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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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FIBONACCI NOTES
4: q-FIBONACCI POLYNOMIALS

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1. We shall make use of the notation of [1]. In addition we define

(1.1)
$$\phi_n(a) = \phi_n(a,q) = \sum_{2k<n} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2} a^{n-2k-1} \quad (n \geq 1).$$

Since

$$\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} = q^{n-2k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix},$$

it is clear that

$$\phi_{n+1}(a) - a\phi_n(a) = \sum_{2k\leq n} \left(\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \right) q^{k^2} a^{n-2k} = \sum_{0<2k\leq n} q^{n-2k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{k^2} a^{n-2k}$$

$$= q^{n-1} \sum_{0<2k\leq n} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{(k-1)^2} a^{n-2k} = q^{n-1} \sum_{2k<n-1} \begin{bmatrix} n-k-2 \\ k \end{bmatrix} q^{k^2} a^{n-2k-2}.$$

Hence

(1.2)
$$\phi_{n+1}(a) - a\phi_n(a) = q^{n-1}\phi_{n-1}(a) \quad (n > 1).$$

The first few values of $\phi_n(a)$ are easily computed by means of (1.1) or (1.2).

$$\phi_1(a) = 1, \quad \phi_2(a) = a, \quad \phi_3(a) = a^2+q, \quad \phi_4(a) = a^3+q(1+q)a,$$
$$\phi_5(a) = a^4+q(1+q+q^2)a^2+q^4, \quad \phi_6(a) = a^5+q(1+q+q^2+q^3)a^3+q^4(1+q+q^2)a.$$
$$\phi_7(a) = a^6+q(1+q+q^2+q^3+q^4)a^4+q^4(1+q+q^2)(1+q^2)a^2+q^9.$$

If we put $\phi_0(a) = 0$ then (1.2) holds for all $n \geq 1$. By means of (1.2) we can define $\phi_n(a)$ for all integral n. It is convenient to put

(1.3)
$$\overline{\phi}_n(a) = \overline{\phi}_n(a,q) = (-1)^{n-1}\phi_{-n}(a).$$

Then (1.2) becomes

(1.4)
$$\overline{\Phi}_n(a) = q^n(a\overline{\Phi}_{n-1}(a) + \overline{\Phi}_{n-2}(a)) \qquad (n \geq 2),$$

where

(1.5)
$$\overline{\Phi}_0(a) = 0, \qquad \overline{\Phi}_1(a) = q.$$

The next few values of $\overline{\Phi}_n(a)$ are

$$\overline{\Phi}_2(a) = q^3a, \quad \overline{\Phi}_3(a) = q^4(1+q^2a^2), \quad \overline{\Phi}_4(a) = q^7((1+q)a+q^3a^3),$$
$$\overline{\Phi}_5(a) = q^9(1+(q^2+q^3+q^4)a^2+q^6a^4),$$
$$\overline{\Phi}_6(a) = q^{13}((1+q+q^2)a+(q^3+q^4+q^5+q^6)a^3+q^8a^5).$$

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Put

(1.6)
$$\Phi(a,x) = \sum_{n=0}^{\infty} \bar{\phi}_n(a)x^n .$$

Then by (1.4) and (1.5),

$$\Phi(a,x) = qx + \sum_{n=2}^{\infty} q^n(a\bar{\phi}_{n-1}(a) + \bar{\phi}_{n-2}(a))x^n ,$$

so that

(1.7)
$$\Phi(a,x) = qx + qx(a+qx)\Phi(a, qx).$$

Thus

$$\begin{aligned}\Phi(a,x) &= qx + qx(a+qx)\left\{q^2x + q^2x(a+q^2x)\Phi(a,q^2x)\right\}\\ &= qx + q^3x^2(a+qx) + q^3x^2(a+qx)(a+q^2x)\Phi(a,q^2x).\end{aligned}$$

Continuing in this way we get

(1.8)
$$\Phi(a,x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1}(a+qx)(a+q^2x)\cdots(a+q^kx).$$

Since

$$(a+qx)(a^2+qx)\cdots(a^2+q^kx) = \sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)}a^{k-j}x^j ,$$

(1.8) becomes

$$\begin{aligned}\Phi(a,x) &= \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)}a^{k-j}x^j\\ &= \sum_{n=0}^{\infty} x^{n+1}\sum_{2j\leqslant n}\begin{bmatrix}n-j\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)+\frac{1}{2}(n-j+1)(n-j+2)}a^{n-2j} .\end{aligned}$$

It follows that

(1.9)
$$\bar{\phi}_{n+1}(a) = \sum_{2j\leqslant n}\begin{bmatrix}n-j\\ j\end{bmatrix}q^{\frac{1}{2}(n+1)(n+2)-nj+j(j-1)}a^{2n-j} .$$

