THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF
THE FIBONACCI ASSOCIATION

VOLUME 11 NUMBER 5

CONTENTS

DECEMBER 1973

THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

*DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

All subscription correspondence should be addressed to Brother Alfred Brousseau, St. Mary's College, California 94575. All checks ($8.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Two copies of manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State University, San Jose, California 95192. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.

The Quarterly is entered as third-class mail at the St. Mary's College Post Office, Calif., as an official publication of the Fibonacci Association.

The Quarterly is published in February, April, October, and December each year.

ARRAYS OF BINOMIAL COEFFICIENTS WHOSE PRODUCTS ARE SQUARES

CALVIN T. LONG
Washington State University, Pullman, Washington and University of British Columbia, Vancouver, B. C.

1. INTRODUCTION

In [1], Hoggatt and Hansell show that the product of the six binomial coefficients surrounding any particular entry in Pascal's triangle is an integral square. In the preceding article in this Journal [2], Moore generalizes this result by showing that the product of the binomial coefficients forming a regular hexagon with sides on the horizontal rows and main diagonals of Pascal's triangle and having $j + 1$ entries per side is an integral square if j is odd. In the present paper, we derive a fundamental lemma which leads to a generalization of Moore's result and enables us to show that a variety of other interesting configurations of binomial coefficients also yield products which are integral squares.

It will suit our purpose to represent Pascal's triangle (or, more precisely, a portion of it) by a lattice of dots as in Fig. 1. We will have occasion to refer to various polygonal figures and when we do, unless expressly stated to the contrary, we shall always mean a simple closed polygonal curve whose vertices are lattice points. Occasionally, it will be convenient to represent a small portion of Pascal's triangle by letters arranged in the proper position.

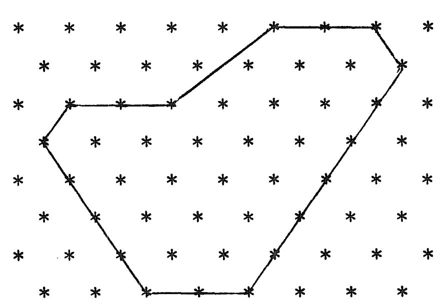


Figure 1

2. THE FUNDAMENTAL LEMMA AND ITS CONSEQUENCES

Lemma 1. The product of the binomial coefficients at the vertices of a pair of parallelograms oriented as in Fig. 2 or Fig. 3 is an integral square. We note that the parallelograms in any pair may overlap and, if they do, the common vertices, if any, must be included twice in the product or, equivalently, must be excluded entirely.

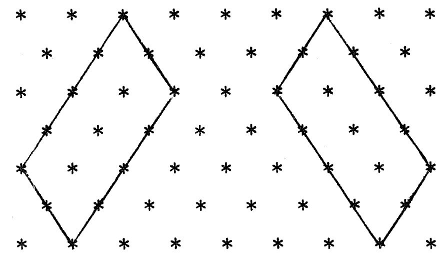


Figure 2

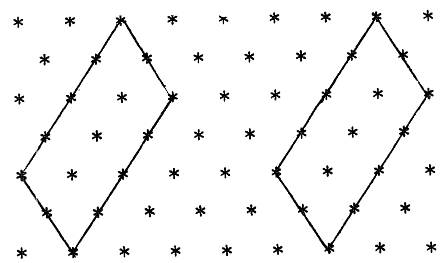


Figure 3

Proof. In the first case, for suitable integers m, n, r, s, and t, the binomial coefficients in question would be

$$\binom{m}{n},\quad \binom{m+r}{n+r},\quad \binom{m+s}{n},\quad \binom{m+s+r}{n+r},$$

$$\binom{m+r}{n+r+t},\quad \binom{m}{n+r+t},\quad \binom{m+s+r}{n+s+r+t},\quad \binom{m+s}{n+s+r+t}.$$

Thus the desired product is

$$\frac{m!}{n!(m-n)!}\cdot\frac{(m+r)!}{(n+r)!(m-n)!}\cdot\frac{(m+s)!}{n!(m-n+s)!}$$
$$\cdot\frac{(m+s+r)!}{(n+r)!(m-n+s)!}\cdot\frac{(m+r)!}{(n+r+t)!(m-n-t)!}\cdot\frac{m!}{(n+r+t)!(m-n-r-t)!}$$
$$\cdot\frac{(m+s+r)!}{(n+s+r+t)!(m-n-t)!}\cdot\frac{(m+s)!}{(n+s+r+t)!(m-n-r-t)!}.$$

This is clearly the square of a rational number. Since it is also an integer, it is an integral square as claimed. The argument for the second case is the same and we omit the details.

As a first consequence of Lemma 1, we now obtain the theorem of Hoggatt and Hansell.

Theorem 2. The product of the six binomial coefficients surrounding $\binom{m}{n}$ in Pascal's triangle is an integral square.

Proof. Let $d = \binom{m}{n}$ and a, b, c, e, f, and g be the six adjacent binomial coefficients as arranged in the array

$$\begin{array}{ccccc} & a & & e & \\ b & & d & & f \\ & c & & g & \end{array} \;.$$

Since a, b, c, d and e, d, g, f form parallelograms as in Lemma 1, it is immediate that both $abcd^2efg$ and $abcefg$ are integral squares as claimed.

By precisely the same argument, we obtain the following generalization of Theorem 2 which is different from the generalization of Moore mentioned above.

Theorem 3. Let $m > 1$ and $n > 1$ be integers and let H be a convex hexagon whose sides lie on the horizontal rows and main diagonals of Pascal's triangle. Let the numbers of elements on the respective sides of H be m, n, m, n, m, and n in that order, with m being the number of elements along the bottom side. Then the product of the binomial coefficients at the vertices of H is an integral square.

Proof. Of course if $m = n = 2$, this reduces to Theorem 2. In any case, we consider two m by n parallelograms with a common vertex and let a, b, c, d, e, f, and g denote the binomial coefficients at the vertices of the rectangles as indicated in Fig. 4. Clearly, a, b, c, g, f, and e lie at the vertices of a hexagon H of the type described and any such H can be obtained in this way. Therefore, it is again immediate from Lemma 1 that $abcd^2efg$ and $abcefg$ are integral squares.

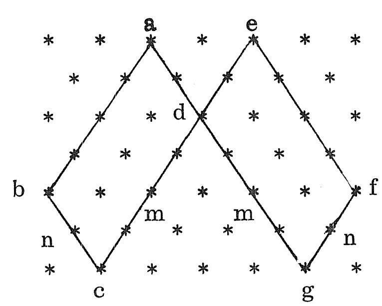


Figure 4

Now let us call the hexagon of Hoggatt and Hansell a fundamental hexagon. Let P be any simple closed polygonal figure. We say that P is tiled with fundamental hexagons if P is "covered" by a set $\mathfrak{F}$ of fundamental hexagons in such a way that

(i) the vertices of each F in $\mathfrak{F}$ are coefficients in P or in the interior of P,

(ii) each boundary coefficient of P is a vertex of precisely one F in $\mathfrak{F}$, and

(iii) each interior coefficient of P is interior to some F in $\mathfrak{F}$ or is a vertex shared by precisely two elements of $\mathfrak{F}$.

For example, in Fig. 5, G can be tiled by fundamental hexagons and H cannot. Now using the result of Theorem 2 and repeating the essentials of its proof we obtain the following quite general result which leads directly to a generalization of the result of Moore.

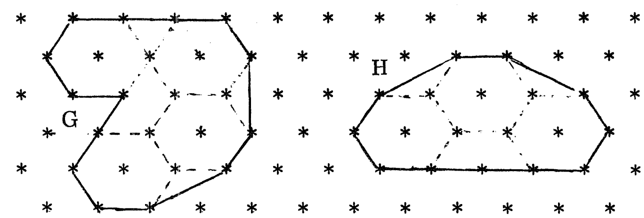


Figure 5

Theorem 4. The product of the binomial coefficients in (the boundary of) any polygonal figure that can be tiled with fundamental hexagons is an integral square.

To see that this generalizes the result of Moore, we prove the following theorem.

Theorem 5. The product of the binomial coefficients in (the boundary of) any convex hexagon with sides oriented along the horizontal rows and main diagonals of Pascal's triangle is an integral square provided the number of coefficients on each side is even.

Proof. In view of Theorem 4, it suffices to show that any hexagon of the type described can be tiled with fundamental hexagons. Let H_n be any such hexagon with n coefficients on its boundary. Plainly, the least possible value of n is 6 which occurs only in the case of a fundamental hexagon. Thus, the result is trivially true in the first possible case. Suppose that it is true for all possible n with $n < k$ where k is any possible value of n with $k > 6$. Since $k > 6$, it follows that at least one side S_1 of H_k must contain at least four coefficients. Without loss of generality, we may presume that S_1 is the lower left-hand side of H_k as indicated in Fig. 6. We may also number the other sides in a counterclockwise direction around H_k. By the induction assumption, it suffices to divide H_k into two hexagons H_i and H_j of the type described and with $i < k$ and $j < k$. We proceed as follows. Let c denote the third coefficient up from the lower end of S_1 and let S be the chord of H_k extending from c and parallel to S_2 as in Fig. 6. Let g be the right-hand end point of S. We distinguish two cases.

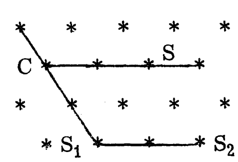


Figure 6

Case 1. If g is on S_3 as in Fig. 7, then the figure a, b, d, h, f, e is an H_i of the desired form since the segment $\overline{dh}$ contains the same number of coefficients as S_2 and the

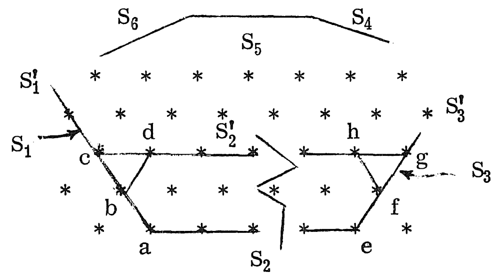


Figure 7

other four sides contain two coefficients each. Also, if we let S_1' denote the upper part of S_1 starting at c, let S_2' denote the line segment $\overline{cg}$, and let S_3' denote the upper part of S_3 starting at g, then S_1' contains two fewer coefficients than S_1, S_2' contains two more coefficients than S_2, and S_3' contains two fewer coefficients than S_3. Thus, the hexagon formed by S_1', S_2', S_3', S_4, S_5, and S_6 is an H_j of the desired type. Finally, since S_2' lies on the interior of H_k (except for its endpoints), it is clear that $i < k$ and $j < k$ as desired.

Case 2. In this case, g lies on S_4 and the appropriate diagram is in Fig. 8. Since the remainder of the argument is essentially the same as for Case 2, we omit the details. This completes the proof.

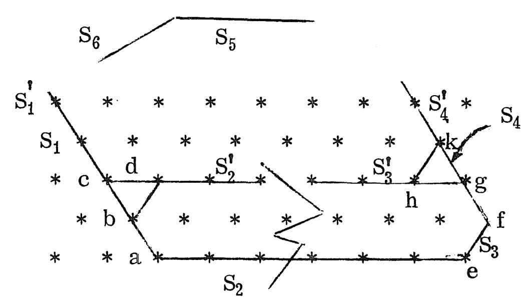


Figure 8

We observe that the convexity conditions of Theorems 3 and 5 are necessary since neither the product of the corner coefficients nor of the boundary coefficients of the hexagon in Fig. 9 is an integral square. Also, it is easy to find examples of convex hexagons where the results of Theorems 3 and 5 do not hold if the condition on the number of elements per side is not met. In fact, we conjecture that the conditions of both theorems are necessary as well as sufficient.

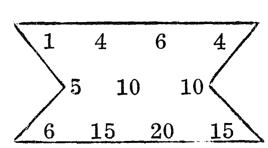


Figure 9

3. SOME ADDITIONAL OBSERVATIONS

In Section 2, we were primarily concerned with hexagons, but it is clear from the fundamental lemma that anything that can be "covered" with pairs of properly oriented parallelograms has the property that the product of those coefficients at the vertices of an odd number of the parallelograms in any such covering is an integral square. Also, if P_1 and P_2 are integral squares which are products of integers and P_3 is the product of those integers common to P_1 and P_2, then P_1P_2/P_3^2 is also an integral square. With these ideas in mind, it is possible to construct an infinite variety of configurations of binomial coefficients whose products are integral squares. The first two examples of such configurations are contained in the following theorems.

Theorem 6. Let K be any convex octagon with sides oriented along the horizontal and vertical rows and main diagonals of Pascal's triangle. Let the number of vertices on the various sides be 2r, 2s, t, 2u, 2v, 2u, t, and 2s as indicated in Fig. 10 where r, s, t, u, and v are positive integers. Then the product of the boundary coefficients is an integral square.

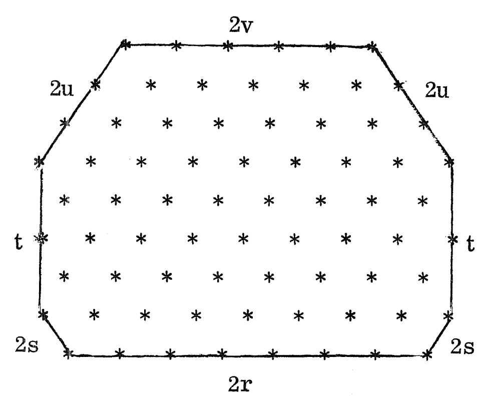


Figure 10

Proof. The proof of this theorem is essentially the same as for Theorem 5 and will be omitted.

In Theorem 6, the convexity condition is not necessary, but it is not presently clear how the theorem should read if this condition is removed. While the octagons of Theorem 6 can be tiled with fundamental hexagons, the octagon of Fig. 11 cannot. It can, however, be tiled with pairs of properly oriented parallelograms (or a combination of parallelograms and fundamental hexagons, if you prefer) and it follows from the fundamental lemma that the product of the boundary coefficients is an integral square.

Also note that the products of the corner coefficients in Fig. 10 of Theorem 6 and in Fig. 11 need not be squares. However, as the following theorem shows, at least one class of octagons exists for which the product of the corner coefficients is always an integral square.

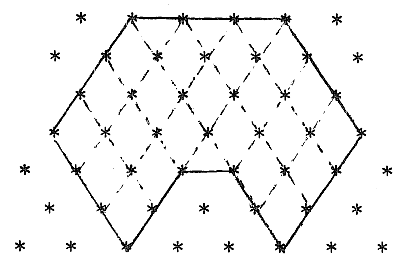


Figure 11

Theorem 7. Let K be a convex octagon formed as in Fig. 12 by adjoining parallelograms with r and s and r and t elements on a side to a parallelogram with r elements on each side. Then the product of the corner coefficients of the octagon is an integral square.

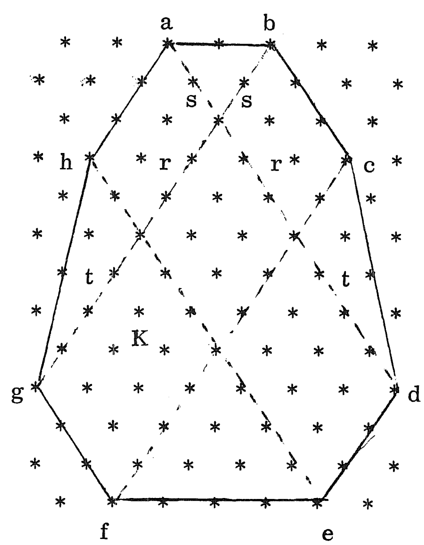


Figure 12

Proof. Let a, b, c, d, e, f, g, and h denote the corner coefficients of the octagon as indicated in Fig. 12. Since a, d, e, and h and b, c, f, and g are the vertices of rectangles oriented as in the fundamental lemma, it is clear that their product is an integral square as claimed.

Again it is clear that the convexity condition of Theorem 7 is not necessary. The most general statement which we can make at the present time is that the product of the corner coefficients of any octagon formed by joining (as in Fig. 13) the vertices of pairs of parallelograms oriented as in the fundamental lemma is an integral square. It is not clear that even this condition is necessary. See Usiskin [3].

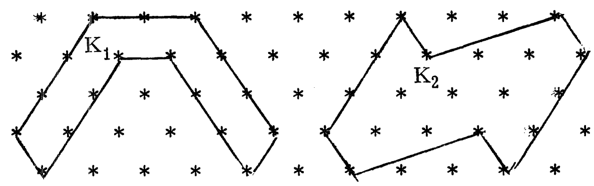
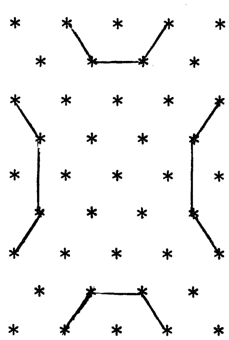
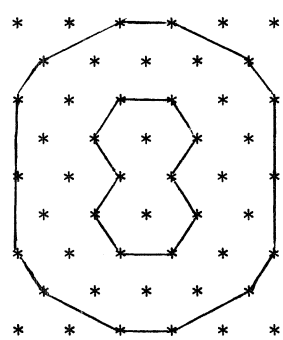


Figure 13

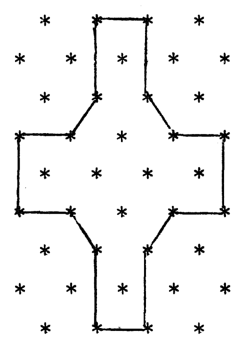
We now give, without proof, several examples of configurations of binomial coefficients whose produces are integral squares. Each example given is a (sometimes not simple, closed, or connected) polygon and it is intended that one consider the product of the boundary coefficients only. Note that it is quite possible to find solid and other non-polygonal arrays whose products are integral squares



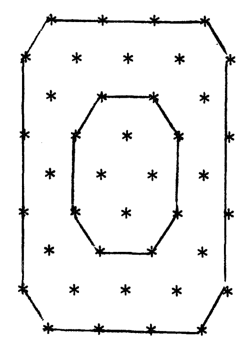
I



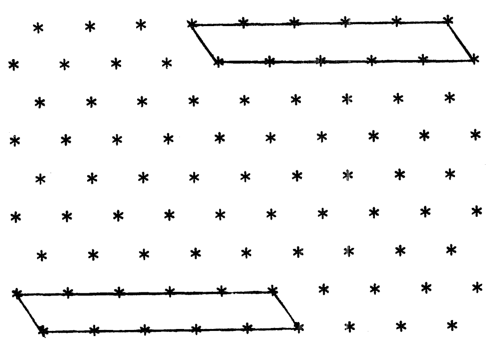
II



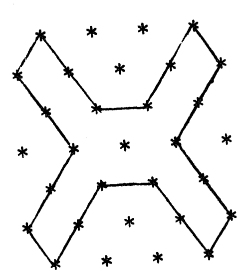
III



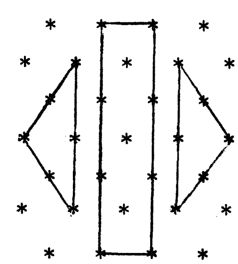
IV



V



VI



VII

REFERENCES

1. V. E. Hoggatt, Jr., and W. Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9, No. 2 (1971), pp. 120-133.
2. C. Moore, "More Hidden Hexagon Squares," The Fibonacci Quarterly,
3. Zalman Usiskin, "Perfect Square Patterns in the Pascal Triangle," Math. Mag., 46 (1973), No. 4, pp. 203-207.

GENERALIZED FIBONACCI POLYNOMIALS

V. E. HOGGATT, JR., and MARJORIE BICKNELL
San Jose State University, San Jose, California

The Fibonacci polynomials and their relationship to diagonals of Pascal's triangle are generalized in this paper. The generalized Q-matrix investigated by Ivie [1] occurs as a special case.

1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials, defined by

$$(1.1)\qquad F_0(x) = 0, \quad F_1(x) = 1, \quad F_2(x) = x, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

are well known to readers of this journal. That the Fibonacci polynomials are generated by a matrix Q_2,

$$(1.2)\qquad Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix},$$

can be verified quite easily by mathematical induction. Also, it is apparent that, when $x = 1$, $F_n(1) = F_n$, the n^{th} Fibonacci number, and when $x = 2$, $F_n(2) = P_n$, the n^{th} Pell number.

Further, when Pascal's triangle is written in left-justified form, the sums of the elements along the rising diagonals give rise to the Fibonacci numbers, and, in fact, those elements are the coefficients of the Fibonacci polynomials. That is,

$$(1.3)\qquad F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1},$$

where $[x]$ is the greatest integer contained in x, and

is a binomial coefficient $\binom{n}{j}$

The first few Fibonacci polynomials are displayed below as well as the array of their coefficients.

Fibonacci Polynomials	Coefficient Array				
$F_1(x) = 1$	1				
$F_2(x) = x$	1				
$F_3(x) = x^2 + 1$	1	1			
$F_4(x) = x^3 + 2x$	1	2			
$F_5(x) = x^4 + 3x^2 + 1$	1	3	1		
$F_6(x) = x^5 + 4x^3 + 3x$	1	4	3		
$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$	1	5	6	1	
$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$	1	6	10	4	
$\cdots \quad \cdots$	...	...	...	...	...

If one observes that, by rule of formation of the Fibonacci polynomials, if one writes the polynomials in descending order, to form the coefficient of the k^{th} term of $F_n(x)$, one adds the coefficients of the k^{th} term of $F_{n-1}(x)$ and the $(k-1)^{st}$ term of $F_{n-2}(x)$, the array of coefficients formed has the same rule of formation as Pascal's triangle when it is written in left-justified form, except that each column is moved one line lower, so that the coefficients formed are those elements that appear along the diagonals formed by beginning in the left-most column and preceding up one and right one throughout the left-justified Pascal triangle. Throughout this paper, this diagonal will be called the <u>rising diagonal</u> of such an array.

2. THE TRIBONACCI POLYNOMIALS

Define the Tribonacci polynomials by

$$T_{-1}(x) = T_0(x) = 0, \qquad T_1(x) = 1, \qquad T_2(x) = x^2\,,$$

$$T_{n+3}(x) = x^2 T_{n+2}(x) + xT_{n+1}(x) + T_n(x). \tag{2.1}$$

When $x = 1$, $T_n(1) = T_n$, the n^{th} Tribonacci number $1, 1, 2, 4, 7, 13, 24, 44, 81, \cdots$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. The first few Tribonacci polynomials follow.

Tribonacci Polynomials

$$T_1(x) = 1$$

$$T_2(x) = x^2$$

$$T_3(x) = x^4 + x$$

$$T_4(x) = x^6 + 2x^3 + 1$$

$$T_5(x) = x^8 + 3x^5 + 3x^2$$

$$T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$$

$$T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1$$

$$T_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2$$

Tribonacci Coefficient Array

1				
1				
1	1			
1	2	1		
1	3	3		
1	4	6	2	
1	5	10	7	1
1	6	15	16	6
...	...	...	...	...

Left-Justified Trinomial Coefficient Array

$(1 + x + x^2)^n, \qquad n = 0, 1, 2, \cdots$

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
...	...	...	...	...	...	...	...	...	...	...	...	...

The Tribonacci coefficient array has the same rule of formation as the trinomial coefficient array, except that each column is placed one line lower. Thus, the sums of the rows are the same as the sums of the rising diagonals of the trinomial coefficient array, both sums yielding the Tribonacci numbers 1, 1, 2, 4, 7, 13, 24, $\cdots$, and the coefficients of the Tribonacci polynomials are the trinomial coefficients found on those same rising diagonals. That is,

$$T_n(x) = \sum_{j=0} \binom{n-j-1}{j}_3 x^{2n-3j-2}, \tag{2.2}$$

where

$$\binom{n}{j}_3$$

is the trinomial coefficient in the n^{th} row and j^{th} column where, as is usual, the left-most column is the zeroth column and the top row the zeroth row, and

$$\binom{n}{j}_3 = 0 \quad \text{if} \quad j > n .$$

The Tribonacci polynomials are generated by the matrix Q_3,

$$Q_3 = \begin{pmatrix} x^2 & 1 & 0 \\ x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

so that

$$Q_3^n = \begin{pmatrix} T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ xT_n(x) + T_{n-1}(x) & xT_{n-1}(x) + T_{n-2}(x) & xT_{n-2}(x) + T_{n-3}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{pmatrix} . \tag{2.3}$$

A proof could be made by mathematical induction. That Q_3^n has the given form for $n = 1$ is apparent by inspection of Q_3, element-by-element. Expansion of the matrix product $Q_3^{n+1} = Q_3 Q_3^n$ gives the elements of Q_3^{n+1} in the required form, making use of the recursion (2.1).

Notice that $\det Q_3^n = 1^n = 1$, analogous to the Fibonacci case. In fact, we can write an interesting determinant identity. Again using (2.1), we multiply row one of Q_3^n by x^2 and add to row 2. Then we exchange rows 1 and 2 to write

$$(-1) = \begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{vmatrix} , \tag{2.4}$$

which becomes an identity for Tribonacci numbers when $x = 1$.

3. THE QUADRANACCI POLYNOMIALS

The Quadranacci polynomials are defined by $T^*_{-2}(x) = T^*_{-1}(x) = T^*_0(x) = 0$, $T^*_1(x) = 1$,

$$T^*_{n+4}(x) = x^3 T^*_{n+3}(x) + x^2 T^*_{n+2}(x) + x T^*_{n+1}(x) + T^*_n(x) . \tag{3.1}$$

The first few values are

$$\begin{aligned}
T_1(x) &= 1 \\
T_2(x) &= x^3 \\
T_3(x) &= x^6 + x^2 \\
T_4(x) &= x^9 + 2x^5 + x \\
T_5(x) &= x^{12} + 3x^8 + 3x^4 + 1 \\
T_6(x) &= x^{15} + 4x^{11} + 6x^7 + 4x^3 \\
T_7(x) &= x^{18} + 5x^{14} + 10x^{10} + 10x^6 + 3x^2 \\
\cdots \quad & \quad \cdots
\end{aligned}$$

Notice that the coefficient of the j^{th} term of $T_n^*(x)$ is the sum of the coefficients of the j^{th} term of $T_{n-1}^*(x)$, $(j-1)^{st}$ term of $T_{n-2}^*(x)$, $(j-2)^{nd}$ term of $T_{n-3}^*(x)$, and $(j-3)^{rd}$ term of $T_{n-4}^*(x)$ when the polynomials are arranged in descending order. Then, the array of coefficients, if each row were moved up one line, would have the same rule of formation as the left-justified array of quadranomial coefficients, arising from expansions of $(1+x+x^2+x^3)^n$, $n = 0, 1, 2, \cdots$. Thus, the coefficients of $T_n^*(x)$ are those found on the n^{th} rising diagonal of the quadranomial triangle. Also, $T_n^*(1) = T_n^*$, the n^{th} Quadranacci number $1, 1, 2, 4, 8, 15, 29, 56, 108, \cdots$, $T_{n+4}^* = T_{n+3}^* + T_{n+2}^* + T_{n+1}^* + T_n^*$.

The Quadranacci polynomials are generated by the matrix Q_4,

$$Q_4 = \begin{pmatrix} x^3 & 1 & 0 & 0 \\ x^2 & 0 & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,$$

so that $Q_4^n = (a_{ij})$ has its j^{th} column given by $a_{1j} = T_{n+2-j}^*(x)$, $a_{2j} = x^2T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-1-j}^*(x)$, $a_{3j} = xT_{n+1-j}^*(x) + T_{n-j}^*(x)$, and $a_{4j} = T_{n+1-j}^*(x)$, $j = 1, 2, 3, 4$. That Q_4^n has the form claimed above can be established by mathematical induction. That Q_4 has the stated form follows by inspection. Let $Q_4^{n+1} = (b_{ij})$ and $Q_4 = (q_{ij})$. Then we expand $Q_4^{n+1} = Q_4 Q_4^n$. The first row of Q_4^{n+1} has the required form, for

$$\begin{aligned} b_{1j} &= q_{11}a_{1j} + q_{12}a_{2j} + q_{13}a_{3j} + q_{14}a_{4j} \\ &= [x^3T_{n+2-j}^*(x)] + [x^2T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-1-j}^*(x)] + 0 + 0 \\ &= T_{(n+1)+2-j}^* , \end{aligned}$$

where we make use of (3.1). Computation of b_{2j}, b_{3j}, and b_{4j} is similar, and shows that Q_4^{n+1} has the required form, which would complete the proof.

