DIVISIBILITY PROPERTIES OF GENERALIZED FIBONACCI POLYNOMIALS

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1. INTRODUCTION

In [2], Webb and Parberry study the divisibility properties of the Fibonacci polynomial sequence $\{f_n(x)\}$ defined by the recursion

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x); \qquad f_0(x) = 0, \quad f_1(x) = 1.$$

As one would expect, these polynomials possess many properties of the Fibonacci sequence which, of course, is just the integral sequence $\{f_n(1)\}$. However, a most surprising result is that $f_p(x)$ is irreducible over the ring of integers if and only if p is a prime. In contrast, for the Fibonacci sequence, the condition that n be a prime is necessary but not sufficient for the primality of $f_n(1) = F_n$. For instance, $F_{19} = 4181 = 37 \cdot 113$.

In the present paper, we obtain a series of results including that of Webb and Parberry for the more general but clearly related sequence $\{u_n(x,y)\}$ defined by the recursion

$$u_{n+2}(x,y) = xu_{n+1}(x,y) + yu_n(x,y); \qquad u_0(x,y) = 0, \quad u_1(x,y) = 1.$$

The first few terms of the sequence are as shown in the following table:

n	$u_n(x,y)$
0	0
1	1
2	x
3	$x^2 + y$
4	$x^3 + 2xy$
5	$x^4 + 3x^2y + y^2$
6	$x^5 + 4x^3y + 3xy^2$
7	$x^6 + 5x^4y + 6x^2y^2 + y^3$
8	$x^7 + 6x^5y + 10x^3y^2 + 4xy^3$
...	...

The basic fact that we will need is that $Z[x,y]$, the ring of polynomials over the integers, is a unique factorization domain. Thus, the greatest common divisor of two elements in $Z[x,y]$ is (essentially uniquely) defined.

Useful Property A: if α, β, and γ are in $Z[x,y]$ and $\gamma \mid \alpha\beta$ with γ irreducible, then $\gamma \mid \alpha$ or $\gamma \mid \beta$.

For simplicity, we will frequently use u_n in place of $u_n(x,y)$ and will let

$$\alpha = \alpha(x,y) = \frac{x + \sqrt{x^2 + 4y}}{2}$$

and

$$\beta = \beta(x,y) = \frac{x - \sqrt{x^2 + 4y}}{2} .$$

2. BASIC PROPERTIES OF THE SEQUENCE

Again, as one would expect, many properties of the Fibonacci sequence hold for the present sequence. In particular, the following two results are entirely expected and are easily proved by induction.

Theorem 1. For $n \geq 0$,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

Theorem 2. For $m \geq 0$ and $n \geq 0$,

$$u_{m+n+1} = u_{m+1}u_{n+1} + yu_m u_n .$$

The next result that one would expect is that $(u_n, u_{n+1}) = 1$ for $n \geq 0$. To obtain this we first prove the following lemma.

Lemma 3. For $n > 0$, $(y, u_n) = 1$.

Proof. The assertion is clearly true for $n = 1$ since $u_1 = 1$. Assume that it is true for any fixed integer $k \geq 1$. Then, since

$$u_{k+1} = xu_k + yu_{k-1} ,$$

the assertion is also true for $n = k + 1$, and hence for all $n \geq 1$ as claimed.

We can now prove

Theorem 4. For $n \geq 0$, $(u_n, u_{n+1}) = 1$.

Proof. Again the result is trivially true for $n = 0$ and $n = 1$ since $u_0 = 0$, $u_1 = 1$, and $u_2 = x$. Assume that it is true for $n = k - 1$ where k is any fixed integer, $k \geq 2$, and let $d(x,y) = (u_k, u_{k+1})$. Since

$$u_{k+1} = xu_k + yu_{k-1},$$

this implies that $d(x,y) \mid u_{k-1}y$. But $(d(x,y), y) = 1$ by Lemma 3 and so $d(x,y) \mid u_{k-1}$. But then $d(x,y) \mid 1$ since $(u_{k-1}, u_k) = 1$ and the desired result holds for all $n \geq 0$ as claimed.

Lemma 5. For $n \geq 0$,

$$u_n(x,y) = \sum_{i=0}^{[(n-1)/2]} \binom{n-i-1}{i} x^{n-2i-1} y^i .$$

Proof. We define the empty sum to be zero, so the result holds for $n = 0$. For $n = 1$, the sum reduces to the single term

$$\binom{0}{0} x^0 y^0 = 1 = u_1 .$$

Assume that the claim is true for $n = k - 1$ and $n = k$, where $k \geq 1$ is fixed. Then

$$\begin{aligned} u_{k+1} &= xu_k + yu_{k-1} \\ &= \sum_{i=0}^{[(k-1)/2]} \binom{k-i-1}{i} x^{k-2i} y^i + \sum_{i=0}^{[(k-2)/2]} \binom{k-i-2}{i} x^{k-2i-2} y^{i+1} \\ &= \sum_{i=0}^{[(k-1)/2]} \binom{k-i-1}{i} x^{k-2i} y^i + \sum_{i=0}^{[k/2]} \binom{k-i-1}{i-1} x^{k-2i} y^i \\ &= \sum_{i=0}^{[k/2]} \binom{k-i}{i} x^{k-2i} y^i . \end{aligned}$$

Thus, the result holds for $n = k + 1$ and hence also for all $n \geq 0$ as claimed.

3. THE PRINCIPAL THEOREMS

Theorem 6. For $m \geq 2$, $u_m \mid u_n$ if and only if $m \mid n$.

Proof. Clearly $u_m \mid u_m$. Now suppose that $u_m \mid u_{km}$ where $k \geq 1$ is fixed. Then, using Theorem 2,

$$\begin{aligned} u_{(k+1)m} &= u_{km+m} \\ &= u_{km}u_{m+1} + yu_{km-1}u_m . \end{aligned}$$

But, since $u_m \mid u_{km}$ by the induction assumption, this clearly implies that $u_m \mid u_{(k+1)m}$. Thus, $u_m \mid u_n$ if $m \mid n$.

Now suppose that $m \geq 2$ and that $u_m \mid u_n$. If $m \nmid n$, then there exist integers q and r with $0 < r < m$, such that $n = mq + r$. Again by Theorem 2, we have that

$$\begin{aligned} u_n &= u_{mq+r} \\ &= u_{mq+1}u_r + yu_{mq}u_{r-1} . \end{aligned}$$

Since $u_m \mid u_{mq}$ by the first part of the proof, this implies that $u_m \mid u_{mq+1}u_r$. But, since $(u_{mq}, u_{mq+1}) = 1$ by Theorem 4, this implies that $u_m \mid u_r$ and this is impossible, since u_r is of lower degree than u_m in x. Therefore, $r = 0$ and $m \mid n$ and the proof is complete.

Theorem 7. For $m \geq 0$, $n \geq 0$, $(u_m, u_n) = u_{(m,n)}$.

Proof. Let $d = d(x,y) = (u_m, u_n)$. Then it is immediate from Theorem 6 that $u_{(m,n)} \mid d$.

Now, it is well known that there exist integers r and s with, say, $r > 0$ and $s < 0$, such that

$$(m,n) = rm + sn .$$

Thus, by Theorem 2,

$$\begin{aligned} u_{rm} &= u_{(m,n)+(-s)n} \\ &= u_{(m,n)}u_{-sn+1} + yu_{(m,n)-1}u_{-sn} . \end{aligned}$$

But then $d \mid u_{-sn}$ and $d \mid u_{rm}$ by Theorem 6 and so $d \mid u_{(m,n)}u_{-sn+1}$. But, $(d, u_{-sn+1}) = 1$ by Theorem 4, and so $d \mid u_{(m,n)}$ by Useful Property A from Section 1. Thus, $d = u_{(m,n)}$ as claimed.

Theorem 8. The polynomial $u_n = u_n(x,y)$ is irreducible over the rational field Q if and only if n is a prime.

Proof. From Lemma 5, if we replace y by y^2 we have

$$u_n(x,y^2) = \sum_{i=0}^{[(n-1)/2]} \binom{n-i-1}{i} x^{n-2i-1} y^{2i}$$

which is clearly homogeneous of degree $n-1$. Now it is well known (see, for example, [1, p. 376, problem 5]) that a homogeneous polynomial $f(x,y)$ over a field F is irreducible if and only if the corresponding polynomial $f(x,1)$ is irreducible over F. Since $u_n(x,1)$ is irreducible by Theorem 1 of [2], it follows that $u_n(x,y^2)$ and hence also $u_n(x,y)$ is irreducible over the rational field and thus is irreducible over the integers.

4. SOME ADDITIONAL THEOREMS

For the Fibonacci sequence $\{F_n\}$, for any nonzero integer r there always exists a positive integer m such that $r \mid F_m$. Also, if m is the least positive integer such that $r \mid F_m$, then $r \mid F_n$ if and only if $m \mid n$. It is natural to seek the analogous results for the sequence of Fibonacci polynomials $\{f_n(x)\}$ considered by Webb and Parberry and the generalized sequence $\{u_n(x,y)\}$ considered here. In a sense, the first problem is solved by Webb and Parberry for the sequence of Fibonacci polynomials, since they give explicitly the roots of each such polynomial. However, it is still not clear exactly which polynomials $r(x)$ possess the derived property. On the other hand, it is immediate that the first result mentioned above does not hold for all polynomials $r(x)$. For example, if c is positive, no linear factor $x - c$ can divide any $f_n(x)$ since this would imply that $f_n(c) = 0$, and this is impossible since $f_n(x)$ has only positive coefficients.

Along these lines, we offer the following theorems which, among other things, show that the second property mentioned above does hold without change for $u_n(x,y)$ and hence also for $f_n(x)$. We give this result first.

Theorem 9. Let $r = r(x,y)$ be any polynomial in x and y. If there exists a least positive integer m such that $r \mid u_m$, then $r \mid u_n$ if and only if $m \mid n$.

Proof. By Theorem 6, if $m \mid n$, then $u_m \mid u_n$. Therefore, if $r \mid u_m$ we have by transitivity that $r \mid u_n$. Now suppose that $r \mid u_n$ and yet $m \nmid n$. Then there exist integers q and s with $0 < s < m$ such that $n = mq + s$. Therefore, by Theorem 2,

$$\begin{aligned} u_n &= u_{mq+s} \\ &= u_{mq+1}u_s + yu_{mq}u_{s-1} . \end{aligned}$$

Since $r \mid u_{mq}$ and $r \mid u_n$, it follows that $r \mid u_{mq+1}u_s$. But $(u_{mq}, u_{mq+1}) = 1$ and this implies that $r \mid u_s$. But this violates the minimality condition on m and so the proof is complete.

Theorem 10. For $n \geq 2$,

$$u_n(x,y) = \prod_{k=1}^{n-1}\left(x - 2i\sqrt{y}\cos\frac{k\pi}{n}\right) .$$

Proof. From the proof of Theorem 8, it follows that

$$u_n(x,y^2) = y^{n-1}u_n\left(\frac{x}{y}, 1\right) = y^{n-1}f_n\left(\frac{x}{y}\right) ,$$

where $f_n(x)$ is the n^{th} Fibonacci polynomial mentioned above. Thus,

$$u_n(x,y) = y^{(n-1)/2} f_n(x/\sqrt{y})$$

and it follows from [2, page 462] that

$$f_n(x/\sqrt{y}) = \prod_{k=1}^{n-1}\left(\frac{x}{\sqrt{y}} - 2i\cos\frac{k\pi}{n}\right) .$$

This, with the preceding equation, immediately yields the desired result.

Corollary 10. For $n \geq 2$, n even,

$$u_n(x,y) = x\prod_{k=1}^{(n-2)/2}\left(x^2 + 4y\cos^2\frac{k\pi}{n}\right)$$

and, for n odd,

$$u_n(x,y) = \prod_{k=1}^{(n-1)/2}\left(x^2 + 4y\cos^2\frac{k\pi}{n}\right) .$$

Proof. This is an immediate consequence of Theorem 10, since, for $1 \leq k < n/2$,

$$\cos\frac{k\pi}{n} = -\cos\frac{(n-k)\pi}{n} .$$

It is clear from the preceding theorems that there is a precise correspondence between the polynomial factors of $u_n(x,y)$ and those of $u_n(x,1) = f_n(x)$. Thus, it suffices to consider only those of $f_n(x)$. Also, it is clear that, except for the factor x, the only polynomial factors of $f_n(x)$ with integral coefficients contain only even powers of x. While we are not able to say in every case which even polynomials are factors of some $f_n(x)$ we offer the following partial results.

Theorem 11.

(i) $x \mid f_n(x)$ if and only if n is even.

(ii) $(x^2 + 1) \mid f_n(x)$ if and only if $3 \mid n$.

(iii) $(x^2 + 2) \mid f_n(x)$ if and only if $4 \mid n$.

(iv) $(x^2 + 3) \mid f_n(x)$ if and only if $6 \mid n$.

(v) $(x^2 + c) \nmid f_n(x)$ if $c \neq 1, 2,$ or 3 and c is an integer.

Proof. Since, except for x only, all polynomials with integral coefficients dividing any $f_n(x)$ must be even, the results (i) through (iv) all follow from Theorem 9 with $y = 1$. One has only to observe that $f_2(x)$ is the first Fibonacci polynomial divisible by x, that $f_3(x)$ is the first Fibonacci polynomial divisible by $x^2 + 1$, and so on. Part (v) follows from the fact that $1 \le 4\cos^2\alpha < 4$ for an α in the interval $(0, \pi/2)$.

Theorem 12. Let m be a positive integer and let $N(m)$ denote the number of even polynomials of degree $2m$ and with integral coefficients which divide at least one (and hence infinitely many) members of the sequence $\{f_n(x)\}$. Then

$$N(m) < \prod_{k=1}^{m} \binom{m}{k} 4^k .$$

Proof. Let $f(x)$ be any polynomial counted by $N(m)$. It follows from Corollary 10 with $y = 1$ that

$$f(x) = x^{2m} + a_{m-1}x^{2m-2} + \cdots + a_1x^2 + a_0$$

$$= \prod_{j=1}^{m} (x^2 + \alpha_j) \qquad ,$$

where $\alpha_j = 4\cos^2\beta_j$ with $0 < \beta_j < \pi/2$ for each j. Therefore, $0 < \alpha_j < 4$ for each j. Since a_{m-k} is the k^{th} elementary symmetric function of the α_j's, it follows that

$$0 < a_{m-k} < \binom{m}{k} 4^k$$

and hence that

$$N(m) < \prod_{k=1}^{m} \binom{m}{k} 4^k$$

as claimed.

Of course, the estimate in Theorem 12 is exceedingly crude and can certainly be improved. It is probably too much to expect that we will ever know the exact value of $N(m)$ for every m.

Our final theorem shows that with but one added condition the generalization to $u_n(a,b)$ of the first result mentioned in this section is valid.

Theorem 13. Let r be a positive integer with $(r,b) = 1$. Then there exists m such that $r \mid u_m(a,b)$.

Proof. Consider the sequence $u_n(a,b)$ modulo r. Since there exist precisely r^2 distinct ordered pairs (c,d) modulo r, it is clear that the set of ordered pairs

$$\{(u_0(a,b),\ u_1(a,b)),\ (u_1(a,b),\ u_2(a,b)),\ \cdots,\ (u_{r^2}(a,b),\ u_{r^2+1}(a,b))\}$$

must contain at least two identical pairs modulo r. That is, there exist s and t with $0 \le s < t \le r^2$ such that

$$u_s(a,b) \equiv u_t(a,b) \pmod r$$

and

$$u_{s+1}(a,b) \equiv u_{t+1}(a,b) \pmod r .$$

But

$$bu_{s-1}(a,b) = u_{s+1}(a,b) - au_s(a,b)$$

and

$$bu_{t-1}(a,b) = u_{t+1}(a,b) - au_t(a,b)$$

and this implies that

$$bu_{s-1}(a,b) \equiv bu_{t-1}(a,b) \pmod r .$$

Since $(r,b) = 1$, this yields

$$u_{s-1}(a,b) \equiv u_{t-1}(a,b) \pmod r .$$

Applying this argument repeatedly, we finally obtain

$$0 = u_{s-s}(a,b) \equiv u_{t-s}(a,b) \pmod r$$

so that $r \mid u_{t-s}(a,b)$ and the proof is complete.

REFERENCES

1. S. MacLane and G. Birkhoff, Algebra, The MacMillan Company, New York, N.Y., 1967.
2. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5 (Dec. 1969), pp. 457-463.

GENERATING IDENTITIES FOR PELL TRIPLES

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This paper is modelled after an article by Hansen [1] dealing with identities for Fibonacci and Lucas triples. Free use has been made of the methods of that article, and this paper follows its format closely. It is hoped that seeing Fibonacci methods used in a slightly different context will lead the reader to a deeper understanding of those methods, in addition to the production of some new Pell identities.

The Pell sequence is closely akin to the Fibonacci sequence; it is defined by $P_0 = 0$, $P_1 = 1$, $P_{n+2} = P_n + 2P_{n+1}$. This gives us the sequence 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, $\cdots$. We may also define a Pell analogue of the Lucas sequence: $R_0 = 2$, $R_1 = 2$, $R_{n+2} = R_n + 2R_{n+1}$. It is simple to show that, with these definitions, $P_{n+1} + P_{n-1} = R_n$. Another useful result, easily proved by the usual Fibonacci methods, gives the Pell sequence and its Lucas analogue as functions of their subscripts:

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad R_n = \alpha^n + \beta^n \;,$$

where

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}\,.$$

Note that α and β are roots of the equation $x^2 - 2x - 1 = 0$, and hence $\alpha\beta = -1$ and $\alpha + \beta = 2$.

Using the generating functions of

$$\{P_{n+m}\}_{n=0}^{\infty} \quad \text{and} \quad \{R_{n+m}\}_{n=0}^{\infty}$$

we shall obtain identities for the triples $P_pP_qP_r$, $P_pP_qR_r$, $P_pR_qR_r$, and $R_pR_qR_r$, where p, q, and r are fixed integers.

To derive the desired generating functions we note that, using the Binet form of the Pell numbers,

$$\sum_{n=0}^{\infty} P_{n+m} x^n = \sum_{n=0}^{\infty} \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} x^n$$

$$= \frac{1}{\alpha - \beta} \left(\alpha^m \sum_{n=0}^{\infty} \alpha^n x^n - \beta^m \sum_{n=0}^{\infty} \beta^n x^n \right)$$

(1)

$$= \frac{1}{\alpha - \beta} \left(\alpha^m \frac{1}{1 - \alpha x} - \beta^m \frac{1}{1 - \beta x} \right)$$

$$= \frac{1}{\alpha - \beta} \left(\frac{(\alpha^m - \beta^m) - \alpha\beta(\alpha^{m-1} - \beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)} \right)$$

$$= \frac{P_m - P_{m-1}x}{1 - 2x - x^2} .$$

In a similar fashion we find

(2)
$$\sum_{n=0}^{\infty} R_{n+m} x^n = \frac{R_m + R_{m-1}x}{1 - 2x - x^2} .$$

We now evaluate formulas (1) and (2) for $-2 \le m \le 4$, letting $1 - 2x - x^2 = D$.

$$\sum_{n=0}^{\infty} P_{n-2} x^n = \frac{P_{-2} + P_{-3}x}{D} = \frac{-2 + 5x}{D} ; \qquad \sum_{n=0}^{\infty} R_{n-2} x^n = \frac{R_{-2} + R_{-3}x}{D} = \frac{6 - 14x}{D}$$

$$\sum_{n=0}^{\infty} P_{n-1} x^n = \frac{P_{-1} + P_{-2}x}{D} = \frac{1 - 2x}{D} ; \qquad \sum_{n=0}^{\infty} R_{n-1} x^n = \frac{R_{-1} + R_{-2}x}{D} = \frac{-2 + 6x}{D}$$

$$\sum_{n=0}^{\infty} P_n x^n = \frac{P_0 + P_{-1}x}{D} = \frac{0 + x}{D} ; \qquad \sum_{n=0}^{\infty} R_n x^n = \frac{R_0 + R_{-1}x}{D} = \frac{2 - 2x}{D}$$

$$\sum_{n=0}^{\infty} P_{n+1}x^n = \frac{P_1 + P_0 x}{D} = \frac{1}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+1}x^n = \frac{R_1 + R_0 x}{D} = \frac{2 + 2x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+2}x^n = \frac{P_2 + P_1 x}{D} = \frac{2 + x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+2}x^n = \frac{R_2 + R_1 x}{D} = \frac{6 + 2x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+3}x^n = \frac{P_3 + P_2 x}{D} = \frac{5 + 2x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+3}x^n = \frac{R_3 + R_2 x}{D} = \frac{14 + 6x}{D} \quad ;$$

$$\sum_{n=0}^{\infty} P_{n+4}x^n = \frac{P_4 + P_3 x}{D} = \frac{12 + 5x}{D} \quad ; \qquad \sum_{n=0}^{\infty} R_{n+4}x^n = \frac{R_4 + R_3 x}{D} = \frac{34 + 14x}{D} \quad .$$

Using the fact that two series are equal if and only if the corresponding coefficients are equal, we now find several elementary identities.

Since

$$\frac{2 - 2x}{D} = \frac{1}{D} + \frac{1 - 2x}{D} \quad ,$$

it follows that

$$\sum_{n=0}^{\infty} R_n x^n = \sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} P_{n-1}x^n$$

$$= \sum_{n=0}^{\infty} (P_{n+1} + P_{n-1})x^n$$

and hence

(3) $$R_n = P_{n+1} + P_{n-1} \; ; \qquad n \text{ a whole number} .$$

Using the Binet forms, it is not difficult to show that $P_{-n} = (-1)^{n+1}P_n$ and $R_{-n} = (-1)^n R_n$ for any positive integer n.

We now observe that

$$\begin{aligned} P_{(-n)+1} + P_{(-n)-1} &= P_{-(n-1)} + P_{-n(n+1)} \\ &= (-1)^{(n-1)+1} P_{n-1} + (-1)^{(n+1)+1} P_{n+1} \\ &= (-1)^n (P_{n-1} + P_{n+1}) \\ &= (-1)^n R_n \\ &= R_{-n} \quad . \end{aligned}$$

Hence Eq. (3) holds for all integers n.

We now proceed with some theorems necessary to the development of Pell triples.

Theorem 1. $P_n R_m + P_{n-1} R_{m-1} = R_{m+n-1}$.

Proof. Let m be any fixed integer. Then

$$\begin{aligned} \sum_{n=0}^{\infty} (P_n R_m + P_{n-1} R_{m-1}) x^n &= R_m \sum_{n=0}^{\infty} P_n x^n + R_{m-1} \sum_{n=0}^{\infty} P_{n-1} x^n \\ &= R_m \frac{x}{D} + R_{m-1} \left(\frac{1 - 2x}{D} \right) \\ &= \frac{R_m x + R_{m-1} - 2R_{m-1} x}{D} = \frac{R_{m-1} + R_{m-2} x}{D} \\ &= \sum_{n=0}^{\infty} \frac{R_{n+m-1}}{D} x^n \end{aligned}$$

and, equating summands,

$$P_n R_m + P_{n-1} R_{m-1} = R_{n+m-1} \quad .$$

Theorem 2. $P_n P_m + P_{n-1} P_{m-1} = P_{n+m-1}$.

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} (P_n P_m + P_{n-1} P_{m-1}) x^n &= P_m \sum_{n=0}^{\infty} P_n x^n + P_{m-1} \sum_{n=0}^{\infty} P_{n-1} x^n \\ &= P_m \frac{x}{D} + P_{m-1} \frac{1 - 2x}{D} \\ &= \frac{P_m x + P_{m-1} - 2P_{m-1} x}{D} = \frac{P_{m-2} x + P_{m-1}}{D} \\ &= \sum_{n=0}^{\infty} \frac{P_{n+m-1}}{D} x^n \quad , \end{aligned}$$

and, equating summands,

$$P_n P_m + P_{n-1} P_{m-1} = P_{n+m-1} .$$

<u>Theorem 3.</u> $R_n R_m + R_{n-1} R_{m-1} = R_{n+m} + R_{n+m-2} = 8 P_{n+m-1}$.
<u>Proof.</u> Let m be any fixed integer. Then

$$\sum_{n=0}^{\infty} (R_n R_m + R_{n-1} R_{m-1}) x^n$$

$$= R_m \sum_{n=0}^{\infty} R_n x^n + R_{m-1} \sum_{n=0}^{\infty} R_{n-1} x^n$$

$$= R_m \frac{2 - 2x}{D} + R_{m-1} \frac{-2 + 6x}{D}$$

$$= \frac{2R_m - 2R_m x - 2R_{m-1} + 6R_{m-1} x}{D}$$

$$= \frac{2(R_m - R_{m-1}) + 2(3R_{m-1} - R_m)x}{D}$$

$$= \frac{R_m + R_{m-2} + (2R_{m-1} - 2R_{m-2})x}{D}$$

$$= \frac{R_m + R_{m-2} + (R_{m-1} + R_{m-3})x}{D}$$

$$= \frac{R_m + R_{m-1} x + R_{m-2} + R_{m-3} x}{D}$$

$$= \sum_{n=0}^{\infty} R_{n+m} x^n + \sum_{n=0}^{\infty} R_{n+m-2} x^n$$

$$= \sum_{n=0}^{\infty} (R_{n+m} + R_{n+m-2}) x^n$$

and hence,

$$R_n R_m + R_{n-1} R_{m-1} = R_{n+m} + R_{n+m-2} .$$

Now,

$$
\begin{aligned}
R_{n-1} + R_{n-1} &= (P_{n+2} + P_n) + (P_n + P_{n-2}) \\
&= 2P_{n+1} + 3P_n + P_{n-2} \\
&= 4P_n + 3P_n + 2P_{n-1} + P_{n-2} \\
&= 8P_n \qquad .
\end{aligned}
$$

We now use a partial fractions technique to find the final necessary result:

$$
\begin{aligned}
\frac{(p + qx)}{D}\,\frac{(r + tx)}{D} &= \frac{pr + (pt + qr)x + qtx^2}{D^2} \\
&= \frac{-qt}{D} + \frac{(pr + qt) + (pt + qr - 2qt)x}{D^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{P_m + P_{m-1}x}{D} \cdot \frac{R_s + R_{s-1}x}{D} &= \sum_{n=0}^{\infty} P_{n+m}x^n \cdot \sum_{n=0}^{\infty} R_{n+s}x^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{k+m}R_{n-k+s}x^n
\end{aligned}
\tag{4}
$$

but also, by Eq. (4),

$$
\begin{aligned}
\frac{P_m + P_{m-1}x}{D} \cdot \frac{R_s + R_{s-1}x}{D} &= \frac{-P_{m-1}R_{s-1}}{D} \\
&\quad + \frac{(P_m R_s + P_{m-1}R_{s-1}) + (P_m R_{s-1} + P_{m-1}R_s - 2P_{m-1}R_{s-1})x}{D^2} \\
&= \frac{-P_{m-1}R_{s-1}}{D} + \frac{R_{m+s-1} + (P_{m-1}R_s + P_{m-2}R_{s-1})x}{D^2} \\
&= (-P_{m-1}R_{s-1})\,\frac{1}{D} + \frac{R_{m+s-1} + (P_{m-1}R_s + P_{m-2}R_{s-1})x}{D^2} \\
&= -P_{m-1}R_{s-1}\sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} R_{n+m+s-1}x^n \sum_{n=0}^{\infty} P_{n+1}x^n \\
&= \sum_{n=0}^{\infty} (-P_{n+1}P_{m-1}R_{s-1})x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{k+1}R_{n-k+m+s-1}x^n \\
&= \sum_{n=0}^{\infty} \left(P_{n+1}P_{m-1}R_{s-1} + \sum_{k=0}^{n} P_{k+1}R_{n-k+m+s-1}\right)x^n \qquad .
\end{aligned}
$$

Hence,

$$\sum_{k=0}^{n} P_{k+m}R_{n-k+s} = -P_{n+1}P_{m-1}R_{s-1} + \sum_{k=0}^{n} P_{k+1}R_{n-k+m+s-1}$$

and

$$P_{n+1}P_{m-1}R_{s-1} = \sum_{k=0}^{n} (P_{k+1}R_{n-k+m+s-1} - P_{k+m}R_{n-k+s}) .$$

Letting $p = m - 1$, $q = n + 1$, and $r = s - 1$, we obtain

Theorem 4.

$$P_pP_qR_r = \sum_{k=0}^{q-1} (P_{k+1}R_{p+q+r-k} + P_{p+k+1}R_{q+r-k}) .$$

Now we convolute

$$\frac{P_m + P_{m-1}x}{D} \quad \text{with} \quad \frac{P_t + P_{t-1}x}{D}$$

and, using the previous procedures, we find

Theorem 5.

$$P_pP_qP_r = \sum_{k=0}^{r-1} (P_{p+q+r-k}P_{k+1} - P_{p+k-1}P_{q+r-k}) .$$

Similarly, we convolute

$$\frac{R_m + R_{m-1}x}{D} \quad \text{with} \quad \frac{R_t + R_{t-1}x}{D}$$

to obtain

Theorem 6.

$$P_pR_qR_r = \sum_{k=0}^{p-1} (8P_{q+r+k+1}P_{p-k} - R_{q+k+1}R_{p+r-k}) .$$

Now,

$$\begin{aligned} R_pR_qR_r &= (P_{p+1} + P_{p-1})R_qR_r \\ &= P_{p+1}R_qR_r + P_{p-1}R_qR_r \end{aligned}$$

$$R_pR_qR_r = \sum_{k=0}^{p} (8P_{q+r+k+1}P_{p-k+1} - R_{q+k+1}R_{p+r-k+1})$$
$$+\sum_{k=0}^{p-2} (8P_{q+r+k+1}P_{p-k-1} - R_{q+k+1}R_{p+r-k-1})$$

$$= \sum_{k=0}^{p-1} \Big[8P_{q+r+k+1}(P_{p-k} + P_{p-k-1})$$
$$- R_{q+k+1}(R_{p+r-k+1} + R_{p+r-k-1})\Big]$$
$$+ (8P_2P_{q+r+p} - R_{q+p}R_{r+2})$$
$$+ (8P_1P_{p+q+r+1} - R_{q+p+1}R_{r+1})$$

$$= \sum_{k=0}^{p-2} 8(P_{q+r+k+1}R_{p-k} - R_{q+k+1}P_{p+r-k})$$
$$+ 8(2P_{p+q+r} + P_{p+q+r+1})$$
$$- (R_{p+q}R_{r+2} + R_{p+q+1}R_{r+1})$$

$$= 8\sum_{k=0}^{p-2} (P_{q+r+k+1}R_{p-k} - R_{q+k+1}P_{p+r-k})$$
$$+ 8P_{p+q+r+2} - (2R_{p+q}R_{r+1} + R_{p+q}R_r + R_{p+q+1}R_{r-1})$$

and, by Theorem 3, we obtain

Theorem 7.

$$R_pR_qR_r = 8\left[\sum_{k=0}^{p-2} (P_{p+q+r+k+1}R_{p-k} - P_{p+r-k}R_{q+k+1}) - P_{p+q+r-1}\right] - 2R_{p+q}R_{r+1} .$$

REFERENCES

1. Rodney T. Hansen, "Generating Identities for Fibonacci and Lucas Triples," Fibonacci Quarterly, Vol. 10, No. 6 (December 1972), pp. 571-578.
2. Carl E. Serkland, "The Pell Sequence and Some Generalizations," Master's Thesis, California State University, San Jose, California, January 1973.

