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SOME ARITHMETIC FUNCTIONS RELATED TO FIBONACCI NUMBERS

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1. INTRODUCTION

As is customary, we define the Fibonacci and Lucas numbers by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad (n \geq 2)$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \qquad (n \geq 2),$$

respectively. It is well known that a positive integer N has the unique representation

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where $r = r(N)$ and the k_j satisfy

$$k_1 \geq 2, \quad k_j - k_{j-1} \geq 2 \qquad (j = 2, 3, \cdots, r). \tag{1.2}$$

The representation (1.1) is called the canonical or Zeckendorf representation of N.

It is proved in [1] that the set A_t of integers $\{N\}$ with $k_1 = t$ can be described in the following way:

$$\begin{cases} A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \\ A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \end{cases} \qquad (t = 1, 2, 3, \cdots), \tag{1.3}$$

where juxtaposition of functions denotes composition and

$$a(n) = [\alpha n], \quad b(n) = [\alpha^2 n], \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}). \tag{1.4}$$

For the Lucas numbers it is known that every positive integer is uniquely representable in either the form

$$N = L_0 + L_{k_1} + L_{k_2} + \cdots + L_{k_r}, \tag{1.5}$$

where

$$k_1 \geq 3, \quad k_j - k_{j-1} \geq 2 \qquad (j = 2, 3, \cdots, r) \tag{1.6}$$

or in the form

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$$N = L_{k_1} + L_{k_2} + \cdots + L_{k_r}, \tag{1.7}$$

where now

$$k_1 \geq 1, \quad k_j - k_{j-1} \geq 2 \quad (j = 2, 3, \cdots, r); \tag{1.8}$$

but not in both (1.5) and (1.7).

Let B_0 denote the set of positive integers representable in the form (1.5) and let B_t denote the set of positive integers representable in the form (1.7) with $k_1 = t$. Then it is proved in [2] that

$$\begin{cases} B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\} \\ B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\} \end{cases}, \tag{1.9}$$

and

$$\begin{cases} B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \\ B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \end{cases} \quad (t = 1, 2, 3, \cdots). \tag{1.10}$$

The functions $a(n)$, $b(n)$ satisfy numerous relations that are consequences of the following.

$$b(n) = a(n) + n = a^2(n) + 1 \tag{1.11}$$

and

$$ab(n) = a(n) + b(n) = ba(n) + 1. \tag{1.12}$$

Moreover if the function $e(n)$ is defined by

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1}, \tag{1.13}$$

where N is defined by (1.1), then we have

$$ea(n) = n, \quad eb(n) = a(n). \tag{1.14}$$

Comparison of (1.9) and (1.10) with (1.3) suggests that it would be of interest to introduce the function

$$c(n) = a(n) + 2n = b(n) + n. \tag{1.15}$$

It is not difficult to show that $bc(n) - cb(n) = 0$ or 1. We accordingly define two strictly monotonic functions $r(n)$, $s(n)$ by means of

$$bcr(n) = cbr(n), \quad bcs(n) = cbs(n) + 1. \tag{1.16}$$

The functions $r(n)$ and $s(n)$ are <u>complementary</u>, that is, the sets $\{r(n)\}$ and $\{s(n)\}$ constitute a (disjoint) partition of the positive integers.

The present paper is concerned with the properties of $r(n)$ and $s(n)$ and various related functions. In particular we define

$$u'(n) = bs(n) + 1 \tag{1.17}$$

and

$$t'(n) = as(n) + n\,. \tag{1.18}$$

It then follows that

$$(s) = (ab) \cup (a^2u')\,; \tag{1.19}$$

more precisely

$$st = ab, \qquad st' = a^2u'\,, \tag{1.20}$$

where t and t' are complementary functions. Also

$$\begin{cases} c(n) \in (a) \rightleftarrows n \in (a^2u) \cup (bs) \\ c(n) \in (b) \rightleftarrows n \in (br) \cup (s) \end{cases}\;;$$

this is equivalent to

$$\begin{cases} ca^2(n) \in (a) & (n \in (r)) \\ cb(n) \in (b) & (n \in (r)) \end{cases}\,. \tag{1.21}$$

It should be noted that the unions above are disjoint unions.

In these formulas we have used the symbol (f) to denote the range of the function f. If f and g are two strictly monotonic functions such that $(f) \subset (g)$, it is clear that there exists a strictly monotonic function h such that $f = gh$. In particular since $(b) \subset (a)$, there exists a function v such that $b = av$. Also since $(cs) \subset (b)$, there exists a function z such that $cs = bz$. Similarly we define functions p, x, y, w by means of

$$es(n) = rp(n) = ux(n) = uwy(n)\,, \tag{1.22}$$

so that $x = wy$. Among various relations among these functions we cite in particular the following.

$$z(n) = c's(n) \tag{1.23}$$

$$zt(n) = ca(n) + 1, \qquad zt'(n) = b^2a(n) \tag{1.24}$$

$$tb^2(n) = t(n) + b^2(n) \tag{1.25}$$

$$t't(n) = tb^2(n) - 1 \tag{1.26}$$

$$yt(n) = 2n \tag{1.27}$$

$$v(n) = w(2n)\,. \tag{1.28}$$

The formula

$$eca(n) = c(n) - 1 \tag{1.29}$$

proved in Section 3 can be thought of as one of the basic results of the paper. It was originally proved in an entirely different way.

We may also note the formula

$$(s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b) \quad , \tag{1.30}$$

which is a consequence of (1.19). There are similar formulas for (r), (u), (u').

For the convenience of the reader a summary of formulas is included at the end of the paper as well as several brief numerical tables.

It should be remarked that almost all the theorems in this paper were suggested by numerical data. Thus it seems plausible that further numerical data may suggest additional theorems. The authors have prepared rather extensive tables which will be available from the Fibonacci Bibliographical and Research Center.

2. NOTATION AND PRELIMINARIES

If f is a function on the set $\mathbf{N}$ of positive integers, we let (f) denote the range of f, that is

$$(f) = \{f(n) \mid n \in \mathbf{N}\} .$$

If $n, m \in \mathbf{N}$, then

$$\eta(n < f < m)$$

is the number of integers j such that $n < f(j) < m$.

If f has the property that $f(n+1) - f(n) > 1$ for all $n \in \mathbf{N}$, then we say that f is <u>separated.</u>

If f is a function such that $\mathbf{N}/(f)$ is infinite, we may define a strictly monotonic function f' by:

$$(f') = \mathbf{N}/(f) .$$

This function f' is called the <u>complement</u> of f.

2.1. <u>Theorem</u>. If f is a strictly monotonic function from $\mathbf{N}$ to $\mathbf{N}$ such that $\mathbf{N}/(f)$ is infinite, then

$$f(n) = n + \eta(f' < f(n))$$

$$f'(n) = n + \eta(f < f'(n)) .$$

<u>Proof</u>. Suppose that $f(n) = k$. Since f is strictly monotonic we have $k \geq n$. Clearly $\eta(f < k) = n - 1$ and by definition of f', some $\eta(f' < k) = k - n$. Thus

$$f(n) = k = n + \eta(f' < k) = n + \eta(f' < f(n)) .$$

A similar argument shows that

$$f'(n) = n + \eta(f < f'(n)) .$$

2.2 Theorem. If f is a strictly monotonic function and, for some n and j we have $f'(j) = f(n) - 1$, then $j = f(n) - n$.

Proof. We have

$$\begin{aligned} f'(j) = f(n) - 1 &= n + \eta(f' < f(n)) - 1 \\ &= n + \eta(f' < (j) + 1) - 1 \\ &= n + j - 1 . \end{aligned}$$

Then $f(n) - 1 = n + j - 1$ and $j = f(n) - n$.

2.3. Corollary. [3, Th. 3.1]. If f is a separated function then for all $n > 1$,

$$f'(f(n) - n) = f(n) - 1$$

and

$$f'(f(n) - n + 1) = f(n) + 1 .$$

Proof. This is a direct consequence of the fact that if f is separated, then $f(n) - 1 \in (f')$ and $f(n) + 1 \in (f')$.

2.4. Theorem. If f is separated, then for all $n > 1$,

$$\eta\bigl(f'(n) < f < f(f'(n)) + 1\bigr) = n .$$

Proof.

$$\begin{aligned} \eta\bigl(f < f(f'(n)) + 1\bigr) = f'(n) &= n + \eta(f < f'(n)) \\ &= \eta(f < f'(n)) + \eta\bigl(f'(n) < f < f(f'(n)) + 1\bigr) \end{aligned}$$

and the theorem follows.

2.5. Definition. If f and g are functions, we use juxtaposition to mean composition of functions, that is,

$$fg(n) \equiv f(g(n)) .$$

2.6. Theorem. If f, g, h and k are all strictly monotonic, and if $f = g'h$ and $g = f'k$, then

$$f'k' = g'h' .$$

Proof. We have

$$(f') = (g) \cup (g'h') = (f'k) \cup (g'h')$$

and

$$(f') = (f'k) \cup (f'k') .$$

Since all functions are strictly monotonic, these are disjoint unions, and hence

$$(f'k') = (g'h') .$$

Again using strict monotonicity, we must have

$$f'k' = g'h' .$$

2.7. Definition. If

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists, we set

$$c_f = \lim_{n\to\infty} \frac{f(n)}{n} .$$

2.8. Theorem. If f and f' are complementary strictly monotonic functions and

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and $\neq 0$, then

$$\lim_{n\to\infty} \frac{f'(n)}{n}$$

exists, and we have

(i) $$\frac{1}{c_f} + \frac{1}{c_{f'}} = 1$$

(ii) $$c_f + c_{f'} = c_f \cdot c_{f'} .$$

2.9. Definition. If ρ is any real number, then $[\rho]$ is defined to be the greatest integer less than or equal to ρ, and $\{\rho\}$ denotes $\rho - [\rho]$.

2.10. We shall make extensive use of the functions a, b, c, e defined in [1]. For convenience, we recall the definitions, and state some properties, of these functions.

(2.11) $$a(n) = [\alpha n] \quad \text{where} \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5})$$

(2.12) $$b(n) = [\alpha^2 n] = a(n) + n$$

(2.13) $$c(n) = [(\alpha + 2)n] = a(n) + 2n .$$

(2.14) $$b(n) = a^2(n) + 1$$

(2.15) $$ab(n) = a(n) + b(n) = ba(n) + 1$$

(2.16) $$ab(n) + 1 = a(b(n) + 1)$$

(2.17) $$a^2(n) + 1 = a(a(n) + 1)$$

(2.18) $$n \in (a) \rightleftarrows \begin{array}{l} a(n + 1) = a(n) + 2 \\ b(n + 1) = b(n) + 3 \\ c(n + 1) = c(n) + 4 \end{array}$$

(2.19) $$n \in (b) \rightleftarrows \begin{array}{l} a(n + 1) = a(n) + 1 \\ b(n + 1) = b(n) + 2 \\ c(n + 1) = c(n) + 3 \end{array} .$$

2.20. Theorem. Suppose that

$$\alpha n = m + \epsilon_1 \qquad (0 < \epsilon_1 < 1)$$

$$\alpha m = k + \epsilon_2 \qquad (0 < \epsilon_2 < 1)$$

then

$$\epsilon_2 + \alpha\epsilon_1 = 1 + \epsilon_1 .$$

Proof. We have

$$\begin{aligned} \alpha^2 n &= \alpha m + \alpha\epsilon_1 = (\alpha + 1)n = \alpha n + n \\ &= m + \epsilon_1 + n \\ &= k + \epsilon_2 + \alpha\epsilon_1 . \end{aligned}$$

From the definition, $m = a(n)$, $k = a^2(n)$ and $k + 1 = b(n) = m + n = [\alpha^2 n]$. Thus $k + 1 = m + n$ and so

$$k + 1 + \epsilon_1 = k + \epsilon_2 + \alpha\epsilon_1$$

and the result follows.

2.21. Theorem [4]. For all n,

(i) $$n \in (a) \rightleftarrows \{\alpha n\} > \frac{1}{\alpha^2}$$

(ii) $$n \in (b) \rightleftarrows \{\alpha n\} < \frac{1}{\alpha^2} .$$

2.22. Theorem. We have

$$\eta(a < n) = a(n) - n .$$

Proof. This follows from the fact that $a(n + 1) - a(n)$ is 1 if $n \in (b)$ and 2 if $n \in (a)$. Since $a(1) = 1$, we have $a(n) = n + \eta(a < n)$.

The following formulas follow from 2.22.

$$\begin{aligned} \eta(b < n) &= n - 1 - \eta(a < n) \\ &= n - 1 - (a(n) - n) \\ &= 2n - 1 - a(n) . \end{aligned} \tag{2.23}$$

$$\eta(b < a(n)) = a(n) - n = \eta(a < n) . \tag{2.24}$$

Recall that the function e was originally defined in terms of the Zeckendorf representation of n. In [1, Th. 6] it is shown that

$$\text{(2.25)} \qquad \begin{aligned} eb(n) &= a(n) \\ ea(n) &= n\ . \end{aligned}$$

We list some properties of e.

$$\text{(2.26)} \qquad e(n) = \eta(a \le n)$$

$$\text{(2.27)} \qquad e(n) = \eta(a < n + 1) = a(n + 1) - (n + 1)$$

$$\text{(2.28)} \qquad e(n) = \left[\frac{n + 1}{\alpha}\right] .$$

3. BASIC RESULTS

3.1. Theorem. For all n, $0 \le bc(n) - cb(n) \le 1$.

Proof. Recall that $b(n) = [\alpha^2 n]$ by (2.9). Thus

$$\begin{aligned} bc(n) &= [\alpha^2 c(n)] = [\alpha^2 (b(n) + n)] \\ &= [\alpha^2 b(n) + \alpha^2 n] \\ &= [\alpha^2[\alpha^2 n] + \alpha^2 n] \end{aligned}$$

and

$$\begin{aligned} cb(n) &= b(b(n)) + b(n) \\ &= [\alpha^2[\alpha^2 n]] + [\alpha^2 n] . \end{aligned}$$

It is evident that $bc(n) \ge cb(n)$, and $0 \le bc(n) - cb(n) \le 1$.

3.2. Corollary. If $cb(n) \in (b)$, then $cb(n) = bc(n)$. If $cb(n) \in (a)$, then $cb(n) = bc(n) - 1 = a^2c(n)$.

Proof. Since for all r, $b(r + 1) - b(r) \ge 2$, then if $cb(n) \in (b)$, it follows that $cb(n) = bc(n)$. If $cb(n) \in (a)$, then $cb(n) = bc(n) - 1 = a^2c(n)$.

3.3. Definition. We define two strictly monotonic complementary functions r and s by means of

$$\text{(3.4)} \qquad \begin{cases} (r) = \{n \mid cb(n) = bc(n)\} \\ (s) = \{n \mid cb(n) = bc(n) - 1\} \end{cases} .$$

3.5. Theorem. For all n, $car(n) = acr(n) - 1$ and $cas(n) = acs(n) - 2$.

Proof. By definition, $cbr(n) = bcr(n)$, that is,

$$abr(n) + 2br(n) = acr(n) + cr(n) .$$

Then

$$\begin{aligned} acr(n) &= abr(n) + 2br(n) - cr(n) = abr(n) + br(n) - r(n) \\ &= abr(n) + ar(n) \end{aligned}$$

and

$$\begin{aligned} car(n) &= bar(n) + ar(n) \\ &= abr(n) - 1 + ar(n) \\ &= acr(n) - 1 \quad . \end{aligned}$$

Similarly, $cas(n) = acs(n) - 2$.

3.6. Theorem. If $ca(n) \in (a)$, then $ca(n) = a(c(n) - 1)$.

Proof. Case 1. if $n \in (r)$, then $ca(n) = ac(n) - 1$, and evidently if $ca(n) \in (a)$, we must have $ca(n) = a(c(n) - 1)$.

Case 2. If $n \in (s)$, then $ca(n) = ac(n) - 2$. Thus if $ca(n) \in (a)$, it must be that $ca(n) + 1 \in (b)$ (by (2.15)), and hence $ca(n) = a(c(n) - 1)$.

3.7. Theorem. For all n, $cs(n) \in (b)$.

For the proof of this theorem, we require some preliminary lemmas.

3.8. Lemma. If $n \in (s)$, then $a(n) \notin (s)$.

Proof. Let $n \in (s)$. Then $cb(n) = a^2c(n)$ and $ac(n) = ca(n) + 2$. Thus

$$a^2c(n) = a(ac(n)) = a(ca(n) + 2) \quad .$$

But also

$$cb(n) = ca^2(n) + 4 = c(ca(n)) + 4 \quad ,$$

by (2.15).

Now suppose that $ca^2(n) \in (a)$. Then by Theorem 3.6, $ca^2(n) = a(ca(n) - 1)$. Since three consecutive integers cannot all be in (a), it must be that $ca^2(n) + 2 \in (b)$, $ca^2(n) + 1 \in (a)$, and $ca^2(n) + 3 \in (a)$. Then

$$\begin{aligned} ca^2(n) + 3 &= a(ca(n) + 1) \\ ca^2(n) + 1 &= a(ca(n)) \\ ca^2(n) &= aca(n) - 1 = ca(a(n)) \end{aligned}$$

and by Theorem 3.5, $a(n) \in (r)$.

On the other hand, if $ca^2(n) \in (b)$, we must have $ca^2(n) + 1 \in (a)$, and precisely one of $ca^2(n) + 2$, $ca^2(n) + 3$ must be in (b). Thus $ca^2(n) + 1 = aca(n)$ and so $aca(n) - 1 = ca(a(n))$ and by Theorem 3.5, $a(n) \in (r)$.

3.9. Lemma. For all n, $cabr(n) = bacr(n)$ and $cabs(n) = bacs(n) - 2$.

Proof. The proof is manipulative. We show first for all n, $cab(n) = 7a(n) + 4n - 1$, as follows:

$$\begin{aligned} cab(n) &= bab(n) + ab(n) && \text{by (2.10)} \\ &= ab^2(n) - 1 + ab(n) && \text{by (2.12)} \\ &= ab(n) + b^2(n) - 1 + ab(n) && \text{by (2.12)} \\ &= 2ab(n) - 1 + ab(n) + b(n) && \text{by (2.9)} \end{aligned}$$

$$\begin{aligned} cab(n) &= 3ab(n) - 1 + a(n) + n && \text{by (2.9)} \\ &= 3(a(n) + b(n)) - 1 + a(n) + n && \text{by (2.12)} \\ &= 4a(n) + 3(a(n) + n) - 1 + n && \text{by (2.9)} \\ &= 7a(n) + 4n - 1 \ . \end{aligned}$$

Similarly, using the fact that $acr(n) = car(n) + 1$, we get $cabr(n) = 7ar(n) + 4r(n) - 1$, and from $acs(n) = cas(n) + 2$, we get $cabs(n) = 7as(n) + 4s(n) + 1$, and the result follows.

3.10. Lemma. For all n, $bcb(n) = cb^2(n)$, that is, $(b) \subset (r)$.

Proof. We first have, for all n,

$$\begin{aligned} cb^2(n) &= ab^2(n) + 2b^2(n) \\ &= ab(n) + 3b^2(n) \\ &= ab(n) + 3(ab(n) + b(n)) \\ &= 4ab(n) + 3b(n) \ . \end{aligned}$$

Case 1. $n \in (r)$. Then $bcb(n) = b^2c(n)$ and

$$\begin{aligned} b^2c(n) &= abc(n) + bc(n) \\ &= ac(n) + 2bc(n) \\ &= ca(n) + 1 + 2cb(n) && \text{(by Corollary 3.2 and Theorem 3.5)} \\ &= ab(n) + a(n) + 2(ab(n) + 2b(n)) \\ &= 3ab(n) + a(n) + 4b(n) \\ &= 4ab(n) + 3b(n) \ . \end{aligned}$$

Case 2. $n \in (s)$. Then $cb(n) = a^2c(n)$ and we have

$$\begin{aligned} bcb(n) &= ba^2c(n) = ba(ac(n)) \\ &= ab(ac(n)) - 1 \\ &= a^2c(n) + bac(n) - 1 \\ &= cb(n) + abc(n) - 2 \\ &= cb(n) + ac(n) + bc(n) - 2 \\ &= cb(n) + ca(n) + cb(n) + 1 && \text{(by Corollary 3.2 and Theorem 3.5)} \\ &= 2(ab(n) + 2b(n)) + (ba(n) + a(n)) + 1 \\ &= 3ab(n) + 4b(n) + a(n) \\ &= 4ab(n) + 3b(n) \ . \end{aligned}$$

3.11. Lemma. We have $(s) \subset (a)$.

The proof follows immediately from Lemma 3.10.

Proof of Theorem 3.7. If $n \in (s)$, then $n = a(j)$, for some integer j, where $j \notin (s)$ by Lemma 3.8. Since $n \in (s)$, we have

$$\begin{aligned} bc(n) - 1 &= cb(n) \\ bc(n) + 2 &= cb(n) + 3 \\ &= c(b(n) + 1) && \text{by (2.16)} \\ &= c(ba(j) + 1) = cab(j) = bac(j) && \text{by (2.12)} . \end{aligned}$$

Thus $bc(n) + 2 \in (b)$, which implies that $c(n) \in (b)$ by (2.19). This completes the proof of Theorem 3.7.

3.12. Corollary. For all n, $cas(n) \in (a)$.

Proof. By Theorem 3.8, $cs(n) = b(j)$ for some integer j. Then

$$\begin{aligned} cas(n) &= acs(n) - 2 \qquad \text{by Theorem 3.5} \\ &= ab(j) - 2 \\ &= a^3(j) . \end{aligned}$$

3.13. Theorem. If $ca(n) \in (b)$, then $ca(n) = a(c(n) - 1) + 1$.

Proof. We need only consider $n \in (r)$, since if $n \in (s)$, $ca(n) \in (a)$. Thus, if $n \in (r)$, $ca(n) = ac(n) - 1$. If $ca(n) \in (b)$, then $ca(n) - 1 = a(c(n) - 1)$ and the result follows.

Recall that the function e satisfies

$$\begin{aligned} e(a(n)) &= n \\ e(b(n)) &= a(n) . \end{aligned}$$

Note that $e(n) = \eta(a \le n)$.

3.14. Theorem. For all n, $eca(n) = c(n) - 1$.

Proof. We have shown that if $ca(n) \in (a)$, then $ca(n) = a(c(n) - 1)$, and if $ca(n) \in (b)$, $ca(n) = a(c(n) - 1) + 1$. In either case, $eca(n) = c(n) - 1$.

3.15. Theorem. For all n, $ecb(n) = ac(n)$.

Proof. Case 1. If $n \in (r)$, $cb(n) = bc(n)$ and $ecb(n) = ebc(n) = ac(n)$.

Case 2. If $n \in (s)$, $cb(n) = a^2c(n)$ and $ecb(n) = e(a^2c(n)) = ac(n)$.

4. THE FUNCTIONS c', ϕ, ϕ', ψ, ψ'

In this section, we consider some functions which arise in a natural way from the results of Section 3, and give some of their properties. Recall that c' denotes the complementary function to c. We shall require some properties of c', given in the following.

4.1. Theorem. We have

(i) $$c'b(n) = c(n) - 1$$
$$c'(b(n) - 1) = c(n) - 2$$
$$c'(b(n) + 1) = c(n) + 1$$
$$c'(b(n) + 2) = c(n) + 2$$

(ii) $$c(n) + c'(n) = 5n - 1$$

(iii) $$c'(n) = n + \eta(b < n)$$

(iv) $$c'a(n) = c'(a(n) + 1) - 1$$

(v) $$c'ab(n) = ca(n) + 1 .$$

Proof. Since $c'(c(n) - n) = c(n) - 1$ (by Theorem 2.2) and $c(n) - n = b(n)$, we have $c'b(n) = c(n) - 1$. The rest of (i) follows from the fact that $c(n + 1) - c(n) \geq 3$ for all n (see (2.15) and (2.16)).

The proof of (ii) is straightforward. For example,

$$\begin{aligned} cb(n) + c'b(n) &= ab(n) + 2b(n) + c(n) - 1 \\ &= a(n) + 4b(n) + n - 1 \\ &= 5b(n) - 1 . \end{aligned}$$

To see (iii), use (i) and the fact that $\eta(b < b(n)) = n - 1$ and $\eta(b < a(n)) = a(n) - n$ (see (2.20)). Both (iv) and (v) follow from (i).

4.2. Theorem. For all n, $ec(n) \in (c')$.

Proof. Case 1. If $n \in (a)$, say $n = a(j)$, then $ec(n) = c(j) - 1 = c'b(j)$.

Case 2. If $n \in (b)$, say $n = b(j)$, then $ec(n) = ac(j)$. We have seen that $ac(j)$ is either $ca(j) + 1$ or $ca(j) + 2$; in either case, $ac(j) \in (c')$.

We may now define a strictly monotonic function ϕ by the equality

$$ec(n) = c'\phi(n) .$$

The complementary function ϕ' is also of interest.

4.3. Theorem. For all n,

(i) $$\phi a(n) = b(n)$$

(ii) $$\phi br(n) = abr(n)$$

(iii) $$\phi bs(n) = abs(n) + 1 .$$

Proof.

(i) $$eca(n) = c(n) - 1 = c'b(n) = c'\phi a(n)$$

(ii) $$\begin{aligned} ecbr(n) &= acr(n) = car(n) + 1 = c'(bar(n) + 1) \\ &= c'(abr(n)) = c'\phi br(n) \end{aligned}$$

(iii)
$$\begin{aligned} ecbs(n) &= acs(n) = cas(n) + 2 = c'(bas(n) + 2) \\ &= c'(abs(n) + 1) = c'\phi bs(n) . \end{aligned}$$

4.4. Theorem. The function ϕ' is separated.

Proof. We show that for all n, $\phi(n+1) - \phi(n) \leq 2$. It then follows that for all n, $\phi'(n+1) - \phi'(n) \geq 2$. Since $(b) \subset (\phi)$, then for all n, $\phi(n+1) - \phi(n) \leq 3$. If $\phi(n) \in (b^2)$, then $\phi(n+1) - \phi(n) = 2$. If $\phi(n) \in (ba)$, then $\phi(n) + 1 \in (ab)$ and $\phi(n) + 2 \in (ab+1)$, and it follows that either $\phi(n) + 1$ or $\phi(n) + 2$ is in (ϕ). If $\phi(n) \in (a^2)$, then $\phi(n) + 1 = \phi(n+1)$ and if $\phi(n) \in (ab)$, then $\phi(n) + 2 = \phi(n+1)$. This completes the proof.

4.5. Theorem. If n is any integer not of the form $n = as(j) + 1$, then $\phi'(n) = a^2(n)$. If $n = as(j) + 1$ for some j, then $\phi'(n) = a(a(n) - 1)$.

Proof. Since by Theorem 4.3 we have

$$(\phi) = (b) \cup (abr) \cup (abs + 1)$$

it follows that

$$(\phi') = (a^2) \cup (abs)/(abs + 1) .$$

We next show that $\phi'(as(j) + 1) = abs(j)$ for all j. We have $\phi bs(j) = abs(j) + 1$. Since $abs(j) \in (\phi')$, then by Theorem 2.2 we have

$$\begin{aligned} \phi bs(j) - 1 &= \phi'(\phi bs(j) - bs(j)) \\ &= \phi'(abs(j) + 1 - bs(j)) \\ &= \phi'(as(j) + 1) \\ &= abs(j) . \end{aligned}$$

Next, if $n = as(j) + 1$, then

$$a(a(n) - 1) = a(a(as(j) + 1) - 1) = a(a^2s(j) + 2 - 1) = a(a^2s(j) + 1) = a(bs(j)) .$$

Thus if $n = as(j) + 1$, $\phi'(n) = a(a(n) - 1)$. Now (ϕ') differs from (a^2) only in that

$$\phi'(as(n) + 1) = a(a(n) - 1) = abs(n)$$

while $a^2(as(n) + 1) = abs(n) + 1$. Thus since ϕ' and a^2 are strictly monotonic, we must have $\phi'(n) = a^2(n)$ for all n not of the form $as(j) + 1$, and $\phi'(n) = a(a(n) - 1)$ for $n = as(j) + 1$ for some j.

It is now possible to define a new strictly monotonic function ψ by $\psi(n) = e\phi'(n)$.

4.6. Theorem. $\psi(n) = a(n)$ for all n not of the form $as(j) + 1$, and $\psi(n) = a(n) - 1$ for $n = as(j) + 1$. Thus $\psi(as(j) + 1) = a(as(j) + 1) - 1 = bs(j)$.

Proof. This follows from Theorem 4.5, and the definition of the function e.

Now ψ', the complementary function to ψ, is also strictly monotonic, and we have

$$(\psi') = (b) \cup (bs + 1)/(bs) .$$

4.7. Theorem. $\psi' r(n) = br(n)$ and $\psi' s(n) = bs(n) + 1$.

Proof. We first show that $\psi' s(n) = bs(n) + 1$. By Corollary 2.3, we have, since $bs(n) + 1 \in (\psi')$, and $bs(n) \in (\psi)$,

$$\begin{aligned} bs(n) + 1 &= \psi'(\psi(as(n) + 1) - (as(n) + 1) + 1) \\ &= \psi'(bs(n) - as(n)) \\ &= \psi'(s(n)) \quad . \end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 4.5.

Using the results of this section, we can easily derive various formulas.

4.8. Theorem.

(i) $$\phi s(n) = bes(n)$$

(ii) $$\phi a(n) + \phi' a(n) = \phi\phi' a(n) - 1$$

(iii) $$\phi'\phi s(n) = abs(n)$$

(iv) $$\phi\phi' s(n) = abs(n) - 1$$

(v) $$\phi\phi' r(n) = bar(n)$$

(vi) $$\phi s(n) + c' s(n) = 3s(n) .$$

5. THE FUNCTIONS r AND s AND SOME RELATED FUNCTIONS

In this section, we consider the functions r and s in detail, and introduce some important auxiliary functions.

5.1. Theorem. The function s is separated.

Proof. Suppose, on the contrary, that consecutive integers $n, n+1$ are in (s). Then both $n, n+1$ must be in (a) by Lemma 3.10, and both $c(n)$ and $c(n+1)$ must be in (b) by Theorem 3.6. Since $n, n+1$ are both in (a), we must have $n = ab(j)$ for some j. Then

$$c(n + 1) = c(ab(j) + 1) = cab(j) + 4 .$$

If $j \in (r)$, then $cab(j) + 4 = bac(j) + 4$, and if $j \in (s)$, then $cab(j) + 4 = bac(j) + 2$. In either case, since for any integer k, $ba(k) + 2$ and $ba(k) + 4$ are both in (a), we must have $c(n+1) \in (a)$, which is a contradiction. Thus s is a separated function.

