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CONTENTS

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A FAREY SEQUENCE OF FIBONACCI NUMBERS

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The Farey sequence is an old and famous set of fractions associated with the integers. We here show that if we form a Farey sequence of Fibonacci Numbers, the properties of the Farey sequence are remarkably preserved (see [2]). In fact we find that with the new sequence we are able to observe and identify "points of symmetry," "intervals," "generating fractions" and "stages." The paper is divided into three parts. In Part 1, we define "points of symmetry," "intervals" and "generating fractions" and discuss general properties of the Farey sequence of Fibonacci numbers. In Part 2, we define conjugate fractions and deal with properties associated with intervals. Part 3 considers the Farey sequence of Fibonacci numbers as having been divided into stages and contains properties associated with "corresponding fractions" and "corresponding stages." A generalization of the Farey sequence of Fibonacci numbers is given at the end of the third part.

The Farey sequence of Fibonacci numbers of order F_n (where F_n stands for the n^{th} term of the Fibonacci sequence) is the set of all possible fractions F_i/F_j, $i = 0, 1, 2, 3, \cdots, n-1$, $j = 1, 2, 3, \cdots, n$ $(i < j)$ arranged in ascending order of magnitude. The last term is $1/1$, i.e., F_1/F_2. The first term is $0/F_{n-1}$. We set $F_0 = 0$ so that $F_0 + F_1 = F_2$, $F_1 = F_2 = 1$.

For convenience we denote a Farey sequence of Fibonacci numbers by $f{\cdot}f$, that of order F_n by $f{\cdot}f_n$ and the r^{th} term in the new Farey sequence of order F_n by $f_{(r)n}$.

PART 1

DEFINITION 1.1. Besides $1/1$ we define an $f_{(r)n}$ to be a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same denominator. We have shown in an appendix the Farey sequence of all Fibonacci numbers up to *34.*

DEFINITION 1.2. We define an interval to be set of all $f{\cdot}f_n$ fractions between two consecutive points of symmetry. The interval may be closed or open depending upon the inclusion or omission of the points of symmetry. A closed interval is denoted by [ ] and an open interval by ( ).

DEFINITION 1.3. The distance between $f_{(r)k}$ and $f_{(k)n}$ is equal to $|r-k|$.

Theorem 1.1. If $f_{(r)n}$ is a point of symmetry then it is of the form $1/F_i$. Moreover $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same denominator if they do not pass beyond the next point of symmetry on either side. The converse is also true.

Proof. In the $f{\cdot}f$ sequence the terms are arranged in the following fashion. The terms in the last interval are of the form F_{j-1}/F_j. The terms in the interval prior to that last are of the form $F_{j-2}/F_j \cdots$. If there are two fractions F_{i-1}/F_{j-1} and F_{i-2}/F_{j-2} then their mediant* F_i/F_j lies in between them. That is,

$$\text{if} \quad \frac{F_{i-1}}{F_{j-1}} < \frac{F_{i-2}}{F_{j-2}} \qquad \text{then} \quad \frac{F_{i-1}}{F_{j-1}} < \frac{F_i}{F_j} < \frac{F_{i-2}}{F_{j-2}}$$

$$\text{if} \quad \frac{F_{i-2}}{F_{j-2}} < \frac{F_{i-1}}{F_{j-1}} \qquad \text{then} \quad \frac{F_{i-2}}{F_{j-2}} < \frac{F_i}{F_j} < \frac{F_{i-1}}{F_{j-1}} .$$

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*If $a/b < c/d$, then $(a+c)/(b+d)$ is the *mediant* fraction to those two fractions.

This inequality can easily be established dealing with the two cases separately.

We shall adopt induction as the method of proof. Our surmise has worked for all $f \cdot f$ sequences up to 34. Let us treat 34 as F_{n-1}. For the next $f \cdot f$ sequence, i.e., of order F_n, fractions to be introduced are:

$$\frac{F_2}{F_n}, \frac{F_3}{F_n}, \cdots, \frac{F_i}{F_n}, \cdots, \frac{F_{n-1}}{F_n}.$$

F_i / F_n will fall in between

$$\frac{F_{i-1}}{F_{n-1}} \quad \text{and} \quad \frac{F_{i-2}}{F_{n-2}}.$$

First assume that $F_{i-1}/F_{n-1} < F_{i-2}/F_{n-2}$. Since our assumption is valid for 34, F_{i-1}/F_{n-1} lies just before F_{i-2}/F_{n-2}. F_{i-3}/F_{n-2} will occur just after F_{i-2}/F_{n-1} from our assumption regarding points of symmetry. But F_{i-1}/F_n lies in between these two fractions. The distance of F_{i-1}/F_n from the point of symmetry, say $1/F_j$, is equal to the distance F_i/F_n from that point of symmetry. Hence this is valid for 55. Similarly it can be made to hold good for 89, $\cdots$. Hence the theorem.

Theorem 1.2. Whenever we have an interval $[1/F_i, 1/F_{i-1}]$ the denominator of term next to $1/F_i$ is F_{i+2}, and the denominator of the next term is F_{i+4}, then $F_{i+6} \cdots$. We have this until we reach the maximum for that $f \cdot f_n$ sequence, i.e., so long as F_{i+2k} does not exceed F_n. Then the denominator of the term after F_{i+2k} will be the maximum possible term not greater than F_n, but not equal to any of the terms formed, i.e., it's either F_{i+2k+1} or F_{i+2k-1}, say F_j. The denominator of the terms after F_j will be $F_{j-2}, F_{j-4}, \cdots$ till we reach $1/F_{i-1}$. (As an example let us take $[1/3, 1/2]$ in the $f \cdot f$ sequence for 55. Then the denominator of the terms in order are 3, 8, 21, 55, 34, 13, 5, 2).

Proof. The proof of Theorem 1.2 will follow by induction on Theorem 1.1.

Theorem 1.3. (a) If $h/k, h'/k', h''/k''$ are three consecutive fractions of an $f \cdot f$ sequence then

$$\frac{h+h''}{k+k''} = \frac{h'}{k'}$$

if h'/k' is not a point of symmetry.

(b) If h'/k' is a point of symmetry, say $1/F_i$, then

$$\frac{F_{i-2}h + F_{i-1}h''}{F_{i-2}k + F_{i-1}k''} = \frac{h'}{k'}.$$

Proof. Case 1. (From Theorem 1.2) We see that

$$\frac{h}{k} = \frac{F_{i-2}}{F_{j-2}}, \frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h''}{k''} = \frac{F_{i+2}}{F_{j+2}}.$$

In this case

$$\frac{F_{i+2}+F_{i-2}}{F_{j+2}+F_{j-2}} = \frac{{}^*3 \cdot F_i}{3 \cdot F_j} = \frac{F_i}{F_j} = \frac{h'}{k'}.$$

($^*F_{n+2} + F_{n-2} = 3F_n$ is a property of the Fibonacci sequence. See Hoggatt [1].)

Case 2.

$$\frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h}{k} = \frac{F_{i-2}}{F_{j-2}} \quad \text{and} \quad \frac{h''}{k''} = \frac{F_{i+1}}{F_{j+1}}$$

(from Theorem 1.2). Then

$$\frac{F_{i+1}+F_{i-2}}{F_{j+1}+F_{j-2}} = \frac{2F_i}{2F_j} = \frac{F_i}{F_j} = \frac{h'}{k'}$$

similarly.

Case 3.

$$\frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h}{k} = \frac{F_{i-2}}{F_{j-2}}, \frac{h''}{k''} = \frac{F_{i-1}}{F_{j-1}}$$

(from Theorem 1.2). Therefore

$$\frac{F_{i-1}+F_{i-2}}{F_{j-1}+F_{j-2}} = \frac{F_i}{F_j} = \frac{h'}{k'} .$$

Hence the result.

Proof of 1.3b. Let $h'/k' = 1/F_i$. From Theorem 1.2 it follows that $h''/k'' = 3/F_{i+2}$ and $h/k = 2/F_{i+2}$. Therefore

$$\frac{F_{i-2}h + F_{i-1}h''}{F_{i-2}k + F_{i-1}k''} = \frac{2F_{i-2}+3F_{i-1}}{F_iF_{i+2}} = \frac{F_{i+2}}{F_iF_{i+2}} = \frac{1}{F_i} .$$

Hence the theorem.

Theorem 1.4. If h/k, and h'/k' are two consecutive fractions of an $f\cdot f_n$ sequence then

$$\left|\frac{h-h'}{k-k'}\right| \in f\cdot f_n \qquad (k-k' \neq 0).$$

Proof. Since $f_{(r)n}$ is of the form F_i/F_j, if Theorem 1.4 is to hold, then it is necessary that $|h-h'|$ be equal to F_i and $|k-k'|$ be equal to F_j. Since h/k and h'/k' are members also,

$$h = F_{i_1} \quad , \quad h' = F_{i_2} \quad , \quad k = F_{j_1} \quad , \quad k' = F_{j_2} \quad .$$

Further

$$|F_{j_1} - F_{j_2}| = F_j \qquad \text{and} \qquad |F_{i_1} - F_{i_2}| = F_i \ .$$

But from the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$ we see that the condition for this is $|i_i - i_2| \leqslant 2$ and $|j_1 - j_2| \leqslant 2$ (but not zero) which follows from Theorem 1.2. Actually

$$\left|\frac{h-h'}{k-k'}\right|$$

are the fractions of the same interval arranged in descending order of magnitude for increasing values of h/k.

Definition 1.4. We now introduce a term "Generating Fraction." If we have a fraction F_i/F_j $(i < j)$. We split F_i/F_j into

$$\frac{F_{i-1}+F_{i-2}}{F_{j-1}+F_{j-2}} .$$

We form from this two fractions F_{i-1}/F_{j-1} and F_{i-2}/F_{j-2} such that F_i/F_j is the mediant of the fractions formed. We continue this process and split the fractions obtained till we reach a state where the numerator is 1. F_i/F_j then amounts to the Generating fraction of the others. We call F_i/F_j as the Generating Fraction of an Interval (G.F.I.) if through this process we are able to get from the G.F.I. all the other fractions of "that" closed interval. We can clearly see a $f\cdot f$ sequence for $F_1, F_2, \cdots, F_n$, F_i/F_n will be a G.F.I. (We also note that F_i/F_j, F_{i-1}/F_{j-1}, $F_{i-2}/F_{j-2}, \cdots$ belong to the same interval because the difference in the suffix of the numerator and denominator is $j-i$). Hence the sequence G.F.I.'s is $F_1/F_n, F_2/F_n, F_3/F_n, \cdots, F_{n-1}/F_n$. We now see some properties concerning G.F.I.'s.

Theorem 1.5. If we form a sequence of the distance between two consecutive G.F.I.'s such a sequence runs thus: $2, 2, 4, 4, 6, 6, 8, 8, \cdots$, i.e., alternate G.F.I.'s are symmetrically placed about a G.F.I.

Theorem 1.6. If we take the first G.F.I., say $f_{(g_1)n}$, then $f_{(g_1+1)n}$ and $f_{(g_1-1)n}$, have the same denominator. For $f_{(g_2)n}$ the second G.F.I. $f_{(g_2+2)n}$, and $f_{(g_2-2)n}$ have the same denominator. In general for $f_{(g_k)n}$ the k^{th} G.F.I. $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator.

The proofs of theorems 1.5 and 1.6 follow from 1.2.

(NOTE: We can verify that for alternate G.F.I.'s $g_{(g_2)n}, f_{(g_4)n}, f_{(g_6)n}, \cdots, f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator for k is even and the sequence of distance shown above is $2, 2, 4, 4, 6, 6, 8, 8, \cdots$).

PART 2

Definition 2.1. We now define F_{i-2} to be the "factor of the interval"

$$\left[\frac{1}{F_1}, \frac{1}{F_{i-1}}\right] .$$

More precisely the factor of a closed interval is that terms F_z where z is suffix of denominator minus suffix of the numerator, of each fraction of that interval. It can be easily seen (Part 1) that z is a constant.

Lemma 2.1. If $j_1 - i_1 = j_2 - i_2 > 0$, then

$$|F_{j_1}F_{i_2} - F_{j_2}F_{i_2}| = |F_{j_2} - F_{j_1}|\,|F_{j_1} - F_{i_1}| = |F_{j_2} - F_{j_1}|\,|F_{j_2} - F_{i_2}|\,.$$

Proof. We apply Binet's formula that

$$F_n = \frac{a^n - b^n}{a-b}\,,$$

where

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2}\,.$$

Then the left-hand side (L.H.S.) of the expression and the right-hand side (R.H.S.) of the expression reduce as follows.

To prove

$$\left|\frac{a^{j_1} - b^{j_1}}{a-b}\cdot\frac{a^{i_2} - b^{i_2}}{a-b} - \frac{a^{j_2} - b^{j_2}}{a-b}\cdot\frac{a^{i_1} - b^{i_1}}{a-b}\right| = \left|\frac{a^{j_2-j_1} - b^{j_2-j_1}}{a-b}\right|\frac{a^{j_1-i_1} - b^{j_1-i_1}}{a-b}$$

because $j_1 - i_1 > 0$, $F_{j_1-i_1}$ is positive and hence can be put within the $|\;|$ sign.

To prove

$$|(a^{j_1} - b^{j_1})(a^{i_2} - b^{i_2}) - (a^{j_2} - b^{j_2})(a^{i_1} - b^{i_1})| = |(a^{j_2-j_1} - b^{j_2-j_1})(a^{j_1-i_1} - b^{j_1-i_1})|$$

the L.H.S. reduces to

$$|a^{j_1+i_2} - a^{j_1}b^{i_2} + b^{j_1+i_2} - b^{j_1}a^{i_2} - a^{j_2+i_1} + a^{j_2}b^{i_1} + b^{j_2}a^{i_1} - b^{j_2+i_1}|$$

$$= |-a^{j_1}b^{i_2} - a^{i_2}b^{j_1} + a^{j_2}b^{i_1} + b^{j_2}a^{i_1}|\,.$$

The R.H.S. reduces to

$$|a^{j_2-i_1} - a^{j_2-j_1}b^{j_1-i_1} + b^{j_2-i_1} - b^{j_2-i_1}a^{j_1-i_1}|\,.$$

This may be simplified further using $ab = -1$ and $j_1 - i_1 = j_2 - i_2$. The R.H.S. is then

$$|a^{j_1}b^{i_2} + b^{j_1}a^{i_2} - a^{j_2}b^{i_1} - b^{j_2}a^{i_1}|\,.$$

We see that L.H.S. = R.H.S. Hence the Lemma.

Corollary. From this we may deduce that if F_{i_1}/F_{j_1} and F_{i_2}/F_{j_2} belong to the same interval, i.e., $j_1 - i_1 = j_2 - i_2$, then

$$F_{j_1}F_{i_2} - F_{j_2}F_{i_1} = F_{|j_2-j_1|}F_{j_2-i_2} = F_{|j_2-j_1|}F_{j_1-i_1}$$

$$F_{i_1}/F_{j_1} < F_{i_2}/F_{j_2}\,.$$

Hence

$$|F_{j_1}F_{i_2} - F_{j_2}F_{i_1}|$$

will be an integral multiple of $F_{j_1-i_1}$ or $F_{j_2-i_2}$ (the factor of that interval) which is the term obtained by the difference in suffixes of the numerator and denominator of each fraction of that interval.

Definition 2.2 We now introduce the term "conjugate fractions." Two fractions h/k and h'/k', h/k and h'/k' are conjugate in an interval

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

if the distance of h/k from $1/F_i$ equals the distance of h'/k' from $1/F_{i-1}$ $(h/k \neq h'/k')$.

Corollary. Two consecutive points of symmetry are conjugate with distance zero.

Theorem 2.2. If h/k and h'/k' are conjugate $[1/F_1, 1/F_{i-1}]$ then $kh' - kh' = F_{i-2}$.

Proof. From Part 1, we can easily see that if h/k is of the form

$$\frac{F_{i_1}}{F_{j_1}} \quad \text{then } h'/k' \text{ is } \quad \frac{F_{i_1-1}}{F_{j_1-1}} \quad \cdots \qquad (*)$$

$1/F_i$, and $1/F_{i-1}$ are conjugate. This agrees with (*) since $F_2 = F_1 = 1$. Since the term after $1/F_i$ is F_4/F_{i+2} and the term before $1/F_{i-1}$ is $2/F_{i+1}$, we see it agrees with the statement (*) above. Proceeding in such a fashion we obtain the result (*). Of course we assume here that there exist at least two terms in

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right] .$$

Hence we can see that any two conjugate to fractions in

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

are given by

$$\frac{F_{j-i+2}}{F_j}, \frac{F_{j-i+1}}{F_{j-1}} .$$

We are required to show $|F_jF_{j-1+1} - F_{j-1}F_{j-i+2}| = F_{i-2}$. This will immediately follow from Lemma 2.1.

Theorem 2.3. (a) If h/k and h'/k' are two consecutive fractions in an $f\cdot f_n$ sequence, which belong to $[1/F_i, 1/F_{i-1}]$, then $kh' - hk' = F_{i-2}$.

(b) If h/k and h'/k' are conjugate in an interval $[1/F_i, 1/F_{i-1}]$ $kh' - hk' = F_{i-2}$.

Proof. Theorem 2.3a and 2.3b can be proved using Lemma and Theorem 1.2.

Definition 2.3. If

$$\frac{h}{k} \in \left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) ,$$

we define the couplet for h/k as the ordered pair

$$\left[\left(\frac{1}{F_i}, \frac{h}{k}\right), \left(\frac{h}{k}, \frac{1}{F_{i-1}}\right)\right] .$$

Theorem 2.4. In the case of couplets we find that

$$(F_ih) - k = F_pF_{i-2}$$

and

$$k - F_{i-1}h = F_{p+1}F_{i-2} ,$$

where F_p is some Fibonacci number.

Proof. Let h/k be

$$\frac{F_{j-i+2}}{F_j} .$$

Then $(F_ih) - k$ is

(1) $$F_iF_{j-i+2} - F_j = F_pF_{i-2}$$

and let $k - F_{i-1}h$ is

(2) $$F_j - F_{i-1}F_{j-1+2} = F_{p+1}F_{i-2} \cdot$$

Adding (1) and (2) we have

$$F_{i-2}F_{j-1+2} = F_{p+2}F_{i-2} .$$

Therefore $F_{j-i+2} = F_{p+2}$ or $j - i = p$; i.e.,

(3) $$F_iF_{j-i+2} - F_j = F_{j-i}F_{i-2} .$$

We can establish (3) using Lemma 2.1. Hence the proof.

Definition 2.4. We define

$$\left[\left(\frac{1}{F_i}, \frac{h}{k}\right) \left(\frac{h}{k}, \frac{1}{F_{i-1}}\right)\right]$$

and

$$\left[\left(\frac{1}{F_i}, \frac{h'}{k'}\right)\left(\frac{h'}{k'}, \frac{1}{F_{i-1}}\right)\right]$$

to be conjugate couplets if h/k and h'/k' are conjugate fractions of the closed interval

$$\left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) .$$

Theorem 2.5. In the case of conjugate couplets if

$$F_ih - k = F_pF_{i-2} \quad \text{and} \quad k - F_{i-1}h = F_{p+1}F_{i-2} ,$$

then

$$F_ih' - k' = F_{p-1}F_{i-2} \quad \text{and} \quad k - F_{i-1}h' = F_pF_{i-2} .$$

Proof. We note that $(j - i)$ in the previous proof is the difference in the suffixes of F_j and F_i. If now

$$h/k = \frac{F_{j-i+2}}{F_j}$$

then $p = j - i$. But since h'/k' is conjugate with h/k,

$$h'/k' = \frac{F_{j-i+1}}{F_{j-1}} .$$

Therefore the constant factor, say F_q in the equation for h'/k', $F_ih' - k = F_qF_{i-2}$ is such that

$$q = j - 1 - i = (j - i) - 1 = p - 1 .$$

Therefore $F_ih' - k' = F_{p-1}F_{i-2}$. Hence $k - F_{i-1}h' = F_pF_{i-2}$ since it follows from Theorem 2.4.

Theorem 2.6. Since we have seen that if h/k and h'/k' are conjugate then the difference in suffixes of their numerators or denominators equals 1, we find

$$\frac{h+h'}{k+k'} \in \left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right] \quad \text{and} \quad \left|\frac{h-h'}{k-k'}\right| \in \left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

if

$$h/k, h'/k' \in \left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) .$$

Moreover

$$\frac{h+h'}{k+k'}$$

are the fractions of the latter half of the interval arranged in descending order while

$$\left|\frac{h-h'}{k-k'}\right|$$

are the fractions of the first half arranged in ascending order, for increasing values of h/k.

PART 3

We now give a generalized result concerning "sequence of distances."

Theorem 3.1a. Points of symmetry if they are of the form $f_{(r)n}$ then

$$r \in \left\{2, 3, 5, 8, 12, 17, \cdots\right\} .$$

Or the sequence of distance between two consecutive points of symmetry will be

$$1, 2, 3, 4, 5, 6, \cdots ,$$

an Arithmetic progression with common difference 1.

Theorem 3.1b. The sequence of distance for fractions with common numerator F_{2n-1} or F_{2n} is

$$2n - 1, 2n, 2n + 1, \cdots .$$

Proof. To prove Theorem 3.1a we have to show that if there are n terms in an interval then there are $(n + 1)$ terms in the next.

Let there be p terms of the form F_i/F_j. It is evident that there are $p+1$ terms of the form F_{i+1}/F_j. But these $(p+1)$ terms ot the form F_{i+1}/F_j are in an interval next to that in which the p terms of the form F_i/F_j lie. So the sequence is an AP with common difference 1. Moreover, the second term is always $1/F_n$ (evident). Hence the result. (Note: $j-i$ is assumed constant.)

If we fix the numerator to be 2 and take the sequence

$$\frac{2}{F_n}, \frac{2}{F_{n-1}}, \frac{2}{F_{n-2}}, \cdots, \frac{2}{3}$$

then the sequence of distance between two consecutive such fractions is $3, 4, 5, \cdots$.

From Theorem 1.2 (Part 1) it follows that $2/F_i$ lies just before a point of symmetry, say $1/F_j$. Since we have seen the sequence of distances concerning points of symmetry it will follow that here too the common difference is 1. The first term is 3 for there are two terms between $2/F_n$ and $2/F_{n-1}$. The inequality

$$\frac{2}{F_n} < \frac{1}{F_{n-2}} < \frac{3}{F_n} < \frac{2}{F_{n-1}}$$

can be established. Hence the result.

In a similar fashion we find that the sequence of distance for numerator 3 is $3, 4, 5, \cdots$.

We shall give a table and the generalization

Numerator	Sequence of Distance
F_1 or F_2	$1, 2, 3, 4, 5, \cdots$
F_3 or F_4	$3, 4, 5, 6, \cdots$
F_5 or F_6	$5, 6, 7, 8, \cdots$
F_{2n-1} or F_{2n}	$2n-1, 2n, 2n+1, 2n+2, \cdots$.

Definition 3.1. Just as we defined an interval, we now define a "stage" as the set of $f \cdot f$ fractions lying between two consecutive G.F.I.'s. The stage may be closed or open depending upon the inclusion or omission of the G.F.I.'s.

Since the sequence of distance of G.F.I.'s is $2, 2, 4, 4, 6, 6, \cdots$, it is possible for two consecutive "stages" to have equal numbers of terms. We define two stages:

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right] \qquad \text{and} \qquad \left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

to be conjugate stages if the distance of F_i/F_n from F_{i-1}/F_n equals the distance of F_{i+1}/F_n from F_i/F_n. That is the number of terms in two conjugate stages are equal. We call a stage comparison of both these stages as a "complex stage." Let us now investigate properties concerning stages. If we have complex stage

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

then we define two fractions h/k and h'/k' to be "corresponding" if

$$\frac{h}{k} \in \left(\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right)$$

and

$$\frac{h'}{k'} \in \left(\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right)$$

and if the distance of h/k from F_{i-1}/F_n is equal to the distance of h'/k' from F_i/F_n.

Theorem 3.2. Two corresponding fractions have the same numerator. If h/k and h'/k' are corresponding fractions then $h = h'$.

Proof. This will follow from 1.2 (part 1).

Let F_{i-1}/F_n be the maximum reached in its interval so that F_{i-1}/F_{n-1} will be the maximum for the interval in which F_i/F_n belongs. (where by maximum we mean the term with denominator F_{i+2k} in the sense of Theorem 1.2). The term next to F_{i-1}/F_n is F_{i-2}/F_{n-1}. Similarly the term next to F_i/F_n is F_{i-2}/F_{n-2}. But these fractions are corresponding in such a fashion that we obtain the result.

Now F_{i-1}/F_n has necessarily to be the maximum in its interval. Since we have considered conjugate stages i is odd. Using Theorem 1.2 it can be established that alternate G.F.I.'s are maximum in their interval and that too, when suffix of numerator is even $(i-1$ is even).

Definition 3.2. Since the number of terms in a stage is odd, we define h/k to be the middle point of a stage

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right]$$

if it is equidistant from both G.F.I.'s. We can deduce from this that h/k is a point of symmetry since F_{i-1}/F_n, and F_i/F_n have the same denominator. So the middle point of a stage is a point of symmetry.

Corollary. If two conjugate stages are taken then their middle points are corresponding. (This follows from the definition). But their numerators should be equal. This is so, for the middle points are points of symmetry whose numerator is 1. This agrees with the result proved.

Definition 3.3. Two fractions h/k and h'/k' are conjugate in a complex stage if the distance of h/k from F_{i-1}/F_n equals the distance of h'/k' from F_{i+1}/F_n, $h/k < h'/k'$ and the complex stage being

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

Taking their middle points

$$\left[\frac{1}{F_p}, \frac{1}{F_{p+1}}\right]$$

we can see that fractions conjugate in this interval are conjugate in the complex stage. Further we saw that for conjugate fractions of the interval, h/k, h'/k',

$$\frac{h+h'}{k+k'}$$

re fractions of the latter half of the interval arranged in descending order, and

$$\left|\frac{h-h'}{k-k'}\right|$$

are fractions of the first half arranged in ascending order for increasing values of h/k.

Theorem 3.3. For conjugate fractions h/k and h'/k' lying in the outer half of the stage we see that

$$\frac{h+h'}{k+k'}$$

are fractions of the latter half of the interval in ascending order while

$$\left|\frac{h-h'}{k-k'}\right|$$

are fractions of the first half in descending order for increasing values of h/k. We here only give a proof to show that

$$\frac{h+h'}{k+k'} \quad \text{and} \quad \frac{h-h'}{k-k'}$$

are in the interval but do not prove the order of arrangement.

Proof. For h/k, h'/k', in the inner half the proof has been given (previous part). The middle point of

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right]$$

is $1/F_{n-i+2}$. Similarly the middle point of

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

is $1/F_{n-i+2}$. That two conjugate fractions of the outer half of a conjugate stage differ in suffix by 1 can be established. That is, to say, if

$$\frac{h}{k} = \frac{F_{j-(n-i)-1}}{F_j}$$

then

$$\frac{h'}{k'} = \frac{F_{j-(n-i)}}{F_{j-1}} \qquad \frac{h+h'}{k+k'} = \frac{F_{j-(n-i)+1}}{F_{j+1}} \in I,$$

where I is the interval $[1/F_p, 1/F_{p+1}]$ and

$$\frac{h-h'}{k-k'} = \frac{F_{j-(n-i)-2}}{F_{j-2}} \in I.$$

Hence the proof.

Definition 3.4. In an $f{\cdot}f$ sequence of order F_n, $[F_i/F_n, F_{i+1}/F_n]$ represents a stage. Let us take an $f{\cdot}f$ sequence of order F_{n+1}. If there we take a stage $[F_i/F_{n+1}, F_{i+1}/F_{n+1}]$, then we say the two stages are corresponding stages. More generally in an $f{\cdot}f$ sequence of order F_n and an $f{\cdot}f$ sequence of order F_{n+k},

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right], \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

are corresponding stages. We stage here properties of corresponding stages. These can be proved using Theorem 1.2.

Theorem 3.4a. If

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right] \quad \text{and} \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

are corresponding stages then the number of terms in both are equal.

Theorem 3.4b. There exists a one-one correspondence between the denominators of these stages. If the denominator of the q^{th} term of $[F_i/F_n, F_{i+1}/F_n]$ is F_j then the denominator of the q^{th} term of

$$\left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

is F_{j+k}.

We can extend this idea further and produce a one-one correspondence between

$$\left[\frac{F_i}{F_n}, \frac{F_{i+m}}{F_n}\right] \quad \text{and} \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+m}}{F_{n+k}}\right], \quad \text{where} \quad \left[\frac{a}{b}, \frac{c}{d}\right]$$

stands for the set of fractions between a/b and c/d inclusive of both. A further extension would give that given two $f{\cdot}f$ sequences, one of order F_n, and the other of order F_{n+k}.

Theorem 3.5a. The numerator of the r^{th} term of the first sequence equals the numerator of the r^{th} term of the second.

Theorem 3.5b. If the denominator of the r^{th} term of the first sequence is F_j, *then the denominator of the* r^{th} term of the second series is F_{j+1}. Precisely

(a) the numerator of $f_{(r)n}$ is equal to the numerator of $f_{(r)n+k}$

(b) if the denominator of $f_{(r)n} = F_j$, the denominator of $f_{(r)n+k} = F_{j+k}$.

This can be proved using 1.2. We can arrive at the same result by defining corresponding intervals.

Definition 3.5. Two intervals, $[1/F_i, 1/F_{i+1}]$ in an $f{\cdot}f$ sequence of order F_n and $[1/F_{i+k}, 1/F_{i+k}]$ in an $f{\cdot}f$ sequence of order F_{n+k} are defined to be corresponding intervals.

The same one-one correspondence as in the case of corresponding stages exists for corresponding intervals. We can extend this correspondence in a similar manner to the entire $f{\cdot}f$ sequence and prove that

(a) the numerator of $f_{(r)n}$ is equal to the numerator of $f_{(r)n+k}$,

(b) if the denominator of $f_{(r)n} = F_j$, the denominator of $f_{(r)n+k} = F_{j+k}$.

(c) GENERALIZED $f{\cdot}f$ SEQUENCE. We defined the $f{\cdot}f$ sequence in the interval $[0,1]$. We now define it in the interval $[0,\infty]$.

Definition 3.6. The $f{\cdot}f$ sequence of order F_n is the set of all functions F_i/F_j, $j \leqslant n$ arranged in ascending order of magnitude $i,j \geqslant 0$. If $i < j$ then the $f{\cdot}f$ sequence is in the interval $[0,1]$. The basic properties of the $f{\cdot}f$ sequence for $[0,1]$ are retained with suitable alterations

Theorem 3.6.1. $f_{(r)n}$ is a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same numerator (beyond $1/1$). If $f_{(r)n}$ is a point of symmetry then $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same numerator, if each fraction does not pass beyond the next G.F.I. in either side (beyond $1/1$).

Theorem 3.6.2. A G.F.I. is a fraction with denominator F_n .

Theorem 3.6.3. A point of symmetry has either numerator or denominator 1.

Theorem 3.6.4. Beyond $1/1$, any interval is given by $[F_{n-1}/1, F_n/1]$. The factor of this interval is again F_{n-2}.

Theorem 3.6.5. The two basic properties

(a)
$$\frac{h+h''}{k+k''} = \frac{h'}{k'}$$

and

(b)
$$kh - hk' = F_{n-2}$$

are retained.

Theorem 3.6.6. If (a) is not good for h'/k' being a point of symmetry then

$$\frac{h'}{k'} = \frac{F_{n-1}h'' + F_{n-2}h}{F_{n-1}k'' + F_{n-2}k} \qquad \text{if} \qquad \frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''};\ \frac{h'}{k'} = \frac{F_n}{1} .$$

For a pertinent article by this author entitled "Approximation of Irrationals using Farey Fibonacci Fractions," see later issues.

f·f Sequence of Order 5

$$\frac{0}{3}, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 8

$$\frac{0}{5}, \frac{1}{8}, \frac{1}{5}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 13

$$\frac{0}{8}, \frac{1}{13}, \frac{1}{8}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 21

$$\frac{0}{13}, \frac{1}{21}, \frac{1}{13}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1} .$$

f·f Sequence of Order 34

$$\frac{0}{21}, \frac{1}{34}, \frac{1}{21}, \frac{2}{34}, \frac{1}{13}, \frac{3}{34}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{5}{34}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13},$$

$$\frac{8}{34}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1} .$$

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2. W.J. LeVeque, *Topics in Number Theory,* Vol. 1, Addison-Wesley, Reading, Mass., 1958, pp. 141–190.

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THE NUMBER OF ORDERINGS OF n CANDIDATES WHEN TIES ARE PERMITTED*

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In a competition it is customary to rank the candidates permitting ties and it is an interesting elementary combinatorial problem to find the number $\omega(n)$ of such orderings when there are n labelled candidates. $\omega(n)$ has curious properties.

Theorem 1. $\omega(n)$ is equal to $n!$ times the coefficient of x^n in the expansion of $(2-e^x)^{-1}$, that is,

$$\sum_{n=0}^{\infty} \frac{\omega(n)x^n}{n!} = \frac{1}{2-e^x} \tag{1}$$

if $\omega(0)$ is defined as 1.

By multiplying by $2-e^x$ and equating coefficients we obtain the recurrence relation

$$\omega(n) = \delta_0^n + \sum_{r=0}^{n-1} \binom{n}{r} \omega(r), \tag{2}$$

where $\delta_0^n = 1$ and $\delta_0^n = 0$ if $n \neq 0$ ("Kronecker's delta").

I mentioned (1) without proof in an appendix to Mayer and Good (1973). [It may be compared with Proposition XXIV in Whitworth (1901/1951) which states that the number of ways in which n different things can be distributed into not more than n indifferent parcels is $n!$ times the coefficient of x^n in the expansion of $\exp(e^x)/e$.]

