TRIANGULAR NUMBERS

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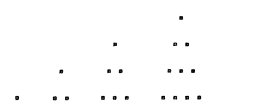
1. INTRODUCTION

To Fibonacci is attributed the arithmetic triangle of odd numbers, in which the nth row has n entries, the center element is n^2 for even n, and the row sum is n^3. (See Stanley Bezuszka [11].)

FIBONACCI'S TRIANGLE	SUMS
1	$1 = 1^3$
3 5	$8 = 2^3$
7 9 11	$27 = 3^3$
13 15 17 19	$64 = 4^3$
21 23 25 27 29	$125 = 5^3$
. . . . .	. . . . .

We wish to derive some results here concerning the triangular numbers $1, 3, 6, 10, 15, \cdots, T_n, \cdots, \cdots$. If one observes how they are defined geometrically,



one easily sees that

(1.1) $$T_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

and

(1.2) $$T_{n+1} = T_n + (n+1) .$$

By noticing that two adjacent arrays form a square, such as

$$3+6 = 9$$

we are led to

(1.3) $$n^2 = T_n + T_{n-1} ,$$

which can be verified using (1.1). This also provides an identity for triangular numbers in terms of subscripts which are also triangular numbers,

(1.4) $$T_n^2 = T_{T_n} + T_{T_n - 1} .$$

Since every odd number is the difference of two consecutive squares, it is informative to rewrite Fibonacci's triangle of odd numbers:

FIBONACCI'S TRIANGLE				SUMS
	(1^2-0^2)			$T_1^2-T_0^2$
(2^2-1^2)	(3^2-2^2)			$T_2^2-T_1^2$
(4^2-3^2)	(5^2-4^2)	(6^2-5^2)		$T_3^2-T_2^2$
(7^2-6^2)	(8^2-7^2)	(9^2-8^2)	(10^2-9^2)	$T_4^2-T_3^2$
. . . . .	. . . . .	. . . . .	. . . . .	. . . . .

Upon comparing with the first array, it would appear that the difference of the squares of two consecutive triangular numbers is a perfect cube. From (1.2),

$$T_{n+1}^2 = (T_n + n + 1)^2 = T_n^2 + 2(n+1)T_n + (n+1)^2$$

But, from (1.1), $T_n = n(n+1)/2$, so that

$$\begin{aligned} T_{n+1}^2 - T_n^2 &= 2(n+1)[n(n+1)/2] + (n+1)^2 \\ &= n(n+1)^2 + (n+1)^2 = (n+1)^3 \quad . \end{aligned}$$

Thus, we do indeed have

$$T_{n+1}^2 - T_n^2 = (n+1)^3, \tag{1.5}$$

which also follows by simple algebra directly from (1.1).

Further,

$$\begin{aligned} T_n^2 &= (T_n^2 - T_{n-1}^2) + (T_{n-1}^2 - T_{n-2}^2) + \cdots + (T_2^2 - T_1^2) + (T_1^2 - T_0^2) \\ &= n^3 + (n-1)^3 + \cdots + 2^3 + 1^3 \quad , \end{aligned}$$

or, again returning to (1.1),

$$T_n^2 = (1+2+3+\cdots+n)^2 = \sum_{k=1}^{n} k^3 \quad . \tag{1.6}$$

For a wholly geometric discussion, see Martin Gardner [10].

Suppose that we now make a triangle of consecutive whole numbers.

WHOLE NUMBER TRIANGLE					SUMS
		0			0
	1	2			3
3	4	5			12
6	7	8	9		30
10	11	12	13	14	60
. . . . .					. . .

If we observe carefully, the row sum of the nth row is nT_{n+1}, or $(n+2)T_n$, which we can easily derive by studying the form of each row of the triangle. Notice that the triangular numbers appear sequentially along the left edge. The nth row, then, has elements

$$T_n \quad T_n+1 \quad T_n+2 \quad T_n+3 \quad \cdots \quad T_n+n$$

so that its sum is

$$(n+1)T_n + (1+2+3+\cdots+n) = (n+1)T_n + T_n = (n+2)T_n \ .$$

Also, the nth row can be written as

$$T_n \quad T_{n+1}-n \quad \cdots \quad T_{n+1}-3 \quad T_{n+1}-2 \quad T_{n+1}-1$$

with row sum

$$T_n + nT_{n+1} - (1+2+3+\cdots+n) = T_n + nT_{n+1} - T_n = nT_{n+1} .$$

Then,

(1.7) $$nT_{n+1} = (n+2)T_n ,$$

which also follows from (1.1), since

$$nT_{n+1} = \frac{n(n+1)(n+2)}{2} = (n+2)T_n .$$

The row sums are also three times the binomial coefficients $1, 4, 10, 20, \cdots$, the entries in the third column of Pascal's left-justified triangle, since

$$nT_{n+1} = \frac{n(n+1)(n+2)}{2} = 3 \cdot \left[\frac{n(n+1)(n+2)}{3 \cdot 2 \cdot 1}\right] = 3 \cdot \binom{n+2}{3} .$$

The numbers $1, 4, 10, 20, \cdots$, are the triangular pyramidal numbers, the three-dimensional analog of the triangular numbers. Of course, the triangular numbers themselves are the binomial coefficients appearing in the second column of Pascal's triangle, so that, by mathematical induction or by applying known properties of binomial coefficients, we can sum the triangular numbers:

(1.8) $$T_n = \binom{n+1}{2} ; \qquad \sum_{k=0}^{n} T_k = \binom{n+2}{3} .$$

Finally, by summing over n rows of the whole number triangle and observing that the number on the right of the nth row is $T_{n+1} - 1$,

(1.9) $$\sum_{j=1}^{n} jT_{j+1} = T_{T_{n+1}-1} ,$$

since, by (1.1), summing all elements of the triangle through the nth row gives

$$0+1+2+3+\cdots+(T_{n+1}-1) = T_{T_{n+1}-1} .$$

Let us start again with

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 7 & & 8 & & 9 & & 10 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

This time we observe the triangular numbers are along the right edge. Each row sum, using our earlier process, is

$$nT_n - T_{n-1} = (n-1)T_{n-1} + n^2 = (n+1)T_n - n .$$

Clearly, the sum over n rows gives us

(1.10) $$T_{T_n} = T_{T_{n-1}} + T_n$$

or, referring again to the row sum of $(n-1)T_{n-1} + n^2$ and to Equation (1.3),

$$T_{T_n} = \sum_{j=1}^{n} [(j-1)T_{j-1} + j^2] = \sum_{j=1}^{n} [(j-1)T_{j-1} + T_j + T_{j-1}]$$

$$= \sum_{j=1}^{n-1} jT_j + \sum_{j=1}^{n} T_j + \sum_{j=1}^{n-1} T_j = \sum_{j=1}^{n-1} (j+2)T_j + T_n .$$

Therefore, from (1.10),

(1.11) $$T_{T_{n-1}} = \sum_{j=1}^{n-1} (j+2)T_j .$$

It is also easy to establish that

(1.12) $$T_{2n} = 3T_n + T_{n-1},$$

and

(1.13) $$T_{2n} - 2T_n = n^2,$$

(1.14) $$T_{2n-1} - 2T_{n-1} = n^2.$$

2. GENERATING FUNCTIONS

Consider the array A

$$\begin{array}{lllll} 1 & & & & \\ 2 & 3 & & & \\ 4 & 5 & 6 & & \\ 7 & 8 & 9 & 10 & \\ 11 & 12 & 13 & 14 & 15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We desire to find the generating functions for the columns. The first column entries are clearly one more than the triangular numbers T_n, $(n = 0, 1, 2, \cdots)$. Thus, since the generating function for triangular numbers (as well as for the other columns of Pascal's triangle) is known,

$$G_0(x) = \sum_{n=0}^{\infty} (T_n + 1)x^n = \frac{x}{(1-x)^3} + \frac{1}{1-x} = \frac{1-x+x^2}{(1-x)^3}.$$

We shall see that generally the column generators are

(2.1) $$G_k(x) = \frac{T_{k+1} - (k+1)^2 x + (T_k + 1)x^2}{(1-x)^3} = \frac{T_{k+1} - (T_{k+1} + T_k)x + (T_k + 1)x^2}{(1-x)^3}$$

PROOF: Clearly, $G_0(x)$ is given by the formula above when $k = 0$. Assume that

$$G_k(x) = \frac{T_{k+1} - (k+1)^2 x + (T_k + 1)x^2}{(1-x)^3}.$$

Then, since each column is formed from the preceding by subtracting the first entry T_{k+1}, and adding one, the $(k+1)^{st}$ column generator is

$$\begin{aligned} G_{k+1}(x) &= \left(\frac{T_{k+1} - (k+1)^2 x + (T_k + 1)x^2}{(1-x)^3} - T_{k+1} \right) \Big/ x + \frac{1}{1-x} \\ &= \frac{T_{k+1} - (k+1)^2 x + (T_k + 1)x^2 - (1 - 3x + 3x^2 - x^3)T_{k+1}}{x(1-x)^3} + \frac{1}{1-x} \\ &= \frac{(3T_{k+1} - (k+1)^2) + (T_k + 1 - 3T_{k+1})x + T_{k+1}x^2 + (1 - 2x + x^2)}{(1-x)^3} \end{aligned}$$

Now, from $(k+1)^2 = T_k + T_{k+1}$ and $T_k = T_{k-1} + k$, this becomes

$$\begin{aligned} G_{k+1}(x) &= [3T_{k+1} + 1 - (T_k + T_{k+1}) + (T_k - 1 - 3T_{k+1})x + (T_{k+1} + 1)x^2]/(1-x)^3 \\ &= [(2T_{k+1} - T_k + 1) - (3T_{k+1} + 1 - T_k)x + (T_{k+1} + 1)x^2]/(1-x)^3 \\ &= \frac{(T_{k+2}) - (T_{k+2} + T_{k+1})x + (T_{k+1} + 1)x^2}{(1-x)^3} = \frac{T_{k+2} - (k+2)^2 x + (T_{k+1} + 1)x^2}{(1-x)^3}. \end{aligned}$$

This may now be exploited as any triangular array.

We now proceed to another array B (Fibonacci's triangle).

$$\begin{array}{lllll} 1 & & & & \\ 3 & 5 & & & \\ 7 & 9 & 11 & & \\ 13 & 15 & 17 & 19 & \\ 21 & 23 & 25 & 27 & 29 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We can tackle this immediately since we have already found the generators for array A, because each entry in array B is twice the corresponding entry in array A, less one. Thus the column generators are

$$G_k^*(x) = \frac{2[T_{k+1} - (k+1)^2 x + (T_k + 1)x^2]}{(1-x)^3} - \frac{1 - 2x + x^2}{(1-x)^3} \tag{2.2}$$
$$= \frac{(2T_{k+1} - 1) - 2[(k+1)^2 - 1]x + (2T_k + 1)x^2}{(1-x)^3} .$$

Now since the row sums of Fibonacci's triangle are the cubes of successive integers, we can find a generating function for the cubes.

$$\sum_{k=0}^{\infty} x^k G_k^*(x) = \left(2 \sum_{k=0}^{\infty} T_{k+1} x^k - \sum_{k=0}^{\infty} x^k - 2x \sum_{k=0}^{\infty} (k+1)^2 x^k \right.$$
$$\left. + 2x \sum_{k=0}^{\infty} x^k + 2x^2 \sum_{k=0}^{\infty} T_k x^k + x^2 \sum_{k=0}^{\infty} x^k \right) \Big/ (1-x)^3 .$$

But

$$\sum_{k=0}^{\infty} T_{k+1} x^k = \frac{1}{(1-x)^3} \quad \text{and} \quad \sum_{k=0}^{\infty} T_k x^k = \frac{x}{(1-x)^3} \tag{2.3}$$

$$\sum_{k=0}^{\infty} (k+1)^2 x^k = \frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty} (T_{k+1} + T_k) x^k \tag{2.4}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} . \tag{2.5}$$

Thus, applying (2.3), (2.4), and (2.5),

$$\sum_{k=0}^{\infty} x^k G_k^*(x) = \frac{2 - (1-x)^2 - 2x(1+x) + 2x(1-x)^2 + 2x^3 + x^2(1-x)^2}{(1-x)^3(1-x)^3} \tag{2.6}$$

$$= \frac{(1+4x+x^2)(1-x)^2}{(1-x)^6} = \frac{1+4x+x^2}{(1-x)^4} = \sum_{k=0}^{\infty} (k+1)^3 x^k .$$

Further extensions of arrays A and B will be found in a thesis by Robert Anaya [1].

Equation (2.6) also says that, for any three consecutive members of the third column of Pascal's triangle, the sum of the first and third, and four times the second, is a cube, or

$$\binom{n}{3} + 4\binom{n-1}{3} + \binom{n-2}{3} = n^3 .$$

Observe that

$$\binom{n}{2} + \binom{n-1}{2} = n^2 \quad \text{and} \quad \binom{n}{1} = n.$$

We can find

$$1\binom{n}{4} + 11\binom{n-1}{4} + 11\binom{n-2}{4} + 1\binom{n-3}{4} = n^4$$

by solving for the coefficients in the beginning values, using column 4 $(1, 5, 15, 35, \cdots)$, in the order given:

$$1 \cdot x_1 = 1^4$$
$$5 \cdot x_1 + 1 \cdot x_2 = 2^4$$
$$15 \cdot x_1 + 5 \cdot x_2 + 1 \cdot x_3 = 3^4$$
$$35 \cdot x_1 + 15 \cdot x_2 + 5 \cdot x_3 + 1 \cdot x_4 = 4^4$$

In the same manner,

$$\binom{n}{5} + 26\binom{n-1}{5} + 66\binom{n-2}{5} + 26\binom{n-3}{5} + \binom{n-4}{5} = n^5 .$$

Applying this method to the kth column, we obtain

$$n^k = \sum_{i=1}^{k} \left[\sum_{j=0}^{i} (i-j)^k (-1)^j \binom{k+1}{k+1-j} \right] \binom{n+1-i}{k} . \tag{2.7}$$

Returning to generating functions, (2.3) is a generating function for the triangular numbers. The triangular numbers generalize to the polygonal numbers $P(n,k)$,

$$P(n,k) = [k(n-1) - 2(n-2)]n/2 , \tag{2.8}$$

the nth polygonal number of k sides. Note that $P(n,3) = T_n$, the nth triangular number, and $P(n,4) = n^2$, the nth square number. A generating function for $P(n,k)$ is

$$\frac{1+(k-3)x}{(1-x)^3} = \sum_{n=0}^{\infty} P(n,k)x^n . \tag{2.9}$$

The sums of the corresponding polygonal numbers are the pyramidal numbers [9] which are generated by

$$\frac{1+(k-3)x}{(1-x)^4} = \sum_{n=0}^{\infty} P^*(n,k)x^n , \tag{2.10}$$

where $P^*(n,k)$ is the nth pyramidal number of order k. Notice that $k=3$ gives the generating function for the triangular numbers and for the triangular pyramidal numbers, which are the sums of the triangular numbers.

3. SOME MORE ARITHMETIC PROGRESSIONS

It is well known that the kth column sequence of Pascal's left-adjusted triangle is an arithmetic progression of order k with common difference of 1. In this section, we discuss subsequences of these whose subscripts are triangular numbers. To properly set the stage, we need first to discuss polynomials whose coefficients are the Eulerian numbers. (See Riordan [2].)

Let

$$\frac{A_k(x)}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} n^k x^n . \tag{3.1}$$

Differentiate and multiply by x, to obtain

$$\frac{x(1-x)A_k'(x) + x(k+1)A_k(x)}{(1-x)^{k+2}} = \sum_{n=0}^{\infty} n^{k+1} x^n .$$

But, by definition,

$$\frac{A_{k+1}(x)}{(1-x)^{k+2}} = \sum_{n=0}^{\infty} n^{k+1} x^n$$

so that

$$A_{k+1}(x) = x(1-x)A_k'(x) + x(k+1)A_k(x) . \tag{3.2}$$

Since, from Section 2,

$$\sum_{n=0}^{\infty} n^1 x^n = \frac{x}{(1-x)^2}, \qquad A_1(x) = x$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \qquad A_2(x) = x + x^2$$

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{x+4x^2+x^3}{(1-x)^4}, \qquad A_3(x) = x + 4x^2 + x^3 .$$

From the recurrence it is easy to see that by a simple inductive argument,

$$A_k(1) = k! .$$

Also, we can easily write $A_4(x) = x^4 + 11x^3 + 11x^2 + x$, which allows us to demonstrate Eq. (1.6) in a second way. Thus, using $T_n = n(n+1)/2$, and the generating functions just listed,

$$\sum_{n=0}^{\infty} T_n^2 x^n = \sum_{n=0}^{\infty} \frac{(n^4 + 2n^3 + n^2)}{4} x^n$$

$$= \frac{1}{4} \cdot \left[\frac{x^4 + 11x^3 + 11x^2 + x}{(1-x)^5} + \frac{2(1-x)(x^3 + 4x^2 + x)}{(1-x)^5} + \frac{(1-x)^2(x^2+x)}{(1-x)^5} \right] = \frac{x^3 + 4x^2 + x}{(1-x)^5}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} k^3 x^n$$

so that

$$T_n^2 = (1 + 2 + 3 + \cdots + n)^2 = \sum_{k=0}^{n} k^3 \quad .$$

Now we can write

$$A_k(x) = \sum_{n=1}^{k} \left[\sum_{j=0}^{n} (n-j)^k (-1)^j \binom{k+1}{k+1-j} \right] x^n \quad , \tag{3.3}$$

from (2.4) by applying the generating function to Pascal's triangle. Notice that $A_1(x)$, $A_2(x)$, $A_3(x)$, and $A_4(x)$ all have the form given in (3.3).

Next, from a thesis by Judy Kramer [3], we have the following theorem.

Theorem 57. If generating function

$$A(x) = \frac{N(x)}{(1-x)^{r+1}} \quad ,$$

where $N(x)$ is a polynomial of maximum degree r, then $A(x)$ generates an arithmetic progression of order r, and the constant of the progression is $N(1)$.

We desire now to look at

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(n+k)(n+k-1)\cdots(n+1)}{k!} x^n = \sum_{n=0}^{\infty} Q(n,k) x^n .$$

Now consider

$$G(x) = \sum_{n=0}^{\infty} Q(T_n, k) x^n \quad ,$$

where T_n is the nth triangular number. Clearly, this is a polynomial in n of degree $2k$. Let us assume it is expanded

$$Q(T_n, k) = \sum_{j=0}^{2k} b_j n^j \qquad \text{and} \qquad \frac{A_j(x)}{(1-x)^{j+1}} = \sum_{n=0}^{\infty} n^j x^n$$

so that

$$G(x) = \sum_{j=0}^{2k} \frac{b_j A_j(x)}{(1-x)^{j+1}} = \frac{N_k(x)}{(1-x)^{2k+1}} \quad .$$

All of the $A_j(x)$ are multiplied by powers of $(1-x)$ in $N_k(x)$ except $A_{2k}(x)$; thus,

$$N_k(1) = A_{2k}(1) = (2k)!/2^k k! \quad ,$$

which is, of course, an integer. Thus $Q(T_n, k)$ is an arithmetic progression of order $2k$ and common difference $d = (2k)!/2^k k!$. The general result is that, for

$$G^*(x) = \sum_{n=0}^{\infty} Q\left(Q(n,m), k\right) x^n$$

$Q(Q(n,m),k)$ is an arithmetic progression of order mk and common difference $d = (mk)!/m^k k!$ which thus must be an integer.

4. PALINDROMIC TRIANGULAR NUMBERS

There are 27 triangular numbers T_n, $n < 151340$, which are palindromes in base 10, as given by Trigg [8]. However, borrowing from Leonard [4] and Merrill [5], every number in array C is a triangular number:

(C)
1
11
111
1111
11111
. . . . .

Clearly, base 10 is ruled out, but array C indeed provides triangular numbers in base 9. Below we discuss some interesting consequences including a proof.

Let $T_{U_n} = (11111 \cdots 1)_9 = C_n$ (n one's) so that

$$C_n = 9^n + 9^{n-1} + 9^{n-2} + \cdots + 9 + 1 = (9^{n+1} - 1)/(9 - 1) .$$

Now

$$T_{U_n} = \frac{U_n(U_n + 1)}{2} ,$$

where U_n, written in base 3 notation, has n one's,

$$U_n = (1111 \cdots 1)_3 = (3^{n+1} - 1)/(3 - 1).$$

Then

$$T_{U_n} = \frac{1}{2} \cdot \left(\frac{3^{n+1} - 1}{3 - 1}\right)\left(\frac{3^{n+1} - 1}{3 - 1} + 1\right) = \frac{(3^{n+1} - 1)(3^{n+1} + 1)}{8} = \frac{9^{n+1} - 1}{9 - 1} = C_n .$$

Also, it is simple to show that if T_n is any triangular number, then so is

$$9T_n + 1 = T_{3n+1} \tag{4.1}$$

since

$$9T_n + 1 = \frac{9n(n+1)}{2} + 1 = \frac{9n^2 + 9n + 2}{2} = \frac{(3n+1)(3n+2)}{2} = T_{3n+1} .$$

This means that, if T_n is any triangular number written in base 9 notation, annexing any number of 1's on the right provides another triangular number, and the new subscript can be found by annexing the same number of 1's to the subscript of T_n, where n is written in base 3 notation. The numbers in array C, then, are a special case of Eq. (4.1).

Three other interesting sets of palindromic triangular numbers occur in bases 3, 5, and 7. In each case below, the triangular number as well as its subscript are expressed in the base given.

Base 3	Base 5	Base 7
$T_1 = 1$	$T_2 = 3$	$T_3 = 6$
$T_{11} = 101$	$T_{22} = 303$	$T_{33} = 606$
$T_{111} = 10101$	$T_{222} = 30303$	$T_{333} = 60606$
$T_{1111} = 1010101$	$T_{2222} = 3030303$	$T_{3333} = 6060606$
. . . . .	. . . . .	. . . . .

Now, base 3 uses only even powers of 3, so the base 9 proof applies. For base 5, if T_n is any triangular number, then

$$25T_n + 3 = T_{5n+2} \tag{4.2}$$

since

$$25T_n + 3 = \frac{25n(n+1)}{2} + 3 = \frac{25n^2 + 25n + 6}{2} = \frac{(5n+2)(5n+3)}{2} = T_{5n+2}$$

so that annexing 03 to any triangular number written in base 5 notation provides another triangular number whose subscript can be found by annexing 2 to the right of the original subscript in base 5 notation. Base 7 is demonstrated similarly from the identity

$$49T_n + 6 = T_{7n+3} \ . \tag{4.3}$$

Using similar reasoning, if any triangular number is written in base 8, annexing 1 to the right will provide a square number, since

$$8T_n + 1 = (2n+1)^2 \ . \tag{4.4}$$

For example, $T_6 = (25)_8$ and $(251)_8 = 169 = 13^2$.

Any odd base $(2k+1)$ has an "annexing property" for triangular numbers, for (4.3) generalizes to

$$T_{(2k+1)n+k} = (2k+1)^2 T_n + T_k \ , \tag{4.5}$$

but other identities of the pleasing form given may require special digit symbols, and T_k must be expressed in base $(2k+1)$. Some examples follow, where both numbers and subscripts are expressed in the base given.

Base 9	Base 17	Base 25 $(t)_{25} = (12)_{10}$
$T_4 = 11$	$T_8 = 22$	$T_t = 33$
$T_{44} = 1111$	$T_{88} = 2222$	$T_{tt} = 3333$
$T_{444} = 111111$	$T_{888} = 222222$	$T_{ttt} = 333333$
... ...	... ...	... ...

Base 33 $(s)_{33} = (16)_{10}$	Base 41 $(q)_{41} = (20)_{10}$	Base 49 $(r)_{49} = (24)_{10}$
$T_s = 44$	$T_q = 55$	$T_r = 66$
$T_{ss} = 4444$	$T_{qq} = 5555$	$T_{rr} = 6666$
$T_{sss} = 444444$	$T_{qqq} = 555555$	$T_{rrr} = 666666$
... ...	... ...	... ...

Base 57 $(m)_{57} = (28)_{10}$	Base 65 $(n)_{65} = (32)_{10}$	Base 73 $(p)_{73} = (36)_{10}$
$T_m = 77$	$T_n = 88$	$T_p = 99$
$T_{mm} = 7777$	$T_{nn} = 8888$	$T_{pp} = 9999$
$T_{mmm} = 777777$	$T_{nnn} = 888888$	$T_{ppp} = 999999$
... ...	... ...	... ...

Base 19 $(t)_{19} = (12)_{10}$

$T_9 = tt$
$T_{99} = tttt$
$T_{999} = tttttt$
... ...

5. GENERALIZED BINOMIAL COEFFICIENTS FOR TRIANGULAR NUMBERS

Walter Hansell [6] formed generalized binomial coefficients from the triangular numbers,

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{T_m T_{m-1} \cdots T_{m-n+1}}{T_n T_{n-1} \cdots T_1}, \qquad 0 < n \leqslant m \ .$$

That these are integers doesn't fall within the scope of Hoggatt [7]. However, it is not difficult to show. Since $T_m = m(m+1)/2$,

$$\left[{m \atop n}\right] = \binom{m}{n}\binom{m+1}{n+1}\frac{1}{m-n+1},$$

where $\binom{m}{n}$ are the ordinary binomial coefficients, so that $\left[{m \atop n}\right]$ are indeed integers if one defined

$$\left[{m \atop 0}\right] = \left[{m \atop m}\right] = 1,$$

as will be seen in the next paragraph or two.

The generalized binomial coefficients for the triangular numbers are

$$\begin{array}{llllll}
1 & & & & & \\
1 & 1 & & & & \\
1 & 3 & 1 & & & \\
1 & 6 & 6 & 1 & & \\
1 & 10 & 20 & 10 & 1 & \\
1 & 15 & 50 & 50 & 15 & 1 \\
1 & 21 & 105 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

If the Catalan numbers $C_n = 1, 1, 2, 5, 14, 42, 132, \cdots$, are given by

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n,$$

then we note that the row sums are the Catalan numbers, C_{n+1}.

We compare elements in corresponding positions in Pascal's triangle of ordinary binomial coefficients and in the triangular binomial coefficient array:

$$\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\qquad
\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 3 & 1 & & \\
1 & 6 & 6 & 1 & \\
1 & 10 & 20 & 10 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Let us examine

$$\left[{m \atop n}\right] = \begin{vmatrix} \binom{m}{n} & \binom{m}{n+1} \\ \binom{m+1}{n} & \binom{m+1}{n+1} \end{vmatrix} = \binom{m}{n}\binom{m+1}{n+1}\cdot\frac{1}{m-n+1}$$

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ON FERNS' THEOREM ON THE EXPANSION OF FIBONACCI AND LUCAS NUMBERS

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Let (F_n) be a Fibonacci-type integer sequence satisfying the recurrence relation $F_n = pF_{n-1} + qF_{n-2}$ $(n \geqslant 2)$ in which $p^2 + 4q \neq 0$, and let (L_n) be the corresponding Lucas-type sequence, as described in [2]. The object of this note is both to generalize Ferns' theorem [1] on the expansion of

$$F_{x_1+x_2+\cdots+x_n} \qquad \text{and} \qquad L_{x_1+x_2+\cdots+x_n}$$

and to simplify the proof. Ferns' theorem was proved for the case when (F_n) and (L_n) were the Fibonacci and Lucas sequences, respectively, so in the statement and proof of the theorem the reader may interpret (F_n) and (L_n) as the ordinary Fibonacci and Lucas sequences, if he so desires.

Let

$$S_k^n = \Sigma F_{x_{i_1}} F_{x_{i_2}} \cdots F_{x_{i_k}} L_{x_{j_1}} \cdots L_{x_{j_{n-k}}}$$

where the sum ranges over all permutations $(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ of $(1, \cdots, n)$ such that

$$1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n \qquad \text{and} \qquad 1 \leqslant j_1 < j_2 < \cdots < j_{n-k} \leqslant n,$$

for $0 \leqslant k \leqslant n$. Let α and β be the roots of $x^2 - px - q$ and let $A = F_1 - F_0\beta$, $B = F_1 - F_0\alpha$. Then $A \neq 0$ and $B \neq 0$ (see [2]) so that

$$\alpha = \left(\frac{L_1 + dF_1}{2A}\right), \qquad \beta = \left(\frac{L_1 - dF_1}{2B}\right),$$

where

$$d = \sqrt{p^2 + 4q}\ .$$

Then the generalized version of Ferns' theorem may be stated in the following way.

Theorem: If

$$\Sigma_e = S_0^n + d^2 S_2^n + d^4 S_4^n + \cdots \qquad \text{and} \qquad \Sigma_o = dS_1^n + d^3 S_3^n + d^5 S_5^n + \cdots$$

then

$$F_{x_1+x_2+\cdots+x_n} = \frac{1}{2^n d}\left\{\left(\frac{1}{A^{n-1}} - \frac{1}{B^{n-1}}\right)\Sigma_e + \left(\frac{1}{A^{n-1}} + \frac{1}{B^{n-1}}\right)\Sigma_o\right\}$$

and

$$L_{x_1+x_2+\cdots+x_n} = \frac{1}{2^n}\left\{\left(\frac{1}{A^{n-1}} + \frac{1}{B^{n-1}}\right)\Sigma_e + \left(\frac{1}{A^{n-1}} - \frac{1}{B^{n-1}}\right)\Sigma_o\right\}\ .$$

Proof: It is well known that if r is a positive integer

$$F_r = \frac{A\alpha^r - B\beta^r}{\alpha - \beta}, \qquad L_r = A\alpha^r + B\beta^r\ .$$

Therefore,

$$\alpha^r = \frac{L_r + dF_r}{2A}, \qquad \beta^r = \frac{L_r - dF_r}{2B}$$

Therefore

$$\begin{aligned}\frac{1}{2A}(L_{x_1+x_2+\cdots+x_n} + dF_{x_1+x_2+\cdots+x_n}) &= \alpha^{x_1+x_2+\cdots+x_n} \\ &= \frac{1}{2^nA^n}(L_{x_1}+dF_{x_1})(L_{x_2}+dF_{x_2})\cdots(L_{x_n}+dF_{x_n}) \\ &= \frac{1}{2^nA^n}(S_0^n + dS_1^n + d^2S_2^n + \cdots + d^nS_n^n).\end{aligned}$$

Similarly

$$\begin{aligned}\frac{1}{2B}(L_{x_1+x_2+\cdots+x_n} - dF_{x_1+x_2+\cdots+x_n}) \\ = \frac{1}{2^nB^n}(S_0^n - dS_1^n + d^2S_2^n - \cdots + (-1)^n d^n S_n^n).\end{aligned}$$

The theorem now follows by addition and subtraction.

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2. A.J.W. Hilton, "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Lucas Sequences," *The Fibonacci Quarterly,* to appear.

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THE FIBONACCI ASSOCIATION

RESEARCH CONFERENCE

PROGRAM OF SATURDAY, MAY 4, 1974

ST. MARY'S COLLEGE

9:00–9:30	PRELIMINARY GATHERING, coffee and rolls.
9:30–10:15	SEQUENCES GENERATED BY LEAST INTEGER FUNCTIONS Brother Alfred Brousseau, St. Mary's College
10:20–11:00	THE SEQUENCES 1, 5, 16, 45, 121, 320, ··· IN COMBINATORICS Ken Rebman, California State University, Hayward
11:05–11:45	REPRESENTATION OF INTEGERS USING FIBONACCI AND LUCAS SQUARES Hardy Reyerson, Masters Student, San Jose State University
12:00–1:30	LUNCH PERIOD
1:30–2:15	RECTANGULAR AND TRIANGULAR PARTITIONS Leonard Carlitz, Duke University
2:20–3:00	GREAT ADVENTURES WITH CATALAN AND LAGRANGE Verner E. Hoggatt, Jr., San Jose State University

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ARGAND DIAGRAMS OF EXTENDED FIBONACCI AND LUCAS NUMBERS

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Numerous extensions of the Fibonacci and Lucas Numbers have been reported in the literature [1–6]. In this paper we present a computer-generated plot of the complex representation of the Fibonacci and Lucas Numbers. The complex representation of the Fibonacci Numbers is given by [5,6].

$$F(x) = \frac{\phi^x - \phi^{-x}[\cos(x\pi) + i\sin(x\pi)]}{\sqrt{5}} ,$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad F(-x) = (-1)^{n+1}F(x),$$

$$Re[F(x)] = \frac{1}{\sqrt{5}}\left\{\phi^x - \phi^{-x}\cos(\pi x)\right\} ;$$

and

$$Im[F(x)] = \frac{1}{\sqrt{5}}\left\{-\phi^{-x}\sin(\pi x)\right\}$$

The Fibonacci identity: $F(x) = F(x-1) + F(x-2)$ is preserved for the complex parts of $F(x)$:

$$Re[F(x)] = Re[F(x-1)] + Re[F(x-2)]$$

and

$$Im[F(x)] = Im[F(x-1)] + Im[F(x-2)] .$$

Figure 1 is a computer-generated Argand plot of $F(x)$ in the range $-5 < x < +5$.