A FIBONACCI ANALOGUE OF GAUSSIAN BINOMIAL COEFFICIENTS

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Gauss, in his work on quadratic reciprocity, defined in [1] an analogue to the binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(x^n - 1)(x^{n-1} - 1)\cdots(x^{n-k+1} - 1)}{(x^k - 1)(x^{k-1} - 1)\cdots(x - 1)} ,$$

n and k positive integers. In order to make the analogy to the binomial coefficients more complete, it is customary to let

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 ,$$

for $n = 0, 1, 2, \cdots$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = 0$$

for $n < k$. We shall call these rational functions in x, Gaussian binomial coefficients. It is shown in [7] that these functions satisfy the recursion formula:

$$\begin{bmatrix} n \\ k \end{bmatrix} = x^k \begin{bmatrix} n - 1 \\ k \end{bmatrix} + \begin{bmatrix} n - 1 \\ k - 1 \end{bmatrix} ,$$

and if we note that as $x \to 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} \to \binom{n}{k},$$

where

$$\binom{n}{k}$$

is the usual binomial coefficient, then the above recursion formula becomes

$$\binom{n}{k} = \binom{n - 1}{k} + \binom{n - 1}{k - 1} ,$$

the recursion formula for the binomial coefficients.

Just as the binomial coefficients are always integers, although they appear to be ratios of integers, the Gaussian binomial coefficients are in fact polynomials rather than rational functions. This is easily seen from the recursion formula and mathematical induction. (See [7].) The Gaussian binomial coefficients and their multinomial analogues have some interesting geometric interpretations and combinatorial applications in counting inversions and special partitions of the integers. Some of these appear in [1] and [6].

There is another well known analogue to the binomial coefficients, the so-called "Fibonomial coefficients:"

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1}$$

n, k positive integers, and

$$\binom{n}{0}_F = \binom{n}{n}_F = 1$$

for $n = 0, 1, 2, \cdots$. It is well known that this is always an integer [5].

Let us now examine the Gaussian analogue of the "Fibonomial coefficient:"

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{(x^{F_n} - 1)(x^{F_{n-1}} - 1) \cdots (x^{F_{n-k+1}} - 1)}{(x^{F_k} - 1)(x^{F_{k-1}} - 1) \cdots (x^{F_1} - 1)} ,$$

n, k positive integers and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$$

for $n = 0, 1, 2, \cdots$. Again it is clear that as $x \to 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_F \to \binom{n}{k}_F .$$

Since

$$F_n = F_{k+1} F_{n-k} + F_k F_{n-k-1} ,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{(x^{F_{k+1}F_{n-k} + F_kF_{n-k-1}} - 1)(x^{F_{n-1}} - 1) \cdots (x^{F_{n-k+1}} - 1)}{(x^{F_k} - 1)(x^{F_{k-1}} - 1) \cdots (x^{F_1} - 1)}$$

$$(1) \qquad = \frac{(x^{F_{k+1}F_{n-k}+F_kF_{n-k-1}} - x^{F_kF_{n-k-1}} + x^{F_kF_{n-k-1}} - 1)}{(x^{F_k} - 1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

$$= \frac{x^{F_kF_{n-k-1}}(x^{F_{k+1}F_{n-k}} - 1) + (x^{F_kF_{n-k-1}} - 1)}{(x^{F_k} - 1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

$$= x^{F_k F_{n-k-1}} \left(\sum_{i=1}^{F_{k+1}} x^{(F_{k+1}-i)F_{n-k}} \right) \begin{bmatrix} n-1 \\ k \end{bmatrix}_F$$

$$+ \left(\sum_{i=1}^{F_{n-k-1}} x^{(F_{n-k-1}-i)F_k} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

so that we have a recursion formula for the "Gaussian Fibonomial coefficients" and this, with mathematical induction, implies the rather remarkable property of these functions: they are polynomials rather than rational functions as they appear to be. Furthermore if we let $x \to 1$ in the recursion formula (1) we obtain

$$\binom{n}{k}_F = F_{k+1} \binom{n-1}{k}_F + F_{n-k-1} \binom{n-1}{k-1}_F$$

the recursion formula for the Fibonomial coefficients. This is the recursion formula used in [3] to prove that the Fibonomial coefficients are integers.

The more general sequence g_n where $g_0 = 0$, $g_1 = 1$, $g_{n+2} = p \cdot g_{n+1} + q \cdot g_n$, $n \geq 0$, $p > 0$, $q \geq 0$, satisfies $g_n = g_{k+1} \cdot g_{n-k} + q \cdot g_k \cdot g_{n-k-1}$ (see [3]) and if we define

$\begin{bmatrix} n \\ k \end{bmatrix}_g$ as follows: $$\begin{bmatrix} n \\ k \end{bmatrix}_g = \frac{(x^{g_n} - 1)(x^{g_{n-1}} - 1) \cdots (x^{g_{n-k+1}} - 1)}{(x^{g_k} - 1)(x^{g_{k-1}} - 1) \cdots (x^{g_1} - 1)}$$

n, k positive integers, and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_g = \begin{bmatrix} n \\ n \end{bmatrix}_g = 1$$

for $n = 0, 1, 2, \cdots$, then it follows, mutatis mutandis, that

$$\begin{bmatrix} n \\ k \end{bmatrix}_g = x^{q \cdot g_k g_{n-k-1}} \left(\sum_{i=1}^{g_{k+1}} x^{(g_{k+1}-i) \cdot g_{n-k}} \right) \begin{bmatrix} n-1 \\ k \end{bmatrix}_g$$

$$+ \left(\sum_{i=1}^{q \cdot g_{n-k+1}} x^{(q \cdot g_{n-k+1}-i) \cdot g_k} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_g .$$

Again,

$$\begin{bmatrix} n \\ k \end{bmatrix}_g$$

are polynomials. Furthermore the functions are again polynomials where $g_n = f_n(t)$, the Fibonacci polynomials, at least for positive integral t, where $f_0(t) = 0$, $f_1(t) = 1$,

$$f_{n+2}(t) = t \cdot f_{n+1}(t) + f_n(t), \qquad n \geq 0 .$$

Since the Pell sequence can be generated as a special case of the Fibonacci polynomials (where $t = 2$), the above "coefficients" are polynomials also when defined in terms of the Pell sequence.

Furthermore, because of the direct analogy between the definitions of the Gaussian binomial coefficients and the related Fibonacci analogues defined above and the expression for the binomial coefficients as ratios of factorials, the polynomials when arranged in a triangular array like Pascal's Triangle will have the beautiful hexagon property described by Hoggatt and Hansell in [4], that the product of the elements "surrounding" an element in the array is a perfect square and the set of six elements can be broken down into two sets of three, the products of the elements in each set being equal. In fact all the perfect square patterns of Usiskin in [8] will appear in these new arrays; the proofs carry over directly.

REFERENCES

1. L. Carlitz, "Sequences and Inversions," Duke Math. J., Vol. 37, No. 1 (Mar. 1970), pp. 193-198.
2. C. F. Gauss, "Summatio quarundam serierum singularium," Werke, Vol. 2, pp. 16-17.
3. V. E. Hoggatt, Jr., "Fibonacci Numbers and Generalized Binomial Coefficients," Fibonacci Quarterly, Vol. 5, No. 4 (Nov. 1967), pp. 383-400.
4. V. E. Hoggatt, Jr., and W. Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly," Vol. 9, No. 2 (April 1971), p. 120.
5. Dov Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem, 1958, pp. 42-45.
6. G. Polya and G. L. Alexanderson, "Gaussian Binomial Coefficients," Elemente der Mathematik, Vol. 26, No. 5 (Sept. 1971), pp. 102-109.
7. Hans Rademacher, Lectures on Elementary Number Theory, Ginn-Blaisdell, New York, 1964, pp. 83-85.
8. Z. Usiskin, "Perfect Square Patterns in the Pascal Triangle," Math. Mag. 46 (1973), pp. 203-208.

A SOLUTION OF ORTHOGONAL TRIPLES IN FOUR SUPERIMPOSED 10 x 10 x 10 LATIN CUBES

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Recently at the 78^{th} Summer Meeting of the American Mathematical Society, Missoula, Montana (August 20-24, 1973), Professor P. Erdös and Professor E. G. Straus proposed the following classical problem to this author: Consider four digits where each digit can have a value of 0, 1, 2, $\cdots$, 9. Divide the four digits into four sets where each set contains three digits in the following way: Set A = 1st, 2nd, 3rd digits; set B = 1st, 2nd, 4th digits; set C = 1st, 3rd, 4th digits; and set D = 2nd, 3rd, 4th digits. For example: if a cell contains the four digits 3742 then 374 would belong in set A, 372 belongs in set B, 342 belongs in set C, and 742 belongs in set D.

Then, using only the digits 0, 1, 2, $\cdots$, 9, is it possible to superimpose four $10 \times 10 \times 10$ Latin Cubes such that (we consider one set at a time) set A, set B, set C, and set D will each contain in some way every one of the following 1000 three-digit numbers 000, 001, 002, $\cdots$, 999, without repetition? (It is, of course, evident there will be four digits in each and every cell of the 1000 cells.) This author has solved the above problem and we are able to construct for the first time orthogonal triples in four $10 \times 10 \times 10$ superimposed Latin Cubes.

Note. With the method of construction shown in this paper, we are also able to construct for the first time orthogonal triples in four $(4m + 2) \times (4m + 2) \times (4m + 2)$ superimposed Latin Cubes, where $3 \leq m = 3, 4, \cdots$.

In Tables 1-10, we have systematically constructed orthogonal triples in four $10 \times 10 \times 10$ superimposed Latin Cubes.

Table 1

Square Number 0

7630	6861	3405	2793	1152	8289	4014	5547	0326	9978
0796	2633	1972	4544	6321	5017	7280	9868	8409	3155
6971	5407	8639	0016	3795	4324	9548	2153	1282	7860
9408	8549	2013	1632	4284	7150	6791	0976	3865	5327
2323	0286	7540	6151	9638	1862	3975	8019	5797	4404
5287	4974	9328	7400	8869	3635	0156	1792	2543	6011
3545	1322	0866	9288	7010	2973	5637	6401	4154	8799
8159	7790	6281	3325	5977	0406	2863	4634	9018	1542
4864	3015	5157	8979	0546	9798	1402	7320	6631	2283
1012	9158	4794	5867	2403	6541	8329	3285	7970	0636

Table 2

Square Number 1

8721	5386	6649	9850	4937	3162	7473	0218	1594	2005
1854	9720	4007	7213	5596	0478	8161	2385	3642	6939
5006	0648	3722	1474	6859	7593	2215	9930	4167	8381
2645	3212	9470	4727	7163	8931	5856	1004	6389	0598
9590	1164	8211	5936	2725	4387	6009	3472	0858	7643
0168	7003	2595	8641	3382	6729	1934	4857	9210	5476
6219	4597	1384	2165	8471	9000	0728	5646	7933	3852
3932	8851	5166	6599	0008	1644	9380	7723	2475	4217
7383	6479	0938	3002	1214	2855	4647	8591	5726	9160
4477	2935	7853	0388	9640	5216	3592	6169	8001	1724

Table 3

Square Number 2

5902	3244	1718	0139	9086	4650	8895	2371	6463	7527
6133	0909	9526	8375	3464	2891	5652	7247	4710	1088
3524	2711	4900	6893	1138	8464	7377	0089	9656	5242
7717	4370	0899	9906	8655	5082	3134	6523	1248	2461
0469	6653	5372	3084	7907	9246	1528	4890	2131	8715
2651	8525	7467	5712	4240	1908	6083	9136	0379	3894
1378	9466	6243	7657	5892	0529	2901	3714	8085	4130
4080	5132	3654	1468	2521	6713	0249	8905	7897	9376
8245	1898	2081	4520	6373	7137	9716	5462	3904	0659
9896	7087	8135	2241	0719	3374	4460	1658	5522	6903

Table 4

Square Number 3

9873	4509	7232	6317	0490	2026	5948	1755	3181	8664
3311	6877	0660	5758	4189	1945	9023	8504	2236	7492
4669	1235	2876	3941	7312	5188	8754	6497	0020	9503
8234	2756	6947	0870	5028	9493	4319	3661	7502	1185
6187	3021	9753	4499	8874	0500	7662	2946	1315	5238
1025	5668	8184	9233	2506	7872	3491	0310	6757	4949
7752	0180	3501	8024	9943	6667	1875	4239	5498	2316
2496	9313	4029	7182	1665	3231	6507	5878	8944	0750
5508	7942	1495	2666	3751	8314	0230	9183	4879	6027
0940	8494	5318	1505	6237	4759	2186	7022	9663	3871

Table 5

Square Number 4

0064	2417	4551	8278	6345	5993	3109	7626	9832	1780
9272	8068	6785	3629	2837	7106	0994	1410	5553	4341
2787	7556	5063	9102	4271	3839	1620	8348	6995	0414
1550	5623	8108	6065	3999	0344	2277	9782	4411	7836
8838	9992	0624	2347	1060	6415	4781	5103	7276	3559
7996	3789	1830	0554	5413	4061	9342	6275	8628	2107
4621	6835	9412	1990	0104	8788	7066	2557	3349	5273
5343	0274	2997	4831	7786	9552	8418	3069	1100	6625
3419	4101	7346	5783	9622	1270	6555	0834	2067	8998
6105	1340	3279	7416	8558	2627	5833	4991	0784	9062

Table 6

Square Number 5

4255	1693	5880	3426	7574	6308	2762	9039	8917	0141
8427	3256	7144	2032	1913	9769	4305	0691	6888	5570
1143	9889	6258	8767	5420	2912	0031	3576	7304	4695
0881	6038	3766	7254	2302	4575	1423	8147	5690	9919
3916	8307	4035	1573	0251	7694	5140	6768	9429	2882
9309	2142	0911	4885	6698	5250	8577	7424	3036	1763
5030	7914	8697	0301	4765	3146	9259	1883	2572	6428
6578	4425	1303	5910	9149	8887	3696	2252	7761	7034
2692	5760	9579	6148	8037	0421	7884	4915	1253	3306
7764	0571	2422	9699	3886	1033	6918	5300	4145	8257

Table 7

Square Number 6

6446	0122	2364	7985	8613	1777	9531	3800	4058	5299
4988	7445	8293	9801	0052	3530	6776	5129	1367	2614
0292	3360	1447	4538	2984	9051	5809	7615	8773	6126
5369	1807	7535	8443	9771	6616	0982	4298	2124	3050
7055	4778	6806	0612	5449	8123	2294	1537	3980	9361
3770	9291	5059	6366	1127	2444	4618	8983	7805	0532
2804	8053	4128	5779	6536	7295	3440	0362	9611	1987
1617	6986	0772	2054	3290	4368	7125	9441	5539	8803
9121	2534	3610	1297	4808	5989	8363	6056	0442	7775
8533	5619	9981	3120	7365	0802	1057	2774	6296	4448

Table 8

Square Number 7

3397	7738	0173	1041	2829	9514	6680	8962	5205	4456
5045	1391	2459	6960	7208	8682	3517	4736	9174	0823
7458	8172	9394	5685	0043	6200	4966	1821	2519	3737
4176	9964	1681	2399	6510	3827	7048	5455	0733	8202
1201	5515	3967	7828	4396	2739	0453	9684	8042	6170
8512	6450	4206	3177	9734	0393	5825	2049	1961	7688
0963	2209	5735	4516	3687	1451	8392	7178	6820	9044
9824	3047	7518	0203	8452	5175	1731	6390	4686	2969
6730	0683	8822	9454	5965	4046	2179	3207	7398	1511
2689	4826	6040	8732	1171	7968	9204	0513	3457	5395

Table 9

Square Number 8

2118	8950	9097	4562	5701	0845	1226	6484	7679	3333
7569	4112	5331	1486	8670	6224	2848	3953	0095	9707
8330	6094	0115	7229	9567	1676	3483	4702	5841	2958
3093	0485	4222	5111	1846	2708	8560	7339	9957	6674
4672	7849	2488	8700	3113	5951	9337	0225	6564	1096
6844	1336	3673	2098	0955	9117	7709	5561	4482	8220
9487	5671	7959	3843	2228	4332	6114	8090	1706	0565
0705	2568	8840	9677	6334	7099	4952	1116	3223	5481
1956	9227	6704	0335	7489	3563	5091	2678	8110	4842
5221	3703	1566	6954	4092	8480	0675	9847	2338	7119

Table 10

Square Number 9

1589	9075	8926	5604	3268	7431	0357	4193	2740	6812
2600	5584	3818	0197	9745	4353	1439	6072	7921	8266
9815	4923	7581	2350	8606	0747	6192	5264	3438	1079
6922	7191	5354	3588	0437	1269	9605	2810	8076	4743
5744	2430	1199	9265	6582	3078	8816	7351	4603	0927
4433	0817	6742	1929	7071	8586	2260	3608	5194	9355
8196	3748	2070	6432	1359	2814	4583	9925	0267	7601
7261	1609	9435	8746	4813	2920	5074	0587	6352	3198
0077	8356	4263	7811	2190	6602	3928	1749	9585	5434
3358	6262	0607	4073	5924	9195	7741	8436	1819	2580

Proof that Construction is Correct. Before going on with the proof, we will set down a few definitions to facilitate our explanation of the proof. It will be noted that the squares in Tables 1-10 are labeled Square 0 through 9. Then suppose we wish to find a certain number of a certain cell — we shall write S (row number, column number, square number) = number in cell. To find a row on a certain square, we write S (row number, *, square number), and S (*, c, s) = column number on a certain square.

The ten columns in each square are considered to be numbered $0, 1, \cdots, 9$ from left to right; the ten rows on each square are considered to be numbered $0, 1, \cdots, 9$ from top to bottom. For example: The number 7630 on Square Number 0 = S(0,0,0); or the row on which 7630 is found may be written as S(0, *, 0); and the column we find 7630 in is S(*, 0, 0). Finally if we refer to a specific square, say square 0, we write S(*,*, 0); if we refer to each and every one of the ten squares we write S(*, *, A); to refer to each and every top row (say) in each and every one of the two squares we write S(0, *, A).

(1) We now consider the 2nd and 3rd digits in each cell of the S(0, *, A), and keeping the cells in the same positions, we place S(0, *, 0), on top of S(0, *, 1), $\cdots$, on top of S(0, *, 9) it is easily verified that we have constructed the following 10×10 square which was formed by superimposing two Latin Squares in such a way that the 100 two-digit numbers are mutually orthogonal.

$$\text{(1a)} \qquad \begin{array}{ccccc} 63 & 86 & 40 & \cdots & 97 \\ 72 & 38 & 64 & \cdots & 00 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 58 & 07 & 92 & \cdots & 81 \end{array} .$$

(1b) It should also be noticed that the 2nd and 3rd digits in each cell of the S(0, *, A) is repeated ten times in its own respective square. For example: The ten cells of 2nd and 3rd digits in S(0, *, 0) are 63 86 40 $\cdots$ 97, and it is seen that in the Square 0, the number 63 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column, the number 86 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column, $\cdots$, the number 97 is repeated (as a 2nd and 3rd digit) ten times in a different row and a different column.

(1c) Now it is easily verified; each and every one of the ten Squares is constructed in the exact way we constructed the Square Number 0 in (1b).

(2) We now look at the first digit in each cell, where it is easily verified that the first digit in each cell of the S(0, *, A) is repeated ten times in a different row and different column on its own respective square.

(2a) For example: the first digit 0 on Square 0 will be found in ten different cells where each cell is in a different row and different column, and this exact arrangement of the first digit 0 is constructed into each and every square 0 through and including Square 9. It is also easily verified that each first digit 0 is on a different file.

(2b) Now, each and every first digit (0, 1, ⋯, 9) in every cell is arranged in the exact way we placed the 0's in our example (2a).

Therefore, there are no two identical first digits in the same row, the same column, or the same file throughout the cube.

(Let the 100 numbers 000, 001, 002, ⋯, 099 = a_0 ;

the 100 numbers 100, 101, 102, ⋯, 199 = a_1 ;

. . . .

the 100 numbers 900, 901, 902, ⋯, 999 = a_9.)

Now, combining (1, a, b, c) with (2, a, b) leads to

(3) The first three digits in each cell in the cube that belongs to ---a_k will have each of its three-digit numbers in a different column, different row, and in a different file, where we replace the subscript k (in a_k) one at a time with the number 0, then 1, ⋯, then 9.

(3a) In (3), we have then satisfied the requirement that set A (set A = the 1st, 2nd, and 3rd digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ⋯, 999, without repetition.

(3b) We now combine in each cell throughout the cube—the second and third digits with the fourth digit — and in the exact way we found (3a) — we find that we have satisfied the requirement that set D (set D = the 2nd, 3rd, and 4th digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, ⋯, 999, without repetition.

(4) Now, it will be noticed that every identical first digit is paired with an identical fourth digit — we inspect one square at a time. For example: In Square 0, every one of the ten cells that have a first digit 0 also have as a fourth digit the number 6; every one of the ten cells that have a first digit 1 also have as a fourth digit the number 2; ⋯; every one of the ten cells that have a first digit 9 also have as a fourth digit the number 8. It should also be noticed that the ten first digits (say 1st digit = A) paired with ten fourth digits (say) B to get the numbers A--B in ten cells on a particular square — shall never again have this particular first and fourth digit combination repeated (that is, the combination A--B) on any one of the nine remaining squares. For example: on Square 0 the first digit 7 is paired with the fourth digit 0, on Square 1 the first digit 7 is paired with the fourth digit 3, ⋯, on Square 9, the first digit 7 is paired with the fourth digit 1. This arrangement for first and fourth digits is ridgidly enforced throughout the construction.

(5) Now, the first and second digits in each square (we consider one square at a time) are mutually (pairwise) orthogonal. For example: The first and second digits in Square 0 are mutually orthogonal and are constructed by superimposing two 10×10 Latin Squares.

(5a) The exact orthogonal properties of digits 1 and 2 in each of the ten squares (we consider one square at a time) that we find to hold true in (5) also are easily verified to hold true for the first and third digits. That is, the first and third digits in each and every one of the ten squares (we consider one square at a time) are mutually (pairwise) orthogonal.

(6) Now, we combine (4) and (5), which leads us to the fact that set B (set B = 1st, 2nd, and 4th digits in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, $\cdots$, 999, without repetition.

(6a) Finally, we combine (4) and (5a), which leads us to the fact that set C (set C = 1st, 3rd, and 4th digit in each and every cell throughout the cube) will contain (in some way) every one of the 1000 three-digit numbers 000, $\cdots$, 999, without repetition.

Remark. We used The Arkin-Hoggatt method [1] to get the 100 mutually **orthogonal** numbers in (1).

Note. For singly-even cubes greater than $10 \times 10 \times 10$ we can combine the above methods with Bose, Shrikande and Parker's work on mutually (pairwise) orthogonal numbers [2] and after the proper extensions of their magnificent theorems — it is easily shown that we can obtain a solution of orthogonal triples in four $(4m + 2) \times (4m + 2) \times (4m + 2)$ superimposed Latin Cubes (where $2 < m = 3, 4, \cdots$).

In conclusion, we discuss (our discussion relies entirely on the construction in this paper) orthogonal triples in Five $10 \times 10 \times 10$ superimposed Latin Cubes.

(7) In our discussion, the ten numbers 7630, 7860, 7400, 7790, 7150, 7280, 7010, 7540, 7320, 7970, that are found in Square Number 0 will be used as an illustrative example.

It is evident that in each of the ten numbers above, the first and fourth digits form the two-digit number 70, and also the second and third digits in the above ten numbers are mutually (pairwise) orthogonal.

(7a) Now, let us add a fifth digit to each of the ten four-digit numbers written above. It is evident that it would be impossible to form orthogonal triples if any two of the ten fifth digits we placed are identical. For example: Say we placed a 0 after (in the fifth position) two of the ten numbers in (7) — say the two numbers are 7630 and 7280. We then have 76300 and 72900 and it is evident that the 700 in 76300 and the 700 in 72800 **are** not in a set of orthogonal triples. Therefore, every one of the ten fifth digits we add to the ten numbers in (7) above must be different and thus the fifth digit in (7) must include each number in 0, 1, $\cdots$, 9. However, since the second and third digits in each of the ten numbers in (7) are mutually (pairwise) orthogonal, it follows that the second, third, and fifth digits in the above ten numbers in (7) are mutually (pairwise) orthogonal.

Then, using the exact method of our example in (7a) we extend our reasoning (step-by-step) to include the entire Square 0, and then Square 1, $\cdots$, and Square 9. In this way, we are easily led to the following.

(7b) IN ORDER TO FIND A SOLUTION OF ORTHOGONAL TRIPLES IN FIVE $10 \times 10 \times 10$ SUPERIMPOSED LATIN CUBES, WE MUST FIRST BE ABLE TO CONSTRUCT A SYSTEM OF THREE MUTUALLY ORTHOGONAL NUMBERS (three pairwise orthogonal) IN A SQUARE MADE OF THREE SUPERIMPOSED $10 \times 10 \times 10$ LATIN SQUARES.

(8) It is easily verified that by combining the NOTE above with (7b), we extend (7b) to read: IN ORDER TO FIND A SOLUTION OF ORTHOGONAL TRIPLES IN FIVE $(4m + 2) \times (4m + 2) \times (4m + 2)$ SUPERIMPOSED CUBES, WE MUST FIRST BE ABLE TO CONSTRUCT A SYSTEM OF THREE MUTUALLY ORTHOGONAL NUMBERS (three pairwise orthogonal) IN A

SQUARE MADE OF THREE SUPERIMPOSED $(4m+2) \times (4m+2) \times (4m+2)$ LATIN SQUARES, where $2 < m = 3, 4, \cdots$.

Remark. It should be noted that the methods of construction of the cube in the above paper are the same methods that were used to construct the cubes in the following two papers (we mention the following two papers, since each paper stated that a method of construction was forthcoming). See [3] and [4].

REFERENCES

1. J. Arkin and V. E. Hoggatt, Jr., "The Arkin-Hoggatt Game and the Solution of a Classical Problem," J. of Recreational Math., 1973 Spring Edition.
2. R. C. Bose and S. S. Shrikande, "On the Falsity of Euler's Conjecture About the Nonexistence of Two Orthogonal Latin Squares of Order 4t + 2," Proc. Nat'l. Acad. Sci., 45 (5), 5/1959, pp. 734-737.

   E. T. Parker, "Orthogonal Latin Squares," Proc. Nat'l. Acad. Sci., 45 (6), 6/59, pp. 859-862.
3. J. Arkin, "The First Solution of the Classical Eulerian Magic Cube Problem of Order Ten," Fibonacci Quarterly, Vol. 11, No. 2 (April 1973), pp. 174-178.
4. J. Arkin, "A Solution to the Classical Problem of Finding Systems of Three Mutually Orthogonal Numbers in a Cube Formed by Three Superimposed $10 \times 10 \times 10$ Cubes," Fibonacci Quarterly, Vol. 11, No. 5 (December 1973), pp. 485-489. This paper was presented in person by the author at The Seventy-Eighth Summer Meeting, Amer. Math. Society, Missoula, Montana (University of Montana), August 20-24, 1973.