5.2. Lemma. $(ab) \subset (s)$.

Proof. The proof is manipulative like the proof of Lemma 3.10. Using the definition $c(n) = a(n) + 2n$, one first shows that, for all n,

$$cbab(n) + 1 = 7ab(n) + 4b(n) - 3 .$$

Case 1. $n \in (r)$. Then we have

$$\begin{aligned} cb(n) &= bc(n) && \text{(Corollary 3.2)} \\ ac(n) &= ca(n) + 1 && \text{(Theorem 3.5)} \\ cab(n) &= bac(n) && \text{(Lemma 3.9)} . \end{aligned}$$

Then

$$\begin{aligned} bcab(n) &= b^2ac(n) \\ &= a^2c(n) + 2bac(n) \\ &= bc(n) - 1 + 2(abc(n) - 1) \\ &= cb(n) - 1 + 2(ac(n) + bc(n) - 1) \\ &= cb(n) - 1 + 2(ca(n) + 1) + 2cb(n) - 2 \\ &= 3(ab(n) + 2b(n)) + 2(ba(n) + a(n) + 1) - 3 \\ &= 5ab(n) + 6b(n) + 2a(n) - 3 \\ &= 7ab(n) + 4b(n) - 3 \\ &= cbab(n) + 1 . \end{aligned}$$

Case 2. $n \in (s)$. Then

$$\begin{aligned} cb(n) &= bc(n) - 1 && \text{(Corollary 3.2)} \\ ac(n) &= ca(n) + 2 && \text{(Theorem 3.5)} \\ cab(n) &= bac(n) - 2 && \text{(Lemma 3.9)} . \end{aligned}$$

By Corollary 3.12 we have $ca(n) \in (a)$, and since $ca(n) + 2 = ac(n)$, we have $ac(n) - 1 \in (b)$. By Theorem 3.7, we have $c(n) \in (b)$, say $c(n) = b(j)$. Then

$$\begin{aligned} cab(n) &= bab(j) - 2 \\ &= b(ba(j) + 1) - 2 \\ &= b^2a(j) + 2 - 2 \\ &= b^2a(j) . \end{aligned}$$

Thus $bac(n) - 2 \in (b)$, and so $bac(n) - 2 = b(ac(n) - 1)$. Now

$$\begin{aligned} bcab(n) &= b^2(ac(n) - 1) \\ &= ab(ac(n) - 1) + b(ac(n) - 1) \\ &= a(ac(n) - 1) + 2b(ac(n) - 1) \\ &= a(ca(n) + 1) + 2b(cs(n) + 1) . \end{aligned}$$

Since $ca(n) \in (a)$, we have $ca(n) = a(c(n) - 1)$, and so

$$\begin{aligned} bcab(n) &= a(a(c(n) - 1) + 1) + 2b(a(c(n) - 1) + 1) \\ &= a^2(c(n) - 1) + 2 + 2[a(a(c(n) - 1) + 1) + a(c(n) - 1) + 1] \\ &= 3(a^2(c(n) - 1) + 2) + 2a(c(n) - 1) + 2 \\ &= 3(b(c(n) - 1) + 1) + 2a(c(n) - 1) + 2 \\ &= 3b(c(n) - 1) + 2a(c(n) - 1) + 5 \\ &= 3b(c(n) - 1) + 2ca(n) + 5 \quad . \end{aligned}$$

Since $c(n) \in (b)$, then $c(n) - 1 \in (a)$ and we have $bc(n) = b(c(n) - 1) + 3$. Then

$$\begin{aligned} bcab(n) &= 3(bc(n) - 3) + 2ca(n) + 5 \\ &= 3(cb(n) - 2) + 2(ba(n) + a(n)) + 5 \\ &= 3(ab(n) + 2b(n) - 2) + 2a(n) + 2ab(n) - 2 + 5 \\ &= 5ab(n) + 6b(n) + 2a(n) - 3 \\ &= 7ab(n) + 4b(n) - 3 \\ &= cbab(n) + 1 \ . \end{aligned}$$

This completes the proof.

5.3. Lemma. For all n, $a^2(bs(n) + 1) \in (s)$.

Proof.

$$\begin{aligned} cb(a^2(bs(n) + 1)) &= cb(b^2s(n) + 1) \\ &= c(b^3s(n) + 2) \\ &= cb^3s(n) + 7 \\ &= b^2cbs(n) + 7 \\ &= [b^2a(acs(n)) + 3] + 4 \\ &= ab^2(acs(n)) + 4 \quad . \end{aligned}$$

But $ab^2(n) + 4 = a(bab(n) + 1)$ for all n, so $cb(a^2(bs(n) + 1)) \in (a)$, and this completes the proof.

5.4. Lemma. If $a^2(n) \in (s)$, then $n = bs(k) + 1$ for some integer k.

Proof. First note that $ab(b(n) + 1) = ab^2(n) + 3$, and $ab(a(n) + 1) = aba(n) + 5$. Since s is separated, if $a^2(n) \in (s)$, there must be some integer $a(j)$ so that $aba(j) + 1 < a^2(n) < ab(a(j) + 1) - 1$. Thus

$$a^2(n) = aba(j) + 3$$

and

$$a^2(n) + 2 = a(a(n) + 1) = ab(a(j) + 1) .$$

Then

$$a(n) + 1 = b(a(j) + 1) = ba(j) + 3 = ab(j) + 2$$

and

$$a(n) = a(b(j) + 1)$$

so that

$$n = b(j) + 1 .$$

We now show that $j = s(k)$ for some integer k. First $ca^2(n) \in (b)$ gives:

$$ca^2(n) = ca^2(b(j) + 1) = a(ca(b(j) + 1) - 1) + 1 \in (b)$$

so that $ca(b(j) + 1) - 1 \in (a)$. We also have:

$$ca(b(j) + 1) = c(ab(j) + 1) = cab(j) + 4$$

and we have seen before that $cab(j) + 4$ is always in (a). Thus

$$ca(b(j) + 1) = a(c(b(j) + 1) - 1) .$$

Now it must be that

$$ca(b(j) + 1) - 1 = a(c(b(j) + 1) - 2)$$

and hence

$$c(b(j) + 1) - 2 \in (b)$$

by (2.7, (iii).). But

$$c(b(j) + 1) - 2 = cb(j) + 3 - 2 = cb(j) + 1 \in (b) .$$

Thus $bc(j) = cb(j) + 1$ and $j \in (s)$. We now have the following:

5.5. Theorem. $(s) = (ab) \cup (a^2(bs + 1))$.

We can now prove

5.6. Theorem. $c(n) \in (a)$ if and only if $n \in [(a) \cup (bs)]/(s)$.

Proof. Clearly if $n \in (bs)$, $c(n) \in (a)$, and if $n \in (s)$, then $c(n) \in (b)$. Also, if $n \in (br)$, then $c(n) \in (b)$. Thus, suppose $n \in (a)/(s)$. Then suppose $ca^2(j) \in (b)$. Then $a(j) \notin (s)$, since $cas(n) \in (a)$ for all n. Thus $cba(j) = bca(j)$. Now if $ca^2(j) \in (b)$, then $ca^2(j) = a(ca(j) - 1) + 1$ so that $ca(j) - 1 \in (a)$. Then $b(ca(j) - 1) + 3 = bca(j)$ and

$$cba(j) = c(a^3(j) + 1) = ca^3(j) + 4 = b(ca(j) - 1) + 3 .$$

Then

$$ca^3(j) = b(ca(j) - 1) - 1 \in (a) .$$

Now, since $a^2(j) \notin (a)$, we have

$$ca(a^2(j)) + 1 = aca^2(j) ,$$

so that if $ca^2(j) \in (b)$, then $ca^3(j) \notin (b)$. This is a contradiction, and the proof is complete.

5.7. <u>Corollary</u>. $c(n) \in (b) \rightleftarrows n \in (br) \cup (s)$.

We now introduce some additional functions, defined as follows:

$$\text{(5.8)} \qquad \begin{aligned} &\text{(i)} & u'(n) &= bs(n) + 1 \\ &\text{(ii)} & t'(n) &= as(n) + n \\ &\text{(iii)} & bz(n) &= cs(n) \ . \end{aligned}$$

We also have the corresponding complementary functions u, t and z'.

The reasons for considering these functions are made evident in the following theorem.

5.9. <u>Theorem</u>. We have

$$\begin{aligned} &\text{(i)} & (u') &= \{n \mid a^2(n) \in (s)\} \\ &\text{(ii)} & (t') &= \{n \mid s(n) \in (a^2)\} \\ & & (t) &= \{n \mid s(n) \in (ab)\} \\ &\text{(iii)} & st(n) &= ab(n) \\ &\text{(iv)} & st'(n) &= a^2(bs(n) + 1) = a^2u'(n) \\ &\text{(v)} & zt(n) &= cs(n) + 1 \\ &\text{(vi)} & zt'(n) &= cbs(n) + 1 \\ &\text{(vii)} & z(n) &= c's(n) \ . \end{aligned}$$

<u>Proof</u>. (i) is clear from Theorem 5.5. To see (ii) we take (ii) as the definition of t and t' and show that we then have $t'(n) = as(n) + n$. In the proof of Lemma 5.4, it was shown that

$$abas(n) < a^2(bs(n) + 1) < ab(as(n) + 1)$$

for all integers n. From (ii), we have $a^2(bs(n) + 1) = st'(n)$, and $stas(n) = abas(n)$ (that is, $a^2(bs(n) + 1)$ is the n^{th} value of s of the form $a^2(bs(j) + 1)$, and $abas(n)$ is the $as(n)^{\text{th}}$ value of s of the form $ab(j)$). Now $stas(n) = s(t'(n) - 1)$, so that $t'(n) = tas(n) + 1$. From Theorem 2.1, we have

$$\begin{aligned} t'(n) &= n + \eta(t < t'(n)) \\ &= n + \eta(t < tas(n) + 1) \\ &= n + as(n) \ . \end{aligned}$$

Parts (iii) and (iv) follow from (ii).

To see (v): Case 1. $n \in (r)$. Then from the definition, $bzt(n) = cst(n) = cab(n) = bac(n)$. Then $zt(n) = ac(n) = ca(n) + 1$ since $n \in (r)$.

Case 2. $n \in (s)$. Then $bzt(n) = cst(n) = cab(n) \in (b)$ and since $n \in (s)$, $cab(n) = bac(n) - 2 = b(ac(n) - 1)$. Then $zt(n) = ac(n) - 1 = ca(n) + 1$ since $n \in (s)$.

To see (vi):

$$\begin{aligned} cst'(n) &= ca^2(bs(n) + 1) \\ &= c(b(bs(n) + 1) - 1) \\ &= c(b^2s(n) + 1) \\ &= cb^2s(n) + 3 \\ &= bcbs(n) + 3 \\ &= ba^2cs(n) + 3 \\ &= b(a^2cs(n) + 1) \quad . \end{aligned}$$

So $zt'(n) = a^2cs(n) + 1 = cbs(n) + 1$.

For (vii), we have first

$$c'st(n) = c'ab(n) = ca(n) + 1 = zt(n) \quad .$$

On the other hand,

$$\begin{aligned} cbs(n) + 1 = c'(b^2s(n) + 1) &= c'(b(bs(n) + 1) - 1) \\ &= c'(a^2(bs(n) + 1)) \\ &= c'st'(n) = zt'(n) \ . \end{aligned}$$

5.10. <u>Theorem</u>. $s(n) = c(n)$ if and only if $t'(n) = b^2(n)$.

<u>Proof</u>. We use the fact that $t'(n) = as(n) + n$, and consider the cases $n \in (r)$ and $n \in (s)$. If $n \in (r)$ and $s(n) = c(n)$, then

$$\begin{aligned} t'(n) = ac(n) + n &= ca(n) + 1 + n \\ &= ba(n) + 1 + a(n) + n \\ &= ab(n) + b(n) \\ &= b^2(n) \quad . \end{aligned}$$

If $n \in (s)$, then $c(n) \in (b)$ and $c(n) = s(n)$ is not possible.

Now suppose $t'(n) = b^2(n)$. Then

$$\begin{aligned} as(n) + n &= ab(n) + b(n) \\ &= ab(n) + a(n) + n \end{aligned}$$

and

$$\begin{aligned} as(n) &= ab(n) + a(n) \\ &= ca(n) + 1 \quad . \end{aligned}$$

Then $ca(n) + 1 \in (a)$. If $ca(n) + 1 = ac(n)$, then we have $c(n) = s(n)$ as required. If $ca(n) + 2 = ac(n)$, then we have $n \in (s)$, and $ca(n) \in (a)$, $ca(n) + 1 \in (b)$ (Theorem 3.5). Thus if $as(n) = ca(n) + 1$, then $n \in (r)$ and $c(n) = s(n)$.

5.11. Corollary. If $s(n) = c(n)$, then $n \in (r)$.

5.12. Theorem. If $s(n) = c(n)$, for some n, then

$$s(t'(n) + 1) = c(t'(n) + 1 .$$

Proof. If $s(n) = c(n)$, then we have $n \in (r)$ and $t'(n) = b^2(n)$. Then

$$\begin{aligned} ca^2b(n) &= a(cab(n) - 1) \\ &= a(bac(n) - 1) = a(a^3c(n)) \\ &= a^2(a^2c(n)) = ba^2c(n) - 1 \\ &= abac(n) - 2 \\ &= stac(n) - 2 \\ &= stas(n) - 2 \\ &= st'(n) - 5 \quad . \end{aligned}$$

On the other hand,

$$c(a^2b(n)) = c(b^2(n) - 1) = cb^2(n) - 4 .$$

Thus we have

$$\begin{aligned} cb^2(n) &= ab^2(n) - 1 \\ cb^2(n) + 3 &= sb^2(n) + 2 \\ &= st'(n) + 2 \\ &= s(t'(n) + 1) \quad \text{(by Theorem 6.5 (iv))} . \end{aligned}$$

Since $cb^2(n) + 3 = c(b^2(n) + 1) = c(t'(n) + 1)$, the proof is complete.

5.13. Theorem. If $s(n) = c(n)$, then $z(n) = 5n - 1$.

Proof. By Theorem 6.1 (iv) we have

$$\begin{aligned} z(n) &= 2s(n) - es(n) = 3s(n) - (s(n) + es(n)) = 3s(n) - (as(n) + 1) \\ &= 3c(n) - ac(n) - 1 = 3a(n) + 6n - (ca(n) + 1) - 1 \quad \text{since } n \in (r) \\ &= 3a(n) + 6n - 1 - b(n) - 2a(n) \\ &= 5n - 1 \quad . \end{aligned}$$

Other results of this nature are easily obtained; for example:

5.14. Corollary. If $s(n) = c(n)$, then

(i) $$rb(n) = c'b(n) = c(n) - 1$$

(ii) $$z'(4n - 1) = 5n - 2 .$$

It should be noted that since $s(1) = c(1)$, for example, it follows that there are infinitely many values of n for which $s(n) = c(n)$. We list the values of $n \leq 101$ for which $s(n) = c(n)$:

Table 1

n	1	6	9	22	40	43	48	56	61	64
s(n) = c(n)	3	21	32	79	144	155	173	202	220	231

We note that $t'(1) + 1 = 6$ and $t'(6) + 1 = 40$, and $t'(9) + 1 = 61$, while $t'(1) + 4 = 9$, $t'(6) + 4 = 43$, $t'(9) + 4 = 64$. One might conjecture that if $s(n) = c(n)$, we have not only $s(t'(n) + 1) = c(t'(n) + 1)$, but also $s(t'(n) + 4) = c(t'(n) + 4)$.

Using the fact that $(s) = (ab) \cup (a^2u')$ where $(u') = (bs + 1)$, we may express (s) as an infinite union as follows:

5.15. Theorem. We have

$$(s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b) .$$

Proof. The proof is by induction. We first show that every $x \in (s)$ satisfies

(5.16) $$x = a(a^2b)^k b(j)$$

for some integers k, j.

For $n = 1$, we have $s(1) = ab(1)$. Suppose n given, and for all $k < n$ we have

(5.17) $$s(k) = a(a^2b)^j b(m)$$

for some integers j and m (depending on k). Now $s(n)$ might be of the form $ab(N)$, for some N, in which case $s(n)$ satisfies (5.16), or else $s(n) = a^2u'(N)$ for some N. In the latter case, we have

$$u'(N) = bs(N) + 1$$

and since $s(n) = a^2u'(N)$, it must be that $N < n$. By the induction assumption,

$$s(N) = a(a^2b)^j b(m)$$

for some integers j and m, and so

$$\begin{aligned} s(n) &= a^2u'(N) \\ &= a^2(bs(N) + 1) \\ &= a^2(ba(a^2b)^j b(m) + 1) \\ &= a^2(ab(a^2b)^j b(m)) \\ &= a(a^2b)^{j+1} b(m) \quad . \end{aligned}$$

This completes the induction, and we have

$$(s) \subseteq \bigcup_{k=0}^{\infty} (a(a^2b)^k b) \quad . \tag{5.18}$$

To show inclusion in the other direction, let m be a fixed integer. We show by induction that every integer of the form

$$K = a(a^2b)^k b(m) \qquad (k = 0, 1, 2, \cdots) \tag{5.19}$$

satisfies $K \in (s)$.

When $k = 1$, we have $ab(m) \in (s)$. Suppose for some integer $k > 1$ we have $a(a^2b)^k b(m) \in (s)$, say

$$s(N) = a(a^2b)^k b(m) .$$

Then $a^2u'(N) \in (s)$, and since

$$\begin{aligned} a^2(bs(N) + 1) &= a^2(ba(a^2b)^k b(m) + 1) \\ &= a^2(ab(a^2b)^k (b(m)) \\ &= a(a^2b)^{k+1} b(m) \end{aligned}$$

we have $a(a^2b)^{k+1} b(m) \in (s)$. Thus for all m, we have

$$\{ a(a^2b)^k b(m) \mid k = 0, 1, 2, \cdots \} \subset (s).$$

This completes the proof.

Using Theorem 5.15 and the fact that $(r) \cup (s) = \mathbf{N}$, it is easy to prove

5.20. Corollary.

$$(r) = (b) \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k a^3) \right] .$$

Since $u'(n) = bs(n) + 1 \quad (n = 1, 2, \cdots)$, we have

5.21. Corollary.

$$(u') = \bigcup_{k=0}^{\infty} (ab(a^2b)^k b) .$$

In a similar fashion, one can find "infinite union" formulas for many of the other functions mentioned in this paper. However, we have not been able to give any such formula for t'.

Theorem 5.15 suggests the definition of a set of functions $\{f_k\}$ as follows:

$$s(f_k(n)) = a(a^2b)^k b(n) . \tag{5.22}$$

It is evident, for example, that $f_0(n) = t(n)$. The functions f_k are completely described in the next theorem.

5.23. Theorem. For all n, $f_k(n) = (t')^k t(n)$.

Proof. The proof is by induction on k. It is clear that $f_0(n) = t(n)$. Suppose for some $k > 0$, we have

$$f_{k-1}(n) = (t')^{k-1}t(n) \qquad (n = 1, 2, 3, \cdots) . \tag{5.24}$$

Then

$$\begin{aligned} st'(f_{k-1}(n)) &= a^2(bsf_{k-1}(n)) + 1) \\ &= a^2(ba(a^2b)^{k-1}b(n) + 1) \end{aligned}$$

by the induction assumption. This gives

$$\begin{aligned} st'(f_{k-1}(n)) &= a^2(ab(a^2b)^{k-1}b(n)) \\ &= a(a^2b)^k b(n) \end{aligned}$$

and it follows that

$$f_k(n) = t'f_{k-1}(n) .$$

Then for all k, we have $f_k(n) = (t')^k t(n)$. This completes the proof.

In Section 6, we shall show that $t't(n) = tb^2(n) - 1$ and also $t't(n) = t(b^2(n) - 1) + 1$ (Theorem 6.3). In view of this, we have the following inequalities for the functions f_k.

5.25. Theorem. For all integers j and k, $k > 0$,

$$f_{k-1}(b^2(j) - 1) < f_k(j) < f_{k-1}(b^2(j)) .$$

In addition, if

$$f_{k-1}(b^2(j) - 1) < f_k(m) < f_{k-1}(b^2(j))$$

then $m = j$.

Proof. Since $t't(n) = t(b^2(n) - 1) + 1$, we have

$$f_{k-1}(b^2(j) - 1) = (t')^{k-1}t(b^2(j) - 1) = (t')^{k-1}(t't(j) - 1) < (t')^k t(j) ,$$

and since $t't(n) = tb^2(n) - 1$, we have

$$f_{k-1}(b^2(j)) = (t')^{k-1}t(b^2(j))$$
$$= (t')^{k-1}(t't(j) + 1) > (t')^k t(j) .$$

To see that $f_k(j)$ is the only value of f_k between $f_{k-1}(b^2(j) - 1)$ and $f_{k-1}(b^2(j))$, consider for example $f_k(j - 1)$. By the preceding argument,

$$f_k(j - 1) < f_{k-1}(b^2(j - 1)) < f_{k-1}(b^2(j) - 1)$$

since f_{k-1} is strictly monotonic and $b^2(j - 1) < b^2(j) - 1$. On the other hand, we have

$$f_k(j + 1) > f_{k-1}(b^2(j + 1) - 1) > f_{k-1}(b^2(j))$$

since f_{k-1} is strictly monotonic, and $b^2(j + 1) > b^2(j) + 1$. This completes the proof.

6. CONTINUATION

In this section, we give various formulas involving the functions introduced in Section 5.

6.1. Theorem.

(i) $$z(n) = c(z(n) - s(n)) + 1$$

(ii) $$s(n) = b(z(n) - s(n)) + 1$$

(iii) $$az(n) - as(n) = es(n)$$

(iv) $$z(n) + es(n) = 2s(n)$$

(v) $$az(n) - as(n) = a(z(n) - s(n)) + 1 .$$

Proof. (i) Case 1. $zt(n) = ca(n) + 1$ and $st(n) = ab(n)$. Then

$$zt(n) - st(n) = ca(n) + 1 - ab(n)$$
$$= ba(n) + a(n) + 1 - ab(n)$$
$$= a(n)$$

and so $zt(n) = c(zt(n) - st(n)) + 1$.

Case 2. $zt'(n) = cbs(n) + 1$ and $st'(n) = a^2(bs(n) + 1)$. As above, we show that $zt'(n) - st'(n) = bs(n)$, and (i) follows.

To see (ii), we have

$$z(n) = c(z(n) - s(n)) + 1$$
$$= b(z(n) - s(n)) + z(n) - s(n) + 1$$

and this proves (ii).

For (iii), we have (since $as(n) = s(n) + es(n) - 1$)

$$\begin{aligned} az(n) &= ebz(n) = ecs(n) = ecaes(n) \\ &= ces(n) - 1 = bes(n) + es(n) - 1 \\ &= aes(n) + es(n) + es(n) - 1 \\ &= (s(n) + es(n) - 1) + es(n) \\ &= as(n) + es(n) . \end{aligned}$$

To see (iv), we use $z(n) = c's(n) = s(n) + \eta(b < s(n))$. Then

$$\begin{aligned} z(n) - s(n) &= (s(n) - 1) - \eta(a < s(n)) \\ &= s(n) - [\eta(a < s(n)) + 1] \\ &= s(n) - es(n) . \end{aligned}$$

Finally, (v) follows from

$$z(n) = a(z(n) - s(n)) + 2(z(n) - s(n)) + 1$$

$$\begin{aligned} a(z(n) - s(n)) &= 2s(n) - z(n) - 1 \\ &= es(n) - 1 \\ &= az(n) - as(n) - 1 . \end{aligned}$$

6.2. Remark. Theorem 6.1 could also have been proved by noting that $z(n) - s(n)$ is a monotonic function satisfying

$$(z - s) = (a) \cup (bs) .$$

6.3. Theorem.

(i) $$t't(n) = a^2b(n) + t(n)$$

(ii) $$tb^2(n) = t(n) + b^2(n)$$

(iii) $$t't(n) = tb^2(n) - 1$$

(iv) $$\eta(t(n) < t' < t(n) + b^2(n)) = n .$$

Proof.

(i): by definition,

$$t't(n) = ast(n) + t(n) = a^2b(n) + t(n) .$$

(ii): We know $t(as(n)) = t(t'(n) - n) = t'(n) - 1$ by Theorem 2.2. Then

$$\begin{aligned} t(ast(n)) &= t't(n) - 1 \\ t(a^2b(n)) &= t't(n) - 1 \\ t(a^2b(n) + 1) &= t't(n) + 1 = a^2b(n) + t(n) + 1 \\ tb^2(n) &= b^2(n) + t(n) . \end{aligned}$$

Statement (iii) follows directly from (i) and (ii), and statement (iv) follows from (iii) and Theorem 2.4.

6.4. Theorem.

(i) $$\mathrm{tsta}(n) = t(n) - 1 + \mathrm{sta}(n)$$

(ii) $$\mathrm{tstb}(n) = \mathrm{ta}(n) + \mathrm{stb}(n)$$

(iii) $$\mathrm{tab}(n) = \mathrm{te}(n) + \mathrm{ab}(n) - \delta$$

where $\delta = 0$ if $n \in (b)$ and $\delta = 1$ if $n \in (a)$.

(iv) $$\mathrm{taba}(n) = t(n) - 1 + \mathrm{aba}(n)$$

(v) $$\mathrm{tab}^2(n) = \mathrm{ta}(n) + \mathrm{ab}^2(n).$$

Proof. For the proof, we require the following identities (See Section 2):

$$b^2(n) = \mathrm{aba}(n) + 2$$

$$\mathrm{ab}^2(n) = b^2a(n) + 3 .$$

Since $t'(n + 1) - t'(n) \geq 4$ for all n, we have

$$t(b^2 - k) = tb^2 - k - 1 \quad \text{for} \quad k = 1, 2, 3$$

and

$$t(b^2 + k) = tb^2 + k \quad \text{for} \quad k = 1, 2, 3 .$$

To see (i), we have

$$\begin{aligned} \mathrm{tsta}(n) &= \mathrm{tab}(n) = t(b^2(n) - 2) \\ &= tb^2(n) - 3 \\ &= t(n) + b^2(n) - 3 \\ &= t(n) + (b^2(n) - 2) - 1 \\ &= t(n) + \mathrm{aba}(n) - 1 \\ &= t(n) + \mathrm{sta}(n) - 1 \quad . \end{aligned}$$

Statement (ii) follows similarly. Statements (iv) and (v) are simply restatements of (i) and (ii). For (iii), note that $\mathrm{ea}(n) = n$ and $\mathrm{eb}(n) = a(n)$ and apply (i) and (ii).

It is of some interest to determine for what values of n the difference $s(n + 1) - s(n)$ takes on the value 2 (or 3, or 5), and similarly for $t'(n + 1) - t'(n)$. The next theorem gives a complete description of this.

6.5. Theorem.

(i) $$s(\mathrm{tb}(n) + 1) = \mathrm{stb}(n) + 3$$

(ii) $$s(\mathrm{tas}(n) + 1) = \mathrm{stas}(n) + 3$$

(iii) $$s(tar(n) + 1) = star(n) + 5$$

(iv) $$s(t'(n) + 1) = st'(n) + 2$$

(v) $$t'(t'(n) + 1) = t't'(n) + 4$$

(vi) $$t'(tar(n) + 1) = t'tar(n) + 9$$

(vii) $$t'(tb(n) + 1) = t'tb(n) + 6$$

(viii) $$t'(tas(n) + 1) = t'tas(n) + 6 .$$

Proof. (i). $stb(n) = ab^2(n)$. Since $ab^2(n) + 3 = ab(b(n) + 1)$, s is a separated function, and $(ab) \subset (s)$, and we must have $ab(b(n) + 1) = s(tb(n) + 1)$.

(ii). $stas(n) + 3 = abas(n) + 3 = a^2(bs(n) + 1)$, and so $stas(n) + 3 = st'(n)$. This proves (ii).

(iii). $star(n) + 5 = abar(n) + 5 = ab(ar(n) + 1)$. We have seen that if $a^2(j) \in (s)$ and $ab(n) < a^2(j) < ab(n + 1)$, then we must have $n \in (as)$. This proves (iii).

(iv). $s(t'(n) + 1) = s(tas(n) + 2)$, and we have

$$stas(n) = abas(n)$$

$$stas(n) + 3 = a^2(bs(n) + 1) = st'(n)$$

$$stas(n) + 5 = ab(as(n) + 1) = s(t'(n) + 1) .$$

This proves (iv).

(v). $t't'(n) = ast'(n) + t'(n)$ and

$$t'(t'(n) + 1) = as(t'(n) + 1) + (t'(n) + 1)$$

$$= a(st'(n) + 2) + t'(n) + 1 .$$

Now $st'(n) \in (a^2)$, so $st'(n) + 1 \in (b)$ and we have

$$a(st'(n) + 2) = a(st'(n) + 1) + 1$$

$$= (ast'(n) + 2) + 1 = ast'(n) + 3 .$$

Then

$$t'(t'(n) + 1) = ast'(n) + t'(n) + 4 = t't'(n) + 4 ,$$

and (v) is proved.

(vi). $t'(tar(n) + 1) = as(tar(n) + 1) + tar(n) + 1$

$$= a(star(n) + 5) + tar(n) + 1$$

$$= a(abar(n) + 5) + tar(n) + 1 .$$

Now $abar(n) + 2 \in (b^2)$, so $abar(n) + 4 \in (b)$ and $abar(n)$, $abar(n) + 1$, $abar(n) + 3$ are all in (a), while $abar(n) + 2$ and $abar(n) + 4$ are in (b). Then by 2.18 and 2.19,

$$a(abar(n) + 5) = a(abar(n)) + 8$$

and we have

$$\begin{aligned} t'(tar(n) + 1) &= a(star(n)) + tar(n) + 9 \\ &= as(tar(n)) + tar(n) + 9 \\ &= t'tar(n) + 9 . \end{aligned}$$

In a similar manner one proves (vii) and (viii).

6.6. Corollary.

(i)
$$\begin{aligned} s(n) = 3 &+ 3\eta(tb < n) + 3\eta(tas < n) \\ &+ 5\eta(tar < n) + 2\eta(t' < n) \end{aligned}$$

(ii)
$$\begin{aligned} t'(n) = 5 &+ 4\eta(t' < n) + 6\eta(tb < n) \\ &+ 6\eta(tas < n) + 9\eta(tar < n) \end{aligned}$$

(iii)
$$2s(n) - t'(n) = 1 + \eta(tar < n) .$$

6.7. Theorem. $$b(n) = r(2n - \eta(u' \le n)) .$$

Proof.

$$\begin{aligned} (r) &= (b) \cup (a^2)/(a^2u') \\ &= (b) \cup (b - 1)/(bu' - 1) . \end{aligned}$$

Thus

$$\begin{aligned} \eta(r < b(n)) &= 2n - 1 - \eta(bu' - 1 < b(n)) \\ &= 2n - 1 - \eta(u' \le b(n)) . \end{aligned}$$

The result follows, since $b(n) \in (r)$.

6.8. Corollary. $bu'(n) = r(2u'(n) - n)$, and

$$b(u'(n) - 1) = r(2u'(n) - n - 1) = b^2s(n) .$$

Proof. The first statement is clear from Theorem 6.5. Since $bu'(n) - 1 = a^2u'(n) \in (s)$, it follows from the definition of r that $r(2u'(n) - n - 1) = b(u'(n) - 1)$. Since $u'(n) - 1 = bs(n)$, we have $r(2u'(n) - n - 1) = b^2s(n)$.

7. PROPERTIES OF OTHER RELATED FUNCTIONS

There are many additional functions which come about naturally from the consideration of relations between the functions r, s, t, t', u, u', and z, and the functions a, b, c, e. In this section we define the most interesting of these functions and list some of their properties.

7.1. Definitions.

(i). Since $(es) \subset (r)$, we define a strictly monotonic function p by: $es(n) = rp(n)$.