We derive a determinant identity for Quadranacci polynomials from Q_4^n by forming the matrix Q_4^{*n} as follows. Add x^3 times row 1 to row 2, making $a'_{2j} = T_{n+3-j}^*(x)$. Add x^2 times row 1 and x^3 times row 2 to row 3, producing $a'_{3j} = T_{n+4-j}^*(x)$. Exchange rows 1 and 3. Then matrix Q_4^{*n} has

$$a_{1j}^* = T_{n+4-j}^*(x), \quad a_{2j}^* = T_{n+3-j}^*(x), \quad a_{3j}^* = T_{n+2-j}^*(x) ,$$

and

$$a_{4j}^* = T_{n+1-j}^*(x), \quad j = 1, 2, 3, 4, \quad \text{and} \quad \det Q_4^{*n} = (-1)^{n+1}$$

because there was one row exchange. That is, for example, when $x = 1$,

$$(3.2)\qquad (-1)^{n+1} = \begin{vmatrix} T^*_{n+3} & T^*_{n+2} & T^*_{n+1} & T^*_{n} \\ T^*_{n+2} & T^*_{n+1} & T^*_{n} & T^*_{n-1} \\ T^*_{n+1} & T^*_{n} & T^*_{n-1} & T^*_{n-2} \\ T^*_{n} & T^*_{n-1} & T^*_{n-2} & T^*_{n-3} \end{vmatrix} ,$$

where T^*_n is the n^{th} Quadranacci number.

4. THE R-BONACCI POLYNOMIALS

Define the r-bonacci polynomials by

$$R_{-(r-2)}(x) = R_{-(r-1)}(x) = \cdots = R_{-1}(x) = R_0(x) = 0, \quad R_1(x) = 1, \quad R_2(x) = x^{r-1} ,$$

$$(4.1)\qquad R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \cdots + R_n(x) .$$

The r-bonacci polynomials, by their recursive definition will have the coefficients of $R_n(x)$, written in descending order, given by the coefficients on the n^{th} rising diagonal of the left-justified r-nomial coefficient array, the coefficients arising from expansions of

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, \cdots .$$

That is,

$$(4.2)\qquad R_n(x) = \sum_{j=0} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj} ,$$

where

$$\binom{n}{j}_r$$

is the element in the n^{th} row and j^{th} column of the left-justified r-nomial triangle, and

$$\binom{n}{j}_r = 0 \quad \text{when} \quad j > n .$$

The r-bonacci polynomials are generated by the $r \times r$ matrix Q_r,

$$Q_r = \begin{pmatrix} x^{r-1} & 1 & 0 & 0 & \cdots & 0 \\ x^{r-2} & 0 & 1 & 0 & \cdots & 0 \\ x^{r-3} & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is an identity matrix of order $(r - 1)$ bordered on the left by a column of descending powers of x and followed by a bottom row of zeros. The matrix Q_r^n has $R_{n+1}(x)$ as the element in the upper left and $R_n(x)$ in the lower left, with general element a_{ij} given by

$$a_{ij} = \sum_{k=1}^{r-1+1} x^{r+1-k-i} R_{n+1-j-k}(x) \; . \tag{4.3}$$

Proof of (4.3) is by mathematical induction. Let $Q_r = (b_{ij})$. Then $b_{i1} = x^{r-i}$, $i = 1, 2, \cdots, r$; $b_{ij} = 1$, $j = i + 1$, $i = 1, 2, \cdots, r$; and $b_{ij} = 0$ whenever $j \neq 1$ and $j \neq i + 1$. Let

$$Q_r^{n+1} = Q_r Q_r^n = (c_{ij}) \; .$$

Then

$$\begin{aligned} c_{ij} = \sum_{k=1}^{r} b_{ik} a_{kj} &= b_{i1} a_{1k} + \sum_{k=2}^{r} b_{ik} a_{kj} \\ &= x^{r-i} R_{n+1-j}(x) + a_{i+1,j} + 0 \\ &= x^{r-i} R_{n+1-j}(x) + \sum_{k=1}^{r-i} x^{r-k-i} R_{n+1-j-k}(x) \\ &= \sum_{k=1}^{r-i+1} x^{r+1-k-i} R_{(n+1)+1-j-k}(x) \; , \end{aligned}$$

which is the required form for the general element of Q_r^{n+1}, completing the proof.

If we operate upon Q_r^n as before, we can again make a determinant identity. Repeatedly add x^{r-1} times the $(i - 1)^{st}$ row, x^{r-2} times the $(i - 2)^{nd}$ row, $\cdots$, x^{r+1-i} times the first row to the i^{th} row, to produce a new i^{th} row with R_{n+i-1} in its first column, for $i = 2, 3, \cdots, r - 1$. Then make $(r - 1)/2$ row exchanges to put the elements in the columns in descending order. The matrix R formed has its general element given by

$$r_{ij} = R_{n+r+1-i-j}(x) \; ,$$

and its determinant has value $(-1)^{(r-1)n+[(r-1)/2]}$. That is, when $x = 1$, the r-bonacci numbers

$$\cdots, 0, 1, 1, 2, \cdots, R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n \; ,$$

have the determinant identity

$$\det R = \begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & \cdots & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & \cdots & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)(2n+1)/2}$$

Notice that Eq. (1.2) gives

$$\det Q_2^n = F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n = \det R$$

for r = 2. Since we recognize

$$\left| F_{n+1}(x)F_{n-1}(x) - F_n^2(x) \right| = 1$$

as the characteristic value [2], [3], [4] of sequences arising from the Fibonacci polynomials, we define $|\det R| = 1$ as the characteristic value of the sequences arising from the generalized Fibonacci polynomials, $r \geq 2$. Then, for example, (2.4) gives the characteristic value $|(-1)| = 1$ for the sequences arising from the Tribonacci polynomials, while (3.2) is the array giving the characteristic value $|(-1)^{n+1}| = 1$ for the Quadranacci numbers.

The matrix R just defined has the interesting property that multiplication by Q_r^n produces a matrix of the same form. To clarify, let $R = R_{r,n} = (r_{ij})$ be the $r \times r$ matrix with $R_n(x)$ appearing in the lower left corner, $r_{ij} = R_{n+r+1-i-j}(x)$. Then

$$R_{r,0}Q_r^n = R_{r,n} \tag{4.4}$$

which is proved by mathematical induction as follows. Consider the matrix product $R_{r,n}Q_r = (p_{ij})$ for any n, where we observe that the first column of Q_r contains the multipliers for the recursion relation for the polynomials $R_n(x)$. The i^{th} row of $R_{r,n}$ multiplied by the first column of Q_r produces

$$p_{i1} = \sum_{k=1}^{r} R_{n+r+1-i-k}(x)\, x^{r+1-k} = R_{n+r+1-i}(x) = R_{(n+1)+r+1-i-1}(x) ,$$

while, when $j \neq 1$, since the only non-zero elements of Q_r occur when $i = j - 1$, the i^{th} row of $R_{r,n}$ times the j^{th} column of Q_r produces

$$p_{ij} = R_{n+r+1-i-(j-1)} = R_{(n+1)+r+1-i-j}, \qquad j = 2, 3, \cdots, r ,$$

so that $R_{r,n}Q_r = R_{r,n+1}$, for any n. Then, we must have that $R_{n,0}Q_r = R_{r,1}$, and, if

$$R_{r,0}Q_r^{k-1} = R_{r,k-1} ,$$

then

$$R_{r,0}Q_r^k = (R_{r,0}Q_r^{k-1})Q_r = R_{r,k-1}Q_r = R_{r,k} ,$$

which completes a proof of (4.4) by mathematical induction.

If we equate the elements in the upper left corner of $R_{r,n}$ and $R_{r,0}Q_r^n$ we obtain the identity

$$
\begin{aligned}
R_{n+r-1}(x) = R_{r-1}(x)R_{n+1}(x) &+ R_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+2}(x)] \\
&+ R_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+3}(x)] \\
&+ \cdots \\
&+ R_1(x)[xR_n(x) + R_{n-1}(x)] + R_0(x)R_n(x) \; .
\end{aligned} \tag{4.5}
$$

Notice that the matrix Q_r provides the multipliers for the recursion relation for the polynomials $R_n(x)$ but does not depend upon the original values of the polynomials in the proof of (4.4). Let $H_n(x)$ be any sequence of polynomials with r arbitrary starting values $H_0(x), H_1(x), \cdots, H_{r-1}(x)$, and with the same recursion relation as the polynomials $R_n(x)$. Form the matrix $R^*_{r,n} = (r^*_{ij})$, $r^*_{ij} = H_{n+r+1-i-j}(x)$. By the arguments used earlier, we can derive $R^*_{r,0}Q_r^n = R^*_{r,n}$ and thus obtain

$$
\begin{aligned}
H_{n+r-1} = H_{r-1}(x)R_{n+1}(x) &+ H_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+2}(x)] \\
&+ H_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) \\
&\qquad + \cdots + R_{n-r+3}(x)] \\
&+ \cdots \\
&+ H_1(x)[xR_n(x) + R_{n-1}(x)] + H_0(x)R_n(x) \; .
\end{aligned} \tag{4.6}
$$

REFERENCES

1. John Ivie, "A General Q-Matrix," Fibonacci Quarterly, Vol. 10, No. 3, April, 1972, pp. 255-261.
2. Eugene Levine, "Fibonacci Sequences with Identical Characteristic Values," Fibonacci Quarterly, Vol. 6, No. 5, Nov., 1968, pp. 75-80.
3. Irving Adler, "Sequences with a Characteristic Number," Fibonacci Quarterly, Vol. 9, No. 2, April, 1971, pp. 147-162.
4. Brother Alfred Brousseau, Fibonacci and Related Number Theoretic Tables, The Fibonacci Association, 1972.

PERIODICITY OVER THE RING OF MATRICES

R. J. DECARLI
Rosary Hill College, Buffalo, New York

Let R be the ring of $t \times t$ matrices with integral entries and identity I. Consider the sequence $\{M_m\}$ of elements of R, recursively defined by

$$M_{m+2} = A_1 M_{m+1} + A_0 M_m \quad \text{for} \quad m \geq 0, \tag{1}$$

where M_0, M_1, A_0, and A_1 are arbitrary elements of R. In [1] we established identities for such a sequence over an arbitrary ring with unity. In this paper we establish an analogue of Robinson's [3] result concerning periodicity modulo k where k is an integer greater than 1. We need the following definitions.

Definition 1. Let $A = [a_{ij}]$ be an element of R. We reduce A modulo k by reducing each entry modulo k. If $B = [b_{ij}] \in R$, then $A \equiv B \pmod{k}$ if and only if $a_{ij} \equiv b_{ij} \pmod{k}$ for all i, j.

Definition 2. We say that the sequence defined by (1) is periodic modulo k if there exists an integer $L \geq 2$ such that $M_i \equiv M_{L+i} \pmod{k}$ for $i = 0, 1, 2, \cdots$. By the nature of the sequence we see that this is equivalent to the existence of an $L \geq 2$ such that $M_0 \equiv M_L \pmod{k}$ and $M_1 \equiv M_{L+1} \pmod{k}$.

We assume for all matrices in the following Theorem that reduction modulo k has already taken place and we employ the usual notation for relative primeness. For $A \in R$ we let det A stand for the determinant of A.

Theorem 1. If $(\det A, k) = 1$, then the $\{M_m\}$ sequence defined by (1) is periodic modulo k.

Proof. Let

$$W_1 = \begin{bmatrix} 0 & I \\ A_0 & A_1 \end{bmatrix}, \tag{2}$$

where the entries are matrices from R. If we set

$$S_m = \begin{bmatrix} M_m \\ M_{m+1} \end{bmatrix}$$

for $n \geq 0$, then a simple induction argument yields

$$S_m = W_1^m S_0 . \tag{3}$$

If we can find an L such that $W_1^L \equiv I \pmod{k}$, then $S_L = W_1^L S_0 \equiv I \cdot S_0 \equiv S_0 \pmod{k}$ and we will have

$$\begin{bmatrix} M_L \\ M_{L+1} \end{bmatrix} \equiv \begin{bmatrix} M_0 \\ M_1 \end{bmatrix} \pmod{k},$$

which gives us periodicity. To show that such an L exists consider the sequence of matrices

$$(4) \qquad I, \quad W_1, \quad W_1^2, \quad \cdots .$$

We first show that each matrix in (4) has an inverse modulo k. Laplace's method for evaluating determinants immediately gives $\det W_1^r = (\det W_1)^r = ((-1)^t \det A_0)^r \not\equiv 0 \pmod{k}$, since $(\det A_0, k) = 1$. Also, $(\det A_0, k) = 1$ implies $(((-1)^t \det A_0)^r, k) = 1$ and thus

$$(5) \qquad (\det W_1^r, k) = 1 .$$

For $r = 0$, $W_1^0 = I$ which is its own inverse. For $r > 0$ we let w_{ij} denote the entries of W_1^r and A_{ij} the cofactor of w_{ij} in $\det W_1^r$. We observe that A_{ij} is always integral. Using matrix methods we have

$$(6) \qquad (W_1^r)^{-1} = \begin{bmatrix} \dfrac{A_{ij}}{\det W_1^r} \end{bmatrix}^T ,$$

where T stands for the transpose. An entry in the right-hand side of (6) is of the form

$$\frac{c}{\det W_1^r} ,$$

where c is an integer. The equation $(\det W_1^r)x \equiv c \pmod{k}$ has a unique solution since from (5) we have $(\det W_1^r, k) = 1$. Thus each entry in the right side of (6) is an integer and W_1^r has an inverse mod k for all $r \geq 0$. Because we only have k distinct integers mod k and $(2t)^2$ places to put them, we have at most $k^{(2t)^2}$ different matrices in (4). Since the sequence is infinite we must have

$$(7) \qquad W_1^{L+r} \equiv W_1^r \pmod{k} \quad \text{for some } L .$$

Multiplying both sides of (7) by $(W_1^r)^{-1}$ yields

$$(8) \qquad W_1^L \equiv I \pmod{k} .$$

Since $W_1 \not\equiv I$ we see that $L \geq 2$. Thus we have $S_L = W_1^L S_0 \equiv IS_0 \equiv S_0 \pmod{k}$ which implies $M_L \equiv M_0$ and $M_{L+1} \equiv M_1$ and establishes periodicity.

The central role played by A_0 is more clearly illustrated if we consider a higher order recurrence defined for a fixed $d \geq 2$ by:

$$M_{m+d} = A_{d-1}M_{m+d-1} + A_{d-2}M_{m+d-2} + \cdots + A_0M_m, \qquad m \geq 0,$$

where the A_i and the M_i, $0 \leq i \leq d-1$, are arbitrary elements from R. Even though there are 2d arbitrary elements that determine this sequence, the question of periodicity still depends on the nature of A_0. If det $(A_0, k) = 1$, then we again have periodicity. This is proved using

$$V = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{d-1} \end{bmatrix}$$

in place of W_1 and

$$S_m = \begin{bmatrix} M_m \\ M_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ M_{m+d-1} \end{bmatrix}.$$

It is easy to show that $S_m = V^m S_0$ and that det V depends on det A_0. The rest of the proof follows as in the proof of Theorem 1. A close look at the position of A_0 in V clearly indicates why it is so important in determining periodicity.

REFERENCES

1. R. J. DeCarli, "A Generalized Fibonacci Sequence Over an Arbitrary Ring," Fibonacci Quarterly, Vol. 8, No. 2 (1970), pp. 182-184.
2. I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, Wiley, New York, 1960.
3. D. W. Robinson, "The Fibonacci Matrix Modulo m," Fibonacci Quarterly, Vol. 1, No. 2 (1963), pp. 29-36.

SPECIAL DETERMINANTS FOUND WITHIN GENERALIZED PASCAL TRIANGLES

MARJORIE BICKNELL and V. E. HOGGATT, JR.,
San Jose State University, San Jose, California

That Pascal's triangle and two classes of generalized Pascal triangles, the multinomial coefficient arrays and the convolution arrays formed from sequences of sums of rising diagonals within the multinomial arrays, share sequences of $k \times k$ unit determinants was shown in [1]. Here, sequences of $k \times k$ determinants whose values are binomial coefficients in the k^{th} column of Pascal's triangle or numbers raised to a power given by the $(k-1)^{st}$ triangular numbers are explored.

1. INTRODUCTION

First, we imbed Pascal's triangle in rectangular form in the $n \times n$ matrix $P = (p_{ij})$, where

$$p_{ij} = \binom{i+j-2}{i-1},$$

(1.1)
$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

Pascal's triangle in left-justified form can be imbedded in the $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \binom{i-1}{j-1},$$

(1.2)
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}$$

To avoid confusion, note that when Pascal's triangle is imbedded in matrices throughout this paper, we will number the rows and columns in the usual matrix notation, with the leftmost column the first column. If we refer to Pascal's triangle itself, however, then the leftmost column is the zeroth column, and the top row is the zeroth row. While we are dealing with infinite matrices here, the multiplication of $n \times n$ matrices provides an easily understood presentation. As in [1], compositions of generating functions actually lie at the heart of the proofs.

Let us define an arithmetic progression of the r^{th} order, denoted by $(AP)_r$, as a sequence of numbers whose r^{th} row of differences is a row of constants, but whose $(r-1)^{st}$ row is not. A row of repeated constants is an $(AP)_0$. The constant in the r^{th} row of differences of an $(AP)_r$ will be called the constant of the progression. That the i^{th} row of Pascal's triangle in rectangular form is an $(AP)_i$ with constant 1 was proved in [1]. We will have need of the following theorem from [1], [2].

Theorem 1.1 (Eves' Theorem). Consider a determinant of order n whose i^{th} row $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an $(AP)_{i-1}$ with constant a_i. Then the value of the determinant is the product $a_1 a_2 \cdots a_n$.

2. BINOMIAL COEFFICIENT DETERMINANT VALUES FROM PASCAL'S TRIANGLE

Return again to matrix P of (1.1). Suppose that we remove the top row and left column, and then evaluate the $k \times k$ determinants containing the upper left corner. Then

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4,$$

and the $k \times k$ determinant has value $(k + 1)$.

Proof is by mathematical induction. Assume that the $(k-1) \times (k-1)$ determinant has value k. In the $k \times k$ determinant, subtract the preceding column from each column successively for $j = k, k-1, k-2, \cdots, 2$. Then subtract the preceding row from each row successively for $i = k, k-1, k-2, \cdots, 2$, leaving

$$\begin{vmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & \cdots \\ 0 & 3 & 6 & 10 & \cdots \\ 0 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= 1 + k .$$

Returning to matrix P, take 2×2 determinants along the 2nd and 3rd rows:

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \quad \cdots,$$

giving the values found in the second column of Pascal's left-justified triangle, for

$$\begin{vmatrix} \binom{j}{1} & \binom{j+1}{1} \\ \binom{j+1}{2} & \binom{j+2}{2} \end{vmatrix} = \binom{j+1}{2}$$

by simple algebra. Of course, 1×1 determinants along the second row of P yield the successive values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10,$$

the successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the 2nd, 3rd, $\cdots$, and $(k+1)^{st}$ rows yields the successive entries of the k^{th} column of Pascal's triangle.

To formalize our statement,

Theorem 2.1. The determinant of the $k \times k$ matrix $R(k,j)$ taken with its first column the j^{th} column of P, the rectangular form of Pascal's triangle imbedded in a matrix, and its first row the second row of P, is the binomial coefficient

$$\binom{j-1+k}{k} .$$

To illustrate,

$$\det R(4,3) = \begin{vmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 6 & 4 & 5 & 6 \\ 10 & 10 & 15 & 21 \\ 15 & 20 & 35 & 56 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 \\ 0 & 6 & 10 & 15 \\ 0 & 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \\ 5 & 15 & 35 & 70 \end{vmatrix}$$

$$= \det R(3,3) + \det R(4,2)$$

$$= \binom{5}{3} + \binom{5}{4} = \binom{6}{4} = 15 .$$

First, the preceding column was subtracted from each column successively, $j = k, k-1, \cdots, 2$, and then the preceding row was subtracted from each row successively for $i = k, k-1, \cdots, 2$. Then the determinant was made the sum of two determinants, one bordering $R(3,3)$ and the other equal to $R(4,2)$ by adding the j^{th} column to the $(j+1)^{st}$, $j = 1, 2, \cdots, k-1$.

By following the above procedure, we can make

$$\det R(k,j) = \det R(k-1, j) + \det R(k, j-1) .$$

We have already proved that

$$\det R(k,1) = 1 = \binom{k+0}{k}, \quad \det R(k,2) = k+1 = \binom{k+1}{k} \text{ for all } k,$$

$$\det R(1,j) = j = \binom{j+0}{1}, \quad \det R(2,j) = \binom{j+1}{2} \text{ for all } j .$$

If

$$\det R(k-1,j) = \binom{j+k-2}{k-1} \quad \text{and} \quad \det R(k, j-1) = \binom{j+k-2}{k} ,$$

then

$$\det R(k,j) = \binom{j+k-2}{k-1} + \binom{j+k-2}{k} = \binom{j+k-1}{k}$$

for all k and all j by mathematical induction.

Since P is its own transpose, Theorem 2.1 is also true if the words "column" and "row" are everywhere exchanged.

Consider Pascal's triangle in the configuration of A^T, which is just Pascal's rectangular array P with the i^{th} row moved $(i-1)$ spaces right, $i = 1, 2, 3, \cdots$. Form $k \times k$ matrices $R'(k,j)$ such that the first row of $R'(k,j)$ is the second row of A^T beginning with the j^{th} column of A^T. Then $AR'(k, j-1) = R(k,j)$ as can be shown by considering their column generating functions, and since $\det A = 1$, $\det R'(k, j-1) = \det R(k,j)$, leading us to the following theorems.

Theorem 2.2. Let A^T be the $n \times n$ matrix containing Pascal's triangle on and above its main diagonal so that the rows of Pascal's triangle are placed vertically. Any $k \times k$ submatrix of A^T selected with its first row along the second row of A^T and its first column the j^{th} column of A^T, has determinant value

$$\binom{k+j-2}{k} .$$

Since A is the transpose of A^T, wording Theorem 2.2 in terms of the usual Pascal triangle provides the following.

Theorem 2.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the i^{th} row has determinant value given by the binomial coefficient

$$\binom{k+i-1}{k} .$$

Returning to the rectangular Pascal matrix P, in Theorem 2.1, the first row of P was omitted to form the $k \times k$ matrix considered. Now we omit any one row.

Theorem 2.4. Let $R_i(k,j)$ be the $k \times k$ matrix formed from the rectangular Pascal matrix P so that its first k rows are the first $(k+1)$ rows of P with the i^{th} row omitted, and its first column is the j^{th} column of P. Then $\det R_i(k,j)$ is given by the binomial coefficient

$$\binom{j-1+k}{k-i+1} .$$

Proof. Notice that $R_1(k,j) = R(k,j)$ of Theorem 2.1. If the first row is not the row omitted, by successively subtracting the p^{th} column of $R_s(k,j)$ from the $(p+1)^{st}$ column, $p = k-1, k-2, \cdots, 1,$ the new array is $R_{s-1}(k-1, j+1)$ bordered by a first row with first element one and all others zero, so that

$$\det R_s(k,j) = \det R_{s-1}(k-1, j+1) .$$

If the theorem holds when $i = s-1$, then

$$\det R_{s-1}(k-1, j+1) = \binom{(j+1)-1+(k-1)}{(k-1)-(s-1)+1} = \binom{j-1+k}{k-s+1} = \det R_s(k,j) ,$$

completing a proof by mathematical induction.

3. OTHER DETERMINANTS WITH SPECIAL VALUES

Suppose we form a matrix using the zeroth, second, fourth, $\cdots$, $(2r)^{th}$, $\cdots$, rows of Pascal's triangle written in rectangular form. Then the columns contain even subscripted elements only. Since the i^{th} column is still an i^{th} order arithmetic progression, Eves' Theorem should apply. The constant for the j^{th} column will be 2^{j-1}, rather than 1, and the determinant of such a $k \times k$ matrix will be $2^0 \cdot 2^1 \cdot 2^2 \cdots \cdot 2^{k-1}$ or $2^{[k(k-1)/2]}$. To clarify this, we present such a 5×5 matrix below, which has determinant $2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4 = 2^{10}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 5 & 15 & 35 & 70 \\ 1 & 7 & 28 & 84 & 210 \\ 1 & 9 & 45 & 165 & 495 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 2 & 9 & 25 & 55 \\ 0 & 2 & 13 & 49 & 140 \\ 0 & 2 & 17 & 81 & 285 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 16 & 41 \\ 0 & 0 & 4 & 24 & 85 \\ 0 & 0 & 4 & 32 & 145 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 7 & 17 \\ 0 & 0 & 0 & 8 & 44 \\ 0 & 0 & 0 & 8 & 60 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 0 & 4 & 7 & 17 \\ 0 & 0 & 0 & 8 & 44 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix} .$$

In this section, we will prove the following:

Theorem 3.1. Form an $n \times n$ matrix in the upper left corner of Pascal's triangle in rectangular form (or in left-justified form) using the rows which are multiples of r so that the $(i + 1)^{st}$ row in the matrix is the first n entries of the $(ir)^{th}$ row of the Pascal array, $i = 0, 1, 2, \cdots, n - 1$. The determinant of the matrix is $r^{[n(n-1)/2]}$.

To prove this theorem, we require more information about r^{th} order arithmetic progressions.

Lemma 3.1. Let $\{c_j\}$, $j = 0, 1, 2, \cdots$, be a sequence of consecutive elements of an $(AP)_i$ with constants a_i. Then the k^{th} difference sequence has elements given by

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{j-p}, \quad \text{and} \quad \sum_{p=0}^{i} (-1)^p \binom{i}{p} c_{j-p} = a_i .$$

Proof. We list successive differences:

1st: $$c_j - c_{j-1}$$

2nd: $$(c_j - c_{j-1}) - (c_{j-1} - c_{j-2}) = c_j - 2c_{j-1} + c_{j-2}$$

3rd: $$(c_j - 2c_{j-1} + c_{j-2}) - (c_{j-1} - 2c_{j-2} + c_{j-3})$$
$$= c_j = 3c_{j-1} + 3c_{j-2} - c_{j-3} .$$

If the $(k - 1)^{st}$ difference has the form of Lemma 3.1, then the k^{th} difference is given by

$$\sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} - \sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p}$$

$$= c_j + \sum_{p=1}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} + \sum_{p=1}^{k-1} (-1)^p \binom{k-1}{p} c_{j-p} + (-1)^k c_{j-k}$$

$$= \sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{j-p} \quad ,$$

which establishes the form given in the Lemma. When $k = i$, then the i^{th} difference is the constant of the sequence.

Now form an $n \times n$ matrix P^* with its $(i+1)^{st}$ row the first n entries of the $(ri)^{th}$ row of Pascal's triangle in rectangular form. Then the elements in its $(k+1)^{st}$ column are given by

$$\binom{(r-i)k}{k} .$$

The k^{th} difference sequence for these elements is

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} c_{r-p} = \sum_{p=0}^{k} (-1)^p \binom{k}{p} \binom{(r-p)k}{k} = r^k$$

applying Lemma 3.1 and a formula given by Knuth [3]. The $(k-1)^{st}$ difference is not a constant, however, so that the sequence is an $(AP)_k$. By Eves' Theorem, the determinant, then, will be $r^0 \cdot r^1 \cdot r^2 \cdots \cdot r^{n-1} = r^{n(n-1)/2}$.

If an $n \times n$ matrix A^* is formed using only the $(ri)^{th}$ rows of Pascal's left-justified triangle, and A^T is the transpose of A defined in (1.2), then $A^*A^T = P^*$, since the row generators of A^* are $(1+x)^{r(i-1)}$, and of A^T,

$$\left(\frac{1}{1-x}\right) \cdot \left(\frac{x}{1-x}\right)^{i-1} ,$$

making the row generators of A^*A^T

$$\left(\frac{1}{1-x}\right) \cdot \left(1 + \frac{x}{1-x}\right)^{r(i-1)} = \left(\frac{1}{1-x}\right)^{r(i-1)+1}$$

which we recognize as the row generators of P^*, making $\det A^* = \det P^* = r^{n(n-1)/2}$. (Here, we apply the method of proof using generating functions as in [1].)

For example, when $n = 4$ and $r = 2$, A^*A^T becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 4 & 6 & 4 \\ 1 & 6 & 15 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 5 & 15 & 35 \\ 1 & 7 & 28 & 84 \end{bmatrix} .$$

In fact, we have the same results if the row numbers taken to form P^* or A^* form an arithmetic progression.

Theorem 3.2. Form an $n \times n$ matrix which has its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of Pascal's triangle in rectangular form, $s \geq 0$, $i = 0, 1, \cdots, n-1$. The determinant of the matrix is $r^{n(n-1)/2}$.

Proof. Subtract the $(k-1)^{st}$ column from the k^{th} column for $k = n, n-1, \cdots, 2$. Repeating this process s times gives the matrix P^*.

Reapplying Eves' Theorem, Theorem 3.1 can be extended to the following.