The branch of the curve for positive x approaches the real axis as x increases. Defining the tangent angle of the curve as:

$$\psi = \tan^{-1}\left\{\frac{Im[F(x)]}{Re[F(x)]}\right\} ;$$

this angle approaches zero for large positive x since

$$\lim_{x\to\infty} = Im[F(x)] = 0.$$

The negative branch of the curve approaches a logarithmic spiral for x large and negative. The modulus r is given by:

$$r = \left\{Re^2[F(x)] + Im^2[F(x)]\right\}^{1/2}$$

in the limit

$$r \approx \frac{\phi - x}{\sqrt{5}} ; \qquad \psi \approx \pi x, \qquad r \approx \frac{1}{\sqrt{5}}\left\{\phi^{-\psi/\pi}\right\} ;$$

therefore,

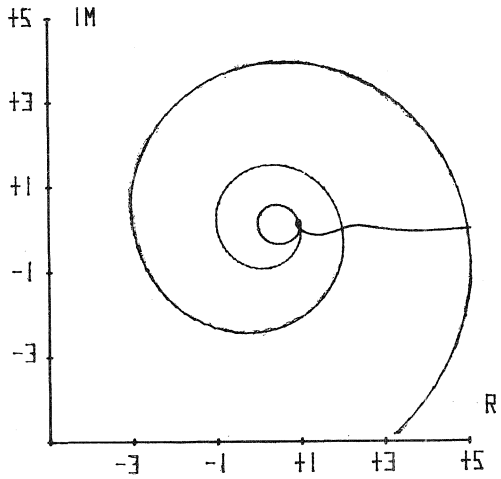

Fig. 1 Computer-Generated Argand Plot of the Fibonacci Function

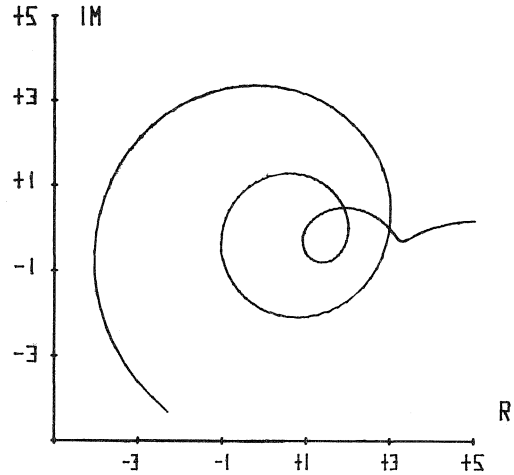

Fig. 2 Computer-Generated Argand Plot of the Lucas Function

$$\ln r \approx (-\psi/\pi)k \ ,$$

where

$$k = \ln(\phi/\sqrt{5}) \qquad \text{and} \qquad r \approx e^{-(\psi k/\pi)} = e^{-kx} \ .$$

Similarly, the Lucas number identity:

$$L(x) = F(x+1) + F(x-1)$$

leads directly to [6]:

$$L(x) = \phi^x + (-1)^x \phi^{-x}$$

and the complex representation of the Lucas Numbers follows

$$L(x) = \phi^x + \phi^{-x}(\cos \pi x + i \sin \pi x)$$

with

$$Re[L(x)] = \phi^x + \phi^{-x} \cos \pi x \qquad \text{and} \qquad Im[L(x)] = \phi^{-x} \sin \pi x \ .$$

Note:

$$Im[L(x)] = \frac{-1}{\sqrt{5}} \, Im[F(x)] \ .$$

As with the previous case for n large and positive, the positive branch of the Lucas number curve approaches the Real axis. Again, the negative branch approaches a logarithmic spiral for n large and negative.

$$\psi \approx \pi x, \quad r \approx \phi^{-(\psi/\pi)}, \quad \ln r \approx -(\psi/\pi)\ln \phi, \quad r \cong e^{-(\psi/\pi)\phi} = e^{-\phi x} \ .$$

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A PENTAGONAL ARCH

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A pentagonal arch can be generated by rolling a regular pentagon along a baseline as follows. In Fig. 1, as the left-hand pentagon is rolled toward the right, the vertex A moves first to B, then to C, D and finally to E as the successive sides touch the baseline. Connecting these points by line segments, the five-sided polygonal arch $ABCDE$ is formed.

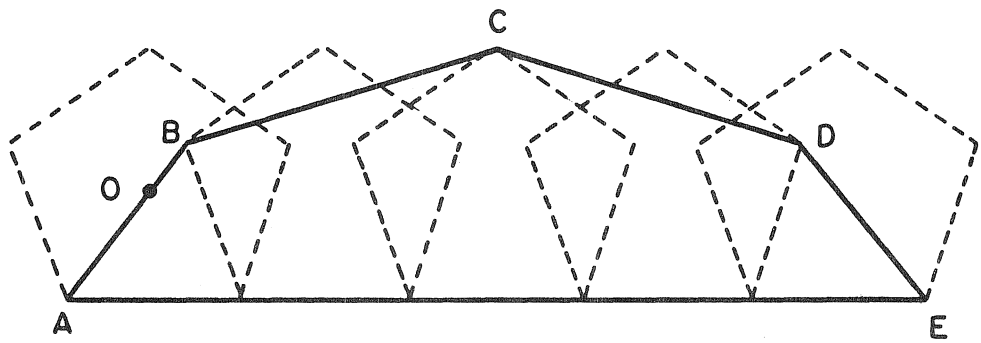

Figure 1

Let s denote the sides of the generating pentagon, and let $\tau = \frac{1}{2}(\sqrt{5}+1)$ denote the golden ratio. It is then easy to show

$$AB = DE = \sqrt{3-\tau}\,s, \qquad BC = CD = \sqrt{2+\tau}\,s$$

$$\angle EAB = \angle AED = 54^\circ, \qquad \angle ABC = \angle BCD = \angle CDE = 144^\circ .$$

Thus the pentagonal arch has some unexpected properties:

(1) Sides AB and BC (and of course DE and CD) are in the proportion of the golden ratio: $\frac{BC}{AB} = \tau$;

(2) The center O of the generating pentagon (in its initial position) lies on the line passing through A and B;

(3) The obtuse angles of the arch are equal.

While these three properties follow directly from the above formulas, a fourth property requires some additional considerations.

(4) $$\frac{\text{area of arch}}{\text{area of generating pentagon}} = 3 .$$

To see this first observe from Fig. 1 that it is enough to show that region $ABCFG$ of Fig. 2a is equal in area to that of the generating pentagon.

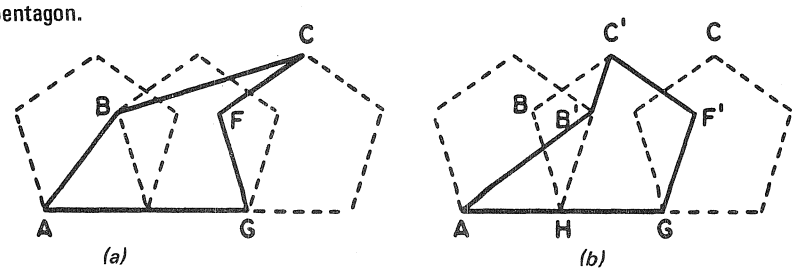

Figure 2

But by referring to Fig. 2b it is seen

$$\text{area } ABCFG = \text{area } AB'C'F'GH = \text{area } HBC'F'G$$

and so property (4) is demonstrated.

In the way of generalization it is natural to ask: Are there analogous properties for the n-sided arch generated by rolling a regular n-gon? The answer is that, upon replacing "pentagon" by "regular polygon," properties (2), (3) and (4) apply equally well to the general case. The two acute base angles are each

$$\left(\frac{1}{2}-\frac{1}{n}\right) \times 180^\circ$$

and the $n-2$ obtuse angles are each equal to

$$\left(1-\frac{1}{n}\right) \times 180^\circ .$$

A proof of (4) for the general case is the main content of [1]; as might be expected the above proof for the pentagonal arch does not generalize, though the ideas are useful for the simpler cases $n = 3, 4, 6$.

There is one aspect of the pentagonal arches which does seem more interesting than for the general arch. By property (2) five arches can be fit together in such a way that their bases form a regular pentagon.

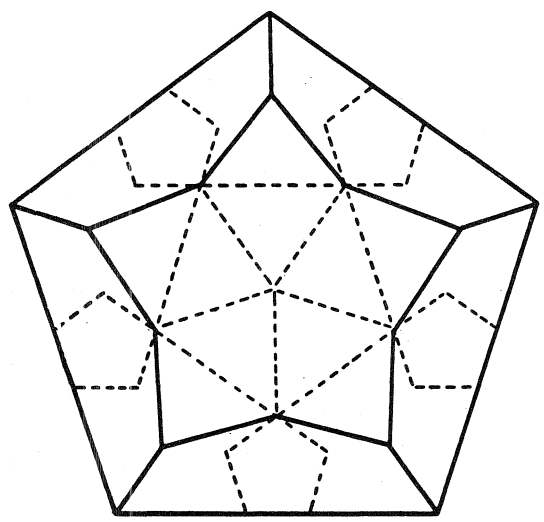

Figure 3

The interior star region can then be partitioned into ten congruent isoceles triangles, each of which has area equal to that of the original generating pentagon. Hence all of the twenty-five elemental polygons of Fig. 3 have equal area.

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A GENERALIZATION OF THE HILTON-FERN THEOREM ON THE EXPANSION OF FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The object of this note is to generalize Hilton's extension [2] of Fern's theorem [1] to sequences of arbitrary order. Ferns found a general method by which products of Fibonacci and Lucas numbers of the form

$$u_{x_1}u_{x_2}\cdots u_{x_n}$$

could be expressed as a linear function of the u_n. Hilton extended Fern's results to include effectively the generalized sequence of numbers of Horadam [3].

We shall extend the result to linear recursive sequences of order r which satisfy the recurrence relation

$$W_{s,n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} W_{s,n+r-j}^{(r)} \qquad (s = 0, 1, \cdots, r-1;\ n \geqslant r) \tag{1.1}$$

where the P_{rj} are arbitrary integers, and for suitable initial values $W_{s,n}^{(r)}$, $n = 0, 1, \cdots, r-1$. When $r = 2$, we have Horadam's sequence. We are in effect supplying an elaboration of the results of Moser and Whitney [4] on weighted compositions.

Modifying Williams [5] let a_{rj} be the r distinct roots of the auxiliary equation

$$x^r = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} x^{r-j} \quad , \tag{1.2}$$

where

$$a_{rj} = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r-1}^{(r)}\, d^k w^{-jk} \qquad (j = 1, 2, \cdots, r) \tag{1.3}$$

in which d is the determinant of the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_{r1} & a_{r2} & \cdots & a_{rr} \\ & \cdots & & \\ a_{r1}^{r-1} & a_{r2}^{r-1} & \cdots & a_{rr}^{r-1} \end{bmatrix}$$

and $w = exp(2i\pi/r)$, $i^2 = -1$. (This is not as general as Williams' definition, but it is adequate for our present purpose.) When $r = 2$,

$$a_{2j} = \tfrac{1}{2}(W_{0,3}^{(2)} + (-1)^j d W_{1,3}^{(2)})$$

which agrees with Hilton.

We shall frequently use the fact that

$$\sum_{j=1}^{r} w^{-ij} = r\delta_{i0},$$

where δ_{ij} is the Kronecker delta.

2. PRELIMINARY RESULTS

The first result we need is that

$$W_{s,r+1}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj} w^{sj} \qquad (s = 0, 1, \cdots, r-1). \tag{2.1}$$

Proof:

$$\sum_{j=0}^{r-1} a_{rj} w^{ij} = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k \sum_{j=0}^{r-1} w^{(i-k)j}$$
$$= \frac{1}{r} W_{i,r+1}^{(r)} d^i r \quad ,$$

from which the result follows.

This suggests that we set

$$W_{s,n+r}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj}^n w^{sj} \qquad (s = 0, 1, \cdots, r-1), \tag{2.2}$$

and it remains to see whether the $W_{s,n}^{(r)}$ of formula (2.2) satisfy the recurrence relation (1.1).

The right-hand side of this recurrence relation is

$$\sum_{k=1}^{r} \sum_{m=0}^{r-1} (-1)^{k+1} d^{-s} a_{rm}^{n-k} w^{sm} P_{rk}$$

(from (2.2))

$$= d^{-s} \sum_{m=0}^{r-1} \left(\sum_{k=1}^{r} (-1)^{k+1} a_{rm}^{r-k} P_{rk} \right) a_{rm}^{n-r} w^{sm}$$

$$= d^{-s} \sum_{m=0}^{r-1} a_{rm}^{r} a_{rm}^{n-r} w^{sm}$$

(from (1.2))

$$= W_{s,n+r}^{(r)} \qquad \text{as required}$$

(from (2.2)). It follows then that

$$a_{rj}^n = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,n+r}^{(r)} d^k w^{-jk} \tag{2.3}$$
$$(j = 1, 2, \cdots, r).$$

Proof: From Eq. (2.2), we have that

$$\sum_{j=0}^{r-1} a_{rj}^n w^{ij} = \frac{1}{r} W_{i,r+1}^{(r)} d^i r$$
$$= \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k \sum_{j=0}^{r-1} w^{(i-k)j}$$
$$= \sum_{j=0}^{r-1} \left(\frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k w^{-jk} \right) w^{ij}$$

from which we obtain the result.

3. HILTON-FERN THEOREM

Following Hilton let

(3.1) $$S_m^n = \sum_{\Sigma k=m} \prod_{i=1}^{n} W_{k,x_i+r}^{(r)} \qquad (k = 0, 1, \cdots, r-1),$$

where we have all permutations of $(x_1, \cdots, x_n)$. For example, when $r = 2$, we get

$$S_0^n = \sum W_{0,x_1+2}^{(2)} W_{0,x_2+2}^{(2)} \cdots W_{0,x_{n-1}+2}^{(2)} W_{0,x_n+2}^{(2)},$$

and

$$S_1^n = \sum W_{0,x_1+2}^{(2)} W_{0,x_2+2}^{(2)} \cdots W_{0,x_{n-1}+2}^{(2)} W_{1,x_n+2}^{(2)},$$

and so on, as in Hilton.

Theorem: For S_m^n defined in formula (3.1),

$$W_{s,x_1+x_2+\cdots+x_n+r}^{(r)} = r^{-n} \sum_{j=0}^{r-1} \sum_{k=0}^{(r-1)n} (dw^{-j})^{k-s} S_k^n .$$

Proof: Let

$$X_n = \sum_{i=1}^{n} x_i .$$

Then

$$a_{rj}^{X_n} = \prod_{x_i=1}^{n} a_{rj}^{x_i} = \frac{1}{r^n} \prod_{x_i=1}^{n} \sum_{k=0}^{r-1} W_{k,x_i+r}^{(r)} d^k w^{-jk}$$

$$= r^{-n}(S_0^n + dw^{-j}S_1^n + \cdots + (dw^{-j})^{(r-1)n} S_{(r-1)n}^n)$$

$$= r^{-n} \sum_{k=0}^{(r-1)n} (dw^{-j})^k S_k^n .$$

Thus

$$W_{s,x_n+r}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj}^{X_n} w^{sj}$$

(from (2.2))

$$= r^{-n} d^{-s} \sum_{j=0}^{r-1} \sum_{j=0}^{(r-1)n} d^k w^{(s-k)j} S_k^n ,$$

as required. For example,

$$W_{0,x_1+x_2+\cdots+x_n+2}^{(2)} = (\tfrac{1}{2})^n \sum_{j=0}^{1} \sum_{k=0}^{n} (dw^{-j})^k S_k^n$$

$$= (\tfrac{1}{2})^n \sum_{k=0}^{n} (d^k + (-d)^k) S_k^n$$

$$= \frac{1}{2^{n-1}} (S_0^n + d^2 S_2^n + \cdots),$$

and

$$W_{1,x_1+x_2+\cdots+x_n+2}^{(2)} = (\tfrac{1}{2})^n \sum_{j=0}^{1} \sum_{k=0}^{n} (dw^{-j})^{k-1} S_k^n$$

$$= (\tfrac{1}{2})^n \sum_{k=0}^{n} (d^{k-1} + (-d)^{k-1}) S_k^n = \frac{1}{2^{n-1} d} (dS_1^n + d^3 S_3^n + \cdots),$$

which agree with Hilton when his $A = B = 1$. These results could be made more general by generalizing the definition of a_{rj} along the lines of Williams.

Thanks are due to Dr. A.J.W. Hilton of Reading University, U.K., for suggesting the problem and for a preprint of his paper.

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TO MARY ON OUR 34th ANNIVERSARY

HUGO NORDEN
Roslindale, Massachusetts 02131

Our wedlock year is thirty-four,
A number Fibo did adore,
He'd say, "Your shape is really great,
A perfect one point six one eight."

As everyone around can see,
You're pure Dynamic Symmetry,
And when demurely you stroll by
All know you are exactly Phi.

Proportions are what makes things run,
Like eight, thirteen and twenty-one,
Then, next in line is thirty-four,
But, wait, there's still a whole lot more.

In nineteen hundred ninety-five
Our wedlock year is fifty-five,
There's much more living yet in store,
Today is only thirty-four!

So stay the way you are today,
Don't work too hard, take time to play,
And stay point six one eight to one
So we can still enjoy the fun.

Hugo

April 7, 1974

SOME ASPECTS OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

In a series of papers, Horadam [8], [9], [10], [11] has obtained many results for the generalized Fibonacci sequence $\{H_n\}$ defined below, which he extended to the more general sequence $\{W_n(a,b;p,q)\}$ in [12], [13].

Additional results for the sequence $\{H_n\}$, which we concentrate on here, have been obtained by, among other authors, Iyer [14], and Zeitlin [20]. Some of the results in §5 have been obtained independently by Iyer [14].

It is the purpose of this paper to add to the literature of properties and identities relating to $\{H_n\}$ in the expectation that they may prove useful to Fibonacci researchers. Further material relating to properties of $\{H_n\}$ will follow in another article.

Though these results may be exhausting to the readers, they are not clearly exhaustive of the rich resources opened up. As Descartes said in another context, we do not give all the facts but leave some so that their discovery may add to the pleasure of the reader.

2. A GENERATION OF H_n

Generalized Fibonacci numbers H_n are defined by the second-order recurrence relation

$$H_{n+2} = H_{n+1} + H_n \quad (n \geqslant 0) \tag{2.1}$$

with initial conditions

$$H_0 = q, \quad H_1 = p \tag{2.2}$$

and the proviso that H_n may be defined for $n < 0$.

(See Horadam [12].)

Standard methods (*e.g.* use of difference equations), allow us to discover that

$$H_n = \frac{1}{2\sqrt{5}}\left(\ell\alpha^n - m\beta^n\right) \tag{2.3}$$

where

$$\left\{\begin{array}{l} \alpha = \dfrac{1+\sqrt{5}}{2}, \ \beta = \dfrac{1-\sqrt{5}}{2} \quad \text{(roots of } x^2 - x - 1 = 0\text{), so that} \\ \alpha+\beta = 1, \ \alpha\beta = -1, \ \alpha-\beta = \sqrt{5}, \ \beta = -\alpha^{-1}\ ; \\ \ell = 2(p-q\beta), \ m = 2(p-q\alpha), \quad \text{so that} \\ \ell+m = 2(2p-q), \ \ell-m = 2q\sqrt{5} \quad \text{and} \\ \tfrac{1}{4}\ell m = p^2 - pq - q^2 = d \quad \text{(say)}. \end{array}\right. \tag{2.4}$$

It is well known that $p = 1, q = 0$ leads to the ordinary Fibonacci sequence $\{F_n\}$, while $p = 2q = -1$ leads to the Lucas sequence $\{L_n\}$.

Following an analytic procedure due to Hagis [5] for generating the ordinary Fibonacci number F_n, we proceed to an alternative establishment of (2.3).

Put $h_n = H_{n+1}$. Let

*Part of the substance of an M. Sc. Thesis presented to the University of New England, Armidale, in 1968.

(2.5)
$$h(x) = \sum_{n=0}^{\infty} h_n x^n$$
$$= h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots$$
$$= h(0) + \frac{h'(0)x}{1!} + \frac{h''(0)x^2}{2!} + \cdots + \frac{h^{(n)}(0)x^n}{n!} + \cdots$$

using a Maclaurin infinite expansion.

With the help of (2.2) one can obtain the generating function

(2.6)
$$h(x) = \frac{p + qx}{1 - x - x^2}$$

Introducing complex numbers, we set

(2.7)
$$h(z) = \frac{p + qz}{1 - z - z^2},$$

where $h(z)$ is an analytic function, whose only singularities are simple poles at the points

$$\frac{-1-\sqrt{5}}{2} = -\alpha \quad \text{and} \quad \frac{-1+\sqrt{5}}{2} = -\beta$$

corresponding to the roots of the equation $z^2 + z - 1 = 0$.

From (2.5), in the complex case, it is clear that

(2.8)
$$h_n = \frac{h^{(n)}(0)}{n!}$$

on comparing coefficients of z^n.

One may follow Hagis, appealing to Cauchy's Integral Theorem and the theory of residues, or argue from (2.7) that, after calculation,

(2.9)
$$h(z) = \frac{1}{2\sqrt{5}}\left\{\frac{\ell}{-\beta - z} + \frac{m}{\alpha + z}\right\}$$

whence, on differentiating n times,

(2.10)
$$h^{(n)}(z) = \frac{1}{2\sqrt{5}}\left\{\frac{\ell n!}{(-z-\beta)^{n+1}} + \frac{(-1)^n m n!}{(z+\alpha)^{n+1}}\right\}$$

so that

(2.11)
$$\frac{h^{(n)}(0)}{n!} = \frac{1}{2\sqrt{5}}\left\{\ell\alpha^{n+1} - m\beta^{n+1}\right\}$$
$$= h_n$$

from (2.8) from which follows the expression for H_{n+1}.

Of course, if we wish to avoid complex numbers altogether, we could simply apply the above argument to (2.6) instead of to (2.7).

3. GENERALIZED "FIBONACCI" ARRAYS

Consider the array (a re-arrangement and re-labelling of Gould [3]):

Row\Col.	0	1	2	3	4	5	6	7	...
0	p								
1	p	q							
2	p	p	q						
3	p	p	p + q	q					
4	p	p	2p + q	p + q	q				
5	p	p	3p + q	2p + q	p + 2q	q			
6	p	p	4p + q	3p + q	3p + 3q	p + 2q	q		
7	p	p	5p + q	4p + q	6p + 4q	3p + 3q	p + 3q	q	
...	...	...	...	...	...	...	...	...	

Letting C_j^n $(j = 0, 1, 2, \cdots, n, \cdots)$ be an element of this array, where the superscript refers to rows and the subscripts to columns, we define the array as in Gould [3] by the conditions:

$$C_0^0 = C_0^1 = p, \quad C_1^1 = q \tag{3.1}$$

$$C_j^n = 0 \quad \text{if} \quad j > n \quad \text{or} \quad j < 0 \quad . \tag{3.2}$$

$$C_j^{n+1} = C_{j-1}^n + \frac{1+(-1)^j}{2} C_j^n \quad \text{if} \quad n \geqslant 1, \ j \geqslant 0. \tag{3.3}$$

The row-sums are given by

$$\begin{aligned} S_n(p,q) &= \sum_{j=0}^{n} C_j^n \qquad (n \geqslant 0) \\ &= pF_{n+1} + qF_n = H_{n+1} \end{aligned} \tag{3.4}$$

by Horadam [8]. Thus the row-sums of this array generate the generalized Fibonacci numbers. As indicated in Gould [3] the given array generalizes two variants of Pascal's triangle which are related to Fibonacci numbers and to Lucas numbers.

It may easily be verified that

$$C_{2k}^n = \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q \tag{3.5}$$

$$C_{2k+1}^n = \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q \tag{3.6}$$

so that

$$\begin{aligned} \sum_{j=0}^{n} C_j^n &= \sum_{k=0}^{[n/2]} C_{2k}^n + \sum_{k=0}^{[(n-1)/2]} C_{2k+1}^n \\ &= H_{n+1} , \end{aligned} \tag{3.7}$$

as expected (*cf.* (3.4)).

Similarly, we can show that

$$\sum_{j=0}^{n} (-1)^j C_j^n = H_{n-2}, \qquad n \geqslant 2 . \tag{3.8}$$

If we define the polynomials $\{C_n(x)\}$ by

$$C_n(x) = \sum_{j=0}^{n} C_j^n x^j , \tag{3.9}$$

then we have on using (3.5) and (3.6) that

$$\begin{aligned} C_n(x) = &\sum_{k=0}^{[n/2]} \left\{ \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q \right\} x^{2k} \\ &+ \sum_{k=0}^{[(n-1)/2]} \left\{ \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q \right\} x^{2k+1} , \end{aligned} \tag{3.10}$$

where it can be shown, as in Gould [3], that the polynomial $C_n(x)$ satisfies the simple recurrence relation

$$2C_{n+1}(x) = (2x+1)C_n(x) + C_n(-x) \tag{3.11}$$

on using (3.3). Similarly, it can be shown that $C_n(x)$ satisfies the second-order recurrence relation

$$C_{n+2}(x) = C_{n+1}(x) + x^2 C_n(x) . \tag{3.12}$$

It may be noted in passing that certain properties of an array involving the elements of $\{H_n\}$ are given in Wall [19].

4. GENERALIZED FIBONACCI FUNCTIONS

Elmore [1] described the concept of Fibonacci functions. Extending his idea, we have a sequence of generalized Fibonacci functions $\{H_n(x)\}$ if we put

$$(4.1)\qquad \begin{cases} H_0(x) = \dfrac{1}{2\sqrt{5}}\left\{\ell e^{\alpha x} - m e^{\beta x}\right\} \\ H_1(x) = H_0'(x) \\ H_2(x) = H_0''(x) \\ \cdots\cdots\cdots\cdots \\ H_n(x) = H_0^{(n)}(x) = \dfrac{1}{2\sqrt{5}}\left\{\ell\alpha^n e^{\alpha x} - m\beta^n e^{\beta x}\right\} \end{cases}$$

so that we have

$$(4.2)\qquad H_{n+2}(x) = H_{n+1}(x) + H_n(x)\ .$$

Obviously,

$$(4.3)\qquad \begin{aligned} &H_0(0) = q = H_0, \quad H_1(0) = p = H_1\ , \\ &H_2(0) = p + q = H_2\ , \cdots, \end{aligned}$$

etc., and

$$(4.4)\qquad H_n(0) = \frac{1}{2\sqrt{5}}\left\{\ell\alpha^n - m\beta^n\right\} = H_n\ .$$

We are able to find numerous identities for these generalized Fibonacci functions, some of which are listed below for reference:

$$(4.5)\qquad H_{n-1}(x)H_{n+1}(x) - H_n^2(x) = (-1)^n de^x$$

$$(4.6)\qquad H_{n-1}(x)F_r(x) + H_n(x)F_{r+1}(x) = H_{n+r}(2x)\ ,$$

where the $F_n(x)$ are the Fibonacci functions corresponding to the $f_n(x)$ of Elmore [1]. Similarly,

$$(4.7)\qquad H_{n-1}(u)F_r(v) + H_n(u)F_{r+1}(v) = H_{n+r}(u+v)$$

$$(4.8)\qquad H_{n-1}^2(x) + H_n^2(x) = (2p-q)H_{2n-1}(2x) - dF_{2n-1}(2x)$$

$$(4.9)\qquad H_{n+1}^2(x) - H_{n-1}^2(x) = (2p-q)H_{2n}(2x) - dF_{2n}(2x)$$

$$(4.10)\qquad H_n^3(x) + H_{n+1}^3(x) = 2H_n(x)H_{n+1}^2(x) + (-1)^n de^x H_{n-1}(x)$$

$$(4.11)\qquad H_{n+1-r}(x)H_{n+1+r}(x) - H_{n+1}^2(x) = (-1)^{n-r} de^x F_r^2$$

$$(4.12)\qquad H_n(x)H_{n+1+r}(x) - H_{n-s}(x)H_{n+r+s+1}(x) = (-1)^{n-s} de^x F_s F_{r+s+1}\ .$$

We note here that (8) to (14) of Horadam [8] are a special case of (4.5) to (4.12) above, since, as we have already shown in (4.3) and (4.4), the generalized Fibonacci functions become the generalized Fibonacci numbers $\{H_n\}$ when $x = 0$.

As in Horadam [8], we also note that (4.5) is a special case of (4.11) when $r = 1$ and n is replaced by $n - 1$. If we put $r = n$ in (4.11) we have

$$(4.13)\qquad H_1(x)H_{2n+1}(x) - H_{n+1}^2(x) = de^x F_n^2\ .$$

Corresponding to the Pythagorean results in Horadam [8], we have, for the generalized Fibonacci function $H_n(x)$

$$(4.14)\qquad \left\{2H_{n+1}(x)H_{n+2}(x)\right\}^2 + \left\{H_n(x)H_{n+3}(x)\right\}^2 = \left\{2H_{n+1}(x)H_{n+2}(x) + H_n^2(x)\right\}^2$$

from which we may derive (16) of Horadam [8], for the special case when $x = 0$.

The above identities are easily established by use of the formula for $H_n(x)$ given in (4.1) with reference to the identities

$$(4.15)\qquad \begin{cases} 1+\alpha^2 = \alpha\sqrt{5}\ , & 1+\beta^2 = -\beta\sqrt{5}\ , \\ \alpha\beta = -1\ , & \tfrac{1}{2}\ell m = d\ , \\ \alpha^3 = 2+\sqrt{5}, & 1+\alpha^3 = 2\alpha^2\ , \\ 2\alpha+\beta = \alpha^2\ , & 1+\alpha = \alpha^2\ , \\ \alpha+\beta = 1\ , & \ell(2p-q)-2d = \tfrac{1}{2}\ell^2\ , \text{ etc.} \end{cases}$$

As in Elmore [1], we can extend this theory of generalized Fibonacci functions to generalized Fibonacci functions of two variables to give a function of two variables, thus:

$$(4.16)\qquad \phi_0 \equiv \phi(x,y) = \sum_{i=0}^{\infty} H_i(x)\,\frac{y^i}{i!} = H_0(x) + H_1(x)y + H_2(x)\frac{y^2}{2!} + \cdots .$$

Differentiating (4.16) term-by-term gives

$$(4.17)\qquad \begin{aligned} \frac{\partial\phi_0}{\partial x} &= \sum_{i=1}^{\infty} H_i(x)\,\frac{y^{i-1}}{(i-1)!} = \sum_{i=0}^{\infty} H_{i+1}(x)\,\frac{y^i}{i!} \\ \frac{\partial\phi_0}{\partial y} &= \sum_{i=0}^{\infty} H_{i+1}(x)\,\frac{y^i}{i!} \end{aligned}$$

i.e.,

$$(4.18)\qquad \frac{\partial\phi_0}{\partial x} = \frac{\partial\phi_0}{\partial y}\ .$$

Similarly, we can verify that all higher power partial derivatives are equal, so that if we denote the kth partial derivative by ϕ_k, we have

$$(4.19)\qquad \phi_k = \frac{\partial^k\phi}{\partial x^r\partial y^s} = \sum_{i=0}^{\infty} H_{k+i}(x)\,\frac{y^i}{i!} = \sum_{i=0}^{\infty} H_{k+i}(y)\,\frac{x^i}{i!}\ ,$$

where r and s are positive integers such that $r+s = k$. Noting that

$$(4.20)\qquad \phi_k(x,0) = H_k(x), \qquad \phi_k(0,y) = H_k(y), \qquad \phi_k(0,0) = H_k\ ,$$

we can expand $\phi_k(x,y)$ as a power series of the two variables x and y at $(0,0)$ so that we have

$$(4.21)\qquad \begin{aligned} \phi_k(x,y) &= \phi_k(0,0) + \left[x\,\frac{\phi_k(0,0)}{\partial x} + y\,\frac{\phi_k(0,0)}{\partial y}\right] \\ &\quad + \frac{1}{2!}\left[x^2\,\frac{\partial^2\phi_k(0,0)}{\partial x^2} + 2xy\,\frac{\partial^2\phi_k(0,0)}{\partial x\partial y} + y^2\,\frac{\partial^2\phi_k(0,0)}{\partial y^2}\right] + \cdots \\ &= H_k + H_{k+1}\,\frac{(x+y)}{1!} + H_{k+2}\,\frac{(x+y)^2}{2!} + \cdots \end{aligned}$$

so that

$$(4.22)\qquad \phi_k(x,y) = H_k(x+y) = \frac{\ell\alpha^k e^{\alpha(x+y)} - m\beta^k e^{\beta(x+y)}}{2(\alpha-\beta)}\ .$$

5. GENERALIZED FIBONACCI NUMBER IDENTITIES

Many other interesting and useful identities may be derived for the sequence $\{H_n\}$ using inductive methods or by argument from (2.1). We list some elementary results without proof:

$$(5.1)\qquad H_{-n} = (-1)^n[qF_{n+1} - pF_n]$$

(5.2) $$\sum_{i=0}^{n} H_i = H_{n+2} - H_1 \; [= H_{n+2} - p]$$

(5.3) $$\sum_{i=0}^{n} H_{2i-1} = H_{2n} - H_{-2} \; [= H_{2n} + (p - 2q)]$$

(5.4) $$\sum_{i=0}^{n} H_{2i} = H_{2n+1} - H_{-1} \; [= H_{2n+1} - (p - q)]$$

(5.5) $$\sum_{i=0}^{2n} (-1)^{i+1} H_i = -H_{2n-1} + p - 2q$$

(5.6) $$\sum_{i=0}^{n} H_i^2 = H_n H_{n+1} - q(p - q)$$

(5.7) $$\sum_{i=0}^{n} iH_i = (n+1)H_{n+2} - H_{n+4} + H_3$$

(5.8) $$\sum_{i=0}^{n} \binom{n}{i} H_i = H_{2n}$$

(5.9) $$\sum_{i=0}^{n} \binom{n}{i} H_{3i} = 2^n H_{2n}$$

(5.10) $$\sum_{i=0}^{n} \binom{n}{i} H_{4i} = 3^n H_{2n} \; .$$

The three summations (5.8), (5.9) and (5.10), which are generalizations of similar results for the ordinary Fibonacci Sequence $\{F_n\}$ as listed in Hoggatt [6], may all be proved by numerical substitution as, for example, in

$$\sum_{i=0}^{n} \binom{n}{i} H_{3i} = \frac{1}{2\sqrt{5}} \left\{ \ell \sum_{i=0}^{n} \binom{n}{i} \alpha^{3i} - m \sum_{i=0}^{n} \binom{n}{i} \beta^{3i} \right\}$$

$$= \frac{1}{2\sqrt{5}} \left\{ \ell (1 + \alpha^3)^n - m(1 + \beta^3)^n \right\}$$

$$= \frac{2^n}{2\sqrt{5}} \left\{ \ell\alpha^{2n} - m\beta^{2n} \right\} = 2^n H_{2n} \; .$$

Some further generalizations of identities listed in Subba Rao [17] are:

(5.11) $$\sum_{i=0}^{n} H_{3i-2} = \tfrac{1}{2}(H_{3n} - H_{-3})$$

Proof: Using identity (3) of Horadam [8], *viz.*,

$$2H_n = H_{n+2} - H_{n-1} \; ,$$

we have

$$2H_{-2} = H_0 - H_{-3}$$
$$2H_1 = H_3 - H_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Adding both sides and then dividing by two gives the desired result. Similarly,

$$\sum_{i=0}^{n} H_{3i-1} = \tfrac{1}{2}(H_{3n+1} - H_{-2}) \tag{5.12}$$

$$\sum_{i=0}^{n} H_{3i} = \tfrac{1}{2}(H_{3n+2} - H_{-1}) \ . \tag{5.13}$$

Some additional identities corresponding to formulae for the sequence $\{F_n\}$ in Siler [16], are

$$\sum_{i=0}^{n} H_{4i-3} = F_{2(n+1)}H_{2n-3} \tag{5.14}$$

$$\sum_{i=0}^{n} H_{4i-1} = F_{2(n+1)}H_{2n-1} \tag{5.15}$$

$$\sum_{i=0}^{n} H_{4i-2} = F_{2(n+1)}H_{2n-2} \tag{5.16}$$

$$\sum_{i=0}^{n} H_{4i} = F_{2(n+1)}H_{2n} \ . \tag{5.17}$$

As in Siler [16], identities (5.4) and (5.11) to (5.17) suggest that we should be able to solve the general summation formula

$$\sum_{i=1}^{n} H_{ai-b} \ . \tag{5.18}$$

Proceeding as in Siler [16], we have:

$$\sum_{i=1}^{n} H_{ai-b} = \frac{1}{2\sqrt{5}} \left\{ \ell \sum_{i=1}^{n} \alpha^{ai-b} - m \sum_{i=1}^{n} \beta^{ai-b} \right\}$$

$$= \frac{(-1)^a H_{an-b} - H_{a(n+1)-b} - (-1)^a H_{-b} + H_{a-b}}{(-1)^a + 1 - L_a}$$

on using the fact that

$$\sum_{i=1}^{n} \alpha^{ai-b} = \alpha^{a-b} \underbrace{[1 + \alpha^a + \cdots + \alpha^{(n-1)a}]}_{n \text{ terms}} = \alpha^{a-b} \frac{\alpha^{na} - 1}{\alpha^a - 1}$$

with a similar expression for the term involving β. Here it should be stated that Siler rediscovered a special case due to Lucas in 1878.

