal representations above use 5 but none use 8. The first eight enumerating maximal polynomials are:

	$F_m - 1 \leq p < F_{m+1} - 1$	$P^*_{m-1}(x)$
$m = 2$	$0 \leq p < 1$	$1 = P^*_1(x)$
$m = 3$	$1 \leq p < 2$	$x = P^*_2(x)$
$m = 4$	$2 \leq p < 4$	$x^2 + x = P^*_3(x)$
$m = 5$	$4 \leq p < 7$	$x^3 + 2x^2 = P^*_4(x)$
$m = 6$	$7 \leq p < 12$	$x^4 + 3x^3 + x^2 = P^*_5(x)$
$m = 7$	$12 \leq p < 20$	$x^5 + 4x^4 + 3x^3 = P^*_6(x)$
$m = 8$	$20 \leq p < 33$	$x^6 + 5x^5 + 6x^4 + x^3 = P^*_7(x)$
$m = 9$	$33 \leq p < 54$	$x^7 + 6x^6 + 10x^5 + 4x^4 = P^*_8(x)$

As before, we now derive the recurrence relation for the polynomials $P^*_n(x)$.

Lemma. Each maximal representation for integers p in the interval $F_m - 1 \leq p < F_{m+1} - 1$ contains explicitly F_{m-1}.

Proof. We can add F_m to each maximal representation in the interval $F_m - 1 \leq p < F_{m+1} - 1$ and these numbers fall in the interval

$$2F_m - 1 \leq p' < F_{m+2} - 1 .$$

We can also add F_m to each maximal representation in the interval $F_{m-1} - 1 \leq p < F_{m+1} - 1$ and these numbers fall in the interval

$$F_{m+1} - 1 \leq p' < 2F_m - 1 .$$

These two intervals are non-overlapping and exhaustive of the interval

$$F_{m+1} - 1 \leq p < F_{m+2} - 1 .$$

Thus, each maximal representation in this interval contains explicitly F_m.

Thus, the enumerating polynomials $P_n^*(x)$ for maximal representations satisfy

$$P_n^*(x) = x[P_{n-1}^*(x) + P_{n-2}^*(x)], \quad P_1^*(x) = 1, \quad P_2^*(x) = x ,$$

and again $P_n^*(1) = F_n$. This establishes that each non-negative integer has at least one maximal representation.

Returning to the table of the first eight polynomials $P_n^*(x)$, by laws of polynomial addition, adding the enumerating maximal polynomials yields a count of how many numbers require k Fibonacci numbers in their maximal representation. So, it appears that

$$\sum_{n=1}^{\infty} P_n^*(x) = P_1^*(x) + P_2^*(x) + P_3^*(x) + P_4^*(x) + P_5^*(x) + \cdots + P_n^*(x) + \cdots$$

$$= 1 + x + (x^2 + x) + (x^3 + 2x^2) + (x^4 + 3x^3 + x^2) + \cdots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^k x^k + \cdots ,$$

(That this is indeed the case is proved in the two lemmas following the Dual Zeckendorf Theorem.) In other words, 2^k non-negative integers require k Fibonacci numbers in their maximal representation. But requiring that each integer has at least one maximal representation exhausts the logical possibilities. Thus, each integer has a unique maximal representation in distinct Fibonacci numbers, which proves the Dual Zeckendorf Theorem [2]:

Theorem. Each positive integer has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are omitted in the representation.

Lemma. Let $f_1(x) = 1$, $f_2(x) = x$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$ be the Fibonacci polynomials. Then

$$P_n^*(x^2) = x^{n-1} f_n(x), \quad n \geq 0 .$$

Proof. We proceed by mathematical induction. Observe that

$$P_1^*(x^2) = 1 = x^0 f_1(x) ,$$

$$P_2^*(x^2) = x^2 = x^1 f_2(x) ,$$

$$P_n^*(x^2) = x^2[P_{n-1}^*(x^2) + P_{n-2}^*(x^2)] .$$

Assume that

$$P^*_{n-1}(x^2) = x^{n-2} f_{n-1}(x) ,$$
$$P^*_{n-2}(x^2) = x^{n-3} f_{n-2}(x) .$$

Thus

$$P^*_n(x^2) = x^2[x^{n-2} f_{n-1}(x) + x^{n-3} f_{n-2}(x)]$$
$$= x^{n-1}[x f_{n-1}(x) + f_{n-2}(x)] = x^{n-1} f_n(x) .$$