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COMBINATORIAL ANALYSIS AND FIBONACCI NUMBERS

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1. INTRODUCTION

The object of this paper is to present a new combinatorial interpretation of the Fibonacci numbers.

There are many known combinatorial interpretations of the Fibonacci numbers (e.g., [9]); indeed, the original use of these numbers was that of solving the rabbit breeding problem of Fibonacci [10]. The appeal of this new interpretation lies in the fact that it provides combinatorial proofs of several well known Fibonacci identities. Among them:

$$\sum_{j=0}^{n} \binom{n}{j} F_j = F_{2n} .$$

These results will be presented in Section 2. In Section 3, we shall describe further possibilities for exploration of Fibonacci numbers via combinatorics.

2. FIBONACCI SETS

Definition 1. We say a finite set S of positive integers is Fibonacci if each element of the set is $\geq |S|$, where $|S|$ denotes the cardinality of S.

Definition 2. We say a finite set S of positive integers is r-Fibonacci if each element of the set is $\geq |S| + r$.

We note that "0-Fibonacci" means "Fibonacci."

Table 1
Subsets of $\{1, 2, \cdots, n\}$ that are r-Fibonacci

n	Fibonacci	1-Fibonacci	2-Fibonacci
1	ϕ, $\{1\}$	ϕ	ϕ
2	ϕ, $\{1\}$, $\{2\}$	ϕ, $\{2\}$	ϕ
3	ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{3,2\}$	ϕ, $\{2\}$, $\{3\}$	ϕ, $\{3\}$
4	ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$	ϕ, $\{2\}$, $\{3\}$, $\{4\}$, $\{3,4\}$	ϕ, $\{3\}$, $\{4\}$

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Theorem 1. There are exactly F_{n+2-r} subsets of $\{1, 2, \cdots, n\}$ that are r-Fibonacci for $n \geq r - 1$.

Proof. When $n = r - 1$ or r, ϕ is the only subset of $\{1, 2, \cdots, n\}$ that is r-Fibonacci, since each element of an r-Fibonacci set must be $>r$. Since $F_1 = F_2 = 1$, we see that the theorem is true for $n = r - 1$ or r.

Assume the theorem true for each n with $r < n \leq n_0$ (and for all r). Let us consider the r-Fibonacci subsets of $\{1, 2, \cdots, n_0, n_0 + 1\}$ that: (1) do not contain $n_0 + 1$, and (2) do contain $n_0 + 1$. Clearly there are F_{n_0+2-r} elements of the first class. If we delete $n_0 + 1$ from each set in the second class, we see that we have established a one-to-one correspondence between the elements of the second class and the $(r + 1)$-Fibonacci subsets of $\{1, 2, \cdots, n_0\}$, hence there are $F_{n_0+2-(r+1)}$ elements of the second class. This means that there are

$$
\begin{aligned}
&F_{n_0+2-r} + F_{n_0+2-(r+1)} \\
&\quad = F_{(n_0+1)+2-r}
\end{aligned}
$$

r-Fibonacci subsets of $\{1, 2, \cdots, n_0 + 1\}$, and this completes Theorem 1.

Theorem 2. For $n \geq 0$,

$$
\begin{aligned}
F_{n+2} &= 1 + \binom{n}{j} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots \\
&= 1 + \sum_{j \geq 1} \binom{n-j+1}{j} \qquad .
\end{aligned}
$$

Proof. By Theorem 1, F_{n+2} is the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$. Of these ϕ is one such subset. There are

$$\binom{n}{1}$$

singleton Fibonacci subsets of $\{1, 2, \cdots, n\}$. The two-element Fibonacci subsets are just the two-element subsets of $\{2, 3, \cdots, n\}$, and there are

$$\binom{n-1}{2}$$

of these. In general, the j-element Fibonacci subsets of $\{1, 2, \cdots, n\}$ are just the j-element subsets of $\{j, j + 1, \cdots, n\}$ and there are exactly

$$\binom{n-j+1}{j}$$

of these. Hence summing over all j and using Theorem 1, we see that

$$F_{n+2} = 1 + \sum_{j \ge 1} \binom{n - j + 1}{j} .$$

Theorem 3. For $n \ge 0$

$$\binom{n+1}{1} F_1 + \binom{n+1}{2} F_2 + \cdots + \binom{n+1}{n} F_n + F_{n+1} = F_{2n+2} ,$$

or

$$\sum_{j=0}^{n} \binom{n+1}{j} F_{n+1-j} = F_{2n+2} .$$

Remark. This is the identity stated in the Introduction with $n + 1$ replacing n.

Proof. By Theorem 1, F_{2n+2} is the number of Fibonacci subsets of $\{1, 2, \cdots, 2n\}$. We first remark that there are at most n elements of a Fibonacci subset of $\{1, 3, \cdots, 2n\}$, for if there were $n + 1$ elements then at least one element would be $\le n$ which is impossible.

Let T_j denote the number of Fibonacci subsets of $\{1, 2, \cdots, 2n\}$ that have exactly j elements $\ge n$. Clearly

$$F_{2n+2} = \sum_{j=0}^{n} T_j .$$

Now to construct the subsets enumerated by T_j, we see that we may select any j-elements in the set $\{n, n + 1, \cdots, 2n\}$ and then adjoin to these j elements a j-Fibonacci subset of $\{1, 2, \cdots, n - 1\}$. Since there are

$$\binom{n+1}{j}$$

choices of the j elements from $\{n, n + 1, \cdots, 2n\}$ and $F_{(n-1)+2-j} = F_{n+1-j}$ j-Fibonacci subsets of $\{1, 2, \cdots, n - 1\}$, we see that

$$T_j = \binom{n+1}{j} F_{n+1-j} .$$

Therefore

$$F_{2n+2} = \sum_{j=0}^{n} T_j = \sum_{j=0}^{n} \binom{n+1}{j} F_{n+1-j} .$$

Theorem 4. For $n \ge 0$,

$$1 + F_1 + F_2 + \cdots + F_n = F_{n+2} .$$

Proof. Let R_j denote the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$ in which the largest element is j. Let $R_0 = 1$ in order to count the empty subset ϕ. Clearly for $j > 0$, R_j equals the number of 1-Fibonacci subsets of $\{1, 2, \cdots, j - 1\}$; thus by Theorem 1, $R_j = F_{(j-1)+2-1} = F_j$. Therefore

$$F_{n+2} = 1 + \sum_{j=1}^{n} R_j = 1 + \sum_{j=1}^{n} F_j .$$

3. CONCLUSION

The genesis of this work lies in the close relationship between the Fibonacci numbers and certain generating functions that are intimately connected with the Rogers-Ramanujan identities. Indeed if $D_{-1}(q) = D_0(q) = 1$, $D_1(q) = 1 + q$, and $D_n(q) = D_{n-1}(q) + q^n D_{n-2}(q)$, then [3; pp. 298-299]

$$D_n(q) = \sum_{j \geq 0} q^{j^2} \begin{bmatrix} n + 1 - j \\ j \end{bmatrix} , \tag{3.1}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{j=1}^{m} (1 - q^{n-j+1})(1 - q^j)^{-1}, \quad \text{for } 0 \leq m \leq n, \quad \begin{bmatrix} n \\ m \end{bmatrix} = 0 \text{ otherwise.}$$

It is not difficult to see that $D_n(q)$ is the generating function for partitions in which each part is larger than the number of parts and $\leq n$. Thus $D_n(1)$ must be F_{n+2}, the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$, and this is clear from (3.1) and Theorem 2 since

$$\begin{bmatrix} n \\ m \end{bmatrix} \quad \text{equals} \quad \binom{n}{m}$$

at $q = 1$. Actually, it is also possible to prove q-analogs of Theorems 3 and 4. Namely,

$$D_{2n}(q) = \sum_{j=0}^{n+1} q^{jn} \begin{bmatrix} n + 1 \\ j \end{bmatrix} D_{n-1-j}(q) , \tag{3.2}$$

and

$$D_n(q) = 1 + \sum_{j=1}^{n} q^j D_{j-2}(q) . \tag{3.3}$$

While (3.3) is a trivial result (3.2) is somewhat tricky although a partition-theoretic analog of Theorem 3 yields the result directly.

Since $D_n(q)$ is also the generating function for partitions in which each part is $\le n$ and each part differs from every other part by at least 2, we might have defined a Fibonacci set in this way also; i.e., a finite set of positive integers in which each element differs from every other element by at least 2. Such a definition provides no new insights and only tends to make the results we have obtained more cumbersome. C. Berge [6; p. 31] gives a proof of our Theorem 2 using this particular approach.

It is to be hoped that the combinatorial approach described in this paper can be extended to prove such appealing identities as

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

[12; p. 7]

$$2^{n-1}F_n = \sum_{j\ge 0} \binom{n}{2j+1} 5^j$$

[8; p. 150, e.q. (10.14.11)].

Presumably a good guide for such a study would be to first attempt (by any means) to establish the desired q-analog for $D_n(q)$. Such a result would then give increased information about the possibility of a combinatorial proof of the corresponding Fibonacci identity. This approach was used in reverse in passing from the formulae [1; p. 113]

$$F_n = \sum_{\alpha=-\infty}^{\infty} (-1)^{\alpha} \binom{n}{[1/2(n-1-5\alpha)]}$$

to new generalizations of the Rogers-Ramanujan identities ([4], [5]). I. J. Schur was the first one to extensively develop such formulas [11] (see also [2], [7]).

REFERENCES

1. George E. Andrews, "Some Formulae for the Fibonacci Sequence with Generalizations," Fibonacci Quarterly, Vol. 7, No. 2 (April 1969), pp. 113-130.
2. George E. Andrews, Advanced Problem H-138, Fibonacci Quarterly, Vol. 8, No. 1 (February 1970), p. 76.
3. George E. Andrews, "A Polynomial Identity which Implies the Rogers-Ramanujan Identities," Scripta Math., Vol. 28 (1970), pp. 297-305.

4. George E. Andrews, "Sieves for Theorems of Euler, Rogers and Ramanujan, from the Theory of Arithmetic Functions," Lecture Notes in Mathematics, No. 251, Springer, New York, 1971.
5. George E. Andrews, "Sieves in the Theory of Partitions," Amer. J. Math. (to appear).
6. C. Berge, Principles of Combinatories, Academic Press, New York, 1971.
7. Leonard Carlitz, Solution to Advanced Problem H-138, Fibonacci Quarterly, Vol. 8, No. 1 (February 1970), pp. 76-81.
8. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th Ed., Oxford University Press, Oxford, 1960.
9. V. E. Hoggatt, Jr., and Joseph Arkin, "A Bouquet of Convolutions," Proceedings of the Washington State University Conf. on Number Theory, March 1971, pp. 68-79.
10. John E. and Margaret W. Maxfield, Discovering Number Theory, W. B. Saunders, Philadelphia, 1972.
11. I. J. Schur, "Ein Beitrag zur additiven Zahlentheorie, Sitzungsber," Akad. Wissensch. Berlin, Phys.-Math. Klasse (1917), pp. 302-321.
12. N. N. Vorobyov, The Fibonacci Numbers, D. C. Heath, Boston, 1963.

FIBONACCI SUMMATIONS INVOLVING A POWER OF A RATIONAL NUMBER
SUMMARY

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The formulas pertain to generalized Fibonacci numbers with given T_1 and T_2 and with

$$T_{n+1} = T_n + T_{n-1} \tag{1}$$

and with generalized Lucas numbers defined by

$$V_n = T_{n+1} + T_{n-1} \quad . \tag{2}$$

Starting with a finite difference relation such as

$$\Delta (b/a)^k T_{2k} T_{2k+2} = (b^k/a^{k+1}) T_{2k+2} (b T_{2k+4} - a T_{2k}) \tag{3}$$

values of b and a are selected which lead to a single generalized Fibonacci or Lucas number for the term in parentheses. Thus for $b = 2$, $a = 13$, the quantity in parentheses is $3 T_{2k-3}$. Using the finite difference approach leads to a formula

$$\sum_{k=1}^{n} (2/13)^k T_{2k} T_{2k+5} = (1/3) \left[(2^{n+1}/13^n) T_{2n+5} T_{2n+7} - 2 T_5 T_7 \right]. \tag{4}$$

Formulas are also developed with terms in the denominator.

(Continued on page 156.)

A PRIMER FOR THE FIBONACCI NUMBERS: PART XIV

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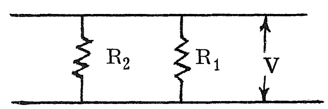
THE MORGAN-VOYCE POLYNOMIALS

1. INTRODUCTION

Polynomial sequences often occur in solving physical problems. The Morgan-Voyce polynomial results when one considers a ladder network of resistances [1], [2], [3]. Let R be the resistance of two resistors R_1 and R_2 in parallel. The voltage drop V across a resistance R due to flow of current I is, of course, $V = IR$.



Now

$$V = I_1R_1 = I_2R_2 = (I_1 + I_2)R$$

Thus

$$\frac{I_1}{V} = R_1, \qquad \frac{I_2}{V} = R_2 ,$$

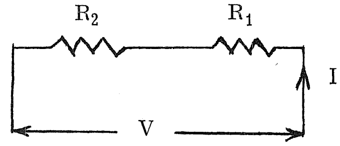
so that

$$\frac{1}{R} = \frac{I_1}{V} + \frac{I_2}{V} = \frac{1}{R_1} + \frac{1}{R_2} .$$

Thus the formula for resistors in parallel is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} .$$

For resistors in series



$$V = I(R_1 + R_2) = IR$$

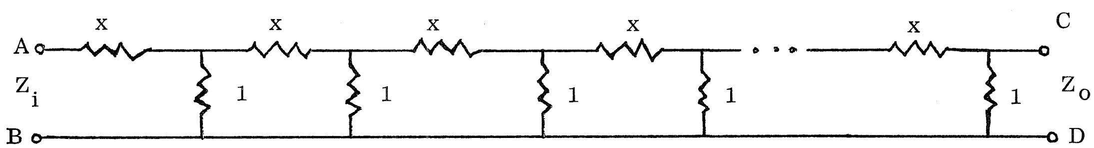
so that the formula relating the resistances is

$$R = R_1 + R_2 .$$

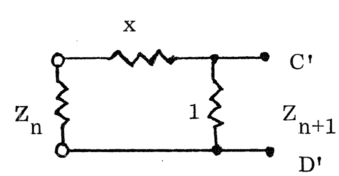
This is all we need to solve the ladder network problem.

2. LADDER NETWORKS

Consider the following:



Assume that the terminals A and B are open. We desire the resistance as measured across terminals C and D. For n ladder sections, let us assume that the resistance is Z_n, and consider the output Z_o .



Since x and Z_n are in series,

$$R = x + Z_n .$$

Now R and 1 are in parallel, so that

$$\frac{1}{Z_{n+1}} = \frac{1}{x + Z_n} + 1 = \frac{x + Z_n + 1}{x + Z_n}$$

$$Z_{n+1} = \frac{x + Z_n}{x + Z_n + 1} .$$

To see what this means, let $Z_n = b_n(x)/B_n(x)$, where $b_n(x)$ and $B_n(x)$ are polynomials.

$$\frac{b_{n+1}(x)}{B_{n+1}(x)} = \frac{x + b_n(x)/B_n(x)}{x + 1 + b_n(x)/B_n(x)} = \frac{xB_n(x) + b_n(x)}{(x + 1)B_n(x) + b_n(x)}$$

so that

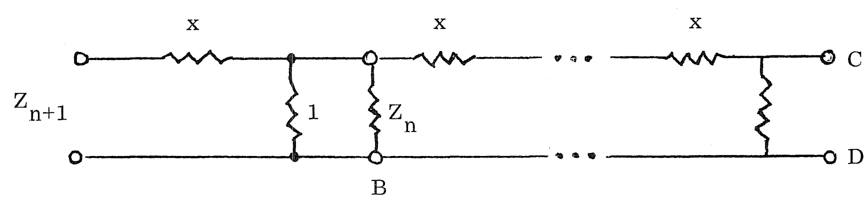
$$\begin{cases} b_{n+1}(x) = xB_n(x) + b_n(x) \\ B_{n+1}(x) = (x+1)B_n(x) + b_n(x) \end{cases} \tag{2.1}$$

which is a mixed recurrence relation for the two polynomial sequences. Clearly, $Z_0 = 1$, so we set $b_0(x) = 1$ and $B_0(x) = 1$. This completely specifies the two sequences which we call the Morgan-Voyce polynomials.

Without too much trouble, one can derive that both sequences $\{b_n(x)\}$ and $\{B_n(x)\}$ satisfy

$$U_{n+2}(x) = (x+2)U_{n+1}(x) - U_n(x) \,. \tag{2.2}$$

This takes care of the resistance as seen from the output end of the ladder network. We now go to the input end, and consider input Z_i.



$$\frac{1}{R} = \frac{1}{Z_n} + \frac{1}{1}, \quad \text{or,} \quad R = \frac{Z_n}{Z_n + 1}$$

$$Z_{n+1} = x + \frac{Z_n}{Z_n + 1} = \frac{xZ_n + x + Z_n}{Z_n + 1} \,.$$

Again let $Z_n = P_n(x)/Q_n(x)$. Then,

$$\frac{P_{n+1}(x)}{Q_{n+1}(x)} = \frac{x(P_n(x) + Q_n(x)) + P_n(x)}{P_n(x) + Q_n(x)} \,.$$

That is,

$$P_{n+1}(x) = (x+1)P_n(x) + xQ_n(x) \,,$$

$$Q_{n+1}(x) = P_n(x) + Q_n(x) \,.$$

Simplifying,

$$P_n(x) = Q_{n+1}(x) - Q_n(x)$$

$$Q_{n+2}(x) - Q_{n+1}(x) = (x+1)(Q_{n+1}(x) - Q_n(x)) + xQ_n(x)$$

or

$$Q_{n+2}(x) = (x+2)Q_{n+1}(x) - Q_n(x) \,.$$

From the case $n = 1$, we see that $P_1(x) = x + 1$, $Q_1(x) = 1$, $Q_2(x) = x + 2$, so that $Q_n(x) \equiv B_n(x)$ from the output considerations earlier, and

$$P_n(x) = Q_{n+1}(x) - Q_n(x) = B_{n+1}(x) - B_n(x) .$$

But, recalling the defining equation (2.1) for the Morgan-Voyce polynomials, a simple subtraction gives us $b_{n+1}(x) = B_{n+1}(x) - B_n(x)$. Thus, $P_n(x) \equiv b_{n+1}(x)$ so that

$$Z_n = \frac{b_{n+1}(x)}{B_n(x)} ,$$

where $b_n(x)$ and $B_n(x)$ are the Morgan-Voyce polynomials. This is the resistance as seen looking into the ladder network from the input end.

There are now several theorems we can prove.

3. THEORETICAL CONSIDERATIONS

Using the recursion (2.2) for $b_n(x)$ and $B_n(x)$, it is a simple matter to compute the first few Morgan-Voyce polynomials.

n	$b_n(x)$	$B_n(x)$
0	1	1
1	$x + 1$	$x + 2$
2	$x^2 + 3x + 1$	$x^2 + 4x + 3$
3	$x^3 + 5x^2 + 6x + 1$	$x^3 + 6x + 10x + 4$
4	$x^4 + 7x^3 + 15x^2 + 10x + 1$	$x^4 + 8x^3 + 21x^2 + 20x + 5$
5	$x^5 + 9x^4 + 28x^3 + 35x^2 + 15x + 1$	$x^5 + 10x^4 + 36x^3 + 56x^2 + 35x + 6$

$$b_{n+2}(x) = (x + 2)b_{n+1}(x) - b_n(x)$$
$$B_{n+2}(x) = (x + 2)B_{n+1}(x) - B_n(x) .$$

Comparing these polynomials to the Fibonacci polynomials $f_n(x)$, $f_0(x) = 0$, $f_1(x) = 1$, $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$, leads to some fascinating results.

FIBONACCI POLYNOMIALS

n	$f_n(x)$									
			1							
1	1	f_1	1	1						
2	x	f_2	1	2	1					
3	$x^2 + 1$	f_3	1	3	3	1				
4	$x^3 + 2x$	f_4	1	4	6	4	1			
5	$x^4 + 3x^2 + 1$	f_5	1	5	10	10	5	1		
6	$x^5 + 4x^3 + 3x$	f_6	1	6	15	20	15	6	1	
7	$x^6 + 5x^4 + 6x^2 + 1$	f_7	1	7	21	35	35	21	7	1
8	$x^7 + 6x^5 + 10x^3 + 4x$	f_8								

Theorem 3.1. See [3], [5]. The polynomial sequences $\{b_n(x)\}$, $\{B_n(x)\}$, and $\{f_n(x)\}$ are related by

$$f_{2n}(x) = xB_{n-1}(x^2)$$
$$f_{2n+1}(x) = b_n(x^2) .$$

Proof 1. By Generating Functions.

It is not difficult to show that

$$\frac{1 - \lambda}{1 - (x + 2)\lambda + \lambda^2} = \sum_{n=0}^{\infty} b_n(x)\lambda^n$$

$$\frac{\lambda}{1 - (x + 2)\lambda + \lambda^2} = \sum_{n=0}^{\infty} B_{n-1}(x)\lambda^n .$$

Hence,

$$\frac{\lambda(1 - \lambda^2)}{1 - (x^2 + 2)\lambda^2 + \lambda^4} = \sum_{n=0}^{\infty} b_n(x^2)\lambda^{2n+1}$$

$$\frac{\lambda^2 x}{1 - (x^2 + 2)\lambda^2 + \lambda^4} = \sum_{n=0}^{\infty} xB_{n-1}(x^2)\lambda^{2n} .$$

Adding these gives

$$\frac{\lambda(1 + \lambda x - \lambda^2)}{1 - 2\lambda^2 + \lambda^4 - x^2\lambda^2} = \frac{\lambda}{1 - x\lambda - \lambda^2} = \sum_{n=0}^{\infty} f_n(x)\lambda^n ,$$

where we recognized the generating function for the Fibonacci polynomials $\{f_n(x)\}$.

Proof 2. By Binét Forms.

Since the Fibonacci polynomials have the auxiliary equation

$$Y^2 = xY + 1 ,$$

which arises from the recurrence relation and which has roots

$$\alpha = \frac{x + \sqrt{x^2 + 4}}{2}, \qquad \beta = \frac{x - \sqrt{x^2 + 4}}{2},$$

it can be shown by mathematical induction that the Fibonacci polynomials have the Binét form

$$f_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta) .$$

Similarly, from the recurrence relation for the Morgan-Voyce polynomials, we have the auxiliary equation

$$Y^2 = (x + 2)Y - 1$$

with roots

$$r = \frac{x + 2 + \sqrt{x^2 + 4x}}{2}, \qquad s = \frac{x + 2 - \sqrt{x^2 + 4x}}{2},$$

leading to, via mathematical induction,

$$B_{n-1}(x) = (r^n - s^n)/(r - s) .$$

Then,

$$f_{2n}(x) = (\alpha^{2n} - \beta^{2n})/(\alpha - \beta) = [(\alpha^2)^n - (\beta^2)^n]/(\alpha - \beta)$$

$$= \left[\left(\frac{x^2 + 2 + x\sqrt{x^2 + 4}}{2}\right)^n - \left(\frac{x^2 + 2 - x\sqrt{x^2 + 4}}{2}\right)^n\right]\Big/\sqrt{x^2 + 4} .$$

On the other hand,

$$B_{n-1}(x^2) = \left[\left(\frac{x^2 + 2 + \sqrt{x^4 + 4x^2}}{2}\right)^n - \left(\frac{x^2 + 2 - \sqrt{x^4 + 4x^2}}{2}\right)^n\right]\Big/\sqrt{x^4 + 4x^2}$$

Notice that, since $\sqrt{x^4 + 4x^2} = |x|\sqrt{x^2 + 4}$,

$$xB_{n-1}(x^2) = f_{2n}(x) .$$

Since $b_{n+1}(x) = B_{n+1}(x) - B_n(x)$,

$$xb_{n+1}(x^2) = xB_{n+1}(x^2) - xB_n(x^2)$$

$$= f_{2n+4}(x) - f_{2n+2}(x) = xf_{2n+3}(x) ,$$

leading to

$$b_{n+1}(x^2) = f_{2n+3}(x) \quad \text{or} \quad b_n(x^2) = f_{2n+1}(x) .$$

Proof 3. By the Recurrence Relations.

Observe that

$$b_0(x) = 1, \quad b_1(x) = x + 1, \quad b_{n+2}(x) = (x + 2)b_{n+1}(x) - b_n(x) ;$$

$$f_1(x) = 1, \quad f_3(x) = x^2 + 1, \quad f_{2n+5}(x) = (x^2 + 2)f_{2n+3}(x) - f_{2n+1}(x) .$$

Thus,

$$b_0(x^2) = 1, \quad b_1(x^2) = x^2 + 1, \quad b_{n+2}(x^2) = (x^2 + 2)b_{n+1}(x^2) - b_n(x^2) .$$

Now, the sequences $\{b_m(x^2)\}$ and $\{f_{2m+1}(x)\}$ have both the same starting pair and the same recurrence relation so that they are the same sequence. Similarly,

$$B_0(x) = 1, \quad B_1(x) = x + 2 , \quad B_{n+2}(x) = (x + 2)B_{n+1}(x) - B_n(x) ;$$

$$f_2(x) = x, \quad f_4(x) = x^3 + 2x, \quad f_{2n+6}(x) = (x^2 + 2)f_{2n+4}(x) - f_{2n}(x) .$$

Next,

$$xB_0(x^2) = x, \quad xB_1(x^2) = x^3 + 2x, \quad xB_{n+2}(x^2) = (x^2+2)xB_{n+1}(x^2) - xB_n(x^2),$$

so that the sequences $\{xB_{n-1}(x^2)\}$ and $\{f_{2n}(x)\}$ are the same.

Several results follow immediately by applying known properties of the Fibonacci polynomials. (See [3], [6], [7].)

Corollary 3.1.1.

$$b_n(1) = F_{2n+1} \quad \text{and} \quad B_{n-1}(1) = F_{2n}$$

for the Fibonacci numbers F_n.

Corollary 3.1.2. The coefficients of $b_n(x)$ and $B_n(x)$ lie on adjacent rising diagonals of Pascal's triangle.

Corollary 3.1.3. The polynomials $\{b_n(x)\}$ are irreducible if and only if $2n+1$ is a prime.

4. FURTHER PROPERTIES OF MORGAN-VOYCE POLYNOMIALS

Let

$$Q = \begin{pmatrix} x+2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$Q^2 = \begin{pmatrix} x+2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x+2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x^2+4x+3 & -(x+2) \\ x+2 & -1 \end{pmatrix} = \begin{pmatrix} B_3(x) & -B_2(x) \\ B_2(x) & -B_1(x) \end{pmatrix}.$$

It can be proved by induction [10] that

$$Q^n = \begin{pmatrix} B_{n+1}(x) & -B_n(x) \\ B_n(x) & -B_{n-1}(x) \end{pmatrix}.$$

Then, since $\det Q^n = (\det Q)^n$,

$$B_{n+1}(x)B_{n-1}(x) - B_n^2(x) = -1.$$

Thus, one can write much by virtue of having $B_n(x)$ trapped in a matrix.

Let

$$R = \begin{pmatrix} x+2 & -2 \\ 2 & -(x+2) \end{pmatrix}, \qquad RQ^n = \begin{pmatrix} C_{n+1}(x) & -C_n(x) \\ C_n(x) & -C_{n-1}(x) \end{pmatrix},$$

so that

$$C_{n+1}(x)C_{n-1}(x) - C_n^2(x) = -(x^2+4x+4) + 4 = -(x^2+4x).$$

Then, $C_n(x)$ corresponds to the Lucas sequence.

Let $\{L_n(x)\}$ be the Lucas polynomial sequence, $L_0(x) = 2$, $L_1(x) = x$, $L_2(x) = x^2 + 2$, $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$. Actually,

$$L_n(x) = f_{n+1}(x) + f_{n-1}(x) ,$$

and for $x = 1$, $L_n(1) = L_n$, the n^{th} member of the Lucas sequence 1, 3, 4, 7, 11, 18, 29, $\cdots$.

Now, $C_{-1}(x) = 2$, $C_0(x) = 2$, $C_1(x) = x + 2$. Thus, since

$$L_{2n+4}(x) = (x^2 + 2)L_{2n+2}(x) - L_{2n}(x) ,$$

we have $L_{2n}(x) = C_{n-1}(x^2)$, and $C_{n-1}(1) = L_{2n}$, a Lucas number with even subscript. Also, since

$$L_{2n}(x) = f_{2n+1}(x) + f_{2n-1}(x), \qquad \text{and} \qquad f_{2n+1}(x) = b_n(x^2),$$

the relationship $L_{2n}(x) = C_{n-1}(x^2)$ implies that

$$C_n(x) = b_n(x) + b_{n+1}(x) .$$

Also,

$$xB_n(x) = b_{n+1}(x) - b_n(x) ,$$

so that

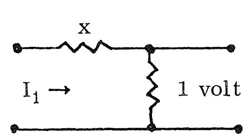
$$b_{n+1}(x) = [C_n(x) + xB_n(x)]/2 .$$

Finally, applying the divisibility properties of Lucas polynomials [6], [8], [9], we have the

Theorem. $C_{2^n}(x)$ is irreducible.