(ii). Since $(b) \subset (u)$, we define a strictly monotonic function v by

$$b(n) = uv(n).$$

(iii). Since $(es) = (b) \cup (abs + 1) \subset (u)$, we define a strictly monotonic function x by

$$es(n) = ux(n) .$$

(iv). Since $(ab)' = (a^2) \cup (b) \subset (u)$, we define a strictly monotonic function w by

$$(ab)' = (uw) .$$

(v). Since $(es) \subset (ab)' = (uw)$, we define a strictly monotonic function y by

$$es(n) = uwy(n) .$$

(vi). Since $(z) \subset (c')$ by Theorem 5.9 (vii), we define a strictly monotonic function λ by

$$c(n) = z'\lambda(n) .$$

(vii). Put $\sigma(n) = pt(n)$. Define a monotonic function τ by:

$$\tau(u(n)) = \sigma(u(n)) - 1$$
$$\tau(u'(n)) = \sigma(u'(n)) \quad .$$

(viii). Define a function K by:

$$K(n) = \eta(b < c(n)) .$$

7.2. <u>Theorem.</u> $pt(n) = 2n - \eta(u' \leq n)$

$$pt'(n) = 2as(n) + 1 - \eta(u' \leq as(n) + 1) .$$

<u>Proof</u>. Since $est(n) = b(n) = rpt(n)$, it follows from Theorem 6.8 that $pt(n) = 2n - \eta(u' \leq n)$. Recall that for all n, $tas(n) = t'(n) - 1$. Thus $t(as(n) + 1) = t'(n) + 1$. We know $rp(tas(n)) = b(as(n))$ and $rpt(as(n) + 1) = b(as(n) + 1)$. Since $as(n) + 1 \notin (u')$, it follows that $b(as(n) + 1) - 1 \in (r)$, and so

$$rp(t(as(n) + 1) - 1) = b(as(n) + 1) - 1$$

that is,

$$rpt'(n) = b(as(n) + 1) - 1.$$

By Theorem 6.8, we have

$$b(as(n) + 1) = r(2(as(n) + 1) - \eta(u' \leq as(n) + 1))$$

and so

$$b(as(n) + 1) - 1 = r(2(as(n) + 1) - \eta(u' \leq as(n) + 1) - 1)$$

which gives

$$pt'(n) = 2as(n) + 1 - \eta(u' \leq as(n) + 1)$$

and this completes the proof.

7.3. Theorem. $v(n) = b(n) - \eta(s < n)$.

Proof. We will show that, for all n,

$$v(b(n) + 1) = vb(n) + 2 \tag{7.4}$$

$$v(s(n) + 1) = vs(n) + 2 \tag{7.5}$$

$$v(a(n) + 1) = va(n) + 3 \quad \text{if} \quad a(n) \notin (s). \tag{7.6}$$

Then since $v(1) = 2$, we have

$$\begin{aligned} v(n) &= 2 + 2\eta(b < n) + 2\eta(s < n) \\ &\quad + 3\eta(a < n) - 3\eta(s < n) \\ &= 2 + 2(n-1) + \eta(a < n) - \eta(s < n) \\ &= 2n + (a(n) - n) - \eta(s < n) \\ &= b(n) - \eta(s < n). \end{aligned} \tag{7.7}$$

We first prove (7.4). Since $b(n) = uv(n)$, we have $b^2(n) = uvb(n)$. Now $b^2(n) + 1 \in (u)$, since $(u') = (bs + 1)$ and $b^2(n) + 1 \neq bs(j) + 1$ for any n, j because $(s) \subset (a)$. Thus

$$u(vb(n) + 1) = uvb(n) + 1$$

and

$$u(vb(n) + 2) = uvb(n) + 2\ .$$

Since $uvb(n) = b^2(n)$ and $uvb(n) + 2 = b^2(n) + 2 = b(b(n) + 1) = uv(b(n) + 1)$, it must be that $v(b(n) + 1) = vb(n) + 2$, as required. To see (7.5), note that $uvs(n) = bs(n)$, and $uvs(n) + 1 = u'(n)$. Then $u(vs(n) + 1) = uvs(n) + 2$ and $u(vs(n) + 2) = bs(n) + 3$. But $bs(n) + 3 = b(s(n) + 1)$ by (2.15), that is,

$$u(vs(n) + 2) = uv(s(n) + 1)$$

and so

$$vs(n) + 2 = v(s(n) + 1)\ .$$

As for (7.6), suppose $a(n) \notin (s)$. Then we show that none of $uva(n) + 1$, $uva(n) + 2$, $uva(n) + 3$ are in (u'). Since $a(n) \notin (s)$, $uva(n) + 1 = ba(n) + 1 \notin (u')$, since $(u') = (bs + 1)$. Also, $uva(n) + 2 = ba(n) + 2 = ab(n) + 1$ by (2.13) and clearly $ab(n) + 1 \neq bs(j) + 1$ for any n, j. Finally $uva(n) + 3 = ba(n) + 3 = b(a(n) + 1) \notin (u')$ and we have $uva(n) + 3 = uv(a(n) + 1)$. Thus $v(a(n) + 1) = va(n) + 3$ for $a(n) \notin (s)$. This completes the proof.

7.8. Theorem.

$$\begin{cases} xt(n) = v(n) \\ xt'(n) = au'(n) - \eta(s < as(n)) . \end{cases}$$

Proof. From the definition, $est(n) = uxt(n)$. Since $st(n) = ab(n)$, we have $b(n) = uxt(n)$. On the other hand, $uv(n) = b(n)$, and so $xt(n) = v(n)$.

For the second statement, we require the fact that $tas(n) + 1 = t'(n)$ (this follows from Theorem 2.2 and the fact that $t'(n) - n = as(n)$). Then

$$xtas(n) = vas(n) = bas(n) - \eta(s < as(n))$$

and also

$$uxtas(n) = uvas(n) = bas(n)$$

$$uxtas(n) + 1 = bas(n) + 1 = abs(n)$$

$$uxtas(n) + 2 = abs(n) + 1 = au'(n) .$$

Since $est'(n) = uxt'(n)$ and $est'(n) = au'(n)$, we must have $uxtas(n) + 2 = uxt'(n)$. Since $uxtas(n) + 1 = abs(n) \in (u)$, we have

$$uxt'(n) = u(xtas(n) + 2) ,$$

and so

$$\begin{aligned} xt'(n) &= xtas(n) + 2 \\ &= bas(n) + 2 - \eta(s < as(n)) \\ &= au'(n) - \eta(s < as(n)) . \end{aligned}$$

This completes the proof.

Table 2

n	1	2	3	4	5	6	7	30	50
$t'(n)$	5	14	20	29	35	39	45	207	342
$xt'(n)$	11	30	42	60	72	79	91	416	686

7.9. Theorem. $w(2n) = v(n)$ and $w(2n - 1) = v(n) - 1$.

Proof. Since $(ab)' = (b) \cup (b - 1)$ and $(ab)' = (uw)$ it is clear that $uw(2n) = b(n)$

and $uw(2n - 1) = b(n) - 1$. On the other hand, $b(n) = uv(n)$, and so $w(2n) = v(n)$. Since $b(n) - 1 \in (n)$ for all n, we have $b(n) - 1 = uv(n) - 1 = u(v(n) - 1) = uw(2n - 1)$, and $w(2n - 1) = v(n) - 1$.

7.10. Theorem. $w(n) = n + \eta(2a^2u < n)$.

Proof. We require the fact that

$$(u') = (abes) = (ab^2) \cup (abau') .$$

It follows that $(u) = (ab)' \cup (abau)$.

Note that $abau(j) = ba^2u(j) + 1$. Then

$$ba^2u(j) = uv(a^2u(j)) = uw(2a^2u(j)) .$$

Since $abau(j) \notin (uw)$ and

$$abau(j) + 1 = b^2u(j) - 1 \in (uw) ,$$

we have

$$w(2a^2u(j) + 1) = w(2a^2u(j)) + 2 . \tag{7.11}$$

On the other hand, since $(u) = (ab)' \cup (abau)$, if n is not of the form $2a^2u(j)$ for some j, then $w(n + 1) = w(n) + 1$. The theorem follows.

7.12. Corollary. $w'(n) = 2a^2u(n) + n$.

Proof. By Theorem 7.10, $w(2a^2u(n)) = 2a^2u(n) + n - 1$ and

$$w(2a^2u(n) + 1) = 2a^2u(n) + n + 1.$$

For all n not of the form $n = 2a^2u(j)$, we have $w(n + 1) = w(n) + 1$ and thus

$$(w') = \{2a^2u(n) + n \mid n = 1, 2, 3, \cdots\} .$$

As usual, w' is taken to be monotonic, and the theorem follows.

7.13. Corollary.

(i) $$b(n) = uv(n)$$

(ii) $$b(n) - 1 = uv'(v(n) - n)$$

(iii) $$b(n) - 1 = u(v(n) - 1)$$

(iv) $$abau(n) = uv'(a^2u(n) + n) .$$

Proof. (i) is evident from the definition of v. Statements (ii) and (iii) follow from the fact that $b(n) - 1 = uv(n) - 1 = u(v(n) - 1)$, and then by Theorem 2.2 $v(n) - 1 = v'(v(n) - n)$ since v is a separated function. To see (iv), note that $abau(n) = ba^2u(n) + 1$. Then

$$\begin{aligned} abau(n) &= uv(a^2u(n)) + 1 \\ &= u(v(a^2u(n)) + 1) . \end{aligned}$$

By Corollary 2.3 we have

$$v'(v(a^2u(n)) - a^2u(n) + 1) = v(a^2u(n)) + 1 .$$

By Theorem 7.9 we have

$$\begin{aligned} v(a^2u(n)) + 1 &= v'(w(2a^2u(n)) - a^2u(n) + 1) \\ &= v'(2a^2u(n) + n - 1 - a^2u(n) + 1) \\ &= v'(a^2u(n) + n) . \end{aligned}$$

This completes the proof.

7.14. Theorem. $yt(n) = 2n$ and $yt'(n) = 2eu'(n) - 1$.

Proof. By the definition of y, we have

$$est(n) = b(n) = uwyt(n) = uv(n) .$$

We know by Theorem 7.9 that $v(n) = w(2n)$, and so $wyt(n) = w(2n)$, that is, $yt(n) = 2n$. Secondly, $est'(n) = au'(n) = a(bs(n) + 1) = b(as(n) + 1) - 1 = uwyt'(n)$. Now

$$b(as(n) + 1) = uv(as(n) + 1) = uw(2as(n) + 2)$$

and since

$$b(as(n) + 1) - 1 \in (uw) ,$$

we have

$$w(2as(n) + 1) = wyt'(n) .$$

Now $eu'(n) = as(n) + 1$, and so

$$2as(n) + 1 = 2eu'(n) - 1 = yt'(n) .$$

This completes the proof.

7.15. Theorem. $\lambda(n) = 3n - \eta(u' \le n)$.

Proof. We show that if $n = bs(j)$ for some j, then $\lambda(n + 1) = \lambda(n) + 2$, and for all other n we have $\lambda(n + 1) = \lambda(n) + 3$. Then clearly

$$\begin{aligned} \lambda(n) &= 3n - \eta(bs < n) \\ &= 3n - \eta(bs + 1 \le n) \qquad (7.16) \\ &= 3n - \eta(u' \le n) . \end{aligned}$$

In the proof, we shall require the fact that $zt(n) = ca(n) + 1$ and $zt'(n) = cbs(n) + 1$.

Case 1. $n = a(j)$. Then $c(n) = ca(j) = z'\lambda a(j)$, and $ca(j) + 1 = zt(j)$. By 2.18,

$$c(a(j) + 1) = ca(j) + 4 = z'\lambda(a(j) + 1) .$$

Since $(z) \subset (c + 1)$, we have $ca(j) + 2$ and $ca(j) + 3 \in (z')$, and so $\lambda(a(j) + 1) = \lambda a(j) + 3$.

Case 2. $n = b(j)$ where $j \notin (s)$. Then $b(j) = a^2(j) + 1$, and $zta(j) = ca^2(j) + 1$. Also $z'\lambda a^2(j) = ca^2(j)$ and $z'\lambda b(j) = cb(j) = ca^2(j) + 4$. Again $ca^2(j) + 2$ and $ca^2(j) + 3 \in (z')$, and we have $\lambda b(j) = \lambda(b(j) - 1) + 3$. Since $j \notin (s)$, $cb(n) + 1 \in (z')$, and so are $cb(n) + 2$ and $cb(n) + 3$. By 2.19, $cb(n) + 3 = c(b(n) + 1) = z'\lambda(b(n) + 1)$. Thus $\lambda(b(n) + 1) = \lambda b(j) + 3$.

Case 3. $n = bs(j)$. Then $cbs(j) + 1 = zt'(j)$, $cbs(j) = z'\lambda bs(j)$, and $cbs(j) + 3 = c(bs(j) + 1) = z'\lambda(bs(j) + 1)$. Since $cbs(j) + 2 \in (z')$, we have $\lambda(bs(j) + 1) = \lambda bs(j) + 2$. This completes the proof.

The function τ is of interest for the following reasons. Recall that the function r satisfies $(r) = (b) \cup (a^2u)$. Thus we get $(er) = (a) \cup (au)$, and er is not strictly monotonic. In particular, if $r(k) = b(n) - 1$ and $r(k + 1) = b(n)$, then $er(k) = er(k + 1) = a(n)$; $\sigma(n) = k + 1$ and $\tau(n) = k$. On the other hand, if $r(k) = bu'(n)$, then $er(k - 1) \neq er(k)$ and $er(k + 1) \neq er(k)$; that is, the value $er(k) = au'(n)$ is not repeated, and we have

$$ptu'(n) = \sigma u'(n) = \tau u'(n) .$$

7.17. <u>Theorem</u>. We have, for all n,

$$\begin{aligned} r\sigma(n) - \sigma(n) &= b(n) - \sigma(n) \\ &= r\tau(n) - \tau(n) . \end{aligned}$$

<u>Proof</u>. We need only note that if $\tau(n) = \sigma(n) - 1$, then $r\tau(n) = r\sigma(n) - 1$.

7.18. <u>Theorem</u>. The function K defined by $K(n) = \eta(b < c(n))$ is strictly monotonic. Furthermore, we have

$$(7.19)\quad \begin{array}{ll} \text{(i)} & Kb(n) = c(n) - 1 \\ \text{(ii)} & Ks(n) = z(n) - 1 \\ \text{(iii)} & Ka^2u(n) = cu(n) - 2 \\ \text{(iv)} & (K') = (z) \cup (cbr) . \end{array}$$

<u>Proof</u>. Since $c(n + 1) - c(n) \geq 3$, and of the three consecutive integers $c(n)$, $c(n) + 1$, $c(n) + 2$ at least one must be in (b), it is evident that $K(n + 1) \geq K(n) + 1$, so that K is strictly monotonic. To see (i), we have

$$\eta(b < cbr(n)) = \eta(b < bcr(n)) = cr(n) - 1$$

$$\eta(b < cbs(n)) = \eta(b < bcs(n) - 1) = cs(n) - 1 .$$

For (ii), $\eta(b < cs(n)) = \eta(b < bz(n)) = z(n) - 1$. For (iii), we have

$$\eta(b < ca^2u(n)) = ca^2u(n) - 1 - \eta(a < ca^2u(n)) .$$

By Theorem 3.6, we know $ca^2u(n) = a(cau(n) - 1)$, and so

$$\begin{aligned}
Ka^2u(n) &= ca^2u(n) - 1 - \eta(a < a(cau(n) - 1)) \\
&= ca^2u(n) - 1 - [cau(n) - 2] \\
&= ca^2u(n) - cau(n) + 1 \\
&= a^3u(n) + 2a^2u(n) - bau(n) - au(n) + 1 \\
&= [bau(n) - 1] + 2a^2u(n) - au(n) - [bau(n) - 1] \\
&= a^2u(n) + [bu(n) - 1] - au(n) \\
&= a^2u(n) + u(n) - 1 \\
&= bu(n) - 1 + u(n) - 1 \\
&= cu(n) - 2 .
\end{aligned}$$

Finally to see (iv), first we note that

$$Kb(n) = c(n) - 1 \notin (z) \quad \text{and} \quad Ks(n) = z(n) - 1 \notin (z).$$

We show that $cu(n) - 2 \notin (z)$. If $u(n) - 1 \in (a)$, then $c(u(n) - 1) + 4 = cu(n)$. Since $cu(n) - 2 \neq c(j) + 1$ for any j, then in this case $cu(n) - 2 \notin (z)$.

Suppose $u(n) - 1 = b(j)$ for some j. Then (since $(u') = (bs + 1)$) we must have $j \in (r)$, say $j = r(k)$ for some k. Then $c(u(n) - 1) + 3 = cu(n)$, and $cu(n) - 2 = cbr(k) + 1$ and this is not a value of z.

Now from (i), (ii) and (iii) we have, for all n,

$$\begin{aligned}
cb(n) + 2 &\in (c - 1) \subset (K) \\
ca(n) + 3 &\in (c - 1) \subset (K) \\
cs(n) + 2 = c(s(n) + 1) - 2 &\in (cu - 2) \subset (K) \\
ca^2u(n) + 2 = c(bu(n) - 2) &\in (cu - 2) \subset (K) \\
ca(n) = zt(n) - 1 &\in (K) \\
cbs(n) = zt'(n) - 1 &\in (K) \\
cbr(n) + 1 \in (cu - 2) &\in (K)
\end{aligned}$$

while $cbr(n) \in (K')$, $ca(n) + 1 \in (K')$, and $cbs(n) + 1 \in (K')$. Thus (iv) holds.

7.20. Theorem.

(i) $$K'br(n) = cbr(n)$$

(ii) $$K'a(n) = ca(n) + 1$$

(iii) $$K'bs(n) = cbs(n) + 1 .$$

Proof. This is evident from the fact that

$$(K') = (cbr) \cup (ca + 1) \cup (cbs + 1) .$$

7.21. Theorem. The following are equivalent.

(a) $$K'(j) = z(n)$$

(b) $$n = j - \eta(br < j)$$

(c) $$n = c(j) + 1 - \lambda(j) .$$

Proof. Since $(K') = (z) \cup (cbr)$, it is evident that (a) and (b) are equivalent. To see that (b) and (c) are equivalent, we show that, for all j,

$$j - \eta(br < j) = c(j) + 1 - \lambda(j) . \tag{7.22}$$

Recall that $\lambda(j) = 3j - \eta(u' \leq j)$ (Theorem 7.15). That is,

$$\begin{aligned} \lambda(j) &= 3j - \eta(bs + 1 \leq j) \\ &= 3j - \eta(bs < j) \quad . \end{aligned}$$

Now

$$\begin{aligned} \eta(bs < j) &= \eta(b < j) - \eta(br < j) \\ &= j - 1 - \eta(a < j) - \eta(br < j) \\ &= j - 1 - (a(j) - j) - \eta(br < j) \\ &= 2j - 1 - a(j) - \eta(br < j) \quad . \end{aligned}$$

Thus

$$\begin{aligned} \lambda(j) &= 3j - [2j - 1 - a(j) - \eta(br < j)] \\ &= j + 1 + a(j) + \eta(br < j) \\ &= b(j) + 1 + \eta(br < j) \quad . \end{aligned}$$

Now $c(j) + 1 - \lambda(j) = j - \eta(br < j)$, and this completes the proof.

7.23 Theorem.

(i) $$tar(n) = br(n) - n$$

(ii) $$K'(br(n) - 1) = ztar(n)$$

(iii) $$cbr(n) = z'(b^2r(n) + n) .$$

Proof. To see (i) we use Theorems 7.20 and 7.21. We have $K'br(n) = cbr(n)$ and

$$\begin{aligned} K'(br(n) + 1) &= c(br(n) + 1) + 1 \\ &= c(a^2r(n) + 2) + 1 \\ &= ca(ar(n) + 1) + 1 \\ &= zt(ar(n) + 1) . \end{aligned}$$

Then by Theorem 7.20(b) we have

$$\begin{aligned} t(ar(n) + 1) &= (br(n) + 1) - \eta(br < br(n) + 1) \\ &= br(n) + 1 - n. \end{aligned}$$

Since $t(n) - t(n - 1) = 1$ unless $n - 1 \in (as)$, we have $t(ar(n)) = br(n) - n$. This proves (i).

To see (ii), we have $K'(br(n) + 1) = zt(ar(n) + 1)$ and $K'(br(n)) = cbr(n)$. Thus $K'(br(n) - 1) \in (z)$ and by (i), $K'(br(n) - 1) = ztar(n)$.

Finally, (iii) follows from the fact that

$$\begin{aligned} ztar(n) - 1 &= z'(ztar(n) - tar(n)) \\ &= z'(ca^2r(n) + 1 - br(n) + n) \\ &= z'(cbr(n) - 4 + 1 - br(n) + n) \\ &= z'(b^2r(n) - 3 + n) , . \end{aligned}$$

Now $ztar(n) + k \in (z')$ for $k = 1, 2, 3$ and so

$$z'(b^2r(n) + n) = ztar(n) + 3 = ca^2r(n) + 4 = cbr(n) .$$

This completes the proof.

7.24. Corollary. $\lambda br(n) = b^2r(n) + n$.

Proof. By definition, $cbr(n) = z'\lambda br(n)$, and the result follows from Theorem 7.23 (iii).

7.25. Theorem. p' is separated.

Proof. We show that $p(n + 1) - p(n) = 1$ or 2.

Case 1. $n = t(j)$ and $n + 1 = t(j + 1)$. Then $p(n) = pt(j) = 2j - \eta(u' \le j)$ by Theorem 7.2, and $p(n + 1) = pt(j + 1) = 2j + 2 - \eta(u' \le j + 1)$. Then $p(n + 1) - p(n) = 2 - \delta$ where $\delta = 1$ if $j + 1 \in (u')$ and $\delta = 0$ if $j + 1 \notin (u')$.

Case 2. $n = t(j)$, and $n + 1 = t'(k)$. Then $j = as(k)$ and we have

$$p(n) = pt(as(k)) = 2as(k) - \eta(u' \le as(k))$$

$$p(n+1) = pt'(k) = 2as(k) + 1 - \eta(u' \le as(k) + 1) .$$

Since $as(k) + 1 \in (b)$, it follows that $as(k) + 1 \notin (u')$ and so $p(n + 1) - p(n) = 1$ in this case.

Case 3. $n = t'(j)$ and $n + 1 = t(k)$. Then $k = as(j) + 1$ and we have

$$p(n) = 2as(j) + 1 - \eta(u' \le as(j) + 1)$$

$$p(n+1) = 2as(j) + 2 - \eta(a' \le as(j) + 1)$$

and

$$p(n+1) - p(n) = 1.$$

This completes the proof.

7.26. Theorem. σ' is separated.

Proof. We show that $\sigma(n + 1) - \sigma(n) \le 2$. By Theorem 7.2, we have

$$\sigma(n) = pt(n) = 2n - \eta(u' \le n).$$

Then

$$\sigma(n+1) - \sigma(n) = 2 - \delta$$

where $\delta = 1$ if $n + 1 \in (u)'$ and $\delta = 0$ if $n + 1 \notin (u')$. This completes the proof.

7.27. Theorem. $\sigma(n) = \lambda(n) - n$.

Proof. This is evident from the fact that $\sigma(n) = 2n - \eta(u' \le n)$, while (by Theorem 7.15) $\lambda(n) = 3n - \eta(u' \le n)$.

7.28. Theorem. $(\tau) = (\sigma') \cup (\sigma u')$.

Proof. If $n > 1$, we have $\sigma u(n) - \sigma(u(n) - 1) = 2$, as above, and thus for all n, $\sigma u(n) - 1 \in (\sigma')$. On the other hand, $\sigma u'(n) - 1 = \sigma(u'(n) - 1)$. Since for all n, $\sigma(n + 1) - \sigma(n) \le 2$, we know $(\sigma') \subset (\sigma - 1)$. It follows that $\tau u(n) = \sigma u(n) - 1 \in (\sigma')$ and further $(\tau u) = (\sigma')$. Since $\tau u'(n) = \sigma u'(n)$, the proof is complete.

7.29. Theorem. (a) $\sigma u(n) = u(n) + n$

(b) $\tau u(n) = u(n) + n - 1$

(c) $\sigma u'(n) = 2u'(n) - n = \tau u'(n)$.

Proof. (a) We have $\sigma(n) = 2n - \eta(u' \le n)$ for all n. Then $\sigma u(n) = 2u(n) - \eta(u' \le u(n))$. By Theorem 2.1, we know $u(n) = n + \eta(u' \le u(n))$. Thus

$$\sigma(u(n)) = 2u(n) - (u(n) - n) = u(n) + n .$$

Statement (b) follows from (a) and the fact that $\tau u(n) = \sigma u(n) - 1$. To see (c), we have

$$\sigma u'(n) = 2u'(n) - \eta(u' \leq u'(n)) .$$

Then $\sigma u'(n) = 2u'(n) - n$.

7.30. Theorem. (a) $$\tau'(n) = \sigma u(n)$$

(b) $$\sigma'(n) = \tau' u(n) .$$

Proof. First, $(\tau) = (\sigma u - 1) \cup (\sigma u')$, in particular $\tau u(n) = \sigma u(n) - 1$ and $\tau u'(n) = \sigma u'(n)$. Since $\sigma u'(n) - 1 = \sigma(u'(n) - 1)$, we have $(\tau') = (\sigma u)$, and (a) follows. Statement (b) is proved similarly.

7.31. Theorem. For each integer $n > 0$, put $J_n = \eta(eu'(j) - j < n)$. Then

(a) y' is a separated function

(b) $$y'(eu'(n) - n) = 2(eu'(n) - 1) - 1$$

(c) $$y'(eu'(n) - n + 1) = y'(eu'(n) - n) + 4$$

(d) If j is not of the form $eu'(n) - n$, then $y'(j + 1) = y'(j) + 2$.

(e) $$y'(n) = 2(n + J_n) - 1 .$$

(f) $$y'(n - \eta(eu' < n)) = 2n - 1.$$

Proof. By Theorem 7.14, we know $yt(n) = 2n$ and $yt'(n) = 2eu'(n) - 1$. It follows that

$$(y') = (2n - 1)/(2eu' - 1)$$

and it is evident that y' is separated. Clearly if $y'(j) = 2eu'(n) - 3$ for some n, then $y'(j + 1) = y'(j) + 4$, and otherwise $y'(j) = y'(j) + 2$. We prove statement (f). Clearly if $y'(j) = 2n - 1$, then since

$$\{y'(k) : y'(k) \leq 2n - 1\} = \{2j - 1 : j = 1, 2, \cdots, n \text{ and } j \neq eu'\}$$

we must have $j = n - \eta(2eu' - 1 < 2n - 1) = n - \eta(eu' < n)$. To see (b), it follows from (f) that

$$y'((eu'(n) - 1) - (n - 1)) = 2(eu'(n) - 1) - 1 = 2eu'(n) - 3 .$$

Statements (c) and (d) are evident from (f) and the fact that numbers of the form $2eu'(n) - 1$ are the only odd numbers in (y). To see (e), suppose that

$$eu'(j) - j < n \leq eu'(j + 1) - (j + 1) ,$$

say $n = eu'(j) - j + k$. Then

$$\begin{aligned} y'(n) &= y'(eu'(j) - j) + 2 + 2k \\ &= 2eu'(j) - 3 + 2 + 2k \\ &= 2(eu'(j) + k) - 1 \\ &= 2([eu'(j) - j] + [k + j]) - 1 \\ &= 2n + 2j - 1 \\ &= 2n + 2\eta(eu'(m) - m < n) - 1 \\ &= 2(n + J_n) - 1 . \end{aligned}$$

This completes the proof.

8. ASYMPTOTIC PROPERTIES

In this section, we show that the function s is asymptotic to the function c. In particular, $s(n) \sim (\alpha + 2)n$. Similar asymptotic results follow at once for all the auxiliary functions introduced so far.

8.1. Theorem. $n \in (s)$ if and only if

$$\frac{\alpha}{\sqrt{5}} \le \{\alpha n\} < 1 .$$

Proof. Recall that $n \in (s)$ if and only if $ac(n) = ca(n) + 2$. By definition, we have

$$ca(n) = [\alpha[\alpha n]] + 2[\alpha n] \tag{8.2}$$

$$ac(n) = [\alpha([\alpha n] + 2n)] . \tag{8.3}$$

Put

$$\begin{aligned} \alpha n &= m + \epsilon_1 \qquad (0 < \epsilon_1 < 1) \\ \alpha m &= k + \epsilon_2 \qquad (0 < \epsilon_2 < 1) . \end{aligned}$$

By 2.20, we have $\epsilon_2 = 1 + (1 - \alpha)\epsilon_1$.

Thus we have

$$ca(n) = [\alpha m] + 2m = k + 2m \tag{8.4}$$

and

$$\begin{aligned} ac(n) &= [\alpha(m + 2n)] = [\alpha m + 2\alpha n] \\ &= [k + \epsilon_2 + 2m + 2\epsilon_1] \\ &= k + 2m + [\epsilon_2 + 2\epsilon_1] . \end{aligned} \tag{8.5}$$

Then $n \in (s)$ if and only if $[\epsilon_2 + 2\epsilon_1] = 2$. Now

$$\begin{aligned}\epsilon_2 + 2\epsilon_1 &= 1 + (1 - \alpha)\epsilon_1 + 2\epsilon_1 \\ &= 1 + (3 - \alpha)\epsilon_1\end{aligned}$$

and so $n \in (s)$ if and only if

$$1 \le (3 - \alpha)\epsilon_1 < 2 , \tag{8.6}$$

that is, if and only if

$$\frac{1}{3 - \alpha} \le \epsilon_1 < \frac{2}{3 - \alpha} .$$

Note that

$$\frac{2}{3 - \alpha} > 1$$

and $\epsilon_1 < 1$, so this reduces to:

$$n \in (s) \leftrightarrows \frac{1}{3 - \alpha} \le \{\alpha n\} < 1 . \tag{8.7}$$

Since $\alpha = \frac{1}{2}(1 + \sqrt{5})$, it is easy to show that

$$\frac{1}{3 - \alpha} = \frac{\alpha}{\sqrt{5}} ,$$

and this completes the proof.

8.8. Theorem. We have $s(n) \sim c(n)$.

Proof. We require the fact that the values of $\{\alpha n\}$ are uniformly distributed in $(0,1)$ (see [5, Th. 6.3]). It follows from the previous theorem that

$$\eta(r < n) \sim \frac{\alpha}{\sqrt{5}} n \tag{8.9}$$

$$\eta(s < n) \sim \left(1 - \frac{\alpha}{\sqrt{5}}\right) n .$$

Since $s(n) = n + \eta(r < s(n))$, we have

$$s(n) \sim n + \frac{\alpha}{\sqrt{5}} s(n) , \quad s(n) \sim \frac{n}{1 - \frac{\alpha}{\sqrt{5}}} , \tag{8.10}$$

$$\frac{s(n)}{n} \to \alpha + 2 ,$$

On the other hand, $c(n) = [(\alpha + 2)n]$, and it follows that $s(n) \sim c(n)$. This completes the proof.

8.11. Corollary. (i) $t'(n) \sim b^2(n)$

(ii) $u'(n) \sim bc(n) + 1$.

Proof. This is evident from the fact that $t'(n) = as(u) + n$ and $u'(n) = bs(n) + 1$. The result follows from Theorem 8.8.

Clearly, similar results could be stated for most of the functions considered previously, since these were defined in terms of a, b, and s.