Lemma.

$$\sum_{n=1}^{\infty} P^*_n(x) = \frac{1}{1 - 2x} .$$

Proof. The Fibonacci polynomials have the generating function

$$\frac{1}{1 - xt - t^2} = \sum_{n=1}^{\infty} f_n(x)\, t^{n-1} .$$

Now let $x = t$, and then by the previous lemma,

$$\frac{1}{1 - x^2 - x^2} = \sum_{n=1}^{\infty} f_n(x) x^{n-1} = \sum_{n=1}^{\infty} P^*_n(x^2) = \frac{1}{1 - 2x^2} .$$

Therefore,

$$\sum_{n=1}^{\infty} P^*_n(x) = \frac{1}{1 - 2x} = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots .$$

Notice that the polynomials $P^*_n(x)$ have as their coefficients the summands along the rising diagonals of Pascal's triangle whose sums are the Fibonacci numbers but in the reverse order of those for $P_n(x)$. In fact, the minimal enumerating polynomials $P_n(x)$ and the maximal enumerating polynomials $P^*_n(x)$ are related as in the following lemma:

Lemma.

$$P_m(x) = x^m P^*_m(1/x) \quad \text{for } m \geq 1 .$$

Proof. This relationship will be proved by mathematical induction.

$$m = 1: \quad P_1(x) = x = x^1[P^*_1(1/x)]$$
$$m = 2: \quad P_2(x) = x = x^2(1/x) = x^2[P^*_2(1/x)]$$
$$m = 3: \quad P_3(x) = x^2 + x = x^3(1/x + 1/x^2) = x^3[P^*_3(1/x)] .$$

Assume that

$$P_{k-1}(x) = x^{k-1}P^*_{k-1}(1/x) \ ,$$
$$P_k(x) = x^k P^*_k(1/x) \ .$$

Then, by the recurrence relations for the polynomials $P_n(x)$ and $P^*_n(x)$,

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + xP_{k-1}(x) \\ &= x^k P^*_k(1/x) + xx^{k-1} P^*_{k-1}(1/x) \\ &= x^{k+1}(1/x)[\, P^*_k(1/x) + P^*_{k-1}(1/x)\,] \\ &= x^{k+1} P^*_{k+1}(1/x) \ , \end{aligned}$$

which establishes the lemma by mathematical induction.

Then, both the minimal and maximal representations of an integer are unique. Then, an integer has a unique representation in Fibonacci numbers if and only if its minimal and maximal representations are the same, which condition occurs only for the integers of the form $F_n - 1$, $n \geq 3$ [3]. In general, the representation of an integer in Fibonacci numbers is not unique, and, from the above remarks, unless the number is one less than a Fibonacci number, it will have at least two representations in Fibonacci numbers. But, one need not stop here. The Fibonacci numbers F_{2n} and F_{2n+1} can each be written as the sum of distinct Fibonacci numbers 1, 2, 3, 5, 8, $\cdots$, in n different ways. For other integers p, the reader is invited to experiment to see what theorems he can produce.

We now turn to representations of integers using Lucas numbers.

4. THE LUCAS CASE

If we change our representative set from Fibonacci numbers to Lucas numbers, we can find minimal and maximal representations of integers as sums of distinct Lucas numbers. The Lucas numbers are 2, 1, 3, 4, 7, 11, $\cdots$, defined by $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 1$. (See Brown [6a].)

The derivation of a recursion formula for the enumerating minimal polynomials $Q_n(x)$ for Lucas numbers is very similar to that for the polynomials $P_n(x)$ for Fibonacci numbers. Details of the proofs are omitted here. Now, for integers p in the interval $L_n \leq p < L_{n+1}$, the enumerating minimal polynomial $Q_{n-1}(x)$ has a term dx^j if $\underline{d}$ of the integers p require $\underline{j}$ Lucas numbers in their minimal representation. For example, the minimal representation in Lucas numbers for integers p in the interval $11 = L_5 \leq p < L_6 = 18$ are:

$$\begin{aligned} 11 &= 11 & \qquad 15 &= 11 + 4 \\ 12 &= 11 + 1 & \qquad 16 &= 11 + 4 + 1 \\ 13 &= 11 + 2 & \qquad 17 &= 11 + 4 + 2 \\ 14 &= 11 + 3 \end{aligned}$$