Since

$$\phi_{n+1}(a) = \sum_{2j\leqslant n}\begin{bmatrix}n-j\\ j\end{bmatrix}q^{j^2}a^{n-2j} ,$$

it is clear that

$$\bar{\phi}_{n+1}(a) = q^{n+1}\phi_n(q^{(n+1)/2}a),$$

that is,

(1.10)
$$\bar{\phi}_n(a) = q^n\phi_n(q^{n/2}a) .$$

2. It is evident that

(2.1)
$$F_n(q) = \phi_n(1,q).$$

Also it follows from

$$F'_{n+1}(q) = \sum_{2k\leqslant n} q^{(k+1)^2}\begin{bmatrix}n-k\\ k\end{bmatrix}$$

that

(2.2)
$$F'_n(q) = q^n\phi_n(q^{-1}, q).$$

We have defined [1] the q-Lucas number

(2.3) $$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q).$$

Hence, by (2.1) and (2.2),

(2.4) $$L_n(q) = \phi_{n+2}(1,q) - q^2\phi_{n-2}(q^{-1}, q).$$

In the next place put

(2.5) $$\phi_n^*(a) = \phi_n^*(a,q) = \phi_n(a,q^{-1}).$$

When q is replaced by q^{-1}, it is easily verified that

$$\begin{bmatrix} n-k \\ k \end{bmatrix} \to q^{k(2k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Hence

$$\phi_{n+1}(a,q^{-1}) = \sum_{2k \leqslant n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2-nk} a^{n-2k},$$

so that

(2.6) $$q^{n^2/2}\phi_{n+1}^*(a,q) = \phi_{n+1}(aq^{-n/2}, q).$$

In particular we have

(2.7) $$q^{n^2/2}F_{n+1}(q^{-1}) = \phi_{n+1}(q^{-n/2}, q)$$

and

(2.8) $$q^{\frac{1}{2}(n^2+1)}F'_n(q^{-1}) = \phi_n(q^{\frac{1}{2}(n+1)}, q).$$

3. Returning to the recurrence (1.2), we have

(3.1) $$a\phi_n(a) = \phi_{n+1}(a) - q^{n-1}\phi_{n-1}(a).$$

Thus

$$a^2\phi_n(a) = \phi_{n+2}(a) - (1+q)q^{n-1}\phi_n(a) + q^{2n-3}\phi_{n-2}(a)$$

and

$$a^3\phi_n(a) = \phi_{n+3}(a) - (1+q+q^2)q^{n-1}\phi_{n+1}(a) + (1+q+q^2)q^{2n-3}\phi_{n-1}(a) - q^{3n-6}\phi_{n-3}(a).$$

This suggests the general formula

(3.2) $$a^k\phi_n(a) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j}(a),$$

where $k \geqslant 0$ but n is an arbitrary integer.

Clearly (3.2) holds for $k = 0, 1, 2, 3$. Assuming that it holds up to and including the value k, we have, by (3.1),

$$a^{k+1}\phi_n(a) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)} \left\{ \phi_{n+k-2j+1}(a) - q^{n+k-2j-1}\phi_{n+k-2j-1}(a) \right\}$$

$$= \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a)$$

$$+ \sum_{j=1}^{k+1} (-1)^j \begin{bmatrix} k \\ j-1 \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)+k-j+1}\phi_{n+k-2j+1}(a)$$

$$= \sum_{j=0}^{k+1} (-1)^j \left\{ \begin{bmatrix} k \\ j \end{bmatrix} + \begin{bmatrix} k \\ j-1 \end{bmatrix} q^{k-j+1} \right\} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a)$$

$$= \sum_{j=0}^{k+1} (-1)^j \begin{bmatrix} k+1 \\ j \end{bmatrix} q^{jn-\frac{1}{2}j(j+1)}\phi_{n+k-2j+1}(a).$$

This completes the proof of (3.2).

Special cases of interest are obtained by taking $n = k, -k, 0, 1$ in (3.2). We get

$$a^k \phi_k(a) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{kj-\frac{1}{2}j(j+1)} \phi_{2k-2j}(a), \tag{3.3}$$

$$a^k \phi_{-k}(a) = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{-kj-\frac{1}{2}j(j+1)} \phi_{-2j}(a), \tag{3.4}$$

$$0 = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{-\frac{1}{2}j(j+1)} \phi_{k-2j}(a), \tag{3.5}$$

$$a^k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{-\frac{1}{2}j(j-1)} \phi_{k-2j+1}(a) . \tag{3.6}$$

Note that in approximately half the terms on the right of (3.6) the subscript $k-2j+1$ is positive but is negative in the remaining terms. Also, if we prefer, we may eliminate negative subscripts in (3.4), (3.5), and (3.6) by making use of (1.10).

