FIBONACCI NOTES
3: q-FIBONACCI NUMBERS

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1. It is well known (see for example [2, p. 14] and [1]) that the number of sequences of zeros and ones of length n:

(1.1) $$(a_1, a_2, \cdots, a_n) \qquad (a_i = 0 \text{ or } 1)$$

in which consecutive ones are forbidden is equal to the Fibonacci number F_{n+2}. Moreover if we also forbid $a_1 = a_n = 1$, then the number of allowable sequences is equal to the Lucas number L_{n-1}. More precisely, for the first problem, the number of allowable sequences with exactly k ones is equal to the binomial coefficient

$$\binom{n-k+1}{k} \quad ;$$

for the second problem, the number of sequences with k ones is equal to

$$\binom{n-k+1}{k} - \binom{n-k-1}{k-2} \quad .$$

We now define the following functions. Let

(1.2) $$f(n,k) = \sum q^{a_1+2a_2+\cdots+na_n} \quad ,$$

where the summation is extended over all sequences (1.1) with exactly k ones in which consecutive ones are not allowable. Also define

(1.3) $$g(n,k) = \sum{}' q^{a_1+2a_2+\cdots+na_n} \quad ,$$

where the summation is the same as in (1.2) except that $a_1 = a_n = 1$ is also forbidden. We shall show that

(1.4) $$f(n,k) = q^{k^2}\begin{bmatrix} n-k+1 \\ k \end{bmatrix}$$

and

(1.5) $$g(n,k) = q^{k^2}\begin{bmatrix} n-k+1 \\ k \end{bmatrix} - q^{n+(k-1)^2}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} \quad ,$$

where

(1.6) $$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} \quad ,$$

the q-binomial coefficient.

These results suggest that we define q-Fibonacci and q-Lucas numbers by means of

(1.7) $$F_{n+1}(q) = \sum_{2k \leq n} q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix} \quad ,$$

(1.8) $$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q) \, ,$$

where

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$$(1.9)\qquad F'_{n+1}(q) = \sum_{2k\leqslant n} q^{(k+1)^2}\begin{bmatrix} n-k \\ k \end{bmatrix} .$$

It follows from the definitions that

$$(1.10)\qquad \begin{cases} F_{n+1}(q) - F_n(q) = q^{n-1}F_{n-1}(q) \\ F'_{n+1}(q) - F'_n(q) = q^{n+1}F'_{n-1}(q) . \end{cases}$$

Thus

$$(1.11)\qquad L_n(q) = F_{n+1}(q) + q^n(F_n(q) - F'_{n-2}(q)) .$$

However, $L_n(q)$ does not seem to satisfy any simple recurrence.

2. For the first problem as defined above it is convenient to define $f_j(n,k)$ as the number of allowable sequences with exactly k ones and $a_n = j$, where $j = 0$ or 1. It then follows at once that

$$(2.1)\qquad f_0(n,k) = f_0(n-1,k) + f_1(n-1,k) \qquad (n > 1)$$

and

$$(2.2)\qquad f_1(n,k) = q^n f_0(n-1,k-1) \qquad (n > 1) .$$

Also it is clear from the definition that

$$(2.3)\qquad f(n,k) = f_0(n,k) + f_1(n,k) .$$

Hence, by (2.1),

$$(2.4)\qquad f(n,k) = f_0(n+1,k) .$$

Combining (2.1) and (2.2) we get

$$(2.5)\qquad f_0(n,k) = f_0(n-1,k) + q^{n-1}f_0(n-2,k-1) \qquad (n > 2) .$$

This formula evidently holds for $k = 0$ if we define $f(n,-1) = 0$.

It is convenient to put

$$(2.6)\qquad f_0(0,k) = \begin{cases} 1 & (k = 0) \\ 0 & (k > 1) . \end{cases}$$

Also, from the definition,

$$(2.7)\qquad f_0(1,k) = \begin{cases} 1 & (k = 0 \\ 0 & (k > 1) \end{cases}$$

and

$$(2.8)\qquad f_0(2,k) = \begin{cases} 1 & (k = 0) \\ q & (k = 1) \\ 0 & (k > 1) \end{cases} .$$

It follows that (2.5) holds for $n \geqslant 2$.

Now put

$$(2.9)\qquad \Phi(x,y) = \sum_{n,k=0}^{\infty} f_0(n,k)x^n y^k .$$

Then, by (2.6), (2.7) and (2.5),

$$\begin{aligned} \Phi(x,y) &= 1 + x + \sum_{n=2}^{\infty}\sum_{k}\left\{ f_0(n-1,k) + q^{n-1}f(n-2,k-1)\right\} x^n y^k \\ &= 1 + x\Phi(x,y) + qx^2y\Phi(qx,y) , \end{aligned}$$

so that

$$(2.10)\qquad \Phi(x,y) = \frac{1}{1-x} + \frac{qx^2y}{1-x}\Phi(qx,y) .$$

Iteration of (2.10) leads to the series

(2.11)
$$\Phi(x,y) = \sum_{k=0}^{\infty} \frac{q^{k^2}x^{2k}y^k}{(x)_{k+1}} ,$$

where
$$(x)_{k+1} = (1-x)(1-qx)\cdots(1-q^kx).$$

Since
$$\frac{1}{(x)_{k+1}} = \sum_{s=0}^{\infty} \begin{bmatrix} k+s \\ k \end{bmatrix} x^s ,$$

where
$$\begin{bmatrix} k+s \\ s \end{bmatrix}$$

is defined by (1.6), it follows that

$$\Phi(x,y) = \sum_{k=0}^{\infty} q^{k^2}x^{2k}y^k \sum_{s=0}^{\infty} \begin{bmatrix} k+s \\ s \end{bmatrix} x^s$$

$$= \sum_{n=0}^{\infty} \sum_{2k \leqslant n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} x^n y^k .$$

Comparison with (2.9) gives

(2.12)
$$f_0(n,k) = q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} .$$

Therefore, by (2.4),

(2.13)
$$f(n,k) = q^{k^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} .$$

3. If we put

(3.1)
$$f(n) = \sum_{2k \leqslant n+1} f(n,k) ,$$

it is evident that
$$f(n) = \sum q^{a_1+2a_2+\cdots+na_n} ,$$

where the summation is over all zero-one sequences of length n with consecutive ones forbidden. This suggests that we define

(3.2)
$$F_{n+1}(q) = f(n-1) = \sum_{2k \leqslant n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \qquad (n \geqslant 0) .$$

We may also define

(3.3)
$$F_0(q) = 0, \qquad F_1(q) = 1 .$$

The next few values are

$$F_2(q) = 1, \qquad F_3(q) = 1+q$$
$$F_4(q) = 1+q+q^2, \qquad F_5(q) = 1+q+q^2+q^3+q^4$$
$$F_6(q) = 1+q+q^2+q^3+2q^4+q^5+q^6$$
$$F_7(q) = 1+q+q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8+q^9 .$$

It is evident from the above that $F_n(1) = F_n$, the ordinary Fibonacci number. To get a recurrence for $F_n(q)$ we use

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} .$$

Then, by (3.2),

$$F_{n+1}(q) - F_n(q) = \sum_k q^{k^2}\left(\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}\right) = \sum_k q^{k^2}\cdot q^{n-2k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}$$

$$= q^{n-1}\sum_k q^{(k-1)^2}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} = q^{n-1}\sum_k q^{k^2}\begin{bmatrix} n-k-2 \\ k \end{bmatrix} ,$$

so that

(3.4) $$F_{n+1}(q) = F_n(q) + q^{n-1}F_{n-1}(q) \qquad (n \geq 1).$$

This of course reduces to the familiar recurrence $F_{n+1} = F_n + F_{n-1}$ when $q = 1$.

It follows easily from (3.4) that $F_n(q)$ is a polynomial in q with positive integral coefficients. If $d(k)$ denotes the degree of $F_k(q)$ then $d(1) = d(2) = 0$, $d(3) = 1$, $d(4) = 2$, $d(5) = 4$, $\cdots$. Generally it is clear from (3.4) that

(3.5) $$d(n+1) = n - 1 + d(n-1) \qquad (n > 1).$$

Thus

$$d(2n+1) = 2n - 1 + d(2n-1), \qquad d(2n) = 2n - 2 + d(2n-2),$$

which yields

(3.6) $$d(2n+1) = n^2, \qquad d(2n) = n(n-1).$$

If we replace q by q^{-1} we find that

$$\begin{bmatrix} n \\ k \end{bmatrix} \to q^{k^2-nk}\begin{bmatrix} n \\ k \end{bmatrix} .$$

Hence

(3.7) $$F_{n+1}(q^{-1}) = \sum_{2k\leq n} q^{k^2-nk}\begin{bmatrix} n-k \\ k \end{bmatrix} .$$

It follows that

(3.8) $$\begin{cases} q^{n^2}F_{2n+1}(q^{-1}) = \displaystyle\sum_{k=0}^{n} q^{(n-k)^2}\begin{bmatrix} 2n-k \\ k \end{bmatrix} \\ q^{n(n-1)}F_{2n}(q^{-1}) = \displaystyle\sum_{k=0}^{n-1} q^{(n-k)(n-k-1)}\begin{bmatrix} 2n-k-1 \\ k \end{bmatrix} \end{cases} .$$

It follows from (2.11) and (3.2) that

(3.9) $$\sum_{n=0}^{\infty} F_{n+1}(q)x^n = \sum_{k=0}^{\infty} \frac{q^{k^2}x^{2k}}{(x)_{k+1}} .$$

G.E. Andrews proposed the following problem. Show that $F_{p+1}(q)$ is divisible by $1 + q + \cdots + q^{p-1}$, where p is any prime $\equiv \pm 2 \pmod 5$. For proof see [3]. This result is by no means apparent from (3.2). The proof depends upon the identity

(3.10) $$F_{n+1} = \sum_{k=-r}^{r} (-1)^k x^{\frac{1}{2}k(5k-1)}\begin{bmatrix} n \\ e(k) \end{bmatrix} ,$$

where

$$e(k) = [\tfrac{1}{2}(n+5k)], \qquad r = [\tfrac{1}{5}(n+2)]$$

In general it does not seem possible to simplify the right member of (3.9). However when $x = q$ it is noted in [3] that

(3.11) $$1 + \sum_{n=1}^{\infty} F_n(q)q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} = \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-1}(1 - x^{5n+4})^{-1} .$$

4. We now turn to the second problem described in the Introduction. To determine $g(n,k)$ as defined in (1.3) it is clear that

(4.1) $$g(n,k) = f(n,k) - h(n,k),$$

where $h(n,k)$ denotes the number of zero-one sequences $(a_1, a_2, \cdots, a_n)$ with k ones, consecutive ones forbidden and in addition $a_1 = a_n = 1$. Then $a_2 = a_{n-1} = 0$ while a_3 and a_{n-2} (if they occur) are arbitrary. Thus, for $n \geqslant 4$, $k \geqslant 2$,

$$h(n,k) = q^{n+1+2(k-2)} f(n-4, k-2) = q^{n+2k-3} f(n-2, k-2),$$

so that (4.1) becomes

$$g(n,k) = f(n,k) - q^{n+2k-3} f(n-4, k-2). \tag{4.2}$$

Combining with (2.13) we get

$$g(n,k) = q^{k^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} - q^{n+(k-1)^2} \begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} \qquad (n \geqslant 4,\ k \geqslant 2). \tag{4.3}$$

As for the excluded values, it is clear that

$$g(n,0) = 1, \qquad g(n,1) = q \begin{bmatrix} n \\ 1 \end{bmatrix} \qquad (n \geqslant 1). \tag{4.4}$$

Also it is easily verified that

$$g(3,k) = 0 \qquad (k \geqslant 2),$$

so that (4.3) holds for all $n \geqslant 1$. It is convenient to define

$$g(0,0) = 1, \qquad g(0,k) = 0 \qquad (k > 0). \tag{4.5}$$

Now put

$$g(n) = \sum_{2k \leqslant n+1} g(n,k). \tag{4.6}$$

Then by (3.2) and (4.3) we have

$$g(n) = f(n) - q^n f'(n-4), \tag{4.7}$$

where

$$f'(n) = \sum_{2k \leqslant n+1} q^{(k+2)^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix}. \tag{4.8}$$

It is easily verified that

$$f'(n) - f'(n-1) = q^{n+1} f'(n-2). \tag{4.9}$$

We now define

$$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q) \qquad (n \geqslant 2), \tag{4.10}$$

$$F'_{n+1}(q) = f'(n-1), \qquad F'_0(q) = 0. \tag{4.11}$$

We have

$$F'_{n+1}(q) - F'_n(q) = q^{n+1} F'_{n-1}(q); \tag{4.12}$$

this recurrence should be compared with (3.4).

The first few values of $L_n(q)$ are

$$L_2(q) = 1 + q + q^2, \qquad L_3(q) = 1 + q + q^2 + q^3,$$
$$L_4(q) = 1 + q + q^2 + q^3 + 2q^4 + q^6,$$
$$L_5(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + q^6 + q^7 + q^8.$$

It follows from (4.8) that

$$\sum_{n=0}^{\infty} F'_{n+1}(q) x^n = \sum_{k=0}^{\infty} q^{(k+1)^2} x^{2k} / (x)_{k+1}. \tag{4.13}$$

The first few values of $F'_n(q)$ are

$$F'_1(q) = q, \qquad F'_2(q) = q, \qquad F'_3(q) = q + q^4, \qquad F'_4(q) = q + q^4 + q^5,$$
$$F'_5(q) = q + q^4(1 + q + q^2) + q^9, \qquad F'_6(q) = q + q^4(1 + q + q^2 + q^3) + q^9(1 + q + q^2).$$

Thus, for example

$$L_4(q) = F_6(q) - q^4 F'_2(q) = (1 + q + q^2 + q^3 + 2q^4 + q^5 + q^6) - q^5,$$
$$L_5(q) = F_7(q) - q^5 F'_3(q) = (1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + q^8 + q^9) - q^5(q + q^4),$$

in agreement with the values previously found.

It would be of interest to find a simple combinatorial interpretation of $F'_n(q)$.

5. By means of the recurrence (3.4) we can define $F_n(q)$ for negative n. Put

$$\bar{F}_n(q) = (-1)^{n-1} F_{-n}(q) . \tag{5.1}$$

Then (3.4) becomes

$$\bar{F}_n(q) = q^n(\bar{F}_{n-1}(q) + \bar{F}_{n-2}(q)) \qquad (n \geq 2). \tag{5.2}$$

where

$$\bar{F}_0(q) = 0, \qquad \bar{F}_1(q) = q .$$

Put

$$\Phi(x) = \sum_{n=0}^{\infty} \bar{F}_n(q)x^n . \tag{5.3}$$

Then

$$\Phi(x) = qx + \sum_{n=2}^{\infty} q^n(\bar{F}_{n-1}(q) + \bar{F}_{n-2}(q))x^n ,$$

so that

$$\Phi(x) = qx + qx(1+qx)\Phi(qx). \tag{5.4}$$

Thus

$$\begin{aligned}\Phi(x) &= qx + qx(1+qx)\left\{ q^2x + q^2x(1+q^2x)\Phi(q^2x) \right\} \\ &= qx + q^3x^2(1+qx) + q^3x^2(1+qx)(1+q^2x)\Phi(q^2x).\end{aligned}$$

At the next stage we get

$$\Phi(x) = qx + q^3x^2(1+qx) + q^6x^3(1+qx)(1+q^2x) + q^6x^3(1+qx)(1+q^2x)(1+q^3x)\Phi(q^3x).$$

The general formula is evidently

$$\Phi(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)} x^{k+1}(1+qx)(1+q^2x)\cdots(1+q^kx). \tag{5.5}$$

Since

$$(1+qx)(1+q^2x)\cdots(1+q^kx) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)} x^j ,$$

(5.5) becomes

$$\Phi(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)} x^{k+1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)} x^j = \sum_{n=0}^{\infty} x^{n+1} \sum_{2j \leq n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)+\frac{1}{2}(n-j+1)(n-j+2)} .$$

Comparison with (5.3) gives

$$\bar{F}_{n+1}(q) = \sum_{2j \leq n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}(n+1)(n+2)-nj+j(j-1)} . \tag{5.6}$$

The first few values of $\bar{F}_n(q)$ are

$$\bar{F}_2(q) = q^3, \qquad \bar{F}_3(q) = q^4(1+q^2), \qquad \bar{F}_4(q) = q^7(1+q+q^3),$$
$$\bar{F}_5(q) = q^9(1+q^2+q^3+q^4+q^6), \qquad \bar{F}_6(q) = q^{13}(1+q+q^2+q^3+q^4+q^5+q^6+q^8) .$$

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P·Q M-CYCLES, A GENERALIZED NUMBER PROBLEM

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In this note all letters will denote non-negative integers. A number

$$N = n_1 \cdot 10^k + n_2 \cdot 10^{k-1} + \cdots + n_{k-1} \cdot 10 + n_k$$

(abbreviated $N = n_1 n_2 \cdots n_k$) will be called a $p \cdot q$ m-cycle whenever

$$p(n_{k-m-1}, n_{k-m-2}, \cdots, n_{k-1} n_k n_1 \cdots n_{k-m}) = q(n_1 n_2 \cdots n_k).$$

Since four parameters $\{p, q, m, k\}$ are involved, some rather interesting questions and conjectures arise naturally. The problem of Trigg [3], for example, yielded 428571, a distinct (i.e., the digits are distinct) $3 \cdot 4$ 3-cycle when $k = 6$, and $1 \cdot 1$ m-cycles which are n-linked were considered in [2]. Klamkin [1] recently characterized the smallest $1 \cdot 6$ 1-cycles. Here we extend some of these concepts, show how to generate various $p \cdot q$ m-cycles, and actually produce the smallest $1 \cdot q$ 1-cycles $(q = 1, 2, \cdots, 9)$ together with some of their properties. As a special case of our more generalized results, we present a much faster method than Wlodarski [4] for obtaining the smallest $1 \cdot q$ 1-cycles with $n_k = q$.

For notation, $n_1 \cdot n_2$ means n_1 times n_2, whereas $n_1 n_2$ will denote the two-digit number $10n_1 + n_2$. For a number $r \cdot s = n_1 n_2$, we shall use $(r \cdot s)_{10} = n_1$ and $(r \cdot s)_1 = n_2$.

1. $1 \cdot q$ 1-CYCLES

We first note that for each q $(q = 1, 2, \cdots, 9)$ and each $n_1 \leqslant 9/q$, there exists a smallest (unique non-repeating) $1 \cdot q$ 1-cycle

$$N_q(n_1) = n_1 n_2 \cdots n_{k_q(n_1)}$$

$(k_q(n_1)$, the number of digits in $N_q(n_1)$ will depend on q and $n_1)$. Indeed, assume that $k_q(n_1)$ is not fixed and note that $n_{k_q(n_1)} = q \cdot n_1 \neq 0$ when $n_1 \neq 0$. Then $N_q(n_1)$ is readily obtained by the following simple multiplication:

$$\begin{array}{rcccc}
 & n_1 & n_{k-2} & n_{k-1} & n_k \\
N = & n \;\cdots & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1 & q \cdot n \\
qN = & q \cdot n \cdots & \{q \cdot n_{k-2} + [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_{10}\}_1 & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1
\end{array}$$

EXAMPLE 1. 025641 and 205128 are $1 \cdot 4$ 1-cycles, whereas 142857 is a $1 \cdot 5$ 1-cycle. These numbers were obtained from

$$\begin{array}{rclcrcl}
 & n_1 & n_{k_4(2)} & & & n_1 & n_{k_5(1)} \\
N = & 2\,0\,5\,1\,2 & 8 & & N = & 1\,4\,2\,8\,5 & 7 \\
4N = & 8\,2\,0\,5\,1 & 2 = (4 \cdot 8)_1 & & 3N = & 7\,1\,4\,2\,8 & 5 = (5 \cdot 7)_1
\end{array}$$

For $n_1 = 1$, the above procedure yields the following $1 \cdot q$ 1-cycles $N_q(1)$. (Note that by simply placing $n_1 = 1$ after $n_{qk(1)}$, one obtains the corresponding $1 \cdot q$ 1-cycles $N_q(0) = 0 n_3 n_4 \cdots q 1$).

q	$N_q(1)$	$k_q(1)$
1	u,u where $u = 0, 1, 2, \cdots, 9$	2
2	105263157894736842	18
3	1034482758620689655172413793	28
4	102564	6
5	102040816326530612244897959183673469387755	42
6	1016949152542372881355932203389830508474576271186440677966	58
7	1014492753623188405797	22
8	1012658227848	13
9	10112359550561797752808988764044943820224719	44

We note here that there does not exist a largest $1 \cdot q$ 1-cycle $N_q(n_1) > 1$ since $n_1 n_2 \cdots n_k n_1 n_2 \cdots n_k$ is a $1 \cdot q$ 1-cycle for each $1 \cdot q$ 1-cycle $N_q(n_1)$.

EXAMPLE 2. The smallest (nonzero) $1 \cdot q$ 1-cycles are given by

$$N_1(1), \quad N_q(0) \quad \text{for } q = 2, 3, 4, 5, 6, 7, 8, 9$$

Indeed,

$$N_2(4) > N_2(3) > N_2(2) > N_2(1)$$

$$N_3(3) > N_3(2) \qquad \text{and} \qquad N_4(2) = 205128 > N_4(1) > N_4(0).$$

For $q \geqslant 5$, the only nonrepeating $1 \cdot q$ 1-cycles are $N_q(0)$ and $N_q(1)$.

We conclude this section by mentioning that the smallest $1 \cdot q$ 1-cycles whose last term $n_{k_q(n_1)} = q$ are precisely the numbers $N_q(1)$ in the above table.

2. $p \cdot q$ 1-CYCLES

Each $1 \cdot q$ 1-cycle is a $p \cdot p \cdot q$ 1-cycle for every integer p, and every $p \cdot q$ 1-cycle is clearly a

$$\frac{p}{(p,q)} \cdot \frac{q}{(p,q)}$$

1-cycle. To obtain $p \cdot q$ 1-cycles $N = n_1 n_2 \cdots n_k$ in general, let

$$N' = n_k n_1 \cdots n_{k-1} .$$

Then $pN' = qN$ requires that $n_k \leqslant n_1$ when $p > q$ and $n_k \geqslant n_1$ for $p < q$, and since

$$(p \cdot n_{k-1})_1 = (q \cdot n_k)_1 ,$$

we use n_k as a sieve for a generalization of the multiplication given in Section 1. Thus, keeping

$$(p \cdot n_{k-1})_1 = (q \cdot n_k)_1 , \quad [p \cdot n_{k-2} + (p \cdot n_{k-1})_{10}]_1 = [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 ,$$

etc., we proceed until the m^{th} position (denoted by a vertical line preceeding the $n_{k-m}{}^{th}$ digit of N), where the sequence of digits begin to repeat anew in the $m + 1^{st}$ position.

$$\begin{array}{rl|ccc}
N' = & n_k & & n_{k-2} & n_{k-1} \\
pN' = & \cdots\cdots\cdots\cdots & \cdot\,\cdot & [p \cdot n_{k-2} + (p \cdot n_{k-1})_{10}]_1 & (p \cdot n_{k-1})_1 \\
qN = & \cdots\cdots\cdots\cdots & \cdot\,\cdot & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1 \\
N = & n_1 & & n_{k-1} & n_k
\end{array}$$

EXAMPLE 3. (i) 162 is a $3 \cdot 4$ 1-cycle.

(ii) 21 is a $7 \cdot 4$ 1-cycle.

(iii) There does not exist a $5 \cdot 8$ 1-cycle.

(i) Since

$$(3 \cdot n_{k-1})_1 = (4 \cdot n_k)_1 ,$$

and since $n_k > n_1$, consider $n_k = 2$ (therefore $n_{k-1} = 6$). Then the above multiplication yields

$$\begin{array}{rl} N' = & 2\ \ 1\ n_2\ n_3 \cdots n_{k-3}\ n_{k-2}\ 6 \\ 3N' = & \qquad\qquad\qquad\qquad\quad 8 \\ 4N = & \qquad\qquad\qquad\qquad 4\ \ 8 \\ N = & 1\ n_2\ n_3\ n_4 \quad n_{k-2}\ \ 6\ \ 2 \end{array}$$

Since

$$(3n_{k-2} + 1)_1 = 4 \Rightarrow n_{k-2} = 1$$

and

$$(3n_{k-3})_1 = 6 \Rightarrow n_{k-3} = 2,$$

$$\begin{array}{rl|l} 3N' = & 3(2\ \ 1\ n_3\ n_4 \cdots & 2\ 1\ 6) \\ & & 6\ 4\ 8 \\ & & 6\ 4\ 8 \\ 4N = & 4(1\ n_2\ n_3\ n_4 \cdots & 1\ 6\ 2) \end{array}$$

from which it readily follows that 162 is a $3 \cdot 4$ 1-cycle.

(ii) $(7n_{k-1})_1 = (4n_k)_1$ is satisfied by the pairs n_{k-1}, n_k

$$\begin{array}{llllllllll} n_k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ n_{k-1} & 2 & 4 & 6 & 8 & 0 & 2 & 4 & 6 & 8 \end{array}.$$

Using the first pair yields

$$\begin{array}{r|l} 7(\ 1\ n_1\ n_2 \cdots & 1\ 2) \\ & 8\ 4 \\ & 8\ 4 \\ 4(n_1\ n_2\ n_3 \cdots & 2\ 1) \end{array},$$

The numbers 42, 63 and 84 are also $7 \cdot 4$ 1-cycles.

(iii) None of the pairs of values satisfying

$$(5n_{k-1})_1 = (8n_k)_1$$

yield $5N' = 8N$.

3. P·Q M-CYCLES

The procedure of Section 2, appropriately modified, also applies to $p \cdot q$ m-cycles in general. We demonstrate this in

EXAMPLE 4. Find a distinct $3 \cdot 4$ 3-cycle for $k = 6$.

For

$$(4n_k)_1 = (3n_{k-3})_1$$

which is satisfied by numerous values, first consider $n_k = 1$ and $n_{k-3} = 8$. Then

$$\begin{array}{rl} 3(n_{k-2}\ n_{k-1}\ \ 1\ n_1\ n_2 \cdots n_{k-6}\ n_{k-5}\ n_{k-4} & 8) \\ & 4 \\ & 4 \\ 4(n_1\ \ n_2\ \ n_3\ n_4\ n_5 \cdots\ \ 8\ \ n_{k-2}\ n_{k-1} & 1) \end{array}$$

yields

$$\begin{array}{r|l} 3(n_{k-2}\ n_{k-1}\ \ 1\ n_1\ n_2 \cdots & 5\ 7\ 1\ 4\ 2\ 8) \\ & 7\ 1\ 4\ 2\ 8\ 4 \\ & 7\ 1\ 4\ 2\ 8\ 4 \\ 4(n_1\ \ \ \ n_2\ n_3\ n_4\ n_5 \cdots & 4\ 2\ 8\ 5\ 7\ 1) \end{array}$$

so that *428571* is a solution to our problem.

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THE APOLLONIUS PROBLEM

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Problem 29 on page 216 of E.W. Hobson's *A Treatise on Plane Trigonometry,"* Cambridge University Press (1918) reads: "Three circles, whose radii are $a, b, c,$ touch each other externally; prove that the radii of the two circles which can be drawn to touch the three are

$$abc/[(bc + ca + ab) \pm 2\sqrt{abc(a + b + c)}].\text{"}$$

Horner [1] states "The formula...is due to Col. Beard" [2]. That the formula is incorrect is evident upon putting $a = b = c,$ whereupon the radii become $a/(3 \pm 2\sqrt{3})$, so that one of them is negative. Horner recognized this when he stated, "The negative sign gives R (absolute value)...".

The correct formula has been shown [3] to be:

$$abc/[2\sqrt{abc(a + b + c)} \pm (ab + bc + ca)].$$

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SOME PROPERTIES OF A FUNDAMENTAL RECURSIVE SEQUENCE OF ARBITRARY ORDER

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1. INTRODUCTION

In this paper, three properties of a fundamental recursive sequence of arbitrary order are examined by analyzing and recombining the zeros of the associated auxiliary equation. The three properties in question are Simson's relation (Sections 2, 3, 4), a Lucas identity discussed by Jarden (Section 5), and Horadam's Pythagorean triples (Section 6).

We define a fundamental i^{th} order linear recursive sequence $\{u_n^{(i)}\}$ in terms of the linear recurrence relation

$$(1.1)\qquad \begin{aligned} u_n^{(i)} &= \sum_{r=1}^{i} P_{ir}u_{n-r}^{(i)} && n > 0, \\ u_n^{(i)} &= 1 && n = 0, \\ u_n^{(i)} &= 0 && n < 0, \end{aligned}$$

in which the P_{ir} are arbitrary integers.

The "fundamental" character of this sequence has been shown elsewhere by the present writer [7].

Associated with the recurrence relation in (1.1) is the auxiliary equation

$$(1.2)\qquad f_i(x) \equiv \prod_{r=1}^{j} (x - a_{ir}) = 0$$

in which it is assumed that the complex numbers a_{ir} are distinct. We shall restrict ourselves to this non-degenerate case, but the basic arguments survive when the zeros of (1.2) are not distinct. In the degenerate case the order of the i-related sequence described below may be reduced.

We define an "i-related sequence of order j," $\{x_n^{(j)}\}$, as one which satisfies the j^{th} order recurrence relation

$$(1.3)\qquad \begin{aligned} x_n^{(j)} &= \sum_{r=1}^{j} (-1)^{r+1} Q_{ir}x_{n-r}^{(j)} && n > 0, \\ x_n^{(j)} &= 1 && n = 0, \quad j = \binom{i}{2}, \\ x_n^{(j)} &= 0 && n < 0, \end{aligned}$$

with an auxiliary equation

$$(1.4)\qquad g_j(x) \equiv \prod_{\substack{r=1 \\ r<m}}^{j} (x - a_{ir}a_{im}) = 0,$$

in which the Q_{ir} are integers and where the $a_{ir}a_{im}$ are the zeros of (1.2). For example, when $i = 3$, $j = 3$, and

$$f_3(x) = (x - a_{31})(x - a_{32})(x - a_{33})$$

$$g_3(x) = (x - a_{31}a_{32})(x - a_{31}a_{33})(x - a_{32}a_{33}) = x^3 - \Sigma a_{31}a_{32}x^2 + \Sigma a_{31}^2a_{32}a_{33}x - (a_{31}a_{32}a_{33})^2.$$

We choose the symbol Q_{ir} rather than Q_{jr} because the Q_{ir} can be expressed in terms of the a_{ir} as we show in Equation (4.7).

2. SIMSON'S RELATION

For the fundamental sequence of Lucas [4], $\{u_n^{(2)}\}$, (in our notation), Simson's relation takes the form

(2.1) $$(u_n^{(2)})^2 - (u_{n-1}^{(2)})(u_{n+1}^{(2)}) = (a_{21}a_{22})^n = x_n^{(1)}$$

since $\binom{2}{2} = 1$.

More generally we assert that

(2.2) $$(u_n^{(i)})^2 - (u_{n-1}^{(i)})(u_{n+1}^{(i)}) = x_n^{(j)}, \qquad j = \binom{i}{2}.$$

To prove this we use the fact that

(2.3) $$u_n^{(i)} = \sum_{r=1}^{i} A_{ir}a_{ir}^n ,$$

wherein the A_{ir} are determined by the initial values of $u_n^{(i)}$, $n = 0, 1, \cdots, i-1$. Thus the left-hand side of (2.2) becomes, after cancellation of terms,

$$-\sum_{r<m} A_{ir}A_{im}(a_{ir} - a_{im})^2(a_{ir}a_{im})^{n-1} = \sum_{r<m} B_{js}\beta_{js}^n$$

in which

$$\beta_{js} = a_{ir}a_{im}, \qquad \text{and} \qquad B_{js}\beta_{js} = -A_{ir}A_{im} \times (a_{ir} - a_{im})^2 .$$

Note that $j = \binom{i}{2}$ since there are ia_{ir} to be taken two at a time. Note further that $A_{ir}A_{im}$ contains $(a_{ir} - a_{im})^2$ in its denominator; see Jarden [3, p. 107].