The identity (5.19) below which arose as a generalization of the combination of (2) and (3) of Sharpe [15], may be established thus:

(5.19) $$H_{n+2k+1}^2 + H_{n+2k}^2 = H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}$$

Proof:

$$\begin{aligned}
20(H_{n+2k+1}^2 + H_{n+2k}^2) &= (\ell\alpha^{n+2k+1} - m\beta^{n+2k+1})^2 + (\ell\alpha^{n+2k} - m\beta^{n+2k})^2 \\
&= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k} - 8d(\alpha\beta)^{n+2k}[1+\alpha\beta] \\
&= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k} \\
20(H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}) &= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k} \\
&\quad - \ell m(\alpha\beta)^{2k+1}[\alpha^{2n} + \beta^{2n}] - \ell m(\alpha\beta)^{2k}[\alpha^{2n} + \beta^{2n}] \\
&= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k} .
\end{aligned}$$

In an attempt to generalize those identities found in Tadlock [18], involving the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ we have

(5.20) $$F_{2j+1} \mid (H_{k+j+1}^2 + H_{k-j}^2)$$

Proof:

$$\begin{aligned}
H_{k+j+1}^2 + H_{k-j}^2 &= \left[\frac{\ell\alpha^{k+j+1} - m\beta^{k+j+1}}{2(\alpha-\beta)}\right]^2 + \left[\frac{\ell\alpha^{k-j} - m\beta^{k-j}}{2(\alpha-\beta)}\right]^2 \\
&= \frac{\ell^2\alpha^{2k+1}(\alpha^{2j+1} + \alpha^{-2j-1} + m^2\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1})}{4(\alpha-\beta)^2} \\
&\quad - \frac{2d(\alpha\beta)^{k+j}[\alpha\beta + (\alpha\beta)^{-2j}]}{(\alpha-\beta)^2} \\
&= \frac{(\alpha^{2j+1} - \beta^{2j+1})(\ell^2\alpha^{2k+1} - m^2\beta^{2k+1})}{(\alpha-\beta)\,4(\alpha-\beta)}
\end{aligned}$$

since

$$\begin{cases} \alpha^{-2j-1} = -\beta^{2j+1} \\ \beta^{-2j-1} = -\alpha^{2j+1} \end{cases}$$

i.e.,

$$H_{k+j+1}^2 + H_{k-j}^2 = F_{2j+1} \cdot \tfrac{1}{4}\,\frac{\ell^2\alpha^{2k+1} - m^2\beta^{2k+1}}{\alpha-\beta}$$

i.e.,

$$F_{2j+1} \mid (H_{k+j+1}^2 + H_{k-j}^2) .$$

Also,

(5.21) $$2[2H_n^2 + (-1)^n d]^2 = H_n^4 + H_{n+1}^4 + H_{n-1}^4 .$$

This identity which is a generalization of Problem H-79 proposed by Hunter [7], may be solved as follows. From the identity (11) of Horadam [8], we have

(5.22) $$\begin{aligned}
2[2H_n^2 + (-1)^n d] &= 2[H_{n-1}H_{n+1} + H_n^2]^2 \\
&= H_n^4 + H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H_{n-1}^2H_{n+1}^2 .
\end{aligned}$$

Now,

(5.23) $$H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H_{n-1}^2H_{n+1}^2 = (H_{n+1} - H_{n-1})^4 + 4(H_{n+1} - H_{n-1})^2H_{n-1}H_{n+1}$$

on calculation, so that (5.21) follows from (5.22) and (5.23).

Two further interesting results are obtained by considering the following generalization of Problem B-9 proposed by Graham [4]. From

$$\frac{1}{H_{n-1}H_{n+1}} = \frac{H_n}{H_{n-1}H_nH_{n+1}} = \frac{H_{n+1} - H_{n-1}}{H_{n-1}H_nH_{n+1}} = \frac{1}{H_{n-1}H_n} - \frac{1}{H_nH_{n+1}}$$

we have, on summing both sides over $n = 2, \cdots, \infty$,

(5.24) $$\sum_{n=2}^{\infty} \frac{1}{H_{n-1}H_{n+1}} = \frac{1}{p(p+q)} .$$

Similarly, from

$$\frac{H_n}{H_{n-1}H_{n+1}} = \frac{H_{n+1} - H_{n-1}}{H_{n-1}H_{n+1}} = \frac{1}{H_{n-1}} - \frac{1}{H_{n+1}}$$

we have

(5.25) $$\sum_{n=2}^{\infty} \frac{H_n}{H_{n-1}H_{n+1}} = \frac{2p+q}{p(p+q)} .$$

6. RECURRENCE RELATIONS FOR $\{H_{H_n}\}$

If we define a sequence $\{G_n\}$ by $G_n = H_{H_n}$, and define $\{X_n\}$ and $\{Y_n\}$ by $X_n = F_{H_n}$ and $Y_n = L_{H_n}$, then we may verify that

(6.1) $$G_{n+3} = G_{n+2}Y_{n+1} - (-1)^{H_{n+1}} G_n ,$$

which corresponds exactly with (1) of Ford [2], and that

(6.2) $$2G_{n+3} = G_{n+1}Y_{n+2} + G_{n+2}Y_{n+1} - (-1)^{H_{n+1}} H_0 Y_n$$

corresponding to (5) of Ford [2].

If we now define the sequence $\{Z_n\}$ by $Z_n = H_{H_n+j}$, then

(6.3) $$\begin{aligned} Z_n &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^{H_n}\alpha^j - m\beta^{H_n}\beta^j \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_n - m\beta^j S_n \right\} \end{aligned} ,$$

where $R_n = \alpha^{H_n}$ (and $S_n = \beta^{H_n}$) for convenience.

(6.4) $$\begin{aligned} \therefore Z_{n+2} &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_{n+2} - m\beta^j S_{n+2} \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_{n+1}R_n - m\beta^j S_{n+1}S_n \right\} \end{aligned}$$

since $R_{n+2} = \alpha^{H_{n+2}} = \alpha^{H_{n+1}}\alpha^{H_n} = R_{n+1}R_n$, and similarly for S_{n+2}.

(6.5) $$\begin{aligned} \therefore Z_{n+2} = \frac{1}{2\sqrt{5}} \Big\{ & R_n(\ell\alpha^j R_{n+1} - m\beta^j S_{n+1}) + S_{n+1}(\ell\alpha^j R_n - m\beta^j S_n) \\ & - R_n S_{n+1}(\ell\alpha^j - m\beta^j) \Big\} = R_n Z_{n+1} + S_{n+1}Z_n - R_n S_{n+1} H_j \end{aligned}$$

i.e.,

(6.6) $$Z_{n+2} = R_n Z_{n+1} + S_{n+1}Z_n - (-1)^{H_n} S_{n-1} H_j$$

since

$$R_n S_{n+1} = \alpha^{H_n}\beta^{H_{n+1}} = (\alpha\beta)^{H_n}\beta^{H_{n-1}} .$$

Similarly,

(6.7) $$Z_{n+2} = S_n Z_{n+1} + R_{n+1}Z_n - (-1)^{H_n} R_{n-1} H_j .$$

Adding Eqs. (6.6) and (6.7) gives

(6.8) $$2Z_{n+2} = Z_{n+1}(R_n + S_n) + Z_n(R_{n+1} + S_{n+1}) - (-1)^{H_n} H_j (R_{n-1} + S_{n-1})$$

i.e.,

$$2Z_{n+2} = Y_n Z_{n+1} + Y_{n+1}Z_n - (-1)^{H_n} Y_{n-1} H_j$$

since

$$R_n + S_n = \alpha^{H_n} + \beta^{H_n} = L_{H_n} = Y_n$$

i.e.

(6.9) $$2H_{H_{n+2}+j} = L_{H_n} H_{H_{n+1}} + L_{H_{n+1}} H_{H_n+j} - (-1)^{H_n} L_{H_{n-1}} H_j$$

which is a generalization of (14) of Ford [2].

One can continue discovering new generalizations *ad infinitum* (but not, we hope, *ad nauseam!*), but the time has come for a halt.

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AN EXTENSION OF FIBONACCI'S SEQUENCE

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Fibonacci's sequence is generally known as the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$ defined by $u_1 = 1$, $u_2 = 1$, $u_{n+1} = u_n + u_{n-1}$, in which n is a positive integer $\geqslant 2$. It is easy to extend this sequence in such a way that n may be any integer number.

We then get:

$$\begin{array}{cccccccccccccccccc} \cdots & -21, & 13, & -8, & 5, & -3, & 2, & -1, & 1, & 0, & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & u_{-8} & u_{-7} & u_{-6} & u_{-5} & u_{-4} & u_{-3} & u_{-2} & u_{-1} & u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & \end{array}$$

In this sequence we have:

(1a) $$u_1 = 1, \quad u_2 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for all} \quad n \in Z.$$

The following definition is known to be equivalent to the previous one:

(1b) $$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \in Z,$$

in which α is the positive root and β the negative root of the equation $x^2 = x + 1$.

We know the following relations involving α and β to be valid:

$$\alpha = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} = 1.6180339 \cdots$$

$$\beta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} = -0.6180339 \cdots$$

$$\alpha^2 = \alpha + 1, \quad \beta^2 = \beta + 1, \quad \alpha\beta = -1, \quad \alpha + \beta = 1, \quad \alpha - \beta = \sqrt{5}\ .$$

The proof of the identities in this paper will in most cases be based upon $\alpha^2 = \alpha + 1$.

The purpose of this article is to study the results of an extension of definition (1b) in such a way that for n not only integers, but also rational numbers, and even all real numbers can be chosen.

If we try $n = \tfrac{1}{2}$ in definition (1b), we get

$$u_{\frac{1}{2}} = \frac{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}}{\alpha - \beta} \quad ,$$

in which $\beta^{\frac{1}{2}} = \sqrt{\beta}$ causes trouble, because β is negative.

To avoid these difficulties, we define:

(2) $$u_n = \frac{\alpha^{2n} - \cos n\pi + i \sin n\pi}{(\alpha - \beta)\alpha^n},$$

or $u_n = x_n + iy_n$, in which

$$x_n = \frac{\alpha^{2n} - \cos n\pi}{(\alpha - \beta)\alpha^n} \qquad \text{and} \qquad y_n = \frac{\sin n\pi}{(\alpha - \beta)\alpha^n}\ .$$

In this definition we have: $n \in R$, $u_n \in C$.

First we shall have to show, of course, that this definition is equivalent to (1b) for $n \in Z$. We calculate:

$$u_1 = \frac{\alpha^2 - \cos\pi + i\sin\pi}{(\alpha-\beta)\alpha} = \frac{\alpha^2+1}{\alpha^2-\alpha\beta} = \frac{\alpha^2+1}{\alpha^2+1} = 1\,,$$

$$u_2 = \frac{\alpha^4 - \cos 2\pi + i\sin 2\pi}{(\alpha-\beta)\alpha^2} = \frac{\alpha^4-1}{(\alpha-\beta)\alpha^2} = \frac{(\alpha^2+1)(\alpha^2-1)}{(\alpha-\beta)\alpha^2} = \frac{(\alpha^2+1)\alpha}{(\alpha-\beta)\alpha^2} = \frac{\alpha^2+1}{\alpha^2-\alpha\beta} = 1.$$

Now we will show that for all n the relation $u_{n+1} = u_n + u_{n-1}$ remains valid.

$$u_{n+1} = \frac{\alpha^{2n+2} - \cos(n+1)\pi + i\sin(n+1)\pi}{(\alpha-\beta)\alpha^{n+1}} = \frac{\alpha^{2n+2} + \cos n\pi - i\sin n\pi}{(\alpha-\beta)\alpha^{n+1}}\,,$$

$$u_{n-1} = \frac{\alpha^{2n-2} - \cos(n-1)\pi + i\sin(n-1)\pi}{(\alpha-\beta)\alpha^{n-1}} = \frac{\alpha^{2n-2} + \cos n\pi - i\sin n\pi}{(\alpha-\beta)\alpha^{n-1}}\,.$$

The identity which we have to prove can now be reduced to:

$$\alpha^{2n+2} + \cos n\pi - i\sin n\pi = \alpha^{2n+1} - \alpha\cos n\pi + \alpha i\sin n\pi + \alpha^{2n} + \alpha^2\cos n\pi - \alpha^2 i\sin n\pi\,,$$

or:

$$(\alpha^2 - \alpha - 1)(\alpha^{2n} - \cos n\pi + i\sin n\pi) = 0\,,$$

which is a proper identity, since $\alpha^2 - \alpha - 1 = 0$.

The numbers, introduced by definition (2) also satisfy identically the relation $u_m u_n + u_{m+1}u_{n+1} = u_{m+n+1}$, which is well known for the ordinary Fibonacci numbers. The truth of this assertion can also be verified without too much difficulty.

Furthermore we can show that for the moduli of the complex numbers the relation $|u_{-n}| = |u_n|$ is valid, just as for the real numbers. For $x_{-n}^2 + y_{-n}^2 = x_n^2 + y_n^2$ is equivalent to

$$\left(\frac{\alpha^{-2n} - \cos n\pi}{(\alpha-\beta)\alpha^{-n}}\right)^2 + \left(\frac{\sin n\pi}{(\alpha-\beta)\alpha^{-n}}\right)^2 = \left(\frac{\alpha^{2n} - \cos n\pi}{(\alpha-\beta)\alpha^{n}}\right)^2 + \left(\frac{\sin n\pi}{(\alpha-\beta)\alpha^{n}}\right)^2\,,$$

and this in its turn is identical to:

$$\frac{\alpha^{-4n} - 2\alpha^{-2n}\cos n\pi + 1}{(\alpha-\beta)^2\alpha^{-2n}} = \frac{\alpha^{4n} - 2\alpha^{2n}\cos n\pi + 1}{(\alpha-\beta)^2\alpha^{2n}}\,,$$

or:

$$\alpha^{-2n} - 2\cos n\pi + \alpha^{2n} = \alpha^{2n} - 2\cos n\pi + \alpha^{-2n} \quad \text{q.e.d.}$$

We now calculate the numerical values of u_n, for n climbing from -4 to $+4$, with intervals of 1/6 as shown in Table 1.

If we take a close look at these numbers, we find that

$$u_{1/2} = iu_{-1/2} = 0.569 + 0.352i\,,$$

$$u_{-1\frac{1}{2}} = iu_{1\frac{1}{2}} = 0.217 + 0.921i\,,$$

$$u_{2\frac{1}{2}} = iu_{-2\frac{1}{2}} = 1.489 + 0.134i\,,$$

etc., etc.

It is simple to prove this property from definition (2), and it is clear that it corresponds with $|u_{-n}| = |u_n|$.

If we make a map of the newly introduced numbers in the complex plane, we get the interesting picture shown in Fig. 1. The curve that we have thus found intersects the x-axis in those real points corresponding with the well-known Fibonacci numbers for $n \in Z$.

For decreasing negative values of n it has the shape of a spiral, and for increasing positive values of n it has the shape of a "sinus-like" curve, with increasing "wave-length" and decreasing "amplitude."

Note how the relation $|u_{-n}| = |u_n|$ is made visible through this graphical representation of u_n.

On differentiating,

$$x_n = \frac{a^{2n} - \cos n\pi}{(a-\beta)a^n}, \qquad y_n = \frac{\sin n\pi}{(a-\beta)a^n}$$

with n as independent variable, we find:

$$\frac{dx_n}{dn} = \frac{\ln a(a^{2n} + \cos n\pi) + \pi \sin n\pi}{(a-\beta)a^n},$$

$$\frac{dy_n}{dn} = \frac{\pi \cos n\pi - \ln a \sin n\pi}{(a-\beta)a^n},$$

so that

$$\frac{dy_n}{dx_n} = \frac{\pi \cos n\pi - \ln a \sin n\pi}{\ln a(a^{2n} + \cos n\pi) + \pi \sin n\pi}.$$

For instance:

$$\frac{dy}{dx_{n=0}} = \frac{\pi}{2\ln a} = \frac{\pi \log e}{2\log a} = \frac{3.1416 \times 0.4343}{2 \times 0.2090} = 3.264.$$

$$\frac{dy}{dx_{n=1}} = -\frac{\pi}{a\ln a} = -\frac{\pi \log e}{a \log a} = -\frac{3.1416 \times 0.4343}{1.618 \times 0.2090} = -4.035.$$

$$\frac{dy}{dx_{n=-1}} = \frac{\pi a}{\ln a} = \frac{\pi a \log e}{\log a} = \frac{3.1416 \times 1.618 \times 0.4343}{0.2090} = 10.56$$

etc., etc.

Among the points in which the curve intersects itself, there is one with $y \neq 0$, a complex number z, so that $z \in C$ but $z \notin R$. With the extension we now have achieved, we can make a similar extension for all Fibonacci-like sequences

If we start with any two complex numbers, say z_1 and z_2, adding them to find the following number we get

$$z_1,\ z_2,\ z_1+z_2,\ z_1+2z_2,\ 2z_1+3z_2,\ 3z_1+5z_2,\ 5z_1+8z_2,\ 8z_1+13z_2,$$

etc., etc. The coefficients are Fibonacci numbers.

To find the extension of this sequence, all we have to do is to apply the extension to the coefficients.

In this manner we will now study the sequence that appears when we start with $z_1 = 1, z_2 = i$. Then we have:

$$1,\ i,\ 1+i,\ 1+2i,\ 2+3i,\ 3+5i,\ 5+8i,$$

etc. It is clear that we can start by extension "to the left," to find:

$$\cdots,\ 5-3i,\ -3+2i,\ 2-i,\ -1+i,\ 1,\ i,\ 1+i,\ 1+2i,\ 2+3i,\ 3+5i,\ \cdots.$$

For reasons of symmetry we shall refer to these terms as v_k, in such a way that $v_{-\frac{1}{2}} = 1$ and

$$v_{+\frac{1}{2}} = i,\ v_{+1\frac{1}{2}} = 1+i,\ v_{+2\frac{1}{2}} = 1+2i,\ v_{+3\frac{1}{2}} = 2+3i, \cdots$$

$$v_{-1\frac{1}{2}} = i-1,\ v_{-2\frac{1}{2}} = 2-i,\ v_{-3\frac{1}{2}} = -3+2i, \cdots.$$

The relation between the v-sequence and the u-sequence is: $v_k = u_{k-\frac{1}{2}} + u_{k+\frac{1}{2}}i$. Therefore:

Table 1

$6n$	$u_n = x_n + i\,y_n$	$6n$	$u_n = x_n + i\,y_n$
	⋮		
−24	−3.000 + 0.000 i	0	0.000 + 0.000 i
−23	−2.380 + 1.415 i	+1	+0.127 + 0.206 i
−22	−1.229 + 2.261 i	+2	+0.335 + 0.330 i
−21	+ 0.083 + 2.410 i	+3	+0.569 + 0.352 i
−20	+ 1.203 + 1.926 i	+4	+0.779 + 0.281 i
−19	+ 1.875 + 1.026 i	+5	+0.927 + 0.150 i
−18	+ 2.000 + 0.000 i	+6	+1.000 + 0.000 i
−17	+ 1.629 + 0.874 i	+7	+1.005 − 0.128 i
−16	+ 0.931 − 1.398 i	+8	+0.967 - 0.204 i
−15	+ 0.134 - 1.489 i	+9	+0.920 - 0.217 i
−14	−0.542 - 1.190 i	+10	+0.897 - 0.174 i
−13	−0.941 - 0.634 i	+11	+0.920 - 0.093 i
−12	−1.000 + 0.000 i	+12	+1.000 + 0.000 i
−11	−0.751 + 0.540 i	+13	+1.132 + 0.079 i
−10	−0.298 + 0.864 i	+14	+1.302 + 0.126 i
−9	+ 0.217 + 0.921 i	+15	+1.489 + 0.134 i
−8	+ 0.661 + 0.736 i	+16	+1.676 + 0.107 i
−7	+ 0.934 + 0.392 i	+17	+1.848 + 0.057 i
−6	+ 1.000 + 0.000 i	+18	+2.000 + 0.000 i
−5	+ 0.878 − 0.334 i	+19	+2.137 − 0.049 i
−4	+ 0.633 − 0.534 i	+20	+2.269 − 0.078 i
−3	+ 0.352 − 0.569 i	+21	+2.410 − 0.083 i
−2	+ 0.118 − 0.455 i	+22	+2.573 − 0.066 i
−1	−0.007 − 0.242 i	+23	+2.768 − 0.035 i
0	0.000 + 0.000 i	+24	+3.000 + 0.000 i
			⋮

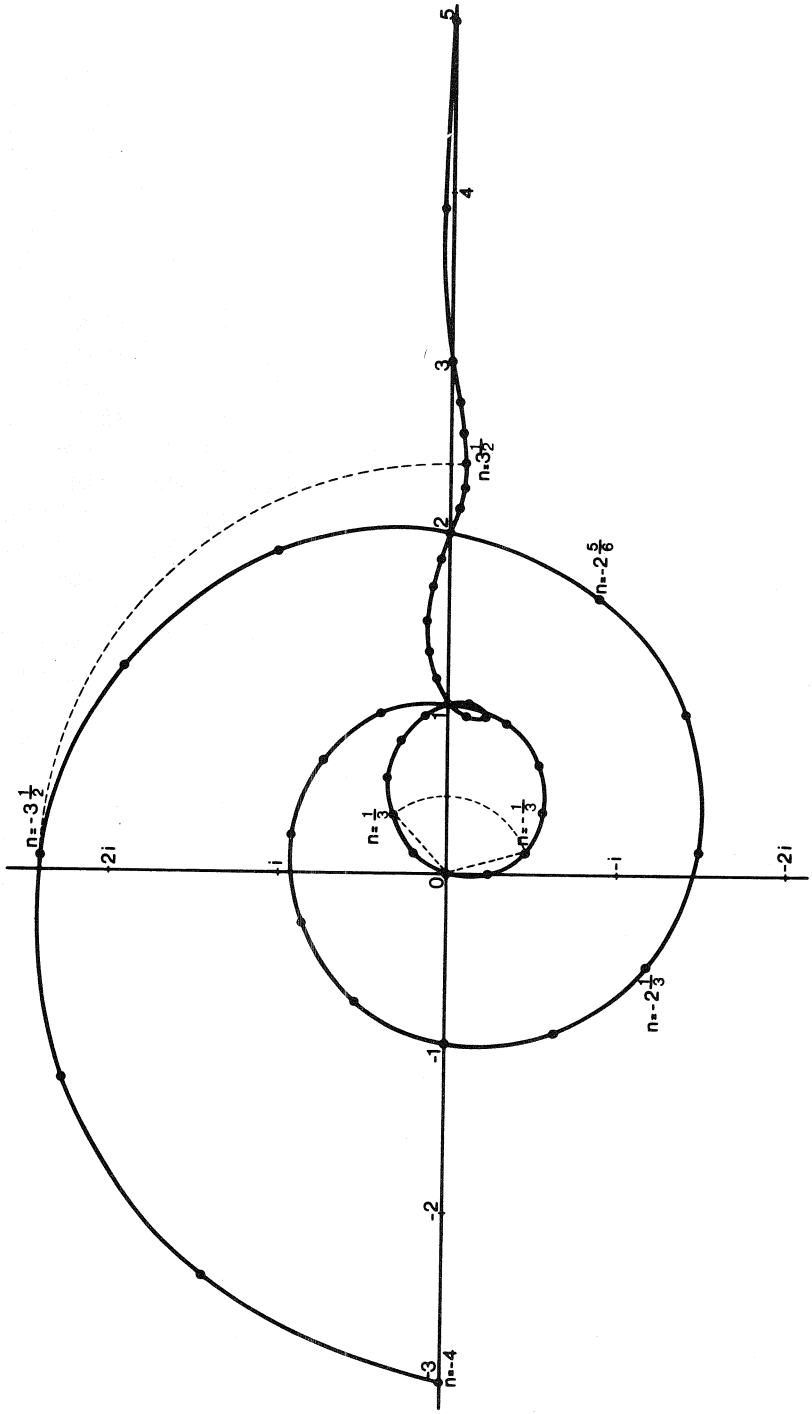

Fig. 1 Graphic Representation of the Complex Numbers of the Extended Fibonacci Sequence, According to Definition (2) for $-4 \leqslant n \leqslant 5$

$$v_k = u_{k-½} + u_{k+½}i = (x_{k-½} + iy_{k-½}) + (x_{k+½} + iy_{k+½})i = (x_{k-½} - y_{k+½}) + i(y_{k-½} + x_{k+½}).$$

We shall now demonstrate that $|v_{-k}| = |v_k|$.

$$|v_k|^2 = (x_{k-½} - y_{k+½})^2 + (y_{k-½} + x_{k+½})^2 = (x_{k-½}^2 + y_{k-½}^2) + (x_{k+½}^2 + y_{k+½}^2) - 2(x_{k-½}y_{k+½} - y_{k-½}x_{k+½}).$$

We can now say that:

$$|v_k|^2 = |u_{k-½}|^2 + |u_{k+½}|^2 - 2(x_{k-½}y_{k+½} - y_{k-½}x_{k+½}).$$

Therefore:

$$|v_{-k}|^2 = |u_{-k-½}|^2 + |u_{-k+½}|^2 - 2(x_{-k-½}y_{-k+½} - y_{-k-½}x_{-k+½}) = |u_{k+½}|^2 + |u_{k-½}|^2 - 2(x_{-k-½}y_{-k+½} - y_{-k-½}x_{-k+½}).$$

so that the relation that we want to prove, namely $|v_{-k}| = |v_k|$, or $|v_{-k}|^2 = |v_k|^2$, is equivalent to

$$x_{k-½}y_{k+½} - y_{k-½}x_{k+½} = x_{-k-½}y_{-k+½} - y_{-k-½}x_{-k+½}.$$

When we now proceed to introduce the index t by means of $k = t + ½$; $-k = -t - ½$, we have to prove that:

$$x_t y_{t+1} - y_t x_{t+1} = x_{-t-1}y_{-t} - y_{-t-1}x_{-t}.$$

Or:

$$\frac{a^{2t} - \cos t\pi}{(a-\beta)a^t} \times \frac{\sin(t+1)\pi}{(a-\beta)a^{t+1}} - \frac{\sin t\pi}{(a-\beta)a^t} \times \frac{a^{2(t+1)} - \cos(t+1)\pi}{(a-\beta)a^{t+1}}$$

$$= \frac{a^{2(-t-1)} - \cos(-t-1)\pi}{(a-\beta)a^{-t-1}} \times \frac{\sin(-t)\pi}{(a-\beta)a^{-t}} - \frac{\sin(-t-1)\pi}{(a-\beta)a^{-t-1}} \times \frac{a^{2(-t)} - \cos(-t)\pi}{(a-\beta)a^{-t}}.$$

This is an identity, if completely worked out.