5. ATTENUATION RESULTS

The attenuation is the ratio of input voltage V_I to output voltage V_O. Since the system is linear, we can assume that the output voltage is 1V. Let us start with no resistive network. There is no current $(I_O = 0)$ and between the terminals is 1 volt $(V_O = 1)$.



$I_1 = 1$ amp $\qquad V_1 = x + 1$ volts

So we see that

$$I_0 = 0 = B_{-1}(x), \qquad V_0 = 1 = b_{-1}(x) ,$$
$$I_1 = 1 = B_0(x) , \qquad V_1 = 1 = B_0(x) .$$

We shall see that

$$I_n = B_{n-1}(x) \quad \text{and} \quad V_n = b_{n-1}(x) .$$

First, we note that from $b_{n+1}(x) = xB_n(x) + b_n(x)$ and from

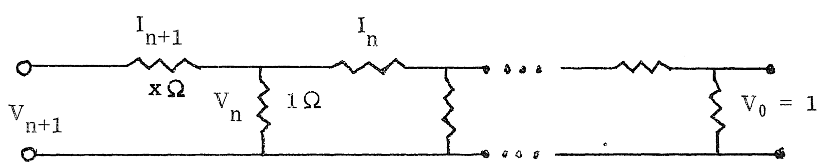
$$B_{n+1}(x) = (x+1)B_n(x) + b_n(x) = B_n(x) + xB_n(x) + b_n(x),$$

we have the lemma,

Lemma 1.

$$B_{n+1}(x) = B_n(x) + b_{n+1}(x) .$$

In the ladder network, the voltage across the n^{th} unit resistance is V_n; hence, the current is also V_n.



Now, the voltage currents obey

$$V_{n+1} = xI_{n+1} + V_n, \qquad I_{n+1} = V_n + I_n .$$

Now assume that $I_n = B_{n-1}(x)$ and $V_n = b_n(x)$. Then,

$$V_{n+1} = xB_n(x) + b_n(x) = b_{n+1}(x) ,$$
$$I_{n+1} = b_n(x) + B_{n-1}(x) = B_n(x) ,$$

applying Lemma 1 to the expression for I_{n+1}, which completes the induction.

We note that

$$V_{n+1} = b_{n+1}(x) = x[B_n(x) + B_{n-1}(x) + \cdots + B_0(x) + 1] ;$$
$$B_n(x) = I_{n+1} = V_n + V_{n-1} + \cdots + V_0 = b_n(x) + b_{n-1}(x) + \cdots + b_0(x) .$$

These follow directly from the special resistive network.

REFERENCES

1. A. M. Morgan-Voyce, "Ladder Network Analysis Using Fibonacci Numbers," IRE Transactions on Circuit Theory, Vol. CT-6, Sept. 1959, pp. 321-322.
2. S. L. Basin, "The Appearance of Fibonacci Numbers and the Q-matrix in Electrical Network Theory," Mathematics Magazine, Vol. 36, Mar.-Apr. 1963, pp. 84-97.
3. Richard A. Hayes, Fibonacci and Lucas Polynomials, Master's Thesis, San Jose State College, San Jose, Calif., 1965.
4. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, No. 1, Feb. 1966, pp. 73-81.
5. V. E. Hoggatt, Jr., Problem H-73, Fibonacci Quarterly, Vol. 3, No. 4, Dec. 1965; Solution by D. A. Lind, Vol. 5, No. 3, Oct. 1967, p. 256.
6. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII — An Introduction to Fibonacci Polynomials and their Divisibility Properties," Fibonacci Quarterly, Vol. 8, No. 4, Oct. 1970, pp. 407-420.
7. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5, Dec. 1969, pp. 457-463.
8. V. E. Hoggatt, Jr., and C. T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," Fibonacci Quarterly, Vol. 12, No. 2 (Apr. 1974), pp. 113-120.
9. V. E. Hoggatt, Jr., and Gerald Bergum, "Irreducibility of Lucas and Generalized Lucas Polynomials," Fibonacci Quarterly, Vol. 12, No. 1 (Feb. 1974), pp. 95-100.
10. M. N. S. Swamy, "Further Properties of Morgan-Voyce Polynomials," Fibonacci Quarterly, Vol. 6, No. 2, Apr. 1968, pp. 167-175.

(Continued from page 146.)

The material consists of two pages of explanation, six pages of tables for systematizing the work of finding the Fibonacci and Lucas expressions in parentheses, and 78 pages of formulas. There are 625 formulas in all arranged in categories according to the difference relation from which they are derived.

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GENERALIZATION OF HERMITE'S DIVISIBILITY THEOREMS AND THE MANN - SHANKS PRIMALITY CRITERION FOR s-FIBONOMIAL ARRAYS

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1. INTRODUCTION

In a previous paper [4] I found that two theorems of Hermite concerning factors of binomial coefficients might be extended to generalized binomial coefficients [2], however one of my proofs imposed severe restrictions on the sequence $\{A_n\}$ used to define the generalized coefficients. Also it was found that the Mann-Shanks primality criterion [6] follows from one of the Hermite theorems and it appeared evident that the criterion also held in the Fibonomial array, but the proof was not completed.

In the present paper I remove all these defects by proving the Hermite theorems in a more elegant manner so that very little needs to be assumed for the generalized array, and the Mann-Shanks criterion is not only proved for the Fibonomial array but for the s-Fibonomial and q-binomial arrays. Some typographical errors in [4] are also corrected.

2. THE GENERALIZED HERMITE THEOREMS

Let $\{A_n\}$ be a sequence of integers with $A_0 = 0$, $A_n \neq 0$ for all $n \geq 1$, and otherwise arbitrary. Define generalized binomial coefficients by

$$\left\{ {n \atop k} \right\} = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1}, \quad \text{with} \quad \left\{ {n \atop 0} \right\} = 1 . \tag{2.1}$$

These generalize the ordinary binomial coefficients which occur for $A_k = k$ identically. Properties of the array and their history may be found in [2]. Our attention here is fixed on the case when these coefficients are all integers. Arithmetic properties are then of primary concern. As usual, (a,b) will mean the greatest common divisor of a and b, and $a|b$ means a divides b. We may now state:

<u>Theorem 1.</u>

$$\frac{A_n}{(A_n, A_k)} \;\Big|\; \left\{ {n \atop k} \right\} \tag{2.2}$$

and

$$\frac{A_{n-k+1}}{(A_{n+1}, A_k)} \;\Big|\; \left\{ {n \atop k} \right\} , \tag{2.3}$$

provided only that in (2.3) we suppose $(A_{n+1}, A_k) | A_{n-k+1}$. Of course, in (2.2) we always have $(A_n, A_k) | A_n$, so that (2.3) is only slightly less general than (2.2).

In [4] I stated that (2.3) holds provided $A_{n+1} - A_k = A_{n+1-k}$ or something close to this. We shall see that no such assumption is necessary.

Proof of (2.2). By the Euclidean algorithm we know that there exist integers x and y such that $(A_n, A_k) = xA_n + yA_k$. Therefore

$$(A_n, A_k)\left\{{n \atop k}\right\} = xA_n\left\{{n \atop k}\right\} + yA_k\left\{{n \atop k}\right\} = xA_n\left\{{n \atop k}\right\} + yA_n\left\{{n-1 \atop k-1}\right\} = A_n \cdot E ,$$

for some integer E. Since $(A_n, A_k) | A_n$ we have proved that (2.2) is true.

Proof of (2.3). Again, for some integers x and y, $(A_{n+1}, A_k) = xA_{n+1} + yA_k$, whence

$$\begin{aligned}(A_{n+1}, A_k)\left\{{n \atop k}\right\} &= xA_{n+1}\left\{{n \atop k}\right\} + yA_k\left\{{n \atop k}\right\}\\ &= xA_{n+1-k}\left\{{n+1 \atop k}\right\} + yA_{n+1-k}\left\{{n \atop k-1}\right\} = A_{n+1-k} \cdot F ,\end{aligned}$$

for some integer F. Thus we have proved in general that

$$A_{n+1-k} \,\Big|\, (A_{n+1}, A_k)\left\{{n \atop k}\right\} , \tag{2.4}$$

and when we suppose that $(A_{n+1}, A_k) | A_{n+1-k}$ we obtain (2.3).

The proof I tried in [4] motivated by Hermite's own argument ran as follows: We have

$$(A_{n+1}, A_k) = xA_{n+1} + yA_k = x(A_{n+1} - A_k) + (x+y)A_k ,$$

whence

$$\begin{aligned}(A_{n+1}, A_k)\left\{{n \atop k}\right\} &= x(A_{n+1} - A_k)\left\{{n \atop k}\right\} + (x+y)A_k\left\{{n \atop k}\right\}\\ &= x\,\frac{A_{n+1} - A_k}{A_{n+1-k}}\,A_{n+1-k}\left\{{n \atop k}\right\} + (x+y)A_{n+1-k}\left\{{n \atop k-1}\right\} ,\end{aligned}$$

and from this, if we suppose that $A_{n+1} - A_k = A_{n+1-k}$, as stated in [4], we could obtain (2.3), because this also implies $(A_{n+1}, A_k) | A_{n+1-k}$. We may also merely suppose that $A_{n+1-k} | A_{n+1} - A_k$ and we shall have proved (2.4), but as seen in our general proof none of these assumptions is necessary. Hermite's device of shifting terms around does not generalize, but then also the shifting is not needed.

In the proof of (2.2) we have used the obvious fact that

$$A_k\left\{{n \atop k}\right\} = A_n\left\{{n-1 \atop k-1}\right\} ,$$

and in our proof of (2.3) we used the obvious relations

$$A_{n+1}\left\{{n \atop k}\right\} = A_{n+1-k}\left\{{n+1 \atop k}\right\} \quad\text{and}\quad A_k\left\{{n \atop k}\right\} = A_{n+1-k}\left\{{n \atop k-1}\right\} ,$$

simple analogues of corresponding formulas for ordinary binomial coefficients.

As our results apply to the Fibonacci numbers, and Fibonomial coefficients, it still seems necessary to know that $(F_a, F_b) = F_{(a,b)}$ if only to get an easy proof that $(F_{n+1}, F_n) | F_{n+1-k}$ so that we can have (2.3) as well as (2.2). Thus we have

$$(F_{n+1}, F_k) = F_{(n+1,k)} = F_{(n+1-k,k)} = (F_{n+1-k}, F_k)$$

which means that $(F_{n+1}, F_k) | F_{n+1-k}$. In any event, our results are obtained more elegantly by our present proofs.

According to Dickson's History [1, p. 265] Th. Schönemann in 1839 proved that

$$\frac{(a,b,\cdots,m)(a+b+\cdots+m-1)!}{a!\,b!\,\cdots\,m!} \tag{2.5}$$

is an integer. The situation for two integers a, b is just that

$$\frac{(a,b)(a+b-1)!}{a!\,b!} \tag{2.6}$$

is an integer. This follows at once from Hermite's original form of (2.2), because by putting

$$H(n,k) = \frac{(n,k)}{n}\begin{Bmatrix} n \\ k \end{Bmatrix},$$

which is an integer, then clearly

$$H(a+b,\,b) = \frac{(a+b,\,b)}{a+b}\begin{Bmatrix} a+b \\ b \end{Bmatrix} = \frac{(a,b)(a+b-1)!}{a!\,b!}$$

must be an integer. The multinomial extension of Schönemann follows readily from Hermite's theorem. I was reminded of these things by a letter from Gupta [5] who remarked that a nice Fibonomial extension of (2.6) would be that

$$\frac{F_{(m,n)}[m+n-1]!}{[m]!\,[n]!} \tag{2.7}$$

is an integer. This, of course, follows at once from (2.2) when $A_n = F_n$ and we define generalized factorials by

$$[n]! = A_n A_{n-1}\cdots A_2 A_1\,, \qquad \text{with} \qquad [0]! = 1\,. \tag{2.8}$$

Indeed, the more general assertion from (2.2) is that since

$$H(n,k) = \frac{(A_n,\,A_k)}{A_n}\begin{Bmatrix} n \\ k \end{Bmatrix}$$

is an integer, so also is

$$H(m+n,\,n) = \frac{(A_{m+n},\,A_n)}{A_{m+n}}\begin{Bmatrix} m+n \\ n \end{Bmatrix} = \frac{(A_{m+n},\,A_n)[m+n-1]!}{[m]!\,[n]!} \tag{2.9}$$

an integer.

According to Dickson [1, p. 265] Cauchy also proved Schönemann's theorem for (2.5), and Catalan (1874) proved that (2.6) is an integer in case $(a,b) = 1$.

Catalan, Segner, Euler, etc., found that $(n+1)\,\Big|\begin{Bmatrix} 2n \\ n \end{Bmatrix}$ by combinatorial or geometrical arguments. See my bibliography [3] for a list of 243 items dealing with the Catalan numbers, ballot numbers, and related matters. A supplement of over 75 items is being prepared.

The fact that $(n+1)\,\Big|\begin{Bmatrix} 2n \\ n \end{Bmatrix}$ follows at once from (2.3) so that the number

$$C(n,k) = \frac{(A_{n+1}, A_k)}{A_{n+1-k}} \left\{ {n \atop k} \right\} \tag{2.10}$$

is a natural generalization. Unfortunately, even in the case $A_n = F_n$ we do not yet have a suitable combinatorial interpretation of this number.

3. THE MANN-SHANKS CRITERION FOR FIBONOMIALS

In [4] we gave some alternative formulations of the elegant Mann-Shanks primality criterion [6]. In particular we noted that their beautiful theorem may be written in the form:

$$\left\{ \begin{array}{l} C = \text{prime if and only if } R \,\Big|\, \binom{R}{C - 2R} \\ \text{for every integer } R \text{ such that } C/3 \le R \le C/2, \; R \ge 1 . \end{array} \right. \tag{3.1}$$

Here R and C are the row and column numbers, respectively, in the original Mann-Shanks shifted binomial array. We showed that when C is a prime the indicated divisibility follows at once from Hermite's form of (2.2).

The corresponding theorem for Fibonomial coefficients (i.e., with $A_n = F_n$ in (2.1)) is also true. That is, we have

Theorem 2. In the Fibonomial coefficient array,

$$\left\{ \begin{array}{l} C = \text{prime if and only if } F_R \,\Big|\, \left\{ {R \atop C - 2R} \right\} \\ \text{for every integer } R \text{ such that } C/3 \le R \le C/2, \; R \ge 1 . \end{array} \right. \tag{3.2}$$

Note that the single difference between this and (3.1) is that the row number R must be replaced by the corresponding Fibonacci number F_R. When C = prime, the divisibility follows from (2.2) since this implies that $F_R/(F_R, F_{C-2R})$ is a factor of the Fibonomial coefficient; however we also have

$$(F_R, F_{C-2R}) = F_{(R,C-2R)} = F_{(R,C)} = F_1 = 1$$

when $C/3 \le R \le C/2$. Thus, we have only to consider the case when C is composite. Our proof is just a slight modification of the proof given by Mann-Shanks. Suppose $C = 2k$, with $k = 0, 2, 3, 4, \cdots$; then the unit $\left\{ {k \atop 0} \right\} = 1$ always occurs in the column, so divisibility cannot occur, and it is sufficient to consider odd composite C. Let p be an odd prime factor of C, and write $C = p(2k + 1)$, with $k \ge 1$. Choose $R = pk$. Then the coefficient in the R-row and C-column is $\left\{ {kp \atop p} \right\}$, and

$$\frac{1}{F_{pk}} \left\{ {kp \atop p} \right\} = \frac{F_{pk} \cdot F_{pk-1} \cdot F_{pk-2} \cdot \cdots \cdot F_{pk-p+1}}{F_{pk} \cdot F_p \cdot F_{p-1} \cdot \cdots \cdot F_1}$$

Cancel F_{pk} with F_{pk}. The factors $F_{p-1}, F_{p-2}, \cdots, F_1$ in the denominator cannot affect the possible divisibility of F_p into the numerator since

$$(F_p, F_{p-r}) = F_{(p,p-r)} = F_{(p,r)} = F_1 = 1 \quad \text{for all} \quad 1 \le r \le p-1,$$

while on the other hand F_p is relatively prime to every factor in the numerator since

$$(F_p, F_{pk-j}) = F_{(p,pk-j)} = F_{(p,j)} = F_1 = 1 \quad \text{for all} \quad 1 \le j \le p-1.$$

and so F_p, which is greater than 1 for odd primes p, cannot divide into the numerator. This means, equivalently, that the row number F_{pk} cannot divide the coefficient $\left\{ {kp \atop p} \right\}$. The proof is complete.

Our proof is a modification of the Mann-Shanks argument using the fact again that

$$(F_a, F_b) = F_{(a,b)} .$$

4. THE MANN-SHANKS CRITERION FOR s-FIBONOMIAL ARRAYS

The s-Fibonomial coefficients follow from (2.1) when we set $A_n = F_{sn}$, s being any positive integer. Our theorem 2 above handles the case $s = 1$. We now have

Theorem 3. In the s-Fibonomial array, the Mann-Shanks criterion is true. That is,

$$(4.1) \qquad \begin{cases} C = \text{prime if and only if } \dfrac{F_{sR}}{F_s} \,\Big|\, \left\{ {R \atop C-2R} \right\}_s \\ \text{for every integer } R \text{ such that } C/3 \le R \le C/2, \ R \ge 1 . \end{cases}$$

To see the motivation, consider Hermite's extended theorem (2.2) with $A_n = F_{sn}$. We see that $F_{sR}/(F_{sR}, F_{sC-2sR})$ is a factor of the coefficient in the R-C position of the Mann-Shanks type array. But when C = prime we have

$$(F_{sR}, F_{sC-2sR}) = F_{(sR, sC-2sR)} = F_{(sR,sC)} = F_{s(R,C)} = F_s ,$$

since C = prime implies $(R,C) = 1$ for each $C/3 \le R \le C/2$, $R \ge 1$. Thus (2.2) yields F_{sR}/F_s as a factor. By the way, it is a known fact that $F_s \mid F_{sR}$. To prove the converse case, when C is composite, first assume $C = 2k$, $k = 0, 2, 3, 4, \cdots$. Then again the unit $\left\{ {k \atop 0} \right\} = 1$ occurs in the column; so that it is sufficient to study the situation for odd composite C. Let p be an odd prime factor of C, and put $C = p(2k+1)$, $k \ge 1$. Choose as before $R = pk$. Then the coefficient in the R–C spot is the s–Fibonomial coefficient $\left\{ {kp \atop p} \right\}$. We find now that

$$\frac{F_s}{F_{spk}} \left\{ {kp \atop p} \right\}_s = \frac{F_s}{F_{spk}} \; \frac{F_{spk} \cdot F_{spk-s} \cdot F_{spk-2s} \cdot \cdots \cdot F_{spk-sp+s}}{F_{sp} \cdot F_{sp-s} \cdot \cdots \cdot F_{3s} F_{2s} F_s} .$$

Cancel F_s and F_{spk}. Now it is easy to see that

$$(F_{sp}, F_{sp-sr}) = F_{(sp, sp-sr)} = F_{(sp, sr)} = F_{s(p, r)} = F_s$$

for all $1 \leq r \leq p - 1$. Also,

$$(F_{sp}, F_{spk-sj}) = F_{(sp, spk-sj)} = F_{(sp, sj)} = F_{s(p, j)} = F_s$$

for all $1 \leq j \leq p - 1$. Remove the common factor F_s throughout. We see now that

$$\left(\frac{F_{sp}}{F_s}, \frac{F_{sp-sr}}{F_s}\right) = 1, \quad \text{for all} \quad 1 \leq r \leq p - 1,$$

and

$$\left(\frac{F_{sp}}{F_s}, \frac{F_{spk-sj}}{F_s}\right) = 1, \quad \text{for all} \quad 1 \leq j \leq p - 1.$$

Also, $F_{sp}/F_s > 1$, and we find that F_{sp}/F_s cannot divide the numerator; equivalently we have shown that F_{spk}/F_s cannot divide the s-Fibonomial coefficient so that our proof is complete.

It would appear that a Fibonacci-type property (a homomorphism)

$$(4.2) \qquad (A_a, A_b) = A_{(a,b)}$$

would be very useful for proving Mann-Shanks type criteria in general arrays.

5. THE MANN-SHANKS CRITERION FOR q-BINOMIAL ARRAYS

The q-binomial or Gaussian coefficients are defined by

$$(5.1) \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{q^{n-j+1} - 1}{q^j - 1}, \quad \text{with} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1 .$$

They are polynomials in q. Since in fact $(q^a - 1, q^b - 1) = q^{(a,b)} - 1$, it is not surprising now that we can assert the Mann-Shanks criterion for the q-binomial array. The q-analogue of (3.1) is motivated by Hermite's generalized theorem (2.2) for we now have that the coefficient in the R-C position is divisible by

$$\frac{q^R - 1}{(q^R - 1, q^{C-2R} - 1)} ,$$

which reduces to

$$\frac{q^R - 1}{q - 1}$$

when C is a prime and $C/3 \le R \le C/2$, $R \ge 1$. Consequently we are led to the following:

Theorem 4. The Mann-Shanks criterion for primality holds in the q-binomial array. That is:

$$(5.2)\qquad \left\{\begin{array}{l} C = \text{prime if and only if } \dfrac{q^R - 1}{q - 1} \,\Big|\, \begin{bmatrix} R \\ C - 2R \end{bmatrix} \\ \text{for every integer } R \text{ such that } C/3 \le R \le C/2, \quad R \ge 1, \\ \text{and where the q-binomial coefficients are defined by (5.1).} \end{array}\right.$$

The proof is left to the reader.

In each of the cases we have presented in this paper, the first non-trivial instance of the non-divisibility by a row number occurs when $C = 25$. The next case is then $C = 35$. Up to this point a row number fails to divide an array number because of the presence of a unit in the column. $C = 25$ and 35 are the first composite numbers where no unit appears. The next such numbers are 49, 55, 65, 77, 85, 95, corresponding to those numbers of form $6j \pm 1$ which are composite.

The column entries for $C = 25$ in the ordinary Pascal case are 36, 252, 165, 12, with corresponding row numbers 9, 10, 11, 12. 10 fails to divide 252, while the other row numbers divide their column entries. Similarly, for the Fibonomial array, the column entries are 714, 136136, 83215, 144, with row numbers 34, 55, 89, 144. Here 55 fails to divide 136136. In the q-binomial array, the column entries are

$$\frac{(q^9 - 1)(q^8 - 1)}{(q^2 - 1)(q - 1)}, \quad \frac{(q^{10} - 1)(q^9 - 1)(q^8 - 1)(q^7 - 1)(q^6 - 1)}{(q^5 - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)},$$

$$\frac{(q^{11} - 1)(q^{10} - 1)(q^9 - 1)}{(q^3 - 1)(q^2 - 1)(q - 1)}, \quad \frac{(q^{12} - 1)}{q - 1}.$$

The corresponding row numbers are

$$(q^9 - 1)/(q - 1), \quad (q^{10} - 1)/(q - 1), \quad (q^{11} - 1)/(q - 1), \quad \text{and} \quad (q^{12} - 1)/(q - 1).$$

It is again, of course, the second row number that fails to divide the coefficient in the column. For arrays of the type we are studying this behavior is typical.

The column entries for $C = 35$ in the Pascal array are 12, 715, 3432, 3003, 560, 17, with row numbers 12, 13, 14, 15, 16, 17. Here $14 \nmid 3432$, and $15 \nmid 3003$. For the Fibonomial array the entries are 144, 27372840, 14169550626, 22890661872, 113490195, 1597, with row numbers 144, 233, 377, 610, 987, 1597, and the row numbers 377 and 610 are the ones which fail to divide their corresponding column entries.

6. GENERALIZED MANN-SHANKS CRITERIA

By placing units in the (R, 2R) and (R, 3R) positions in their rectangular array and carefully choosing the other entries (which turned out to be binomial coefficients) Mann and Shanks developed a kind of sieve which tests numbers of the form $6j \pm 1$ for primality. This suggests that there may be ways to devise similar sieves based on other arithmetic progressions. After all, it is a very old theorem of Dirichlet that if $(a,b) = 1$ then there are infinitely many primes of the form $a + bt$, where t ranges over the integers. We might expect then to find a criterion similar to that of Mann-Shanks by using the progressions $4j \pm 1$ for example. Although I have not found any simple formula for generating the entries in an array, I can suggest some obvious necessary properties of such an array, by analogy with the original Mann-Shanks array. Below is presented an outline for such an array:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1																					
1			1	-	1																	
2					1	2	-	2	1													
3							1	3	-	*	-	3	1									
4									1	*	-	4	-	4	-	*	1					
5											1	5	-	5	-	*	-	5	-	5	1	
6													1	6	-	*	-	6	-	6	-	*
7															1	*	-	7	-	7	-	*
8																	1	8	-	8	-	*
9																			1	9	-	*
10																					1	*

Numbers listed above are the smallest factors which an entry must have in order to be allowed, so that the row number will divide each entry in a prime column. This guarantees that a prime will correspond to the row-column divisibility property desired. However, of the remaining entries, those spots marked by a dash (-) can be filled arbitrarily, while those marked by a star (*) must be chosen so that at least one of the starred numbers in each column will not be divisible by the row number. Such special column numbers are 9, 15, 21, 25, 27, etc. One may imagine that it would be desirable to have a symmetrical row, in analogy to the binomial coefficients, though this may not be desired. However, it seems worth exploring. The first few rows suggest such symmetry. For this reason, I place a factor of 7 in the $R = 7$, $C = 25$ position to preserve symmetry in that row, etc. It would be very remarkable if we could determine simple formulas for generating such generalized Mann-Shanks arrays based on Dirichlet progressions.

In the outline array based on $4j \pm 1$, it is easy to see that the bottom star in the special columns will always occur for row number $(K - 1)/2$, where $K = 4j \pm 1 \neq$ prime. If we choose an entry for that position which is not divisible by the row number and otherwise fill open spots in the array by the row number in any given row, we shall obtain the following array having the Mann-Shanks property:

$$\begin{array}{c}
1 \\
1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 1 \\
1 \quad a \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 1 \\
1 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 1 \\
1 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 1 \\
1 \quad b \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 1 \\
1 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 1 \\
1 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 1 \\
1 \quad c \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 1 \\
1 \quad 1
\end{array}$$

where $4 \nmid a$, $7 \nmid b$, $10 \nmid c$, $12 \nmid d$, etc. For example, we could simply choose $a = b = c = d = \cdots = 1$ throughout. We summarize in the following:

Theorem 5. Let an array be defined by

$$A(n,0) = A(n,2n) = 1, \qquad n \geq 0\,,$$

$$A(n,k) = n, \quad 2 \leq k \leq 2n-1\,,$$

$$A(n,1) = n, \quad \text{if} \quad n \neq \frac{K-1}{2}\,, \quad \text{where } K = 4j \pm 1 \neq \text{prime},$$

$$A(n,1) = x, \quad \text{with} \quad n \nmid x, \quad \text{if} \quad n = \frac{K-1}{2}\,.$$

Then this array has the Mann-Shanks property when shifted in the way of the original Mann-Shanks array.

Similarly, the binomial coefficients in the original Mann-Shanks array may be replaced by numbers chosen in the same way. We have

Theorem 6. Let an array be defined by

$$A(n,0) = A(n,n) = 1, \qquad n \geq 0\,,$$

$$A(n,k) = n, \qquad 2 \leq k \leq n-1,$$

$$A(n,1) = n, \quad \text{if} \quad n \neq \frac{K-1}{2}\,, \quad \text{where } K = 6j \pm 1 \neq \text{prime},$$

$$A(n,1) = x, \quad \text{with} \quad n \nmid x, \quad \text{if} \quad n = \frac{K-1}{2}\,.$$

Then this array, when shifted as prescribed by Mann-Shanks has the Mann-Shanks primality criterion property.

These two examples are not what we mean by a 'simple formula' of course, because we must prescribe and know the prime nature of $a + bt$ in advance, whereas the beauty of the binomial coefficients, Fibonomial coefficients, or q-binomial coefficients is that they automatically take care of the situation. Nevertheless, it is felt that Theorems 5 and 6 shed further light on the nature of the Mann-Shanks property.

Another intriguing problem would be to find out whether any similar extensions to higher dimensions might be possible, using multinomial coefficients and variations.

7. TYPOGRAPHICAL ERRORS IN PREVIOUS PAPER

In [4] the following errors have been noted: p. 356, in (2.3), for "mod ⋯" read "(mod ⋯)"; p. 359, line 4, for

$$\left(\frac{n-1}{3}\right) \quad \text{read} \quad \left[\frac{n-1}{3}\right] \;;$$

p. 360, lines 6 and 8 from bottom, for "Erdos" read "Erdös"; p. 372, in Ref. 2, for "Institute" read "Institution."