Theorem 3.3. Form an $n \times n$ matrix such that its $(i+1)^{st}$ row consists of any n successive entries whose subscripts differ by r from the i^{th} row of Pascal's triangle written in rectangular form, $i = 0, 1, \cdots, n-1$. The determinant of the matrix is $r^{n(n-1)/2}$.

The theorems of this section are special cases of the more general theorem which follows.

Theorem 3.4. Form an $n \times n$ matrix which has its $(i+1)^{st}$ row the subsequence $\{c_{ir+s}\}$, s arbitrary, of an $(AP)_i\ \{c_i\}$ with constant a_i, $i = 0, 1, \cdots, n-1$. Then the determinant is $r^{n(n-1)/2}\ a_0 a_1 a_2 \cdots a_{n-1}$.

The proof, which is omitted, hinges upon showing that $\{c_{ir+s}\}$ is an $(AP)_i$ with constant $r^i a_i$ and applying Eves' Theorem.

The theorems of this section can also be stated for columns. Next, the results can be extended to convolution arrays and to multinomial coefficient arrays by considering certain matrix products.

4. MULTINOMIAL COEFFICIENT ARRAYS

Let the array of multinomial coefficients arising from expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, be called the m-multinomial coefficient array. Let the left-justified m-multinomial coefficient array be imbedded in an $n \times n$ matrix A_m. Let the $n \times n$ matrix F_m contain the rows of A_m as the columns of F_m written on and below the main diagonal. Let A^T be the transpose of the $n \times n$ matrix A of (1.2). Then the matrix equation

$$F_{m-1}A^T = A_m^T, \qquad m = 2, 3, \cdots$$

was proved in [1]. Since any $k \times k$ submatrix of A_m^T having its first row along the second row of A_m^T and its first column the j^{th} column of A_m^T is the product of a submatrix of

F_{m-1} with a unit determinant and a $k \times k$ submatrix of A^T satisfying Theorem 2.2, its determinant will be given by

$$\binom{k + j - 2}{k} .$$

Since the transpose of A_m^T is A_m, we restate these results in terms of the m-multinomial coefficient array.

Theorem 4.1. The determinant of the $k \times k$ matrix formed with its first column the first column of any multinomial coefficient array in left-justified form (the column of successive whole numbers) and its first row the i^{th} row of that multinomial coefficient array, has value given by the binomial coefficient

$$\binom{k + i - 1}{k} .$$

Now let $(A^*)^T$ be the transpose of the $n \times n$ matrix A^* formed with its $(i + 1)^{st}$ row the first n entries of the $(ri)^{th}$ row of Pascal's left-justified triangle, $i = 0, 1, \cdots, n - 1$. Then the matrix product $F_{m-1}(A^*)^T = (A_m^*)^T$, where A_m^* is the $n \times n$ matrix formed using only the $(ri)^{th}$ rows of the m-multinomial coefficient array, $i = 0, 1, \cdots, n - 1$, as can be proved by examining the column generating functions. For, the column generators of F_{m-1} are $G_j(x) = [x(1 + x + \cdots + x^{m-1})]^{j-1}$ and of $(A^*)^T$, $H_j(x) = (1 + x)^{r(j-1)}$, $j = 1, 2, \cdots, n$, making the column generators of $F_{m-1}(A^*)^T$ be $H_j(G_j(x)) = (1 + x + x^2 + \cdots + x^m)^{r(j-1)}$, which we recognize as the column generators for the matrix $(A_m^*)^T$ claimed above. Again, considering the very special products of submatrices involved, we are led to the following result.

Theorem 4.2. If any multinomial coefficient array is written in left-justified form, the determinant of the $k \times k$ matrix formed with its $(i + 1)^{st}$ row the first n entries of the $(ri)^{th}$ row of the multinomial array, $i = 0, 1, 2, \cdots, k - 1$, is given by $r^{k(k-1)/2}$.

As an example, for $n = 5$ and $r = 2$, $F_1(A^*)^T = (A_2^*)^T$ becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 1 & 6 & 15 & 28 \\ 0 & 0 & 4 & 20 & 56 \\ 0 & 0 & 1 & 15 & 70 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 3 & 10 & 21 & 36 \\ 0 & 2 & 16 & 50 & 112 \\ 0 & 1 & 19 & 90 & 266 \end{bmatrix}$$

where $(A_2^*)^T$ has alternate rows of the trinomial coefficient array appearing as its columns, and the determinant equals 2^{10}.

Further examination of a matrix product, $F_{m-1}(A_m^{**})^T$, where the $n \times n$ matrix A_m^{**} is formed with its $(i+1)^{st}$ row the first n entries of the $(ri + s)^{th}$ row of the m-multinomial coefficient array, $s \geq 0$, $i = 0, 1, 2, \cdots, n - 1$, shows that Theorem 3.2 can be extended to the multinomial coefficients.

Theorem 4.3. Consider any left-justified multinomial coefficient array. Form a $k \times k$ matrix with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of the multinomial array, $i = 0, 1, \cdots, k-1$, $s \geq 0$. The determinant of that matrix is given by $r^{k(k-1)/2}$.

5. THE FIBONACCI CONVOLUTION ARRAY AND RELATED ARRAYS

The Fibonacci sequence, when convolved with itself $j - 1$ times, forms the sequence in the j^{th} column of the matrix C below (see [1] and [4])

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 3 & 10 & 22 & 40 & 65 & 98 & \cdots \\ 5 & 20 & 51 & 105 & 190 & 315 & \cdots \\ 8 & 38 & 111 & 256 & 511 & 924 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the original sequence is in the leftmost column and the column generators are given by $[1/(1 - x - x^2)]^j$, $j = 1, 2, \cdots, n$. If F_1 is the $n \times n$ matrix formed as in Section 3 with the rows of Pascal's triangle in vertical position on and below the main diagonal, and P is Pascal's rectangular array (1.1), then $F_1P = C$ as proved in [1]. Now, since here submatrices of C taken along the second row of C are the product of submatrices of F_1 with unit determinants and similarly placed submatrices of P whose determinants are given in Theorem 2.1, these submatrices of C have determinant values given by the same binomial coefficients found for P.

The generalization to convolution triangles for sequences which are found as sums of rising diagonals of m-multinomial coefficient arrays written in left-justified form is not difficult, since the matrix product F_mP yields just those arrays as shown in [1]. We thus write the following theorem.

Theorem 5.1. Let the convolution triangle for the sequences of sums found along the rising diagonals of the left-justified m-multinomial coefficient array be written in rectangular form and imbedded in an $n \times n$ matrix C^*. Then the determinant of any $k \times k$ submatrix of C^* selected with its first row along the second row of C^* and its first column the j^{th} column of C^* has determinant value given by the binomial coefficient

$$\binom{k+j-1}{k} .$$

Now, let P^* be the $n \times n$ matrix with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of Pascal's rectangular array P, $i = 0, 1, \cdots, n-1$, $s \geq 0$. Paralleling the development given for Theorem 5.1 but considering the matrix product F_mP^* which is the $n \times n$ matrix containing the $(ri + s)^{th}$ rows of the convolution array for the rising diagonals of the given m-multinomial coefficient array, we find that Theorem 3.2 extends to the following.

Theorem 5.2. If a $k \times k$ matrix is formed with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of the rectangular convolution array for the rising diagonals of any left-justified multinomial coefficient array, $i = 0, 1, \cdots, k-1$, $s \geq 0$, then its determinant has value $r^{k(k-1)/2}$.

Lastly, consider the sequence of sums of elements $u_m(n; p, 1)$ found on the rising diagonals formed by beginning at the leftmost column of a left-justified m-multinomial coefficient array and going up p and to the right one throughout the array. (For the Pascal triangle, these numbers are the generalized Fibonacci numbers $u(n; p, 1)$ of Harris and Styles [5].) Form the matrix $D_m(p, 1)$ so that the sequence of elements having $u_m(n; p, 1)$ as its sum lies (in reverse order) on its rows. $D_m(p, 1)$ will have a one for each element on its main diagonal and each column will contain the corresponding row of the m-multinomial array but with $(p-1)$ zeros between entries, so that the generating functions for its columns are $[x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^j$, $j = 1, 2, \cdots, n$. It was shown in [1] that $D_m(p, 1)P$ gives the convolution triangle in rectangular form for the sequence $u_m(n; p, 1)$. By examining the column generators, we also have that $D_m(p, 1)P^*$ gives the array containing the $(ir + s)^{th}$ rows of the convolution triangle for the sequence $u_m(n, p, 1)$. Putting all of this together, we write the following theorem.

Theorem 5.3. Write the convolution triangle in rectangular form imbedded in an $n \times n$ matrix C_m^* for the sequence of sums found on the rising diagonals formed by beginning at the leftmost column and moving up p and right one throughout any left-justified multinomial coefficient array. The $k \times k$ submatrix formed with its first row the second row of C_m^* and its first column the j^{th} column of C_m^* has determinant given by the binomial coefficient

$$\binom{k + j - 1}{k} .$$

The $n \times n$ matrix formed with its $(i+1)^{st}$ row the first n entries of the $(ri+s)^{th}$ row of the convolution triangle, $i = 0, 1, \cdots, n-1$, $s \geq 0$, has determinant equal to $r^{n(n-1)/2}$.

REFERENCES

1. Marjorie Bicknell and V. E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 11, No. 2, pp. 131-144.
2. Howard Eves in a letter to V. E. Hoggatt, Jr., January, 1972.
3. Donald E. Knuth, Fundamental Algorithms, Vol. 1, Addison-Wesley, Menlo Park, Calif., 1969, p. 64, Eq. 36, Sec. (1.2.6).
4. Verner E. Hoggatt, Jr., and Marjorie Bicknell, "Convolution Triangles," Fibonacci Quarterly, Vol. 10, No. 6, pp. 599-608.
5. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 4, pp. 277-289.

NOTE ON A COMBINATORIAL ALGEBRAIC IDENTITY AND ITS APPLICATION

L. C. HSU*
Department of Mathematics, Jilin University, Changchun, People's Republic of China

The identity

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{sr+t}{m} = \begin{cases} 0 & (m < n) \\ (-s)^n & (m = n) \end{cases} , \tag{1}$$

is well-known (cf. Schwatt [4, p. 104] and Gould [3, Formula (3.150)]) and has been utilized by Gould [1], [2] in proving some elegant combinatorial identities, e.g.,

$$\sum_{k=0}^{n} \binom{a+b(n-k)}{n-k} \binom{bk+c}{k} \frac{c}{bk+c} = \binom{a+c+bn}{n} , \tag{2}$$

and

$$\sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{k+j} \binom{n}{k} \binom{n}{j} \binom{kj+t}{n} = n! . \tag{3}$$

In what follows, we shall establish a combinatorial algebraic identity which involves a wider generalization of (1). We offer the following

Theorem. Let $F(X)$ be a polynomial of degree $m \le n$ in X having the leading term $P_0 X^m$. Then for arbitrary quantities $P_1, \cdots, P_n$ and Q we have

$$F(Q) + \sum_{r=1}^{n} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le n} F(P_{k_1} + \cdots + P_{k_r} + Q) = \binom{m}{n} (-1)^n n!\, P_0 P_1 \cdots P_n , \tag{4}$$

where the inner summation extends over all the r-combinations $(k_1, \cdots, k_r)$ of the integers $1, 2, \cdots, n$, and $\binom{m}{n}$ is 0 or 1 according as $m < n$ or $m = n$.

As a consequence of (4) we have a pair of generalized Euler identities (with $m \le n$) as follows:

---

*Communication concerning this paper should be directed to H. W. Gould, Mathematics Department, West Virginia University, Morgantown, West Virginia 26506.

$$(5)\qquad \binom{Q}{m} + \sum_{r=1}^{n} (-1)^r \sum_{1\le k_1<\cdots<k_r\le n} \binom{P_{k_1} + \cdots + P_{k_r} + Q}{m} = \binom{m}{n} (-1)^n P_1P_2\cdots P_n ,$$

and

$$(6)\qquad Q^m + \sum_{r=1}^{n} (-1)^r \sum_{1\le k_1<\cdots<k_r\le n} (P_{k_1} + \cdots + P_{k_r} + Q)^m = \binom{m}{n} (-1)^n n!\, P_1P_2\cdots P_n .$$

Clearly (1) is a special case of (5) with $P_1 = \cdots = P_n = s$. For $P_1 = \cdots = P_n = 1$, $Q = 0$, and $m = n$ we find that (6) implies the familiar Euler theorem about the n^{th} difference of x^n at $x = 0$, viz.

$$\Delta^n 0^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^n = n! .$$

Gould [3, Formula (Z.8)] has remarked about the use of this to determine certain combinatorial identities easily.

With other choices of the P_k's and Q these identities (5) and (6) may give somewhat "strange" but elementary identities such as

$$(7)\qquad \sum_{r=1}^{n} (-1)^{n-r} \sum_{1\le k_1<\cdots<k_r\le n} \binom{k_1^2 + \cdots + k_r^2}{n} = (n!)^2$$

and

$$(8)\qquad \sum_{r=1}^{n} (-1)^{n-r} \sum_{1\le k_1<\cdots<k_r\le n} (k_1^m + \cdots + k_r^m)^n = (n!)^{m+1} .$$

Since every polynomial $F(x)$ of degree m can be expressed as a linear combination of

$$\binom{x}{0},\ \binom{x}{1},\ \ldots,\ \binom{x}{m} ,$$

it is easily observed that (4), (5) and (6) are implied by each other. In other words, (4), (5), and (6) are logically equivalent.

For the proof of (4) it suffices to verify (6). Actually (6) can be verified by means of the principle of inclusion and exclusion in combinatorial analysis. Let us expand

$$(P_{i_1} + \cdots + P_{i_r} + Q)^m$$

in accordance with the multinomial theorem and consider a typical term of the form with exponents $a_1 \geq 1, \cdots, a_r \geq 1, \; b \geq 0$:

$$C P_{i_1}^{a_1} \cdots P_{i_r}^{a_r} Q^b, \qquad \left(C = \frac{m!}{a_1! \cdots a_r! \, b!}, \quad a_1 + \cdots + a_r + b = m \right),$$

where $(i_1, \cdots, i_r)$ is an r-subset of $(1, 2, \cdots, n)$. First consider the case $r < n$. In this case the difference set $(j_1, \cdots, j_{n-r}) = (1, 2, \cdots, n) - (i_1, \cdots, i_r)$ is non-empty, so that the typical term occurs in the inner sum

$$(-1)^r \sum (P_{k_1} + \cdots + P_{k_r} + Q)^m$$

and also in all those inner sums of (6) following this one. Consequently, the total number of occurrences of the term is given by

$$(-1)^r \left\{ \binom{n-r}{0} - \binom{n-r}{1} + \binom{n-r}{2} - \cdots + (-1)^{n-r} \binom{n-r}{n-r} \right\} = 0 .$$

This means that every term with $r < n$ vanishes always by cancellation, and this is generally true for $m < n$. For the case $m = n$, the only exceptional term is

$$(-1)^n n! \, P_1 P_2 \cdots P_n Q^0$$

which cannot be cancelled out anyway. Finally, the number of occurrences of the particular term Q^m is seen to be

$$1 - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0 .$$

Thus (6) is completely verified.

Similarly, a direct verification of (5) can be accomplished by using Vandermonde's multiple convolution formula (instead of the multinomial theorem) for expansion of the summands.

APPLICATION

For $m = n$ and $Q = 0$ the identities (5) and (6) imply that every integer $N = P_1 P_2 \cdots P_n$ with n relatively prime factors $P_1, P_2, \cdots, P_n$ can always be represented as an algebraic sum of

$$\binom{\sum P}{n} \text{'s} ,$$

and that $N = n! \, P_1 P_2 \cdots P_n$ as an algebraic sum of the n^{th} powers.

It is known that there are infinitely many solutions of the equation $A^3 + B^3 + C^3 = D^3$ in positive integers (see Shanks [5, p. 157]). Here as a simple application of (6) we shall construct certain sets of non-trivial positive integral solutions of the 2-sided 3-cube equation

$$X_1^3 + X_2^3 + X_3^3 = Y_1^3 + Y_2^3 + Y_3^3 \; . \tag{9}$$

Making use of (6) with $m = n = 3$ and $Q = 0$ we have

$$\begin{aligned} P_1^3 + P_2^3 + P_3^3 + (P_1 + P_2 + P_3)^3 &= (P_1 + P_2)^3 + (P_2 + P_3)^3 \\ &\quad + (P_3 + P_1)^3 + 6P_1P_2P_3 \; . \end{aligned} \tag{10}$$

Let $P_1^3 = 6P_1P_2P_3$ so that $P_1^2 = 6P_2P_3$, and we may put $P_2 = 2a^2c$, $P_3 = 3b^2c$, or $P_2 = a^2c$, $P_3 = 6b^2c$ (a, b, c being arbitrary positive integers) in order to make $6P_2P_3$ a perfect square. By substitution we find $P_1 = 6abc$, and then dropping the common factor c we get two identities as follows:

$$\begin{aligned} &(2a^2)^3 + (3b^2)^3 + (2a^2 + 3b^2 + 6ab)^3 \\ &\qquad = (2a^2 + 3b^2)^3 + (2a^2 + 6ab)^3 + (3b^2 + 6ab)^3 \quad , \end{aligned} \tag{11}$$

and

$$\begin{aligned} (a^2)^3 + (6b^2)^3 + (a^2 + 6b^2 + 6ab)^3 &= (a^2 + 6b^2)^3 + (a^2 + 6ab)^3 \\ &\quad + (6b^2 + 6ab)^3 \; . \end{aligned} \tag{12}$$

These two identities provide (9) with two sets of positive integral solutions involving two arbitrary integer parameters a and b. Similarly we can make use of (6) with $m = n = 4$ and $Q = 0$ to obtain infinitely many integral solutions of the equation

$$\sum_{i=1}^{7} X_i^4 = \sum_{i=1}^{7} Y_i^4 \; .$$

In classical number theory

$$\binom{N}{2} = N(N - 1)/2$$

is usually called a "triangular number." It is obvious that not every such number can be expressed as a sum of two triangular numbers. Simple examples $N = 5, 6, 8$ explain this point. These integers are of the form $N \equiv 0, 1, 3 \pmod 5$. Now as an immediate application of (5) we easily show the small

Theorem. Every triangular number $\binom{N}{2}$ with $N \equiv 2, 4 \pmod 5$ can always be expressed as a sum of two triangular numbers.

These numbers may be listed as a sequence:

$$\binom{4}{2}, \binom{7}{2}, \binom{9}{2}, \binom{12}{2}, \binom{14}{2}, \binom{17}{2}, \binom{19}{2}, \binom{22}{2}, \binom{24}{2}, \binom{27}{2}, \binom{29}{2}, \cdots$$

In fact, we have explicit relations for $N = 5P + 2$ and $N = 5P - 1$:

$$\binom{5P+2}{2} = \binom{3P+1}{2} + \binom{4P+2}{2}, \qquad \binom{5P-1}{2} = \binom{3P}{2} + \binom{4P-1}{2} .$$

These are easily obtained from (5) by taking $m = n = 2$ and letting $Q = 2P_1 + 1$ or $Q = 2P_1$ in order to delete the two equal terms

$$\binom{Q}{2} = P_1P_2 .$$

These relations may be compared with the formulas

$$\binom{3k+1}{2} + \binom{4k+2}{2} = \binom{5k+2}{2}, \qquad \binom{5k+5}{2} + \binom{12k+10}{2} = \binom{13k+11}{2},$$

and

$$\binom{8k+5}{2} + \binom{15k+10}{2} = \binom{17k+11}{2}, \qquad k = 0, 1, 2, \cdots,$$

of M. N. Khatri, cited by Sierpiński [6, pp. 84-86]. Sierpiński proves that there exist infinitely many pairs of natural numbers x, y satisfying the system of equations

$$\binom{x+1}{2} + \binom{2y+1}{2} = \binom{3y+1}{2}, \qquad \binom{x+1}{2} - \binom{2y+1}{2} = \binom{y}{2} .$$

Each of these equations is equivalent to the Diophantine equation $x^2 + x = 5y^2 + y$.

REFERENCES

1. H. W. Gould, "Some Generalizations of Vandermonde's Convolution," Amer. Math. Monthly, Vol. 63 (1956), pp. 84-91.
2. H. W. Gould, "Note on a Combinatorial Identity in the Theory of Bi-Colored Graphs," Fibonacci Quarterly, Vol. 5 (1967), pp. 247-250.
3. H. W. Gould, "Combinatorial Identities, Revised Edition," Published by the Author, Morgantown, W. Va., 1972.
4. I. J. Schwatt, An Introduction to the Operations with Series, Univ. of Pennsylvania Press, 1924; Chelsea Reprint, New York, 1962.
5. D. Shanks, Solved and Unsolved Problems in Number Theory, Vol. 1, Spartan Books, Wash., D.C., 1962.
6. W. Sierpiński, Elementary Theory of Numbers, Warsaw, Poland, 1964. (= Monografie Matematyczne, Vol. 42).

A SOLUTION TO THE CLASSICAL PROBLEM OF FINDING SYSTEMS OF THREE MUTUALLY ORTHOGONAL NUMBERS IN A CUBE FORMED BY THREE SUPERIMPOSED 10 X 10 X 10 CUBES

JOSEPH ARKIN
Spring Valley, New York

INTRODUCTION

In 1779, Euler conjectured that no pair of orthogonal squares exist for $n \equiv 2 \pmod 4$. Then in 1959, the Euler conjecture was shown to be incorrect by Bose, Shrikande and Parker [1]. Recently (in 1972), Hoggatt and this author extended Bose, Shrikande and Parker's work by finding a way to make the 10×10 square pairwise orthogonal as well as magic (for a square to be magic, each of the two diagonals must have the same sum as in every row and in every column) [2]. The work done on this difficult problem was then extended by this author, who found a solution to the classical Eulerian magic cube problem of order ten [3]. Then this author was fortunate enough to receive some letters from the great mathematician, Professor Erdös. Professor Erdös introduced me to one of the most difficult and unsolved problems of our time: namely, the 200-year-old question of whether it is possible to find systems of three mutually orthogonal numbers in a cube of three superimposed $10 \times 10 \times 10$ Latin cubes.

ABSTRACT

In this paper, we have succeeded in constructing for the first time certain systems of three mutually orthogonal numbers in a cube of three superimposed $10 \times 10 \times 10$ Latin cubes (the letters used are A, B, C, $\cdots$, J).

In our construction (Tables 1 through 10), we find ten separate groups (where each group consists of 100 cells and each cell contains three letters) such that each and every cell in a single group (we consider one group at a time) is in a different file, different column, and different row; and also (this is the major requirement) in any one group when we compare each and every one of the 100 cells to one another, the three letters in each and every cell in the group are mutually (three pairwise) orthogonal. In the construction of our cubes in Tables 1-10, we find in each cell three capital letters of the alphabet followed by a comma and then a digit (the digits range through 0, 1, 2, $\cdots$, 9). The digits on the right denote the group to which the three letters in the cell belong. For example: the three letters in each of the 100 cells throughout the cube that end in ,0 denote a single group (say) G(,0) and in this group G(,0) when we compare each and every one of the 100 cells to one another, the three letters in each and every cell in the group G(,0) are mutually (three pairwise) orthogonal. In the exact way we found the orthogonal properties of group G(,0) we find the identical orthogonal properties in the remaining nine groups G(,1), G(,2), $\cdots$, G(,9).

In our construction, it is also possible to find three pairwise orthogonal letters in a system of files where each file is in a different row, and different column (we use our top 10×10 square (square number 0, Table 1) as a reference for the coordinates of our rows and columns). An example of a single file (all files are considered to begin on square number 0, abbreviated SN0) is the ten cells in f(HGD, 0) = (HGD, 0) on square number 0 + (IHC, 1) on SN1 + (FJA, 2) on SN2 + $\cdots$ + (BFI, 9) on SN9. Then we define a group of files (say F(, 0)) ending in , 0 as the 100 cells in F(, 0) = f(HGD, 0) + f(HCI, 0) + $\cdots$ + f(HJH, 0). Now in F(, 0) when we compare each and every one of the cells (100 cells) to one another, the three letters in each and every cell in F(, 0) are mutually (three pairwise) orthogonal. In the exact way we found the orthogonal properties of F(, 0), we find the identical orthogonal properties in the remaining F(, 1), F(, 2), $\cdots$, F(, 9).

Remark. Using the exact methods that were used to construct the cubes in Tables 1-10, this author has extended the remarkable results on singly-even orthogonal squares by Bose, Shrikande and Parker [1], since we have generalized the construction technique and are able to find systems (exactly like the systems in this paper) of three pairwise orthogonal numbers in all (except 2^3 and 6^3) cubes formed by three superimposed Latin cubes. It is also possible to show: if a construction for a square $(2P)^2$ is known we are then always able to construct a cube $(2P(2m+1))^3$ with the exact three pairwise orthogonal properties we have shown in this paper ($P > 3$ is an odd prime and $m = 0, 1, \cdots$). However, since this author has not resolved (to his satisfaction) the question: "Is it possible to superimpose three mutually orthogonal 10×10 Latin squares?," we shall discuss our methods in a future paper.