Proof. Let r denote the number of distinct *positions* in an ordering of n candidates; for example, if among five candidates two tied for the first place, one was "third," and the other two were "fourth and fifth equal" we would say that the number of distinct positions is 3. We shall prove that the number $g(n,r)$ of orderings of n candidates having just r distinct "positions" is equal to $n!$ times the coefficient of x^n in $(e^x-1)^r$. (This is Whitworth's Proposition XXII whose proof is different.) Equation (1) then follows from the identity

$$(2-e^x)^{-1} = \sum_{r=0}^{\infty} (e^x-1)^r .$$

When there are just r "positions" for the n candidates, let us adopt the unconventional terminology of calling these positions *first, second,* $\cdots$, r^{th} and let us imagine that, for a specific ordering, there are n_1 candidates who are first, n_2 who are second, $\cdots$, and n_r who are r^{th}, where necessarily

$$n_1 \geqslant 1,\ \ n_2 \geqslant 1, \cdots, n_r \geqslant 1,\ \ n_1+n_2+\cdots+n_r = n.$$

The sequence of numbers $n_1, n_2, \cdots, n_r$ can be regarded as defining the *structure* of an ordering that has just r "positions." The number of orderings having just this structure (which incidentally is clearly a multiple of $r!$) is equal to the number of ways of throwing n labelled objects into r pigeon holes in such a way that there are n_1 in the first pigeon hole, n_2 in the second one, and so on. But this is equal to the multinomial coefficient $n!/(n_1! \cdots n_r!)$ Hence $g(n,r)$ is equal to $n!$ times the coefficient of x^n in

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*For some overlooked references, see Sloan (1973), p. 109.

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$$(3)\qquad \left(x+\frac{x2}{2!}+\frac{x3}{3!}+\cdots\right)\left(x+\frac{x2}{2!}+\frac{x3}{3!}+\cdots\right)\cdots\left(x+\frac{x2}{2!}+\frac{x3}{3!}+\cdots\right),$$

where there are r factors. The reason for putting in the x's here is that they automatically take care of the constraint $n_1+\cdots+n_r=n$. Equation (1) then follows immediately.

Theorem 2.

$$(4)\qquad \omega(n) = \sum_{r=0}^{\infty} \frac{r^n}{2^{r+1}} \quad .$$

Proof. We have

$$(2-e^x)^{-1} = 2^{-1}\sum_{r=0}^{\infty}\frac{e^{rx}}{2^r} \qquad (|x| < \log_e 2)$$

and the result follows at once from Theorem 1.

Theorem 3.

$$(5)\qquad \omega(n) = \sum_{r=0}^{n} r!S_n^{(r)} = \sum_{r=0}^{n} \Delta^r 0^n$$

$$(6)\qquad = \sum_{r=0}^{n}\left\{r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^r 0^n\right\},$$

where $S_n^{(r)}$ is a Stirling integer (number) of the second kind defined, for example, by Abramowitz and Stegun (1964, p. 824) or David and Barton (1962, p. 294), and tabulated in these two books on pages 835 and 294, respectively, and more completely in Fisher and Yates (1953, p. 78). Another notation for $S_n^{(r)}$ is $S(n,r)$, e.g. Riordan (1958). We could define $S_n^{(r)}$ by

$$(7)\qquad r!S_n^{(r)} = \Delta^r 0^n \, .$$

(Note the conventions $0^0=1$, $S_n^{(0)}=0$ if $n \geqslant 1$, $S_0^{(0)}=1$.)

Proof. It follows either from the proof of Theorem 1, or from Whitworth's Proposition XXII, that the term corresponding to a given value of r is equal to the contribution to $\omega(n)$ arising from those orderings of the n candidates having just r "positions." Equations (5) and (6) then follow at once. The "incidental" remark in the proof of Theorem 1 shows that $S_n^{(r)}$ is an integer.

An alternative proof of Theorem 3 follows from Theorem 2 by using the relationship between ordinary powers and factorial powers,

$$(8)\qquad r^n = \sum_{m=0}^{n} S_n^{(m)} r(r-1)\cdots(r-m+1),$$

combined with the binomial theorem for negative integral powers.

Theorem 3 provides one way of computing $\omega(n)$, given tables of $S_n^{(r)}$. The calculations can be partly checked by the special case of (8),

$$(9)\qquad \sum_{r=0}^{n}(-1)^r r!S_n^{(r)} = \sum_{r=1}^{n}(-1)^r\Delta^r 0^n = (-1)^n \, .$$

Theorem 4.

$$(10)\qquad \omega(n) = \frac{n!}{2}\left\{\tfrac{1}{2}\delta_0^n + \sum_{m=-\infty}^{\infty}\frac{1}{(\log_e 2 + 2\pi i m)^{n+1}}\right\}$$

(11) $$\omega(n) - \tfrac{1}{4}\delta_0^n = \frac{n!}{2}\left\{\frac{1}{(\log_e 2)^{n+1}} + 2\sum_{m=1}^{\infty} \frac{\cos[(n+1)\theta_m]}{[(\log_e 2)^2 + 4\pi^2 m^2]^{(n+1)/2}}\right\}$$

(12) $$= n!(\log_2 e)^{n+1}\left\{\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos^{n+1}\theta_m \cos[(n+1)\theta_m]\right\},$$

where

$$\theta_m = \tan^{-1}(2\pi m \log_2 e)$$

and the sum in (8) is a Cauchy principal value when $n = 0$.

Corollary.

(13) $$\omega(n) \sim n!\,(\log_2 e)^{n+1}/2$$

when n tends to infinity.

This asymptotic formula gives the answer to the nearest integer (and hence exactly) when $n < 16$ (see Table 1). It is curious that $n!\,(\log_2 e)^{n+1}/2$ is within 1/ 50 of an odd integer, namely $\omega(n)$, when $2 \leqslant n \leqslant 13$. We can obtain $\omega(n)$ exactly by taking the series of Theorem 4 as far as the first term for which $m > n/(2\pi e)$.

Proof of Theorem 4. By, say Titchmarch (1932, p. 113),

$$(1 - e^{-z})^{-1} = \tfrac{1}{2} + \lim_{M\to\infty} \sum_{m=-M}^{m} \frac{1}{z + 2m\pi i},$$

where z is a real or complex number, not a multiple of $2\pi i$. Put $z = u - x$ and we can deduce that the coefficient of x^n in the power series expansion of $(1 - e^{x-u})^{-1}$ at $x = 0$ (when $Re(u) > o$) is

(14) $$\tfrac{1}{2}\delta_0^n + \lim_{M\to\infty} \sum_{m=-M}^{m} \frac{1}{(u + 2m\pi i)^{n+1}}.$$

Theorem 4 follows on putting $u = \log_e 2$.

TABLE 1

Fractional part of $a_{n,0}$ (denoted by $\{a_{n,0}\}$), and the values of $a_{n,1}, a_{n,2}$, and $a_{n,3}$, where $a_{n,m}$ denotes the terms of formula (11). The sum column gives the total to be added to the integral part of $a_{n,0}$.

n	$\{a_{n,0}\}$	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$	*Sum*
1	.0406844905	–0.0244239291	–0.0062750652	–0.0028030856	.007
2	.0027807072	–0.0025628988	–0.0001650968	–0.0000327956	.000020
5	.0015185164	–0.0014866887	–0.0000285616	–0.0000026000	.00000067
10	.0052710420	–0.0052693807	–0.0000016476	–0.0000000133	.0000000004
16	.5130767435	0.4869198735	0.0000033805	0.0000000025	1.0000000000
20	.5284857660	27.4714964238	0.0000178075	0.0000000028	28.0000000000
25	.4328539621	22480.5672001073	–0.0000540633	–0.0000000061	22481.0000000000

Theorem 5. (i) If $n \equiv n' \pmod{p-1}$, where $n \geqslant 1$, $n' \geqslant 1$, we have

(15) $$\omega(n) \equiv \omega(n') \pmod{p},$$

where p is any prime. (ii) If $n \equiv 0 \pmod{p-1}$, where $n \geqslant 1$, then

(16) $$\omega(n) \equiv 0 \pmod{p},$$

where p is any odd prime.

COMMENT. If we had defined $\omega(0) = 0$, Part (ii) would have been a special case of Part (i), but unfortunately the convention $\omega(n) = 1$ is more convenient for Theorems 2 and 3.

Proof. To prove Theorem 5 we first give the following properties of the differences of powers at zero.

Lemma.

(17) (i) $$\Delta^a 0^b = 0 \quad \text{if} \quad a > b \quad (a,b = 1,2,3,\cdots)$$

(18) (ii) $$\Delta^r 0^n \equiv \Delta^r 0^{n'} \pmod p \quad \text{if} \quad n \equiv n' \pmod{p-1},\ n \geqslant 1,\ n' \geqslant 1$$

(19) (iii) $$\Delta^r 0^n \equiv (-1)^{r-1} \pmod p \quad \text{if} \quad n \equiv 0 \pmod{p-1},\ r \neq 0,\ n \neq 0\ .$$

Equation (17) is a special case of the fact that the a^{th} difference of a polynomial of degree b is zero if $a < b$. To prove (18) we first note that

(20) $$\Delta^r 0^n = \begin{cases} r^n - \binom{r}{1}(r-1)^n + \cdots + r(-1)^{r-1}1^n & (r > 0, n > 0) \\ 0 \quad (r = 0,\ n > 0) \\ 1 \quad (r = n = 0)\ . \end{cases}$$

But, by Fermat's theorem,

$$a^n \equiv a^{n'} \pmod p,$$

so that (18) follows at once from (20). If $n \equiv 0 \pmod{p-1},\ n \neq 0,\ r \neq 1,$ it follows from (20) and Fermat's theorem that

$$\Delta^r 0^n \equiv 1 - \binom{r}{1} + \cdots + \binom{r}{r-1}(-1)^{r-1} \pmod p$$

and this gives (19) by the binomial theorem.

To deduce Theorem 5, we now see from Eq. (5) that

$$\omega(n) = \sum_{r=0}^{n} \Delta^r 0^n \equiv \sum_{r=0}^{n} \Delta^r 0^{n'} \pmod p$$

by (18). Hence, by (5), with n replaced by n',

$$\omega(n) \equiv \omega(n') + \sum_{r=n'+1}^{n} \Delta^r 0^{n'} = \omega(n')$$

by (17). To prove Part (ii), where $n \equiv 0 \pmod{p-1},\ n \neq 0,$ we have

$$\omega(n) = \sum_{r=0}^{n} \Delta^r 0^n \equiv \sum_{r=1}^{n} (-1)^{r-1}$$

by (19), and this vanishes because n is even when p is odd.

SOME DEDUCTIONS FROM THEOREM 5

(a) Taking $p = 2$ in Part (i) we see that $\omega(n)$ is always odd.
(b) Given any odd prime p, there are an infinity of values for n for which p divides ω (n).
(c) When n is even, 3 divides $\omega(n)$.
(d) 59 divides ω (69) and 78803 divides ω (78813). (See the factorization of ω (11) in Table 2.)
(e) $2^{11213} - 1$ divides $\omega(2^{11213} - 2)$, but the division will never be done!
(f) $\omega(sp) \equiv \omega(s) \pmod p \ (s = 1,2,3,\cdots)$. [Here, and in (f), ···, (k), p is any prime number.]
(g) $\omega(p) \equiv 1 \pmod p$. (Also deducible easily from (2).)
(h) $\omega(p^k) \equiv 1 \pmod p \ (k = 1,2,3,\cdots)$.
(i) $\omega(2p^k) \equiv 3 \pmod p \ (k = 1,2,3,\cdots)$.
(k) $\omega(3p^k) \equiv 13 \pmod p \ (k = 1,2,3,\cdots)$.

In Table 2, some prime factorizations of $\omega(n)$ are shown, and (g) is also exemplified. Large primes seem to have a propensity to appear as factors of $\omega(n)$.

Conjecture 1. Part (i) of Theorem 5 shows that the sequence $\omega(1), \omega(2), \omega(3), \cdots$ has period $p - 1$ when p is a prime. It may be conjectured that it never has a shorter period (properly dividing $p - 1$). If this is true then the

TABLE 2

Some Values of $\omega(n)$, and Some Prime Factorizations of $\omega(n)$ and of $\omega(n) - 1$

The values of $\omega(n)$ were computed independently by (2), (6), and (11). x denotes any number known to have no factor less than 214500.

n	$\omega(n)$	Factorization of $\omega(n)$ into Primes	Factorization of $\omega(n) - 1$ into Primes
1	1	—	—
2	3	Prime	2
3	13	Prime	2·2·3
4	75	3·5·5	2·37
5	541	Prime	2·2·3·3·3·5
6	4683	3·7·223	2·2341
7	47293	Prime	2·2·3·7·563
8	5 45835	3·5·36389	2·272917
9	70 87261	Prime	2·2·3·5·41·43·67
10	1022 47563	3·11·41·75571	2·29·263·6703
11	16226 32573	59·349·78803	2·2·3·3·11·283·14479
12	2 80915 67595	3·5·7·13·20579903	2·2969·4730813
13	52 68583 48381	Prime	2·2·3·5·7·13·13·113·65687
14	1064 13429 70443	3·x	2·37·x
15	23028 31909 77853	13·23·23·271·123564559	2·2·3·59·325258744319
16	5 31565 46819 81355	3·5·17·x	2·3659·4327·167871689
17	130 37076 70291 35901	101·157·809·10162736677	2·2·3·3·5·5·17·37·131·3889·452041
18	3385 53466 32568 45323	3·7·19·x	2·251·x
19	92801 58731 93284 11133	59·13723·25579·4480956911	2·2·3·7·19·x
20	26 77687 79624 43842 03115	3·5·11·37·229·x	2·x
21	811 24824 99850 40738 81821	x	2·2·3·5·11·3083·x
22	25748 44419 80319 03845 44203	3·23·x	2·59·167·1489·x
23	8 54384 51336 74570 92945 80413	14081·x	2·2·3·3·3·23·71·271·6263·x
24	295 82791 21074 14547 26506 48875	3·5·5·5·7·13·x	2·31·43·x
25	10669 73654 38475 77582 55834 98141	x	2·2·3·5·7·7·13·x
26	4 00222 57598 44168 49248 61275 39083	3·31·149·2969·x	2·x
27	155 89776 39186 21623 24927 62262 53693	13·131·x	2·2·3·29·x
28	6297 56206 49500 66033 51837 39353 34635	3·5·29·37·x	2·71·18223·119173·x
29	2 63478 38526 30236 90020 89332 90445 76861	x	2·2·3·3·5·29·43·997·1447·38167·x
30	114 03568 79401 18804 83742 46419 61849 01963	3·7·11·31·4289·85247·x	2·x
31	5100 08036 57426 93884 30841 02407 59181 18973	41·523·12829·103291·x	2·2·3·7·11·31·73·127·269·150907·x
32	2 35451 54085 73489 66491 84490 63714 48554 76395	3·5·17·66643·x	2·109·151·x
33	112 09597 42203 05626 82674 94209 29300 68825 89981	137·x	2·2·3·5·17·x
34	5498 49040 77825 68486 24268 68390 30104 97501 04843	3·x	2·59·3209·x
35	2 77642 56952 89206 00263 03102 19593 68549 61635 84253	x	2·2·3·3·37·x
36	144 19928 09516 55469 62836 09781 09406 91758 35130 90155	3·5·7·7·13·19·37·449·36017·x	2·199·x

converse of Part (ii) would be true; that is p could divide $\omega(n)$ only if $n \equiv 0 \pmod{p-1}$. I have verified the conjecture for all primes less than 73, but I do *not* regard this as strong evidence. In fact I estimate that the probability that the conjecture would have survived the tests, if it is false, is about 0.18.

If this conjecture is true then we can deduce that $\omega(n)$ is never a multiple of n, for any integer n greater than 1. Since $\omega(n)$ is always odd we need consider only odd values of n. *Suppose* then that n divides $\omega(n)$ and let p be a prime factor of n. Let the highest power of p that divides n be p^m. By repeated application of (f) we have $\omega(n) \equiv \omega(n/p^m) \pmod{p}$, and therefore by the converse of Part (ii) of Theorem 5 (which is true if the conjecture is) we see that n/p^m is a multiple of $p-1$ and is therefore even. But n is odd by assumption and we have arrived at a contradiction. So the conjecture implies that n cannot divide $\omega(n)$.

Conjecture 2. Modulo 2, 4, 8, 16, 32, 64, 128, 256, 512, ⋯ the sequence $\{\omega(n)\}$ runs into cycles of lengths 1, 2, 2, 2, 2, 4, 8, 16, 32, ⋯. That is the period modulo 2^k appears to be 2^{k-4} when $k \geq 5$, and, for k = 1, 2, 3, 4 is 1, 2, 2, and 2. This conjecture would follow from the following one.

Conjecture 3. If $\omega(n)$ is expressed in the binary system as

$$a_{n0} + 2a_{n1} + 2^2 a_{n2} + 2^3 a_{n3} + \cdots,$$

then the sequence of r^{th} least significant digits, $a_{1r}, a_{2r}, a_{3r}, \cdots$ runs into a cycle whose lengths, for $r = 0, 1, 2, 3, 4, \cdots$ are respectively 1, 2, 2, 1, 2, 4, 8, 16, ⋯. That is, the period is 2^{r-3} for $r \geq 3$ and for $r = 0, 1, 2$ is $1, 2,$ and 2. This conjecture is formulated on the basis of the columns of Table 3.

Conjecture 4. If $\omega(n)$ is expressed in the scale of p, where p is an odd prime,

$$\omega(n) = b_{n0} + pb_{n1} + p^2 b_{n2} + \cdots$$

then the sequence $b_{1r}, b_{2r}, b_{3r}, \cdots$ runs into a cycle of length $p^r(p-1)$. This has been verified empirically for $p^{r+1} = 9, 27,$ and 25 (and $n \leq 36$). For $r = 0$ we know the result is true by Theorem 5, as we said before. A feasible conjecture is that the periods are never less than the ones stated.

Conjecture 5. Modulo p^r, where p is an odd prime, and $r \geq 1$, the sequence $\{\omega(n)\}$ runs into a cycle of length $p^{r-1}(p-1)$ and no less. This would follow from Conjecture 4. It generalizes Conjecture 1.

From Conjectures 2 and 5, if they are true, we can deduce that, modulo $m = 2^k p_1^{k_1} p_2^{k_2} \cdots$, the sequence $\{\omega(n)\}$ runs into a cycle of length

$$\begin{array}{ll} \phi(m) & \text{if } k = 0, 1, \text{ or } 2 \\ \phi(m)/2 & \text{if } k = 3 \\ \phi(m)/4 & \text{if } k = 4 \\ \phi(m)/8 & \text{if } k \geq 5, \end{array}$$

where ϕ denotes Euler's arithmetic function.

Conjecture 6. Parts of Conjectures 2 to 5 could perhaps be proved inductively, by using Eq. (2) combined with the use of m^{th} roots of unity.

Conjecture 7. For each n, $\omega(n)$ and $\omega(n+1)$ have no common factor, and the highest common factor of $\omega(n) - 1$ and $\omega(n+1) - 1$ is 2. This follows from Conjecture 1.

GENERALIZATION OF SOME OF THE RESULTS

The proof of Theorem 4 suggests correctly that several formulae that we have mentioned can be generalized by replacing $\log_e 2$ by u. By making this change we see that, in addition to (14), we have:

The coefficient of x^n in $(1 - e^{x-u})^{-1}$ (where $Re(u) > 0$) is equal to

$$\frac{1}{n!}\left(-\frac{d}{du}\right)^n \frac{1}{1-e^{-u}} \tag{21}$$

$$= \tfrac{1}{2}\delta_0^n + \frac{(-1)^n}{n!} \sum_{m=[n+1/2]}^{\infty} \frac{B_{2m}u^{2m-n-1}}{2m(2m-n-1)!} \quad (u < 2\pi) \tag{22}$$

TABLE 3

The Ten Least Significant Binary Digits a_{nr} of $\omega(n)$ $(n = 1, 2, \cdots, 36)$

$n \backslash r$	9	8	7	6	5	4	3	2	1	0
1										1
2									1	1
3							1	1	0	1
4				1	0	0	1	0	1	1
5	1	0	0	0	0	1	1	1	0	1
6	1	0	0	1	0	0	1	0	1	1
7	0	0	1	0	1	1	1	1	0	1
8	0	0	0	0	1	0	1	0	1	1
9	0	0	1	0	0	1	1	1	0	1
10	0	0	1	0	0	0	1	0	1	1
11	0	0	0	1	1	1	1	1	0	1
12	0	1	1	1	1	0	1	0	1	1
13	1	1	0	1	0	1	1	1	0	1
14	1	0	0	1	0	0	1	0	1	1
15	0	1	0	0	1	1	1	1	0	1
16	0	1	1	0	1	0	1	0	1	1
17	1	0	0	0	0	1	1	1	0	1
18	0	0	0	0	0	0	1	0	1	1
19	0	1	1	1	1	1	1	1	0	1
20	0	1	0	1	1	0	1	0	1	1
21	0	0	1	1	0	1	1	1	0	1
22	0	1	1	1	0	0	1	0	1	1
23	1	0	1	0	1	1	1	1	0	1
24	0	1	0	0	1	0	1	0	1	1
25	1	1	1	0	0	1	1	1	0	1
26	1	1	1	0	0	0	1	0	1	1
27	1	1	0	1	1	1	1	1	0	1
28	0	0	1	1	1	0	1	0	1	1
29	1	0	0	1	0	1	1	1	0	1
30	0	1	0	1	0	0	1	0	1	1
31	0	0	0	0	1	1	1	1	0	1
32	0	0	1	0	1	0	1	0	1	1
33	0	1	0	0	0	1	1	1	0	1
34	1	1	0	0	0	0	1	0	1	1
35	0	0	1	1	1	1	1	1	0	1
36	0	0	0	1	1	0	1	0	1	1
Period		32	16	8	4	2	1	2	2	1
Antiperiod		16	8	4	2	1	–	1	1	–

$$(23)\qquad = \frac{1}{n!}\sum_{r=0}^{\infty} r^n e^{-ru} \qquad (Re(u) > 0)$$

$$(24)\qquad = \frac{1}{n!}e^u \sum_{m=0}^{n} S_r^{(m)} m!(e^u - 1)^{-m-1}$$

$$(25)\qquad = \frac{1}{n!}e^u \sum_{m=0}^{n} (e^u - 1)^{-m-1}\Delta^m 0^n$$

$$(26)\qquad = \tfrac{1}{2}\delta_0^n + \sum_{m=-\infty}^{\infty} \frac{\cos\,[(n+1)\tan^{-1}(2\pi m/u)]}{(u^2 + 4\pi^2 m^2)^{(n+1)/2}}\,.$$

For example,

$$\frac{1}{7!}\sum_{r=0}^{\infty} r^7 e^{-r} = \frac{e}{7!}\sum_{m=0}^{7} (e-1)^{-m-1}\Delta^m 0^7 = 1.00000023$$

and the coefficients of $1, x, x^2, x^3, \cdots$ in $(1 - e^{x-1})^{-1}$ are respectively

$$1.58,\ 0.92,\ 0.9962,\ 1.0011,\ 1.00014,\ 0.999982,\ 0.9999957,\ 1.00000023,\ \cdots,$$

tending rapidly to 1.

Formula (26) is always very effective for summing the series

$$\sum_{r=0}^{\infty} r^n z^r$$

numerically when $|z|$ is close to 1.

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CHEBYSHEV POLYNOMIALS AND RELATED SEQUENCES

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1. A COMBINATORIAL APPROACH

In [3], the nonzero coefficients of the Chebyshev polynomials $T_n(x) = \cos n\theta$, $\cos\theta = x$, which satisfy the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ since $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$, are arranged in left-adjusted triangular form. The first seven rows of the array are

$n \backslash k$	0	1	2	3
0	1			
1	1			
2	2	−1		
3	4	−3		
4	8	−8	1	
5	16	−20	5	
6	32	−48	18	−1

Furthermore, letting $a_{n,k}$ be the element in the n^{th} row and k^{th} column, it is shown in [3] that

$$a_{n,k} = (-1)^k \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k-1} \tag{1.1}$$

and

$$a_{n,k} = 2a_{n-1,k} - a_{n-2,k-1}\ . \tag{1.2}$$

In this section, we discuss several linear recurrences which arise as a result of a careful examination of the triangular array. The validity of these linear recurrences is established by means of common combinatorial identities.

Summing along the rising diagonals, we obtain the sequence $1, 1, 2, 3, 5, 8, 13, \cdots$, which appears to be the sequence of Fibonacci numbers. To show that this is in fact the case, we first observe that the sum of the n^{th} rising diagonal is given by

$$f_n = \begin{cases} 1, & n = 1 \text{ or } 2 \\ \displaystyle\sum_{k=0}^{M} a_{n-k-1,k}, & M = \left[\dfrac{n-1}{3}\right],\quad n \geq 3\ . \end{cases} \tag{1.3}$$

We now verify that $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

In [2], we find the following combinatorial identities

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{1.4}$$

and

(1.5)
$$\binom{n-k}{k} + \binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k} .$$

Using (1.1) together with (1.3) and applying (1.5) and then (1.4) twice, we have,

$$f_n = \sum_{k=0}^{M} (-1)^k \frac{n-k-1}{n-2k-1}\binom{n-2k-1}{k} 2^{n-3k-2}$$

$$= \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-2}{k} + 2\binom{n-2k-2}{k-1}\right] 2^{n-3k-2}$$

$$= \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-2}{k} + \binom{n-2k-3}{k-1}\right] 2^{n-3k-3}$$

(1.6)
$$+ \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-3}{k} + 4\binom{n-2k-2}{k-1}\right] 2^{n-3k-3}$$

$$= f_{n-1} + \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-3}{k} + \binom{n-2k-4}{k-1}\right] 2^{n-3k-4}$$

$$+ \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-4}{k} + 8\binom{n-2k-2}{k-1}\right] 2^{n-3k-4}$$

$$= f_{n-1} + f_{n-2} + \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-4}{k} + 8\binom{n-2k-2}{k-1}\right] 2^{n-3k-4} .$$

Since the first and last terms cancel for successive integral values in the last sum, and because

$$n-4 < n-1 \leqslant 3M \quad \text{implies that} \quad n-2M-4 < M,$$

the last sum has value zero so that

(1.7)
$$f_n = f_{n-1} + f_{n-2}, \qquad n \geqslant 3.$$

The sequence of the sums of the rising diagonals in absolute value, denoted by $\{u_n\}_{n=1}^{\infty}$, is $1,1,2,5,11,24,53,\cdots$ and it appears to satisfy the recurrence relation

(1.8)
$$u_1 = u_2 = 1, \quad u_3 = 2, \quad 2u_{n-1} + u_{n-3} = u_n, \quad n \geqslant 4.$$

By the definition of u_n, (1.1), and (1.3), we see for $n \geqslant 4$, following an argument similar to that of (1.6), that,

$$u_n = \sum_{k=0}^{M} \frac{n-k-1}{n-2k-1}\binom{n-2k-1}{k} 2^{n-3k-2} = \sum_{k=0}^{M} \left[\binom{n-2k-2}{k} + 2\binom{n-2k-2}{k-1}\right] 2^{n-3k-2}$$

(1.9)
$$= 2\sum_{k=0}^{M} \left[\binom{n-2k-2}{k} + \binom{n-2k-3}{k-1}\right] 2^{n-3k-3} + \sum_{k=0}^{M} \left[2\binom{n-2k-2}{k-1} - \binom{n-2k-3}{k-1}\right] 2^{n-3k-2}$$

$$= 2u_{n-1} + \sum_{k=0}^{M-1} \left[2\binom{n-2k-4}{k} - \binom{n-2k-5}{k}\right] 2^{n-3k-5} = 2u_{n-1} + \sum_{k=0}^{M} \left[\binom{n-2k-4}{k}\right.$$

$$\left. + \binom{n-2k-5}{k-1}\right] 2^{n-3k-5} = 2u_{n-1} + u_{n-3}$$

and (1.8) is proved.

Let w_n be the sum of the terms along the n^{th} falling diagonal. The terms of $\{w_n\}_{n=1}^{\infty}$ appear to be given by

$$w_n = \begin{cases} 1, & n = 1 \\ 0, & n \geq 2 \end{cases} \tag{1.10}$$

To show that $w_n = 0$ for $n \geq 2$, we observe that

$$w_n = \sum_{k=0}^{n-1} a_{n+k-1,k} = \sum_{k=0}^{n-1} (-1)^k \left[\binom{n-1}{k} + \binom{n-2}{k-1}\right] 2^{n-k-2}$$

$$= \tfrac{1}{2}\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} 2^{n-k-1} - \tfrac{1}{2}\sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} 2^{n-k-2} \tag{1.11}$$

$$= \frac{(2-1)^{n-1}}{2} - \frac{(2-1)^{n-2}}{2} = 0$$

and (1.10) is proved.

Letting q_n be the sum of the absolute value of the terms along the n^{th} falling diagonal, we see that the terms of $\{q_n\}_{n=1}^{\infty}$ are $1, 2, 6, 18, 54, 162, 486, \cdots$ and it appears as if we have

$$q_n = \begin{cases} 1, & n = 1 \\ 2 \cdot 3^{n-2}, & n \geq 2 \end{cases}. \tag{1.12}$$

By the definition of q_n and (1.11), we have

$$q_n = \sum_{k=0}^{n-1} |a_{n+k-1,k}| = \tfrac{1}{2}\sum_{k=0}^{n-1} \binom{n-1}{k} 2^{n-k-1} + \tfrac{1}{2}\sum_{k=0}^{n-2} \binom{n-2}{k} 2^{n-k-2} \tag{1.13}$$

$$= \frac{(2+1)^{n-1}}{2} + \frac{(2+1)^{n-2}}{2} = 2 \cdot 3^{n-2}$$

so that (1.12) is true.

It is easy to determine the row sum r_n because, as is pointed out in [3], the sums are all one since $\cos n0 = 1$. The last sequence of this section, denoted by $\{p_n\}_{n=1}^{\infty}$, deals with the sums of the absolute values of the terms of the rows, and the first few terms of the sequence are $1, 1, 3, 7, 17, 41, 91, \cdots$. It appears as if we have

$$p_1 = p_2 = 1, \quad p_n = 2p_{n-1} + p_{n-2}, \quad n \geq 3, \tag{1.14}$$

which is a generalized Pell sequence where the Pell numbers P_n are given by the recurrence relation

$$P_1 = 1, \quad P_2 = 2, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3. \tag{1.15}$$

The first few terms of the sequence are $1, 2, 5, 12, 29, 70, 169, \cdots$. Letting $P_{-1} = 1$ and $P_0 = 0$, it is easy to establish by mathematical induction that

$$p_n = P_{n-1} + P_{n-2} = P_n - P_{n-1} \tag{1.16}$$

and

$$P_n = \sum_{i=1}^{n} p_n . \tag{1.17}$$

To verify (1.14), we use (1.2) and observe that

$$|a_{n,k}| = 2|a_{n-1,k}| + |a_{n-2,k-1}| \tag{1.18}$$

so that with $N = [n/2]$, we have

$$p_n = \sum_{k=0}^{N} |a_{n,k}| = 2\sum_{k=0}^{N} |a_{n-1,k}| + \sum_{k=0}^{N} |a_{n-2,k-1}| = 2p_{n-1} + \sum_{k=0}^{N-1} |a_{n-2,k}| . \tag{1.19}$$

However, $|a_{n-2,N}| = 0$ because $n - 2 < n \leq 2N$ implies that $n - 2 - N < N$. Hence, (1.20) $$p_n = 2p_{n-1} + p_{n-2}\,.$$

2. GENERATING FUNCTIONS

In a personal correspondence, V.E. Hoggatt, Jr., pointed out that the relationships of Section 1 could be established by means of generating functions.

Let $G_k(x)$ be the generating function for the k^{th} column. Following standard techniques, it is easy to show that

(2.1) $$G_0(x) = \frac{1-x}{1-2x}$$

and, with the aid of (1.2) that

(2.2) $$G_k(x) = \frac{-G_{k-1}(x)}{1-2x}\,.$$

Employing mathematical induction together with (2.1) and (2.2), we have

(2.3) $$G_k(x) = \left(\frac{-1}{1-2x}\right)^k \left(\frac{1-x}{1-2x}\right), \quad k \geqslant 0.$$

Adding along the rising diagonals is equivalent to

(2.4) $$\begin{aligned}\sum_{k=0}^{\infty} x^{3k} G_k(x) &= \sum_{k=0}^{\infty} \left(\frac{1-x}{1-2x}\right)\left(\frac{-x^3}{1-2x}\right)^k \\ &= \left(\frac{1-x}{1-2x}\right) \div \left(1 + \frac{x^3}{1-2x}\right) \\ &= (1-x-x^2)^{-1}\,.\end{aligned}$$

Since

$$(1-x-x^2)^{-1}$$

is the generating function for the Fibonacci sequence, we have an alternate proof of (1.7).

Letting

(2.5) $$G_k^*(x) = \left(\frac{1-x}{1-2x}\right)\left(\frac{1}{1-2x}\right)^k,$$

we see that adding along rising diagonals with all signs positive is equivalent to

(2.6) $$\sum_{k=0}^{\infty} x^{3k} G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1 - \frac{x^3}{1-2x}\right) = \frac{1-x}{1-2x-x^3}$$

which verifies (1.8) since $(1-x)(1-2x-x^3)^{-1}$ is the generating function for $\{u_n\}_{n=1}^{\infty}$.

To verify (1.10) and (1.12), we recognize that

(2.7) $$\sum_{k=0}^{\infty} x^k G_k(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1 + \frac{x}{1-2x}\right) = 1,$$

where 1 is the generating function for $\{w_n\}_{n=1}^{\infty}$ while

(2.8) $$\sum_{k=0}^{\infty} x^k G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1 - \frac{x}{1-2x}\right) = \frac{1-x}{1-3x},$$

where $(1-x)(1-3x)^{-1}$ is the generating function for $\{q_n\}_{n=1}^{\infty}$.

Since

(2.9) $$\sum_{k=0}^{\infty} x^{2k} G_k(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1 + \frac{x^2}{1-2x}\right) = (1-x)^{-1}$$

we have an alternate proof that the row sums are all one. Furthermore,

$$(2.10)\qquad \sum_{k=0}^{\infty} x^{2k} G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1-\frac{x^2}{1-2x}\right) = \frac{1-x}{1-2x-x^2},$$

where $(1-x)(1-2x-x^2)^{-1}$ is the generating function for $\{p_n\}_{n=1}^{\infty}$. Hence, we have an alternate proof of (1.14).

In conclusion, we note that

$$(2.11)\qquad \sum_{n=0}^{\infty} P_{n-1}x^n + \sum_{n=0}^{\infty} P_n x^n = \frac{1-2x}{1-2x-x^2} + \frac{x}{1-2x-x^2} = \frac{1-x}{1-2x-x^2} = \sum_{n=0}^{\infty} p_{n+1}x^n$$

and we have a generating function proof of (1.16).

3. ANOTHER ARRAY

If we let

$$Q_n(x) = \frac{\sin n\theta}{\sin\theta}, \qquad x = \cos\theta,$$

and use

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta,$$

we see that

$$Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x)$$

and $Q_n(x)$ is a polynomial in x.