The result (2.2) does not tell us much about the specific terms of $\{x_n^{(j)}\}$. We can find the initial terms by substituting successively the first $j+1$ values of $\{u_n^{(i)}\}$ in (2.2). For example, the first three terms can be found as follows:

$$(u_0^{(i)})^2 - (u_{-1}^{(i)})(u_1^{(i)}) = 1 = x_0^{(j)} .$$

$$(u_1^{(i)})^2 - (u_0^{(i)})(u_2^{(i)}) = P_{i1}^2 - P_{i1}^2 - P_{i2} = \sum_{r<m} a_{ir}a_{im}$$

$$= Q_{i1} = Q_{i1}x_0^{(j)} = x_1^{(j)} .$$

$$(u_2^{(i)})^2 - (u_1^{(i)})(u_3^{(i)}) = P_{i2}^2 - P_{i1}P_{i3} = Q_{i1}x_1^{(j)} - Q_{i2}x_0^{(j)} = x_2^{(j)} .$$

One can examine the nature of $\{x_n^{(j)}\}$ by the use of the multinomial expression for $u_n^{(i)}$, namely,

(2.4) $$u_n^{(i)} = \sum_{\Sigma r\lambda_r = n} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_n!} \prod_{r=1}^{i} P_{ir}^{\lambda_r} ,$$

and we shall do that in Section 4. We first consider the auxiliary equation for $\{x_n^{(j)}\}$ and the coefficient, Q_{ir}, of the recurrence relation separately.

Equation (2.4) follows if we adapt Macmahon [5, pp. 2–4], because $u_n^{(i)}$ is in fact the homogeneous product sum of weight n of the quantities a_{ij}. It is the sum of a number of symmetric functions formed from a partition of the number n. The first three cases are

$$u_1^{(i)} = P_{i1} = \sum a_{i1} ,$$

$$u_2^{(i)} = P_{i1}^2 + P_{i2} = \sum a_{i1}^2 + \sum a_{i1}a_{i2},$$

$$u_3^{(i)} = P_{i1}^3 + 2P_{i1}P_{i2} + P_{i3} = \sum a_{i1}^3 + \sum a_{i1}^2 a_{i2} + \sum a_{i1}a_{i2}a_{i3} .$$

In general,

$$u_n^{(i)} = \sum_{\Sigma\lambda=n} a_{i1}^{\lambda_1}a_{i2}^{\lambda_2}\cdots = \sum_{\Sigma\lambda=n} \prod_{r=1}^{i} a_{ir}^{\lambda_r} .$$

It is of interest to note that another formula for $u_n^{(i)}$ can be given by

$$u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{i+n-1} \Big/ \prod_{r \neq s} (a_{ir} - a_{is}) \,. \tag{2.5}$$

From Jarden [3, p. 107] we have that

$$u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{n} D_r / D \,, \tag{2.6}$$

where D is the Vandermonde of the roots given by

$$D = \sum_{r=1}^{i} a_{ir}^{i-1} \prod_{\substack{r \neq s \neq t \\ s<t}} (a_{it} - a_{is}) = \prod_{r>s} (a_{ir} - a_{is}) \prod_{s<t} (a_{it} - a_{is}) \tag{2.7}$$

and D_r is the determinant of order i obtained from D on replacing its r^{th} column by the initial terms of $\left\{ u_n^{(i)} \right\}$, $n = 0, 1, 2, \cdots, i-1$. It thus remains to prove that

$$D_r = a_{ir}^{i-1} \prod_{\substack{r \neq s \neq t \\ s<t}} (a_{it} - a_{is}) = D a_{ir}^{i-1} \Big/ \prod_{r>s} (a_{ir} - a_{is}) . \tag{2.8}$$

We use the method of the contrapositive. If

$$D_r \neq a_{ir}^{i-1} \prod_{\substack{r \neq s \neq t \\ s<t}} (a_{it} - a_{is}) ,$$

then

$$D = \sum_{r=1}^{i} D_r \qquad \text{(from (2.6) with } n = 0)$$

$$\neq \sum_{r=1}^{i} a_{ir}^{i-1} \prod_{\substack{r \neq s \neq t \\ s<t}} (a_{it} - a_{is})$$

which contradicts (2.7). This proves (2.8) and we have established that

$$u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{n} D_r / D = \sum_{r=1}^{i} a_{ir}^{i+n-1} D_r / D a_{ir}^{i-1} = \sum_{r=1}^{i} a_{ir}^{i+n-1} / \prod_{r>s} (a_{ir} - a_{is}) ,$$

as required.

3. AUXILIARY EQUATIONS

van der Poorten [6] has proved that if $f(x)$ is a polynomial with complex coefficients, and $\{U_n\}$, $\{V_n\}$ denote sequences of elements of C, and if

$$\prod_{r=1}^{i} (E - a_r) U_n = 0, \qquad f(E) V_n = 0 ,$$

then

$$h(E) U_n V_n = 0, \qquad n \geqslant 0 ,$$

where E is the operator on sequences which performs the action

$$E U_n = V_{n+1} , \qquad E V_n = V_{n+1}, \cdots \qquad n \geqslant 0 ,$$

and $H(x)$ denotes the monic polynomial which is the least common multiple of the polynomials

$$f(x/a_1),\ f(x/a_2),\ \cdots,\ f(x/a_i),$$

in which it is assumed that $a_1, a_2, \cdots, a_i$ are non-zero and distinct.

We now consider $\Pi(E - a_{ir})u_n^{(i)} = 0$ in place of both $\Pi(E - a_r)U_n$ and $f(E)V_n$. Then it follows from above that

$$h(E)(u_n^{(i)})^2 = 0, \tag{3.1}$$

where $H(x)$ is the l.c.m. of

$$\prod_{s,r=1}^{i} ((x/a_{is}) - a_{ir})$$

which can be re-written as $P_{ii}^{-1}\Pi(x - a_{ir}a_{is})$ since

$$P_{ii} = \prod_{s=1}^{i} a_{is}.$$

Thus the zeros of $h(x)$ are $a_{i1}, \cdots, a_{ii}$ taken 2 at a time.

In (3.1) we have established that the sequence

$$\left\{ (u_n^{(i)})^2 \right\}$$

satisfies a linear recurrence relation of order $\binom{i+1}{2}$ with auxiliary equation

$$F_{i+j}(x) = \prod_{\lambda_1+\lambda_2=2} (x - a_{ir}^{\lambda_1} a_{im}^{\lambda_2}), \tag{3.2}$$

where $j = \binom{i}{2}$ as before since

$$\binom{i+1}{2} = \binom{i}{1} + \binom{i}{2} = i + j.$$

Note that r may equal m in (3.2), and so

$$F_{i+j}(x) = \prod_{r=1}^{i} (x - a_{ir}^2) \prod_{m<s} (x - a_{im}a_{is}).$$

If we let

$$F_i(x) = \prod_{r=1}^{i} (x - a_{ir}^2), \tag{3.3}$$

which is the auxiliary equation associated with the sequence $\left\{ s_{2n}^{(j)} \right\}$, then we have proved

$$g_j(x) = F_{i+j}(x)/F_i(x). \tag{3.4}$$

The auxiliary equation for $\left\{ x_n^{(j)} \right\}$ can also be represented in terms of the coefficients of the corresponding recurrence relation by

$$g_j(x) = x^j + \sum_{r=1}^{j} (-1)^r Q_{ir} x^{j-r}. \tag{3.5}$$

We now seek an expression for the Q_{ir} in terms of the zeros of the auxiliary equation of the fundamental sequence.

4. RECURRENCE RELATION COEFFICIENTS

From (1.3) and (1.4), we see that $\left\{ x_n^{(j)} \right\}$ is the product sum of weight j of the quantities $a_{ir}a_{im}$ $(r < m)$. Thus

$$x_n^{(j)} = u_{2n}^{(i)} - \sum_{\lambda>n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots = \sum_{\Sigma\lambda=2n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots - \sum_{\lambda>n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r}. \tag{4.1}$$

For example, when $i = 3$, $j = 3$, and

$$x_1^{(3)} = \Sigma a_{31}a_{32}$$

$$x_2^{(3)} = \Sigma a_{31}^2 a_{32}^2 + \Sigma a_{31}^2 a_{32} a_{33}$$

$$x_3^{(3)} = \Sigma a_{31}^3 a_{32}^3 + \Sigma a_{31}^3 a_{32}^2 a_{32} + \Sigma a_{31}^2 a_{32}^2 a_{33}^2.$$

Furthermore, by analogy with (2.4)

(4.2)
$$x_n^{(j)} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{i} Q_{ir}^{\lambda_r} ,$$

the first few terms of which are

$$x_0^{(j)} = 1$$
$$x_1^{(j)} = Q_{i1}$$
$$x_2^{(j)} = Q_{i1}^2 - Q_{i2}$$
$$x_3^{(j)} = Q_{i1}^3 - 2Q_{i1}Q_{i2} + Q_{i3} ,$$
$$x_4^{(j)} = Q_{i1}^4 - 3Q_{i1}^2Q_{i2} + 2Q_{i1}Q_{i3} + Q_{i2}^2 - Q_{i4} .$$

Write

(4.3)
$$\prod_{\substack{r,m=1\\ r<m}}^{\infty} (1 - a_{ir}a_{im}x) = \sum_{n=0}^{\infty} Q_{in}(-x)^n$$

and then put

(4.4)
$$\sum_{n=0}^{\infty} K_{in}x^n = 1 \Big/ \sum_{n=0}^{\infty} Q_{in}(-x)^n .$$

Thus

$$\sum_{n=0}^{\infty} K_{in}x^n = \prod_{r<m} \sum_{n=0}^{\infty} a_{ir}^n a_{im}^n x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\Sigma\lambda=2n} (a_{i1}a_{i2})^{\lambda_{11}}(a_{i1}a_{i3})^{\lambda_{12}} \cdots (a_{i1}a_{ii})^{\lambda_{1,i-1}}(a_{i2}a_{i3})^{\lambda_{21}} \cdots (a_{i2}a_{ii})^{\lambda_{2,i-2}}(a_{i3}a_{i4})^{\lambda_{31}} \cdots x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\Sigma\lambda_r = rn} \prod_{r=1}^{i} a_{ir}^{\lambda_r} x^n ,$$

in which

$$\lambda_r = \sum_{m+v=r} \lambda_{mv} + \sum_{s=1}^{i-r} \lambda_{rs} ,$$

so that

(4.5)
$$K_{in} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r} .$$

In other words, K_{in} is the product sum of weight n of the quantities $a_{ir}a_{im}$ $(r < m)$, and so $K_{in} = x_n^{(j)}$.

If we write $-x$ for x in (4.4) we get

$$\Sigma Q_{in}x^n = 1/\Sigma K_{in}(-x)^n ,$$

which can also be obtained by leaving x unchanged in (4.4) and simply interchanging the symbols Q and K.

We next expand the right-hand side of (4.4) by the multinomial theorem to obtain

(4.6)
$$x_n^{(j)} = K_{in} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{n} Q_{ir}^{\mu_r} .$$

An interchange of symbols yields

(4.7)
$$Q_{in} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{n} \left(x_r^{(j)}\right)^{\mu_r} ,$$

which is an expression for Q_{ir} in terms of a_{ir}, since from (4.5),

$$x_n^{(j)} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r} ,$$

where

$$\lambda_r = \sum_{m+s=r} \lambda_{ms} + \sum_{w=1}^{i-r} \lambda_{rw} .$$

For example,

$$Q_{31} = x_1^{(3)} = \Sigma a_{31}a_{32} ,$$

$$Q_{32} = -(x_1^{(3)})^2 + x_2^{(3)} = -(\Sigma a_{31}^2 a_{32} + 2\Sigma a_{31}^2 a_{32}a_{33}) + (\Sigma a_{31}^2 a_{32}^2 + \Sigma a_{31}^2 a_{32}a_{33}) = -\Sigma a_{31}^2 a_{32}a_{33} .$$

Q_{in} can also be expressed in terms of $u_n^{(i)}$ from (4.1), and $u_n^{(i)}$ can be expressed in terms of P_{in} in (2.4), so that Q_{in} can be expressed in terms of P_{in} if desired. This has already been illustrated for (2.2).

Another formula for $x_n^{(j)}$ can be given by analogy with (2.5). Since

$$x_n^{(j)} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r}$$

and

$$u_n^{(i)} = \sum_{\Sigma\lambda=n} \prod_{r=1}^{i} a_{ir}^{\lambda_r} ,$$

then

$$x_n^{(j)} = \sum_{r=1}^{i} a_{ir}^{i+2n-1} \Big/ \prod_{r>s} (a_{ir} - a_{is}) ,$$

which is somewhat surprising since it is expressed entirely in terms of the zeros of $f_i(x)$ rather than $g_j(x)$.

5. JARDEN'S QUERY

Corresponding to the "fundamental" sequence $\{u_n^{(i)}\}$ and by analogy with Lucas' second-order "primordial" sequence [4], we define an i^{th} order primordial sequence by

$$v_n^{(i)} = \sum_{r=1}^{i} P_{ir} v_{n-r}^{(i)} \qquad n > 0,$$

(5.1)
$$v_n^{(i)} = i \qquad n = 0 ,$$
$$v_n^{(i)} = 0 \qquad n < 0,$$

so that

$$v_n^{(i)} = \sum_{r=1}^{i} a_{ir}^n .$$

Jarden [3, p. 88] suggests that it would be interesting to determine (in our notation)

(5.2)
$$u_{2n}^{(i)} - u_n^{(i)} v_n^{(i)}$$

since

$$u_{2n}^{(2)} - u_n^{(2)} v_n^{(2)}$$

is of great importance in the arithmetic of second-order sequences. We have already seen the auxiliary equation for $\{u_{2n}^{(i)}\}$ in (3.3). Thus

$$u_{2n}^{(i)} - u_n^{(i)} v_n^{(i)} = \sum_{r=1}^{i} A_{ir}a_{ir}^{2n} - \sum_{r=1}^{i} A_{ir}a_{ir}^n \sum_{m=1}^{i} a_{im}^n = -\sum_{\substack{r,s=1\\ r<s}}^{i} (A_{ir} + A_{is})(a_{ir}a_{is})^n = \sum_m C_{jm}\beta_{jm}^n = y_n^{(j)} ,$$

where $\beta_{jm} = a_{ir}a_{is}$, $r < s$, and $C_{jm} = -(A_{ir} + A_{is})$. Note that since

$$u_0^{(i)} = 1 = \sum_{r=1}^{i} A_{ir}$$

$$v_0^{(j)} = \sum_m C_{jm} = -2\sum_{r=1}^{i} A_{ir} = 2\,.$$

Furthermore, the zeros of the auxiliary equations of

$$\left\{x_n^{(j)}\right\} \quad \left\{y_n^{(j)}\right\}$$

are the same, namely β_{jr}. The β_{jr} also come into other properties of recurrence relations such as the quadratic forms of divisors of $v_{2n}^{(2)}$ determined by Lucas [4, p. 43].

The mention of these examples is made to point out that though we have restricted our study of these "i-related sequences of order j" to expressions for auxiliary equations (3.4) and (3.5) and for recurrence relation coefficients (4.3), (4.5) and (4.7), they can be used in other situations.

6. HORADAM'S PYTHAGOREAN TRIPLES

This basic approach of analyzing and recombining the zeros of the auxiliary equation might be the only fruitful one in studying other properties of recurrence relations of arbitrary order. For instance, Shannon and Horadam [8] proved a general Pythagorean theorem for

$$f_n^{(i)} = \sum_{r=1}^{i} f_{n-r}^{(i)}$$

with suitable initial values. It was shown that

$$(f_n^{(i)} f_{n+i+1}^{(i)})^2 + (2f_{n+i}^{(i)}(f_{n+i}^{(i)} - f_n^{(i)}))^2 = ((f_n^{(i)})^2 + 2f_{n+i}^{(i)} - f_n^{(i)}))^2\,, \tag{6.1}$$

and that all Pythagorean triples can be formed from such recurrence triples. The case $i = 2$ is the situation studied first by Horadam [2].

The proof of (6.1) cannot be extended to a similar expression for $\left\{u_n^{(i)}\right\}$ because of the presence of the coefficients P_{ir} in the recurrence relation for $\left\{u_n^{(i)}\right\}$. An essential feature of the proof of (6.1) was the result

$$2f_{n+i}^{(i)} - f_{n+i+1}^{(i)} = f_n^{(i)}\,.$$

This suggests that we consider

$$2u_{n+i}^{(i)} - u_{n+i+1}^{(i)} = 2\sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+i} - \sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+i+1} \tag{6.2}$$

which follows from (2.3).

The right-hand side of (6.2) becomes

$$\begin{aligned}\sum_{r=1}^{i} A_{ir}\left(2\alpha_{ir}^{n+i} - \alpha_{ir}^{n+i+1}\right) &= \sum_{r=1}^{i} A_{ir}\sum_{s=1}^{i} P_{is}\alpha_{ir}^{n+i-s}(2-\alpha_{ir})\\ &= \sum_{r=1}^{i}\sum_{s=1}^{i} A_{ir}P_{is}\alpha_{ir}^{n+i-s}(2-\alpha_{ir}) = \sum_{s=1}^{i} P_{is}\sum_{r=1}^{i} A_{ir}(2-\alpha_{ir})\alpha_{ir}^{n+i-s}\\ &= \sum_{s=1}^{i} P_{is}\left(\sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}\right)\,,\end{aligned}$$

where we have set

$$B_{ir} = A_{ir}(2-\alpha_{ir})\,.$$

Suppose further that

$$z_n^{(i)} = \sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}$$

so that $\{z_n^{(i)}\}$ satisfies the same recurrence relation as $\{u_n^{(i)}\}$ but has different initial conditions (which give rise to the B_{ir}). Then

$$z_n^{(i)} = \sum_{r=1}^{i} P_{ir} z_{n-r}^{(i)} \quad .$$

Proof:

$$z_{n-r}^{(i)} = \sum_{s=1}^{i} B_{is}\alpha_{is}^{n-r} .$$

$$\sum_{r=1}^{i} P_{ir}z_{n-r}^{(i)} = \sum_{r=1}^{i} P_{ir} \sum_{s=1}^{i} B_{is}\alpha_{is}^{n-r} = \sum_{r=1}^{i} P_{ir} \sum_{s=1}^{i} (2A_{is}\alpha_{is}^{n-r} - A_{is}\alpha_{is}^{n-r+1}) = 2\sum_{r=1}^{i} P_{ir}u_{n-r}^{(i)} - \sum_{r=1}^{i} P_{ir}u_{n-r+1}^{(i)}$$

$$= 2u_n^{(i)} - u_{n+1}^{(i)} = \sum_{r=1}^{i} 2A_{ir}\alpha_{ir}^{n} - \sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+1} = \sum_{r=1}^{i} A_{ir}(2-\alpha_{ir})\alpha_{ir}^{n} = \sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n} = z_n^{(i)} ,$$

as required. So

$$\sum_{s=1}^{i} P_{is}\left(\sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}\right) = \sum_{s=1}^{i} P_{is}z_{n+i-s}^{(i)} = z_{n+i}^{(i)} .$$

Thus we have proved

(6.3) $$2u_{n+i}^{(i)} - u_{n+i+1}^{(i)} = z_{n+i}^{(i)} ,$$

from which it follows immediately that

$$2u_{n+i}^{(i)} + u_{n+i+1}^{(i)} = 4u_{n+i}^{(i)} - z_{n+i}^{(i)} .$$

Thus we have

$$(2u_{n+i}^{(i)} - u_{n+i+1}^{(i)})(2u_{n+i}^{(i)} + u_{n+i+1}^{(i)}) = z_{n+i}^{(i)}(4u_{n+i}^{(i)} - z_{n+i}^{(i)})$$

which becomes

$$4(u_{n+i}^{(i)})^2 - (u_{n+i+1}^{(i)})^2 = z_{n+i}^{(i)}(4u_{n+i}^{(i)} - z_{n+i}^{(i)}) .$$

This can be rearranged as

$$(u_{n+i+1}^{(i)})^2 = (z_{n+i}^{(i)})^2 + 4u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}) .$$

Multiply each side of this equation by $(z_{n+i}^{(i)})^2$ and

$$(z_{n+i}^{(i)}u_{n+i+1}^{(i)})^2 = (z_{n+i}^{(i)})^4 + 4(z_{n+i}^{(i)})^2 u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}) .$$

Add

$$(2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2$$

to each side to get

(6.4) $$(z_{n+i}^{(i)}u_{n+i+1}^{(i)})^2 + (2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2 = ((z_{n+i}^{(i)})^2 + 2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2 .$$

Equation (6.4) may be considered as an extension of (6.1) and a generalization of Horadam's Pythagorean theorem, since (6.4) reduces to (6.1) when $P_{ir} = 1$ $(r = 1, 2, \cdots, i)$ because $z_{n+i}^{(i)} = u_n^{(i)}$ then (from (6.3) above and Eq. 9 of [7]).

Thus we have shown how three properties of a fundamental recursive sequence of arbitrary order can be generalized by analyzing and recombining the zeros of the auxiliary equation so that the essential features of the properties are revealed.

It is worth noting that Marshall Hall [1] looked at the divisibility properties of a third-order sequence with auxiliary equation roots $\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2$ formed from a second-order sequence with auxiliary equation roots α_1 and α_2.

Thanks are due to Professor A.F. Horadam of the University of New England, New South Wales, for his comments on a draft of this paper.

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LETTER TO THE EDITOR

January 1, 1973

Dear Prof. Hoggatt:

HAPPY NEW YEAR. Here is a problem:

Let $p_1, p_2, \cdots, p_s$ be given primes and let $a_1 < a_2 < \cdots$ be the integers composed of the primes $p_1, p_2, \cdots p_r$. Put

$$A_k = [a_1, a_2, \cdots, a_k]$$

(least common multiple), then

$$\sum_{k=1}^{\infty} \frac{1}{A_k}$$

is irrational. (Conjecture) This is undoubtedly true, but I cannot prove it. All I can show is that

$$\sum_{k=1}^{\prime} \frac{1}{A_k}$$

is irrational, where in Σ' the summation is extended only over the distinct A_k's (i.e., if

$$[a_1, \cdots, a_k] = [a_1, \cdots, a_{k+1}] ,$$

then we count only one of the $1/[a_1, \cdots, a_k]$).

Regards to all,

Paul Erdös

LATTICE PATHS AND FIBONACCI AND LUCAS NUMBERS

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Several papers have been presented in this quarterly relating lattice paths and Fibonacci numbers: [1], [5], and [6]. In [1] Greenwood remarked about a certain artificialness in his approach. Here we present what we believe is a more natural approach which gives direct derivations of the formulae. We also obtain the Lucas numbers and some generalizations.

1. INTRODUCTION

By a lattice point in the plane is meant a point with integral coordinates. Unless otherwise stated we take these to be non-negative integers. By a path (or lattice path) is meant a minimal path via lattice points taking unit horizontal and unit vertical steps.

It is well known [2, p. 167] that the number of paths from $(0,0)$ to (p,q) is

$$\binom{p+q}{p}. \tag{1}$$

If we associate a plus sign with each horizontal step and a minus sign with each vertical step, there is a one-to-one correspondence between the paths from $(0,0)$ to (p,q) and the arrangements of p pluses and q minuses on a line.

Another well known result [2, p. 127] is that the number of paths from $(0,0)$ to (p,q), $p \geqslant q$, which touch but do not cross the line $y = x$ is

$$\frac{p-q+1}{p+1}\binom{p+q}{q}. \tag{2}$$

In other words (2) gives the number of paths from $(0,0)$ to (p,q) such that at any stage the number of vertical steps never exceeds the number of horizontal steps.

For $p = q$, (2) gives

$$\frac{1}{p+1}\binom{2p}{p},$$

the Catalan numbers. These have a number of combinatorial applications [3, p. 192].

Note that if (1) is summed over all $p + q = n$, we get the number of paths from $(0,0)$ to the line $x + y = n$. In this case we get

$$\sum_{p=0}^{n}\binom{n}{p} = 2^n .$$

If each of these paths is reflected in the line $x + y = n$, we have all the symmetric paths from $(0,0)$ to (n,n).

If (2) is summed in the same way, we get the paths from $(0,0)$ to $x + y = n$ which may touch but not cross $y = x$. Reflect each of these to get the symmetric paths from $(0,0)$ to (n,n) which do not cross $y = x$. This is a larger collection than Greenwood's.

2. FIBONACCI NUMBERS

The following problem appears in [4, p. 14]. In how many ways can p pluses and q minuses be placed on a line so that no two minuses are together? In our problem we shall also require an initial plus sign to keep the path from crossing $y = x$.

First, we solve the more general problem of finding the number of arrangements of p pluses and q minuses on a line so that before the first minus and between any two minuses there are at least b pluses, $b \geqslant 0$, $p \geqslant bq$.

Arrange the q minuses and bq of the pluses on a line with b pluses before the first minus and b pluses between each pair of minuses. Distribute the remaining $p-bq$ pluses in the $q+1$ cells determined by the q minuses. This can be done in

$$\binom{p-(b-1)q}{q} \tag{3}$$

ways [4, p. 92].

For the original problem put $b=1$ to get

$$\binom{p}{q}. \tag{4}$$

Summed over $p+q=n,\ p \geqslant q,$ (4) gives, with q replaced by $k,$ that there are

$$F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \tag{5}$$

paths with the stated conditions from $(0,0)$ to $x+y=n.$ These paths begin with a horizontal step and can have no two consecutive vertical steps, so they cannot cross $y=x.$ Now reflect each path in $x+y=n$ to get the symmetric paths from $(0,0)$ to $(n,n).$

Here we have replaced each diagonal step of Greenwood with a horizontal step followed by a vertical step. Thus each of our paths crosses $x+y=n$ on a lattice point. As indicated by Stocks [6, p. 83], this accounts for the fact that we have F_{n+1} such paths where Greenwood gets F_{n+2}. That is, of the $h(n)$ paths of Greenwood only $h(n-1)$ cross $x+y=n$ on a lattice point.

Similarly, (3) summed over $p+q=n,\ p \geqslant bq,$ gives

$$F_{n+1}(b) = \sum_{k=0}^{\left[\frac{n}{b+1}\right]} \binom{n-bk}{k} \tag{6}$$

which has the analogous interpretation with respect to the line $by=x.$ These numbers have a Fibonacci character, for it is easy to show that

$$F_{n+1}(b) = \begin{cases} 1, & 0 \leqslant n \leqslant b, \\ F_n(b) + F_{n-b}(b), & n \geqslant b+1 \end{cases}$$

For the enumeration of paths without subpaths [5, p. 143] we note that in Greenwood's terminology these are simply those paths which begin with a diagonal step, and the paths to be deleted are those that begin with a horizontal step. By his proof of the recurrence this is precisely $h(n)-h(n-1)=h(n-2).$ In our terminology the paths without subpaths are those that begin with one horizontal step followed by a vertical step, i.e., paths from (1,1). Thus directly by (5) or a recurrence argument similar to Greenwood's we find that these are F_{n-1} in number.

Analogous results can be gotten for the paths enumerated in (6).

3. LUCAS NUMBERS

Again consider the problem in Riordan [4, p. 14] of arranging p pluses and q minuses on a line with no two minuses together. There are

$$\binom{p+1}{q} \tag{7}$$

such arrangements. These are the paths from $(0,0)$ to (p,q) which do not cross $y=x+1.$ That is, a path may start with a vertical step, but there will be no two in succession.

Now look at the paths as enumerated in (7), but with the added restriction that if the first and last steps are both vertical, we consider them as being consecutive. We thus have two mutually exclusive cases. In the first case, the paths start with a vertical step and must end with a horizontal step. These are the paths from $(1,1)$ to $(p-1,q).$ By (7) there are

$$\binom{p-1}{q-1} = \frac{q}{p}\binom{p}{q} \tag{8}$$

of these. In the second case, the paths start with a horizontal step and end with either. These are the paths from $(1,0)$ to (p,q). By (7), there are

$$\binom{p}{q} \tag{9}$$

of these. Note that the second case is really only (4).

Add (8) and (9) to get the solution

$$\frac{p+q}{p}\binom{p}{q}\;. \tag{10}$$

Summed over $p+q=n$, with $k=q$, (10) gives

$$L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k}\binom{n-k}{k}\;. \tag{11}$$

Reflect each of these paths in $x+y=n$ to get the symmetric paths from $(0,0)$ to (n,n).

Again it is easy to see that the number of paths without subpaths is L_{n-2}.

In analogy with (3), these results can also be generalized for arbitrary $b \geqslant 0$. In fact, (7) becomes

$$\binom{p-(b-1)(q-1)+1}{q}\;,$$

(10) becomes

$$\frac{p+q}{p+q-bq}\binom{p+q-bq}{q}\;,$$

and (11) becomes

$$L_n(b) = \sum_{k=0}^{\left[\frac{n}{b+1}\right]} \frac{n}{n-bk}\binom{n-bk}{k}\;.$$

Again parallel results follow with respect to the line $y=x+b$.

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ON THE PARTITION OF HORADAM'S GENERALIZED SEQUENCES INTO GENERALIZED FIBONACCI AND GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

If p,q are integers, $p^2+4q \neq 0$, let $\omega = \omega(p,q)$ be the set of those second-order integer sequences

$$(W_n) = (W_0, W_1, W_2, \cdots)$$

satisfying the relationship

$$W_n = pW_{n-1} + qW_{n-2} \quad (n \geqslant 2)$$

which are not also first-order sequences; i.e., they do not satisfy $W_n = cW_{n-1}$ $(\forall n)$ for some c. In Horadam's papers ([3], [4], [5], [6]) our W_n is denoted by $W_n(a,b;p,-q)$. In this paper we show that ω can be partitioned naturally into a set $\mathcal{F}$ of generalized Fibonacci sequences and a set $\mathcal{L}$ of generalized Lucas sequences; to each $F \in \mathcal{F}$ there corresponds one $L \in \mathcal{L}$ and vice-versa. We also indicate how very many of the well-known identities may be generalized in a simple way.

2. THE PARTITION OF $\omega(p,q)$

If α,β are the roots of $x^2 - px - q = 0$, $d = +\sqrt{p^2+4q}$ then the following relationships are true:

$$\alpha = \frac{p+d}{2}, \qquad \beta = \frac{p-d}{2},$$

$$\alpha+\beta = p, \qquad \alpha\beta = -q, \qquad \alpha-\beta = d,$$

(1) $$W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta},$$

where $A = W_1 - W_0\beta$, $B = W_1 - W_0\alpha$. Since (W_n) is not a first-order sequence it follows that $\alpha \neq 0, \beta \neq 0$, $A \neq 0$, $B \neq 0$. When W_n is represented as in (1) we say that W_n is in *Fibonacci form.* On the other hand, with different constants C and D, W_n could be represented as

$$W_n = C\alpha^n + D\beta^n .$$

In this case, we say that W_n is in *Lucas form.*

When W_n is in Fibonacci form (1) we may perform an operation ($'$) to obtain a number W'_n, where

$$W'_n = A\alpha^n + B\beta^n .$$

We say that the sequence (W'_n) is *derived from* the sequence (W_n). The sequence (W'_n) is a sequence of integers since

(2) $$W'_0 = A+B = W_1 - W_0\beta + W_1 - W_0\alpha = 2W_1 - W_0(\alpha+\beta) = 2W_1 - pW_0$$

and

(3) $$W'_1 = A\alpha + B\beta = (W_1 - W_0\beta)\alpha + (W_1 - W_0\alpha)\beta = W_1(\alpha+\beta) - 2W_0\alpha\beta = pW_1 + 2qW_0.$$

W'_n may now be expressed in Fibonacci form. In that case

$$W'_n = \frac{[A(\alpha-\beta)]\,\alpha^n - [-B(\alpha-\beta)]\,\beta^n}{\alpha-\beta} .$$

If we perform the operation ($'$) on W'_n we obtain

$$\begin{aligned} W''_n &= [A(\alpha-\beta)]\,\alpha^n + [(-B)(\alpha-\beta)]\,\beta^n \\ &= (\alpha-\beta)^2 \left[\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right] \\ &= d^2 W_n . \end{aligned}$$

We have proved

Theorem 1. $W_n'' = d^2 W_n$ for all $n \geqslant 0.$

It is not hard to verify that the equation $W_n = W_n' \; (\forall_n)$ cannot be true if (W_n) is not a first-order sequence.