Table 1

Square Number 0

HGD, 0	GIG, 1	DEA, 5	CHJ, 3	BBF, 2	ICI, 9	EAB, 4	FFE, 7	ADC, 6	JJH, 8
AHJ, 6	CGD, 3	BJH, 2	EFE, 4	GDC, 1	FAB, 7	HCI, 0	JIG, 8	IEA, 9	DBF, 5
GJH, 1	FEA, 7	IGD, 9	AAB, 6	DHJ, 5	EDC, 4	JFE, 8	CBF, 3	BCI, 2	HIG, 0
JEA, 8	IFE, 9	CAB, 3	BGD, 2	ECI, 4	HBF, 0	GHJ, 1	AJH, 6	DIG, 5	FDC, 7
CDC, 3	ACI, 6	HFE, 0	GBF, 1	JGD, 8	BIG, 2	DJH, 5	IAB, 9	FHJ, 7	EEA, 4
FCI, 7	EJH, 4	JDC, 8	HEA, 0	IIG, 9	DGD, 5	ABF, 6	BHJ, 2	CFE, 3	GAB, 1
DFE, 5	BDC, 2	AIG, 6	JCI, 8	HAB, 0	CJH, 3	FGD, 7	GEA, 1	EBF, 4	IHJ, 9
IBF, 9	HHJ, 0	GCI, 1	DDC, 5	FJH, 7	AEA, 6	CIG, 3	EGD, 4	JAB, 8	BFE, 2
EIG, 4	DAB, 5	FBF, 7	IJH, 9	AFE, 6	JHJ, 8	BEA, 2	HDC, 0	GGD, 1	CCI, 3
BAB, 2	JBF, 8	EHJ, 4	FIG, 7	CEA, 3	GFE, 1	IDC, 9	DCI, 5	HJH, 0	AGD, 6

Table 2

Square Number 1

IHC,1	FDI,6	GGE,9	JIF,0	EJD,7	DBG,2	HEH,3	ACB,8	BFJ,4	CAA,5
BIF,4	JHC,0	EAA,7	HCB,3	FFJ,6	AEH,8	IBG,1	CDI,5	DGE,2	GJD,9
FAA,6	AGE,8	DHC,2	BEH,4	GIF,9	HFJ,3	CCB,5	JJD,0	EBG,7	IDI,1
CGE,5	DCB,2	JEH,0	EHC,7	HBG,3	IJD,1	FIF,6	BAA,4	GDI,9	AFJ,8
JFJ,0	BBG,4	ICB,1	FJD,6	CHC,5	EDI,7	GAA,9	DEH,2	AIF,8	HGE,3
ABG,8	HAA,3	CFJ,5	IGE,1	DDI,2	GHC,9	BJD,4	EIF,7	JCB,0	FEH,6
GCB,9	EFJ,7	BDI,4	CBG,5	IEH,1	JAA,0	AHC,8	FGE,6	HJD,3	DIF,2
DJD,2	IIF,1	FBG,6	GFJ,9	AAA,8	BGE,4	JDI,0	HHC,3	CEH,5	ECB,7
HDI,3	GEH,9	AJD,8	DAA,2	BCB,4	CIF,5	EGE,7	IFJ,1	FHC,6	JBG,0
EEH,7	CJD,5	HIF,3	ADI,8	JGE,0	FCB,6	DFJ,2	GBG,9	IAA,1	BHC,4

Table 3

Square Number 2

FJA,2	DCE,4	BHB,8	ABD,9	JAI,6	EGF,0	IIJ,5	CDH,1	GEG,3	HFC,7
GBD,3	AJA,9	JFC,6	IDH,5	DEG,4	CIJ,1	FGF,2	HCE,7	EHB,0	BAI,8
DFC,4	CHB,1	EJA,0	GIJ,3	BBD,8	IEG,5	HDH,7	AAI,9	JGF,6	FCE,2
HHB,7	EDH,0	AIJ,9	JJA,6	IGF,5	FAI,2	DBD,4	GFC,3	BCE,8	CEG,1
AEG,9	GGF,3	FDH,2	DAI,4	HJA,7	JCE,6	BFC,8	EIJ,0	CBD,1	IHB,5
CGF,1	IFC,5	HEG,7	FHB,2	ECE,0	BJA,8	GAI,3	JBD,6	ADH,9	DIJ,4
BDH,8	JEG,6	GCE,3	HGF,7	FIJ,2	AFC,9	CJA,1	DHB,4	IAI,5	EBD,0
EAI,0	FBD,2	DGF,4	BEG,8	CFC,1	GHB,3	ACE,9	IJA,5	HIJ,7	JDH,6
ICE,5	BIJ,8	CAI,1	EFC,0	GDH,3	HBD,7	JHB,6	FEG,2	DJA,4	AGF,9
JIJ,6	HAI,7	IBD,5	CCE,1	AHB,9	DDH,4	EEG,0	BGF,8	FFC,2	GJA,3

Table 4

Square Number 3

JIH,3	EFA,9	HCD,2	GDB,7	AEJ,0	CAC,6	FJE,8	BHF,5	DBI,1	IGG,4
DDB,1	GIH,7	AGG,0	FHF,8	EBI,9	BJE,5	JAC,3	IFA,4	CCD,6	HEJ,2
EGG,9	BCD,5	CIH,6	DJE,1	HDB,2	FBI,8	IHF,4	GEJ,7	AAC,0	JFA,3
ICD,4	CHF,6	GJE,7	AIH,0	FAC,8	JEJ,3	EDB,9	DGG,1	HFA,2	BBI,5
GBI,7	DAC,1	JHF,3	EEJ,9	IIH,4	AFA,0	HGG,2	CJE,6	BDB,5	FCD,8
BAC,5	FGG,8	IBI,4	JCD,3	CFA,6	HIH,2	DEJ,1	ADB,0	GHF,7	EJE,9
HHF,2	ABI,0	DFA,1	IAC,4	JJE,3	GGG,7	BIH,5	ECD,9	FEJ,8	CDB,6
CEJ,6	JDB,3	EAC,9	HBI,2	BGG,5	DCD,1	GFA,7	FIH,8	IJE,4	AHF,0
FFA,8	HJE,2	BEJ,5	CGG,6	DHF,1	IDB,4	ACD,0	JBI,3	EIH,9	GAC,7
AJE,0	IEJ,4	FDB,8	BFA,5	GCD,7	EHF,9	CBI,6	HAC,2	JGG,3	DIH,1

Table 5

Square Number 4

AAG,4	CEB,7	EFF,1	ICH,8	GDE,5	FJJ,3	DBA,9	HGC,6	JID,2	BHI,0
JCH,2	IAG,8	GHI,5	DGC,9	CID,7	HBA,6	AJJ,4	BEB,0	FFF,3	EDE,1
CHI,7	HFF,6	FAG,3	JBA,2	ECH,1	DID,9	BGC,0	IDE,8	GJJ,5	AEB,4
BFF,0	FGC,3	IBA,8	GAG,5	DJJ,9	ADE,4	CCH,7	JHI,2	EEB,1	HID,6
IID,8	JJJ,2	AGC,4	CDE,7	BAG,0	GEB,5	EHI,1	FBA,3	HCH,6	DFF,9
HJJ,6	DHI,9	BID,0	AFF,4	FEB,3	EAG,1	JDE,2	GCH,5	IGC,8	CBA,7
EGC,1	GID,5	JEB,2	BJJ,0	ABA,4	IHI,8	HAG,6	CFF,7	DDE,9	FCH,3
FDE,3	ACH,4	CJJ,7	EID,1	HHI,6	JFF,2	IEB,8	DAG,9	BBA,0	GGC,5
DEB,9	EBA,1	HDE,6	FHI,3	JGC,2	BCH,0	GFF,5	AID,4	CAG,7	IJJ,8
GBA,5	BDE,0	DCH,9	HEB,6	IFF,8	CGC,7	FID,3	EJJ,1	AHI,4	JAG,2

Table 6

Square Number 5

ECF,5	BGJ,3	FII,0	DEC,6	HFH,4	GDA,8	CHG,2	JAD,9	IJB,7	ABE,1
IEC,7	DCF,6	HBE,4	CAD,2	BJB,3	JHG,9	EDA,5	AGJ,1	GII,8	FFH,0
BBE,3	JII,9	GCF,8	IHG,7	FEC,0	CJB,2	AAD,1	DFH,6	HDA,4	EGJ,5
AII,1	GAD,8	DHG,6	HCF,4	CDA,2	EFH,5	BEC,3	IBE,7	FGJ,0	JJB,9
DJB,6	IDA,7	EAD,5	BFH,3	ACF,1	HGJ,4	FBE,0	GHG,8	JEC,9	CII,2
JDA,9	CBE,2	AJB,1	EII,5	EGJ,8	FCF,0	IFH,7	HEC,4	DAD,6	BHG,3
FAD,0	HJB,4	IGJ,7	ADA,1	EHG,5	DBE,6	JCF,9	BII,3	CFH,2	GEC,8
GFH,8	EEC,5	BDA,3	FJB,0	JBE,9	III,7	DGJ,6	CCF,2	AHG,1	HAD,4
CGJ,2	FHG,0	JFH,9	GBE,8	IAD,7	AEC,1	HII,4	EJB,5	BCF,3	DDA,6
HHG,4	AFH,1	CEC,2	JGJ,9	DII,6	BAD,3	GJB,8	FDA,0	EBE,5	ICF,7

Table 7

Square Number 6

GEE,6	ABC,2	CDG,4	HJI,5	IGB,3	BHH,7	JFD,1	DIA,0	EAF,8	FCJ,9
EJI,8	HEE,5	ICJ,3	JIA,1	AAF,2	DFD,0	GHH,6	FBC,9	BDG,7	CGB,4
ACJ,2	DDG,0	BEE,7	EFD,8	CJI,4	JAF,1	FIA,9	HGB,5	IHH,3	GBC,6
FDG,9	BIA,7	HFD,5	IEE,3	JHH,1	GGB,6	AJI,2	ECJ,8	CBC,4	DAF,0
HAF,5	EHH,8	GIA,6	AGB,2	FEE,9	IBC,3	CCJ,4	BFD,7	DJI,0	JDG,1
DHH,0	JCJ,1	FAF,9	GDG,6	BBC,7	CEE,4	EGB,8	IJI,3	HIA,5	AFD,2
CIA,4	IAF,3	EBC,8	FHH,9	GFD,6	HCJ,5	DEE,0	ADG,2	JGB,1	BJI,7
BGB,7	GJI,6	AHH,2	CAF,4	DCJ,0	EDG,8	HBC,5	JEE,1	FFD,9	IIA,3
JBC,1	CFD,4	DGB,0	BCJ,7	EIA,8	FJI,9	IDG,3	GAF,6	AEE,2	HHH,5
IFD,3	FGB,9	JJI,1	DBC,0	HDG,5	AIA,2	BAF,7	CHH,4	GCJ,6	EEE,8

Table 8

Square Number 7

DDJ,7	HHD,8	ABH,3	BAE,1	CIC,9	JFB,4	GGI,0	IJG,2	FCA,5	EEF,6
FAE,5	BDJ,1	CEF,9	GJG,0	HCA,8	IGI,2	DFB,7	EHD,6	JBH,4	AIC,3
HEF,8	IBH,2	JDJ,4	FGI,5	AAE,3	GCA,0	EJG,6	BIC,1	CFB,9	DHD,7
EBH,6	JJG,4	BGI,1	CDJ,9	GFB,0	DIC,7	HAE,8	FEF,5	AHD,3	ICA,2
BCA,1	FFB,5	DJG,7	HIC,8	EDJ,6	CHD,9	AEF,3	JGI,4	IAE,2	GBH,0
IFB,2	GEF,0	ECA,6	DBH,7	JHD,4	ADJ,3	FIC,5	CAE,9	BJG,1	HGI,8
AJG,3	CCA,9	FHD,5	EFB,6	DGI,7	BEF,1	IDJ,2	HBH,8	GIC,0	JAE,4
JIC,4	DAE,7	HFB,8	ACA,3	IEF,2	FBH,5	BHD,1	GDJ,0	EGI,6	CJG,9
GHD,0	AGI,3	IIC,2	JEF,4	FJG,5	EAE,6	CBH,9	DCA,7	HDJ,8	BFB,1
CGI,9	EIC,6	GAE,0	IHD,2	BBH,1	HJG,8	JCA,4	AFB,3	DEF,7	FDJ,5

Table 9

Square Number 8

CBB,8	IJF,0	JAJ,7	EFG,2	FHA,1	AIE,5	BCC,6	GEI,4	HGH,9	DDD,3
HFG,9	EBB,2	FDD,1	BEI,6	IGH,0	GCC,4	CIE,8	DJF,3	AAJ,5	JHA,7
IDD,0	GAJ,4	ABB,5	HCC,9	JFG,7	BGH,6	DEI,3	EHA,2	FIE,1	CJF,8
DAJ,3	AEI,5	ECC,2	FBB,1	BIE,6	CHA,8	IFG,0	HDD,9	JJF,7	GGH,4
EGH,2	HIE,9	CEI,8	IHA,0	DBB,3	FJF,1	JDD,7	ACC,5	GFG,4	BAJ,6
GIE,4	BDD,6	DGH,3	CAJ,8	AJF,5	JBB,7	HHA,9	FFG,1	EEI,2	ICC,0
JEI,7	FGH,1	HJF,9	DIE,3	CCC,8	EDD,2	GBB,4	IAJ,0	BHA,6	AFG,5
AHA,5	CFG,8	IIE,0	JGH,7	GDD,4	HAJ,9	EJF,2	BBB,6	DCC,3	FEI,1
BJF,6	JCC,7	GHA,4	ADD,5	HEI,9	DFG,3	FAJ,1	CGH,8	IBB,0	EIE,2
FCC,1	DHA,3	BFG,6	GJF,4	EAJ,2	IEI,0	AGH,5	JIE,7	CDD,8	HBB,9

Table 10

Square Number 9

BFI,9	JAH,5	IJC,6	FGA,4	DCG,8	HED,1	ADF,7	EBJ,3	CHE,0	GIB,2
CGA,0	FFI,4	DIB,8	ABJ,7	JHE,5	EDF,3	BED,9	GAH,2	HJC,1	ICG,6
JIB,5	EJC,3	HFI,1	CDF,0	IGA,6	AHE,7	GBJ,2	FCG,4	DED,8	BAH,9
GJC,2	HBJ,1	FDF,4	DFI,8	AED,7	BCG,9	JGA,5	CIB,0	IAH,6	EHE,3
FHE,4	CED,0	BBJ,9	JCG,5	GFI,2	DAH,8	IIB,6	HDF,1	EGA,3	AJC,7
EED,3	AIB,7	GHE,2	BJC,9	HAH,1	IFI,6	CCG,0	DGA,8	FBJ,4	JDF,5
IBJ,6	DHE,8	CAH,0	GED,2	BDF,9	FIB,4	EFI,3	JJC,5	ACG,7	HGA,1
HCG,1	BGA,9	JED,5	IHE,6	EIB,3	CJC,0	FAH,4	AFI,7	GDF,2	DBJ,8
AAH,7	IDF,6	ECG,3	HIB,1	CBJ,0	GGA,2	DJC,8	BHE,9	JFI,5	FED,4
DDF,8	GCG,2	AGA,7	EAH,3	FJC,4	JBJ,5	HHE,1	IED,6	BIB,9	CFI,0

[Continued on page 494.]

A FURTHER ANALYSIS OF BENFORD'S LAW

W. A. SENTANCE
The City University, London, England

In a recent paper [1] J. Wlodarski noted the interesting fact that Benford's "Law of anomalous numbers" was obeyed very closely by the first 100 Fibonacci numbers and the first 100 Lucas numbers. The same paper ended with the suggestion, taken up by the present author, that many more than the first 100 Fibonacci and Lucas numbers should be used for the purpose of analyzing Benford's Law more closely.

In a list of random numbers, one would normally expect to find that the distribution of the initial digit would have an approximately equal spread over the nine integers 1 to 9. However, it is an observed fact that in many tabulations the digit 1 occurs almost three times more often than any of the other eight digits. It was this that led Frank Benford in 1938 to enunciate his "law of anomalous numbers" that the probability of a random decimal number beginning with digit p is $\log(p+1) - \log(p)$ where the logarithms are expressed to the base 10. [2]

Using a computer, it has been possible to extend the study to cover the first 1000 Fibonacci and the first 1000 Lucas numbers. Such a study would be perhaps unfeasible and certainly very tedious without the aid of a computer since F_{100} has 209 digits. Normally, numbers are held within the computer to an accuracy of so many digits, usually within the range of 10 to 20, and any arithmetic performed on such numbers will only be correct to this accuracy. However, by assigning one computer word for each digit of any particular number, we are able to store exactly large integer numbers. It is a relatively easy matter to simulate the operation of addition between any two such numbers. Addition is the only operation we need since the two sequences in which we are interested are defined by the additive recurrence formula

$$A_{n+1} = A_n + A_{n-1}$$

different initial conditions giving rise to the Fibonacci and Lucas sequences. To give some idea of the time involved, the additions which were needed to produce F_{1000}, took approximately 18 seconds. Such a method has other distinct advantages besides its great speed and ease as is shown later in this paper.

Table 1

Digit	1	2	3	4	5	6	7	8	9
N_F	30	18	13	9	8	6	5	7	4
N_L	31	16	14	10	8	5	8	4	4
N_B	30.1	17.6	12.5	9.7	7.9	6.7	5.8	5.1	4.6

$$G_F = 0.657 \times 10^{-4}$$

$$G_L = 1.673 \times 10^{-4}$$

N_F : Number of times the digit occured as initial digit in the Fibonacci sequence.

N_L : Same as N_F but for the Lucas sequence.

N_B : Expected value, given by Benford's Law, of the digit to be the initial digit.

Table 1 reproduces the figures from [1] for the distribution of the initial digits of the first 100 Fibonacci numbers and the first 100 Lucas numbers, together with the expected value given by Benford's Law. In order to effect a comparison with later results, we have calculated "goodness of fit" constants G_F and G_L where

$$G_F = \sum_{i=1}^{9} \left(\frac{N_F}{100} - \frac{N_B}{100} \right)^2 \Big/ 9$$

$$G_L = \sum_{i=1}^{9} \left(\frac{N_L}{100} - \frac{N_B}{100} \right)^2 \Big/ 9 \quad .$$

Table 2 is exactly the same as Table 1 except that it gives the results pertaining to the first 1000 Fibonacci numbers and the first 1000 Lucas numbers, again with "goodness of fit" constants. It is readily seen that the behaviour exhibited by the small set of numbers has been propagated by the large set of numbers. The goodness of fit constant is in both cases considerably reduced indicating that the distribution of initial digits is approximating more closely to that predicted by Benford's Law as more numbers in the respective sequence are taken into account.

Table 2

Digit	1	2	3	4	5	6	7	8	9
N_F	301	177	125	96	80	67	56	53	45
N_L	301	174	127	97	79	66	59	51	46
N_B	301.0	176.1	124.9	96.9	79.1	66.9	58.0	51.2	45.8

Note: N_B correct only to 1D. Accurate values used in calculating G_F and G_L

$$G_F = \sum_{i=1}^{9} \left(\frac{N_F}{1000} - \frac{N_B}{1000} \right)^2 \Big/ 9 = 0.0114 \times 10^{-4}$$

$$G_L = \sum_{i=1}^{9} \left(\frac{N_L}{1000} - \frac{N_B}{1000} \right)^2 \Big/ 9 = 0.0118 \times 10^{-4}$$

The point could be made at this stage that the reduction in the values of G_F and G_L is purely fortuitous and that the author was fortunate in finding that G_F and G_L for the first 1000 numbers of each sequence were considerably smaller than for the first 100 numbers. To counteract this argument we give in Table 3, the values of G_F and G_L for the first i of the Fibonacci numbers and for the first i of the Lucas numbers where i takes the values 100 to 1000 in steps of 100. Although there are fluctuations in these values they do exhibit in general a downward trend.

Table 3

i	$G_F \times 10^4$	$G_L \times 10^4$
100	0.656	1.673
200	0.260	.261
300	0.139	.104
400	0.037	.031
500	0.026	.035
600	0.025	.013
700	0.036	.028
800	0.021	.007
900	0.012	.008
1000	0.011	.012

Again one may try to explain this strange distribution by the hypothesis that for these two sequences of numbers, the frequency of occurrence of each of the digits 1 to 9 throughout the numbers follows this pattern. However, Table 4 shows this not to be the case.

Table 4

	0	1	2	3	4	5	6	7	8	9
F	10474	10696	10495	10476	10431	10516	10433	10576	10350	10369
L	10393	10690	10783	10519	10699	10278	10507	10524	10285	10420

For the Fibonacci sequence the total number of digits in the first 1000 numbers is 104818. Assuming that each digit is distributed randomly then we expect each digit to occur with the same frequency. In this case the expectation for each digit is 10481.8. It is seen that the actual occurrence for each digit is very close to this expected value. Similar remarks apply to the Lucas sequence, too. The digit 1 therefore does not have an overall distribution different to any of the other digits.

This paper ends with a proposal to extend Benford's Law so that it now reads:

"The probability that a random number expressed in the number base b begins with digit p is $\log(p + 1) - \log p$, where the logarithms are to the base b."

Benford's Law is a particular case of this with b equal to 10. The idea behind such a proposal is that if it is true then it means that the distribution of initial digits seems to be some function inherent within the number system itself.

The method we have used to implement the addition of large integers is capable of being adapted to give results expressed with respect to any number base. Table 5 reproduces the

Table 5

Digit		Base 4	5	6	7	8	9
	N_F	501	430	389	356	336	314
1	N_L	502	430	385	355	336	318
	N_B	500	430.1	386.9	356.2	333.3	315.5
		291	253	227	211	193	187
2		292	251	226	207	193	181
		292.5	251.9	226.3	208.4	195.0	184.5
		208	178	160	146	140	132
3		206	180	162	151	139	134
		207.5	178.7	160.6	147.8	138.3	130.9
			139	123	114	105	99
4			139	125	113	106	108
			138.6	124.5	114.7	107.3	101.6
				101	93	90	83
5				102	94	89	82
				101.8	93.7	87.7	83.0
					80	73	69
6					80	73	70
					79.2	74.1	70.2
						63	62
7						64	60
						64.2	60.8
							54
8							52
							53.6
G of	F	.0114	.0057	.0157	.0198	.0390	.0220
Fit $\times 10^4$	L	.0218	.0075	.0116	.0280	.0233	.0432

computer results for the first 1000 Fibonacci and Lucas numbers using bases 4 to 9 inclusive together with the theoretical expectation based on the extension to Benford's Law. Again we include a goodness-of-fit constant.

It can be seen that the distribution of initial digits in the other number bases closely resembles that predicted by this extension of Benford's Law.

In conclusion then, as far as the sequences of Fibonacci and Lucas numbers are concerned, the frequency of occurrence of the digits 1-9 as initial digits is an excellent illustration of Benford's Law. The distribution would seem to approach that given by Benford as more and more numbers are taken into account. If we choose to express them in any other base, then there is a very strong indication that the initial digits occur in a distribution given by the extension to Benford's Law proposed earlier in this paper.

REFERENCES

1. J. Wlodarski, "Fibonacci and Lucas Numbers Tend to Obey Benford's Law," Fibonacci Quarterly, Vol. 9, No. 1 (1971), pp. 87-88.
2. F. Benford, Proceedings of the American Philosophical Society, Vol. 78, No. 4, pp. 551-572 (March 31, 1938).

[Continued from page 489.]

REFERENCES

1. A. R. C. Bose and S. S. Shrikande, "On the Falsity of Euler's Conjecture About the Non-Existence of Two Orthogonal Latin Squares of Order 4t + 2," Proc. Nat'l. Acad. Sci., 45 (5), May 1959, pp. 734-737.

   B. E. T. Parker, "Orthogonal Latin Squares," Proc. Nat'l. Acad. Sci., 45 (6), June, 1959, pp. 859-862.
2. J. Arkin and V. E. Hoggatt, Jr., "The Arkin-Hoggatt Game and the Solution of a Classical Problem," Jour. Recreational Math., Vol. 6, No. 2 (Spring, 1973), pp. 120-122.
3. J. Arkin, "The First Solution of the Classical Eulerian Magic Cube Problem of Order Ten," Fibonacci Quarterly, Vol. 11, No. 2 (April, 1973), pp. 174-178.

A FIBONACCI-RELATED SERIES IN AN ASPECT OF INFORMATION RETRIEVAL

MICHAEL F. LYNCH
University of Sheffield, Western Bank, Sheffield, England S10 2TN

A continuing objective of research in the field of information science is a better understanding of the structure of subject indexes, and of methods of preparing and using them. Most of us depend on these tools for access to the steadily increasing flow of publications in science and technology, yet for the most part their preparation is still an art rather than a science. It was not a little surprising, therefore, to discover that a familiar linguistic device that is widely used in indexes, catalogs, and directories could be formalized, and that this formalization had connotations which included a Fibonacci-related series. The linguistic device is that of inversion of prepositional phrases, such as "England, Kings of," which is encountered in such diverse sources as back-of-the-book indexes and the Library of Congress catalog.

The process of inversion of phrases reaches its peak in complex subject indexes such as those to Nuclear Science Abstracts and Chemical Abstracts, the latter currently including about 300,000 scientific papers, books and patents each year. The magnitude of the task of publishing and searching such amounts of literature has called for the increasing application of computer technology during the past decade, and it was in the context of one such investigation that the process of inversion came to be more clearly scrutinized [1]. In these indexes, entries are made under a series of subject headings, which serve as the primary entry points for the user. The entries themselves consist of prepositional phrases, highly convoluted, but organized in such a way as to enable the reader to scan them rapidly and to extract the essential content during a rapid scan of the entry. The following example, taken from a recent index to Chemical Abstracts, illustrates the point (the numerical reference is the abstract number):

Coal
    flotation of, hydrocarbon agent activity in, oxygen compd. formation
        in relation to, 89893W.

It is clear that, without particular training, the reader can reconstitute the sense of the original phrase as it was first conceived by the indexer. This is an intuitive process, not immediately formalizable. With computer techniques in view, however, it was necessary to define the procedure in symbol-manipulative terms. It was noted that the entries consisted, in the main, of sequences of phrases either beginning or ending with prepositions, and it was this which provided the necessary clue. In the case of an entry such as "England, Kings of," it is clear that the natural order would read "Kings of England," while if the entry read

"Kings, of England," no alteration in sequence would be required. So too with highly complex index entries, provided that the constituent phrases can be suitably identified. Fortunately, this delimitation is provided by the sequence of commas within the entry, which usually serve to separate the component phrases from one another. Thus, extending the rule which gives us "Kings of England," we can say that if we take the component phrases of an entry in sequence, then, according as the phrase begins with a preposition (or connective such as "and"), or ends with one, it is to be placed so as either to precede the subject heading or to follow it, as the case may be. Applying this to each component in turn, and adding successive phrases at one end or the other of that part of the sequence built up always produces the intended result, i.e., the normal form of the description as originally derived by the indexer. In practice, the rule cannot be applied to all entries, since commas may also occur in the normal form of the expression; however, for those entries in which each component phrase either begins or ends with a preposition or other function word, the rule is absolutely consistent, and is illustrated by its application to the example noted above:

"oxygen compd. formation in relation to hydrocarbon activity
in flotation of coal."

While interesting, this formalization has not yet been widely utilized in computer studies of index structure. Its usefulness seemed to us to lie rather in the fact that its obverse offered the possibility of taking natural language title-like phrases, and automatically producing an index of high quality from them. This reverse transformation, from natural language phrase to index entry, presented particular problems, since it became apparent that it produced not a single result, but rather a variety of possible forms of entries, that is, that while the transformation from entry to the normal form of the description is single-valued, the transformation from normal format to entry is many-valued. This became clear while the selection rule for entry production was being elaborated — a process which the indexer carries out intuitively, and which has now been termed articulation.

It is useful at this point to consider a simple model for these transformations. The model necessarily ignores certain complexities which are encountered in practice, notably those due to the proportion "of," as illustrated below. It consists of a formalized descriptive phrase composed of a sequence of nouns or noun phrases separated by function words:

——————
—————— o —————— o —————— o —————— o ——————

An entry consists of an articulated form of these, in the following fashion:

——————
o ——————, —————— o , o —————— , o ——————

in which the pairs of function words/nouns form the components of the entry. The selection rule is as follows. A noun or noun phrase is selected to act as a subject heading from any position in the sequence. As a result, equal numbers of nouns/noun phrases and function words remain. The entry may then be formed by successive selection of components from positions adjacent to the subject heading, either to the right or to the left of it, a kind of decision tree resulting from the multiplicity of choices that are open. The following example illustrates the point:

rains on plains in Spain

Heading: Plains

1st Component:

Plains
    rains on,
	Plains
	    in Spain

2nd Component:

Plains
    rains on, in Spain
	Plains,
	    in Spain, rains on

Heading: Spain

1st Component:

Spain
    plains in
	Spain
	    rains on

2nd Component:

Spain
    plains in, rains on
	Spain
	    rains on plains in

The complication caused by the preposition "of" can be illustrated by the following example:

"production of indexes by computer;"

when "indexes" is selected as the subject heading, two entries are provided by the simple model:

Indexes
    by computer, production of,
	Indexes
	    production of, by computer

Of these, only the second is acceptable, the first seeming ill-formed, due to separation of the phrase "production of" from the noun which it qualifies directly. In practice, this can be accommodated by simple additional rules.

Again, in practical terms, economic factors, both of production and of size of the resulting index for users, do not permit the inclusion in a printed index of all of the variant forms of entry which the model permits. Further characteristics of printed subject indexes, including the use of indentation to enhance the ease of scanning of the printed display, have enabled us to adduce further rules which are now incorporated within a useful program suite for the automatic production of printed subject indexes [2, 3]. The advantages of this technique are that the indexer need concern himself solely with providing an accurate and consistent record of the content of the subject matter of the document being indexed, and can economize on the time needed to make an entry under each heading in articulated form, which is required in the traditional index-production method.

It is nonetheless interesting to pursue the implications of the simple model somewhat further, particularly in terms of the great variety of variant entries which can be formed from a single title-like phrase describing the subject content of an article or book. It is clear that if the first noun or noun phrase of a longer description is chosen as the subject heading, only a single form of entry is possible. Taking the earlier example:

"rains on plains in Spain"

when "rains" is selected as the heading, only a single form of entry is possible, i.e.,

Rains
  on plains in Spain .

This is termed an invariant phrase. When the last noun, Spain, is chosen, either of the nouns preceding it may form the first component of the entry, while if a noun occurring at an intermediate position is selected, the first component can be formed from any of the nouns preceding it or from the one following it. Using a different symbolism, in which the components are denoted by alphabetical symbols, a sequence of three nouns can, in theory, give rise to the following entries:

A · B · C

A	B	C
BC	AC	AB
	CA	BA

A sequence of four noun phrases, A. B. C. D can produce a greater variety:

A	B	C	D
BCD	ACD	ABD	ABC
	CAD	BAD	BCA
	CDA	BDA	CAB
		DAB	CBA
		DBA	

Tabulating these graphically for phrases of lengths 1 to 4 provides the following possibilities:

No. of headings	Phrase	Possible Entries			
1	A	A			
2	A. B.	A_B	B_A		
3	A. B. C.	A_{BC}	B_{CA}	C_{BA}	
4	A. B. C. D.	A_{BCD}	B_{ACD}	C_{ABD}	D_{ABC}
			CAD	BAD	BCA
			CDA	BDA	CAB
				DAB	CBA
				DBA	

Replacing now the particular articulated arrangements by the numbers of variant entries possible under each heading in turn, we obtain the following table:

n	No. of entries under n^{th} heading							Total
	1	2	3	4	5	6	7	
1	1							1
2	1	1						2
3	1	2	2					5
4	1	3	5	4				13
5	1	4	9	12	8			34
6	1	5	14	25	28	16		89
7	1	6	20	44	66	64	32	233

This series proves to be of more than casual interest. Not only are the row sums the alternate terms of the Fibonacci series, the internal structure of the table also provides an algorithmic extension, other than by an exhaustive examination of all the possibilities provided by the selection rule. Thus any entry in the table may be computed by taking the entry above it, and adding to it the entry immediately to the left of it and all those on the left-hand diagonal of the latter.