so that $Q_4(x) = 2x^3 + 4x^2 + x$ since 2 integers require 3 Lucas numbers, 4 integers require 2 Lucas numbers, and one integer requires one Lucas number. Notice that $L_5 = 11$ is included in each representation, but that $L_4 = 7$ does not appear in any representation in this inverval. Also notice that we could have written $16 = 11 + 3 + 2$. To make the minimal representation unique, it is necessary to avoid one of the combinations $L_0 + L_1$ or $L_1 + L_3$: we agree not to use the combination $L_0 + L_2 = 2 + 3$ in any minimal representation unless one or both of L_1 and L_3 also appear. The first nine Lucas enumerating minimal polynomials follow.

	$L_m \le p < L_{m+1}$	$Q_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q_0(x)$
$m = 2$	$3 \le p < 4$	$x = Q_1(x)$
$m = 3$	$4 \le p < 7$	$2x^2 + x = Q_2(x)$
$m = 4$	$7 \le p < 11$	$3x^2 + x = Q_3(x)$
$m = 5$	$11 \le p < 18$	$2x^3 + 4x^2 + x = Q_4(x)$
$m = 6$	$18 \le p < 29$	$5x^3 + 5x^2 + x = Q_5(x)$
$m = 7$	$29 \le p < 47$	$2x^4 + 9x^3 + 6x^2 + x = Q_6(x)$
$m = 8$	$47 \le p < 76$	$7x^4 + 14x^3 + 7x^2 + x = Q_7(x)$
$m = 9$	$76 \le p < 123$	$2x^5 + 16x^4 + 20x^3 + 8x^2 + x = Q_8(x)$

Similarly to $P_n(x)$, by the rules of polynomial addition and because of the way the polynomials $Q_n(x)$ are defined,

$$Q_{n+1}(x) = Q_n(x) + xQ_{n-1}(x), \quad Q_0(x) = 2x, \quad Q_1(x) = x,$$

is the recursion relation satisfied by the polynomials $Q_n(x)$. Here we have the same recursion formula satisfied by the polynomials $P_n(x)$, but with different starting values. Notice that $Q_n(1) = L_n$. As before, $Q_{n-1}(1)$ is the sum of the coefficients of $Q_{n-1}(x)$, or, the count of the numbers for which a minimal representation exists in the interval $L_n \le p < L_{n+1}$, which contains exactly $L_{n+1} - L_n = L_{n-1}$ integers. Thus, each integer has at least one minimal representation in distinct Lucas numbers.

Now, let us reconsider the n boxes. To have a minimal representation, we wish to fill the n boxes with zeros or ones such that no two ones are adjacent and to discard the arrangement using all zeros. As before, there are $F_{n+2} - 1$ such arrangements. Now, establish a Lucas number positional notation by putting the Lucas numbers $L_0, L_1, L_2, L_3, \cdots, L_{n-1}$ into the n boxes. Again, the significance of the ones and zeros is determination of which Lucas numbers are used in the sum. But, notice that $L_0 + L_2 = L_1 + L_3$, which would make more than one minimal representation of an integer possible. To avoid this problem, we consider the first four boxes and reject $L_0 + L_2$ whenever that combination occurs without L_1 or L_2. If such four boxes hold then there are $(n - 4)$ remaining boxes which

0	1	0	1
L_3	L_2	L_1	L_0

can hold F_{n-2} compatible arrangements. Thus, rejecting these endings eliminates F_{n-2} arrangements, making the number of admissible arrangements $F_{n+2} - F_{n-2} - 1 = L_n - 1$. But the Lucas sequence begins with $L_0 = 2$, so that the number L_n is in the box numbered $(n + 1)$. Therefore, using the first n Lucas numbers and the two constraints, we can have at most $L_n - 1$ different numbers represented, for

$$L_1 + L_3 + \cdots + L_{2k-1} = L_{2k} - 2 ,$$
$$L_2 + L_4 + \cdots + L_{2k-2} = L_{2k-1} - 1 ,$$

and the $L_0 + L_2$ ending was rejected.