The first eight rows of the nonzero coefficients of the polynomials $Q_n(x)$ in left-adjusted triangular form are

$n \backslash k$	0	1	2	3
1	1			
2	2			
3	4	−1		
4	8	−4		
5	16	−12	1	
6	32	−32	6	
7	64	−80	24	−1
8	128	−192	80	−8

Letting $b_{n,k}$ be the element in the n^{th} row and k^{th} column, it can be shown, as in [3], that

$$(3.1)\qquad b_{n,k} = 2b_{n-1,k} - b_{n-2,k-1}$$

and

$$(3.2)\qquad b_{n,k} = (-1)^k \binom{n-k-1}{k} 2^{n-2k-1}.$$

The six linear recurrences of Section 1, relative to the $Q_n(x)$ array, are

$$(3.3)\qquad F_1 = 1,\quad F_2 = 2,\quad F_n = F_{n-1} + F_{n-2} + 1,\quad n \geq 3$$

$$(3.4)\qquad U_1 = 1,\quad U_2 = 2,\quad U_3 = 4,\quad U_n = 2U_{n-1} + U_{n-3},\quad n \geq 4$$

$$(3.5)\qquad W_n = 1,\quad n \geq 1$$

$$(3.6)\qquad Q_n = 3^{n-1},\quad n \geq 1$$

$$(3.7)\qquad R_n = n,\quad n \geq 1,$$

and

$$(3.8)\qquad P_1 = 1,\quad P_2 = 2,\quad P_n = 2P_{n-1} + P_{n-2},\quad n \geq 3$$

which is the sequence of Pell numbers given in (1.15).

The preceding six linear recurrences can be verified by using combinatorial arguments like those of Section 1 or by means of generating functions as in Section 2 where the column generators of the $Q_n(x)$ table are given by

$$(3.9)\qquad H_k(x) = \frac{1}{1-2x}\left(\frac{-1}{1-2x}\right)^k,\quad k \geq 0$$

and

$$H_k^*(x) = \frac{1}{1-2x}\left(\frac{1}{1-2x}\right)^k, \quad k \geqslant 0 \tag{3.10}$$

if we want all positive values. Hence, the details are omitted.

4. CONCLUDING REMARKS

Equations (1.16) and (1.17) relate the sequences of (1.14) and (3.8). Similar relationships, which can be proved by mathematical induction, also hold for the other five recurrences. That is,

$$f_n = F_n - F_{n-1} \quad \text{and} \quad F_n = \sum_{i=1}^{n} f_i \tag{4.1}$$

$$u_n = U_n - U_{n-1} \quad \text{and} \quad U_n = \sum_{i=1}^{n} u_i \tag{4.2}$$

$$w_n = W_n - W_{n-1} \quad \text{and} \quad W_n = \sum_{i=1}^{n} w_i \tag{4.3}$$

$$q_n = Q_n - Q_{n-1} \quad \text{and} \quad Q_n = \sum_{i=1}^{n} q_i \tag{4.4}$$

$$r_n = R_n - R_{n-1} \quad \text{and} \quad R_n = \sum_{i=1}^{n} r_i \,. \tag{4.5}$$

Since Eq. (3.9) is $(1-x)^{-1}$ times Eq. (2.3), it can be shown that the entries in the $Q_n(x)$ table are partial sums of the column entries of the $T_n(x)$ table. Hence,

$$b_{n+2k,k} = \sum_{j=0}^{n-1} a_{j+2k,k} \tag{4.6}$$

which gives rise to the combinatorial identity

$$2^n\binom{n+k}{k} = \sum_{j=0}^{n}\left(\frac{j+2k}{j+k}\right)\binom{j+k}{k}2^{j-1}\,. \tag{4.7}$$

An interesting consequence of (4.6) since the $b_{n,k}$ and $a_{n,k}$ are respectively the coefficients of the polynomials $Q_n(x)$ and $T_n(x)$ is the identity

$$\sum_{j=0}^{n} \cos^{n-j}\theta \cos j\theta = \frac{\sin (n+1)\theta}{\sin\,\theta}\,. \tag{4.8}$$

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SOME RESULTS CONCERNING THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF THE FORM $p^{\alpha}M^{2\beta}$

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ABSTRACT

It is shown here that if n is an odd number of the form $p^{\alpha}M^{10}$, $p^{\alpha}M^{24}$, $p^{\alpha}M^{34}$, $p^{\alpha}M^{48}$ or $p^{\alpha}M^{124}$, where M is square-free and p is a prime which does not divide M, then n is not perfect.

1. INTRODUCTION

Euler (see page 19 in [1]) proved that if n is odd and perfect (that is, if n has the property that its positive divisor sum $\sigma(n)$ is equal to $2n$) then $n = p^{\alpha}N^2$ where $p \nmid N$ and $p \equiv \alpha \equiv 1 \pmod 4$. In considering the still unanswered question as to whether or not an odd perfect number exists, several investigators have focused their attention on the conditions which must be satisfied by the exponents in the prime decomposition of N. If M is square-free and β is a natural number then it is known that $n = p^{\alpha}M^{2\beta}$ is not perfect if β has any of the following values: 1 (Steuerwald in [8]), 2 (Kanold in [3]), 3 (Hagis and McDaniel in [2]), $3k + 1$ where k is a non-negative integer (McDaniel in [5]). Our purpose here is to show that n is not perfect for five additional values of β. Thus, we shall prove the following result.

Theorem. Let $n = p^{\alpha}M^{2\beta}$ where M is an odd square-free number, $p \nmid M$, and $p \equiv \alpha \equiv 1 \pmod 4$. Then n is not perfect if (A) $\beta = 5$, (B) $\beta = 12$ or 62, (C) $\beta = 24$, (D) $\beta = 17$.

2. SOME PRELIMINARY RESULTS AND REMARKS

For the reader's convenience we list several well-known facts concerning the sigma function, cyclotomic polynomials, and odd perfect numbers which will be needed. If q is a prime the notation $q^C \| K$ means that $q^C | K$ but $q^{C+1} \nmid K$.

(1) If P is a prime, then

$$\sigma(P^s) = \prod_m F_m(P),$$

where $F_m(x)$ is the m^{th} cyclotomic polynomial and m ranges over the positive divisors other than 1 of $s + 1$. (See Chapter 8 in [7].) If n is odd and perfect and q is an odd prime then it is immediate, since $\sigma(n) = 2n$, that $q | n$ if and only if $q | F_m(P)$ where P^s is a prime power such that $P^s \| n$ and $m | (s + 1)$.

(2) If $m = q^C$ where q is a prime then $q | F_m(P)$ if and only if $P \equiv 1 \pmod q$. Furthermore, if $q | F_m(P)$ and $m > 2$, then $q \| F_m(P)$. (See Theorem 95 in [6].)

(3) If $q | F_m(P)$ and $q \nmid m$, then $q \equiv 1 \pmod m$. (See Theorem 94 in [6].)

(4) If $n = p^{\alpha}p_1^{2\beta_1} \cdots p_t^{2\beta_t}$ is odd and perfect then the fourth power (at least) of any common divisor of the numbers $2\beta_i + 1$ $(i = 1, 2, \cdots, t)$ divides n. (See Section III in [3].)

(5) If n is an odd perfect number then n is divisible by $(p + 1)/2$.

We shall also require the following lemma which, to the best of our knowledge, is new.

Lemma. Let $n = p^{\alpha}M^{2\beta}$ be an odd perfect number with M square-free. If $2\beta + 1 = RQ^a$ where Q is a prime different from p and $Q \nmid R$, then at most $2\beta/a$ distinct prime factors of M are congruent to 1 modulo Q.

Proof. Since $Q^4 | n$, by (4), and $Q \neq p$ we have $Q^{2\beta} \| n$. If P is a prime factor of M then from (1) we see that $F_{Q^j}(P) | n$ for $j = 1, 2, \cdots, a$.

Thus, if $P \equiv 1 \pmod Q$ then $Q^a | n$, by (2). It now follows that if M is divisible by C distinct primes, each congruent to 1 modulo Q, then $Q^{aC} | n$. Since $Q^{2\beta} \| n$, $C \leqslant 2\beta / a$.

We are now prepared to prove our theorem. Our proof utilizes the principle of *reductio ad absurdum* with Kanold's result (4) furnishing a starting point and our lemma providing a convenient "target" for contradiction. The prime factors of the cyclotomic polynomials encountered in the sequel were obtained using the CDC 6400 at the Temple University Computing Center. For the most part only those prime factors of $F_m(P)$ were sought which did not exceed 10^5.

3. THE PROOF OF (A)

We begin by noting that

$$F_{11}(199) = 11R_1 \quad \text{and} \quad F_{11}(463) = 11 \cdot 23 \cdot 5479 R_2 ,$$

where every prime which divides $R_1 R_2$ exceeds 10^5. Since

$$R_1 / R_2 \doteq (8.899 \cdot 10^{21}) / (3.273 \cdot 10^{20}) \doteq 27.2$$

we see that $R_2 \nmid R_1$ from which it follows that $R_1 R_2$ has at least two distinct prime divisors P_1 and P_2, both greater than 10^5. By (3), $P_1 \equiv P_2 \equiv 1 \pmod{11}$. We also remark that if

$$P_3 = 1806113 \quad \text{and} \quad P_4 = 3937230404603 = F_{11}(23)/11$$

then it can be verified that neither of the primes P_3 or P_4 divides either R_1 or R_2.

Now assume that $n = p^\alpha M^{10}$ is perfect. From (4) we see that $11^4 | n$ and, therefore, that

$$F_{11}(11) = 15797 \cdot 1806113 | n.$$

We now consider three possibilities.

CASE 1. $p = 15797$. By (5), $3 \cdot 2633 | n$. It was found that

$$2113 | F_{11}(2633), \quad 683 \cdot 7459 | F_{11}(2113), \quad 23 \cdot 99859 | F_{11}(683), \quad \text{and} \quad 3719 \cdot 8999 | F_{11}(99859).$$

Also,

$$463 | F_{11}(3719) \quad \text{and} \quad 199 | F_{11}(1806113).$$

It follows from (1) that n is divisible by each of the following eleven primes, all congruent to 1 modulo 11:

$$23,\ 199,\ 463,\ 683,\ 2113,\ 3719,\ 7459,\ 8999,\ 99859,\ P_3,\ P_4 .$$

But this is impossible since, according to our lemma, M has at most 10 prime divisors congruent to 1 modulo 11.

CASE 2. $p = 1806113$. By (5), $3 \cdot 17 \cdot 17707 | n$. $1013 | F_{11}(17707)$ and $199 | F_{11}(1013)$; while

$$463 | F_{11}(15797), \quad 23 \cdot 5479 | F_{11}(463), \quad \text{and} \quad 1277 \cdot 18701 | F_{11}(5479).$$

From (1) and the discussion in the first paragraph of this section we see that each of the eleven primes

$$23,\ 199,\ 463,\ 1013,\ 1277,\ 5479,\ 15797,\ 18701,\ P_1,\ P_2,\ P_4$$

divides n. Our lemma has been contradicted again.

CASE 3. $p \neq 15797$ and $p \neq 1806113$. Since $199 | F_{11}(1806113)$ and $463 | F_{11}(15797)$ we see from the discussion thus far that n is divisible by the following eleven primes:

$$23,\ 199,\ 463,\ 1277,\ 5479,\ 15797,\ 18701,\ P_1,\ P_2,\ P_3,\ P_4 .$$

If $p = 18701$ then $3 | n$ and, therefore, 3851 (a factor of $F_{11}(3)$) divides n. If $p \neq 18701$ then n is divisible by 34607, a factor of $F_{11}(18701)$. In either case n is divisible by twelve primes, each congruent to 1 modulo 11, at most one of which is p. This contradiction to our lemma completes the proof of (A).

4. THE PROOF OF (B)

If we assume that $n = p^\alpha M^{2\beta}$ is perfect, where $\beta = 12$ or 62, then $5^4 | n$ by (4). If $p \equiv 2 \pmod 3$ then from (5) we have $3 | n$, and since $F_5(3) = 11^2$ it follows from (1) that $3 \cdot 5^2 \cdot 11 | n$. But this contradicts a well known result of Kanold's ((2) Hilfssatz in [4]). We conclude, since $p \equiv 1 \pmod 4$, that $p \equiv 1 \pmod{12}$.

Since $5^4 | n$ we have $5^{24} \| n$ (or $5^{124} \| n$), and from (1) we see that

$$F_5(5) = 11\cdot 71 \quad \text{and} \quad F_{25}(5) = 101\cdot 251\cdot 401\cdot 9384251$$

both divide n.

Proceeding as in the proof of (A) and referring to Table 1 we see that n is divisible by at least 43 different primes congruent to 1 modulo 5. (Here, and in our other tables, the presence of an asterisk indicates that the prime *might* be p.) Since at most one of these primes can be p, and since our lemma implies that M has at most 12 (or 41) prime factors congruent to 1 modulo 5, we have a contradiction.

TABLE 1

Selected Prime Factors of $F_5(q)$ and $F_{25}(q)$

q	$F_5(q)$	$F_{25}(q)$
5	11, 71	101, 251, 401, 9384251
11	3221	3001*, 24151
71	211, 2221*	
101	31, 491, 1381*	
401	1231	
9384251	181*, 191	1051, 70051
3221		151, 601*, 1301, 1601
211	1361	
31	17351	4951
1231	3491	55351
191	1871, 13001	5101*, 10151, 38351
1051	241*	2351, 19751
1301	61*	701, 6451
13001	1801*, 5431, 17981, 32491	

5. THE PROOF OF (C)

Assume that $n = p^{\alpha}M^{48}$ is perfect. Then $7^{48}\| n$ by (4), and if $p \equiv 2 \pmod 3$ then $3^{48}\| n$ by (5). (We note that $p \neq 29$ since otherwise $3\cdot 5\cdot 7 | n$ which is impossible.) According to Table 2, in which the upper half is applicable if $p \equiv 2 \pmod 3$ and the bottom half if $p \equiv 1 \pmod 3$, we see that n is divisible by at least 26 primes congruent to 1 modulo 7, at most one of which can be p. This is a contradiction since, by our lemma. M is divisible by at most 24 such primes.

6. THE PROOF OF (D)

We shall prove a more general result which includes (D) as a special case. Thus, suppose that

$$n = p^{\alpha}p_1^{2\beta_1} \cdots p_t^{2\beta_t} \quad \text{and that} \quad 35 | (2\beta_i + 1) \quad \text{for} \quad i = 1, 2, \cdots, t.$$

If n is perfect then $35^4 | n$ by (4). As in the proof of (B), $p \equiv 1 \pmod{12}$, and from (1) we see that $F_5(5) = 11\cdot 71$ and $F_7(7) = 29\cdot 4733$ each divides n. Referring to Table 3 and noting that either 181 or 86353 is *not* p we see that n is divisible by the primes

$$5, 7, 11, 29, 31, 41, 43, 61^*, 71, 101, 113, 127, 131, 151, 191, 197, 211, 241^*, 251, 271, 281, 491, 911.$$

If m is the product of the primes in this list which are not congruent to 1 modulo 12, then

$$\sigma(n)/n > \sigma(61\cdot 241 m^4)/(61\cdot 241 m^4) > 2.$$

This contradiction shows that n is not perfect.

7. CONCLUDING REMARKS

From the results obtained to date we see that if $n = p^{\alpha}M^{2\beta}$ is perfect then either $2\beta + 1 = q \geqslant 13$ where q is a prime, or $2\beta + 1 = m \geqslant 55$ where m is composite. Thus, it seems reasonable to conjecture that an odd number of the form $p^{\alpha}M^{2\beta}$, M square-free, cannot be perfect. It is clear, however, that the proof must await the development of a new approach: the magnitude of the numbers encountered for which factors must be found makes the attack of

TABLE 2

Selected Prime Factors of $F_7(q)$ and $F_{49}(q)$

q	$F_7(q)$	$F_{49}(q)$
7	29, 4733*	3529
3	1093	491, 4019, 8233, 51157, 131713
29	88009573	197*
3529	7883	16759
1093	14939	883
491	617*, 1051	
131713	43, 239	8527
88009573	71, 22807	4999
16759	701* 6959	6763
7	29, 4733	3529*
29	88009573*	197
4733	70001	83203
197	97847, 2957767	1373
70001	50359, 263621	
83203	43	83497*
2957767	127	
1373	281, 659	
50359	71, 1093*	16759
43	5839	491
16759	701, 6959	883, 6763

TABLE 3

Selected Prime Factors of $F_5(q)$ and $F_7(q)$

q	$F_5(q)$	$F_7(q)$
5	11, 71	
7		29, 4733
71	211	
4733	41, 101	70001
211	292661	
101	31, 491	
70001	61*, 181*	
292661	191, 241*	
191	1871	127, 197
1871	151	911
127		43, 86353*
181*		281
86353*	281	
151		1499
281	271	
1499	131	113
113	251	

the present paper impractical for "large" *deficient* values of $2\beta + 1$ (m is deficient of $\sigma(m) < 2m$), even with the aid of a high-speed computer. Six is perhaps the only value of β for which $2\beta + 1$ is a prime power within reach at present. If, on the other hand, $2\beta + 1 = m$ is abundant (that is, $\sigma(m) > 2m$) then it is trivial that $n = p^{\alpha}M^{2\beta}$ cannot be perfect; for by (4), $m | n$ and this implies that $\sigma(n)/n > \sigma(m)/m > 2$.

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AN INTERESTING SEQUENCE OF FIBONACCI SEQUENCE GENERATORS

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An observation that certain sequences of power residues modulo some primes were generalized Fibonacci sequences led to the investigation of the positive sequence with general term $n^2 - n - 1$. This sequence was found to have some interesting properties.

For example,

$$3^k \equiv 3^{k-1} + 3^{k-2} \pmod 5, \qquad 4^k \equiv 4^{k-1} + 4^{k-2} \pmod{11},$$

$\{5^k\}$ is similarly defined mod 19, etc. If we take as initial values $1, n,$ and define a Fibonacci sequence based on these values, the r^{th} term is given by $nf_{r-1} + f_{r-2}$, where f_r is the r^{th} Fibonacci number. It is then a simple matter to show that $n^2 - n - 1$ divides $n^r - nf_{r-1} - f_{r-2}$. Thus,

$$n^k \equiv n^{k-1} + n^{k-2} \pmod{n^2 - n - 1}.$$

THE SEQUENCE $\{n^2 - n - 1\}$

1. Let $m(n) = n^2 - n - 1$. Let p be prime, and let $p|m(N)$. Then there is a unique partition of p, $p = a + b$, such that $p|m(N + kp)$ and $p|m(N + kp + a)$.

i. That $p|m(N + kp)$ is easily verified

ii. $p|m(N + kp + a)$

$$m(N + kp + a) = N^2 + 2Nkp + 2Na + k^2p^2 + 2kpa + a^2 - N - kp - a - 1 .$$

This is divisible by p if $p|2N + a - 1$.

There is some smallest value of a for which this is true, and this value of a is independent of N. For let $p|m(n)$ $n \ne N$. Then $p|m(N + kp + a')$ for a' such that $p|2n + a' - 1$.

Thus,

$$pk' = a - 1 + 2N, \qquad pk''' = a' = 1 + 2n.$$

Subtracting and adding:

$$pk'' = (a' - a) + 2(n - N) \qquad \text{and} \qquad pk^* = a + a' + 2(N + n - 1).$$

Since

$$p|N^2 - N - 1 \qquad \text{and} \qquad p|n^2 - n - 1 ,$$

then

$$p|(N^2 - N - 1) - (n^2 - n - 1) ,$$

that is, $p|(N - n)(N + n - 1)$.

Either $p|N - n$ or $p|N + n - 1$.

In the former instance it follows that $p|a' - a$, and since both are less than p, $a = a'$. In the latter case $p|a + a'$, and $a + a' = p$, that is, $a' = b$.

2. If $p|m(N)$, then $p|m(N - b)$.

$$m(N - b) = m(N) + b(b - 2N + 1).$$

But

$$b - 2N + 1 = p - a - 2N + 1 = p - (a - 1 + 2N), \qquad \text{and} \qquad p|(a - 1 + 2N).$$

3. If a prime p appears as a factor in the sequence it does appear at these regular intervals of a and b, and only then. For let

$$p\,|\,m(N),\quad p\,|\,m(N+a)\quad\text{and}\quad p\,|\,m(N+a+x),\quad a+x \leqslant p.$$

$$m(N+a+x) = m(N+a)+x(2N+a-1)+(a+x).$$

Since $p\,|\,m(N+a)$ and $p\,|\,2N+a-1$, p must divide $a+x$. But this is possible only if $p=a+x$, and $x=b$.

4. Let

$$m(N) = p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t},$$

p_i prime, $t>1$. We have $N^2 > m(N) > (N-1)^2$. No $p=N$, for if $m(N) = p\cdot Q$ with $p=N$, we have

$$Q = N-1-\frac{1}{N},$$

which is impossible. Thus some $p<N$. But in that event $N-p>0$ and $p\,|\,m(N-p)$, yielding: if $p\,|\,m(N)$, then

$$p = m(N)\ \text{or}\ p\,|\,m(n)$$

for some $n<N$.

5. All factors of $m(N)$ terminate in $1, 5$ or 9. The period for $m(N)$ *modulo* 10 is $1, 5, 1, 9, 9$. The product of such elements terminates in $1, 5$ or 9. Since $N^2 > m(N)$, at most one p can exceed N, and by (4) at most one prime factor new to the sequence can be introduced per term. If we assume for $n<k$ all factors terminate in $1, 5$ or 9, and if $m(N) = p\cdot Q$ for $N \geqslant k$, with p a new factor, then since Q terminates in $1, 5$ or 9 so must p.

6. Further, it is true that every prime of the form $10n \pm 1$ is a member of the sequence.

i. First we establish that 5 is a quadratic residue of every prime of the form $10n\pm 1$. If p is an odd prime $(p \neq 5)$, then by the Law of Quadratic Reciprocity,

$$\left(\frac{5}{p}\right)\left(\frac{p}{5}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{p-1}{2}} = +1.$$

Thus $(p/5) = (5/p)$, and if 5 is a quadratic residue of p, p is also a quadratic residue of 5, that is, $5\,|\,x^2-p$ for some x. It is easily verified that $p \equiv \pm 1 \mod 10$.

ii. There are two incongruent solutions to $x^2-5 \equiv 0 \mod p$, z and $p-z$. One is odd, the other even. Let z be odd, and let $N = (z+1)/2$.

$$N^2-N-1 = \tfrac{1}{4}(z^2-5).\quad p\,|\,z^2-5 \quad\therefore\ p\,|\,N^2-N-1.$$

7. An examination of the sequence reveals an unexpected number of terms which are prime. However, this situation cannot be expected to continue. It is known that primes of the form $10n\pm 1$ and $10n\pm 3$ are equinumerous [1], and that $\sum 1/p$, p prime, diverges.

$$\sum_{n=2}^{\infty} 1/n^2-n-1$$

converges, as must the subseries consisting of terms which are prime. The implication being, terms, n^2-n-1, which are prime must become rarer as n increases.

SOME TERMS OF $m(n) = n^2-n-1$

n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$
2	1	12	131	22	461	32	991	42	1721	52	11·241	62	19·199	72	19·269	82	29·229	92	11·761
3	5	13	5·31	23	5·101	33	5·211	43	$5\cdot 19^2$	53	5·19·29	63	5·11·71	73	5·1051	83	5·1361	93	5·29·59
4	11	14	181	24	19·29	34	19·59	44	31·61	54	2861	64	29·139	74	11·491	84	6971	94	8741
5	19	15	11·19	25	599	35	29·41	45	1979	55	2969	65	4159	75	31·179	85	$11^2\cdot 59$	95	8929
6	29	16	239	26	11·59	36	1259	46	2069	56	3079	66	4289	76	41·139	86	7309	96	11·829
7	41	17	271	27	701	37	11^3	47	2161	57	3191	67	4421	77	5851	87	7481	97	9311
8	5·11	18	5·61	28	5·151	38	5·281	48	5·11·41	58	5·661	68	5·911	78	5·1201	88	5·1531	98	5·1901
9	71	19	11·31	29	811	39	1481	49	2351	59	11·311	69	4691	79	61·101	89	41·191	99	89·109
10	89	20	379	30	11·79	40	1559	50	31·79	60	3539	70	11·439	80	71·89	90	8009	100	19·521
11	109	21	419	31	929	41	11·149	51	2549	61	3659	71	4969	81	11·19·31	91	19·431		

REFERENCE

1. Daniel Shanks, *Solved and Unsolved Problems in Number Theory,* Vol. 1, p. 22.

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A RAPID METHOD TO FORM FAREY FIBONACCI FRACTIONS

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One question that might be asked after discussing the properties of Farey Fibonacci fractions [1] is the following: Is there any rough and ready method of forming the Farey sequence of Fibonacci numbers of order F_n, given n, however large? The answer is in the affirmative, and in this note we discuss the method. To form a standard Farey sequence of arbitrary order is no easy job, for the exact distribution of numbers coprime to an arbitrary integer cannot be given. The advantage of the Farey sequence of Fibonacci numbers is that one has a regular method of forming $f\cdot f_n$ without knowledge of $f\cdot f_m$ for $m < n$. We demonstrate our method with $F_9 = 34$; that is, we form $f\cdot f_9$.

STEP 1: Write down in ascending order the points of symmetry—fractions with numerator 1. (We use Theorem 1.1 here.)

$$\frac{1}{34},\ \frac{1}{21},\ \frac{1}{13},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{1}.$$

STEP 2: Take and interval $(1/2, 1/1)$. Write down successively as demonstrated the alternate members of the Fibonacci sequence in increasing magnitude beginning with 2, less than or equal to F_n, for a prescribed $f\cdot f_n$. This will give a sequence of denominators

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34}.$$

STEP 3: Choose the maximum number of the Fibonacci sequence $\leqslant F_n$ not written in Step 2, and with this number as starting point write down successively the alternate numbers of the Fibonacci sequence in descending order of magnitude until 1.

$$\frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}.$$

STEP 4: Put these two sequences together, the latter written later. (Theorem 1.2 has been used.)

$$\frac{1}{2},\ \frac{}{5},\ \frac{}{13},\ \frac{}{34},\ \frac{}{21},\ \frac{}{8},\ \frac{}{3},\ \frac{1}{1}.$$

STEP 5: Use the fact that $f_{(r+k)n}$, $f_{(r-k)n}$ have same denominators (Theorem 1.1) to get the sequence of denominators in all other intervals.

$$\frac{}{21},\frac{1}{34},\frac{1}{21},\frac{}{34},\frac{1}{13},\frac{}{34},\frac{}{21},\frac{1}{8},\frac{}{21},\frac{}{34},\frac{}{13},\frac{1}{5},\frac{}{13},\frac{}{34},\frac{}{21},\frac{}{8},\frac{1}{3},\frac{}{8},\frac{}{21},\frac{}{34},\frac{}{13},\frac{}{5},\frac{1}{2},\frac{}{5},\frac{}{13},\frac{}{34},\frac{}{21},\frac{}{8},\frac{}{3},\frac{1}{1}$$

STEP 6: Use the concept of factor of an interval to form numerators. The numerators of $(1/2, 1/1)$ will differ in suffix one from the corresponding denominators. The numerators of $(1/3, 1/1)$ will differ by suffix 2 from the corresponding denominators, $\cdots$. Use the above to form numerators and hence the Farey sequence in $[0,1]$. The first fraction is $0/F_{n-1}$.

$$\frac{0}{21},\frac{1}{34},\frac{1}{21},\frac{2}{34},\frac{1}{13},\frac{3}{34},\frac{2}{21},\frac{1}{8},\frac{3}{21},\frac{5}{34},\frac{2}{13},\frac{1}{5},\frac{3}{13},\frac{8}{34},\frac{5}{21},$$

$$\frac{2}{8},\frac{1}{3},\frac{3}{8},\frac{8}{21},\frac{13}{34},\frac{5}{13},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{8}{13},\frac{21}{34},\frac{13}{21},\frac{5}{8},\frac{2}{3},\frac{1}{1}.$$

To form the fractions in the intervals $(1,2), (2,3), (3,5), \cdots$, write the reciprocals in reverse order of the fractions in $(1/2, 1)$ in $f \cdot f_{n+1}$, of $(1/3, 1/2)$ in $f \cdot f_{n+2}, \cdots$, respectively. This gives $f \cdot f_n$ as far as we want it.

In fact, one of the purposes of investigating the symmetries of Farey Fibonacci sequences was to develop easy methods to form them.

REFERENCE

1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 13, No. 1 (Feb. 1975), pp.

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A SIMPLE PROOF THAT PHI IS IRRATIONAL

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Most proofs of the irrationality of phi, the golden ratio, involve the concepts of number fields and the irrationality of $\sqrt{5}$. This proof involves only very simple algebraic concepts.

Denoting the golden ratio as ϕ, we have

$$\phi^2 - \phi - 1 = 0 .$$

Assume $\phi = p/q$, where p and q are integers with no common factors except 1. For if p and q had a common factor, we could divide it out to get a new set of numbers, p' and q'.

Then

$$(p/q)^2 - p/q - 1 = 0$$

$$(p/q)^2 - p/q = 1$$

$$p^2 - pq = q^2$$

$$p(p-q) = q^2 \tag{1}$$

Equation (1) implies that p divides q^2, and therefore, p and q have a common factor. But we already know that p and q have no common factor other than 1, and p cannot equal 1 because this would imply $q = 1/\phi$, which is not an integer. Therefore, our original assumption that $\phi = p/q$ is false and ϕ is irrational.

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SYMMETRIC SEQUENCES

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This paper deals with integer sequences governed by linear recursion relations. To avoid useless duplication, sequences with terms having a common factor greater than one will be considered equivalent to the sequence with the greatest common factor of the terms eliminated. The recursion relation governing a sequence will be taken as the recursion relation of lowest order which it obeys.

Symmetric sequences are of two types:

A. Sequences with an Unmatched Zero Term

(1) $$\cdots T_{-3}, T_{-2}, T_{-1}, T_0, T_1, T_2, T_3, \cdots$$

with

$$T_n = T_{-n}$$

B. Sequences with All Matched Terms

(2) $$\cdots T_{-3}, T_{-2}, T_{-1}, T_1, T_2, T_3, \cdots$$

FIRST-ORDER SEQUENCES

The recursion relation of the first order is:

(3) $$T_{n+1} = aT_n$$

which will have all terms integers only if $a = \pm 1$. The only sequences governed by such relations subject to the initial restrictions given above are:

$$\cdots 1, 1, 1, 1, 1, 1, \cdots$$

$$\cdots -1, 1, -1, 1, -1, 1, \cdots$$

These sequences and the sequence $\cdots 0, 0, 0, 0, \cdots$ will be eliminated from consideration in the work that follows.

SECOND-ORDER SEQUENCES

For a recursion relation

$$T_{n+1} = aT_n + bT_{n-1}$$

to have all integer terms, the quantity b must be $+1$ or -1. The same applies to sequences of higher order. These will be denoted Case I $(+1)$ and Case II (-1).

Case I. $$T_{n+1} = aT_n + T_{n-1}$$

A. Zero Term

$$T_0 = T_2 - aT_1, \qquad T_{-1} = T_1 - aT_0 = T_1 - aT_2 + a^2T_1 = T_1, \qquad a(aT_1 - T_2) = 0.$$

Thus either $a = 0$ or $T_0 = 0$. $a = 0$ leads to sequences such as:

$$\cdots 2, 3, 2, 3, 2, 3, 2, 3, \cdots$$

If $T_0 = 0$,

$$T_{-2} = T_2 = T_0 - aT_{-1} = -aT_1 .$$

Hence $T_2 = aT_1$ and $T_2 = -aT_1$ with the result that $a = 0$.

B. No Zero Term

$$T_{-1} = T_2 - aT_1 = T_1, \qquad (a+1)T_1 = T_2, \qquad T_{-2} = T_2 = T_1 - aT_{-1} = (1-a)T_1 .$$

Therefore $aT_1 = 0$. If $T_1 = 0$, all the terms are zero. If $a = 0$, we have the type of sequence given above for this value.

Case II. $$T_{n+1} = aT_n - T_{n-1} .$$

A. Zero Term

(4) $$T_0 = aT_1 - T_2, \qquad T_{-1} = T_1 = aT_0 - T_1 = a^2T_1 - aT_2 - T_1$$
$$(a^2-2)T_1 - aT_2 = 0, \qquad T_{-2} = T_2 = aT_{-1} - T_0 = aT_{-1} - aT_1 + T_2 = T_2 .$$

If symmetry holds up to T_n, then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and hence the entire sequence will be symmetrical.

EXAMPLES

For any value of a, select T_1 and T_2 to satisfy (4) in order to generate a symmetric sequence. Thus for $a = 3$, $7T_1 = 3T_2$, giving the sequence:

$$\cdots 47, 18, 7, 3, 2, 3, 7, 18, 47, \cdots$$

governed by

$$T_{n+1} = 3T_n - T_{n-1} .$$

For $a = 8$, $62T_1 = 8T_2$, giving the sequence:

$$\cdots 1921, 244, 31, 4, 1, 4, 31, 244, 1921, \cdots$$

governed by $T_{n+1} = 8T_n - T_{n-1}$.

B. No Zero Term

The relations

$$T_{-1} = T_1 = aT_1 - T_2 \quad \text{and} \quad T_{-2} = aT_{-1} - T_1$$

both lead to

$$(a-1)T_1 = T_2 .$$

If $T_{-n} = T_n$ holds up to n, then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and the symmetry will be maintained throughout the sequence.

For $a = 5$, $T_2 = 4T_1$ giving a sequence

$$\cdots 19, 4, 1, 1, 4, 19, 91, 436, \cdots$$

governed by

$$T_{n+1} = 5T_n - T_{n-1} .$$

THIRD-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + T_{n-2} .$$

A. Zero Term

$$T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \qquad T_0 = T_3 - aT_2 - bT_1,$$
$$T_{-1} = T_1 = T_2 - aT_1 - bT_0 = T_2 - aT_1 - bT_3 + abT_2 + b^2T_1$$

(5) $$(b^2 - a - 1)T_1 + (ab+1)T_2 = bT_3 .$$

Also

$$T_{-2} = T_2 = T_1 - aT_0 - bT_{-1} = T_1 - aT_3 + a^2T_2 + abT_1 - bT_1$$

from which

(6) $$(ab - b + 1)T_1 + (a^2-1)T_2 = aT_3$$

$$T_{-3} = T_3 = T_0 - aT_{-1} - bT_{-2} = T_3 - aT_2 - bT_1 - aT_1 - bT_2$$

so that

(7) $$(a+b)(T_1+T_2) = 0\;.$$

Equation (7) will hold if $b=-a$ which makes (5) and (6):

(5′) $$(a^2-a-1)T_1+(1-a^2)T_2 = -aT_3$$

(6′) $$(-a^2+a+1)T_1+(a^2-1)T_2 = aT_3$$

which are the same relation. Since

$$T_4 = aT_3-bT_2+T_1 \quad \text{and} \quad T_{-4} = T_{-1}-aT_{-2}-bT_{-3} = T_1-aT_2+aT_3 = T_4$$

the symmetry persists up to this point. An entirely similar argument shows that it holds in general.