Finally, a general expression for computing the row sums for each value of n takes the following form:

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{2n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1}\right] = F_{2n-1} .$$

ACKNOWLEDGEMENTS

Financial assistance from the Office for Scientific and Technical Information, London, in support of this work is gratefully acknowledged, as also the capable help of J. E. Ash, J. H. Petrie, I. J. Palmer and M. J. Snell in the computational work. Dr. I. J. Good provided the expression for the row sum series.

REFERENCES

1. M. F. Lynch, "Subject Indexes and Automatic Document Retrieval: The Structure of Entries in Chemical Abstracts Subject Indexes," J. Documentation, 22 (1966), pp. 167-185.
2. J. E. Armitage, M. F. Lynch, J. H. Petrie and M. Belton, "Experimental use of a Program for Computer-Aided Subject Index Production," Information Storage and Retrieval, 6 (1970), pp. 79-87.
3. M. F. Lynch, J. H. Petrie, "A Program Suite for the Production of Articulated Subject Indexes," Computer Journal (in the press).

LETTER TO THE EDITOR

Dear Editor:

Professor Dr. Tibor Šalát of Bratislava has pointed out two corrigenda to my article on arithmetic progression, April, 1973, Fibonacci Quarterly, pp. 145-152.

In the proof of Lemma 2.2, one may not assume that ad and $c/(a,c)$ are relatively prime. After the second display in the proof, proceed as follows:

$$(i - i')ad \equiv (j' - j)bc \pmod{c}$$

$$(i - i')ad \equiv 0 \pmod{c}.$$

Since $(c,d) = 1$, we get $(i - i')a \equiv 0 \pmod{c}$. Division by (a,c) yields

$$(i - i')(a/(a,c)) \equiv 0 \pmod{c/(a,c)},$$

hence

$$i - i' \equiv 0 \pmod{c/(a,c)}.$$

On page 151, insert a "1 -" before Π in the second, third, and fourth displays.

How far can Theorem 4.1 be generalized to other polynomials?

Sherman K. Stein
University of California,
Davis, Calif. 95616

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-227 Proporsed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj+ck)^m(bj+dk)^n$$

$$= m!n!\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n}{r}a^{m-r}d^{n-r}(bc)^r .$$

In particular, show that the Legendre polynomial $P_n(x)$ satisfies

$$(n!)^2P_n(x) = \sum_{j,k=0}^{n}(-1)^{j+k}\binom{n}{j}\binom{n}{k}(aj+ck)^n(bj+dk)^n ,$$

where

$$ad = \frac{1}{2}(x+1), \qquad bc = \frac{1}{2}(x-1) .$$

H-228 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the sequence $\{u_n\}_{n=1}^{\infty}$ as follows: $u_n = (F_n)^{F_n}$ $(n \geq 1)$, where F_n denotes the n^{th} Fibonacci number.

(1). Find a recurrence relation for $\{u_n\}_{n=1}^{\infty}$ and

(2). Find a generating function for the sequence, $\{u_n\}_{n=1}^{\infty}$.

H-229 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array

$$A(n,k) \qquad (0 \le k \le n)$$

is defined by means of

$$(*) \qquad \begin{cases} A(n+1,\ 2k) = A(n,\ 2k-1) + aA(n,\ 2k) \\ A(n+1,\ 2k+1) = A(n,\ 2k) + bA(n,\ 2k+1) \end{cases}$$

together with

$$A(0,0) = 1, \qquad A(0,k) = 0 \qquad (k \ne 0) .$$

Find $A(n,k)$ and show that

$$\sum_k A(n,\ 2k)(ab)^k = a(a+b)^{n-1}, \qquad \sum_k A(n,\ 2k+1)(ab)^k = (a+b)^{n-1} .$$

SOLUTIONS

ARRAY OF HOPE

H-195 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California

Consider the array indicated below:

1	1						
1	2						
2	4	1	1				
5	9	3	4				
13	22	7	11	1	1		
34	56	16	27	5	6		
89	145	38	65	16	22	1	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

(i) Show that the row sums are F_{2n}, $n \ge 2$.

(ii) Show that the rising diagonal sums are the convolution of

$$\{F_{2n-1}\}_{n=0}^{\infty} \quad \text{and} \quad \{u(n;\ 2,2)\}_{n=0}^{\infty} ,$$

the generalized numbers of Harris and Styles.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let $A(n,k)$ denote the element in the n^{th} row and k^{th} column. Then (presumably)

$$A(n,1) = F_{2n-3} \qquad (n > 1)$$

and

$$\begin{cases} A(n,\ 2k) = A(n,\ 2k-1) + A(n-1,\ 2k) \\ A(n,\ 2k+1) = A(n-1,\ 2k+1) + A(n-2,\ 2k) \end{cases} \qquad (k \geq 1) \quad .$$

Put

$$F(x,y) = \sum_{n=1}^{\infty} \sum_{k} A(n,\ 2k)\, x^n y^{2k}$$

$$G(x,y) = \sum_{n=1}^{\infty} \sum_{k} A(n,\ 2k+1)\, x^n y^{2k+1}$$

$$A(x) = \sum_{n=1}^{\infty} A(n,1)\, x^n \quad .$$

Then

$$A(x) = x + \sum_{n=2}^{\infty} F_{2n-3}\, x^n = x + x \sum_{n=1}^{\infty} F_{2n-1}\, x^n$$

$$= x + x\, \frac{x - x^2}{1 - 3x + x^2} = \frac{x - 2x^2}{1 - 3x + x^2} \quad .$$

Next

$$F(x,y) = \sum_{n} \sum_{k} (A(n,\ 2k-1) + A(n-1,\ 2k))\, x^n y^{2k}$$

$$= xF(x,y) + yG(x,y) \quad ,$$

so that

(1) $$(1 - x)F(x,y) = yG(x,y) \ .$$

Also,

$$G(x,y) = y\sum_1^\infty A(n,1)x^n + \sum_{n=2}^\infty\sum_{k>0} (A(n-1,\ 2k+1) + A(n-2,\ 2k))\,x^n y^{2k+1}$$

$$= yA(x) + x\sum_{n=1}^\infty\sum_{k>0} A(n,\ 2k+1)x^n y^k + x^2y\sum_{n=1}^\infty\sum_{k>0} A(n,\ 2k)x^n y^{2k}$$

$$= y(1-x)A(x) + xG(x,y) + x^2yF(x,y)\ ,$$

so that

(2) $$(1-x)G(x,y) = x^2yF(x,y) + \frac{x(1-x)(1-2x)y}{1-3x+x^2}\ .$$

It follows from (1) and (2) that

(3) $$\begin{cases} ((1-x)^2 - x^2y^2)F(x,y) = \dfrac{x(1-x)(1-2x)y^2}{1-3x+x^2} \\ ((1-x)^2 - x^2y^2)G(x,y) = \dfrac{x(1-x)^2(1-2x)y}{1-3x+x^2} \end{cases}\ .$$

Hence

(4) $$((1-x)^2 - x^2y^2)\sum_n\sum_k A(n,k)x^n y^k = \frac{xy(1-x)(1-2x)(1-x+y)}{1-3x+x^2}\ .$$

For $y = 1$ this reduces to

$$\sum_n x^n \sum_k A(n,k) = \frac{x(1-x)(2-x)}{1-3x+x^2}$$

$$= x + \frac{x}{1-3x+x^2}$$

$$= x + \sum_1^\infty F_{2n}x^n\ ,$$

so that

$$\sum_k A(n,k) = F_{2n} \qquad (n > 1)\ .$$

If we take $y = x$, Eq. (4) reduces to

$$\sum_{n=2}^{\infty} x^n \sum_k A(n-k,\ k) = \frac{x^2(1-x)(1-2x)}{(1-x-x^2)(1-x+x^2)(1-3x+x^2)}$$

$$= \sum_{1}^{\infty} F_{2n-1} x^n \ \frac{x(1-2x)}{(1-x-x^2)(1-x+x^2)}\ .$$

This expresses the rising diagonal sums

$$\sum_{k=1}^{n-1} A(n-k,\ k)$$

as convolutions as stated.

Remark. It follows from (3) that

$$\begin{cases} F(x,y) = \dfrac{1-2x}{1-3x+x^2} \displaystyle\sum_{k=1}^{\infty} \frac{x^{2k-1}y^{2k}}{(1-x)^{2k-1}} \\ G(x,y) = \dfrac{1-2x}{1-3x+x^2} \displaystyle\sum_{k=1}^{\infty} \frac{x^{2k-1}y^{2k-1}}{(1-x)^{2k-2}} \end{cases},$$

so that

$$(5)\qquad \begin{cases} \displaystyle\sum_{n=2k-1}^{\infty} A(n,2k)x^n = \frac{(1-2x)x^{2k-1}}{(1-3x+x^2)(1-x)^{2k-1}} \\ \displaystyle\sum_{n=2k-1}^{\infty} A(n,\ 2k-1)x^n = \frac{(1-2x)x^{2k-1}}{(1-3x+x^2)(1-x)^{2k-2}} \end{cases}.$$

By means of (5) we can obtain explicit formulas for $A(n,k)$. Since

$$\frac{1-2x}{1-3x+x^2} = \sum_{r=0}^{\infty} F_{2r-1}\, x^r\ ,$$

it follows that

$$\sum_{n=2k-1}^{\infty} A(n,2k)x^n = x^{2k-1} \sum_{r=0}^{\infty} F_{2r-1}\, x^r \sum_{s=0}^{\infty} \binom{2k+s-2}{s} x^s\ .$$

Therefore,

$$A(n,2k) = \sum_{r=0}^{n-2k+1} \binom{n - r - 1}{2k - 2} F_{2r-1} .$$

Similarly

$$A(n,\ 2k - 1) = \sum_{r=0}^{n-2k+1} \binom{n - r - 2}{2k - 3} F_{2r-1} \qquad (k > 1) .$$

Also solved by the Proposer.

PARTITION

H-196 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

(a) Let A_0 be the set of integral parts of the positive integral multiples of τ, where

$$\tau = \frac{1 + \sqrt{5}}{2} \quad ,$$

and let A_{m+1}, $m = 0, 1, 2, \cdots$, be the set of integral parts of the numbers $n\tau^2$ for $n \in A_m$. Prove that the collection of Z^+ of all positive integers is the disjoint union of the A_j.

(b) Generalize the proposition in (a).

Solution by L. Carlitz, Duke University, Durham, North Carolina.

1. Put

$$a(n) = [n\tau] , \qquad b(n) = [n\tau^2] = [n(\tau + 1)] = a(n) + n .$$

Also for brevity put

$$(a) = \{a(n)\,|n = 1, 2, 3, \cdots\} ,$$

$$(b) = \{b(n)\,|n = 1, 2, 3, \cdots\} .$$

It is well known that*

$$(*) \qquad Z^+ = (a) \cup (b) .$$

Put

$$(b^k a) = \{b^k a(n)\,|n = 1, 2, 3, \cdots\} ,$$

where juxtaposition denotes composition. Then it follows at once from (*) that

$$\begin{aligned} Z^+ &= (a) \cup (ba) \cup (b^2) \\ &= (a) \cup (ba) \cup (b^2a) \cup (b^3), \end{aligned}$$

and so on. Clearly this implies

$$Z^+ = \bigcup_{k=0}^{\infty} (b^k a) = \bigcup_{k=0}^{\infty} A_k .$$

2. Let α, β be positive irrational numbers such that

$$(1/\alpha) + (1/\beta) = 1$$

and put

$$a(n) = [\alpha n], \qquad b(n) = [\beta n] .$$

Then it is well known that

$$Z^+ = (a) \cup (b) ,$$

where, as above,

$$(a) = \{a(n) | n = 1, 2, 3, \cdots\}, \qquad b(n) = \{b(n) | n = 1, 2, 3, \cdots\} .$$

Hence

$$\begin{aligned} Z^+ &= (a) \cup (ba) \cup (b^2) \\ &= (a) \cup (ba) \cup (b^2a) \cup (b^3), \end{aligned}$$

and so on. Thus

$$Z^+ = \bigcup_{k=0}^{\infty} (b^k a) .$$

Remark. The functions $a(n)$, $b(n)$ in 1 are studied in considerable detail in the paper by L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville: "Fibonacci Representations," Fibonacci Quarterly, Vol. 10, No. 1, pp. 1-28.

Also solved by the Proposer.

Editorial Note: See Beatty's Theorem (American Math. Monthly, 33 (1926), 159, and 34 (1927) 159.)

The editor wishes to acknowledge solutions to H-194 by L. Frohman, P. Bruckman, and J. Ivie.

Editorial Note: The following list represents previous problem proposals (less than or equal to H-100) which, to date, have not been solved: 22, 23, 40, 43, 46, 60, 61, 73, 76, 77, 84, 87, 90, 91, 94, and 100. Starting in the next section, we shall re-run some of these proposals.

ERRATA

On Problem H-218, April, 1973, please change the matrix to read:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ 0 & 1 & 1 & 0 & \cdots \\ \cdots & 0 & 2 & \cdots & \end{pmatrix}_{n \times n}$$

ON THE NUMBER OF DIVISIONS NEEDED IN FINDING THE GREATEST COMMON DIVISOR

DALE D. SHEA
Student, San Diego State College, San Diego, California

Let $n(a,b)$ and $N(a,b)$ be the number of divisions needed in finding the greatest common divisor of positive integers a,b using the Euclidean algorithm and the least absolute value algorithm, respectively. In addition to showing some properties of periodicity of $n(a,b)$ and $N(a,b)$, the paper gives a proof of the following theorems:

Theorem 1. If $n(a,b) = k > 1$, then $a + b \geq f_{k+3}$ and the pair (a,b) with the smallest sum such that $n(a,b) = k$ is the pair (f_{k+1}, f_{k+2}), where

$$f_1 = 1, \quad f_2 = 1 \quad \text{and} \quad f_{n+2} = f_{n+1} + f_n, \quad n = 1, 2, 3, \cdots .$$

Theorem 2. If $N(a,b) = k > 1$, then $a + b \geq x_{k+1}$ and the pair (a,b) with smallest sum such that $N(a,b) = k$ is the pair $(x_k, x_k + x_{k-1})$, where $x_1 = 1$, $x_2 = 2$, and $x_k = 2x_{k-1} + x_{k-2}$, $k = 3, 4, \cdots$. These results may be compared with other results found in [1], [2].

Since $n(a,b) = n(b,a)$ we can assume $a \leq b$. To prove the first theorem, let $n(a,b) = k$ and assume the k steps in finding (a,b) are

$$\begin{aligned} b &= q_1 a + r_1 \\ a &= q_2 r_1 + r_2 \\ &\cdots \\ r_{k-3} &= q_{k-1} r_{k-2} + r_{k-1} \\ r_{k-2} &= q_k r_{k-1} \quad . \end{aligned}$$

If $k = 1$, then $r_1 = 0$ so $b = q_1 a$ and the smallest pair (a,b) is $(1,1)$ so

$$a = f_1 , \qquad b = f_2 , \qquad a + b = f_3 = 2.$$

Note this case is not included in the theorem. In case $k > 1$ it is evident the smallest values of a,b will be obtained for $r_{k-1} = 1$ and all the q's $= 1$ except q_k, which cannot be 1 but is 2. Thus the pairs $(r_{k-1}, r_{k-2}), \cdots, (a,b)$ are $(1,2), \cdots, (f_{k+1}, f_{k+2})$. Since $a + b = f_{k+1} + f_{k+2} = f_{k+3}$, the theorem is proved.

We have

Corollary 1. If $a + b < f_{k+3}$, then $n(a,b) < k$ for $k > 1$.

For $b = a + i$, i a fixed positive integer so that $b < 2a$, the quantities satisfy

$$n(a + mi,\ a + [m + 1]i) = n(a,\ a + i), \qquad m = 0, 1, 2, \cdots . \tag{1}$$

This follows from the remark that if $n(a,b) = k$, then $n(a + b,\ 2a + b) = k + 1$, $k = 1, 2, 3, \cdots$. This is evident since the first division would be $(2a + b) = 1(a + b) + a$ and

$$n(a,\ a + b) = n(a,\ b) = k .$$

Equation (1) is a consequence since each n is one more than $n(i,\ a + mi) = n(i,a)$. The periodicity is evident in the table of values of $n(a,b)$ for $a \le b < 2a$. (See Fig. 1.)

a =	
1	1
2	1 2
3	1 2 3
4	1 2 2 3
5	1 2 3 4 3
6	1 2 2 2 3 3
7	1 2 3 3 4 4 3
8	1 2 2 4 2 5 3 3
9	1 2 3 2 3 4 3 4 3
10	1 2 2 3 3 2 4 4 3 3
11	1 2 3 4 4 3 4 5 5 4 3
12	1 2 2 2 2 4 2 5 3 3 3 3
13	1 2 3 3 3 5 3 4 6 4 4 4 3
14	1 2 2 4 3 4 3 2 4 5 4 5 3 3
15	1 2 3 2 4 2 3 3 4 4 3 5 3 4 3

Figure 1

$n(a,b)$ for $b = a,\ a + 1,\ \cdots,\ 2a - 1$.

To prove Theorem 2, assume the steps in finding (a,b) with $N(a,b) = k$ are

$$\begin{aligned} b &= q_1 a \pm r_1 \\ a &= q_2 r_1 \pm r_2 \\ &\cdots \\ r_{k-3} &= q_{k-1} r_{k-2} \pm r_{k-1} \\ r_{k-2} &= q_k r_{k-1} , \end{aligned}$$

where

$$0 < r_1 \le \frac{1}{2}a, \quad 0 < r_2 \le \frac{1}{2}r_1, \cdots, \quad 0 < r_{k-1} \le \frac{1}{2}r_{k-2} .$$

Because of the restriction on the remainders, we must have $q_2, q_3, \cdots, q_k$ equal to or greater than 2. But since $2r_i + r_{i+1} \le 3r_i - r_{i+1}$, $i = 1, \cdots, k - 1$, in each case we obtain the smallest sum $a + b$ with $q_2 = \cdots = q_k = 2$ and with $q_1 = 1$. For $k = 1$, we have $1 = 1 \cdot 1$ so $a = b = 1$. Set $x_i = r_{k-i}$. For $k > 1$, $a = x_k = 2x_{k-1} + x_{k-2}$ and $b = x_{k+1} = x_k + x_{k-1}$. Then $a + b = 2x_k + x_{k-1} = x_{k+1}$. This completes the proof of the theorem.

Corollary 2. If $a + b < x_{k+1}$, then $N(a,b) < k$ for $k > 1$.

Figure 2 exhibits the periodicity for i fixed):

$$N(a,\ a + i) = N(a + mi,\ a + [m + 1]i), \qquad 1 \le i \le a/2 \tag{2}$$

and the symmetry:

$$N(a,\ a + i) = N(a,\ 2a - i), \qquad 1 \le i \le a - 1\ . \tag{3}$$

a = 1	1
2	2
3	2 2
4	2 2 2
5	2 3 3 2
6	2 2 2 2 2
7	2 3 3 3 3 2
8	2 2 3 2 3 2 2
9	2 3 2 3 3 2 3 2
10	2 2 3 3 2 3 3 2 2
11	2 3 3 3 3 3 3 3 3 2
12	2 2 2 2 4 2 4 2 2 2 2
13	2 3 3 3 4 3 3 4 3 3 3 2
14	2 2 3 3 3 3 2 3 3 3 3 2 2
15	2 3 2 3 2 3 3 3 3 2 3 2 3 2
16	2 2 3 2 3 2 4 2 4 2 3 2 3 2 2
17	2 3 3 3 4 3 4 3 3 4 3 4 3 3 2 2
18	2 2 2 3 4 2 4 2 2 2 4 2 4 3 2 2 2
19	2 3 3 3 3 3 4 4 3 3 4 4 3 3 3 3 3 2
20	2 2 3 2 2 3 3 3 4 2 4 3 3 3 2 2 3 2 2
21	2 3 2 3 3 3 2 4 3 3 3 3 4 2 3 3 3 2 3 2
22	2 2 3 3 4 2 3 3 4 3 2 3 4 3 3 2 4 3 3 2 2
23	2 3 3 3 4 3 4 3 4 4 3 3 4 4 3 4 3 4 3 3 3 2

Figure 2

$N(a,b)$ for $b = a + 1, \cdots, 2a - 1$

I wish to acknowledge the assistance of Professor V. C. Harris in shortening the proofs.

REFERENCES

1. R. L. Duncan, "Note on the Euclidean Algorithm," The Fibonacci Quarterly, 4 (1966), pp. 367-368.
2. A. W. Goodman and W. M. Zaring, "Euclid's Algorithm and the Least Remainder Algorithm," The Amer. Math. Monthly, 59 (1952), pp. 156-159.

A PRIMER FOR THE FIBONACCI NUMBERS: PART XIII

MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara California

THE FIBONACCI CONVOLUTION TRIANGLE, PASCAL'S TRIANGLE, AND SOME INTERESTING DETERMINANTS

The simplest and most well-known convolution triangle is Pascal's triangle, which is formed by convolving the sequence $\{1, 1, 1, \cdots\}$ with itself repeatedly. The Fibonacci convolution triangle [1] is formed by repeated convolutions of the sequence $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$ with itself. We now show three different ways to obtain the Fibonacci convolution triangle, as well as some interesting sequences of determinant values found in Pascal's triangle, the Fibonacci convolution triangle, and the trinomial coefficient triangle.

1. CONVOLUTION OF SEQUENCES

If $\{a_n\}$ and $\{b_n\}$ are two sequences, then the convolution of the two sequences is another sequence $\{c_n\}$ which is calculated as shown:

$$\begin{aligned}
c_1 &= a_1b_1 \\
c_2 &= a_1b_2 + a_2b_1 \\
c_3 &= a_1b_3 + a_2b_2 + a_3b_1 \\
&\cdots\cdots\cdots \\
c_n &= a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_nb_1 = \sum_{k=1}^{n} a_kb_{n-k+1} .
\end{aligned}$$

If we convolve the Fibonacci sequence with itself, we obtain the <u>First Fibonacci Convolution Sequence</u> $\{1, 2, 5, 10, 20, 38, 71, \cdots\}$, as follows:

$$\begin{aligned}
F_1^{(1)} &= F_1F_1 &&= 1\cdot1 &&= 1 \\
F_2^{(1)} &= F_1F_2 + F_2F_1 &&= 1\cdot1 + 1\cdot1 &&= 2 \\
F_3^{(1)} &= F_1F_3 + F_2F_2 + F_3F_1 &&= 1\cdot2 + 1\cdot1 + 2\cdot1 &&= 5 \\
F_4^{(1)} &= F_1F_4 + F_2F_3 + F_3F_2 + F_4F_1 &&= 1\cdot3 + 1\cdot2 + 2\cdot1 + 3\cdot1 &&= 10 \\
&\cdots\cdots\cdots
\end{aligned}$$

Next we can obtain the <u>Second Fibonacci Convolution Sequence</u> $\{1, 3, 9, 22, 51, 111, \cdots\}$ as indicated below.

$$
\begin{array}{lllll}
F_1^{(2)} & = F_1F_1^{(1)} & = 1\cdot 1 & = & 1 \\
F_2^{(2)} & = F_2F_1^{(1)} + F_1F_2^{(1)} & = 1\cdot 1 + 1\cdot 2 & = & 3 \\
F_3^{(2)} & = F_3F_1^{(1)} + F_2F_2^{(1)} + F_1F_3^{(1)} & = 2\cdot 1 + 1\cdot 2 + 1\cdot 5 & = & 9 \\
F_4^{(4)} & = F_4F_1^{(1)} + F_3F_2^{(1)} + F_2F_3^{(1)} + F_1F_4^{(1)} & = 3\cdot 1 + 2\cdot 2 + 1\cdot 5 + 1\cdot 10 & = & 22 \\
\cdots & \cdots & \cdots & & \cdots
\end{array}
$$

by writing the convolution of the first Fibonacci convolution sequence with the Fibonacci sequence. To obtain the succeeding Fibonacci convolution sequences, we continue writing the convolution of a Fibonacci convolution sequence with the Fibonacci sequence. A second method follows.

The Fibonacci sequence is obtained from the generating function

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots ,$$

which provides Fibonacci numbers as coefficients of successive powers of x as far as one pleases to carry out a long division. The k^{th} convolution of the Fibonacci numbers appears as the coefficients of successive powers of x in the generating function

$$\frac{1}{(1 - x - x^2)^{k+1}} = F_1^{(k)} + F_2^{(k)}x + F_3^{(k)}x^2 + \cdots + F_{n+1}^{(k)}x^n + \cdots ,$$

$k = 0, 1, 2, \cdots$. For $k = 0$, we get just the Fibonacci numbers. In the next section, we shall see yet another way to find the convolved Fibonacci sequences.

3. THE FIBONACCI CONVOLUTION TRIANGLE

Suppose someone writes a column of zeroes. To the right and one space down place a one. To generate the elements below the one we add the one element directly above and the one element diagonally left of the element to be written. Such a rule generates a convolution triangle. This rule, of course, generates Pascal's triangle in left-justified form:

$$
\begin{array}{ccccccc}
0 & & & & & & \\
0 & 1 & & & & & \\
0 & 1 & 1 & & & & \\
0 & 1 & 2 & 1 & & & \\
0 & 1 & 3 & 3 & 1 & & \\
0 & 1 & 4 & 6 & 4 & 1 & \\
0 & 1 & 5 & 10 & 10 & 5 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

The columns of Pascal's triangle give convolution sequences for the sequence $\{1, 1, 1, \cdots\}$. Notice that the row sums give powers of two, and the sums of rising diagonals formed by beginning in the column of ones and going up one and to the right one throughout the array give the Fibonacci numbers $1, 1, 2, 3, 5, \cdots, F_n, \cdots$, where $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, 5, \cdots$.

Next suppose we change the rule of formation. Begin as before, but to generate elements below the one, add the two elements directly above and the element diagonally left of the element to be generated. Now we have the Fibonacci convolution triangle in left-justified form,

$$\begin{array}{ccccccc}
0 & & & & & & \\
0 & 1 & & & & & \\
0 & 1 & 1 & & & & \\
0 & 2 & 2 & 1 & & & \\
0 & 3 & 5 & 3 & 1 & & \\
0 & 5 & 10 & 9 & 4 & 1 & \\
0 & 8 & 20 & 22 & 14 & 5 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

The columns give the convolution sequences for the Fibonacci sequence. The row sums are the Pell numbers $1, 2, 5, 12, 29, 70, \cdots, p_n, \cdots$, where $p_n = 2p_{n-1} + p_{n-2}$. The rising diagonal sums are $1, 1, 3, 5, 11, 21, \cdots, r_n, \cdots$, where $r_n = r_{n-1} + 2r_{n-2}$. The diagonal sums found by beginning in the column of Fibonacci numbers and going up two and right one throughout the array are $1, 1, 2, 4, 7, 13, 24, \cdots, T_n, \cdots, T_n = T_{n-1} + T_{n-2} + T_{n-3}$, the Tribonacci numbers.

If one changes the rule of formation yet again, so that the elements below the initial one are found by adding the one element directly above and the two elements diagonally left of the element to be generated, the array obtained is the trinomial coefficient triangle. The coefficients in successive rows are the same as those found in expansions of the trinomial $(1 + x + x^2)^n$, $n = 0, 1, 2, \cdots$. The columns do not form convolution sequences as before, but the row sums are now the powers of three, and the sums of elements appearing on the rising diagonals are $1, 1, 2, 4, 7, 13, \cdots$, the Tribonacci numbers just defined. To illustrate, the trinomial triangle is formed as follows:

$$\begin{array}{cccccccccc}
0 & & & & & & & & & \\
0 & 1 & & & & & & & & \\
0 & 1 & 1 & 1 & & & & & & \\
0 & 1 & 2 & 3 & 2 & 1 & & & & \\
0 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\
0 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

3. SOME SPECIAL MATRICES

If one looks again at how convoluted sequences are formed, the arithmetic is much like matrix multiplication. Suppose that we define three matrices. Let P be the $n \times n$ matrix formed by using as elements the first n rows of Pascal's triangle in rectangular form. Let F be the $n \times n$ matrix formed by writing the first n rows of Pascal's triangle in vertical position on and below the main diagonal, which makes the row sums of F be Fibonacci numbers. Let C be the $n \times n$ matrix whose elements are the first n rows of the Fibonacci convolution triangle written in rectangular form. Then it can be proved that $FP = C$ (see [1], [2].) To illustrate, for $n = 6$,

$$FP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

$$(3.1) \qquad = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 9 & 14 & 20 & 27 \\ 3 & 10 & 22 & 40 & 65 & 98 \\ 5 & 20 & 51 & 105 & 190 & 315 \\ 8 & 38 & 111 & 256 & 511 & 924 \end{bmatrix} = C$$

Suppose that, instead of multiplying matrix F by the rectangular Pascal array P, we use an $n \times n$ matrix A whose elements are given by the first n rows of Pascal's triangle in left-justified form on and below its main diagonal, and zero elsewhere. Let F^t be the transpose of F. Then the matrix product $AF^t = T$, where T is the $n \times n$ matrix whose elements are found in the left-justified trinomial coefficient triangle given in Section 2. We illustrate for $n = 6$:

$$AF^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(3.2) \qquad = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 1 & 3 & 6 & 7 & 6 & 3 \\ 1 & 4 & 10 & 16 & 19 & 16 \\ 1 & 5 & 15 & 30 & 45 & 51 \end{bmatrix} = T \;.$$

4. SPECIAL DETERMINANTS IN PASCAL'S TRIANGLE

A multitude of unit determinants can be found in Pascal's triangle. The following theorems are proved in [2].