Throughout this paper let $(X_n), (Y_n) \in \omega(p,q)$, let $X_n' = Y_n \; (n = 0, 1, 2, \cdots)$ and let $X_0 = a$, $X_1 = b$. Then, from (2) and (3),

$$Y_0 = 2b - ap, \qquad Y_1 = pb + 2qa.$$

By theorem 1, therefore, or directly, it follows that

$$ad^2 = 2Y_1 - pY_0, \qquad bd^2 = pY_1 + 2qY_0.$$

The following theorem now follows easily:

Theorem 2. (i)

(4) $$d^2 \mid 2Y_n - pY_{n-1} \quad \text{and} \quad d^2 \mid pY_n + 2qY_{n-1} \quad \text{for all} \quad n \geqslant 1.$$

(ii) If $(W_n) \in \omega(p,q)$, $d^2 \mid 2W_1 - pW_0$ and $d^2 \mid pW_1 + 2qW_0$ then $(W_n) = (X_n')$ for some $(X_n) \in \omega(p,q)$.

Proof of (ii). If

$$X_0 = \frac{2W_1 - pW_0}{d^2}, \quad X_1 = \frac{pW_1 + 2qW_0}{d^2} \quad \text{and} \quad (X_n) \in \omega(p,q),$$

then

$$X_0' = 2\left(\frac{pW_1 + 2qW_0}{d^2}\right) - p\left(\frac{2W_1 - pW_0}{d^2}\right) = W_0 \quad \text{and} \quad X_1' = p\left(\frac{pW_1 + 2qW_0}{d^2}\right) + 2q\left(\frac{2W_1 - pW_0}{d^2}\right) = W_1$$

which proves part (ii).

The basic linear relationships connecting (X_n) and (Y_n) are described in the following theorem.

Theorem 3. The following are equivalent:

(i) $$(X_n') = (Y_n),$$

(ii) $$Y_n = 2X_{n+1} - pX_n \quad \text{for all} \quad n \geqslant 0,$$

(iii) $$Y_{n+1} = pX_{n+1} + 2qX_n \quad \text{for all} \quad n \geqslant 0,$$

(iv) $$Y_n = X_{n+1} + qX_{n-1} \quad \text{for all} \quad n \geqslant 1,$$

(v) $$X_n = \frac{2Y_{n+1} - pY_n}{d^2} \quad \text{for all} \quad n \geqslant 0,$$

(vi) $$X_{n+1} = \frac{pY_{n+1} + 2qY_n}{d^2} \quad \text{for all} \quad n \geqslant 0,$$

(vii) $$X_n = \frac{Y_{n+1} + qY_{n-1}}{d^2} \quad \text{for all} \quad n \geqslant 1.$$

NOTE: For each of (ii), $\cdots$, (vii) we need only require that the expression is true for two adjacent values of n.

Proof. (i) $\Rightarrow$ (ii). If $(X_n') = (Y_n)$, then from (2) and (3), $Y_0 = 2X_1 - pX_0$ and $Y_1 = pX_1 + q2X_0 = 2X_2 - pX_1$ since $X_2 = pX_1 + qX_0$. Let $m \geqslant 2$ and assume (ii) is true for $0 \leqslant n < m$. Then

$$Y_m = pY_{m-1} + qY_{m-2} = p(2X_m - pX_{m-1}) + q(2X_{m-1} - pX_{m-2}) = 2X_{m+1} - pX_m.$$

The result now follows by induction.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow \cdots \Leftrightarrow$ (vii). This follows easily using

$$X_{n+1} = pX_n + qX_{n-1} \quad \text{and} \quad Y_{n+1} = pY_n + qY_{n-1} \quad (n \geqslant 1).$$

[(ii), (iii), $\cdots$ (vii)] $\Rightarrow$ (i). Since

$$X_0 = \frac{2Y_1 - pY_0}{d^2} \quad \text{and} \quad X_1 = \frac{pY_1 + 2qY_0}{d^2}$$

it follows from (2) and (3) that

$$X'_0 = 2\left(\frac{pY_1 + 2qY_0}{d^2}\right) - p\left(\frac{2Y_1 - pY_0}{d^2}\right) = Y_0$$

and similarly $X'_1 = Y_1$. Hence $(X'_n) = (Y_n)$. This completes the proof of Theorem 3.

We now describe the partition of $\omega(p,q)$ previously referred to:

If $(W_n) \in \omega(p,q)$ and $d \neq 1$ let $W_n = d^{2m}\omega_n$ for all $n \geq 0$, where $m \geq 0$ is an integer, $(\omega_n) \in \omega$ and $d^2 \nmid \omega_n$ for at least one $n \geq 0$. Then

$$(W_n) \in L \quad \text{if} \quad d^2 | 2\omega_1 - p\omega_0 \quad \text{and} \quad d^2 | p\omega_1 + 2q\omega_0 ,$$
$$(W_n) \in F \quad \text{if either} \quad d^2 \nmid 2\omega_1 - p\omega_0 \quad \text{or} \quad d^2 \nmid p\omega_1 + 2q\omega_0 .$$

If $(W_n) \in \omega(p,q)$ and $d = 1$ let

$$(W_n) \in L \quad \text{if} \quad W_1 - W_0\alpha < 0,$$
$$(W_n) \in F \quad \text{if} \quad W_1 - W_0\alpha > 0.$$

The assignment of (W_n) to L or F is natural in the case $d \neq 1$, but if $d = 1$, although the partition itself is natural, it is not true to say that a sequence is "like" the Lucas sequence rather than the Fibonacci sequence or vice-versa. In view of Theorem 3 if (W_n) is a member of F (or L) then any "tail" of (W_n) is also a member of F (or L, respectively).

Theorem 4. $(X_n) \in F$ if and only if $(Y_n) \in L$.

Proof. Case 1. $d = 1$. $(X_n) \in F$

$$\Leftrightarrow X_n = A\alpha^n - B\beta^n, \quad \text{where} \quad B < 0$$
$$\Leftrightarrow Y_n = A\alpha^n + B\beta^n$$
$$\Leftrightarrow (Y_n) \in L .$$

Case 2. $d \neq 1$. (i) If $(X_n) \in F$ suppose that $X_n = d^{2m}x_n$ for all $n \geq 0$, where $m \geq 0$ is an integer, $(x_n) \in F$ and $d^2 \nmid x_n$ for at least one $n \geq 0$. Clearly $d^2 \nmid x_0$ or $d^2 \nmid x_1$. By Theorem 3, $Y_0 = 2X_1 - pX_0$ and $Y_1 = pX_1 + 2qX_0$. Let $Y_n = d^{2m}y_n$ for all $n \geq o$. Then $y_0 = 2x_1 - px_0$ and $y_1 = px_1 + 2qx_0$. Since $(x_n) \in F$, either $d^2 \nmid 2x_1 - px_0$ or $d^2 \nmid px_1 + 2qx_0$. Therefore either $d^2 \nmid y_0$ or $d^2 \nmid y_1$. But it is easy to verify that

$$2y_1 - py_0 = d^2x_0 \quad \text{and} \quad py_1 + 2qy_0 = d^2x_1 .$$

Therefore $(y_n) \in L$ and so $(Y_n) \in L$.

(ii) If $(Y_n) \in L$ suppose that $Y_n = d^{2m}y_n$ for all $n \geq 0$, where $m \geq 0$ is an integer, $(y_n) \in L$ and $d^2 \nmid y_n$ for at least one $n \geq 0$. Clearly $d^2 \nmid y_0$ or $d^2 \nmid y_1$. By Theorem 3,

$$X_0 = \frac{2Y_1 - pY_0}{d^2}, \qquad X_1 = \frac{pY_1 + 2qY_0}{d^2} .$$

Let $X_n = d^{2m}x_n$ for all $n \geq 0$. Then

$$x_0 = \frac{2y_1 - py_0}{d^2}, \qquad x_1 = \frac{py_1 + 2qy_0}{d^2} .$$

Since $(y_n) \in L$,

$$d^2 | 2y_1 - py_0 \quad \text{and} \quad d^2 | py_1 + 2qy_0 ,$$

so x_0 and x_1 are integers, so $(x_n) \in \omega$. But

$$2x_1 - px_0 = y_0 \quad \text{and} \quad px_1 + 2qx_0 = y_1 ,$$

and since $d^2 \nmid y_0$ or $d^2 \nmid y_1$ it follows that either $d^2 \nmid 2x_1 - px_0$ or $d^2 \nmid px_1 + 2qx_0$. Therefore $(x_n) \in F$ and so $(X_n) \in F$. This completes the proof of Theorem 4.

Here are some examples of members of F alongside the corresponding member of L.

$0, 1, 1, 2, 3, 5, 8, 13, \cdots$	$2, 1, 3, 4, 7, 13, \cdots$
$0, 1, p, p^2 + q, \cdots$	$2, p, p^2 + 2q, \cdots$
$0, 1, 3, 7, 15, \cdots, 2^n - 1, \cdots$	$2, 3, 5, 9, 17, \cdots, 2^n + 1, \cdots$
$0, 1, 2, 5, 12, 29, \cdots$ (Pell's sequences)	$2, 2, 6, 14, \cdots$
$a, b, qa, qb, q^2a, q^2b, \cdots$	$2b, 2qa, 2qb, 2q^2a, 2q^2b, \cdots$

3. BINOMIAL IDENTITIES

Many identities involving Fibonacci and Lucas numbers are readily derived from the binomial theorem; for example see [1], [2] or [8]. They can nearly always be generalized to become identities involving generalized Fibonacci and Lucas numbers.

In this section we could derive a long list of such identities; but this seems unnecessary in view of the proofs in [2] and [8], and also it would take up a lot of space, as the constant multipliers which have to be introduced seem to make the generalized formulae up to twice as long as the formulae in [2] and [8]. Instead we derive one set of identities as an example and show how further identities may be derived.

There often seem to be two very similar identities, one featuring Fibonacci numbers, the other Lucas numbers. When there are two such identities they may often be derived from one identity by using the fact that 1 and $\sqrt{5}$ are linearly independent over the rationals, although this is not the procedure adopted in [2] or [8]. With generalized Fibonacci and Lucas numbers such a process would not be appropriate, but, as the examples show, the method of proof which is natural does lead to a single identity, from which the two identities may be obtained by specialization.

For this section (F_n) and (L_n) denote a pair of sequences such that $(F_n) \in F$, $(L_n) \in L$ and $(F_n)' = (L_n)$. Also, $C = F_1 - F_0\beta$, $D = F_1 - F_0\alpha$.

The natural method of proof is firstly to derive a single identity involving (X_n) and (Y_n). Then either of the following sets of substitutions may be made:

I.
$$X_n = F_{n+r}$$
$$Y_n = L_{n+r}$$
$$A = X_1 - X_0\beta = F_{r+1} - F_r\beta = C\alpha^r$$
$$B = X_1 - X_0\alpha = F_{r+1} - F_r\alpha = D\beta^r .$$

(The third of these follows since

$$F_{r+1} = \frac{C\alpha^{r+1} - D\beta^{r+1}}{\alpha - \beta} = \frac{(C\alpha^r - D\beta^r)\beta - C\alpha^r\beta + C\alpha^{r+1}}{\alpha - \beta} = F_r\beta + C\alpha^r ,$$

and the fourth follows similarly.)

Or

II.
$$X_n = L_{n+r}$$
$$Y_n = d^2F_{n+r}$$
$$A = X_1 - X_0\beta = L_{r+1} - L_r\beta = Cd\alpha^r$$
$$B = X_1 - X_0\alpha = L_{r+1} - L_r\alpha = -Dd\beta^r .$$

Then each of these sets of substitutions leads to one of the two derived identities mentioned above.

EXAMPLES OF BINOMIAL IDENTITIES

EXAMPLE 1. Since

$$\alpha^m = \frac{Y_m + dX_m}{2A}, \qquad \beta^m = \frac{Y_m - dX_m}{2B}$$

it follows that

$$\alpha^{mn} = (2A)^{-n} \sum_{i=0}^{n} d^iX_m^iY_m^{n-i}\binom{n}{i}, \quad \text{and} \quad \beta^{mn} = (2B)^{-n} \sum_{i=0}^{n} (-1)^id^iX_m^iY_m^{n-i}\binom{n}{i} .$$

Therefore,

$$Y_{mn} + dX_{mn} = (2A)^{1-n} \sum_{i=0}^{n} d^iX_m^iY_m^{n-i}\binom{n}{i}, \quad \text{and} \quad Y_{mn} - dX_{mn} = (2B)^{1-n} \sum_{i=0}^{n} (-1)^id^iX_m^iY_m^{n-i}\binom{n}{i}.$$

Therefore,

$$X_{mn} = 2^{-n}d^{-1} \sum_{i=0}^{n} (dX_m)^iY_m^{n-i}\binom{n}{i} [A^{1-n} - (-1)^iB^{1-n}] .$$

A similar formula may be derived for Y_{mn}.

Making the first set of substitutions, we obtain

$$F_{mn+r} = 2^{-n}d^{-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} - (-1)^i[D\beta^r]^{1-n}).$$

But

$$C^{1-n}\alpha^{r-rn} - (-1)^i D^{1-n}\beta^{r-rn} = C^{1-n}\left(\frac{L_{r-rn} + dF_{r-rn}}{2C}\right) - \frac{(-1)^i D^{1-n}(L_{r-rn} - dF_{r-rn})}{2D}$$

$$= \frac{L_{r-rn}}{2}\left(\frac{1}{C^r} - (-1)^i\frac{1}{D^r}\right) + \frac{dF_{r-rn}}{2}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right).$$

Therefore

$$F_{nm+r} = 2^{-n-1}d^{-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}\left\{L_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right)\right\}.$$

Making the second set of substitutions we obtain

$$L_{nm+r} = 2^{-n}d^{-1}\sum_{i=0}^{n}(dL_{m+r})^i(d^2F_{m+r})^{n-i}\binom{n}{i}([Cd\alpha^r]^{1-n} - (-1)^i[-Dd\beta^r]^{1-n})$$

$$= 2^{-n}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} - (-1)^{n-i}(-1)^{1-n}[D\beta^r]^{1-n})$$

$$= 2^{-n}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} + (-1)^i[D\beta^r]^{1-n}).$$

But

$$C^{1-n}\alpha^{r-rn} + (-1)^i D^{1-n}\beta^{r-rn} = \frac{L_{r-rn}}{2}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right).$$

Therefore

$$L_{mn+r} = 2^{-n-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}\left\{L_{r-rn}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right)\right\}.$$

EXAMPLE 2. Since

$$dX_m = 2A\alpha^m - Y_m \quad\text{and}\quad dX_m = -(2B\beta^m - Y_m)$$

it follows that

$$\alpha^k d^n X_m^n = \sum_{i=0}^{n}(-Y_m)^i(2A)^{n-i}\binom{n}{i}\alpha^{mn-mi+k} \quad\text{and}\quad \beta^k d^n X_m^n = (-1)^n\sum_{i=0}^{n}(-Y_m)^i(2B)^{n-i}\binom{n}{i}\beta^{mn-mi+k}.$$

Therefore

$$Y_k d^n X_m^n + X_k d^{n+1}X_m^n = \sum_{i=0}^{n}(-Y_m)^i(2A)^{n-i}\binom{n}{i}(Y_{mn-mi+k} + dX_{mn-mi+k})$$

and

$$Y_k d^n X_m^n - X_k d^{n+1}X_m^n = (-1)^n\sum_{i=0}^{n}(-Y_m)^i(2\beta)^{n-i}\binom{n}{i}(Y_{mn-mi+k} - dX_{mn-mi+k}).$$

Therefore

$$X_kX_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^iY_m^i2^{n-i}\binom{n}{i}[Y_{mn-mi+k}(A^{n-i}-(-1)^n\beta^{n-i})+dX_{mn-mi+k}(A^{n-i}+(-1)^n\beta^{n-i})]$$

and

$$Y_kX_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^iY_m^i2^{n-i}\binom{n}{i}[Y_{mn-mi+k}(A^{n-i}+(-1)^n\beta^{n-i})+dX_{mn-mi+k}(A^{n-i}-(-1)^n\beta^{n-i})].$$

Making the first set of substitutions we obtain

$$F_kF_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^iL_m^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(C^{n-i}-(-1)^nD^{n-i})+dF_{mn-mi+k}(C^{n-i}+(-1)^nD^{n-i})]$$

and

$$L_kF_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^iL_m^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(C^{n-i}+(-1)^nD^{n-i})+dF_{mn-mi+k}(C^{n-i}-(-1)^nD^{n-i})].$$

Making the second set of substitutions we obtain

$$d^2F_kL_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^i(d^2F_m)^i2^{n-i}\binom{n}{i}[d^2F_{mn-mi+k}(d^{n-i}C^{n-i}+(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})$$
$$+dL_{mn-mi+k}(d^{n-i}C^{n-i}-(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})]$$

so that

$$F_kL_m^n = \frac{1}{2d}\sum_{i=0}^{n}(dF_m)^i2^{n-i}\binom{n}{i}[dF_{mn-mi+k}(D^{n-i}+(-1)^iC^{n-i})-L_{mn-mi+k}(D^{n-i}-(-1)^iC^{n-i})]$$

and

$$L_kL_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^i(d^2F_m)^i2^{n-i}\binom{n}{i}[d^2F_{mn-mi+k}(d^{n-i}C^{n-i}-(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})$$
$$+dL_{mn-mi+k}(d^{n-i}C^{n-i}+(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})]$$

so that

$$L_kL_m^n = \frac{1}{2}\sum_{i=0}^{n}(dF_m)^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(D^{n-i}+(-1)^iC^{n-i})-dF_{mn-mi+k}(D^{n-i}-(-1)^iC^{n-i})].$$

Further three term identities from which binomial identities may be derived in the way described are

$$dX_n = A\alpha^n - B\beta^n\ ,$$
$$Y_n = A\alpha^n + B\beta^n\ ,$$
$$A\alpha^{m+n} = X_m\alpha^{n+1} + qX_{m-1}\alpha^n\ , \tag{5}$$
$$B\beta^{m+n} = X_m\beta^{n+1} + qX_{m-1}\beta^n\ , \tag{6}$$
$$\alpha^2 = p\alpha + q\ ,$$
$$\beta^2 = p\beta + q\ ,$$
$$Y_n^2 = d^2X_n^2 + 4AB(-q)^j\ , \tag{7}$$
$$A\alpha^{2m} = Y_m\alpha^m - B(-q)^m\ ,$$
$$A\alpha^{2m} = dX_m\alpha^m + B(-q)^m$$
$$B\beta^{2m} = Y_m\beta^m - A(-q)^m\ , \qquad B\beta^{2m} = -dX_m\beta^m + A(-q)^m\ .$$

Most of these identities are obvious, or nearly so. Identity (5) may be proved as follows:

$$A\alpha^m = \tfrac{1}{2}Y_m + \tfrac{1}{2}dX_m = \tfrac{1}{2}(pX_m + 2qX_{m-1} + dX_m) = X_m\left(\frac{p+d}{2}\right) + qX_{m-1} = X_m\alpha + qX_{m-1},$$

and identity (6) is proved similarly. Identity (7) is proved as follows:

$$Y_n^2 = (A\alpha^n + B\beta^n)^2 = (A\alpha^n - B\beta^n) + 4AB(\alpha\beta)^n = (\alpha-\beta)^2\left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right)^2 + 4AB(-q)^n = d^2X_n^2 + 4AB(-q)^n.$$

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ERRATA

Please make the following corrections to "Fibonacci Sequences Modulo *M,*" appearing in the February 1974 (Vol. 12, No. 1) issue of *The Fibonacci Quarterly,* pp. 51–64.

On page 52, last line, last sentence, change "If $2/f(p)$," to read "If $2\nmid f(p)$."

On page 53, change the fourth line of the third paragraph from "which $(a,b,p^e) = 1$," to: "which $(a,b,p^e) \neq 1$."

On page 56, third paragraph of proof, tenth line should read:

"...is given by $5^{2e} - 5^{2e-2} - 4\cdot 5^{2e-2} = 4\cdot 5^{2e-1}$ ... ."

On page 61, change the second displayed equation to read:

$$n(k) = \frac{p^{2t}-1}{k}.$$

Line 7 from the bottom should read:

" for $i = t, \cdots, e-1$. "

A RECIPROCAL SERIES OF FIBONACCI NUMBERS

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Theorem

$$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\frac{1}{F_8}+\frac{1}{F_{16}}+\cdots=\frac{7-\sqrt{5}}{2}\,.$$

Proof

$$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\cdots+\frac{1}{F_{2^n}}=3-F_{2^n-1}/F_{2^n}$$

is easily proved by induction using Binet's formula, and the theorem follows by letting $n\to\infty$. The result resembles the formula

$$\sqrt{m}=\frac{(m-1)\alpha_n}{4\beta_{n-1}}-\frac{m-1}{2}\left(\frac{1}{\beta_n}+\frac{1}{\beta_{n+1}}+\cdots\right)\,,$$

where

$$m>1,\ \alpha_1=2\,\frac{m+1}{m-1},\ \alpha_{n+1}=\alpha_n^2-2,\ \beta_0=1,\ \beta_n=\alpha_1\alpha_2\cdots\alpha_n\,.$$

(Reference 1.

Some curious properties of Fibonacci numbers appeared in [2]; for example,

$$\Delta_{48}^2 5^{F_n}=5^{F_{n+96}}-2\cdot 5^{F_{n+48}}+5^{F_n}$$

is a multiple of $2^{12}3^5 7^3=341,397,504$ for $n=1,2,3,\cdots$.

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Received by the Editors May, 1972. See H-237, Oct. 1974 *Fibonacci Quarterly*, p. 309.

POWER SERIES AND CYCLIC DECIMALS

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There is an interesting relation between series based on the powers of an integer, and infinitely repeating decimal reciprocals whereby the sum of the powers of a single integer give not one, but two reciprocals. Figures 1 and 2 illustrate this in the case of the two integers 3 and 19, which yield respectively the decimal reciprocals *1/29, 1/7;* and *1/189, 1/81.* The left-hand member in each instance starts at the decimal point and develops (in reverse) to the left. Although it is obviously not a decimal, it is purely cyclic, and has the repetend of its decimal version. Since shifting the decimal by a suitable divisor rectifies this, and for the sake of simplicity, it is treated here as a decimal.

If M is any integer having k digits, the following equations apply:

$$1/(10M - 1) = \sum_{n=1}^{\infty} M^{n-1} \times 10^{n-1} \tag{1}$$

and

$$1/(10^k - M) = \sum_{n=1}^{\infty} M^{n-1} \times 10^{-kn} \; . \tag{2}$$

Equation (1) is limited by the expression $(10M - 1)$ to a fraction having a denominator with the last digit 9, and will thus be odd and yield a cyclic decimal fraction having a repetend with the terminal digit 1. Equation (2) is limited by the expression $(10^k - M)$ to a denominator which is the complement of M and will thus be odd, or even, and will not be limited as to type of repeating decimal. In the preparation of Figs. 1 and 2, zeros not contributing to the relations shown have been omitted.