REFERENCES

1. L. E. Dickson, History of the Theory of Numbers, Carnegie Institution, Washington, D.C., Vol. I, 1919. Reprinted by Chelsea Publ. Co., New York, 1952.
2. H. W. Gould, "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients," Fibonacci Quarterly, Vol. 7 (1969), No. 1, pp. 23-40, 55.
3. H. W. Gould, "Research Bibliography of Two Special Number Sequences," Mathematica Monongaliae, No. 12, May, 1971, Morgantown, W. Va. iv + 25 pp.
4. H. W. Gould, "A New Primality Criterion of Mann and Shanks and its Relation to a Theorem of Hermite with Extension to Fibonomials," Fibonacci Quarterly, Vol. 10 (1972), No. 4, pp. 355-364, 372.
5. H. Gupta, Personal Correspondence, 19 March, 1972.
6. Henry B. Mann and Daniel Shanks, "A Necessary and Sufficient Condition for Primality, and its Source," J. Combinatorial Theory, Ser. A, 13 (1972), 131-134.

ALGORITHMS FOR THIRD-ORDER RECURSION SEQUENCES

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Given a third-order recursion relation

(1) $$T_{n+1} = a_1 T_n - a_2 T_{n-1} + a_3 T_n \quad .$$

Let the auxiliary equation

(2) $$x^3 - a_1 x^2 + a_2 x - a_3 = 0$$

have three distinct roots r_1, r_2, r_3. Then any term of a sequence governed by this recursion relation can be expressed in the form

(3) $$T_n = A_1 r_1^n + A_2 r_2^n + A_3 r_3^n \quad .$$

THE SEQUENCE $S_n = \sum r_i^n$

Since the individual elements of these sums are powers of the roots, the sums obey the given recursion relation. Hence it is possible to determine a few terms of S_n by means of symmetric functions and thereafter generate additional terms of the S sequence. Since this sequence is basic to all the algorithms, its generation constitutes the first algorithm. (Note. This use of the S sequence is exemplified in [1].)

ALGORITHM FOR FINDING THE TERMS OF S_n

Three consecutive terms of the sequence are:

(4) $$\begin{cases} S_1 = a_1 \\ S_2 = a_1^2 - 2a_2 \\ S_3 = a_1^3 - 3a_1 a_2 + 3a_3 \end{cases} \quad .$$

Then use the recursion relation to obtain positive and negative subscript terms of the sequence.

The algorithm will be illustrated for two recursion relations which will be used to check other algorithms numerically.

EXAMPLE 1: $x^3 - x^2 - x - 1 = 0$

n	S_n	n	S_n	n	S_n	n	S_n	n	S_n
-30	-14429	-18	47	-6	11	6	39	18	58035
-29	13223	-17	271	-5	-1	7	71	19	106743
-28	-3253	-16	-253	-4	-5	8	131	20	196331
-27	-4459	-15	65	-3	5	9	241	21	361109
-26	5511	-14	83	-2	-1	10	443	22	664183
-25	-2201	-13	-105	-1	-1	11	815	23	1221623
-24	-1149	12	43	0	3	12	1499	24	2246915
-23	2161	-11	21	1	1	13	2757	25	4132721
-22	-1189	-10	-41	2	3	14	5071	26	7601259
-21	-177	-9	23	3	7	15	9327	27	13980895
-20	795	-8	3	4	11	16	17155	28	25714875
-19	-571	-7	-15	5	21	17	31553	29	47297029
								30	86992799

EXAMPLE 2: $x^3 - 7x^2 + 5x + 4 = 0$

n	$S_n(-4)^n$
-23	2450995949 6004997927 85
-22	2879858678 8067714806 5
-21	3383761613 1827843249
-20	3975834906 620902593
-19	4671506147 59541201
-18	5488902409 1011041
-17	6449322392 180465
-16	7577792077 14561
-15	8903714463 1313
-14	1046164399 5681
-13	1229215792 433
-12	1444301540 49
-11	1697004500 9
-10	1993985121
-9	234271601
-8	27532161
-7	3232913
-6	380577
-5	44465
-4	5313
-3	593
-2	81
-1	5

n	S_n
0	3
1	7
2	39
3	226
4	1359
5	8227
6	49890
7	302659
8	1836255
9	11140930
10	67594599
11	410112523
12	2488250946
13	1509681561 1
14	9159600445 5
15	5557349493 46
16	3371777360 703
17	2045738276 0371
18	1241198527 21698
19	7530649458 07219
20	4569025826 000559
21	2772137664 2081026
22	1681922475 81335511
23	1020462746 554941211
24	6191392481 409586818
25	3756466464 6767059627
26	2279138391 3410171845 5
27	1382807980 7792383837 78

RECURSION RELATIONS FOR SPACED TERMS OF A SEQUENCE

Given a sequence T_n satisfying the given recursion relation. It is desired to find the recursion relation for a spacing of k among the terms, namely, for the sequence T_{nk+a}.

Since

$$T_{nk+a} = A_1 r_1^{nk+a} + A_2 r_2^{nk+a} + A_3 r_3^{nk+a} \tag{5}$$

and since there is a change of r_i^k from one term to the next, the recursion relation is that whose roots correspond to r_i^k. Let the coefficients be given in the relation

$$x^3 - B_1 x^2 + B_2 x - B_3 = 0 .$$

Then

$$B_1 = \sum r_i^k = S_k$$
$$B_2 = \sum r_i^k r_j^k = a_3^k \sum r_i^{-k} = a_3^k S_{-k}$$
$$B_3 = a_3^k ,$$

Hence the recursion relation is given by

$$x^3 - S_k x^2 + a_3^k S_{-k} - a_3^k = 0 . \tag{6}$$

EXAMPLE FOR $x^3 - x^2 - x - 1 = 0$ with $k = 5$.

$$T_{n+5} = 21 T_n + T_{n-5} + T_{n-10} .$$

Using the sequence S_n with $n = 20$,

$$T_{25} = 21*196331 + 9327 + 443 = 4132721 .$$

EXAMPLE FOR $x^3 - 7x^2 + 5x + 4 = 0$ using the terms of the S sequence.

$$T_{-5} = (-593 T_{-2} + 226 T_1 - T_4)/64$$
$$T_{-5} = (-593*81/16 + 226*7 - 1359)/64 = -44465/1024 .$$

SECOND-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

If there are several sequences satisfying the given recursion relation, a sum of terms of the form $T_{m_1}^{(1)} T_{m_2}^{(2)}$ would form a homogeneous sequence function of the second degree. Such terms if expanded using the roots of the auxiliary equation would yield terms of the form $B_i r_i^{m_1+m_2}$ and others of the form $C_{ij} r_i^{m_1} r_j^{m_2}$. The first type obey the recursion relation for r_i^2 since there is a change of 2 in the power in going from one term in the product to the next as the m's change by 1. The second type obey the recursion relation for the quantities $r_i r_j$.

ALGORITHM FOR THE SECOND-DEGREE FUNCTIONS

The recursion relation governing the quantities r_i^2 has already been obtained and is given by:

$$x^3 - S_2 x^2 + a_3^2 S_{-2} x - a_3^2 = 0 . \tag{7}$$

For the second we need to find the symmetric functions of the roots $r_i r_j$.

$$B_1 = \sum r_i r_j = a_2$$
$$B_2 = \sum r_i^2 r_j r_k = a_3 a_1$$
$$B_3 = r_i^2 r_j^2 r_k^2 = a_3^2 \; .$$

Hence the recursion relation is

$$x^3 - a_2 x^2 + a_3 a_1 x - a_3^2 = 0 \; . \tag{8}$$

The total recursion relation is the product of (7) and (8):

$$(x^3 - S_2 x^2 + a_3^2 S_{-2} x - a_3^2)(x^3 - a_2 x^2 + a_3 a_1 x - a_3^2) = 0 \; . \tag{9}$$

EXAMPLE FOR $x^3 - 7x^2 + 5x + 4 = 0$.

$$\begin{aligned} S_5^2 &= 44 S_4^2 - 248 S_3^2 - 655 S_2^2 + 1564 S_1^2 + 848 S_0^2 - 256 S_{-1}^2 \\ &= 44*1359^2 - 248*226^2 - 655*39^2 + 1564*7^2 + 848*3^2 + 256*(5/4)^2 \\ &= 67683529 = 8227^2 \; . \end{aligned}$$

THIRD-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

An expression of the form

$$T_{m_1}^{(1)} T_{m_2}^{(2)} T_{m_3}^{(3)}$$

gives rise to terms of the form

$$r_i^{m_1+m_2}, \qquad r_i^{m_1+m_2} r_j^{m_3}, \qquad r_i^{m_1} r_j^{m_2} r_k^{m_3} \; .$$

The first type corresponds to the recursion relation for r_i^3, the second to the recursion relation for $r_i^2 r_j$, and the third to the recursion relation for a_3. The first relation is:

$$x^3 - S_3 x^2 + a_3^3 S_{-3} x - a_3^3 = 0 \; . \tag{10}$$

The last relation is:

$$x - a_3 = 0 \; . \tag{11}$$

For the second we have a relation of the sixth degree with coefficients symmetric functions of the roots

$R_1 = r_1^2 r_2$, $R_2 = r_2^2 r_1$, $R_3 = r_1^2 r_3$, $R_4 = r_3^2 r_1$, $R_5 = r_2^2 r_3$, $R_6 = r_3^2 r_2$.

$$B_1 = \sum R_i = (21) = -3a_3 + a_2 a_1 ,$$

where the notation $(21) = \sum r_i^2 r_j$ taken as a symmetric function.

(12)
$$B_2 = \sum R_i R_j = (41^2) + (3^2) + (321) + 3(222)$$
$$B_2 = 6a_3^2 - 5a_3 a_2 a_1 + a_3 a_1^3 + a_2^3$$
$$B_3 = \sum R_i R_j R_k = (531) + 2(432) + 2(3^3)$$
$$B_3 = -7a_3^3 + 6a_3^2 a_2 a_1 - 2a_3^2 a_1^3 - 2a_3 a_2^3 + a_3 a_2^2 a_1^2$$
$$B_4 = (63^2) + (5^2 2) + (543) + 3(444) = a_3^3(3) + a_3^2(3^2) + a_3^3(21) + 3(4^3)$$
$$B_4 = 6a_3^4 - 5a_3^3 a_2 a_1 + a_3^3 a_1^3 + a_3^2 a_2^3$$
$$B_5 = (654) = a_3^4(21) = -3a_3^5 + a_3^4 a_2 a_1$$
$$B_6 = a_3^6$$

The product of (10), (11) and the polynomial whose coefficients are given by (12) is the required recursion relation for the third degree. APPLIED TO $x^3 - x^2 - x - 1 = 0$, we have

$$(x^3 - 7x^2 + 5x - 1)(x - 1)(x^6 + 4x^5 + 11x^4 + 12x^3 + 11x^2 + 4x + 1) = 0$$

or

$$x^{10} - 4x^9 - 9x^8 - 34x^7 + 24x^6 - 2x^5 + 40x^4 - 14x^3 - x^2 - 2x + 1 = 0 .$$

Starting with $S_9 = 241$ we have:

$$4*241^3 + 9*131^3 + 34*71^3 - 24*39^3 + 2*21^3 - 40*11^3 + 14*7^3 + 3^3 + 2*1^3 - 3^3$$
$$= 86938307 = 443^3 \cdot S_{10}^3 .$$

FOURTH-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

We proceed as before but without going through the preliminary details we arrive at the conclusion that the symmetric functions of the roots are given by the partitions (4), (31), (22), (211) of four into three parts or less. We determine the recursion relations or equivalently the coefficients for each of these.

<u>(4)</u>

(13) $$x^3 - S_4 x^2 + a_3^4 S_{-4} x - a_3^4 = 0 .$$

<u>(211)</u> Since this symmetric function is equivalent to $a_3 r_i$ in its terms, we have the relation

(14) $$x^3 - a_3 a_1 x^2 + a_3^2 a_2 x - a_3^4 = 0 .$$

<u>(31)</u>

$$A_1 = (31) = -a_3 a_1 - 2a_2^2 + a_2 a_1^2$$

$$A_2 = (611) + (44) + (431) + (332)$$

$$= a_3(5) + (44) + a_3(32) + a_3^2 a_2$$

$$A_2 = -a_3^2 a_2 + 5a_3^2 a_1^2 + 2a_3 a_2^2 a_1 - 5a_3 a_2 a_1^3 + a_3 a_1^5 + a_2^4$$

$$A_3 = (741) + (642) + (543) + 2(444)$$

$$= a_3(63) + a_3^2(42) + a_3^3(21) + 2a_3^4$$

(15)
$$A_3 = 2a_3^4 - 13a_3^3 a_2 a_1 + a_3^3 a_1^3 + a_3^2 a_2^3 + 10a_3^2 a_2^2 a_1^2 - 3a_3^2 a_2 a_1^4 - 3a_3 a_2^4 a_1 + a_3 a_2^3 a_1^3$$

$$A_4 = a_3^2(5^2) + a_3^4(4) + a_3^4(31) + a_3^5(1)$$

$$A_4 = -a_3^5 a_1 + 5a_3^4 a_2^2 + 2a_3^4 a_2 a_1^2 - 5a_3^3 a_2^3 a_1 + a_3^2 a_2^5 + a_3^4 a_1^4$$

$$A_5 = a_3^5(32) = -a_3^6 a_2 - 2a_3^6 a_1^2 + a_3^5 a_2^2 a_1$$

$$A_6 = a_3^8$$

<u>(22)</u>

$$B_1 = (2^2) = -2a_3 a_1 + a_2^2$$

(16)
$$B_2 = (422) = a_3^2(2) = -2a_3^2 a_2 + a_3^2 a_1^2$$

$$B_3 = a_3^4 .$$

The product of the polynomials given by (13), (14), (15), and (16) gives the required recursion relation for the fourth degree.

APPLICATION TO $x^3 - x^2 - x - 1 = 0$.

$$(x^3 - 11x^2 - 5x - 1)(x^6 + 4x^5 + 15x^4 - 24x^3 + 7x^2 + 1)(x^3 + x^2 + 3x - 1) \times (x^3 - x^2 - x - 1) = 0$$

or

$$x^{15} - 7x^{14} - 33x^{13} - 223x^{12} + 197x^{11} + 41x^{10} + 1559x^9 - 451x^8 - 373x^7 - 637x^6 + 269x^5 + 131x^4 + 47x^3 - 5x^2 - 3x - 1 = 0 .$$

REMARKS

The determination of the coefficients of the polynomials for higher degrees in terms of the coefficients of the original recursion relation leads to expressions of ever greater complexity which make calculations tedious and present a greater possibility of error. A simpler approach is to use symmetric functions of the roots which in turn can be calculated by means of the S sequence of the given recursion relation. For three roots all such symmetric functions can be reduced to one of the forms (ab), (a^2) or (a). The last is simply S_a while the others are given by:

(17) $$(ab) = S_a S_b - S_{a+b}$$

(18) $$(a^2) = (S_a^2 - S_{2a})/2 \ .$$

FIFTH-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

On the basis of partitions we consider symmetric functions of the roots of the forms (5), (41), (32), (311), (221).

(5) $$x^3 - S_5 x^2 + a_3^5 S_{-5} x - a_3^5 \ .$$

(311)
$$B_1 = a_3(2)$$
$$B_2 = a_3^2(2^2)$$
$$B_3 = a_3^5 \ .$$

(221)
$$C_1 = a_3 a_2$$
$$C_2 = a_3^3 a_1$$
$$C_3 = a_3^5 \ .$$

(41)
$$D_1 = (41)$$
$$\begin{aligned} D_2 &= (81^2) + (5^2) + (541) + (442) \\ &= a_3(7) + (5^2) + a_3(43) + a_3^2(2^2) \\ D_3 &= (951) + (852) + (654) + 2(5^3) \\ &= a_3(84) + a_3^2(63) + a_3^4(21) + 2a_3^5 \\ D_4 &= (992) + (10,55) + (965) + (866) \\ &= a_3^2(7^2) + a_3^5(5) + a_3^5(41) + a_3^6(2) \end{aligned}$$
$$D_5 = (10,96) = a_3^6(43)$$
$$D_6 = a_3^{10} \ .$$

(32)
$$E_1 = (32)$$
$$\begin{aligned} E_2 &= (622) + (55) + (532) + (433) \\ &= a_3^2(4) + (5^2) + a_3^2(31) + a_3^3 a_1 \\ E_3 &= (852) + (654) + 2(555) + (753) \\ &= a_3^2(63) + a_3^4(21) + 2a_3^5 + a_3^3(42) \\ E_4 &= (884) + (10,55) + (875) + (776) \\ &= a_3^4(4^2) + a_3^5(5) + a_3^5(32) + a_3^6 a_2 \end{aligned}$$
$$E_5 = (10,87) = a_3^7(31)$$
$$E_6 = a_3^{10} \ .$$

APPLICATION TO $x^3 - x^2 - x - 1 = 0$.

$$(x^3 - 21x^2 - x - 1)(x^3 - 3x^2 - x - 1)(x^3 + x^2 + x - 1)(x^6 + 0x^5 + 7x^4 - 24x^3 + 15x^2 + 4x + 1)$$
$$\times (x^6 + 10x^5 + 75x^4 + 28x^3 - x^2 - 6x + 1) = 0 .$$

The product is

$$x^{21} - 13x^{20} - 110x^{19} - 1374x^{18} \pm 2425x^{17} + 543x^{16} + 60340x^{15} - 3976x^{14}$$
$$- 43106x^{13} - 149310x^{12} + 137592x^{11} + 88200x^{10} + 63126x^9 - 21742x^8 - 13076x^7$$
$$- 8932x^6 + 1041x^5 - 37x^4 + 150x^3 - 10x^2 + x - 1 = 0 .$$

CONCLUDING NOTES

1. That the symmetric functions of the roots can always be expressed in terms of the quantities S_n is an elementary proposition in combinatorial analysis. (See [2; p. 7].)

2. For the n^{th} degree, the recursion relation has degree

$$\binom{n+2}{2} .$$

This follows from the fact that the number of terms involving the roots is equivalent to the solution of $x + y + z = n$ in positive integers and zero.

3. For the sixth-degree relations, the coefficients A_1 and A_5, A_2 and A_4, are complementary, the respective quantities in the symmetric functions adding up to $2n$.

4. Each term in a coefficient has a weight. The coefficient A_k would have its terms of weight kn where n is the degree being considered for the terms of the original recursion relation. Thus for $n = 8$, E_4 has a term $a_3^6(7^2)$ which has a weight $6 \times 3 + 2 \times 7 = 32 = 4 \times 8$.

5. If $a_3 = 1$, all the factors for the n^{th} degree are found for degree $n + 3$.

6. With some modifications on the symmetric functions involved, this approach could be used to produce algorithms relating to recursion relations of higher order.

7. The algorithms were checked numerically by using a relation with roots 1, 2, and 4, finding the symmetric functions directly and comparing the result with that given by the algorithms.

REFERENCES

1. Trudy Y. H. Tong, "Some Properties of the Tribonacci Sequence and the Special Lucas Sequence," Master's Thesis, San Jose State University, August 1970.
2. P. A. MacMahon, Combinatory Analysis, Cambridge University Press, 1915, 1916. Reprinted by Chelsea Publishing Company, 1960.

Editor's Note: There are an additional twelve pages on this subject, going through the tenth degree. If you would like a Xerox copy of the additional material at four cents a page (which includes postage, materials and labor), send your request to:

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ON THE DIVISORS OF SECOND-ORDER RECURRENCES

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1. INTRODUCTION AND NOTATIONS

In this note, we shall give a criterion to determine whether a given prime p divides terms of the second-order recurrence

(1) $$A_{n+2} = PA_{n+1} - QA_n ,$$

with arbitrary initial values A_0 and A_1, and we shall give several applications.

A particular case of (1) is the recurrence

(2) $$U_{n+2} = PU_{n+1} - QU_n, \quad U_0 = 0, \quad U_1 = 1 .$$

We shall denote by Δ the discriminant $P^2 - 4Q$ of the recurrence. The general term U_n of (2) may be denoted by

$$(a^n - b^n)/(a - b) ,$$

where

$$a = \frac{P + \sqrt{\Delta}}{2}$$

and

$$b = \frac{P - \sqrt{\Delta}}{2} .$$

There is an integer $k(m)$ such that m divides U_n if and only if $k(m) \mid n$. p will denote a prime not dividing Q. In this paper, we shall be working in the field of integers modulo p.

2. THE CRITERION FOR DIVISIBILITY

Let R_n be the quotient U_{n+1}/U_n (mod p): i.e., the solution X of

$$XU_n \equiv U_{n+1} \pmod{p} .$$

R_n exists, unless p divides U_n, in which case the value of R_n will be denoted by ∞. (All quotients which have a zero divisor will be denoted ∞.) If R_n exists and is nonzero, then

(3) $$R_{n+1} \equiv U_{n+2}/U_{n+1} \equiv P - QR_n^{-1} \pmod{p} ;$$

if $p | R_n$ then $R_{n+1} \equiv \infty$; if $R_n \equiv \infty$ then $p | U_n$, so $R_{n+1} \equiv P \pmod p$.

Theorem 1. (R_n) is a first-order recurrence mod p and is periodic with primitive period $k(p)$.

Proof. We have already shown that (R_n) is a first-order recurrence (3). That it has primitive period $k(p)$ follows from the definition of k and the fact that $R_n \equiv 0$ if and only if $p | U_{n+1}$.

The following theorem gives a criterion for determining whether p is a divisor of terms of (A_n). It is known that if a number m divides some term A_n of (1), then m divides $A_{n+tk(m)}$ for any integer t for which the subscript is nonnegative, and only those terms.

Theorem 2. (Divisibility criterion). p is a divisor of $A_{tk(p)-n}$ (for any t for which the subscript is nonnegative) if and only if

$$A_1/A_0 \equiv R_n \pmod p .$$

Proof. By Eq. (8) of [6].

$$Q^n A_m = U_{n+1}A_{k(p)} - U_n A_{k(p)+1} ,$$

where $m + n = k(p)$. Thus, $p | A_m$ if and only if

$$A_{k(p)+1}/A_{k(p)} \equiv R_n ,$$

and it is known that

$$A_{k(p)+1}/A_{k(p)} \equiv A_1/A_0 .$$

Furthermore, $p | A_m$ if and only if $p | A_{tk(p)-n}$, and the theorem follows.

3. APPLICATIONS OF THE CRITERION

It is well known that $k(p) | p - (\Delta/p)$. A proof is given in [4] for the Fibonacci series, and it may be easily generalized to the recurrence (2). For most recurrences, there are many primes p such that $k(p) = p - (\Delta/p)$. In the first two theorems in this section, we consider such primes.

The following result was proved in [1] and [2] for the Fibonacci series.

Theorem 3. If

$$k(p) = p + 1$$

then p divides some terms of (A_n) regardless of the initial values A_0 and A_1, and conversely.

Proof. It follows from Theorem 1 that if

$$k(p) = p + 1 ,$$

then for any residue class c there is an n such that $c \equiv R_n \pmod{p}$. Therefore, there is an n such that

$$A_1/A_0 \equiv R_n \pmod{p} ,$$

and the first part follows by the criterion of Theorem 2. If $k(p)$ is less than $p + 1$ then not every residue class is included in (R_n), and the converse follows.

Theorem 4. p is a divisor of terms of (A_n) for any initial values A_0 and A_1, excepting when $A_1/A_0 \equiv a$ or b, if and only if $k(p) = p - 1$.

Proof. Since

$$k(p) = p - 1 ,$$

we have

$$(\Delta/p) = 1 ,$$

so a and b are in the field of integers modulo p and $p \nmid \Delta$. By definition,

$$R_n \equiv (a^{n+1} - b^{n+1})/(a^n - b^n) .$$

If $R_n \equiv a$ (or b) $\pmod{p}$ then it follows that $a \equiv b$, whence $p \mid \Delta$, giving a contradiction. Thus, $R_n \not\equiv a$ or b. By Theorem 2 and the fact that $R_n \equiv A_1/A_0$ for some n when

$$k(p) = p - 1$$

and

$$A_1/A_0 \not\equiv a \text{ or } b \pmod{p} ,$$

we see that p divides terms of (A_n). If $k(p)$ is less than $p - 1$, then not every residue class can be included in (R_n), whence the converse follows.

Theorem 5. If

$$A_1/A_0 \equiv a \text{ or } b \pmod{p}$$

then p divides no term of (A_n).

Proof. If

$$A_1/A_0 \equiv a \text{ or } b$$

then

$$(\Delta/p) = 1$$

and $p \nmid \Delta$. If

$$R_n \equiv a \ (\text{or } b) \pmod p$$

then

$$(a^{n+1} - b^{n+1})/(a^n - b^n) \equiv a \ (\text{or } b)$$

so that $a \equiv b$ and $p|\Delta$, giving a contradiction. Thus, $R_n \not\equiv a$ (or b) $\equiv A_1/A_0$, and so $p \nmid A_n$ for any n, by Theorem 2.

4. CONCLUDING REMARKS

Hall [3] has given a different criterion for whether a prime p divides some terms of (1). Bloom [2] has studied the related question of which composite numbers (as well as which primes) are divisors of recurrences of the form (1) with $P = 1$, $Q = -1$.

Ward [5] has pointed out that the question of whether or not there are infinitely many primes for which $k(p) = p + 1$ or $p - 1$ is a generalization of Artin's conjecture that an integer not -1 or a square is a primitive root of infinitely many primes. For recurrences in which Δ is a square and a or b is 1, the question is equivalent to Artin's conjecture.

REFERENCES

1. Brother U. Alfred, "Primes which are Factors of all Fibonacci Sequences," Fibonacci Quarterly, Vol. 2, No. 1 (February 1964), pp. 33-38.
2. D. M. Bloom, "On Periodicity in Generalized Fibonacci Sequences," Amer. Math. Monthly, 72 (1965), pp. 856-861.
3. Marshall Hall, "Divisors of Second-Order Sequences," Bull. Amer. Math. Soc., 43 (1937), pp. 78-80.
4. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Third Ed., Clarendon Press, Oxford, 1954.
5. Morgan Ward, "Prime Divisors of Second-Order Recurrences," Duke Math. J., 21 (1954) pp. 607-614.
6. Oswald Wyler, "On Second-Order Recurrences," Amer. Math. Monthly, 72 (1965), pp. 500-506.

FIBONACCI NOTES
2: MULTIPLE GENERATING FUNCTIONS

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1. The Hermite polynomial $H_n(x)$ may be defined by means of

$$\sum_{n=0}^{\infty} H_n(a) \cdot \frac{z^n}{n!} = e^{2az-z^2} .$$

The writer [1] has proved formulas of the following kind.

(1.1)
$$\sum_{m,n=0}^{\infty} H_{m+n}(a)H_m(b)H_n(c) \frac{x^m y^n}{m!\,n!}$$
$$= (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2}\right\} ,$$

(1.2)
$$\sum_{m,n,p=0}^{\infty} H_{m+n+p}(a)H_m(b)H_n(c)H_p(d) \frac{x^m y^n z^p}{m!\,n!\,p!}$$
$$= (1 - 4x^2 - 4y^2 - 4z^2)^{-\frac{1}{2}} \cdot \exp\left\{\frac{-4a^2(x^2 + y^2 + z^2) + 4a(bx + cy + dz) - 4(bx + cy + dz)^2}{1 - 4x^2 - 4y^2 - 4z^2}\right\},$$

(1.3)
$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a)H_{p+m}(b)H_{m+n}(c) \frac{x^m y^n z^p}{m!\,n!\,p!}$$
$$= d^{-\frac{1}{2}} \exp\left\{\Sigma a^2 - \frac{\Sigma a^2 - 4\Sigma a^2x^2 - 4\Sigma abz + 8\Sigma abxy}{d}\right\} ,$$

where

$$d = 1 - 4x^2 - 4y^2 - 4z^2 + 16xyz$$

and Σa^2, Σa^2x^2, Σabz, $\Sigma abxy$ are symmetric functions in the indicated parameters.

The object of the present note is to prove formulas of a similar kind for the Fibonacci and Lucas numbers.