We have already seen that if $v_k = a_k + ib_k$, then $a_k = x_{k-½} - y_{k+½}$ and $b_k = y_{k-½} + x_{k+½}$. Thus:

$$a_k = x_{k-½} - y_{k+½} = \frac{a^{2k-1} - \cos(k\pi - ½\pi)}{a^{k-½}(a-\beta)} - \frac{\sin(k\pi + ½\pi)}{(a-\beta)a^{k+½}}$$

Or:

$$a_k = \frac{a^{2k} - a\sin k\pi - \cos k\pi}{(a-\beta)a^{k+½}}.$$

In the same way we derive from $b_k = y_{k-½} + x_{k+½}$:

$$b_k = \frac{a^{2k+1} - a\cos k\pi + \sin k\pi}{(a-\beta)a^{k+½}}.$$

It is now fairly easy to calculate some values of v_k, simply by choosing different values of k; we find

$$v_{½} = i, \quad v_{1½} = 1 + i, \quad v_{2½} = 1 + 2i,$$

as it should be. We also have:

$$v_1 = \frac{1}{\sqrt{a}} + i\sqrt{a}, \quad v_{-1} = \frac{1}{\sqrt{a}} + i\sqrt{a},$$

(so that $v_{-1} = v_1$), and $v_0 = 0$. Also

$$v_2 = \frac{1}{\sqrt{a}} + i\sqrt{a} \ (= v_{-1} = v_1) \text{ and } v_{-2} = -\frac{1}{\sqrt{a}} - i\sqrt{a};\ v_3 = \frac{2}{\sqrt{a}} + 2i\sqrt{a} \text{ and } v_{-3} = \frac{2}{\sqrt{a}} + 2i\sqrt{a},\ v_4 = \frac{3}{\sqrt{a}} + 3i\sqrt{a}.$$

It now seems very likely that

$$v_k = \left(\frac{1}{\sqrt{a}} + i\sqrt{a}\right) u_k,$$

for all values of k. Indeed we have:

$$(a^{-½} + ia^{½}) \times u_k = (a^{-½} + ia^{½})(x_k + iy_k) = (a^{-½}x_k - a^{½}y_k) + i(a^{-½}y_k + a^{½}x_k),$$

whereas

$$a^{-1/2}x_k - a^{1/2}y_k = \frac{a^{-1/2}(a^{2k} - \cos k\pi)}{(a-\beta)a^k} - \frac{a^{1/2}\sin k\pi}{(a-\beta)a^k} = \frac{a^{2k} - \cos k\pi - a\sin k\pi}{(a-\beta)a^{k+1/2}} = a_k,$$

and in the same way we prove that $a^{-1/2}y_k + a^{1/2}x_k = b_k$, so that $(a^{-1/2} + ia^{1/2})u_k = a_k + ib_k = v_k$, which had to be proved. The relation

$$v_k = \left(\frac{1}{\sqrt{a}} + i\sqrt{a}\right) u_k$$

implies that the graphic representation of the numbers v_k in the complex plane has the same shape as the one that we have found previously for u_k:

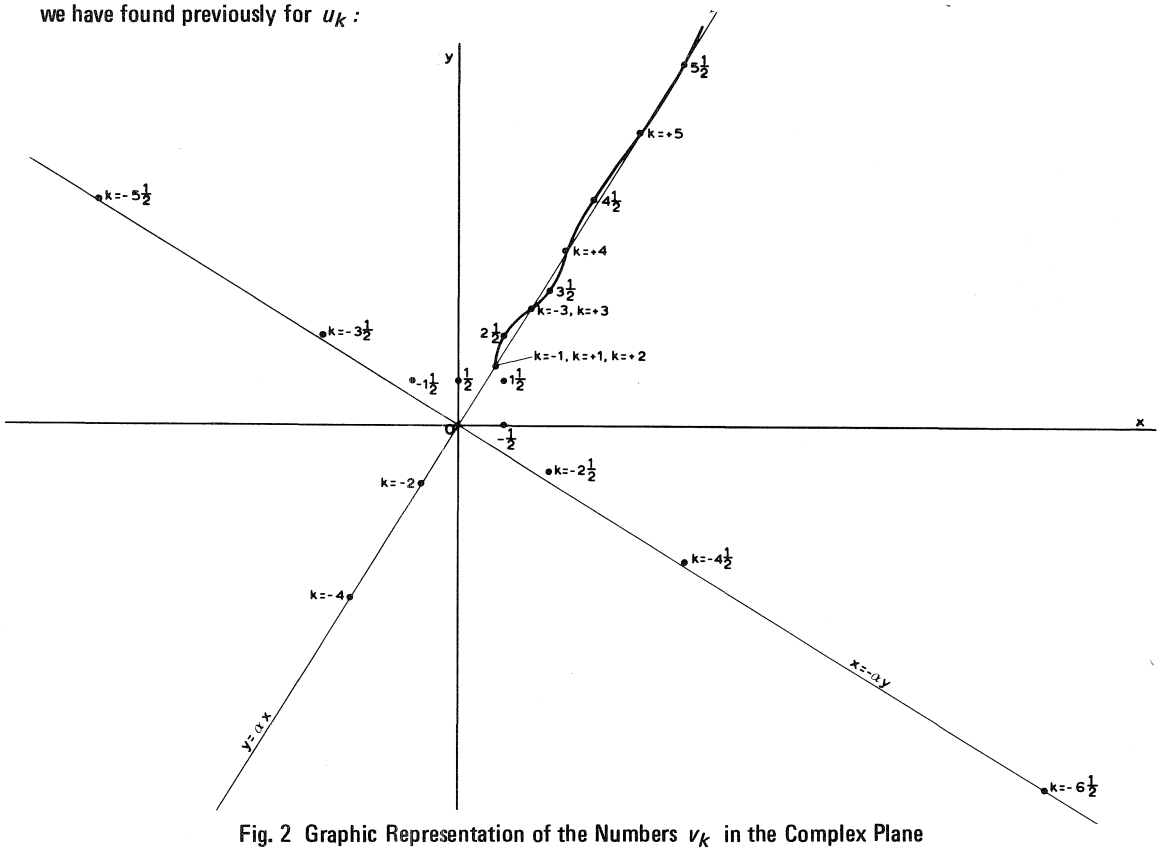

Fig. 2 Graphic Representation of the Numbers v_k in the Complex Plane

There is one continuous curve going through all these points, a curve that originates from the one in Fig. 1 by multiplication with

$$\frac{1}{\sqrt{a}} + i\sqrt{a}\ .$$

It is clearly shown how the points *(0,1); (1,1); (1,2); (2,3); (3,5); (5,8); (8,13);* ··· belonging to the index-values *½, 1½, 2½, 3½, 4½, 5½, 6½,* ··· of k are lying closer to the asymptote $y = ax$ as k increases, thus indicating that

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = a\ .$$

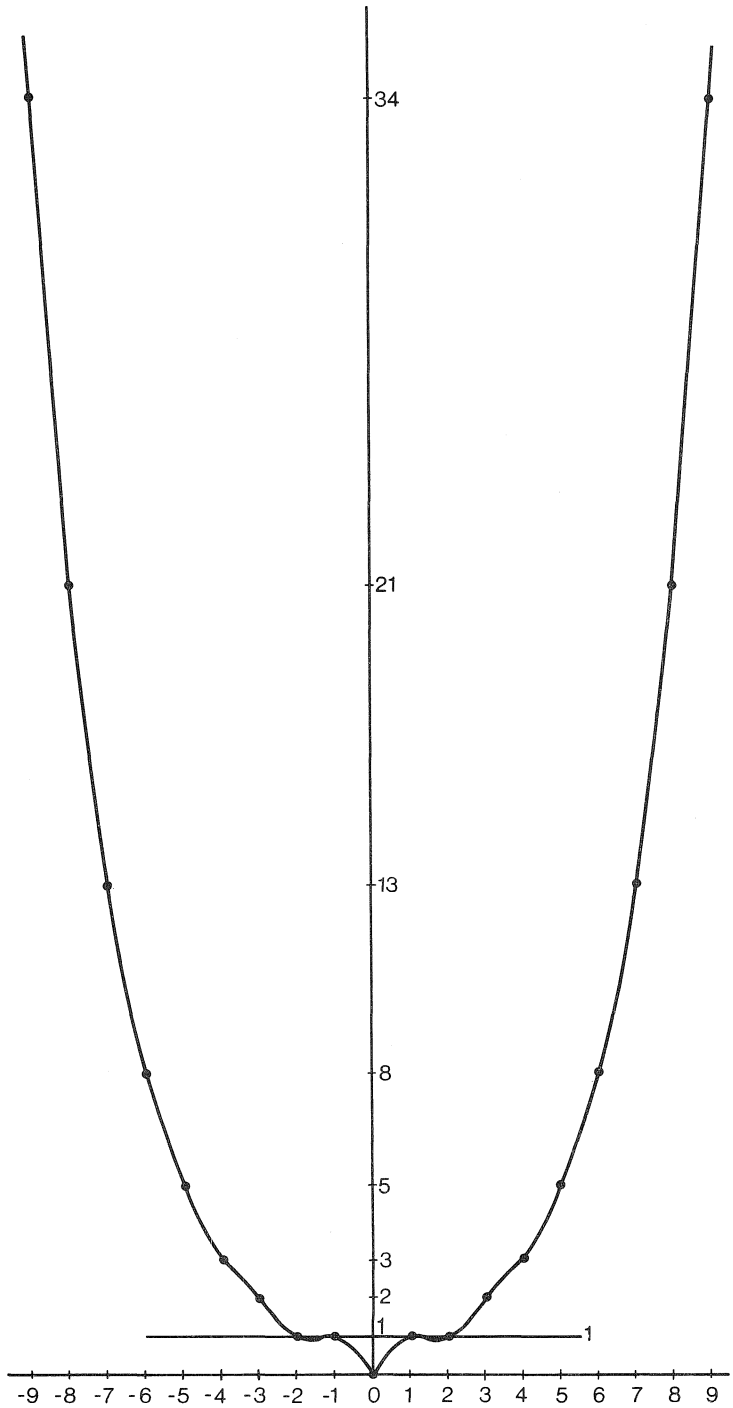

Fig. 3 Graph of $|u_n|$ as a function of n

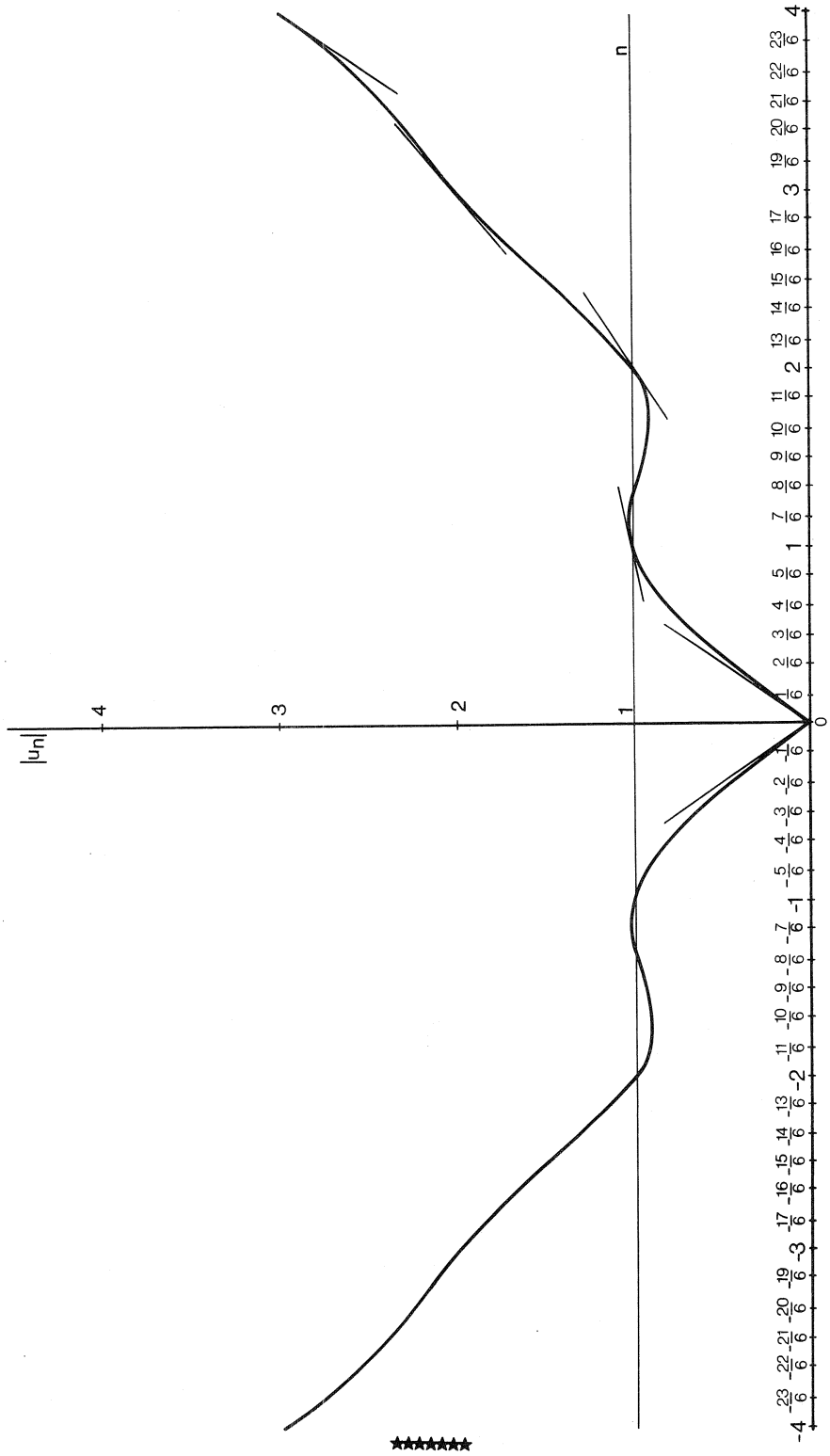

Fig. 4 Fig. 3 Enlarged to Show the Behavior of $|u_n|$ for $n = 0$.

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SPANNING TREES AND FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The Fibonacci numbers F_n are defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geqslant 1),$$

and the Lucas numbers L_n by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geqslant 1).$$

We shall use the graph theoretic terminology of Harary [2]. A *wheel* on $n+1$ points is obtained from a cycle on n points by joining each of these n points to a further point. This cycle is known as the *rim* of the wheel, the other edges are the *spokes*, and the further point is the *hub*. A *fan* is what is obtained when one edge is removed from the rim of a wheel. We also refer to the rim and the spokes of a fan, but use the word *pivot* instead of hub. We give now an illustration of a labelled wheel and a labelled fan on 9 points.

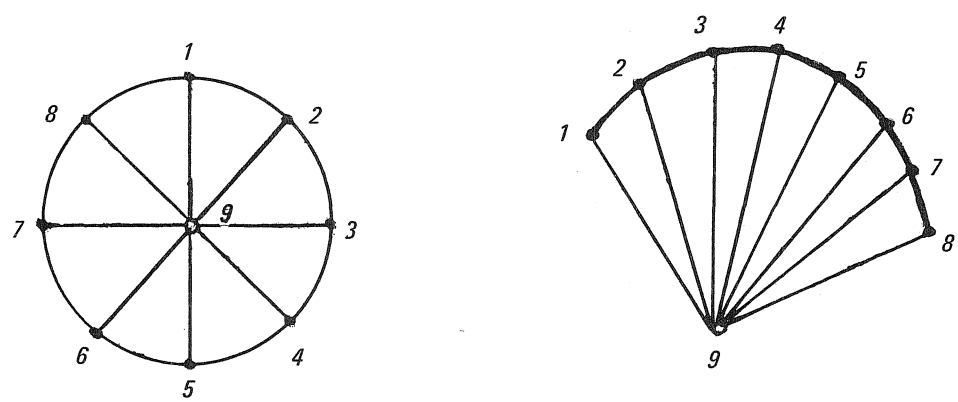

Figure 1

A *composition* of the positive integer n is a vector $(a_1, a_2, \cdots, a_k)$ whose components are positive integers such that $a_1 + a_2 + \cdots + a_k = n$. If the vector has order k then the composition is a k-part composition.

For $n \geqslant 2$ the number of spanning trees of a labelled wheel on $n+1$ points is $L_{2n} - 2$, and the number of spanning trees of a labelled fan on $n+1$ points is F_{2n}. References concerning the first of these results may be found in [3]; both results are proved simply in [4].

In this paper, by simple new combinatorial arguments, we derive both old and new formulae for the Fibonacci and Lucas numbers.

2. A SIMPLE COMBINATORIAL PROOF THAT $F_{2n+2m} = F_{2n+1}F_{2m} + F_{2n}F_{2m-1}$

Let the number of spanning trees of a labelled fan on $n+1$ points be f_n, and the number of those spanning trees

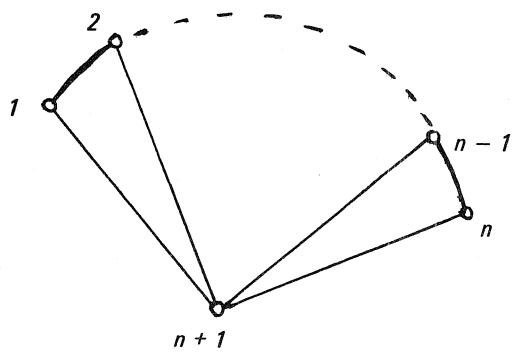

Figure 2

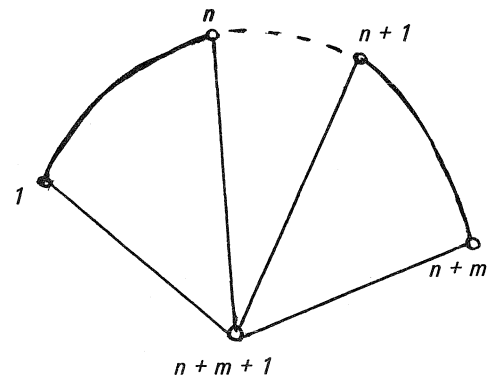

Figure 3

of a labelled fan on $n+1$ points which include a specified leading edge ($\{1, n+1\}$ in Fig. 2) be e_n. Clearly

(1) $$e_{n+1} = e_n + f_n \qquad (n \geqslant 1).$$

Now consider a fan on $n+m+1$ points. This may be thought of as two fans A and B, connected at the pivot and at two points labelled n and $n+1$ as indicated in Fig. 3. Then

(2) $$f_{n+m} = f_n f_m + f_n e_m + e_n f_m \qquad (n, m \geqslant 1)$$

so

(3) $$f_{n+m} = e_{n+1} f_m + f_n e_m \qquad (n, m \geqslant 1)$$

by (1). In formula (2) $f_n f_m$ is the number of those spanning trees which do not include $\{n, n+1\}$. The restrictions of a spanning tree which includes $\{n, n+1\}$ to A and to B are either a spanning tree of A and a spanning subgraph of B consisting of two trees, one including $\{n+1\}$, the other including $\{n+m+1\}$, or are a spanning tree of B and a spanning subgraph of A consisting of two trees, one including $\{n\}$, the other including $\{n+m+1\}$. Therefore, the number of spanning trees which include $\{n, n+1\}$ is $f_n e_m + e_n f_m$. But $f_n = F_{2n}$, and it is shown in [4] that $e_n = F_{2n-1}$. Therefore, from (3),

$$F_{2n+2m} = F_{2n+1} F_{2m} + F_{2n} F_{2m-1} \qquad (n, m \geqslant 1).$$

The corresponding formula for L_{2n+2m} does not appear to come through so readily from this type of argument.

3. COMPOSITION FORMULAE FOR F_{2n}

If $(a_1, \cdots, a_k)$ is a composition of n, then the number of spanning trees of the fan in Fig. 2 which exclude

$$\{a_1, a_1+1\}, \{a_1+a_2, a_1+a_2+1\}, \cdots \{a_1+\cdots+a_{k-1}, a_1+\cdots+a_{k-1}+1\}$$

but include all other edges of the rim is $a_1 a_2 \cdots a_k$, for this is the number of different combinations of spokes which such a spanning three may include. Therefore

(4) $$F_{2n} = \sum_{\gamma(n)} a_1 a_2 \cdots a_k ,$$

where $\gamma(n)$ indicates summation over all compositions $(a_1 \cdots, a_k)$ of n, the number of components being variable. This formula is due to Moser and Whitney [6].

Hoggatt and Lind [5] have shown that this formula may be inverted to give

$$-n = \sum_{\gamma(n)} (-1)^k F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

This may be demonstrated combinatorially as follows. The number of spanning trees of the fan in Fig. 2 which do not have any rim edges missing is n. The total number of spanning trees is F_{2n}. For a given composition $(a_1, \cdots, a_k)$ of n with $k \geqslant 2$, the number of spanning trees which do not contain the edges $\{a_1, a_1 + 1\}$,

$$\{a_1 + a_2, a_1 + a_2 + 1\}, \cdots, \{a_1 + \cdots + a_{k-1}, a_1 + \cdots + a_{k-1} + 1\}$$

is $F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$. Therefore, by the Principle of Inclusion and Exclusion (see Riordan [7], Chapter 3)

$$n = \sum_{\gamma(n)} (-1)^{k-1} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

Of course it now follows that

$$F_{2n} = n + \sum_{k=2}^{n} \sum_{\gamma_k(n)} (-1)^k F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} , \tag{5}$$

where $\gamma_k(n)$ denotes summation over all k-part compositions of n.

4. COMPOSITION FORMULAE FOR $L_{2n} - 2$.

The formulae in this section are analogous to the formulae (4) and (5) of the previous section. The main difference is that the formulae in this section are obtained from the wheel in Fig. 4, whereas in the last section they were obtained from the fan in Fig. 2.

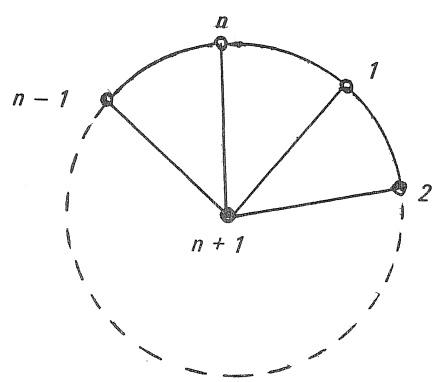

Figure 4

If $(a_1, \cdots, a_k)$ is a composition of n, and j is an integer, $0 \leqslant j < n$, then the number of spanning trees of the wheel in Fig. 4 which exclude the edges

$$\{a_1 + j, a_1 + j + 1\}, \{a_1 + a_2 + j, a_1 + a_2 + j + 1\}, \cdots, \{a_1 + \cdots + a_k + j, a_1 + \cdots + a_k + j + 1\}$$

[the integers here being taken modulo n], but include all the remaining edges in the rim, is $a_1 a_2 \cdots a_n$. If we sum over all such compositions into k parts and all possible values of j, we obtain

$$n \sum_{\gamma_k(n)} a_1 a_2 \cdots a_k .$$

But this sum counts each spanning tree which has exactly k specified edges on the rim excluded, precisely k times. Therefore the number of spanning trees which exclude exactly k edges of the rim is

$$\frac{n}{k} \sum_{\gamma_k(n)} a_1 a_2 \cdots a_n .$$

Therefore

$$L_{2n} - 2 = n \sum_{k=1}^{n} \frac{1}{k} \sum_{\gamma_k(n)} a_1 a_2 \cdots a_k .$$

i.e.,

$$L_{2n} - 2 = \sum_{\gamma(n)} \frac{n a_1 a_2 \cdots a_k}{k} ,$$

a formula which is analogous to (4).

We now find a formula for $L_{2n} - 2$ which is analogous to (5). The number of spanning trees of a wheel which do not have any rim edges missing is 0. The total number of spanning trees of a wheel is $L_{2n} - 2$. For a given composition $(a_1, a_2, \cdots, a_k)$ of n, and a given integer j, $0 \leq j < n$, the number of spanning trees which do not contain the edges

$$\left\{ a_1 + j, a_1 + j + 1 \right\}, \left\{ a_1 + a_2 + j, a_1 + a_2 + j + 1 \right\}, \cdots, \left\{ a_1 + \cdots + a_k + j, \ a_1 + \cdots + a_k + j + 1 \right\}$$

is $F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$. By a similar argument to that just used above, the sum

$$\frac{n}{k} \sum_{\gamma_k(n)} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$$

is the sum taken over all combinations of k edges from the rim of the number of spanning trees which do not contain any of the k rim edges of the combination. Therefore, by the Principle of Inclusion and Exclusion

$$0 = L_{2n} - 2 + \sum_{k=1}^{n} (-1)^k \frac{n}{k} \sum_{\gamma_k(n)} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

Therefore

$$L_{2n} - 2 = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} ,$$

a formula which is analogous to (5).

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ON POLYNOMIALS RELATED TO TCHEBICHEF POLYNOMIALS OF THE SECOND KIND

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1. Tchebichef polynomials of the second kind have been defined by

$$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x)\ ,$$
$$U_0 = 1, \quad U_1 = 2x\ .$$

It is known [1] that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\ ,$$

and

$$U_n(x) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} (-1)^r (2x)^{n-2r}\ .$$

Also [2]

$$F_{n+1} = i^{-n}\, U_n(i/2)\ ,$$

where F_n represents the nth Fibonacci number.

The first few polynomials are

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \\ U_4(x) &= 16x^4 - 12x^2 + 1\ . \end{aligned}$$

Figure 1

If we take the sums along the rising diagonals in the expression on the right-hand side, we obtain an interesting polynomial $p_n(x)$, which is closely related to Fibonacci numbers.

The first few polynomials are

$$\begin{aligned} &p_1(x) = 1, \qquad p_2(x) = 2x, \qquad p_3(x) = 4x^2\ , \\ &p_4(x) = 8x^3 - 1, \qquad p_5(x) = 16x^4 - 4x\ . \end{aligned} \tag{1.1}$$

In this note we shall derive the generating function, recurrence relation and a few interesting properties of these polynomials.

2. On putting $2x = y$ in the expansion on the right-hand side in Figure 1 we obtain

		Columns 0	1	2	3
Rows	1	1			
	2	y			
	3	y^2	-1		
	4	y^3	$-2y$		
	5	y^4	$-3y^2$	1	
		$\vdots$			

Figure 2

The generating function for the kth column in Figure 2 is $(-1)^k(1-ty)^{-(k+1)}$. Since we are summing along the rising diagonals, the row adjusted generating function for the kth column becomes

$$h_k(y) \equiv (-1)^k(1-ty)^{-(k+1)}t^{3k+1} .$$

Since

$$\sum_{k=0}^{\infty} h_k(y) = \frac{1}{1-ty}\sum_{k=0}^{\infty}\left(\frac{-t^3}{1-ty}\right)^k$$

$$= \frac{t}{1-ty+t^3} ,$$

we have

$$G(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n = \frac{t}{1-2xt+t^3} . \tag{2.1}$$

From (2.1) we obtain

$$\sum_{n=1}^{\infty} p_n(x)t^n = t(1-2xt+t^3)^{-1}$$

On expanding the right-hand side and comparing the coefficients of t^{n+1}, we obtain

$$p_{n+1}(x) = (2x)^n - \binom{n-2}{1}(2x)^{n-3} + \binom{n-4}{2}(2x)^{n-6} + \dots = \sum_{r=0}^{[n/3]}\binom{n-2r}{r}(-1)^r(2x)^{n-3r} . \tag{2.2}$$

Again from (2.1) we have

$$(1-2xt+t^3)\sum_{n=1}^{\infty} p_n(x)t^n = t .$$

On equating coefficient of t^{n+3} on both sides, we obtain the recurrence relation

$$p_{n+3}(x) = 2xp_{n+2}(x) - p_n(x), \quad n > 1, \quad p_1(x) = 1, \quad p_2(x) = 2x, \quad p_3(x) = 4x^2 . \tag{2.3}$$

Extending (2.3) we find that $p_0(x) = 0$.

From (2.1) we have

$$G(x,t) = tF(2xt-t^3), \quad F(u) = (1-u)^{-1} . \tag{2.4}$$

Differentiating (2.4) partially with respect to x and t, we find that $G(x,t)$ satisfies the partial differential equation

$$2t\frac{\partial G}{\partial t} - (2x-3t^2)\frac{\partial G}{\partial x} - 2G = 0 .$$

Since

$$\frac{\partial G}{\partial t} = \sum_{n=1}^{\infty} np_n(x)t^{n-1}, \quad \frac{\partial G}{\partial x} = \sum_{n=1}^{\infty} p_n'(x)t^n ,$$

it follows that

$$2xp_{n+2}'(x) - 3p_n'(x) = 2(n+1)p_{n+2}(x) . \tag{2.5}$$

3. On substituting $x = 1$ in the polynomials $p_n(x)$, we obtain the sequence $\{P_n\}$ which has a recurrence relation

$$P_{n+2} = P_{n+1} + P_n + 1, \quad P_0 = 0. \quad P_1 = 1. \tag{3.1}$$

The sequence $\{P_n\}$ is related to the Fibonacci sequence $\{F_n\}$ by the relation

$$P_n - P_{n-1} = F_n ,$$

which leads to

$$P_n = \sum_{k=0}^{n} F_k . \tag{3.4}$$

From (3.4) several interesting properties of the sequence $\{P_n\}$ can be derived. A few of them are

(3.5)

(1) $$P_n = F_{n+2} - 1$$

(2) $$\sum_{k=1}^{n} P_k = F_{n+4} - (n+3)$$

(3) $$\sum_{k=1}^{n} P_k^2 = F_{n+2}F_{n+3} - 2F_{n+4} + (n+4)$$

(4) with $$\prod_{i=1}^{n} (1 + x^{L_i}) = a_0 a_1 x + \cdots + a_m x^m, \quad m = L_1 + L_2 + \cdots + L_n .$$

and q_n equal to the number of integers k such that both $0 < k < m$ and $a_k = 0$, Leonard [3] has proposed a problem to find a recurrence relation for q_n. The author [4] has shown that the recurrence relation is

$$q_{n+2} = q_{n+1} + q_n + 1, \quad q_1 = 0, \quad q_2 = 1 .$$

Comparing this result with (3.1) we observe that

$$P_n = q_{n+1} .$$

On using (3.5)–(1) and (2.2) we obtain

$$F_{n+3} = 1 + \sum_{r=0}^{[n/3]} \binom{n-2r}{r} (-1)^r 2^{n-3r}, \quad n \geqslant 0 , \tag{3.6}$$

a result which is believed to be undiscovered so far.

I am grateful to Dr. V. M. Bhise, G.S. Technological Institute, for his help and guidance in the preparation of this paper.

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CORRIGENDUM TO: ENUMERATION OF TWO-LINE ARRAYS

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The proof of (2.5) and (2.7) in the paper: "Enumeration of Two-Line Arrays" [1] is incorrect as it stands. A corrected proof follows.

Let $g(n,k)$ denote the number of two-line arrays of positive integers

$$\boxed{\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{matrix}}$$

satisfying the inequalities

$$\begin{aligned} &\max(a_i,b_i) \leqslant \min(a_{i+1}, b_{i+1}) && (1 \leqslant i < n), \\ &\max(a_i,b_i) \leqslant i && (1 \leqslant i < n) \end{aligned}$$

and

$$\max(a_n, b_n) = k.$$

We wish to show that

$$g(n+k,k) = \sum_{j=1}^{k} g(j,j)g(n+k-j, k-j+1) \qquad (n \geqslant 1). \tag{2.7}$$

Let j be the greatest integer $\leqslant k$ such that

$$\max(a_j,b_j) = j.$$

It follows that $a_{j+1} = b_{j+1} = j$.

Consider the array

$$\boxed{\begin{matrix} a_1 & \cdots & j \\ 1 & \cdots & \cdot \end{matrix} \;\Big|\; \begin{matrix} j & \cdots & k \\ j & \cdots & \cdot \end{matrix}}$$

Put

$$\begin{aligned} a_i' &= a_{j+i} - (j-1) \\ b_i' &= b_{j+i} - (j-1) \end{aligned} \qquad (1 \leqslant i \leqslant n+k-j).$$

It follows from the conditions satisfied by a_i, b_i that

$$\begin{aligned} &\max(a_i', b_i') \leqslant \min(a_{i+1}', b_{i+1}') && (1 \leqslant i < n+k-j), \\ &\max(a_i', b_i') \leqslant i && (1 \leqslant i \leqslant n+k-j), \\ &\max(a_{n+k-j}', b_{n+k-j}') = k-j+1. \end{aligned}$$

This evidently yields (2.7).

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ORESME NUMBERS

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1. INTRODUCTION

The purpose of this article is to make known some properties of an interesting sequence of numbers which I believe has not received much (if any) attention.

In the mid-fourteenth century, the scholar and cleric, Nicole Oresme, found the sum of the sequence of rational numbers

(1) $$\frac{1}{2},\ \frac{2}{4},\ \frac{3}{8},\ \frac{4}{16},\ \frac{5}{32},\ \frac{6}{64},\ \frac{7}{128},\ \frac{8}{256},\ \cdots\ .$$

Unfortunately, Oresme's original calculations were not published.

Such a sequence is of considerable biological interest. As Hogben [3] remarks: "...what is of importance to the biologist is an answer to the question: if we know the first two terms, *i.e.*, the proportion of grandparents and parents of different genotypes, how do we calculate the proportions in any later generations?"

2. ORESME NUMBERS

The sequence (1) of Oresme can be extended "to the left" to include negative numbers if we see the pattern of the sequence, which is easily discernible. More is gained by recognizing the sequence (1) as a special case of a general sequence discussed by Horadam [4], [5] and [6].

This general sequence $\{w_n(a, b; p, q)\}$ is defined by

(2) $$w_{n+2} = pw_{n+1} - qw_n\ ,$$

where

(3) $$w_0 = a, \quad w_1 = b$$

and p, q are arbitrary integers at our disposal. To achieve our purpose, we now extend the values of p, q to be arbitrary rational numbers.

Taking $a = 0$, $b = 1$, $p = 1$, $q = \frac{1}{4}$, and denoting a term of the special sequence by O_n $(n = \cdots, -2, -1, 0, 1, 2, \cdots)$, we write the sequence $\{O_n\} \equiv \{w_n(0, \frac{1}{2}, 1, \frac{1}{4})\}$ as

(4) $$\begin{array}{ccccccccccccccccc} \cdots & O_{-7} & O_{-6} & O_{-5} & O_{-4} & O_{-3} & O_{-2} & O_{-1} & O_0 & O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & \cdots \\ \cdots & -896 & -384 & -160 & -64 & -24 & -8 & -2 & 0 & \frac{1}{2} & \frac{2}{4} & \frac{3}{8} & \frac{4}{16} & \frac{5}{32} & \frac{6}{64} & \frac{7}{128} & \cdots \end{array}$$

The extension (4) of the original sequence (1) studied by Oresme we will call the *Oresme sequence.* Terms of this sequence are called *Oresme numbers.* Thus, Oresme numbers are, by (2), (3), (4), given by the second-order relation

(5) $$O_{n+2} = O_{n+1} - \frac{1}{4}O_n$$

with

(6) $$O_0 = 0\ ,\ O_1 = O_2 = \frac{1}{2}\ .$$

An interesting feature of the Oresme sequence is that it is a degenerate case of $\{w_n\}$ occurring when $p^2 - 4q = 0$ *(i.e.,* $1^2 - 4 \times \frac{1}{4} = 0$*).* Further comments will be made on this aspect later in §6.

A number which characterizes special cases of $\{w_n(a,b;p,q)\}$ is $e = pab - qa^2 - b^2$ which depends on the initial values a, b and on p, q. For the Oresme sequence,

(7) $$e = -\frac{1}{4}.$$

Immediate observations from (4) include these facts:

(8) $$O_n = n2^{-n} \quad (n \geqslant 0)$$

(9) $$O_{-n} = -n2^n \quad (n \leqslant 0)$$

i.e.,

$$O_m = m2^{-m} \quad (m \text{ integer})$$

whence

(10) $$O_{-n}O_n = -n^2$$

(11) $$\frac{O_{-n}}{O_n} = -2^{2n}$$

and

(12) $$\lim_{n\to\infty} O_n \to 0, \quad \lim_{n\to\infty} O_n \to -\infty$$

(13) $$\lim_{m\to\infty} \frac{O_m}{O_{m-1}} \to \tfrac{1}{2}.$$

Two well-known sequences, associated with the researches of Lucas [9] are:

(14) $$\{U_n\} \equiv \{w_n(1,p;p,q)\}$$

(15) $$\{V_n\} \equiv \{w_n(2,p;p,q)\}.$$

When $p = q = -1$, (14) gives the ordinary Fibonacci sequence and (15) the ordinary Lucas sequence.