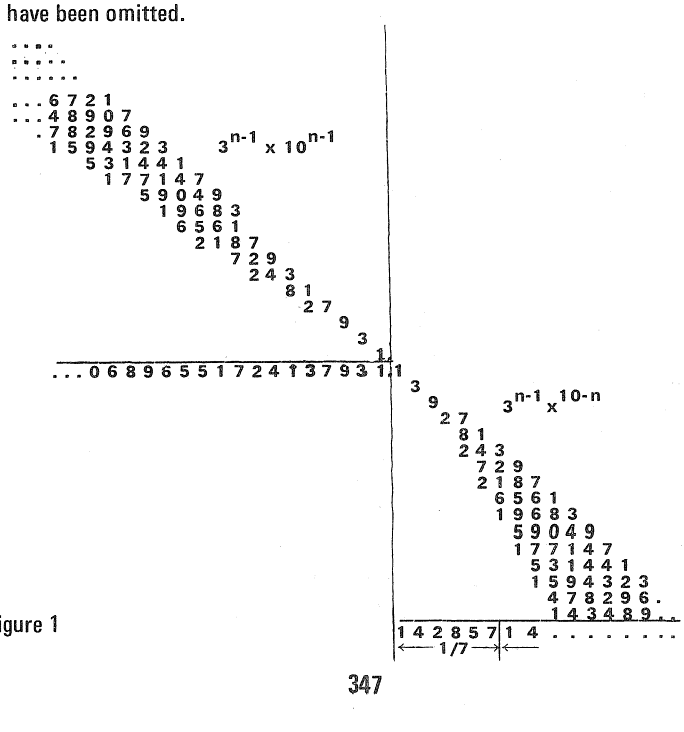

Figure 1

```
                    .
                    . .
                    . . .
                    . . . 9
                    . . . 0 1
                    . . . 7 7 9
                    . . . 3 0 4 1
                    . . . 7 1 7 3 9
                      . . 0 4 5 8 8 1
                          2 4 7 6 0 9 9  19^(n-1) x 10^(n-1)
                              1 3 0 3 2 1
                                  6 8 5 9
                                      3 6 1
                                          1 9
                                              1|
                    . . . 0 0 5 2 9 1 0 0 5 2 9 1|0 1
                                                 |  1 9
                                                 |    3 6 1      19^(n-1) x 10^(-2n)
                                                 |      6 8 5 9
                                                 |      1 3 0 3 2 1
                                                 |        2 4 7 6 0 9 9
                                                 |          4 7 0 4 5 8 8 .
                                                 |            8 9 3 8 7 1 . . .
                                                 |            1 6 9 8 3 5 . . .
                                                 |              3 2 2 6 8 . . .
                                                 |                6 1 3 1 . . .
                                                 |                  5 5 1 . . .
                                                 |0 1 2 3 4 5 6 7 9|. . . . . . .
                                                 |<------1/81------>|
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Figure 2

★★★★★★

ON GENERATING FUNCTIONS FOR POWERS OF A GENERALIZED SEQUENCE OF NUMBERS

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GENERATING FUNCTIONS

For the record, some results are presented here which arose many years ago (1965) in connection with the author's paper [3]. Familiarity with the notation and results of Carlitz [1], Riordan [6], and the author [2], [3] and [4], are assumed in the interests of brevity. Note, however, that h_n in [3] has been replaced by H_n to avoid ambiguity. Our results and techniques parallel those of Riordan.

Calculations yield

$$
(1)\qquad \left\{\begin{aligned}
H_n^2 - 3H_{n-1}^2 + H_{n-2}^2 &= 2(-1)^n e \\
H_n^3 - 4H_{n-1}^3 - H_{n-2}^3 &= 3(-1)^n eH_{n-1} \\
H_n^4 - 7H_{n-1}^4 + H_{n-2}^4 &= 2e^2 + 8(-1)^n eH_{n-1}^2 \\
H_n^5 - 11H_{n-1}^5 - H_{n-2}^5 &= 5e^2 H_{n-1} + 15(-1)^n eH_{n-1}^3 \,.
\end{aligned}\right. \qquad (e = r^2 - rs - s^2)
$$

and so on. Corresponding generating functions for the k^{th} power of H_n,

[Continued on page 350.]

A LOWER BOUND FOR THE PERIOD OF THE FIBONACCI SERIES MODULO M

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In this note we shall determine a nontrivial lower bound for the period of the Fibonacci series modulo m. This problem was posed by D. D. Wall [2], p. 529.

Let $a(m)$ denote the subscript of the first term of the Fibonacci series

(1) $$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1,$$

which is divisible by m. Let $k(m)$ denote the period of $\{F_n\}$ modulo m. Define the sequence $\{L_n\}$ so that

(2) $$L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 2, \qquad L_1 = 1.$$

Then our main result is the following theorem:

Theorem. Let t be any natural number such that $L_t \leqslant m$, where $m > 2$. Then $k(m) \geqslant 2t$, with equality if and only if $L_t = m$ and t is odd.

Wall posed the question for prime values of m. It is not known whether or not there are infinitely many prime m such that $L_t = m$ when t is odd.

For the proof of the theorem we need some preliminary results. The following theorem is proved in [1] (Th. 3).

Vinson's Theorem. Let m be any integer greater than 2. If $a(m)$ is odd, then $k(m) = 4a(m)$; if $8 \nmid m$ and $a(p) \equiv 2 \pmod 4$ for all odd prime divisors of m, then $k(m) = a(m)$; in any other case, $k(m) = 2a(m)$.

In addition to well known identities, the following is useful:

(3) $$F_i \equiv -(-1)^i F_{k(m)-i} \pmod m.$$

Equation (3) follows by induction on i, using (1), $F_0 \equiv F_{k(m)} \equiv 0$, and $F_1 \equiv F_{k(m)+1} \equiv 1 \pmod m$.

Lemma. $k(L_n) = 4n$ when n is even; $k(L_n) = 2n$ when n is odd.

Proof: $a(L_n) = 2n$ is known, and may be proved using $F_{2n} = F_n L_n$, $F_t < L_n$ for $t \leqslant n+1$, and the fact that subscripts n for which $m \mid U_n$ form an ideal. The lemma follows by an application of Vinson's theorem.

Proof of Theorem: It is known [2] that $k(m)$ is even. Using the identities (3) and

(4) $$L_n = F_{n-1} + F_{n+1},$$

we see that if $k(m) = 2t$ then $F_t \equiv -(-1)^t F_t \pmod m$, so t is odd or $F_t \equiv 0 \pmod m$. If t is odd then by (3) we have $F_{t+1} \equiv -F_{t-1} \pmod m$ implying (by (4)) that

(5) $$L_t \equiv F_{t+1} + F_{t-1} \equiv 0 \pmod m.$$

Otherwise, if $F_t \equiv 0 \pmod m$ then by (1),

(6) $$F_{t+1} - F_{t-1} \equiv 0 \pmod m.$$

Clearly, if $t \leqslant n$ then

(7) $$0 < F_{t+1} + F_{t-1} \leqslant F_{n+1} + F_{n-1} = L_n \leqslant m,$$

by the hypothesis of the theorem. By (4)

$$F_{t+1} - F_{t-1} = 2F_{t+1} - L_t,$$

and since $m \geqslant L_t > F_{t+1}$ when $t \leqslant n$, we have

(8) $$F_{t+1} - F_{t-1} < m.$$

Now, (5) and (7) imply that $t = n$ and $L_t = m$, and (6) and (8) are never simultaneously true. Thus $t \geqslant n$, with equality only if $L_n = m$. By the lemma,

$$k(m) = 2t = 2n$$

if and only if n and t are odd and $L_n = m$. The conclusion of the theorem follows.

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[Continued from page 348.]

$$H_k(x) = \sum_{n=0}^{\infty} H_n^k x^n \qquad (H_k(0) = (H_o)^k = r^k),$$

where

$$H_o(x) = f_0(x) = \sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

and

$$H_1(x) = (r+sx)(1-x-x^2)^{-1}$$

are

$$(2)\qquad \begin{cases} (1-3x+x^2)H_2(x) = r^2 - s^2x - 2exH_0(-x) \\ (1-4x-x^2)H_3(x) = r^3 + s^3x - 3exH_1(-x) \\ (1-7x+x^2)H_4(x) = r^4 - s^4x + 2e^2xH_0(x) - 8exH_2(-x) \\ (1-11x-x^2)H_5(x) = r^5 + s^5x + 5e^2xH_1(x) - 15exH_3(-x)\,. \end{cases}$$

The general expression for the generating function is (see [3])

$$(3)\qquad (1-a_kx+(-1)^kx^2)H_k(x) = r^k - (-s)^kx + kx\sum_{j=1}^{[k/2]} \frac{(-1)^j}{j}\, e^j a_{kj} H_{k-2j}((-1)^j x)\,,$$

where

$$(1-x-x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj}x^{k-2j}\,,$$

that is, a_{kj} are generated by the j^{th} power of the generating function for Fibonacci numbers f_n. Note the occurrence in (3) of the Lucas numbers a_n.

FUNCTIONS ASSOCIATED WITH THE GENERATING FUNCTIONS

In the process of obtaining (3), we use

$$(4)\qquad g_k(x) = \sqrt{5}\,H_k(x) = \sum_{j=0}^{[k/2]} \binom{k}{j} e^j F_{k-2j}((-1)^j x) \qquad (F_0(x) = H_0(x))\,,$$

where

$$F_k(x) = [(r-sb)a]^k(1-a^kx)^{-1} + [(sa-r)b]^k(1-b^kx)^{-1} \qquad (k = 1, 2, 3, \cdots)$$

and

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2} \qquad (a,b \text{ roots of } x^2 - x - 1 = 0),$$

leading to the general inverse

[Continued on page 354.]

SOME CONGRUENCES FOR FIBONACCI NUMBERS

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1. INTRODUCTION

The first congruence in this paper arose in an effort to extend a result of Collings [1] and the second congruence is merely an elaboration of part of a theorem of Wall [5]. In the final section we look at some congruences modulo m^2.

Some of the symbols involved are: $D(m)$, the period of divisibility modulo m (or rank of apparition of m or entry point of m), the smallest positive integer z such that $F_z \equiv 0 \pmod{m}$ (see Daykin and Dresel [2]); $C(m)$, the period of cycle modulo m, the smallest positive integer k: $F_{k+n} \equiv F_n \pmod{m}$, $n \geqslant 0$; $T(m)$, the smallest positive integer ℓ: $F_{z+1}^{\ell} \equiv 1. \pmod{m}$. In fact, $z\ell = k$. (See Wyler [6].)

Collings' result was than when m is prime, ℓ is even,

$$F_r + F_{\frac{1}{2}\ell z+r} \equiv 0 \pmod{m}, \tag{1.1}$$

where

$$F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 3), \qquad F_1 = F_2 = 1 .$$

We show that m can be composite if $F_{z+1}^{\frac{1}{2}\ell} \equiv -1 \pmod{m}$.

2. LEMMAS

Lemma 2.1: (see Vinson [5].)

For $m > 2$, $D(m)$ is odd implies that $T(m) = 4$; and $D(m)$ is even implies that $T(m) = 1$ or 2.

Proof: Simson's relation can be expressed as

$$\begin{aligned} F_{z+1}^2 &= F_{z+2}F_z + (-1)^{z+2} \\ &\equiv (-1)^{z+2} \quad \text{since} \quad F_z \equiv 0 \pmod{m}, \\ &\equiv 1 \pmod{m} \quad \text{if} \quad z = D(m) \quad \text{is even}, \\ &\equiv -1 \pmod{m} \quad \text{if} \quad z = D(m) \quad \text{is odd.}\ . \end{aligned}$$

When

$$\begin{aligned} &F_{z+1}^2 \equiv 1 \pmod{m}, \\ &T(m) = 2 \quad \text{if} \quad F_{z+1} \not\equiv 1 \pmod{m}, \\ &T(m) = 1 \quad \text{if} \quad F_{z+1} \equiv 1 \pmod{m}. \end{aligned}$$

When

$$\begin{aligned} &F_{z+1}^2 \equiv -1 \pmod{m}, \\ &F_{z+1}^2 \equiv 1 \pmod{m} \quad \text{if} \quad m > 2; \end{aligned}$$

so

$$F_{z+1} \not\equiv \pm 1 \pmod{m}.$$
$$F_{z+1}^3 = F_{z+1}^2 F_{z+1} \equiv -F_{z+1} \pmod{m};$$
$$F_{z+1}^4 = [F_{z+1}^2]^2 \equiv 1 \pmod{m},$$

and

$$T(m) = 4.$$

Lemma 2.2: $$F_{k-1} \equiv 1 \pmod{m}.$$

Proof: $$F_{k-1} = F_{k+1} - F_k \equiv F_1 - 0 \pmod{m}$$
$$\equiv 1 \pmod{m}.$$

3. THEOREMS

Theorem 3.1: If $\ell \neq 1$ and $F_{z+1}^{\frac{1}{2}\ell} \equiv -1 \pmod{m}$, then

$$F_r + F_{\frac{1}{2}\ell z+r} \equiv 0 \pmod{m} \quad \text{for all} \quad r > 0.$$

Proof: $\ell = T(m)$ which takes only the values $1, 2, 4$ (Lemma 2.1). But $\ell \neq 1$ (given). Therefore ℓ is even. Therefore, $F_{z+1}^{\frac{1}{2}\ell}$ exists and is unique. Moreover,

$$F_{\frac{1}{2}\ell z+r} \equiv F_{z+1}^{\frac{1}{2}} F_r \pmod{m} \quad \text{(see Eq. (8) of [4])}$$
$$\equiv -F_r \pmod{m} \quad \text{as} \quad F_{z+1}^{\frac{1}{2}\ell} \equiv -1$$
$$\therefore F_r + F_{\frac{1}{2}\ell z+r} \equiv 0 \pmod{m}.$$

NOTE. (i) Conversely, if for $\ell \neq 1$ we are given that

$$F_r + F_{\frac{1}{2}\ell z+r} \equiv 0 \pmod{m},$$

for all r, this congruence must hold for $r = 1$.

$$\therefore 1 = F_1 \equiv -F_{\frac{1}{2}\ell z+1} \pmod{m}$$
$$\equiv -F_{z+1}^{\frac{1}{2}\ell} F_1 \pmod{m}$$
$$\equiv -F_{z+1}^{\frac{1}{2}\ell}.$$

On the other hand, it is possible for

$$F_r + F_{\frac{1}{2}\ell z+r}$$

to be congruent to zero for some particular r without $F_{z+1}^{\frac{1}{2}\ell}$ being congruent to -1. Thus, when $m = 12$,

$$F_{12} = 144 \equiv 0 \pmod{12} \quad \text{and} \quad z = 12.$$
$$F_{z+1} = F_{13} = 233 \equiv 5 \pmod{12}$$
$$\therefore \ell = 2$$
$$\therefore F_{z+1}^{\frac{1}{2}\ell} = F_{13} \not\equiv -1 \pmod{12}.$$

Despite this,

$$F_3 + F_{\frac{1}{2}\ell z+3} = F_3 + F_{15} = 2 + 610 = 612 \equiv 0 \pmod{12}.$$

(ii) When $\ell = 1$ the situation is very untidy. If z is odd, $F_{\frac{1}{2}\ell z+r}$ does not exist. Even when z is even, we have trouble with $F_{z+1}^{\frac{1}{2}\ell}$. As $\ell = 1$, $F_{z+1} \equiv 1 \pmod{m}$. Therefore

$$F_{z+1}^{\frac{1}{2}} = \sqrt{F_{z+1}} \equiv \sqrt{1} = \pm 1$$

(and possibly other values as well). -1 is always a possible value for $F_{z+1}^{\frac{1}{2}}$, but never the exclusive value.

(iii) Although -1 is always a possible value for $F_{z+1}^{\frac{1}{2}\ell}$ $(\ell = 1)$, it is not necessarily true that

$$F_r + F_{\frac{1}{2}\ell z+r} \equiv 0 \pmod{m} \quad \text{for all} \quad r > 0.$$

Thus, when $m = 4$, $z = 6$.

$$\therefore F_{z+1} \equiv 1 \pmod{m}, \quad \therefore \ell = 1.$$
$$\therefore F_2 + F_{\frac{1}{2}z+2} = F_2 + F_5 = 6 \equiv 2 \pmod{4}.$$

Theorem 3.2: $$F_r + (-1)^r F_{k-r} \equiv 0 \pmod m.$$

Proof: $-F_k \equiv 0 = F_0$ and $F_{k-1} \equiv 1 = F_1 \pmod m$, by Lemma 2.2

$$-F_{k-2} = -F_k + F_{k-1} \equiv F_0 + F_1 \equiv F_2 \pmod m.$$

It follows by induction on k that

$$\begin{aligned}(-1)^{r-1} F_{k-r} &= (-1)^{r-1} F_{k-r+2} + (-1)^r F_{k-r+1}\\ &\equiv F_{r-2} + F_{r-1} \pmod m\\ &\equiv F_r \pmod m,\end{aligned}$$

which gives the required result.

4. CONGRUENCES MODULO m^2

Here we use the results (see Hoggatt [3])

(4.1) $$F_{nr+1} = F_{(n-1)r}F_r + F_{(n-1)r+1}F_{r+1}$$

and

(4.2) $$F_{2n+1} = F_n^2 + F_{n+1}^2 .$$

If $a \pmod m \equiv F_{z+1} \equiv b \pmod{m^2}$, then b is of the form $Bm + a$, for some B. For example, $F_5 \equiv 0 \pmod 5$, $3 \pmod 5 \equiv F_6 \equiv 8 \pmod{5^2}$, and $8 = 1 \times 5 + 3$.

Using $F_z \equiv 0 \pmod m$ and (4.1) and (4.2) we find

$$F_{2z+1} \equiv F_{z+1}^2 \pmod{m^2} \equiv b^2 \pmod{m^2},$$

and

$$F_{3z+1} \equiv F_{2z+1}F_{z+1} \pmod{m^2} \equiv b^3 \pmod{m^2},$$

which, by the use of (4.1), can be generalized to

(4.3) $$F_{nz+1} \equiv b^n \pmod{m^2}.$$

Furthermore, since $F_z = Am$ for some A, then

$$F_{z-1} \equiv b - Am \pmod{m^2}$$

and

$$\begin{aligned}F_{2z} &= F_{z-1}F_z + F_zF_{z+1}\\ &\equiv (b - Am)Am + Amb \pmod{m^2}\\ &\equiv 2bAm \pmod{m^2}.\end{aligned}$$

Also,

$$\begin{aligned}F_{3z} &= F_{2z-1}F_z + F_{2z}F_{z+1} \quad \text{(from (4.1))}\\ &\equiv (b^2 - 2bAm)Am + 2bAm \cdot b \pmod{m^2}\\ &\equiv 3b^2Am \pmod{m^2}.\end{aligned}$$

Similarly, $F_{4z} \equiv 4b^3Am \pmod{m^2}$. Thus

(4.4) $$F_{nz} \equiv nb^{n-1}Am \pmod{m^2}.$$

When $F_{nz} \equiv 0$ the congruence $nb^{n-1}A \equiv 0 \pmod m$ reduces to $nA \equiv 0 \pmod m$, because, from (4.3) and (4.4), if b and m have any factor in common, so have F_{nz} and F_{nz+1}, which is impossible as adjacent Fibonacci numbers are always co-prime. Thus, if we solve $nA \equiv 0 \pmod m$ for n, then $Z = nz$ gives that F_Z which is zero $\pmod{m^2}$.

Let us apply these methods to find which Fibonacci numbers are divisible by convenient powers of 10. Instead of working with $m = 10$, we shall find the equations simpler if we write $10 = m_1 \cdot m_2$, where $m_1 = 2$, $m_2 = 5$, and $100 = 2^2 \cdot 5^2$. $m_1 = 2$, $z = 3$, $F_3 = 1 \cdot 2$ and so $A = 1$. The equation $nA \equiv 0 \pmod m$ reduces to $n \equiv 0 \pmod 2$, which gives $n = 2$, so that $Z = 2z = 6$. Similarly with $m_2 = 5$, $z = 5$, and we find that $Z = 5z = 25$.

If we take $m_1 = 4$, $z = 6$, $F_6 = 2 \cdot 4$ and so $A = 2$. Thus $2n \equiv 0 \pmod 4$ which gives $n = 2$ and $Z = 2z = 12$. Similarly, with $m_2 = 25$, $z = 25$ and $F_{25} = 75025 = 3001 \cdot 25$ which yields $A \equiv 1 \pmod{25}$. So $n = 25$ and $Z = 25z = 625$.

Relying on the known result that the period of divisibility by m_1m_2 (m_1, m_2 co-prime) is given by $D(m_1m_2) = LCM(z_1, z_2)$ (see Wall [6]), we get the results:

$LCM(3,5) = 15$, and so F_{15} is the first Fibonacci number to be divisible by 10. $lcm(6,25) = 150$, and so F_{150} is divisible by 100. $LCM(12,625) = 7{,}500$ and so F_{7500} is divisible by 10^4.

This has been an exercise in finding the z numbers. By an extension of the argument we can produce the corresponding k numbers—the period of recurrence of the Fibonacci numbers (mod m^2).

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[Continued from page 350.]

$$F_k(x) = \sum_{j=0}^{[k/2]} (-1)^j e^j \frac{k}{k-j} \binom{k-j}{j} g_{k-2j}((-1)^j x)\,. \tag{5}$$

Write

$$\begin{cases} h_k(x) = (1 - a_k x + (-1)^k x^2) g_k(x) \\ c_k \quad = [(r-sb)a]^k + [(sa-r)b]^k\,. \end{cases} \tag{6}$$

Following Riordan [6], with $a_0 = 2$ and $h_0(x) = 1 - x$, we eventually derive

$$\begin{cases} c_1 + s\sqrt{5}\,x = h_1(x) \\ c_2 - x(2e + 5s^2) = h_2(x) - 2e\left\{h_0(-x) - (a_0 + a_2)xg_0(-x)\right\} \\ c_3 + s\sqrt{5}\,x(3e + 5s^2) = h_3(x) - 3e\left\{h_1(-x) - (a_1 + a_3)xg_1(-x)\right\} \\ c_4 - x(2e^2 + 20s^2e + 25s^4) = h_4(x) - 4e\left\{h_2(-x) - (a_2 + a_4)xg_2(-x)\right\} \\ \qquad + 2e^2\left\{h_0(x) - (a_4 - a_0)xg_0(x)\right\} \\ c_5 - e_1 = h_5(x) - 5e\left\{h_3(-x) - (a_3 + a_5)xg_3(-x)\right\} + 5e^2\left\{h_1(x) - (a_5 - a_1)xg_1(x)\right\} \end{cases} \tag{7}$$

where

$$e_1 = 2r^5 - 5r^4s + 30r^2s^2 - 40r^2s^3 + 35rs^4 - 10s^5\,.$$

Substituting values of $a_k = a^k + b^k$, we have

$$\begin{cases} h_1(x) = \sqrt{5}\,(r + sx) \\ h_2(x) = 5(r^2 - s^2x) - 10exg_0(-x) \\ h_3(x) = 5\sqrt{5}\,(r^3 + s^3x) - 15exg_1(-x) \\ h_4(x) = 25(r^4 - s^4x) - 40exg_2(-x) + 50e^2xg_0(x) \\ h_5(x) = 25\sqrt{5}\,(r^5 + s^5x) - 75exg_3(-x) + 125e^2xg_1(x)\,. \end{cases} \tag{8}$$

These functions lead back to (2).

[Continued on page 362.]

FIBONACCI NUMBERS IN TREE COUNTS FOR SECTOR AND RELATED GRAPHS

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1. INTRODUCTION

Consider a set of $(n+1)$ vertices of a nonoriented graph with vertices $1, 2, \cdots, n$ adjacent to vertex $(n+1)$ and with vertex i adjacent to vertex $(i+1)$ for $1 \leqslant i \leqslant n-1$. The graph described is called a *sector* graph herein. If the first n vertices are equi spaced in clockwise ascending order on the circumference of a circle with the $(n+1)^{st}$ vertex at the center, the geometry justifies the choice of name.

If the first and n^{th} vertices were made adjacent, the result would be the well known *wheel**, W_{n+1} described by Harary [1]. For this reason, the lucidly descriptive terminology of W_{n+1} is applicable to a sector graph as well. Vertex $(n+1)$ is a *hub* vertex with *spokes* radiating outward to the n *rim vertices* which are adjacent by virtue of *rim* edges. Multiple spokes and/or rim edges are admissible.

The designation used herein for a sector graph with n rim vertices is S_n followed in parentheses by spoke and rim edge multiplicity information. (In particular, rim edge position i is between vertices i and $(i+1)$. This would also specify the position of sector i.) Thus, the designation $S_8(1(2), 6(3), \boxed{3}(2))$ would describe a sector graph of nine vertices total having double spokes in the first spoke position, triple spokes in the sixth spoke position, and double rim edges in the third rim edge position. A simple sector graph would only require the designation S_n. The same terminology applies to wheels after a rim vertex is designated as vertex 1. An example is given in Figure 1. The

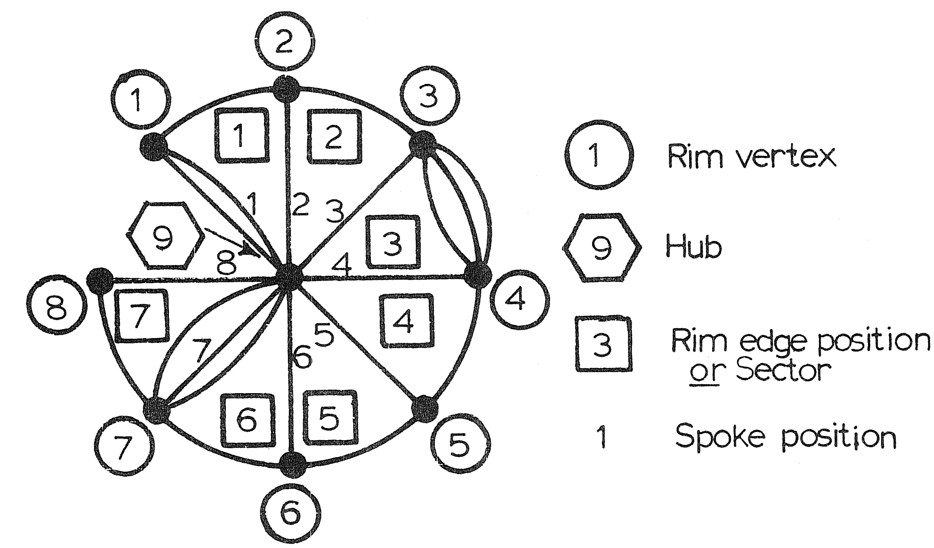

Figure 1 Example of $S_8(1(2), 6(3), \boxed{3}(3))$

number of trees is indicated by prefixing a T. Thus, TS_n is the number of trees in a simple sector graph. (Unless otherwise stated, trees will refer to *spanning* trees.)

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*The subscript for the wheel customarily denotes the total number of vertices including the hub. The subscript $n+1$ is used here to retain identification with the n rim vertices.

2. THE COUNT OF $TS_n(1(2), n(2))$ AND A BASIC DETERMINANT

If a graph has some measure of symmetry, an algebraic approach to counting of trees is often feasible. If one row of the incidence matrix A of the graph is suppressed to obtain the reduced incidence matrix A_n (of rank n), it is known [2] that the number of trees is given by $det(A_nA_n^t)$, where t indicates the transpose operation. In the case of $S_n(1(2), n(2))$, suppressing the hub vertex row yields

(1)
$$det(A_nA_n^t) = TS_n(1(2), n(2)) = \underbrace{\begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}}_{n \times n} = a_n .$$

The determinant a_n of (1) is basic to succeeding work.

The recurrence realtion from (1) is easily found to be

(2)
$$a_n = 3a_{n-1} - a_{n-2}$$

whose solution is [3]

(3)
$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right] .$$

Physically, (3) is valid for $n \geqslant 2$. However, $a_0 = 1$, $a_1 = 3$ are consistent mathematically. The resulting numerical sequence of tree counts is

(4)
$$1, 3, 8, 21, 55, 144, \cdots \qquad (n = 0, 1, 2, 3, 4, 5, \cdots).$$

It is evident that (4) gives alternate numbers of the Fibonacci sequence

(5)
$$F_1, F_2, F_3, F_4, F_5, F_6, \cdots \rightarrow 1, 1, 2, 3, 5, 8, \cdots .$$

Upon comparing (5) with (4), it is seen that

(6)
$$a_n = TS_n(1(2), n(2)) = F_{2n+2} .$$

This result is not surprising, of course, since it is well known [4] that electrical ladder networks have graphs of the sector type and immittance calculations on unit element ladders involve tree-derived numerators and denominators of Fibonacci numbers.

Application of the Z-Transform [5] to (2), results in

(7)
$$Z(a_n) = \frac{z^2a_0 + z(a_1 - 3a_0)}{z^2 - 3z + 1} .$$

By dividing the numerator of (7) by the denominator, the values of a_n are found as coefficients of $1/z^n$. By setting $a_0 = 1$, $a_1 = 3$,

(8)
$$Z(a_n) = \frac{z^2}{z^2 - 3z + 1}$$

is found as the generating function in powers of $1/z$ of the sequence (4).

3. THE COUNT OF $TS_n(1(2))$

Next, consider $TS_n(1(2))$ (by symmetry, $TS_n(n(2))$. $Det(A_nA_n^t)$ is the same as that of (1) except the first 3 on the main diagonal is replaced by 2. Thus, in terms of a_n and through the use of (2) and (6),

(9)
$$TS_n(1(2)) = 2a_{n-1} - a_{n-2} = a_n - a_{n-1} = F_{2n+2} - F_{2n} = F_{2n+1} .$$

The Fibonacci numbers not in (4) satisfy the same recurrence relation (2) as those in (4). Use of new initial conditions with (2), say, $a_n = 5$ for $n = 2$ and $a_n = 13$ for $n = 3$ yield

$$(10) \qquad TS_n(1(2)) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n = F_{2n+1}\,.$$

The resulting sequence of tree count numbers is

$$(11) \qquad 1, 2, 5, 13, 34, 89, \cdots \qquad (n = 0, 1, 2, 3, 4, 5, \cdots),$$

where physical validity applies for $n \geq 2$.

By letting $a_0 = 1$, $a_1 = 2$ in (7), the generating function for the sequence (11) becomes

$$(12) \qquad Z(TS_n(1(2))) = \frac{z^2 - z}{z^2 - 3z + 1}\,.$$

4. THE COUNT OF TS_n

In S_n, the degree of rim vertices 1 and n is two. Hence, the $det\,(A_nA_n^t)$ for S_n is the same as (1) except that the 3's in the $(1,1)$ and (n,n) positions are replaced by 2's. There results

$$(13) \qquad TS_n = \underbrace{\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & & & & & & 0 \\ 0 & & & & & & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & a_{n-2} & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & & & & & & 0 \\ 0 & & & & & & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix}}_{n \times n} = 2\begin{vmatrix} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & a_{n-2} & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & -1 \\ 0 & 0 & 0 \cdots & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & \cdots & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & \cdot & \cdot \\ & & a_{n-3} & & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & 1 & 0 \\ & & & & 3 & -1 \\ 0 & 0 & \cdots 0 & -1 & & 2 \end{vmatrix}$$

$$= 4a_{n-2} - 4a_{n-3} + a_{n-4} = a_{n-1} = F_{2n}\,.$$

This means that

$$(14) \qquad TS_n = TS_{n-1}(1(2), n-1(2)) = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]\,.$$

An index shift by one can be accomplished in (8) by multiplication by $1/z$. Hence, the generating function for TS_n becomes

$$(15) \qquad Z(TS_n) = \frac{z}{z^2 - 3z + 1}\,.$$

In terms of *sectors*, the simple sector graph of k sectors has the tree count given by (6) with n replaced by k.

5. EXTENSION TO TW_{n+1}

In S_n, by additionally making rim vertices 1 and n simply adjacent, the simple wheel W_{n+1} is obtained. $Det\,(A_nA_n^t)$ is the same as (1) except that -1's replace 0's in the $(1,n)$ and $(n,1)$ positions. There results

$$(16) \qquad TW_{n+1} = \begin{vmatrix} & & & & & -1 \\ & & & & & 0 \\ & & & & & 0 \\ & & a_{n-1} & & & \vdots \\ & & & & & 0 \\ & & & & & -1 \\ -1 & 0 & 0 \cdots & 0 & -1 & 3 \end{vmatrix} = 3a_{n-1} - 2a_{n-2} - 2 = a_n - a_{n-2} - 2$$

$$= \frac{3}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n - \frac{2\sqrt{5}}{3}\right] = F_{2n+2} - F_{2n-2} - 2.$$

6. COUNT OF TREES WHICH INCLUDE INDIVIDUAL SPOKES OR RIM EDGES

One way to find the number of trees which contain a particular graph edge is to coalesce the vertices of the edge and count the trees of the vertex-reduced graph, the count being the desired number of trees in the unreduced graph [6] containing the edge. The self loop into which the edge degenerates can be disregarded for tree counting.

If a connected graph is separable, the number of trees is equal to the product of the trees of the separable subgraphs. When removal of a graph edge produces two separable but connected components, the difference between the product tree count and the number of trees of the original graph provides an additional way of finding the number of trees containing a particular graph edge. A few easily extended illustrative examples follow.

COUNT OF TREES WITH A GIVEN SPOKE. Consider the h^{th} spoke of S_n. By coalescing vertex h with hub vertex, two edge-disjoint subgraphs appear so that the vertex-reduced graph is separable with the hub vertex being a cut vertex. Each subgraph is a sector graph having a double end spoke. One subgraph has $(h-1)$ vertices and the other has $(n-h)$ vertices. Through use of (9) and the product rule for separable graphs, it is seen that the number of trees of S_n which contain spoke h is

(17) $$T_{h-1}(1(2))\cdot T_{n-h}(1(2)) = F_{2h-1}\cdot F_{2n-2h+1}, \qquad (1 \leqslant h \leqslant n).$$

Consider *any* spoke of W_n. Coalescing the rim vertex to the hub yields $S_{n-1}(1(2), n-1(2))$ which, by (6), has

(18) $$TS_{n-1}(1(2), n-1(2)) = F_{2n}$$

trees. Thus, any spoke of W_n is in F_{2n} trees.

COUNT OF TREES WITH A GIVEN RIM EDGE. Let rim edge k be the edge of S_n which is incident with rim vertices k and $(k+1)$. Removal of rim edge k reduces S_n to a separable graph having the hub vertex as the cut vertex. The subgraphs are the sector graphs S_k and S_{n-k}. They are

(19) $$TS_k\cdot TS_{n-k} = F_{2k}\cdot F_{2n-2k}$$

trees in the reduced graph. Since S_n has F_{2n} trees, the number of trees of S_n in which rim edge k appears is

(20) $$TS_n - TS_k\cdot TS_{n-k} = F_{2n} - F_{2k}\cdot F_{2n-2k}.$$

If any rim section is removed from W_n, S_n results. Therefore, any rim selection of W_n must be in

(21) $$TW_n - TS_n = F_{2n+2} - F_{2n-2} - F_{2n} - 2$$

trees.

7. GRAPHS WITH MULTIPLE SPOKES AND RIM EDGES

TREE COUNT WITH MULTIPLE SPOKES. Suppose that the number of spokes in position h of S_n is increased to j. Since a spoke cannot be in a tree with any other spoke in the same position (resulting loop could not be part of a tree), the number of trees would be (with the aid of (13) and (17)),

(22) $$TS_n + (j-1)T_{h-1}(1(2))\cdot T_{n-h}(1(2)) = F_{2n} + (j-1)F_{2h-1}\cdot F_{2n-2h+1}, \qquad (1 \leqslant h \leqslant n).$$

Correspondingly, the increase of the number of spokes in any position of W_{n+1} results in a total of trees given by (see (16) and (18))

(23) $$TW_n + (j-1)TS_{n-1}(1(2), n-1(2)) = F_{2n+2} - F_{2n-2} + (j-1)F_{2n} - 2.$$

TREE COUNT WITH MULTIPLE RIM EDGES. If the number of rim edges for sector k of S_n is increased to j, the number of trees would become (with the aid of (13) and (20)),

(24) $$jTS_n - (j-1)TS_k\cdot TS_{n-k} = jF_{2n} - (j-1)F_{2k}\cdot F_{2n-2k}.$$

Also, if the number of rim edges for any given sector of W_{n+1} were increased to j, the number of trees becomes (see (16) and (21))

(25) $$jTW_{n+1} - (j-1)TS_n = j(F_{2n+2} - F_{2n-2} - F_{2n} - 2) + F_{2n}.$$

Extensions to additional multiple edges are available only for the trying. One obvious use for the tree count formulas is the evaluation in general Fibonacci terms special determinants whose form fits $det\,(A_nA_n^t)$ for the multiple edge-modified sector graph or wheel.

Examples showing numbers of trees containing various edges are given in Figure 2.

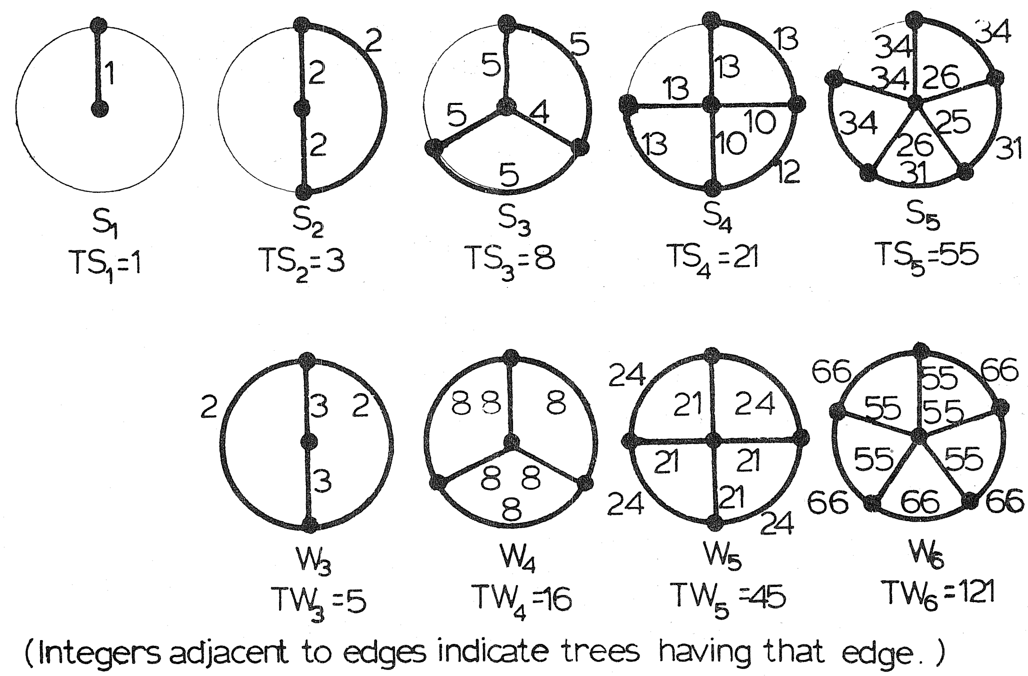

Figure 2. Examples of S_n and W_{n+1}

8. SOME OBSERVATIONS

From Figure 2, it can be surmised that the sum of the number of trees containing edge one, edge two, etc., of W_{n+1} is exactly n times TW_{n+1}. Since there are n spokes and n rim edges in W_{n+1}, multiplication of the sum of (18) and (21) by n does yield n times (16), which verifies the surmise.

Also, from Figure 2 it can be surmised that S_n has this same property. The surmise again is true and rests eventually on the identity

$$F_{2n-2} = \sum_{h=1}^{n-1} [F_{2h-1} \cdot F_{2n-2h+1} - F_{2h} \cdot F_{2n-2h}] \tag{26}$$

which is left as a search or research exercise for the reader.

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COMBINATORIAL INTERPRETATION OF AN ANALOG OF GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Defining $f_{j,k}(n;r,s)$ as the number of sequences of nonnegative integers

$$\left\{a_1, a_2, \cdots, a_n\right\} \tag{1.1}$$

such that

$$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1), \tag{1.2}$$

where r and s are arbitrary positive integers, and

$$a_1 = j, \qquad a_n = k, \tag{1.3}$$

the author [2] has shown that the generating function

$$\phi_{j,r,s}(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\min\left\{n(r+s), j+nr\right\}} f_{j,j+nr-m}(n+1;r,s)x^n y^m$$

can be expressed in terms of generalized binomial coefficients $c_{r+s}(n,k)$ defined by

$$\left(\sum_{h=0}^{r+s} x^h\right)^n = \sum_{k=0}^{\infty} c_{r+s}(n,k)x^k \; . \tag{1.4}$$

For the cases $r = 1$ or $s = 1$ we have explicit formulas for $f_{j,k}(n;r,s)$, namely

$$f_{j,k}(n+1;1,s) = \sum_{t=0}^{j} c_{s+1}(-t-1, j-t)\left[c_{s+1}(n+t, n+t-k) - \sum_{h=0}^{s-1} (h+1)c_{s+1}(n+t, n+t-k-h-2)\right] \; . \tag{1.5}$$

and

$$f_{j,k}(n+1;r,1) = \sum_{t=0}^{k} c_{r+1}(-t-1, k-t)\left[c_{r+1}(n+t, n+t-j) - \sum_{h=0}^{r-1} (h+1)c_{r+1}(n+t, n+t-j-h-2)\right]. \tag{1.6}$$

These formulas generalize a result of Carlitz [1] for $r = s = 1$.

We now define an analog of $c_{r+s}(n,k)$, $n > 0$, by

$$\prod_{j=1}^{n}\left(\sum_{h=0}^{r+s} q^{(r-h)j} x^h\right) = \sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q)x^k \; . \tag{1.7}$$

Letting $f_k(m,n;r,s)$ denote the number of sequences of integers

$$\left\{a_1, a_2, \cdots, a_n\right\} \tag{1.8}$$

satisfying

$$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1), \tag{1.9}$$

where r and s are nonnegative integers,

$$a_1 = 0, \qquad a_n = k \tag{1.10}$$

and

(1.11) $$\sum_{i=1}^{n} a_i = m ,$$

we show in this paper that

(1.12) $$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m, n+1; r,s)q^m .$$

From (1.12) we obtain a partition identity.

2. COMBINATORIAL INTERPRETATION OF $c_{r+s}(n,k;q)$

From the definition of $f_k(m,n;r,s)$ it follows that

(2.1) $$f_k(m,1;r,s) = \delta_{k,0}\delta_{m,0}$$

and

(2.2) $$f_k(m,n+1;r,s) = \sum_{h=0}^{r+s} f_{k+s-h}(m-k, n; r,s) .$$

Now (2.1) had (2.2) imply respectively

(2.3) $$\sum_m f_k(m,1;r,s)q^m = \delta_{k,0}$$

and

(2.4) $$\sum_m f_k(m,n+1;r,s)q^m = \sum_{h=0}^{r+s}\sum_m f_{k+s-h}(m,n;r,s)q^{m+k} .$$

Let

$$\phi(x,y;q) = \sum_{n=0}^{\infty}\sum_{k=0}^{n(r+s)}\sum_m f_{nr-k}(m,n+1;r,s)q^m x^k y^n .$$

Using (2.3) and (2.4) we get

$$\phi(x,y;q) = 1 + y\sum_{h=0}^{r+s}\sum_{n=0}^{\infty}\sum_{k=0}^{(n+1)(r+s)}\sum_m f_{nr-k+h}(m,n+1;r,s)q^{m+nr-k}x^k y^r = 1 + y\left(\sum_{h=0}^{r+s} q^{r-h}x^h\right)\phi(xq^{-1},yq^r;q).$$

By iteration

$$\phi(x,y;q) = \sum_{n=0}^{\infty}\prod_{j=1}^{n}\left(\sum_{h=0}^{r+s} q^{(r-h)j}x^h\right)y^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q)x^k y^n .$$

Equating coefficients we have

(2.5) $$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m,n+1;r,s)q^m .$$

3. APPLICATION TO PARTITIONS

Assuming the parts of a partition to be written in ascending order, let $u_r(k,m,n)$ denote the number of partitions of m into at most n parts with the minimum part at most r, the maximum part k and the difference between consecutive parts at most r. Define $v_r(k,m,n)$ to be the number of partitions of m into k parts with each part at most n and each part occurring at most r times. We show that

(3.1) $$u_r(k,m,n) = v_r(k,m,n) \qquad (r \geq 1).$$

Proof. It is easy to see that

(3.2) $$u_r(k,m,n) = f_k(m,n+1;r,0) .$$

By (2.5) and (1.7) we have

$$\sum_{k=0}^{nr}\sum_m f_k(m,n+1;r,0)q^m x^k = \sum_{k=0}^{nr} c_r(n,nr-k;q)x^k = \prod_{j=1}^{n}\left(\sum_{h=0}^{r} q^{hj}x^h\right)$$

Thus the generating function for $u_r(k,m,n)$ is

$$\prod_{j=1}^{n} \left(\sum_{h=0}^{r} q^{hj} x^h \right) . \tag{3.3}$$

But it is well known (see for example [3, p. 10] for $r = 1$) that the generating function for $v_r(k,m,n)$ is also (3.3). Hence we have (3.1). This identity is also evident from the Ferrers graph.

To illustrate (3.1) and (3.2) let $m = 7,\ n = 4,\ k = 3$ and $r = 2$. The sequences enumerated by $f_3(7,5;2,0)$ are $0,0,1,3,3$, $0,0,2,2,3$ and $0,1,1,2,3$. The function $u_2(3,7,4)$ counts the corresponding partitions, namely 13^2, 2^23 and 1^223. The partitions which $v_2(3,7,4)$ enumerates are 2^23, 13^2 and 124. From the graphs

```
. . .     . . .     . . . .
. .       . . .     . .
. .       .         .
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we observe that 13^2 is the conjugate of 2^23, 2^23 is the conjugate of 13^2 and 1^223 is the conjugate of 124.