Theorem 4.1. The determinant of any $k \times k$ array taken with its first column along the column of ones and its first row the i^{th} row of Pascal's triangle written in left-justified form, has value one.

Theorem 4.2. The determinant of any $k \times k$ array taken with its first row along the row of ones or with its first column along the column of ones in Pascal's triangle written in rectangular form, is one.

For example,

$$1 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 10 & 15 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \\ 1 & 6 & 21 & 56 \end{vmatrix} .$$

Pascal's triangle also has sequences of determinants which have binomial coefficients for their values. Here we have to number the rows and columns of Pascal's triangle; the row of ones is the zeroth row; the column of ones the zeroth column. To illustrate some of the sequences of determinants considered here, we look back at the matrix P of (3.1) which contains the first n rows and columns of Pascal's triangle written in rectangular form. When 2×2 determinants are taken across the first and second rows of Pascal's rectangular array,

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \quad \cdots,$$

giving values found in the second column of Pascal's triangle. Of course, the 1×1 determinants along the first row give the values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10, \quad \cdots,$$

successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the first, second, $\cdots$, and k^{th} rows yields the successive entries of the k^{th} column of Pascal's triangle.

The following theorems are proved in [3].

Theorem 4.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the i^{th} row has determinant value given by the binomial coefficient

$$\binom{i + k - 1}{k} .$$

Theorem 4.4. The determinant of the $k \times k$ matrix taken with its first column the j^{th} column of Pascal's triangle written in rectangular form, and its first row the first row of the rectangular Pascal array, has values given by the binomial coefficient

$$\binom{j+k-1}{k} .$$

5. SPECIAL DETERMINANTS IN THE FIBONACCI CONVOLUTION TRIANGLE AND IN THE TRINOMIAL TRIANGLE ARRAYS

Now we are ready to prove that the unit determinants and binomial coefficient determinants of Section 4 are also found in the Fibonacci convolution triangle and in the trinomial coefficient triangle. Returning to (3.1), the first n entries of the first n rows of the Fibonacci convolution triangle are given by the matrix product $FP = C$. But, notice that $k \times k$ submatrices of C taken along either the first or second matrix row are the product of a $k \times k$ submatrix of F with a unit determinant and a similarly placed $k \times k$ submatrix of P which has been evaluated in Theorem 4.2 or Theorem 4.4. Let us also number the Fibonacci convolution triangle as Pascal's triangle, with the top row the zeroth row. Thus, we have

Theorem 5.1. Let a $k \times k$ matrix M be selected from the Fibonacci convolution triangle in rectangular form. If M includes the row of ones, then $\det M = 1$. If M has its first column the j^{th} column and its first row along the first row of the Fibonacci array, then

$$\det M = \binom{j+k-1}{k} .$$

Reasoning in a similar fashion from (3.2), the matrix product AF^t and Theorems 4.1 and 4.3 yield the following, where the trinomial coefficient triangle is numbered as Pascal's triangle, with the left-most column the zeroth column.

Theorem 5.2. Let a $k \times k$ matrix N be selected from the trinomial triangle written in left-justified form. If N includes the column of ones, then $\det N = 1$. If N has its first row the i^{th} row and its first column along the first column of the trinomial triangle, then

$$\det N = \binom{i+k-1}{k}.$$

These results are generalized in [2] and [3]. Other classes of determinants are also developed there. The reader should verify the results given here numerically.

REFERENCES

1. Verner E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 8, No. 2, March 1970, pp. 158-171.
2. Marjorie Bicknell and V. E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 11, No. 2, April, 1973, pp. 131-144.
3. Marjorie Bicknell and V. E. Hoggatt, Jr., "Special Determinants Found within Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 11, No. 5, Dec., 1973, pp. 469-479.

A FIBONACCI PROBABILITY FUNCTION

HAROLD D. SHANE
Baruch College of CUNY, New York, New York

1. THE FIBONACCI DISTRIBUTION

Consider the following Markov Process. To begin, a marker is placed in slot number zero. At each minute thereafter, a coin is flipped. If it comes up heads, the marker is moved up one slot. If it comes up tails, the marker is moved back to position zero. Let X_n be the number of flips needed to advance the marker to position n. We would like to investigate the distribution of the random variable, X_n. For the case $n = 1$, the random variable is simply geometric (i.e., X_1 = number of trials until the first success occurs). Let us therefore, start with the case $n = 2$ and probability of a head, $p = 1/2$.

Let $P(X_2 = k) = p_2(k)$, $k = 2, 3, 4, \cdots$. Now,

$$p_2(2) = P(HH) = 1/2^2, \quad p_2(3) = P(THH) = 1/2^3$$

and

$$\begin{aligned} p_2(k+3) &= P(k \text{ trials with no run of two heads}) \cdot P(THH) \\ &= (A_{2,k}/2^k)\cdot(1/2^3) = A_{2,k}/2^{k+3}, \quad k = 1,2,3,\cdots, \end{aligned} \tag{1}$$

where $A_{2,k}$ = number of arrangements of k heads and tails with no two consecutive heads. In order to evaluate $A_{2,k}$, we note that we may classify the allowable arrangements according to whether the last tail is in the k^{th} or $(k-1)^{st}$ position. Letting $a_{2,k,i}$ = number of arrangements of k heads and tails having no two consecutive heads and having a tail in the i^{th} position, $i = k, k-1$, gives $A_{2,k} = a_{2,k,k} + a_{2,k,k-1}$. But, $a_{2,k,k} = A_{2,k-1}$ and $a_{2,k,k-1} = A_{2,k-2}$, yielding

$$A_{2,k} = A_{2,k-1} + A_{2,k-2}\,. \tag{2}$$

For $k = 1$, the possible arrangements are simply H and T. Thus, $A_{2,1} = 2$. For $k = 2$, the possible arrangements are HT, TH, TT. Thus, $A_{2,2} = 3$. Combining (1), (2) and the preceding, we have

$$p_2(k) = F_{k-2}/2^k \qquad k = 2, 3, 4, \cdots, \tag{3}$$

where F_k = k^{th} Fibonacci number (with $F_0 = F_1 = 1$). Certainly, a good name for this is the Fibonacci Probability Distribution.

The cumulative distribution function of X_2 is given by

$$G_2(x) = P(X_2 \le x) = \sum_{k=2}^{[x]} F_{k-2}/2^k \quad \text{for } x \geq 2, \text{ and zero otherwise,} \tag{4}$$

where $[x]$ = largest integer less than or equal to x. In order to close this sum, we simply note the following

Lemma.

$$\sum_{j=0}^{n} 2^{n-j}F_j = 2^{n+2} - F_{n+3} .$$

Proof. By induction, if $n = 0$, the left-hand side is simply $F_0 = 1$ and the right-hand side is $2^2 - F_3 = 4 - 3 = 1$. Now assuming the result for n, consider

$$\sum_{j=0}^{n+1} 2^{n+1-j}F_j = F_{n+1} + 2\sum_{j=0}^{n} 2^{n-j}F_j = F_{n+1} + 2(2^{n+2} - F_{n+3})$$

$$= F_{n+1} + 2^{n+3} - 2F_{n+3} = 2^{n+3} - (F_{n+3} + F_{n+3} - F_{n+1})$$

$$= 2^{n+3} - F_{n+4} = 2^{(n+1)+2} - F_{(n+1)+3} \qquad \text{q. e. d.}$$

Applying this Lemma, we see that $G_2(x)$ is simply

$$G_2(x) = \sum_{k=2}^{[x]} F_{k-2}/2^k = 2^{-[x]}\sum_{k=0}^{[x]-2} 2^{[x]-2-k}F_k = 2^{-[x]}(2^{[x]} - F_{[x]+1}). \tag{5}$$

So,

$$G_2(x) = 1 - F_{[x]+1}/2^{[x]} \quad \text{if } x \geq 0 \text{ and } 0 \text{ otherwise.} \tag{6}$$

The factorial moment generating function $M_2(t) = Et^{X_2}$, is easily obtained,

$$M_2(t) = \sum_{k=2}^{\infty} t^k p_2(k) = \sum_{k=2}^{\infty} t^k F_{k-2}/2^k = (t/2)^2 \sum_{k=0}^{\infty} F_k (t/2)^k = \tfrac{1}{4}t^2 g(\tfrac{1}{2}t) ,$$

where

$$g(x) = \sum_{k=0}^{\infty} F_k x^k ,$$

the generating function for the Fibonacci numbers, that is, $g(x) = (1 - x - x^2)^{-1}$. Therefore,

$$M_2(t) = t^2/(4 - 2t - t^2) \tag{7}$$

and

$$\frac{d^m}{dt^m} M_2(t)\Big|_{t=1} = EX_2(X_2 - 1)\cdots(X_2 - m + 1) = f_{2,m} ,$$

the m^{th} factorial moment. The usual moment generating function is, of course, $M_2(e^t)$. Making the substitution, $u = t - 1$, we produce $m_2(u) = M_2(u + 1)$, for which $m_2^{(m)}(0) = f_{2,m}$. A partial fraction decomposition of the preceding yields

$$m_2(u) = -1 + \frac{2u - 2}{u^2 + 4u - 1} = -1 + \left(\frac{5 + 3\sqrt{5}}{5}\right)\left(\frac{1}{u + \alpha}\right) + \left(\frac{5 - 3\sqrt{5}}{5}\right)\left(\frac{1}{u + \beta}\right) ,$$

where $\alpha = 2 + \sqrt{5}$ and $\beta = 2 - \sqrt{5}$. Expanding both fractions as power series, elementary computations yield

$$m_2(u) = -1 + \sum_{j=0}^{\infty} \left[\frac{3(\alpha^j + \beta^j) + (\alpha^{j+1} + \beta^{j+1})}{5}\right] u^j . \tag{8}$$

Since the coefficient of u^j is $m^{(j)}(0)/j!$, comparing terms in (8), we have

$$f_{2,m} = m!\,[3(\beta^j + \alpha^j) + (\beta^{j+1} + \alpha^{j+1})]/5 . \tag{9}$$

2. THE POLY-NACCI DISTRIBUTION

Let us now proceed along the lines of section one, to develop the situation for the case of n greater than or equal to two. Let $P(X_n = k) = p_n(k)$ $k = n, n+1, \cdots$. Here we have $p_n(n) = P$ (n heads in a row) $= (1/2)^n$, $p_n(n+1) = P$ (one tail followed by n consecutive heads) $= (1/2)^{n+1}$ and

$$\begin{aligned} p_n(k + n + 1) &= P(k \text{ trials with no run of } n \text{ heads}) \cdot p_n(n + 1) \\ &= (A_{n,k}/2^k)\cdot(\tfrac{1}{2})^{n+1} = A_{n,k}/2^{n+k+1} \quad k = 1, 2, 3, \cdots, \end{aligned}$$

where $A_{n,k}$ = number of arrangements of heads and tails with no run of n heads. Again, we may evaluate $A_{n,k}$ by letting $a_{n,k,i}$ = number of arrangements of k heads and tails having no run of n heads and the last tail in the i^{th} position, $i = k, k-1, \cdots, k-n+1$. Thus,

$$\sum_{j=0}^{n-1} a_{n,k,k-j} ,$$

but

$$a_{n,k,k-j} = A_{n,k-(j+1)} \qquad j = 0, 1, \cdots, n-1 .$$

So, analogously to (2),

$$A_{n,k} = \sum_{j=0}^{n-1} A_{n,k-(j+1)} \quad \text{where } A_{n,i} = 2^i \quad 1 = 0, 1, 2, \cdots, n-1 . \tag{10}$$

At this point, it is convenient to define the k^{th} poly-nacci number of order n, $F_{n,k}$, by the recurrence

$$F_{n,k} = F_{n,k-1} + F_{n,k-2} + \cdots + F_{n,k-n} \qquad k = 1, 2, 3, \cdots, \tag{11}$$

where $F_{n,0} = 1$ and $F_{n,-r} = 0$.

Using this notation, we may write

$$p_n(k) = F_{n,k-n}/2^k \qquad k = n, n+1, n+2, \cdots. \tag{12}$$

The cumulative distribution function

$$G_n(x) = P(X_n \leq x) = \sum_{k=n}^{[x]} F_{n,k-n}/2^k \qquad \text{for } x \geq n. \tag{13}$$

As in Section One, we state

<u>Lemma.</u>

$$\sum_{j=0}^{N} 2^{N-j} F_{n,j} = 2^{N+n} - F_{n,N+n+1}.$$

<u>Proof.</u> By induction on N, when $N = 0$, the left-hand side is simply $F_{n,0} = 1$. The right-hand side is $2^n - F_{n,n+1}$, but

$$F_{n,n+1} = \sum_{k=1}^{n} F_{n,k} = \sum_{k=0}^{n-1} 2^k = 2^n - 1,$$

establishing the result for $N = 0$. Assuming the result for N, let us consider

$$\begin{aligned}\sum_{j=0}^{N+1} 2^{N+1-j} F_{n,j} &= F_{n,N+1} + 2\sum_{j=0}^{N} 2^{N-j} F_{n,j} = F_{n,N+1} + 2^{N+n+1} - 2F_{n,N+n+1} \\ &= 2^{n+(N+1)} - (F_{n,N+n+1} + F_{n,N+n+1} - F_{n,N+1}).\end{aligned}$$

Since

$$F_{n,N+n+1} = \sum_{j=0}^{n-1} F_{n,N+1+j}, \qquad F_{n,N+n+1} - F_{n,N+1} = \sum_{j=1}^{n-1} F_{n,N+1+j},$$

and

$$\begin{aligned}\sum_{j=0}^{N+1} 2^{N+1-j} F_{n,j} &= 2^{n+(N+1)} - \left(F_{n,N+1+n} + \sum_{j=1}^{n-1} F_{n,N+1+j}\right) \\ &= 2^{n+(N+1)} - F_{n,N+1+n+1} = 2^{n+(N+1)} - F_{n,n+(N+1)+1}, \quad \text{q.e.d.}\end{aligned}$$

Applying the Lemma,

$$G_n(x) = \sum_{k=n}^{[x]} F_{n,k-n}/2^k = 2^{-[x]} \sum_{k=n}^{[x]} 2^{[x]-k} F_{n,k-n}$$

$$= 2^{-[x]} \sum_{r=0}^{[x]-n} 2^{[x]-n-r} F_{n,r} = 2^{-[x]}(2^{[x]} - F_{n,[x]+1}) .$$

Thus, Eq. (13) reduces to a form almost identical to (6), namely,

$$(14) \qquad G_n(x) = 1 - 2^{-[x]} F_{n,[x]+1} \quad \text{if } x \geq n \text{ and } 0 \text{ otherwise.}$$

Finally, the factorial moment generating function

$$M_n(t) = Et^{X_n} = \sum_{k=n}^{\infty} t^k p_n(k) = \sum_{k=n}^{\infty} t^k F_{n,k-n}/2^k = (\tfrac{1}{2}t)^n \sum_{k=0}^{\infty} F_{n,k}(\tfrac{1}{2}t)^k$$

$$= (\tfrac{1}{2}t)^n g_n(\tfrac{1}{2}t) ,$$

where

$$g_n(x) = \sum_{k=0}^{\infty} F_{n,k} x^k ,$$

the generating function for the n^{th} order poly-nacci numbers. Since $g_n(x)$ is easily seen to be $g_n(x) = (1 - x - x^2 - \cdots - x^n) = (1 - x)/(1 - 2x + x^{n+1})$, we obtain

$$(15) \qquad M_n(t) = t^n/(2^n - 2^{n-1}t - \cdots - t^n) = t^n(2 - t)/(2^{n+1}(1 - t) + t^{n+1}) .$$

Unfortunately, a closed form expression for the $f_{n,m}$ the m^{th} factorial moment of X_n, is not readily available.

3. THE GENERALIZED POLY-NACCI DISTRIBUTION

Let us briefly apply the methods of Section 2 to the case where the probability of a head is p, $0 < p < 1$, and not necessarily 1/2. Let $q = 1 - p$ and as before let X_n = number of trials needed to reach position n. Letting $p_n(k) = P(X_n = k)$, $p_n(n) = p^n$, $p_n(n+1) = qp^n$, $p_n(n + j + 1) = qp^n p_{n,j}$ $j = 1, 2, 3, \cdots$, where $p_{n,j} = P$(j trials with no run of n heads). Now, $p_{n,j} = 1$ for $j = 0, 1, 2, 3, \cdots, n - 1$ and breaking down the probability according to the number of the last tail, we obtain

$$p_{n,j} = \sum_{r=1}^{n} qp^{r-1} p_{n,j-r} , \qquad j = n, n+1, n+2, \cdots .$$

Thus, if we define

$$F_p(n;j) = q\sum_{r=1}^{n} p^{r-1}F_p(n;\, j - r) \qquad j = 0,\ 1,\ 2,\ \cdots$$

with $F_p(n;0) = 1$ and $F_p(n;-k) = 0$, we may write $p_n(k) = p^nF_p(n;k-n)$, $k = n,\ n+1, \cdots$. The $F_p(n;j)$ being the "Poly-nacci Polynomials of order n in p." For example, the first few Fibonacci Polynomials $F_x(2;j)$ are given by: $1,\ 1 - x,\ 1 - x,\ (1-x)^2(1+x),\ (1-x)^3(1+x) + (1-x)^2x, \cdots$. The cumulative distribution function of X_n is

$$G_n(x) = \sum_{k=n}^{[x]} p_n(k) = \sum_{k=n}^{[x]} p^nF_p(n;\, k - n) \qquad \text{for} \qquad x \geq n\,.$$

That is,

$$G_n(x) = p^n\sum_{i=0}^{[x]-n} F_p(n;\, i)\ .$$

It is easy to show by induction that

$$\sum_{i=0}^{M} F_p(n;i) = (q - F_p(n,\, M + n + 1))/qp^n$$

so that

(16) $$G_n(x) = 1 - q^{-1}F_p(n;\, [x] + 1) \quad \text{if} \quad x \geq n \text{ and } 0 \text{ otherwise.}$$

The generating function for the $F_p(n;i)$,

$$g_n(x;p) = \sum_{i=0}^{\infty} F_p(n;i)x^i = \left[1 - qx\sum_{j=0}^{n-1}(px)^j\right]^{-1} = (1 - px)/(1 - x + qp^nx^{n+1})\,.$$

Thus, the factorial moment generating function for X is

(17) $$M_n(t;p) = \sum_{k=n}^{\infty} t^kp_n(k) = p^nt^n\sum_{i=0}^{\infty} F_p(n;i)\,t^i = p^nt^n(1 - pt)/(1 - t + qp^nt^{n+1}).$$

So, for instance,

$$\frac{d}{dt}M_n(t;p)\Big|_{t=1} = EX_n = (1 - p^n)/qp^n\ ,$$

which for $p = 1/2$ yields $E_{1/2}X_n = 2^{n+1} - 2$. Of course, results concerning the mean are easily obtained by developing the recurrence for EX_{n+1} in terms of EX_n but the same is not true for the higher moments. Lastly, the analysis of the probabilistic situations such as the preceding may well reveal insights into the Fibonacci numbers and their extensions.

SOME GENERAL FIBONACCI SHIFT FORMULAE

FRANK J. D. TRUMPER
Seismograph Services, Ltd., Holwood, Keston, Kent, England

The reader is probably aware of such formulae as: [1]

(a) $$F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$$

(b) $$F_n = (-1)^{k-1}(F_kF_{n+k+1} - F_{k+1}F_{n+k}) \; .$$

The object of this paper is to prove more general shift formulae. For this purpose, the following notation will be used:

$$F_nF_m = (n:m), \qquad F_n = (n:1) = (1:n), \quad \text{etc.}$$

Let

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} \; ,$$

then $\alpha\beta = -1$. Now,

$$F_n = (n:1) = \frac{\alpha^n - \beta^n}{\sqrt{5}} \; .$$

So

$$(n:m) = \frac{1}{5}(\alpha^{n+m} + \beta^{n+m} - \alpha^n\beta^m - \alpha^m\beta^n) \; .$$

Replace n by x, and m by $n + m - x$. Then,

$$(x:n + m - x) = \frac{1}{5}(\alpha^{n+m} + \beta^{n+m} - \alpha^x\beta^{n+m-x} - \alpha^{n+m-x}\beta^x)$$

$$(n:m) - (x:n + m - x) = \frac{1}{5}(\alpha^x\beta^{n+m-x} + \alpha^{n+m-x}\beta^x - \alpha^n\beta^m - \alpha^m\beta^n)$$

(c) $$= \frac{(\alpha\beta)^m}{5}(\alpha^{x-m}\beta^{n-x} + \alpha^{n-x}\beta^{x-m} - \alpha^{n-m} - \beta^{n-m})$$

$$= \frac{(-1)^{m+1}}{5}(\alpha^{n-m} + \beta^{n-m} - \alpha^{x-m}\beta^{n-x} - \alpha^{n-x}\beta^{x-m})$$

$$= (-1)^{m+1}(x - m:n - x) \; .$$

If x is replaced by $-x$, we get

(d) $$(n:m) = (-x:n + m + x) + (-1)^{m+1}(-x - m:n + x) .$$

Equations (c) and (d) may be combined into the one formula:

(1) $$(n:m) = (\pm x:n + m \mp x) + (-1)^{m+1}(\pm x - m:n \mp x) .$$

By the same method, the following formulae may be proved:

(2) $$(n:m) = (\pm x + m:n \mp x) + (-1)^{m+1}(\pm x:n - m \mp x)$$

(3) $$(n \pm x:m) = (\pm x:n + m) + (-1)^{m+1}(\pm x - m:n)$$

(4) $$(n \pm x:m) = (\pm x + m:n) + (-1)^{m+1}(\pm x:n - m)$$

(5) $$(n[x \pm 1]:m) = (\pm n:nx + m) + (-1)^{m+1}(\pm n - m:nx)$$

(6) $$(n[x \pm 1]:m) = (\pm n + m:nx) + (-1)^{m+1}(\pm n:nx - m)$$

(7) $$(n:m) = (-1)^{x+1}[(x:n + m + x) - (m + x:n + x)]$$

Clearly, Equations (a) and (b) are special cases of Equations (2) and (7), respectively.

REFERENCE

1. Brother Alfred Brousseau, An Introduction to Fibonacci Discovery, P. 46, page 11, and P. 48, page 12.

ERRATA

Please make the following corrections on "A Generalized Fibonacci Numeration," by E. Zeckendorf, appearing on pp. 365-372 of the October, 1972 Fibonacci Quarterly:

p. 366, line 15: Please change the third word from: sequencex to sequences.

p. 368, line 13: Read: $t_{6,3,-1,-4,-6} = F_5 + F_2 + F_{-2} + F_{-5} + F_{-7}$.

line 8 from bottom: Underscore: symmetric pairs .

line 6 from bottom: Read: metric pairs may join up into one symmetric group (e.g., $t_{6,0,-6}$, $t_{8,4,-4,-8}$) .

line 4 from bottom: Underscore: saturated symmetric groups .

MORE HIDDEN HEXAGON SQUARES

CARL F. MOORE
Tacoma, Washington

In [1], Hoggatt and Hansell prove the following remarkable result.

Theorem 1. Let $\binom{m}{n}$ be such that $0 < n < m$ and $2 \le m$. Then the product of the six binomial coefficients surrounding $\binom{m}{n}$ is a perfect integral square.

In this paper, we show that this theorem is a special case of a more general result. In particular, we prove the following theorem.

Theorem 2. Let H_j, for j odd, be a hexagon of entries from Pascal's triangle with $j + 1$ entries per side and with the sides lying along main diagonal and horizontal rows of the triangle. Then the product of the entries forming H_j is an integral square.

Proof. Let j be a positive odd integer and let n and r be integers with $1 \le n - j$, $j \le r \le n$, and $0 \le r \le n - j$. If H_j is centered at $\binom{n}{r}$, then it can be displayed in the following way where we label the sides I, $\cdots$, VI.

$$\binom{n-j}{r-j}\binom{n-j}{r-j+1}\cdots\binom{n-j}{r-1}\binom{n-j}{r}$$

$$\binom{n-j+1}{r-j} \qquad\qquad \mathrm{I} \qquad\qquad \binom{n-j+1}{r+1}$$

$$\cdot^{\cdot^{\cdot}} \quad \mathrm{VI} \qquad\qquad\qquad\qquad \mathrm{II} \quad \ddots$$

$$\binom{n-1}{r-j} \qquad\qquad\qquad\qquad\qquad\qquad \binom{n-1}{r+j-1}$$

$$\binom{n}{r-j} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \binom{n}{r+j}$$

$$\binom{n+1}{r-j+1} \qquad\qquad\qquad\qquad\qquad\qquad \binom{n+1}{r+j}$$

$$\ddots \quad \mathrm{V} \qquad\qquad\qquad\qquad \mathrm{III} \quad \cdot^{\cdot^{\cdot}}$$

$$\binom{n+j-1}{r-1} \qquad\qquad \mathrm{IV} \qquad\qquad \binom{n+j-1}{r+j}$$

$$\binom{n+j}{n}\binom{n+j}{r+1}\cdots\binom{n+j}{r+j-1}\binom{n+j}{r+j}$$

Of course, each coefficient is of the form $\frac{a}{bc}$ where a, b, and c are the appropriate factorials. We prove that the desired product is a square by proving that the product of the a's is a square and similarly for the b's and c's. The products of the a's in sides I and IV, respectively, are clearly $[(n - j)!]^{j+1}$ and $[(n + j)!]^{j+1}$ and both are squares since j is odd. Also, the product of the a's in II, III, V, and VI and not in I or IV is clearly

$$[(n - j + 1)!(n - j + 2)! \cdots (n + j - 1)!]^2 .$$

Similarly, the products of the b's in III and VI, respectively, are $[(r + j)!]^{j+1}$ and $[(r - j)!]^{j+1}$, and the product of the b's in I, II, IV and V and not in III and VI is

$$[(r - j + 1)!(r - j + 2)! \cdots (r + j - 1)!]^2 .$$

Finally, the products of the c's in II and V, respectively, are $[(n - r - j)!]^{j+1}$ and $[(n - r + j)!]^{j+1}$ and the product of the c's in I, III, IV and VI and not in II and V is

$$[(n - j - r + 1)!(n - j - r + 2)! \cdots (n + j - r - 1)!]^2 .$$

Therefore, the product of the coefficients in question is a rational square and, since the product is a product of integers, it is also an integral square as claimed.

REFERENCE

1. V. E. Hoggatt, Jr., and Walter Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9 (1971), pp. 120, 133.

THE BALMER SERIES AND THE FIBONACCI NUMBERS

J. WLODARSKI
Proz-Westhoven, Federal Republic of Germany

In 1885, J. J. Balmer discovered that the wave lengths (λ) of four lines in the hydrogen spectrum (now known as "Balmer Series") can be expressed by the multiplication of a numerical constant $k = 364.5$ nm (1 nm = 1 nanometre = 10^{-9} m) by the simple fractions as follows:

$$656 = \frac{9}{5} \times 364.5 \tag{1}$$

$$486 = \frac{4}{3} \times 364.5 = \frac{16}{12} \times 364.5 \tag{2}$$

$$434 = \frac{25}{21} \times 364.5 \tag{3}$$

$$410 = \frac{9}{8} \times 364.5 = \frac{36}{32} \times 364.5 . \tag{4}$$

By extending both fractions, 4/3 and 9/8, he recognized the successive numerators as the squares 3^2, 4^2, 5^2 and 6^2, and the denominators as the square-differences $3^2 - 2^2$, $4^2 - 2^2$, $5^2 - 2^2$, $6^2 - 2^2$.

From this he developed his famous formula:
[Continued on page 540.]