EXAMPLE. For a given value of a, many symmetric sequences can be determined. For $a=5$,

$$19T_1-24T_2 = -5T_3$$

from which one may derive any number of symmetric sequences obeying the relation

$$T_{n+1} = 5T_n-5T_{n-1}+T_{n-2}\,.$$

Examples are:

$$\cdots 1350, 361, 96, 25, 6, 1, 0, 1, 6, 25, 96, 361, 1350, \cdots$$

$$\cdots 363, 98, 27, 8, 3, 2, 3, 8, 27, 98, 363, \cdots, \qquad \cdots 362, 97, 26, 7, 2, 1, 2, 7, 26, 97, 362, \cdots$$

B. No Zero Term

$$T_{n+1} = aT_n+bT_{n-1}+T_{n-2}\,, \quad T_{n-2} = T_{n+1}-aT_n-bT_{n-1}\,, \quad T_{-1} = T_1 = T_3-aT_2-bT_1$$

(8) $$(b+1)T_1+aT_2 = T_3$$

$$T_{-2} = T_2 = T_2-aT_1-bT_{-1}$$

(9) $$(a+b)T_1 = 0$$

which is satisfied if $b=-a$

$$T_{-3} = T_3 = T_1-aT_{-1}-bT_{-2}$$

(10) $$T_3 = (1-a)T_1+aT_2$$

which agrees with (8) when $b=-a$.

If the symmetry holds to $T_n = T_{-n}$, then

$$T_{-n-1} = T_{-n+2}-aT_{-n+1}+aT_{-n} = T_{n-2}-aT_{n-1}+aT_n = T_{n+1}$$

so that all corresponding pairs are equal.

EXAMPLES. For $a=4$, $T_3 = 4T_2-3T_1$ yields many sequences governed by

$$T_{n+1} = 4T_n-4T_{n-1}+T_{n-2}$$

$$\cdots 233, 89, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, 89, 233, \cdots$$

$$\cdots 177, 67, 25, 9, 3, 1, 1, 3, 9, 25, 67, 177, \cdots$$

$$\cdots 265, 100, 37, 13, 4, 1, 1, 4, 13, 37, 100, 265, \cdots$$

Case II. $$T_{n+1} = aT_n+bT_{n-1}-T_{n-2}\,, \qquad T_{n-2} = aT_n+bT_{n-1}-T_{n+1}$$

A. Zero Term

$$T_0 = aT_2+bT_1-T_3\,,\; T_{-1} = T_1 = aT_1+bT_0-T_2 = aT_1+baT_2+b^2T_1-bT_3-T_2$$

(11) $$(a+b^2-1)T_1+(ba-1)T_2-bT_3 = 0$$

$$T_{-2} = T_2 = aT_0-bT_{-1}-T_1 = a^2T_2+abT_1-aT_3+bT_{-1}-T_1$$

(12) $$(ab+b-1)T_1+(a^2-1)T_2-aT_3 = 0$$

$$T_{-3} = T_3 = aT_1+bT_2-aT_2-bT_1+T_3$$

(13) $$(a-b)(T_1-T_2) = 0$$

so that $b=a$ satisfies this relation.

Equations (11) and (12) both become for $b = a$:

(14) $$(a^2 + a - 1)T_1 + (a^2 - 1)T_2 - aT_3 = 0.$$

For $a = 2$, $2T_3 = 5T_1 + 3T_2$ yields an infinity of sequences satisfying

$$T_{n+1} = 2T_n + 2T_{n-1} - T_{n-2}$$

$$\cdots 64, 25, 9, 4, 1, 1, 0, 1, 1, 4, 9, 25, 64, \cdots$$

$$\cdots 129, 49, 19, 7, 3, 1, 1, 1, 3, 7, 19, 49, 129, \cdots$$

$$\cdots 194, 73, 29, 10, 5, 1, 2, 1, 5, 10, 29, 73, 194, \cdots$$

$$\cdots 259, 97, 39, 13, 7, 1, 3, 1, 7, 13, 39, 97, 259, \cdots$$

B. No Zero Term

$$T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \qquad T_{-1} = T_3 - aT_2 - bT_1$$

(15) $$(b+1)T_1 + aT_2 = T_3$$

$$T_{-2} = T_2 = T_2 - aT_1 - bT_{-1}$$

(16) $$(a+b)T_1 = 0\ .$$

Equation (15) becomes $T_3 = (1-a)T_1 + aT_2$ for $b = -a$. Now, $T_{-3} = T_3 = T_1 - aT_{-1} - bT_{-2}$

(17) $$T_3 = (1-a)T_1 + aT_2$$

in agreement with (15) if $b = -a$.

$$T_{-4} = T_{-1} - aT_{-2} + aT_{-3} = aT_3 - aT_2 + T_1$$

whereas

$$T_4 = aT_3 - aT_2 - T_1$$

so that $T_1 = 0$ if $T_{-4} = T_4$.

Similarly setting $T_{-5} = T_5$ makes $T_2 = 0$, etc. Hence this case yields nothing more than the trivial result $\cdots 0,0,0,0,0,\cdots$.

FOURTH-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + T_{n-3}$$

A. Zero Term

$$T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \qquad T_0 = T_4 - aT_3 - bT_2 - cT_1$$

$$T_{-1} = T_1 = T_3 - aT_2 - bT_1 - cT_0 = T_3 - aT_2 - bT_1 - cT_4 + acT_3 + bcT_2 + c^2T_1$$

(18) $$(c^2 - b - 1)T_1 + (bc - a)T_2 + (ac+1)T_3 - cT_4 = 0$$

$$T_{-2} = T_2 = T_2 - aT_1 - bT_0 - cT_{-1} = T_2 - aT_1 - bT_4 + abT_3 + b^2T_2 + bcT_1 - cT_1$$

(19) $$(bc - c - a)T_1 + b^2T_2 + abT_3 - bT_4 = 0$$

$$T_{-3} = T_3 = T_1 - aT_0 - bT_{-1} - cT_{-2} = T_1 - aT_4 + a^2T_3 + abT_2 + acT_1 - bT_1 - cT_2$$

(20) $$(ac - b + 1)T_1 + (ab - c)T_2 + (a^2 - 1)T_3 - aT_4 = 0$$

$$T_{-4} = T_4 = T_0 - aT_{-1} - bT_{-2} - cT_{-3} = T_4 - aT_3 - bT_2 - cT_1 - aT_1 - bT_2 - cT_3$$

(21) $$(a+c)T_1 + 2bT_2 + (a+c)T_3 = 0\ .$$

If this set of four equations in T_1, T_2, T_3, T_4 is to have a non-zero solution, the determinant of the coefficients must be zero.

$$\begin{vmatrix} c^2 - b - 1 & bc - a & ac + 1 & -c \\ bc - c - a & b^2 & ab & -b \\ ac - b + 1 & ab - c & a^2 - 1 & -a \\ a + c & 2b & a + c & 0 \end{vmatrix} = 0$$

from which

(22) $$(a + b + c)(-a + b - c)(a^2 - c^2 + 4b) = 0\ .$$

Before proceeding to further analysis some relations will be derived from equations (18) to (20). From (18) and (19)

(23) $$(c^2+ac-b^2-b)T_1-abT_2+bT_3 = 0\,.$$

From (19) and (20)

(24) $$(b^2-b-ac-a^2)T_1+bcT_2+bT_3 = 0$$

and from (23) and (24)

(25) $$(c^2+a^2+2ac-2b^2)T_1 = b(a+c)T_2\,.$$

THE CONDITION $a+b+c=0$

$b=-a-c$ substituted into (25) gives

$$(c^2+a^2+2ac-2c^2-2a^2-4ac)T_1 = -(a+c)^2T_2$$

so that $T_1=T_2$. Then by (21)

$$(a+c)T_1+2(-a-c)T_1+(a+c)T_3 = 0$$

so that $T_3=T_1$. By (18),

$$(c^2+a+c-1-c^2-ac-a+ac+1)T_1 = cT_4$$

so that $T_4 = T_1$. If the terms up to T_n are all equal to T_1, then

$$T_{n+1} = aT_1+(-a-c)T_1+cT_1+T_1 = T_1$$

so that all terms of the sequence are the same.

THE CONDITION $-a+b-c=0$

$b=a+c$ leads to

$$T_2 = -T_1\,, \quad T_3 = T_1\,, \quad T_4 = -T_1\,.$$

If this alternation holds up to T_n, then

$$T_{n+1} = [a(-1)^{n-1}+(a+c)(-1)^n+c(-1)^{n-1}+(-1)^n]T_1 = (-1)^nT_1$$

so that the alternation continues.

THE CONDITION $a^2-c^2+4b=0$

a and c must be of the same parity.

EXAMPLE: $a = 1, \quad b = 12, \quad c = 7\,.$

Using Eqs. (18), (19) and (20) we obtain:

$$36T_1+83T_2+8T_3-7T_4 = 0, \qquad 76T_1+144T_2+12T_3-12T_4 = 0, \qquad -4T_1+5T_2+0T_3-T_4 = 0\,.$$

from which $T_1:T_2:T_3:T_4 = 3:-7:18:-47$.

Using the recursion relation

$$T_{n+1} = T_n+12T_{n-1}+7T_{n-2}+T_{n-3}$$

and a corresponding backward recursion relation, the following terms were obtained:

$$\cdots 843, -322, 123, -47, 18, -7, 3, -2, 3, -7, 18, -47, 123, -322, 843, \cdots.$$

Second-Order Factor

If the symmetry is to continue beyond a term T_{-n}, the condition for this would be:

$$T_{-n-1} = T_{n+1} = T_{-n+3}-aT_{-n+2}-bT_{-n+1}-cT_{-n} = T_{n-3}-aT_{n-2}-bT_{n-1}-cT_n\,.$$

But

$$T_{n+1} = aT_n+bT_{n-1}+cT_{n-2}+T_{n-3}\,.$$

Hence there is a relation

$$(a+c)T_n+2bT_{n-1}+(a+c)T_{n-2} = 0\,.$$

But since $4b=(c-a)(c+a)$ we have in fact

$$T_n = (a-c)T_{n-1}/2-T_{n-2}\,.$$

Thus if the symmetry is to continue the terms must satisfy a second-order recursion relation. That they do so can be seen from factoring

$$x^4 - ax^3 - bx - c - 1 = 0 \quad \text{into factors} \quad (x^2 + Ex + 1)(x^2 + Fx - 1) = 0,$$

where E is $(c-a)/2$. The conditions would be:

$$(c-a)/2 + F = -a \quad \text{or} \quad F = -(a+c)/2$$

from the coefficient of x cubed and the same value of F comes from the coefficient of x. Then the coefficient of x^2 would be:

$$EF = (-c^2 + a^2)/4 = -b$$

as required. Hence the terms obey this second-order relation and this insures the continuation of symmetry beyond T_{-4}. Note that this is not a proper fourth-order symmetric sequence.

B. No Zero Term

$$T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \qquad T_{-1} = T_1 = T_4 - aT_3 - bT_2 - cT_1$$

(26) $$(c+1)T_1 + bT_2 + aT_3 - T_4 = 0$$

$$T_{-2} = T_2 = T_3 - aT_2 - bT_1 - cT_{-1}$$

(27) $$(b+c)T_1 + (a+1)T_2 - T_3 = 0$$

$$T_{-3} = T_3 = T_2 - aT_1 - bT_{-1} - cT_{-2}$$

(28) $$(a+b)T_1 + (c-1)T_2 + T_3 = 0$$

$$T_{-4} = T_4 = T_1 - aT_{-1} - bT_{-2} - cT_{-3}$$

(29) $$(a-1)T_1 + bT_2 + cT_3 + T_4 = 0.$$

To have a non-zero solution the following determinant must be zero.

$$\begin{vmatrix} c+1 & b & a & -1 \\ b+c & a+1 & -1 & 0 \\ a+b & c-1 & 1 & 0 \\ a-1 & b & c & 1 \end{vmatrix} = 0$$

or

(30) $$(a+b+c)(c^2 - a^2 - 4b) = 0.$$

As in the zero case, the condition $a+b+c=0$ leads to a sequence where all terms are the same. The other condition requires that the fourth-order recursion relation have a second-order factor which the terms of the symmetric sequence must obey. Hence this is a degenerate case also.

Case II. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

A. Zero Term

$$T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n+1}$$

If the symmetry is to continue indefinitely

$$T_{-n-1} = aT_{-n+2} + bT_{-n+1} + cT_{-n} - T_{-n+3}$$

$$T_{n+1} = aT_{n-2} + bT_{n-1} + cT_n - T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

$$(a-c)(T_{n-2} - T_n) = 0$$

so that $a = c$ unless there is to be a recursion relation of lower order.

$$T_0 = aT_3 + bT_2 + aT_1 - T_4, \qquad T_{-1} = T_1 = aT_2 + bT_1 + aT_0 - T_3$$

from which

(31) $$(a^2 + b - 1)T_1 + a(1+b)T_2 + (a^2 - 1)T_3 = aT_4$$

$$T_{-2} = T_2 = aT_1 + b(aT_3 + bT_2 + aT_1 - T_4) + aT_{-1} - T_2$$

from which

(32) $$a(2+b)T_1 + (b^2 - 2)T_2 + abT_3 = bT_4.$$

Other relations simply repeat one of the above. Eliminating T_4 from (31) and (32):

(33) $$(b^2 - b - 2a^2)T_1 + a(b+2)T_2 - bT_3 = 0$$

For given a and b, a suitable selection of T_1 and T_2 will given an integral value for T_3. Thus for $a = 7$, $b = -5$,

$$-68T_1 - 21T_2 = -5T_3 .$$

$$T_1 = 1, \quad T_2 = 2, \quad T_3 = 22 .$$

Then from (31), $T_4 = 149$. The symmetric sequence:

$$\cdots 38494, 6029, 946, 149, 22, 2, 1, 2, 1, 2, 22, 149, 946, 6029, 38494, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 7T_n - 5T_{n-1} + 7T_{n-2} - T_{n-3} .$$

B. No Zero Term

As before the continuation of symmetry for all terms requires that $a = c$ in the relation

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3} .$$

Two relations are obtained from the requirement $T_{-1} = T_1$ and $T_{-2} = T_2$, namely:

(34) $$(a-1)T_1 + bT_2 + aT_3 = T_4$$

(35) $$(b+a)T_1 + (a-1)T_2 = T_3$$

The relations for T_{-3} and T_{-4} repeat these in inverse order.

EXAMPLE: $$a = -2, \quad b = 5, \quad -3T_1 - 3T_2 = T_3$$

$$T_1 = 4, \quad T_2 = 7, \quad T_3 = -9 .$$

Then from (34), $T_4 = 41$.

The symmetric sequence:

$$\cdots 6399, -1810, 506, -145, 41, -9, 7, 4, 4, 7, -9, 41, -145, 506, -1810, 6399, \cdots$$

obeys the recursion relation:

$$T_{n+1} = -2T_n + 5T_{n-1} - 2T_{n-2} - T_{n-3}$$

FIFTH-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} + T_{n-4}$$

A. Zero Term

To insure symmetry for all n we set:

$$T_{-n-1} = T_{n+1} = T_{-n+4} - aT_{-n+3} - bT_{-n+2} - cT_{-n+1} - dT_{-n} = T_{n-4} - aT_{n-3} - bT_{n-2} - cT_{n-1} - dT_n .$$

Combining this with the original recursion relation:

$$(a+d)(T_n + T_{n-3}) + (b+c)(T_{n-1} + T_{n-2}) = 0$$

so that $d = -a$ and $b = -c$ are necessary conditions to prevent reduction to a lower order recurrence relation.

Using the same techniques as previously we have the relations:

(36) $$(a^2 + b - 1)T_1 + (ab - b)T_2 + (-ab - a)T_3 + (1 - a^2)T_4 + aT_5 = 0$$

(37) $$(ab - b + a)T_1 + (b^2 - a - 1)T_2 + (1 - b^2)T_3 - abT_4 + bT_5 = 0 .$$

Eliminating T_5 from (36) and (37) gives:

(38) $$(b^2 - b + ab - a^2)T_1 + (a^2 + a - b^2)T_2 + (-ab - a)T_3 + bT_4 = 0 .$$

EXAMPLE: $a = 5$, $b = -3$ from which

$$-28T_1 + 21T_2 + 10T_3 = 3T_4$$

which is satisfied by $T_1 = 1$, $T_2 = 3$, $T_3 = 4$, $T_4 = 25$. Then from (36)

$$21T_1 - 12T_2 + 10T_3 - 24T_4 = -5T_5$$

which gives $T_5 = 115.$

The sequence

$$\cdots 190299, 43060, 9745, 2203, 498, 115, 25, 4, 3, 1, -2, 1, 3, 4, 25, 115, 498, 2203, 9745, 43060, 190299, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

B. No Zero Term

An entirely similar analysis leads to two relations:

$$T_5 = (1-a)T_1 - bT_2 + bT_3 + aT_4 \tag{39}$$

$$T_4 = (-b-a)T_1 + (b+1)T_2 + aT_3 \tag{40}$$

EXAMPLE. $a = 5,\ b = -3.$ From (40),

$$T_4 = -2T_1 - 2T_2 + 5T_3$$

which is satisfied by $T_1 = 1,\ T_2 = 3,\ T_3 = 4,\ T_4 = 12.$

Then by (39), $T_5 = -4T_1 + 3T_2 - 3T_3 + 5T_4 = 53.$ The sequence

$$\cdots 19428, 4397, 995, 227, 53, 12, 4, 3, 1, 1, 3, 4, 12, 53, 227, 995, 4397, 19428, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

Case II. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} - T_{n-4} .$$

In this case symmetry in the sequence requires that $a = d$ and $b = c.$

A. Zero Case

The final relations obtained from the analysis are:

$$(a^2 + b - 1)T_1 + (ab + b)T_2 + (ab + a)T_3 + (a^2 - 1)T_4 = aT_5 \tag{41}$$

$$(ab + a + b)T_1 + (b^2 + a - 1)T_2 + (b^2 - 1)T_3 + abT_4 = bT_5 \tag{42}$$

from which

$$(b^2 - b - a^2 - ab)T_1 + (b^2 - a^2 + a)T_2 + (ab + a)T_3 = bT_4 . \tag{43}$$

EXAMPLE. $a = 3,\ b = -7.$ (43) becomes

$$68T_1 + 43T_2 - 18T_3 = -7T_4$$

which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 9, \quad T_4 = -5 .$$

Then from (41),

$$T_1 - 28T_2 - 18T_3 + 8T_4 = 3T_5 \qquad \text{gives} \qquad T_5 = -95 .$$

The symmetric sequence:

$$\cdots 2203, -191, -305, -95, -5, 9, 3, 1, -1, 1, 3, 9, -5, -95, -305, -191, 2203, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 3T_n - 7T_{n-1} - 7T_{n-2} + 3T_{n-3} - T_{n-4}$$

B. No Zero Term

The relations obtained are:

$$(a-1)T_1 + bT_2 + bT_3 + aT_4 = T_5 \tag{44}$$

$$(a+b)T_1 + (b-1)T_2 + aT_3 = T_4 \tag{45}$$

$$bT_1 + aT_2 = T_3 \tag{46}$$

EXAMPLE. $a = -5,\ b = 7.$ (46) becomes $7T_1 - 5T_2 = T_3$ which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = -8 .$$

Then (45)

$$2T_1 + 6T_2 - 5T_3 = T_4 \qquad \text{gives} \qquad T_4 = 60..$$

Finally (44)

$$-6T_1 + 7T_2 + 7T_3 - 5T_4 = T_5$$

gives a value $T_5 = -341$. The symmetric sequence:

$$\cdots 72667, -12195, 2053, -341, 60, -8, 3, 1, 1, 3, -8, 60, -341, 2053, -12195, 72667, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = -5T_n + 7T_{n-1} + 7T_{n-2} - 5T_{n-3} - T_{n-4}\,.$$

CONCLUSION

From this investigation the following general approach to creating symmetric sequences of integers governed by linear recursion relations emerges.

(1) Given a linear recursion relation of order k,

$$T_{n+1} = a_1T_n + a_2T_{n-1} + \cdots + a_{k-1}T_{n-k+2} + T_{n-k+1}$$

the condition of symmetry in the sequence requires that:

$$a_j = -a_{k-j}$$

and for the recursion relation:

$$T_{n+1} = a_1T_n + a_2T_{n-1} + \cdots + a_{k-1}T_{n-k+2} - T_{n-k+1}$$

symmetry requires that $a_j = a_{k-j}$.

(2) For the reduced number of parameters a_i, set up a corresponding number of symmetry conditions using the first few terms of the sequence.

(3) Using these conditions, select values for the parameters a_i and then find starting values in integers that satisfy the given conditions.

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SOME INTERESTING NECESSARY CONDITIONS FOR $(a-1)^n + (b-1)^n - (c-1)^n = 0$

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In the present note we obtain certain inequalities which are necessary for the equation of the title to hold for positive integral n and real $a,b,$ and c satisfying $1 < a \leqslant b < c,$ and illustrate with several examples. Several preliminary lemmas are required.

Lemma 1. $(a-1)^x + (b-1)^x - (c-1)^x$ vanishes at $x = n$ if and only if

$$a^x + b^x - c^x = P_{n-1}(x)$$

at $x = 0, 1, \cdots, n,$ where $P_{n-1}(x)$ is a polynomial of degree $n-1$.

Proof. Apply the n^{th} order difference operator Δ^n to $a^x + b^x - c^x$ to obtain

$$\Delta^n(a^x + b^x - c^x) = (a-1)^n a^x + (b-1)^n b^x - (c-1)^n c^x \;,$$

which vanishes at $x = 0$ if and only if $a^x + b^x - c^x$ behaves as a polynomial of degree $n-1$ at $x = 0, 1, \cdots, n.$

A result in Pólya and Szegö [1] is needed for the next lemma and may be stated as follows for present purposes: If $a < b < c$ and $\mu_1, \mu_2,$ and μ_3 are positive, then

$$\mu_1 a^x + \mu_2 b^x - \mu_3 c^x$$

has exactly one real simple zero. As an immediate consequence of this and other elementary considerations we have the following result.

Lemma 2. Let

$$f(x) = a^x + b^x - c^x \;,$$

where $1 < a \leqslant b < c$. Then $f^{(k)}(x)$ has exactly one real simple zero, one stationary point at which $f^{(k)}$ has a positive maximum and to the right of which $f^{(k)}$ is monotone decreasing.

In the following we will always let $f(x)$ and $P_{n-1}(x)$ be as stated in Lemmas 1 and 2.

Lemma 1 says that

$$F(x) \equiv f(x) - P_{n-1}(x)$$

has at least $n+1$ zeros. That this is the exact number is assured by the next result.

Lemma 3. $F(x) \equiv f(x) - P_{n-1}(x)$ has at most $n+1$ zeros (counting multiplicity).

Proof. Assume that F has at least $n+2$ zeros. Then $F^{(n)}$ has at least 2 zeros. Since $P_{n-1}^{(n)} \equiv 0$ this implies that $f^{(n)}$ has 2 zeros in contradiction to Lemma 2.

Write

$$P_{n-1}(x) = c_1 + c_2 x + \cdots + c_n x^{n-1} \,.$$

Our final preliminary result may be stated as follows.

Lemma 4. $c_n > 0.$

Proof. We know that

$$f(x) - P_{n-1}(x) = 0$$

at the $n+1$ points $x = 0, 1, \cdots, n.$ Thus

$$f^{(n-1)}(x) = (n-1)!c_n$$

at two points which because of Lemma 2 implies that c_n is positive.

Now consider the special case when $n = 2$.

Theorem 1. If $(a-1)^2+(b-1)^2-(c-1)^2 = 0$ then

(1) $$ab/c < e^{a+b-c-1},$$

(2) $$a^ab^b/c^c > e^{a+b-c-1},$$

and

(3) $$a^{a^2}b^{b^2}/c^{c^2} < e^{a+b-c-1}.$$

Proof. By the preceding lemmas we know that in $P_1(x) = c_1 + c_2x$ we have $c_2 > 0$, that

$$f(x) = a^x + b^x - c^x$$

is monotone decreasing for all sufficiently large x, and that $f(x) - P_1(x)$ has simple zeros at precisely $x = 0, 1, 2$. This requires that $f'(2) < P_1'(2)$ and in turn $f'(1) > P_1'(1)$ and $f'(0) < P_1'(0)$. In other words, using the last of the three inequalities, we have $\ln(ab/c) < c_2$. c_2 can be easily determined from the coincidence of $f(x)$ and $P_1(x)$ at $x = 0, 1, 2$ to give $c_2 = a+b-c-1$. Hence, finally, $ab/c < e^{a+b-c-1}$. The inequalities (2) and (3) follow in a similar manner from $f'(1) > P_1'(1)$ and $f'(2) < P_1'(2)$.

For the case of $n = 3$, the following result can be obtained by arguments similar to those used above for Theorem 1. The proof is therefore omitted.

Theorem 2. If $(a-1)^3+(b-1)^3-(c-1)^3 = 0$, then

(1) $$ab/c > e^{a+b-c-1-c_3},$$

(2) $$a^ab^b/c^c < e^{a+b-c-1+c_3},$$

(3) $$a^{a^2}b^{b^2}/c^{c^2} > e^{a+b-c-1+3c_3},$$

and

(4) $$a^{a^3}b^{b^3}/c^{c^3} < e^{a+b-c-1+5c_3},$$

where

$$c_3 = \tfrac{1}{2}[a^2+b^2-c^2+1-2a-2b+2c].$$

Inequalities of a similar nature may be found for any given value of n, however let us proceed to a result for arbitrary n. By $L_n(a)$ we shall mean the partial sum of the first $n-1$ terms of the formal Maclaurin series for log a, i.e.,

$$L_n(a) = \sum_{k=1}^{n-1} (-1)^{k+1}\frac{a^k}{k}.$$

Theorem 3. Let $(a-1)^n+(b-1)^n-(c-1)^n = 0$. Then

$$(-1)^n(\log a + \log b - \log c) < (-1)^n[L_n(a)+L_n(b)-L_n(c)].$$

Proof. Proceeding as for Theorem 1, we find that

$$(-1)^nf'(0) < (-1)^nP'_{n-1}(0).$$

Write

$$P_{n-1}(x) = \sum_{k=0}^{n-1} c_kx^{(k)},$$

where

$$x^{(k)} = x(x-1)\cdots(x-n+1).$$

Gregory-Newton interpolation gives

$$c_k = \Delta^kf(0)/k!.$$

Now

$$\Delta^ka^x = (a-1)^ka^x,$$

from which it follows that

$$\Delta^kf(0) = (a-1)^k+(b-1)^k-(c-1)^k.$$

Therefore, since

$$\frac{d}{dx} x^{(k)}\Big|_{x=0} = (-1)^{k-1}(k-1)!\,,$$

we have

$$(-1)^n(\ln a + \ln b - \ln c) < (-1)^n \sum_{k=1}^{n-1} (-1)^{k+1} \frac{(k-1)!}{k!} [(a-1)^k + (b-1)^k - (c-1)^k]\,,$$

as desired.

We give an indication, in the following examples, of the sharpness of the inequalities obtained above. First we take $n = 2,\ a = 4,\ b = 5,$ in which case inequalities (2) and (3) of Theorem 1 yield $c < 6.5$ and $c > 5.9,$ respectively, bracketing the known solution $c = 6$. This example corresponds to the well-known Pythagorean triple $3,4,5$ which satisfies $3^2 + 4^2 = 5^2$. If we now take $n = 3,\ a = 2,\ b = 3,$ then inequalities (2) and (4) of Theorem 2 give $c < 3.2$ and $c > 3,$ whereas the actual solution of

$$1 + 2^3 - (c-1)^3 = 0$$

is

$$c = 1 + \sqrt[3]{9} \cong 3.08.$$

The sharpness of these results seems rather surprising when one considers that they are based on such simple considerations as the relative slope of two curves at their points of intersection.

REFERENCE

1. G. Pólya and G. Szegö, *Aufgahen und Lehrsätze aus der Analysis,* Berlin, 1925.

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FIBONACCI TILES

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1. INTRODUCTION

The conventional method of tiling the plane uses congruent geometric figures. That is, the plane is covered with non-overlapping translates of a given shape or tile [1]. Such tilings have interesting algebraic models in which the centers of each tile play an important role.

The plane can also be tiled with squares whose sides are in 1:1 correspondence with the Fibonacci numbers in the manner shown in Fig. 1 and such patterns can be used to demonstrate interesting algebraic properties of the Fibonacci numbers [2].

Similar spiral patterns can be obtained with squares whose sides are in 1:1 correspondence with similar recursive sequences of positive real numbers as in Fig. 2.

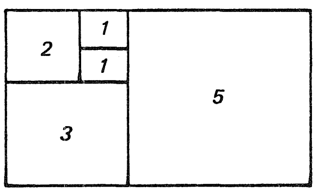


Figure 1

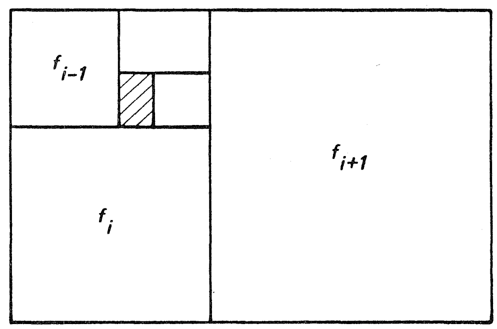


Figure 2

We will show that the centers of the squares in such a pattern all lie on two perpendicular straight lines and the slopes of these lines are independent of the choice of f_1 and f_2. Furthermore, the distances of the centers from the intersection of these two lines also form a recursive sequence.

2. CONSTRUCTION OF THE PATTERN

The pattern in Fig. 2 is a counter-clockwise spiral of squares which fills the plane except for a small initial rectangle. The side of the i^{th} square is denoted by f_i and the f_i are defined by

(1) $$f_{i+2} = f_{i+1} + f_i \quad \text{for} \quad i \geq 1 \quad \text{and} \quad 0 < f_1 \leq f_2 .$$

The side of the first square is f_1 and for notational convenience we define

$$f_i = f_{i+2} - f_{i+1} \quad \text{for} \quad i \leq 0 .$$

The position of successive squares in the spiral can be conveniently expressed in terms of an appropriate corner point of each square and a sequence of vectors which are parallel to the sides of the squares. Consider the sequence of vectors V_i defined by

$$V_1 = (1,0) \qquad V_{i+1} = V_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for} \quad i \geq 1.$$

This sequence consists of four distinct vectors:

(2) $$V_i \in \{(1,0), (0,1), (-1,0), (0,-1)\} \quad .$$

The vestors in this sequence have the property that $V_{i+2} = -V_i$.

If P_1 denotes the lower right corner point of the first square (see Fig. 3) then successive corner points are given by

(3) $$P_i = P_{i-1} + f_{i+1} V_i \ .$$

The center C_i of the i^{th} square is obtained from the corresponding corner point (see Fig. 4) by means of the equation

(4) $$C_i = P_i + \frac{f_i}{2}\,(V_{i+1} - V_i) \ .$$

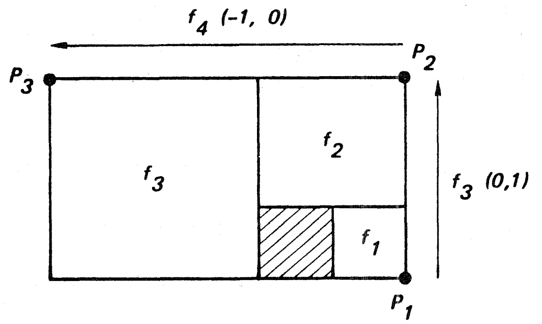


Figure 3

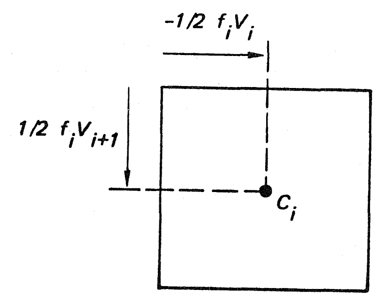


Figure 4

We now proceed to obtain an expression for the vector between alternate centers. Some sample values for P_i and C_i, are given in Tables 1 and 2.