It is clear from (1.1) that we may put

$$a^k = \sum_{2j \leqslant k} (-1)^j q^j C_{k,j} \phi_{k-2j+1}(a) , \tag{3.7}$$

where the coefficients $C_{k,j}$ are independent of a. This formula is of course not the same as (3.6). To determine $C_{k,j}$ we multiply both sides of (3.6) by a and then apply (3.1). We get

$$a^{k+1} = \sum_{2j \leqslant k} (-1)^j q^j C_{k,j} \left\{ \phi_{k-2j+2}(a) - q^{k-2j} \phi_{k-2j}(a) \right\}$$

$$= \sum_{2j \leqslant k} (-1)^j q^j C_{k,j} \phi_{k-2j+2}(a) + \sum_{2j \leqslant k+1} (-1)^j q^{k-j+1} C_{k,j-1} \phi_{k-2j+2}(a).$$

It follows that

$$C_{k+1,j} = C_{k,j} + q^{k-2j+1} C_{k,j-1} \qquad (2j \leqslant k). \tag{3.8}$$

The first few values of $C_{k,j}$ are easily computed by means of (3.8).

$n \backslash k$	0	1	2	3
0	1			
1	1			
2	1	1		
3	1	$1+q$		
4	1	$1+q+q^2$	$1+q$	
5	1	$1+q+q^2+q^3$	$1+2q+q^2+q^3$	
6	1	$1+q+q^2+q^3+q^4$	$1+2q+2q^2+2q^3+q^4+q^5$	$1+2q+q^2+q^3$
7	1	$1+q+q^2+q^3+q^4+q^5$	$1+2q+2q^2+3q^3+2q^4+2q^5+q^6+q^7$	$1+3q+3q^2+3q^3+2q^4+q^5+q^6$

It is evident from (3.8) that $C_{k,j}$ is a polynomial in q with nonnegative coefficients and that

(3.9) $$C_{k,0} = 1 \qquad (k = 0, 1, 2, \cdots),$$

(3.10) $$C_{k,j} = 0 \qquad (2j > k).$$

Also it is easily seen that

(3.11) $$C_{k,1} = \frac{1-q^{k-1}}{1-q} \qquad (k \geqslant 1).$$

To get $C_{k,2}$ we take $j = 2$ in (3.8). Thus

$$C_{k+1,2} - C_{k,2} = q^{k-3}C_{k,1} = q^{k-3}\,\frac{1-q^{k-1}}{1-q},$$

which holds for $k \geqslant 3$. Hence

$$C_{k+1,2} = \frac{1}{1-q}\sum_{j=3}^{k} q^{j-3}(1-q^{j-1}),$$

which reduces to

(3.12) $$C_{k+1,2} = \begin{bmatrix} k-2 \\ 1 \end{bmatrix} + q\begin{bmatrix} k-1 \\ 2 \end{bmatrix}.$$

In the next place, taking $j = 3$ in (3.8),

$$C_{k+1,3} - C_{k,3} = q^{k-5}C_{k,2} \qquad (k \geqslant 5).$$

We find that

(3.13) $$C_{k+1,3} = q^{-1}\begin{bmatrix} k-2 \\ 2 \end{bmatrix} + \begin{bmatrix} k-1 \\ 3 \end{bmatrix} - q^{-1} - 1.$$

By means of (3.8) it can be proved that

(3.14) $$\deg C_{k,j} = jk - \tfrac{1}{2}j(3j+1).$$

The proof is by induction on k. The second term on the right of (3.8) is of higher degree than the first term, so that

$$\deg C_{k+1,j} = k - 2j + 1 + \deg C_{k,j-1} = (k-2j+1) + (j-1)k - \tfrac{1}{2}(j-1)(3j-2) = j(k+1) - \tfrac{1}{2}j(3j+1).$$

It would be of interest to find a simple explicit formula for $C_{k,j}$. The problem is equivalent to inverting

(3.15) $$u_n = \sum_{2k \leqslant n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2} v_{n-2k} \qquad (n = 0, 1, 2, \cdots).$$

In this connection the following two inversion theorems may be mentioned:

I. $$u_r = \sum_{2s \leqslant r} \begin{bmatrix} r \\ s \end{bmatrix} v_{r-2s} \qquad (r = 0, 1, 2, \cdots)$$

if and only if

$$v_r = \sum_{2s \leqslant r} (-1)^s q^{\frac{1}{2}s(s-1)}\,\frac{1-q^r}{1-q^{r-s}} \begin{bmatrix} r-s \\ s \end{bmatrix} v_{r-2s} \qquad (r = 0, 1, 2, \cdots).$$

II. $$u_r = \sum_{2s \leqslant r} \left\{ \begin{bmatrix} r \\ s \end{bmatrix} - \begin{bmatrix} r \\ s-1 \end{bmatrix} \right\} v_{r-2s} \qquad (r = 0, 1, 2, \cdots)$$

if and only if

$$v_r = \sum_{2s \leqslant r} (-1)^s q^{\frac{1}{2}s(s+1)} \begin{bmatrix} r-s \\ s \end{bmatrix} u_{r-2s} \qquad (r = 0, 1, 2, \cdots).$$

For proof of these and some related inversion theorems see [2].