2. Consider first the sum

$$S = \sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n .$$

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Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha + \beta = 1, \qquad \alpha\beta = -1,$$

we get

$$S = \frac{1}{\alpha - \beta} \sum_{m,n=0}^{\infty} F_m F_n (\alpha^{m+n} - \beta^{m+n}) x^m y^n$$

$$= \frac{1}{\alpha - \beta} \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2} \frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} - \frac{\beta x}{1 - \beta x - \beta^2 x^2} \frac{\beta y}{1 - \beta y - \beta^2 y^2}$$

$$= \frac{1}{\alpha - \beta} \left\{ \frac{\alpha^2 xy}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \alpha^2 y)(1 - \alpha\beta y)} - \frac{\beta^2 xy}{(1 - \alpha\beta x)(1 - \beta^2)(1 - \alpha\beta y)(1 - \beta^2 y)} \right\}$$

$$= \frac{xy}{(\alpha - \beta)(1 + x)(1 + y)} \frac{\alpha^2 [1 - \beta^2(x + y) + \beta^4 xy] - \beta^2 [1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Thus

$$\sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n = \frac{xy - x^2 y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} . \tag{2.1}$$

Similarly we find that

$$\sum_{m,n=0}^{\infty} L_{m+n} F_m F_n x^m y^n = \frac{3 - 2(x + y) + 3xy}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} . \tag{2.2}$$

The sum

$$\sum_{m,n=0}^{\infty} L_{m+n} L_m L_n x^m y^n$$

is somewhat more complicated. We get

$$\frac{(2 - \alpha x)(2 - \alpha y)[1 - \beta^2(x + y) + \beta^4 xy] + (2 - \beta x)(2 - \beta y)[1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

After some manipulation we find that

$$\sum_{m,n=0}^{\infty} L_{m+n} L_m L_n x^m y^n \tag{2.3}$$

$$= \frac{8 - 14(x + y) - 2(x^2 + y^2) + 27xy + 7xy(x + y) + 3x^2 y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Recurrences of an unusual kind are implied by these formulas. In particular (2.1) yields

$$F_{m+n}F_mF_n + F_{m+n-1}F_{m-1}F_n + F_{m+n-1}F_mF_{n-1} + F_{m+n-2}F_{m-1}F_{n-1} = F_{2m}F_{2n} - F_{2m-2}F_{2n-2}, \tag{2.4}$$

while (2.2) gives

$$L_{m+n}F_mF_n + L_{m+n-1}F_{m-1}F_n + L_{m+n-1}F_mF_{n-1} + L_{m+n-2}F_{m-1}F_{n-1} = 3F_{2m+2}F_{2n+2} - 2F_{2m}F_{2n+2} - 2F_{2m+2}F_{2n} + 3F_{2m}F_{2n}. \tag{2.5}$$

It may be of interest to mention that the generating functions (2.1), (2.2), (2.3) can be extended in various ways. For example we have

$$\sum_{m,n=0}^{\infty} F_{m+n+p}F_mF_nx^my^n = \frac{F_{p+2}xy - F_pxy(x+y) + F_{p-2}x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)}. \tag{2.6}$$

This in turn leads to the following extension of (2.4):

$$F_{m+n+p}F_mF_n + F_{m+n+p-1}(F_{m-1}F_n + F_mF_{n-1}) + F_{m+n+p-2}F_{m-1}F_{n-1} = F_{p+2}F_{2m}F_{2n} - F_p(F_{2m-2}F_{2n} + F_{2m}F_{2n-2}) + F_{p-2}F_{2m-2}F_{2n-2}. \tag{2.7}$$

Since $L_n = F_{n+1} + F_{n-1}$, it is evident that (2.6) and (2.7) imply

$$\sum_{m,n=0}^{\infty} L_{m+n+p}F_mF_nx^my^n = \frac{L_{p+2} - L_pxy(x+y) + L_{p-2}x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)} \tag{2.8}$$

and

$$L_{m+n+p}F_mF_n + L_{m+n+p-1}(F_{m-1}F_n + F_mF_{n-1}) + L_{m+n+p-2}F_{m-1}F_{n-1} = L_{p+2}F_{2m}F_{2n} - L_p(F_{2m-2}F_{2n} + F_{2m}F_{2n-2}) + L_{p-2}F_{2m-2}F_{2n-2}, \tag{2.9}$$

respectively.

3. We consider next the triple sum

$$\sum_{m,n,p=0}^{\infty} F_{m+n+p}F_mF_nF_px^my^nz^p. \tag{3.1}$$

Exactly as above we find that (3.1) is equal to

$$\frac{1}{\alpha - \beta}\left\{\frac{\alpha^3 xyz}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \alpha^2 y)(1 - \alpha\beta y)(1 - \alpha^2 z)(1 - \alpha\beta z)}\right.$$

$$\left. - \frac{\beta^3 xyz}{(1 - \alpha\beta x)(1 - \beta^2 x)(1 - \alpha\beta y)(1 - \beta^2 y)(1 - \alpha\beta z)(1 - \beta^2 z^2)}\right\}$$

$$= \frac{xyz}{\alpha - \beta}\left\{\frac{\alpha^3[1 - \beta^2(x + y + z) + \beta^4(yz + zx + xy) - \beta^6 xyz]}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}\right.$$

$$\left. - \beta^3 \frac{[1 - \alpha^2(x + y + z) + \alpha^4(yz + zx + xy) - \alpha^6 xyz]}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}\right\}.$$

Simplifying we get

(3.2)
$$\sum_{m,n,p=0} F_{m+n+p} F_m F_n F_p x^m y^n z^p$$

$$= \frac{2 - (x + y + z) + (yz + zx + xy) - 2xyz}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

$$= \frac{(1 - x)(1 - y)(1 - z) + 1 - xyz}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

The general formula of this kind can now be stated, namely

(3.3)
$$\sum_{n_1,\cdots,n_k=0}^{\infty} F_{n_1+\cdots+n_k} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k}$$

$$= \frac{\sum_{0 \le 2j \le k} (-1)^j F_{k-2j}(c_j - c_{k-j})}{\prod_{j=1}^{k} (1 + x_j) \cdot \prod_{j=1}^{k} (1 - 3x_j + x_j^2)} ,$$

where c_j is the j^{th} elementary symmetric function of $x_1, x_2, \cdots, x_k$.

To prove (3.3) it is enough to observe that the numerator is equal to

$$\frac{1}{\alpha - \beta}\left\{\alpha^k \prod_{j=1}^{k} (1 - \beta^2 x_j) - \beta^k \prod_{j=1}^{k} (1 - \alpha^2 x_j)\right\}$$

$$= \frac{1}{\alpha - \beta}\left\{\alpha^k \sum_{j=0}^{k} (-1)^j c_j \beta^{2j} - \beta^k \sum_{j=0}^{k} (-1)^j c_j \alpha^{2j}\right\}$$

$$= \frac{1}{\alpha - \beta} \sum_{j=0}^{k} (-1)^j c_j (\alpha^k \beta^{2j} - \alpha^{2j} \beta^k)$$

$$= \sum_{2j<k} (-1)^j c_j F_{k-2j} - (-1)^k \sum_{2j>k} (-1)^j c_j F_{2j-k}$$

$$= \sum_{2j<k} (-1)^j c_j F_{k-2j} - \sum_{2j<k} (-1)^j c_{k-j} F_{k-2j}$$

$$= \sum_{2j<k} (-1)^j (c_j - c_{k-j}) F_{k-2j} \quad .$$

In exactly the same way we can prove the more general result

$$\sum_{n_1,\cdots,n_k=0}^{\infty} F_{n_1+\cdots+n_k+p} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k} = \frac{\sum_{j=0}^{k} (-1)^j c_j F_{k+p-2j}}{\prod_{j=1}^{k} (1 + x_j) \cdot \prod_{j=1}^{k} (1 - 3x_j + x_j^2)} \quad . \tag{3.4}$$

Hence we also have

$$\sum_{n_1,\cdots,n_k=0}^{\infty} L_{n_1+\cdots+n_k+p} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k} = \frac{\sum_{j=0}^{k} (-1)^j c_j L_{k+p-2j}}{\prod_{j=1}^{k} (1 + x_j) \cdot \prod_{j=1}^{k} (1 - 3x_j + x_j^2)} \quad . \tag{3.5}$$

4. We consider next the series

$$\sum_{m,n,p=0}^{\infty} F_{n+p} F_{p+m} F_{m+n} x^m y^n z^p = \frac{1}{(\alpha-\beta)^3} \sum_{m,n,p=0}^{\infty} (\alpha^{n+p} - \beta^{n+p})(\alpha^{p+m} - \beta^{p+m})(\alpha^{m+n} - \beta^{m+n}) x^m y^n z^p$$

$$= \frac{1}{(\alpha-\beta)^3} \left\{ \frac{1}{(1-\alpha^2 x)(1-\alpha^2 y)(1-\alpha^2 z)} - \sum \frac{1}{(1-\alpha^2 x)(1+y)(1+z)} + \sum \frac{1}{(1-\beta^2 x)(1+y)(1+z)} - \frac{1}{(1-\beta^2 x)(1-\beta^2 y)(1-\beta^2 z)} \right\}$$

$$= \frac{1}{(\alpha-\beta)^3} \frac{(1-\beta^2 x)(1-\beta^2 y)(1-\beta^2 z) - (1-\alpha^2 x)(1-\alpha^2 y)(1-\alpha^2 z)}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)} - \frac{1}{(\alpha-\beta)^3} \sum \frac{(\alpha^2 - \beta^2) x}{(1-3x+x^2)(1+y)(1+z)} \quad .$$

It follows that

$$(4.1)\qquad \sum_{m,n,p=0}^{\infty} F_{n+p}F_{p+m}F_{m+n}x^m y^n z^p$$

$$= \frac{1}{5}\,\frac{\sum x - 3\sum xy + 8xyz}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)} - \frac{1}{5}\sum \frac{x}{(1-3x+x^2)(1+y)(1+z)}\,.$$

It can be shown that the right member of (4.1) is equal to

$$(4.2)\qquad \frac{q - 5r + 2pr + 2qr + r^2 - q^2}{(1+x)(1+y)(1+z)(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)}\,,$$

where

$$p = \sum x, \qquad q = \sum xy, \qquad r = xyz.$$

A somewhat more general result than (4.1) is

$$(4.3)\qquad \sum_{m,n,p=0}^{\infty} F_{n+p+r}F_{p+m+r}F_{m+n+r}x^m y^n z^p$$

$$= \frac{1}{5}\,\frac{F_{3r} - F_{3r-2}\sum x + F_{3r-4}\sum xy - F_{3r-6}\,xyz}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)}$$

$$- \frac{(-1)^r}{5}\sum \frac{F_r - F_{r-2}x}{(1-3x+x^2)(1+y)(1+z)}\,.$$

Similarly we can show that

$$(4.4)\qquad \sum_{m,n,p=0}^{\infty} L_{n+p+r}L_{p+m+r}L_{m+n+r}x^m y^n z^p$$

$$= \frac{L_{3r} - L_{3r-2}\sum x + L_{3r-4}\sum xy - L_{3r-6}\,xyz}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)}$$

$$+ (-1)^r\sum \frac{L_r - L_{r-2}x}{(1-3x+x^2)(1+y)(1+z)}\,,$$

We remark that the left member of (4.3) can be transformed in a rather interesting way. Put

$$m' = n + p, \quad n' = p + m, \quad p' = m + n .$$

Then

$$\left\{\begin{array}{l} -m' + n' + p' = 2m \\ m' - n' + p' = 2n \\ m' + n' - p' = 2p \end{array}\right. , \tag{4.5}$$

so that

$$m' + n' + p' \equiv 0 \pmod 2 \tag{4.6}$$

and

$$m' \le n' + p', \quad n' \le p' + m', \quad p' \le m' + n' . \tag{4.7}$$

Conversely if m', n', p' are nonnegative integers satisfying (4.6) and (4.7) then m, n, p as defined by (4.5) are also nonnegative integers. Hence replacing x, y, z by vw, wu, uv, (4.3) becomes

$$\sum_{m',n',p'} F_{m'+r}F_{n'+r}F_{p'+r}u^{m'}v^{n'}w^{p'} \tag{4.8}$$

$$= \frac{1}{5}\,\frac{F_{3r} - F_{3r-2}\sum uv + F_{3r-4}\,uvw\sum u - F_{3r-6}\,u^2v^2w^2}{(1 - 3vw + v^2w^2)(1 - 3wu + w^2u^2)(1 - 3uv + u^2v^2)}$$

$$- \frac{(-1)^r}{5}\sum \frac{F_r - F_{r-2}\,vw}{(1 - 3vw + v^2w^2)(1 + wu)(1 + uv)} .$$

A similar result can be stated for (4.4).

REFERENCE

1. L. Carlitz, "Some Extensions of the Mehler Formula," Collectanea Mathematica, Vol. 21 (1970), pp. 117-130.

A COMBINATORIAL IDENTITY

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Define

$$f(n,k) = 2^n \sum_{i=k}^{c} (-1)^i \binom{n-i}{i}\binom{i}{k} 2^{-2i} , \tag{1}$$

where

$$c = \begin{cases} n/2, & n \text{ even} \\ (n-1)/2, & n \text{ odd} \end{cases} .$$

By induction, it is proved that

$$f(n,k) = (-1)^k \binom{n+1}{2k+1} = (-1)^k \binom{n+1}{n-2k} \quad \text{for} \quad 0 \le k \le c . \tag{2}$$

The usual induction procedure must be modified since the identity involves both n and k but only restricted values of k associated with each n. Figure 1 illustrates how the induction proceeds. For the n and k shown, the identity is valid at the darkened grid points. The letter label on a grid point or on an arrow refers to part A, B, C, or D of the proof.

Part A of the proof shows that when n even, assuming (2) is true for (n,k), $(n-1,k)$, and $(n-1, k-1)$, then (2) is true for $(n+1, k)$. This applies to all k associated with n and $n+1$ except for $k = 0$ and $k = n/2$. Part B shows that for n even, $k \neq 0$, $k \neq (n+2)/2$, assuming as in A that (2) is true for (n,k), adding the assumption that (2) is true for $(n, k-1)$, and using the result of A that (2) is true for $(n+1, k)$, then (2) is true for $(n+2, k)$. Part C shows that (2) is true for $(n,0)$ and Part D deals with the special cases of $(n, n/2)$ and $(n+1, n/2)$ for n even.

A. Starting with

$$\binom{n+1-i}{i}\binom{i}{k} \equiv \binom{n-i}{i}\binom{i}{k} + \binom{n-i}{i-1}\binom{i-1}{k} + \binom{n-i}{i-1}\binom{i-1}{k-1} \tag{3}$$

for $1 \le k \le i-1$, $i \le n/2$, n even, a factor of $(-1)^i 2^{n-2i}$ is introduced into each term. Each term in the equation is summed over $i = k+1, \cdots, n/2$. For notational convenience, call the result

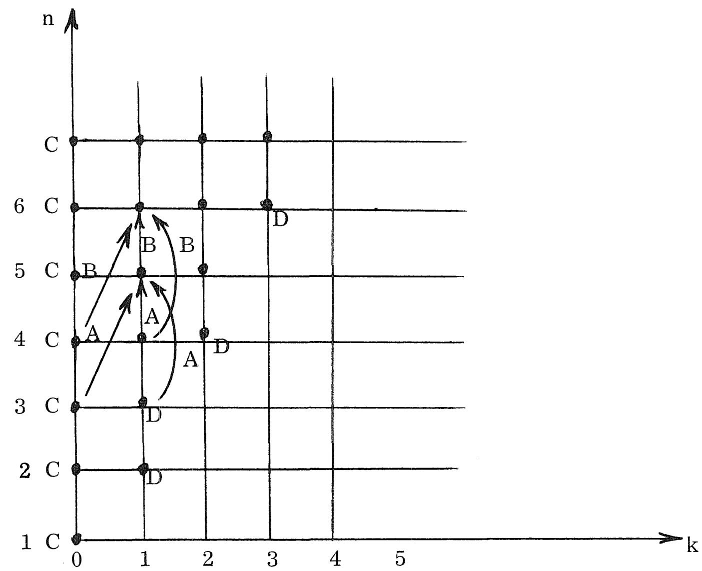

Figure 1

(4)
$$S_{1,n} = S_{2,n} + S_{3,n} + S_{4,n} .$$

It is found that, for n even,

(5)
$$S_{1,n} = \left[f(n+1, k) - 2^{n+1-2k}(-1)^k \binom{n+1-k}{k}\right] \Big/ 2$$
$$S_{2,n} = f(n,k) - 2^{n-2k}(-1)^k \binom{n-k}{k}$$
$$S_{3,n} = -f(n-1, k)/2$$
$$S_{4,n} = -\left[f(n-1, k-1) + 2^{n+1-2k}(-1)^k \binom{n-k}{k-1}\right] \Big/ 2 .$$

If (2) is true for (n,k), $(n-1, k)$, and $(n-1, k-1)$, (4) can be solved for $f(n+1, k)$ and

(6)
$$f(n+1, k) = (-1)^k \binom{n+2}{n+1-2k}$$

for $1 \le k \le (n-2)/2$, n even.

B. Using (3) modified such that each n is replaced by $n+1$, a factor of

$$(-1)^i \, 2^{n+1-2i}$$

is introduced into each term and each term of the equation is summed over $i = k+1, \cdots, n/2$. The result is

$$\left[S_{1,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n+2}{2}}{k}\right]$$

$$(4') \qquad = S_{2,n+1} + \left[S_{3,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n}{2}}{k}\right] + \left[S_{4,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n}{2}}{k-1}\right] .$$

If (2) is true for (n,k), $(n, k-1)$, and $(n+1, k)$, (4') can be solved for $f(n+2, k)$ and

$$(5') \qquad f(n+2, k) = (-1)^k \binom{n+3}{n+2-2k}$$

for $1 \le k \le n/2$, n even.

C. When $k = 0$, (3) reduces to the familiar identity

$$(3'') \qquad \binom{n+1-i}{i} \equiv \binom{n-i}{i} + \binom{n-i}{i-1}$$

for $1 \le i \le n/2$, n even, and (4) reduces to

$$(4'') \qquad S_{1,n} = S_{2,n} + S_{3,n} ,$$

where $S_{1,n}$, $S_{2,n}$, $S_{3,n}$ are as defined in (5).

Hence, if $f(n,0) = n+1$ and $f(n-1, 0) = n$, then $f(n+1, 0) = n+2$ for n even.

Similar modification of Part B leads to $f(n+2, 0) = n+3$ if $f(n,0) = n+1$ and $f(n+1, 0) = n+2$ for n even. Verifying by substitution into (1) that $f(2,0) = 3$ and $f(1,0) = 2$ completes the case of $k = 0$.

D. Finally by substitution into (1), it is verified that (2) is true for $(n, n/2)$ and $(n+1, n/2)$ for n even.

DIVISIBILITY AND CONGRUENCE RELATIONS

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In [1], we find three well known divisibility properties which exist between the Fibonacci and Lucas numbers. They are

$$F_n \mid F_m \quad \text{iff} \quad m = kn; \tag{1}$$

$$L_n \mid F_m \quad \text{iff} \quad m = 2kn, \quad n > 1; \tag{2}$$

$$L_n \mid L_m \quad \text{iff} \quad m = (2k-1)n, \quad n > 1. \tag{3}$$

The primary intention of this paper is to investigate the decomposition of Fibonacci and Lucas numbers in that we are interested in finding n such that $n|F_m$ or $n|L_m$. As a result of this investigation, we will also illustrate several interesting congruence relationships which exist between the elements of the sequences $\{F_n\}$ and $\{L_n\}$.

The first result, due to Hoggatt, is

Theorem 1. If $n = 2\cdot 3^k$, $k \geq 1$, then $n|L_n$.

Proof. Using α and β as the roots of the equation $x^2 - x - 1 = 0$ and recalling that $L_n = \alpha^n + \beta^n$, we have

$$\begin{aligned} L_{3n} &= \alpha^{3n} + \beta^{3n} \\ &= (\alpha^n + \beta^n)(\alpha^{2n} - \alpha^n\beta^n + \beta^{2n}) \\ &= L_n(L_{2n} - (-1)^n) = L_n(L_{2n} - 1). \end{aligned}$$

However, $L_n^2 = L_{2n} + 2$ if n is even so that

$$L_{3n} = L_n(L_n^2 - 3). \tag{4}$$

The theorem is true if $k = 1$ because $n = 6$ and $L_6 = 18$. The result now follows by induction on k together with (4).

Curiosity leads one to ask if there are other sequences $\{n_k\}$ such that $n_k|L_{n_k}$. The authors were unable to find other such sequences until they obtained the computer results of Mr. Joseph Greener from which they were able to make several conjectures and establish several results. Before stating the results, we establish the following theorem which was discovered independently by Carlitz and Bergum.

Theorem 2. If p is an odd prime and $p|L_n$ then $p^k|L_{np^{k-1}}$, $k \geq 1$.

Proof. By hypothesis, the theorem is true for $k = 1$. Assume $p^k|L_{np^{k-1}}$ and let $t = p^{k-1}$ then $pt|L_{nt}$. We shall show that $p^2t|L_{npt}$.

Using the factorization of $x^p + y^p$, we have

$$L_{npt} = (\alpha^{nt})^p + (\beta^{nt})^p = L_{nt}\left(\sum_{i=1}^{p} (-1)^{i+1}\alpha^{nt(p-i)}\beta^{nt(i-1)}\right) . \tag{5}$$

The middle term of the summation is

$$(-1)^{(p+3)/2}(\alpha\beta)^{nt(p-1)/2} = (-1)^{(n+1)(p-1)/2} . \tag{6}$$

The sum of the q^{th} and $(p + 1 - q)^{th}$ terms, where $q \neq (p + 1)/2$, is

$$\begin{aligned}(-1)^{q+1}\alpha^{nt(p-q)}\beta^{nt(q-1)} &+ (-1)^{p-q}\alpha^{nt(q-1)}\beta^{nt(p-q)} \\ &= (-1)^{q+1}(\alpha\beta)^{nt(q-1)}(\alpha^{nt(p-2q+1)} + \beta^{nt(p-2q+1)}) \\ &= (-1)^{(n+1)(q-1)}L_{nt(p-2q+1)} .\end{aligned} \tag{7}$$

Using (6) and (7) in (5) with $p = 4k + 1$, we have

$$\begin{aligned} L_{npt} &= L_{nt}\left(\sum_{q=1}^{2k} (-1)^{(n+1)(q-1)}L_{nt(4k-2q+2)} + 1\right) \\ &= L_{nt}\left(\sum_{q=0}^{k-1} L_{4nt(k-q)} + \sum_{q=1}^{k} (-1)^{n+1}L_{2nt(2k-2q+1)} + 1\right) \\ &= L_{nt}\left(\sum_{q=0}^{k-1} [5F^2_{2nt(k-q)} + 2] + \sum_{q=1}^{k} (-1)^{n+1}[L^2_{nt(2k-2q+1)} - 2(-1)^n] + 1\right) \\ &= L_{nt}\left(\sum_{q=0}^{k-1} 5F^2_{2nt(k-q)} + \sum_{q=1}^{k} (-1)^{n+1}L^2_{nt(2k-2q+1)} + p\right)\end{aligned} \tag{8}$$

since $L_{4r} = 5F^2_{2r} + 2$, $L^2_r = L_{2r} + 2(-1)^r$, and $t(2k - 2q + 1)$ is odd.

Now $pt|L_{nt}$, $(2k - 2q + 1)$ is odd, and $2(k - q)$ is even so that by (2) and (3) one sees that p is a factor of the expression in the parentheses of (8). Hence, $p^2t|L_{npt}$ and the theorem is proved if we have $p \equiv 1 \pmod 4$.

Suppose $p = 4k + 3$. An argument similar to the above yields

$$L_{npt} = L_{nt}\left(\sum_{q=1}^{k+1} L^2_{nt(2k-2q+3)} + \sum_{q=0}^{k-1} (-1)^{n+1}5F^2_{2nt(k-q)} - p(-1)^n\right) \tag{9}$$

and we see, as before, that $p^2t \mid L_{npt}$ if $p \equiv 3 \pmod 4$.

Since $3 \mid L_2$, we have

$$3^k \mid L_{2\cdot 3^{k-1}} \quad \text{or} \quad 3^k \mid L_{2\cdot 3^k} \quad \text{for} \quad k \ge 1 .$$

However, $2 \mid L_{2\cdot 3^k}$ for $k \ge 1$. But $(2,3) = 1$ and we have an alternate proof of Theorem 1 so that Theorem 1 is now an immediate consequence of Theorem 2. Furthermore, this procedure can be used to establish sequences $\{n_k\}$ such that $n_k \mid L_{n_k}$. We have

__Theorem 3.__ Let p be any odd prime different from 3 and such that $p \mid L_{2\cdot 3^k}$, $k \ge 1$. Let $n = 2\cdot 3^k p^t$ where $t \ge 1$; then $n \mid L_n$.

__Proof.__ By Theorem 1 and (3), we see that $2\cdot 3^k \mid L_{2\cdot 3^k p^t}$ for all $t \geq 1$. However, by Theorem 2 and (3), one has $p^t \mid L_{2\cdot 3^k p^t}$ for $t \ge 1$. Since $(2\cdot 3^k, p^t) = 1$, one has $2\cdot 3^k p^t \mid L_{2\cdot 3^k p^t}$ for $t \ge 1$.

By an argument similar to that of Theorem 3, it is easy to see that the following are true.

__Corollary 1.__ If p and q are distinct odd primes such that $p \mid L_n$ and $q \mid L_m$ where m and n are odd, then $(pq)^k \mid L_{mn(pq)^{k-1}}$ for all $k \ge 1$.

and

__Corollary 2.__ If p and q are distinct odd primes different from 3 such that $p \mid L_{2\cdot 3^k}$ and $q \mid L_{2\cdot 3^k}$ where $k \ge 1$ and $n = 2\cdot 3^k p^t q^r$ then $n \mid L_n$ for $t \ge 0$ and $r \ge 0$.

Using $F_{2r} = F_r L_r$, we have

__Corollary 3.__ If p is an odd prime and $p \mid L_n$ then $p^k \mid F_{2np^{k-1}}$ for $k \ge 1$.

and

__Corollary 4.__ If p and q are distinct odd primes such that $p \mid L_n$ and $q \mid L_m$ where m and n are odd integers then $(pq)^k \mid F_{2mn(pq)^{k-1}}$ for $k \ge 1$.

Corollaries 3 and 4 can be strengthened if we know that p is an odd prime and $p \mid F_n$. To do this, we show another theorem discovered independently by Carlitz and Bergum.

__Theorem 4.__ If p is an odd prime and $p \mid F_n$ then $p^k \mid F_{np^{k-1}}$ for all $k \ge 1$.

__Proof.__ By hypothesis, the theorem is true for $k = 1$. Assume $p^k \mid F_{np^{k-1}}$ and let $t = p^{k-1}$ then $pt \mid F_{nt}$. We shall show that $p^2t \mid F_{npt}$. Using Binet's formula together with the factorization of $x^p - y^p$, we have

$$F_{npt} = F_{nt} \sum_{i=1}^{p} \alpha^{nt(p-i)} \beta^{nt(i-1)} . \tag{10}$$

The middle term of the summation is $(-1)^{n(p-1)/2}$ while the sum of the q^{th} and $(p+1-q)^{th}$ terms, where $q \neq (p+1)/2$, using the formula $L_{2r} = 5F_r^2 + 2(-1)^r$, is

$$\alpha^{nt(p-q)}\beta^{nt(q-1)} + \alpha^{nt(q-1)}\beta^{nt(p-q)} = (-1)^{n(q-1)} L_{2nt(p-2q+1)/2} \tag{11}$$
$$= (-1)^{n(q-1)} 5F^2_{nt(p-2q+1)/2} + 2(-1)^{n(p-1)/2} .$$

By substitution into (10), we obtain

$$F_{npt} = F_{nt}\left(\sum_{q=1}^{p-1/2} (-1)^{n(q-1)} 5F^2_{nt(p-2q+1)/2} + p(-1)^{n(p-1)/2}\right) . \tag{12}$$

Using $pt \mid F_{nt}$ and (1), we see that p is a factor of the expression in the parentheses of (12) so that $p^2t \mid F_{npt}$ and the theorem is proved.

Let $F_n(L_n)$ be the least such that $p \mid F_n (p \mid L_n)$ then it is still unresolved if $p^k \mid F_m (p^k \mid L_m)$ or $p^k \nmid F_m (p^k \nmid L_m)$ for $np^{k-2} < m < np^{k-1}$ and $k \geq 2$.

An immediate consequence of Theorem 4, by use of (1), is

Corollary 5. If p and q are distinct odd primes such that $p \mid F_n$ and $q \mid F_m$ then $(pq)^k \mid F_{mn(pq)^{k-1}}$ for $k \geq 1$.

Another result of Theorem 4 which was already discovered by Kramer and Hoggatt and occurs in [2] is

$$5^k \mid F_{5^k}, \quad \text{for} \quad k \geq 1 \tag{13}$$

since $F_5 = 5$. Note that this result also gives us a sequence $\{n_k\}$ such that $n_k \mid F_{n_k}$.

Just as the authors could find several sequences $\{n_k\}$ such that $n_k \mid L_{n_k}$ they were also able to show that there are several other sequences $\{n_k\}$ such that $n_k \mid F_{n_k}$. With this in mind, we prove the next four theorems.

Theorem 5. If $n = 3^m 2^{r+1}$ where $m \geq 1$ and $r \geq 1$ then $n \mid F_n$.

Proof. By the discussion following Theorem 2 and Corollary 3, we have $3^m \mid F_{4\cdot 3^m}$ for $m \geq 1$. But $4 \mid F_6$ so that $4 \mid F_{4\cdot 3^m}$ for $m \geq 1$. Since $(4, 3^m) = 1$, we have $4\cdot 3^m \mid F_{4\cdot 3^m}$ for $m \geq 1$ and the theorem is proved if $r = 1$.

Since

$$F_{3^n 2^{r+2}} = F_{3^m 2^{r+1}} L_{3^m 2^{r+1}} = F_{3^m 2^{r+1}}(5F^2_{3^m 2^r} + 2)$$

and $2 \mid F_3$, we have by induction on r that $3^m 2^{r+2} \mid F_{3^m 2^{r+2}}$.

Theorem 6. If

$$n = 2^{r+1} 3^m 5^k,$$

where $r \geq 1$, $m \geq 1$, and $k \geq 1$ then $n \mid F_n$.