TABLE 1

i	f_i	P_i	C_i	$d_i\sqrt{10}$
1	1	(1, −1)	(0.5, −0.5)	3
2	2	(1, 2)	(0, 1)	4
3	3	(−4,2)	(−2.5, 0.5)	7
4	5	(−4,−6)	(−1.5, −3.5)	11
5	8	(9, −6)	(5, −2)	18
6	13	(9, 15)	(2.5, 8.5)	29
7	21	(−25, 15)	(−14.5, 4.5)	47
8	34	(−25, 40)	(−8, −23)	76
9	55	(64, −40)	(36.5, −12.5)	123
10	89	(64, 104)	(19.5, 59.5)	199
11	144	(−169, 104)	(−97, 32)	322
12	233	(−169, −273)	(−52.5, −156.5)	521

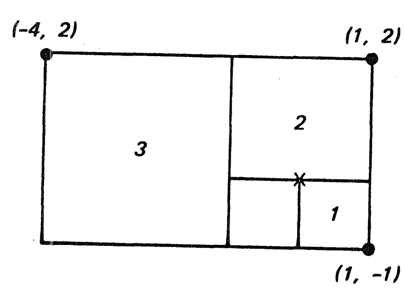
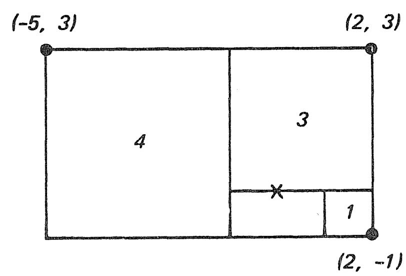


TABLE 2

i	f_i	P_i	C_i	$d_i\sqrt{10}$
1	1	(2, −1)	(1.5, −0.5)	5
2	3	(2, 3)	(0.5, 1.5)	10
3	4	(−5, 3)	(−3. 1)	15
4	7	(−5. 8)	(−1.5, −4.5)	25
4	11	(13, −8)	(7.5, −2.5)	40
6	18	(13, 21)	(4,12)	65
7	29	(−34, 21)	(−19.5, 6.5)	105
8	47	(−34, −55)	(−10.5, −31.5)	170
9	76	(89, −55)	(51, −17)	275
10	123	(89, 144)	(27.5, 82.5)	445
11	199	(−233, 144)	(−133.5, 44.5)	720
12	322	(−233, −377)	(−72, −216)	1165



3. STRUCTURAL PROPERTIES

Lemma 1.

$$C_i - C_{i-2} = \frac{f_{i-1}}{2}\,(3V_i - V_{i+1})\,.$$

Proof. From Eq. (4), we have

$$C_i = P_i + \frac{f_i}{2}\,(V_{i+1} - V_i)$$

$$C_{i-2} = P_{i-2} + \frac{f_{i-2}}{2}\,(V_{i-1} - V_{i-2}) = P_{i-2} + \frac{f_{i-2}}{2}\,(V_i - V_{i+1})$$

(5) $$C_i - C_{i-2} = P_i - P_{i-2} + \frac{f_i}{2}\,(V_{i+1} - V_i) - \frac{f_{i-2}}{2}\,(V_i - V_{i+1})\,.$$

Combining Eqs. (5) and (6) and collecting terms in V_i and V_{i+1} we have

$$C_i - C_{i-2} = \tfrac{1}{2}(2f_{i+1} - f_i - f_{i-2})V_i + \tfrac{1}{2}(f_{i-2} - f_i)V_{i+1}\quad .$$

Using the recursive definition of the f_i (see Eq. (1)), this reduces to

$$C_i - C_{i-2} = \frac{3f_{i-1}}{2}V_i - \frac{f_{i-1}}{2}\,V_{i+1}\,\,.$$

Corollary 1.1. The distance between alternating centers is given by :

$$|C_i - C_{i-2}| = \frac{f_i\sqrt{10}}{2}\qquad .$$

Proof. From the definition of the V_i we have

$$V_i \cdot V_i = 1 \qquad \text{and} \qquad V_i \cdot V_{i+1} = 0$$

$$|C_i - C_{i-2}|^2 = (C_i - C_{i-2})\cdot(C_i - C_{i-2}) = \frac{9}{4}f_{i-1}^2 + \frac{1}{4}f_{i-1}^2 = \frac{10}{4}f_{i-1}^2 .$$

Lemma 2. C_i, C_{i+2}, and C_{i+4} are colinear for all $i \geq 1$.

Proof. From Lemma 1 we have

$$C_{i+4} - C_{i+2} = \frac{f_{i+5}}{2}(3V_{i+4} - V_{i+5}) = -\frac{f_{i+5}}{2}(3V_{i+2} - V_{i+3}) = -\frac{f_{i+5}}{f_{i+3}}\cdot\frac{f_{i+3}}{2}(3V_{i+2} - V_{i+3}) = -\frac{f_{i+5}}{f_{i+3}}(C_{1+2} - C_i).$$

Hence $C_{i+4} - C_{i+2}$ is a multiple of $C_{i+2} - C_i$ and both vectors have the point C_{i+2} in common.

Theorem 1. The C_i all lie on two perpendicular straight lines. The slopes of these lines are 3 and $-(1/3)$ independent of the choice of f_1 and f_2.

Proof. By Lemma 2 we need only consider the slopes of $C_4 - C_2$ and $C_3 - C_1$.

$$C_4 - C_2 = \left(-\frac{f_3}{2}, -\frac{3f_3}{2}\right) \quad \text{and} \quad C_3 - C_1 = \left(-\frac{3f_2}{2}, \frac{f_2}{2}\right) .$$

Hence the slopes are 3 and $-(1/3)$.

Definition 1. Let I be the point of intersection for the two lines in Theorem 1, then the distance from C_i to I will be denoted by d_i. That is $d_i = |C_i - I|$. (Sample values are given in Tables 1 and 2.)

Lemma 3.

$$d_i + d_{i-2} = \frac{f_{i-1}\sqrt{10}}{2}, \quad d_i^2 + d_{i-1}^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2) .$$

Proof. By the definition of d_i we have

$$d_i + d_{i-2} = |C_i - C_{i-2}|$$

and hence the first equation follows from Corollary 1.1.

From Equation 4, we have

$$C_{i-1} = P_{i-1} + \frac{f_{i-1}}{2}(V_i - V_{i-1}) = P_{i-1} + \frac{f_{i-1}}{2}(V_i + V_{i+1})$$

$$C_i - C_{i-1} = P_i - P_{i-1} + \frac{f_1}{2}(V_{i+1} - V_i) - \frac{f_{i-1}}{2}(V_i + V_{i+1}).$$

Since $P_i - P_{i-1} = f_{i+1}V_i$ we have

$$C_i - C_{i-1} = \tfrac{1}{2}(2f_{i+1} - f_i - f_{i-1})V_i + \tfrac{1}{2}(f_i - f_{i-1})V_{i+1} = \frac{f_{i+1}}{2}V_i + \frac{f_{i-2}}{2}V_{i+1} .$$

$$|C_i - C_{i-1}|^2 = (C_i - C_{i-1})(C_i - C_{i-1}) = \tfrac{1}{4}(f_{i+1} + f_{i-2}) .$$

By Theorem 1 the triangle formed by the points C_i, C_{i-1}, and I is a right triangle.

$$d_i^2 + d_{i-1}^2 = |C_i - C_{i-1}|^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2) .$$

We now proceed to find an explicit expression for the d_i which leads to the fact that the d_i form a recursive sequence.

Theorem 2.

$$d_i = \frac{f_{i+3} + f_{i-3}}{2\sqrt{10}} .$$

Proof. Let C_{i-2}, C_{i-1}, and C_i be three consecutive centers

$$d_i^2 + d_{i-1}^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2)$$

$$d_{i-1}^2 + d_{i-2}^2 = \tfrac{1}{4}(f_i^2 + f_{i-3}^2)$$

(7) $$d_i^2 - d_{i-2}^2 = \tfrac{1}{4}(f_{i+1}^2 - f_i^2 + f_{i-2}^2 - f_{i-3}^2) = \tfrac{1}{4}(f_{i+2}f_{i-1} + f_{i-4}f_{i-1})$$

Also,

$$(8) \qquad d_i^2 - d_{i-2}^2 = (d_i + d_{i-2})(d_i - d_{i-2}) = \frac{f_{i-1}\sqrt{10}}{2}\,(d_i - d_{i-2})\,.$$

Combining (7) and (8) we have

$$d_i - d_{i-2} = \frac{1}{2\sqrt{10}}\,(f_{i+2} + f_{i-4})$$

and from Lemma 3

$$d_i + d_{i-2} = \frac{f_{i-1}\sqrt{10}}{2}\,.$$

Adding the last two equations we obtain

$$d_i = \frac{f_{i+2} + f_{i-4} + 10f_{i-1}}{4\sqrt{10}}\,.$$

It is a straightforward albeit tedious exercise to verify from Equation (1) that

$$f_{i+2} + f_{i-4} + 10f_{i-1} - 2f_{i+3} - 2f_{i-3} = 0$$

$$f_{i+2} + f_{i-4} + 10f_{i-1} = 2(f_{i+3} + f_{i-3})$$

$$\therefore\ d_i = \frac{f_{i+3} + f_{i-3}}{2\sqrt{10}}$$

Theorem 3.

$$d_{i+2} = d_{i+1} + d_i\,.$$

Proof.

$$d_{i+1} + d_i = \frac{1}{2\sqrt{10}}\,(f_{i+4} + f_{i-2} + f_{i+3} + f_{i-3})$$

$$= \frac{1}{2\sqrt{10}}\,(f_{i+5} + f_{i-1}) = d_{i+2}$$

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EMBEDDING A SEMIGROUP IN A RING

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Let S be a set of arbitrary cardinality. For each element $s \in S$, define a function $a_s : S \to Z_2$ by

$$a_s(t) = \begin{cases} 0 \text{ if } s \neq t \\ 1 \text{ if } s = t \end{cases} .$$

Denote the set of all such functions by $X(S)$. There is obviously a 1-1 correspondence between S and $X(S)$ by mapping $s \to a_s$.

Let $f: S \to S$ be an arbitrary map. Define a map $m_f : S \times S \to Z_2$ by

$$m_f(t,s) = \begin{cases} 1 \text{ if } f(s) = t \\ 0 \text{ otherwise} \end{cases} ,$$

and define a map $\bar{f} : X(S) \to X(S)$ by

$$\bar{f}(a_s)(v) = \sum_{u \in s} m_f(v,u) a_s(u) .$$

Clearly,

$$\bar{f}(a_s) = a_{f(s)} ,$$

and there is a 1-1 correspondence between S^s = the set of all functions of S into itself and

$$M = \{ m_f \mid f \in S^s$$

under the mapping $f \to m_f$. M is actually a semigroup if we define multiplication on M by

$$m_f m_g(u,v) = \sum_{s \in S} m_f(u,s) m_g(s,v) .$$

This semigroup is clearly isomorphic to the semigroup S^s under composition of mappings.

With the above considerations, we can prove the following:

Theorem. Every semigroup may be embedded in a ring.

Proof. Let G be a semigroup. It is isomorphic to a semigroup of mappings G_x on a set S, i.e., a subsemigroup of S^s, hence a subsemigroup of M [1, p. 20].

If we define $+$ and $\cdot$ on $Z_2^{S \times S}$ by $(i+j)(u,v) = i(u,v) + j(u,v)$,

$$(i \cdot j)(u,v) = \sum_{s \in S} i(u,s) j(s,v) .$$

This clearly makes $Z_2^{S \times S}$ a ring, and M is a subsemigroup of its multiplicative semigroup.

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THE SEQUENCE: $1\ 5\ 16\ 45\ 121\ 320\ \cdots$ IN COMBINATORICS

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Extremely dedicated Fibonaccists might possibly recognize that this sequence can be derived by subtracting 2 from every other Lucas number. The purpose of this note is to describe how this rather bizarre sequence arises naturally in two quite disparate areas of combinatorics. For completeness, and to guarantee uniformity of notation, all basic definitions will be given.

A. FIBONACCI SEQUENCES

Any sequence $\{x_1, x_2, x_3, \cdots\}$ that satisfies $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$ will be called a Fibonacci sequence; such a sequence is completely determined by x_1 and x_2. The Fibonacci sequence $\{F_n\}$ with $F_1 = F_2 = 1$ is the sequence of Fibonacci numbers; the Fibonacci sequence $\{L_n\}$ with $L_1 = 1$, $L_2 = 3$ is the sequence of Lucas numbers. For reference, the first few numbers of these two sequences are given as follows:

n:	1	2	3	4	5	6	7	8	9	10	11	12
F_n:	1	1	2	3	5	8	13	21	34	55	89	144
L_n:	1	3	4	7	11	18	29	47	76	123	199	322

There are of course many identities involving these numbers; two which will be used here are:

$$F_{k+2} = 3F_k - F_{k-2} \qquad k \geq 3 .$$

$$L_k = 3F_k - 2F_{k-2} \qquad k \geq 3 .$$

Both of these identities can be verified by a straightforward induction argument.

B. THE FUNDAMENTAL MATRIX

In both of the combinatorial examples to be discussed, it will be important to evaluate the determinant of the $n \times n$ matrix A_n which is defined as:

$$A_n = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 3 \end{bmatrix} .$$

In words, A_n has 3's on the digaonal, -1's on the super- and sub-diagonals, -1's in the lower left and upper right-hand corners, and 0's elsewhere. This description explains why we set

$$A_1 = [1], \qquad \text{and} \qquad A_2 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} .$$

To facilitate the evaluation of $\det A_n$, define T_n to be the $n \times n$ continuant with 3's on the diagonal, -1's on the super- and sub-diagonals, and 0's elsewhere. That is:

$$T_n = \begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 3 \end{bmatrix}$$

Lemma. $\det T_n = F_{2n+2}$.

Proof. The lemma is certainly true for $n = 1$ and $n = 2$, since

$$T_1 = [3], \qquad \text{and} \qquad T_2 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Thus we will assume that the lemma is true for all $k < n$, and expand $\det T_n$ by the first row:

$$\det T_n = 3 \det T_{n-1} - (-1) \det \left[\begin{array}{c|c} -1 & -1 \\ \hline & T_{n-2} \end{array}\right] = 3 \det T_{n-1} - \det T_{n-2} = 3F_{2n} - F_{2n-2} = F_{2n+2}.$$

We are now able to verify that the sequence $\{\det A_1, \det A_2, \det A_3, \cdots\}$ is the sequence in the title.

Theorem. $\det A_n = L_{2n} - 2$.

Proof. The theorem is true for A_1 and A_2 as defined above; this can be easily verified. Now for $n > 2$, we can expand $\det A_n$ by its first row to obtain:

(1) $$\det A_n = 3 \det T_{n-1} - (-1) \det R_{n-1} + (-1)^{n+1}(-1) \det S_{n-1},$$

where R_n and S_n are $n \times n$ matrices defined by:

$$R_n = \left[\begin{array}{c|c} -1 & -1 \\ \hline & T_{n-1} \\ -1 & \end{array}\right] \qquad \text{and} \qquad S_n = \left[\begin{array}{c|c} -1 & T_{n-1} \\ \hline -1 & -1 \end{array}\right].$$

Notice that T_{n-1} is symmetric, so we have

$$S_n^t = \left[\begin{array}{c|c} -1 & -1 \\ \hline T_{n-1} & \\ & -1 \end{array}\right].$$

Thus:

(2) $$\det S_n = \det S_n^t = (-1)^{n-1} \det R_n.$$

Now, expanding $\det R_n$ by the first column, we obtain:

$$\det R_n = (-1) \det T_{n-1} + (-1)^{n+1}(-1) \det \begin{bmatrix} -1 & & & \\ 3 & \ddots & & \\ -1 & \ddots & \ddots & \\ & -1 & 3 & -1 \end{bmatrix} = -\det T_{n-1} + (-1)^{n+2}(-1)^{n-1}.$$

Thus:

(3) $$\det R_n = -\det T_{n-1} - 1.$$

We can now substitute (2) and (3) into (1), and we obtain:

$$\det A_n = 3 \det T_{n-1} + (-\det T_{n-2} - 1) + (-1)^{n+2}(-1)^{n-2}(-\det T_{n-2} - 1) = 3 \det T_{n-1} - 2 \det T_{n-2} - 2.$$

Then by using the Lemma and an identity mentioned earlier, we have:

$$\det A_n = 3F_{2n} - 2F_{2n-2} - 2 = L_{2n} - 2.$$

C. SPANNING TREES OF WHEELS

This section begins with some very basic definitions from graph theory. The reader uninitiated in this subject is urged to consult one of the many texts in this field (for example, [1] or [2]).

A graph on n vertices is a collection of n points (called vertices), some pairs of which are joined by lines (called edges).

A subgraph of a graph consists of a subset of the vertices, together with some (perhaps all or none) of the edges of the original graph that connect pairs of vertices in the chosen subset.

A subgraph containing all vertices of the original graph is called a spanning subgraph.

A graph is connected if every pair of vertices is joined by a sequence of edges.

A cycle is a sequence of three or more edges that goes from a vertex back to itself.

A tree is a connected graph containing no cycles. It is easy to verify that any tree with n vertices must have exactly $n-1$ edges.

A spanning tree of a graph is a spanning subgraph of the graph that is in fact a tree. Two spanning trees are considered distinct if there is at least one edge not common to them both.

Given a graph G, the complexity of the graph, denoted by $k(G)$, is the number of distinct spanning trees of the graph.

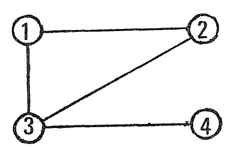
If a graph G has n vertices, number them $1, 2, \cdots, n$. The adjacency matrix of G, denoted by $A(G)$, is an $n\times n$ $(0,1)$ matrix with a 1 in the (i,j) position if and only if there is an edge joining vertex i to vertex j.

For any vertex i, the degree of i, denoted by $\deg i$, is the number of edges that are joined to i. Let $D(G)$ be the $n\times n$ diagonal matrix whose (i,i) entry is $\deg i$.

We are now able to state a quite remarkable theorem, attributed in [2] to Kirkhoff. For a proof of this theorem, see [1], page 159, or [2], page 152.

For any graph G, $k(G)$ is equal to the value of the determinant of any one of the n principal $(n-1)$-rowed minors of the matrix $D(G)-A(G)$.

As a simple example to illustrate this theorem, consider the graph G:

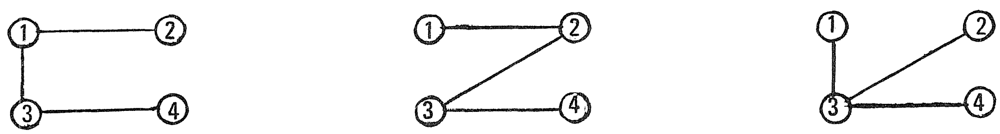


$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and thus

$$D(G)-A(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

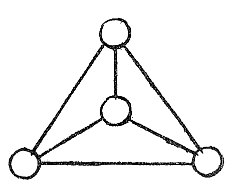
Each of the four principal 3-rowed minors of $D(G)-A(G)$ has determinant 3. The 3 spanning trees of G are:



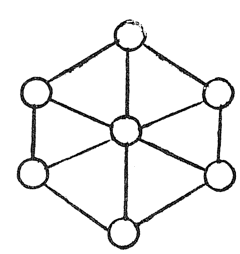
The relevance of these ideas to the title sequence will be established after making one more definition.

For $n \geqslant 3$, the n-wheel, denoted by W_n, is a graph with $n+1$ vertices; n of these vertices lie on a cycle (the rim) and the $(n+1)^{st}$ vertex (the hub) is connected to each rim vertex.

W_3:



W_6:



Theorem. $$k(W_n) = L_{2n}-2.$$

Proof. Number the rim vertices $1, 2, \cdots, n$; the hub vertex is $n+1$. Each rim vertex i has degree 3; it is adjacent to vertices $i-1$ and $i+1 \pmod n$ and to vertex $n+1$. The hub vertex has degree n and is adjacent to all other vertices. Thus

$$D(W_n) - A(W_n) = \left[\begin{array}{ccc|c} & & & -1 \\ & A_n & & \vdots \\ & & & -1 \\ \hline -1 & -1 & \cdots -1 & n \end{array}\right] .$$

To compute $k(W_n)$, any n-rowed principal minor will do. So delete row and column $n+1$. Then we have, by previous results:

$$k(W_n) = \det A_n = L_{2n} - 2 .$$

This result can be found in [4] and in [7], but in neither instance is the number expressed explicitly in terms of the Lucas numbers. In [7], the formula for $k(W_n)$ is given by:

$$k(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

while in [4] the result is expressed:

$$k(W_n) = F_{2n+2} - F_{2n-2} - 2 .$$

Readers familiar with Fibonacci identities will have no trouble verifying that both of these expressions are equivalent to the value given in the theorem.

D. GENERALIZING TOTAL UNIMODULARITY

A matrix M is said to be totally unimodular if every non-singular submatrix of M has determinant ± 1. Since the individual entries are 1×1 submatrices, they must necessarily be $0, \pm 1$. The following theorem, found in [3], provides sufficient conditions for total unimodularity:

Let M be a matrix satisfying the following four conditions:

(1) All entries of M are $0, \pm 1$.
(2) The rows of M are partitioned into two disjoint sets T_1 and T_2.
(3) If any column has two non-zero entries of the same sign, then one is in a row of T_1 and the other in a row of T_2.
(4) If any column has two non-zero entries of opposite sign, then they are both in rows of T_1 or both in rows of T_2.

Then M is totally unimodular.

This result usually includes the additional condition that there be at most two non-zero entries per column; this, however, is actually a consequence of conditions (3) and (4).

We are thus motivated to consider the class $\mathcal{M}$ of matrices which satisfy conditions (1), (2), and (3), but not (4). If $M \in \mathcal{M}$, then as an immediate consequence of (3), we see that there are at most four non-zero entries in any column of M; at most two non-zero entries (with opposite sign) in rows of T_1, and at most two non-zero entries (with opposite sign) in rows of T_2.

It is then natural to define the subclasses: $\mathcal{M}'' \subset \mathcal{M}' \subset \mathcal{M}$, where any matrix in $\mathcal{M}'$ satisfies conditions (1), (2), and (3) and has at most three non-zero entries per column; any matrix in $\mathcal{M}''$ satisfies (1), (2), and (3), and has at most two non-zero entries per column. An obvious problem is to find the maximum determinantal value of an $n \times n$ matrix in any one of these three classes. This problem is completely solved only for the class $\mathcal{M}''$; the following theorem appears in [6]:

If M is any $n \times n$ matrix in the class $\mathcal{M}''$, then $\det M \leqslant 2^{[n/2]}$. Moreover, for each $n \geqslant 1$, there is an $n \times n$ matrix in $\mathcal{M}''$ whose determinant achieves this upper bound.

The title sequence is relevant in considering the class $\mathcal{M}'$. For any $k \geqslant 1$, let I_k be the $k \times k$ identity matrix, and define J_k to be the $k \times k$ matrix with 1's on the diagonal, -1's on the super-diagonal, and a -1 in the lower left-hand corner. That is,

$$J_k = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ -1 & & & & 1 \end{bmatrix} .$$

Then for n even, say $n = 2k$, we can define the $n \times n$ matrices H_n and G_n as follows:

$$H_n = \begin{bmatrix} I_k & -J_k^t \\ J_k & I_k \end{bmatrix} \qquad G_n = \begin{bmatrix} I_k & 0 \\ -J_k & I_k \end{bmatrix} .$$

Notice first that $H_n \in \mathcal{M}'$. Now since $det\, G_n = 1$, we have:

$$det\, H_n = det\,(H_n G_n) = det \begin{bmatrix} I_k + J_k^t J_k & -J_k^t \\ 0 & I_k \end{bmatrix} = det\,(I_k + J_k^t J_k) .$$

But the (i,j) entry of $J_k^t J_k$ is simply the inner product of the i^{th} and j^{th} columns of J_k. It is thus not difficult to verify that

$$I_k + J_k^t J_k = A_k ,$$

where A_k is the fundamental matrix of this paper. We have thus verified the following result:

For n even, there is an $n \times n$ matrix in $\mathcal{M}'$ with determinant $L_n - 2$. A comparable result for odd n is proved in [5].

For n odd, there is an $n \times n$ matrix in $\mathcal{M}'$ with determinant $2F_n - 2$. It is my present conjecture that, for any given n, these determinantal values are the maximum possible for an $n \times n$ matrix in the class $\mathcal{M}'$, or in the class $\mathcal{M}$.

Finally, it should be noted that totally unimodular matrices occur naturally in the formulation of a problem in optimization theory known as the transportation problem. In [6], it is shown that matrices from class $\mathcal{M}$ arise in a discussion of the two-commodity transportation problem.

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ON NON-BASIC TRIPLES

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Definition 1. A set of integers $\{b_i\}_i \geqslant 1$ will be called a base for the set of all integers whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} a_i b_i ,$$

where $a_i = 0$ or 1 and

$$\sum_{i=1}^{\infty} a_i < \infty .$$

Thus, a base is obtained by taking $b_i = \pm 2^i$ for each i so long as terms of each sign are used infinitely often. Also, a sequence $\{d_i\}_i \geqslant 1$ of odd numbers will be called basic whenever the sequence

$$\{d_i 2^{i-1}\} i \geqslant 1$$

is a base. If the sequence $\{d_i\}_i \geqslant 1$ of odd integers is such that $d_{i+s} = d_i$ for all i's, then the sequence is said to be periodic mod s and is denoted by $\{d_1, d_2, d_3, \cdots, d_s\}$.

Theorem 1. A basic sequence remains basic whenever a finite number of odd numbers is added, omitted, or replaced by other odd numbers.

Proof. This is proved in [1].

Theorem 2. A necessary and sufficient condition for the sequence $\{d_i\}_i \geqslant 1$ of odd integers, which is periodic mod s, to be basic is that

$$0 \neq \sum_{i=1}^{m} a_i 2^{i-1} d_i \equiv 0 \pmod{2^{ns} - 1}$$

is impossible for $n \geqslant 1$, and $a_i = 0$ or 1 for all $i \geqslant 1$.

Proof. This is also proved in [1].

Theorem 3. Let a, b, c be a periodic mod 3. If $a = d(2^{3K} + 1)$, where d is an integer and

(1) $$d + 2b + 4c \equiv 0 \pmod 7,$$

or

(2) $$b + 2d + 4c \equiv 0 \pmod 7,$$

or

(3) $$c + 2d + 4b \equiv 0 \pmod 7,$$

or

(4) $$c + 2b + 4d \equiv 0 \pmod 7,$$

or

(5) $$d + 2c + 4b \equiv 0 \pmod 7,$$

or

(6) $$b + 2c + 4d \equiv 0 \pmod 7,$$

then a,b,c is non-basic.

Proof. In case (1) holds, consider the expression

$$\begin{aligned} u &= a + 2b + 2^2c + \cdots + 2^{3K-3}a + 2^{3K-2}b + 2^{3K-1}c + 2^{3K+1}b + 2^{3K+2}c + \cdots + 2^{6K-2}b + 2^{6K-1}c \\ &= a(1 + 2^3 + \cdots + 2^{3K-3}) + 2b(1 + 2^3 + \cdots + 2^{6K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}) \\ &= a \cdot \frac{2^{3K-1}}{2^3 - 1} + 2b \cdot \frac{2^{6K-} - 1}{2^3 - 1} + 2^2c \cdot \frac{2^{6K} - 1}{2^3 - 1} \\ &= d(2^{3K} + 1) \cdot \frac{2^{3K} - 1}{2^3 - 1} + 2b \cdot \frac{2^{6k} - 1}{2^3 - 1} + 2^2c \cdot \frac{2^{6k} - 1}{2^3 - 1} = \frac{(d + 2b + 2^2c)(2^{6K} - 1)}{2^3 - 1} \;. \end{aligned}$$

It follows that u is divisible by $2^{6K} - 1$ since, by hypothesis,

$$(2^3 - 1) \mid (d + 2b + 2^2c).$$

Hence, by applying Theorem 2 with $n = 3$ and $s = 2k$, $\{a,b,c\}$ is not basic.

Suppose now that (2) holds and that $\{a,b,c\}$ is basic. By Theorem 1, we may interchange a with b the first $3K$ times these numbers appear in the sequence $\{a,b,c\}$ and still have a basic sequence. Consider

$$\begin{aligned} v &= b + 2a + 2^2c + \cdots + 2^{3K-3}b + 2^{3K-2}a + 2^{3K-1}c + 2^{3K}b + 2^{3K+2}c + \cdots + 2^{6K-3}b + 2^{6K-1}c \\ &= b(1 + 2^3 + \cdots + 2^{6K-3}) + 2a(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}). \end{aligned}$$

As above, this reduces to

$$v = \frac{(b + 2d + 2^2c)(2^{6K} - 1)}{2^3 - 1}$$

and since $(2^3 - 1) \mid (b + 2d + 2^2c)$, v is divisible by $2^{6K} - 1$. But then, as before $\{a,b,c\}$ is not basic.

The remaining cases are handled in the same way, with an appropriate permutation of the first few terms in the sequence $\{a,b,c\}$ and so the proof is complete.

Theorem 4. Let

$$a = \frac{e(2^{6K} - 1)}{2^{2K} - 1} \quad \text{and} \quad b = \frac{d(2^{6K} - 1)}{2^{3K} - 1}\,,$$

where e and d are integers, $K \neq 0$, and $3/K$. If $e + 2d + 2^2c$ is divisible by 7, then $\{a,b,c\}$ is non-basic.

Proof. Consider the expression

$$\begin{aligned} w &= a + 2b + 2^2c + \cdots + 2^{2K-3}a + 2^{2K-2}b + 2^{2K-1}c + 2^{2K+1}b + 2^{2K+2}c + \cdots + 2^{3K-2}b + 2^{3K-1}c + \cdots + 2^{6K-1}c \\ &= a(1 + 2^3 + \cdots + 2^{2K-3}) + 2b(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}) \\ &= a \cdot \frac{(2^{2K} - 1)}{2^3 - 1} + 2b \cdot \frac{(2^{3K} - 1)}{2^3 - 1} + 2^2c \cdot \frac{(2^{6K} - 1)}{2^3 - 1} \\ &= e \cdot \frac{(2^{6K} - 1)}{2^{2K} - 1} \cdot \frac{(2^{2K} - 1)}{2^3 - 1} + 2d \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1} \cdot \frac{(2^{3K} - 1)}{2^3 - 1} + 2^2c \cdot \frac{(2^{6K} - 1)}{2^3 - 1} = \frac{(e + 2d + 2^2c)(2^{6K} - 1)}{2^3 - 1} \;. \end{aligned}$$

Since $e + 2d + 2^2c$ is divisible by 7, w is divisible by $2^{6K} - 1$, and $\{a,b,c\}$ is non-basic by Theorem 2.

Theorem 5. Let

$$a = e \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1} \quad \text{and} \quad b = d \cdot \frac{(2^{6K} - 1)}{2^{3K} - 1}\,,$$

where e and d are integers, $K \neq 0$, $3/K$. If

$$e + 2d + 2^2c$$

is divisible by 7, then $\{a,b,c\}$ is non-basic.

Proof. This time we set

$$
\begin{aligned}
v &= a + 2b + 2^2c + \cdots + 2^{3K-3}a + 2^{3K-2}b + 2^{3K-1}c + 2^{3K+2}c + \cdots + 2^{6K-1}c \\
&= a(1 + 2^3 + \cdots + 2^{3K-3}) + 2b(1 + 2^3 + \cdots + 2^{3K-3}) + 2^2c(1 + 2^3 + \cdots + 2^{6K-3}) \\
&= a \cdot \frac{2^{3K}-1}{2^3-1} + 2b \cdot \frac{2^{3K}-1}{2^3-1} + 2^2c \cdot \frac{2^{6K}-1}{2^3-1} \\
&= e \cdot \frac{2^{6K}-1}{2^{3K}-1} \cdot \frac{2^{3K}-1}{2^3-1} + 2d \cdot \frac{2^{6K}-1}{2^{3K}-1} \cdot \frac{2^{3K}-1}{2^3-1} + 2^2c \cdot \frac{2^{6K}-1}{2^3-1} \\
&= \frac{(e+2d+2^2c)(2^{6K}-1)}{2^3-1} \,.
\end{aligned}
$$

Since

$$e + 2d + 2^2c$$

is divisible by 7, v is divisible by $2^{6K}-1$ and as before $\{a,b,c\}$ is non-basic. In a similar way, we obtain the following theorem.

Theorem 6. Let

$$a = \frac{e(2^{6K}-1)}{2^{2K}-1} \qquad \text{and} \qquad b = \frac{d(2^{6K}-1)}{2^{2K}-1} \,,$$

where e and d are integers, $K \neq 0,\ 3/k.$ If

$$e + 2d + 2^2c$$

is divisible by 7, then $\{a,b,c\}$ is non-basic.

Other similar interesting results may be found in another article in [2].

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NEW RELATIONS BETWEEN FIBONACCI AND BERNOULLI NUMBERS

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1. INTRODUCTION

There seems to be no end to the number or variety of identities involving the Fibonacci sequence and/or its relatives. During the past decade, hundreds of such relations have been published in this journal alone. Those interesting identities, however, are mostly "pure"—containing terms within the same family; that is, not many of them are relations that involve a Fibonacci-type sequence together with some *other* classical sequence having different properties.

The family of Fibonacci-like numbers, for example, satisfies simple recurrence relations with constant coefficients while such famous sequences as those of Bernoulli satisfy more complicated difference equations having variable coefficients. It is thus of interest to pursue the questions: Can these sequences nevertheless be expressed simply in terms of each other? What kinds of identities can one easily find that involve both of them, etc.? Some relations answering such questions have been developed by Gould in [6] and by Kelisky in [8].

This article gives further answers in a systematic way with the use of several simple techniques. The paper will present various explicit relations between Fibonacci numbers and the number sequences of Bernoulli.[1] Relations involving the *generalized* Bernoulli numbers will represent a one-parameter, infinite class of such identities. Little detailed discussion, however, is given of the many special properties of the Bernoulli numbers themselves, for they have been the object of much published research for two hundred years.

2. BACKGROUND PRELIMINARIES

BERNOULLI POLYNOMIALS AND BERNOULLI NUMBERS

We begin by reviewing some prpperties of Bernoulli numbers and polynomials that will be needed for our purpose. The *Bernoulli polynomials* $B_n(x)$ of the n^{th} degree and *first order*[2] may be defined by the exponential generating function

$$(1) \qquad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{t^n}{n!}\,, \qquad |t| < 2\pi\,.$$

(See, for instance, [4] and [10].) More explicitly, these polynomials are given by the equation

$$(2) \qquad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}\,,$$

where B_k are the so-called *Bernoulli numbers.* One definition of the Bernoulli number sequence[3]

$$\left\{ 1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66, \right\} \cdots$$

---

[1]A subsequent paper will explicitly relate the Fibonacci and Lucas sequences to the famous numbers of Euler.

[2]Bernoulli numbers of *higher order* will be defined later.