Recall (Definition 2.7) if

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and $\neq 0$ we set

$$c_f = \lim_{n\to\infty} \frac{f(n)}{n} .$$

In view of Theorem 8.8, we have

$$\lim_{n\to\infty} \frac{s(n)}{n}$$

exists, and is not 0. Then all of the functions introduced so far also satisfy

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists, since they are defined in terms of s, a, b, c, and e. Then we have the following:

8.12. Theorem. We have

(i) $$c_a = \alpha, \quad c_b = \alpha + 1$$

(ii) $$c_c = \alpha + 2, \quad c_{c'} = 3 - \alpha$$

(iii) $$c_s = \alpha + 2, \quad c_r = 3 - \alpha$$

(iv) $$c_w = \frac{\alpha + 1}{\sqrt{5}} = \frac{\alpha^2}{\sqrt{5}}, \quad c_{w'} = 3\alpha + 1$$

(v) $$c_u = \frac{\sqrt{5}}{2}, \quad c_{u'} = 4\alpha + 3$$

(vi) $$c_v = \frac{3 + \sqrt{5}}{\sqrt{5}}, \quad c_{v'} = \frac{2\alpha^2}{3}$$

(vii) $$c_t = \frac{2\alpha + 1}{\alpha + 2}, \quad c_{t'} = \alpha^4$$

(viii) $$c_z = 5, \quad c_{z'} = 5/4$$

(ix) $$c_\sigma = \frac{2 + \sqrt{5}}{\sqrt{5}}, \qquad c_{\sigma'} = \frac{1}{2} + \alpha$$

(x) $$c_\tau = \frac{2 + \sqrt{5}}{\sqrt{5}}, \qquad c_{\tau'} = \frac{1}{2} + \alpha$$

(xi) $$c_\lambda = \frac{4}{5}(\alpha + 2) = \frac{4\alpha}{\sqrt{5}}, \qquad c_{\lambda'} = \frac{4(\alpha + 2)}{4\alpha + 3}$$

9. CONJECTURES

Many of the results in the preceding sections were first arrived at empirically, using extensive numerical data. We list here some conjectures, also arrived at "by inspection," which remain unproved.

(9.1) $$tt'(n) = t(n) + t'(n) \qquad \text{except for} \quad n \in (bs)$$

(9.2) $$ts(n) = et'(n) \qquad \text{except for} \quad n \in (bs)$$

(9.3) $$t'b(n) \in (a^2) \qquad \text{except for} \quad n \in (s)$$

(9.4) $$|xt'(n) - 2t'(n)| \leq 2 .$$

It has been shown that the functions s and c are asymptotic (Theorem 8.8) and also that there are infinitely many values of n for which $s(n) = c(n)$ (Theorem 5.12). It remains an open question whether the difference $|s - c|$ is bounded. More generally, what is the smallest $H \geq 0$ such that

$$|s(n) - c(n)| = 0(n^{H+\epsilon}) \qquad (\epsilon > 0) .$$

Numerically, we have for $n \leq 101$, $|s(n) - c(n)| \leq 5$.

Of course any such result for s and c implies corresponding results for other pairs (e.g., for t' and b^2, or for u' and bc).

We could define another function, say g(n), where g(n) is the n^{th} integer k such that $s(k) = c(k)$. It is evident from Theorem 5.12 that if $n \in (g)$, then $t'(n) + 1 \in (g)$. The numerical data indicate a possibility that if $n \in (g)$, then also $t'(n) + 4 \in (g)$, but this remains unproved.

Finally, it would be very interesting to have an "infinite union" formula for the function t', similar to that for s given by Theorem 5.15.

10. SUMMARY

1. $$cbs(n) = bcs(n) - 1 = a^2cs(n)$$

2. $$cbr(n) = bcr(n)$$

3. $$cs(n) = bz(n)$$

4. $$cas(n) = a(cs(n) - 1) = acs(n) - 2$$

5. $$car(n) = acr(n) - 1$$

6. $$ca(n) = \begin{cases} a(c(n) - 1) & \text{if } ca(n) \in (a) \\ a(c(n) - 1) + 1 & \text{if } ca(n) \in (b) \end{cases}$$

7. $$cabr(n) = bacr(n)$$

8. $$cabs(n) = bacs(n) - 2$$

9. $$cab(n) = b(ca(n) + 1)$$

10. $$cb^2(n) = bcb(n)$$

11. $$ecb(n) = ac(n)$$

12. $$eca(n) = c(n) - 1$$

13. $$\begin{cases} c'b(n) = c(n) - 1 \\ c'(b(n) - 1) = c(n) - 2 \\ c'(b(n) + 1) = c(n) + 1 \\ c'(b(n) + 2) = c(n) + 2 \end{cases}$$

14. $$c(n) + c'(n) = 5n - 1$$

15. $$c'(n) = n + \eta(b \lessdot n)$$

16. $$c'a(n) = c'(a(n) + 1) - 1$$

17. $$c'ab(n) = ca(n) + 1$$

18. $$ec(n) = c'\phi(n)$$

19. $$\phi a(n) = b(n)$$

20. $$\phi br(n) = abr(n)$$

21. $$\phi bs(n) = abs(n) + 1$$

22. $$\phi s(n) = bes(n) = as(n) + 1$$

23. $$\phi'(n) = \begin{cases} a^2(n), & n \in (as + 1) \\ a(a(n) - 1), & n \in (as + 1) \end{cases}$$

24. $$\phi a(n) + \phi' a(n) = \phi\phi' a(n) - 1$$

25. $$\psi(n) = e\phi'(n)$$

26. $$\psi' r(n) = br(n)$$

27. $$\psi' s(n) = bs(n) + 1 = u'(n)$$

28. $$\phi br(n) - \phi' ar(n) = 2$$

29. $$\phi'\phi s(n) = abs(n)$$

30. $$\phi\phi' s(n) = abs(n) - 1$$

31. $$\phi\phi' r(n) = bar(n)$$

32. $$\phi s(n) + c' s(n) = 3s(n)$$

33. $$ab(n) = st(n)$$

34. $$st'(n) = a^2u'(n) = a^2(bs(n) + 1)$$

35. $$z(n) = c' s(n)$$

36. $$es(n) = rp(n)$$

37. $$aes(n) = s(n) = arp(n)$$

38. $$b(n) = rpt(n)$$

39. $$pt(n) = \sigma(n)$$

40. $$b(n) = r\sigma(n)$$

41. $$b(n) = uv(n)$$

42. $$u'(n) = bs(n) + 1$$

43. $$t'(n) = as(n) + n$$

44. $$tas(n) = t'(n) - 1$$

45. $$z(n) = c(z(n) - s(n)) + 1$$

46. $$s(n) = b(z(n) - s(n)) + 1$$

47. $$az(n) - as(n) = a(z(n) - s(n)) + 1$$

48. $$zt(n) = ca(n) + 1$$

49. $$zt'(n) = cbs(n) + 1 = b^2z(n)$$

50. $$s(n) = c(n) \Longleftrightarrow t'(n) = b^2(n)$$

51. $$zt(n) - st(n) = a(n)$$

52. $$zt'(n) - st'(n) = bs(n)$$

53. $$bas(n) = rp(t'(n) - 1)$$

54. $$az(n) - as(n) = es(n)$$

55. $$z(n) + es(n) = 2s(n)$$

56. $$u'(n) + z(n) = 4s(n)$$

57. $$tb^2(n) = t(n) + b^2(n)$$

58. $$t't(n) = t(n) + b^2(n) - 1 = tb^2(n) - 1$$

59. $$t't(n) = a^2b(n) + t(n)$$

60. $$tab(n) = te(n) + ab(n) - \delta$$

where $\delta = 0$ if $n \in (b)$ and $\delta = 1$ if $n \in (a)$

61. $tst'(n) = ts(n) + bu'(n) - 1$

62. $tst'(n) - st'(n) = ts(n)$

63. $r(2u'(n) - n) = bu'(n) = st'(n) + 1$

64. $b(n) = r(2n - \eta(u' \leq n))$

65. $c^2r(n) = 5br(n)$

66. $c^2s(n) = 5bs(n) + 1$

67. $es(n) = ux(n)$

68. $\tau u(n) = \sigma u(n) - 1$

69. $\tau u'(n) = \sigma u'(n)$

70. $K(n) = \eta(b \leq c(n))$

71. $(ab)'(n) = uw(n)$

72. $es(n) = uwy(n)$

73. $c(n) = z'\lambda(n)$

74. $pt(n) = 2n - \eta(u' \leq n) = \sigma(n) = \lambda(n) - n$

75. $pt'(n) = 2as(n) + 1 - \eta(u' \leq as(n) + 1)$

76. $v(n) = b(n) - \eta(s < n)$

77. $xt(n) = v(n)$

78. $xt'(n) = au'(n) - \eta(s < as(n))$

79. $v(n) = w(2n)$

80. $v(n) - 1 = w(2n - 1)$

81. $w(n) = n + \eta(2a^2u < n)$

82. $w'(n) = 2a^2u(n) + n$

83. $b(n) - 1 = uv'(v(n) - n) = u(v(n) - 1)$

84. $abau(n) = uv'(a^2u(n) + n)$

85. $yt(n) = 2n$

86. $yt'(n) = 2eu'(n) - 1$

87. $\lambda(n) = 3n - \eta(u' \leq n)$

88. $r\sigma(n) - \sigma(n) = b(n) - \sigma(n) = r\tau(n) - \tau(n)$

89. $Kb(n) = c(n) - 1$

$Ks(n) = z(n) - 1$

$Ka^2u(n) = cu(n) - 2$

90. $$K'br(n) = cbr(n)$$

$$K'a(n) = ca(n) + 1$$

$$K'bs(n) = cbs(n) + 1$$

91. $$K'(j) = z(n) \Longleftrightarrow n = j - \eta(br < j) = c(j) + 1 - \lambda(j)$$

92. $$tar(n) = br(n) - n$$

93. $$K'(br(n) - 1) = ztar(n)$$

94. $$cbr(n) = z'(b^2r(n) + n)$$

95. $$\lambda br(n) = b^2r(n) + n$$

96. $$s(n) \sim c(n)$$

$$t'(n) \sim b^2(n)$$

$$u'(n) \sim bc(n)$$

97. $$\lambda u(n) = 2u(n) + n$$

98. $$\lambda u'(n) = 3u'(n) - n$$

99. $$vs(n) = u'(n) - n$$

100. $$vr(n) = ar(n) + n$$

101. $$(s) = (ab) \cup (a^2u')$$

102. $$(r) = (b) \cup (bu - 1) = (b) \cup (a^2u)$$

103. $$(es) = (au)' = (au') \cup (b)$$

104. $$(u') = (ab^2) \cup (abau')$$

105. $$(u') \cup (abau) = (ab)$$

106. $$(u) = (a^2) \cup (b) \cup (abau) = (b) \cup (b - 1) \cup (abau)$$

107. $$(u) = (uv) \cup (uv') = (b) \cup (uv')$$

108. $$(es) = (b) \cup (au') = (ux)$$

109. $$(ux) = (uxt) \cup (au') = (uxt) \cup (uxt')$$

110. $$(\phi) = (b) \cup (abr) \cup (abs + 1) = (b) \cup (abr) \cup (au')$$

111. $$(\phi') = (abs) \cup (a^2(bes)') = (abs) \cup (a^2(eu')')$$

112. $$(\psi) = (e\phi') = (bs) \cup (a(eu')')$$

113. $$(eu') = (b^2) \cup (bau')$$

114. $$(eu') \cup (bau) = (b^2) \cup (ba) = (b)$$

115. $$(eu')' = (a) \cup (bau)$$

116. $$(\psi) = (a^2) \cup (abau) \cup (bs)$$

117. $(\psi') = (u') \cup (br)$

118. $(e\psi') = (eu') \cup (ar)$

119. $(u') = ab(az - as)$

120. $(a) = \{ n \mid aen = n \}$

$(b) = \{ n \mid aen = n - 1 \}$

121. $c(n) \in (a) \Longleftrightarrow n \in [(a)/(s)] \cup (bs) = (a^2u) \cup (bs)$

122. $a(n) = n + \eta(a < n)$

123. $e(n) = \eta(a \leq n)$

124. $a(n) = t(n) + \eta(ar < n)$

125. $t(n) = n + \eta(as < n)$

126. $t(n) = \eta(as < b^2(n))$

127. $(\eta(a < s(n)))) = (a^2) \cup (abs) = (a(z - s))$

128. $z(n) = as(n) - n + 1 + \eta(t' < n)$

129. $z(n) + 2n = t'(n) + 1 + \eta(t' < n)$

130. $\eta(t(n) < t' < t(n) + b^2(n)) = n$

131. $\lambda(n) = 3n - \eta(bs < n)$

132. $\lambda(n) = n + 1 - \eta(r < b(n))$

133. $\eta(r < b(n)) = 2n - 1 - \eta(u' \leq n)$

134. $2s(n) = t'(n) + 1 + \eta(tar < n)$

135. $2ab(n) = t(n) + b^2(n) + \eta(ar < n)$

136. $\eta(t' < t(n)) = \eta(as < n)$

137. $$(s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b)$$

138. $$(r) = (b) \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k a^3) \right]$$

139. $$(u') = \bigcup_{k=0}^{\infty} (ab(a^2b)^k b)$$

140. $$(u) = (b) \cup (b - 1) \cup \left[\bigcup_{k=0}^{\infty} (ab(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (ab(a^2b)^k a^3) \right]$$

141. $s(f_k(n)) = a(a^2b)^k b(n)$

142. $$s(t')^k t(n) = a(a^2b)^k b(n)$$

143. $$f_k(n) = (t')^k t(n) \qquad (k = 0, 1, 2, \cdots)$$

144. $$u'(f_k(n)) = ab(a^2b)^k b(n)$$

145. $$f_{k-1}(b^2(n) - 1) < f_k(n) < f_{k-1}(b^2(n)) \qquad (k = 1, 2, 3, \cdots) .$$

The following functions are separated:

$$b, c, s, \phi', \psi', t', z, u',$$
$$K', v, y', w', \lambda, \rho', \sigma', \tau' .$$

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27
b	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44
c	3	7	10	14	18	21	25	28	32	36	39	43	47	50	54	57	61
s	3	8	11	16	19	21	24	29	32	37	42	45	50	53	55	58	63
r	1	2	4	5	6	7	9	10	12	13	14	15	17	18	20	22	23
z	4	11	15	22	26	29	33	40	44	51	58	62	69	73	76	80	87
z'	1	2	3	5	6	7	8	9	10	12	13	14	16	17	18	19	20
t'	5	14	20	29	35	39	45	54	60	69	78	84	93	99	103	109	118
t	1	2	3	4	6	7	8	9	10	11	12	13	15	16	17	18	19
u'	8	21	29	42	50	55	63	76	84	97	110	118	131	139	144	152	164
u	1	2	3	4	5	6	7	9	10	11	12	13	14	15	16	17	18

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
p	2	4	6	8	9	10	12	14	15	17	19	21	23	24	25	27	29
p'	1	3	5	7	11	13	16	18	20	22	26	28	30	32	36	38	41
v	2	5	7	9	12	14	17	19	21	24	26	28	31	33	36	38	40
v'	1	3	4	6	8	10	11	13	15	16	18	20	22	23	25	27	29
w	1	2	4	5	6	7	8	9	11	12	13	14	16	17	18	19	20
w'	3	10	15	22	29	34	41	52	59	64	71	78	83	90	95	102	109

(continued)

Table 2 (continued)

x	2	5	7	9	11	12	14	17	19	21	24	26	28	30	31	33	36
x'	1	3	4	6	8	10	13	15	16	18	20	22	23	25	27	29	32
y	2	4	6	8	9	10	12	14	16	18	20	22	24	25	26	28	30
y'	1	3	5	7	11	13	15	17	19	21	23	27	29	31	33	37	39
λ	3	6	9	12	15	18	21	23	26	29	32	35	38	41	44	47	50
λ'	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	24	25

Table 3

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
e	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11
ϕ	2	3	5	7	8	10	12	13	15	16	18	20	21	23	24	26	28
ϕ'	1	4	6	9	11	14	17	19	22	25	27	30	32	35	38	40	43
ψ	1	3	4	6	7	9	11	12	14	16	17	19	20	22	24	25	27
ψ'	2	5	8	10	13	15	18	21	23	26	29	31	34	36	39	42	44
σ	2	4	6	8	10	12	14	15	17	19	21	23	25	27	29	31	33
σ'	1	3	5	7	9	11	13	16	18	20	22	24	26	28	30	32	34
τ	1	3	5	7	9	11	13	15	16	18	20	22	24	26	28	30	32
τ'	2	4	6	8	10	12	14	17	19	21	23	25	27	29	31	33	35
K	1	2	3	5	6	8	9	10	12	13	14	16	17	19	20	21	23
K'	4	7	11	15	18	22	26	29	33	36	40	44	47	51	54	58	62

Fifty pages of extended data tables are available (for $2.50) from Brother Alfred Brousseau, St. Mary's College, Maraga, California 94575.

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COMPLETE SEQUENCES OF FIBONACCI POWERS

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1. INTRODUCTION

We define the Fibonacci sequence in the usual manner by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for each k in the set N of positive integers. An integer is said to have a representation with respect to a sequence if it is the sum of some finite subsequence. A sequence is complete if every positive integer has a representation with respect to the sequence.

In [2], Hoggatt and King showed that the sequence $\{F_k\}$ of Fibonacci numbers is complete, and O'Connell showed in [3] that the sequence 1, 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ of two of each of the Fibonacci squares F_k^2 is complete. In Theorem 1 of this paper we show that the sequence of 2^{n-1} of each of the Fibonacci n^{th} powers F_k^n is complete, and (as is obvious) that fewer than 2^{n-1} copies of $\{F_k^n\}$ does not yield a complete sequence.

Hoggatt and King [2] showed that the Fibonacci sequence with any term deleted is complete, but is no longer complete if any two terms are omitted. O'Connell [3] showed that the twofold sequence of Fibonacci squares mentioned above with any one of the first six terms deleted is still complete, but that the deletion of any term after the sixth or of any two terms will destroy completeness. In treating sequences of Fibonacci n^{th} powers, this led us to define a minimal sequence.

Definition. A minimal sequence is a complete sequence which is no longer complete if any element is deleted.

For each positive integer k, the sequence $\{F_i\}$ with F_k removed is a minimal sequence of Fibonacci numbers. Although only two minimal sequences 1, 1, 1, 4, 4, 9, 9, 25, 25, $\cdots$ and 1, 1, 1, 1, 4, 9, 9, 25, 25, $\cdots$ can be obtained by deleting elements from the twofold sequence of Fibonacci squares, there are infinitely many minimal sequences comprised of Fibonacci squares. One can simply replace some term F_k^2 in a minimal sequence of Fibonacci squares by several terms F_i^2 whose sum is less than or equal to F_k^2 in a way which preserves minimality, as illustrated by the minimal sequences 1, 1, 1, 1, 1, 1, 1, 1, 9, 9, 25, 25, $\cdots$ (replacing 4 by 1, 1, 1, 1) or 1, 1, 1, 4, 4, 4, 9, 25, 25, $\cdots$ (replacing 9 by 4). For each n larger than 3, infinitely many minimal sequences can be obtained by deleting terms from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. However, one can obtain a particular minimal sequence from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers by deleting as many terms F_k^n as possible without destroying completeness before deleting any terms F_{k+1}^n, for $k = 2, 3, 4, \cdots$. We show in Sec. 4 that this yields the unique minimal sequence defined below.

Definition. A distinguished sequence of Fibonacci n^{th} powers is a sequence a which maps N onto $\{F_k^n \mid k \in N\}$ satisfying

(a) $$a_k \leq a_{k+1} \quad \text{for every} \quad k \in N,$$

(b) $$a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i \quad \text{for every} \quad k \in N,$$

(c) $$a_{k+1} \neq a_k \quad \text{implies} \quad 1 + \sum_{i=1}^{k-1} a_i < a_{k+1} \quad \text{for every} \quad k \in N.$$

In Theorem 4, we show that if

$$r = \left[\!\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]\!\right],$$

where $[\![\cdots]\!]$ is the greatest integer function, and $\{a_k\}$ is the distinguished sequence of Fibonacci n^{th} powers, then from some point on each F_i^n occurs exactly r or $r - 1$ times. Theorem 5 sharpens this result to show that for even n each F_i^n appears exactly r times from some point on.

2. BASIC IDENTITIES

The identities

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k \quad \text{and} \quad F_{k+1}F_{k+2} = F_kF_{k+3} + (-1)^k$$

follow easily by induction on k and show that

(1) $$\frac{F_{2k}}{F_{2k-1}} < \frac{F_{2k+1}}{F_{2k}}, \quad \frac{F_{2k}}{F_{2k-1}} < \frac{F_{2k+2}}{F_{2k+1}}, \quad \frac{F_{2k+1}}{F_{2k}} > \frac{F_{2k+3}}{F_{2k+2}}$$

for any positive integer k. Thus $\{F_{2k}/F_{2k-1}\}$ is an increasing sequence, $\{F_{2k+1}/F_{2k}\}$ is a decreasing sequence, and

(2) $$\frac{F_{2j}}{F_{2j-1}} < \frac{F_{2k+1}}{F_{2k}}$$

for all positive integers j and k. Since

$$\frac{F_{2k+1}}{F_{2k}} - \frac{F_{2k}}{F_{2k-1}} = \frac{1}{F_{2k}F_{2k-1}}$$

approaches zero as k approaches infinity, the sequences

$$\{F_{2k}, F_{2k-1}\}, \qquad \{F_{2k+1}/F_{2k}\}, \qquad \text{and} \qquad \{F_{k+1}/F_k\}$$

have a common limit α. The identity

$$\frac{F_{k+2}}{F_{k+1}} = \frac{F_{k+1} + F_k}{F_{k+1}} = 1 + \frac{F_k}{F_{k+1}}$$

implies that $\alpha = 1 + 1/\alpha$, and α is clearly positive, so $\alpha = (1 + \sqrt{5})/2$ is the common limit. These are all well known properties of the Fibonacci numbers.

From (1) and (2), it is clear that $5/3 = F_5/F_4 \geq F_{k+1}/F_k$ except when $k = 2$. It follows by induction that

$$\left(\frac{F_{k+1}}{F_k}\right)^n \leq \left(\frac{5}{3}\right)^n < 1 + 2^{n-1} \qquad (k \neq 2) \tag{3}$$

for all positive integers n and k with $k \neq 2$. Now the inequality

$$F_{k+1}^n \leq 1 + 2^{n-1}\sum_{i=1}^{k} F_i^n \qquad (n, k \in N) \tag{4}$$

is true for $k = 1$ and $k = 2$, and for $k > 2$,

$$F_{k+1}^n < (1 + 2^{n-1})F_k^n = F_k^n + 2^{n-1}F_k^n \leq 1 + 2^{n-1}\sum_{i=1}^{k-1} F_i^n + 2^{n-1}F_k^n = 1 + 2^{n-1}\sum_{i=1}^{k} F_i^n$$

follows by induction on k.

3. COMPLETE SEQUENCES

It will frequently be helpful to use the following criterion due to Brown [1] in considering the completeness of various sequences.

Completeness Criterion: A non-decreasing sequence $\{a_i\}$ of positive integers is complete if and only if

$$a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i \tag{5}$$

for every non-negative integer k.

Brown's criterion and the inequality (4) are instrumental in proving the next theorem.

Theorem 1. For any positive integer n, the sequence of 2^{n-1} of each of the Fibonacci n^{th} powers is complete, but the sequence of $2^{n-1} - 1$ of each of the Fibonacci n^{th} powers is not complete.

Proof. Let $\{a_i\}$ be the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. That is,

$$a_k = F_m^n \quad \text{for} \quad (m - 1) \cdot 2^{n-1} < k \le m \cdot 2^{n-1} .$$

If $k = 2^{n-1} \cdot m$ for some $m \in N$, then $a_{k+1} = F_{m+1}^n$ and

$$\sum_{i=1}^{k} a_i = 2^{n-1} \sum_{i=1}^{m} F_i^n ,$$

so the inequality (5) follows from (4). Otherwise, $a_{k+1} = a_k$ and the inequality (5) is clear. Hence $\{a_i\}$ is complete by the Completeness Criterion. $2^{n-1} - 1$ copies of $\{F_k^n\}$ is obviously not complete, since

$$F_3^n = 2^n > 1 + (2^{n-1} - 1)(F_1^n + F_2^n) = 2^n - 1 .$$

It is easy to see that the 2^{n-1}-fold sequence of Fibonacci n^{th} powers is not minimal, since in any case, one of the 2^n ones can be omitted. For $n \ge 4$, infinitely many terms can be omitted without destroying completeness, as is shown by the following theorem.

Theorem 2. Let n be a positive integer and let $r_2 = 2^n - 2$, $r_k = [\![(F_{k+1}/F_k)^n]\!]$ for each positive integer $k \ne 2$. Then the non-decreasing sequence of Fibonacci n^{th} powers in which, for each positive integer k, F_k^n occurs exactly r_k times, is complete.

Proof. The sequence $\{a_i\}$ defined in this theorem is given by taking $a_i = F_k^n$ when

$$\sum_{j=1}^{k-1} r_j < i \le \sum_{j=1}^{k} r_j .$$

The condition (5) is clear if $a_{k+1} = a_k$. Otherwise, we have $a_k = F_m^n$, $a_{k+1} = F_{m+1}^n$, and

$$\begin{aligned} a_{k+1} &= F_{m+1}^n \le (1 + r_m)F_m^n = F_m^n + r_m F_m^n \\ &\le 1 + \sum_{i=1}^{m-1} r_i F_i^n + r_m F_m^n = 1 + \sum_{i=1}^{m} r_i F_i^n . \end{aligned}$$

Therefore (5) follows by induction on m. The Completeness Criterion gives the theorem.

If $n \ge 4$ and $k \ge 3$,

$$r_k = [\![(F_{k+1}/F_k)^n]\!] \le [\![(5/3)^n]\!] \le 2^{n-1} - 1$$

by (3), and so at least one term F_k^n for each $k = 3, 4, 5, \cdots$ can be omitted from the 2^{n-1}-fold sequence of Fibonacci n^{th} powers. There is still no guarantee that the sequence given in Theorem 2 is minimal. (It is, if $n = 1$ or 2.) However, we will see in the next section that if n is even, then the sequence given in Theorem 2 is almost minimal, in the sense that a minimal sequence can be obtained from it by deleting a finite subsequence.

4. THE DISTINGUISHED SEQUENCES

We will now specialize our study to a particular minimal sequence of Fibonacci n^{th} powers. Let us recall our definition of a distinguished sequence.

Definition. A distinguished sequence of Fibonacci n^{th} powers is a sequence $\{a_k\}$ which maps N onto $\{F_k^n \mid k \in N\}$ satisfying

(a) $$a_k \le a_{k+1} \quad \text{for every} \quad k \in N\,,$$

(b) $$a_{k+1} \le 1 + \sum_{i=1}^{k} a_i \quad \text{for every} \quad k \in N\,,$$

(c) $$a_{k+1} \ne a_k \quad \text{implies} \quad 1 + \sum_{i=1}^{k-1} a_i < a_{k+1} \quad \text{for every} \quad k \in N\,.$$

We would like to show that for each positive integer n, a distinguished sequence exists and is unique. Starting with the complete 2^{n-1}-fold sequence of Fibonacci n^{th} powers, deleting one $F_2^n = 1$, and consecutively deleting enough F_j^n so that (b) and (c) are satisfied for $j = 3, 4, 5, \cdots$, it is clear that one can construct a sequence satisfying (a), (b), and (c). The inequalities $F_{j-1}^n + F_j^n \le (F_{j-1} + F_j)^n = F_{j+1}^n$, (b), and (c) insure that $\{a_k\}$ is onto $\{F_i^n \mid i \in N\}$. Properties (b) and (c) also guarantee uniqueness. Henceforth, we will call the unique distinguished sequence of Fibonacci n^{th} powers the n^{th} distinguished sequence.

The work [2] of Hoggatt and King shows that the first distinguished sequence $\{a_i\}$ is the sequence $1, 2, 3, 5, 8, 13, \cdots$ defined by taking $a_i = F_{i+1}$, and O'Connells' work [3] shows that the second distinguished sequence $\{a_i\}$ is the sequence $1, 1, 1, 4, 4, 9, 9, 25, 25, 64, 64, \cdots$ defined by taking $a_i = F_j^2$ for $j = [\![i/2]\!] + 1$.

Theorems 4 and 5 give information about the n^{th} distinguished sequence for any positive integer n. Before stating these theorems we will introduce some notation and recall some well known facts concerning Fibonacci and Lucas numbers.

Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ be the roots of the equation $x^2 = x + 1$, and note that $\alpha\beta = -1$. Recall that

$$\alpha = \lim_{k\to\infty} (F_{k+1}/F_k)\,.$$

Multiplying $x^2 = x + 1$ by x^n, we see that

$$\begin{aligned} \alpha^{n+2} &= \alpha^{n+1} + \alpha^n\,, \\ \beta^{n+2} &= \beta^{n+1} + \beta^n\,, \\ (\alpha^{n+2} + \beta^{n+2}) &= (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n)\,. \end{aligned} \tag{6}$$

Defining

(7) $$L_n = \alpha^n + \beta^n = \alpha^n + (-1/\alpha)^n$$

for $n = 0, 1, 2, 3, \cdots$, we see that $L_0 = 2$, $L_1 = 1$ (since $\alpha = 1 + 1/\alpha$), and $L_{n+1} = L_n + L_{n-1}$ for any positive integer n. The integers L_n defined in this way are the usual Lucas numbers. Now $\alpha > 1$, so $0 < 1/\alpha^n < 1$ for each positive integer n, and it follows from (7) that

(8) $$[\![\alpha^n]\!] = \alpha^n - \frac{1}{\alpha^n} = L_n \quad (n \text{ odd}),$$

$$[\![\alpha^n]\!] = \alpha^n + \frac{1}{\alpha^n} - 1 = L_n - 1 \quad (n \text{ even}),$$

and thence

(9) $$[\![\alpha^n]\!] < \alpha^n < [\![\alpha^n]\!] + 1 \quad (n \in N).$$

Also, $\alpha^2 = \alpha + 1 > 2$, so $0 < 2/\alpha^n < 1$ for any even positive integer n, and

(10) $$1/\alpha^n + [\![\alpha^n]\!] < \alpha^n < [\![\alpha^n]\!] + 1 \quad (n \text{ even})$$

follows from (8).

Lemma 3. For each positive integer n there is a positive integer M such that, for $k \geq M$,

$$[\![\alpha^n]\!] < (F_{k+1}/F_k)^n < [\![\alpha^n]\!] + 1,$$

and, if n is even,

$$(F_{k-1}/F_k)^n + [\![\alpha^n]\!] < (F_{k+1}/F_k)^n < [\![\alpha^n]\!] + 1.$$

Proof. This lemma is immediate from (9), (10), and the limits,

$$\lim_{k\to\infty} (F_{k+1}/F_k)^n = \alpha^n, \qquad \lim_{k\to\infty} (F_{k-1}/F_k)^n = 1/\alpha^n.$$

We are now ready to prove Theorems 4 and 5.

Theorem 4. Let $\{a_i\}$ be the n^{th} distinguished sequence, and let $r = [\![\alpha^n]\!]$, where $\alpha = (1 + \sqrt{5})/2$. Then there is a positive integer M such that for each $k \geq M$, $a_i = F_k^n$ for exactly r or r - 1 values of i.