It is a ready consequence of (4) and (14) that

(16) $$O_n = \frac{1}{2}U_{n-1},$$

where, for this $\{U_n\}$, $p = 1$, $q = \tfrac{1}{4}$.

That is, the Oresme sequence turns out to be a special case of the sequence $\{U_n\}$ after division by 2.

3. LINEAR RELATIONS FOR ORESME NUMBERS

Two simple expressions derived readily from (5) are:

(17) $$O_{n+2} - \frac{3}{4}O_n + \frac{1}{4}O_{n-1} = 0$$

(18) $$O_{n+2} - \frac{3}{4}O_{n+1} + \frac{1}{16}O_{n-1} = 0.$$

Sums of interest are:

(19) $$\sum_{j=0}^{n-1} O_j = 4(\tfrac{1}{2} - O_{n+1})$$

(20) $$\sum_{j=0}^{\infty} O_j = 2$$

(21) $$\sum_{j=0}^{n-1} (-1)^j O_j = \frac{4}{9}[-\tfrac{1}{2} + (-1)^n(O_{n+1} - 2O_n)]$$

(22) $$\sum_{j=0}^{n-1} O_{2j} = \frac{4}{9}[2 + O_{2n-1} - 5O_{2n}]$$

(23) $$\sum_{j=0}^{n-1} O_{2j+1} = \frac{1}{9}(10 + 5O_{2n-1} - 16O_{2n}).$$

Also,

(24) $$O_{n+r} = O_r U_n - \frac{1}{4} O_{r-1} U_{n-1}$$
$$= O_n U_r - \frac{1}{4} O_{n-1} U_{r-1}$$

(25) $$O_{n+r} = O_{r-j} U_{n+j} - \frac{1}{4} O_{r-j-1} U_{n+j-1}$$
$$= O_{n+j} U_{r-j} - \frac{1}{4} O_{n+j-1} U_{r-j-1}$$

(26) $$\frac{O_{n+r} + 4^{-r} O_{n-r}}{O_n} = V_r \text{ (independent of } n)$$

(27) $$\frac{O_{n+r} - 4^{-r} O_{n-r}}{O_{n+s} - 4^{-s} O_{n-s}} = \frac{U_{r-1}}{U_{s-1}}$$

(28) $$O_{2n} = (-\tfrac{1}{4})^n \sum_{j=0}^{n} \binom{n}{j} (-4)^{n-j} O_{n-j} .$$

4. NON-LINEAR PROPERTIES OF ORESME NUMBERS

A basic quadratic expression, corresponding to Simson's result for Fibonacci numbers, is

(29) $$O_{n+1} O_{n-1} - O_n^2 = -(\tfrac{1}{4})^n .$$

This result is the basis of a geometric paradox of which the general expression is given in Horadam [5].

A specially interesting result is the "Pythagorean" theorem of which the generalization is discussed in Horadam [5]:

(30) $$(O_{n+2}^2 - O_{n+1}^2)^2 + (2O_{n+2} O_{n+1})^2 = (O_{n+2}^2 + O_{n+1}^2)^2$$

For instance, $n = 3$ leads to the Pythagorean triple 39, 80, 89 after we have ignored a common denominator $(= 1024)$; $n = 4$ leads to the Pythagorean triple 8, 6, 10 after simplification (and division by 64, which we ignore).

Some other quadratic properties are:

(31) $$\frac{1}{2} O_{m+n-1} = O_m O_n - \frac{1}{4} O_{m-1} O_{n-1}$$

(32) $$\frac{1}{2} O_{2n-1} = O_n^2 - \frac{1}{4} O_{n-1}^2$$
$$= O_{n+1} O_{n-1} - \frac{1}{4} O_n O_{n-2}$$

(33) $$O_{n+r} O_{n-r} - O_n^2 = -(\tfrac{1}{4})^{n-r+1} U_{r-1}^2$$
(an extension of (29))

(34) $$O_{n+1}^2 - (\tfrac{1}{4})^2 O_{n-1}^2 = \frac{1}{2} O_{2n+1} + \frac{1}{8} O_{2n-1}$$

(35) $$O_{n-r} O_{n+r+t} - O_n O_{n+t} = -(\tfrac{1}{4})^{n-r+1} U_{r-1} U_{r+t-1}$$
(an extension of (33)).

Many other results can be obtained, if we use the fact that $\{O_n\}$ is a special case of $\{w_n\}$. Rather than produce numerous identities here, we suggest (as we did in [7] with Pell identities) that the reader may entertain himself by discovering them. Recent articles by Zeitlin [11], [12] and [13] give many properties of $\{w_n\}$ which may be of assistance.

Some of the distinguishing features of $\{O_n\}$ arise from the fact that it is a degenerate case of (2), occurring when $p^2 - 4q = 0$.

5. GENERATING FUNCTION

A generating function for the Oresme numbers O_n $(n \geqslant 1)$ is

$$\sum_{n=1}^{\infty} O_n x^n = \frac{\frac{1}{2}x}{1 - x + \frac{1}{4}x^2} \tag{36}$$

This may be obtained from the general result for w_n in Horadam [6], by the appropriate specialization.

6. COMMENTS ON THE DEGENERACY PROPERTY

Since the general term of $\{w_n\}$ is

$$w_n = A\alpha^n + B\beta^n , \tag{37}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \beta = \frac{p - \sqrt{p - 4q}}{2} \tag{38}$$

are the roots of $x^2 - px + q = 0$, and

$$A = \frac{b - a\beta}{\alpha - \beta}, \qquad B = \frac{a\alpha - \beta}{\alpha - \beta}, \qquad (\alpha - \beta = \sqrt{p^2 - 4q}), \tag{39}$$

it follows that in the degenerate case, O_n cannot be expressed in the form (36), as we have seen earlier in (8) and (9). An interesting derivation from Eq. (4.6) of Horadam [4] is the relationship $O_n^2 - \frac{1}{4}U_{n-1}^2 = 0$, leading back to (16).

Carlitz [2], acknowledging the work of Riordan, established an interesting relationship between the sum of kth powers of terms of the degenerate sequence $\{U_n\}$ (for which $q = p^2/4$) and the Eulerian polynomial $A_k(x)$ which satisfies the differential equation

$$A_{n+1}(x) = (1 + nx)A_n(x) + x(1 - x)\frac{d}{dx}A_n(x) , \tag{40}$$

where

$$A_0(x) = A_1(x) = 1, \quad A_2(x) = 1 + x, \quad A_3(x) = 1 + 4x + x^2 . \tag{41}$$

This result specializes to the Oresme case where $p = 1$.

7. HISTORICAL

It is thought that Nicole Oresme was born in 1323 in the small village of Allemagne, about two miles from Caen, in Normandy. Records show that in 1348 he was a theology student at the College of Navarre—of which he became principal during the period 1356–1361—and that he attended Paris University.

His star in the Church rose quickly. Successively he became archdeacon of Bayeux (1361), then caron (1362), and later dean (1364) of Rouen Cathedral. In this period, he journeyed to Avignon with a party of royal emissaries and preached a sermon at the papal court of Urban V. While dean of Rouen, Oresme translated several of Aristotle's works, at the request of Charles V.

Thanks to his imperial patron (Charles V), Oresme was made bishop of Lisieux in 1377, being enthroned in Rouen Cathedral the following year. In 1382, Oresme died at Lisieux and was buried in his cathedral church.

Mathematically, Oresme is important for at least three reasons. Firstly, he expounded a graphic representation of of qualities and velocities, though there is no mention of the (functional) dependence of one quality upon another, as found in Descartes. Secondly, he was the first person to conceive the notion of fractional powers (afterwards rediscovered by Stevin), and suggested a notation.

In Oresme's notation, $4^{1\frac{1}{2}}$ is written as

$$\boxed{1p \cdot \frac{1}{2}}\ 4 \text{ or } \boxed{\frac{p \cdot 1}{1 \cdot 2}}\ 4\ .$$

Thirdly, in an unpublished manuscript, Oresme found the sum of the series derived from the sequence (1). Such recurrent infinite series did not generally appear again until the eighteenth century.

In all, Oresme was one of the chief medieval theological scholars and mathematical innovators. It is the writer's hope that something of Oresme's intellectual capacity has been appreciated by the reader. With this in mind, we honor his name by associating him with the extended recurrence sequence (4), of which he had a glimpse so long ago.

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INCREDIBLE IDENTITIES

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Consider the algebraic numbers

$$A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$$

$$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}$$

To 25 decimals they both equal

$$7.38117\ 59408\ 95657\ 97098\ 72669\ .$$

Either this is an incredible coincidence or

(1) $$A = B$$

is an incredible identity, since A and B do not appear to lie in the same algebraic field. But they do. One has

(2) $$A = B = 4X - 1\ ,$$

[Continued on page 280.]

SOME FURTHER IDENTITIES FOR THE GENERALIZED FIBONACCI SEQUENCE $\{H_n\}$

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1. INTRODUCTION

In this paper we are concerned with developing and establishing further identities for the generalized Fibonacci sequence $\{H_n\}$, with particular emphasis on summation properties. First we obtain a number of power identities by substitution into some known identities and then we establish a number of summation identities. Next we proceed to derive some further summation identities involving the fourth power of generalized Fibonacci numbers $\{H_n\}$ from a consideration of the ordinary Pascal triangle. Finally, we arrive at some additional summation identities by applying standard difference equation theory to the sequence $\{H_n\}$. Notation and definitions of Walton and Horadam [9] are assumed.

2. POWER IDENTITIES FOR THE SEQUENCE $\{H_n\}$

In this section a number of new power identities for the generalized Fibonacci numbers $\{H_n\}$ have been obtained by following the reasoning of Zeitlin [10], for similar identities relating to the ordinary Fibonacci sequence $\{F_n\}$.

Use will be made of identities (11) and (12) of Horadam [6], viz.,

$$H_nH_{n+2} - H_{n+1}^2 = (-1)^{n+1}d \tag{2.1}$$

$$H_{m+h}H_{m+k} - H_mH_{m+h+k} = (-1)^{m+2h}dF_hF_k\ , \tag{2.2}$$

(where we have substituted $n = m+h$, $h = s$ and $k = r+s+1$), and the identity

$$H_{k+1}H_{m-k} + H_kH_{m-k-1} = (2p-q)H_m - dF_m\ , \tag{2.3}$$

where the right-hand side of (2.3) is derived from (9) of Horadam [6].

Re-writing (2.1) in the form

$$H_n^2 - H_{n+1}^2 = (-1)^{n+1}d - H_nH_{n+1} \tag{2.4}$$

yields

$$H_{n+1}^4 + H_n^4 = (H_n^2 - H_{n+1}^2)^2 + 2H_n^2H_{n+1}^2 = d^2 + 2(-1)^n dH_nH_{n+1} + 3H_n^2\ H_{n+1}^2 \tag{2.5}$$

$$\begin{aligned} -2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_{n+1}H_n^3 &= 2H_nH_{n+1}[(-1)^{n+1}d - H_nH_{n+1}] - H_n^2H_{n+1}^2 \\ &= -2(-1)^n dH_nH_{n+1} - 3H_n^2H_{n+1}^2\ . \end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6) gives

$$H_{n+1}^4 - 2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_{n+1}\,H_n^3 + H_n^4 = d^2\ . \tag{2.7}$$

If we now substitute the identities

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*Part of the substance of an M.Sc. thesis presented to the University of New England in 1968.

$$\begin{cases} H_{n+4} = 3H_{n+1} + 2H_n \\ H_{n+3} = 2H_{n+1} + H_n \\ H_{n+2} = H_{n+1} + H_n \end{cases} \tag{2.8}$$

into the expression

$$H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4$$

we have -6 times the left-hand side of (2.7), *i.e.*,

$$H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4 = -6d^2 . \tag{2.9}$$

Re-arranging (2.9) and substituting $n = n + 1$ yields

$$H_{n+5}^4 = 4H_{n+4}^4 + 19H_{n+3}^4 + 4H_{n+2}^4 - H_{n+1}^4 - 6d^2 \tag{2.10}$$

so that substitution for $-6d^2$ from (2.9) gives

$$H_{n+5}^4 = 5H_{n+4}^4 + 15H_{n+3}^4 - 15H_{n+2}^4 - 5H_{n+1}^4 + H_n^4 . \tag{2.11}$$

We note here that (2.9) is a verification of (4.6) of Zeitlin [11].

If we now let $V_n = H_{n+1}^4 - H_n^4$, we may re-write (2.9) in the form

$$V_{k+3} - 3V_{k+2} - 22V_{k+1} - 26V_k - 25H_k^4 = -6d^2 , \tag{2.12}$$

where

$$\sum_{k=0}^{n} V_{k+j} = H_{n+j+1}^4 - H_j^4 .$$

Summing both sides of (2.12) over k, where $k = 0, 1, \cdots, n$, gives

$$25 \sum_{k=0}^{n} H_k^4 = H_{n+4}^4 - 3H_{n+3}^4 - 22H_{n+2}^4 - 26H_{n+1}^4 + 6(n+1)d^2 + \delta , \tag{2.13}$$

where

$$\delta = 9p^4 - 20p^3q - 6p^2q^2 + 4pq^3 + 28q^4 .$$

($\delta = 9$ for the Fibonacci numbers $\{F_n\}$.)

Substituting for H_{n+4}^4 in (2.13) by using (2.9) gives

$$25 \sum_{k=0}^{n} H_k^4 = H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6nd^2 + \delta \tag{2.14}$$

which yields the obvious result

$$H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6nd^2 + \delta' \equiv 0 \mod 25 , \tag{2.15}$$

where

$$\delta' = 9p^4 - 20p^3q - 6p^2q^2 + 4pq^3 + 3q^4 .$$

($\delta' = 9$ for the Fibonacci numbers $\{F_n\}$.)

Multiplying (2.11) by $(-1)^{n+5}$ and replacing n by k gives

$$W_{k+4} + 6W_{k+3} - 9W_{k+2} - 24W_{k+1} - 19W_k = 18(-1)^k H_k^4 , \tag{2.16}$$

where

$$W_n = (-1)^{n+1} H_{n+1}^4 - (-1)^n H_n^4 . \tag{2.17}$$

Summing over both sides of (2.16) for $k = 0, 1, \cdots, n$, and using

$$\sum_{k=0}^{n} W_{k+j} = (-1)^{n+j+1} H_{n+j+1}^4 - (-1)^j H_j^4 \tag{2.18}$$

gives

(2.19)
$$18\sum_{k=0}^{n}(-1)^k H_k^4 = (-1)^n[-H_{n+5}^4 + 6H_{n+4}^4 + 9H_{n+3}^4 - 24H_{n+2}^4 + 19H_{n+1}^4] + 6\epsilon$$
$$= (-1)^n[H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4] + 6\epsilon \quad \text{by (2.11)}$$
$$= (-1)^n[-2H_{n+3}^4 + 10H_{n+2}^4 + 28H_{n+1}^4 - 2H_n^4 - 6d^2] + 6\epsilon \quad \text{by (2.9)},$$

where

$$\epsilon = 2p^3q - 3p^2q^2 - 2pq^3 + 3q^4 \left(= q(2p^3 - 3p^2q - 2pq^2 + 3q^3)\right).$$

($\epsilon = 0$ for the Fibonacci numbers $\{F_n\}$.)

Therefore, on using (2.11), we have

(2.20)
$$18\sum_{k=0}^{n}(-1)^k H_k^4 = (-1)^n[H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4] + 6\epsilon$$
$$= 2\left\{(-1)^n[-H_{n+3}^4 + 5H_{n+2}^4 + 14H_{n+1}^4 - H_n^4 - 3d^2] + 3\epsilon\right\}$$

on using (2.9). Now (2.20) implies that

(2.21)
$$H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4 \equiv 0 \mod 6$$

from which we conclude that

(2.22)
$$H_{n+4}^4 - 9H_{n+2}^4 - H_n^4 \equiv 0 \mod 6$$

so that

(2.23)
$$H_{n+4}^4 - H_n^4 \equiv 0 \mod 3 \quad .$$

We will now use the identity

(2.24)
$$H_{k+1}H_{k+2}H_{k+4}H_{k+5} = H_{k+3}^4 - d^2$$

(which is a generalization of an identity for the sequence $\{F_n\}$ stated by Gelin and proved by Cesàro – see Dickson [2]) to establish the two results

(2.25)
$$25\sum_{k=0}^{n} H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 26H_{n+3}^4 + 22H_{n+2}^4 + 3H_{n+1}^4 - H_n^4 - 19nd^2 - 25d^2 + \delta - 50t^2$$

(2.26)
$$9\sum_{k=0}^{m}(-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = (-1)^m[-H_{m+6}^4 + 5H_{m+5}^4 + 14H_{m+4}^4 - H_{m+3}^4 - 3d^2] - 3\epsilon - 9d^2g(m) + 18\gamma \ ,$$

where

$$g(m) = \begin{cases} 0 & \text{if } m = 2n-1, \ n = 1, 2, \cdots \\ 1 & \text{if } m = 2n, \quad n = 0, 1, \cdots \end{cases}$$

and

$$\begin{cases} \gamma = q^4 + 2q^3p + 3q^2p^2 + 2qp^3 \left(= q(q^3 + 2q^2p + 3qp^2 + 2q^3)\right) \\ t = p^2 + pq + q^2 \ . \end{cases}$$

for the Fibonacci numbers $\{F_n\}$, $\gamma = 0$, $t = 1$.

Proof: Sun both sides of (2.24) with respect to k. Then

(2.27)
$$25\sum_{k=0}^{n} H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 25\sum_{k=0}^{n} H_{k+3}^4 - 25(n+1)d^2$$

(2.28)
$$9\sum_{k=0}^{m}(-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 9\sum_{k=0}^{m}(-1)^k H_{k+3}^4 - 9d^2g(m) \ ,$$

where

$$g(m) = \sum_{k=0}^{m}(-1)^k \ .$$

Now,

$$\sum_{k=0}^{n} H_{k+3}^4 = \sum_{j=0}^{n+3} H_j^4 - 2t^2 ,$$

where

$$t = p^2 + pq + q^2 ,$$

so that on using (2.14), with n replaced by $n+3$, the right-hand side of (2.27) reduces to

$$H_{n+6}^4 - 3H_{n+5}^4 - 22H_{n+4}^4 - H_{n+3}^4 - 19nd^2 - 7d^2 + \delta - 50t^2$$

Eliminating H_{n+6}^4, H_{n+5}^4 and H_{n+4}^4 by using (2.9) gives (2.25). Since

$$\sum_{k=0}^{m} (-1)^k H_{k+3}^4 = -\sum_{j=0}^{m+3} (-1)^j H_j^4 + 2\gamma ,$$

where

$$\gamma = q^4 + 2q^3p + 3q^2p^2 + 2pq^3 ,$$

use of (2.20), where $m+3$ replaces n, and of (2.28) yields (2.26).

From (2.2) with $m = n-j$, $h = j$ and $k = 1$, we obtain

$$H_nH_{n-j+1} - H_{n-j}H_{n+1} = (-1)^{n+j}dF_jF_1 = (-1)^{n+j}dF_j . \tag{2.29}$$

Now

$$H_n = H_{n+2} - H_{n+1} ,$$

so that (2.29) simplifies to

$$H_{n+2}H_{n+1-j} - H_{n+1}H_{n+2-j} = (-1)^{n+j}dF_j . \tag{2.30}$$

From (2.3), with $m = 2n+4-j$ and $k = n+2$, we obtain

$$(2p-q)H_{2n+4-j} - dF_{2n+4-j} = H_{n+3}H_{n+2-j} + H_{n+2}H_{n+1-j} . \tag{2.31}$$

Substituting for $H_{n+2}H_{n+1-j}$ in (2.30) by means of (2.31) gives

$$\begin{aligned}(2p-q)H_{2n+4-j} - dF_{2n+4-j} &= H_{n+3}H_{n+2-j} + H_{n+1}H_{n+2-j} + (-1)^{n+j}dF_j \\ &= (pL_{n+3} + qL_{n+2})H_{n+2-j} + (-1)^{n+j}dF_j\end{aligned} \tag{2.32}$$

which may be written as

$$\begin{aligned}&(-1)^{j+1}H_{j+1}\left\{(2p-q)H_{2n+4-j} - dF_{2n+4-j}\right\} \\ &\quad = (-1)^{j+1}(pL_{n+3} + qL_{n+2})H_{n+2-j}H_{j+1} + (-1)^{n+1}dH_{j+1}F_j .\end{aligned} \tag{2.33}$$

From (2.2) with $m = j+1$, $h = n+1-j$ and $k = n+2-j$, we obtain

$$H_{n+2}H_{n+3} - H_{j+1}H_{2n+4-j} = (-1)^{j+1}dF_{n+1-j}F_{n+2-j} \tag{2.34}$$

so that

$$(-1)^{j+1}H_{j+1}(2p-q)H_{2n+4-j} = (-1)^{j+1}(2p-q)H_{n+2}H_{n+3} - d(2p-q)F_{n+1-j}F_{n+2-j} . \tag{2.35}$$

Substituting (2.35) into (2.33) gives

$$\begin{aligned}&(2p-q)dF_{n+1-j}F_{n+2-j} + (-1)^{j+1}(pL_{n+3} + qL_{n+2}) \cdot H_{n+2-j}H_{j+1} + (-1)^{j+1}dH_{j+1}F_{2n+4-j} \\ &\qquad + (-1)^{n+1}H_{j+1}F_j = (-1)^{j+1}(2p-q)H_{n+2}H_{n+3} .\end{aligned} \tag{2.36}$$

The following identities may be proved by induction:

(2.37) $$2\sum_{k=0}^{n}(-1)^k H_{m+3k} = (-1)^n H_{m+3n+1} + H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.38) $$3\sum_{k=0}^{n}(-1)^k H_{m+4k} = (-1)^n H_{m+4n+2} + H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.39) $$11\sum_{k=0}^{n}(-1)^k H_{m+5k} = (-1)^n [5H_{m+5n+1} + 2H_{m+5n}] + 4H_m - 5H_{m-1} \qquad (m = 1, 2, \cdots)$$

(2.40) $$4\sum_{k=0}^{n} H_k H_{2k+1} = H_{2n+3}H_n + H_{2n}H_{2n+3} - 2q^2$$

(2.41) $$3\sum_{k=0}^{n}(-1)^k H_{m+2k}^2 = (-1)^n H_{m+2n}H_{m+2n+2} + H_m H_{m-2} \qquad (m = 2, 3, \cdots)$$

(2.42) $$7\sum_{k=0}^{n}(-1)^k H_{m+4k}^2 = (-1)^n H_{m+4n}H_{m+4n+4} + H_m H_{m-4} \qquad (m = 4, 5, \cdots)$$

(2.43) $$2\sum_{k=0}^{n} H_{k+2}H_{k+1}^2 = H_{n+3}H_{n+2}H_{n+1} - pq(p+q)$$

(2.44) $$2\sum_{k=0}^{n}(-1)^k H_k H_{k+1}^2 = (-1)^n H_{n+2}H_{n+1}H_n + pq(p-q)\,.$$

Zeitlin [11] has also examined numerous power identities for the sequence $\{H_n\}$ as special cases of even power identities found for the generalized sequence $\{\omega_n\}$ used in Horadam [7], and earlier by Tagiuri (Dickson [2]).

As seen in Horadam [7], the generalized Fibonacci sequence $\{H_n\}$ is a particular case of generalized sequence $\{\omega_n\}$ for $a = q$, $b = p$, $r = 1$ and $s = -1$. Hence applying these results to (3.1), Theorem I, of Zeitlin [11] yields, for $n = 0, 1, \cdots$ (see (2.47) below):

(2.45) $$(-1)^{mrn}\sum_{k=0}^{2t}(-1)^{mrt} b_k^{(2t)}\left(-\frac{i}{2}\right) H_{m(n+2t-k)+n_0}^{2r} \qquad (i = \sqrt{-1}\,)$$

$$= (-1)^{rn_0+mt(4r-t-1)/2}\binom{2r}{r}(-5)^{t-r} d^r \prod_{k=1}^{t} F_{mk}^2\,.$$

However,

$$\begin{aligned}(-1)^{mt(4r-t-1)/2} &= (-1)^{2mtr-mt(t+1)/2}\\ &= (-1)^{2mtr-mt(t+1)+mt(t+1)/2}\\ &= (-1)^{mt(t+1)/2}\end{aligned}$$

since $2mtr$ and $mt(t+1)$* are always even. Hence, we may rewrite (2.45) as

*This result for $mt(t+1)$ may be easily verified by considering the table

m	t	$t+1$	$mt(t+1)$
odd	odd	even	even
even	even	odd	

$$(2.46)\qquad (-1)^{mrn}\sum_{k=0}^{2t}(-1)^{mrt}b_k^{(2t)}\left(-\frac{i}{2}\right)H^{2r}_{m(n+2t-k)+n_0}$$

$$= (-1)^{rn_0+mt(t+1)/2}\binom{2r}{r}(-5)^{t-r}d^r\prod_{k=1}^{t}F^2_{mk}\ ,$$

where $n_0 = 0, 1, \cdots;\ m, t = 1, 2, \cdots,\ r = 0, 1, \cdots, t,$ and where the

$$b_k^{(2t)}\left(-\frac{i}{2}\right),\qquad k = 0, 1, \cdots, 2t,$$

are defined (as a special case of (2.9) of Zeitlin [11]) by

$$(2.47)\qquad \sum_{k=0}^{2t} b_k^{(2t)}\left(-\frac{i}{2}\right)y^{2t-k} = \prod_{k=1}^{t}(y^2-(-1)^{mk}L_{2mk}y+1)\ .$$

If we now consider $r = t = 1$ in (2.46) and then (2.47), then (2.46) reduces to

$$(2.48)\qquad (-1)^{mn}[H^2_{m(n+2)+n_0} - L_{2m}H^2_{m(n+1)+n_0} + H^2_{mn+n_0}] = 2(-1)^{m+n_0}dF_n^2\ .$$

on calculation. This corresponds to (4.5) of Zeitlin [11].

Similarly, we can obtain (4.6) to (4.16) of Zeitlin [11] by the correct substitutions into (2.46) and (2.47), where as already mentioned, (4.6) is our previous identity, (2.9). Identities (4.7) to (4.16) of Zeitlin should be noted for reference and comparison.

3. FOURTH POWER GENERALIZED FIBONACCI IDENTITIES

Hoggatt and Bicknell [5] have derived numerous identities involving the fourth power of Fibonacci numbers $\{F_n\}$ from Pascal's triangle.

By considering the same matrices S and U where $u_1 = H_0 = q$ and $u_2 = H_1 = p$, *i.e.*,

$$(3.1)\qquad S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $U = (a_{ij})$ is the column matrix defined by

$$(3.2)\qquad a_{i1} = \binom{4}{i-1}H_0^{5-i}H_1^{i-1}\ ,\qquad i = 1, 2, \cdots, 5,$$

the following identities for the fourth power of generalized Fibonacci numbers may easily be verified by proceeding as in Hoggatt and Bicknell [5]:

$$(3.3)\qquad \sum_{i=0}^{4n+1}(-1)^i\binom{4n+1}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - H^4_{2n+j+1}) = A_j \quad \text{(say)}$$

$$(3.4)\qquad \sum_{i=0}^{4n+2}(-1)^i\binom{4n+2}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 2H^4_{2n+j+1} + H^4_{2n+j+2}) = A_j - A_{j+1}$$

$$(3.5)\qquad \sum_{i=0}^{4n+3}(-1)^i\binom{4n+3}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 3H^4_{2n+j+1} + 3H^4_{2n+j+2} - H^4_{2n+j+3}) = A_j - 2A_{j+1} + A_{j+2}$$

$$(3.6)\qquad \sum_{i=0}^{4n+4}(-1)^i\binom{4n+4}{i}H^4_{i+j} = 25^n(H^4_{2n+j} - 4H^4_{2n+j+1} + 6H^4_{2n+j+2} - 4H^2_{2n+j+3} + H^4_{2n+j+4})$$
$$= A_j - 3A_{j+1} + 3A_{j+2} - A_{j+3}\ .$$

Noting that the coefficients of the terms involving the A's on the right-hand side of the above equations are the first four rows of Pascal's triangle, we deduce the general identity

$$\sum_{i=0}^{4n+k} (-1)^i \binom{4n+k}{i} H_{i+j}^4 = 25^n(H_{2n+j}^4 - (k-1)H_{2n+j+1}^4 + \cdots + (-1)^{k-1}H_{2n+j+k}^4) \tag{3.7}$$
$$= A_j - (k-1)A_{j+1} + \cdots + (-1)^{k-1}A_{j+k}\,.$$

Similarly, we have

$$\sum_{i=0}^{4n+5} (-1)^i \binom{4n+5}{i} H_{i+j}^4 = 25^{n+1}(H_{2n+j+2}^4 - H_{2n+j+3}^4) = 25A_{j+2}\,, \tag{3.8}$$

which results in the recurrence relation

$$A_j - 4A_{j+1} + 6A_{j+2} - 4A_{j+3} + A_{j+4} = 25A_{j+2} \tag{3.9}$$

i.e.,

$$A_j - 4A_{j+1} - 19A_{j+2} - 4A_{j+3} + A_{j+4} = 0 \tag{3.10}$$

on equating (3.8) and (3.7) with $k = 5$. Defining

$$G(j) = H_{n+j}^4 - 4H_{n+j+1}^4 - 19H_{n+j+2}^4 - 4H_{n+j+3}^4 + H_{n+j+4}^4 \tag{3.11}$$

yields

$$25^n\{G(j) - G(j+1)\} = A_j - 4A_{j+1} - 19A_{j+2} - 4A_{j+3} + A_{j+4} \tag{3.12}$$
$$= 0 \quad \text{on using (3.10)}\,.$$

Hence, $G(j)$ is a constant.

When $n = j = 0$, (3.11) reduces to

$$G(0) = -6d^2\,, \tag{3.13}$$

which leads to identity (2.9) which is in turn a generalization of a result due to Zeitlin [10] while also being a verification of a result due to Hoggatt and Bicknell [5] and also Zeitlin [11].

4. FURTHER GENERALIZED FIBONACCI IDENTITIES

In addition to the numerous identities of, say, Carlitz and Ferns [1], Iyer [4], Zietlin [10], [11], Subba Rao [8] and Hoggatt and Bicknell [5], Harris [3] has also listed many identities for the Fibonacci sequence $\{F_n\}$ which may be generalized to yield new identities for the generalized Fibonacci sequence $\{H_n\}$.

$$\sum_{k=0}^{n} kH_k = nH_{n+2} - H_{n+3} + H_3 \tag{4.1}$$

Proof: If

$$u_k\Delta v_k = \Delta(u_kv_k) - v_{k+1}\Delta u_k$$

(Δ is the difference operator) then

$$\sum_{k=0}^{n} u_k\Delta v_k = [u_kv_k]_0^{n+1} - \sum_{k=0}^{n} v_{k+1}\Delta u_k\,.$$

Let $u_k = k$ and $\Delta v_k = H_k$. Then

$$\Delta u_k = 1 \quad \text{and} \quad v_k = \sum_{i=0}^{k-1} H_i = H_{k+1} - p\,.$$

Omitting the constant $-p$ from v_k, we find

$$\sum_{k=0}^{n} kH_k = [kH_{k+1}]_0^{n+1} - \sum_{k=0}^{n} 1\cdot H_{k+2} = (n+1)H_{n+2} - H_{n+4} - p - H_1 - H_0 = nH_{n+2} - H_{n+3} + (2p+q).$$

Using this technique, we also have the following identities:

$$\sum_{k=0}^{n} (-1)^k kH_k = (-1)^n(n+1)H_{n-1} + (-1)^{n-1}H_{n-2} - H_{-3} \tag{4.2}$$

$$\sum_{k=0}^{n} kH_{2k} = (n+1)H_{2n+1} - H_{2n+2} + H_0 \tag{4.3}$$

$$\sum_{k=0}^{n} kH_{2k+1} = (n+1)H_{2n+2} - H_{2n+3} + H_1 \tag{4.4}$$

$$\sum_{k=0}^{n} k^2H_{2k} = (n^2+2)H_{2n+1} - (2n+1)H_{2n} - (2o-q) \tag{4.5}$$

$$\sum_{k=0}^{n} k^2H_{2k+1} = (n^2+2)H_{2n+2} - (2n+1)H_{2n+1} - (p+2q) \tag{4.6}$$

$$\sum_{k=0}^{n}\sum_{j=0}^{k} H_j = H_{n+4} - (n+3)p - q \tag{4.7}$$

$$\sum_{k=0}^{n} k^2H_k = (n^2+2)H_{n+2} - (2n-3)H_{n+3} - H_6 \tag{4.8}$$

$$\sum_{k=0}^{n} k^3H_k = (n^3+6n-12)H_{n+2} - (3n^2-9n+19)H_{n+3} + (50p+31q) \tag{4.9}$$

$$\sum_{k=0}^{n} k^4H_k = (n^4+12n^2-48n+98)H_{n+2} + (4n^3-18n^2+76n-159)H_{n+3} - (416p+257q) \tag{4.10}$$

$$5\sum_{k=0}^{n} (-1)^k H_{2k} = (-1)^n(H_{2n+2}+H_{2n}) - (p-3q) \tag{4.11}$$

$$5\sum_{k=0}^{n} (-1)^k H_{2k+1} = (-1)^n(H_{2n+3}+H_{2n+1}) + (2p-q) \tag{4.12}$$

$$5\sum_{k=0}^{n} (-1)^k kH_{2k} = (-1)^n(nH_{2n+2}+(n+1)H_{2n}) - q \tag{4.13}$$

(4.14) $$5\sum_{k=0}^{n}(-1)^k kH_{2k+1} = (-1)^n(nH_{2n+3}+(n+1)H_{2n+1})-p$$

(4.15) $$4\sum_{k=0}^{n}(-1)^k kH_{m+3k} = 2(-1)^n(n+1)H_{m+3n+1}-(-1)^n H_{m+3n+2}-H_{m-1} \quad (m = 2, 3, \cdots)$$

and so on.