Proof. This result follows immediately from Theorem 5, (1), and (13) because

$$(5^k,\ 2^{r+1}3^m) = 1 .$$

By using Theorem 4 and Corollary 5 in an argument similar to that of Theorem 6, we have

Theorem 7. Let p be any odd prime different from 3 and such that $p \mid F_{2^{r+1}3^m}$ where $r \geq 1$ and $m \geq 1$. Let $n = 2^{r+1}3^m p^k$ where $k \geq 1$, then $n \mid F_n$.

and

Theorem 8. Let $s = 2^{r+1}3^m$. Let p and q be distinct odd primes such that $p \mid F_s$ and $q \mid F_s$. Let $n = sp^kq^t$ where $k \geq 0$ and $t \geq 0$ then $n \mid F_n$.

For our next divisibility property, we establish

Theorem 9. If $k \geq 1$ then $2^{k+2} \mid F_{3\cdot 2^k}$.

Proof. Since $8 \mid F_6$, the theorem is true for $k = 1$. Suppose $s = 2^{k-1}$ and $8s \mid F_{6s}$. Since $F_{12s} = F_{6s}L_{6s} = F_{6s}(5F_{3s}^2 + 2)$ and $2 \mid F_3$, the result follows by induction with the use of (1).

Throughout the remainder of this paper, we analyze the prime decomposition of L_n where n is odd and establish several congruence relations between the elements of $\{F_n\}$ and $\{L_n\}$. With this in mind, we first establish

Lemma 1. If n is odd then $L_n = 4^tM$ where $t = 0$ or 1 and M is odd.

Proof. Since n is odd, we have (1) $L_n = L_{3m+1}$ where m is even, (2) $L_n = L_{3m+2}$ where m is odd, or (3) $L_n = L_{3m}$ where m is odd.

If $L_n = L_{3m+1}$ and $m = 2r$ then $L_n = L_{6r+1}$. Since $2 \mid F_{3r}$, $L_{6r} = 5F_{3r}^2 + 2(-1)^r$, and $(L_{6r}, L_{6r+1}) = 1$, we have L_{3m+1} is odd or that $L_{3m+1} = 4^0M$ where M is odd.

By a similar argument, it is easy to show that $L_{3m+2} = 4^0M$ where M is odd.

Suppose $L_n = L_{3m}$ where $m = 2r + 1$. By an argument similar to that of Theorem 2, it is easy to show that

$$L_n = L_{6r+3} = \begin{cases} 4\left(\sum_{q=0}^{r-1} 5F_{3(r-q)}^2 + 1\right) & \text{if } r \text{ is even;} \\ 4\left(\sum_{q=0}^{r-1} 5F_{3(r-q)}^2 - 1\right) & \text{if } r \text{ is odd .} \end{cases} \tag{14}$$

Now $2 \mid F_{3(r-q)}$ so that the terms in the parentheses are odd and $L_n = 4M$ where M is odd.

The following theorem is due to Hoggatt while the proof is that of Brother Alfred Brousseau.

Theorem 10. The Lucas numbers L_n with n odd have factors 4^tM where $t = 0$ or 1 and the prime factors of M are primes of the form $10m \pm 1$.

Proof. The first part of the theorem is a result of Lemma 1.

From $L_n^2 - L_{n-1}L_{n+1} = (-1)^n 5$, we have that $L_{n-1}L_{n+1} \equiv 5 \pmod p$ for any odd prime divisor p of L_n. However, $L_{n+1} = L_n + L_{n-1}$ so that $L_{n+1} \equiv L_{n-1} \pmod p$.

Therefore, $L_{n-1}^2 \equiv 5 \pmod{p}$ and 5 is a quadratic residue modulo p. Since the only primes having 5 as a quadratic residue are of the form $10m \pm 1$, we are through.

Using Binet's formula, it can be shown that

$$L_{12t+j} = 5F_{(12t+j-1)/2}F_{(12t+j+1)/2} + (-1)^{(j-1)/2}, \quad j \text{ odd}. \tag{15}$$

Combining the results of Lemma 1 with (15), we have

Theorem 11. There exists an integer N such that

(a) $$L_{12t+1} = 10N + 1,$$

(b) $$L_{12t+3} = 4(10N + 1),$$

(c) $$L_{12t+5} = 10N + 1,$$

(d) $$L_{12t+7} = 10N - 1,$$

(e) $$L_{12t+9} = 4(10N - 1),$$

and

(f) $$L_{12t+11} = 10N - 1.$$

Since the proof of Theorem 11 is trivial, it has been omitted. However, a word of caution about the results is essential. Even though $L_{12t+3} = 4(10N + 1)$ and $L_{12t+5} = 10N + 1$, not all prime factors are of the form $10n + 1$ since $19^2 \mid L_{12\cdot 14+3}$ and $199^2 \mid L_{12\cdot 182+5}$. However, the number of prime factors of the form $10n - 1$ which divide L_{12t+3} or L_{12t+5} must be even.

Since $11^2 \mid L_{4\cdot 12+7}$, $211 \mid L_{12\cdot 1+9}$ and $11^2 \mid L_{12\cdot 22+11}$, we see that there can be primes of the form $10n + 1$ which divide L_{12t+j} for $j = 7, 9$, or 11. In fact, the number of primes of the form $10n - 1$ which divide L_{12t+j} where $j = 7, 9$, or 11 must be odd.

Examining [4], we see that $L_{49} = 29\cdot 599786069$ so that L_{12t+1} may have prime factors of the form $10n \pm 1$.

By Binet's formula, we have

$$F_{n+6} - F_{n-2} = L_n + L_{n+4} = L_{n+2}L_2. \tag{16}$$

Hence, by expanding and substitution of (16), we have

$$\sum_{i=0}^{2^j-1} L_{n+4i} = F_{n+2^{j+2}-2} - F_{n-2}. \tag{17}$$

Using (16) and induction, it can be shown that

$$\sum_{i=0}^{2^j-1} L_{n+4ki} = L_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \qquad j \geq 1. \tag{18}$$

Hence, by (17) and (18) with $k = 1$ and n replaced by $n + 2$, we have

$$L_{n+2^j+1} \prod_{i=1}^{j} L_{2^i} = F_{n+2^{j+2}} - F_n \tag{19}$$

so that

$$F_{n+2^{j+2}} \equiv F_n \pmod{L_{2^i}} \quad \text{for} \quad 1 \leq i \leq j \tag{20}$$

and

$$F_{n+2^{j+2}} \equiv F_n \pmod{L_{n+2^j+1}} \quad \text{if} \quad j \neq 0. \tag{21}$$

In papers to follow, the authors will generalize, where possible, the results of this paper to the generalized sequence of Fibonacci numbers as well as to several general linear recurrences. They will also investigate sums and products of the form occurring in (18).

REFERENCES

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Co., 1969.
2. Judy Kramer and V. E. Hoggatt, Jr., "Special Cases for Fibonacci Periodicity," Fibonacci Quarterly, Vol. 10, No. 5, pp. 519-522.
3. I. Niven and H. Zuckerman, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc., 1960.
4. Fibonacci and Related Number Theoretical Tables, edited by Brother Alfred Brousseau, Fibonacci Association, 1972, p. 11.
5. Larry Lang, Fibonacci Quarterly, Elementary Problem No. B-247, Vol. 10, No. 5 (Dec. 1972).

COMBINATIONS AND SUMS OF POWERS

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We adopt the following notation and conventions:

1. n and Q are non-negative integers.

2. $$S_Q = \sum_{i=1}^{n} i^Q .$$

3. $$\sum_{i=a}^{b} F(i) = 0 \quad \text{for} \quad a > b .$$

4. $$\prod_{i=a}^{b} F(i) = 1 \quad \text{for} \quad a > b .$$

5. $B_1 = 1/6$, $B_2 = -1/30$, $B_3 = 1/42$, etc., are the non-zero Bernoulli numbers.

6. $$g_Q(x_1, x_2, \cdots, x_m) = \left[\prod_{i=1}^{m} x_i^{-1}\right] \cdot \left[\prod_{j=1}^{m-1} \binom{x_{j+1}}{x_{j-1}}\right] \cdot \binom{Q+1}{x_m - 1} .$$

For example,

$$g_4(1) = 1^{-1}\binom{5}{0}$$
$$g_4(1,3) = (1\cdot 3)^{-1}\binom{3}{0}\binom{5}{2}$$
$$g_4(1,3,4) = (1\cdot 3\cdot 4)^{-1}\binom{3}{0}\binom{4}{2}\binom{5}{3} .$$

7. $$d_Q(x_1, x_2, \cdots, x_m) = g_Q(x_1, x_2, \cdots, x_m)\cdot n^{x_1} .$$

<u>Theorem 1.</u> Say $Q \geq 0$. Then

$$(Q + 1)S_Q = n^{Q+1} + (Q + 1)n^Q - 1 + \prod_{i=1}^{Q} (1 - r_i) ,$$

where

$$\prod_{i=2}^{Q} (1 - r_i)$$

is expressed in terms of sums of products of the r_i, and for each such product, e.g., $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m}$, where $x_1 < x_2 < \cdots < x_m$ for $m \geq 2$, we let $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m} = dQ(x_1, x_2, \cdots, x_m)$.

Theorem 2. Say $Q \geq 1$. Then

$$(2Q + 1)B_Q = -r_1 \prod_{i=2}^{2Q} (1 - r_i) ,$$

where

$$-r_1 \prod_{i=2}^{2Q} (1 - r_i)$$

is expressed in terms of sums of products of the r_i, and for each such product, e.g., $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m}$, where $x_1 < x_2 < \cdots < x_m$ for $m \geq 2$, we let $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m} = g_{2Q}(x_1, x_2, \cdots, x_m)$.

Theorem 3. Say $Q \geq 1$. Then

$$(S + 1)^Q - S^Q = (n + 1)^Q - 1 ,$$

where S^i is formally replaced by S_i when the left-hand side of this equation is expanded; e.g., $1S_0 + 3S_1 + 3S_2 = (n + 1)^3 - 1$. Hence, starting with $S_0 = n$, this theorem can be used to find S_Q in a recursive fashion.

Theorem 4.

$$S_1 = \frac{1}{2!} \begin{vmatrix} 1 & n \\ 1 & n^2 \end{vmatrix} + n$$

$$S_2 = \frac{1}{3!} \begin{vmatrix} 1 & 0 & n \\ 1 & 2 & n^2 \\ 1 & 3 & n^3 \end{vmatrix} + n^2$$

$$S_3 = \frac{1}{4!} \begin{vmatrix} 1 & 0 & 0 & n \\ 1 & 2 & 0 & n^2 \\ 1 & 3 & 3 & n^3 \\ 1 & 4 & 6 & n^4 \end{vmatrix} + n^3 ,$$

etc., where the entries in the determinants are binomial coefficients, zeros, and powers of n.

We now illustrate two more methods for finding S_Q.

Method 1. The "$(i+1)^Q - (i-1)^Q$" method. For example,

$$\sum_{i=1}^{n}\left[(i+1)^2 - (i-1)^2\right] = \sum_{i=1}^{n} 4i .$$

$$\therefore \qquad (n+1)^2 + n^2 - 1 = \sum_{i=1}^{n} 4i .$$

$$\therefore \qquad 4\sum_{i=1}^{n} i = 2n^2 + 2n .$$

$$\therefore \qquad \sum_{i=1}^{n} i = \frac{n^2+n}{2} = \frac{n(n+1)}{2} .$$

Method 2. Lagrange interpolation. Assuming that S_Q is a polynomial of degree $Q + 1$ in n, we now compute S_1. Let $f(n) = S_1 = 1 + 2 + \cdots + n$. Then, by Lagrange interpolation, we have $f(n) = f(1)P_1 + f(2)P_2 + f(3)P_3$, where, letting $t_i = i$,

$$P_1 = \frac{(n-t_2)(n-t_3)}{(t_1-t_2)(t_1-t_3)} = \frac{(n-2)(n-3)}{(-1)(-2)}$$

$$P_2 = \frac{(n-t_1)(n-t_3)}{(t_2-t_1)(t_2-t_3)} = \frac{(n-1)(n-3)}{(1)(-1)}$$

$$P_3 = \frac{(n-t_1)(n-t_2)}{(t_3-t_1)(t_3-t_2)} = \frac{(n-1)(n-2)}{(2)(1)} .$$

Editors' Note: This abstract qualifies for the Fibonacci Note Service. It is an abstract of a paper which is fifty pages long. If you would like a Xerox copy of the entire article at four cents a page (which includes postage, materials and labor), send your request to:

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FUNCTIONAL EQUATIONS WITH PRIME ROOTS FROM ARITHMETIC EXPRESSIONS FOR $\mathcal{G}_\alpha$

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1. In this article, a generalized form of Euler's law concerning the sigma function will be obtained and used to derive expressions for $\mathcal{G}_\alpha$ which contain just functions involving addition and multiplication. These will be substituted in the equations

$$\mathcal{G}_\alpha(n) - n^\alpha - 1 = 0 \tag{1}$$

to obtain equations with classes of solutions identical with the class of prime numbers.

2. Let

$$F(n) = \sum_{d|n} f(d) .$$

Proposition 1. If

$$\sum_{n=1}^{\infty} F(n)\, x^n$$

converges on some interval about 0, then

$$0 = nR(n) + \sum_{a=1}^{n} F(a)R(n-a) , \tag{2}$$

where

$$\sum_{n=0}^{\infty} R(n)\, x^n = \prod_{n=1}^{\infty} (1 - x^n)^{f(n)/n} . \tag{3}$$

The proof mimics Euler's for the case f = identity, which is the recursive expression for sum of divisors he obtained by describing R. [1]

Proof.

$$\sum_{n=1}^{\infty} f(n)\,x^n/(1 - x^n) = \sum_n f(n) \sum_k x^{nk}$$

$$\begin{aligned} = f(1)x &+ f(1)x^2 + f(1)x^3 + f(1)x^4 + f(1)x^5 + f(1)x^6 + \cdots \\ &+ f(2)x^2 \qquad\qquad + f(2)x^4 \qquad\qquad + f(2)x^6 + \cdots \\ &\qquad\qquad + f(3)x^3 \qquad\qquad\qquad\qquad + f(3)x^6 + \cdots \\ &\qquad\qquad\qquad\qquad + f(4)\,x^4 + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad + f(5)x^5 + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f(6)x^6 + \cdots \end{aligned}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{d|n} f(d) = \sum_{n=1}^{\infty} F(n)\,x^n .$$

That is,

$$(4) \qquad \sum_{n=1}^{\infty} f(n)\,x^n/(1 - x^n) = \sum_{n=1}^{\infty} F(n)\,x^n .$$

Suppose

$$(5) \qquad 0 < \Pi\,(1 - x^n)^{f(n)/n} < \infty$$

on some interval about 0. We show that (2) holds under (5) and then that (5) holds when

$$\sum_{n=1}^{\infty} F(n)\,x^n$$

converges on some interval about 0.

Let (5) hold. We have the identity:

$$\log \prod_{n=1}^{\infty} (1 - x^n)^{f(n)/n} = \sum_{1}^{\infty} f(n)/n \log (1 - x^n) .$$

Differentiating, and substituting from (3) as (5) permits:

$$\sum_1^\infty -f(n)x^{n-1}/(1 - x^n) = \frac{\frac{d}{dx}\left[\prod_1^\infty (1 - x^n)^{f(n)/n}\right]}{\prod_1^\infty (1 - x^n)^{f(n)/n}}$$

$$= \left(\frac{d}{dx}\sum_0^\infty R(m)x^m\right) / \sum_0^\infty R(m)x^m$$

$$= \sum_0^\infty mR(m)x^{m-1} / \sum_0^\infty R(m)x^m \; .$$

Hence, by (4),

$$-\sum_0^\infty mR(m)x^m / \sum_0^\infty R(m)x^m = \sum_1^\infty f(n)x^n / (1 - x^n) = \sum_1^\infty F(n)x^n \; . \tag{6}$$

and Eq. (6) gives:

$$0 = \left(\sum_1^\infty F(n)x^n\right)\left(\sum_0^\infty R(m)x^m\right) + \sum_0^\infty mR(m)x^m \; .$$

So, for each $n \geq 0$, the coefficient x^n is 0:

$$0 = \sum_{a=1}^n F(a)R(n - a) + nR(n) \; .$$

It remains to show that (5) holds when

$$\sum_{n=1}^\infty F(n)\, x^n$$

converges on some interval about 0. By Eq. (6),

$$\sum_1^\infty F(n)x^n = -xd/dx \log P(x) \quad ,$$

where

$$P(x) = \prod_{1}^{\infty} (1 - x^n)^{f(n)/n} .$$

Therefore,

$$P(x) = \exp \int - \sum_{1}^{\infty} F(n)x^{n-1}\,dx . \tag{7}$$

Hence $P(x) = 0$ iff

$$\int \sum_{1}^{\infty} F(n)x^{n-1}\,dx = \infty$$

iff

$$\sum_{1}^{\infty} \frac{F(n)x^n}{n} = \infty$$

and $P(x) = \infty$ iff

$$\sum_{1}^{\infty} (1/n)F(n)x^n = -\infty .$$

Thus (5) holds iff

$$\left| \sum_{1}^{\infty} (1/n)F(n)x^n \right| < \infty$$

on some interval about 0, and this is the case when

$$\left| \sum_{1}^{\infty} F(n)\,\kappa^n \right| < \infty$$

on the same interval. Q. E. D.

Now it is necessary to show that the conditions of Proposition 1 apply to $\mathcal{G}_\alpha$. Actually, we show a little more.

Proposition 2. Let

$$\sum_{d|n} f(d) = F(n) .$$

Then,

$$\left| \sum F(n)\,x^n \right| < \infty$$

on some interval about 0 if and only if

$$\left|\sum f(n)x^n\right| < \infty$$

on some interval about 0.

Proof.

$$\left|\sum F(n)x^n\right| < \infty \to \left|\sum f(n)x^n/1 - x^n\right| < \infty \to \left|\sum f(n)x^n\right| < \infty$$

by (4) and comparison. For the other direction, let

$$\left|\sum f(n)x^n\right| < \infty \quad .$$

By the root test,

$$\limsup |f(n)|^{\frac{1}{n}} < \infty \; .$$

That is, $\sup L_i < \infty$, where

$$L_i = \lim_k |f(a_{ik})|^{\frac{1}{a_{ik}}}$$

on some sequence $\{a_{ik}\}$.

Define $\{c_k\}$ by:

$$|f(c_k)| = \max_{d|a_k} |f(d)|$$

for a sequence $\{a_k\}$. For each $\underline{k}$, c_k is one of the divisors of a_k. Then,

$$\lim |f(c_k)|^{\frac{1}{c_k}} \leq \sup L_i < \infty,$$

and over all sequences $\{a_k\}$ the $\{c_k\}$ are bounded by:

$$\sup_{\{a_k\}} \lim |f(c_k)|^{\frac{1}{c_k}} \leq \sup L_i < \infty \quad .$$

That is,

$$\sup_{\{a_k\}} \lim \left[\max_{d|a_k} |f(d)|\right]^{\frac{1}{c_k}} \leq \sup L_i \; .$$

Now

$$\left[\max_{d \mid a_k} |f(d)|\right]^{\frac{1}{a_k}} \le \left[\max_{d \mid a_k} |f(d)|\right]^{\frac{1}{c_k}} .$$

So:

$$\sup_{\{a_k\}} \lim \left[\max_{d \mid a_k} |f(d)|\right]^{\frac{1}{a_k}} \le \sup L_i < \infty .$$

That is,

$$\limsup \max_{d \mid n} |f(d)|^{\frac{1}{n}} \le \sup L_i < \infty .$$

Now, we demonstrate below that

$$\left|\sum \tau(n)\, x^n\right| < \infty$$

on some interval about 0, where τ is the number-of-divisors function. The demonstration below is valid but clearly circuitous. Thus,

$$\limsup |\tau(n)|^{\frac{1}{n}} < \infty$$

by the root test, and

$$\limsup \left[\tau(n) \max_{d \mid n} |f(d)|\right]^{\frac{1}{n}} = \limsup |\tau(n)|^{\frac{1}{n}} \left[\max_{d \mid n} |f(d)|\right]^{-\frac{1}{n}}$$

$$\le \limsup |\tau(n)|^{\frac{1}{n}} \limsup \left[\max_{d \mid n} |f(d)|\right]^{\frac{1}{n}} < \infty .$$

Thus,

$$\sum_n x^n \tau(n) \max_{d \mid n} |f(d)| < \infty$$

on some interval about 0. Then,

$$\left|\sum F(n) x^n\right| \le \sum |F(n)| x^n \le \sum \sum_{d \mid n} |f(d)| x^n$$

$$\le \sum \tau(n) \max_{d \mid n} |f(d)|\, x^n < \infty$$

q. e. d.

We repair the gap in the proof of Proposition 2, the assertion without demonstration that

$$\sum \tau(n)\, x^n$$

converges on some interval about 0, by comparing this sum with another. The result is obvious on comparing $\tau(n)$ with the identity function:

$$\left|\sum nx^n\right| < \infty$$

on (-1, 1).

One more proposition is needed to finish the background for a demonstration that Proposition 1 applies to $\mathscr{G}_\alpha$.

Proposition 3:

$$\sum 1/n\ F(n)\,x^n$$

converges on some interval about 0 iff

$$\sum F(n)\,x^n$$

converges on some interval about 0.

Proof. Under the hypothesis that

$$\sum 1/n\ F(n)\,x^n$$

converges we have by the root test:

$$\limsup \left|F(n)(1/n)\right|^{\frac{1}{n}} < \infty .$$

That is,

$$\sup_{\{a_k\}} \lim \left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}} < \infty .$$

Now, clearly when

$$\left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}}$$

converges, its limit is

$$\lim \left|F(a_k)\right|^{\frac{1}{a_k}}$$

Also, it is clear that

$$\left|F(a_k)\right|^{\frac{1}{a_k}}$$

converges if and only if

$$\left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}} ,$$

too, converges. So

$$\limsup \left|F(n)\right|^{\frac{1}{n}} = \sup_{\{a_k\}} \lim \left|F(a_k)\right|^{\frac{1}{a_k}} = \sup_{\{a_k\}} \lim \left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}}$$
$$= \limsup \left|F(n)(1/n)\right|^{\frac{1}{n}} .$$

So
$$\sum F(n)\, x^n$$
converges on some interval about 0. The other direction is similar, or by comparison, q. e. d.

Now we prove that Proposition 1 may be applied to $\mathcal{G}_\alpha$.

<u>Proposition 4.</u>
$$\sum \mathcal{G}_\alpha(n)\, x^n$$
converges on some interval about 0.

<u>Proof.</u> $\sum x^n$ converges on $[0,1)$. Apply Proposition 3 inductively: for each α,
$$\sum n^\alpha x^n$$
converges on some interval. Then, by Proposition 2,
$$\sum \mathcal{G}_\alpha(n)\, x^n$$
converges. q. e. d. Proposition 1 now yields a recursive relation on $\mathcal{G}_\alpha$ in terms of the coefficients of the power series for $P(x)$ with $f(n) = n^\alpha$. $P(x)$ is an infinite product and, in order to determine an expression for $\mathcal{G}_\alpha$ which is recursive in addition and multiplication, we express the coefficients of the power series for $P(x)$ as the coefficients of the expansion of a finite product.

<u>Proposition 5.</u>
$$0 = nR(n) + \sum_{a=1}^{n} \mathcal{G}_\alpha(a) R(n - a)\ ,$$
where $R(k)$ = coefficient of x^k in
$$\prod_{n=1}^{k} (1 - x^n)^{n^{\alpha-1}} .$$

<u>Proof.</u> Applying Proposition 1, to
$$\prod_{n=1}^{\infty} (1 - x^n)^{n^{\alpha-1}} = \sum_{n=0}^{\infty} S(n) x^n\ :$$
Let
$$\prod_{1}^{k} (1 - x^n)^{n^{\alpha-1}} = \sum_n \overline{R}_k(n) x^n$$
(Definition). Then
$$\sum_n \overline{R}_{k+1}(n)\, x^n = \prod_{n=1}^{k+1} (1 - x^n)^{n^{\alpha-1}} = (1 - x^{k+1})^{[k+1]^{\alpha-1}} \times \sum_n \overline{R}_k(n) x^n$$
$$= \sum_{r=0}^{[k+1]^{\alpha-1}} \binom{[k+1]^{\alpha-1}}{r} (-1)^r x^{(k+1)r} \times \sum_n \overline{R}_k(n) x^n =$$

$$= \sum_n \overline{R}_k(n)\,x^n + \sum_{r=1}^{[k+1]^{\alpha-1}} \binom{[k+1]^{\alpha-1}}{r} (-1)^r x^{(k+1)r} x \sum_n \overline{R}_k(n)\,x^n$$

$$= \sum_n \overline{R}_{k+1}(n)\,x^n \; .$$

None of the terms in the second summand have exponents $\leq k$. Thus

$$\overline{R}_k(i) = \overline{R}_{k+1}(i)$$

for all $i \leq k$. Indeed, $\overline{R}_k(i) = \overline{R}_1(i)$ for all i and 1 such that $i \leq k \leq 1$. Thus

$$\sum S(n)\,x^n = \lim_k \prod_1^k (1 - x^n)^{n^{\alpha-1}} = \lim_k \sum_n \overline{R}_k(n)\,x^n = \sum_n \lim_k \overline{R}_k(n)\,x^n = \sum_n \overline{R}_n(n)\,x^n,$$

and $\overline{R}_n(n) = S(n)$. q.e.d.

It is now possible to define a function, which turns out to be $\mathcal{G}_\alpha$, which is expressible in terms of just addition and multiplication, and which leads to the equation mentioned in the title.

Define $F_\alpha(1) = 1$ and, supposing F_α defined on $1, 2, \cdots, n-1$, let $F_\alpha(n)$ satisfy

$$0 = nR(n) + \sum_{a=1}^{n} F_\alpha(a)R(n-a) \; ,$$

where R is defined as in the statement of Proposition 5. Then, by Proposition 5, $F_\alpha = \mathcal{G}_\alpha$, and F_α satisfies $0 = F_\alpha(n) - n^\alpha - 1$ just when $\underline{n}$ is a prime number.

REFERENCE

1. Euler, Opera Omnia, Series 1, Vol. 2, pp. 241-253, "Discovery of a Most Extraordinary Law of the Numbers Concerning the Sum of Their Divisors."

AN EXPANSION OF e^x OFF ROOTS OF ONE

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The proposition below is proved in [1].

Let Δ be the operator on arithmetical functions such that

(1) $$\Delta F(n) = \sum_{d|n} F(d) .$$

Let

$$\sum_{n=0}^{\infty} x^n \Delta f(n)$$

converge. Let

(2) $$\prod_{n=1}^{\infty} (1 - x^n)^{\frac{f(n)}{n}} = \sum_{n=0}^{\infty} R_f(n)\, x^n .$$

Then for all n:

(3) $$0 = n R_f(n) + \sum_{a=1}^{n} \Delta f(a)\, R_f(n - a)$$

when x is not a root of one.

Now, let $f = \mu$ (the Mobius function) and let

$$\eta \begin{array}{l} = 1 \text{ on } 1, \\ = 0, \text{ elsewhere} . \end{array}$$

It is well known that $\Delta\mu = \eta$. Now, $\Sigma x^n \eta(n)$ converges. It follows immediately (by induction) from (3) that $R_\mu(n) = (-1)^n/n!$ and hence that

$$\prod_{n=1}^{\infty} (1 - x^n)^{\mu(n)/n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$$

(when x is not a root of 1); thus

$$e^x = \prod_{n=1}^{\infty} (1 - x^n)^{\frac{-\mu(n)}{n}}$$

off roots of 1.

REFERENCE

1. Barry Brent, "Functional Equations with Prime Roots from Arithmetical Expressions for $\mathcal{G}_\alpha$," Fibonacci Quarterly, Vol. 12, No. 2 (April 1974), pp. 199-207.

INFINITE SEQUENCES OF PALINDROMIC TRIANGULAR NUMBERS

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A triangular number, $\Delta(n) = n(n+1)/2$, is palindromic if it is identical with its reverse. It has been established that an infinity of palindromic triangular numbers exists in bases three [1], five [2], and nine [5]. Also, it has been shown [3] that, in a system of numeration with base $(2k+1)^2$, when $k(k+1)/2$ is annexed to $n(n+1)/2$ then

$$[n(n+1)/2](2k+1)^2 + k(k+1)/2 = [(2k+1)n + k][(2k+1)n + k + 1]/2 ,$$

another triangular number. If the first value of n is k, then an infinite sequence of triangular numbers can be generated, each consisting of like "digits," $k(k+1)/2$, so that each member of the sequence is palindromic.

In the following discussion, n and $\Delta(n)$ are expressed in the announced base. An abbreviated notation is employed, wherein a subscript in the decimal system following an expression indicates the number of times it is repeated in the integer containing it. Thus, the repdigit $333333 = 3_6$, $21111000 = 21_4 0_3$, and $1010101 = (10)_3 1$.

The base $(2k+1)^2 = 8[k(k+1)/2] + 1$ is of the form $8m+1$, where m itself is a triangular number. It is not necessary to restrict m to this extent. In general, if n has the form $(10^k - 1)/2$, then $\Delta(n) = (10^{2k} - 1)/2^3$. It follows that in any system of notation with a base, $b = 8m+1$, a palindromic $\Delta(n) = m_{2k}$ corresponds to the palindromic $n = \overline{4m_k}$.