Proof. Let M be as in Lemma 3. It suffices to show that if $k \geq M$ and

$$h = \min\{i \mid a_i = F_k^n\},$$

then

$$1 + \sum_{i=1}^{h-1} a_i + (r-2)F_k^n < F_{k+1}^n \leq 1 + \sum_{i=1}^{h-1} a_i + rF_k^n.$$

We know by property (c) of the distinguished sequence that

$$1 + \sum_{i=1}^{h-2} a_i = 1 + \sum_{i=1}^{h-1} a_i - F_{k-1}^n < F_k^n ,$$

and so

$$1 + \sum_{i=1}^{h-1} a_i + (r-2)F_k^n < F_{k-1}^n + (r-1)F_k^n < rF_k^n < F_{k+1}^n$$

since

$$r < (F_{k+1}/F_k)^n .$$

Also,

$$F_k^n \le 1 + \sum_{i=1}^{h-1} a_i$$

by property (b) of the distinguished sequence, so

$$F_{k+1}^n < (r+1)F_k^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n ,$$

since

$$(F_{k+1}/F_k)^n < r + 1 .$$

We can sharpen this result for even values of n.

Theorem 5. Let n be an even positive integer, let $\{a_i\}$ be the n^{th} distinguished sequence, and let $r = [\![\alpha^n]\!]$ for $\alpha = (1+\sqrt{5})/2$. Then there is a positive integer M such that for each $k \ge M$, $a_i = F_k^n$ for exactly r values of i.

Proof. Let M be as in Lemma 3. It suffices to show that if $k \ge M$ and

$$h = \min\{i \mid a_i = F_k^n\} ,$$

then

$$1 + \sum_{i=1}^{h-1} a_i + (r-1)F_k^n < F_{k+1}^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n .$$

By property (c) of the distinguished sequence $\{a_i\}$,

$$1 + \sum_{i=1}^{h-2} a_i = 1 + \sum_{i=1}^{h-1} a_i - F_{k-1}^n < F_k^n$$

so

$$1 + \sum_{i=1}^{h-1} a_i + (r-1)F_k^n < F_{k-1}^n + rF_k^n < F_{k+1}^n ,$$

since

$$(F_{k-1}/F_k)^n + r < (F_{k+1}/F_k)^n .$$

By property (b),

$$F_k^n \le 1 + \sum_{i=1}^{h-1} a_i, \quad \text{so} \quad F_{k+1}^n < (r+1)F_k^n \le 1 + \sum_{i=1}^{h-1} a_i + rF_k^n ,$$

since

$$(F_{k+1}/F_k)^n < r + 1 .$$

Now we define a complete sequence to be <u>almost minimal</u> if it can be made minimal by deleting a finite subsequence.

<u>Corollary 6.</u> If n is even, the sequence defined in Theorem 2 is almost minimal.

<u>Proof</u>. The sequence $\{a_i\}$ defined in Theorem 2 is given by taking

$$r_k = [\![(F_{k+1}/F_k)^n]\!]$$

for each positive integer $k \neq 2$, $r_2 = 2^n - 2$, and taking $a_i = F_k^n$ when

$$\sum_{j=1}^{k-1} r_j < i \le \sum_{j=1}^{k} r_j .$$

Since $r_k \ge 1$ for every $k \neq 2$, the sequence $\{a_i\}$ is onto the set $\{F_k^n \mid k \in N\}$. Since $\{a_i\}$ is a complete sequence of non-decreasing terms, it satisfies properties (a) and (b) of the n^{th} distinguished sequence. For M as in Lemma 3, $r_k = [\![\alpha^n]\!] = r$ when $k \ge M$. Thus, the minimal n^{th} distinguished sequence can be obtained from the sequence $\{a_i\}$ by deleting finitely many terms F_k^n with $k \le M$.

When n is even, Corollary 6 provides a fairly efficient means of constructing the n^{th} distinguished sequence in a finite number of steps. First M is determined by inspection, for example, and then enough of the r_k terms F_k^n are deleted to make the sequence satisfy property (c) of the distinguished sequence, for $k = 3, 4, \cdots, M - 1$. For example, when $n = 4$, we have $M = 5$ and $r_1 = 1$, $r_2 = 14$, $r_3 = 5$, $r_4 = 7$, $r_k = 6$ $(k \ge 5)$. So the sequence of Theorem 2 has fifteen 1's, five 16's, seven 81's, and six F_k^4's for all $k \ge 5$. Since none of the F_k^4 can be deleted for $k < 5$, this sequence is already the fourth distinguished sequence.

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ANALYTICAL VERIFICATION OF AN "AT SIGHT" TRANSFORMATION

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There has been interest recently [1] - [3] in finding easy, at sight ways of transforming

(1) $$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

to

(2) $$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 = 0$$

subject to

(3) $$z = (s + 1)/(s - 1) .$$

Power [1], [2] discovered an $(n + 1) \times (n + 1)$ X matrix applicable for transforming the a's of (1) to the b's of (2), and Fielder [3] developed a somewhat different Q matrix for the same purpose. In each case, the most rewarding feature of the <u>application</u> matrix (X or Q) was the apparent presence of an extremely simple combinatorial scheme for constructing the application matrix from the barest minimum of information. While many trial values of n assured that either application matrix was safe to use for practical purposes, the analytic validity of the combinatorial schemes defied verification, leaving an understandably disturbing theoretical situation.

In this note, a very simple proof of the validity of the combinatorial method for constructing the Q matrix is presented. The proof is effected through application of generating functions.*

It has been shown [3] that the coefficients of (1) and (2) are related through

(4) $$b = \frac{1}{A_0} Q a ,$$

where a is the $(n + 1) \times 1$ column matrix $(1, a_{n-1}, \cdots, a_0)$, b is the $(n + 1) \times 1$ column matrix $(1, b_{n-1}, \cdots, b_0)$, Q is an $(n + 1) \times (n + 1)$ application matrix of integers, and A_0 is the sum of the elements of a. It has also been shown [3] that Q is the product of two $(n + 1) \times (n + 1)$ matrices of integers PN. Briefly, the elements of P and N are given, respectively, by

(5) $$p_{i,j} = (-1)^{n-i}(-2)^{j-1}\binom{n - j + 1}{n - i + 1}, \qquad (i,j = 1, 2, \cdots, n + 1) ,$$

*In private correspondence with Fielder, Power outlines an analytic verification of the combinatorial process for constructing his X matrix. The proof presented herein for the Q matrix is an independent development and is, of course, different from Power's.

$$\nu_{i,j} = \binom{n - j + 1}{i - 1}, \qquad (i, j = 1, 2, \cdots, n + 1) . \tag{6}$$

As a brief digression, Q_5 is exemplified to illustrate the conjectured combinatorial construction method.

$$Q_5 = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 5 & 3 & 1 & -1 & -3 & -5 \\ 10 & 2 & -2 & -2 & 2 & 10 \\ 10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{array}\right] \tag{7}$$

The conjectured combinatorial pattern is

$$q_{i+1,j+1} + q_{i,j+1} + q_{i,j} = q_{i+1,j}, \qquad (i, j = 1, 2, \cdots, n) \tag{8}$$

and is illustrated for $10 - 5 - 3 = 2$. Once (8) is established, all that is needed to assure a valid Q matrix in general is a first row of $(n + 1)$ ones and a last column of alternating sign binomial coefficients for the appropriate index n.

From a consideration of (5) and (6), it is seen that the general element of Q is given by

$$q_{i,j} = (-1)^{i-1} \sum_{k=1}^{\rho_{i,j}} (-2)^{k-1} \binom{n - k + 1}{n - i + 1} \binom{n - j + 1}{k - 1}, \qquad (i, j = 1, 2, \cdots, n + 1) . \tag{9}$$

Since P has all zero elements above the main diagonal with non-zero elements elsewhere, and N has all-zero elements below the secondary diagonal with non-zero elements [3], it can readily be seen that the upper summation index $\rho_{i,j}$ of (9) is the lesser of the number of non-zero elements of row i of P or column j of N. This value can be established as

$$\rho_{i,j} = \tfrac{1}{2}(n + 1 + i - j - |n + 2 - i - j|) . \tag{10}$$

In any event, if $\rho_{i,j}$ is replaced by $(n + 1)$ the laws of matrix multiplication assure that $q_{i,j}$ so determined is the correct value for the element of Q. Subsequent work, however, is based on $q_{i,j}$ having the correct value as the upper summation index approaches infinity. This extension is covered by Lemma 1.

<u>Lemma 1.</u>

$$q_{i,j} = (-1)^{i-1} \sum_{k=1}^{\infty} (-2)^{k-1} \binom{n - k + 1}{n - i + 1} \binom{n - j + 1}{k - 1}, \qquad (i, j = 1, 2, \cdots, n + 1) . \tag{11}$$

<u>Proof.</u> For an upper index of $(n + 1)$, $q_{i,j}$ is valid. For $k > (n + 1)$, the upper term of the first binomial coefficient of (11) is negative. The value of this binomial coefficient is thereby zero. Thus, for any upper index greater than $(n + 1)$ no contribution to $q_{i,j}$ is

made.* Q.E.D.

Lemma 2. Given the generating functions

$$G_j = \sum_{i=1}^{\infty} g_{i,j} x^{i-1}, \qquad G_{j+1} = \sum_{i=1}^{\infty} g_{i,j+1} x^{i-1} . \tag{12}$$

Then,

$$g_{i+1,j+1} + g_{i,j+1} + g_{i,j} = g_{i+1,j} \tag{13}$$

iff

$$G_{j+1} = \frac{1 - x}{1 + x} G_j .$$

Proof. For necessity, assume (13) is true. A relation for the generating functions is

$$G_{j+1} + xG_{j+1} + xG_j = G_j , \tag{15}$$

from which it is seen that (14) follows. For sufficiency, assume (14) true. Form (15) and equate coefficients according to (12) to establish (13) in general.

The following theorem verifies (8).

Theorem. With matrices P and N as defined above, $Q = PN$ has elements $q_{i,j}$ which satisfy

$$q_{i+1,j+1} + q_{i,j+1} + q_{i,j} = q_{i+1,j}, \qquad (i,j = 1, 2, \cdots, n + 1) . \tag{16}$$

Proof. Let W be a $1x(n + 1)$ row matrix

$$W = (1, x, x^2, \cdots, x^n) . \tag{17}$$

Then, WP is a row matrix whose j^{th} element is the generating function for the j^{th} column of P, i.e., the $(n + 1) \times 1$ column vectors of P, the powers of x increasing down the column.

$$WP = \left\{ (-2)^{j-1} \sum_{k=1}^{n+1} (-1)^{k-2} \binom{n - j + 1}{n - k + 1} x^{k-1} \right\} = \left\{ (2x)^{j-1}(1 - x)^{n-j+1} \right\} . \tag{18}$$

Because of the associative property of matrix multiplication, $(WP)N = W(PN) = WQ$. Hence, with use of Lemma 1

$$\begin{aligned}(WP)N &= \left\{ \sum_{k=1}^{\infty} \binom{n - j + 1}{k - 1} (2x)^{k-1}(1 - x)^{n-k+1} \right\} \\ &= \left\{ (1 - x)^{j-1} \sum_{k=1}^{\infty} \binom{n - j + 1}{k - 1} (2x)^{k-1}(1 - x)^{n-k-j+2} \right\} \\ &= \left\{ (1 - x)^{j-1}([1 - x] + 2x)^{n-j+1} \right\} = \left\{ (1 - x)^{j-1}(1 + x)^{n-j+1} \right\} \\ &= WQ = \left\{ Q_j \right\} , \qquad (j = 1, 2, \cdots, n + 1).\end{aligned} \tag{19}$$

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*Note that for either i or j (or both) greater than $n + 1$, the value $q_{i,j}$ is zero regardless of the upper index.

But WQ is a row matrix whose elements are the generating functions of the column elements of Q. From (19), it is seen that

$$Q_{j+1} = (1 - x)^j (1 + x)^{n-j} = \frac{1 - x}{1 + x} Q_j . \tag{20}$$

Thus, by Lemma 2,

$$q_{i+1,j+1} + q_{j,j+1} + q_{i,j} = q_{j+1,j} , \qquad (i,j = 1, 2, \cdots, n + 1) . \tag{21}$$

Q.E.D.

Corollary to the Theorem. The matrix $Q = \{q_{i,j}\}$, $(i, j = 1, 2, \cdots, n+1)$ is such that

$$q_{1,j} = 1 \qquad (j = 1, 2, \cdots, n + 1) , \tag{22}$$

$$q_{i,n+1} = (-1)^{i-1}\binom{n}{n - i + 1} \qquad (i = 1, 2, \cdots, n + 1) . \tag{23}$$

Proof. Since $Q_j = (1 - x)^{j-1}(1 + x)^{n-j+1}$, it follows that the constant term of the generating function is always unity regardless of j. This proves (22). It can be seen from (20) that

$$Q_{n+1} = (1 - x)^n . \tag{24}$$

The coefficients of the generating functions are thereby identically those values given by (23).

The establishment of specific forms for (22) and (23) means that the first row and last column can be immediately written down, and the remainder of the elements of Q follow from application of the combinatorial rule.

As a check on the operational calculations, the bottom row of Q starting with the left element should consist of $1, -1, 1, -1, \cdots$ and the leftmost column starting with the upper element should be the coefficients of increasing powers of x in $(1 + x)^{n+1}$.

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GENERALIZED FIBONACCI POLYNOMIALS AND ZECKENDORF'S THEOREM

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1. INTRODUCTION

The Zeckendorf Theorem states that every positive integer can be uniquely represented as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in any given sum. In fact, in an earlier paper [1], it was shown that every positive integer can be uniquely represented as the sum from k copies of distinct members of the generalized Fibonacci sequence formed by evaluating the Fibonacci polynomials at $x = k$, if no two consecutive members of the sequence with coefficient k are used in any given sum. Now, this result is extended to include sequences formed from generalized Fibonacci polynomials evaluated at $x = k$.

2. THE GENERALIZED ZECKENDORF THEOREM FOR THE TRIBONACCI POLYNOMIALS

The Tribonacci polynomials have been defined in [2] as $T_{-1}(x) = T_0(x) = 0$, $T_1(x) = 1$, $T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x)$. Let us say that the Tribonacci polynomials are evaluated at $x = k$. Then the number sequence is $U_{-1} = U_0 = 0$, $U_1 = 1$, $U_2 = k^2$,

$$U_{n+3} = k^2U_{n+2} + kU_{n+1} + U_n .$$

Theorem 2.1. Let U_n be the n^{th} member of the sequence formed when the Tribonacci polynomials are evaluated at $x = k$. Then every positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

with the constraints

$$\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1,$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^2 ;$$

If $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, \cdots, k - 1$;
If $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;
If $\epsilon_{i+1} = k^2$ and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

We begin with three useful lemmas, which can be proved by mathematical induction or by considering how to represent the specific integers given using the constraints of Theorem 2.1.

Lemma 1. For $k \geq 2$,

$$\begin{aligned} U_{3n} - 1 &= k^2(U_{3n-1} + U_{3n-4} + \cdots + U_2) + k(U_{3n-2} + U_{3n-5} + \cdots + U_1) - 1 \\ &= k^2(U_{3n-1} + \cdots + U_2) + k(U_{3n-2} + U_{3n-5} + \cdots + U_4) + (k-1)U_1 . \end{aligned}$$

Lemma 2. For $k \geq 2$,

$$U_{3n+1} - 1 = k^2(U_{3n} + U_{3n-3} + \cdots + U_3) + k(U_{3n-1} + U_{3n-4} + \cdots + U_2) .$$

Lemma 3. For $k \geq 2$,

$$U_{3n+2} - 1 = k^2(U_{3n+1} + U_{3n-2} + \cdots + U_4) + k(U_{3n} + U_{3n-3} + \cdots + U_3) + (k^2 - 1)U_1 .$$

These lemmas are almost self-explanatory. Now for the utility of the three lemmas in the proof of Theorem 2.1.

Assume that every integer $s \leq U_{3n+2} - 1$ has a unique admissible representation. By using rU_{3n+2} for $r = 1, 2, \cdots, k^2$, one can get a representation for $s \leq (k^2 - 1)U_{3n+2} + (U_{3n+2} - 1)$ without using k^2U_{3n+2}. If we now add another U_{3n+2}, then k^2U_{3n+2} is representable but now the representation for $U_{3n+2} - 1$ cannot be used since U_{3n+1} has too large a coefficient. Let $k^2U_{3n+2} \geq s$ be representable in admissible form. Now we gradually build up to $(k - 1)U_{3n+1}$, and since $k^2 = \epsilon_{i+1}$ and $\epsilon_i = k - 1$, so that there are no restrictions on the earlier coefficients because we can still obtain $U_{3n+1} - 1$ without further conflict.

Thus $s \leq k^2U_{3n+2} + (k - 1)U_{3n+1} + U_{3n+1} - 1 = k^2U_{3n+2} + kU_{3n+1} - 1$ is obtainable. We cannot now add another U_{3n+1} since $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, so that $\epsilon_{i-1} = 0$ and we cannot now use the representation of $U_{3n+1} - 1$, but we now achieve, without U_{3n}, the sum as great as $U_{3n} - 1$. Now we have a representation up to $s \leq k^2U_{3n+2} + kU_{3n+1} + U_{3n} - 1 = U_{3n+3} - 1$, which completes the proof of Theorem 2.1 by mathematical induction.

3. HIERARCHY OF RESULTS: ZECKENDORF'S THEOREM FOR THE GENERALIZED FIBONACCI POLYNOMIAL SEQUENCES

Define the generalized Fibonacci polynomials as in [2] by

$$P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0, \quad P_1(x) = 1, \quad P_2(x) = x^{r-1},$$
$$P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x) .$$

Let $U_n = P_n(k)$, the n^{th} member of the sequence formed by evaluating the generalized Fibonacci polynomials at $x = k$. We state the Zeckendorf Theorem for selected values of r.

Theorem 3.1. The Binary Case, $r = 1$. Let $P_n(x) = x^{n-1}$, or, $P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x$, $P_{n+1}(x) = xP_n(x)$. Now, if $U_n = P_n(k) = k^{n-1}$, then any positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if and only if $\epsilon_i = 0, 1, 2, \cdots, k - 1$.

Theorem 3.1 provides, for example, the representation of a number in decimal notation. Theorem 3.2 is the generalized Zeckendorf Theorem proved in [2], and Theorem 3.3 is Theorem 2.1 restated.

Theorem 3.2. The Fibonacci Case, $r = 2$. Let $P_{n+2}(x) = xP_{n+1}(x) + P_n(x)$, $P_0(x) = 0$, $P_1(x) = 1$. Let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form $N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$ if and only if $\epsilon_1 = 0, 1, 2, \cdots, k - 1$, and for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k ;$$

If $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.3. The Tribonacci Case, $r = 3$. Let $P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, and $P_{n+3}(x) = x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x)$, and let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form $N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$ if and only if

$$\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1$$

and for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^2;$$

If $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, \cdots, k - 1$;
If $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;
If $\epsilon_{i+1} = k^2$ and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.4. The Quadranacci Case, $r = 4$. Let $P_{-2}(x) = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^3$, $P_{n+4}(x) = x^3P_{n+3}(x) + x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x)$. Let $U_n = P_n(k)$, $k \geq 1$. Then any positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^3 - 1,$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^3;$$

If $\epsilon_2 = k^3$, then $\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1$;
If $\epsilon_2 = k^3$ and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;
If $\epsilon_{i+2} = k^3$, then $\epsilon_{i+1} = 0, 1, 2, \cdots, k^2$;
If $\epsilon_{i+2} = k^3$ and $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;
If $\epsilon_{i+2} = k^3$, $\epsilon_{i+1} = k^2$, and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.5. The Pentanacci Case, $r = 5$. Let $P_{-3}(x) = P_{-2}(x) = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^4$, and

$$P_{n+5}(x) = x^4P_{n+4}(x) + x^3P_{n+3}(x) + x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x),$$

and then let $U_n = P_n(k)$. Then every positive integer N can be represented uniquely in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^4 - 1$$

and, for $i \geq 2$, $\epsilon_i = 0, 1, 2, \cdots, k^4$

where

If $\epsilon_2 = k^4$, then $\epsilon_1 = 0, 1, 2, \cdots, k^3 - 1$;
If $\epsilon_3 = k^4$ and $\epsilon_2 = k^3$, then $\epsilon_1 = 0, 1, \cdots, k^2 - 1$;
If $\epsilon_4 = k^4$ and $\epsilon_3 = k^3$ and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;
If $\epsilon_{i+3} = k^4$, then $\epsilon_{i+2} = 0, 1, \cdots, k^3$;
If $\epsilon_{i+3} = k^4$ and $\epsilon_{i+2} = k^3$, then $\epsilon_{i+1} = 0, 1, 2, \cdots, k^2$;

If $\epsilon_{i+3} = k^4$, $\epsilon_{i+2} = k^3$, and $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, \cdots, k$;
If $\epsilon_{i+3} = k^4$, $\epsilon_{i+2} = k^3$, $\epsilon_{i+1} = k^2$, and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

A proof by mathematical induction of Theorem 3.5 requires five lemmas given below.

Lemma 1.

$$\begin{aligned} U_{5n} - 1 = {} & k^4U_{5n-1} + k^3U_{5n-2} + k^2U_{5n-3} + kU_{5n-4} \\ & + k^4U_{5n-6} + k^3U_{5n-7} + k^2U_{5n-8} + kU_{5n-9} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_4 + k^3U_3 + k^2U_2 + (k-1)U_1 . \end{aligned}$$

Lemma 2.

$$\begin{aligned} U_{5n+1} - 1 = {} & k^4U_{5n} + k^3U_{5n-1} + k^2U_{5n-2} + kU_{5n-3} \\ & + k^4U_{5n-5} + k^3U_{5n-6} + k^2U_{5n-8} + kU_{5n-9} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_5 + k^3U_4 + k^2U_3 + kU_2 + 0(U_1 - 1) . \end{aligned}$$

Lemma 3.

$$\begin{aligned} U_{5n+2} - 1 = {} & k^4U_{5n+1} + k^3U_{5n} + k^2U_{5n-1} + kU_{5n-2} \\ & + k^4U_{5n-4} + k^3U_{5n-5} + k^2U_{5n-6} + kU_{5n-7} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_6 + k^3U_5 + k^2U_4 + kU_3 + (k^4 - 1)U_1 . \end{aligned}$$

Lemma 4.

$$\begin{aligned} U_{5n+3} - 1 = {} & k^4U_{5n+2} + k^3U_{5n+1} + k^2U_{5n} + kU_{5n-1} \\ & + k^4U_{5n-3} + k^3U_{5n-4} + k^2U_{5n-5} + kU_{5n-6} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_7 + k^3U_6 + k^2U_5 + kU_4 \\ & + k^4U_2 + (k^3 - 1)U_1 . \end{aligned}$$

Lemma 5.

$$\begin{aligned} U_{5n+4} - 1 = {} & k^4U_{5n+3} + k^3U_{5n+2} + k^2U_{5n+1} + kU_{5n} \\ & + k^4U_{5n-1} + k^3U_{5n-2} + k^2U_{5n-3} + kU_{5n-4} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_8 + k^3U_7 + k^2U_6 + kU_5 \\ & + k^4U_3 + k^3U_2 + (k^2 - 1)U_1 . \end{aligned}$$

Theorem 3.6. Let $P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^{r-1}$, and $P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x)$. and then let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form

$$N = \epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^{r-1} - 1$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^{r-1},$$

where

If $\epsilon_2 = k^{r-1}$, then $\epsilon_1 = 0, 1, 2, \cdots, k^{r-2} - 1$;
If $\epsilon_3 = k^{r-1}$ and $\epsilon_2 = k^{r-2}$, then $\epsilon_1 = 0, 1, 2, \cdots, k^{r-3} - 1$;
$\cdots$
If $\epsilon_{r-1} = k^{r-1}$, $\epsilon_{r-2} = k^{r-2}$, $\cdots$, and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;
If $\epsilon_{i+r-2} = k^{r-1}$, then $\epsilon_{i+r-3} = 0, 1, \cdots, k^{r-2}$;
If $\epsilon_{i+r-2} = k^{r-1}$ and $\epsilon_{i+r-3} = k^{r-2}$, then $\epsilon_{i+r-4} = 0, 1, \cdots, k^{r-3}$;
$\cdots$
If $\epsilon_{i+r-2} = k^{r-1}$, $\epsilon_{i+r-3} = k^{r-2}$, $\cdots$, $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, \cdots, k$;
If $\epsilon_{i+r-2} = k^{r-1}$, $\epsilon_{i+r-3} = k^{r-2}$, $\cdots$, $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, then $\epsilon_{i=1} = 0$.

The number of conditions increases, of course, as r increases. For $r = 2$, the Fibonacci case needs 3 constraints; for $r = 3$, 5 constraints; for $r = 4$, 7 constraints; for $r = 5$, 9 constraints, and for the r^{th} case, $2r - 1$ constraints are needed and r identities must be used in the inductive proof.

4. THE ZECKENDORF THEOREM FOR SIMULTANEOUS REPRESENTATIONS

Klarner has proved the following theorem in [3]:

Klarner's Theorem. Given non-negative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \cdots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$
$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1},$$

for $|k_i - k_j| \geq 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_n\}_{-\infty}^{+\infty}$, the double-ended Fibonacci sequence, $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.

This is a new Zeckendorf theorem for simultaneous representation. Actually, integers A and B are so representable if and only if

$$\alpha B + A \geq 0, \quad 1 < \alpha < 2, \quad \alpha^2 = \alpha + 1, \quad \alpha = (1 + \sqrt{5})/2,$$

with equality if and only if $A = B = 0$, the vacuous representation of $(0, 0)$ using no representing Fibonacci numbers.

From the fact that $\alpha B + A \geq 0$ is a condition for representability, it follows that every integer can be either an A or a B and can have a proper representation. For instance,

$$-100 = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

for some $\{k_1, k_2, \cdots, k_r\}$, $|k_i - k_j| \geq 2$, $i \neq j$. The line $x = -100$ cuts the line $\alpha x + y = 0$, say, in $(-100, y_0)$. Then let $y_1 > y_0$ be an integer, and $-100\alpha + y_1 > 0$, and so -100 has a representation, and indeed has an infinite number of such representations as all integers $y_k > y_1$ give rise to admissible representations.

Now, given positive integers A and B, $B > A$, how does one find the simultaneous representation of Klarner's Theorem? To begin, write the Zeckendorf minimal representation for $A + B$,

$$A + B = F_{m_1} + F_{m_2} + \cdots + F_{m_n} ,$$

where $|m_i - m_j| \geq 2$, $i \neq j$, $m_1 > m_2 > m_3 > \cdots > m_n$. Then $B = F_{m_1-1} + R_B$ and $A = F_{m_1-2} + R_A$. The next Fibonacci numbers in the representations of B and A are F_{m_2-1} and F_{m_2-2} if $R_B \geq R_A \geq 0$, not both R_B and $R_A = 0$. When m_2 is odd and $R_B < R_A$ or $R_A < 0$, then the next Fibonacci numbers in the representations of B and A are F_{-m_2-1} and F_{-m_2-2}. The process continues, so that the sums of successive terms in the representations of A and of B give the successive terms in $A + B$, except that the last terms may have a zero sum, and the subscripts in the representations of A and B may not be ordered.

We give a constructive proof of Klarner's Theorem using mathematical induction. First, $A = 0$ and $B = 1$ is given uniquely by $A = F_0$ and $B = F_1$, while $A = 1$ and $B = 0$ is given uniquely by $A = F_{-1}$ and $B = F_0$. Here, of course, we seek minimal representations in the form

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} ,$$

for $F_i \in \{F_n\}_{-\infty}^{+\infty}$ with $|k_i - k_j| \geq 2$, $i \neq j$ (the conditions for the original Zeckendorf Theorem), and, of course, we assume that

$$k_1 < k_2 < k_3 < \cdots < k_r .$$

If we make the inductive assumption that all integers $A \geq 0$, $B \geq 0$, $0 \leq A + B \leq n$ can be so represented, then we must secure compatible pairs $A + 1, B$ and $A, B + 1$ each in admissible form from those of the pair A, B. We do this as follows. Let

$$A + 1 = F_{k_1} + \cdots + F_{k_r} + F_{-1} ,$$

$$B = F_{k_1+1} + \cdots + F_{k_r+1} + F_0 ;$$

then we must put $A + 1$ and B into admissible form. We will show how to put these into admissible form in general by putting

$$F_{k_1} + F_{k_2} + \cdots + F_{k_r} + F_m$$

into admissible form. Now, if F_m is detached, there is no problem. If F_m and F_{k_j} are adjacent, then simply use the formula $F_{v+1} = F_v + F_{v-1}$ to work upward in the subscripts until the stress is relieved. Since we have only a finite number r, there is no problem.

Now, if $F_m = F_{k_j}$, then $2F_m = F_m + F_{m-1} + F_{m-2} = F_{m+1} + F_{m-2}$. This may cause $2F_{m-2}$ and also the condition $F_{m+2} + F_{m+1}$. If the latter, use $F_{v+1} = F_v + F_{v-1}$ to move upward in the subscripts to relieve the stress. We notice now that the $2F_{m-2}$ has a smaller subscript than before. Repeat the process. This ultimately terminates or forms two consecutive Fibonacci numbers, where we can use $F_{v+1} = F_v + F_{v-1}$ to relieve the stress, and we are done.

Notice that this same procedure on B leaves the relation between A + 1 and B intact. You can also consider $F_m = F_{-1}$ or $F_m = 0$, etc.

Next, form the sequence $G_0 = A$, $G_1 = B$, $G_{n+2} = G_{n+1} + G_n$, and assume that A (and hence B) has two distinct admissible forms. Then let n become so large that all Fibonacci subscripts are positive, and we will violate the original Zeckendorf Theorem, for G_n would have two distinct representations. Then, A and B must have unique representations in the admissible form.

But, all of this is extendable. The double-ended Lucas sequence, $\{L_n\}_{-\infty}^{+\infty}$, $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, also enjoys the representation property of A and B, for $\alpha B + A \geq 0$, except that, additionally, A and B are chosen such that $5 \mid (A^2 + B^2,\ A^2 + 2AB)$, so that not every integer pair qualifies.

We can generalize Klarner's Theorem to apply to the sequences formed when the Fibonacci polynomials are evaluated at $x = k$ as follows.

Theorem 4.1. Given non-negative integers A and B, there exists a unique set of integers, $\{\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_r, j\}$ such that

$$A = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r},$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1},$$

where each U_i is an element of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence formed from the Fibonacci polynomials, $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, when $x = k$, so that $U_n = F_n(k)$, and the ϵ_i satisfy the constraints $\epsilon_i = 0, 1, 2, \cdots, k$, and if $\epsilon_i = k$, then $\epsilon_{i-1} = 0$, $-\infty < i < +\infty$.

Proof. When $k = 1$, we have Klarner's Theorem. We take $k \geq 2$. First, we can represent uniquely $A = 0 = U_0$ and $B = 1 = U_1$ or $A = 1 = U_{-1}$ and $B = 0 = U_0$. If we make the inductive assumption that all integers $A \geq 0$, $B \geq 0$, $0 \leq A + B \leq n$ can be so represented, then we must secure compatible pairs A + 1, B and A, B + 1 each in admissible form from the pair A, B. Let

$$A + 1 = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} + U_{-1},$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1} + U_0.$$

We show how to put these into admissible form by working with

$$\epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} + U_m.$$

Case 1. U_m is away from any other U_{j+i} or no $\epsilon_i = k$. We are done; there is no interference unless we create $\epsilon_i + 1 = k$, in which case we may have to use $U_{j+1} = kU_j + U_{j-1}$.