[3]Rather than $B_n(0)$, some authors prefer to call b_n the ordinary Bernoulli numbers, where $b_n = (-1)^{n+1} B_{2n}$, $n > 1$. The numbers b_n are essentially the absolute values of the non-zero elements in the B_n sequence. All the numbers are *rational;* they have applications in several branches of mathematics, appearing in the theory of numbers in the remarkable theorem of von Staudt-Clausen. (See, for example, [2], [3], and [5].)

is

(3) $$B_k \equiv B_k(0).$$

Alternately, the numbers B_k may be defined by means of the generating formula

(4) $$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad |t| < 2\pi .$$

Using combinatorial techniques given by Riordan in [11], one can *invert* Eq. (2) to obtain

(5) $$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k(x)}{n-k+1} .$$

It can also be shown that special values of $B_n(x)$ are

(6) $$\begin{cases} B_n(0) = (-1)^n B_n(1) = B_n, \; n = 0, 1, 2, \cdots \\ B_1(0) = B_1(1) - 1, \; B_0 = 1 \\ B_{2n+1}(0) = 0, \; n = 1, 2, \cdots \end{cases}$$

and that $B_1 = -\frac{1}{2}$ is the only non-zero Bernoulli number with odd index. We can thus write (2) as

(7) $$B_n(x) = x^n - \frac{n}{2}x^{n-1} + \sum_{k=1}^{[n/2]} \binom{n}{2k} B_{2k}x^{n-2k} .$$

The $(2k)^{th}$ Bernoulli number is computed by means of the recurrence relation

(8) $$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} \sum_{m=0}^{k-1} \binom{2k+1}{2m} B_{2m}, \quad k \geqslant 1$$

with $B_0 = 1$, or explicitly by use of the little-known formula

(9) $$B_{2k} = \sum_{n=0}^{2k} \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{2k}, \quad k \geqslant 0.$$

With this finite sum substituted into (7), it is possible to express the Bernoulli polynomials in a closed form not involving the Bernoulli numbers themselves. In fact (see [7]),

$$B_k(x) = \sum_{n=0}^{k} \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x+j)^k .$$

FIBONACCI POLYNOMIALS AND FIBONACCI NUMBERS

We recall that the *Fibonacci polynomials* $F_n(x)$ of degree $(n-1)$ are solutions of the recurrence relation

(10) $$F_{k+1}(x) = xF_k(x) + F_{k-1}(x), \quad k \geqslant 1$$

with $F_1(x) = 1$ and $F_2(x) = x$. More explicitly, we have

(11) $$F_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} x^{k-2m} ,$$

and note that the numbers

(12) $$F_{k+1}(1) \equiv F_k$$

are the *Fibonacci numbers.* These numbers, and their closest relative, the *Lucas numbers* L_n, are often defined by the familiar *generating functions*

(13) $$\frac{e^{at}-e^{bt}}{\sqrt{5}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad e^{at}+e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

or, in the so-called Binet forms, by the formulas

(14) $$F = \frac{a^n-b^n}{a-b}, \qquad L = a^n+b^n,$$

where

(15) $$a = (1+\sqrt{5})/2, \qquad b = (1-\sqrt{5})/2.$$

3. RELATIONS BETWEEN FIBONACCI AND BERNOULLI NUMBERS

With the above preliminaries, an explicit relation

(16) $$F_{2N+1}(x) = \sum_{k=0}^{2N} C_{k,N}B_k(x) \qquad N \geqslant 0$$

expressing the Fibonacci polynomials of even degree in terms of Bernoulli polynomials, can now be developed in the following simple way. Equation (11) gives

(17) $$F_{2N+1}(x) = \sum_{k=0}^{N} \binom{2N-n}{n} x^{2N-2n},$$

so with the inversion formula (5) inserted in (17), we have

$$F_{2N+1}(x) = \sum_{n=0}^{N} \binom{2N-n}{n} \sum_{k=0}^{2N-2n} \binom{2N-2n}{k} \frac{B_k(x)}{2N-2n-k+1},$$

or, on reversing order of summation,

(18) $$F_{2N+1}(x) = \sum_{k=0}^{2N} B_k(x) \sum_{n=0}^{\left[\frac{2N-k}{2}\right]} \binom{2N-n}{n}\binom{2N-2n}{k} \frac{1}{2N-2n-k+1}.$$

Thus, with coefficients $C_{k,N}$ given by

(19) $$C_{k,N} = \sum_{n=0}^{\left[\frac{2N-k}{2}\right]} \binom{2N-n}{n}\binom{2N-2n}{k} \frac{1}{2N-2n-k+1},$$

we have the desired relation

(20) $$F_{2N+1}(x) = \sum_{k=0}^{2N} C_{k,N}B_k(x).$$

Similarly, for Fibonacci polynomials $F_{2N+2}(x)$ of odd degree, it is easy to show that expressed by

(21) $$F_{2N+2}(x) = \sum_{k=0}^{2N+1} A_{k,N}B_k(x)$$

with coefficients

(22) $$A_{k,N} = \sum_{n=0}^{\left[\frac{2N+1-k}{2}\right]} \binom{2N+1-n}{n}\binom{2N+1-2n}{k} \frac{1}{2N-2n-k+2}.$$

Since $F_n(1) \equiv F_n$, and

$$\left\{\begin{array}{ll} B_k(1) = B_k(0) = B_k, & (k \geq 2) \\ B_{2m+1}(1) \equiv B_{2m+1}(0) \equiv B_{2m+1} = 0, & (m \geq 1) \end{array}\right. \tag{23}$$

the equations (20) and (21) will immediately furnish explicit relations which express Fibonacci numbers in terms of Bernoulli numbers. From (20), with $x = 1$, we thus have

$$F_{2N+1} = C_{1,N}B_1(1) + \sum_{k=0}^{N} C_{2k,N}B_{2k}\,. \tag{24}$$

But, $B_1(1) = -B_1 = \frac{1}{2}$, and

$$C_{1,N} = \sum_{n=0}^{N-1} \binom{2N-n}{n} = -1 + F_{2N+1}\,. \tag{25}$$

Hence,

$$F_{2N+1} \equiv -1 + 2\sum_{k=0}^{N} C_{2k,N}B_{2k}\,, \tag{26}$$

where

$$C_{2k,N} = \sum_{n=0}^{N-k} \binom{2N-n}{n}\binom{2N-2n}{2k}\frac{1}{2N+1-2k-2n}\,. \tag{27}$$

With the same procedure, using (21) and (23), we find that

$$F_{2N+2} = 2\sum_{k=0}^{N} A_{2k,N}B_{2k}\,, \tag{28}$$

where

$$A_{2k,N} = \sum_{n=0}^{N-k} \binom{2N+1-n}{n}\binom{2N+1-2n}{2k}\frac{1}{2N-2n-2k+2}\,. \tag{29}$$

Inverse relations (expressing the Bernoulli polynomials and numbers in terms of those of Fibonacci) are equally important. In [1], the author showed how an analytic function can be expanded in polynomials associated with Fibonacci numbers, so the details of carrying this out in the special case of Bernoulli polynomials will be left to the reader.

4. SOME NEW IDENTITIES

With a little inventive manipulation* and the application of Cauchy's rule for multiplying power series, many new relations between Fibonacci and Bernoulli numbers can be easily obtained. Although these are all special examples of the general case presented later, there may be an advantage to many readers of this Journal to consider them in some detail.

EXAMPLE 1. Starting with Eq. (13), we have

$$e^{at} - e^{bt} = e^{bt}[e^{(a-b)t} - 1] = e^{bt}[e^{t\sqrt{5}} - 1] = \sqrt{5}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!}\,, \tag{30}$$

or

$$te^{bt} = \frac{t\sqrt{5}}{e^{t\sqrt{5}} - 1}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!}\,. \tag{31}$$

*Series manipulation has long been a most powerful fundamental tool for obtaining or operating with generating functions as we shall be doing throughout this article.

Expanding the left-hand side, and noting from (4) that

$$\frac{t\sqrt{5}}{e^{t\sqrt{5}}-1} = \sum_{n=0}^{\infty} B_n \frac{(t\sqrt{5})^n}{n!} \tag{32}$$

one sees that (31) becomes

$$t\sum_{n=0}^{\infty} b^n \frac{t^n}{n!} = \left[\sum_{s=0}^{\infty} B_s(\sqrt{5})^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} F_n \frac{t^n}{n!}\right]. \tag{33}$$

If we make use of Cauchy's rule and equate coefficients, we find the identity

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \frac{F_{n-k+1}}{n-k+1} B_k = b^n , \tag{34}$$

which holds for all $n \geqslant 0$. (It may appear simpler to use

$$\frac{1}{n-k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k} .$$

This can apply to some subsequent formulas presented here.)

EXAMPLE 2. On the other hand, if we write

$$e^{at} - e^{bt} = -e^{at}[e^{-t\sqrt{5}} - 1] = \sqrt{5}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} , \tag{35}$$

we get

$$\sum_{n=0}^{\infty} a^n \frac{t^n}{n!} = \left[\sum_{s=0}^{\infty} (-\sqrt{5})^s B_s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} F_n \frac{t^{n-1}}{n!}\right] ,$$

and thus obtain, since $F_0 = 0$, the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{F_{n-k+1}}{n-k+1} B_k = a^n, \qquad n \geqslant 0. \tag{36}$$

EXAMPLE 3. Recalling that the Lucas numbers are given by

$$L_m = a^m + b^m, \qquad [a = (1+\sqrt{5})/2,\ b = (1-\sqrt{5})/2] , \tag{37}$$

one can add equations (34) and (36) to attain the more interesting identity

$$L_n = 2\sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \frac{F_{n-2k+1}}{n-2k+1} B_{2k} , \qquad n \geqslant 0, \tag{38}$$

which contains three different number sequences—Lucas, Fibonacci, and Bernoulli. [However, subtracting (34) from (36) only gives the trivial identity $F_n = F_n$.]

5. AN EXTENSION

We now make a generalization involving Bernoulli numbers of *higher order.* Use of the same procedures just given will furnish a whole class of new identities.

DEFINITION

*Generalized Bernoulli numbers** $B_n^{(m)}$ of the m^{th} *order* are generated by the expansion

$$\frac{t^m}{(e^t-1)^m} = \sum_{n=0}^{\infty} B_n^{(m)} \frac{t^n}{n!}, \qquad |t| < 2\pi , \tag{40}$$

where m is any positive or negative integer. (If $m = 1$, one writes $B_n^{(m)} \equiv B_n$, omitting the superscript as we did before.) Thus,

*A thorough discussion of the properties of these numbers is given in [9].

$$B_n^{(m)} = \frac{d^n}{dt^n}\left[\left(\frac{t}{e^t-1}\right)^m\right]_{t=0} \tag{41}$$

and we obtain the number sequence

$$B_0^{(m)} = 1,\quad B_1^{(m)} = -\tfrac{1}{2}m,\quad B_2^{(m)} = \frac{1}{12}m(3m-1),\quad B_3^{(m)} = -\frac{1}{8}m^2(m-1), \tag{42}$$

$$B_4^{(m)} = \frac{1}{240}m(15m^3-30m^2+5m+2),\ \cdots .$$

The sequence satisfies the *partial difference equation*

$$mB_n^{(m+1)} - (m-n)B_n^{(m)} + mnB_{n-1}^{(m)} = 0 . \tag{43}$$

If m is a negative integer, i.e., $m = -p,\ p \geqslant 1$, an explicit formula for the numbers is given by

$$B_n^{(-p)} = \frac{n!}{(n+p)!}\sum_{r=0}^{p}(-1)^r\binom{p}{r}(p-r)^{n+p} . \tag{44}$$

SPECIAL CASE OF SECOND ORDER

Let us first consider the case when $m = 2$, and quickly obtain four new identities expressed in Eqs. (47)–(50) below. Note that

$$(e^{at}-e^{bt})^2 = e^{2bt}(e^{t\sqrt{5}}-1)^2$$

and also, using (13) that

$$(e^{at}-e^{bt})^2 = [e^{2at}+e^{2bt}] - 2e^t = \sum_{n=0}^{\infty}[2^nL_n-2]\frac{t^n}{n!}, \tag{45}$$

where L_n are again the Lucas numbers. One may thus write

$$t^2e^{2bt} = \frac{5t^2}{(e^{t\sqrt{5}}-1)^2}\cdot\frac{1}{5}\sum_{n=0}^{\infty}[2^nL_n-2]\frac{t^n}{n!} . \tag{46}$$

Since, from Eq. (40),

$$\frac{(t\sqrt{5})^2}{(e^{t\sqrt{5}}-1)^2} = \sum_{n=0}^{\infty}(\sqrt{5})^nB_n^{(2)}\frac{t^n}{n!},$$

and since

$$e^{2bt} = \sum_{n=0}^{\infty}(2b)^n\frac{t^n}{n!},$$

relation (46) gives

$$5\sum_{n=0}^{\infty}(2b)^n\frac{t^n}{n!} = \left[\sum_{s=0}^{\infty}[2^sL_s-2]\frac{t^{s-2}}{s!}\right]\left[\sum_{n=0}^{\infty}(\sqrt{5})^nB_n^{(2)}\frac{t^n}{n!}\right]$$

We note that $2^sL_s - 2 = 0$ for $s = 0$ and $s = 1$, and we then use Cauchy's rule. For each value of $n \geqslant 0$, there results the identity

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k\frac{[2^{n-k+2}L_{n-k+2}-2]}{(n-k+1)(n-k+2)}B_k^{(2)} = 5(2b)^n \tag{47}$$

involving Lucas numbers and Bernoulli numbers of the second order.*

On the other hand, taking

$$(e^{at}-e^{bt})^2 \quad\text{as}\quad e^{2at}(e^{-t\sqrt{5}}-1)^2$$

leads to the identity

*The Bernoulli number sequence of order 2 is $\left\{1, -1, 5/6, -1/2, 1/10, -1/6, \cdots\right\}$.

(48) $$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{[2^{n-k+2}L_{n-k+2}-2]}{(n-k+1)(n-k+2)} B_k^{(2)} = 5(2a)^n .$$

If Eqs. (47) and (48) are added, one obtains the identity

(49) $$L_n = \frac{2}{5(2)^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{[2^{n-2k+2}L_{n-2k+2}-2]}{(n-2k+1)(n-2k+2)} 5^k B_{2k}^{(2)} ,$$

while subtraction yields the identity

(50) $$F_n = \frac{2}{5(2)^n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k \frac{2-2^{n-2k+1}L_{n-2k+1}}{(n-2k)(n-2k+1)} B_{2k+1}^{(2)} ,$$

both relations being valid for all $n > 0$.

SPECIAL CASE OF NEGATIVE ORDER

Before discussing the most general case, let us take $m = -2$. Now, from (40), it is seen that

$$(t\sqrt{5})^{-2}(e^{t\sqrt{5}}-1)^2 = \sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!} .$$

Thus

$$(e^{at}-e^{bt})^2 = [(t\sqrt{5})^2 e^{2bt}][(t\sqrt{5})^{-2}(e^{t\sqrt{5}}-1)^2] = (t\sqrt{5})^2 \left[\sum_{s=0}^{\infty} (2b)^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!}\right].$$

On the other hand, in view of (13), we have

$$(e^{at}-e^{bt})^2 = [e^{2at}+e^{2bt}] - 2e^t = \sum_{n=0}^{\infty} [2^n L_n - 2] \frac{t^n}{n!} ,$$

and therefore,

$$\frac{1}{5}\sum_{n=0}^{\infty} [2^{n+2}L_{n+2}-2] \frac{t^n}{(n+2)} = \left[\sum_{s=0}^{\infty} (2b)^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!}\right] .$$

From this equation there immediately results the identity

(51) $$L_{n+2} = \frac{1}{2^{n+2}}\left[2+5(n+1)(n+2)\sum_{k=0}^{n} \binom{n}{k} (2b)^{n-k}(\sqrt{5})^k B_k^{(-2)}\right] , \quad n \geqslant 0 .$$

Similarly, starting with

$$(e^{at}-e^{bt})^2 = [(t\sqrt{5})^2 e^{2at}][(t\sqrt{5})^{-2}(e^{-t\sqrt{5}}-1)^2] ,$$

we are led to the identity

(52) $$L_{n+2} = \frac{1}{2^{n+2}}\left[2+5(n+1)(n+2)\sum_{k=0}^{n} \binom{n}{k} (2a)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-2)}\right] .$$

If Eq. (51) is subtracted from (52), one obtains the identity

(53) $$5\sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k B_{2k}^{(-2)} F_{n-2k} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^k B_{2k-1}^{(-2)} L_{n-2k-1} , \quad n \geqslant 1$$

which involves Fibonacci numbers, Lucas numbers, and Bernoulli numbers of negative second order.

GENERAL CASE WHEN m IS AN ARBITRARY NEGATIVE INTEGER

Let $m = -p$ with p being a positive integer. Notice that

$$(54)\qquad (e^{at}-e^{bt})^p = [e^{pat}+(-1)^p e^{pbt}] + \sum_{r=1}^{p-1} (-1)^r \binom{p}{r} e^{[pa+(b-a)r]t}$$

and that

$$[e^{pat}+(-1)^p e^{pbt}] = \sum_{n=0}^{\infty} p^n L_n \frac{t^n}{n!} \qquad \text{if } p \text{ is even,}$$

$$= \sqrt{5}\sum_{n=0}^{\infty} p^n F_n \frac{t^n}{n!} \qquad \text{if } p \text{ is odd.}$$

It is also clear that

$$(55)\quad (e^{at}-e^{bt})^p = [(t\sqrt{5})^p e^{pbt}][(t\sqrt{5})^{-p}(e^{t\sqrt{5}}-1)^p] = (t\sqrt{5})^p \left[\sum_{r=0}^{\infty} (pb)^r \frac{t^r}{r!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-p)} \frac{t^n}{n!}\right].$$

Equating (54) and (55) results in the following two identities:

$$(56)\quad L_{n+p} = \frac{1}{p^{n+p}}\left\{-\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p} + (\sqrt{5})^p \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pb)^{n-k}(\sqrt{5})^k B_k^{(-p)}\right\}$$

if p is even, and

$$(57)\quad F_{n+p} = \frac{1}{p^{n+p}}\left\{-\frac{1}{\sqrt{5}}\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p} + (\sqrt{5})^{p-1} \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pb)^{n-k}(\sqrt{5})^k B_k^{(-p)}\right\}$$

when p is odd. If $p = 1$, the first summation does not appear, and (57) reduces to

$$(58)\qquad F_{n+1} = (n+1)\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(\sqrt{5})^k B_k^{(-1)}.$$

In all these formulas

$$a = (1+\sqrt{5})/2, \quad b = (1-\sqrt{5})/2, \quad \text{and} \quad b-a = -\sqrt{5}.$$

The identities (56) and (57) give new relations for each p, and thus represent a whole class of identities.

Another infinite class of such relations is obtained by beginning with

$$(e^{at}-e^{bt})^p = (-1)^p e^{pat}(e^{-t\sqrt{5}}-1)^p = (-1)^p[(t\sqrt{5})^p e^{pat}][(t\sqrt{5})^{-p}(e^{-t\sqrt{5}}-1)^p]$$

instead of with (55). This consideration yields

$$(59)\qquad L_{n+p} = \frac{1}{p^{n+p}}\left\{-\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p}\right.$$

$$\left. + (\sqrt{5})^p \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pa)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-p)}\right\}$$

if p is even, and

$$(60)\qquad F_{n+p} = \frac{1}{p^{n+p}}\left\{-\frac{1}{\sqrt{5}}\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p}\right.$$

$$\left. + (\sqrt{5})^{p-1} \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pa)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-p)}\right\}$$

when p is odd. For $p = 1$, (60) reduces to

$$F_{n+1} = (n+1)\sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k}(\sqrt{5})^k B_k^{(-1)}. \tag{61}$$

Subtracting relation (56) from (59) yields

$$5\sum_{k=0}^{[n/2]} \binom{n}{2k} p^{n-2k}5^k F_{n-2k}B_{2k}^{(-p)} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} p^{n-2k+1}5^k L_{n-2k+1}B_{2k-1}^{(-p)} \tag{62}$$

while subtracting (57) from (60) gives the same thing. Thus the identity (62) holds for all non-negative p, and for $n \geqslant 1$.

GENERAL CASE WHEN m IS AN ARBITRARY POSITIVE INTEGER

The same techniques of a little creative manipulation and the application of Cauchy's rule is used here. Without giving the details of the development, we shall just present the results.

For *even* positive values of m, one obtains the identities

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \left\{ \frac{m^{n-k+m}L_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m}}{(n-k+m)!} \right\} (n-k)!B_k^{(m)} = (\sqrt{5})^m (mb)^n, \qquad n \geqslant 0 \tag{63}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \left\{ \frac{m^{n-k+m}L_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m}}{(n-k+m)!} \right\} (n-k)!B_k^{(m)} = (\sqrt{5})^m (ma)^n, \tag{64}$$

Adding these two identities yields

$$L_n = \frac{2}{(\sqrt{5})^m m^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \frac{(n-2k)!}{(n-2k+m)!} \left\{ m^{n-2k+m}L_{n-2k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+m} \right\} B_{2k}^{(m)}, \qquad n \geqslant 0 \tag{65}$$

while subtraction gives

$$F_n = \frac{-2}{(\sqrt{5})^m m^n} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^{k-1} \frac{(n-2k+1)!}{(n+2k+1+m)!} \left\{ m^{n-2k+1+m}L_{n-2k+1+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+1+m} \right\} B_{2k-1}^{(m)}, \qquad n \geqslant 1. \tag{66}$$

For *odd* positive values of m, there result the identities

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \frac{(n-k)!}{(n-k+m)!} \left\{ \sqrt{5}m^{n-k+m}F_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m} \right\} B_k^{(m)} = (\sqrt{5})^m (mb)^n, \qquad n \geqslant 0, \tag{67}$$

and

$$(68)\quad \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{(n-k)!}{(n-k+m)!} \left\{ \sqrt{5}\, m^{n-k+m} F_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m} \right\} B_k^{(m)} = (\sqrt{5})^m (ma)^n, \qquad n \geq 0.$$

If (67) and (68) are added or subtracted, we get, respectively,

$$(69)\quad L_n = \frac{2}{(\sqrt{5})^m m^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \left\{ \sqrt{5}\, m^{n-2k+m} F_{n-2k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+m} \right\} \frac{(n-2k)!}{(n-2k+m)!} B_{2k}^{(m)} \qquad n \geq 0$$

and

$$(70)\quad F_n = \frac{-2}{(\sqrt{5})^m m^n} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^{k-1} \left\{ \sqrt{5}\, m^{n-2k+1+m} F_{n-2k+1+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+1+m} \right\} \frac{(n-2k+1)!}{(n-2k+1+m)!} B_{2k-1}^{(m)} \qquad n \geq 1$$

We note that the identities given by each of the above eight relations, involving Bernoulli numbers of positive order, constitute one-parameter infinite classes since a different identity results for each value of $m > 0$.

6. REMARKS

Making a direct connection of Stirling numbers of the second kind to Bernoulli generalized numbers permits one to immediately utilize some of the above results in order to find explicit relations between Stirling numbers and those of Fibonacci or Lucas.

Stirling numbers of the second kind $S(n,j)$, which represent the number of ways of partitioning a set of n elements into j non-empty subsets, are the coefficients in the expansion

$$(71)\qquad x^n = \sum_{j=1}^{n} S(n,j)(x)_j ,$$

where $(x)_j$ is the factorial polynomial

$$(72)\qquad (x)_j = x(x-1)(x-2)\cdots(x-j+1).$$

(See, for example, [11].) Since these numbers are also defined by the generating function

$$(73)\qquad (e^t-1)^m = m! \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!} ,$$

it is easy to show, in view of (40), that they are related to generalized Bernoulli numbers by the simple formula

$$(74)\qquad \frac{(n+p)!}{n!} \binom{n}{k} B_k^{(-p)} = \binom{n+p}{k+p} S(k+p,p).$$

Substitution of this in to relations (56, (57), (59), (60), and (61) will immediately furnish identities involving Stirling numbers together with those of Fibonacci and Lucas. Although the resulting identities would essentially be the *same* (except for new notation or symbolism), they may nevertheless be interesting to those interested in Stirling numbers.

We have developed the identities in this article in a formal way without attempting to explore their implication or to find applications for them. Perhaps this paper will interest some reader to do so, as well as to make simplifications and further extensions. However, as interesting as such formulas may seem, one should pursue the more important question of whether or not they imply any new arithmetical properties, or more beautiful number theoretic theorems, of the various sequences involved.

It was pointed out by Zeitlin, a referee of this paper, that all the results here can be generalized to apply to sequences defined by

$$W_{n+2} = pW_{n+1} - qW_n$$

(See [12] for some properties of such sequences.)

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STAR OF DAVID THEOREM (I)

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The greatest Common divisor property of the binomial coefficients, namely,

$$✡ \qquad GCD\left\{\binom{n-1}{k-1},\ \binom{n}{k+1},\ \binom{n+1}{k}\right\} = GCD\left\{\binom{n+1}{k+1},\ \binom{n}{k-1},\ \binom{n-1}{k}\right\}$$

was conjectured and named as the Star of David Property by H. Gould in 1972 [1]. So far, three solutions appeared [2, 3, 4]. All three proofs were based on the exponents of primes in binomial coefficients of ✡.

An integer matrix multiplication of the integer vectors,

$$\begin{bmatrix}\binom{n-1}{k-1}\\ \binom{n}{k+1}\\ \binom{n+1}{k}\end{bmatrix} = \begin{bmatrix} k+1 & k-n-1 & -n-1\\ -k & n-k+1 & n\\ k+1 & k-n & -n\end{bmatrix}\begin{bmatrix}\binom{n+1}{k+1}\\ \binom{n}{k-1}\\ \binom{n-1}{k}\end{bmatrix},$$

which together with its inverse, i.e.,

$$\begin{bmatrix}\binom{n+1}{k+1}\\ \binom{n}{k-1}\\ \binom{n-1}{k}\end{bmatrix} = \begin{bmatrix} -n & -k & n-k+1\\ n & k+1 & k-n\\ -n-1 & -k-1 & n-k+1\end{bmatrix}\begin{bmatrix}\binom{n-1}{k-1}\\ \binom{n}{k+1}\\ \binom{n+1}{k}\end{bmatrix}$$

shows that a common factor of numbers that appear on one side of ✡ also divides each number of the other side. This proves the Star of David property ✡.

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EULERIAN NUMBERS AND OPERATORS

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1. INTRODUCTION

The Eulerian numbers $A_{n,k}$ are usually defined by means of the generating function

$$\frac{1-y}{e^{x(y-1)}-y} = 1+\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k}y^{k-1} \tag{1.1}$$

or equivalently

$$\frac{1-y}{1-ye^{x(1-y)}} = 1+\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k}y^{k} . \tag{1.2}$$

From either generating function we can obtain the recurrence

$$A_{n+1,k} = (n-k+2)A_{n,k-1} + kA_{n,k} \tag{1.3}$$

and the symmetry relation

$$A_{n,k} = A_{n,n-k+1} . \tag{1.4}$$

For references see [5, pp. 487–491], [6], [7], [8, Ch. 8].

In an earlier expository paper [1] one of the writers has discussed algebraic and arithmetic properties of the Eulerian numbers but did not include any combinatorial properties. The simplest combinatorial interpretation is that A_{nk} is the number of permutations of

$$Z_n = \left\{ 1, 2, \cdots, n \right\}$$

with k rises, where we agree to count a conventional rise to the left of the first element. Conversely if we define $A_{n,k}$ as the number of such permutations, the recurrence (1.3) and the symmetry relation (1.4) follow almost at once but it is not so easy to obtain the generating function.

The symmetry relation (1.4) is by no means obvious from either (1.1) or (1.2). This suggests the introduction of the following symmetrical notation:

$$A(r,s) = A_{r+s+1,s+1} = A_{r+s+1,r+1} = A(s,r). \tag{1.5}$$

It is then not difficult to verify that (1.1) implies

$$\sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} = \frac{e^x - e^y}{xe^y - ye^x} \tag{1.6}$$

from which the symmetry is obvious. Moreover there is a second generating function

$$\sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s)!} = (1+xF(x,y))(1+yF(x,y)), \tag{1.7}$$

where

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} .$$

The generating function (1.7) suggests the following generalization.

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$$\sum_{r,s=0}^{\infty} A(r,s|\alpha,\beta)\,\frac{x^r y^s}{(r+s)!} = (1+xF(x,y))^{\alpha}(1+yF(x,y))^{\beta}\ , \tag{1.8}$$

where the parameters α,β are unrestricted. Clearly

$$A(r,s|1,1) = A(r,s)$$

and

$$A(r,s|\alpha,\beta) = A(s,r|\beta,\alpha)\,.$$

Moreover $A(r,s|\alpha,\beta)$ satisfies the recurrence

$$A(r,s|\alpha,\beta) = (r+\beta)A(r,s-1|\alpha,\beta) + (s+\alpha)A(r-1,s|\alpha,\beta). \tag{1.9}$$

It follows from (1.9) and $A(0,0|\alpha,\beta) = 1$ that $A(r,s|\alpha,\beta)$ is a polynomial in α,β and that the numerical coefficients in this polynomial are positive integers. Algebraic properties of $A(r,s|\alpha,\beta)$ corresponding to the known properties of $A(r,s)$ have been obtained in [3]; also this paper includes a number of combinatorial applications. We shall give a brief account of these results in the present paper. Of the combinatorial applications we mention in particular the following two.

Let $P(r,s,k)$ denote the number of permutations of Z_{r+s-1} with r rises, s falls and k maxima; we count a conventional fall on the extreme right as well as a conventional rise on the left. We show

$$P(r+1,s+1,k+1) = \binom{r+s-2k}{r-k}\,C(r+s,k)\,, \tag{1.10}$$

where

$$A(r,s) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j}\,C(r+s,j)\,; \tag{1.11}$$

$C(r+s,s)$ is equal to the number of permutations of Z_{r+s+1} with $r+1$ rises, $s+1$ falls and $s+1$ maxima. Also we obtain a generating function for $P(r,s,k)$.

The element a_k in the permutation $(a_1a_2\cdots a_n)$ is called a *left upper record* if

$$a_i < a_k \qquad (1 \leqslant i < k)\,;$$

it is a *right upper record* if

$$a_i > a_k \qquad (k < i \leqslant n)\,.$$

Let $A(r,s;t,u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls, t left and u right upper records. Then we show that

$$A(r,s|\alpha,\beta) = \sum_{t,u} A(r,s;t,u)\alpha^{t-1}\beta^{u-1}\ , \tag{1.12}$$

so that the coefficients in the polynomial $A(r,s|\alpha,\beta)$ have a simple combinatorial description.

If we put

$$A_n(x,y|\alpha,\beta) = \sum_{r+s=n} A(r,s|\alpha,\beta)x^r y^s\ ,$$

it follows from the recurrence (1.9) that

$$A_n(x,y|\alpha,\beta) = [\alpha x + \beta y + xy(D_x + D_y)]\,A_{n-1}(x,y|\alpha,\beta).$$

Hence

$$A_n(x,y|\alpha,\beta) = [\alpha x + \beta y + xy(D_x + D_y)]^n \cdot 1. \tag{1.13}$$

Thus it is of interest to expand the operator

$$\Omega^n_{\alpha,\beta}\,[\alpha x + \beta y + xy(D_x + D_y)]^n\ .$$

We show that

$$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x,y)(xy)^k(D_x + D_y)^k\,, \tag{1.14}$$

where

$$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(\alpha+\beta)_k}\,(D_x + D_y)^k A_n(x,y)\ , \tag{1.15}$$

where

$$(\alpha+\beta)_k = (\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1).$$

The case $\alpha+\beta$ equal to zero or a negative integer requires special treatment.

As an application of (1.9) we cite

$$(1.16)\qquad A_{m+n}(x,y\,|\,\alpha,\beta) = \sum_{k=0}^{\min(m,n)} \frac{1}{k!(\alpha+\beta)_k}(xy)^k(D_x+D_y)^k A_m(x,y\,|\,\alpha,\beta)(D_x+D_y)^k A_n(x,y\,|\,\alpha,\beta).$$

For additional results see §8 below.

2. THE NUMBERS $A(r,s)$

Let

$$\pi = (a_1 a_2 \cdots a_n)$$

denote an arbitrary permutation of Z_n. A *rise* is a pair of consecutive elements a_i, a_{i+1} such that $a_i < a_{i+1}$; a *fall* is a pair a_i, a_{i+1} such that $a_1 > a_{i+1}$. In addition we count a conventional rise to the left of a_1 and a conventional fall to the right of a_n. If π has $r+1$ rises and $s+1$ falls, it is clear that

$$(2.1)\qquad r+s = n+1.$$

Let $A(r,s)$ denote the number of permutations of Z_{r+s+1} with $r+1$ rises and $s+1$ falls. Let π be a typical permutation with $r+1$ rises and $s+1$ falls and consider the effect of inserting the additional element $n+1$. If it is inserted in a rise, the number of rises remains unchanged while the number of falls is increased by one; if it is inserted in a fall, the number of rises is increased by one while the number of falls is unchanged. This implies

$$(2.2)\qquad A(r,s) = (r+1)A(r,s-1) + (s+1)A(r-1,s).$$

Next if $\pi = (a_1 a_2 \cdots a_n)$ and we put

$$b_i = n - a_i + 1 \qquad (i = 1, 2, \cdots, n),$$

then corresponding to the permutation π we get the permutation

$$\pi' = (b_1 b_2 \cdots b_n)$$

which has $r+1$ falls and $s+1$ rises. It follows at once that

$$(2.3)\qquad A(r,s) = A(s,r).$$

Another recurrence that is convenient for obtaining a generating function is

$$(2.4)\qquad A(r,s) = A(r,s-1) + A(r-1,s) + \sum_{j<r}\sum_{k<s}\binom{r+s}{j+k+1}A(j,k)A(r-j-1,s-k-1).$$

This recurrence is obtained by deleting the element $r+s+1$ from a typical permutation with $r+1$ rises and $s+1$ falls.

Now put

$$(2.5)\qquad F(z) = \sum_{r,s=0}^{\infty} A(r,s)\frac{x^r y^s z^{r+s+1}}{(r+s+1)!}.$$

By (2.4)

$$\sum_{r,s=0}^{\infty} A(r,s)\frac{x^r y^s z^{r+s}}{(r+s)!} = 1 + \sum_{r,s=0}^{\infty} A(r,s)\frac{x^r y^{s+1} z^{r+s+1}}{(r+s+1)!} + \sum_{r,s=0}^{\infty} A(r,s)\frac{x^{r+1} y^s z^{r+s+1}}{(r+s+1)!}$$

$$+ \sum_{j,k=0}^{\infty} A(j,k)\frac{x^j y^k z^{j+k+1}}{(j+k+1)!}\sum_{r,s=0}^{\infty} A(r,s)\frac{x^{r+1} y^{s+1} z^{r+s+1}}{(r+s+1)!}.$$

This implies

$$(2.6)\qquad F'(z) = 1 + (x+y)F + xyF^2.$$

Since $F(0) = 1$, it is easily verified that the differential equation (2.6) has the solution

$$F(z) = \frac{e^{xz} - e^{yz}}{xe^{yz} - ye^{xz}}.$$

Hence, taking $z = 1$, we get the generating function

(2.7) $$\frac{e^x - e^y}{xe^y - ye^x} = \sum_{r,s=0}^{\infty} A(r,s)\,\frac{x^r y^s}{(r+s+1)!}\quad .$$

It is convenient to put

(2.8) $$F = F(x,y) = \frac{e^x - e^y}{xe^y - ye^x}\quad .$$

It is easily verified that

(2.9) $$(D_x + D_y)F = F^2\ ,$$

(2.10) $$(1 + xD_x + yD_y)F = (1+xF)(1+yF),$$

where $D_x = \partial/\partial x,\ D_y = \partial/\partial y$.