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[Continued from Page 271.]

where X is the largest root of

(3) $$x^4 - x^3 - 3x^2 + x + 1 = 0.$$

The astonishing appearance of (1) stems from a peculiarity of (3). The Galois group of this quartic is the octic group (the symmetries of a square), and its resolvent cubic is therefore reducible:

(4) $$z^3 - 8z - 7 = (z+1)(z^2 - z - 7) = 0.$$

The common discriminant of (3) and (4) equals $725 = 5^2 \cdot 29$. While the quartic field $Q(X)$ contains $Q(\sqrt{5})$ as a subfield it does not contain $Q(\sqrt{29})$. Yet X can be computed from any root of (4). The rational root $z = -1$ gives $X = (A+1)/4$ while $z = (1+\sqrt{29})/2$ gives $X = (B+1)/4$.

It is clear that we can construct any number of such incredible identities from other quartics having an octic group. For example

$$x^4 - x^3 - 5x^2 - x + 1 = 0$$

has the discriminant $4205 = 29^2 \cdot 5$, and so the two expressions involve $\sqrt{5}$ and $\sqrt{29}$ once again. But this time $Q(\sqrt{29})$ is in $Q(X)$ and $Q(\sqrt{5})$ is not .

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EXPLICIT EXPRESSIONS FOR POWERS OF LINEAR RECURSIVE SEQUENCES

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1. DEFINITIONS

Van der Poorten [6] in a generalization of a result of Shannon and Horadam [8] has shown that (in my notation) if $\{w_n^{(i)}\}$ is a linear recursive sequence of orbitrary order i defined by the recurrence relation

$$w_n^{(i)} = \sum_{j=1}^{i} P_{ij} w_{n-j}^{(i)}, \quad n \geqslant i, \tag{1.1}$$

where the P_{ij} are arbitrary integers, with suitable initial values $w_0^{(i)}, w_1^{(i)}, \cdots, w_{i-1}^{(i)}$, then the sequence of powers $\{w_n^{(i)r}\}$, for integers $r \geqslant 1$, satisfies a similar recurrence relation of order at most

$$\binom{r+i-1}{r} .$$

In other words, he has established the existence of generating functions

$$w_r^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)r} x^n, \quad (w_n^{(i)r} \equiv (w_n^{(i)})^r). \tag{1.2}$$

The aim here is to find the recurrence relation for $\{w_n^{(i)r}\}$ and an explicit expression for $w_r^{(i)}(x)$. We shall concern ourselves with the non-degenerate case only; the degenerate case is no more difficult because the order of the recurrence relation for $\{w_n^{(i)r}\}$ is then lower than

$$\binom{r+i-1}{r} .$$

It is worth noting in passing that Marshall Hall [1] looked at the divisibility properites of a third-order sequence by a similar approach. From a second-order sequence with auxiliary equation roots a_1 and a_2 he formed a third-order sequence with auxiliary equation roots $a_1^2, a_2^2, a_1 a_2$.

2. RECURRENCE RELATION FOR SEQUENCE OF POWERS

Van der Poorten proved that if the auxiliary equation for $\{w_n^{(i)}\}$ is

$$g(x) \equiv x^i - \sum_{j=1}^{i} P_{ij} x^{i-j} = \prod_{t=1}^{i} (x - a_{it}) = 0 , \tag{2.1}$$

then the sequence $\{w_n^{(i)r}\}$ satisfies a linear recurrence relation of order

$$\binom{r+i-1}{r}$$

with auxiliary equation

$$g_r(x) \equiv \prod_{\Sigma\lambda_n = r} (x - a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots a_{ii}^{\lambda_i}) = 0 , \tag{2.2}$$

the zeros of which are exactly the zeros of $g(x)$ taken r at a time.

We now set

$$g_r(x) = x^u - \sum_{j=1}^{u} R_{uj} x^{u-j}, \qquad u = \binom{r+i-1}{r} \quad , \tag{2.3}$$

and we seek the R_{uj}.

Macmahon [5, p. 3] defines h_j, the homogeneous product sum of weight j of the quantities a_{ir}, as the sum of a number of symmetric functions, each of which is denoted by a partition of the number j. He showed that in our notation

$$h_j = \sum_{\Sigma n\lambda_n = j} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2! \cdots \lambda_j!} P_{i1}^{\lambda_1} P_{i2}^{\lambda_2} \cdots P_{ij}^{\lambda_j} \quad .$$

The first three cases of h_j are

$$h_1 = P_{i1} = \Sigma a_{i1} \quad ,$$

$$h_2 = P_{i1}^2 + P_{i2} = \Sigma a_{i1}^2 + \Sigma a_{i1} a_{i2} \quad ,$$

$$h_3 = P_{i1}^3 + 2P_{i1}P_{i2} + P_{i3} = \Sigma a_{i1}^3 + \Sigma a_{i1}^2 a_{i2} + \Sigma a_{i1} a_{i2} a_{i3} \quad .$$

Now $g_r(x) = 0$ is the equation whose zeros are the several terms of h_r with $a_{ij} = 0$ for $j > i$, since from its construction its zeros are a_{ij} taken r at a time; that is,

$$R_{u1} = h_r \quad \text{with} \quad a_{ij} = 0 \quad \text{for} \quad j > i,$$

since we have supposed that there are

$$\binom{r+i-1}{r} = u$$

distinct zeros of $g_r(x) = 0$.

Macmahon has proved [5, p. 19] that $H_{\substack{r\\j}}$ the homogeneous product sum, j together, of the whole of the terms of h_r, can be represented in terms of the symmetric functions (denoted by [ ]) of the roots of

$$x^i - h_1 x^{i-1} + h_2 x^{i-2} - \cdots = 0$$

by

$$H_{\substack{r\\j}} = \sum_{\Sigma n\mu_n = j} (-1)^{r(3\mu_2 + 5\mu_4 + \cdots)} \frac{[1^r]^{\mu_1} [2^r]^{\mu_2} [3^r]^{\mu_3} \cdots}{1^{\mu_1} \cdot 2^{\mu_2} \cdot 3^{\mu_3} \cdots \mu_1! \mu_2! \mu_3! \cdots} \quad . \tag{2.4}$$

Some examples of $H_{\substack{r\\j}}$ are (with $a_{ij} = 0$ for $j > i$)

$$H_{\substack{2\\1}} = a_{21}^2 + a_{22}^2 + a_{21}a_{22} \quad ,$$

$$H_{\substack{2\\2}} = a_{21}^4 + a_{22}^4 + 2a_{21}^2 a_{22}^2 + a_{21}^3 a_{22} + a_{21} a_{22}^3 \quad ,$$

$$H_{\substack{2\\3}} = a_{21}^6 + a_{22}^6 + 2a_{21}^3 a_{22}^3 + a_{21}^5 a_{22} + a_{21} a_{22}^5 + 2a_{21} a_{22}^2 + 2a_{21}^2 a_{22}^4 \quad ,$$

$$H_{\substack{2\\1}}^2 = H_{\substack{2\\1}} H_{\substack{2\\1}} = a_{21}^4 + a_{22}^4 + 3a_{21}^2 a_{22}^2 + 2a_{21}^3 a_{22} + 2a_{21} a_{22}^3 \quad ,$$

$$H_{\substack{2\\1}}^3 = a_{21}^6 + a_{22}^6 + 7a_{21}^3 a_{22}^3 + 3a_{21}^5 a_{22} + 3a_{21} a_{22}^5 + 6a_{21}^4 a_{22}^2 + 6a_{21}^2 a_{22}^4 \quad .$$

h_m is the homogeneous product sum of weight m of the terms of P_{i1}. $H_{\substack{r\\m}}$ is the homogeneous product sum of weight m of the terms of R_{u1}.

$(-1)^{j+1} P_{ij}$ is the product sum, j together, of the terms of P_{i1}.

$(-1)^{j+1} R_{uj}$ is the product sum, j together, of the terms of R_{u1}. It follows directly from Macmahon [5, p. 4] that

$$P_{ij} = \sum_{\Sigma n\lambda_n = j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2! \cdots \lambda_j!} \prod_{m=1}^{j} h_m^{\lambda_m} ,$$

and so

$$R_{uj} = \sum_{\Sigma n\lambda_n = j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2! \cdots \lambda_j!} \prod_{m=1}^{j} H_{r_m}^{\lambda_m} .$$

For example,

$$R_{31} = H_{\substack{2\\1}} = a_{21}^2 + a_{22}^2 + a_{21}a_{22} ,$$

$$\begin{aligned} R_{32} = -H_{\substack{2\\1}}^2 + H_{\substack{2\\2}} &= -(\Sigma a_{21}^4 + 2\Sigma a_{21}^3 a_{22} + 3a_{21}^2 a_{22}^2) \\ &\quad + (\Sigma a_{21}^4 + \Sigma a_{21}^3 a_{22} + 2a_{21}^2 a_{22}^2) \\ &= -\Sigma a_{21}^3 a_{22} - \Sigma a_{21}^2 a_{22}^2 , \end{aligned}$$

$$R_{33} = H_{\substack{2\\1}}^3 + H_{\substack{2\\3}} - 2H_{\substack{2\\2}}H_{\substack{2\\1}} = a_{21}^3 a_{22}^3 .$$

We can verify these results by utilizing some of the properties of the generalized sequence of numbers $\{ w_n^{(2)} \}$ developed by Horadam [3].

From Eq. (27) of Horadam's paper we have that

$$w_n^{(2)} w_{n-2}^{(2)} - w_{n-1}^{(2)^2} = (-P_{22})^{n-2} e , \tag{2.5}$$

where

$$e = P_{21} w_0^{(2)} w_1^{(2)} + P_{22} w_0^{(2)^2} - w_1^{(2)^2} .$$

Thus

$$w_{n-1}^{(2)} w_{n-3}^{(2)} - w_{n-2}^{(2)^2} = (-P_{22})^{n-3} e$$

and

$$P_{22} w_{n-2}^{(2)^2} - P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = (-P_{22})^{n-2} e. \tag{2.6}$$

Subtracting (2.5) from (2.6), we get

$$P_{22} w_{n-2}^{(2)} + w_{n-1}^{(2)} = P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)} . \tag{2.7}$$

But

$$w_n^{(2)} - P_{22} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)} ,$$

and

$$w_{n-1}^{(2)} - P_{22} w_{n-3}^{(2)} = P_{21} w_{n-2}^{(2)} ,$$

so

$$w_n^{(2)} + P_{22}^2 w_{n-2}^{(2)^2} - 2P_{22} w_n^{(2)} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)^2} ,$$

and

$$P_{22} w_{n-1}^{(2)^2} + P_{22} w_{n-3}^{(2)^2} - 2P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = P_{21}^2 P_{22} w_{n-2}^{(2)^2} .$$

Adding the last two equations we obtain

$$w_n^{(2)^2} + P_{22} w_{n-1}^{(2)^2} + P_{22}^2 w_{n-2}^{(2)^2} + P_{22}^3 w_{n-3}^{(2)^2} - 2P_{22}(P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)^2}) = P_{21}^2 w_{n-1}^{(2)^2} + P_{21}^2 P_{22} w_{n-2}^{(2)^2} .$$

Combining this with (2.7) we then have

(2.8) $$w_n^{(2)^2} = (P_{21}^2 + P_{22})w_{n-1}^{(2)^2} + (P_{22}^2 + P_{21}^2 P_{22})w_{n-2}^{(2)^2} + (-P_{22}^3)w_{n-3}^{(2)^2} ,$$

so

$$R_{31} = P_{21}^2 + P_{22} = a_{21}^2 + a_{22}^2 + a_{21}a_{22} ,$$
$$R_{32} = P_{22} + P_{21}P_{22} = -a_{21}^3 a_{22} - a_{21}a_{22}^3 - a_{21}^2 a_{22}^2,$$
$$R_{33} = -P_{22}^3 = a_{21}^3 a_{22}^3 ,$$

as required.

To obtain an expression for $H_{\substack{r\\m}}$ in terms of a_{ij}, we now use a result of Macmahon, namely,

$$[u^r] = (-1)^{r(u+1)}\sigma_u ,$$

where σ_u denotes the sum of the uth powers of the roots of $g_r(x) = 0$. It is sufficient for our purposes to state that Macmahon has shown that σ_u is the homogeneous product sum of order r of the quantities a_{ij}^u. It is thus given by

$$\sigma_u = \sum_{\Sigma t=r} \prod_m a_{im}^{ut_m}$$

by analogy with

$$h_r = \sum_{\Sigma t=r} \prod_m a_{im}^{t_m} ,$$

the homogeneous product sum of order r of the quantities a_{ij}. We now define σ_{iu}, the homogeneous product sum of order r of the quantities a_{ij}^u such that $a_{ij} = 0$ for $j > i$:

$$\sigma_{iu} = \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{uv_j} ,$$

and we introduce the term

$$\sigma_{iur} = (-1)^{r(u+1)}\sigma_{iu} .$$

We have thus established that for

$$w_n^{(i)^r} = \sum_{j=1}^{u} R_{uj} w_{n-j}^{(i)^r} ,$$

(2.9) $$R_{uj} = \sum_{\Sigma n\lambda_n=j} (-1)^{1+\Sigma\lambda} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_j!} \prod_{m=1}^{j} H_{\substack{r\\m}}^{\lambda_m} ,$$

where

$$H_{\substack{r\\m}} = \sum_{\Sigma n\mu_n=m} (-1)^{r(3\mu_2+5\mu_4+\cdots)} \prod_{v=1}^{m} \frac{(\sigma_{iur})^{\mu_v}}{v^{\mu_v}\cdot\mu_v} ,$$

and

$$\sigma_{iur} = (-1)^{r(u+1)} \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{uv_j} ,$$

and

$$u = \binom{i+r-1}{r} .$$

It is of interest to note that another formula for σ_{iur} can be given by

(2.9) $$\sigma_{iur} = (-1)^{r(u+1)} \sum_{j=1}^{i} a_{ij}^{(j+r-1)} \Big/ \prod_{j>k} (a_{ij}^u - a_{ik}^u) .$$

We prove this by noting that

$$\sigma_{iu} = \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{uv_j} = (-1)^{r(u+1)} \sigma_{iur}$$

and defining

$$h_r' = \sum_{\Sigma v=r} \prod_{j=1}^{i} a_{ij}^{v_j}$$

and showing that

$$h_r' = \sum_{j=1}^{i} a_{ij}^{j+r-1} \Big/ \prod_{j>k} (a_{ij} - a_{ik}) .$$

It follows from Macmahon [5, p. 4] that h_r' satisfies a linear recurrence relation of order i given by

$$\begin{aligned} h_r' &= \sum_{n=1}^{i} P_{in} h_{r-n}', & r &> 0, \\ h_r' &= 1, & r &= 0 \\ h_r' &= 0, & r &< 0 ; \end{aligned}$$

the P_{ir} and a_{ir} are those of (2.1). We again assume that the a_{ir} are distinct so that from Jarden [4, p. 107]

$$h_r' = \sum_{j=1}^{i} a_{ij}^{r} D_j / D , \tag{2.10}$$

where D is the Vandermonde of the roots, given by

$$D = \sum_{j=1}^{i} a_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ n<m}} (a_{im} - a_{in}) = \prod_{j>n} (a_{ij} - a_{in}) \prod_{\substack{j \neq n \neq m \\ n<m}} (a_{im} - a_{in}) \tag{2.11}$$

and D_j is the determinant of order i obtained from D on replacing its jth column by the initial terms of the sequence, $h_0', h_1', \cdots h_{i-1}'$. It thus remains to prove that

$$D_j = a_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ n<m}} (a_{im} - a_{in}) = D a_{ij}^{i-1} \Big/ \prod_{j>n} (a_{ij} - a_{in}) . \tag{2.12}$$

We use the method of the contrapositive. If

$$D_j \neq a_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ m>n}} (a_{im} - a_{in}) ,$$

then

$$D = \sum_{j=1}^{i} D_j$$

(from (2.10) with $n = 0$)

$$\neq \sum_{j=1}^{i} a_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ m>n}} (a_{im} - a_{in})$$

which contradicts (2.11). This proves (2.12) and we have established that

$$h_r' = \sum_{j=1}^{i} a_{ij}^r D_j/D = \sum_{j=1}^{i} a_{ij}^{i+n-1} D_j/D a_{ij}^{i-1} = \sum_{j=1}^{i} a_{ij}^{i+r-1} \Big/ \prod_{j>n} (a_{ij} - a_{in}),$$

as required.

3. GENERATING FUNCTION FOR SEQUENCE OF POWERS

Van der Poorten [6] further proved that if

$$w^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)} x^n = f(x)/x^i g(x^{-1}), \tag{3.1}$$

then there exists a polynomial $f_r(x)$ of degree at most $u-1$, such that

$$w_r^{(i)}(x) = f_r(x)/x^u g_r(x^{-1}), \qquad u = \binom{r+i-1}{r}. \tag{3.2}$$

We first seek an expression for $f_r(x)$.

$$w_r^{(i)}(x) = w_0^{(i)^r} + w_1^{(i)^r} x + w_2^{(i)^r} x^2 + \cdots + w_{u-1}^{(i)^r} x^{u-1} + w_u^{(i)^r} x^u + \cdots$$

$$-R_{u1} x w_r^{(i)}(x) = -R_{u1} w_0^{(i)^r} x - R_{u1} w_1^{(i)^r} x^2 - \cdots - R_{u1} w_{n-2}^{(i)^r} x^{u-1} - R_{u1} w_{u-1}^{(i)^r} x^u - \cdots$$

$$-R_{u2} x^2 w_r^{(i)}(x) = -R_{u2} w_0^{(i)^r} x^2 - \cdots - R_{u2} w_{n-3}^{(i)^r} x^{u-1} - R_{u2} w_{u-1}^{(i)^r} x^u - \cdots$$

$$\vdots$$

$$-R_{u,u-1} x^{u-1} w_r^{(i)}(x) = -R_{u,u-1} w_0^{(i)^r} x^{u-1} - R_{u,u-1} w_1^{(i)^r} x^u - \cdots$$

$$-R_{uu} x^u w_r^{(i)}(x) = -R_{uu} w_0^{(i)^r} x^u - \cdots$$

We then sum both sides of these equations. On the left we have

$$w_r^{(i)}(x)\left(1 - \sum_{j=1}^{u} R_{uj} x^j\right) = w_r^{(i)}(x) x^u \left(x^{-u} - \sum_{j=1}^{u} R_{uj} x^{-(u-j)}\right) = w_r^{(i)}(x) x^u g_r(x^{-1}),$$

as in van der Poorten.

On the right we obtain

$$f_r(x) = \sum_{j=0}^{u-1} T_{uj} x^j, \tag{3.3}$$

where

$$T_{uj} = w_j^{(i)^r} - \sum_{m=0}^{j} R_{um} w_{j-m}^{(i)^r}, \qquad R_{u0} \equiv 0,$$

since

$$w_n^{(i)^r} x^n = \sum_{j=1}^{u} R_{uj} w_{n-j}^{(i)^r} x^n.$$

Thus we have

$$w_r^{(i)}(x) = \left(\sum_{j=0}^{u-1} \left\{ w_j^{(i)^r} - \sum_{m=1}^{j} R_{um} w_{j-m}^{(i)^r} \right\} x^j\right) \Big/ x^u g_r(x^{-1}). \tag{3.4}$$

We now show how (3.4) agrees with Eq. (33) of Horadam [3] when $i=2$ and $r=2$. We first multiply each side of the equation by $x^3 g_2(x^{-1})$.

The left-hand side of (3.4) is then

$$x^3 g_2(x^{-1}) w_2^{(2)}(x) = (-1(P_{21}^2 + P_{22})x - (P_{22}^2 + P_{21}^2 P_{22})x^2 + P_{22}^3 x^3) w_2^{(2)}(x)$$
$$= (1 + P_{22}x)(1 - (P_{21}^2 + 2P_{22})x + P_{22}^2 x^2) w_2^{(2)}(x).$$

When $i=2$, the right-hand side of (3.4) is

$$\sum_{j=0}^{2}\left\{ w_j^{(2)^2} - \sum_{m=1}^{j} R_{3m} w_{j-m}^{(2)^2} \right\} x^j = w_0^{(2)^2} + w_1^{(2)^2} x + w_2^{(2)^2} x^2 - R_{31} w_0^{(2)} x^2 - R_{31} w_1^{(2)} x^2 - R_{32} w_0^{(2)^2} x^2$$

$$= w_0^{(2)^2} + w_1^{(2)^2} x + P_{21} w_1^{(2)^2} x^2 + P_{22}^2 w_0^{(2)^2} x^2 + 2P_{21}P_{22} w_0^{(2)} w_1^{(2)} x^2$$

$$- P_{21} w_0^{(2)^2} x - P_{22} w_0^{(2)^2} x - P_{21} w_1^{(2)^2} x^2 - P_{22} w_1^{(2)^2} x^2$$

$$- P_{22}^2 w_0^{(2)^2} x^2 - P_{21}^2 P_{22} w_0^{(2)^2} x^2$$

$$= (1 + P_{22}x) w_0^{(2)^2} - (1 + P_{22}x)(P_{21} w_0^{(2)} - w_1^{(2)})^2 x$$

$$- 2x(P_{21} w_0^{(2)} w_1^{(2)} + P_{22} w_0^{(2)^2} - w_1^{(2)^2}) \frac{(1 + P_{22}x)}{(1 + P_{22}x)}$$

$$= (1 + P_{22}x)(w_0^{(2)^2} - x(P_{21} w_0^{(2)} - w_1^{(2)})^2 - 2x\, e\, w_0^{(2)}(-P_{22}x))$$

(since $w_0(-P_{22}x)$ $= (1 + P_{22}x)^{-1}$).

This agrees with Horadam's Eq. (33) if we multiply that equation through by $(1 + P_{22}x)$ and note that $a_{21}^2 + a_{22} = P_{21}^2 + 2P_{22}$. When $r = 1$, we get $u = i$, $R_{im} = P_{im}$. If we consider the special case of $\{w_n^{(i)}\}$:

$$w_n^{(i)} = 0, \qquad n < 0$$

$$w_n^{(i)} = 1, \qquad n = 0$$

$$w_n^{(i)} = \sum_{r=1}^{i} P_{ir} w_{n-r}^{(i)}, \qquad n > 0,$$

then $\{w_n^{(i)}\} \equiv \{u_n^{(i)}\}$, the fundamental sequence discussed by Shannon [7], and (3.4) becomes

$$u^{(i)}(x) = \left\{ \sum_{j=0}^{n-1} \left\{ u_j^{(i)} - \sum_{m=1}^{j} P_{im} u_{j-m}^{(i)} \right\} x^j \right\} \Big/ x^i g(x^{-1}) = \left\{ u_0^{(i)} + \sum_{j=1}^{n-1} \left(u_j^{(i)} - u_j^{(i)} \right) x^j \right\} \Big/ x^i g(x^{-1})$$

$$= 1/x^i g(x^{-1}), \qquad \text{where} \qquad n = \binom{i+r-1}{r},$$

which is effectively Eq. (1) of Hoggatt and Lind [2]. (Equation (2) of Hoggatt and Lind [2] is essentially the same as Eq. (2.4) of Shannon.)

Thus in (2.9) we have found the coefficients in the recurrence relation for $\{w_n^{(i)^r}\}$ and in (3.4) an explicit expression for the generating function for $\{w_n^{(i)r}\}$.

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LATIN k-CUBES

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1. LATIN SQUARES

A Latin square of order n is an $n \times n$ square in which each of the numbers $0, 1, \cdots, n-1$ occurs exactly once in each row and exactly once in each column. For example

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{matrix}$$

are Latin squares of order $2, 3, 4,$ respectively. Two Latin squares of order n are orthogonal, if when one is superimposed on the other, every ordered pair $00, 01, \cdots, n-1\ n-1$ occurs. Thus

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{matrix} \quad \text{superimpose to} \quad \begin{matrix} 0\,0 & 1\,1 & 2\,2 \\ 1\,2 & 2\,0 & 0\,1 \\ 2\,1 & 0\,2 & 1\,0 \end{matrix}$$

and therefore are orthogonal squares of order 3. A set of Latin squares of order n is orthogonal if every two of them are orthogonal. As an example the 4×4 square of triples

$$\begin{matrix} 0\,0\,0 & 1\,1\,1 & 2\,2\,2 & 3\,3\,3 \\ 1\,2\,3 & 0\,3\,2 & 3\,0\,1 & 2\,1\,0 \\ 2\,3\,1 & 3\,2\,0 & 0\,1\,3 & 1\,0\,2 \\ 3\,1\,2 & 2\,0\,3 & 1\,3\,0 & 0\,2\,1 \end{matrix}$$

represents three mutually orthogonal squares of order 4 since each of the 16 pairs $00, 01, \cdots, 33$ occurs in each of the three possible positions among the 16 triples.

There cannot exist more than $n-1$ mutually orthogonal Latin squares of order n, and the existence of such a complete system is equivalent to the existence of a finite projective plane of order n, that is a system of n^2+n+1 points and n^2+n+1 lines with $n+1$ points on each line. If n is a power of a prime there exist finite fields of order n which can be used to construct finite projective planes of order n. So, for $n = 2, 3, 4, 5, 7, 8, 9$ there exist complete systems of $n-1$ orthogonal Latin squares of order n. We have listed the examples $n = 2, 3, 4,$ above. It is known [2] that there are no orthogonal Latin squares of order 6 and that there are at least two orthogonal Latin squares of every order $n > 2$, $n \neq 6$. In fact, the number of mutually orthogonal Latin squares of order n goes to infinity with n [3]. However no case of a complete system of $n-1$ orthogonal Latin squares is known for any n which is not a power of a prime.

2. LATIN CUBES

We can generalize all these concepts to $n \times n \times n$ cubes and cubes of higher dimensions.

A Latin cube of order n is an $n \times n \times n$ cube (n rows, n columns and n files) in which the numbers $0, 1, \cdots, n-1$ are entered so that each number occurs exactly once in each row, column and file. If we list the cube in terms of the n squares of order n which form its different levels we can list the cubes

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{matrix} \qquad \begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix}$$

as Latin cubes of order two and three, respectively. Since even this method of listing becomes unwieldy for higher dimensions we also use a listing by indices. Thus we write the first cube as $A = (a_{ijk})$ with $a_{000} = 1, a_{010} = 1, a_{011} = 0, a_{100} = 1, a_{101} = 0, a_{110} = 0, a_{111} = 1$. In a similar manner we can describe four-dimensional cubes $A = (a_{ijk\ell})$ or order n, where each of the indices, i,j,k,ℓ ranges from 1 to n. Generally we can discuss k-cubes $A = (a_{i_1 i_2 \cdots i_k})$ with k indices ranging from 1 to n. These cubes will be Latin k-cobes of order n if each of the n^k entries $a_{i_1 \cdots i_k}$ is one of the numbers $0, 1, \cdots, n-1$ so that $a_{i_1 \cdots i_k}$ ranges over all these numbers as one of the indices varies from 1 to n while the other indices remain fixed.

Orthogonality of Latin cubes is now a relation among three cubes, or in general among k Latin k-cubes. That is, three Latin cubes of order n are orthogonal if, when superimposed, each ordered triple will occur. Similarly k Latin k-cubes are orthogonal if, when superimposed, each ordered k-tuple will occur. A set of at least k Latin k-cubes is orthogonal if every k of its cubes are orthogonal.

Theorem. If there exist two orthogonal Latin squares of order n then there exist 4 orthogonal Latin cubes of order n and k orthogonal Latin k-cubes for each $k > 3$.

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$ be orthogonal Latin squares of order n.

Define 4 cubes C, D, E, F of order n by

$$c_{ijk} = a_{a_{ij},k}, \; d_{ijk} = a_{b_{ij},k}, \; c_{ijk} = b_{a_{ij},k}, \; f_{ijk} = b_{b_{ij},k}, \; i,j,k = 0, 1, \cdots, n-1.$$

Note that the squares A,B are used both as entries and as indices in the construction of the cubes. For example the pair of 3×3 Latin squares

$$\begin{matrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{matrix} \qquad \begin{matrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{matrix}$$

leads to the four $3 \times 3 \times 3$ cubes

$$\text{C:} \quad \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{matrix} \qquad \begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix}$$

$$\text{D:} \quad \begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{matrix} \qquad \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{matrix}$$

$$\text{E:} \quad \begin{matrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{matrix} \qquad \begin{matrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{matrix}$$

$$\text{F:} \quad \begin{matrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{matrix} \qquad \begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{matrix}$$

Superimposed these lead to a cube of quadruples

$$\text{CDEF:} \quad \begin{matrix} 0000 & 1122 & 2211 \\ 1221 & 2010 & 0102 \\ 2112 & 0201 & 1020 \end{matrix} \qquad \begin{matrix} 1111 & 2200 & 0022 \\ 2002 & 0121 & 1210 \\ 0220 & 1012 & 2101 \end{matrix} \qquad \begin{matrix} 2222 & 0011 & 1100 \\ 0110 & 1202 & 2021 \\ 1001 & 2120 & 0212 \end{matrix}$$

where each ordered triple occurs in every one of the four possible positions in the quadruples.

It is easy to see that C,D,E,F are Latin cubes. For example, for fixed i,j the values $c_{ijk} = a_{a_{ij},k}$ go through the a_{ij}^{th} row of A, that is, through the values $0, 1, \cdots, n-1$. For fixed i,k the index a_{ij} goes through all the values in the ith row of A, that is, through all values $0, 1, \cdots, n-1$ and hence c_{ijk} goes through all values in the kth column of A. Finally for fixed j,k the index a_{ij} goes through all values in the jth column of A and therefore c_{ijk} again goes through all values in the kth column of A.

To prove the orthogonality of, say, C,D,E we have to prove that for every triple (x,y,z) from $\{0, 1, \cdots, n-1\}$ the equations

$$c_{ijk} = x, \qquad d_{ijk} = y, \qquad e_{ijk} = z$$

have a solution i,j,k. By the orthogonality of A and B the pair (x,z) occurs exactly once in the superimposed square AB so that the equations $a_{\ell,k}=x$, $b_{\ell,k}=z$ determine k and ℓ. Thus the equations

$$c_{ijk} = a_{a_{ij},k} = x \qquad \text{and} \qquad e_{ijk} = b_{a_{ij},k} = z$$

determine a_{ij} and k. Now, since A is a Latin square, there is exactly one occurrence of y in the kth column of A so the equation

$$d_{ijk} = a_{b_{ij},k} = y$$

determines b_{ij} and the pair (a_{ij},b_{ij}) determines i,j;

Thus for every triple (x,y,z) there is a unique triple (i,j,k).

This construction is essentially that given by Arkin for 4 orthogonal $10\times 10\times 10$ cubes [1].

To prove the last part of the theorem we proceed by induction on k. Let $A^1,\cdots,A^k$ be orthogonal Latin k-cubes of order n, and write the entries of A^j as $a^j_{i_1,\cdots,i_k}$. We now define $k+1$ orthogonal Latin $(k+1)$-cubes $B^1,\cdots,B^{k+1}$ by

$$b^1_{i_1,\cdots,i_{k+1}} = a^1_{a^1_{i_1,\cdots,i_k},i_{k+1}}$$

$$\vdots$$

$$b^k_{i_1,\cdots,i_{k+1}} = a^k_{a^1_{i_1,\cdots,i_k},i_{k+1}}$$

$$b^{k+1}_{i_1,\cdots,i_{k+1}} = b^1_{a^1_{i_1,\cdots,i_k},i_{k+1}} \quad .$$

We omit the proof that the B^j are Latin cubes, which is the same as before. In order to prove orthogonality we have to solve

$$B^j_{i_1\cdots i_{k+1}} = x_j \qquad j = 1,\cdots,k+1 .$$

For any $(k+1)$-tuple $(x_1,\cdots,x_{k+1})$ from $\{0,1,\cdots,n-1\}$. Now, by the orthogonality of A and B the two equations

$$A^1_{a^1_{i_1,\cdots,i_k},i_{k+1}} = x_1, \qquad B^1_{a^1_{i_1,\cdots,i_{k+1}},i_{k+1}} = x_{k+1}$$

determine $a^1_{i_1\cdots i_k}$ and i_{k+1}. Once i_{k+1} is determined the equations

$$A^j_{a^j_{i_1,\cdots,i_k},i_{k+1}} = x_j \qquad j = 2,\cdots,k$$

determine

$$a^j_{i_1\cdots i_k} \qquad (j = 2,\cdots,k).$$

Once the elements

$$a^j_{i_1\cdots i_k} \qquad (j = 1,\cdots,k)$$

are determined it follows from the orthogonality of the k-cubes $A^1,\cdots,A^k$ that the indices $i_1,\cdots,i_k$ are determined. Thus for every $(k+1)$-tuple $(x_1,\cdots,x_{k+1})$ there is a unique $(k+1)$-tuple $(i_1,\cdots,i_{k+1})$ with

$$B^j_{i_1\cdots i_{k+1}} = x_j \qquad j = 1,\cdots,k+1.$$

Since, as we mentioned, there are orthogonal Latin squares of every order $n\geq 2$, $n\neq 6$ we have the following.