Then, we can have at most $L_n - 1$ different numbers represented using $L_0, L_1, \cdots, L_{n-1}$, but the enumerating minimal polynomial guarantees that each of the numbers 1, 2, 3, $\cdots, L_n - 1$, has at least one minimal representation. Thus, the minimal representation of an integer in Lucas numbers, subject to the two constraints given, is unique. This is the Lucas Zeckendorf Theorem.

For the maximal representation of an integer using distinct Lucas numbers, again we will need to use adjacent Lucas numbers whenever possible. In our n boxes, then, we will want to place the ones and zeros so that there never are two consecutive zeros. Also, we need to exclude the ending $L_1 + L_3$ in our representations to exclude the possibility of two maximal representations for an integer, one using $L_0 + L_2 = 5$ and the other $L_1 + L_3 = 5$. We will use the combination $L_1 + L_3$ only when one of L_0 or L_2 occurs in the same maximal representation.

Now, let the enumerating maximal polynomials for the Lucas case for the interval $L_n \le p < L_{n+1}$ be $Q^*_{n-1}(x)$, where dx^j is a term of $Q^*_{n-1}(x)$ if d of the integers p require j Lucas numbers in their maximal representation. For example, the maximal representation in Lucas numbers for integers p in the interval $11 = L_5 \le p < L_6 = 18$ are:

$$11 = 7 + 3 + 1$$
$$12 = 7 + 3 + 2$$
$$13 = 7 + 3 + 2 + 1$$
$$14 = 7 + 4 + 2 + 1$$
$$15 = 7 + 4 + 3 + 1$$
$$16 = 7 + 4 + 3 + 2$$
$$17 = 7 + 4 + 3 + 2 + 1$$

so that $Q^*_4(x) = x^5 + 4x^4 + 2x^3$, since one integer requires 5 Lucas numbers, 4 integers require 4 Lucas numbers, and 2 integers require 3 Lucas numbers in their maximal representation. The first nine polynomials $Q^*_n(x)$ follow.

	$L_m \le p < L_{m+1}$	$Q^*_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q^*_0(x)$
$m = 2$	$3 \le p < 4$	$x^2 = Q^*_1(x)$
$m = 3$	$4 \le p < 7$	$x^3 + 2x^2 = Q^*_2(x)$
$m = 4$	$7 \le p < 11$	$x^4 + 3x^3 = Q^*_3(x)$
$m = 5$	$11 \le p < 18$	$x^5 + 4x^4 + 2x^3 = Q^*_4(x)$
$m = 6$	$18 \le p < 29$	$x^6 + 5x^5 + 5x^4 = Q^*_5(x)$
$m = 7$	$29 \le p < 47$	$x^7 + 6x^6 + 9x^5 + 2x^4 = Q^*_6(x)$
$m = 8$	$47 \le p < 76$	$x^8 + 7x^7 + 14x^6 + 7x^5 = Q^*_7(x)$
$m = 9$	$76 \le p < 123$	$x^9 + 8x^8 + 20x^7 + 16x^6 + 2x^5 = Q^*_8(x)$

The recursion relation for the polynomials $Q^*_n(x)$ can be derived in a similar fashion to $P^*_n(x)$, becoming

$$Q^*_{n+1}(x) = x[\,Q^*_n(x) + Q^*_{n-1}(x)\,], \quad Q^*_0(x) = 2x, \quad Q^*_1(x) = x^2 .$$

Notice that the same coefficients occur in the enumerating minimal Lucas polynomial $Q_n(x)$ and in the enumerating maximal Lucas polynomial $Q^*_n(x)$. The relationship in the lemma below could be proved by mathematical induction, paralleling the proof of the similar property of $P_n(x)$ and $P^*_n(x)$ given in the preceding section.

Lemma.

$$Q_m(x) = x^{m+1} Q^*_m(1/x) \quad \text{for } m \ge 1 .$$

Also, the polynomials $P^*_n(x)$ and $Q^*_n(x)$ are related as follows:

Lemma.

$$Q^*_{n-1}(x) = x P^*_n(x) + x^2 P^*_{n-2}(x), \quad n \ge 1 ,$$

which could be proved by mathematical induction. Notice that the lemma above becomes the well known identity, $L_{n-1} = F_n + F_{n-2}$, when $x = 1$.

Now we return to our main problem.