Case 2. $\epsilon_i = k$ and $U_m = U_{j+i-1}$; then replace $kU_j + U_{j-1}$ by U_{j+1} and work the subscripts upward to relieve the stress. Notice that this process always terminates, thus minimizing the number of terms. Suppose that $U_m = U_{j+i}$ with $\epsilon_i = k$; then

$$\begin{aligned}(k + 1)U_{j+i} &= (U_{j+i+1} - U_{j+i-1}) + kU_{j+i-1} + U_{j+i-2} \\ &= U_{j+i+1} + (k - 1)U_{j+i-1} + U_{j+i-2} .\end{aligned}$$

Now $\epsilon_{j+1} \neq k$ (since $\epsilon_j = k$), so, if $\epsilon_{j+2} = k$, then $kU_t + U_{t-1} = U_{t+1}$ can be used and the subscripts can be worked upwards to relieve the stress. If the coefficient of U_{j+i-2} is now $(k + 1)$, we note that we can repeat the process and ultimately work it out downward while using $kU_t + U_{t-1} = U_{t+1}$ to relieve the stress upward. In any case, this algorithm will reduce $A + 1$ to an admissible form. Thus, we can represent A' and B' for $0 \le A' + B' \le n + 1$, finishing a proof that pairs of integers are so representable.

Next, to show uniqueness, form the sequence $V_0 = A$, $V_1 = B$, $V_{n+2} = kV_{n+1} + V_n$, and assume that A, and hence B, has two distinct admissible forms. Then, V_n has two distinct representations from those of, say A. Then, if n is large enough, V_n must have terms $\epsilon_i U_i$ which all have positive subscripts, and V_n has two distinct representations, which violates the generalized Zeckendorf Theorem for the Fibonacci polynomials evaluated at $x = k$ given in [1], which guarantees a unique representation.

The condition for representability for the Fibonacci polynomials evaluated at $x = k$ is

$$B(k + \sqrt{k^2 + 4})/2 + A \ge 0$$

with equality only if $A = B = 0$ is vacuously represented.

Now, if we use the Tribonacci numbers, 1, 1, 2, $\cdots$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, then we can represent any three non-negative integers A, B, and C as in Klarner's Theorem. For the Tribonacci numbers,

$$C\lambda^2 + B(\lambda + 1) + A \ge 0, \quad 1 < \lambda < 2, \quad \lambda^3 = \lambda^2 + \lambda + 1 ,$$

is the condition for the ordered triple A, B, C to be a lattice point in the representation half-space, and

$$x\lambda^2 + y(\lambda + 1) + z = 0$$

is the separator plane containing $(0, 0, 0)$ to be represented vacuously using no Tribonacci numbers. We now generalize Klarner's Theorem to Tribonacci numbers.

Theorem 4.2. Given three non-negative integers A, B, and C, there exists a unique set of integers $\{k_1, k_2, \cdots, k_r\}$ such that

$$A = T_{k_1} + T_{k_2} + \cdots + T_{k_r} ,$$

$$B = T_{k_1+1} + T_{k_2+1} + \cdots + T_{k_r+1} ,$$
$$C = T_{k_1+2} + T_{k_2+2} + \cdots + T_{k_r+2} ,$$

where $k_1 < k_2 < k_3 < \cdots < k_r$ and no three k_i, k_{i+1}, k_{i+2} are consecutive integers, and where the T_i are members of the sequence $\{T_n\}_{-\infty}^{+\infty}$, the double-ended sequence of Tribonacci numbers, $T_{-1} = T_0 = 0$, $T_1 = 1$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$.

Proof. First, $A = 1 = T_{-2}$, $B = 0 = T_{-1}$, $C = 0 = T_0$; and $A = 0 = T_{-3} + T_{-2}$, $B = 1 = T_{-2} + T_{-1}$, $C = 0 = T_{-1} + T_0$; and $A = 0 = T_{-1}$, $B = 0 = T_0$, $C = 1 = T_1$ are given uniquely. We make the inductive assumption that all integers $A \ge 0$, $B \ge 0$, $C \ge 0$, $0 \le A + B + C \le n$ can be represented uniquely in the form of the theorem. We must show that we can secure the compatible triples $A + 1, B, C$; $A, B + 1, C$; and $A, B, C + 1$ in admissible form from the triple A, B, C to get the representations for $(A + 1) + B + C \le n + 1$, $A + (B + 1) + C \le n + 1$, $A + B + (C + 1) \le n + 1$. To get $A + 1, B, C$, we add T_{-2} to A, T_{-1} to B, and T_0 to C, and then work upwards in the subscripts if necessary. To get $A, B + 1, C$ from A, B, and C, we add $T_{-3} + T_{-2}$ to the representation for A, $T_{-2} + T_{-1}$ to B, and $T_{-1} + T_0$ to C. To get $A, B, C + 1$ from the representations for A, B, and C, we add respectively T_{-1}, T_0, T_1. Thus, given the representations for A, B, C, $0 \le A + B + C \le n$, we can always make the representation for one member of the triple to be increased by 1, so that we can represent all numbers whose sum is less than or equal to $n + 1$.

Uniqueness follows from Theorem 3.3 with $k = 1$.

Theorem 4.2 can be generalized to the general Tribonacci numbers obtained when the Tribonacci polynomials are evaluated at $x = k$. In that proof, one would obtain $A + 1, B, C$ from A, B, C by adding T_{-2}, T_{-1}, and T_0 to A, B, and C, respectively; $A, B + 1, C$ by adding $T_{-3} + kT_{-2}$ to A, $T_{-2} + kT_{-1}$ to B, and $T_{-1} + kT_0$ to C; and $A, B, C + 1$ by adding T_{-1}, T_0, T_1 to A, B, and C, respectively. Theorem 4.3 contains the generalization, as

Theorem 4.3. Given three non-negative integers A, B, and C, there exists a unique set of integers $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_r, j\}$ such that

$$A = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} ,$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1} ,$$
$$C = \epsilon_1 U_{j+3} + \epsilon_2 U_{j+4} + \cdots + \epsilon_r U_{j+r+2} ,$$

where each U_i is an element of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence given by $U_n = T_n(k)$, $T_{-1}(x) = T_0(x) = 0$, $T_1(x) = 1$, $T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x)$, where the ϵ_i satisfy the constraints $\epsilon_i = 0, 1, 2, \cdots, k^2$, and if $\epsilon_i = k^2$, then $\epsilon_{i-1} = 0, 1, \cdots, k^2 - 1$, and if $\epsilon_i = k^2$ and $\epsilon_{i-1} = k$, then $\epsilon_{i-2} = 0$.

Finally, we can generalize Klarner's Theorem to apply to the sequences which arise when the generalized Fibonacci polynomials are evaluated at $x = k$.

Theorem 4.4. Given r non-negative integers $N_1, N_2, \cdots, N_r$, there exists a unique set of integers $\{k_1, k_2, \cdots, k_s\}$ such that

$$N_i = U_{k_1+i-1} + U_{k_2+i-1} + \cdots + U_{k_s+i-1} \,,$$

where $k_1 < k_2 < k_3 < \cdots < k_s$ and no r $k_i, k_{i+1}, \cdots, k_{i+r-1}$ are consecutive integers, where the U_i are members of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence of r-nacci numbers, $U_{-r} = -1$, $1 = U_{-r+1}$, $U_{-r+2} = U_{-r+3} = \cdots = U_{-1} = U_0 = 0$, $U_1 = 1$, $U_{n+r} = U_{n+r-1} + U_{n+r-2} + \cdots + U_{n+1} + U_n$.

Clearly, an inductive proof of Theorem 4.3, or of the theorem when generalized to the generalized Fibonacci polynomials evaluated at $x = k$, hinges upon being able to add one to one number represented and to again have all r numbers in admissible form. We examine the additions necessary for the inductive step for the generalized Fibonacci polynomials $P_n(x)$ for some small values of r. The induction has been done for $r = 2$ and $r = 3$. For $r = 4$, the Quadranacci polynomials evaluated at $x = k$, where admissible forms are known for A, B, C and D, the additions before adjustment of subscripts are as follows:

$$\begin{aligned}
A + 1 &= A + U_{-3} \,, \\
B &= B + U_{-2} \,, \\
C &= C + U_{-1} \,, \\
D &= D + U_0 \;;
\end{aligned}$$

$$\begin{aligned}
A &= A + U_{-4} + kU_{-3} \,, \\
B + 1 &= B + U_{-3} + kU_{-2} \,, \\
C &= C + U_{-2} + kU_{-1} \,, \\
D &= D + U_{-1} + kU_0 \;;
\end{aligned}$$

$$\begin{aligned}
A &= A + U_{-5} + kU_{-4} + k^2U_{-3} \,, \\
B &= B + U_{-4} + kU_{-3} + k^2U_{-2} \,, \\
C + 1 &= C + U_{-3} + kU_{-2} + k^2U_{-1} \,, \\
D &= D + U_{-2} + kU_{-1} + k^2U_0 \;;
\end{aligned}$$

$$\begin{aligned}
A &= A + U_{-2} \,, \\
B &= B + U_{-1} \,, \\
C &= C + U_0 \,, \\
D + 1 &= D + U_1 \,.
\end{aligned}$$

Note that the Quadranacci polynomials extend to negative subscripts as $U_1 = 1$, $U_0 = U_{-1} = U_{-2} = 0$, $U_{-3} = 1$, $U_{-4} = -k$, $U_{-5} = U_{-6} = 0, \cdots$ when evaluated at $x = k$.

For $r = 5$, the Pentanacci polynomials evaluated at $x = k$ are $U_1 = 1$, $U_0 = U_{-1} = U_{-2} = U_{-3} = 0$, $U_{-4} = 1$, $U_{-5} = -k$, $U_{-6} = U_{-7} = 0$, $U_{-8} = 0$ using the relation $U_n =$

$-kU_{n+1} - k^2U_{n+2} - k^3U_{n+3} - k^4U_{n+4} + U_{n+5}$ to move to values for the negative subscripts. The inductive one step pieces for the Pentanacci case, where representations for A, B, C, D and E are given, are

$$\begin{aligned} A + 1 &= A + U_{-4} & A &= A + U_{-5} + kU_{-4} \\ B &= B + U_{-3} & B + 1 &= B + U_{-4} + kU_{-3} \\ C &= C + U_{-2} & C &= C + U_{-3} + kU_{-2} \\ D &= D + U_{-1} & D &= D + U_{-2} + kU_{-1} \\ E &= E + U_0 & E &= E + U_{-1} + kU_0 \end{aligned}$$

$$\begin{aligned} A &= A + U_{-6} + kU_{-5} + k^2U_{-4}\,, \\ B &= B + U_{-5} + kU_{-4} + k^2U_{-3}\,, \\ C + 1 &= C + U_{-4} + kU_{-3} + k^2U_{-2}\,, \\ D &= D + U_{-3} + kU_{-2} + k^2U_{-1}\,, \\ E &= E + U_{-2} + kU_{-1} + k^2U_0\;; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-7} + kU_{-6} + k^2U_{-5} + k^3U_{-4}\,, \\ B &= B + U_{-6} + kU_{-5} + k^2U_{-4} + k^3U_{-3}\,, \\ C &= C + U_{-5} + kU_{-4} + k^2U_{-3} + k^3U_{-2}\,, \\ D + 1 &= D + U_{-4} + kU_{-3} + k^2U_{-2} + k^3U_{-1}\,, \\ E &= E + U_{-3} + kU_{-2} + k^2U_{-1} + k^3U_0\;; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-3}\,, \\ B &= B + U_{-2}\,, \\ C &= C + U_{-1}\,, \\ D &= D + U_0\,, \\ E + 1 &= E + U_1\;. \end{aligned}$$

Thus the pattern from the first cases is clear. The recurrence relation backward for U_n for general r is

$$U_n = -kU_{n+1} - k^2U_{n+2} - k^3U_{n+3} - \cdots - k^{r-1}U_{n+r-1} + U_{n+r}\,,$$

which leads to the lemma for general r and $k \ge 2$;

Lemma. $U_{-(2r-2)} = \cdots = U_{-(r+1)} = 0$ ($r - 2$ zeroes), $U_{-r} = -k$, $U_{-r+1} = 1$, $U_{-(r-2)} = \cdots = U_0 = 0$ ($r - 1$ zeroes), $U_1 = 1$.

Using the lemma and the pattern made clear by the earlier cases, one could prove the final generalized theorem given below.

Theorem 4.5. Let

$$P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0, \quad P_1(x) = 1, \quad P_2(x) = x^{r-1},$$

and

$$P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x) \, ,$$

and let

$$U_n = P_n(k) \, .$$

Then, given r non-negative integers $N_1, N_2, \cdots, N_r$, there exists a unique set of integers $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_s, j\}$ such that

$$N_i = \epsilon_1 U_{j+i+1} + \epsilon_2 U_{j+i+2} + \cdots + \epsilon_s U_{j+i+s}$$

for $i = 1, 2, \cdots, r$, where

$$U_i \in \{U_n\}_{-\infty}^{+\infty}$$

and ϵ_i satisfies the constraints $\epsilon_i = 0, 1, 2, \cdots, k^{r-1}$, $-\infty < i < +\infty$; where if $\epsilon_{i+r-2} = k^{r-1}$, then $\epsilon_{i+r-3} = 0, 1, \cdots, k^{r-2}$; if $\epsilon_{i+r-2} = k^{r-1}$ and $\epsilon_{i+r-3} = k^{r-2}, \cdots,$ and $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

5. CONDITIONS FOR REPRESENTABILITY

In Section 4, a necessary and sufficient condition for representability of an integer pair A, B by Klarner's Theorem was given as

$$\alpha B + A \geq 0, \qquad \alpha = (1 + \sqrt{5})/2 \, ,$$

where α is the positive root of $\lambda^2 - \lambda - 1 = 0$. Here a proof is provided, as well as statement and proof in the general case.

First, the Fibonacci polynomials have the recursion relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

and hence the associated polynomial

$$\lambda^2 - x\lambda - 1 = 0$$

with roots λ_1 and λ_2, $\lambda_1 > \lambda_2$, $\lambda_1 = (x + \sqrt{x^2 + 4})/2$. If $F_1(x)$ is written as a linear combination of the roots, $F_1(x) = A_1\lambda_1 + A_2\lambda_2$, then $F_n(x) = A_1\lambda_1^n + A_2\lambda_2^n$. We consider the limiting ratio of successive Fibonacci polynomials, which becomes

$$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \lim_{n\to\infty} \frac{A_1\lambda_1^{n+1} + A_2\lambda_2^{n+1}}{A_1\lambda_1^n + A_2\lambda_2^n} = \lambda_1 = \frac{x + \sqrt{x^2 + 4}}{2}$$

upon dividing through by λ_1^n, since $|\lambda_2/\lambda_1| < 1$.

Now let $H_0 = A$ and $H_1 = B$, linear combinations of elements of the sequence of Fibonacci polynomials evaluated at $x = k$ as defined in Theorem 4.1, and let $H_n = kH_{n-1} + H_{n-2}$ be the recursion relation for $\{H_n\}$. Then, as the special case of identity (4.6) proved in [2] where $r = 2$,

$$H_{n+1} = H_1F_{n+1}(k) + H_0F_n(k)$$

$$\frac{H_{n+1}}{F_n(k)} = \frac{H_1F_{n+1}(k)}{F_n(k)} + H_0 \; .$$

For sufficiently large n, we have $H_{n+1}/F_n(k) > 0$. Thus, taking the limit of the expression above as n tends to infinity,

$$0 \le \lambda_1 H_1 + H_0 = B(k + \sqrt{k^2 + 4})/2 + A \;, \tag{5.1}$$

the condition for representability of an integer pair A, B by Theorem 4.1, with equality only if $A = B = 0$ is vacuously represented. The conditions for Klarner's Theorem follow when $k = 1$.

In the Tribonacci case, we let $H_0 = A$, $H_1 = B$, $H_2 = C$, linear combinations of the elements of the sequence of Tribonacci polynomials evaluated at $x = k$ as defined in Theorem 4.3, and let $H_n = k^2H_{n-1} + kH_{n-2} + H_{n-3}$ be the recursion relation for the sequence $\{H_n\}$, the same recursion as for the Tribonacci polynomials evaluated at $x = k$, the sequence $\{T_n(k)\}$. Both $\{T_n(k)\}$ and $\{H_n\}$ have, then, the associated polynomial

$$\lambda^3 - k^2\lambda^2 - k\lambda - 1 = 0$$

with roots $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 > |\lambda_2| \ge |\lambda_3|$, $k^2 < \lambda_1 < k^3$ for $k \ge 2$, and $1 < \lambda_1 < 2$ for $k = 1$; and λ_1 is the root greatest in absolute value. Analogous to the Fibonacci case, we can prove that

$$\lim_{n\to\infty} \frac{T_{n+m}(k)}{T_n(k)} = \lambda_1^m \; .$$

Again applying the identity (4.6) from [2], where $r = 3$, we write

$$H_{n+2} = H_2T_{n+1}(k) + H_1[kT_n(k) + T_{n-1}(k)] + H_0T_n(k) \; .$$

Upon division by $T_{n-1}(k)$, for n sufficiently large, $H_{n+2}/T_{n-1}(k) > 0$. Then we evaluate the limit as n approaches infinity to obtain

$$(5.2)\qquad 0 \le \lambda_1^2 H_2 + H_1[k\lambda_1 + 1] + H_0\lambda_1 = \lambda_1^2 C + B[k\lambda_1 + 1] + \lambda_1 A ,$$

with equality only if $A = B = C = 0$, the vacuous representation. Thus, we have the conditions for representability of an integer triple A, B, C in terms of Tribonacci polynomials evaluated at $x = k$ as in Theorem 4.3. The conditions for Theorem 4.2 for representation using Tribonacci numbers follow when $k = 1$.

Now, for representability in the general case, we consider a sequence $\{H_n\}$ having the same recursion as the generalized Fibonacci polynomials $\{P_n(k)\}$,

$$H_{n+r} = k^{r-1}H_{n+r-1} + k^{r-2}H_{n+r-2} + \cdots + H_n ,$$

and take as its initial values $H_0 = N_1,\ H_1 = N_2, \cdots, H_{r-1} = N_r$, the r integers represented in Theorem 4.5. Now, the generalized Fibonacci polynomials evaluated at $x = k$ have the associated polynomial

$$(5.3)\qquad \lambda^r = (k\lambda)^{r-1} + (k\lambda)^{r-2} + \cdots + k\lambda + 1$$

with roots $\lambda_1 > |\lambda_2| \ge |\lambda_3| \ge \cdots \ge |\lambda_r|$, where $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$, $k \ge 2$, and where λ_1 is the root of greatest modulus. (If $k = 1$, then $1 < \lambda_1 < 2$.)

We next prove that there is a root of greatest modulus for (5.3), that the roots are distinct, and that the root of greatest modulus is positive and lies in the interval described.

<u>Lemma 1.</u> Let

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1 .$$

Then, for $r \ge 2$ and $k \ge 2$, $f(k^{r-1}) < 0$ and $f(k^{r-1} + 1/k) > 0$.

<u>Proof.</u> Let $\lambda^* = k\lambda$, so that

$$h(\lambda^*) = k^r f(\lambda) = \lambda^{*r} - k^r(\lambda^{*r-1} + \lambda^{*r-2} + \cdots + 1) .$$

Then,

$$\begin{aligned} h(k^r) &= k^{r^2} - k^r(k^{r^2-r} + k^{r^2-2r} + \cdots + 1) \\ &= k^{r^2} - k^{r^2} - k^{r^2-r} - \cdots - k^r < 0 , \end{aligned}$$

and this implies that $f(k^{r-1}) < 0$.

Now, let $\lambda_1 > 1/k$ be a zero of $g(\lambda)$, where

$$g(\lambda) = (\lambda k - 1)f(\lambda) = (\lambda k - 1)\left(\lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1}\right)$$

by summing the geometric series formed by all but the first term of $f(\lambda)$. Then,

$$-g(\lambda_1) = \lambda_1^r(k^r + 1 - k\lambda_1) - 1 = 0$$

so that

$$\lambda_1^r(k^r + 1 - k\lambda_1) = 1 .$$

Thus

$$k^r + 1 - k\lambda_1 > 0 ,$$

or

$$\lambda_1 < k^{r-1} + \frac{1}{k} .$$

We note that $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$ for $k \geq 2$, $r > 2$, agrees with $1 < \lambda_1 < 2$ for the case $k = 1$, $r \geq 2$.

Lemma 2. Take $f(\lambda)$ as defined in Lemma 1, and let

$$g(\lambda) = (\lambda k - 1)f(\lambda) = \lambda^{r+1}k - \lambda^r - \lambda^r k^r + 1 .$$

Then, $g(\lambda)$ and $g'(\lambda)$ have no common zeros.

Proof. Since

$$g'(\lambda) = \lambda^{r-1} [\lambda k(r + 1) - r(1 + k^r)] ,$$

$\lambda = 0$ is an $(r - 1)$-fold zero of $g'(\lambda)$, and the other root is

$$\lambda = \frac{r}{k(r + 1)} (1 + k^r) .$$

We observe that $\lambda = 0$ is not a zero of $g(\lambda)$, and

$$\lambda g'(\lambda) = rg(\lambda) + \lambda^{r+1}k - r .$$

Let λ_0 be a common root of $g'(\lambda) = 0$ and $g(\lambda) = 0$ so that

$$\lambda_0^{r+1}k - r = 0, \quad \text{or} \quad \lambda_0^{r+1} = \frac{r}{k} .$$

We note in passing that if $k\lambda_0^{r+1} = r$ and $g(\lambda_0) = 0$, then

$$g(\lambda_0) = \lambda_0^{r+1}k - \lambda_0^r(1 + k^r) + 1 = 0$$

so that

$$g(\lambda_0) = r - \lambda_0^r(1 + k^r) + 1 = 0, \quad \text{or} \quad \lambda_0^r = \frac{r + 1}{1 + k^r} .$$

We now solve for λ_0 :

$$\lambda_0^{r+1} = \frac{(\lambda_0)(r+1)}{1+k^r} = \frac{r}{k}$$

or

(a) $$\lambda_0 = \frac{r(1+k^r)}{(r+1)k} \quad .$$

We now show that $k\lambda_0^{r+1} = r$ is inconsistent with (a), by demonstrating that

$$\left(\frac{r(k^r+1)}{(r+1)k}\right)^{r+1} > \frac{r}{k} \ .$$

For $k \geq 2$, $r \geq 3$, $k^{r^2-1}/r > 4$ and

$$1 < \left(\frac{1+k^r}{k^r}\right)^{k^r+1} < 4$$

so that

$$1 < \left(\frac{1+k^r}{k^r}\right)^{r+1}$$

$$4 < \left(\frac{k^{r^2-1}}{r}\right) \cdot \left(\frac{1+k^r}{k^r}\right)^{r+1}$$

while

$$4 > \left(\frac{r+1}{r}\right)^{r+1} > e \ .$$

The fact that

$$\left(\frac{k^{r^2-1}}{r}\right) \cdot \left(\frac{1+k^r}{k^r}\right)^{r+1} > \left(\frac{r+1}{r}\right)^{r+1}$$

is equivalent to the stated inequality. Thus we conclude that there are no common zeros between the functions $g(\lambda)$ and $g'(\lambda)$, for if there would be at least one repeated root λ_0, then the two expressions for λ_0^{r+1} would be equal, which has been shown to be impossible.

Comments. For all integers $r \geq 2$ and $k \geq 1$, the

Theorem. The roots of

$$\lambda^r = \lambda^{r-1} + \lambda^{r-2} + \cdots + \lambda + 1$$

are distinct. The root λ_1, of greatest modulus, lies in the interval $1 < \lambda_1 < 2$, and the remaining $r - 1$ roots $\lambda_2, \lambda_3, \cdots, \lambda_r$ satisfy $|\lambda_j| < 1$ for $j = 2, 3, \cdots, r$.

was proved by E. P. Miles, Jr., in [4]. The case $k \geq 2$ and $r = 2$ is very easy to prove, involving only a quadratic. The general case for $k \geq 2$ and $r \geq 3$ now follows.

Theorem 5.1. For $r \geq 3$, $k \geq 2$, the roots $\lambda_1, \lambda_2, \cdots, \lambda_r$ of the polynomial

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1$$

are distinct, and λ_1, the root of greatest modulus, satisfies

$$k^{r-1} < \lambda_1 < k^{r-1} + \frac{1}{k} .$$

Proof. Let

$$g(\lambda) = (\lambda k - 1)f(\lambda) = \lambda^{r+1}k - \lambda^r - \lambda^r k^r + 1 .$$

Clearly, $g(\lambda)$ has the same zeros as $f(\lambda)$ except that $g(\lambda) = 0$ also when $\lambda = 1/k$. By Lemma 2, the polynomial $g(\lambda)$ has no repeated zeros and thus for $k \geq 2$, $r \geq 3$, the polynomial $f(\lambda)$ has no repeated zeros.

We now show that the root λ_1 of Lemma 1,

$$k^{r-1} < \lambda_1 < k^{r-1} + \frac{1}{k} ,$$

is the zero of greatest modulus for the polynomial $f(\lambda)$. We make use of the theorem appearing in Marden [5]:

Theorem (32, 1) (Montel): At least p zeros of the polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

lie in the circular disk

$$|z| < 1 + \max\left|\frac{a_j}{a_n}\right|^{\frac{1}{n-p+1}} , \qquad j = 0, 1, 2, \cdots, p .$$

As applied to our $f(\lambda)$,

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1 ,$$

$a_r = 1$ and $a_j = k^j$, $j = 0, 1, \cdots, r - 1$. Thus $(r - 1)$ of the zeros of $f(\lambda)$ lie inside the disk

$$|\lambda| < 1 + k^{(r-1)/2} .$$

To show that $\lambda_1 > |\lambda|$, we simply compare the two. From Lemma 1,

$$k^{r-1} < \lambda_1 < k^{r-1} + 1/k ,$$

$$|\lambda| < 1 + k^{(r-1)/2} .$$

The quadratic $x^2 - x - 1 > 0$ if $x > (1 + \sqrt{5})/2$; thus, $k^{(r-1)/2} = x > (1 + \sqrt{5})/2$ if $k \geq 2$ and $r \geq 3$. Therefore, the $(r - 1)$ zeros of $f(\lambda)$ distinct from λ_1 have modulus less than that of λ_1. We conclude that $f(\lambda)$ does indeed have a positive root and this root is the one of greatest modulus.

Corollary 5.1.1. For all real numbers $k \geq 2$ and all positive integers $r \geq 2$, the polynomial $f(\lambda)$ of Theorem 5.1 has distinct zeroes and λ_1, the zero of greatest modulus, is positive and satisfies $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$.

Corollary 5.1.2. The only positive root λ_1 of the polynomial $f(\lambda)$ of Theorem 5.1 lies in the interval

$$k^{r-1} + \frac{1}{k} - \frac{1}{k^{r^2-r+1}} < \lambda_1 < k^{r-1} + \frac{1}{k} .$$

Proof. We have only to show that

$$\lambda_1 > k^{r-1} + \frac{1}{k} - \frac{1}{k^{r^2-r+1}} = a .$$

Calculating $f(a)$ from the following form,

$$f(\lambda) = \lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1} = \frac{k\lambda^{r+1} - (k^r + 1)\lambda^r + 1}{k\lambda - 1}$$

it is not difficult to show that $f(a) < 0$. But $f(\lambda) > 0$ whenever $\lambda > \lambda_1$, and λ_1 is the only positive root. Also, it is not difficult to show that $a > 0$. Therefore, we must have $\lambda_1 > a$.

Corollary 5.1.2 still yields $1 < \lambda_1 < 2$ for $k = 1$, $r \geq 2$. For $k = 10$ and $r = 10$, the root can vary only in an interval $\Delta = 1/10^{91}$; if $k = 10$ and $r = 100$, then $\Delta = 1/10^{9901}$ making an extremely accurate approximation for large values of r.

The following improved proof of Theorem 5.1 was given by A. P. Hillman [7]. First,

$$f(\lambda) = \lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1} = \frac{k\lambda^{r+1} - (k^r + 1)\lambda^r + 1}{k\lambda - 1} = \frac{k\lambda^r(\lambda - k^{r-1}) - (\lambda^r - 1)}{k\lambda - 1}$$

$$f(k^{r-1}) = -\frac{k^{r(r-1)} - 1}{k^r - 1} < 0$$

and

$$f(k^{r-1} + (1/k)) = \frac{k(k^{r-1} + (1/k))^r \cdot (1/k) - [(k^{r-1} + (1/k))^r - 1]}{k^r} = \frac{1}{k^r} > 0.$$

It now follows from the Intermediate Value Theorem that $f(\lambda_1) = 0$ for some λ_1 with $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$. But Descartes' Rule of Signs tells us that $f(\lambda) = 0$ has only one positive root. Hence, λ_1 is the only positive root. Since the coefficient of the highest power, λ^r, is positive in $f(\lambda)$, we know that $f(\lambda) > 0$ for λ very large. But $f(\lambda)$ does not change sign for $\lambda > \lambda_1$. Hence $f(\lambda) > 0$ for $\lambda > \lambda_1$.

Now let $|\lambda| = p$ with $p > \lambda_1$. Then $p^r - (kp)^{r-1} - (kp)^{r-2} - \cdots - kp - 1 > 0$, or $p^r > (kp)^{r-1} + (kp)^{r-2} + \cdots + kp + 1$, and

$$\begin{aligned} |\lambda^r| = p^r > (kp)^{r-1} + \cdots + kp + 1 &= |(k\lambda)^{r-1}| + \cdots + |k\lambda| + 1 \\ &\geq |(k\lambda)^{r-1} + \cdots + k\lambda + 1| . \end{aligned}$$

Hence, $\lambda^r \neq (k\lambda)^{r-1} + (k\lambda)^{r-2} + \cdots + k\lambda + 1$ and so $f(\lambda) \neq 0$ for $\lambda > \lambda_1$.

Next let $|\lambda| = \lambda_1$ with $\lambda \neq \lambda_1$. Since

$$|z_1| + |z_2| + \cdots + |z_n| > |z_1 + z_2 + \cdots + z_n|$$

if the z_i are not all on the same ray from the origin,

$$\begin{aligned} |\lambda^r| = \lambda_1^r &= (k\lambda_1)^{r-1} + \cdots + k\lambda_1 + 1 \\ &= |(k\lambda)^{r-1}| + \cdots + |k\lambda| + 1 > |(k\lambda)^{r-1} + \cdots + k\lambda + 1| . \end{aligned}$$

Thus, for $|\lambda| = \lambda_1$, $\lambda \neq \lambda_1$, we have $\lambda^r \neq (k\lambda)^{r-1} + \cdots + k\lambda + 1$, or $f(\lambda) \neq 0$.

All that remains is to show that λ_1 is not a multiple root of $f(\lambda) = 0$, i.e., not a root of $f'(\lambda) = 0$. Since $f(\lambda_1) = 0$, we have

$$(k^r + 1)\lambda_1^r = k\lambda_1^{r+1} + 1 .$$

Then

$$\begin{aligned} f'(\lambda_1) = \frac{(r + 1)k\lambda_1^r - r(k^r + 1)\lambda_1^{r-1}}{k\lambda_1 - 1} &= \frac{(r + 1)k\lambda_1^{r+1} - r(k^r + 1)\lambda_1^r}{\lambda_1(k\lambda_1 - 1)} \\ &= \frac{(2rk + k + r)\lambda_1^{r+1}}{\lambda_1(k\lambda_1 - 1)} . \end{aligned}$$

Hence, $f'(\lambda_1) \neq 0$ and the proof is finished.

Corollary. Theorem 5.1 holds for any $k > 0$.

Proof. Examine the Hillman proof of the theorem and see the fact that $k \geq 2$ was not explicitly used as in the earlier proof. This extends Theorem 5.1 to include E. P. Miles' theorem.