Corollary. There exist orthogonal k-tuples of Latin k-cubes of order n for every $n>2$, $n\neq 6$.

3. FINITE FIELDS

A field is a system of elements closed under the rational operations of addition, subtraction, multiplication and division (except by 0) subject to the usual commutative, associative and distributive laws. There exist finite fields with n elements if and only if n is a power of a prime p. The prime p is the characteristic of the field and we have $p\alpha=0$ for every α in the field. Following are the addition and multiplication tables for the fields with 3 and 4 elements:

F_3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

F_4

$+$	0	1	a	$1+a$
0	0	1	a	$1+a$
1	1	0	$1+a$	a
a	a	$1+a$	0	1
$1+a$	$1+a$	a	1	0

$\times$	0	1	a	$1+a$
0	0	0	0	0
1	0	1	a	$1+a$
a	0	a	$1+a$	1
$1+a$	0	$1+a$	1	a

.

If there is a field F_n with n elements, that is if n is a power of a prime, we use the elements $\{f_1, f_2, \cdots, f_n\}$ of F_n as indices to construct Latin squares, cubes, etc. We give the construction for cubes, but the generalization to k-cubes is easily seen.

Let α, β, γ be three nonzero elements of F_n then we can define the Latin cube $A = (a_{ijk})$ by

$$a_{ijk} = \alpha f_i + \beta f_j + \beta f_k \ .$$

To see that A is a Latin cube consider, say, fixed i,j and see that γf_k runs through all elements of F_n as f_k does. Hence a_{ijk} runs through F_n as $k = 1, \cdots, n$.

Now let $(\alpha,\beta,\gamma), (\alpha',\beta',\gamma')$ and $(\alpha'',\beta'',\gamma'')$ be three triples of nonzero elements of F_n so that the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \neq 0.$$

Then the three Latin cubes

$$A = (a_{ijk}), \quad A' = (a'_{ijk}), \quad A'' = (a''_{ijk})$$

with

$$a_{ijk} = \alpha f_1 + \beta f_j + \gamma f_k, \quad a'_{ijk} = \alpha' f_i + \beta' f_j + \gamma' f_k, \quad a''_{ijk} = \alpha'' f_i + \beta'' f_j + \gamma'' f_k$$

are orthogonal. This follows from the fact that for any triple (x,y,z) from F_n the three equations

$$a_{ijk} = x, \quad a'_{ijk} = y, \quad a''_{ijk} = z$$

have a unique solution f_i, f_j, f_k.

Now the Vandermonde determinants

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$$

are different from zero for any three distinct elements α,β,γ of F_n. Thus, letting α run through the nonzero elements of F_n we get $n-1$ orthogonal Latin cubes of order n,

$$A^\alpha = (a^\alpha_{ijk}), \quad a^\alpha_{ijk} = f_i + \alpha f_j + \alpha^2 f_k \ .$$

The construction for a system of $n-1$ orthogonal Latin k-cubes of order n proceeds in exactly the same way if we set

$$A^\alpha = (a^\alpha_{i_1 \cdots i_k}), \quad a^\alpha_{i_1 \cdots i_k} = f_{i_1} + \alpha f_{i_2} + \cdots + \alpha^{k-1} f_{i_k}$$

where α runs through the nonzero elements of F_n.

Theorem. If n is a power of a prime and $k < n$, then there exists a system of $n-1$ orthogonal k-cubes of order n.

Our previous examples constructing four orthogonal cubes of orders 3 and 4 show that $n-1$ is not necessarily the maximal number of orthogonal k-cubes of order n for $k > 2$. However, the orthogonal cubes constructed with the aid of finite fields satisfy additional properties. For each fixed value of k the squares

$$A^\alpha_{\cdot\cdot k} = (a^\alpha_{ijk}) \qquad i,j = 1, 2, \cdots, n$$

form a complete system of $n-1$ orthogonal Latin squares as α ranges through the nonzero elements of F_n, and similarly for each fixed i the squares

$$A^\alpha_{i\cdot\cdot} = (a^\alpha_{ijk}) \qquad j,k = 1, 2, \cdots, n$$

form a complete system of orthogonal Latin squares. If n is a power of 2 then the third family of cross-sections

$$A^\alpha_{\cdot j\cdot} = (a^\alpha_{ijk}) \qquad i,k = 1, 2, \cdots, n$$

form a complete system of orthogonal Latin squares for each fixed j, while for n odd we get a system of $(n-1)/2$ orthogonal Latin squares, each square occurring twice.

Theorem. If n is a power of 2 then there exist $n-1$ orthogonal Latin cubes of order n with the property that the corresponding plane sections form systems of $n-1$ orthogonal Latin squares.

If n is a power of an odd prime then there exist $n-1$ orthogonal Latin cubes with the property that the corresponding plane cross-sections in two directions form complete systems of orthogonal Latin squares, while the plane cross-sections in the third direction form a system of $(n-1)/2$ orthogonal Latin squares, each square occurring twice.

Finally we observe that if we have orthogonal k-cubes of orders m and n then we can form their Kronecker products to obtain orthogonal k-cubes of order mn. That is from orthogonal k-cubes

$$A^1 = (a^1_{i_1\cdots i_k}), \cdots, A^\ell = (a^\ell_{i_1\cdots i_k}); \qquad B^1 = (b^1_{i_1\cdots i_k}), \cdots, \qquad B^\ell = (b^\ell_{i_1\cdots i_k}),$$

where the a's run from 1 to m and the b's from 1 to n we can form the orthogonal k-cubes $C^1, \cdots, C^\ell$, where

$$C^j = (c^j_{i_1\cdots i_k}) \qquad \text{and} \qquad c^j_{i_1\cdots i_k} = (a^j_{i_1\cdots i_k}, b^j_{i_1\cdots j_k})$$

so that the c's run through all ordered pairs $(1,1), \cdots, (m,n)$ as the pairs $(i_1,j_1), \cdots, (i_k,j_k)$ run through these ordered pairs. Thus we have the following.

Corollary. If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \qquad \text{and} \qquad q = \min_{1 \leqslant j \leqslant s} p_j^{\alpha_j}$$

then for any $k < q$ there exist at least $q-1$ orthogonal Latin k-cubes of order n.

The relation to finite k-dimensional projective spaces is not as immediate as it is for Latin squares, and we shall not discuss it here.

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ON EXTENDING THE FIBONACCI NUMBERS TO THE NEGATIVE INTEGERS

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A sequence of positive integers defined by the formula

(1) $$x_{n+1} = ax_n + bx_{n-1}, \quad n \text{ a positive integer,}$$

is said to be extendable to the negative integers if (1) holds for n any integer. See page 28 of [1]. The purpose of this note is to show that the Fibonacci numbers form a sequence which is extendable to the negative integers in a unique way. In this note only nontrivial integral sequences will be considered.

[Continued on Page 308.]

A METHOD OF CARLITZ APPLIED TO THE K^{TH} POWER GENERATING FUNCTION FOR FIBONACCI NUMBERS

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1. INTRODUCTION

If we consider $f(x)$ such that the power series expansion of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \ , \tag{1.1}$$

then $f(x)$ is called the ordinary generating function of the sequence $\{ f_n \}$.

We define the generating function for the kth power of f_n as

$$f_k(x) = \sum_{n=0}^{\infty} f_n^k x^n \ . \tag{1.2}$$

The complexity of expressions which involve f_n^k increases as k increases. This makes it increasingly difficult to determine $f_k(x)$ by the methods described by Hoggatt and Lind [2]. Riordan [5] devised a method to overcome this. His approach depended basically on the expansion of f_n^k by the binomial theorem and subsequent examination of the coefficients. Carlitz [1] applied this to the more general relation

$$u_n = pu_{n-1} + qu_{n-2} \quad (n > 2), \qquad u_0 = 1, \qquad u_1 = p \ . \tag{1.3}$$

He then developed an elegant approach which employed a special function of x and z and depended for success on the identity $u_{n+1}u_{n-1} - u_n^2 = q^n$. Because it is so elegant and because it has appeared hitherto in abbreviated form in papers by Carlitz, Riordan, and Horadam [3], it is proposed here to apply it to the Fibonacci sequence and to expound it in sufficient detail for the general reader to be able to follow it. It is worth pointing out that Kolodner [4] used another approach in which he exploited the fact that the zeros of $z^2 - 2z\cos\theta + 1$, with any θ real or complex, are $e^{i\theta}$ and $e^{-i\theta}$, the powers of which are easily managed.

2. CARLITZ' METHOD

Following Carlitz, we write

$$F(x,z) = \sum_{k=1}^{\infty} (1 - a^k x)(1 - b^k x) f_k(x) \frac{z^k}{k} \ , \tag{2.1}$$

where $a = \frac{1}{2}(1 + \sqrt{5})$ and $b = \frac{1}{2}(1 - \sqrt{5})$ satisfy the auxiliary equation $x^2 - x - 1 = 0$. If we expand this, $F(x,z)$ using the power series expansion of $\log(1 + z)$, we find that

$$F(x,z) = \sum_{k=1}^{\infty}(1-a^k+b^k)x+(ab)^k x^2)\frac{z^k}{k}\sum_{j=0}^{\infty} f_j^k x^j$$

$$= -\sum_{j=0}^{\infty} x^j \log(1-f_j z)+\sum_{j=0}^{\infty} x^{j+1}\log(1-af_j z)$$

$$+\sum_{j=0}^{\infty} x^{j+1}\log(1-bf_j z)-\sum_{j=0}^{\infty} x^{j+2}\log(1+f_j z)$$

$$= -\log(1-f_0 z)+x\log(1+f_{-1}z)$$

$$+x\sum_{j=0}^{\infty} x^j \log(1-(a+b)f_j z+abf_j^2 z^2)$$

$$-x\sum_{j=0}^{\infty} x^j \log(1-f_{j+1}z)-x\sum_{j=0}^{\infty} x^j \log(1+f_{j-1}z).$$

Since $f_{j+1}f_{j-1}-f_j^2=(-1)^{j-1}$, it follows that

$$(1-f_{j+1}z)(1+f_{j-1}z) = 1-(f_{j+1}-f_{j-1})z-f_{j+1}f_{j-1}z^2$$
$$= 1-f_j z-(f_j^2-(-1)^j)z^2 .$$

These last two lines are the crucial steps because they make it possible to eliminate terms in z from the numerator in the next few lines. It is the inability to do this with higher degree equations that seems to make the method break down then as will be pointed out later.

$$F(x,z) = -\log(1-f_0 z)+x\log(1+f_{-1}z)$$

$$+x\sum_{j=0}^{\infty} x^j \log(1-f_j z-f_j^2 z^2) \tag{2.2}$$

$$-x\sum_{j=0}^{\infty} x^j \log(1-f_j z-(f_j^2-(-1)^j)z^2).$$

The last two terms can be combined to give

$$x\sum_{j=0}^{\infty} x^j\left\{-\log\left[1+\frac{(-1)^j z^2}{1-f_j z-f_j^2 z^2}\right]\right\},$$

where there is no z in the numerator. This becomes

$$x\sum_{j=0}^{\infty} x^j \sum_{r=1}^{\infty}\frac{(-1)^r}{r}\,\frac{(-1)^{rj}z^{2r}}{(1-f_j z-f_j^2 z^2)^r}$$

$$= x\sum_{j=0}^{\infty} x^j \sum_{r=1}^{\infty}\frac{(-1)^{r+rj}}{r}\,z^{2r}\sum_{k=2r}^{\infty} a_{kr}(f_j z)^{k-2r} . \tag{2.3}$$

The numbers a_{kr} are, in a sense, the "r^{th} convoluted Fibonacci numbers;" they are generated by the rth power of the ordinary generating function for Fibonacci members. They will be considered in more detail in Section 4. (2.3) becomes

$$x \sum_{j=0}^{\infty} x^j \sum_{r=1}^{\infty} \frac{(-1)^{r+rj}}{r} \sum_{k=2r}^{\infty} a_{kr} f_j^{k-2r} z^k$$

$$= x \sum_{j=0}^{\infty} x^j \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^{r+rj}}{r} a_{kr} f_j^{k-2r}$$

in which $[k/2]$ is the greatest integer function: it represents the integral part of the real number $k/2$.

If we replace this in (2.2) we get

$$F(x,z) = -\log(1 - f_0 z) + x \log(1 + f_{-1} z)$$

$$+ x \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} \sum_{j=0}^{\infty} f_j^{k-2r} ((-1)^r x)^j \tag{2.4}$$

$$= -\log(1 - f_0 z) + x \log(1 + f_{-1} z) + x \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} f_{k-2r}((-1)^r x) .$$

Comparing coefficients of z^k we get

$$\frac{1}{k}(1 - \ell_k x + (-1)^k x^2) f_k(x) = \frac{f_0^k}{k} - x \frac{(-f_{-1})^k}{k} + x \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} f_{k-2r}((-1)^r x) ,$$

where ℓ_k is the kth Lucas number. Thus,

$$(1 - \ell_k x + (-1)^k x^2) f_k(x) = 1 + kx \sum_{r=1}^{[k/2]} (-1)^r (a_{kr}/r) f_{k-2r}((-1)^r x) , \tag{2.5}$$

which agrees with the result obtained by Riordan's method [5]. For example, put $k=2$, and

$$(1 - 3x + x^2) f_2(x) = 1 + 2x(-1)(1) f_0(-x) = 1 - \frac{2x}{1+x}$$

which gives

$$f_2(x) = \frac{1 - x}{1 - 2x - 2x^2 + x^3} .$$

3. THE COEFFICIENTS OF $f_k(x)$

It is still necessary to look more closely at the coefficients, especially for high k. Carlitz' approach here is also rewarding to study. Applying his method to the Fibonacci coefficients we get from before

$$f_k(x) = \sum_{n=0}^{\infty} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}} \right)^k x^n$$

$$= 5^{k/2} \sum_{s=0}^{k} \binom{k}{s} \left\{ a^{k-s} b^s + a^{2k-2s} b^{2s} x + a^{3k-3s} b^{3s} x^2 + \dots \right\} \tag{3.1}$$

$$= 5^{k/2} \sum_{s=0}^{k} \binom{k}{s} a^{k-s} b^s (1 - a^{k-s} b^s x)^{-1} .$$

Define,

$$D_k(x) = \prod_{s=0}^{k} (1 - a^{k-s}b^s x)$$

and write $f_k(x) = F_k(x)/D_k(x)$, where $F_k(x)$ is a polynomial of degree $< k$ $(k \geqslant 1)$. We show that the coefficients of these polynomials satisfy certain recurrence relations and can be determined explicitly.

$$f_{k+1}(x) = \sum_{n=0}^{\infty} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right) x^n$$

(3.2)
$$= \sum_{n=0}^{\infty} \frac{a}{\sqrt{5}} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k (ax)^n - \frac{b}{\sqrt{5}} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k (bx)^n$$

$$= \frac{a}{\sqrt{5}} f_k(ax) - \frac{b}{\sqrt{5}} f_k(bx) .$$

Then

(3.3)
$$\frac{F_{k+1}(x)}{D_{k+1}(x)} = \frac{a}{\sqrt{5}} \frac{F_k(ax)}{D_k(ax)} - \frac{b}{\sqrt{5}} \cdot \frac{F_k(bx)}{D_k(bx)} .$$

Now,

$$\frac{D_{k+1}(x)}{D_k(ax)} = \frac{\prod_{s=0}^{k+1} (1 - a^{k+1-s}b^s x)}{\prod_{s=0}^{k} (1 - a^{k+1-s}b^s x)} = (1 - b^{k+1}x) .$$

Similarly,

$$\frac{D_{k+1}(x)}{D_k(bx)} = (1 - a^{k+1}x)$$

Whence from (3.3) we get

(3.4)
$$F_{k+1}(x) = \frac{a}{\sqrt{5}} (1 - b^{k+1}x)F_k(ax) - \frac{b}{\sqrt{5}} (1 - a^{k+1}x)F_k\, bx) .$$

Put

(3.5)
$$F_k(x) = \sum_{s=0}^{k} F_{ks}x^s$$

and it follows from (3.4) if we equate coefficients of x^j that

(3.6)
$$F_{k+1,j} = \frac{a^{j+1}}{\sqrt{5}} F_{kj} - \frac{a^j b^{k+1}}{\sqrt{5}} F_{k,j-1} - \frac{b^{j+1}}{\sqrt{5}} F_{kj} + \frac{a^{k+1}b^j}{\sqrt{5}} F_{k,j-1}$$
$$= f_j F_{kj} + (-1)^k f_{-(k-j+2)} F_{k,j-1}$$

which is an expression that enables us to find $F_k(x)$ explicitly. We still need to find D_k and to do this we need the following piece of algebra.

It can be shown easily that

$$\prod_{s=0}^{3}(1-z^s x) = (-1)^0 z^0 x^0 + (-1)^1 \frac{(z^4-1)}{(z-1)} z^0 x^1$$

$$+(-1)^2 \frac{(z^4-1)(z^3-1)}{(z-1)(z^2-1)} z^1 x^2 + (-1)^3 \frac{(z^4-1)(z^3-1)(z^2-1)}{(z-1)(z^2-1)(z^3-1)} z^3 x^3 \tag{3.7}$$

$$+(-1)^4 \frac{(z^4-1)(z^3-1)(z^2-1)(z-1)}{(z-1)(z^2-1)(z^3-1)(z^4-1)} z^6 x^4 = \sum_{s=0}^{4} (-1) z^{\frac{1}{2}s(s-1)} \begin{bmatrix} 4 \\ s \end{bmatrix} x^s ,$$

where

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} 4 \\ s \end{bmatrix} = \frac{(z^4-1)(z^3-1)\cdots(z^{4-s+1}-1)}{(z-1)(z^2-1)\cdots(z^s-1)} \quad (s>0) .$$

More generally we have that

$$\prod_{s=0}^{k}(1-z^s x) = \sum_{s=0}^{k+1} (-1)^s z^{\frac{1}{2}s(s-1)} \begin{bmatrix} k+1 \\ s \end{bmatrix} x^s , \tag{3.8}$$

where

$$\begin{bmatrix} k+1 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} k+1 \\ s \end{bmatrix} = \frac{(z^{k+1}-1)(z^k-1)\cdots(z^{k-s+2}-1)}{(z-1)(z^2-1)\cdots(z^s-1)} \quad (s>0) .$$

In

$$\begin{bmatrix} k+1 \\ s \end{bmatrix}$$

replace z by b/a, and

$$\begin{bmatrix} k+1 \\ s \end{bmatrix} = \frac{((b/a)^{k+1}-1)((b/a)^k-1)\cdots((b/a)^{k-s+2}-1)}{((b/a)-1)((b/a)^2-1)\cdots((b/a)^s-1)}$$

$$= \frac{a^{\frac{s}{2}(1+s)}(b^{k+1}-a^{k+1})(b^k-a^k)\cdots(b^{k-s+2})}{a^{\frac{s}{2}(2k-s+3)}(b-a)(b^2-a^2)\cdots(b^s-a^s)}$$

$$= a^{-ks+s(s-1)} \frac{f_k f_{k-1}\cdots f_{k-s+1}}{f_0 f_1 \cdots f_{s-1}} = a^{-ks+s(s-1)} \left\{ {k \atop s} \right\} .$$

Thus if we replace x by $a^k x$ in (3.8) we get

$$D_k(x) = \sum_{s=0}^{k+1} (-1)^{\frac{1}{2}s(s+1)} \left\{ {k \atop s} \right\} x^s , \tag{3.9}$$

since $ab = -1$. This completes the examination of the nature of the coefficients of $f_k(x)$.

4. CONVOLUTED FIBONACCI NUMBERS

We shall now review briefly the so-called "convoluted" Fibonacci numbers [5]. a_{kj} satisfies the recurrence relation

$$a_{kj} - a_{k-1,j} - a_{k-2,j} = a_{k-2,j-1}, \quad k > 2j+2 . \tag{4.1}$$

Moreover, it is convenient to write

$$a_{kj} = 0, \quad k < 2j .$$

By definition,

$$a_j(x) = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j} .$$

Consider

$$\begin{aligned}(1-x-x^2(a_j(x) &= a_{2j,j}+(a_{2j+1,j}-a_{2j,j})x+(a_{2j+2,j}-a_{2j+1,j}-a_{2j,j})x^2+\cdots\\ &= a_{2j,j}+a_{2j-1,j}x+a_{2j-1,j-1}x+a_{2j,j-1}x^2+\cdots\\ &= a_{2j,j}+a_{2j-1,j}x-a_{2j-2,j-1}+a_{2j-2,j-1}+a_{2j-1,j-1}x+a_{2j,j-1}x^2+\cdots\\ &= a_{j-1}(x)\end{aligned}$$

since $a_{kj} = 0,\ k < 2j$. Thus

(4.2) $$(1-x-x^2)^j a_j(x) = (1-x-x^2)^{j-1} a_{j-1}(x) = (1-x-x^2)^{j-2} a_{j-2}(x) = (1-x-x^2)a_1(x) = 1\ .$$

Hence

(4.3) $$a_j(x) = (1-x-x^2)^{-j} = \left\{f(x)\right\}^j\ ,$$

where $f(x)$ is the ordinary generating function for Fibonacci numbers.

5. PROBLEMS FOR FURTHER STUDY

Consider the third-order recurrence relation.

(5.1) $$K_n = K_{n-1}+K_{n-2}+K_{n-3} \quad (n > 3)$$

and the sequences

$$\begin{array}{l} 0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ \cdots,\ K_n,\ \cdots\\ 1,\ 0,\ 1,\ 2,\ 3,\ 6,\ 11,\ 20,\ 37,\ \cdots,\ L_n,\ \cdots\end{array}$$

in which

$$L_0 = K_1-K_0,\quad L_1 = K_2-K_1\ ,$$

and for $n > 2$,

$$L_n = K_{n-1}+K_{n-2}\ .$$

Using a simple induction proof and matrix and determinant theory, we can show that

(5.2) $$\begin{vmatrix} K_{n+1} & K_{n-1} & K_n\\ K_n & K_{n-2} & K_{n-1}\\ K_{n-1} & K_{n-3} & K_{n-2}\end{vmatrix} = \begin{vmatrix} 1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}^n = 1\ .$$

Similar treatment with a fourth-order recurrence relation and the sequences

$$\begin{array}{l} 0,\ 0,\ 1,\ 1,\ 2,\ 4,\ 8,\ 15,\ 29,\ 56,\ \cdots,\ M_n,\ \cdots\\ 0,\ 1,\ 0,\ 1,\ 2,\ 4,\ 7,\ 14,\ 27,\ 52,\ \cdots,\ N_n,\ \cdots\\ 1,\ 0,\ 0,\ 1,\ 2,\ 3,\ 6,\ 12,\ 23,\ 43,\ \cdots,\ O_n,\ \cdots\end{array}$$

yields

(5.3) $$\begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+1} & M_n\\ M_{n+2} & M_{n+1} & M_n & M_{n-1}\\ M_{n+1} & M_n & M_{n-1} & M_{n-2}\\ M_n & M_{n-1} & M_{n-2} & M_{n-3}\end{vmatrix} = (-1)^n$$

Ordinary generating functions for these are easily found, but what about generating functions for the powers of the numbers? The forms of (5.2) and (5.3) by comparison with

$$u_{n+1}u_{n-1}-u_n^2 = q^2 \quad \text{and} \quad f_{n+1}f_{n-1}-f_n^2 = (-1)^{n-1}$$

rule out Carlitz' method for finding the kth power generating function for third- and fourth-order recurrence relations. The complexity of the multinomial coefficients would seem to make Riordan's approach break down. Kolodner's dependence on quadratic equation theory makes it difficult to extend his method to general cubic and quartic equations. What approaches then can be used for recurrence relations of order greater than the second?

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A CONSTRUCTED SOLUTION OF $\sigma(n) = \sigma(n+1)$

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With $\sigma(n)$ the sum of the positive divisors of n, one finds that

$$\sigma(n) = \sigma(n+1) \tag{1}$$

for

$$n = 14, 206, \cdots, 18873, 19358, \cdots, 174717, \cdots. \tag{2}$$

Sierpinski [1] asked if (1) has infinitely many solutions. Earlier, Erdös had conjectured [2] that it does, but the answer is unknown. Makowski [3] listed the nine solutions of (1) with $n < 10^4$ and subsequently Hunsucker *et al* continued and found 113 solutions with $n < 10^7$. See [4] for a reference to this larger table.

It is unlikely that there are only finitely many solutions but, in any case, there is a much larger solution, namely,

$$n = 5559060136088313. \tag{3}$$

It is easily verified that the first, second, and fourth examples in (2) are given by

$$n = 2p, \quad n+1 = 3^m q, \tag{4}$$

where

$$q = 3^{m+1} - 4, \qquad p = (3^m q - 1)/2 \tag{4a}$$

are both prime, and m equals $1, 2,$ or 4. One finds that

$$\sigma(n) = \sigma(n+1) = \frac{1}{2}(9^{m+1} + 3) - 6 \cdot 3^m. \tag{4b}$$

The third and fifth examples in (2) are given by

$$n = 3^m q, \qquad n+1 = 2p \tag{5}$$

with the primes

$$q = 3^{m+1} - 10, \qquad p = (3^m q + 1)/2 \tag{5a}$$

for $m = 4$ and 5. Then

$$\sigma(n) = \sigma(n+1) = \frac{1}{2}(9^{m+1} + 9) - 15 \cdot 3^m. \tag{5b}$$

Our new solution (3) is given by *(5 – 5a)* for $m = 16$. But there are no other examples of (5) or (4) for $m < 44$. While we do conjecture that there are infinitely many solutions of (1) we do not think that infinitely many solutions can be constructed in this way. D.H. and Emma Lehmer assisted us in these calculations.

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THE DESIGN OF THE FOUR BINOMIAL IDENTITIES: MORIARTY INTERVENES

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We have seen in a previous episode [3] some of the artful disguises of the Moriarty identities. With skillful detective work we may unmask Moriarty in many situations. The case we are about to discuss arose in a study of questions asked me by David Zeitlin (personal correspondence of 9 August 1972), and reveals Moriarty in a fourfold fantasy; for there are actually a full dozen formulas to be analyzed. As corollaries we find other interesting sums. The objective in our study is pedagogical, viz. to show how to handle Moriarty. But let us hear Zeitlin's question.

"Are the following two related identities,

$$\sum_{k=j}^{m-1} \binom{k}{j}\binom{m+k}{2k+1}(-4)^{k-j} = (-1)^{m+j+1}\binom{m+j}{2j+1}, \tag{1}$$

$$\sum_{k=j}^{m-1} \binom{k}{j}\binom{m+k-1}{2k}(-4)^{k-j} = (-1)^{m+j+1}\binom{m+j}{2j+1}\frac{2m-1}{m+j} \tag{2}$$

listed (or special cases) in your tables [4]?" asked Zeitlin. "I am convinced that (1) and (2) are correct, but I am unable to prove it so."

Zeitlin stumbled onto these formulas as a consequence of several Fibonacci identities. Naturally no set of tables is ever complete; but the careful reader will ascertain at once that relation (2) is precisely (3.162) in my tables...precisely upon changing a few letters and shifting m to $m+1$. Relation (1) is not listed. However, relations (3.160) and (3.161) are obviously related to (1) and (2) in some manner, as we shall see.

We are therefore concerned at the outset with the four identities

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} = (-1)^n \binom{n+a}{2a} 2^{2a}\frac{2n+1}{2a+1}, \tag{3} \qquad (3.162)$$

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k} = (-1)^{n-1} \binom{n+a}{2a+1} 2^{2a}, \tag{4} \qquad \text{(Zeitlin)}$$

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k}\frac{n}{n+k} = (-1)^n \binom{n+a}{2a} 2^{2a}\frac{n}{n+a}, \tag{5} \qquad (3.160)$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k}\frac{2n+1}{2k+1} = (-1)^n \binom{n+a}{2a} 2^{2a}, \tag{6} \qquad (3.161).$$

Here a is a non-negative integer; the range of summation in each case may start with $k=a$ if one prefers, since $\binom{k}{a} = 0$ for $0 \leqslant k < a$. However, we state the four in a more elegant form as above.

Relations (3) and (6) are inverses of each other; this is so because of the easy and well known inversion principle that

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a} f(k) = (-1)^n g(n)$$

if and only if

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a} g(k) = (-1)^n f(n).$$

Thus we have only to prove the one to obtain the other. Observe that (4) and (5) are self-inverses.

There are various ways to prove (3)–(6) directly; to this we shall give attention. But the main object of our work will be to show that these four sums are equivalent to the following four sums:

$$\sum_{k=0}^{n} \binom{2n+1}{2k}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a} \frac{2n+1}{2a+1}, \tag{7}$$

$$\sum_{k=0}^{n} \binom{2n}{2k+1}\binom{k}{n-1-a} = \binom{n+a}{2a+1} 2^{2a+1}, \tag{8}$$

$$\sum_{k=0}^{n} \binom{2n}{2k}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a} \frac{n}{n+a}, \tag{9}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a}. \tag{10}$$

These are the *four* relations of Moriarty. The attentive reader of [3] may at first think we proved *two* relations, and indeed we did. They were:

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{k}{r} = 2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r}, \tag{11}$$ (3.120) in [4],

and

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1}\binom{k}{r} = 2^{n-2r}\binom{n-r}{r}, \tag{12}$$ (3.121) in [4].

To see how we get (7)–(10) from these, proceed as follows. In (11) put $2n+1$ for n, and recall that $[n+\frac{1}{2}] = n$. Replace r by $n-a$. The result is (7). In (12) put $2n-1$ for n, and note that $[n-\frac{1}{2}] = n-1$. Replace r by $n-a-1$. The result is (8). In (11), put $2n$ for n and replace r by $n-a$. The result is (9). Finally, in (12), put $2n$ for n and replace r by $n-a$. The result is (10).

What we have done above is reveal the fourfold design of the Moriarty identities. These formulas occur frequently in trigonometric identities.

We shall need the easy formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{r} = (-1)^n \binom{x}{r-n} \tag{13}$$

valid for all real x; this is formula (3.47) in [4] and can be proved from the Vandermonde convolution, for example.

To carry out the proofs that the fourfold Moriarty (7)–(10) imply and are implied by (3)–(6), we need to note the following four sums:

$$\sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{a-k} = \binom{n+a}{2a} 2^{2a}, \tag{14}$$ (3.149 in [4],

(15) $$\sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{a-k} = \binom{n+a}{2a+1} 2^{2a+1} \qquad \text{(3.158) in [4].}$$

(16) $$\sum_{k=0}^{a} \binom{2n}{2k}\binom{n-k}{a-k} = \frac{n}{n+a}\binom{n+a}{2a} 2^{2a}, \qquad \text{(3.26) in [4],}$$

and

(17) $$\sum_{k=0}^{a} \binom{2n+1}{2k+1}\binom{n-k}{a-k} = \frac{2n+1}{2a+1}\binom{n+a}{2a} 2^{2a}, \qquad \text{(3.27) in [4].}$$

By the way, formula (3.157) in [4] is redundant, being equivalent to (3.158) by a simple change of variable.

Relations (14)–(17) may be proved directly as we could even prove the original (3)–(6). They occur quite naturally in work with trigonometric identities, and I first came on them some years ago while studying Bromwich [1] wherein they are implicit...some other time we may discuss this case. Note how (14)–(17) differ from the corresponding (7)–(10) in that 'k' has been replaced by '$n-k$' in each case, or '$n-k-1$' for the transition from (8) to (15). The relations (14)–(17) may be called another of Moriarty's disguises. The design of the four changes here. For proofs of (14)–(17), see [5].

PROOFS

We turn now to the proofs. To begin with, we show that (3) may be found from (7) using (14) and (13). Here are the step-by-step details:

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n+1}{2j}\binom{n-j}{k-j}, \qquad \text{by (14),}$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j} \sum_{k=j}^{n} (-1)^k \binom{k}{a}\binom{n-j}{k-j}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k}\binom{k+j}{a}, \qquad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} (-1)^{n-j} \binom{j}{a-n+j}, \qquad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n+1}{2j}\binom{j}{n-a} = (-1)^n \binom{n+a}{2a} 2^{2a} \frac{2n+1}{2a+1}, \qquad \text{by (7).}$$

The proofs that (8) and (15) imply (4), that (9) and (16) imply (5), and that (10) and (17) imply (6) are done in similar fashion, using (13), and we give the details so the reader will have no mystery left to solve.

The steps may be reversed so that (7) follows from (3) using (14) and (13), etc., so that we find relations (3)–(6) equivalent to relations (7)–(10) assuming relations (14)–(17).