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[Continued from page 354.]

SPECIAL CASES

Putting $r = 1,\ s = 0$, we obtain the generating function for the Fibonacci sequence (see [3] and Riordan [6]). Putting $r = 2,\ s = -1$, we obtain the generating function for the Lucas sequence (see [3] and Carlitz [1]).

Other results in Riordan [6] carry over to the H-sequence. The H-sequence (and the Fibonacci and Lucas sequences), and the generalized Fibonacci and Lucas sequences are all special cases of the W-sequence studied by the author in [4]. More particularly,

$$\{H_n\} = \{w_n(r, r+s; 1, -1)\}$$

and so

$$\{f_n\} = \{w_n(1, 1; 1, -1)\}, \qquad \{a_n\} = \{w_n(2, 1; 1, -1)\} .$$

Interested readers might consult the article by Kolodner [5] which contains material somewhat similar to that in [3], though the methods of treatment are very different.

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ON THE SET OF DIVISORS OF A NUMBER

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If z is a natural number and if $z = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_j^{\lambda_j}$ is its factorization into primes, then the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_j$ will be called the *degree* of z. Let m be a squarefree natural number of degree n, i.e., m is the product of n different primes. Let the set of all divisors of m of degree k be denoted by D_k, $k = 0, 1, \cdots, n$; clearly, the cardinality of D_k is equal to $C(n,k)$, where $C(n,k)$ denotes the binomial coefficient, $n!/[k!(n-k)!]$. Two natural numbers δ and ζ are said to *differ in exactly one factor* if $\delta = rp_1$ and $\zeta = rp_2$, where p_1 and p_2 are prime numbers, with $p_1 \neq p_2$. Let α be a natural number that is a divisor of m. A natural number β is said to be an *extension* of α if β is a divisor of m, α is a divisor of β and the degree of β is one more than the degree of α. A natural number γ is said to be a *restriction* of α if γ is a divisor of m, γ is a divisor of α and the degree of γ is one less than the degree of α. If A is a non-empty set of divisors of m, we shall denote by A^+ the set of all extensions of the divisors in A; if $A = \phi$, we define $A^+ = \phi$. The cardinality of any set A will be denoted by $|A|$ and we use the superscript "c" to denote complementation.

In this note, the author gives a relatively short and interesting proof of the following theorem:

Theorem. Let A be a collection of divisors of a squarefree natural number m such that each divisor in A has degree k, $0 \leqslant k \leqslant n$. Then

$$|A^+| \geqslant \frac{|A|C(n,k+1)}{C(n,k)}, \tag{1}$$

and for $A \neq \phi$ equality holds if and only if $|A| = C(n,k)$.

Before proving the theorem, we need to prove one lemma that is also of independent interest.

Lemma. Let A be a non-empty collection of divisors of a squarefree natural number m such that each number in A has degree k, $0 < k < n$, and $|A| < C(n,k)$. Then there exists natural numbers $\alpha \in A$ and $\beta \in A^c \cap D_k$ such that α and β differ in exactly one factor.

Proof. Let ν_0 be an arbitrary number in A. Since $|A| < C(n,k)$, there exists a number $\delta \in A^c$ with the degree of δ equal to k. Let q be the greatest common divisor of ν_0 and of δ and let the degree of q be equal to ω. Then

$$\frac{\nu_0}{\delta} = \frac{t_1 t_2 \cdots t_{k-\omega}}{s_1 s_2 \cdots s_{k-\omega}}, \quad t_i \neq s_j,$$

where $i,j = 1, 2, \cdots, k-\omega$. We now define recursively a finite sequence of numbers by setting

$$\nu_j = \nu_{j-1}\left(\frac{s_j}{t_j}\right), \qquad j = 1, 2, \cdots, k-\omega.$$

Plainly, $\nu_j \in D_k$, ν_{j-1} and ν_j differ in exactly one factor and $\nu_{k-\omega} = \delta$. Since the first number in the sequence $\nu_0, \nu_1, \cdots, \nu_{k-\omega}$ is in A and the last number is in A^c, there exist consecutive numbers ν_{j_0-1}, ν_{j_0} such that $\nu_{j_0-1} \in A$ and $\nu_{j_0} \in A^c$; these can be taken to be, respectively, the numbers α and β of the lemma.

We now prove the previously stated theorem.

Proof. Since (1) holds trivially when either $A = \phi$ or $k = n$, we may assume that $A \neq \phi$ and $k < n$. Consider the set of ordered pairs,

$$E = \left\{(\alpha,\beta): \alpha \in A,\ \beta \text{ is an extension of } \alpha\right\}$$

Since each number $\alpha \in A$ has precisely $n-k$ extensions, $|E| = |A|(n-k)$. If we now set

$$F = \left\{ (\alpha,\beta) : \beta \in A^+,\ \alpha \text{ is a restriction of } \beta \right\},$$

it is clear that $E \subseteq F$ and $|F| = (k+1)|A^+|$. Hence,

$$(k+1)|A^+| \geqslant |A|(n-k),$$

which is equivalent to (1).

If $|A| = C(n,k)$, then

$$C(n,k+1) \geqslant |A^+| \geqslant C(n,k+1),$$

so that equality holds in (1).

Suppose conversely that $A \neq \phi$ and

$$|A^+| = \frac{|A|C(n,k+1)}{C(n,k)} = \frac{|A|(n-k)}{k+1} \quad . \tag{2}$$

We wish to prove that $|A| = C(n,k)$; since this is trivial for the cases $k = 0$ and $k = n$, we may restrict attention to integers k such that $0 < k < n$. If $|A| < C(n,k)$, by the lemma there are numbers $\alpha \in A$, $\beta \in A^c \cap D_k$ such that α and β differ in exactly one factor. Let $\alpha = rp_1$ and $\beta = rp_2$, with $p_1 \neq p_2$, and put $\gamma = rp_1p_2$. Then $\gamma \in A^+$ and

$$(\beta,\gamma) \in E^c \cap F. \tag{3}$$

On the other hand, (2) implies that

$$|F| = (k+1)|A^+| = |A|(n-k) = |E| \ .$$

Since $E \subseteq F$, we conclude that $E = F$, which contradicts (3). Thus, $|A| = C(n,k)$.

Recently, it was communicated to the author that the second part of the theorem with m any integer and with $|D_k|$ in place of $C(n,k)$ is false. For example, if $m = 12$, $k = 1$, $A = \{3\}$, then $|D_k| = |D_{k+1}| = 2$, $A^+ = \{6\}$. Thus,

$$|A^+| = (|A|\,|D_{k+1}|)/\,|D_k| \qquad \text{and yet}$$

$A \neq D_k$. Nevertheless, it is the author's conjecture that the first part of the theorem remains true if one omits the hypothesis that m is a squarefree number and if one substitutes $|D_k|$ for $C(n,k)$. However, the above assertion has not been proved completely by the author.

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ON THE MULTIPLICATION OF RECURRENCES

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In this note we shall consider recurrences of the form

$$A_{n+2} = A_{n+1} + A_n \ , \tag{1}$$

with initial values A_0 and A_1. The special case $A_0 = 0,\ A_1 = 1$ in (1) is the well known Fibonacci series (F_n), and $A_0 = 2,\ A_1 = 1$ is the Lucas series (L_n). The integer $N(A) = A_1^2 - A_0A_2$ is the *norm* (also known as the *characteristic number)* of (1). When recurrences (A_n) and (B_n) are multiplied (the multiplication of recurrences, which is defined below, was developed in [5]), we have that $N(A)N(B) = N(AB)$. This multiplicative property is the justification of the use of the word norm. In this paper, we shall derive some basic properties of recurrences under multiplication. Our main result will be that recurrences can be factored uniquely (up to order and sign) into recurrences whose norms are prime.

Let $A_0^* = A_0,\ A_1^* = A_0 - A_1$. The recurrence (A_n^*), obtained by using A_0^* and A_1^* as initial values in (1), will be called the *dual* recurrence of (A_n), and the asterisk will be used to denote dual recurrences. The notion of dual recurrences was introduced in [3]. It may be shown by induction that

$$A_n^* = F_{n+1}A_0^* - F_nA_1^* \ . \tag{2}$$

$t(A_n)$ will denote the *scalar product* (taken term-wise) of an integer t and (A_n). If $(A_0, A_1) = t > 1$, we can express the recurrence as a scalar product $t(B_n) = (A_n)$, where $tB_i = A_i$ for all i. It is only necessary to consider such reduced recurrences.

We define the *product* $(A_n)(B_n)$ of two recurrences to be the recurrence (C_n) (of the form (1)) such that

$$C_1 - aC_0 = (A_1 - aA_0)(B_1 - aB_0). \tag{3}$$

where a is a zero of $x^2 - x = 1$, the characteristic polynomial of (1) (a is adjoined to the integers, and (3) is an equation in the extension). It follows (see [5]) that

$$C_{m+n} = A_mB_{n+1} + A_{m+1}B_n - A_mB_n \ , \tag{4}$$

and

$$C_{m+n+1} = A_{m+1}B_{n+1} + A_mB_n \ . \tag{5}$$

As stated before (and in [5]), $N(A)N(B) = N(C)$.

We point out that the value of $N(A)$ changes only in sign as the starting point A_0 of the recurrence (A_n) is translated one term at a time: the value of $N(A) = A_1^2 - A_0A_2$ alternates in sign. This follows from the identity

$$(A_{n+1})^2 - A_nA_{n+2} = (-1)^n N(A) \ , \tag{6}$$

which may be proved by induction. Henceforth, we shall disregard the sign when we discuss the norm; we shall only use its absolute value. Also, the signs of terms of (A_n) will be disregarded in the sense that (A_n) and $-(A_n)$ will be considered equivalent. Thus, for the Fibonacci and Lucas series, we have that $N(F) = 1$ and $N(L) = 5$.

It has been shown (see [1]) that a recurrence other than (F_n) can be translated so that $|A_0| > |A_1|$, and that this representation is unique. For the purposes of this paper, however, we shall make no such assumption.

It follows from (4), (5) and the definitions of the norm and dual recurrences that

$$(A_n)(A_n^*) = N(A)(F_n) = N(A^*)(F_n). \tag{7}$$

Since $(L_n^*) = (L_n)$, it follows from (7) that

$$(L_n)^2 = 5(F_n).$$

The sum, taken termwise, of (A_n) and (A_n^*) is $A_0(L_n)$. Of course, $((A_n^*)^*) = (A_n)$. Several identities involving (A_n) and its dual can be derived as special cases of general identities in [5]; among them are the following, which are generalizations of well known identities for (F_n) and (L_n).

$$A_nA_n^* + N(A)F_n^2 = (-1)^nA_0^2 ,$$

$$F_{2n}A_0 = F_n(A_n + A_n^*) = A_0F_nL_n .$$

Using the theory of binary quadratic forms, it may be shown that distinct recurrences of norm m (where distinct recurrences are recurrences which are not translates or scalar multiples of each other) are in a one-one correspondence with roots n of

$$n^2 \equiv 5 \pmod{4m} ,$$

where $0 \leqslant n < 2m$. It follows that there are recurrences with norm m if and only if $(p/5) = 0$ or 1 for all prime factors p of m, and that the number of distinct recurrences of norm m is 2^r where r is the number of prime factors p of m such that $(p/5) = 1$ (i.e., $p = 10k \pm 1$). Also, it is not possible for 25 to divide the norm. These results may be found in [2] and [4]. In [1] there is a table of the recurrences having a given norm for all possible norms up to 1000.

We remark that multiplication of recurrences with a given discriminant d ($d = 5$ in this paper) corresponds to the composition of binary quadratic forms of the same discriminant; in fact, (4) and (5) are used in the definition of composition of forms.

The following theorem shows that (A_n^*) is in a sense the inverse of (A_n), since (F_n) is the multiplicative identity.

Theorem 1. $X = (A_n^*)$ is the only recurrence satisfying $(A_n)X = N(A)(F_n)$.

Proof. By setting $C_0 = 0$, $C_1 = N(A)$, $m = n = 0$ in (4) and (5) and solving simultaneously for B_0 and B_1, we find that $B_0 = A_0 = A_0^*$ and $B_1 = A_1 - A_0 = -A_1^*$. Thus, if signs are disregarded, $(B_n) = (A_n^*)$, proving the theorem.

Theorem 2. The dual map is an automorphism of the group of recurrences under multiplication.

Proof. By (7), if $(A_n)(B_n) = (C_n)$ then

$$N(A)N(B)(F_n) = (A_n)(B_n)(A_n^*)(B_n^*) = (C_n^*)((A_n^*)(B_n^*)) ,$$

whence $(A_n^*)(B_n^*) = (C_n^*)$ by Theorem 1. Since the dual map is bijective, the theorem follows.

Theorem 3. Any automorphism of the multiplicative group of recurrences preserves the value of the norm.

Proof. By (7),

$$(A_n)(A_n^*) = N(A)(F_n).$$

Let $(A_n) \to (A_n')$ be an automorphism. Then

$$(A_n')(A_n^{*\prime}) = N(A)(F_n') = N(A)(F_n) ,$$

since an automorphism must map the multiplicative identity onto itself. Thus, by Theorem 1, $(A_n^{*\prime}) = (A_n'^*)$, so that $N(A') = N(A'^*) = N(A^{*\prime})$, and the theorem follows by the multiplicative property of the norm.

Let $S = \{p_1, p_2, \cdots\}$ be a subset of the set Q of primes which are quadratic residues of 5 and let $S' = Q - S$. Then the function T mapping recurrences onto recurrences such that

$$T((A_n)) = \begin{cases} (A_n^*) & \text{if } N(A) \in S \\ (A_n) & \text{if } N(A) \in S' \end{cases}$$

determines an automorphism, and any automorphism of the multiplicative group of recurrences can be so characterized. The proof, which uses Theorem 3 and the Unique Factorization Theorem to be proved later, is left to the reader.

Theorem 4. Consider recurrences (G_n) and (H_n) such that

$$N(G) = m_1^2, \qquad N(H) = m_2^2, \qquad (m_1, m_2) = 1.$$

Then

$$(G_n)(H_n) = m_1m_2(F_n)$$

if and only if

$$(G_n) = m_1(F_n) \qquad \text{and} \qquad (H_n) = m_2(F_n).$$

Proof. Suppose $m_1m_2(F_n) = (G_n)(H_n)$. Multiplying by (G_n^*),

$$m_1m_2(G_n^*) = (G_n)(G_n^*)(H_n) = m_1^2(H_n) \quad .$$

Thus,

$$m_2(G_n^*) = m_1(H_n).$$

It follows that

$$m_2G_i^* = m_1H_i$$

for all i. Since $(m_1, m_2) = 1$, then $m_1 \mid G_i^*$ and $m_2 \mid H_i$. Therefore,

$$m_2(G_n^*) = m_1(H_n) = m_1m_2(E_n)$$

for some recurrence (E_n), whence

$$m_1^2 = N(G) = N(G^*) = N(m_1E) = m_1^2N(E) \ ,$$

so that $N(E) = 1$. It has been shown in [2] that there is only one recurrence whose norm is 1: namely (F_n). Hence, $(E_n) = (F_n)$.

The converse is obvious.

Theorem 5. (Unique Facorization). Recurrences of a given norm whose terms have no common divisor factor uniquely up to order and sign into recurrences whose norms are the prime divisors of m.

Proof. First we shall show that a recurrence (E_n) can be factored uniquely into recurrences whose norms are (relatively prime) maximal prime power divisors of m. It is only necessary to prove uniqueness for $m = m_1m_2$ with $(m_1,m_2) = 1$, and uniqueness for prime power divisors follows.

If $N(E)$ has only one prime power factor or if $(E_n) = (F_n)$, we are done. Otherwise, let (E_n) have at least two relatively prime factors m_1 and m_2, and assume that factorization is unique for recurrences whose norms are those relatively prime factors. We shall show that (E_n) factors uniquely.

Since there are 2^r recurrences with norm m_1, where r is the number of prime divisors p of m_1 satisfying $(p/5) = 1$ and assuming that $(p/5) = -1$ for no divisors of m_1 (see [2]), and since, under similar conditions, there are 2^s recurrences with norm m_2, then the set of recurrences obtained by taking products of recurrences, one with each norm is contained in the set of 2^{r+s} recurrences of norm m_1m_2, with equality of sets if and only if any pair of products is distinct. Thus, we must show that if $(A_n)(B_n) = (C_n)(D_n)$ with

$$N(A) = N(C) = m_1, \qquad N(B) = N(D) = m_2, \qquad (m_1,m_2) = 1,$$

then $(A_n) = (C_n)$, $(B_n) = (D_n)$.

Under the conditions stated, there exists a recurrence (G_n), equal to $(A_n^*)(C_n)$ such that $N(G) = m_1^2$ and

$$(A_n)(G_n) = m_1(C_n).$$

Likewise, there is an (H_n) such that

$$N(H) = m_2^2 \qquad \text{and} \qquad (B_n)(H_n) = m_2(D_n).$$

Substituting these relations into

$$(A_n)(B_n) = (C_n)(D_n)$$

we get

$$m_1m_2(A_n)(B_n) = (A_n)(G_n)(B_n)(H_n) \ ,$$

and multiplying by $(A_n^*)(B_n^*)$ and applying (7) obtain

$$m_1m_2(F_n) = (G_n)(H_n).$$

Since $(m_1,m_2) = 1$, we have that $(G_n) = m_1(F_n)$ and $(H_n) = m_2(F_n)$ by Theorem 4. Thus,

$$m_1(C_n) = (A_n)(G_n) = m_1(A_n),$$

whence $(A_n) = (C_n)$, and $(B_n) = (D_n)$, likewise.

Next we show that each of the two recurrences of prime power norm p^k factors uniquely into k recurrences of norm p. Let (A_n) be a recurrence of norm p. Then the only other recurrence of the same norm is (A_n^*) and no recurrence (except the identity recurrence (F_n)) has a norm dividing p. We shall proceed by induction.

For $k = 1$, the theorem is obviously true. Assume truth for all exponents not greater than k. Then there are two recurrences of norm p^k which factor uniquely, and since $(A_n)^k$ and $(A_n^*)^k$ are factorizations of the recurrences of norm p^k, they are unique factorizations. Multiplying $(A_n)^k$ and $(A_n^*)^k$ by each of the recurrences of norm p and using (7), we get the products

$$(A_n)^{k+1}, \qquad (A_n^*)^{k+1}, \qquad (A_n)^k(A_n^*) = N(A)(A_n)^{k-1}, \qquad \text{and} \qquad (A_n^*)^k(A_n) = N(A)(A_n^*)^{k-1},$$

and the last two products fail to satisfy the requirement that the terms have no common factor. Thus, $(A_n)^{k+1}$ and $(A_n^*)^{k+1}$ are two factorizations of recurrences of norm p^{k+1}, and they are the only two meeting the requirement that the terms of the product have no common factor. Since there are two recurrences of norm p^{k+1} (see [2]), $(A_n)^{k+1}$ and $(A_n^*)^{k+1}$ must be their factorizations. This completes the proof.

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A NOTE ON FERMAT'S LAST THEOREM

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In this note, $n, m, x, y,$ and z are all positive integers, with $x < y < z$.

Theorem 1. For $n \geq 2$, the equation $x^n + y^n = z^n$ has no solutions whenever $x + ny \leq nz$.

Corollary. For $m \geq 1$ and $n \geq 2$, $x^{mn} + y^{mn} = z^{mn}$ has no solutions whenever $x^m + ny^m \leq nz^m$.

Proof. Suppose $x^n + y^n = z^n$ has a solution with $y = x + a, z = x + b$, where $b > a > 0$ are integers. Then, by using the binomial theorem, we have

$$x^n = z^n - y^n = (x+b)^n - (x+a)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}(b^i - a^i) = nx^{n-1}(b-a) + Q(n,x,b,a), \quad Q > 0 .$$

Thus

$$x^{n-1}(x - n(b-a)) = Q ,$$

and so $x - n(b-a) > 0$ is a necessary condition for a solution. Since

$$b - a = (x+b) - (x+a) = z - y, \quad x - n(z-y) \leq 0$$

is the stated result.

REMARKS. Since $nz < ny + x$ is a necessary condition for a solution and since $y < z$, we see that

[Continued on Page 402.]

A q-IDENTITY

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1. The object of this note is to prove the following q-identity:

$$(*)\qquad \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k} (a)_k (b)_k q^{\frac12(n-k)(n+k-1)} (ab)^{n-k} = (a)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{b^k}{1-q^{n-k}a}$$

$$= (b)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{a^k}{1-q^{n-k}b},$$

where

$$(a)_k = (a,q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a), \qquad (a)_0 = 1,$$

$$(q)_k = (q,q)_k = (1-q)(1-q^2)\cdots(1-q^k), \qquad (q)_0 = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix} \qquad (0 \le k \le n)$$

and q is not a t^{th} root of unity, $1 \le t \le n$.

Since each side of

$$(1)\qquad \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k} (a)_k (b)_k q^{\frac12(n-k)(n+k-1)} (ab)^{n-k} = (b)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{a^k}{1-q^{n-k}b}$$

is a polynomial in b of degree $\le n$, it will suffice to show that (1) holds for $b = q^{-r}$, $0 \le r \le n$.

We have

$$\left.\frac{(b)_{n+1}}{1-q^r b}\right|_{b=q^{-r}} = (1-q^{-r})\cdots(1-q^{-1})(1-q)(1-q^2)\cdots(1-q^{n-r}) = (-1)^r q^{-\frac12 r(r+1)} (q)_r (q)_{n-r}.$$

Thus the right-hand side of (1) reduces to

$$(2)\qquad (-1)^{n-r} \begin{bmatrix} n \\ n-r \end{bmatrix} q^{\frac12(n-r)(n-r-1)} (-1)^r q^{-\frac12 r(r+1)} (q)_r (q)_{n-r} a^{n-r} = (-1)^n (q)_n q^{\frac12 n(n-1)-nr} a^{n-r}.$$

As for the left-hand side, since

$$(q^{-r})_k = (1-q^{-r})(1-q^{-r+1})\cdots(1-q^{-r+k-1}) = (-1)^k q^{-rk+\frac12 k(k-1)} (1-q^r)(1-q^{r-1})\cdots(1-q^{r-k+1})$$

$$= \begin{cases} (-1)^k q^{-rk+\frac12 k(k-1)} (q)_r/(q)_{r-k} & (0 \le k \le r) \\ 0 & (k > r) \end{cases},$$

we get

$$\sum_{k=0}^{r} (-1)^{n-k} \frac{(q)_n}{(q)_k} a^{n-k} q^{\frac12(n-k)(n+k-1)} \cdot (-1)^k q^{-rk+\frac12 k(k-1)} \frac{(q)_r}{(q)_{r-k}} q^{-r(n-k)} = (-1)^n (q)_n q^{\frac12 n(n-1)-nr} a^{n-r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k}$$

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We shall now show that

$$\sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = 1 \qquad (r = 0, 1, 2, \cdots), \tag{3}$$

so that the left-hand side of (1) is equal to

$$(-1)^n (q)_n q^{\frac{1}{2}n(n-1)-nr} a^{n-r}$$

in agreement with (2).

To prove (3) we take

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k \sum_{r=0}^{\infty} \frac{a^r x^r}{(q)_r}.$$

By a well known identity

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k = \frac{e(x)}{e(ax)},$$

where

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \prod_{n=0}^{\infty} (1 - q^n x)^{-1}. \tag{4}$$

Thus

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \frac{e(x)}{e(ax)} e(ax) = e(x)$$

and (3) follows at once.

This evidently completes the proof of (*).

2. The identity (*) can also be proved by making use of the q-analog of Gauss's theorem (see for example [1, p. 68]):

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (x)_k} \left(\frac{x}{ab} \right)^k = \frac{e(x) e(x/ab)}{e(x/a) e(x/b)}, \tag{5}$$

where $e(x)$ is defined by (4).

Define the operator E by means of

$$E^n f(x) = f(q^n x) \qquad (n = 0, 1, 2, \cdots)$$

and Δ^n by means of the operational formula

$$\Delta^n = (1 - E)(q - E) \cdots (q^{n-1} - E).$$

Then it is easily verified that

$$\Delta^n = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} E^{n-r}.$$

It follows that

$$\Delta^n x^k = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} q^{(n-r)k} x^k = (q^k - 1)(q^k - q) \cdots (q^k - q^{n-1}) x^k,$$

so that

$$\Delta^n x^k = \begin{cases} 0 & (n > k) \\ (-1)^k q^{\frac{1}{2}k(k-1)} (q)_k x^k & (n = k) \end{cases} \tag{6}$$

Now multiply both sides of (5) by $(x)_n$ and apply Δ^n. Then divide by x^n and put $x = 0$. In view of (6) the LHS becomes

$$\sum_{k=0}^{n} \frac{(a)_k(b)_k}{(q)_k}(ab)^{-k}\cdot(-1)^{n-k}q^{\frac12(n-k)(n-k-1)+k(n-k)}\cdot(-1)^n q^{\frac12 n(n-1)}(q)_n$$

$$= (-1)^n q^{\frac12 n(n-1)}\sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac12(n-k)(n+k-1)}(ab)^{-k}\,. \tag{7}$$

As for the RHS, we have first

$$(x)_n\,\frac{e(x)e(x/ab)}{e(x/a)e(x/b)} = \frac{e(q^n x)e(x/ab)}{e(x/a)e(x/b)}$$

$$= \sum_{j=0}^{\infty}\frac{(q^{-n}/a)_j}{(q)_j}(q^n x)^j\sum_{k=0}^{\infty}\frac{(a)_k}{(q)_k}\left(\frac{x}{ab}\right)^k$$

$$= \sum_{r=0}^{\infty}\frac{x^r}{(q)_r}\sum_{k=0}^{r}\begin{bmatrix} r\\ k\end{bmatrix}(a)_k(ab)^{-k}(q^{-n}/a)_{r-k}q^{n(r-k)}\,.$$

Apply Δ^n, divide by x^n and put $x = 0$. We get

$$(-1)^n q^{\frac12 n(n-1)}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(a)_k(ab)^{-k}(q^{-n}/a)_{n-k}q^{n(n-k)}\,. \tag{8}$$

Since

$$(q^{-n}/a)_{n-k} = (1-q^{-n}/a)(1-q^{-n+1}/a)\cdots(1-q^{-k-1}/a) = (-1)^{n-k}a^{-n+k}q^{-\frac12 n(n+1)+\frac12 k(k+1)}$$
$$\cdot(1-q^{k+1}a)(1-q^{k+2}a)\cdots(1-q^n a)\,,$$

(8) becomes

$$q^{\frac12 n(n-1)}(ab)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 n(n-1)-nk+\frac12 k(k+1)}b^{n-k}\,\frac{(a)_{n+1}}{1-q^k a}$$

$$= (-1)^n q^{\frac12 n(n-1)}(ab)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 k(k-1)}b^k\,\frac{(a)_{n+1}}{1-q^{n-k}a}\,.$$

Comparing this with (7) it is clear that we have proved (*).

3. We have

$$\sum_{n=0}^{\infty}\frac{x^n}{(q)_n}\sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac12(n-k)(n+k-1)}(ab)^{n-k}$$

$$= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(q)_k}x^k\sum_{n=0}^{\infty}(-1)^n q^{\frac12 n(n-1)}(q^k abx)^n\,.$$

Also, since

$$(a)_{n+1} = (a)_{n-k}(1-q^{n-k}a)(q^{n-k+1}a)_k\,,$$

$$\sum_{n=0}^{\infty}\frac{(a)_{n+1}}{(q)_n}x^n\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac12 k(k-1)}\frac{b^k}{1-q^{n-k}a} = \sum_{k=0}^{\infty}(-1)^k q^{\frac12 k(k-1)}\frac{b^k x^k}{(q)_k}\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}(q^{n+1}a)_k x^n$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac12 k(k-1)}\frac{(q^{n+1}a)_k}{(q)_k}(bx)^k\,.$$

Thus (*) is equivalent to the identity

$$\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} (abx)^n \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k} (q^n x)^k = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{(q^{n+1}a)_k}{(q)_k} (bx)^k$$

$$(9) \qquad = \sum_{n=0}^{\infty} \frac{(b)_n}{(q)_n} x^n \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{(q^{n+1}b)_k}{(q)_k} (ax)^k,$$

where now $|q| < 1$.