It is evident from (2.7) that

$$(1 + xD_x + yD_y)F = \sum_{r,s=0}^{\infty} A(r,s)\,\frac{x^r y^s}{(r+s)!}\quad .$$

We therefore have the second generating function

(2.11) $$(1 + xF(x,y))(1+yF(x,y)) = \sum_{r,s=0}^{\infty} A(r,s)\,\frac{x^r y^s}{(r+s)!}\quad .$$

We note that iteration of (2.9) gives

(2.12) $$(D_x + D_y)^k F = k!\,F^{k+1}\quad .$$

3. GENERALIZED EULERIAN NUMBERS

Put

(3.1) $$\Phi_{\alpha;\beta} = \Phi_{\alpha,\beta}(x,y) = (1+xF(x,y))^{\alpha}(1+yF(x,y))^{\beta}$$

and define $A(r,s|\alpha,\beta)$ by means of

(3.2) $$\Phi_{\alpha,\beta} = \sum_{r,s=0}^{\infty} A(r,s|\alpha,\beta)\,\frac{x^r y^s}{(r+s)!}\quad .$$

Then we have

$$A(r,s|1,1) = A(r,s)\ ,$$

$$A(r,s|1,0) = A(r-1,s),\qquad A(r,s|0,1) = A(r,s-1)\ ,$$

(3.3) $$A(r,s|\alpha,\beta) = A(s,r|\beta,\alpha)\,,$$

also

(3.4) $$A(r,o|\alpha,\beta) = \alpha^r,\qquad A(o,s|\alpha,\beta) = \beta^s\ .$$

It is easily verified that

(3.5) $$(D_x + D_y)\Phi_{\alpha,\beta} = (\alpha+\beta)F\Phi_{\alpha,\beta}$$

and generally

(3.6) $$(D_k + D_y)^k\,\Phi_{\alpha,\beta} = (\alpha+\beta)_k F^k\,\Phi_{\alpha,\beta}\ ,$$

where

$$(\alpha+\beta)_k = (\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1)\,.$$

In the next place we have

$$(xD_x + yD_y)\Phi_{\alpha,\beta} = \alpha(1+xF)^{\alpha-1}(1+yF)^{\beta}(x + x^2D_x + xyD_y)F + \beta(1+xF)^{\alpha}(1+yF)^{\beta-1}(y+xyD_x+y^2D_y)F$$
$$= [\alpha x + \beta y + (\alpha+\beta)xtF]\,\Phi_{\alpha,\beta}\ .$$

Hence by (3.5)

(3.7) $$(xD_x + yD_y)\Phi_{\alpha,\beta} = [\alpha x + \beta y + xy(D_x + D_y)]\,\Phi_{\alpha,\beta}\ .$$

This yields the recurrence

(3.8) $$A(r,s|\alpha,\beta) = (r+\beta)A(r,s-1|\alpha,\beta) + (s+\alpha)A(r-1,s|\alpha,\beta)\ .$$

We can also show, after some manipulation, that

$$A(r,s|a+k,\beta) = \frac{k!}{(a+\beta)_k}\sum_{t=0}^{r}\binom{s+t}{t}\frac{(a+\beta+r)_{k-t}}{(k-t)!}A(r-t,s+t|a,\beta)\,. \tag{3.9}$$

If we take $s=0$ and make use of (3.4) we get

$$(a+k)^r\binom{a+\beta+k-1}{k} = \sum_{t=0}^{r}\binom{a+\beta+k+t-1}{k-r+t}A(t,r-t|a,\beta)\,. \tag{3.10}$$

If $a+\beta$ is a positive integer, Eq. (3.10) becomes

$$(a+x)^r\binom{a+\beta+x-1}{a+\beta-1} = \sum_{t=0}^{r}\binom{a+\beta+x+t-1}{a+\beta+r-1}A(t,r-t|a,\beta)\,. \tag{3.11}$$

For $a=\beta=1$, Eq. (3.11) reduces to the known formula

$$(x+1)^{r+1} = \sum_{t=0}^{r}\binom{x+t+1}{r+1}A(t,r-t) = \sum_{t=0}^{r}\binom{x+t+1}{r+1}A_{r+1,t+1}\,. \tag{3.12}$$

In order to get an explicit expression for $A(r,s|a,\beta)$ we take

$$1+xF = \frac{(x-y)e^x}{xe^y-ye^x}, \qquad 1+yF = \frac{(x-y)e^y}{xe^y-ye^x}\,.$$

Then

$$\Phi_{\alpha,\beta} = \frac{(x-y)^{\alpha+\beta}e^{\alpha x+\beta y}}{(xe^y-ye^x)^{\alpha+\beta}} = \left(\frac{x-y}{x-y-x(1-e^{y-x})}\right)^{\alpha+\beta}e^{\beta(y-x)} = \sum_{k=0}^{\infty}\frac{(a+\beta)_k}{k!}\frac{x^k}{(x-y)^k}(1-e^{y-x})^k e^{\beta(y-x)}$$

$$= \sum_{k=0}^{\infty}\frac{(a+\beta)_k}{k!}\frac{x^k}{(x-y)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{(\beta+j)(y-x)} = \sum_{n=0}^{\infty}\frac{(y-x)^n}{n!}\sum_{k=0}^{n}\frac{(a+\beta)_n}{k!}\frac{x^k}{(x-y)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\beta+j)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{(a+\beta)_k}{k!}\sum_{t=0}^{n-k}(-1)^t\binom{n-k}{t}y^{n-k-t}x^{k+t}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\beta+j)^n$$

$$= \sum_{r,s=0}^{\infty}\frac{x^ry^s}{(r+s)!}\sum_{j=0}^{r}(-1)^{r-j}(\beta+j)^{n+j}\sum_{k=j}^{r+s}\frac{(a+\beta)_k}{j!(k-j)!}\binom{r+s-k}{s}\,.$$

The sum on the extreme right is equal to

$$\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j},$$

so that

$$\Phi_{\alpha,\beta} = \sum_{r,s=0}^{\infty}\frac{x^ry^s}{(r+s)!}\sum_{j=0}^{r}(-1)^{r-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j}(\beta+j)^{r+s}\,.$$

Therefore

$$A(r,s|a,\beta) = \sum_{j=0}^{r}(-1)^{r-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{r-j}(\beta+j)^{r+s}\,. \tag{3.13}$$

In view of (3.3) we have also

$$A(r,s|a,\beta) = \sum_{j=0}^{s}(-1)^{s-j}\binom{a+\beta+j-1}{j}\binom{a+\beta+r+s}{s-j}(a+j)^{r+s} \tag{3.14}$$

For $a=\beta=1$, Eq. (3.14) reduces to

$$A(r,s) = \sum_{j=0}^{s}(-1)^{s-j}\binom{r+s+2}{s-j}(j+1)^{r+s+1} = \sum_{j=1}^{s+1}(-1)^{s-j+1}\binom{r+s+2}{s-j+1}j^{r+s+1} \tag{3.15}$$

in agreement with a known formula for $A_{n,k}$.

Returning to the recurrence (3.8), iteration gives

$$A(r,s|\alpha,\beta) = (r+\beta)^2A(r,s-2|\alpha,\beta) + [(r+\beta)(s+\alpha-1)+(s+\alpha)(r+\beta-1)]A(r-1,s-1|\alpha,\beta) + (s+\alpha)^2A(r-2,s|\alpha,\beta).$$

This suggests a formula of the type

$$A(r,s|\alpha,\beta) = \sum_{j=0}^{k} B(j,k-j)A(r-j,s-k+j|\alpha,\beta) \qquad (0 \leqslant k \leqslant r+s), \tag{3.16}$$

where $B(j,k-j)$ depends also on r,s,α,β and is homogeneous of degree k in r,s,α,β. Applying (3.8) to (3.11) we get

$$B(j,k-j+1) = (r-j+\beta)B(j,k-j) + (s-k+j+\alpha-1)B(j-1,k-j+1).$$

Replacing k by $j+k-1$ this reduces to

$$B(j,k) = (r-j+\beta)B(j,k-1) + (s-k+\beta)B(j-1,k). \tag{3.17}$$

If we put

$$B(j,k) = (-1)^{j+k}\bar{B}(j,k),$$

(3.17) becomes

$$\bar{B}(j,k) = (j-r-\beta)\bar{B}(j,k-1) + (k-s-\alpha)\bar{B}(j-1,k). \tag{3.18}$$

Since, by (3.17),

$$B(j,0) = (r+\beta)^j, \qquad B(o,k) = (s+\alpha)^k,$$

it follows that

$$\bar{B}(j,o) = (-r-\beta)^j, \qquad \bar{B}(o,k) = (-s-\alpha)^k.$$

Hence

$$\bar{B}(j,k) = A(j,k|-s-\alpha,-r-\beta)$$

and (3.16) becomes

$$A(r,s|\alpha,\beta) = (-1)^k \sum_{j=0}^{K} A(j,k-j|-s-\alpha,-r-\beta)A(r-j,s-k+j|\alpha,\beta) \qquad (0 \leqslant k \leqslant r+s). \tag{3.19}$$

For $k = r+s$ Eq. (3.19) reduces to

$$A(r,s|\alpha,\beta) = (-1)^{r+s}A(r,s|-s-\alpha,-r-\beta) \tag{3.20}$$

which can also be proved by using (3.13). Substituting from (3.20) in (3.19) we get

$$A(r,s|\alpha,\beta) = \sum_{j=0}^{k} A(j,k-j|s-k+j+\alpha,r-j+\beta)A(r-j,s-k+j|\alpha,\beta) \qquad (o \leqslant k \leqslant r+s). \tag{3.21}$$

We remark that (3.21) is equivalent to

$$\Phi_{\alpha,\beta}\left\{x(1+z),y(1+z)\right\} = \Phi_{\alpha,\beta}\left\{x+xyzF(xz,yz),y+xyzF(xz,yz)\right\}\Phi_{\alpha,\beta}(xz,yz). \tag{3.22}$$

4. THE SYMMETRIC CASE

When $\alpha = \beta$ we define

$$A(r,s|\alpha) = A(r,s|\alpha,\alpha) = A(r,s|\alpha,\alpha) \tag{4.1}$$

and

$$\Phi_\alpha(x,y) = \Phi_{\alpha,\alpha}(x,y) = \Phi_\alpha(y,x).$$

Since $\Phi_\alpha(x,y)$ is symmetric in x,y we may put

$$\Phi_\alpha(x,y) = \sum_{n=0}^{\infty} \sum_{2j \leqslant n} C(n,j|\alpha) \frac{(xy)^j(x+y)^{n-2j}}{n!}. \tag{4.2}$$

Since

$$(xD_x + yD_y)\Phi_\alpha = \alpha(x+y)\Phi_\alpha + xy(D_x+D_y)\Phi_\alpha$$

and

$$(xD_x + yD_y)\Phi_\alpha = \sum_{n=1}^{\infty} \sum_{2j \leqslant n} C(n,j|\alpha) \frac{(xy)^j(x+y)^{n-2j}}{(n-1)!},$$

$$(x+y)\Phi_\alpha = \sum_{n=1}^{\infty} \sum_{2j<n} C(n-1,j|\alpha) \frac{(xy)^j(x+y)^{n-2j}}{(n-1)!},$$

$$xy(D_x+D_y)\Phi_\alpha = \sum_{n=1}^{\infty} \sum_{2j\leqslant n} C(n-1,j-1|\alpha) \frac{2(n-2j)(xy)^j(x+y)^{n-2j}}{(n-1)!} + \sum_{n=1}^{\infty} \sum_{2j<n} C(n-1,j|\alpha) \frac{j(xy)^j(x+y)^{n-2j}}{(n-1)!},$$

it follows that

(4.3) $$C(n,j|\alpha) = 2(n-2j+1)C(n-1,j-1|\alpha) + (\alpha+j)C(n-1,j|\alpha).$$

The special case

(4.4) $$F(x,y) = \sum_{n=0}^{\infty} \sum_{2n\leqslant j} C(n,j) \frac{(xy)^j(x+y)^{n-2j}}{n!}$$

is of interest. It is easily seen that

(4.5) $$C(n,j) = C(n,j|1).$$

In the next place it follows from (4.2) that

(4.6) $$A(r,s|\alpha) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j} C(r+s,j|\alpha)$$

and in particular, for $\alpha = 1$,

(4.7) $$A(r,s) = \sum_{j=0}^{\min(r,s)} \binom{r+s-2j}{r-j} C(r+s,j).$$

To invert (4.7) we use the identity

$$x^n + y^n = \sum_{2j\leqslant n} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j(x+y)^{n-2j}.$$

We find that

(4.8) $$\begin{cases} C(n,k|\alpha) = \sum_{r=0}^{r} (-1)^{k-r} \dfrac{n-2r}{n-k-r} \dbinom{n-k-r}{k-r} A(r,n-r|\alpha) & (n \neq 2k), \\ C(2k,k|\alpha) = 2\sum_{r=0}^{k-1} (-1)^{k-r} A(r,2k-r|\alpha) + A(k,k|\alpha). \end{cases}$$

To get a generating function for $C(n,j|\alpha)$ put $u = x+y$, $v = xy$ in (4.2). We get after some manipulation

(4.9) $$\sum_{n,j=0}^{\infty} C(n+2j,j|\alpha) \frac{u^n v^j}{(n+2j)!} = \left\{ \cosh \tfrac{1}{2}\sqrt{u^2-4v} - u\, \frac{\sinh \tfrac{1}{2}\sqrt{u^2-4v}}{\sqrt{u^2-4v}} \right\}^{-2\alpha}$$

The following values of $A(r,s)$, $C(n,j)$ are easily computed.

$A(r,s)$

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1

$C(n,j)$

1		
1		
1	2	
1	8	
1	22	16
1	52	136

5. ENUMERATION BY RISES, FALLS AND MAXIMA

We consider first the enumeration of permutations by number of maxima. Let $M(n,k)$ denote the number of permutations of Z_n with k maxima. Since we count a conventional fall on the right there is no ambiguity in counting the number of maxima. For example the permutation (1243) has one maxima while (3241) has two.

Let π denote an arbitrary permutation of Z_n with k maxima. If the element $n+1$ is inserted immediately to the left or right of a maximum the number of maxima does not change. If however it is inserted in any other position, the number of maxima becomes $k+1$. Therefore we have

(5.1) $$M(n+1,k) = (n-2k+3)M(n,k-1) + 2kM(n,k).$$

If we put

$$M(n,k) = 2^{n-2k+1}\bar{M}(n,k),$$

(5.1) becomes

(5.2) $$\bar{M}(n+1,k) = 2(n-2k+3)\bar{M}(n,k-1) + k\bar{M}(n,k) \qquad (1 \leqslant k \leqslant n)\ .$$

If we take $a = 1$ in (4.3) we get

(5.3) $$C(n,j) = 2(n-2j+1)C(n-1,j-1) + (j+1)C(n-1,j) \qquad (0 \leqslant j \leqslant n)\ .$$

It follows that

$$\bar{M}(n+1,k+1) = C(n,k),$$

so that

(5.4) $$M(n+1,k+1) = 2^{n-2k}C(n,k).$$

Thus (4.9) yields the generating function

(5.5) $$\sum_{n,j=0}^{\infty} M(n+2j+1,j+1)\frac{u^n v^j}{(n+2j)!} = \left\{ \cosh\sqrt{u^2-v} - \frac{u}{\sqrt{u^2-v}}\sinh\sqrt{u^2-v} \right\}^{-2}\ .$$

This result may be compared with [4].

We now consider the enumeration of permutations by rises, falls and maxima. Let $P(r,s,k)$ denote the number of permutations with r rises, s falls and k maxima, subject to the usual conventions. Let π be an arbitrary permutation with r rises, s falls and k maxima and consider the effect of inserting the additional element $r+s$. There are four possibilities depending on the location of the new element.

(i) immediately to the right of a maximum:

$$r \to r+1, \quad s \to s, \quad k \to k;$$

(ii) Immediately to the left of a maximum:

$$r \to r, \quad s \to s+1, \quad k \to k;$$

(iii) in any other rise:

$$r \to r, \quad s \to s+1, \quad k \to k+1;$$

(iv) in any other fall:

$$r \to r+1, \quad s \to s, \quad k \to k+1\ .$$

We accordingly get the recurrence

(5.6) $$P(r,s,k) = kP(r-1,s,k) + kP(r,s-1,k) + (r-k+1)P(r,s-1,k-1) + (s-k+1)P(r-1,s,k-1).$$

It is convenient to put

(5.7) $$P(r,s,k) = \binom{r+s-2k}{r-k}B(r,s,k).$$

Then (5.6) becomes

(5.8) $$B(r,s,k) = \frac{k(r-k)}{r+s-2k}B(r-1,s,k) + \frac{k(s-k)}{r+s-2k}B(r,s-1,k)$$
$$+ (r+s-2k+1)(B(r-1,s,k-1) + B(r,s-1,k)).$$

We then show by induction that

$$B(r,s,k) = \phi(r+s,k),$$

that is, $B(r,s,k)$ is a function of $r+s$ and k. Indeed we show that

(5.9) $$B(r+1,s+1,k+1) = C(r+s,k),$$

where $C(r+s,k)$ has the same meaning as in (5.3).

Substituting from (5.9) in (5.7) we get

(5.10) $$P(r+1,s+1,k+1) = \binom{r+s-2k}{r-k}C(r+s,k).$$

It follows from (5.10) that

$$M(n+1,k+1) = \sum_{r+s=n} P(r+1,s+1,k+1) = \sum_{r+s=n}\binom{r+s-2k}{r-k}C(r+s,k) = 2^{n-2k}C(n,k)$$

in agreement with (5.4)

We remark that for $r = s = k$

(5.11) $$P(k+1,k+1,k+1) = C(2k,k) = A(2k+1),$$

the number of down-up (or up-down) permutations of Z_{2k+1}. It is well known that

(5.12) $$\sum_0^\infty A(2k+1)\frac{x^{2k+1}}{(2k+1)!} = \tan x.$$

Generating functions for $P(r,s,k)$ are furnished by

(5.13) $$\sum_{r,s=0}^\infty \sum_{k=0}^{\min(r,s)} P(r+1,s+1,k+1)\frac{x^r y^s z^k}{(r+s+1)!} = F(U,V),$$

and

(5.14) $$\sum_{r,s=0}^\infty \sum_{k=0}^{\min(r,s)} P(r+1,s+1,k+1)\frac{x^r y^s z^k}{(r+s)!} = (1+UF(U,V))(1+VF(U,V)),$$

where

(5.15) $$\begin{cases} U = \tfrac{1}{2}(x+y+\sqrt{(x+y)^2-4xyz}) \\ V = \tfrac{1}{2}(x+y-\sqrt{(x+y)^2-4xyz}) \end{cases}$$

and

$$F(U,V) = \frac{e^U-e^V}{Ue^V-Ve^U}.$$

6. (α,β)-SEQUENCES

Let α,β be fixed positive integers. We shall generalize rises, falls and maxima in the following way. In addition to the "real" elements $1, 2, \cdots, n$ we introduce two kinds of "virtual" elements which will be denoted by the symbols 0, 0′. There are α symbols 0 and β symbols 0′. To begin with $(n = 1)$ we have

(6.1) $$\underbrace{0\cdots 0}_{\alpha}\ 1\ \underbrace{0'\cdots 0'}_{\alpha}.$$

We then insert the symbols $2, 3, \cdots, n$ in all possible ways subject to the requirement that there is at least one 0 on the extreme left and at least one 0′ on the extreme right. The resulting sequence is called an (α,β)-sequence. A rise is defined as a pair of consecutive elements a, b with $a < b$; here a may be 0. A fall is as a pair of consecutive elements a, b with $a > b$; now b may be 0′. The element b is a maximum if a, b, c are consecutive and a,b is a rise while b, c is a fall. For example in

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we have

$$\alpha = 2,\quad \beta = 3,\quad r = 4,\quad s = 3,\quad k = 1.$$

Let $P(r,s,k|\alpha,\beta)$ denote the number of (α,β)-sequences with r rises, s falls and k maxima. Then we have the recurrence

(6.2) $$\begin{aligned} P(r,s,k|\alpha,\beta) &= (k+-1)P(r-1,s,k|\alpha,\beta) + (k+-1)P(r,s-1,k|\alpha,\beta) \\ &\quad + (r-k+1)P(r,s-1,k-1|\alpha,\beta) + (s-k+1)P(r-1,s,k-1|\alpha,\beta). \end{aligned}$$

In the special case $\alpha = \beta$ we put

(6.3) $$P(r,s,k|\alpha) = P(r,s,k|\alpha,\alpha).$$

We also put

(6.4) $$P(r,s,k|\alpha) = \binom{r+s-2k}{r-k} Q(r,s,k|\alpha).$$

Now let $M(n,k|\alpha,\beta)$ denote the number (α,β)-sequences with n real elements and k maxima. Then we have the recurrence

(6.5) $$\begin{aligned} M(n+1,k|\alpha,\beta) &= (2k+\alpha+\beta-2)M(n,k|\alpha,\beta) \\ &\quad + (n-2k+3)M(n,k-1|\alpha,\beta). \end{aligned}$$

In particular, for

$$M(n,k|\alpha) = M(n,k|\alpha,\alpha),$$

(6.5) reduces to

(6.6) $$M(n+1,k|\alpha) = 2(k+\alpha-1)M(n,k|\alpha) + (n-2k+3)M(n,k-1|\alpha).$$

We find that

(6.7) $$M(n+1,k+1|\alpha) = 2^{n-2k}C(n,k|\alpha),$$

and

(6.8) $$Q(r+1,s+1,k+1|\alpha) = C(r+s,k|\alpha).$$

Hence, by (6.4) and (6.8),

(6.9) $$P(r+1, s+1, k+1\,|\,\alpha) = \binom{r+s-2k}{r-2k} C(r+s, k\,|\,\alpha).$$

A generating function for $P(r+1, s+1, k+1\,|\,\alpha)$ is given by

(6.10) $$\sum_{r,s=0}^{\infty} \sum_{k=0}^{\min(r,s)} P(r+1, s+1, k+1\,|\,\alpha) \frac{x^r y^s z^k}{(r+s)!} = (1+UF(U,V))^{\alpha}(1+VF(U,V))^{\beta},$$

where U, V are given by (5.15).

For a generating function for $P(r+1, s+1, k+1\,|\,\alpha, \beta)$ see [3].

7. UPPER RECORDS

Returning to ordinary permutations, let $\pi = (a_1 a_2 \cdots a_n)$ be a permutation of Z_n. The element a_k is called a *left upper record* if

$$a_i < a_k \qquad (1 \leqslant i < k);$$

it is called a *right upper record* if

$$a_k > a_i \qquad (k < i \leqslant n).$$

Let $A(r, s; t, u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls, t left and u right upper records. We make the usual conventions about rises and falls. Also let $A(r, s; t)$ denote the number of permutations with $r+1$ rises, $s+1$ falls and t left upper records; let $\bar{A}(r, s, u)$ denote the number of permutations with $r+1$ rises, $s+1$ falls and u right upper records.

To begin with we have

(7.1) $$A(r, s; t+1) = \sum_{j=0}^{r-1} \sum_{k=0}^{s-1} \binom{r+s}{j+k+1} A(j, k; t) A(r-j-1, s-k-1) + A(r-1, s; t) \qquad (t > 0)$$

and

(7.2) $$A(r, s; 1) = A(r, s-1) \qquad (s \geqslant 1).$$

Put

$$F_t(z) = \sum_{r,s=0}^{\infty} A(r, s; t) \frac{x^r y^s z^{r+s+1}}{(r+s+1)!}.$$

Then, for $t > 0$,

$$F'_{t+1}(z) = \sum_{r,s=0}^{\infty} A(r, s; t) \frac{x^{r+1} y^s z^{r+s+1}}{(r+s+1)!} + \sum_{j,k=0}^{\infty} A(j, k; t) \frac{x^j y^k z^{j+k+1}}{(j+k+1)!} \cdot \sum_{r,s=0}^{\infty} A(r,s) \frac{x^{r+1} y^{s+1} z^{r+s+1}}{(r+s+1)!},$$

so that

(7.3) $$F'_{t+1}(z) = F_t(z)(x + xyF(z)),$$

where

$$F(z) = \frac{e^{xz} - e^{yz}}{xe^{yz} - ye^{xz}}.$$

Also, by (7.2),

(7.4) $$F'_1(z) = 1 + yF(z).$$

If we put

$$G(z) = \sum_{t=1}^{\infty} F_t(z) \lambda^t,$$

it follows from (7.3) and (7.4) that

$$G'(z) = \lambda G(z)(x + xyF(z)) + \lambda(1 + yF(z)).$$

The solution of this differential equation is

(7.5) $$G(z) = \frac{1}{x}\left\{ (1 + xF(z))^{\lambda} - 1 \right\}.$$

Similarly if we put

$$\bar{F}_u(z) = \sum_{r,s=0}^{\infty} \bar{A}(r, s; u) \frac{x^r y^s z^{r+s+1}}{(r+s+1)!}, \qquad \bar{G}(z) = \sum_{u=1}^{\infty} \bar{F}_u(z) \lambda^u,$$

we have

(7.6)
$$\bar{G}(z) = \frac{1}{y}\left\{(1+yF(z))^{\lambda} - 1\right\} .$$

We now consider the general case. It follows from the definition that

(7.7)
$$A(r,s;t+1,u+1) = \sum_{j,k} \binom{r+s}{j+k+1} A(j,k;t)\bar{A}(r-j-1,s-k-1;u) \qquad (t>0,\ u>0)$$

and
$$\begin{cases} A(r,s;1,u+1) = \bar{A}(r,s-1;u) & (s>0,\ u>0) \\ A(r,s;t+1,1) = A(r-1,s;t) & (r>0,\ t>0) \end{cases} .$$

Now put
$$F_{t,u}(z) = \sum_{r,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s z^{r+s}}{(r+s)!} .$$

Then
$$\begin{cases} F'_{t+1,u+1}(z) = xyF_t(z)\bar{F}_u(z) & (t>0,\ u>0) \\ F'_{1,u+1}(z) = y\bar{F}_u(z) & (u>0) \\ F'_{t+1,1}(z) = xF_t(z) & (t>0) \\ F'_{1,1}(z) = 1 \end{cases}$$

Therefore, by (7.5) and (7.6),
$$\sum_{t,u=1}^{\infty} \alpha^t\beta^u \sum_{r,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s z^{r+s+1}}{(r+s+1)!} = \alpha\beta + \alpha\beta[(1+xF(z))^{\alpha} - 1] + \alpha\beta[(1+yF(z))^{\beta} - 1]$$
$$+ \alpha\beta[(1+xF(z))^{\alpha} - 1][(1+yF(z))^{\beta} - 1] = \alpha\beta(1+xF(z))^{\alpha}(1+yF(z))^{\beta} .$$

Taking $z = 1$ we get

(7.8)
$$\sum_{t,u=1}^{\infty} \alpha^t\beta^u \sum_{t,s=0}^{\infty} A(r,s;t,u)\frac{x^r y^s}{(r+s)!} = \alpha\beta(1+xF(x,y))^{\alpha}(1+yF(x,y))^{\beta} ,$$

where
$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} .$$

It follows that

(7.9)
$$A(r,s|\alpha,\beta) = \sum_{t,u} A(r,s;t,u)\alpha^{t-1}\beta^{u-1} .$$

Thus the generalized Eulerian number $A(r,s|\alpha,\beta)$ has the explicit polynomial expansion (7.9).

If we put
$$R(n+1;t,u) = \sum_{r+s=n+1} A(r,s;t,u)$$

it is evident that $R(n+1;t,u)$ is the number of permutations of Z_{n+1} with t left and u right upper records. By taking $y = x$ in (7.8) we find that

(7.10)
$$R(n+1;t+1,u+1) = \binom{t+u}{t} S_1(n,t+u) ,$$

where $S_1(n,t+u)$ denotes a Stirling number of the first kind.

In particular, if we put
$$R(n+1;t) = \sum_{r+s=n} A(r,s;t), \qquad \bar{R}(n+1;t) = \sum_{r+s=n} \bar{A}(r,s;t) ,$$

we get

(7.11)
$$R(n;t) = \bar{R}(n;t) = S_1(n,t) .$$

It is easy to give a direct proof of (7.11).

8. EULERIAN OPERATORS

Put

(8.1)
$$A_n(x,y) = \sum_{r+s=n} A(r,s)x^r y^s .$$

It follows from recurrence (2.2) that

$$A_n(x,y) = (x+y+xy(D_x+D_y))A_{n-1}(x,y). \tag{8.2}$$

Iteration of (8.2) gives

$$A_n(x,y) = (x+y+xy(D_x+D_y))^n \cdot 1. \tag{8.3}$$

It is accordingly of interest to consider the expansion of the operator

$$\Omega^n \equiv [x+y+xy(D_x+D_y)]^n . \tag{8.4}$$

We find that

$$\Omega^n = \sum_{k=0}^{n} C_{n,k}(x,y)(xy)^k(D_x+D_y)^k , \tag{8.5}$$

where

$$C_{n,k}(x,y) = \frac{1}{k!(k+1)!}(D_x+D_y)^k A_n(x,y). \tag{8.6}$$

More generally if we put

$$A_n(x,y|\alpha,\beta) = \sum_{r+s=n} A(r,s|\alpha,\beta)x^r y^s , \tag{8.7}$$

it follows from (3.8) that

$$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]A_{n-1}(x,y|\alpha,\beta). \tag{8.8}$$

Thus

$$A_n(x,y|\alpha,\beta) = [\alpha x+\beta y+xy(D_x+D_y)]^n \cdot 1, \tag{8.9}$$

so that it is of interest to expand the operator

$$\Omega^n_{\alpha,\beta} \equiv [\alpha x+\beta y+xy(D_x+D_y)]^n . \tag{8.10}$$

We find that

$$\Omega^n_{\alpha,\beta} = \sum_{k=0}^{n} C^{(\alpha,\beta)}_{n,k}(x,y)(xy)^k(D_x+D_y)^k , \tag{8.11}$$

where

$$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(\alpha+\beta)_k}(D_x+D_y)^k A_n(x,y|\alpha,\beta) \tag{8.12}$$

provided $\alpha+\beta$ is not equal to zero or a negative integer. Note that

$$\Omega = \Omega_{1,1}, \qquad C_{n,k}(x,y) = C^{(1,1)}_{n,k}(x,y).$$

As an application of (8.8) and (8.11) we have

$$A_{m+n}(x,y|\alpha,\beta) = \sum_{k=0}^{\min(m,n)} \frac{1}{k!(\alpha+\beta)_k}(xy)^k(D_x+D_y)^k A_m(x,y|\alpha,\beta)\cdot(D_x+D_y)^k\cdot A_n(x,y|\alpha,\beta), \tag{8.13}$$

where again $\alpha+\beta$ is not equal to zero or a negative integer.

When $\alpha=\beta=0$, (8.11) becomes

$$(xy(D_x+D_y))^n = \sum_{k=1}^{\infty} C^{(0,0)}_{n,k}(x,y)(xy)^k(D_x+D_y)^k \qquad (n \geqslant 1). \tag{8.14}$$

We find that

$$C^{(0,0)}_{n,k}(x,y) = \frac{1}{k!(k-1)!}(D_x+D_y)^{k-1}A_{n-1}(x,y) \qquad (1 \leqslant k \leqslant n). \tag{8.15}$$

The formula

$$C^{(\alpha,\beta)}_{n,k}(x,y) = \frac{1}{k!(k-1)!}\sum_{j=k}^{n}\binom{n}{r}(D_x+D_y)^{k-1}A_{r-1}(x,y)\cdot A_{n-r}(x,y|\alpha,\beta) \qquad (1 \leqslant k \leqslant n) \tag{8.16}$$

holds for arbitrary α,β. When $\alpha=\beta=0$, (8.16) reduces to (8.15).

In the next place we consider the inverse of (8.11), that is,

(8.17)
$$(xy)^n(D_x+D_y)^n = \sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)}(x,y)\Omega_{\alpha,\beta}^{k} .$$

We find that

(8.18)
$$(D_x+D_y)B_{n,k}^{(\alpha,\beta)}(x,y) = n(\alpha+\beta+n-1)B_{n-1,k}^{\alpha,\beta}(x,y)$$

and

(8.19)
$$\sum_{n=0}^{\infty} \frac{u^n}{n!}\sum_{k=0}^{n} B_{n,k}^{(\alpha,\beta)}(x,y)(x-y)^k v^k = (1-xu)^{-\alpha-v}(1-yu)^{-\beta+v} .$$

In the special case $\alpha=\beta=0$ we put

(8.20)
$$b_{n,k} = \frac{1}{(n-1)!}B_{n,k}^{(0,0)}(x,y) \qquad (n \geqslant 1).$$

Then we have

(8.21)
$$b_{n,1} = \frac{x^n-y^n}{x-y} \equiv \sigma_n ,$$

(8.22)
$$b_{n+1,2} = \sum_{j=1}^{n} \frac{1}{j}\sigma_j\sigma_{n-j+1}$$

and generally

(8.23)
$$b_{n+1,k} = \sum_{j=k-1}^{n} \frac{1}{j} b_{j,k-1}\sigma_{n-j+1} .$$

This may also be written in the form

(8.24)
$$b_{n+k,k} = \sum_{j=0}^{n} \frac{1}{j+k-1} b_{j+k-1,k-1}\sigma_{n-j+1} .$$

Thus for example

$$b_{n+3,n} = \sum_{0\leqslant i\leqslant j\leqslant n} \frac{1}{(i+1)(j+2)}\sigma_{i+1}\sigma_{j-i+1}\sigma_{n-j+1}$$

$$b_{n+4,n} = \sum_{0\leqslant i\leqslant j\leqslant k\leqslant n} \frac{1}{(i+1)(j+2)(k+3)}\sigma_{i+1}\sigma_{j-i+1}\sigma_{k-j+1}\sigma_{n-k+1}$$

and so on.

For proof of the formulas in this section the reader is referred to [2].