Theorem 5.1 states that the zeros of $f(\lambda)$ are distinct. Something of this kind is needed since if a root, say λ_2, is repeated $(r-1)$ times, then

$$Q_n = A_1\lambda_1^n + \lambda_2^n(B_2 n^{r-2} + B_3 n^{r-3} + \cdots + B_r)$$

and the existence of

$$\lim_{n\to\infty} \frac{\lambda_2^{n+1}(B_2 n^{r-2} + B_3 n^{r-3} + \cdots + B_r)}{\lambda_1^n}$$

may be in doubt or at least it raises some questions.

Now, for the generalized Fibonacci polynomials $\{P_n(k)\}$, since we can write

$$P_n(k) = A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_r\lambda_r^n ,$$

a linear combination of the roots of the associated polynomial (5.3),

$$\frac{P_{n+m}(k)}{P_n(k)} = \frac{A_1\lambda_1^{n+m} + A_2\lambda_2^{n+m} + \cdots + A_r\lambda_r^{n+m}}{A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_r\lambda_r^n} .$$

Upon division by λ_1^n, since $|\lambda_i/\lambda_1| < 1$, $i = 2, 3, \cdots, r$,

$$\lim_{n\to\infty} \frac{P_{n+m}(k)}{P_n(k)} = \lambda_1^m$$

so that the ratio of a pair of successive generalized Fibonacci polynomials evaluated as $x = k$ approaches the greatest positive root of its associated polynomial as n approaches infinity.

Now, the following was proved as identity (4.6) in [2]:

$$(5.4)\qquad \begin{aligned} H_{n+r-1} = H_{r-1}P_{n+1}(k) &+ H_{r-2}[k^{r-2}P_n(k) + k^{r-3}P_{n-1}(k) \\ &\qquad + \cdots + P_{n-r+2}(k)] \\ &+ H_{r-3}[k^{r-3}P_n(k) + k^{r-4}P_{n-1}(k) \\ &\qquad + \cdots + P_{n-r+3}(k)] \\ &+ \cdots \\ &+ H_1[xP_n(k) + P_{n-1}(k)] + H_0P_n(k) . \end{aligned}$$

Upon division by $P_{n-r+2}(k)$, if $H_{n+r-1}/P_{n-r+2}(k) > 0$ for all sufficiently large values of n, (5.4) becomes

$$\begin{aligned} 0 \le H_{r-1}\lambda_1^{r-1} &+ H_{r-2}[(k\lambda_1)^{r-2} + (k\lambda_1)^{r-3} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1 H_{r-3}[(k\lambda_1)^{r-3} + (k\lambda_1)^{r-4} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1^2 H_{r-4}[(k\lambda_1)^{r-4} + (k\lambda_1)^{r-5} + \cdots + k\lambda_1 + 1] \\ &+ \cdots + \lambda_1^{r-3} H_1(k\lambda_1 + 1) + \lambda_1^{r-2} H_0 \ . \end{aligned}$$

Thus, the representability condition for Theorem 4.5, for the generalized Fibonacci polynomials evaluated at $x = k$, becomes

$$\begin{aligned} 0 \le N_r\lambda_1^{r-1} &+ N_{r-1}[(k\lambda_1)^{r-2} + (k\lambda_1)^{r-3} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1 N_{r-2}[(k\lambda_1)^{r-3} + (k\lambda_1)^{r-4} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1^2 N_{r-3}[(k\lambda_1)^{r-4} + (k\lambda_1)^{r-5} + \cdots + k\lambda_1 + 1] \\ &+ \cdots + \lambda_1^{r-3} N_2(k\lambda_1 + 1) + \lambda_1^{r-2} N_1 \ , \end{aligned}$$

where λ_1 is the positive root of greatest absolute value of the associated polynomial (5.3), with equality only if $N_1 = N_2 = \cdots = N_r = 0$, the vacuous representation.

When $k = 1$, we have the representation conditions for r integers $N_1, N_2, \cdots, N_r$ in terms of the r-bonacci numbers as in Theorem 4.4.

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A DISCUSSION OF SUBSCRIPT SETS WITH SOME FIBONACCI COUNTING HELP

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1. INTRODUCTION

If the elements of continued fraction-oriented physical and mathematical systems are systematically arranged with respect to subscripts attached to the elements, the choice of order and parity for the subscripts often leads to easily implemented algorithms for the combinatorial determination of the subscripts. All the essential information of the problem can be carried by the subscripts since integer manipulation of the subscripts can substitute for algebraic manipulation of the elements of the system. Specific and general sets of subscripts are discussed, together with the application of Fibonacci methods for the counting of members of subscript sets.

2. "BASIC" SUBSCRIPT SETS AND THEIR GENERATION

The Euler-Minding formulas are introduced early in Perron's classic "Die Lehre von den Kettenbrüchen" [1] and figure prominently in much of the subsequent continued fraction discussions. If Perron's notation is altered slightly to eliminate (for convenience) the zero subscript, the Euler-Minding formulas appear as

$$
\begin{aligned}
S_n = a_1a_2\cdots a_n\Bigg(1 &+ \sum_{j}^{1,n-1}\frac{c_j}{a_j a_{j+1}} + \sum_{j<k}^{1,n-2}\frac{c_j}{a_j a_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}} \\
&+ \sum_{j<k<\ell}^{1,n-3}\frac{c_j}{a_j a_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}}\,\frac{c_\ell}{a_{\ell+2}a_{\ell+3}} + \cdots\Bigg)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
T_{n-1} = a_2a_3\cdots a_n\Bigg(1 &+ \sum_{j}^{k,n-2}\frac{c_j}{a_j a_{j+1}} + \sum_{j<k}^{1,n-3}\frac{c_j}{a_j a_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}} \\
&+ \sum_{j<k<\ell}^{1,n-4}\frac{c_j}{a_j a_{j+1}}\,\frac{c_k}{a_{k+1}a_{k+2}}\,\frac{c_\ell}{a_{\ell+2}a_{\ell+3}} + \cdots\Bigg)\,.
\end{aligned} \tag{2}
$$

There are $\left[\frac{n}{2}\right]$ summations plus the one in the parentheses of (1) and $\left[\frac{n-1}{2}\right]$ summations plus the one in the parentheses of (2).*

*The brackets specify the largest integer less than or equal to the number bracketed.

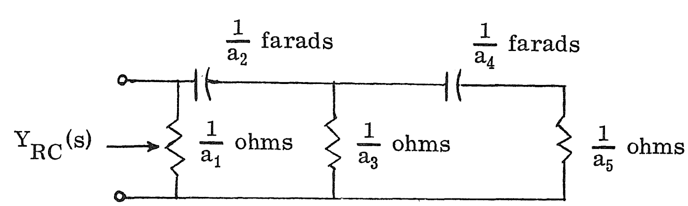
By letting the c's of (1) and (2) assume particular values, the ratio S_n/T_{n-1} can be used to describe various rational fraction forms of continued fractions some of which are directly related to physical structures. For example, if the c's are all equal to one, the ratio S_n/T_{n-1} is the rational fraction equivalent of the continued fraction [1]

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\cdots\,\frac{1}{+ a_n} = \frac{S_n}{T_{n-1}} . \tag{3}$$

More concretely, for n equal five,

$$\begin{aligned} &a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\,\frac{1}{a_5} = \\ &\frac{a_5a_4a_3a_2a_1 + (a_5a_4a_3 + a_5a_4a_1 + a_5a_2a_1 + a_3a_2a_1) + (a_5 + a_3 + a_1)}{a_5a_4a_3a_2 + (a_5a_4 + a_5a_2 + a_3a_2) + 1} . \end{aligned} \tag{4}$$

Salzer [2] in an interpolation problem sets all c's equal to $(x - x_i)$ in the ratio S_n/T_{n-1} and uses the continued fraction process to retrieve $a_1, a_2, \cdots$. As a further example, by letting the c's equal the complex frequency variable $s = \sigma + j\omega$, the impedance or admittance of two-element kind electrical ladder networks can be described by S_n/T_{n-1}. For instance, the resistance-capacitance network



has the S_n/T_{n-1} ratio [3]

$$Y_{RC}(s) = \frac{(a_5 + a_3 + a_1)s^4 + (a_5a_4a_3 + a_5a_4a_1 + a_5a_2a_1 + a_3a_2a_1)s^2 + a_5a_4a_3a_2a_1}{s^4 + (a_5a_4 + a_5a_2 + a_3a_2)s^2 + a_5a_4a_3a_2} . \tag{5}$$

It is seen that the ascending subscript arrangement in the continued fraction of (4) and in the physical network above both lead to rational fractions having numerators and denominators with sums of products of n or less coefficients with the sums of products of <u>no</u> coefficients being interpreted as the numeric one. Features immediately apparent with each sum of h coefficients are the lexicographical order of subscripts, the absence of repeats, and the presence of a leading $a_na_{n-1} \cdots a_{n+1-h}$ and a final $a_ha_{h-1} \cdots a_1$ or $a_{h+1}a_h \cdots a_2$.

It is seen that equationwise all the information needed for the construction of the rational fraction is contained in the subscripts alone.

The subscripts of the coefficients of a sum of products of h coefficients thus constitute a <u>subscript set.</u> The numerator and denominator of the rational fraction can thereby be represented as a collection of subscript sets. Because of the basic nature of (3) and because of the basic role played by the subscripts exemplified by (4) in specifying properties of more general subscript sets, the subscripts of a sum of products of h coefficients determined from a continued fraction as in (3) are called <u>basic subscript sets</u> and are given the symbol $\{N_n^h\}_0$ where n is the largest subscript of the set, h is the number coefficients in each product, and the 0 subscript on the braces identifies the set as "basic." A typical basic subscript set from (4) is $\{5, 4, 3;\ 5, 4, 1;\ 5, 2, 1;\ 3, 2, 1\}$.

What are the precise properties of basic subscript sets? How can they be generated <u>easily</u>, and what is the power of a basic subscript set? A discussion follows.

Consider a sequence of h non-zero, non-repeating integers, called <u>subscripts.</u> The subscripts in the sequence are arranged in alternating parity and descending size with the largest subscript (on the left) assigned a specific parity. A basic subscript set has as members all possible such sequences with the largest subscript in any sequence not exceeding n. The subscript sets are represented as

$$\{N_n^h\}_0 = \boxed{;}_{f=0}^{n/2} (n - 2f), \quad \{N_{h-2f-1}^{h-1}\}, \qquad n \text{ even}, \tag{6}$$

$$\{N_n^h\}_0 = \boxed{;}_{f=0}^{(n-1)/2} (n - 2f), \quad \{N_{h-2f-1}^{h-1}\}, \qquad n \text{ odd}. \tag{7}$$

$\{N_n^0\}_0$ stands for <u>no</u> subscripts and is associated with the numeric one or a single term with no coefficients. (See, for example, the denominators of (4) and (5).) $\{N_n^k\}_0$ for $k > n$ is the null set with no value. The boxed semicolon $\boxed{;}$ is a symbol for collecting the sequences of a subscript set.

If n is odd (even), the largest subscript of any sequence has odd (even) parity. The smallest subscript of any sequence has odd (even) parity if $n - h + 1$ is odd (even).

From (6) and (7), it can be determined that the <u>starting</u>* sequence-<u>last</u> sequence pair of $\{N_n^h\}_0$ assume either (8) and (9) or (10) and (11).

$$\left.\begin{array}{ll} (8) & n, n-1, n-2, \cdots, n-h+1 \\ (9) & h, h-1, h-2, \cdots, 1 \end{array}\right\} \quad n - h + 1 \text{ odd}$$

*No other sequence with the prescribed properties can be found which has a larger subscript in a given position than the subscript in that position in the starting sequence. If "less than" is substituted for "larger than," the last sequence is described.

$$\left.\begin{array}{ll} (10) & n,\ n-1,\ n-2,\ \cdots,\ n-h+1 \\ (11) & h+1,\ h,\ h-1,\ \cdots,\ 2 \end{array}\right\},\quad n-h+1 \text{ even}.$$

Note that the difference between given position subscripts in the starting and last sequences is a constant q, where $q = (n - h)$ for $(n - h) + 1$ odd and $q = (n - h - 1)$ for $(n - h) + 1$ even. In either case, q is even. This is a property which is valid for the more general subscript sets discussed later.

An algorithm to generate basic subscript sets can be deduced from an inspection of (1) and (2) once the starting and last sequence have been established. Assume that the f^{th} member of a subscript set is known. To find the $(f + 1)^{st}$ member, start at the right side of the f^{th} member and scan the subscripts toward the left until the first subscript is found which has a value of at least two greater than the corresponding position subscript of the last sequence. Subtract two from this subscript to obtain the subscript for the $(f + 1)^{st}$ member and complete the $(f + 1)^{st}$ member by filling all positions to the right with the largest possible subscripts consistent with size-order and position parity. Note that subtraction of two's is necessary to retain position parity.

The implementation of the algorithm is even simpler than the description as is illustrated in the "by hand" generation of $\{N_8^4\}_0$ in (12).

$$(12)\qquad \begin{array}{ccccc} 8,\ 7,\ 6,\ 5 & 8,\ 7,\ 4,\ 3 & 8,\ 5,\ 4,\ 3 & 8,\ 3,\ 2,\ 1 & 6,\ 5,\ 2,\ 1 \\ \underline{\quad\quad\ -2} & \underline{\quad\quad\ -2} & \underline{\quad\quad\ -2} & \underline{-2\quad\quad\ } & \underline{\quad -2\quad\ } \\ 8,\ 7,\ 6,\ 3 & 8,\ 7,\ 4,\ 1 & 8,\ 5,\ 4,\ 1 & 6,\ 5,\ 4,\ 3 & 6,\ 3,\ 2,\ 1 \\ \underline{\quad\quad -2\ \ } & \underline{\quad -2\quad\ } & \underline{\quad -2\quad\ } & \underline{\quad\quad\ -2} & \underline{-2\quad\quad\ } \\ 8,\ 7,\ 6,\ 1 & 8,\ 7,\ 2,\ 1 & 8,\ 5,\ 2,\ 1 & 6,\ 5,\ 4,\ 1 & 4,\ 3,\ 2,\ 1 \\ \underline{\quad -2\quad\ } & \underline{\ -2\quad\quad} & \underline{\ -2\quad\quad} & \underline{\quad\quad -2\ \ } & \\ 8,\ 7,\ 4,\ 3 & 8,\ 5,\ 4,\ 3 & 8,\ 3,\ 2,\ 1 & 6,\ 5,\ 2,\ 1 & \end{array}$$

What is the power of a basic subscript set? It can be shown by comparison with a physical model that the power of the collection of either numerator subscript sets or denominator subscript sets is Fibonaccian and this, in turn, provides a clue to the answer.

It is well established [4]-[6] that the resistance or conductance of electrical ladder networks has as the ratio of numerator terms to denominator terms a ratio of Fibonacci numbers. For example, if a ladder network is composed of n unit conductances with a shunt conductance at the input end and either a shunt conductance (n odd) or a short circuit (n even) at the output end, the conductance measured at the input terminals is given by*

---

*Several other forms in terms of resistance or conductance are, of course, possible. For example, Basin [6] states the input resistance of the dual of the above network with n even as F_{2n+1}/F_{2n}. However, Basin's n is half the n of this paper because of a choice in size of his unit network.

$$G_n = \frac{F_{n+1}}{F_n} \text{ mhos }, \tag{13}$$

where $F_1, F_2, F_3, F_4, \cdots = 1, 1, 2, 3, \cdots$ are the well-known Fibonacci numbers. Moreover, if the shunt arms of the ladder network are replaced and described by odd subscripted admittances (y's) and the series arms are replaced and described by even subscripted impedances (z's) with the numbering increasing away from the input terminals, (4) exemplifies the continued fraction and rational fraction form of the input admittance. To complete the identification, odd subscripted a's of (4) are interpreted as y's, and even subscripted a's are interpreted as z's. It can be seen that the power of a collection of basic subscript sets is given by

$$\begin{aligned} \left|\{N_n^{n-1}\}_0 + \{N_n^{n-3}\}_0 + \cdots + \{N_n^0\}_0\right| &= \left|\{N_n^{n-1}\}_0\right| + \left|\{N_n^{n-3}\}_0\right| + \cdots + \left|\{N_n^0\}_0\right| \\ &= \left|\{N_{n-1}^{n-1}\}_0 + \{N_{n-1}^{n-3}\}_0 + \cdots + \{N_{n-1}^0\}_0\right| \\ &= \left|\{N_{n-1}^{n-1}\}_0\right| + \left|\{N_{n-1}^{n-1}\}_0\right| + \cdots + \left|\{N_{n-1}^0\}_0\right| = F_n, \quad n \text{ odd}, \end{aligned} \tag{14}$$

$$\begin{aligned} \left|\{N_n^{n-1}\}_0 + \{N_n^{n-3}\}_0 + \cdots + \{N_n^1\}_0\right| &= \left|\{N_n^{n-1}\}_0\right| + \left|\{N_n^{n-3}\}_0\right| + \cdots + \left|\{N_n^1\}_0\right| \\ &= \left|\{N_{n-1}^{n-1}\}_0 + \{N_{n-1}^{n-3}\}_0 + \cdots + \{N_{n-1}^1\}_0\right| \\ &= \left|\{N_{n-1}^{n-1}\}_0\right| + \left|\{N_{n-1}^{n-3}\}_0\right| + \cdots + \left|\{N_{n-1}^1\}_0\right| = F_n, \quad n \text{ even}. \end{aligned} \tag{15}$$

That $\{N_n^h\}_0$ might be equal to a Fibonacci-related binomial coefficient is suggested in a paper by Raab [9] in this Journal. Raab shows that by selecting the entries of a certain diagonal of the Pascal triangle array, the Fibonacci numbers are given by

$$F_n = \sum_{\delta=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-\delta}{\delta}. \tag{16}$$

However, Perron [1] lists term-by-term the identical binomial coefficients obtained in the expansions of (1) and (2). This verifies, as was suspected, that

$$\left|\{N_n^h\}_0\right| = \binom{\left[\frac{n+h}{2}\right]}{\left[\frac{n+h}{2}\right] - h}. \tag{17}$$

3. GENERAL SUBSCRIPT SETS

It is apparent that the basic subscript sets belong to a more general class of subscript sets. Consider a set of all possible sequences of h, non-zero, non-repeating, positive integers called subscripts, having the properties that no subscript exceeds M or is less than m and that each sequence within a set has the same parity order. Let it be further specified that each sequence be arranged in descending size order from left to right. Thus, there is a unique starting sequence and a unique last sequence. The leftmost position of the starting sequence is occupied by a subscript $\leq M$ (depending on mutual parities), and the remaining (h - 1) positions are filled with the largest subscripts possible consistent with size-order and parity. Similarly, the rightmost position of the last sequence is occupied by a subscript $\geq m$ (depending on mutual parities), and the remaining (h - 1) positions are occupied by the smallest consistent subscripts. For example, if $h = 6$, $M = 20$, $m = 3$ and position parity is even, odd, even, even, even, even, the starting sequence must be 20, 19, 18, 16, 14, 12, and the last sequence must be 12, 11, 10, 8, 6, 4. Because the position parity must be the same for the starting and last sequence and because of the compacting of subscripts to the left in the starting sequence and to the right in the last sequence, the difference between the same position subscripts within the starting and last sequences is the same. From this fact, it can be seen that there is a constant difference q between corresponding position subscripts in the starting and last sequences, and moreover, q must be even as the result of position parity. Once a starting and last sequence are determined, the generation of subscript sets in general follows the algorithm given for basic subscript sets. Of course, parity must be strictly observed.

While (17) applies in particular to basic subscript sets and is useful for counting them without first determining the starting and last sequences, it is possible to use (17) to obtain a new form suitable for counting all subscript sets.

Consider $\left|\{N_n^h\}_0\right|$. If n and h are both odd or both even (i.e., $n + h$ is even),

$$\left[\frac{n+h}{2}\right] = \frac{n+h}{2} \quad . \tag{18}$$

Since the last member of the starting sequence is $(n - h + 1)$, it must be odd. This makes q

$$q = (n - h + 1) - 1 = (n - h) . \tag{19}$$

If n is odd and h even or vice versa (i.e., $n + h$ is odd),

$$\left[\frac{n+h}{2}\right] = \frac{n+h-1}{2} \quad . \tag{20}$$

In this case, the value of q is

$$q = (n - h + 1) - 2 = (n - h - 1) . \tag{21}$$

Elimination of n between either (18) and (19) or between (19) and (20) results in the single equation

$$\{N_n^h\}_0 = \binom{h + q/2}{q/2} \tag{22}$$

which is independent of n and the parity of $(n + h)$.

Next, consider the sequences of differences between any sequence and the last sequence of $\{N_n^h\}_0$. This set of differences starts with a sequence of h q's, $(q, q, q, \cdots, a)$ and ends with the sequence of h zeros $(0, 0, 0, \cdots, 0)$. The same algorithm applied to the sequence of differences produces members of the difference set in one-to-one correspondence with the members of the basic subscript set, and thereby (22) is applicable for counting them. However, a little reflection reveals that the same $(q, q, q, \cdots, q)$ to $(0, 0, 0, \cdots, 0)$ sequences apply to any subscript set having the given q and h. Thus, (22) can be recast more generally as

$$R_{h,q} = \binom{h + q/2}{q/2}. \tag{23}$$

4. SOME USEFUL NON-BASIC SUBSCRIPT SETS

It was noted earlier that $\{N_n^h\}_0$ provided subscripts for a sum of products of coefficients such as $a_xa_ya_z \cdots$ (see (4)). If the even subscripted a's represent one kind of item (as in (5)) and the odd subscripted a's represent another, the sequences of the basic subscript set represent sums of products of kinds of things in a fixed alternation pattern. For example, in another of the physical systems described earlier, the odd subscripted a's were shunt arm admittances (y's) and the even subscripted a's were series arm impedances (z's). In the case of a lumped element ladder network, a product has a specific $\cdots zyz \cdots$ order. In the study of certain cascaded distributed element transmission systems, a mathematical interaction takes place which, in effect, keeps the $\cdots zyz \cdots$ order the same but introduces additional sums of products in which even subscript positions replace some or all of the formerly odd subscript positions of the basic subscript set [10], [11].

Let $\{N_n^h\}_\ell$ be a subscript set whose subscripts describe the same element product order as is described by the basic subscript set but whose sequences each have ℓ of the odd subscript positions of $\{N_n^h\}_0$ replaced by ℓ even subscript positions. If g is the number of odd parity positions in a sequence of $\{N_n^h\}_0$, there are $\binom{g}{\ell}$ distinct types of parity arrangement for the sequences of $\{N_n^h\}_\ell$. To obtain $\{N_n^h\}$, it is feasible to form $\binom{g}{\ell}$ subsets each having its own starting sequence and last sequence. The subsets are designated $\{N_n^h\}_{\ell_1}$, $\{N_n^h\}_{\ell_2}$, etc., and are generated and or counted just like any subscript set. Let the position of the rightmost odd subscript of $\{N_n^h\}_0$ be designated odd position 1, next on the left odd position 2, etc., up to and including g. Determine the names of the $\binom{g}{\ell}$ combinations of the odd position numbers 1, 2, $\cdots$, g taken ℓ at a time. For each combination of odd position numbers, the sequences of the subsets have the parity arrangement of $\{N_n^h\}_0$ except for ℓ former odd subscript positions replaced by ℓ even subscript positions. The subscripts of the starting sequence should be as large as consistently possible and those of the last sequence as small as consistently possible. While the power of the individual subsets can be found from (23), the power of $\{N_n^h\}_\ell$ is given by

(24)
$$\left|\{N_n^h\}_\ell\right| = \sum_{i=1}^{\binom{g}{\ell}} \binom{h + \frac{q_i}{2}}{\frac{q_i}{2}} .$$

5. DERIVATION OF ℓ_{MAX}

For the physical systems which utilize $\{N_n^h\}_\ell$, the value of ℓ_{max} for each h is of great use in determining the number of coefficients, and hence size, of governing equations. Certainly ℓ_{max} cannot exceed g and there are many possible situations in which ℓ_{max} cannot even equal g. It is shown below, in fact, that ℓ_{max} is equal to the lesser of $q/2$ or g of $\{N_n^h\}_0$.

The starting and last sequences, respectively, of $\{N_n^h\}_0$ take on either of the two forms given by (8), (9) or (10), (11). Since corresponding position subscripts are of the same parity, n and h in (8) and (9) can be either both even or both odd. In (10) and (11), if n is even, h is odd, and if n is odd, h is even.

(a) <u>n, h both even (Eqs. (8) and (9)).</u> There are $h/2$ even and $h/2 = g$ odd subscripts in any sequence. If $n \geq 2h$, there are exactly (equals sign) or more than h even subscripts available between n and 1 (including n). Thus, if $n - h = q$ is divided by two, and thereby $q/2 \geq h/2$, a sequence with all even subscripts can be found. Thus ℓ_{max} is not limited by $q/2$ since $h/2$ odd positions have been filled with even subscripts. If $n < 2h$, there are less than h even subscripts available between n and 1 (including n). This is reflected by $q/2 < h/2$. The value for ℓ_{max} must be $q/2$.

(b) <u>n, h both odd (Eqs. (8) and (9)).</u> There are $(h - 1)/2$ even and $(h + 1)/2 = g$ odd subscripts in any sequence. If $n \geq 2h + 1$, there are exactly (equals sign) or more than h odd subscripts between n and 1 (<u>exclusive</u> of 1). Thus, if $q/2 \geq (h + 1)/2$, there are at least h odd subscripts between n and 1 (exclusive of 1) which can be reduced by one to give at least h even subscripts. Such a sequence would have $(h + 1)/2$ former odd positions filled by even subscripts. Therefore, ℓ_{max} is not limited by $q/2$ since $(h + 1)/2$ odd positions have been filled by even subscripts. If $q/2 < (h + 1)/2$, the value for ℓ_{max} must be $q/2$.

(c) <u>n even, h odd (Eqs. (10) and (11)).</u> There are $(h + 1)/2$ even and $(h - 1)/2 = g$ odd subscripts in any sequence. If $n \geq 2h$, there are h <u>distinct</u> even subscripts between n and 2 (including n and 2). The condition can be arranged as $n - 1 \geq 2h - 1$ or $n - 1 - h \geq h - 1$ or $(n - 1 - h)/2 \geq (h - 1)/2$, where $(n - 1 - h) = q$. Since fulfillment of this condition fills $(h - 1)/2$ odd positions with even subscripts, ℓ_{max} is not limited by $q/2$. If $q/2 < (h - 1)/2$, the value for ℓ_{max} must be $q/2$.

(d) <u>n odd, h even (Eqs. (10) and (11)).</u> There are $h/2$ even and $h/2 = g$ odd subscripts in any sequence of the basic set. If $n \geq 2h + 1$ there are exactly (equals sign) or more than h odd subscripts between n and 2 (including n) which can be reduced by one to give at least h even subscripts. Therefore, $n - h - 1 \geq h$, $(n - h - 1)/2 \geq h/2$, and ℓ_{max} is not limited by $q/2$. If $q/2 < h/2$, the value for ℓ_{max} must be $q/2$.

From (a), (b), (c), and (d), it is seen that in all cases $q/2$ is the value for ℓ_{max} if $q/2$ is less than or equal to g, the number of odd positions in a sequence, and g is the value for ℓ_{max} if $q/2$ is greater than or equal to g. A sufficient condition for $q/2$ to be the greatest ℓ_{max} for a given n and <u>any</u> h occurs when $q/2 = g$.

6. EXAMPLE OF $\{N_n^h\}_\ell$

$\{N_8^4\}_0$			$\{N_8^4\}_{1,1}$	$\{N_8^4\}_{1,2}$	$\{N_8^4\}_2$
8765	8721	6543	8764	8643	8642
8763	8543	6521	8762	8641	
8761	8541	6521	8742	8621	
8743	8521	6321	8542	8421	
8741	8321	4321	6542	6421	

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SOME DOUBLY EXPONENTIAL SEQUENCES

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1. INTRODUCTION

Let $x_0, x_1, x_2, \cdots$ be a sequence of natural numbers satisfying a nonlinear recurrence of the form $x_{n+1} = x_n^2 + g_n$, where $|g_n| < \frac{1}{4}x_n$ for $n \ge n_0$. Numerous examples of such sequences are given, arising from Boolean functions, graph theory, language theory, automata theory, and number theory. By an elementary method it is shown that the solution is x_n = nearest integer to k^{2^n}, for $n \ge n_0$, where k is a constant. That is, these are doubly exponential sequences. In some cases k is a "known" constant (such as $\frac{1}{2}(1 + \sqrt{5})$), but in general the formula for k involves $x_0, x_1, x_2, \cdots$!

2. EXAMPLES OF DOUBLY EXPONENTIAL SEQUENCES

2.1 BOOLEAN FUNCTIONS

The simplest example is defined by

(1) $$x_{n+1} = x_n^2, \quad n \ge 0; \quad x_0 = 2$$

so that the sequence is 2, 4, 16, 256, 65536, 4294967296, $\cdots$ and $x_n = 2^{2^n}$. This is the number of Boolean functions of n variables ([12], p. 47) or equivalently the number of ways of coloring the vertices of an n-dimensional cube with two colors.

2.2 ENUMERATING PLANAR TREES BY HEIGHT

The recurrence

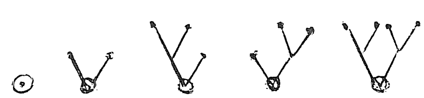
(2) $$x_{n+1} = x_n^2 + 1, \quad n \ge 0; \quad x_0 = 1$$

generates the sequence 1, 2, 5, 26, 677, 458330, 210066388901, $\cdots$. This arises, for example, in the enumeration of planar binary trees.

We assume the reader knows what a rooted tree ([10]) is. (The drawings below are of rooted trees.) A binary rooted tree is a rooted tree in which the root node has degree 2 and all other nodes have degree 1 or 3 (or else is the trivial tree consisting of the root node alone). A planar binary rooted tree is a particular embedding of a binary rooted tree in the plane.

The height of a rooted tree is the maximum length of a path from any node to the root.

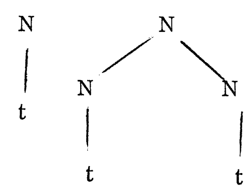
For example here are the planar binary rooted trees of heights 0, 1 and 2. (Here the root is drawn at the bottom.)



Let x_n be the number of planar binary rooted trees of height at most n, so that $x_0 = 1$, $x_1 = 2$, $x_2 = 5$. Deleting the root node either leaves the empty tree or two trees of height at most $n - 1$, from which it follows that x_n satisfies (2).

Planar binary rooted trees arise in a variety of splitting processes. We give three illustrations.

a. In parsing certain context-free languages [1], [13], [18]. For example, consider a context-free grammar G with two productions $N \to NN$ and $N \to t$ where N is a nonterminal and t a terminal symbol. Derivation trees for the sentences t and tt are shown below.* Deleting the terminal symbols



and their adjacent edges converts a derivation tree into a planar binary rooted tree. Thus x_n represents the number of derivation trees for G of height at most $n + 1$.

b. Using the natural correspondence ([4], Vol. 1, p. 65) between planar binary rooted trees and the parenthesizing of a sentence, x_n is the number of ways of parenthesizing a string of symbols of any length so that the parentheses are nested to depth at most n.

c. If, in a planar binary rooted tree, we write a 0 when the path branches to the left and a 1 when the path branches to the right, the set of all paths from the root to the nodes of degree 1 forms a variable length binary code ([7]). Thus x_n is the number of variable length binary codes of maximum length at most n.