By laws of polynomial addition, if we add all polynomials $Q^*_n(x)$, the coefficients in the sum will provide a count of how many integers require k Lucas numbers in their maximal representation. Then, it would appear that

$$\sum_{n=0}^{\infty} Q^*_n(x) = Q^*_0(x) + Q^*_1(x) + Q^*_2(x) + Q^*_3(x) + Q^*_4(x) + \cdots + Q^*_k(x) + \cdots$$

$$= 2x + x^2 + (x^3 + 2x^2) + (x^4 + 3x^3) + (x^5 + 4x^4 + 2x^3) + \cdots$$

$$= 2x + 3x^2 + 6x^3 + 12x^4 + 24x^5 + \cdots + 3 \cdot 2^{k-2} x^k + \cdots ,$$

so that $3 \cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, $k \geq 2$. A proof that this is the correct computation of the sum of the polynomials $Q_n^*(x)$ follows.

Lemma. If

$$Q_0^*(x) = 2x, \quad Q_1^*(x) = x^2, \quad \text{and} \quad Q_n^*(x) = x[Q_{n-1}^*(x) + Q_{n-2}^*(x)],$$

then

$$Q_{n-1}^*(x^2) = x^{n+1}[f_n(x) + f_{n-2}(x)],$$

where $f_n(x)$ are the Fibonacci polynomials.

Proof. To begin a proof by mathematical induction, observe that

$$n = 1: \quad Q_0^*(x^2) = 2x^2 = x^2(1 + 1) = x^{1+1}[f_1(x) + f_{-1}(x)]$$

$$n = 2: \quad Q_1^*(x^2) = x^4 = x^3(x + 0) = x^{2+1}[f_2(x) + f_0(x)] .$$

Assume that the lemma holds for $(n - 1)$ and $(n - 2)$. Then

$$\begin{aligned} Q_n^*(x^2) &= x^2[Q_{n-1}^*(x^2) + Q_{n-2}^*(x^2)] \\ &= x^2\{x^{n+1}[f_n(x) + f_{n-2}(x)] + x^n[f_{n-1}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}\{[xf_n(x) + f_{n-1}(x)] + [xf_{n-2}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}[f_{n+1}(x) + f_{n-1}(x)], \end{aligned}$$

establishing the lemma by mathematical induction for $n \geq 1$.

Using known generating functions for the Fibonacci polynomials as before,

$$\sum_{n=1}^{\infty} f_n(x)\, t^{n+1} = \frac{t^2}{1 - xt - t^2} ,$$

$$\sum_{n=1}^{\infty} f_{n-2}(x) t^{n+1} = \frac{t^2(1 - xt)}{1 - xt - t^2} .$$

Adding,

$$\frac{t^2(2 - xt)}{1 - xt - t^2} = \sum_{n=1}^{\infty} [f_n(x) + f_{n-2}(x)]\, t^{n+1} .$$

Setting $t = x$,

$$\frac{2x^2 - x^4}{1 - 2x^2} = \sum_{n=1}^{\infty} x^{n+1}[f_n(x) + f_{n-2}(x)] = \sum_{n=1}^{\infty} Q_{n-1}^*(x^2) .$$

Therefore,

$$\sum_{n=0}^{\infty} Q_n^*(x) = \frac{2x - x^2}{1 - 2x} = 2x + \frac{3x^2}{1 - 2x} = 2x + \sum_{n=2}^{\infty} 3 \cdot 2^{n-2} x^n .$$

To see the reason for the peculiar coefficients $3 \cdot 2^{k-1}$, examine the eight possible ways to fill the first four boxes with zeros and ones. Then see how many numbers requiring n Lucas numbers in their maximal representation could be written. In other words, consider how to distribute n ones without allowing two consecutive zeros. The eight cases follow.

L_3	L_2	L_1	L_0	Count of Possibilities ($n \geq 4$)
1	1	1	1	2^{n-4}
1	0	1	0	excluded
0	1	0	1	2^{n-3}
1	1	0	1	2^{n-3}
1	0	1	1	2^{n-3}
1	1	1	0	2^{n-3}
0	1	1	1	2^{n-4}
0	1	1	0	2^{n-3}

Summing the seven useable cases gives

$$5 \cdot 2^{n-3} + 2 \cdot 2^{n-4} = 6 \cdot 2^{n-3} = 3 \cdot 2^{n-2}, \quad n \geq 4 ,$$

possible maximal representations. The endings with a zero in the left-most box would require that the L_4 box contain a one, while all would have either an L_4 or an L_5 appearing in the representation. The endings listed above do not give the numbers requiring 1, 2, or 3 Lucas numbers in their maximal representation. So, the endings given above do not include the representations of 1 through 9, 11 and 12, which give the first three terms $2x + 3x^2 + 6x^3$ of the enumerating maximal Lucas polynomial sum and explain the irregular first term in the sum of the polynomials $Q_n^*(x)$. The numbers not included in the count of possibilities above follow.