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DIOPHANTINE REPRESENTATION OF THE FIBONACCI NUMBERS

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In the year 1202, the Italian mathematician Leonardo of Pisano, or *Fibonacci* as he is known today, gave the sequence $1, 1, 2, 3, 5, 8, 13, 21, \cdots,$ in his book *Liber Abacci.* The numbers occurred in connection with a problem concerning the number of offspring of a pair of rabbits. The sequence has many interesting properties, and has fascinated mathematicians for over 700 years. It is usually defined recursively by means of the equations

$$\phi_1 = 1, \quad \phi_2 = 1, \quad \text{and} \quad \phi_{n+2} = \phi_n + \phi_{n+1} .$$

These equations permit us to obtain the n^{th} Fibonacci number, ϕ_n, by computing all smaller Fibonacci numbers. Many formulas are known which permit calculation of the n^{th} Fibonacci number directly from n. J.P.M. Binet found [1] the well known formula

$$\phi_n = \frac{1}{\sqrt{5}}\left[\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right] .$$

E. Lucas [6] noticed that the Fibonacci numbers were the sums of the binomial coefficients on the "rising diagonals" of Pascal's triangle.

$$\phi_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots .$$

We shall prove here that the set of Fibonacci numbers is identical with the set of positive values of a polynomial of the fifth degree in two variables:

$$(1) \qquad 2y^4x + y^3x^2 - 2y^2x^3 - y^5 - yx^4 + 2y .$$

To construct the polynomial (1), we shall need three lemmas. These lemmas assert that pairs of adjacent Fibonacci numbers, and only these, are to be found among the points with integer coordinates on the hyperbolas

$$y^2 - yx - x^2 = \pm 1 .$$

(L.E. Dickson [4] credits E. Lucas [7] and J. Wasteels [13] with this observation.)

Lemma 1. For any positive integer i,

$$\phi_{i+1}^2 - \phi_{i+1}\phi_i - \phi_i^2 = (-1)^i .$$

Proof. By induction on i. Plainly, the statement is true if $i = 1$. Suppose it holds for i. Then

$$\begin{aligned}\phi_{i+2}^2 - \phi_{i+2}\phi_{i+1} - \phi_{i+1}^2 &= (\phi_i + \phi_{i+1})^2 - (\phi_i + \phi_{i+1})\phi_{i+1} - \phi_{i+1}^2 \\ &= -(\phi_{i+1}^2 - \phi_{i+1}\phi_i - \phi_i^2) = -(-1)^i = (-1)^{i+1} .\end{aligned}$$

This completes the proof of the lemma.

Lemma 2. For any positive integers x and y, if $y^2 - yx - x^2 = 1$ then it is possible to find a positive integer i such that $x = \phi_{2i}$ and $y = \phi_{2i+1}$.

Proof. By induction on x. If $x = 1$ then necessarily $y = 2$. In this case we may take $i = 1$.

Suppose that x and y are numbers satisfying the equation of the lemma and that $1 < x$. Then $2 \leqslant y$. Assume that the statement of the lemma holds for all pairs, (x_0, y_0), of positive integers for which $x_0 < x$. Let us set $x_0 = 2x - y$, and $y_0 = y - x$. Since $2 \leqslant y$,

$$(x+1)^2 = x^2 + 2x + 1 \leqslant x^2 + yx + 1 = y^2 ,$$

hence $y > x$. And since $1 < x$,

$$y^2 = yx + x^2 + 1 < yx + x^2 + x = yx + (x+1)x \leqslant yx + yx = 2yx,$$

hence $y < 2x$. Therefore

(2) $$0 < x_0 < x, \qquad \text{and} \qquad 0 < y_0.$$

Furthermore,

(3) $$y_0^2 - y_0 x_0 - x_0^2 = (y-x)^2 - (y-x)(2x-y) - (2x-y)^2 = y^2 - yx - x^2 = 1\ .$$

The induction hypothesis, together with (2) and (3) implies that it is possible to find a positive integer i such that $x_0 = \phi_{2i}$ and $y_0 = \phi_{2i+1}$. Then

$$x = x_0 + y_0 = \phi_{2i} + \phi_{2i+1} = \phi_{2(i+1)} \qquad \text{and} \qquad y = y_0 + x = \phi_{2i+1} + \phi_{2i+2} = \phi_{2(i+1)+1}\ .$$

This completes the proof of the lemma.

Lemma 3. For any positive integers x and y, if

$$y^2 - yx - x^2 = -1,$$

then it is possible to find a positive integer i such that $x = \phi_{2i-1}$ and $y = \phi_{2i}$.

Proof. Let x and y be numbers satisfying the conditions of the lemma. Then

$$(x+y)^2 - (x+y)(y) - y^2 = x^2 + 2xy + y^2 - xy - y^2 - y^2 = -(y^2 - xy - x^2) = -(-1) = 1.$$

According to Lemma 2 it is possible to find a positive integer i such that

$$y = \phi_{2i} \qquad \text{and} \qquad x + y = \phi_{2i+1}\ .$$

Hence

$$x = \phi_{2i+1} - \phi_{2i} = \phi_{2i-1} \qquad \text{and} \qquad y = \phi_{2i}\ .$$

This completes the proof of the lemma.

Lemmas 1, 2 and 3 imply that the set of all Fibonacci numbers has a very simple Diophantine defining equation. [A relation in positive integers is said to be *Diophantine* if it is equal to the set of values of parameters for which a polynomial equation is solvable in positive integers.]

Theorem 1. For any positive integer y, in order that y be a Fibonacci number, it is necessary and sufficient that there exist a positive integer x such that

(4) $$(y^2 - yx - x^2)^2 = 1\ .$$

Proof. We have only to use Lemmas 1, 2, and 3.

Lemma 4. If x and y are positive integers, then $y^2 - yx - x^2 \neq 0$.

Proof. Multiplying by 4 and completing the square, we find that

$$4y^2 - 4yx - 4x^2 = (2y - x)^2 - 5x^2\ .$$

If the right side of this expression were zero, for positive integers x and y, then $\sqrt{5}$ would be a rational number. The lemma is proved.

Theorem 2. The set of all Fibonacci numbers is identical with the set of positive values of the polynomial

(1) $$y(2 - (y^2 - yx - x^2)^2)$$

for $(x = 1, 2, \cdots, y = 1, 2, \cdots)$.

Proof. According to Theorem 1, if y is a Fibonacci number then a positive integer x may be found to satisfy equation (4). For such an x, (1) assumes the value y. Therefore all Fibonacci numbers are values of the polynomial (1).

To see that *only* Fibonacci numbers are assumed as values of (1), suppose that x, y and w are positive integers and that

(5) $$w = y(2 - (y^2 - yx - x^2)^2)\ .$$

Then, since y and w are positive, we see that

(6) $$0 < (y^2 - yx - x^2)^2 < 2,$$

using Lemma 4, to obtain the lower inequality.

Since x and y are integers, (6) implies that equation (4) must hold. According to Theorem 1, y must be a Fibonacci number. Equations (4) and (5) imply that $w = y$. Therefore w is a Fibonacci number.

This completes the proof of the theorem. (Putnam's method [10] would produce a polynomial of degree 9.)

The polynomial (1), which represents the set of Fibonacci numbers, assumes in addition certain negative values such as -28 $(x = 2,\ y = 2)$. The appearance of non-Fibonacci numbers cannot be prevented, for we can prove

Theorem 3. The set of Fibonacci numbers is not the exact range of any polynomial.

Proof. We shall show that a polynomial $P(x_1, x_2, \cdots, x_k)$ which assumes only Fibonacci number values must be constant. The proof will be carried out by induction on the number k of variables.

If $k = 0$, there is nothing to prove. Let us assume that the result holds for k and consider a polynomial

$$P(x_1, x_2, \cdots, x_k, x_{k+1})$$

in $k+1$ variables. If this polynomial is not identically zero then we may write

$$P(x_1, x_2, \cdots, x_k, x_{k+1}) = \sum_{i=0}^{m} P_i(x_1, x_2, \cdots, x_k)x_{k+1}^i, \qquad P_m(x_1, x_2, \cdots, x_k) \not\equiv 0 .$$

If $m = 0$, then $P(x_1, x_2, \cdots, x_k, x_{k+1})$ is a polynomial in $x_1, x_2, \cdots, x_k$ only. If not we may find positive integers $a_1, a_2, \cdots, a_k$ for which the polynomial

$$Q(x) = P(a_1, a_2, \cdots, a_k, x)$$

is not constant. In this event we must have one or the other of two cases:

$$\text{(i)}\ \lim_{x\to+\infty} Q(x) = +\infty, \qquad \text{or (ii)}\ \lim_{x\to+\infty} Q(x) = -\infty .$$

Assuming there are no negative Fibonacci numbers (see remark following), we have only case (i) to deal with. Since $Q(x)$ is a polynomial, a positive integer b may be found such that

$$Q(b) < Q(b+1) < Q(b+2) < Q(b+3) < \cdots . \tag{7}$$

By assumption, $Q(x)$ assumes only Fibonacci number values. Choose a positive integer c such that $\phi_c = Q(b)$. Condition (7) implies that for each positive integer y

$$\phi_{c+y} \leqslant Q(b+y). \tag{8}$$

The formula of Binet may be used to prove that for each positive integer n,

$$\left| \phi_n - \frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2}\right]^n \right| < \frac{1}{2} . \tag{9}$$

Conditions (8) and (9) imply that for each positive integer y

$$\frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2}\right]^{(c+y)} \leqslant Q(b+y) + \frac{1}{2} . \tag{10}$$

Inequality (10) implies that the polynomial $Q(b+y) + \tfrac{1}{2}$ grows exponentially, which is, of course, impossible.

This completes the proof of the theorem.

REMARK. The sequence of Fibonacci numbers is sometimes continued into the negative:

$$\cdots, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots .$$

The assertion of Theorem 3 remains correct for this enlarged set. We need only modify the proof to deal with case (ii) as was done with case (i). Also, it is not difficult to see that the number of variables in the polynomial (1) cannot be further decreased. Thus Theorem 2 is best possible.

THE RELATION $v = \phi_u$

In 1970 Ju. V. Matijasevič made ingenious use of the Fibonacci numbers to solve Hilbert's tenth problem. In his famous address of 1900 [5], David Hilbert posed the problem of finding an algorithm to decide of an arbitrary polynomial equation, in several variables, with integer coefficients, whether or not the equation was solvable in integers.

Matijasevič [8], [9] showed that no such algorithm exists. He proved this by proving that *every recursively enumerable set is Diophantine.*

The Fibonacci numbers were important in Matijasevič's proof, because the sequence of Fibonacci numbers grows exponentially. Martin Davis, Julia Robinson and Hilary Putnam [3] had nearly solved Hilbert's tenth problem in 1961, when they succeeded in proving that the stated result would follow from the existence of a single Diophantine predicate with exponential growth. Matijasevič completed the solution of Hilbert's tenth problem by proving that the relation $v = \phi_{2u}$ is Diophantine.

In [8], [9], Matijasevič gives an explicit system of ten Diophantine equations such that, for any given positive integers u and v, the equations are solvable in the other variables if and only if $v = \phi_{2u}$. Of course it follows from the central result of [8], [9] that the relation $v = \phi_u$ is also Diophantine. However, an explicit system of equations for this relation is not written out in [9].

We shall give here an explicit system of Diophantine equations for the relation $v = \phi_u$. Our equations may conveniently be based upon Lemmas 1 and 2 and the equations of Matijasevič [9].

Theorem 4. For any positive integers t and w, in order that $w = \phi_t$, it is necessary and sufficient that there exist positive integers $a, b, c, d, e, g, h, l, m, p, r, u, v, x, y, z$ such that

$$u + a = l, \tag{11}$$

$$v + b = l, \tag{12}$$

$$l^2 - lz - z^2 = 1, \tag{13}$$

$$g^2 - gh - h^2 = 1, \tag{14}$$

$$l^2 c = g, \tag{15}$$

$$ld = m - 2, \tag{16}$$

$$(2h + g)e = m - 3, \tag{17}$$

$$x^2 - mxy + y^2 = 1, \tag{18}$$

$$l(p - 1) = x - u, \tag{19}$$

$$(2h + g)(r - 1) = x - v, \tag{20}$$

$$((2u - t)^2 + (w - v)^2)((2u + 1 - t)^2 + (w^2 - wv - v^2 - 1)^2) = 0. \tag{21}$$

Proof. For the proof we refer the reader to [9], proof of Theorem 1. There it is shown that equations (11)–(20) are solvable in positive integers if and only if $v = \phi_{2u}$. (In the necessity part of this proof we find that $3 < m$ and also $u \leqslant v \leqslant x$, so that conditions (40), (41), (43) and (44) there, may be replaced by equations (16), (17), (19) and (20) above.) When $v = \phi_{2u}$, Lemma 2 implies that the condition $w^2 - wv - v^2 = 1$ is equivalent to $w = \phi_{2u+1}$. Thus equation (21) holds if and only if

$$t = 2u \quad \text{and} \quad w = \phi_{2u}, \qquad \text{or} \qquad t = 2u + 1 \quad \text{and} \quad w = \phi_{2u+1}$$

Thus equations (11)–(21) are solvable if and only if $w = \phi_t$.

Theorem 4 makes it possible to give a polynomial formula for the t^{th} Fibonacci number, ϕ_t. We shall prove

Theorem 5. There exists a polynomial $P(t, x_1, \cdots, x_{12})$, of degree 13, with the property that, for any positive integers t and s,
$$\phi_t = s \leftrightarrow (\exists x_1, \cdots, x_{12})[P(t, x_1, \cdots, x_{12}) = s].$$

Proof. The variables l, g, m and x are easily eliminated from the system (11)–(21) by means of Eqs. (11), (15), (16) and (19). Also, the variables b and c may be replaced by a single variable. (We need only use the fact that when α and β are positive integers, and γ is any integer, $\alpha | \beta$ and $0 < \gamma$ is equivalent to $(\exists\lambda)[\alpha\beta\gamma = \beta + \lambda\alpha]$.) If we now transpose all terms in the equations to the left side and sum the squares of the equations, we obtain the polynomial $Q(t, w, a, \cdots, z)$ with the property that $\phi_t = w$ if and only if $Q(t, w, a, \cdots, z) = 0$ for some positive integers $a, \cdots, z$. Q will be a polynomial of the 12^{th} degree. For P we may take the polynomial $w(1 - Q(t, w, a, \cdots, z))$.

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ADVANCED PROBLEMS AND SOLUTIONS

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H-245 Proposed by P. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove the identity

$$\sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} = \frac{2\prod_{r=1}^{n-1}(1+x^r)}{(x)_n}, \quad n = 1, 2, \cdots,$$

where

$$(x)_n = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n), \quad n = 1, 2, \cdots; \ (x)_0 = 1 .$$

H-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} F_{i+j}F_{m-i+j}F_{i+n-j}F_{m-i+n-j}$$

$$L(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{i+j}L_{m-i+j}L_{i+n-j}L_{m-i+n-j} .$$

Show that

$$L(m,n) - 25F(m,n) = 8L_{m+n}F_{m+1}F_{n+1} .$$

H-247 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that for each Fibonacci number F_r, there exist an infinite number of positive nonsquare integers, D, such that

$$F_{r+s}^2 - F_r^2 D = 1 .$$

H-248 Proposed by F.D. Parker, St. Lawrence University, New York.

A well known identity for the Fibonacci numbers is

$$F_n^2 - F_{n-1}F_{n+1} = -(-1)^n$$

and a less well known identity for the Lucas numbers is

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n .$$

More generally, if a sequence $\{y_0, y_1, \cdots\}$ satisfies the equation

$$y_n = y_{n-1} + y_{n-2},$$

and if y_0 and y_1 are integers, then there exists an integer N such that

$$y_n^2 - y_{n-1}y_{n+1} = N(-1)^n .$$

Prove this statement and show that N cannot be of the form $4k+2$, and show that $4N$ terminates in 0, 4, or 6.

SOLUTIONS

SUM SEQUENCE

H-216 Proposed by Guy A.R. Guillotte, Cowansville, Quebec, Canada.

Let G_m be a set of rational integers such that

$$\sum_{n=1}^{\infty}\left[\log_e\left(\sum_{m=0}^{\infty}\frac{G_m}{(m)!(F_{2n+1})^m}\right)\right] = \frac{\pi}{4} .$$

Find a formula for G_m.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$e^{\arctan x} = \sum_{m=0}^{\infty} G_m \frac{x^m}{m!},\quad G_0 = G_1 = 1 .$$

Then, by differentiation

$$e^{\arctan x} = (1+x^2)\sum_{m=0}^{\infty} G_{m+1}\frac{x^m}{m!},$$

so that

$$G_m = G_{m+1} + m(m-1)G_{m-1} \qquad (m \geqslant 1) .$$

It follows that the G_m are rational integers.

Consider

$$S \equiv \sum_{n=1}^{\infty}\log\left[\sum_{m=0}^{\infty}\frac{G_m}{m!F_{2n+1}^m}\right] = \sum_{n=1}^{\infty}\log\left[\exp\left(\arctan\frac{1}{F_{2n+1}}\right)\right] = \sum_{n=1}^{\infty}\arctan\frac{1}{F_{2n+1}} .$$

Since

$$\arctan\frac{1}{F_{2n}} - \arctan\frac{1}{F_{2n+2}} = \arctan\left(\frac{F_{2n+2}-F_{2n}}{F_{2n}F_{2n+1}+1}\right) = \arctan\frac{1}{F_{2n+1}}$$

it follows that

$$\sum_{n=1}^{\infty}\arctan\frac{1}{F_{2n+1}} = \arctan\frac{1}{F_2} = \arctan 1 = \frac{\pi}{4} .$$

Hence $S = \pi/4$.

To get an explicit formula for G_m we proceed as follows. Put

$$x = \tan u = \frac{1}{i}\frac{e^{iu}-e^{-iu}}{e^{iu}+e^{-iu}} = \frac{1}{i}\frac{e^{2iu}-1}{e^{2iu}+1}, \qquad e^{2iu} = \frac{1+ix}{1-ix},$$

that is,

$$e^{2i\arctan x} = \frac{1+ix}{1-ix}\,.$$

Thus

$$e^{\arctan x} = \left(\frac{1+ix}{1-ix}\right)^{-\frac{1}{2}i} = (1+ix)^{-\frac{1}{2}i}(1-ix)^{\frac{1}{2}i}$$

$$= \sum_{r=0}^{\infty}\binom{-\frac{1}{2}i}{r}(ix)^r\sum_{s=0}^{\infty}\binom{\frac{1}{2}i}{s}(-ix)^s = \sum_{m=0}^{\infty} i^m x^m \sum_{r+s=m}(-1)^s\binom{-\frac{1}{2}i}{r}\binom{\frac{1}{2}i}{s}\,.$$

It follows that

$$G_m = i^m m! \sum_{r+s=m}(-1)^s\binom{-\frac{1}{2}i}{r}\binom{\frac{1}{2}i}{s}$$

$$= (-1)^m \sum_{r+s=m}\binom{m}{r}(\tfrac{1}{2}i)(\tfrac{1}{2}i+1)\cdots(\tfrac{1}{2}i+r-1)(\tfrac{1}{2}i)(\tfrac{1}{2}i-1)\cdots(\tfrac{1}{2}i-s+1)\,.$$

A simpler formula for G_m would be desirable.

Also partially solved by P. Bruckman.

PRIME ASSUMPTION

H-217 (corrected) Proposed by S. Krishnan, Orissa, India.

(a) Show that

$$2^{4n-4x-4}\binom{2x+2}{x+1} \equiv \binom{4n-2x-2}{2n-x-1} \pmod{4n+1}\,,$$

where n is a positive integer and $-1 \leqslant x \leqslant 2n-1$, x is an integer, and $4n+1$ is prime.

(b) Show that

$$2^{4n-4x-6}\binom{2x+4}{x+2} + \binom{4n-2x-2}{2n-x-1} \equiv 0 \pmod{4n+3}\,,$$

where n is a positive integer, $-2 \leqslant x \leqslant 2n-1$, x is an integer, and $4n+3$ is prime.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Assertions (a) and (b) are *false* for general n; we may make them true assertions by adding the hypothesis that $4n+1$ is prime, for part (a), and $4n+3$ is prime, for part (b). We may combine the two assertions as follows:

If p is a positive odd prime and x is an integer with $0 \leqslant x \leqslant \frac{1}{2}(p-1)$, then

$$2^{p-1-4x}\binom{2x}{x} \equiv (-1)^{\frac{1}{2}(p-1)}\binom{p-1-2x}{\frac{1}{2}(p-1)-x} \pmod p\,.$$

The following lemma is useful in the proof:

Lemma. If p is an odd prime, then

$$(\tfrac{1}{2})^{p-1}\binom{p-1}{\frac{1}{2}(p-1)} = \frac{1\cdot 3\cdot 5\cdots(p-2)}{2\cdot 4\cdot 6\cdots(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \pmod p\,.$$

Proof.

$$\frac{1\cdot 3\cdots(p-2)}{2\cdot 4\cdots(p-1)} = \frac{1^2 3^2\cdots(p-2)^2}{(p-1)!} \equiv \frac{1\cdot 3\cdots(p-2)(-2)(-4)\cdots(1-p)}{(p-1)!} \pmod p$$

$$\equiv (-1)^{\frac{1}{2}(p-1)}\frac{(p-1)!}{(p-1)!} \pmod p \equiv (-1)^{\frac{1}{2}(p-1)} \pmod p,$$

as asserted.

Now, let

$$U = 2^{p-1-4x}\binom{2x}{x}, \qquad V = \binom{p-1-2x}{\frac{1}{2}(p-1)-x},$$

where p and x are as stated above. Thus,

$$U = 2^{p-1-2x}\left\{\frac{1\cdot 3\cdots(2x-1)}{2\cdot 4\cdots(2x)}\right\}, \qquad V = 2^{p-1-2x}\left\{\frac{1\cdot 3\cdots(p-2-2x)}{2\cdot 4\cdots(p-1-2x)}\right\}.$$

Therefore,

$$V \equiv 2^{p-1-2x}\left\{\frac{(-2x-2)(-2x-4)\cdots(-p+1)}{(-2x-1)(-2x-3)\cdots(-p+2)}\right\} \pmod p \equiv 2^{p-1-2x}\left\{\frac{(2x+2)(2x+4)\cdots(p-1)}{(2x+1)(2x+3)\cdots(p-2)}\right\} \pmod p.$$

Since all the factors in the last expression are relatively prime to p, $V \not\equiv 0 \pmod p$; therefore, V^{-1} exists, and

$$UV^{-1} \equiv \frac{2^{p-1-2x}}{2^{p-1-2x}}\left\{\frac{1\cdot 3\cdots(2x-1)(2x+1)(2x+3)\cdots(p-2)}{2\cdot 4\cdots(2x)(2x+2)(2x+4)\cdots(p-1)}\right\} \pmod p.$$

Thus,

$$UV^{-1} \equiv \frac{1\cdot 3\cdots(p-2)}{2\cdot 4\cdots(p-1)} \pmod p \equiv (-1)^{\frac{1}{2}(p-1)} \pmod p,$$

by the lemma. Therefore,

$$U \equiv (-1)^{\frac{1}{2}(p-1)}V \pmod p,$$

which is equivalent to our assertion.

Also solved by P. Tracy.

STAGGERING PASCAL

H-218 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ 0 & 1 & 1 & 0 & \cdots \\ \cdots & \cdots & 2 & 1 & \cdots \\ & \cdots & & & \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the staggered Pascal Triangle and

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ & & \cdots & & \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the Pascal Binomial Array.

Finally let

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \\ & & \cdots & & \end{pmatrix}_{n\times n}$$

represent the matrix corresponding to the Fibonacci Convolution Array. Prove $A\cdot B = C$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Presumably, the matrix A should look as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

By inspection, or otherwise, we obtain the formulas

(1) $$a_{ij} = \binom{j-1}{i-j}, \quad \text{for } j \le i \le 2j-1; \quad a_{ij} = 0 \text{ otherwise}$$

(2) $$b_{ij} = \binom{i+j-2}{j-1} .$$

Let $D = AB.$ Then,

$$d_{ij} = \sum_{k=1+[\frac{1}{2}i]}^{i} \binom{k-1}{i-k}\binom{k+j-2}{j-1} .$$

For convenience, let $i-1=r$ and $j-1=a;$ also, let $m=i-k.$ Then,

$$d_{ij} = \theta_{rs} = \sum_{m=0}^{[\frac{1}{2}r]} \binom{r-m}{m}\binom{r+s-m}{s} .$$

Now, let

$$f_j(x) = \sum_{i=1}^{\infty} d_{ij}x^{i-1} = \sum_{r=0}^{\infty} \theta_{rs}x^r ;$$

then $f_j(x)$ is the generating function for the j^{th} column of $D.$

Thus,

$$f_j(x) = \sum_{r=0}^{\infty} x^r \sum_{m=0}^{[\frac{1}{2}r]} \binom{r-m}{m}\binom{r+s-m}{r-m} = \sum_{m=0}^{\infty} x^{2m} \sum_{r=0}^{\infty} \binom{r+m}{m}\binom{r+s+m}{r+m} x^r$$

$$= \sum_{m=0}^{\infty} x^{2m} \sum_{r=0}^{\infty} \binom{s+m}{m}\binom{r+s+m}{r} x^r = \sum_{m=0}^{\infty} \binom{-s-1}{m} (-x^2)^m \sum_{r=0}^{\infty} \binom{-s-m-1}{r} (-x)^r$$

$$= \sum_{m=0}^{\infty} \binom{-s-1}{x} (-x^2)^m (1-x)^{-s-m-1} = (1-x)^{-s-1}\left(1 - \frac{x^2}{1-x}\right)^{-s-1} = (1-x-x^2)^{-s-1},$$

i.e.,

$$f_j(x) = (1-x-x^2)^{-j} .$$

Since

$$f_1(x) = (1-x-x^2)^{-1} ,$$

the familiar generating function for the Fibonacci numbers, $f_j(x)$ is the column generator for the Fibonacci convolution matrix, i.e., C. Thus, $D = AB = C$.

Also solved by the Proposer.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A.P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-298 Proposed by Richard Blazej, Queens Village, New York.

Show that

$$5F_{2n+3} \cdot F_{2n-3} = L_{4n} + 18.$$

B-299 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Establish a simple closed form for

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k}.$$

B-300 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Establish a simple closed form for

$$L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k}.$$

B-301 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $[x]$ denote the greatest integer in x, i.e., the integer m with $m \leqslant x < m+1$. Also let

$$A(n) = (n^2 + 6n + 12)/12 \qquad \text{and} \qquad B(n) = (n^2 + 7n + 12)/6.$$

Does

$$[A(n)] + [A(n+1)] = [B(n)]$$

for all integers n? Explain.

B-302 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Prove that $F_n - 1$ is a composite integer for $n \geqslant 7$ and that $F_n + 1$ is composite for $n \geqslant 4$.

B-303 Proposed by David Singmaster, Polytechnic of the South Bank, London, England.

In B-260, it was shown that

$$\sigma(mn) > \sigma(m) + \sigma(n)\ ,$$

where $\sigma(n)$ is the sum of the positive integral divisors of n. What relation holds between $\sigma(mn)$ and $\sigma(m)\sigma(n)$?

SOLUTIONS

3 SYMBOL GOLDEN MEAN

B-274 Proposed by C.B.A. Peck, State College, Pennsylvania.

Approximate $(\sqrt{5}-1)/2$ to within 0.002 using at most three distinct familiar symbols. (Each symbol may represent a number or an operation and may be repeated in the expression.)

I. Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

We may use the well-known continued fraction expansion for

$$\theta = \tfrac{1}{2}(\sqrt{5}-1): \qquad \theta = 1/1 + 1/1 + 1/+ \cdots$$

with convergents:

$$0/1,\ 1/1,\ 1/2,\ 2/3,\ 3/5,\ 5/8,\ 8/13,\ 13/21, \cdots .$$

Clearly, any such expression satisfies the conditions of the problem, since it uses only the three symbols "1," "+" and "/" (or "_," for the last symbol, representing division). To obtain any desired degree of accuracy, we may use the inequality:

$$|\theta - p_n/q_n| < 1/q_nq_{n+1}\ ,$$

where p_n/q_n is the n^{th} convergent of the continued fraction. For this problem, we desire $1/q_nq_{n+1}$ to be less than $.002$, i.e., q_nq_{n+1} must exceed 500. Now

$$13 \cdot 21 = 273 < 500, \quad \text{while} \quad 21 \cdot 34 = 714 > 500\ ,$$

so we may take the continued fraction expression for $13/21$ as one solution (the simplest solution), although the corresponding expression for any higher convergent is also a solution.

II. The Proposer gave the solution in I and also noted that

$$(\sqrt{5}-1)/2 \doteq \pi^2/2^2 \doteq 0.6169$$

is easily obtained from

$$\pi \doteq \sqrt{8(\sqrt{5}-1)}\ ,$$

given in P. Poulet, *C'est Encore* π, Sphinx, Vol. 6, No. 12, Dec. 1936, pp. 208–212.

TWO IN ONE

B-275 Proposed by Warren Cheves, Littleton, North Carolina.

Show that

$$F_{mn} = L_mF_{m(n-1)} + (-1)^{m+1}F_{m(n-2)}\ .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

The required equation is a condensation into one identity of the two identities (I_{21}) and (I_{23}) on page 59 of Hoggatt's book, *Fibonacci and Lucas Numbers, viz.,*

$$F_{n+p} + F_{n-p} = F_nL_p\ ,\ p \text{ even},$$
$$F_{n+p} - F_{n-p} = F_nL_p\ ,\ p \text{ odd}.$$

In these two equations, replace n by $mn-m$ and p by m.

Also solved by Paul S. Bruckman, Wray G. Brady, Herta T. Freitag, John W. Milsom, C.B.A. Peck, A.G. Shannon (New South Wales), and the Proposer.

ONLY TWO SOLUTIONS

B-276 Proposed by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Find all the triples of positive integers m, n, and x such that

$$F_h = x^m, \quad \text{where} \quad h = 2^n \quad \text{and} \quad m > 1.$$

Solution by Phil Tracy, Lexington, Massachusetts.

It has only the trivial solutions $n = 0$ and $n = 1$ since F_{2n} is an integral multiple of 3 but not of 9 when $n > 1$. One can see this as follows. Modulo 9, the Fibonacci numbers repeat in blocks of 24. Examining the block, one finds $3|F_m$ if and only if $4|m$ while $9|F_m$ if and only if $12|m$. Finally, 2^n is an integral multiple of 4 but not of 12, when $n > 1$.

Also solved by Paul S. Bruckman, Herta T. Freitag, and the Proposer.

A LUCAS-FIBONACCI CONGRUENCE

B-277 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}}$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Using the Binet formulas

$$F_n = (a^n - b^n)/(a-b) \quad \text{and} \quad L_n = a^n + b^n,$$

one easily shows that

$$\text{(1)} \qquad L_{m+p} - L_{m-p} = 5F_mF_p, \quad p \text{ even}.$$

Set $m = 2n(k+1)$ and $p = 2nk$ in (1) to obtain

$$L_{2n(2k+1)} - L_{2n} = 5F_{2n(k+1)}F_{2nk}.$$

Since $F_{2n}|F_{2nk}$, the result follows.

REMARK. Since $F_{2n}|F_{2n(k+1)}$, the result can be stronger, i.e.,

$$L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}^2}.$$

Also solved by Gregory Wulczyn and the Proposer.

ANOTHER LUCAS-FIBONACCI CONGRUENCE

B-278 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $L_{(2n+1)(4k+1)} \equiv L_{2n+1} \pmod{F_{2n+1}}$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$\text{(1)} \qquad L_{(2n+1)(4k+1)} - L_{2n+1} = a^{(2n+1)(4k+1)} - b^{(2n+1)(4k+1)} - b^{2n+1}.$$

The quotient of (1) by

$$\frac{a^{2n+1} - b^{2n+1}}{\sqrt{5}} = 5\left[\frac{a^{4n+2} - b^{4n+2}}{\sqrt{5}} + \frac{a^{2(4n+2)} - b^{2(4n+2)}}{\sqrt{5}} + \cdots + \frac{a^{4k(2n+1)} - b^{4k(2n+1)}}{\sqrt{5}}\right]$$

$$= 5(F_{4n+2} + F_{4(2n+1)} + \cdots + F_{4k(2n+1)}),$$

an integer.

Also solved by David Zeitlin and the Proposer.

CORRECTED AND REINSERTED

Due to the typographical error in the original statement of B-279, the deadline for receipt of solutions has been extended. The error was corrected and the correct problem solved by Paul S. Bruckman, Charles Chouteau, Edwin T. Hoefer, and the Proposer. The error was also noted by Wray G. Brady. The corrected version is:

B-279 Find a closed form for the coefficient of x^n in the Maclaurin series expansion of $(x + 2x^2)/(1 - x - x^2)^2$.

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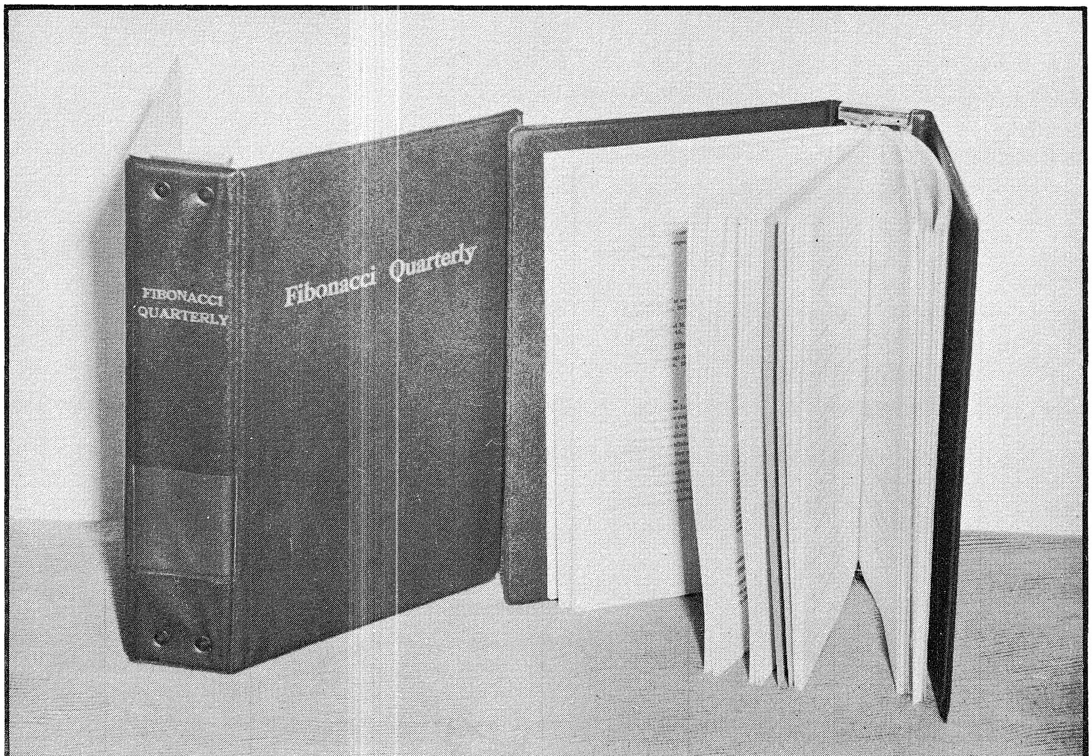
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