4. The following special cases of (*) may be noted. For $b = q$ we have

$$\sum_{k=0}^{n} (-1)^{n-k} (a)_k q^{\frac{1}{2}k(n-k)(n+k+1)} a^{n-k} = \frac{(a)_{n+1}}{(q)_n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{1}{2}k(k+1)}}{1 - q^{n-k}a}$$

$$(10) \qquad = (1 - q^{n+1}) \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \frac{a^k}{1 - q^{n-k+1}}.$$

For $a = q$ this reduces to

$$(11) \qquad \sum_{k=0}^{n} (-1)^{n-k} (q)_k q^{\frac{1}{2}(n-k)(n+k+3)} = (1 - q^{n+1}) \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{1}{2}k(k+1)}}{1 - q^{n-k+1}}.$$

Since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \frac{1 - q^{n+1}}{1 - q^{n-k+1}} \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\frac{1}{2}k(k+1)} = (q)_{n+1} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)},$$

(11) becomes

$$(12) \qquad \sum_{k=0}^{n} (-1)^{n-k} (q)_k q^{\frac{1}{2}(n-k)(n+k+3)} = (q)_{n+1} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)}.$$

Somewhat more generally, it follows from (10) that

$$(13) \qquad \sum_{k=0}^{n} (-1)^{n-k} (a)_k q^{\frac{1}{2}(n-k)(n+k+1)} a^{n-k} = (a)_{n+1} + (-1)^n q^{\frac{1}{2}n(n+1)} a^{n+1}.$$

We shall give a direct proof of (13). The formula evidently holds for $n = 0$. Assuming that it holds up to and including the value n, we replace a by qa and multiply both sides by $1 - a$. Thus

$$\sum_{k=0}^{n} (-1)^{n-k} (a)_{k+1} q^{\frac{1}{2}(n-k)(n+k+3)} a^{n-k} = (a)_{n+2} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)} a^{n+1} (1 - a).$$

Hence

$$\sum_{k=0}^{n+1} (-1)^{n-k+1} (a)_k q^{\frac{1}{2}(n-k+1)(n+k+2)} a^{n-k+1} = (a)_{n+2} + (-1)^n q^{\frac{1}{2}(n+1)(n+2)} a^{n+1} (1 - a) + (-1)^{n+1}$$

$$\cdot\, q^{\frac{1}{2}(n+1)(n+2)} a^{n+1} = (a)_{n+2} + (-1)^{n+1} q^{\frac{1}{2}(n+1)(n+2)} a^{n+2}.$$

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THE EVALUATION OF CERTAIN ARITHMETIC SUMS

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1. In this paper we evaluate certain cases of the expression

$$\sum \max(A_1, A_2, \cdots, A_n), \tag{1.1}$$

where

$$A_k = a_{k1} + \cdots + a_{km_k}$$

and where the sum is over all the a's, each ranging from zero to some positive integer. We also consider analogous sums for min. For example we obtain, from some general results which we establish, the formula

$$\sum_{a,b,c,d=0}^{r} \max(a+b, c+d) = 22\binom{r}{1} + 170\binom{r}{2} + 420\binom{r}{3} + 420\binom{r}{4} + 148\binom{r}{5}.$$

Some general properties of more general cases of (1.1) are established.

Solutions of many problems, particularly combinatorial problems, are often expressed in terms of such sums. For example, without going into detail, we frequently encounter sums of max and min in problems of enumerating arrays. See H. Anand *et al.* [1] and Carlitz [2] for details.

In a related work, Carlitz [3] and [4] obtains, and relates to other problems, generating functions for $\max(n_1, \cdots, n_k)$ and $\min(n_1, \cdots, n_k)$. More generally, he evaluates

$$\sum_{n_1,\cdots,n_k=0}^{\infty} M_r(n_1, \cdots, n_k) x_1^{n_1} \cdots x_k^{n_k} \qquad (r = 1, \cdots, k),$$

where $M_r(n_1, \cdots, n_k)$ is defined by the following two properties: (a) it is symmetric in $n_1, \cdots, n_k$; (b) if $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ then

$$M_r(n_1, \cdots, n_k) = n_r \qquad (r = 1, \cdots, k).$$

He also evaluates the related series

$$\sum_{n_1,\cdots,n_k=0}^{\infty} x_1^{n_1} \cdots x_k^{n_k} z^{M_r(n_1,\cdots,n_k)} \qquad (r = 1, \cdots, k).$$

Roselle [6] examines the relationship between this series and the Eulerian function.

Other than [3], [4], [6] and this paper, there apparently has been very little published on problems of this nature.

A number of techniques are employed to solve various aspects of the problem and computer computation was necessary in some instances.

The main results of this paper are (3.3), (4.3), (4.7), (4.8), (4.11), (4.12), (4.13), (4.14), (4.15), (5.7), (5.9), (5.10), (5.13), (5.14), (5.15) and (5.16).

2. Preliminary to our discussion we need some basic properties of Eulerian polynomials. The n^{th} Eulerian polynomial, $a_n(x)$, is defined, following Riordan [5], by

$$a_n(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} k^n x^k .$$

From this definition we get

$$a_n(x) = nxa_{n-1}(x) + x(1-x)Da_{n-1}(x) ,$$

where D is the differential operator. Hence, the first few polynomials, which we will use later, are

$$a_0(x) = 1, \qquad a_1(x) = x, \qquad a_2(x) = x^2 + x,$$
$$a_3(x) = x^3 + 4x^2 + x, \qquad a_4(x) = x^4 + 11x^3 + 11x^2 + x.$$

A recurrence and a table for the coefficients (Eulerian numbers) of Eulerian polynomials and a generating function for $a_n(x)$ may be found in [5; pp. 39, 215].

As for convenient notation we write

$$\begin{matrix} max \\ min \end{matrix} (a,b) ,$$

where we wish to discuss both $max\,(a,b)$ and $min\,(a,b)$.

Also we adopt the convention

$$\phi = \phi_n = (1-x_1)(1-x_2) \cdots (1-x_n).$$

3. Taking the simplest case first we evaluate the sum

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1, \cdots, i_n) .$$

To do this, we put

$$F(x_1, \cdots, x_n) = \sum_{r_1,\cdots,r_n=0}^{\infty} \sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1,\cdots,i_n)x_1^{r_1} \cdots x_n^{r_n} , \tag{3.1}$$

which becomes

$$F(x_1, \cdots, x_n) = \phi^{-1} \sum_{i_1,\cdots,i_n=0}^{\infty} min\,(i_1, \cdots, i_n)x_1^{i_1} \cdots x_n^{i_n} .$$

Now since

$$min\,(i_1, \cdots, i_n) = \sum_{j=0}^{min(i_1-1,\cdots,i_n-1)} 1 = \sum_{\substack{k_1+j+1=i_1 \\ \cdots \\ k_n+j+1=i_n}} 1,$$

it follows immediately that

$$\sum_{i_1,\cdots,i_n=0}^{\infty} min\,(i_1, \cdots, i_n)x_1^{i_1} \cdots x_n^{i_n} = \phi^{-1} \frac{x_1x_2 \cdots x_n}{(1-x_1x_2\cdots x_n)} .$$

Therefore,

$$F(x_1, \cdots, x_n) = \phi^{-2} \frac{x_1x_2\cdots x_n}{1-x_1x_2\cdots x_n} . \tag{3.2}$$

Comparing the coefficients in the expansion of (3.2) with those of (3.1) gives

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} min\,(i_1, \cdots, i_n) = \sum_{j \le min(r_1,\cdots,r_n)} \prod_{i=1}^{n} (r_i - j) . \tag{3.3}$$

If $r_1 = \cdots = r_n = r$ then the right side of (3.3) reduces to the familiar series

$$\sum_{j=0}^{r} j^n .$$

4. The sum

$$\sum_{i_1,\cdots,i_n=0}^{r_1,\cdots,r_n} \max(i_1,\cdots,i_n) \tag{4.1}$$

is apparently more difficult to evaluate. If $n = 2$ we may use (3.3) and the identity

$$\min(i,j) + \max(i,j) = i + j \tag{4.2}$$

to get

$$\sum_{i,j=0}^{r,s} \max(i,j) = \tfrac{1}{2}r(r+1)(s+1) + s(s+1)(r+1) - \sum_{j=0}^{\min(r,s)} (r-j)(s-j)\,. \tag{4.3}$$

Now, considering the case of (4.1) where all the r's are equal we define

$$F_n(r) = \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)\,. \tag{4.4}$$

Then by an inclusion-exclusion argument,

$$F_n(r) = \binom{n}{1} \sum_{i_1,\cdots,i_{n-1}=0}^{r} \max(i_1,\cdots,i_{n-1},r) - \binom{n}{2} \sum_{i_1,\cdots,i_{n-2}=0}^{r} \max(i_1,\cdots,i_{n-2},r,r)$$

$$+ \cdots + (-1)^{n+1}\binom{n}{n} \max(r,\cdots,r) + F_n(r-1)$$

or

$$F_n(r) = r\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1}(r+1)^{n-k} + F_n(r-1) \qquad (r \geqslant 1)\,. \tag{4.5}$$

The expression

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1}(r+1)^{n-k} \tag{4.6}$$

may be simplified, according to the binomial theorem, giving

$$(r+1)^n - r^n$$

so that (4.5) becomes

$$F_n(r) = r(r+1)^n - r^{n+1} + F_n(r-1) \qquad (r \geqslant 1)\,. \tag{4.7}$$

Applying (4.7) to $F_n(r-1)$ we get

$$F_n(r) = r(r+1)^n - r^{n+1} + (r-1)r^n - (r-1)^{n+1} + F_n(r-2)$$

and continuing in this manner we eventually arrive at

$$F_n(r) = \sum_{k=0}^{r-1} (r-k)(r+1-k)^n - \sum_{k=0}^{r-1} (r-k)^{n+1}\,.$$

$$F_n(r) = \sum_{k=0}^{r} k(k+1)^n - \sum_{k=0}^{r} k^{n+1} = \sum_{k=0}^{r} k((k+1)^n - k^n) = \sum_{k=0}^{r} k \sum_{j=0}^{n-1} \binom{n}{j} k^j\,.$$

Now by comparing the last expression with (4.4) we have

$$F_n(r) = \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n) = \sum_{k=0}^{r}\sum_{j=0}^{n-1} \binom{n}{j} k^{j+1} \tag{4.8}$$

or, in the usual notation for Bernoulli polynomials,

$$\sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n) = \sum_{j=0}^{n-1} \binom{n}{j} \frac{B_{j+2}(r+1)-B_{j+2}(0)}{j+2} \tag{4.9}$$

The first few special cases of $F_n(r)$, obtained from (4.7) are

$$F_n(1) = 2^n - 1, \quad F_n(2) = 2\cdot 3^n - 2^n - 1, \quad \text{and} \quad F_n(3) = 3\cdot 4^n - 3^{n+1} + 2\cdot 3^n - 2^n - 1.$$

Next we evaluate the generating function

$$\sum_{n=0}^{\infty} F_n(r)x^n . \tag{4.10}$$

From (4.8), (4.10) becomes

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}\binom{n}{j}\sum_{k=0}^{r} k^{j+1}x^n = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{r} k^{j+1}x^n - \sum_{n=0}^{\infty}\binom{n}{n}\sum_{k=0}^{r} k^{n+1}x^n$$

$$= \sum_{k=0}^{r} k \sum_{n=0}^{\infty} x^n \sum_{j=0}^{\infty}\binom{n+j}{j}(kx)^j - \sum_{k=0}^{r} k \sum_{n=0}^{\infty}(kx)^n$$

$$= \sum_{k=0}^{r} k \sum_{n=0}^{\infty} x^n(1-kx)^{-n-1} - \sum_{k=0}^{r} k(1-kx)^{-1}$$

$$= \sum_{k=0}^{r} k(1-kx-x)^{-1} - \sum_{k=0}^{r} k(1-kx)^{-1}$$

$$= x\sum_{k=0}^{r} k(1-(k+1)x)^{-1}(1-kx)^{-1} .$$

We have therefore proved

$$\sum_{n=0}^{\infty} F_n(r)x^n = \sum_{n=0}^{\infty}\sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)x^n = x\sum_{k=0}^{r} k(1-(k+1)x)^{-1}(1-kx)^{-1} . \tag{4.11}$$

From (4.11) it is easy to see that

$$\sum_{n,r=0}^{\infty} F_n(r)x^ny^r = \frac{x}{(1-y)}\sum_{k=0}^{\infty}\frac{ky^k}{(1-(k+1)x)(1-kx)} .$$

If, on the other hand, we define

$$G_n(y) = \sum_{r=0}^{\infty} F_n(r)y^r$$

then using the recurrence (4.7) and setting $F_n(-1) = 0$ we have

$$G_n(y) = \sum_{r=0}^{\infty}(r(r+1)^n - r^{n+1})y^r + \sum_{r=0}^{\infty} F_n(r-1)y^r = \sum_{r=0}^{\infty}(r(r+1)^n - r^{n+1})y^r + y\sum_{r=0}^{\infty} F_n(r)y^r .$$

Then

$$(1-y)G_n(y) = \sum_{r=0}^{\infty} (r(r+1)^n - r^{n+1})y^r = \sum_{r=0}^{\infty} r \sum_{k=0}^{n} \binom{n}{k} r^k y^r - \sum_{r=0}^{\infty} r^{n+1} y^r$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{r=0}^{\infty} r^{k+1} y^r - \sum_{r=0}^{\infty} r^{n+1} y^r$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+2}} - \frac{a_{n+1}(y)}{(1-y)^{n+2}} = \sum_{k=0}^{n-1} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+2}} ,$$

where $a_p(x)$ is the p^{th} Eulerian polynomial introduced in Section 2.

Therefore, we have

$$G_n(y) = \sum_{r=0}^{\infty} \sum_{i_1,\cdots,i_n=0}^{r} \max(i_1,\cdots,i_n)\, y^r = \sum_{k=0}^{n-1} \binom{n}{k} \frac{a_{k+1}(y)}{(1-y)^{k+3}} . \tag{4.12}$$

From (4.12) and from the list of Eulerian polynomials in Section 2 we easily arrive at the special cases

$$G_2(y) = \frac{3y+y^2}{(1-y)^4}, \qquad G_3(y) = \frac{7y+10y^2+y^3}{(1-y)^5} , \quad \text{and} \quad G_4(y) = \frac{15y+55y^2+25y^3+y^4}{(1-y)^6} .$$

The expansions of each of these generating functions yield, for $r \geqslant 0$,

$$\sum_{a,b=0}^{r} \max(a,b) = 3\binom{r+2}{3} + \binom{r+1}{3} , \tag{4.13}$$

$$\sum_{a,b,c=0}^{r} \max(a,b,c) = 7\binom{r+3}{4} + 10\binom{r+2}{4} + \binom{r+1}{4} , \tag{4.14}$$

and

$$\sum_{a,b,c,d=0}^{r} \max(a,b,c,d) = 15\binom{r+4}{5} + 55\binom{r+3}{5} + 25\binom{r+2}{5} + \binom{r+1}{5} , \tag{4.15}$$

respectively.

In general, from (4.9) it may be seen that $F_n(r)$ is a polynomial in r of degree $n+1$.

5. In this section, we consider $A(r,m,n)$ and $B(r,m,n)$, where

$$A(r,m,n) = \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n=0}^{r} \max(a_1+\cdots+a_m,\, b_1+\cdots+b_n) , \tag{5.1}$$

and

$$B(r,m,n) = \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n=0}^{r} \min(a_1+\cdots+a_m,\, b_1+\cdots+b_n) . \tag{5.2}$$

It is convenient to let the expression $a,b=0$ mean $a_1,\cdots,a_m,b_1,\cdots,b_n = 0$.

Using the formula

$$\begin{matrix}\max\\ \min\end{matrix}(a,b) = \tfrac{1}{2}(a+b \pm |a-b|) ,$$

(5.2) becomes

$$(5.3)\qquad B(r,m,n) = \tfrac{1}{2}\sum_{a,b=0}^{r} (a_1+\cdots+a_m+b_1+\cdots+b_n) - \tfrac{1}{2}\sum_{a,b=0}^{r} \left|a_1+\cdots+a_m-b_1-\cdots-b_n\right|$$

$$= \frac{(m+n)r(r+1)^{m+n}}{4} - \tfrac{1}{2}\sum_{a,b=0}^{r}\left|a_1+\cdots+a_m-b_1-\cdots-b_n\right| .$$

Now

$$(5.4)\qquad \sum_{a,b=0}^{r} a_1+\cdots+a_m-b_1-\cdots-b_n = -\sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n)+\sum_{II}(a_1+\cdots+a_m-b_1-\cdots-b_n),$$

where I and II stand for

$$a_1+\cdots+a_m \leqslant b_1+\cdots+b_n \leqslant r\max(m,n)$$

and

$$b_1+\cdots+b_n \leqslant a_1+\cdots+a_m \leqslant r\max(m,n).$$

respectively, and where it is understood that $a_i, b_i \leqslant r$.

Moreover,

$$\sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{\substack{b_1+\cdots+b_n=k\\ b_i\leqslant r}}\ \sum_{\substack{a_1+\cdots+a_m\leqslant k\\ a_i\leqslant r}} (a_1+\cdots+a_m-k)$$

$$= \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}\ \sum_{\substack{b_1+\cdots+b_n=k\\ b_i\leqslant r}}\ \sum_{\substack{a_1+\cdots+a_m=j\\ a_i\leqslant r}} (j-k) .$$

Now let $P(a,b,c)$ be the number of partitions of $a>0$ into at most b parts, each part $\leqslant c$, and let $P(0,b,c)=1$.

$$(5.5)\qquad \sum_{I}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)P(k,n,r)P(j,m,r) .$$

By a similar argument we find

$$(5.6)\qquad \sum_{II}(a_1+\cdots+a_m-b_1-\cdots-b_n) = \sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)P(k,m,r)P(j,n,r) .$$

By substituting (5.5) and (5.6) into (5.4) and referring to (5.3), and then by applying this same argument for $A(r,m,n)$ we find that $A(r,m,n)$ and $B(r,m,n)$ are given by

$$(5.7)\qquad \sum_{a_1,\cdots,a_m,b_1,\cdots,b_n\leqslant r} \substack{\max\\ \min}(a_1+\cdots+a_m,\ b_1+\cdots+b_n) = \frac{(m+n)r(r+1)^{m+n}}{4}$$

$$\pm\ \tfrac{1}{2}\sum_{k=0}^{r\max(m,n)}\ \sum_{j=0}^{k}(k-j)\left[P(k,n,r)P(j,m,r)+P(k,m,r)P(j,n,r)\right].$$

respectively.

Now, although (5.7) expresses our problem in terms of a difficult partition problem the formula is nevertheless very useful in that it affords us a method of determining the generating function. Here we observe, from (5.7), that $A(r,m,n)$ and $B(r,m,n)$ are polynomials in r. Furthermore, their degrees are less than or equal to $m+n+1$, for if not, then for large values of r, $B(r,m,n)$ will be negative. In fact, in view of special cases, we may conjecture that the degree of $A(r,m,n)$ and $B(r,m,n)$ is precisely $m+n+1$.

We may now evaluate several cases. First, we consider $B(r,2,1)$.

$$B(r,2,1) = \sum_{a,b,c=0}^{r} \min(a+b,c) = \sum_{\substack{a+b\le c\\ a,b,c\le r}} (a+b) + \sum_{\substack{a+b\ge c\\ a,b,c\le r}} c - \sum_{c=0}^{r}\sum_{a+b=c} c\,.$$

After some manipulation it is seen that

$$\sum_{\substack{a+b\le c\\ a,b,c\le r}} (a+b) = \sum_{c=0}^{r}\sum_{k=0}^{c}\sum_{a+b=k} k = \frac{1}{3}\sum_{c=0}^{r}(c^3+3c^2+2c)$$

and

$$\sum_{c=0}^{r}\sum_{a+b=c} c = \sum_{c=0}^{r} c(c+1)\,.$$

Then we have

$$B(r,2,1) = \sum_{c=0}^{r}\left(\frac{1}{3}k^3 - \frac{1}{3}k\right) + \sum_{\substack{a+b\ge c\\ a,b,c\le r}} c = \frac{1}{12}r^2(r+1)^2 - \frac{1}{6}r(r+1) + \sum_{a,b=0}^{r}\sum_{c=0}^{\min(a+b,r)} c.$$

Since the degree of $B(r,2,1)$ is at most $2+1+1$, it suffices to compute $B(r,2,1)$ for $r = 1, 2, 3, 4$. If we put

$$K_r = \sum_{a,b=0}^{r}\sum_{c=0}^{\min(a+b,r)} c$$

so that

(5.8) $$B(r,2,1) = \frac{1}{12}r^2(r+1)^2 - \frac{1}{6}r(r+1) + K_r$$

then it is easy to compute $K_1 = 3$, $K_2 = 20$, $K_3 = 71$ and $K_4 = 185$. Then from (5.8), we have

$B(1,2,1) = 3$, $B(2,2,1) = 22$, $B(3,2,1) = 81$ and $B(4,2,1) = 215$.

Then we find the differences

$\Delta B(r,2,1) = 3$, $\Delta^2 B(r,2,1) = 16$, $\Delta^3 B(r,2,1) = 24$, $\Delta^4 B(r,2,1) = 11$.

Substituting these values into Newton's expansion for the generating function we have

$$\sum_{k=0} B(r,2,1)x^k = \sum_{k=0}^{4} \Delta^k B(0,2,1)\frac{x^k}{(1-x)^{k+1}} = \frac{x}{(1-x)^5}(3+7x+x^2)\,.$$

Upon expanding we get

(5.9) $$B(r,2,1) = \sum_{a,b,c=0}^{r} \min(a+b,c) = 3\binom{r+3}{4} + 7\binom{r+2}{4} + \binom{r+1}{4}\,.$$

From (4.2),

$$\sum_{a,b,c=0}^{r} (\max(a+b,c) + \min(a+b,c)) = \sum_{a,b,c=0}^{r}(a+b+c)$$

which, from (5.9) reduces to

(5.10) $$\sum_{a,b,c=0}^{r} \max(a+b,c) = 3\binom{r+3}{4} + 7\binom{r+2}{4} + \binom{r+1}{4} + \frac{3}{2}r(r+1)^3\,.$$

Next we put $B(r,2,2) = B(r)$ for brevity, and consider

$$B(r) = \sum_{a,b,c=0}^{r} \min(a+b,\, c+d) \tag{5.11}$$

From (5.7) we have

$$B(r) = r(r+1)^4 - \sum_{k=0}^{2r} \sum_{j=0}^{k} (k-j)P(k,2,r)P(j,2,r) .$$

Hence $B(r)$ is a polynomial in r of degree $\leqslant 5$. Therefore the generating function is

$$\sum_{r=0}^{\infty} B(r)x^r = \sum_{i=0}^{5} \Delta^i B(0) \frac{x^i}{(1-x)^{i+1}} .$$

Resorting to a computer to calculate $\Delta^i B(0)$ for $i = 1, 2, 3, 4, 5$, (5.12) becomes

$$\sum_{r=0}^{\infty} B(r)x = \frac{10x}{(1-x)^2} + \frac{90x^2}{(1-x)^3} + \frac{240x^3}{(1-x)^4} + \frac{252x^4}{(1-x)^5} + \frac{92x^5}{(1-x)^6} .$$

Expanding for the coefficients we find

$$\sum_{a,b,c,d=0}^{r} \min(a+b,\, c+d) = 10\binom{r}{1} + 90\binom{r}{2} + 240\binom{r}{3} + 252\binom{r}{4} + 92\binom{r}{5} . \tag{5.13}$$

In the very same way that we obtained (5.13) we get

$$\sum_{a,b,c,d=0}^{r} \max(a+b,\, c+d) = 22\binom{r}{1} + 170\binom{r}{2} + 420\binom{r}{3} + 420\binom{r}{4} + 148\binom{r}{5} , \tag{5.14}$$

$$\sum_{a,b,c,d=0}^{r} \min(a+b+c,d) = 7\binom{r}{1} + 61\binom{r}{2} + 159\binom{r}{3} + 164\binom{r}{4} + 59\binom{r}{5} \tag{5.15}$$

and

$$\sum_{a,b,c,d=0}^{r} \max(a+b+c,d) = 25\binom{r}{1} + 199\binom{r}{2} + 501\binom{r}{3} + 508\binom{r}{4} + 181\binom{r}{5} . \tag{5.16}$$

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MATRICES AND GENERALIZED FIBONACCI SEQUENCES

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Horadam [4] has pointed out that generalizations of the Fibonacci sequence $\{F_n\}$ fall in either of two categories: (1) an alteration of the recurrence relation of the sequence, and (2) an alteration of the first two terms of the sequence. He further states that these two techniques may be combined, and in this paper we follow this suggestion by considering the sequence $\{U_n\}$ defined as follows: Let U_0, U_1 be arbitrary integers, not both zero; let r,s be non-zero integers, and let

$$U_n = rU_{n-1} + sU_{n-2}, \qquad n \geqslant 2. \tag{1}$$

This sequence has been considered by Buschman [2], Horadam [5], and Raab [7]. If $r = s = 1,$ the sequence $\{U_n\}$ becomes the sequence considered by Horadam in [4]. Quite clearly, if $r = s = 1$ and $U_0 = 0, U_1 = 1,$ then $\{U_n\}$ becomes the Fibonacci sequence $\{F_n\}$.

King [6], Bicknell and Hoggatt [1], and others have used the Q-matrix to generate, so to speak, the Fibonacci sequence where

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

It is routine to show that

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3}$$

In order to generate the sequence $\{U_n\}$ we define the R-matrix,

$$R = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}. \tag{4}$$

It is also useful to define what we call the sequence $\{K_n\}$ as the special case of $\{U_n\}$ where $U_0 \equiv K_0 = 0,$ $U_1 \equiv K_1 = 1,$ and $K_n = rK_{n-1} + sK_{n-2}$. With these stipulations, it follows that

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = \begin{bmatrix} K_n & sK_{n-1} \\ K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_o \end{bmatrix}. \tag{5}$$

In (5) if we replace n by $n+p,\ p > 0,$ then

$$\begin{bmatrix} U_{n+p} \\ U_{n+p+1} \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n+p+1} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{p} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} K_n & sK_{n-1} \\ K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_{p+1} \\ U_p \end{bmatrix}.$$

Now by equating corresponding elements in (6), we obtain

$$U_{n+p} = K_n U_{p+1} + sK_{n-1} U_p. \tag{7}$$

Similarly, it may be shown that for any p,q such that $0 \leqslant q \leqslant n-1$ and $0 \leqslant q \leqslant p,$

$$U_{n+p} = K_{n+q} U_{p-q+1} + sK_{n+q-1} U_{p-q}. \tag{8}$$

Using (5), (7) and (8) we derive a number of vector-matrix relations which are listed here since they will be used in the sequel.

We have

$$(9)\qquad \begin{bmatrix} U_n \\ -U_{n-1} \end{bmatrix} = \begin{bmatrix} r & -s \\ -1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} = \begin{bmatrix} K_n & -sK_{n-1} \\ -K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix},$$

$$(10)\qquad \begin{bmatrix} U_{n-1} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 \\ \pm s & r \end{bmatrix}^{n-1} \begin{bmatrix} U_0 \\ \pm U_1 \end{bmatrix}$$

$$(11)\qquad \begin{bmatrix} U_{n+p} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} K_p & \pm sK_{p-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n+1} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} K_{p+1} & \pm sK_p \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} U_n \\ \pm U_{n-1} \end{bmatrix},$$

$$(12)\qquad \begin{bmatrix} U_n \\ \pm U_{n-p} \end{bmatrix} = \begin{bmatrix} K_p & \pm sK_{p-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n-p+1} \\ \pm U_{n-p} \end{bmatrix} = \begin{bmatrix} K_{p+1} & \pm sK_p \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} U_{n-p} \\ \pm U_{n-p-1} \end{bmatrix}$$

$$(13)\qquad \begin{bmatrix} U_n \\ U_{n+p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ sK_p & K_{p+1} \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sK_{p-1} & K_p \end{bmatrix} \begin{bmatrix} U_n \\ U_{n+1} \end{bmatrix},$$

$$(14)\qquad \begin{bmatrix} U_n \\ sU_{n-1} \end{bmatrix} = \begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ sU_0 \end{bmatrix},$$

$$(15)\qquad \begin{bmatrix} 0 & s \\ 1 & r \end{bmatrix}^{n-1} = \begin{bmatrix} sK_{n-2} & sK_{n-1} \\ K_{n-1} & K_n \end{bmatrix}.$$

When considering generalizations of the Fibonacci sequence, one of the natural questions to investigate is which, if any, of the Fibonacci identities may be generalized to identities for the generalized sequence. In many cases identities can be modified to generalized identities which, as special cases, reduce to Fibonacci identities. For example, Horadam has shown [4] that the well known Fibonacci identity,

$$(16)\qquad F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1},$$

becomes

$$(17)\qquad H_n^2 - H_{n-1}H_{n+1} = (-1)^{n-1}(H_1^2 - H_1H_0 - H_0^2),$$

where H_0, H_1 are arbitrary integers and

$$H_n = H_{n-1} + H_{n-2}.$$

Other well known Fibonacci identities have been generalized in [4] also.

In [5], Horadam has given the generalization of (16) for the $\{U_n\}$ sequences as well as the generalization of several other identities. We show here a derivation and proof of these generalizations using appropriate matrices and vectors. This method not only provides a very clear proof, but it also *derives* the generalized expression. This latter task is not always easy if we have to rely on "guessing" what the generalized form should be.

If we consider the following vector dot product and use (5) and (10), we have

$$U_n^2 - U_{n-1}U_{n+1} = [U_n, U_{n-1}] \cdot \begin{bmatrix} U_n \\ -U_{n+1} \end{bmatrix} = [U_1, U_0] \begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix}^n \begin{bmatrix} U_0 \\ -U_1 \end{bmatrix}$$

$$= (-s)^{n-1}[U_1, U_0] \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix} \begin{bmatrix} U_0 \\ -U_1 \end{bmatrix},$$

since

$$\begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix} = \begin{bmatrix} -s & 0 \\ 0 & -s \end{bmatrix}.$$

Now if we multiply out these three matrices, we get

$$(18)\qquad U_n^2 - U_{n-1}U_{n+1} = (-s)^{n-1}(U_1^2 - rU_1U_0 - sU_0^2).$$

If $U_1 = 1$, $U_0 = 0$, we have

$$K_n^2 - K_{n-1}K_{n+1} = (-s)^{n-1}\,, \tag{19}$$

an expression independent of r. Thus, we conclude that if $s = 1$, the $\{K_n\}$ sequence satisfies (16) without alteration regardless of what value r assumes.

The method above may be used to show that

$$U_n^2 - U_{n-q}U_{n+q} = (-s)^{n-q}K_q^2(U_1^2 - rU_1U_0 - sU_0^2)\,, \tag{20}$$

$$U_{n+p}U_{n+q} - U_nU_{n+p+q} = (-s)^nK_pK_q(U_1^2 - rU_1U_0 - sU_0^2)\,. \tag{21}$$

These identities also appear in [5], but the method used there to derive them is quite different. Since the proof of (20) and (21) is more involved than the proof of (18), we give the proof of (20) here. Using (12), (13), and then (15), we have,

$$\begin{aligned}
U_n^2 - U_{n-q}U_{n+q} &= [U_n,\, U_{n+q}]\begin{bmatrix} U_n \\ -U_{n-q}\end{bmatrix} \\
&= [U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r\end{bmatrix}^n\begin{bmatrix} 1 & sK_{q-1} \\ 0 & K_q\end{bmatrix}\begin{bmatrix} K_q & -sK_{q-1} \\ 0 & 1\end{bmatrix}\begin{bmatrix} r & -s \\ -1 & 0\end{bmatrix}^{n-q}\begin{bmatrix} U_1 \\ -U_0\end{bmatrix} \\
&= K_q[U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r\end{bmatrix}^n\begin{bmatrix} r & -s \\ -1 & 0\end{bmatrix}^{n-q}\begin{bmatrix} U_1 \\ -U_0\end{bmatrix} \\
&= (-s)^{n-q}K_q[U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r\end{bmatrix}^q\begin{bmatrix} U_1 \\ -U_0\end{bmatrix} \\
&= (-s)^{n-q}K_q[U_0,\, U_1]\begin{bmatrix} sK_{q-1} & sK_q \\ K_q & K_{q+1}\end{bmatrix}\begin{bmatrix} U_1 \\ -U_0\end{bmatrix}\,.
\end{aligned}$$

If we multiply these three matrices, rearrange terms properly and observe that

$$K_{q+1} - sK_{q-1} = rK_q\,,$$

we have

$$U_n^2 - U_{n-q}U_{n+q} = (-s)^{n-q}K_q^2[U_1^2 - rU_1U_0 - sU_0^2]\,.$$

Again if we let $U_0 = 0,\ U_1 = 1,$ (20) becomes

$$K_n^2 - K_{n-q}K_{n+q} = (-s)^{n-q}\,, \tag{22}$$

an expression independent of r.