L_4	L_3	L_2	L_1	L_0	representing:	
	0	0	1	0	1	$2x$
	0	0	0	1	2	
	0	0	1	1	3	$3x^2$
	0	1	1	0	4	
	0	1	0	1	5	
	0	1	1	1	6	$6x^3$
	1	0	1	1	7	
	1	1	1	0	8	
	1	1	0	1	9	
1	0	1	1	0	11	
1	0	1	0	1	12	

Now, the enumerating maximal polynomial guarantees that $3 \cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, but examining the possible maximal representations which could be written using k Lucas numbers shows that at most $3 \cdot 2^{k-2}$ different representations could be formed. That is exactly one apiece, so the maximal representation of an integer using Lucas numbers subject to the two constraints, that no two consecutive Lucas numbers are omitted and that the combination $L_3 + L_1$ is not used unless L_0 or L_2 also appear, is unique.

5. CONCLUDING REMARKS

Much interest has been shown in the subject of representations of integers in recent years. Some of the many diverse new results which arise naturally from this paper are recorded here with references for further reading.

That the Fibonacci and Lucas sequences are complete has been shown in this paper, although the property was not named. A sequence of positive integers, $a_1, a_2, \cdots, a_n, \cdots$, is complete with respect to the positive integers if and only if every positive integer m is the sum of a finite number of the members of the sequence, where each member is used at most once in any given representation. (See [4], [5].) For example, the sequence of powers of two is complete; any positive integer can be represented in the binary system of numeration. However, if any power of 2, for example, $1 = 2^0$, is omitted, the new sequence is not complete. It is surprising that, for the Fibonacci sequence where $a_n = F_n$, $n \geq 1$, if any one arbitrary number F_k is missing, the sequence is still complete, but if any two arbitrary Fibonacci numbers F_p and F_q are missing, the sequence is incomplete [4].

The Dual Zeckendorf Theorem has an extension that characterizes the Fibonacci numbers. Brown in [2] proves that, if each positive integer has a unique representation as the sum of distinct members of a given sequence when no two consecutive members of the sequence are omitted in the representation, then the given sequence is the sequence of Fibonacci numbers.

Generalized Fibonacci numbers can be studied in a manner similar to the Lucas case. A set of particularly interesting sequences arising in Pascal's triangle appears in [6]: the sequences formed as the sums of elements of the diagonals of Pascal's left-justified triangle, beginning in the left-most column and going right one and up p throughout the array. (The Fibonacci numbers occur when p = 1.) Or, the squares of Fibonacci numbers may be used (see [7]), which gives a complete sequence if members of the sequence can be used twice. Other ways of studying generalized Fibonacci numbers include those given in [8], [9], [10], and [11].

To return to the introduction, Carlitz [12] and Klarner [13] have studied the problem of counting the number of representations possible for a given integer. Tables of the number of representations of integers as sums of distinct elements of the Fibonacci sequence as well as other related tables appear in [14]. The general problem of representations of integers using the Fibonacci numbers, enumerating intervals, and positional binary notation for the representations were given by Ferns [15] while [16] is one of the earliest references following Daykin [8]. The suggested readings and the references given here are by no means exhaustive. The range of representation problems is bounded only by the imagination.

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YE OLDE FIBONACCI CURIOSITY SHOPPE

Edited by
BROTHER ALFRED BROUSSEAU
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Let $S(X^2)_q$ symbolize the sum of the digits of X^2 on the base q. For example, $S(9^2)_5 = S(14^2)_5 = 5$ since $9_5^2 = 311$.

The following is a method for finding q such that $S(X^2)_q = X$ when X is given. For example $S(7^2)_8 = 7$ since $7_8^2 = 61$.

Step 1. List all the factors of X except X itself.

Step 2. List all the factors of X - 1.