To show that (4) may be found from (8) using (15) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k+1} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n}{2j+1}\binom{n-1-j}{k-j}, \qquad \text{by (15)}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j+1} \sum_{k=j}^{n} (-1)^k \binom{k}{a}\binom{n-1-j}{k-j}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} \sum_{k=0}^{n-j} (-1)^k \binom{k+j}{a} \binom{n-1-j}{k}, \qquad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} \sum_{k=0}^{n-j-1} (-1)^k \binom{n-1-j}{k} \binom{k+j}{a},$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} (-1)^{n-j-1} \binom{j}{a-(n-j-1)}, \qquad \text{by (13),}$$

$$= (-1)^{n-1} \sum_{j=0}^{n} \binom{2n}{2j+1} \binom{j}{a-n+j+1}$$

$$= (-1)^{n-1} \sum_{j=0}^{n} \binom{2n}{2j+1} \binom{j}{n-1-a}$$

$$= (-1)^{n-1} \binom{n+a}{2a+1} 2^{2a+1}, \qquad \text{by (8).}$$

To show that (5) may be found from (9) using (16) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n}{2j} \binom{n-j}{k-j}, \qquad \text{by (16),}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j} \sum_{k=j}^{n} (-1)^k \binom{k}{a} \binom{n-j}{k-j}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j} (-1)^j \sum_{k=0}^{n-j} (-1)^k \binom{k+j}{a} \binom{n-j}{k}, \qquad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (-1)^{n-j} \binom{j}{a-n+j}, \qquad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} \binom{j}{a-n+j} = (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} \binom{j}{n-a}$$

$$= (-1)^n \frac{n}{n+a} \binom{n+a}{2a} 2^{2a}, \qquad \text{by (9).}$$

To show that (6) may be found from (10) using (17) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a} \frac{2n+1}{2k+1} \binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n+1}{2j+1} \binom{n-j}{k-j}, \qquad \text{by (17),}$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j+1} \sum_{k=j}^{n} (-1)^k \binom{k}{a} \binom{n-j}{k-j},$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j+1} (-1)^n \binom{j}{n-a}, \qquad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n+1}{2j+1}\binom{j}{n-a} = (-1)^n \binom{n+a}{2a} 2^{2a}, \qquad \text{by (10).}$$

PROOFS USING GENERATING FUNCTIONS

From the binomial theorem we have

$$\sum_{n=0}^{\infty} \binom{n+a}{a} x^n = (1-x)^{-a-1} \quad \text{or} \quad \sum_{n=a}^{\infty} \binom{n}{a} x^n = x^a(1-x)^{-a-1}.$$

In particular

$$\sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} x^n = x^{a+1}(1-x)^{-2a-2}. \tag{18}$$

We first use (18) to prove (4) of Zeitlin, as follows:

$$\begin{aligned}
\sum_{n=a}^{\infty} t^n \sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k} &= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k}^{\infty} t^n \binom{n+k}{2k+1} \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k+1}^{\infty} t^n \binom{n+k}{2k+1} \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^{k+1}(1-t)^{-2k-2} \\
&= \frac{t}{(1-t)^2} \sum_{k=a}^{\infty} \binom{k}{a} \left\{\frac{-4t}{(1-t)^2}\right\}^k \\
&= \frac{t}{(1-t)^2} \left\{\frac{-4t}{(1-t)^2}\right\}^a \left\{1 + \frac{4t}{(1-t)^2}\right\}^{-a-1} \\
&= (-1)^a 2^{2a} t^{a+1}(1+t)^{-2a-2}.
\end{aligned}$$

But also

$$-2^{2a} \sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} (-t)^n = -2^{2a} t^{a+1}(-1)^{a+1}(1+t)^{-2a-2},$$

so that each side of (4) gives the same generating function, whence, by uniqueness of the expansion, (4) is proved.

The generating function for (3) is similar, and is in fact

$$(-1)^a 2^{2a} t^a (1-t)(1+t)^{-2a-2}.$$

We have on the one hand

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} &= \sum_{k=0}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k}^{\infty} \binom{n+k}{2k} t^n \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^k \sum_{n=0}^{\infty} \binom{n+2k}{2k} t^n
\end{aligned}$$

$$= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^k (1-t)^{-2k-1} = \frac{1}{1-t} \sum_{k=a}^{\infty} \binom{k}{a} \left\{ \frac{-4t}{(1-t)^2} \right\}^k = (-1)^a \frac{2^{2a} t^a (1-t)}{(1+t)^{2a+2}} .$$

On the other hand

$$(-1)^a \frac{2^{2a} t^a (1-t)}{(1+t)^{2a+2}} = (-1)^a 2^{2a} t^a (1-t) \sum_{n=0}^{\infty} \binom{n+2a+1}{2a+1} (-t)^n = 2^{2a}(1-t) \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n$$

$$= 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n - 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-1)^n t^{n+1}$$

$$= 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n + 2^{2a} \sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} (-t)^n$$

$$= 2^{2a} \sum_{n=a}^{\infty} \left\{ \binom{n+a+1}{2a+1} + \binom{n+a}{2a+1} \right\} (-t)^n = 2^{2a} \sum_{n=a}^{\infty} (-t)^n \binom{n+a}{2a+1} \frac{2n+1}{2a+1} ,$$

so that (3) is proved.

PROOFS USING HYPERGEOMETRIC FUNCTIONS

The ordinary hypergeometric function is given by

$$F(a,b;c;x) = \sum_{k=0}^{\infty} (-1)^k \binom{-a}{k} \binom{-b}{k} \binom{-c}{k}^{-1} x^k . \tag{19}$$

Since it is easy to verify that

$$\binom{n+k}{2k} 2^{2k} = \binom{n}{k} \binom{-n-1}{k} \binom{-\frac{1}{2}}{k}^{-1} , \tag{20}$$

it is easy to see that series (3) may be put in hypergeometric form using a^{th} derivatives; in fact because

$$D_x^a x^k = a! \binom{k}{a} x^{k-a} ,$$

$$D_x^a F(-n, n+1; \tfrac{1}{2}; x)\big|_{x=1} = a! \sum_{k=0}^{n} (-1)^k \binom{k}{a} \binom{n+k}{2k} 2^{2k} = a!S .$$

Now a standard result about the hypergeometric function is that

$$D_x^m F(a,b;c;x) = m! \binom{a+m-1}{m} \binom{b+m-1}{m} \binom{c+m-1}{m}^{-1} F(a+m, b+m; c+m; x) ,$$

and thus

$$a!S = a! \binom{-n+a-1}{a} \binom{n+a}{a} \binom{\frac{1}{2}+a-1}{a}^{-1} F(-n+a, n+1+a; \tfrac{1}{2}+a; 1)$$

$$= a! \binom{n}{a} \binom{n+a}{a} \binom{-\frac{1}{2}}{a}^{-1} \frac{(-\frac{1}{2}+a)!(-3/2-a)!}{(-\frac{1}{2}+n)!(-3/2-n)!} ,$$

by Gauss' formula for a terminating $F(-m, b; c; 1)$, since $a \leqslant n$,

$$= (-1)^a 2^{2a} \frac{(n+a)!(-\frac{1}{2}+a)!(-3/2-a)!a!}{(n-a)!(-\frac{1}{2}+n)!(-3/2-n)!(2a)!}$$

$$= (-1)^a 2^{2a} \frac{(n+a)!(-\frac{1}{2}+a)!(-\frac{1}{2}-a)!a!}{(n-a)!(-\frac{1}{2}+n)!(-\frac{1}{2}-n)!(2a)!} \cdot \frac{2n+1}{2a+1} .$$

Making use of the formula $(-\frac{1}{2}+m)!(-\frac{1}{2}-m)! = (-1)^m \pi$, this then reduces to

$$(-1)^n \frac{(n+a)!(2n+1)a!}{(n-a)!(2a+1)!} 2^{2a} ,$$

which proves (3).

Somewhat similar proofs may be given for (4)–(6). Because

$$\binom{n+k}{2k+1} 2^{2k} = \frac{n}{2k+1}\binom{n-1}{k}\binom{-n-1}{k}\binom{-\frac{1}{2}}{k}^{-1}, \tag{21}$$

some proofs of relations like (4) using hypergeometric series will involve integration techniques as well.

OTHER PROOFS BY DIFFERENTIATION

For any function f we have trivially

$$\sum_{k=0}^{n}\binom{n+k}{2k} f(k) = \sum_{k=0}^{n}\binom{n+k}{n-k} f(k) = \sum_{k=0}^{n}\binom{2n-k}{k} f(n-k). \tag{22}$$

Thus, for example,

$$a!\sum_{k=0}^{n}\binom{n+k}{2k}\binom{k}{a} x^{k-a} = a!\sum_{k=0}^{n}\binom{2n-k}{k}\binom{n-k}{a} x^{n-k-a} = D_x^a \sum_{k=0}^{n}\binom{2n-k}{k} x^{n-k}. \tag{23}$$

The series

$$\sum_{k=0}^{[n/2]}\binom{n-k}{k} z^k$$

can be written in a (complicated) closed form. See relation (1.70)–(1.71) in [4]. In principle then, one can obtain (23) in closed form. The form of the series again shows how our work is related to Fibonacci numbers since we know that

$$\sum_{k=0}^{[n/2]}\binom{n-k}{k} = F_{n+1} = F_n + F_{n-1}, \qquad F_0 = 0, \quad F_1 = 1.$$

A RECURRENCE RELATION

Some other interesting things can be deduced by looking briefly at a recurrence relation for (4). Since

$$\binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+k+1}{2k+1},$$

we find easily

$$\sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k} 2^{2k} + \sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k+1} 2^{2k} = \sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k+1}{2k+1} 2^{2k}$$

or, in virtue of (3), then

$$S_{n+1} - S_n = (-1)^n\binom{n+a}{2a} 2^{2a}\frac{2n+1}{2a+1}, \tag{24}$$

where S_n is Zeitlin's series in (4).

Recalling that

$$\sum_{j=0}^{n-1}(S_{j+1} - S_j) = S_n - S_0,$$

we next find, since $S_0 = 0$, that for arbitrary S_j,

$$S_n = \frac{2^{2a}}{2a+1}\sum_{j=0}^{n-1}(-1)^j\binom{j+a}{2a}(2j+1).$$

and unless we know how to sum this in closed form the method yields nothing. But since we do know the value of S_n, we may look on this as a way to have evaluated a new series, and so we have found in fact

$$\sum_{k=0}^{n}(-1)^k\binom{k+a}{2a}(2k+1) = (-1)^n(n+a+1)\binom{n+a}{2a}. \tag{25}$$

INVERSION

As the reader will recall from the previous Moriarty episode [3], a good detective learns something by adroit use of inversion. Indeed, we now make use of the following inversion principle, that

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{2k} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+1}{n-k} \frac{2k+1}{2n+1} f(k) .$$

This is relation (21) on p. 67 of [6]. Applying this principle to (3), we find by inversion that

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} \binom{k+a}{2a} (2k+1)^2 = (2n+1)(2a+1) \binom{n}{a} 2^{2n-2a} . \tag{26}$$

This relation might be somewhat difficult to come by without the inversion application and may possibly serve in some way to indicate the fondness with which I like to use inversion techniques to establish new identities.

Riordan gives another inversion formula, same page, which is

$$f(n) = \sum_{k=0}^{n} \frac{2n}{n+k} \binom{n+k}{2k} g(n-k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{n-k} f(k) .$$

This may be used to obtain other interesting series.

A FINAL REMARK

The four series (14)–(17) were posed as a problem in the *American Math. Monthly* [5] and the solution by M.T.L. Bizley used just simple coefficient comparison in suitable generating functions. We asked there to sum

$$\sum_{k=0}^{n} \binom{2x+i}{2k+j} \binom{x-k}{n-k} \tag{27}$$

for all real x and for $i = 0, 1,$ and $j = 0,1.$ Our question as to whether the series can be summed for *all* integers i,j remains unanswered.

It seems of value to remark also that in the case of (16) and (17) we have factorizations that are of interest:

$$\sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{2^{2n}}{(2n)!} \prod_{k=0}^{n-1} (x^2 - k^2), \tag{28}$$

and

$$\sum_{k=0}^{n} \binom{2x+1}{2k+1} \binom{x-k}{n-k} = \frac{2x+1}{(2n+1)!} \prod_{k=0}^{n-1} \left\{ (2x+1)^2 - (2k+1)^2 \right\} . \tag{29}$$

We leave it as an exercise for the reader to determine whether factorizations exist for (14) and (15). This has an easy affirmative answer.

Sherlock Holmes [2, p. 470] remarked about the original Professor Moriarty that "the man pervades London, and no one has heard of him...I tell you Watson, in all seriousness, that if I could beat that man, if I could free society of him, I should feel that my own career had reached its summit, and I should be prepared to turn to some more placid line in life." Our mathematical Moriarty formulas pervade mathematics and his formulas are the secret behind half of the conspiracy of formulas we meet with in our work. Moriarty is everywhere Watson, everywhere! Look closely and you cannot help seeing him and his formulas.

EPILOGUE

As if to show the force of the remark that Moriarty is everywhere, if we just look for him, it is instructive to say now that relations (14)–(17) are nothing in the world but relations (7)–(10) of Moriarty viewed in a slightly different way. An easy way to see this is to make sufficient use of the following simple operations on series and binomial coefficients:

$$\binom{m}{k} = 0$$

for $k > m,$ and, typically,

$$\binom{n-k}{a-k} = \binom{n-k}{n-a}, \binom{2n}{2k} = \binom{2n}{2n-2k}, \sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} f(n-k).$$

Illustration. We show that (14) is equivalent to (10):

$$\sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{a-k} = \sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{n-a} = \sum_{k=0}^{n} \binom{2k+1}{2k}\binom{n-k}{n-a}$$

$$= \sum_{k=0}^{n} \binom{2n+1}{2n-2k}\binom{k}{n-a} = \sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{k}{n-a} .$$

Similarly (15) is equivalent to (8):

$$\sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{a-k} = \sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{n-1-a} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}\binom{n-1-k}{n-1-a}$$

$$= \sum_{k=0}^{n-1} \binom{2n}{2n-2k-1}\binom{k}{n-1-a} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}\binom{k}{n-1-a} .$$

The reader should now have no difficulty in showing that (16) is equivalent to (9), and that (17) is equivalent to (7). The equivalences are so complete and obvious that we wonder how anyone could miss them. Thus we have used the Moriarty formulas twice in our proofs of (3)–(6). Moriarty, Moriarty, all is Moriarty! "Indubitably my Dear Watson, indubitably."

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[Continued from Page 292.]

Theorem. The Fibonacci numbers form the only sequence of integers for which its extended sequence satisfies:

(i) $$x_{-n} = (-1)^{n+1}x_n, \quad n \text{ an integer},$$

(ii) any three consecutive terms of the sequence are relatively prime.

Proof. Let x_n be a sequence which satisfies (i) and (ii); then,

$$x_1 = ax_0 + bx_{-1} = ax_0 + bx_1 .$$

Hence,

(*) $$ax_0 = (1-b)x_1 .$$

Now,

$$x_0 = ax_{-1} + bx_{-2} = ax_1 - b(ax_1 + bx_0),$$

which implies that

$$(1+b^2)x_0 = ax_1(1-b) = a^2x_0 ,$$

using (*). Since the sequence is nontrivial x_0 and x_1 cannot both be 0. If $x_0 \neq 0$; then $a^2 = 1+b^2$, which implies that $a = \pm 1$ and $b = 0$. In either of these cases, (ii) will not hold. Hence, $x_0 = 0$. From (*) it follows that $b = 1$.

Thus far, the sequence hs the form $x_0 = 0, x_1, ax_1, \cdots$; hence, in order to satisfy (ii), x_1 must equal 1. This yields a sequence of the form

$$x_0 = 0, 1, a, a^2+1, a^3+2a, \cdots,$$

[Continued on Page 316.]

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-237 Proposed by D. A. Millin, High School Student, Annville, Pennsylvania.

Prove

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2} .$$

H-238 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum_{m,n,p=0}^{\infty}{}' \, x^m y^n z^p ,$$

where the summation is restricted to m,n,p such that

$$m \leqslant n+p, \qquad n \leqslant p+m, \qquad p \leqslant m+n .$$

SOLUTIONS

FIBONACCI COMBINATION

H-202 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\left\{ {k \atop j} \right\} = \frac{F_k F_{k-1} \cdots F_{k-j+1}}{F_1 F_2 \cdots F_j} , \quad \left\{ {k \atop 0} \right\} = 1 .$$

Show that

$$(*) \quad \begin{cases} \displaystyle\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j+1)} \left\{ {2k \atop j+k} \right\} = \prod_{j=1}^{k} L_{2j-1} \\ \displaystyle\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j-1)} \left\{ {2k \atop j+k} \right\} = (-1)^k \prod_{j=1}^{k} L_{2j-1} , \end{cases}$$

$$(**) \quad \begin{cases} \displaystyle\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} L_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}k} F_1 F_3 \cdots F_{2k-1} & (k \text{ even}) \\ \displaystyle\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} F_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}(k-1)} F_1 F_3 \cdots F_{2k-1} & (k \text{ odd}) . \end{cases}$$

Solution by the Proposer.

1. We use the well known identity

$$\sum_{j=0}^{k}(-1)^j\begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)}x^j = \prod_{j=0}^{k-1}(1-q^jx), \tag{1}$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}.$$

Put $q = \alpha/\beta$, where $\alpha+\beta = 1$, $\alpha\beta = -1$. It is easily verified that

$$\begin{bmatrix} k \\ j \end{bmatrix} \to \frac{(\beta^k-\alpha^k)(\beta^{k-1}-\alpha^{k-1})\cdots(\beta^{k-j+1}-\alpha^{k-j+1})}{(\beta-\alpha)(\beta^2-\alpha^2)\cdots(\beta^j-\alpha^j)}\beta^{j^2-jk} = \begin{Bmatrix} k \\ j \end{Bmatrix}\beta^{j^2-jk}.$$

Next, replace k by $2k$ and x by $\alpha^{1-k}\beta^kx$. Then (1) becomes

$$\prod_{j=0}^{2k}(\beta^j-\alpha^{j-k+1}\beta^{k-j}x) = \sum_{j=0}^{2k}(-1)^j\begin{Bmatrix} 2k \\ j \end{Bmatrix}(\alpha\beta)^{\frac{1}{2}j(j+1)-jk}x^j. \tag{2}$$

Since

$$\begin{aligned}\prod_{j=1}^{k}(\alpha^{j-1}-\beta^jx)(\beta^{j-1}-\alpha^jx) &= (\alpha\beta)^{\frac{1}{2}k(k-1)}\prod_{j=1}^{k}(1-\alpha^{-j+k}\beta^jx)(1-\alpha^j\beta^{-j+1}x)\\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)}\prod_{j=0}^{k-1}(1-\alpha^j\beta^{k-j}x)(1-\alpha^{k-j}\beta^{j-k+1}x)\\ &= (\alpha\beta)^{\frac{1}{2}k(k-1)}\prod_{j=0}^{2k-1}(1-\alpha^{j-k+1}\beta^{k-j}x),\end{aligned}$$

(2) reduces to

$$\begin{aligned}\sum_{j=0}^{2k}(-1)^{\frac{1}{2}j(j-1)+jk}\begin{Bmatrix} 2k \\ j \end{Bmatrix}x^j &= (-1)^{\frac{1}{2}k(k-1)}\prod_{j=1}^{k}(\alpha^{j-1}-\beta^jx)(\beta^{j-1}-\alpha^jx)\\ &= (-1)^{\frac{1}{2}k(k-1)}\prod_{j=1}^{k}((-1)^{j-1}-L_{2j-1}x+(-1)^jx^2).\end{aligned}$$

Hence for $x = 1$ we get

$$\sum_{j=0}^{2k}(-1)^{\frac{1}{2}j(j-1)+jk}\begin{Bmatrix} 2k \\ j \end{Bmatrix} = (-1)^{\frac{1}{2}k(k+1)}\prod_{j=1}^{k}L_{2j-1}, \tag{3}$$

while for $x = -1$,

$$\sum_{j=0}^{2k}(-1)^{\frac{1}{2}j(j+1)+jk}\begin{Bmatrix} 2k \\ j \end{Bmatrix} = (-1)^{\frac{1}{2}k(k+1)}\prod_{j=1}^{k}L_{2j-1}. \tag{4}$$

Finally, replacing j by $j+k$, (3) and (4) become

$$\sum_{j=-k}^{k}(-1)^{\frac{1}{2}(j+k)(j+k-1)}\begin{Bmatrix} 2k \\ j+k \end{Bmatrix} = (-1)^k\prod_{j=1}^{k}L_{2j-1},$$

$$\sum_{j=-k}^{k} (-1)^{\frac{1}{2}(j+k)(j+k+1)} \left\{ {2k \atop j+k} \right\} = \prod_{j=1}^{k} L_{2j-1} ,$$

respectively. This completes the proof of (*).

2. To prove (**), we use Gauss's identity

$$\sum_{j=0}^{2k} (-1)^j \left[{2k \atop j} \right] = \prod_{j=1}^{k} (1 - q^{2j-1}) \tag{5}$$

(for proof see for example G.B. Mathews, *Theory of Numbers,* Stechert, New York, 1927, p. 209). Replacing q by α/β; we find that (5) reduces to

$$\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \beta^{(j-k)^2} = (-1)^k (\alpha - \beta)^k \prod_{j=1}^{k} F_{2j-1} . \tag{6}$$

Similarly, if q is replaced by β/α, we get

$$\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} \alpha^{(j-k)^2} = (-1)^k (\beta - \alpha)^k \prod_{j=1}^{k} F_{2j-1} . \tag{7}$$

When k is even, we add (6) to (7) to get

$$\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} L_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}k} \prod_{j=1}^{k} F_{2j-1} .$$

When k is odd, we subtract (6) from (7) and get

$$\sum_{j=0}^{2k} (-1)^j \left\{ {2k \atop j} \right\} F_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}(k-1)} \prod_{j=1}^{k} F_{2j-1} .$$

This completes the proof of (**).

ON Q

H-205 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Evaluate the determinants of nth order

$$D_n = \begin{vmatrix} z & 1 & & & & \\ -1 & qz & 1 & & & \\ & -1 & q^2 z & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & -1 & q^{n-2}z & 1 \\ & & & & -1 & q^{n-1}z \end{vmatrix} ,$$

$$\Delta_n = \begin{vmatrix} z & 1 & & & & \\ -1 & z & q & & & \\ & -1 & z & q^2 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & -1 & z & q^{n-2} \\ & & & & -1 & z \end{vmatrix} .$$

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

If we expand the last row of each determinant by minors, we may readily obtain the following recursions:

(1) $$D_n = q^{n-1}zD_{n-1} + D_{n-2} \quad (n \geqslant 3); \quad D_1 = z; \quad D_2 = qz^2 + 1$$

(2) $$\Delta_n = z\Delta_{n-1} + q^{n-2}\Delta_{n-2} \quad (n \geqslant 3); \quad \Delta_1 = z; \quad \Delta_2 = z^2 + 1 .$$

The first recursion readily admits expression in continued fraction form. D_n is equal to the numerator of the nth convergent of the simple continued fraction:

$$z + 1/qz + 1/q^2z + 1/q^3z + \cdots .$$

An alternative notation for this infinite simple continued fraction is:

$$[z,\ qz,\ q^2z,\ q^3z,\ \cdots,\ q^{n-1}z,\ \cdots].$$

Recursion (2) may also be expressed in continued fraction form, but as it stands, it cannot be expressed in the form of a simple continued fraction, i.e., one with continued numerators of unity. If, however, we make the substitution:

(3) $$\Delta_n = q^{\frac{1}{4}(n^2-2n)}C_n \qquad (n = 1, 2, 3, \cdots) ,$$

then (2) reduces to a form similar to that of (1), namely:

(4) $$C_n = zq^{-\frac{1}{4}(2n-3)}C_{n-1} + C_{n-2} \quad (n \geqslant 3); \quad C_1 = zq^{\frac{1}{4}}; \quad C_2 = z^2 + 1 .$$

Thus, C_n is equal to the numerator of the nth convergent of the simple continued fraction:

$$[zq^{\frac{1}{4}},\ zq^{-\frac{1}{4}},\ zq^{-3/4},\ \cdots,\ zq^{-\frac{1}{4}(2n-3)},\ \cdots],$$

Δ_n is then found, by using (3).

Also solved by the Proposer.

UNITY OF ROOTS

H-206 Proposed by P. Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity:

$$\frac{1}{1-x^n} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-xe^{2k\pi i/n}} .$$

Solution by C. Bridger, Springfield, Illinois

Let $a, b, c, \cdots k, \cdots$ be the n n^{th} roots of unit. Among them, say the kth, is $e^{2k\pi i/n}$. Put $x = 1/y$ and set $y^n - 1 = (y-a)(y-b)(y-c)\cdots(y-k)\cdots$. The logarithmic derivative is

$$\frac{ny^{n-1}}{y^n-1} = \frac{1}{y-a} + \frac{1}{y-b} + \cdots + \frac{1}{y-k} + \cdots .$$

But this is exactly what the identity becomes when x is replaced by $1/y$ and the extra y is discarded. The next and final step is to replace y in the logarithmic derivative with $1/x$, discard the extra x and divide both sides by n.

Also solved by G. Lord and the Proposer.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

CORRECTED PROBLEM

B-279 Correction of typographical error in Vol. 12, No. 1 (February 1974).

Find a closed form for the coefficient of x^n in the Maclaurin series expansion of $(x+2x^2)/(1-x-x^2)^2$.

PROBLEMS PROPOSED IN THIS ISSUE

B-286 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be the "golden ratio" defined by $g = \lim_{n\to\infty} (F_n/F_{n+1})$. Simplify

$$\sum_{i=0}^{n} \binom{n}{i} g^{2n-3i} .$$

B-287 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be as in B-286. Simplify

$$g^2 \left\{ (-1)^{n-1} [F_{n-3} - gF_{n-2}] + g + 2 \right\} .$$

B-288 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{2n(4k+1)} \equiv F_{2n} \pmod{L_{2n}}$ for all integers n and k.

B-289 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{(2n+1)(2k+1)} \equiv F_{2n+1} \pmod{L_{2n+1}}$ for all integers n and k.

B-290 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Obtain a closed form for

$$2n+1+\sum_{k=1}^{n} (2n+1-2k)F_{2k} .$$

B-291 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the second-order recursion relation for $\{ z_n \}$ given that

$$z_n = \sum_{k=0}^{n} \binom{n}{k} y_k \quad \text{and} \quad y_{n+2} = ay_{n+1} + by_n ,$$

where a and b are constants.

SOLUTIONS

LUCAS SUM MULTIPLES OF 5 AND 10

B-262 Proposed by Herta T. Freitag, Roanoke, Virginia.

(a) Prove that the sum of n consecutive Lucas numbers is divisible by 5 if and only if n is a multiple of 4.
(b) Determine the conditions under which a sum of n consecutive Lucas numbers is a multiple of 10.

Composite of Solutions by Graham Lord, Temple University, Philadelphia, Pennsylvania, and Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

The sum $S = L_{a+1} + L_{a+2} + \cdots + L_{a+n}$ of n consecutive Lucas numbers is equal to $L_{a+n+2} - L_{a+2}$; hence $d|s$ if and only if $L_{a+n+2} \equiv L_{a+2} \pmod{d}$.

(a) Modulo 5, the Lucas sequence is the block of four numbers 1, 3, 4, 2 repeated endlessly. Thus $5|S$ if and only if $4|n$.

(b) Modulo 10, the Lucas sequence is the block of twelve numbers

$$1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2$$

repeated endlessly. From this one sees that $10|S$ [or equivalently, $L_{a+n+2} \equiv L_{a+2}$ (mod 10)] if and only if either (i) $12|n$, or (ii) $12|(n-4)$ and $3|(a+1)$, or (iii) $12|(n-8)$ and $3|a$.

Also solved by C.B.A. Peck and the Proposer. Partial solutions were received from Paul S. Bruckman, Ralph Garfield, and David Zeitlin.

LUCASLIKE SEQUENCE

B-263 Proposed by Timothy B. Carroll, Graduate Student, Western Michigan University, Kalamazoo, Michigan.

Let $S_n = a^n + b^n + c^n + d^n$, where $a, b, c,$ and d are the roots of $x^4 - x^3 - 2x^2 + x + 1 = 0$.

(a) Find a recursion formula for S_n.
(b) Express S_n in terms of the Lucas number L_n.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

(a) Since $a^4 - a^3 - 2a^2 + a + 1 = 0$, then, for $n = 0, 1, 2, \cdots$,

$$a^{n+4} - a^{n+3} - 2a^{n+2} + a^{n+1} + a^n = 0;$$

a similar relation holds for $b, c,$ and d. Adding these four equations, we obtain the recursion:

$$S_{n+4} - S_{n+3} - 2S_{n+2} + S_{n+1} + S_n = 0 \qquad (n = 0, 1, 2, \cdots)$$

(b)
$$x^4 - x^3 - 2x^2 + x + 1 = (x^2 - 1)(x^2 - x - 1) = (x-1)(x+1)(x-\alpha)(x-\beta).$$

So

$$S_n = 1 + (-1)^n + \alpha^n + \beta^n = 1 + (-1)^n + L_n .$$

Also solved by Clyde A. Bridger, Herta T. Freitag, Ralph Garfield, Graham Lord, Jeffrey Shallit, Paul Smith, M.N.S. Swamy, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI PRODUCT

B-264 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Use the identities $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and $F_n^2 - F_{n-2}F_{n+2} = (-1)^n$ to obtain a factorization of $F_n^4 - 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

We note that

$$F_n^4 - 1 = \left\{ F_n^2 + (-1)^n \right\} \left\{ F_n^2 - (-1)^n \right\} = F_{n-1}F_{n+1}F_{n-2}F_{n+2} .$$

In the paper by D. Zeitlin, "Generating Functions for Products of Recursive Sequences," *Transactions of the Amer. Math. Soc.,* 116 (April, 1965), pp. 300-315, it was shown on p. 304 that if $H_{n+2} = H_{n+1} + H_n$, then for $n = 0, 1, \cdots$,

(1) $$H_{n-2}H_{n-1}H_{n+1}H_{n+2} = H_n^4 - (H_2^4 - H_0H_1H_3H_4) .$$

Thus, if $H_0 = 0$ and $H_1 = 1$, the $H_n = F_n$ and (1) gives the above result. If $H_0 = 2$ and $H_1 = 1$, then $H_n = L_n$ and (1) gives

(2) $$L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25 \qquad (n = 0, 1, \cdots).$$

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Ralph Garfield, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

FIBONACCI NUMBERS FOR POWERS OF 3

B-265 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois.

Let F_n and L_n be designated as $F(n)$ and $L(n)$. Prove that

$$F(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1] .$$

Composite of solutions by Ralph Garfield, College of Insurance, N.Y., N.Y., and David Zeitlin, Minneapolis, Minn.

Using the Binet formulas $F(n) = (a^n - b^n)/(a - b)$ and $L(n) = a^n + b^n$, one easily shows that

$$F(3m)/F(m) = L(2m) + (-1)^m .$$

This with $m = 3^k, 0 \leqslant k \leqslant n - 1$, and the facts that $F(1) = 1$ and 3^k is odd, help us obtain

$$F(3^n) = \prod_{k=0}^{n-1} \frac{F(3^{k+1})}{F(3^k)} = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1] .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

LUCAS NUMBERS FOR POWERS OF 3

B-266 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois.

Let L_n be designated as $L(n)$. Prove that

$$L(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) + 1] .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since $L(3m) = L(m)[L(2m) - (-1)^m]$, we have, for $m = 3^k$, $0 \leqslant k \leqslant n - 1$,

$$L(3^n) = \prod_{k=0}^{n-1} \frac{L(3^{k+1})}{L(3^k)} = \prod_{k=0}^{n-1} [L(2\cdot 3^k)+1].$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

REGULAR POLYGON RELATION

B-267 Proposed by Marjorie Bicknell, Wilcox High School, Santa Clara, California.

Let a regular pentagon of side p, a regular decagon of side d, and a regular hexagon of side h be inscribed in the same circle. Prove that these lengths could be used to form a right triangle; i.e., that $p^2 = d^2 + h^2$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Hobson, in *Plane and Advanced Trigonometry,* on page 31 states:

$$\sin 18° = \frac{\sqrt{5}-1}{4}, \qquad \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

$$p = 2r\sin 36°, \qquad h = r, \qquad d = 2r\sin 18°$$

$$h^2+d^2 = r^2 + \frac{4r^2}{16}(6-2\sqrt{5}) = \frac{(5-\sqrt{5})}{2}r^2$$

$$p^2 = \frac{4r^2}{16}(10-2\sqrt{5}) = \frac{5-\sqrt{5}}{2}r^2$$

$$\therefore p^2 = h^2+d^2.$$

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Graham Lord, C.B.A. Peck, Paul Smith, M.N.S. Swamy, David Zeitlin, and the Proposer.

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and in order for (ii) to be satisfied a must equal 1. Therefore, the given sequence must be the Fibonacci sequence.

NOTE: The most general sequence satisfying (i) has the form

$$\cdots, ax_1, x_1, x_0 = 0, x_1, ax_1, (a^2+1)x_1, \cdots.$$

Also, if condition (ii) is weakened to the restriction that two consecutive terms be relatively prime, then the most general sequence would have the form

$$\cdots, -a, 1, x_0 = 0, 1, a, a^2+1, \cdots.$$

REFERENCE

1. V.E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton-Mifflin Co., New York, 1969.

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