Another well known Fibonacci identity is

$$F_{n+1}^2 + F_n^2 = F_{2n+1}\,. \tag{23}$$

Matrix methods are again especially helpful in not only proving a generalization of (23) but in discovering what this generalization ought to be.

Using (10) and (14), we have

$$\begin{aligned}
U_{n+1}^2 + sU_n^2 &= [U_{n+1}, sU_n]\begin{bmatrix} U_{n+1} \\ U_n\end{bmatrix} = [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^n\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^n\begin{bmatrix} U_1 \\ U_0\end{bmatrix} \\
&= [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^{2n}\begin{bmatrix} U_1 \\ U_0\end{bmatrix} = [U_1, sU_0]\begin{bmatrix} U_{2n+1} \\ U_{2n}\end{bmatrix} = U_1U_{2n+1} + sU_0U_{2n}\,.
\end{aligned}$$

Hence, we have

$$U_{n+1}^2 + sU_n^2 = U_1U_{2n+1} + sU_0U_{2n}\,, \tag{24}$$

which is again an expression independent of r. For the $\{K_n\}$ sequence, this becomes

$$K_{n+1}^2 + sK_n^2 = K_{2n+1}\,. \tag{25}$$

As an alternate way of writing the right side of (24), we observe in the proof that

$$\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^n\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^n = \begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^{n-q}\begin{bmatrix} r & s \\ 1 & 0\end{bmatrix}^{n+q}\,.$$

Substituting this expression into the above proof, we see that we may write

$$(26)\qquad U_1U_{2n+1}+sU_0U_{2n} = U_{n-q+1}U_{n+q+1}+sU_{n-q}U_{n+q}\ .$$

As a further exercise in identities we see that if we replace n by $n+1$ in (8), let $p=n$, and $U_i=K_i$, we have

$$(27)\qquad K_{2n+1} = K_{n+q+1}K_{n-q+1}+sK_{n+q}K_{n-q}\ .$$

We may also obtain (27) as a special case of (26) by simply replacing U_i by K_i. However, (8) cannot be obtained from (27).

The Fibonacci identity

$$(28)\qquad F_{n+1}^2-F_{n-1}^2 = F_{2n}\ ,$$

generalizes to

$$(29)\qquad U_{n+1}^2-s^2U_{n-1}^2 = r(U_1U_{2n}+sU_0U_{2n-1})\ .$$

We may prove (29) by using (24) or by using matrices as follows:

$$\begin{aligned} U_{n+1}^2-s^2U_{n-1}^2 &= U_{n+1}^2+sU_n^2-(sU_n^2+s^2U_{n-1}^2)\\ &= [U_1,sU_0]\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^{2n-2}\left[\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^2-s\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right]\begin{bmatrix} U_1\\ U_0\end{bmatrix}\\ &= r[U_1,sU_0]\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^{2n-2}\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}\begin{bmatrix} U_1\\ U_0\end{bmatrix} = r(U_1U_{2n}+sU_0U_{2n-1})\ . \end{aligned}$$

Again, the identities (24) and (29) are found in [5] and perhaps elsewhere in the literature, although the alternate way of expressing the right side of (24) which appears in (26) is apparently not known.

The method used in the proof of (29) may be generalized to find and prove numerous other identities for the sequence $\{U_n\}$. As an illustration, we note that in the proof of (29) we needed and used the fact that

$$\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^2-s\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} = r\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}\ .$$

Using this as a clue, we can show, for example, that

$$\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^4-r\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^3-s^2\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix} = rs\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}\ .$$

Therefore, since

$$\begin{aligned} &U_{n+2}^2+sU_{n+1}^2-r(U_{n+2}U_{n+1}+sU_{n+1}U_n)-s^2(U_n^2+sU_{n+1}^2)\\ &\quad = [U_1,sU_0]\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^{2n-2}\left[\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^4-r\begin{bmatrix} r & s\\ 1 & 0\end{bmatrix}^3-s^2\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right]\begin{bmatrix} U_1\\ U_0\end{bmatrix}, \end{aligned}$$

we conclude in the manner used above that

$$(29)\qquad U_{n+2}^2+sU_{n+1}^2-rU_{n+2}U_{n+1}-rsU_{n+1}U_n-s^2U_n^2-s^3U_{n+1}^2 = rs(U_1U_{2n}+sU_0U_{2n-1})\ .$$

The use of matrices adapts itself very nicely for generalizing some of the identities involving sums of Fibonacci numbers. One such identity is

$$(30)\qquad \sum_{i=1}^{n} F_i = F_{n+2}-1\ .$$

In order to generalize this identity for the sequence $\{U_n\}$, we first prove that

$$(31)\qquad s\sum_{i=1}^{n} r^{n-i}K_i = K_{n+2}-r^{n+1}\ .$$

The method of derivation and proof is a generalization of a method used by Hoggatt and Ruggles [3]. We first observe that for the matrix R as defined by (4) and the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(32) $$R^{n+1} - r^{n+1}I = (r^nI + r^{n-1}R + \cdots + r^2R^{n-2} + rR^{n-1} + R^n)(R - rI).$$

Furthermore, it is easy to show that

$$R^2 - rR - sI = 0,$$

or

$$R(R - rI) = sI.$$

Hence, we see that

$$(R - rI)^{-1} = s^{-1}R.$$

If we multiply both sides of (32) by $s^{-1}R$ and then subtract r^nI from both sides, we obtain

(33) $$r^{n-1}R + r^{n-2}R^2 + \cdots + rR^{n-1} + R^n = s^{-1}(R^{n+2} - r^nR^2).$$

Writing out the matrices in (33), we have

$$r^{n-1}\begin{bmatrix} K_2 & sK_1 \\ K_1 & sK_0 \end{bmatrix} + r^{n-2}\begin{bmatrix} K_3 & sK_2 \\ K_2 & sK_1 \end{bmatrix} + r^{n-3}\begin{bmatrix} K_4 & sK_3 \\ K_3 & sK_2 \end{bmatrix} + \cdots + r\begin{bmatrix} K_n & sK_{n-1} \\ K_n & sK_{n-2} \end{bmatrix} + \begin{bmatrix} K_{n+1} & sK_n \\ K_n & sK_{n-1} \end{bmatrix}$$
$$= s^{-1}\left[\begin{bmatrix} K_{n+3} & sK_{n+2} \\ K_{n+2} & sK_{n+1} \end{bmatrix} - r^n\begin{bmatrix} K_3 & sK_2 \\ K_2 & sK_1 \end{bmatrix}\right].$$

Now equating elements in the upper right corner of this matrix equation, we obtain (recall that $K_2 = r$),

$$sr^{n-1}K_1 + sr^{n-2}K_2 + \cdots + srK_{n-1} + sK_n = K_{n+2} - r^{n+1},$$

which is (31).

In order to generalize this identity for arbitrary U_0, U_1, we use (7) with $p = 0$ to get

$$s\sum_{i=1}^{n} r^{n-i}U_i = s\sum_{i=1}^{n} r^{n-2}(U_1K_i + sU_0K_{i-1})$$

$$= U_1 s\sum_{i=1}^{n} r^{n-i}K_i + s^2U_0\sum_{i=1}^{n} r^{n-i}K_{i-1}$$

$$= U_1\left(s\sum_{i=1}^{n} r^{n-i}K_i\right) + \frac{sU_0}{r}\left(s\sum_{i=2}^{n+1} r^{n-(i-1)}K_{i-1}\right) - \frac{s^2U_0}{r}K_n.$$

Now we use (31) on these two sums to obtain

$$U_1\left(s\sum_{i=1}^{n} r^{n-i}K_i\right) + \frac{sU_0}{r}\left(s\sum_{i=2}^{n+1} r^{n-(i-1)}K_{i-1}\right) - \frac{s^2U_0}{r}K_n = U_1(K_{n+2} - r^{n+1}) + \frac{sU_0}{r}(K_{n+2} - r^{n+1})$$

$$- \frac{s^2U_0K_n}{r} = U_1K_{n+2} - U_1r^{n+1} + \frac{sU_0}{r}(rK_{n+1} + sK_n - r^{n+1}) - \frac{s^2U_0K_n}{r}$$

$$= U_1K_{n+2} + sU_0K_{n+1} - r^{n+1}U_1 - sr^nU_0 = U_{n+2} - r^nU_2.$$

Hence we find that the generalized form of (30) is

(34) $$s\sum_{i=1}^{n} r^{n-i}U_i = U_{n+2} - r^nU_2 \quad .$$

By factoring the expression

$$(R^2)^{n+1} - (r^2)^{n+1}I \; ,$$

and proceeding as above, we find

(35) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}K_{2i} = (r^2-s)K_{2n+2} + rsK_{2n+1} - r^{2n+3}$$

and

(36) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}K_{2i-1} = (r^2-s)K_{2n+1} + rsK_{2n} - r^{2n+2} + sr^{2n} \; .$$

If we use (35) and (36) in the same manner as we did in proving (34), we get

(37) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}U_{2i} = (r^2-s)U_{2n+2} + rsU_{2n+1} + s^2r^{2n}U_0 - r^{2n+2}U_2$$

(38) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}U_{2i-1} = (r^2-s)U_{2n+1} + rsU_{2n} + r^{2n}(sU_1 - rU_2) \; .$$

It is quite likely that many other well known identities can be generalized in ways similar to those used above. It is not our purpose to provide an exhaustive list, but to illustrate the method and in particular the usefulness of the R–matrix.

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ANTIMAGIC SQUARES DERIVED FROM THE THIRD-ORDER MAGIC SQUARE

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In a third-order antimagic square, the three elements along each of the eight lines—three rows, three columns, and two unbroken diagonals—have different sums. An antimagic square, its rotations and reflections are equivalent and count as only one square.

It is not difficult to modify the distribution of the digits around the central 5 of the unique nine-digit third-order magic square

8	1	6
3	5	7
4	9	2

so as to convert it into the antimagic square

1	2	3
6	5	8
9	4	7

while preserving an odd-even alternation of digits around the perimeter. Nor, to set up a sequence of antimagic squares,

1 3 2	1 3 2	1 3 2	1 3 2	1 3 2	1 3 2
9 5 8	9 5 8	9 5 7	9 5 6	7 5 6	7 5 6
6 7 4	4 7 6	4 8 6	4 8 7	4 8 9	4 9 8

still around the central 5, in which each square results from the interchange of two digits in the previous square.

The *complements* of the squares in this sequence, obtained by subtracting each of the digits from 10, form the similar sequence

9 7 8	9 7 8	9 7 8	9 7 8	9 7 8	9 7 8
1 5 2	1 5 2	1 5 3	1 5 4	3 5 4	3 5 4
4 3 6	6 3 4	6 2 4	6 2 3	6 2 1	6 1 2

Of course, if a square is antimagic, its complement also is. The eight sums of the complementary square may be obtained by subtracting each sum of the parent square from 30.

The question naturally arises, what is the minimum number of digits that need to change position and what is the minimum number of moves, or interchanges, necessary in order to convert the magic square into an antimagic square?

THE CRITERIA

To convert the magic square into an antimagic square by interchanging digits, two conditions must be met:

(1) Not more than one line (sum = 15) can remain unaltered;

(2) If two or more lines contain the same single changed element, only one of those lines can be left without another change.

There are 16 distinct ways in which three markers (×) can be distributed on a nine-cell 3-by-3 array. Thus

XOO	OXO	XOX	XOX	OXO	XXX	OOO	XXO
OXO	OXO	OXO	OOO	XOO	OOO	XXX	XOO
OOX	XOO	OOO	OXO	OOX	OOO	OOO	OOO

XOO	OXO	XXO	XXO	XXO	XXO	XOX	OXO
XXO	XXO	OOX	OOO	OOO	OOO	OOO	XOX
OOO	OOO	OOO	XOO	OXO	OOX	XOO	OOO

If the x's indicate the elements to be moved by interchange, then only the first five configurations meet the first condition, and none of those five meet the second condition. So at least four digits must be involved in the interchange.

There are 23 distinct ways in which four markers (x) can be distributed on a nine-cell 3-by-3 array. Thus

xxx	xxx	xoo	oxo	xxo	xxo	xxo	xxo
xoo	ooo	xxx	xxx	xxo	ooo	ooo	xox
ooo	oxo	ooo	ooo	ooo	xxo	oxx	ooo
xxo	xxo	xxo	xxo	oxo	xox	xox	oxo
xoo	oox	oox	oxo	xxo	oxo	ooo	xox
oox	xoo	oxo	oox	oox	oox	xox	oxo
xxx	xxx	xxo	xxo	xxo	xxo	xox	
oxo	ooo	oxo	oxx	ooo	oox	oxo	
ooo	xoo	xoo	ooo	xox	oox	oxo	
A	*B*	*C*	*D*	*E*	*F*	*G*	

If the x's indicate the elements to be moved in the interchanging, then only the last seven meet both conditions. Each of the three symmetrical arrangements (A, F and G) can be applied to the magic square in four ways, and the four asymmetrical configurations (B, C, D and E) and their mirror images can each be applied in four ways. So there are 44 applications of change patterns to consider.

TWO INTERCHANGES

Four elements can be divided into two pairs in three distinct ways. These pairings are applied to the seven change patterns. If both members of a pair fall on the same line, interchange of their positions does not affect the sum of the elements of that line. Each of the patterns A, C, D and E has one *ooo* line unaffected by interchange of the x's. In any pairing of their x elements, any interchange between members of the pairs leaves the sum of the elements unchanged in one of the lines involved. Thus two lines of the square retain their original, and hence equal, sums.

In patterns F and G interchange between members of the pairs leaves the pairs in the original lines or interchanges the elements of a column and a row. Thus that column and row retain their original equal sums.

In the asymmetrical B, the interchange

$$\begin{matrix} abc & & cda \\ ooo & \to & ooo \\ doo & & boo \end{matrix}$$

leaves only one line sum unmodified. However, when applied to the nine-digit magic square in each of the eight possible ways, duplicate sums of 12, 14, 16 or 18 appear after the interchange.

Consequently, no antimagic square can be created by interchange of the members in each of two pairs of the elements of the nine-digit third-order magic square.

THREE INTERCHANGES

There are 24 permutations of the four elements M, N, P, Q. In 15 of these at least one of the elements has not moved from its original position. In 3 others, there have been two interchanges of positions. The other 6 can be attained from $MNPQ$ by three successive interchanges, namely:

U – $NPQM$: MN, MP, MQ	X – $PQNM$: MP, NQ, MN
V – $NQMP$: MN, MP, PQ	Y – $QMNP$: PQ, MQ, NM
W – $PMQN$: MP, NM, NQ	Z – $QPMN$: MQ, NP, NM

These six permutations, identified by the prefaced letters U, V, W, X, Y, Z can be applied to each of the seven patterns A, B, C, D, E, F, G in their various orientations on the magic square. The letters M, N, P, Q may be arbitrarily assigned to the four x elements of the pattern without affecting the ultimate position of the interchanged digits.

Pattern B is asymmetrical, so it and its mirror image both are applied to the magic square in the four orientations, proceeding clockwise around the square. Thus the magic square operated on by pattern B becomes

$$\begin{matrix} M & N & P \\ 3 & 5 & 7 \\ Q & 9 & 2 \\ & B_1 & \end{matrix} \qquad \begin{matrix} Q & 1 & M \\ 3 & 5 & N \\ 4 & 9 & P \\ & B_2 & \end{matrix} \qquad \begin{matrix} 8 & 1 & Q \\ 3 & 5 & 7 \\ P & N & M \\ & B_3 & \end{matrix} \qquad \begin{matrix} P & 1 & 6 \\ N & 5 & 7 \\ M & 9 & Q \\ & B_4 & \end{matrix}$$

P	N	M		8	1	P		Q	1	6		M	1	Q
3	5	7		3	5	N		3	5	7		N	5	7
4	9	Q		Q	9	M		M	N	P		P	9	2
	B_5				B_6				B_7				B_8	

The particular orientation of the operating pattern is indicated by the numerical subscript. This notation will be followed with subsequent pattern operators.

In B_1, $M = 8$, $N = 1$, $P = 6$, $Q = 4$, so for the six permutations we have:

$U - NPQM$	$V - NQMP$	$W - PMQN$
1 6 4 8	1 4 8 6	6 8 4 1
$X - PQMN$	$Y - QMNP$	$Z - QPMN$
6 4 1 8	4 8 1 6	4 6 8 1

The digits in each of the permutations are placed, in sequential order, in the M, N, P, Q positions of the square array, B_1. Not all of the permutations will yield antimagic squares. No sum of 15 remains, but other duplicate sums may appear in the process. In B_1, no antimagic squares are produced by V, W, and Y. Indeed, V does not produce an antimagic square in any B_i.

When the pattern orientations are 180° apart, as in B_1B_3, B_2B_4, B_5B_7, and B_6B_8, complementary antimagic squares are produced. (The $MNPQ$ sets are complementary.) Only one of each complementary pair is recorded below in identifying the twenty antimagic squares produced by pattern B. In general, the orientation of the pattern which has operated on the magic square can be identified by the digits which occupy the same position in the antimagic square as they did in the original magic square.

1	6	4		6	4	1		4	6	8		6	1	7		6	1	2
3	5	7		3	5	7		3	5	7		3	5	2		3	5	8
8	9	2		8	9	2		1	9	2		4	9	8		4	9	7
8	1	4		2	8	1		1	2	8		1	6	2		6	8	2
3	5	2		3	5	7		3	5	7		3	5	7		3	5	7
7	9	6		4	9	6		4	9	6		4	9	8		4	9	1

Since pattern B always leaves a mid-row or mid-column undisturbed, each of these antimagic squares has a central 5. The last square is particularly noteworthy in that its sums are 10, 11, 12, 13, 14, 15, 16, and 22, where seven of the eight are consecutive numbers. No antimagic square has all eight sums in arithmetic progression. [1,2]

In *Pattern C,*

$$\begin{matrix} MNo \\ oPo \\ Qoo \end{matrix},$$

application of X or Z leaves the sums of the row and diagonal which have the upper right-hand element in common unchanged and hence equal. U, V, W, and Y also fail to produce an antimagic square in the eight orientations of C.

In *Pattern D,*

$$\begin{matrix} MNo \\ oPQ \\ ooo \end{matrix},$$

application of X or Z fails to produce an antimagic square in any of the eight orientations. Nor do any of the permutations produce one with D_2 or D_4. As with pattern B, complementary antimagic squares are produced by pattern orientations 180° apart. One of each of the eight complementary pairs resulting from pattern D is recorded below:

1	5	6		1	7	6		5	8	6		7	8	6
3	7	8		3	8	5		3	7	1		3	1	5
4	9	2		4	9	2		4	9	2		4	9	2
8	5	6		8	7	6		8	5	6		8	5	1
3	2	1		3	1	2		3	7	2		6	3	7
4	9	7		4	9	5		4	9	1		4	9	2

Pattern E fails to yield any antimagic squares when X and Z are used as operators, nor from E_2 or E_4 with any operator. As in B and D, the antimagic squares formed are in complementary pairs. One member of each of the eight pairs from pattern E follows:

```
1 4 6    1 2 6    4 8 6    2 8 6
3 5 7    3 5 7    3 5 7    3 5 7
2 9 8    8 9 4    2 9 1    1 9 4

8 6 2    2 1 6    7 1 6    4 1 6
3 5 7    3 5 4    3 5 2    3 5 2
1 9 4    8 9 7    8 9 4    7 9 8
```

In common with the squares from B, all these squares have 5 as a central digit.

Pattern A,

```
MNP
oQo ,
ooo
```

being symmetrical, can be applied to the magic square in only four orientations. Of the permutations, only X fails to produce an antimagic square from some orientation. Those formed when the pattern orientations are 180° apart are complements. One of each of the six complementary pairs is given here:

```
1 6 5    1 5 8    6 8 5    5 8 1    5 6 8    8 1 5
3 8 7    3 6 7    3 1 7    3 6 7    3 1 7    3 7 2
4 9 2    4 9 2    4 9 2    4 9 2    4 9 2    4 9 6
```

The sums of the fourth square in this set are 10, 11, 12, 13, 14, 15, 16, and 23, another case where seven of the sums are consecutive integers.

In *patterns F and G,* the other two symmetrical patterns, X and Z merely interchange a row and column and leave the sums equal. The other four permutations fail to produce an antimagic square with any of the four orientations.

SUMMARY

In order to convert the nine-digit third-order magic square into an antimagic square by interchange of digits, not less than four digits must be moved in three successive interchanges. The four digits must fall into one of four basic patterns *(B, D, E, A)* to which one of the six permutations *U, V, W, X, Y, Z* is applied. The 64 antimagic squares which can be produced in this manner fall into 32 complementary pairs. Complementary pairs are produced by patterns 180° apart in orientation. Six of these pairs come from a symmetrical pattern. The 26 pairs that are the result of applying asymmetrical patterns are produced in equal quantities by the patterns and their mirror images. The central digit of 36 of the antimagic squares is 5. The frequency of occurrence of the other central digits follows each of the following digits in parentheses: 1(4), 2(3), 3(5), 4(2), 6(2), 7(5), 8(3), 9(4). Two of the squares have seven of their sums in arithmetic progression, with $d = 1$.

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CONCERNING AN EQUIVALENCE RELATION FOR MATRICES

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Let each of s and n be a positive integer, p an arbitrary prine, Λ the field of integers modulo p and S the set of all s by n matrices over Λ. Let each of A and B be in S. We say that A is equivalent to B (written $A \sim B$) if and only if there is a non-singular matrix X over Λ and a matrix $Y = (y_{ij})$ in S with

$$y_{i1} \equiv y_{i2} \equiv \cdots \equiv y_{in} \pmod p, \qquad i = 1, 2, \cdots, s$$

such that

$$A = XB + Y.$$

It is easy to show that $\sim$ is an equivalence relation on S. Let $L_p(s,n)$ be the smallest non-negative number not greater than $p - 1$ such that each equivalence class contains a member $X = (x_{ij})$ with the property that

$$0 \leqslant x_{ij} \leqslant L_p(s,n) \qquad 1 \leqslant i \leqslant s, \qquad 1 \leqslant j \leqslant n.$$

We shall give an elementary proof of the

Theorem.

$$L_p(s,n) \leqslant 2[p^{(ns-t-1)/(ns-t)}], \qquad n = 2, 3, \cdots,$$

where

(1) $$t = s^2 \text{ if } s \leqslant [n/2] \quad \text{and} \quad t = [n/2]^2 - n[n/2] + ns \text{ if } s > [n/2].$$

Here $[x]$ is the greatest integer $\leqslant x$.

For the case $s = 1$ the theorem gives

$$L_p(1,n) \leqslant 2[p^{(n-2)/(n-1)}], \qquad n = 2, 3, \cdots.$$

L. Redei [3] has shown, using the geometry of numbers, that

$$L_p(1,n) \leqslant 2n^{-1/(n-1)}p^{(n-2)/(n-1)}, \qquad n = 2, 3, \cdots.$$

Using elementary methods (a theorem of Thue [4]), Redei has also shown that

$$L_p(1,n) \leqslant 2([p^{1/(n-1)}] + 1)^{n-2}, \qquad n = 2, 3, \cdots.$$

Our theorem then generalizes the results of Redei and improves his weaker inequality, by elementary methods.

We shall make use of the following theorem which has an elementary proof.

Theorem A. (A. Brauer and R.L. Reynolds [1]). Let r and s be rational integers $r < s$ and let f_δ be positive numbers less than m $(\delta = 1, 2, \cdots, s)$ such that

$$\prod_{\delta=1}^{s} f_\delta > m^r.$$

Then the system of r linear congruences

$$y_\rho = \sum_{\delta=1}^{s} a_{\rho\delta} x_\delta \equiv 0 \pmod m \qquad (\rho = 1, 2, \cdots, r)$$

has a non-trivial solution in integers $x_1, x_2, \cdots, x_s$ such that

$$|x_\delta| < f_\delta \qquad (\delta = 1, 2, \cdots, s)\ .$$

We note that the hypothesis of this theorem can be weakened by letting the numbers f_δ $(\delta = 1, 2, \cdots, s)$ be positive numbers *not greater than* m. The proof is the same as in [1]. We follow, in part, the method of Redei [3], as given when $s = 1$.

Now let $Y = (y_{ij})$ be a member of S. The matrix $Z = (z_{ij})$, where $Z = IY + B$, I is the identity matrix and $B = (b_{ij})$ is the matrix with

$$b_{i1} = b_{i2} = \cdots = b_{in} = -y_{in} \qquad (i = 1, 2, \cdots, s),$$

is equivalent to Y. Note that $z_{in} = 0$, $i = 1, 2, \cdots, s$.

Let r be the rank of the matrix Z. It is well known that there is a non-singular matrix C over Λ, such that the matrix $U = CZ$ has $s - r$ zero rows and has r columns each with exactly one non-zero element (see for example [2]). The matrix U then has at least

$$f(r) = r^2 - nr + ns, \qquad 0 \leqslant r \leqslant s$$

zero elements. The minimum value for $f(r)$ is given by t in (1). Thus Y is equivalent to a matrix U that has at most $ns - t$ non-zero elements.

Let $u_1, u_2, \cdots, u_\lambda$ be the non-zero elements of U. Consider the system

$$x_i \equiv au_i \pmod p, \qquad i = 1, 2, \cdots, \lambda \tag{2}$$

of λ congruences in the $\lambda + 1$ variables a, x_i $(i = 1, 2, \cdots, \lambda)$. Setting $f_0 = p$ and $f_\delta = [p^{(\lambda-1)/\lambda}] + 1$, $(\delta = 1, 2, \cdots, \lambda)$, we have

$$\prod_{\delta=0}^{\lambda} f_\delta = p([p^{(\lambda-1)/\lambda}] + 1)^\lambda > p(p^{(\lambda-1)/\lambda})^\lambda = p^\lambda\ . \tag{3}$$

Using Theorem A, the remark following it, together with (3), it follows that the system of linear congruences (2) has a non-trivial solution a, x_i $(i = 1, 2, \cdots, \lambda)$ with

$$|a| \leqslant p - 1 \qquad \text{and} \qquad |x_i| \leqslant [p^{(\lambda-1)/\lambda}], \qquad i = 1, 2, \cdots, \lambda\ .$$

Since the solution is non-trivial, $a \not\equiv 0 \pmod p$; and since $\lambda \leqslant ns - t$,

$$|x_i| \leqslant [p^{(ns-t-1)/(ns-t)}], \qquad i = 1, 2, \cdots, \lambda\ . \tag{4}$$

The s by n matrix $X = (x_{ij})$ with entries x_i $(i = 1, 2, \cdots, \lambda)$ in the same position as u_i $(i = 1, 2, \cdots, \lambda)$ of U, and zero elsewhere, satisfies the equation $X = AU$, where A is the diagonal matrix with all diagonal entries equal to a. Naturally, since $a \not\equiv 0 \pmod p$, A is non-singular.

Set

$$t = \max_{i,j} |x_{ij}|\ .$$

If T is the s by n matrix all of whose entries are t, then $W = (w_{ij})$, where $W = IX + T$ is equivalent to X, and

$$0 \leqslant w_{ij} \leqslant 2[p^{(ns-t-1)/(ns-t)}], \qquad 1 \leqslant i \leqslant s, \quad 1 \leqslant j \leqslant n. \tag{5}$$

Since $Y \sim W$, we have, using (5) together with the definition of $L_p(s,n)$, proved the theorem.

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SOME SEQUENCES GENERATED BY SPIRAL SIEVING METHODS

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The object of this note is to point out some curious sequences which may be generated by natural number spirals and rotating grids. The method is a combination of the spiral introduced by Ulam [2] in his studies of prime number distribution and a well known technique employed in cryptographic work. We illustrate with Fibonacci numbers.

Ulam considers a spiral numbering of the lattice points in the plane, i.e., by starting at (0,0) and proceeding counterclockwise in a spiral so that

$$(0,0) \to 1,\quad (1,0) \to 2,\quad (1,1) \to 3,\quad (0,1) \to 4,\quad (-1,1) \to 5,\quad (-1,0) \to 6,\quad (-1,-1) \to 7,\quad (0,-1) \to 8,$$
$$(1,-1) \to 9,\quad (2,-1) \to 10,\quad (2,0) \to 11,\quad (2,1) \to 12,\quad (0,2) \to 13,\quad (-1,2) \to 14,\quad etc.$$

This mapping gives us the spiral below.

(1)

17	16	15	14	13	
18	5	4	3	12	
19	6	1	2	11	
20	7	8	9	10	27
21	22	23	24	25	26

A nice illustration of the basic Ulam spiral makes up the front cover of the March 1964 *Scientific American.* In the same issue Martin Gardner [1] gives an account of Ulam's work. Briefly, Ulam marks the primes $(1,2,3,5,7,11,\cdots)$ in the spiral and studies the visual display for patterns or almost-patterns in the prime number sequence. By use of a computer at Los Alamos he is able to generate displays having around 65,000 points in them. It would be of interest to try something of the same sort for the Fibonacci, Lucas, and other recurrent sequences, however the writer does not have available such versatile equipment as that used by Ulam and his colleagues at Los Alamos, so we have little to suggest about possible patterns in a spiral display of Fibonacci numbers. Of course, the fact that we now know [3], [4] that 1 and 144 are the only square Fibonacci numbers does tell us that the diagonals $1, 9, 25, 49, \cdots$ and $4, 16, 36, 64, \cdots$ will be conspicuously blank in such a display.

Now, there is a technique in cryptographic work which makes use of a rotating grid. We can best illustrate by means of an example. Consider the message, "INTUITION LIKE A FLASH OF LIGHTNING LASTS ONLY FOR A SECOND." We write this in a square array

I	N	T	U	I	T	I
O	N	L	I	K	E	A
F	L	A	S	H	O	F
L	I	G	H	T	N	I
N	G	L	A	S	T	S
O	N	L	Y	F	O	R
A	S	E	C	O	N	D

and then impose a prepunched grid, e.g., of the form (where an X indicates a hole)

(2)

X		X			X	
			X			
		X			X	
			X	X		
						X
X		X			X	
			X			

and copy out the visible letters, which are (serially, row by row) ITTIAOHTSOLOC. We then rotate the grid counterclockwise through 90° and again copy out the visible letters, which are IOLESIHINLTAN. Two more rotations gives us UNKAFGHGSYSOD and NILHFLHNANFRE. Running these four groups together and breaking the whole up into convenient blocks then gives us the enciphered message. To decipher, one merely places the grid on a sheet of paper, writes in the letters serially, row by row, thirteen at a time here, rotating the grid until all four positions are used, removes the grid and reads off the message. Here we have used a 7 by 7 grid which leaves the middle point fixed (H). This is unsatisfactory for cryptographic work in some cases and most ordinary uses involve an even-order grid.

The effect of an odd-order grid in the case of superposition on the natural number spiral is to partition the natural numbers into four sets, any two of which have only the number 1 in common.

It is clear that the very special cryptographic grid cannot be made from the Fibonacci sequence (or the prime number sequence) without adding and/or deleting elements, since any given square annulus of the grid must be so designed that one-fourth of its lattice points are punched, and in such a way that the same hole does not appear under successive rotations of 90° until the original position is assumed. We shall not discuss how this can be effected.

We modify the rotating grid as follows. On the original natural number spiral (1) superimpose a square sheet of paper which will just cover the first $(2n-1)^2$ natural numbers, unity being kept at the center. Make a grid by punching the sheet wherever an element a_k $(k = 1, 2, 3, \cdots)$ of a given sequence appears in the natural number spiral. We shall call this the (counterclockwise) spiral grid of the sequence $\{a_k\}$. We next rotate the spiral grid through 90° and read off from the natural number spiral a new sequence generated by the spiral grid of our original sequence. With any given sequence there will be associated three new sequences, and by turning the grid over (making it a clockwise spiral grid) we can generate four other sequences. Clearly all these eight sequences will be somehow related.