Step 3. Multiply each factor of X by one of the factors of X - 1, discarding all products greater than X - 1. The retained products are the ten's digits of the X_q^2 that we seek.

Step 4. The unit's digits can be obtained by simple subtraction of the quantities in three from X.

Step 5. q can now be computed by simple arithmetic.

Example. $S(21^2)_q = 21$. Find all values of q.

Step I: 1 3 7

Step II: 1 2 4 5 10 20

Step III: 1 2 4 5 10 20

3 6 12 15

7 14

Step IV: 1(20) 2(19) 4(17) 5(16) 10(11) 20(1)

3(18) 6(15) 12(9) 15(6) 7(14) 14(7)

The quantities in parentheses are the unit's digits.

Step V: For example, for 5(16), $5b + 16 = 441$ in base ten so that $b = 85$ expressed as a base ten number. The bases taken in order are

421 211 106 85 43 22

141 71 36 29 61 31

The problem is: Why does this method work?.

Harlan L. Umansky, Emerson High School, Union City, N. J.

* * *

If eleven alternate terms of any Fibonacci sequence are added and divided by $L_{11}(199)$, the result is the middle term of the group of eleven terms added together.

Example. Using the series beginning 1, 4, $\cdots$,

$157 + 411 + 1076 + 2817 + 7375 + 19308 + 50549 + 132339 + 346468 + 907065 + 2374727 = 3942292$

Dividing by 199 gives 19308.

Brother Alfred Brousseau, St. Mary's College, California

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico 87131

Each proposed problem or solution should be submitted on a separate sheet or sheets, preferably typed in double spacing, in the format used below, to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131.

Solutions should be received within four months of the publication date of the proposed problem.

DEFINITIONS

$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n;\ L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-262 Proposed by Herta T. Freitag, Roanoke, Virginia

(a) Prove that the sum of n consecutive Lucas numbers is divisible by 5 if and only if n is a multiple of 4.

(b) Determine the conditions under which a sum of n consecutive Lucas numbers is a multiple of 10.

B-263 Proposed by Timothy B. Carroll, Graduate Student, Western Michigan University, Kalamazoo, Michigan.

Let $S_n = a^n + b^n + c^n + d^n$ where a, b, c, and d are the roots of $x^4 - x^3 - 2x^2 + x + 1 = 0$.

(a) Find a recursion formula for S_n.

(b) Express S_n in terms of the Lucas number L_n.

B-264 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Use the identities $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and $F_n^2 - F_{n-2}F_{n+2} = (-1)^n$ to obtain a factorization of $F_n^4 - 1$.

B-265 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois

Let F_n and L_n be designated as F(n) and L(n). Prove that

$$F(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1].$$

B-266 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois

Let L_n be designated as $L(n)$. Prove that

$$L(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) + 1] \quad .$$

B-267 Proposed by Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, California.

Let a regular pentagon of side p, a regular decagon of side d, and a regular hexagon of side h be inscribed in the same circle. Prove that these lengths could be used to form a right triangle; i.e., that $p^2 = d^2 + h^2$.

SOLUTIONS

OF THREE, WHO IS SHE?

B-238 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Can you guess WHO IS SHE? This is an easy simple addition and SHE is divisible by 29.

$$\begin{array}{r} \text{WHO} \\ \text{IS} \\ \hline \text{SHE} \end{array}$$

Solution by John W. Milsom, Butler County Community College, Butler, Pennsylvania.

Although it is not stated in the problem, assume (as is customary) that distinct letters represent distinct digits.

The letter I must be replaced with the number 9 (base 10). W is one less than S. Examining the three-digit numbers which are divisible by 29, there are three sets of numbers which satisfy the conditions imposed by the problem.

$$\begin{array}{rrrr} \text{WHO} & 628 & 714 & 743 \\ \text{IS} & 97 & 98 & 98 \\ \hline \text{SHE} & 725 & 812 & 841 \end{array}$$

Thus WHO IS SHE can be

1. 628 97 725
2. 714 98 812
3. 743 98 841 .

Also solved by Harold Don Allen, Paul S. Bruckman, J. A. H. Hunter, Robert Kaplar, Jr., Edgar Karst, David Zeitlin, and the Proposer. At least one of the solutions was found by Kim Bachick, Warren Cheves, and Herta T. Freitag. A solution with W = E was found by Richard W. Sielaff.