2.3 THE RECURRENCE

$$x_{n+1} = x_n^2 - 1, \quad n \geq 0; \quad x_0 = 2 \tag{3}$$

generates the sequence 2, 3, 8, 63, 3968, 15745023, 247905749270528, $\cdots$.

2.4 THE RECURRENCE

$$y_{n+1} = y_n^2 - y_n + 1, \quad n \geq 1; \quad y_1 = 2 \tag{4}$$

generates the sequence 2, 3, 7, 43, 1807, 3263443, 10650056950807, $\cdots$. This sequence occurs (a) in Lucas' test for the primality of Mersenne numbers ([11], p. 233) and (b) in approximating numbers by sums of reciprocals. Any positive real number $y < 1$ admits a unique expansion of the form

$$y = \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \cdots ,$$

*In language theory, it is customary to draw trees with the root at the top.

where the y_i are integers so chosen that after i terms, when the sum s_i has been obtained, y_{i+1} is the least integer such that $s_i + 1/y_{i+1}$ does not exceed y ([16]). It follows that $y_{i+1} = y_i^2 - y_i + \epsilon_i$, $\epsilon_i \geq 1$. The most slowly converging such series is

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \cdots$$

when $\epsilon_i = 1$ for $i \geq 1$; this converges to 1, and the denominators satisfy (4). Recurrence (4) is a special case of the next example.

2.5 GOLOMB'S NONLINEAR RECURRENCES

For $r = 1, 2, \cdots$, Golomb [9] has defined a sequence $[y_n^{(r)}]$ by

$$y_{n+1}^{(r)} = y_0^{(r)} y_1^{(r)} \cdots y_n^{(r)} + r, \qquad n \geq 0; \qquad y_0^{(r)} = 1 \;. \tag{5}$$

Equivalent definitions are

$$y_0^{(r)} = 1, \qquad y_1^{(r)} = r + 1$$

$$y_{n+1}^{(r)} = \left(y_n^{(r)}\right)^2 - r y_n^{(r)} + r, \qquad n \geq 1 \tag{6}$$

and

$$y_0^{(r)} = 1, \qquad y_1^{(r)} = r + 1$$

$$y_{n+1}^{(r)} = \left(y_n^{(r)} - \rho\right)^2 + (2\rho - \rho^2), \qquad n \geq 1 \;, \tag{7}$$

where $\rho = \frac{r}{2}$.

From (6) $[y_n^{(1)}]$ is the sequence of example 2.4. The Fermat numbers are $y_n^{(2)}$. The sequences $[y_n^{(2)}]$ - $[y_n^{(5)}]$ begin:

1, 3, 5, 17, 257, 65537, 4294967297, $\cdots$

1, 4, 7, 31, 871, 756031, 571580604871, $\cdots$

1, 5, 9, 49, 2209, 4870849, 23725150497409, $\cdots$

1, 6, 11, 71, 4691, 21982031, 483209576974811, $\cdots$.

(Note that the value of $y_6^{(3)}$ given in [9] is incorrect.)

The substitution $x_n = y_n^{(r)} - \rho$, $n \geq 1$, converts (7) to

$$x_{n+1} = x_n^2 + \rho(1 - \rho), \qquad n \geq 0; \qquad x_0 = (1 + \rho^2)^{\frac{1}{2}} \tag{8}$$

2.6 THE RECURRENCE

$$y_0 = 1, \qquad y_1 = 2 \;,$$

$$y_{n+1} = 2y_n(y_n - 1), \qquad n \geq 1 \tag{9}$$

generates the sequence 1, 2, 4, 24, 1104, 2435424, 11862575248704, $\cdots$, which also arises in approximating numbers by sums of reciprocals [16]. The substitution $x_n = 2y_n - 1$, $n \geq 1$, converts (9) to

$$x_0 = \sqrt{5},$$

$$x_{n+1} = x_n^2 - 2, \qquad n \geq 0 \,. \tag{10}$$

Sequences generated by (10) with different initial values are also used in primality testing. With the initial value $x_0 = 3$ we obtain the sequence 3, 7, 47, 2207, 4870847, 23725150497407, $\cdots$ ([17], p. 280), and with $x_0 = 4$ the sequence 4, 14, 194, 37634, 1416317954, $\cdots$ ([19]).

2.7 THE RECURRENCE

$$y_0 = 1, \qquad y_1 = 2$$

$$y_{n+1} = y_n^2 - y_{n-1}^2, \qquad n \geq 1 \tag{11}$$

generates the sequence 1, 2, 3, 5, 16, 231, 53105, 2820087664, $\cdots$. In [3] it was given as a puzzle to guess the recurrence satisfied by this sequence.

The substitution $x_n = y_n - \frac{1}{2}$, $n \geq 0$, converts (11) to

$$x_0 = \tfrac{1}{2}, \qquad x_1 = 1\tfrac{1}{2}, \qquad x_2 = 2\tfrac{1}{2}$$

$$x_{n+1} = x_n^2 - x_{n-2}^2 - x_{n-2} - 1, \qquad n \geq 2 \,. \tag{12}$$

3. SOLVING THE RECURRENCES

Recurrences (1)-(3), (8), (10) and (12) all have the form

$$x_{n+1} = x_n^2 + g_n, \qquad n \geq 0 \tag{13}$$

with boundary conditions, and are such that

(i) $x_n > 0$

(ii) $|g_n| < \frac{1}{4} x_n$ and $1 \leq x_n$ for $n \geq n_0$ and

(iii) g_n satisfies condition (16) below.

Let

$$y_n = \log x_n, \qquad \alpha_n = \log\left(1 + \frac{g_n}{x_n^2}\right) \,.$$

Then by taking logarithms of (13) we obtain

$$y_{n+1} = 2y_n + \alpha_n, \qquad n \geq 0 \,. \tag{14}$$

For any sequence $\{\alpha_n\}$, the solution of (14) is (see for example [15], p. 26)

$$y_n = 2^n\left(y_0 + \frac{\alpha_0}{2} + \frac{\alpha_1}{2^2} + \cdots + \frac{\alpha_{n-1}}{2^n}\right)$$
$$= Y_n - r_n \quad ,$$

where

(15)
$$Y_n = 2^n y_0 + \sum_{i=0}^{\infty} 2^{n-1-i}\alpha_i$$
$$r_n = \sum_{i=n}^{\infty} 2^{n-1-i}\alpha_i \quad .$$

Assuming that the g_n are such that

$$|\alpha_n| \geq |\alpha_{n+1}| \quad \text{for} \quad n \geq n_0 \ , \tag{16}$$

it follows from (15) that $|r_n| \leq |\alpha_n|$. Then

$$x_n = e^{y_n} = e^{Y_n - r_n} = X_n e^{-r_n} \quad , \tag{17}$$

where

$$X_n = e^{Y_n} = k^{2^n} \quad , \tag{18}$$

$$k = x_0 \exp\left(\sum_{i=0}^{\infty} 2^{-i-1}\alpha_i\right) \quad . \tag{19}$$

Also

$$X_n = x_n e^{r_n} \leq x_n e^{|\alpha_n|}$$
$$\leq x_n\left(1 + \frac{2|g_n|}{x_n^2}\right) \quad \text{for} \quad n \geq n_0 \ ,$$

using (ii), and the fact that $(1-u)^{-1} \leq 1 + 2u$ for $0 \leq u \leq \frac{1}{2}$,

$$= x_n + \frac{2|g_n|}{x_n}$$

and

$$X_n \geq x_n e^{-|\alpha_n|} \geq x_n\left(1 - \frac{|g_n|}{x_n^2}\right) = x_n - \frac{|g_n|}{x_n} \quad .$$

From assumption (ii), this means that

$$|x_n - X_n| < \tfrac{1}{2} \quad \text{for } n \geq n_0 .$$

If x_n is an integer, as in recurrences (1)-(3), (8) for r even, and (10), then the solution to the recurrence (13) is

$$x_n = \text{nearest integer to } k^{2^n}, \qquad \text{for } n \geq n_0 \tag{20}$$

while if x_n is half an odd integer, as in (8) for r odd and (12), the solution is

$$x_n = (\text{nearest integer to } k^{2^n} + \tfrac{1}{2}) - \tfrac{1}{2}, \qquad \text{for } n \geq n_0 , \tag{21}$$

where k is given by (19).

Note that if g_n is always positive, then $\alpha_n > 0$, $r_n > 0$, $X_n > x_n$, and (20) may be replaced by

$$x_n = [k^{2^n}] \qquad \text{for } n \geq n_0 , \tag{22}$$

where $[a]$ denotes the integer part of a. Similarly if g_n is always negative then $X_n < x_n$ and

$$x_n = [k^{2^n}] \qquad \text{for } n \geq n_0 , \tag{23}$$

where $[a]$ denotes the smallest integer $\geq a$.

In some cases (see below) k turns out to be a "known" constant (such as $\frac{1}{2}(1 + \sqrt{5})$). But in general Eqs. (20)-(23) are not legitimate solutions to the recurrence (13), since the only way we have to calculate k involves knowing the terms of the sequence. Nevertheless, they accurately describe the asymptotic behavior of the sequence.

We now apply this result to the preceding examples. For all except 2.7 the proofs of properties (ii) and (iii) are by an easy induction, and are omitted.

Example 2.1.

Here $g_n = 0$, $k = 2$ and (20) correctly gives the solution $x_n = 2^{2^n}$.

Example 2.2.

Condition (ii) holds for $n_0 = 2$, and (iii) requires $x_n \leq x_{n+1}$, which is immediate. From (20) $x_n = [k^{2^n}]$ for $n \geq 1$, where

$$k = \exp\left(\frac{1}{2}\log 2 + \frac{1}{4}\log\frac{5}{4} + \frac{1}{8}\log\frac{26}{25} + \frac{1}{16}\log\frac{677}{676} + \cdots\right)$$
$$= 1.502837 \cdots .$$

The comparison of k^{2^n} with x_n is as follows:

n	0	1	2	3	4	5
x_n	1	2	5	26	677	458330
k^{2^n}	1.50284	2.25852	5.10091	26.01924	677.00074	458330.00000

Example 2.3 is similar, and $x_n = [k^{2^n}]$ where $k = 1.678459 \cdots$.

Example 2.5.

It is found that (ii) is valid for $n_0 = 1$ if $r = 1$ and for $n_0 = 3$ if $r \geq 3$. The solution of (5) for $r = 1$ (and of example 2.4) is

$$y_n^{(1)} = [k^{2^n} + \tfrac{1}{2}], \quad n \geq 0,$$

and for $r \geq 3$ is

$$y_n^{(r)} = [k^{2^n} + \tfrac{r}{2}], \quad n \geq 3,$$

where k is given by (19). The first few values of k are as follows.

r	1	3	4	5
k	1.264085	1.526526	1.618034	1.696094

For $r = 4$, the value of k is seen to be very close to the "golden ratio"

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}) = 1.6180339887 \cdots .$$

In fact we may take $k = \varphi$, for

$$y_1^{(4)} = 5,$$
$$y_{n+1}^{(4)} = (y_n^{(4)} - 2)^2, \quad n \geq 1$$

is solved exactly by

$$y_n^{(4)} = \varphi^{2^n} + \varphi^{-2^n} + 2, \quad n \geq 1,$$

and so

$$y_n^{(4)} = [\varphi^{2^n} + 2], \quad n \geq 1.$$

(This was pointed out to us by D. E. Knuth.) So far, none of the other values of k have been identified. Golomb [9] has studied the solution of (5) by a different method.

Example 2.6.

The solution to (9) is

$$y_n = [\tfrac{1}{2}(1 + k^{2^n})] \quad \text{for } n \geq 1,$$

where $k = 1.618034 \cdots$, and again, as pointed out by D. E. Knuth, we may take

$$k = \varphi = \tfrac{1}{2}(1 + \sqrt{5}),$$

since

$$x_n = \varphi^{2^n} + \varphi^{-2^n}, \quad n \geq 0$$

solves (10) exactly. A similar exact solution can be given for (10) for any initial value x_0.

Example 2.7.

This is the only example for which (ii) and (iii) are not immediate. Bounds on x_n and y_n are first established by induction:

$$2^{2^{n-2.1}} \le x_n \le y_n \le 2^{2^{n-2}} \quad \text{for } n \ge 4 .$$

then

$$g_n = -(x_{n-2} + \tfrac{1}{2})^2 - \frac{3}{4} = -y_{n-2}^2 - \frac{3}{4}$$

and

$$2^{2^{n-3.1}} \le g_n \le 2^{2^{n-3}} \quad \text{for} \quad n \ge 7 .$$

It is now easy to show that (ii) and (iii) hold for $n \ge n_0 = 5$. The solution is

$$y_n = [k^{2^n} + \tfrac{1}{2}] , \qquad n \ge 1 ,$$

where $k = 1.185305 \cdots$.

EXERCISES

The technique may sometimes be applied to recurrences not having the form of (13). We invite the reader to tackle the following.

(1) $$y_{n+1} = y_n^3 - 3y_n, \quad n \ge 0; \quad y_0 = 3 ,$$

which generates the sequence 3, 18, 5778, 192900153618, $\cdots$ used in a rapid method of extracting a square root ([5]).

(2) $$y_0 = 1, \quad y_1 = 3$$
$$y_{n+1} = y_n y_{n-1} + 1, \quad n \ge 1 ,$$

which generates the sequence 1, 3, 4, 13, 53, 690, 36571, 25233991, 922832284862, $\cdots$ ([2]).

(3) $$y_0 = 1$$
$$y_{n+1} = y_0 + y_0y_1 + \cdots + y_0y_1 \cdots y_n, \quad n \ge 0$$

which generates the sequence 1, 1, 2, 4, 12, 108, 10476, 108625644, 11798392680793836, $\cdots$.

(4) $$y_0 = 1$$
$$y_{n+1} = y_n^2 + y_n + 1, \quad n \ge 0$$

which generates the sequence 1, 3, 13, 183, 33673, 1133904603, $\cdots$, the coefficients of the least rapidly converging continued cotangent ([14]).

(5) $$y_0 = 1$$
$$y_{n+1} = (y_n + 1)^2, \quad n \ge 0$$

which generates the sequence 1, 4, 25, 676, 458329, 210066388900, $\cdots$ ([8]).

(6)
$$y_0 = y_1 = 1$$
$$y_{n+1} = y_n^2 + 2y_n(y_0 + y_1 + \cdots + y_{n-1}), \qquad n \geq 1 .$$

which generates the sequence 1, 1, 3, 21, 651, 457653, 210065930571, $\cdots$, arising in the enumeration of shapes ([6]).

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ON STORING AND ANALYZING LARGE STRINGS OF PRIMES

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By the prime number theorem, the number of primes less than x is asymptotic to x/log x. A short table of actual counts follows.

RANGE	NO. OF PRIMES
$0 - 2.5 \times 10^6$	183,072
$10^8 - 10^8 + 2.5 \times 10^6$	135,775
$10^{10} - 10^{10} + 2.5 \times 10^6$	108,527
$10^{12} - 10^{12} + 2.5 \times 10^6$	90,509
$10^{14} - 10^{14} + 2.5 \times 10^6$	77,254
$10^{16} - 10^{16} + 2.5 \times 10^6$	68,081

Computer runs for finding the larger numbers are very time-consuming and it is often desirable to store the primes on magnetic tape or punched cards for use in certain statistical routines. Many users also store the lower primes for computing the higher ones, applying some variation of the sieve of Eratosthenes.

Assume we want to store the 68,081 primes in the interval from 10^{16} to $10^{16} + 2.5 \times 10^6$ on punched cards. How many cards are required? The first prime is 10 000 000 000 000 061 (17 digits) and if all digits are used, we would require $68{,}081 \times 17/80$ (a card can hold 80 alphanumeric characters) or 14,468 cards.

Obviously, we don't need to record the value of 10^{16} for every prime. We can store only the last seven digits (since we have an interval of 2,500,000) and keep in mind that every number is to be augmented by 10^{16}. Using only the last 7 digits requires $68{,}081 \times 7/80$ or 5958 cards.

Now, we don't have to store the actual primes. If we record the first one we need simply store the difference to the next one. For example, the second prime in this interval is 10 000 000 000 000 069 and so we just record the number 8. The next one is $10^{16} + 79$ and we record the number 10. Allowing for a 3-digit maximum difference (the actual maximum

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difference is 432, i.e., 431 consecutive composites) we now need $68{,}080 \times 3/80$ or 2553 cards.

It is desirable to cut this number down stil further. There are 68,081 primes in this interval of 2,500,000 numbers. Thus the average difference is about 36. Indeed, a computer count reveals that 52,273 of the 68,081 primes have gaps (differences) less than or equal to 52. Moreover, since all primes are odd (except for the number 2) all differences are even; and we need to store only half the difference (keeping in mind that when reconstructing the primes from the differences, we will double the gap). Thus for most of the gaps we could use a number from 1 to 26 or a single letter from A to Z.

What about a gap of 54? This would be stored as 1A. The numeric 1 signifying 52 and A a difference of 2. A gap of 104 would appear as 1Z and 106 would be 2A. This method allows for a difference up to 572 using the ten numeric digits and 26 alphabetics. (It could be extended in an obvious fashion by having two numerics precede the alphabetic, etc.) A numeric digit is present only if it precedes an alphabetic, never by itself.

As an example, consider the first three cards for the primes after 10^{16}. The first prime $(10^{16} + 61)$ is recorded elsewhere and the first letter (D) gives the increment to the next prime, and so on.

DEJ6UN2FHMQ1PTMURFKIDQS1JN2C1AIE1BW1A1JAH1SA1DBDFBLIVT1G1KRBO1A1G1G
F1MU1JCSOK1EF

1RSTGNMOLIB1PF1A2FLML1LVCTAFNLJTRDC1DIRHYXILI1IU1BTL1G2RE1EHMHG1GEL
LUFJHA2JLJEY1

DHYBF1E1VUKACLT1QFXUTRJ1ILC1TB2FNMN1SCRDCCRI1LC2Q1GIA1DH1PCO1AL2COE
M1SC1D1AE1NQA

These three cards* translate to the 190 primes:

$$(61),\ 69,\ 79,\ 453,\ \cdots,\ 7357,\ 7359\ .$$

The last A on the third card indicating the twin prime (a gap of 2). With this system the number of cards needed to store the primes between 10^{16} and $10^{16} + 2.5$ million reduces to 1048. (About half of a box.) Of course, cards are only an illustration. The same economy is effected using magnetic tape, terminal display, or any other device.

Based on the above rules, a computer program could easily construct and reconstruct the primes in any given interval. (It is desirable to store the last prime, as well as the first, for a check.)

*Comment. Two lines represent one card. Our margins required putting each card in two lines.

For many applications, however, it is not necessary to reconstruct the primes. For example, if one wishes to find the number of twin primes in an interval one simply looks for isolated A's (A's not preceded by a numeric character). Or one could have the program search for the combination ABA signifying a quadruple of primes within a span of eight integers; this occurs for example at $10^{16} + 2,470,321,\ 323,\ 327,\ 329$; as indicated in the following line:

ONM1V1FAXQA1ATR1SY1C<u>ABA</u>2GOABJRICOLQILDU1VI1V2EWJIFQFSHRAFONAQMHPRH

M1F2TVOK1AFJOE

Similarly, one can search for any permissible combination of letters. Certain sequences are obviously forbidden; such as AA which would mean that p, $p + 2$, $p + 4$ are all primes and evidently one of these is divisible by 3. FIBONACCI, for example, is also forbidden. An interesting problem is: what is the probability that a random sequence of N letters is permissible? Is Shakespeare's Macbeth, word for permissible word, somewhere amongst the primes? After all, as x goes to infinity, so does $x/\log x$.

Finally, is there a way of storing primes (or any similar string of numbers) using fewer characters? How close can one come to using only <u>one</u> binary bit (0 or 1) for each prime?

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COUNTING OF CERTAIN PARTITIONS OF NUMBERS

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It can be shown from a recent discussion of subscript sets [1] that the counting of certain restricted, but useful, partitions can be made by means of binomial coefficients. In [1], these binomial coefficients were those which sum to a Fibonacci number.

A brief description of a subscript set is repeated here for continuity. A subscript set is defined as the set of all sequences of h, non-zero, non-repeating positive integers called subscripts, having the properties that no subscript exceeds M, no subscript is less than m, and that for all sequences one fixed parity order applies. Let it further be specified that each sequence be arranged in descending size order from the left. Under these conditions, there are unique starting and last sequences. The leftmost position of the starting sequence is occupied by a subscript $k \le M$ (depending on parity) with the other $(h - 1)$ positions filled with the largest permissible subscripts. Correspondingly, the rightmost position of the last sequence is occupied by a subscript $p \ge m$ with all other $(h - 1)$ positions filled with the smallest subscripts possible.

A practical method for generating the sequences from the starting sequence is described in the reference [1] and, briefly, consists of progressive and exhaustive reduction of subscripts by two. The number of sequences in a subscript set is [1]

$$R_{h,q} = \binom{h + q/2}{q/2} \quad , \tag{1}$$

where q is the necessarily even difference between the rightmost subscripts of the starting and last sequences. If collections of subscript sets described as basic [1] are enumerated, the $R_{h,q}$ for each set is one of a sum of binomial coefficients which sum to a Fibonacci number.

Suppose a new set of sequences called the q-set is formed whose sequences are formed by positionwise subtraction of the last sequence of a subscript set from all the others (including the last). The q-set sequences start with h q's, $(q, q, \cdots, q)$ and end with h zeros, $(0, 0, \cdots, 0)$ and are in one-to-one correspondence with the sequences of the subscript set. All the sequences of a q-set contain even numbers only.

Next, divide all integers of a q-set by two. It is seen that the set of sequences so produced are the h and less part partitions of $(h\,q/2)$ with no integer exceeding $q/2$, plus the null partition $(0, 0, \cdots, 0)$ which can be conveniently discarded. Accordingly, in Chrystal's notation* for partition counting,

$$(2) \qquad P(\leq hq/2 | \leq h | \leq q/2) = \binom{h + q/2}{q/2} - 1 ,$$

where the subtracted one accounts for the discarded null partition. A slightly different version of (2) states

$$(3) \qquad P(\leq ab - b^2 | \leq a - b | \leq b) = \binom{a}{b} - 1 .$$

To each integer of the partitions counted by (2) or (3) including the null partition add one. The result is the h-part partitions of $h(1 + q/2)$ with no member exceeding $(1 + q/2)$. Accordingly,

$$(4) \qquad P(h + hq/2 | \leq h | \leq 1 + q/2) = \binom{h + q/2}{q/2} ,$$

and

$$(5) \qquad P(a + ab - b - b^2 | \leq a - b | \leq b + 1) = \binom{a}{b} .$$

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*In Chrystal's notation $P(A|B|C)$ represents a number of B part partitions of A with C the largest integer possible for any partition with no restrictions on A, B, and C except as they occur naturally. Inequalities $\leq$ affixed to some or all of the quantities specifies not greater than.

COUNTING OMITTED VALUES

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1. INTRODUCTION

H. L. Alder in [1] has extended J. L. Brown, Jr.'s result on complete sequences [2] by showing that if $\{P_i\}_{i=1,2,\cdots}$ is a non-decreasing sequence and $\{k_i\}_{i=1,2,\cdots}$ is a sequence of positive integers, then with $P_1 = 1$ every natural number can be represented as

$$\sum_{i=1}^{\infty} \alpha_i P_i ,$$

where $0 \le \alpha_i \le k_i$ if and only if

$$P_{k+1} \le 1 + \sum_{i=1}^{n} k_i P_i$$

for $n = 1, 2, \cdots$. He also proves for a given sequence $\{k_i\}_{i=1,2,\cdots}$ there is only one non-decreasing sequence of positive integers $\{P_i\}_{i=1,2,\cdots}$ for which the representation is unique for every natural number, namely the set $\{P_i\} = \{\phi_i\}_{i=1,2,\cdots}$ where $\phi_1 = 1$, $\phi_2 = 1 + k_1$, $\phi_3 = (1 + k_1)(1 + k_2)$, $\cdots$, $\phi_i = (1 + k_1)(1 + k_2) \cdots (1 + k_{i-1})$, $\cdots$.

This paper investigates those natural numbers not represented by the form

$$\sum_{i=1}^{\infty} \alpha_i P_i$$

for $0 \le \alpha_i \le k_i$ where $\{k_i\}$ is as above and $\{P_i\}_{i=1,2,\cdots}$ is a necessarily increasing sequence of positive integers satisfying

$$(1) \qquad P_{n+1} \ge 1 + \sum_{i=1}^{n} k_i P_i \qquad n = 1, 2, \cdots .$$

When specialized to $k_i = 1$ for all i the results obtained include those in Hoggatt's and Peterson's paper [3].

2. UNIQUENESS OF REPRESENTATION

Theorem 1. For P_i satisfying (1), the representation of the natural number N as

$$\sum_{i=1}^{\infty} \alpha_i P_i ,$$

where $0 \le \alpha_i \le k_i$ is unique.

Proof. Let N be the smallest integer with possibly two representations

$$N = \sum_{s=1}^{n} \alpha_s P_s = \sum_{t=1}^{n} \beta_t P_t ,$$

where $\alpha_n \ne 0 \ne \beta_m$.

If $m \ne n$ assume $m > n$. Then by (1)

$$\sum_{s=1}^{n} \alpha_s P_s \le \sum_{s=1}^{n} k_s P_s \le P_{n+1} - 1 \le P_m - 1 \le \sum_{t=1}^{m} \beta_t P_t - 1 < \sum_{t=1}^{m} \beta_t P_t .$$

Thus, $m = n$. Either $\alpha_n \ge \beta_n$ or $\alpha_n \le \beta_n$. Suppose without loss of generality $\alpha_n \ge \beta_n$. The natural number

$$\sum_{t=1}^{n-1} \beta_t P_t = \sum_{s=1}^{n-1} \alpha_s P_s + (\alpha_n - \beta_n) P_n$$

and since it is less than N it has only one representation. Hence $\alpha_s = \beta_s$ for $s = 1, 2, \cdots, n$, i.e., N has a unique representation.

3. OMITTED VALUES

Definition. For $x \ge 0$ let $M(x)$ be the number of natural numbers less than or equal to x which are not represented by

$$\sum_{i=1}^{\infty} \alpha_i P_i \ .$$

Theorem 2. If

$$N = \sum_{i=1}^{n} \alpha_i P_i, \qquad \alpha_n \neq 0 \ ,$$

is the largest representable integer not exceeding the positive number x then

$$M(x) = [x] - \sum_{i=1}^{n} \alpha_i \phi_i \ ,$$

where $[\ ]$ is the greatest integer function.

Proof. By Theorem 1, it is sufficient to show the number, $R(x)$, of representable integers not exceeding x, equals

$$\sum_{i=1}^{n} \alpha_i \phi_i \ .$$

But $R(x) = R(N)$ from the definition of N. Now all integers of the form

$$\sum_{i=1}^{n} \beta_i P_i$$

with the only restriction that $0 \leq \beta_n < \alpha_n$ are less than N since:

$$\begin{aligned}\sum_{i=1}^{n} \beta_i P_i &\leq \sum_{i=1}^{n-1} k_i P_i + \beta_n P_n \\ &\leq \{1 + \beta_n\} P_n - 1 \\ &< \alpha_n P_n + \sum_{i=1}^{n-1} \alpha_i P_i = N \ .\end{aligned}$$

Again by the uniqueness of representation to form

$$\sum_{i=1}^{n} \beta_i P_i, \qquad 0 \le \beta_n < \alpha_n ,$$

there are α_n choices for β_n, $\{1 + k_{n-1}\}$ choices for β_{n-1}, $\{1 + k_{n-2}\}$ choices for β_{n-2}, $\cdots$, and $\{1 + k_1\}$ choices for β_1; in all there are $\alpha_n \phi_n$ numbers.

It remains to count numbers of the form

$$\alpha_n P_n + \sum_{i=1}^{n-1} \beta_i P_i$$

which do not exceed

$$N = \alpha_n P_n + \sum_{i=1}^{n-1} \alpha_i P_i .$$

That is the number of integers

$$\sum_{i=1}^{n-1} \beta_i P_i \le \sum_{i=1}^{n-1} \alpha_i P_i .$$

Hence

$$R\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \alpha_n \phi_n + R\left(\sum_{i=1}^{n-1} \alpha_i P_i\right)$$

and because $R\{\alpha_1 P_1\} = \alpha_1 = \alpha_1 \phi_1$ then

$$R\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i \phi_i .$$

[The representable positive integers less than or equal to $\alpha_1 P_1$ are P_1, $2P_1$, $\cdots$, $\alpha_1 P_1$.] This completes the proof.

As P_i is representable, the theorem give $M(P_i) = P_i - \phi_i$ and the following result is immediate.

Corollary.

$$M\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i M(P_i) = \sum_{i=1}^{n} M(\alpha_i P_i) .$$

Note that if $k_i = 1$ for all $i = 1, 2, \cdots$, Theorems 3 and 4 in [3] are special cases of the above theorem and corollary.

4. SOME APPLICATIONS

The two sequences $P_n = F_{2n}$ and $P_n = F_{2n-1}$, $n = 1, 2, \cdots$ mentioned in [3] satisfy Theorems 1 and 2 for $k_i = 1$ $i = 1, 2, \cdots$. However,

$$1 + 2(F_2 + F_4 + \cdots + F_{2n}) = F_{2n+2} + F_{2n-1} - 1 \geq F_{2n+2}$$

with equality only when $n = 1$, and

$$1 + 2(F_1 + \cdots + F_{2n-1}) = F_{2n+1} + F_{2n-2} + 1 > F_{2n+1} .$$

Consequently, by Alder's result,

Theorem 3. Every natural number can be expressed as

$$\sum_{i=1}^{\infty} \alpha_i F_{2i}$$

and as

$$\sum_{i=1}^{\infty} \beta_i F_{2i-1} ,$$

where α_i and β_i are 0, 1, or 2.

To return to the general case, let $\{k_i\}$ be a fixed sequence of positive integers; then any sequence $\{P_i\}$ satisfying (1) also satisfies $P_n \geq \phi_n$ for all n. This follows from $P_1 \geq 1 = \phi_1$ and induction:

$$P_n \geq 1 + \sum_{i=1}^{n-1} k_i P_i \geq 1 + \sum_{i=1}^{n-1} k_i \phi_i = \phi_n .$$

Hence by the corollary

$$M\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i \{P_i - \phi_i\} \geq 0 ,$$

with equality iff $P_i = \phi_i$. Furthermore, since $k_i \geq 1$ then $\{\phi_i\}$ is an increasing sequence and so for every natural number N there exist α_i such that

$$N < \sum_{i=1}^{\infty} \alpha_i \phi_i .$$

Therefore

$$M(N) \leq M\left(\sum_{i=1}^{\infty} \alpha_i \phi_i\right) = 0 ,$$

i.e., N has a representation in the form

$$\sum_{i=1}^{\infty} \beta_i \phi_i .$$

These facts, together with Theorem 1, give

Theorem 4. If $\{k_i\}$ is any sequence of positive integers, then every natural number has a unique representation as

$$\sum_{i=1}^{\infty} \alpha_i \phi_i ,$$

where $0 \leq \alpha_i \leq k_i$ and $\phi_1 = 1$, $\phi_2 = (1 + k_1)$, $\cdots$, $\phi_i = (1 + k_1) \cdots (1 + k_{i-1})$.

Corollary. If r is a fixed integer larger than 1 then every natural number has a unique representation in base r.

Proof. In Theorem 4, take $1 + k_i = r$ for all i.

5. REFERENCES

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