For a grid measuring $2n-1$ by $2n-1$ $(n \geq 2)$ there will be the natural numbers from 1 through $(2n-1)^2$ with the outer square annulus containing the successive natural numbers from $(2n-3)^2+1$ to $(2n-1)^2$. If an element a_k of our given sequence lies in the outer square annulus, then so will the corresponding element b_k^i of any of the associated sequences obtained by use of the grid. It is possible to work out complicated formulas relating b_k to a_k depending upon the position of an element in the annulus. For example, any two diagonally opposite elements in the outer annulus have numerical difference $4n$.

We give below, in Table 1, a few values for the sequences generated by the counterclockwise spiral grid of the Fibonacci sequence (I, II, III, IV) and also for the clockwise grid (I′, II′, III′, IV′).

Here, $d = d_k$ is the minimum positive difference between terms in the sequences, or

$$d = d_k = \min_{i,j} (b_k^i - b_k^j) > 0,$$

$$\text{(with } d_0 \text{ def.} = 0)$$

for Counterclockwise (I – IV), or for Clockwise (I′ – IV′).

In our table, $a_k = F_{k+1}$, with

$$F_{k+1} = F_k + F_{k-1}, \qquad F_0 = 0, \qquad F_1 = 1.$$

It is convenient to begin our sequence with F_2 instead of making some rules about how to interpret $0, 1, 1, 2, 3, \cdots$. (The indistinguishability of F_1 and F_2 prevents us from calling the ordinary Fibonacci sequence a subset of the set of all natural numbers.)

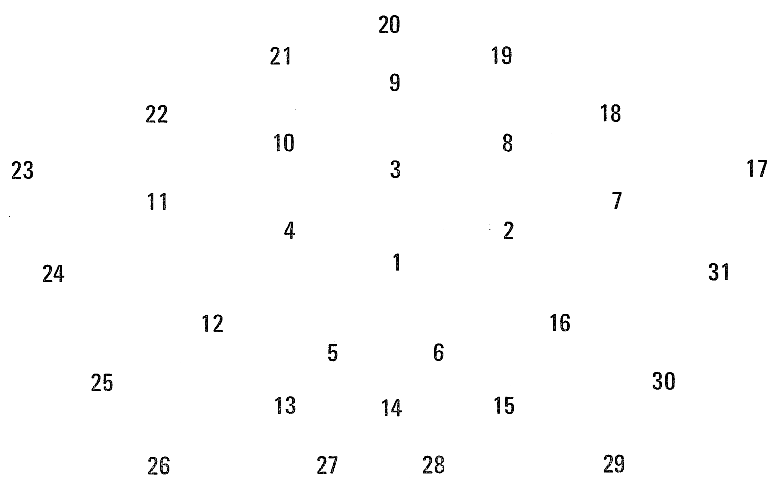
There is no reason to confine our attention to spirals based on a square. Ulam's work with the sequence of primes quite naturally fits in well with such a spiral because quadratic polynomials $Ax^2 + Bx + C$ are often so rich in

Table 1

	a_k	b_k^2	b_k^3	··· etc.					
k	I	II	III	IV	I′	II′	III′	IV′	d
1	1	1	1	1	1	1	1	1	0
2	2	2	4	3	3	2	2	3	1
3	3	4	6	6	5	5	4	4	1
4	5	5	7	8	6	7	7	6	1
5	8	7	9	9	8	8	9	9	1
6	13	17	13	17	17	13	17	13	4
7	21	25	21	25	25	21	25	21	4
8	34	40	46	28	34	40	46	28	6
9	55	63	71	79	67	75	51	59	8
10	89	99	109	119	103	113	83	93	10
11	144	156	168	132	134	146	158	122	12
12	233	249	265	281	265	281	233	249	16
13	377	397	417	437	405	425	365	385	20

primes for integral values of x (Euler's polynomial x^2+x+41 being the most well-known example). However, to exhibit other properties of a sequence, as well as to generate variations of a given sequence, it is natural to pass on to figurate numbers as the basis of our spirals. That is, we may consider a polygon of m sides.

Consider, for example, a pentagonal spiral as shown below.



It would be of interest to examine the distribution of primes, Fibonacci numbers, etc., in an extended pentagon with thousands of points, and of course this would require quite an elaborate computer set-up.

It is fairly easy to type out a pentagonal spiral on ordinary typing paper with 456 points and this is sufficient to give an idea of how the pentagonal spiral grid of the Fibonacci sequence can be used to generate curious sequences. Here, of course, we shall have in all ten sequences. The sequences are tabulated below in Table 2.

The number d tabulated in the last column is defined as before by

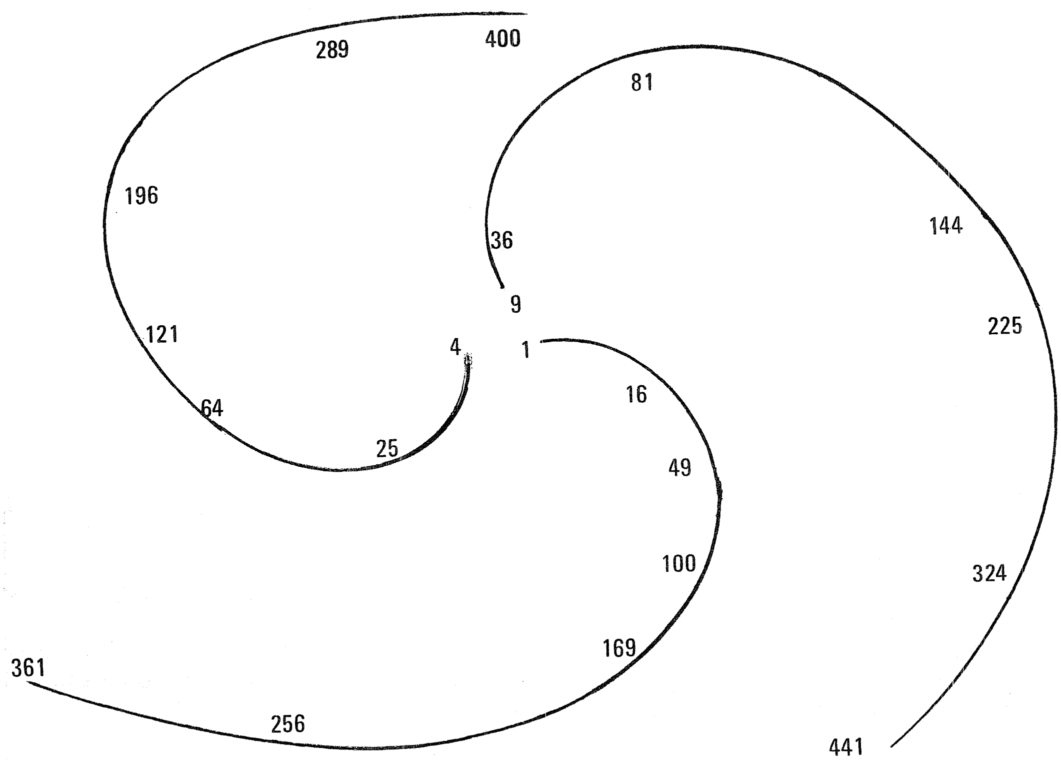
$$d = \min_{i,j}\,(b_k^i - b_k^j) > 0 \tag{3}$$

(for I – V or I′ – V′), and it is not difficult to see that for any given value of k the numbers II – V determined by our grid will differ from the Fibonacci number a_k by a multiple of the number d. The reader may find it of interest to try and develop a general formula for d in terms of k and m (generalizing to an m-gon).

Table 2

	a_k	b_k	b_k	··· etc.							
k	I	II	III	IV	V	I′	II′	III′	IV′	V′	d
1	1	1	1	1	1	1	1	1	1	1	0
2	2	3	2	3	2	3	2	3	2	2	1
3	3	4	4	5	4	4	4	5	4	3	1
4	5	6	5	6	6	6	5	6	6	5	1
5	8	10	7	9	11	10	7	9	11	8	1
6	13	15	12	14	16	15	12	14	16	13	1
7	21	24	27	30	18	19	22	25	28	31	3
8	34	38	42	46	50	38	42	46	50	34	4
9	55	60	65	70	75	59	64	69	74	54	5
10	89	95	101	77	83	77	83	89	95	101	6
11	144	152	160	168	176	156	164	172	180	148	8
12	233	243	253	263	273	241	251	261	271	231	10
13	377	389	341	353	365	371	383	335	347	359	12

The visual display of *perfect squares* in a pentagonal spiral turns out to be a simple *trefoil* spiral appearing somewhat as diagrammed below.



This is easily verified to be in accord with the fact that the three arms of the spiral are formed by squares of form $(3n)^2$, $(3n+1)^2$, and $(3n+2)^2$, respectively.

Finally, we turn to the case of a *triangular* spiral grid. Because of the hexagonal rotational character in this case, one may generate 12 sequences for a given spiral grid, 6 counterclockwise and 6 clockwise. A portion of the triangular spiral appears below.

39		7		35	
	15		13		50
26		3		22	
	8		6		34
16		1		12	
	4		2		21
9		10		5	
	18		19		11

The 12 sequences generated by a triangular spiral grid based on the Fibonacci numbers are tabulated in Table 3.

Table 3

I	II	III	IV	V	VI	I′	II′	III′	IV′	V′	VI′	d	d*
1	1	1	1	1	1	1	1	1	1	1	1	0	0
2	4	3	2	2	3	3	3	2	4	2	2	1	1
3	6	4	8	4	6	4	8	4	6	3	6	1	2
5	8	7	10	6	10	6	10	5	10	7	8	1	2
8	15	10	18	9	12	9	16	8	19	10	13	1	3
13	22	16	26	19	30	15	29	18	21	12	25	3	4
21	23	25	27	29	31	27	30	31	22	23	26	4	4
34	35	39	40	44	45	40	44	45	34	35	39	5	5
55	57	61	63	49	51	51	55	57	61	63	49	6	6
89	70	97	77	105	84	99	81	107	67	91	74	8	7

Here, d is based on either I – III – V or I′ – III′ – V′ while d^* is based on II – IV – VI or II′ – IV′ – VI′. This is because II, IV, and VI arise from the hexagonal effect. Thus it seems of interest to list d as based on both triangular pattern and hexagonal.

With this much as an introduction to the notion of a spiral grid for generating variations of a given sequence, we shall close this account. Our purpose has been mainly to exhibit the results of some calculations and suggest possible avenues of research. Various questions could be posed. For example: What can be said about divisibility properties of the new sequences? What can be said about when such sequences will satisfy simple recurrence relations? Does any of this shed light on when a Fibonacci number may be a figurate number? Can simple formulas be written for the various derived sequences? What is a simple formula for the number we have called d?

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2. M.L. Stein, S.M. Ulam, and M.B. Wells, "A Visual Display of Some Properties of the Distribution of Primes," *Amer. Math. Monthly,* 71 (1964), pp. 516–520.
3. Oswald Wyler, Solution of Problem 5080, *Amer. Math. Monthly,* 71 (1964), pp. 220–222.
4. John H.E. Cohn, "Square Fibonacci Numbers, etc.," *The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 109–113.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-239 Proposed by D. Finkel, Brooklyn, New York.

If a Fermat number $2^{2^n}+1$ is a product of precisely two primes, then it is well known that each prime is of the form $4m+1$ and each has a unique expression as the sum of two integer squares. Let the smaller prime be a^2+b^2, $a>b$; and the larger prime be c^2+d^2, $c>d$. Prove that

$$\left|\frac{c}{d}-\frac{d}{b}\right| \leqslant \frac{1}{100}.$$

Also, given that

$$2^{2^6}+1 = (274,177)(67,280,421,310,721),$$

and that

$$274,177 = 516^2+89^2,$$

express the 14-digit prime as a sum of two squares.

H-240 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let

$$S(m,n,p) = (q)_n(q)_p \sum_{i=0}^{\min(n,p)} \frac{q^{mi+(n-i)(p-i)}}{(q)_i(q)_{n-i}(q)_{p-i}},$$

where

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j), \qquad (q)_o = 1.$$

Show that $S(m,n,p)$ is symmetric in m,n,p.

H-241 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\frac{1}{1-x^n} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-xe^{\frac{2k\pi}{n}i}}.$$

SOLUTIONS

GEE!

H-207 Proposed by C. Bridger, Springfield, Illinois

Define $G_k(x)$ by the relation

$$\frac{1}{1-(x^2+1)s^2-xs^3} = \sum_{n=0}^{\infty} G_k(x)s^k ,$$

where x is independent of s. (1) Find a recurrence formula connecting the $G_k(x)$. (2) Put $x = 1$ and find $G_k(1)$ in terms of Fibonacci numbers. (3) Also with $x = 1$, show that the sum of any four consecutive G numbers is a Lucas number.

Solution by the Proposer.

After carrying out the indicated division, we find

$$G_0(x) = 1,\ G_1(x) = 0,\ G_2(x) = x^2+1,\ G_3(x) = x,\ G_4(x) = (x^2+1)^2,$$

etc.

(1) Assume the recursion formula of the type

$$G_{k+3}(x) = pG_{k+2}(x) + qG_{k+1}(x) + rG_k(x),$$

and put $k = 0$, $k = 1$, and $k = 2$. The solution of the resulting equations gives $p = 0$, $q = x^2+1$, and $r = x$. So the recursion formula is

$$G_{k+3}(x) = (x^2+1)G_{k+1}(x) + xG_k(x).$$

(2) Put $x = 1$ to obtain

$$G_{k+3} = 2G_{k+1} + G_k .$$

This has the characteristic equation $z^3 - 2z - 1 = 0$, whose roots are

$$a = \frac{1+\sqrt{5}}{2},\quad b = \frac{1-\sqrt{5}}{2},\quad c = -1 .$$

Now,

$$\frac{a^k - b^k}{a-b} = F_k, \qquad \text{so} \qquad G_k(1) = F_k + (-1)^k .$$

(3) Use $F_{k+1} + F_{k-1} = L_k$ and $F_{k+2} + F_k = L_{k+1}$, replace F by G and add to obtain

$$G_{k+2} + G_{k+1} + G_k + G_{k-1} = L_{k+2} .$$

Also solved by G. Wulczyn, P. Tracy, P. Bruckman.

BOUNDS FOR A SUM

H-208 Proposed by P. Erdös, Budapest, Hungary.

Assume

$$\frac{n!}{a_1!a_2!\cdots a_k!} \qquad (a_i \geq 2,\ 1 \leq i \leq k),$$

is an integer. Show that the

$$\max \sum_{i=1}^{k} a_i < \frac{5}{2}n ,$$

where the maximum is to be taken with respect to all choices of the a_i's and k.

Solution by O.P. Lossers, Technological University Eindhoven, the Netherlands.

From the well known fact that the number $c_p(m)$ of prime factors p in $m!$ equals

$$c_p(m) = \left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots$$

($[x]$ denotes greatest integer in x), it follows that

$$c_2(2) = c_2(3) = 1,\ \frac{m}{2} \leq c_2(m) < m \quad (m \geq 4) \qquad c_3(m) < \frac{1}{2}m \quad (m \geq 2).$$

Now writing

$$a_1!a_2!\cdots a_k! = (2!)^{\alpha}(3!)^{\beta}b_1!\cdots b_{\ell}!\,,$$

where $b_i \geqslant 4$ $(i = 1, \cdots, \ell)$ lower bounds for the number of factors 2 and 3 in $a_1!\cdots a_k!$ and *a fortiori* for $c_2(n)$ and $c_3(n)$ are found to be $\alpha+\beta+\tfrac{1}{2}\Sigma b_i$ and β, respectively. So

$$\Sigma a_j = 2\alpha+3\beta+\Sigma b_i \leqslant 2c_2(n)+c_3(n) < 2n+\frac{1}{2}n = \frac{5}{2}n\,.$$

Also solved by V. E. Hoggatt, Jr.

SEARCH!

H-209 (Corrected). Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$u_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\,,$$

where $\alpha+\beta = \alpha\beta = z$. determine the coefficients $C(n,k)$ such that

$$Z^n = \sum_{k=1}^{n} C(n,k)U_{n-k+1} \qquad (n \geqslant 1)\,.$$

Solution by the Proposer.

It is easily verified that

$$\begin{aligned} z &= u_1 \\ z^2 &= u_2+u_1 \\ z^3 &= u_3+2u_2+2u_1 \\ z^4 &= u_4+3u_3+5u_2+5u_1 \\ z^5 &= u_5+4u_4+5u_3+14u_2+14u_1\,. \end{aligned}$$

Put

$$z^n = \sum_{k=1}^{n} C(n,k)u_{n-k+1}\,.$$

Since

$$(\alpha+\beta)u_k = (\alpha+\beta)\,\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} = \frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta}+\alpha\beta\,\frac{\alpha^k-\beta^k}{\alpha-\beta} = u_{k+1}+(\alpha+\beta)u_{k-1}\,,$$

it follows that

$$(\alpha+\beta)u_k = u_1+u_2+\cdots+u_{k+1}\,.$$

Hence

$$z^{n+1} = \sum_{k=1}^{n} C(n,k)(\alpha+\beta)u_{n-k+1} = \sum_{k=1}^{n} C(n,k)\sum_{j=1}^{n-k+2} u_j = \sum_{j=1}^{n+1} u_j \sum_{k=1}^{n-j+2} C(n,k) = \sum_{j=1}^{n+1} u_{n-j+2}\sum_{k=1}^{j} C(n,k)\,.$$

It follows that $C(n,k)$ satisfies the recurrence

$$C(n+1,k) = \sum_{j=1}^{k} C(n,k)\,.$$

The first few values are easily computed $(1 \leqslant k \leqslant n \leqslant 5)$.

$$\begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 & 5 \\ 1 & 4 & 9 & 14 & 14\,. \end{array}$$

Thus $C(n,k)$ can be identified with the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$\begin{cases} a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \\ \quad a_i \leqslant i \qquad (i = 1, 2, \cdots, n) . \end{cases}$$

It is known (see for example L. Carlitz and J. Riordan, "Two Element Lattice Permutation Numbers and Their q-Generalization," *Duke Math. Journal,* Vol. 31 (1964), pp. 371–388) that

$$C(n,k) = \binom{n+k-2}{k-1} - \binom{n+k-2}{k-2} .$$

LUCAS CONDITION

H-210 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that a positive integer n is a Lucas number if and only if $5n^2+20$ or $5n^2-20$ is a square.

Solution by the Proposer.

I. (a) Let $n = L_{2m+1}$

$$5n^2+20 = 5(\alpha^{2m+1}+\beta^{2m+1})^2+20 = 5[\alpha^{4m+2}-2(\alpha\beta)^{2m+1}+\beta^{4m+2}] = 25F_{2m+1}^2 .$$

(b) Let $n = L_{2m}$

$$5n^2-20 = 5(\alpha^{2m}+\beta^{2m})^2-20 = 5[\alpha^{4m}-2(\alpha\beta)^{2m}+\beta^{4m}] = 25F_{2m}^2 .$$

II. $s^2 = 5n^2+20.$

(a) One solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(5+\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots$$

with the irrational part also identical to L_{6t+1}. Let

$$(5+\sqrt{5})(9+4\sqrt{5})^t = s_t + L_{6t+1}\sqrt{5}, \qquad s_t^2 = 5L_{6t+1}^2+20.$$

$$(5+\sqrt{5})(9+4\sqrt{5})^{t+1} = 9s_t+20L_{6t+1}+\sqrt{5}\,(9L_{6t+1}+4s_t) .$$

$$9L_{6t+1}+4\sqrt{5L_{6t+1}^2+20} = 9L_{6t+1}+4\sqrt{5(\alpha^{6t+1}-\beta^{6t+1})^2} = 9L_{6t+1}+20F_{6t+1}$$

$$\begin{aligned} L_{6t+7} = \alpha^{6t+7}+\beta^{6t+7} &= \alpha^{6t+1}(9+4\sqrt{5})+\beta^{6t+1}(9-4\sqrt{5}) \\ &= 9L_{6t+1}+20F_{6t+1} \end{aligned},$$

(b) A second solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(10+4\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the rational part of

$$(10+4\sqrt{5})(9+4\sqrt{5})^t$$

is identically L_{6t+3} is similar to that used in II (a).

(c) A third solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(25+11\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots \quad .$$

The proof that the irrational part of

$$(25+11\sqrt{5})(9+4\sqrt{5})^t$$

is identically L_{6t+5} is similar to that used in II (a).

III. $s^2 = 5n^2-20.$

(a) One solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(5+3\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

Assume

$$(5+3\sqrt{5})(9+4\sqrt{5})^t = s_t + L_{6t+2}5, \qquad s_t^2 = 5L_{6t+2}^2 - 20.$$

$$(5+3\sqrt{5})(9+4\sqrt{5})^{t+1} = 9s_t + 20L_{6t+20} + \sqrt{5}(9L_{6t+2} + 4s_t)$$

$$9L_{6t+2} + 4s_t = 9L_{6t+2} + 4\sqrt{5[(\alpha^{6t+2}+\beta^{6t+2})^2 - 4]} = 9L_{6t+2} + 20F_{6t+2}$$

$$L_{6t+8} = \alpha^{6t+8} + \beta^{6t+8} = (9+4\sqrt{5})\alpha^{6t+2} + (9-4\sqrt{5})\beta^{6t+2}$$

$$= 9L_{6t+2} + 20F_{6t+2} \quad .$$

(b) A second solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(15+7\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the irrational part of

$$(15+7\sqrt{5})(9+4\sqrt{5})^t$$

is identical to L_{6t+4} is similar to that used in III (a).

(c) A third solution chain is given by the rational part (for s) and the irrational part (for n) of

$$(40+18\sqrt{5})(9+4\sqrt{5})^t, \qquad t = 0, 1, 2, \cdots .$$

The proof that the irrational part of

$$(40+18\sqrt{5})(9+4\sqrt{5})^t$$

is identical to L_{6t} is similar to that used in III (a).

Also solved by P. Bruckman, P. Tracy, and J. Ivie.

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[Continued from Page 368.]

$$y+1 \leqslant z < y + (x/n)$$

is a necessary condition for a solution. Thus, we see that there can be no solution for integer x, $1 \leqslant x \leqslant n$, a well known result (see [1, p. 744]). Again, if $y = n$, there is no solution for $1 \leqslant x \leqslant n$, a well known result (see [1, p. 744]). Our proof can also be used to establish the following general result.

Theorem 2. For $n \geqslant m \geqslant 2$ and integers $A \geqslant 1$, $B \geqslant 1$, the equation

$$Ax^n + By^m = Bz^m$$

has no solution whenever $Ax^{n-m+1} + Bmy \leqslant Bmz$.

REMARK. Theorem 2 gives Theorem 1 for $A = B$ and $n = m$.

REFERENCE

1. L.E. Dickson, *History of the Theory of Numbers,* Vol. 2, Diophantine Analysis, Carnegie Institute of Washington, 1919, Reprint by Chelsea, 1952.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-292 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain and prove a formula for the number $S(n,t)$ of terms in

$$(x_1 + x_2 + \cdots + x_n)^t,$$

where n and t are integers with $n > 0$ and $t \geq 0$.

B-293 Proposed by Harold Don Allen, Nova Scotia Teachers College, N.S., Canada.

Identify T, W, H, R, E, F, I, V, and G as distinct digits in $\{1, 2, \cdots, 9\}$ such that we have the following sum (in which 1 and 0 are the digits 1 and 0):

$$\begin{array}{r} 1 \\ 1 \\ TWO \\ THREE \\ \underline{FIVE} \\ EIGHT \end{array}$$

B-294 Proposed by Richard Blazej, Queens Village, New York.

Show that $F_n L_k + F_k L_n = 2F_{n+k}$ for all integers n and k.

B-295 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Find a closed form for

$$\sum_{k=1}^{n} (n+1-k)F_{2k} = nF_2 + (n-1)F_4 + \cdots + F_{2n}.$$

B-296 Proposed by Gary Ford, Vancouver, British Columbia, Canada.

Find constants a and b and a transcendental function G such that

$$G(y_{n+3}) + G(y_n) = G(y_{n+2})G(y_{n+1})$$

whenever y_n satisfies

$$y_{n+2} = ay_{n+1} + by_n.$$

B-297 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Obtain a recursion formula and a closed form in terms of Fibonacci and Lucas numbers for the sequence $\{G_n\}$ defined by the generating function:

$$(1-3x-x^2+5x^3+x^4-x^5)^{-1} = G_0+G_1x+G_2x^2+\cdots+G_nx^n+\cdots .$$

SOLUTIONS

FIBONACCI COMPLEX NUMBERS

B-268 Proposed by Warren Cheves, Littleton, North Carolina.

Define a sequence of complex numbers $\{C_n\}$, $n = 1, 2, \cdots$, where $C_n = F_n + iF_{n+1}$. Let the conjugate of C_n be $\bar{C}_n = F_n - iF_{n+1}$. Prove

(a) $$C_n\bar{C}_n = F_{2n+1};$$

(b) $$C_n\bar{C}_{n+1} = F_{2n+2} + (-1)^n i.$$

Solution by J. L. Hunsucker, University of Georgia, Athens, Georgia.

In solving this problem we quote identities by number from V.E. Hoggatt's *Fibonacci and Lucas Numbers.*

First

$$C_n\bar{C}_n = F_n^2 + (F_{n+1})^2 = F_{2n+1}$$

by I_{11} in Hoggatt and (a) is proved. Second,

$$C_n\bar{C}_{n+1} = (F_nF_{n+1} + F_{n+1}F_{n+2}) + i(F_{n+1}^2 - F_nF_{n+2}) .$$

Then

$$F_nF_{n+1} + F_{n+1}F_{n+2} = F_{n+1}(F_n + F_{n+2}) = (F_{n+2} - F_n)(F_{n+2} + F_n) = F_{n+2}^2 - F_n^2 = F_{2n+2}$$

by I_{10}. Also, by I_{13},

$$F_{n+1}^2 - F_nF_{n+2} = (-1)(-1)^{n+1} = (-1)^n ,$$

and (b) is proved.

Also solved by Wray G. Brady, Herta T. Freitag, Ralph Garfield, John W. Milsom, C.B.A. Peck, M.N.S. Swamy, P. Thrimurthy, Gregory Wulczyn, David Zeitlin, and the Proposer.

DIAGONALIZING THE Q MATRIX

B-269 Proposed by Warren Cheves, Littleton, North Carolina.

Let Q be the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The eigenvalues of Q are α and β, where

$$\alpha = (1+\sqrt{5})/2 \qquad \text{and} \qquad \beta = (1-\sqrt{5})/2 .$$

Since the eigenvalues of Q are distinct, we know that Q is similar to a diagonal matrix A Show that A is either

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} .$$

Solution by P. Thrimurthy, Gujarat University, Ahmedabad, India.

The eigenvectors corresponding to the two eigenvalues α and β are

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta \\ 1 \end{pmatrix} ,$$

respectively. Hence the transforming matrix is either

$$P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad R = \begin{pmatrix} \beta & \alpha \\ 1 & 1 \end{pmatrix} .$$

Now

$$P^{-1}QP = \begin{pmatrix} 1/(\alpha-\beta) & -\beta/(\alpha-\beta) \\ -1/(\alpha-\beta) & \alpha/(\alpha-\beta) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad R^{-1}QR = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} .$$

Also solved by David Zeitlin and the Proposer.

A MULTIPLE OF L_{2m+1}

B-270 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is odd,

$$L_k \,\big|\, [F_{(n+2)k} - F_{nk}] .$$

Solution by C.B.A. Peck, State College, Pennsylvania.

$$F_{(n+2)k} - F_{nk} = L_k F_{(n+1)k}$$

for k odd (see *The Fibonacci Quarterly,* Vol. 7, No. 5 (Dec. 1969), p. 486).

Also solved by W.G. Brady, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIND THE MULTIPLE OF $L_{2m} - 2$

B-271 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is even, $L_k - 2$ is an exact divisor of:

(a) $$F_{(n+2)k} + 2F_k - F_{nk} \;;$$

(b) $$F_{(n+2)k} - 2F_{(n+1)k} + F_{nk} \,;$$

and

(c) $$2[F_{(n+2)k} - F_{(n+1)k} + F_k] .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since

$$F_{(n+2)k} - L_k F_{(n+1)k} + (-1)^k F_{nk} = 0 ,$$
$$F_{(n+2)k} - 2F_{(n+1)k} + (-1)^k F_{nk} = (L_k - 2)F_{(n+1)k} .$$

For k even,

$$(L_k - 2) \mid (F_{(n+2)k} - 2F_{(n+1)k} + F_{nk})$$

and (b) is true. (a) False. For $n = 0$,

$$(L_k - 2) \nmid (F_{2k} + 2F_k)$$

when $k = 4$. (c) False. For $n = 0$,

$$(L_k - 2) \nmid 2F_{2k}$$

when $k = 4$.

Also solved by C.B.A. Peck, Gregory Wulczyn, and the Proposer.

A NONLINEAR RECURRENCE

B-272 Proposed by Gary G. Ford, Vancouver, British Columbia, Canada.

Find at least some of the sequences $\{ y_n \}$ satisfying

$$y_{n+3} + y_n = y_{n+2}y_{n+1} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Three solutions are given by:

(1) $y_n = 0$ for all n.

(2) $y_n = 2$ for all n.

(3) Let b denote a parameter, independent of n. Then one may let $y_{4m} = b$, $y_{4m+1} = -1 = y_{4m+3}$, $y_{4m+2} = 1 - b$, for all integers m.

NOTE: Herta T. Freitag and P. Thrimurthy each pointed out that any three consecutive terms may be chosen arbitrarily, and then the recurrence determines the other terms. Another version of this problem is proposed in this issue as B-296.

GOLDEN MINIMUM PERIMETER

B-273 Proposed by Marjorie Bicknell, A.C. Wilcox High School, Santa Clara, California.

Construct any triangle $\triangle ABC$ with vertex angle $A = 54°$ and median $\overline{AM}$ to the side opposite A such that $AM = 1$. Now, inscribe $\triangle XYM$ in $\triangle ABC$ so that M is the midpoint of $\overline{BC}$, and X and Y lie between A and B and between A and C, respectively. Find the minimum perimeter possible for the inscribed triangle, $\triangle XYM$.

Solution by the Proposer.

Construct $\angle MAB \cong \angle M'AB$, $AM' = AM$; $\angle MAC \cong \angle M''AC$, $AM'' = AM$. Then draw $M'M''$, intersecting $\overline{AB}$ at X and $\overline{AC}$ at Y. Since, by S.A.S., $\triangle M'AX \cong \triangle MAX$ and $\triangle M''AY \cong \triangle MAY$, $MX + XY + YM = M'X + XY + YM''$, which is a minimum when M', X, Y, and M'' are colinear. So, the minimum perimeter is given by the length $M'M''$. Also, $\angle M'AM'' = 2A$. (This construction was given by Samuel L. Greitzer, Rutgers University, as solution to a problem appearing in *Summation*, Association of Teachers of Mathematics of New York City, Spring, 1972.)

By the Law of Cosines,

$$(M'M'')^2 = (AM')^2 + (AM'')^2 - 2(AM')(AM'')\cos 2A = 2(AM)^2(1-\cos 2A) = 4(AM)^2\sin^2 A.$$

Thus, $M'M'' = 2(AM)(\sin A)$.

Now, that $\sin 54° = (1+\sqrt{5})/4 = \phi/2$ is easily seen from the following:

$$\sin 36° = \cos 54° = \cos 18° \cos 36° - \sin 18° \sin 36°$$

$$2\cos 18° \sin 18° = \cos 18° \cos 36° - 2\sin^2 18° \cos 18°$$

leading to

$$4\sin^2 18° + 2\sin 18° - 1 = 0$$

so that

$$\sin 18° = (\sqrt{5}-1)/4 = 1/2\phi \,.$$

Then,

$$\cos 36° = 1 - 2\sin^2 18° = (1+\sqrt{5})/4 = \phi/2 = \sin 54° \,.$$

Therefore, the minimum perimeter is given by

$$M'M'' = 2(AM)(\sin 54°) = 2(1)(\phi/2) = \phi \,,$$

the Golden Section Ratio.

Notice that nowhere was the fact that $\overline{AM}$ was a median required. If M is any point between B and C such that $AM = 1$, we have the same minimum perimeter.

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VOLUME INDEX

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