A POLYNOMIAL WITH GENERALIZED FIBONACCI COEFFICIENTS

BRUCE W. KING
Adirondack Community College, Glens Falls, New York

In Elementary Problem B-135 (this Quarterly, Vol. 6, No. 1, p. 90), L. Carlitz asks readers to show that

$$\sum_{k=0}^{n-1} F_k 2^{n-k-1} = 2^n - F_{n+2} , \tag{1}$$

and that

$$\sum_{k=0}^{n-1} L_k 2^{n-k-1} = 3(2^n) - L_{n+2} . \tag{2}$$

The problem invites generalization in at least two ways. It is natural to investigate

$$\sum_{k=0}^{n-1} T_k 2^{n-k-1} ,$$

where T_k is the generalized Fibonacci sequence with $T_1 = a$ and $T_2 = b$. It is not difficult to show by induction that

$$\sum_{k=0}^{n-1} T_k 2^{n-k-1} = T_2 (2^n) - T_{n+2} . \tag{3}$$

The relations given in (1) and (2) are, thus, a consequence of (3).

A second generalization may be obtained by trying to determine whether anything worthwhile can be said about the polynomial

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} . \tag{4}$$

This seems to be a more difficult problem than that posed by the first generalization, and the rest of this note is devoted to it.

To begin with, evaluating (4) for several values of n suggests that

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} = a \sum_{k=0}^{n-1} F_{k-2} x^{n-k-1} + b \sum_{k=0}^{n-1} F_{k-1} x^{n-k-1} \quad . \tag{5}$$

This can be proved by induction. For $n = 1$, both members of (5) reduce to $b - a = T_0$. (We use $x^0 \equiv 1$ here.) If we now suppose (5) true for some integer $n \geq 1$, then

$$\begin{aligned} \sum_{k=0}^{n} T_k x^{n-k} &= x \sum_{k=0}^{n-1} T_k x^{n-k-1} + T_n \\ &= x \left[a \sum_{k=0}^{n-1} F_{k-2} x^{n-k-1} + b \sum_{k=0}^{n-1} F_{k-1} x^{n-k-1} \right] + T_n \end{aligned}$$

and, since $T_n = aF_{n-2} + bF_{n-1}$,

$$\begin{aligned} \sum_{k=0}^{n} T_k x^{n-k} &= a \sum_{k=0}^{n-1} F_{k-2} x^{n-k} + aF_{n-2} \\ &\qquad + b \sum_{k=0}^{n-1} F_{k-1} x^{n-k} + bF_{n-1} \\ &= a \left[\sum_{k=0}^{n-1} F_{k-2} x^{n-k} + F_{n-2} \right] + b \left[\sum_{k=0}^{n-1} F_{k-1} x^{n-k} + F_{n-1} \right] \\ &= a \sum_{k=0}^{n} F_{k-2} x^{n-k} + b \sum_{k=0}^{n} F_{k-1} x^{n-k} \quad . \end{aligned}$$

This completes the proof of (5). The problem has, thus, been reduced slightly to the problem of evaluating an expression such as

$$\sum_{k=1}^{n} F_k x^{n-k}$$

in closed form, for such a result would lend some significance to the right member of (5).

Let us define

$$f_n(x) = \sum_{k=1}^{n} F_k x^{n-k} = x^n \sum_{k=1}^{n} \frac{F_k}{x^k} .$$

Now, it is known [1, p. 43] that the power series

$$\sum_{k=1}^{\infty} F_k t^{k-1}$$

converges to

$$\frac{1}{1 - t - t^2} .$$

The radius of convergence is

$$\lim_{k\to\infty} \frac{F_k}{F_{k+1}} = \frac{1}{\phi} ,$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

is the Golden Ratio. Thus, for a fixed value of t in the interval of convergence

$$-\frac{\sqrt{5} - 1}{2} < t < \frac{\sqrt{5} - 1}{2} ,$$

it follows that

$$\frac{1}{1 - t - t^2} = \sum_{k=1}^{\infty} F_k t^{k-1} = \sum_{k=1}^{n} F_k t^{k-1} + R_n ,$$

where $R_n \to 0$ as $n \to \infty$. Thus,

$$\sum_{k=1}^{n} F_k t^{k-1} = \frac{1}{1 - t - t^2} - R_n$$

or, what is the same,

$$\sum_{k=1}^{n} F_k t^k = \frac{t}{1 - t - t^2} - tR_n \; .$$

If we now let $t = 1/x$ then, for $x < -\phi$ or $x > \phi$,

$$\sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{\frac{1}{x}}{1 - \frac{1}{x} - \frac{1}{x^2}} - \frac{1}{x} R_n = \frac{x}{x^2 - x - 1} - \frac{1}{x} R_n \, ,$$

and

$$x^n \sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1} R_n \; .$$

We have, therefore,

$$f_n(x) = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1} R_n \; . \tag{6}$$

The problem is thus essentially reduced to finding the remainder R_n in some suitable form. Investigating (6) for the first few values of n suggests that

$$R_n = \frac{F_{n+1} x + F_n}{x^{n-1}(x^2 - x - 1)} \; .$$

This, in turn, suggests that

$$f_n(x) = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1}\left[\frac{F_{n+1} x + F_n}{x^{n-1}(x^2 - x - 1)}\right] \; .$$

That is,

$$\sum_{k=1}^{n} F_k x^{n-k} = \frac{x^{n+1} - F_{n+1} x - F_n}{x^2 - x - 1} \quad \text{for} \quad x \neq \frac{1 \pm \sqrt{5}}{2} \; . \tag{7}$$

We will prove (7) by induction. For $n = 1$, both members reduce to 1. If (7) is true for some integer $n \geq 1$, then

$$\sum_{k=1}^{n+1} F_k x^{n-k+1} = \sum_{k=1}^{n} F_k x^{n-k+1} + F_{n+1}$$

$$= x\sum_{k=1}^{n} F_k x^{n-k} + F_{n+1}$$

$$= x\,\frac{x^{n+1} - F_{n+1}x - F_n}{x^2 - x - 1} + F_{n+1}$$

$$= \frac{x^{n+2} - F_{n+1}x^2 - F_n x + F_{n+1}x^2 - F_{n+1}x - F_{n+1}}{x^2 - x - 1}$$

$$= \frac{x^{n+2} - (F_n + F_{n+1})x - F_{n+1}}{x^2 - x - 1}$$

$$= \frac{x^{n+2} - F_{n+2}x - F_{n+1}}{x^2 - x - 1} .$$

This completes the proof of (7).

Now, returning to the summations in (5),

$$\sum_{k=0}^{n-1} F_{k-2}x^{n-k-1} = F_{-2}x^{n-1} + F_{-1}x^{n-2} + F_0 x^{n-3} + \sum_{k=3}^{n-1} F_{k-2}x^{n-k-1}$$

$$= -x^{n-1} + x^{n-2} + \frac{1}{x^3}\sum_{k=3}^{n-1} F_{k-2}\,x^{n-k+2}$$

$$= -x^{n-1} + x^{n-2}$$
$$+ \frac{1}{x^3}\left[\sum_{k=3}^{n+2} F_{k-2}x^{n-k+2} - F_{n-2}x^2 - F_{n-1}x - F_n\right].$$

Using the change of variable $j = k - 2$ in the summation on the right, we have

$$\sum_{k=0}^{n-1} F_{k-2}x^{n-k-1} = -x^{n-1} + x^{n-2} + \frac{1}{x^3}\sum_{j=1}^{n} F_j x^{n-j}$$
$$- \frac{F_{n-2}x^2 + F_{n-1}x + F_n}{x^3} .$$

After substituting from (7), combining fractions and simplifying, the result is that

$$\sum_{k=0}^{n-1} F_{k-2}\, x^{n-k-1} = \frac{x^n(2 - x) - F_{n-2}x - F_{n-3}}{x^2 - x - 1} . \tag{8}$$

In a similar manner, we can use (7) to show that

$$\sum_{k=0}^{n-1} F_{k-1}\, x^{n-k-1} = \frac{x^n(x - 1) - F_{n-1}x - F_{n-2}}{x^2 - x - 1} . \tag{9}$$

Now substitute (8) and (9) into (5), combine fractions and arrange the numerator in powers of x. The result is

$$\sum_{k=0}^{n-1} T_k\, x^{n-k-1} = \frac{1}{x^2 - x - 1}\Big\{x^n\big[(b - a)x + (2a - b)\big] - \big[aF_{n-2} + bF_{n-1}\big]x - \big[aF_{n-3} + bF_{n-2}\big]\Big\}.$$

Consequently, we have the following generalization from Carlitz' problem:

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} = \frac{(T_0 + T_{-1})x^n - T_n x - T_{n-1}}{x^2 - x - 1} . \tag{10}$$

It is not difficult to see that (10) reduces to (3) when $x = 2$. Other results of interest can be obtained by letting $x = \pm 1$ in (10).

REFERENCE

1. Brother Alfred Brousseau, An Introduction to Fibonacci Discovery, The Fibonacci Association, 1965.

ANOTHER PROOF FOR A CONTINUED FRACTION IDENTITY

FURIO ALBERTI
University of Illinois, Chicago, Illinois

Denote the convergents of the continued fraction (Pringsheim's notation [2]),

$$\left[0;\ a_n/b_n\right]_{n=1}$$

by P_n/Q_n, $n = 0, 1, 2, \cdots$, where $P_0/Q_0 = 0/1$. Denote the convergents of the "cut off" continued fraction

$$\left[0;\ a_n/b_n\right]_{n=m+1}$$

by $P_{m,k}/Q_{m,k}$, where $P_{m,o}/Q_{m,o} = 0$, $P_{m,1}/Q_{m,1} = a_{m+1}/b_{m+1}$, etc. Now,

$$P_{m+k} = \left|\begin{array}{cccccc|ccccc}
0 & -1 & 0 & 0 & \cdots & & & & & & 0 \\
a_1 & b_1 & -1 & 0 & \cdots & & & & & & 0 \\
0 & a_2 & b_2 & -1 & \cdots & & & & & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & a_m & b_m & -1 & 0 & & & \\
\hline
 & & & & 0 & a_{m+1} & b_{m+1} & -1 & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & & & & & & a_{m+k} & b_{m+k}
\end{array}\right|_{(m+k+1)} .$$

LaPlace's expansion applied to the last k columns gives

$$P_{m+k} = P_m Q_{m,k} - \left|\begin{array}{cccccc}
0 & -1 & & & & \\
a_1 & b_1 & -1 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & a_{m-1} & b_{m-1} & -1 \\
 & & & & & a_{m+1}
\end{array}\right| \left|\begin{array}{ccccc}
-1 & & & & \\
a_{m+2} & b_{m+2} & -1 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
 & & a_{m+k-1} & b_{m+k-1} & -1 \\
 & & & a_{m+k} & b_{m+1}
\end{array}\right|_{(k)}$$

or

$$P_{m+k} = P_m Q_{m,k} + a_{m+1} P_{m-1} \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & * & * & \cdots & * \\ 0 & a_{m+2} & b_{m+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & \cdots\cdots & & b_{m+k} \end{vmatrix}_{(k+1)}$$

where the places denoted by the asterisks may be filled in by any quantities desired. Hence, a_{m+1} is introduced in this last determinant by choosing the second row to be

$$a_{m+1}, \quad b_{m+1}, \quad -1, \quad 0, \quad 0, \quad \cdots, \quad 0$$

and get

$$P_{m+k} = Q_{m,k} P_m + P_{m,k} P_{m-1} \ ,$$

Similarly,

$$Q_{m+k} = Q_{m,k} Q_m + P_{m,k} Q_{m-1} \ .$$

These results may be derived without the use of determinants [1, p. 40] but the procedure is rather lengthy.

REFERENCES

1. Alexey N. Khovanskii, "The Applications of Continued Fractions," translated to English by Peter Wynn, P. Noordhoff, Ltd., Groningen: The Netherlands, 1963.

2. A. Pringsheim, "Ueber die Convergence unendlicher Kettenbruche," Sitzungsber. der Math. Phys. Klasse der Kgl. Bayer. Akad. Wiss., Munchen 28 (1898), pp. 295-324.

ON THE PERIODICITY OF THE TERMINAL DIGITS IN THE FIBONACCI SEQUENCE

DANIEL LANCE HERRICK
Barrington College, Barrington, Rhode Island

In the Fibonacci Quarterly, Vol. 1, No. 2, page 84, Stephen P. Geller reported on a computation (using an IBM 1620) in which he established the period of the Fibonacci numbers modulo 10^n for $n = 1, 2, 3, 4, 5, 6$. For example, the last digit of the decimal numeral for F_k is periodic with period 60, and the last six digits are periodic with period 1,500,000. Mr. Geller closed his report by saying, "There does not yet seem to be any way of guessing the next period," and expresses a hope that a clever computer program could be designed for skipping part of the sequence. And Mr. Geller and R. B. Wallace proposed the finding of an expression for these periods as Problem B15.

In the Quarterly, Vol. 1, No. 4, page 21, Dov Jarden, with all of the scorn of the theoretician for the empiricist, brings out the big guns and batters the problem to pieces, showing that F_k is periodic modulo 10^n with period $15 \cdot 10^{n-1}$ if $n \geq 3$, for $n = 1, 2$ the periods are 60 and 300.

And in the Quarterly, Vol. 2, No. 3, page 211, Richard L. Heimer reported on a calculation examining the same problem in numerals of radix $2, 3, 4, 5, \cdots, 16$. (In his article he does not mention a machine and probably did the calculation by hand.) He writes that his interest was aroused by the eccentricity of the first two periods for decimal numerals.

At the same time as I recently read these articles, I stumbled on the big guns necessary to almost completely reduce the problem, "What is the period of the last j digits of the numeral of radix n of F_k, the k^{th} term in the Fibonacci Sequence?," to a routine computation. (I say almost completely because, for example, $n = 241$ would require extended calculations with large numbers or the use of tables that I don't have available.) The problem is equivalent to:

What is the period of the Fibonacci sequence modulo n^j?

Definition 1. The period of the Fibonacci sequence modulo m, which we write $P(m)$, is the smallest natural number k such that $F_{n+k} \equiv F_n \pmod{m}$ for every natural number n.

We start the subscripts of the Fibonacci sequence in the usual place; that is, $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

All of the theorems necessary to solve this problem have been proven already. We will quote them here as we develop the need for them and close the paper by commenting on where proofs can be found.

Theorem 1. F_k is periodic modulo m for every natural number $m > 1$.

Hence, there is a solution.

To solve the problem for all natural numbers n, it will suffice to solve it for prime numbers, for

Theorem 2. If $m = p_1^{j_1} \cdot p_2^{j_2} \cdot \cdots \cdot p_i^{j_i}$ where the p's are distinct primes, then

$$P(m) = \mathrm{LCM}(P(p_1^{j_1}), \cdots, P(p_i^{j_i}))$$

and we can find the period modulo m if we know the periods of the powers of the prime factors of m. We need one more technical term to talk easily about the problem:

Definition 2. If p is a prime number, the rank of apparition of p, R(p), is the subscript of the first Fibonacci number divisible by p. That is, R(1) is the least natural number k such that p/F_k.

There is a reasonably nice relationship between R(p) and P(p):

Theorem 3. If $p > 2$ is prime,

$$\frac{P(p)}{R(p)} = \begin{cases} 1 & \text{if } R(p) \equiv 2 \pmod 4 \\ 2 & \text{if } R(p) \equiv 0 \pmod 4 \\ 4 & \text{if } R(p) \equiv \pm 1 \pmod 4 \end{cases} .$$

Thus, if we can find the rank of p, we have the period. For many primes, we can find this ratio without knowing the rank of p.

Theorem 4.

$$\frac{P(p)}{R(p)} \begin{cases} = 1 & \text{if } p \equiv 11 \text{ or } 19 \pmod{20} \\ = 2 & \text{if } p \equiv 3 \text{ or } 7 \pmod{20} \\ = 4 & \text{if } p \equiv 13 \text{ or } 17 \pmod{20} \\ \neq 2 & \text{if } p \equiv 21 \text{ or } 29 \pmod{40} \end{cases} .$$

There is a limit to the amount of work involved in finding R(p).

Theorem 5.

$$R(p) \mid (p - 1) \text{ if } p \equiv \pm 1 \pmod{10}$$

$$R(p) \mid (p + 1) \text{ if } p \equiv \pm 3 \pmod{10}$$

so that checking somewhat fewer than $p/2$ Fibonacci numbers is guaranteed to find the first Fibonacci number divisible by p.

Theorem 6. If $P(p^2) \neq P(p)$ then $P(p^j) = p^{j-1}P(p)$.

Thus, subject to a rather odd condition, if we know P(p) we know $P(p^j)$. So far as I know, neither has $P(p^2) \neq P(p)$ been proved nor has a counter-example been found. Just in case, there are theorems to take care of odd situations that might arise:

Theorem 7.

$$\frac{P(p^k)}{R(p^k)} = \frac{P(p)}{R(p)}$$

for prime $p > 2$.

Theorem 8. If t is the largest integer such that $P(p^t) = P(p)$ then

$$P(p^k) = p^{k-t}P(p) \quad \text{for} \quad k > t .$$

Table

t(m) denotes P(m)/R(m); in the last three columns, $n > 2$

m	t(m)	R(m)	P(m)	$t(m^2)$	$R(m^2)$	$P(m^2)$	$t(m^n)$	$R(m^n)$	$P(m^n)$
2	1	3	3	1	6	6	2	$3 \cdot 2^{n-2}$	$3 \cdot 2^{n-1}$
3	2	4	8	2	12	24	2	$4 \cdot 3^{n-1}$	$8 \cdot 3^{n-1}$
4		6	6		12	24		$3 \cdot 2^{2n-2}$	$3 \cdot 2^{2n-1}$
5	4	5	20	4	25	100	4	5^n	$4 \cdot 5^n$
6		12	24		12	24		$3 \cdot 6^{n-2}$	6^{n-1} *
7	2	8	16	2	56	112	2	$8 \cdot 7^{n-1}$	$16 \cdot 7^{n-1}$
8		6	12		48	96		$3 \cdot 2^{3n-2}$	$3 \cdot 2^{3n-1}$
9		12	24		108	216		$4 \cdot 3^{2n-1}$	$8 \cdot 3^{2n-1}$
10		15	60		150	300		$75 \cdot 10^{n-2}$	$15 \cdot 10^{n-1}$
11	1	10	10	1	110	110	1	$10 \cdot 11^{n-1}$	$10 \cdot 11^{n-1}$
12		12	24		12	24		12^{n-1}	$2 \cdot 12^{n-1}$
13	4	7	28	4	91	364	4	$7 \cdot 13^{n-1}$	$28 \cdot 13^{n-1}$
14		24	48		168	336		$21 \cdot 14^{n-2}$	$3 \cdot 14^{n-1}$ **
15		20	40		300	600		$20 \cdot 15^{n-1}$	$40 \cdot 15^{n-1}$
16		12	24		192	384		$3 \cdot 2^{4n-2}$	$3 \cdot 2^{4n-1}$
17	4	9	36	4	153	612	4	$9 \cdot 17^{n-1}$	$36 \cdot 17^{n-1}$
18		12	24		108	216		$27 \cdot 18^{n-2}$	$3 \cdot 18^{n-1}$ †
19	1	18	18	1	342	342	1	$18 \cdot 19^{n-1}$	$18 \cdot 19^{n-1}$
20		30	60		300	600		$15 \cdot 20^{n-1}$	$30 \cdot 20^{n-1}$
21		8	16		168	336		$8 \cdot 21^{n-1}$	$16 \cdot 21^{n-1}$
22		30	30		330	330		$165 \cdot 22^{n-2}$	$15 \cdot 22^{n-1}$
23	2	24	48	2	552	1104	2	$24 \cdot 23^{n-1}$	$48 \cdot 23^{n-1}$
24		12	24		48	96		$2 \cdot 24^{n-1}$	$4 \cdot 24^{n-1}$
25		25	100		625	2500		5^{2n}	$4 \cdot 5^{2n}$
26		21	84		546	1092		$273 \cdot 26^{n-2}$	$21 \cdot 26^{n-1}$ ‡
27		36	72		972	1944		$4 \cdot 3^{3n-1}$	$8 \cdot 3^{3n-1}$
28		24	48		84	168		$3 \cdot 28^{n-1}$	$6 \cdot 28^{n-1}$
29		14	14	1	406	406	1	$14 \cdot 29^{n-1}$	$14 \cdot 29^{n-1}$

*holds for $n > 3$; for $n > 2$, $R(6^n) = 3^{n-1}\,\mathrm{LCM}(2^{n-2}, 4)$ and $P(6^n) = 2R(6^n)$

**holds for $n > 4$; for $n > 2$, $R(14^n) = 3 \cdot 7^{n-1}\,\mathrm{LCM}(8, 2^{n-2})$, and $P(14^n) = 2R(14^n)$

†holds for $n > 3$; for $n > 2$, $R(18^n) = 3^{2n-1}\,\mathrm{LCM}(4, 2^{n-2})$, and $P(18^n) = 2R(18^n)$

‡R holds for $n > 2$; P holds only for $n > 3$, for $n > 2$

$$P(26^n) = 21 \cdot 13^{n-1}\,\mathrm{LCM}(4,\ 2^{n-1})$$

The original problem can, in principle, be solved for any natural number m by, first, using the fundamental theorem of arithmetic to write m as a product of powers of distinct primes,

$$m = p_1^{j_1} \cdot p_2^{j_2} \cdot \cdots \cdot p_i^{j_i} \;;$$

second, finding $R(p_k)$, $1 \le k \le i$, using Theorem 5 to save labor; third, checking whether $R(p_k^2) = R(p_k)$ and using Theorem 3 or 4, Theorem 7 and Theorem 6 or 8 to find

$$P(p_k^{j_k}) \;;$$

and, finally, using Theorem 2 to find $P(m)$. The same algorithm works for $m = n, n^2, n^3, \ldots$.

After learning these strange things, I constructed a table, starting with $m = 2$ because 2 was the natural place to start and going to 28 because my paper had 27 lines and then adding 29 because it seemed a shame to stop when the next entry would be prime.

We can now shed light on the question that aroused Mr. Heimer — why are the first few periods for decimal numerals irregular? The answer appears when we construct

$$\mathrm{LCM}\,(P(2^k),\ P(5^k)) = \mathrm{LCM}\,(3\cdot 2^{k-1},\ 4\cdot 5^{k-1})$$

in which the exponent of 2 does not start to grow until the 2^2 in $P(5^k)$ is used up. The same thing happens when $m = 18$, for example. See the notes for the table.

I suspect that there is not much more to say about the periodicity of the terminal digits of F_k. The matter of the periodicity of F_k modulo p is an interesting one for labor-saving purposes when one is seeking the prime factorizations of large Fibonacci numbers. In order that this article contain all of the elementary machinery for working on this problem, I quote one more theorem.

Theorem 9. If a is a divisor of F_k then a is a divisor of F_{nk} for every natural number n.

In particular F_k/F_{nk} and p/F_k where k is a multiple of $R(p)$.

Theorems 1 and 2 are theorems 1 and 2, respectively, in Wall. Theorems 3, 4, and 5 are Theorem 2, Theorem 4, and Lemma 3, respectively, in Vinson; Theorems 6 and 8 are Theorem 5 in Wall; Theorem 7 is a Corollary of Vinson's Theorem 2, and Theorem 9 is a Corollary of Theorem 3 in Wall.

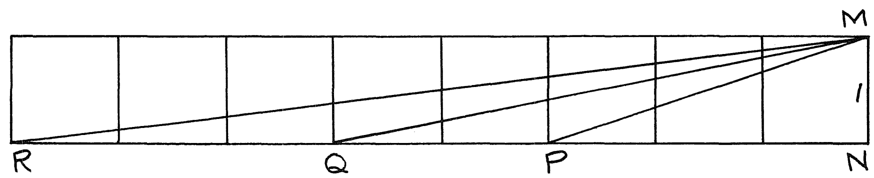
REFERENCES

1. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," The Fibonacci Quarterly, Vol. 1, No. 2, 1963, p. 37.
2. D. D. Wall, "Fibonacci Series Modulo m," The American Math. Monthly, Vol. 67, 1960, pp. 525-532.

GEOMETRIC PROOF OF A RESULT OF LEHMER'S

CHARLES W. TRIGG
San Diego, California

The rectangle in the figure is composed of unit squares: $NP = F_{2n}$, $NQ = F_{2n+1}$, $NR = F_{2n+2}$ and $MN = 1$. It follows that $MP = (F_{2n}^2 + 1)^{1/2}$, $PQ = F_{2n+1} - F_{2n}$, and $PR = F_{2n+2} - F_{2n}$.



Starting with the well-known identity,

$$F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3} = 1,$$

we have

$$F_{2n+1}F_{2n+2} - F_{2n}(F_{2n+1} + F_{2n+2}) + F_{2n}^2 = F_{2n}^2 + 1$$

$$(F_{2n+1} - F_{2n})(F_{2n+2} - F_{2n}) = F_{2n}^2 + 1$$

$$(F_{2n+1} - F_{2n}) : (F_{2n}^2 + 1)^{1/2} = (F_{2n}^2 + 1)^{1/2} : (F_{2n+2} - F_{2n}) .$$

Therefore, triangles QPM and MPR are similar, since the sides including their common angle are proportional. Therefore $\angle MRP = \angle QMP$. It follows that $\angle MPN = \angle QMP + \angle MQP = \angle MRP + \angle MQP$. That is, $\operatorname{arccot} F_{2n} = \operatorname{arccot} F_{2n+1} + \operatorname{arccot} F_{2n+2}$. Thus we write:

$$\begin{aligned}
\operatorname{arccot} 1 &= \operatorname{arccot} 2 + \operatorname{arccot} 3 \\
&= \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 8 \\
&= \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 13 + \operatorname{arccot} 21 \\
&= \cdots \\
&= \sum_{i=1}^{n} \operatorname{arccot} F_{2i+1} + \operatorname{arccot} F_{2n+2} \\
&= \sum_{i=1}^{\infty} \operatorname{arccot} F_{2i+1} \quad .
\end{aligned}$$

This result was announced by D. H. Lehmer [1] in 1936, and proved in different ways by M. A. Heaslet [2] and V. E. Hoggatt, Jr. [3,4]. The first value of arccot 1 above applies to Gardner's three-square problem [5] which has been proven synthetically in 54 ways [6]. Proof of the second value of arccot 1 is asked for in [7].

REFERENCES

1. D. H. Lehmer, Problem Proposal 3801, American Math. Monthly, 43 (Nov., 1936), p. 580.
2. M. A. Heaslet, Problem Solution 3801, American Math. Monthly, 45 (Nov., 1938), pp. 636-637.
3. Verner E. Hoggatt, Jr., and I. D. Ruggles, "A Primer for the Fibonacci Numbers, Part V," The Fibonacci Quarterly, 2 (Feb., 1964), pp. 59-65.
4. Verner E. Hoggatt, Jr., "Fibonacci Trigonometry," The Mathematical Log, Vol. XIII, No. 2, Dec., 1968, p. 3.
5. Martin Gardner, "Mathematical Games," Scientific American, 222, No. 2 (Feb., 1970), pp. 112-114; No. 3 (Mar., 1970), pp. 121-125.
6. Charles W. Trigg, "A Three-Square Geometry Problem," Journal of Recreational Math., April, 1971, pp. 90-99.
7. Alfred E. Neuman, Problem Proposal 243, Pi Mu Epsilon Journal, 6 (Fall, 1970), p. 133.

[Continued from page 526.]

$$\lambda = k\frac{n^2}{n^2 - 2^2} \qquad (n = 3, 4, 5, 6)$$

or in the better known form:

$$\nu = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) ,$$

where ν is the frequency and R the "Rydberg's constant."

It may be of interest to note that all denominators of the simple fractions used by Balmer for deriving his formula, i.e., 3, 5, 8 and 21, are Fibonacci numbers.

FIBONACCI AND APOLLONIUS

WALTER W. HORNER
Pittsburgh, Pennsylvania

Apollonius proposed the problem: Given three fixed circles, to find a circle which touches all of them. In general, there are eight solutions. Obviously, if the given circles are mutually tangent, the number of solutions is reduced to two. This case is a favorite with problemists for creating puzzlers and formulas have been found for their solution. In this note, we shall consider only the case where the given circles are mutually tangent and have their centers on the vertices of a Pythagorean triangle. The purpose of this note is to point out a relation between these five circles and any four consecutive Fibonacci numbers. Let r_1, r_2, r_3 denote the given radii; R and r denote the required radii; and F_n, F_{n+1}, F_{n+2}, F_{n+3} any four consecutive Fibonacci numbers. Assume $r_1 < r_2 < r_3$ and $R > r$.