BASE NINE

The smallest base of the form $8m+1$ is nine, for $m = 1$. Hence $n = 4_k$ generates the palindromic $\Delta(n) = 1_{2k}$, $k = 1, 2, 3, \cdots$. Nine also is of the form $(2k+1)^2$. The above argument regarding the existence of an infinity of palindromic triangular numbers in bases of this type does not deal with the nature of the corresponding n's.

In base nine, for $k = 0, 1, 2, \cdots$, $n = 14_k$ may also be written as

$$n = 10^k + (10^k - 1)/2 = [3(10^k) - 1]/2 = (10^{k+1} - 3)/6 .$$

Then

$$\Delta(n) = (10^{k+1} - 3)(10^{k+1} + 3)/2(6^2) = (10^{2k+2} - 10)/80 = (10^{2k+1} - 1)/8 = 1_{2k+1} .$$

These two results reestablish that, in the scale of nine, any repunit, 1_p, with $p = 1, 2, 3, \cdots$, is a palindromic triangular number.

Furthermore, for $k = 0, 1, 2, \cdots$, we have

$$n = 24_k6 = 2(10^{k+1}) + (10^k - 1)(10)/2 + 6 = [5(10^{k+1}) + 3]/2 .$$

It follows that

$$\begin{aligned}\Delta(n) &= [5(10^{k+1}) + 3][5(10^{k+1}) + 5]/8 \\ &= 5^2(10^{2k+2})/8 + 8(5)(10^{k+1})/8 + 3(5)/8 \\ &= 3(10^{2k+2}) + 10^{k+2}(10^k - 1)/8 + 6(10^{k+1}) + 10(10^k - 1)/8 + 3 \\ &= 31_k61_k3 .\end{aligned}$$

Thus there are two infinite sequences of palindromic triangular numbers in base nine. These do not include all the palindromic $\Delta(n)$ for $n < 42161$. Also, there are:

$$\Delta(2) = 3,\ \Delta(3) = 6,\ \Delta(35) = 646,\ \Delta(115) = 6226,\ \Delta(177) = 16661,\ \Delta(353) = 64246$$

(the distinct digits are consecutive even digits),

$$\Delta(1387) = 1032301$$

(the distinct digits are consecutive),

$$\Delta(1427) = 1075701,\ \Delta(2662) = 3678763,\ \Delta(3525) = 6382836,\ \Delta(3535) = 6428246$$

(the distinct digits are consecutive even digits),

$$\Delta(4327) = 10477401,\ \Delta(17817) = 167888761,\ \Delta(24286) = 306272603,\ \Delta(24642) = 316070613,$$
$$\Delta(26426) = 362525263,\ \Delta(36055) = 666707666 .$$

BASES OF FORM $2m + 1$

In bases of the form $2m + 1$, if $n = m_k = (10^k - 1)/2$, then

$$\Delta(n) = (10^{2k} - 1)/2^3 = (\overline{2m}_{2k})/2^3 .$$

Now, if

$$[2m(2m + 1) + 2m]/2^3 = m(m + 1)/2 < 2m + 1 ,$$

then $\Delta(n)$ is palindromic. Thus, in base three, $\Delta(1_k) = \overline{01}_k$. In base 5, $\Delta(2_k) = \overline{03}_k$. In base seven, $\Delta(3_k) = \overline{06}_k$. In base nine, $\Delta(4_k) = \overline{11}_k$. In base eleven, $\Delta(5_k) = \overline{14}_k$.

That is, in every odd base not of the form $8m + 1$ there is an infinity of triangular numbers that are smoothly undulating (composed of two alternating unlike digits). In these odd bases < nine, these triangular numbers are palindromic with $2k - 1$ digits. In such odd bases > nine, these triangular numbers consist of repeated pairs of unlike digits, so they are not palindromic.

In bases of the form $8m + 1$ (including nine), these triangular numbers are repdigits with $2k$ digits, and are palindromic.

In base three, all of the palindromic triangular numbers for $n < 11(10^4)$ are of the $\Delta(1_1) = \overline{01}_k$ type.

In base five, for $n < 102140$, the other palindromic triangular numbers are

$$\Delta(1) = 1, \quad \Delta(3) = 11, \quad \Delta(13) = 121, \quad \Delta(102) = 3003,$$

$$\Delta(1303) = 1130311, \quad \Delta(1331) = 1222221, \quad \Delta(10232) = 30133103,$$

$$\Delta(12143) = 102121201, \quad \Delta(12243) = 103343301, \quad \Delta(31301) = 1022442201.$$

In base seven, for $n < 54145$, the other palindromic numbers are:

$$\Delta(1) = 1, \quad \Delta(2) = 3, \quad \Delta(15) = 141, \quad \Delta(24) = 333, \quad \Delta(135) = 11211,$$

$$\Delta(242) = 33033, \quad \Delta(254) = 36363, \quad \Delta(1301) = 1012101,$$

$$\Delta(1611) = 1525251, \quad \Delta(2414) = 3251523, \quad \Delta(2424) = 3306033,$$

$$\Delta(2442) = 3352533, \quad \Delta(2522) = 3546453, \quad \Delta(12665) = 100646001,$$

$$\Delta(13065) = 102252201, \quad \Delta(13531) = 112050211, \quad \Delta(15415) = 142323241,$$

$$\Delta(16055) = 15202051, \quad \Delta(23462) = 312444213,$$

$$\Delta(24014) = 321414123, \quad \Delta(25412) = 363030363.$$

Thus, in bases five, seven, and nine (but evidently not in base three) there are palindromic $\Delta(n)$ for which n is palindromic and palindromic $\Delta(n)$ for which n is nonpalindromic.

BASE TWO

In base two, for $k > 1$, if $n = 10^k + 1$, then

$$\begin{aligned}\Delta(n) &= (10^k + 1)(10^k + 10)/10 = (10^k + 1)(10^{k-1} + 1) \\ &= 10^{2k-1} + 10^k + 10^{k-1} + 1 = 10^{2k-1} + 11(10^{k-1}) + 1 \\ &= 10_{k-2}110_{k-2}1 .\end{aligned}$$

For $n < 101101$, in the binary system, palindromic $\Delta(n)$ not contained in this infinite sequence are:

$$\Delta(1) = 1, \quad \Delta(10) = 11, \quad \Delta(110) = 10101, \quad \Delta(10101) = 11100111,$$

$$\Delta(11001) = 101000101, \quad \Delta(101010) = 1110000111\,.$$

No infinite sequence of palindromic triangular numbers has been found in base ten [4] or in other even bases > two.

REFERENCES

1. Charles W. Trigg and Bob Prielipp, "Solution to Problem 3413," School Science and Mathematics, 72 (April 1972), p. 358.
2. Charles W. Trigg and E. P. Starke, "Triangular Palindromes," Solution to Problem 840, Mathematics Magazine, 46 (May 1973), p. 170.
3. Charles W. Trigg, Mathematical Quickies, McGraw Hill Book Co. (1967), Q112 p. 127.
4. Charles W. Trigg, "Palindromic Triangular Numbers," Journal of Recreational Mathematics, 6 (Spring 1973), pp. 146-147.
5. G. W. Wishard and Helen A. Merrill, "Solution to Problem 3480," American Mathematical Monthly, 39 (March 1932), p. 179.

A NOTE ON THE FERMAT - PELLIAN EQUATION $x^2 - 2y^2 = 1$

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It is a well known fact that $3 + 2\sqrt{2}$ is the fundamental solution of the Fermat-Pellian equation $x^2 - 2y^2 = 1$. Hence, if $u + v\sqrt{2}$ is any other solution then there exists an integer n such that $u + v\sqrt{2} = (3 + 2\sqrt{2})^n$. Let $T = (a_{ij})$ be the 3-by-3 matrix where $a_{12} = a_{21} = 1$, $a_{33} = 3$, and $a_{ij} = 2$ for all other values. It is interesting to observe that there exists a relationship between the integral powers of T and $3 + 2\sqrt{2}$. In fact, a necessary and sufficient condition for $M = T^n$ is that $M = (b_{ij})$ with $b_{33} = 2m + 1$, $b_{12} = b_{21} = m$, $b_{11} = b_{22} = m + 1$ and $b_{13} = b_{23} = b_{31} = b_{32} = v$, where $(2m + 1)^2 - 2v^2 = 1$. If $n \geq 0$ both the necessary and sufficient condition follow by induction. Using this fact, it then follows for $n < 0$.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-234 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Suppose an alphabet, $A = \{x_1, x_2, x_3, \cdots\}$, is given along with a binary connective, P (in prefix form). Define a well formed formula (wff) as follows: a wff is

(1) x_i for $i = 1, 2, 3, \cdots$, or

(2) If A_1, A_2 are wff's, then PA_1A_2 is a wff and

(3) The only wff's are of the above two types.

A wff of order n is a wff in which the only alphabet symbols are $x_1, x_2, \cdots, x_n$ in that order with each letter occurring exactly once. There is one wff of order 1, namely x_1. There is one wff of order 2, namely Px_1x_2. There are two wff's of order 3, namely $Px_1Px_2x_3$ and $PPx_1x_2x_3$, and there are five wff's of order 4, etc.

Define a sequence

$$\{G_i\}_{i=1}^{\infty}$$

as follows:

g_i is the number of distinct wff's of order i.

a. Find a recurrence relation for $\{G_i\}_{i=1}^{\infty}$ and

b. Find a generating function for $\{G_i\}_{i=1}^{\infty}$.

H-235 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

a. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} F_{n+1} x^n .$$

b. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} L_{n+1} x^n .$$

H-236 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1) \qquad \sum_{n=0}^{\infty} (-1)^n x^{n^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{k=1}^{\infty} (1 - x^k) ,$$

$$(2) \qquad \sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{k=1}^{\infty} (1 - x^k) ,$$

where $(x)_k = (1 - x)(1 - x^2) \cdots (1 - x^k)$, $(x)_0 = 1$.

SOLUTIONS

TO COIN A THEOREM

H-199 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

A certain country's coinage consists of an infinite number of types of coins: $\cdots, C_{-2}, C_{-1}, C_0, C_1, \cdots$. The value V_n of the coin C_n is related to the others as follows: for <u>all</u> n,

$$V_n = V_{n-3} + V_{n-2} + V_{n-1} .$$

Show that any (finite) pocketful of coins is equal in value to a pocketful containing at most one coin of each type.

Solution by the Proposer.

Call a pocketful Q canonical if it consists entirely of coins of different types and such that no three coins of "adjacent" types (e.g., c_n, c_{n+1} and c_{n+2}) are present. Call two pocketfuls equivalent if they have the same value.

We will prove for any pocketful P the statement:

S: P is equivalent to a canonical pocketful.

Note that any pocketful containing only differing types is equivalent to a canonical pocketful since the three adjacent coins of highest value, C_n, C_{n+1}, C_{n+2} can be replaced by C_{n+3}, etc.

Assume for the moment, the following statement:

R: S is true for any canonical pocketful to which one extra coin has been added.

Then the general result follows by induction on the number of coins for any pocketful P: Remove a coin to get P', apply the induction hypothesis to P' to get a canonical pocketful Q, return the removed coin and apply R.

Now to prove R, let us prove by induction on j the series of statements R_j:

R_j If Q is any canonical pocketful in which the coin of least value is a C_n, then if a C_{n+j} or a C_{n+j} and a C_{n+j+1} be added to Q to get a pocketful P', then S is true for P'.

Assume R_k for all $k < j$ (it is obvious if $k \le -3$). Now let Q be canonical. We can suppose that $n + j = 0$. Suppose Q contains δ_i coins of type C_i, $\delta_i = 0$ or 1, $\delta_i\delta_{i+1}\delta_{i+2} = 0$ for all i. Then

$$* \qquad Q \cup C_0 \equiv \cdots \delta_{-3},\ \delta_{-2},\ \delta_{-1},\ \delta_0 + 1,\ \delta_1,\ \delta_2,\ \delta_3,\ \cdots .$$

If $\delta_0 = 0$, we are finished, so assume $\delta_0 = 1$. Then

$$Q \cup C_0 \equiv \cdots \delta_{-3} + 1,\ \delta_{-2},\ \delta_{-1},\ 0,\ \delta_1 + 1,\ \delta_2,\ \delta_3,\ \cdots .$$

Again, if $\delta_1 = 0$, by induction, we are finished, so assume $\delta_1 = 1$. Then

$$Q \cup C_0 \equiv \cdots \delta_{-3} + 1,\ \delta_{-2} + 1,\ \delta_{-1},\ 0,\ 0,\ \delta_2 + 1,\ \delta_3 \cdots .$$

Now, since $\delta_0\delta_1\delta_2 = 0$, $\delta_2 = 0$ so we are finished.

For the next part,

$$Q \cup C_0 \cup C_1 \equiv \cdots \delta_{-3},\ \delta_{-2},\ \delta_{-1},\ \delta_0 + 1,\ \delta_1 + 1,\ \delta_2,\ \delta_3 \cdots .$$

If either δ_0 or $\delta_1 = 0$, this case can be handled as above, so suppose δ_0 and δ_1 are 1. Then

$$
\begin{array}{lllllllll}
Q \cup C_0 \cup C_1 \equiv \cdots & \delta_{-3} & \delta_{-2} & \delta_{-1} & 2 & 2 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+1 & \delta_{-1}+1 & 1 & 2 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+2 & \delta_{-1}+2 & 2 & 1 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+2 & \delta_{-1}+1 & 1 & 0 & \delta_2+1 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+1 & \delta_{-1} & 0 & 1 & \delta_2+1 & \delta_3 & \cdots
\end{array}
$$

and again, by induction, we are finished. This completes the proof.

We note, without proof, that no two canonical pocketfuls are equivalent.

Editorial Note: The given sequences identify the elements of the union.

ASYMPTOTIC PI

H-200 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Let $M(n)$ be the number of primes (distinct) which divide the binomial coefficient,

$$C_k^n \equiv \binom{n}{k}^* .$$

Clearly, for $1 \le n \le 15$, we have $M(1) = 0$, $M(2) = M(3) = 1$, $M(4) = M(5) = 2$, $M(6) = M(7) = M(8) = M(9) = 3$, $M(10) = 4$, $M(11) = M(12) = M(14) = 5$, $M(13) = M(15) = 6$, etc. Show that

$$\{m(n)\}_{n=1}^{\infty}$$

has an upper bound and find an asymptotic formula for $M(n)$.

*Divide at least one C_k^n, where $0 \le k \le n$.

Solution by D. Singmaster, Instituto Mathematica, Pisa, Italy.

For a prime p, if

$$p \,\Big|\, \binom{n}{k}$$

for some k, $0 \le k \le n$, then $p | n!$ and so $p \le n$. Hence $M(n) \le \pi(n)$, where $\pi(n)$ is the number of primes less than or equal to n. We claim $M(n) \sim \pi(n)$. To see this, we can use the following result of B. Ram. (See: L. E. Dickson, History of the Theory of Numbers, Vol. 1; Chelsea, 1952; p. 274, item 98.) There is at most one prime $p < n$ such that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ and such a prime p exists if and only if $n + 1 = ap^s$ with $1 \le a < p < n$.

Since Ram's paper is somewhat inaccessible, I will prove a slight sharpening of it, using an accessible result. N. J. Fine ("Binomial Coefficients modulo a Prime," Amer. Math. Monthly, 54 (1947), 589-592, Theorem 4) has shown that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ if and only if $n = ap^s - 1$ with $1 \le a < p$ and $s \ge 0$. Now suppose we have two primes p_1 and p_2 with $p_1 < p_2 \le n + 1$ and

$$p_i \nmid \binom{n}{k}$$

for $0 \le k \le n$. By Fine's result, we have

$$n + 1 = a_1 p_1^{s_1} = a_2 p_2^{s_2}$$

with $1 \le a_1 < p_1$ and $1 \le a_2 < p_2$. But $a_1 < p_1 < p_2$ implies that $p_2 \nmid a_1 p_1^{s_1}$, so $s_2 = 0$ and $n + 1 = a_2 < p_2$, contrary to $p_2 \le n + 1$. Hence there is at most one prime $p \le n + 1$ such that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ and such a p exists if and only if $n + 1 = ap^s$ with $1 \le a < p$. (More discussion related to this may be found in my survey paper: "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers," (to appear).)

By carefully examining the role of $n + 1$, we can deduce the following formulas for $M(n)$.

$$M(n) = \begin{cases} \pi(n+1) & \text{if } n+1 \neq ap^s \text{ with } 1 \le a < p \\ \pi(n+1) - 1 & \text{otherwise}. \end{cases}$$

$$M(n) = \begin{cases} \pi(n) & \text{if } n+1 \neq ap^s \text{ with } 1 \le a < p \le n \\ \pi(n) - 1 & \text{otherwise}. \end{cases}$$

Hence $M(n) \sim \pi(n) \sim [(\log n)/n]^{-1}$.

Incidentally, $M(13) = M(15) = 5$, contrary to what was asserted in the statement of the problem. The first place where $M(n) > M(n+1)$ is $n = 83$, where $M(83) = 23$ and $M(84) = 22$. The next cases are $n = 89$ and $n = 104$. From the expression for $M(n)$, we have the following necessary conditions for such an n: $n+1$ must have three distinct prime factors and $n+2$ must not be prime.

Also solved by the Proposer.

DISPLAY CASE

H-201 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Copy 1, 1, 3, 8, $\cdots$, F_{2n-2} $(n \ge 1)$ down in staggered columns as in display C:

$$C \qquad \begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 3 & 1 & 1 & & \\ 8 & 3 & 1 & 1 & \\ 21 & 8 & 3 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdot \end{array} .$$

(i) Show that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$.

(ii) Show that, if the columns are multiplied by 1, 2, 3, $\cdots$, sequentially to the right, then the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

(iii) Show that the rising diagonal sums ($\nearrow$) are F^2_{n+1} $(n = 0, 1, 2, \cdots)$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

(i) Let R_n denote the row sum of the $(n+1)^{\text{th}}$ row, $(n = 0, 1, 2, \cdots)$, with $R_0 = 1$.

$$R_n = 1 + \sum_{k=0}^{n-1} F_{2n-2k} = 1 + \sum_{k=1}^{n} F_{2k} = 1 + \sum_{k=1}^{n} (F_{2k+1} - F_{2k-1}) = 1 + F_{2n+1} - 1 = F_{2n+1},$$

as asserted.

(ii) Let S_n denote the sum as defined in the problem, for the $(n+1)^{\text{th}}$ row, $(n = 0, 1, 2, \cdots)$, with $S_0 = 1$. Then, if $n \geq 1$,

$$S_n = \sum_{k=1}^{n} k F_{2n+2-2k} + (n+1) = \sum_{k=0}^{n-1} (n-k) F_{2k+2} + (n+1) = (n+1) + \sum_{k=0}^{n-1} F_{2k+2} \sum_{i=0}^{n-k-1} 1$$

$$= (n+1) + \sum_{i=0}^{n-1} 1 \sum_{k=0}^{n-i-1} F_{2k+2} = (n+1) + \sum_{i=0}^{n-1} (F_{2n-2i+1} - 1) = (n+1) + \sum_{i=1}^{n} F_{2i+1} - n$$

$$= 1 + \sum_{i=1}^{n} (F_{2i+2} - F_{2i}) = 1 + F_{2n+2} - F_2 = F_{2n+2},$$

as asserted. This is also true for $n = 0$.

(iii) Let T_n denote the rising diagonal sums. Then, if $n \geq 2$,

$$T_n = \sum_{k=1}^{\frac{1}{2}n} F_{4k} + 1, \text{ if } n \text{ is even}; \quad T_n = \sum_{k=1}^{\frac{1}{2}(n+1)} F_{4k-2}, \text{ if } n \text{ is odd}; \quad T_0 = T_1 = 1.$$

$$T_{2m} = \sum_{k=1}^{m} F_{4k} + 1 = F_1 + \sum_{k=1}^{m} (F_{4k+1} - F_{4k-1}) = \sum_{i=0}^{2m} (-1)^i F_{2i+1};$$

also,

$$T_{2m+1} = \sum_{k=1}^{m+1} F_{4k-2} = \sum_{k=1}^{m+1} (F_{4k-1} - F_{4k-3}) = \sum_{i=0}^{2m+1} (-1)^{i+1} F_{2i+1}.$$

Combining these results, we have

$$T_n = \sum_{i=0}^{n} (-1)^{n-i} F_{2i+1} = \sum_{i=0}^{n} (-1)^{n-i} (F_{i+1}^2 + F_i^2)$$

$$= \sum_{i=0}^{n} (-1)^{n-i} F_{i+1}^2 - (-1)^{n-i+1} F_i^2 = (-1)^{n-n} F_{n+1}^2 - (-1)^{n+1} \cdot 0 = F_{n+1}^2.$$

This last result is also true for $n = 0$ and $n = 1$.

Also solved by the Proposer and one unsigned solver.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS PROPOSED IN THIS ISSUE

B-280 Proposed by Maxey Brooke, Sweeney, Texas.

Identify A, E, G, H, J, N, O, R, T, V as the ten distinct digits such that the following holds with the dots denoting some seven-digit number and ϕ representing zero:

$$\begin{array}{rl} & \mathrm{V\,E\,R\,N\,E\,R} \\ \times & \underline{\qquad\qquad\mathrm{E}} \\ & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ - & \underline{\mathrm{R}\,\phi\,\phi\,\phi\,\phi\,\mathrm{J}\,\mathrm{R}} \\ & \mathrm{H\,O\,G\,G\,A\,T\,T} \end{array}$$

B-281 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $T_n = n(n + 1)/2$. Find a positive integer b such that for all positive integers m, $T_{11\cdots1} = 11\cdots1$, where the subscript on the left side has m 1's as the digits in base b and the right side has m 1's as the digits in base b^2.

B-282 Proposed by Herta T. Freitag, Roanoke, Virginia.

Characterize geometrically the triangles that have

$$L_{n+2}L_{n-1}, \qquad 2L_{n+1}L_n, \qquad \text{and} \qquad 2L_{2n} + L_{2n+1}$$

as the lengths of the three sides.

B-283 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the ordered triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$, a odd, $c < 1000$, and c/a as close to 2 as possible. [This approximates the sides of a 30°, 60°, 90° triangle with a Pythagorean triple.]

B-284 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $z^2 - xy - y = 0$ and let k, m, and n be nonnegative integers. Prove that:

(a) $z^n = p_n(x,y)z + q_n(x,y)$, where p_n and q_n are polynomials in x and y with integer coefficients and p_n has degree $n - 1$ in x for $n > 0$;

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0 .$$

B-285 Proposed by Barry Wolk, University of Manitoba, Winnipeg, Manitoba, Canada.

Show that

$$F_{k(n+1)}/F_k = \sum_{r=0}^{[n/2]} (-1)^{r(k-1)} \binom{n-r}{r} L_k^{n-2r} .$$

SOLUTIONS

A LUCAS PRODUCT

B-256 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2n} - 3(-1)^n$ is the product of two Lucas numbers.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$\begin{aligned} L_{n-1}L_{n+1} &= (\alpha^{n+1} + \beta^{n+1})(\alpha^{n-1} + \beta^{n-1}) \\ &= L_{2n} + (\alpha^2 + \beta^2)(-1)^{n-1} \\ &= L_{2n} - 3(-1)^n . \end{aligned}$$

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, Tim Carroll, Juliana D. Chan, Warren Cheves, Ralph Garfield, John E. Homer, Graham Lord, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, William E. Thomas, Jr., David Zeitlin, and the Proposer.

A FIBONACCI PRODUCT

B-257 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $[L_{2n}+3(-1)^n]/5$ is the product of two Fibonacci numbers.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

$$\begin{aligned} F_{n-1}\cdot F_{n+1} &= (\alpha^{n-1} - \beta^{n-1})(\alpha^{n+1} - \beta^{n+1})/5 \\ &= (\alpha^{2n} + \beta^{2n} - (\alpha\beta)^{n-1}(\alpha^2 + \beta^2))/5 \\ &= (L_{2n} + 3(-1)^n)/5 \quad . \end{aligned}$$

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, Tim Carroll, Juliana D. Chan, Warren Cheves, Ralph Garfield, John E. Homer, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, William E. Thomas, Jr., Gregory Wulczyn, David Zeitlin, and the Proposer.

GOLDEN RATIO FORMULA

B-258 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Let $[x]$ denote the greatest integer in x, $a = (1 + \sqrt{5})/2$, and $e_n = (1 + (-1)^n)/2$. Prove that for all positive integers m and n

(a) $$nF_{n+1} = [naF_n] + e_n$$

(b) $$nF_{m+n} = F_m([naF_n] + e_n) + nF_{m-1}F_n \ .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Since $aF_n = F_{n+1} - b^n$, where $b = (1 - \sqrt{5})/2$, to prove (a) it suffices to show $|nb^n| < 1$. But

$$1\cdot(\sqrt{5} - 1)/2 < .65 < 1$$

and

$$2\cdot(\sqrt{5} - 1)^2/4 < 2\cdot(.65)^2 < 1 \ .$$

The latter inequality verifies the case $n = 2$ of the induction hypothesis: if $n \geq 2$ then $n|b^n| < 1$. Then

$$(n + 1)|b^{n+1}| < (n + 1)(.65)^{n+1} < (n + 1)(.65)/n < 1 \ ,$$

for $n \geq 2$, which completes the induction and the proof of (a).

Equality (b) comes from substituting (a) in the known identity:

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n \ .$$

Also solved by C. B. A. Peck and the Proposer.

A. P. OF BINOMIAL COEFFICIENTS

B-259 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Characterize the infinite sequence of ordered pairs of integers (m, r) with $4 \le 2r \le m$, for which the three binomial coefficients

$$\binom{m-2}{r-2}, \quad \binom{m-2}{r-1}, \quad \binom{m-2}{r}$$

are in arithmetic progression.

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

Equivalently, find all solutions of:

$$\binom{m-2}{r} + \binom{m-2}{r-2} = 2\binom{m-2}{r-1}.$$

A simple computation yields $m = (m - 2r)^2$, whence $m = n^2$ and $r = (m - \sqrt{m})/2$; $2r$ is strictly less than m. The required sequence is thus

$$\{(n^2, \ (n^2 - n)/2)\}_{n>2} \ .$$

Also solved by Wray G. Brady, Paul S. Bruckman, Tim Carroll, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

SUMS OF DIVISORS

B-260 Proposed by John L. Hunsucker and Jack Nebb, University of Georgia, Athens, Georgia.

Let $\sigma(n)$ denote the sum of the positive integral divisors of n. Show that

$$\sigma(mn) > \sigma(m) + \sigma(n)$$

for all integers $m > 1$ and $n > 1$.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

We may write

$$m = \prod_{k=1}^{r} p_k^{e_k}, \qquad n = \prod_{k=1}^{r} p_k^{f_k}, \qquad mn = \prod_{k=1}^{r} p_k^{e_k+f_k},$$

where the p_k are distinct primes and the e_k and f_k are nonnegative integers. Since

$$\sigma(m) = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{e_k}),$$

one has

$$\sigma(m)/m = \prod_{k=1}^{r} \sum_{j=0}^{e_k} p^{-j} .$$

Then it follows that $\sigma(mn)/mn > \sigma(m)/m$ and $\sigma(mn)/mn > \sigma(n)/n$. We may add these inequalities and multiply by mn, which yields:

$$2\sigma(mn) > n\sigma(m) + m\sigma(n) \geq 2\sigma(m) + 2\sigma(n)$$

and the desired result follows.

Also solved by Wray G. Brady, Tim Carroll, Graham Lord, C. B. A. Peck, Philip Tracy, and the Proposer.

CYCLIC GROUP MODULO D

B-261 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let d be a positive integer and let S be the set of all non-negative integers n such that $2^n - 1$ is an integral multiple of d. Show that either $S = \{0\}$ or the integers in S form an infinite arithmetic progression.

Solution by Tim Carroll, Western Michigan University, Kalamazoo, Michigan.

$0 \in S$ since $d \mid (2^0 - 1)$. Let n be the least positive integer in S when $S \neq \{0\}$. For any positive integer k,

$$2^{kn} - 1 = (2^n - 1)(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 2^n + 1) .$$

Since d divides $2^n - 1$, d divides $2^{kn} - 1$ for all positive k. Therefore $kn \in S$ for all positive integers k. We now show there are no other integers in S. Suppose $m \in S$ and $m = qn + r$, $0 < r < n$.

$$\begin{aligned} 2^m - 1 &= 2^{qn}2^r - 1 \\ &= 2^{qn}2^r - 2^{qn} + 2^{qn} - 1 \\ &= 2^{qn}(2^r - 1) + (2^{qn} - 1) . \end{aligned}$$

Since $q_n \in S$, $m \in S$, and d does not divide 2^{qn}, d divides $2^r - 1$. But this is impossible by our choice of n. Therefore, $S = \{0\}$ or $S = \{0, n, 2n, 3n, \cdots\}$.

Also solved by Wray G. Brady, Paul S. Bruckman, Warren Cheves, Herta. T. Freitag, Graham Lord, Richard W. Sielaff, Paul Smith, David Zeitlin, and the Proposer.