INEQUALITY ON GENERALIZED BINOMIALS

B-239 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let $p > 0$, $q > 0$, $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = pu_n + qu_{n-1}$ $(n \geq 1)$. Put

$$\left\{ {n \atop k} \right\} = u_n u_{n-1} \cdots u_{n-k+1} / u_1 u_2 \cdots u_k, \qquad \left\{ {n \atop 0} \right\} = 1 .$$

Show that

$$(*) \qquad \left\{ {n \atop k} \right\}^2 - p^2 \left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} > 0 \qquad (0 \le k \le n) .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Let

$$L = \left\{ {n \atop k} \right\}^2 - p^2 \left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} .$$

When $n = 2$ and $k = 1$ the inequality is not strict since $L = 0$. Now

$$L = (u_{n-k+1} u_{k+1} - p^2 u_{n-k} u_k) u_n^2 \cdots u_{n-k+2}^2 u_{n-k+1} / u_1^2 \cdots u_k^2 u_{k+1} .$$

Also

$$\begin{aligned} u_{n-k+1} u_{k+1} - p^2 u_{n-k} u_k &= (pu_{n-k} + qu_{n-k-1})(pu_k + qu_{k-1}) - p^2 u_{n-k} u_k \\ &= pq(u_{n-k} u_{k-1} + u_{n-k-1} u_k) + q^2 u_{n-k-1} u_{k-1} . \end{aligned}$$

Since u_k for $k > 0$, p, and q are positive, L is a product of positive numbers except for $n = 2$, $k = 1$.

Also solved by H. W. Gould and the Proposer.

THE MISSING LUCAS FACTOR

B-240 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California.

Prove that, for all positive integers n, $3F_{n+2}F_{n+3}$ is an exact divisor of

$$7F_{n+2}^3 - F_{n+1}^3 - F_n^3 .$$

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let F_m be denoted by a, b, c, and d when m is n, $n+1$, $n+2$, and $n+3$, respectively. Let $E = 7c^3 - b^3 - a^3$. Then $E = 7c^3 - (d-c)^3 - (2c-d)^3 = 3cd(3c-d)$ and so $(3cd) \mid E$ as desired. (One may note that the remaining factor $3c - d$ equals L_{n+1}.)

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, L. Carlitz, Warren Cheves, Herta T. Freitag, J. A. H. Hunter, Edgar Karst, Graham Lord, F. D. Parker, David Zeitlin, and the Proposer.

THREE FACES OF A POSSIBLE PRIME

B-241 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

If $2F_{2n-1}F_{2n+1} - 1$ and $2F_{2n}^2 + 1$ are both prime numbers, then prove that

$$F_{2n}^2 + F_{2n-1}F_{2n+1}$$

is also a prime number.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Since

$$F_{2n+1}F_{2n-1} - F_{2n}^2 = 1, \quad 2F_{2n-1}F_{2n+1} - 1 = 2F_{2n}^2 + 1 = F_{2n}^2 + F_{2n-1}F_{2n+1}.$$

If any one of these three equal expressions represents a prime, so do the other two.

Also solved by James D. Bryant, Edgar Karst, David Zeitlin, and the Proposer.

FIBONACCI-PASCAL PROPORTION

B-242 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n}{k-1} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k < n$).

Solution by the Proposer.

Let

$$R = \binom{n}{k} \div \binom{n}{k-1}.$$

Then

$$R = [n!/k!(n-k)!][(k-1)!(n-k+1)!/n!] = (n-k+1)/k.$$

Then we can make R equal to F_m/F_{m+1} by choosing k as tF_{m+1} and n as $tF_{m+2} - 1$, with t any positive integer.

ANOTHER ELUSIVE PLEASING PROPORTION

B-243 Proposed by J. Wlodarski, Proz-Westhoven, Federal Republic of Germany.

Prove that

$$\binom{n}{k} \div \binom{n+1}{k} = F_m \div F_{m+1}$$

for infinitely many values of the integers m, n, and k (with $0 \le k \le n$).

Solution by the Proposer.

Here the given ratio of binomial coefficients equals $(n-k+1)/(n+1)$ and this becomes F_m/F_{m+1} when $n = tF_{m+1} - 1$ and $k = tF_{m-1}$, with t any positive integer.