For convenience in computation, we shall denote our Fibonacci numbers by a, b, c, d. Then using b, c as generators, we get the Pythagorean triplets:

$$c^2 - b^2; \qquad 2bc; \qquad c^2 + b^2.$$

Then by the condition of our problem, we get

$$\begin{aligned} r_1 + r_2 &= c^2 - b^2 \\ r_1 + r_3 &= 2bc \\ r_2 + r_3 &= c^2 + b^2 . \end{aligned}$$

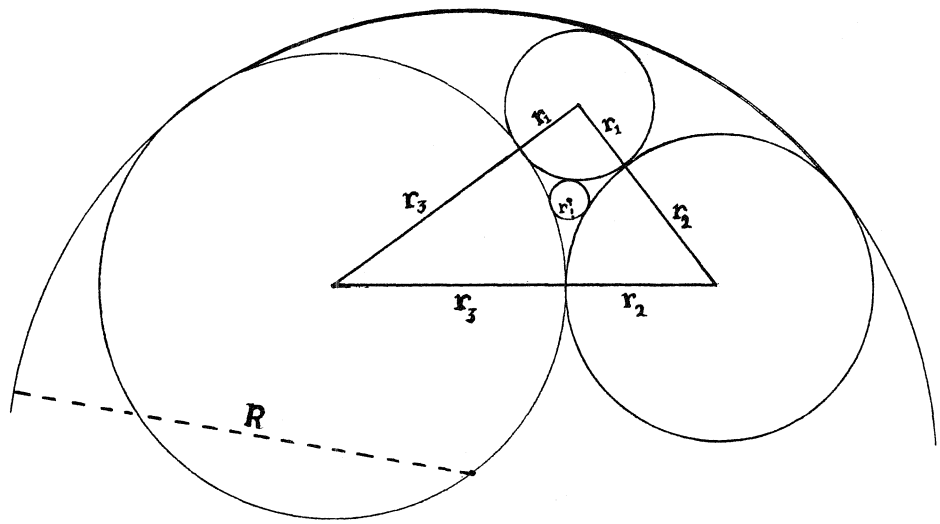
Solving we get

$$\begin{aligned} r_1 &= b(c - b) = ab \\ r_2 &= c(c - b) = ac \\ r_3 &= b(c + b) = bd \\ r_1 + r_2 + r_3 &= c(b + c) = cd . \end{aligned}$$

Then

$$\begin{aligned} r_1 r_2 r_3 &= a^2 b^2 c d \\ r_1 r_2 &= a^2 b c \\ r_1 r_3 &= a b^2 d \\ r_2 r_3 &= a b c d . \end{aligned}$$

The formula below is due to Col. Beard and applies to all cases where the given circles are mutually tangent.



$$R \text{ or } r = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 + 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} .$$

The negative sign gives R (absolute value) and positive sign gives r.

Substituting the values already found for r_1, r_2, r_3 we get

$$R \text{ or } r = \frac{a^2 b^2 cd}{a^2 bc + ab^2 d + abcd \mp 2\sqrt{a^2 b^2 cd \cdot cd}}$$

$$R = \frac{a\,b\,c\,d}{ac + bd - cd} = -c\,d$$

$$r = \frac{a\,b\,c\,d}{4\,cd - ab} .$$

Hence in Fibonacci numbers we have

$$r_1 = F_n F_{n+1}$$

$$r_2 = F_n F_{n+2}$$

$$r_3 = F_{n+1} F_{n+3}$$

$$R = F_{n+2} F_{n+3}$$

$$r = \frac{F_n F_{n+1} F_{n+2} F_{n+3}}{4F_{n+2} F_{n+3} - F_n F_{n+1}} .$$

All this holds for Lucas numbers, also.

REFERENCES

1. Kasner and Newman, Mathematics and the Imagination, pp. 13 and 14.
2. Col. R. S. Beard, "A Variation of the Apollonius Problem," Scripta Mathematica, Vol. 21 (1955), pp. 46-47.

A METHOD FOR CONSTRUCTING SINGLY EVEN MAGIC SQUARES

JEROME ROTHSTEIN
The Ohio State University, Columbus, Ohio

In a recent note* we described a method for constructing magic squares of order $n = 2(2m + 1)$ based on systematic alteration of 2×2 blocks of integers substituted for the integers of any odd square of order $2m + 1$. The present note derives a convenient alternative rule starting from a block of four odd squares of order $2m + 1$. Its derivation shows the existence of a very large number of similar rules.

Divide the square of order $n = 2(2m + 1)$, with sum

$$S_n = n(n^2 + 1)/2 = 2S_{2m+1} + 3(2m + 1)^3$$

into four squares of order $2m + 1$. Label them I, II, III, IV as shown in Fig. 1, filling the cells of I with integers of any magic square of order $2m + 1$, filling II with any square of the same order whose integers have each been augmented by $(2m + 1)^2$, likewise III and IV,

I	III
IV	II

Figure 1

where the augmentations are respectively by $2(2m + 1)^2$ and $3(2m+1)^2$, and the unaugmented squares of IV and I are identical, likewise those of II and III. Clearly the column sums each add up to S_n, and this property is not destroyed by interchanges within a column.

The upper $(2m + 1)$ rows sum to $2S_{2m+1} + 2(2m + 1)^3$, while the lower $(2m + 1)$ rows sum to $2S_{2n+1} + 4(2m + 1)^3$. Exchanges within columns which reduce the lower rows by $(2m + 1)^3$ and increase the upper rows by the same amount will thus bring the row sum to S_n. If p interchanges are made between I and IV and q between II and III, all of them in the same row, then the upper row increases by $(3p - q)(2m + 1)^2$, the lower row decreasing by the same amount. Any p and q less than $2m + 1$ satisfying $3p - q = 2m + 1$ will bring the row sum to S_n. For k an integer, positive, zero or negative, and satisfying $-2m + 1 \leq 3k \leq m + 2$ we have $p = m + k$, $q = m + 3k - 1$ as the possible cases. The case $p = m$, $q = m - 1$ is the simplest.

The two diagonal sums differ by $4(2m + 1)^3$, or twice the row difference. As the row sum adjustments are independent of which cells in a row are selected for the $p + q$ inter-

*J. Rothstein, American Math. Monthly, Vol. 67, No. 6, pp. 583-585 (June-July, 1960).

changes, we select them to bring the diagonal sums to S_n. If m diagonal cells of I interchange with the corresponding (non-diagonal) cells of IV, likewise m diagonal cells of IV with the corresponding (non-diagonal) cells of I, and the center cells of I and IV are also interchanged, then the I-II diagonal increases by $(2m + 1)2(2m + 1)^2$ and the III-IV diagonal decreases by the same amount, thus bringing them to S_n. This diagonal correction, which uses only I-IV interchanges, applies only if $p \geq m$. Other rules, involving II-III interchanges also, can easily be worked out.

Figure 2 gives a pictorial representation of a simple rule for $p = m$, with I-IV diagonal correction, illustrated for the case $m = 2$. The numbers assigned to the empty cells of the squares of order $2m + 1$ are left undisturbed. Those assigned to cells with + or - are interchanged with the numbers in the corresponding cells, i.e., the number in cell (i, j) of I exchanges with that in cell (i, j) of IV, likewise II and III. A - label can be moved anywhere in its row (in its square of order $2m + 1$) except to a cell on a diagonal. A + label, except for those in the center cells, which are fixed, can be displaced to the other diagonal position in its row as long as the same number of mobile + labels are on the diagonal of the square of order n as off it (these are still on the diagonals of I and IV, of course). It is understood that when a label moves, the corresponding label moves correspondingly. In Fig. 2, it can be seen that a simple rule can be expressed as follows. After I, II, III, IV have been written down, interchange the center elements and the m columns on the left, with the exception of the center cell of one column, between I and IV. Perform the same interchanges between II and III except that diagonal cells are not interchanged.

+	-					-			
-	+				-				
	-	+				-			
-	+				-				
+	-					-			
+	-					-			
-	+				-				
	-	+				-			
-	+				-				
+	-					-			

Figure 2

THE Z TRANSFORM AND THE FIBONACCI SEQUENCE

WILLIAM L. MATHIS
Texas Tech University, Lubbock, Texas

Definition. The z transform of f(n) is the function

$$\zeta[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}, \qquad |z| > \frac{1}{\rho}$$

where z is a complex variable and ρ is the radius of convergence of the series.

Applying the z transform to the recursion relation

$$f_{n+2} = f_{n+1} + f_n,$$

we obtain

$$\zeta[f_{n+2}] = \zeta[f_{n+1} + f_n] = \zeta[f_{n+1}] + \zeta[f_n] \ .$$

Using the shifting theorem for z transforms,

$$\zeta[f(n+m)] = z^m[F(z) - F_m(z)] \ ,$$

where

$$F_m(z) = \sum_{k=0}^{m-1} f(k) z^{-k} \ ,$$

which yields

$$z^2[F(z) - F_2(z)] = z[F(z) - F_1(z)] + F(z)$$

and

$$(z^2 - z - 1)F(z) = z^2 F_2(z) - zF_1(z) \ .$$

Hence

$$F(z) = \frac{z^2[f(0) - f(1)z^{-1}] - z[f(0)]}{z^2 - z - 1} ,$$

where

$$z^2 - z - 1 \neq 0 .$$

Since $f_0 = 0$ and $f_1 = 1$, we have

$$F(z) = \frac{z}{z^2 - z - 1} .$$

$F(z)$ is a Laurent series. Therefore, we can multiply $F(z)$ by z^{n-1} and integrate it around a circle for which $|z| > R$. This gives

$$\int_\Gamma F(z)z^{n-1}dz = 2\pi i f(n)$$

or

$$f(n) = \frac{1}{2\pi i} \int_\Gamma F(z)z^{n-1}dz = \Sigma \text{ Residues of } F(z)z^{n-1} .$$

Hence

$$f(n) = \Sigma \text{ Residues} \left[\frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}\right] z^{n-1}$$

$$= \lim_{z \to \frac{1+\sqrt{5}}{2}} \left[\frac{z^n}{z - \frac{1-\sqrt{5}}{2}}\right] + \lim_{z \to \frac{1-\sqrt{5}}{2}} \left[\frac{z^n}{z - \frac{1+\sqrt{5}}{2}}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^n \Big/ \sqrt{5} - \left(\frac{1-\sqrt{5}}{2}\right)^n \Big/ \sqrt{5} .$$

Therefore

$$f(n) = (\alpha^n - \beta^n)/\sqrt{5} ,$$

where

$$\alpha^n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

and

$$\beta^n = \left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

which is Binet's formula.

ON GENERALIZED FIBONACCI QUARTERNIONS

M. N. S. SWAMY
Nova Scotia Technical College, Halifax, Canada

Horadam [1] defined and studied in detail the generalized Fibonacci sequence defined by

(1) $$H_n = H_{n-1} + H_{n-2} \quad (n \geq 3), \quad \text{with} \quad H_1 = p, \quad H_2 = p + q,$$

p and q being arbitrary integers. In a later article [2], he defined Fibonacci and generalized Fibonacci quaternions as follows, and established a few relations for these quaternions:

(2) $$P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}$$

(3) $$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

where

(4) $$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$

and F_n is the n^{th} Fibonacci number. He also defined the conjugate quaternion as

(5) $$\overline{P}_n = H_n - iH_{n+1} - jH_{n+2} - kH_{n+3}$$

and $\overline{Q}_n$ in a similar way.

We shall now establish a few interesting relations for these quaternions. Let R_n be the quaternion for the generalized sequence M_n defined by

(6) $$M_n = M_{n-1} + M_{n-2} \quad (n \geq 3), \quad \text{with} \quad M_1 = r, \quad M_2 = r + s.$$

Then from (2) and (5),

(7) $$\overline{P}_n = 2H_n - P_n.$$

Also,

(8) $$\overline{R}_n = 2M_n - R_n.$$

Hence

$$P_n\overline{R}_n - \overline{P}_n R_n = 2(M_n P_n - H_n R_n) . \tag{9}$$

Similarly, the following results may be obtained:

$$P_n\overline{R}_n + \overline{P}_n R_n = 2(2M_n P_n + 2H_n R_n - P_n R_n)$$

$$P_n R_n - \overline{P}_n\overline{R}_n = 2(H_n R_n - 2H_n M_n + M_n P_n)$$

$$P_n\overline{R}_n + P_n\overline{R}_n = \overline{R}_n P_n + \overline{P}_n R_n$$

$$P_n\overline{R}_n - \overline{P}_n R_n = \overline{R}_n P_n - R_n\overline{P}_n = 2(M_n P_n - H_n R_n)$$

$$P_n\overline{R}_n - \overline{R}_n P_n = \overline{P}_n R_n - R_n\overline{P}_n = R_n P_n - P_n R_n .$$

It may also be seen that $P_n R_n \neq R_n P_n$ unless $P_n = R_n$, whereas,

$$P_n\overline{P}_n = \overline{P}_n P_n = 2H_n P_n - P_n^2 . \tag{10}$$

Some of these results have been obtained earlier [3] for P_n and Q_n, which may be deduced by assuming $r = 1$, $s = 0$ in which case $M_n = F_n$ and $R_n = Q_n$. Now consider

$$\begin{aligned} F_{m+1}P_{n+1} + F_m P_n & \\ &= (F_{m+1}H_{n+1} + F_m H_n) + i(F_{m+1}H_{n+2} + F_m H_{n+1}) \\ &\quad + j(F_{m+1}H_{n+3} + F_m H_{n+2}) + k(F_{m+1}H_{n+4} + F_m H_{n+3}). \end{aligned}$$

It is also known [1] that

$$H_{m+n+1} = F_{m+1}H_{n+1} + F_m H_n = F_{n+1}H_{m+1} + F_n H_m . \tag{11}$$

Hence we have

$$\begin{aligned} F_{m+1}P_{n+1} + F_m P_n &= H_{m+n+1} + iH_{m+n+2} + jH_{m+n+3} + kH_{m+n+4} \\ &= P_{m+n+1} . \end{aligned}$$

Thus,

$$P_{m+n+1} = F_{m+1}P_{n+1} + F_m P_n = F_{n+1}P_{m+1} + F_n P_m . \tag{12}$$

Also

$$P_{2n+1} = F_{n+1}P_{n+1} + F_n P_n \tag{13}$$

and

(14) $$P_{2n} = F_{n+1}P_n + F_nP_{n-1} = F_nP_{n+1} + F_{n-1}P_n .$$

It may also be verified that

(15) $$P_n\overline{P}_n = \overline{P}_nP_n = 3(2p - q)H_{2n+3} - (p^2 - pq - q^2)F_{2n+3} ,$$

where use has been made of the relation [1]

(16) $$H_{n+1} = qF_n + pF_{n+1} .$$

Hence from (15) and (16),

(17) $$\begin{aligned} P_n\overline{P}_n = \overline{P}_nP_n &= 3(2pq - q^2)F_{2n+2} + (p^2 + q^2)F_{2n+3} \\ &= 3(p^2F_{2n+3} + 2pqF_{2n+2} + q^2F_{2n+1}) . \end{aligned}$$

Hence

(18) $$P_n\overline{P}_n + P_{n-1}\overline{P}_{n-1} = 3(p^2L_{2n+2} + 2pqL_{2n+1} + q^2L_{2n}) .$$

Also from (12) we have

$$P_n^2 + P_{n-1}^2 = 2(H_nP_n + H_{n-1}P_{n-1}) - (P_n\overline{P}_n + P_{n-1}\overline{P}_{n-1}) .$$

Using (13) and (21) we get

(19) $$P_n^2 + P_{n-1}^2 = 2P_{2n-1} - 3(p^2L_{2n+2} + 2pqL_{2n+1} + q^2L_{2n}) .$$

If $p = 1$, $q = 0$ then we have the Fibonacci sequence F_n and the corresponding quarternion Q_n for which we may write the following results:

(20) $$Q_n\overline{Q}_n = \overline{Q}_nQ_n = 3F_{2n+3}$$

(21) $$Q_n\overline{Q}_n + Q_{n-1}\overline{Q}_{n-1} = 3L_{2n+2}$$

(22) $$Q_n^2 + Q_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2} .$$

Similar results may be obtained for the Lucas numbers and its quarternion by letting $p = 1$ and $q = 2$ in the various results derived in this article. Also, many other interesting results for these quarternions P_n and M_n may be obtained.

REFERENCES

1. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly, 68 (1961), pp. 455-459.
2. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quarternions," Amer. Math. Monthly, 70 (1963), pp. 289-291.
3. M. R. Iyer, "A Note on Fibonacci Quarternions," to be published.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

Definitions. The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS PROPOSED IN THIS ISSUE

B-268 Proposed by Warren Cheves, Littleton, North Carolina.

Define a sequence of complex numbers $\{C_n\}$, $n = 1, 2, \cdots$, where $C_n = F_n + iF_{n+1}$. Let the conjugate of C_n be $\overline{C}_n = F_n - iF_{n+1}$. Prove

(a) $C_n\overline{C}_n = F_{2n+1}$
(b) $C_n\overline{C}_{n+1} = F_{2n+2} + (-1)^n i$.

B-269 Proposed by Warren Cheves, Littleton, North Carolina.

Let Q be the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of Q are α and β, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Since the eigenvalues of Q are distinct, we know that Q is similar to a diagonal matrix A. Show that A is either

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

B-270 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is odd,

$$L_k \mid [F_{(n+2)k} - F_{nk}] \ .$$

B-271 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refutue the following: If k is even, $L_k - 2$ is an exact divisor of

(a) $$F_{(n+2)k} + 2F_k - F_{nk}\ ;$$

(b) $F_{(n+2)k} - 2F_{(n+1)k} + F_{nk}$; and

(c) $2[F_{(n+2)k} - F_{(n+1)k} + F_k]$.

B-272 Proposed by Gary G. Ford, Vancouver, British Columbia, Canada.

Find at least some of the sequences $\{y_n\}$ satisfying

$$y_{n+3} + y_n = y_{n+2}\, y_{n+1} .$$

B-273 Proposed by Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, California.

Construct any triangle ΔABC with vertex angle $A = 54^\circ$ and median $\overline{AM}$ to the side opposite A such that $AM = 1$. Now, inscribe ΔXYM in ΔABC so that M is the midpoint of $\overline{BC}$, and X and Y lie between A and B and between A and C, respectively. Find the minimum perimeter possible for the inscribed triangle, ΔXYM.

SOLUTIONS

POLYNOMIALS IN THE Q MATRIX

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let Q be the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the sum of a finite number of matrices chosen from the sequence $Q, Q^2, Q^3, \cdots$. Prove that $b = c$ and $a = b + d$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

It will be sufficient to show for $n = 1, 2, 3, \cdots$ that if

$$Q^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $b = c$ and $a = b + d$. For if each Q^n has this property then the sum of a finite number of terms from the sequence $Q, Q^2, Q^3, \cdots$ will retain the same property.

However, if

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then it is easily shown by induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for $n \geq 1$, and clearly this latter matrix has the required property.

Also solved by Richard Blazej, Wray G. Brady, Paul S. Bruckman, Warren Cheves, C. B. A. Peck, Richard W. Sielaff, Tony Waters, Gregory Wulczyn, David Zeitlin, and the Proposer.

SUMS AND DIFFERENCES OF FIBONACCI SQUARES

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term F_n with $n > 0$ in the sequence $F_0, F_1, F_2, \cdots$ is expressible as $x^2 + y^2$ or $x^2 - y^2$ with x and y terms of the sequence with distinct subscripts.

Solution by David Zeitlin, Minneapolis, Minnesota.

The result follows by noting that $F_{2n} = F^2_{n+1} - F^2_{n-1}$ and $F_{2n-1} = F^2_n + F^2_{n-1}$.

Also solved by Richard Blazej, W. G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Graham Lord, C. B. A. Peck, Gregory Wulczyn, and the Proposer.

AT MOST ONE IS RATIONAL

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational.

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}.$$

Solution by C. B. A. Peck, State College, Pennsylvania.

Since (FQ, Vol. 5, pp. 469-471) sum I is $\sqrt{5}$ times sum II, sum I is irrational if sum II is rational, completing the proof.

Also solved by Paul S. Bruckman and the Proposer.

LUCAS MULTIPLES OF FIBONACCI NUMBERS

B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that m and n are integers with $0 < n < m$ and $F_n | L_m$, prove that n is 1, 2, 3, or 4.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $m = qn + r$ with m, n, and q positive integers and $0 \le r < n$. Since

$$\gcd(F_n, F_{n+1}) = 1 \quad \text{and} \quad L_m = L_{m-n}F_{n+1} + L_{n-n-1}F_n ,$$

$F_n | L_m$ implies $F_n | L_{m-n}$. Continuing this way, one shows that $F_n | L_m$ implies $F_n | L_{m-qn}$, i.e., $F_n | L_r$. Then $F_n < L_r$, $r < n$, and $n > 4$ imply $r = n - 1$ since it is easily shown by induction that $F_n > L_r$ for $n > 4$ and $r < n - 1$. Since $L_{n-1} = F_n + F_{n-2}$, $F_n | L_{n-1}$ implies $F_n | F_{n-2}$. This is impossible for $n > 2$, completing the proof.

SOME CASES OF $n|F_n$

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $h = 5^k$. Prove that $h|F_h$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Proof by induction;

Let $h|F_h$ for $k = n$, and note that for $n = 1$, $h = 5|F_5 = 5$. The factorization $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ with $x = \alpha^h$ and $y = \beta^h$ yields

$$F_{5h} = F_h(L_{4h} - L_{2h} + 1) .$$

But $L_{4h} - L_{2h} + 1 = (5F_{2h}^2 + 2) - (5F_h^2 - 2) + 1 \equiv 0 \pmod 5$. ($I_{16}$, I_{17}, p. 59 of Hoggatt's book). Hence F_{5h} is divisible by $5h$ if F_h is divisible by h, which completes the induction.

Also solved by W. G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Gregory Wulczyn, and the Proposer.

EXAMPLES OF $n|L_n$

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $g = 2\cdot 3^k$. Prove that $g|L_g$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

It will be shown that if k is a positive integer and $g = 2\cdot 3^k$ then $(3g)|L_g$ but $(9g)\nmid L_g$, which implies the property asked in B-249.

Proof by induction.

Let the induction hypothesis be for $k = n$, $(3g)|L_g$ but $(9g)\nmid L_g$. For $n = 1$ the hypothesis is true since $3g = 18 = L_6$. From the induction hypothesis $L_g = 3gt$, where 3 and t are coprime. Then

$$\begin{aligned} L_{3g} &= L_g(L_{2g} - 1) && [\text{from } x^3 + y^3 = (x + y)(x^2 + xy + y^2)] \\ &= 3gt(L_g^2 - 3) && (I_{15}, \text{ p. 59, of Hoggatt's book}) \\ &= 9gt(3g^2t^2 - 1) , \end{aligned}$$

which shows that $[3(3g)]|L_{3g}$ but $[9(3g)]\nmid L_{3g}$.

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Gregory Wulczyn, David Zeitlin, and the Proposer.

VOLUME INDEX

SUSTAINING MEMBERS

*H. L. Alder
V. V. Alderman
G. L. Alexanderson
R. H. Anglin
*J. Arkin
L. L. Badii
Leon Bankoff
W. C. Barley
Murray Berg
D. C. Beverage
*Marjorie Bicknell
J. H. Biggs
E. W. Boehne
J. L. Bohnert
M. B. Boisen, Jr.
C. A. Bridger
*Maxey Brooke
*Bro. A. Brousseau
*J. L. Brown, Jr.
C. R. Burton
*P. F. Byrd
L. Carlitz
D. Chakerian
P. V. Charland
Jacques Chone
P. J. Cocuzza
Nanette Cox
Calvin Crabill
T. J. Cullen
D. E. Daykin
M. J. DeLeon
J. E. Desmond
M. H. Diem
N. A. Draim
D. C. Duncan

*Charter Members

Merritt Elmore
J. P. Ercolano
R. S. Erlien
H. W. Eves
R. A. Fairbairn
D. R. Farmer
*H. H. Ferns
D. C. Fielder
E. T. Frankel
H. Gabai
C. L. Gardner
R. M. Giuli
G. H. Glabe
*H. W. Gould
Nicholas Grant
George Greene
R. T. Hansen
V. C. Harris
F. K. Higashihara
*A. P. Hillman
A. W. Hilton
Bruce Hoelter
*V. E. Hoggatt, Jr.
*A. F. Horadam
J. A. H. Hunter
F. R. Jachim
*Dov Jarden
J. H. Jordan
Virginia Kelemen
R. P. Kelisky
*Kenneth Kloss
*D. E. Knuth
Sidney Kravitz
J. Lahr
*George Ledin, Jr.
H. T. Leonard, Jr.

*C. T. Long
J. S. Madachy
*James Maxwell
C. A. B. McArthur
*Sister M. DeSales McNabb
L. P. Meissner
L. R. Morton
Stephen Nytch
Roger O'Connell
P. B. Onderdonk
F. J. Ossiander
Ann Pape
C. A. Pasma
R. J. Pegis
M. M. Risueno
F. G. Rothwell
J. A. Schumacher
H. D. Seielstad
A. G. Shannon
B. B. Sharpe
L. R. Shenton
D. Singmaster
D. C. Stone
C. C. Styles
M. N. S. Swamy
*D. E. Thoro
H. L. Umansky
J. VanWassenhoven
Marcellus Waddill
*L. A. Walker
R. E. Whitney
C. F. Winans
P. A. Willis
David Zeitlin
Charles Ziegenfus

ACADEMIC OR INSTITUTIONAL MEMBERS

THE BAKER STORE
EQUIPMENT COMPANY

DUKE UNIVERSITY
Durham, North Carolina

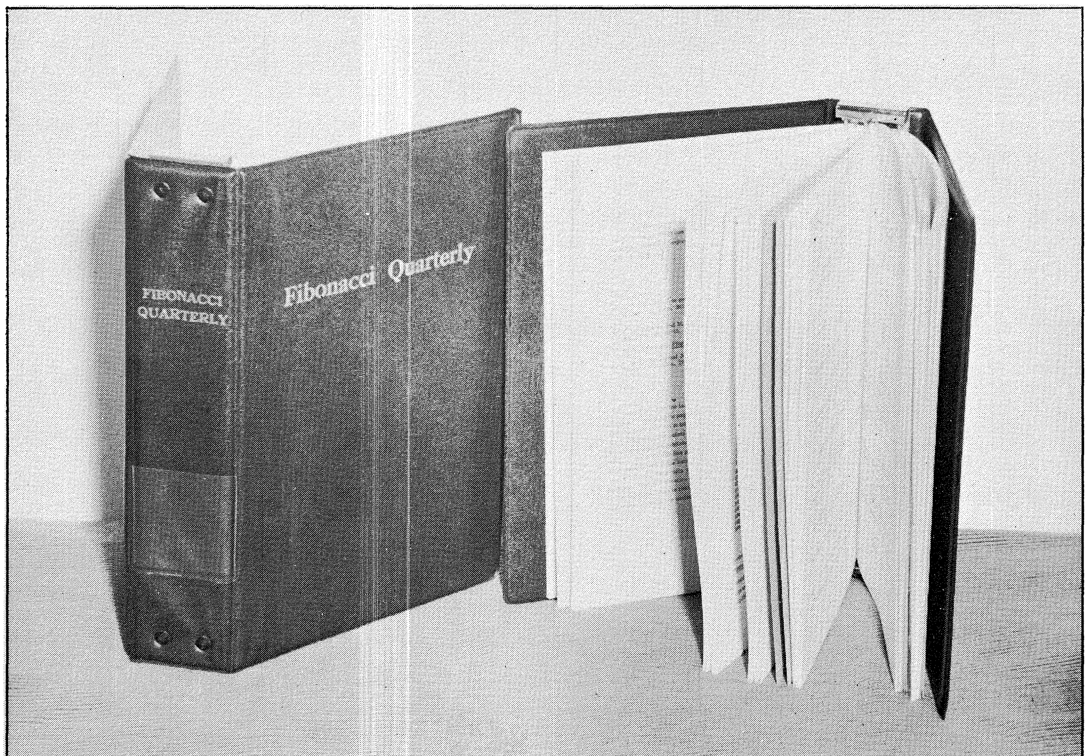
SACRAMENTO STATE COLLEGE
Sacramento, California

SAN JOSE STATE UNIVERSITY
San Jose, California

ST. MARY'S COLLEGE
Moraga, California

WASHINGTON STATE UNIVERSITY
Pullman, Washington

THE CALIFORNIA MATHEMATICS COUNCIL



BINDERS NOW AVAILABLE

The Fibonacci Association is making available a binder which can be used to take care of one volume of the publication at a time. This binder is described as follows by the company producing it:

> "....The binder is made of heavy weight virgin vinyl, electronically sealed over rigid board equipped with a clear label holder extending 2 -3/4" high from the bottom of the backbone, round cornered, fitted with a 1 1/2 " multiple mechanism and 4 heavy wires."

The name, FIBONACCI QUARTERLY, is printed in gold on the front of the binder and the spine. The color of the binder is dark green. There is a small pocket on the spine for holding a tab giving year and volume. These latter will be supplied with each order if the volume or volumes to be bound are indicated.

The price per binder is $3.50 which includes postage (ranging from 50¢ to 80¢ for one binder). The tabs will be sent with the receipt or invoice.

All orders should be sent to: Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif. 94575