4. Returning to the recurrence (1.2) we now construct a second solution $\psi_n(a) = \psi_n(a,q)$ such that

(4.1) $$\psi_0(a) = 1, \qquad \psi_1(a) = a$$

and of course

(4.2) $$\psi_{n+1}(a) = a\psi_n(a) + q^{n-1}\psi_{n-1}(a) \qquad (n \geq 1).$$

Put

(4.3) $$\Psi(a,x) = \sum_{n=0}^{\infty} \psi_n(a)x^n .$$

Then

$$\Psi(a,x) = 1 + ax + \sum_{n=2}^{\infty} (a\psi_{n-1}(a) + q^{n-2}\psi_{n-2}(a))x^n = 1 + ax\Psi(a,x) + x^2\Psi(a,qx),$$

so that

(4.4) $$\Psi(a,x) = \frac{1}{1-ax} + \frac{x^2}{1-ax}\Psi(a,qx).$$

Iteration of (4.4) yields

(4.5) $$\Psi(a,x) = \sum_{r=0}^{\infty} \frac{q^{r(r-1)}x^{2r}}{(ax)_{r+1}} .$$

Hence

$$\Psi(a,x) = \sum_{r=0}^{\infty} q^{r(r-1)}x^{2r} \sum_{s=0}^{\infty} \begin{bmatrix} r+s \\ s \end{bmatrix} a^s x^s = \sum_{n=0}^{\infty} x^n \sum_{2r \leq s} \begin{bmatrix} n-r \\ r \end{bmatrix} q^{r(r-1)}a^{n-2r} ,$$

which implies

(4.6) $$\psi_n(a) = \sum_{2r \leq n} \begin{bmatrix} n-r \\ r \end{bmatrix} q^{r(r-1)}a^{n-2r} .$$

We have therefore

(4.7) $$q^{\frac{n}{2}}\psi_n(a) = \phi_{n+1}(q^{1/2}a) .$$

Finally we mention the following continued fraction formula.

(4.8) $$a + \frac{q}{a+}\frac{q^2}{a+}\cdots\frac{q^n}{a} = \frac{\phi_{n+2}(a)}{q^{n/2}\phi_{n+1}(q^{-1/2}a)} = \sum_{2k \leq n+1} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} q^{k^2}a^{n-2k+1} \Big/ \sum_{2k \leq n} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k(k+1)}a^{n-2k}.$$

An equivalent result has been obtained by Hirschhorn [3].

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A GENERAL IDENTITY FOR MULTISECTING GENERATING FUNCTIONS

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Consider the general power series:

$$(1)\qquad f(x) = \sum_{n=0}^{\infty} a_n x^n$$

(defined for some radius of convergence R, whenever $|x| < R$).

It is desired to find an expression, preferably in terms of $f(x)$, for the so-called multisecting generating function, defined as follows:

$$(2)\qquad g(r,s,x) = \sum_{n=0}^{\infty} a_{nr+s} x^{nr+s}$$

(where r and s are integers satisfying $0 \leqslant s < r$).

We shall suppose that $f(x)$, and therefore $g(r,s,x)$ satisfy appropriate convergence requirements, so that the following development may have validity.

The problem indicated above has been solved by various investigators, for certain special cases. For example, Gould [1] has obtained the following results, for the case where $a_n = F_n$ (the n^{th} Fibonacci number):

$$(3)\qquad f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}\,;\; g(r,s,x) = \sum_{n=0}^{\infty} F_{nr+s} x^{nr+s} = \frac{F_s x^s + (-1)^s F_{r-s} x^{r+s}}{1 - L_r x^r + (-1)^r x^{2r}}\,.$$

Also, Hoggatt and Anaya, in a recent joint paper [2], derived a comparable relation for the column generators of Pascal's left-justified triangle. Actually, the definition of the multisecting generating function of $f(x)$ used by these writers was the following:

$$(4)\qquad h(r,s,x) = \sum_{n=0}^{\infty} a_{nr+s} x^n\,.$$

The modification of the latter definition given by (2) is slight, since $g(r,s,x)$ and $h(r,s,x)$ are related as follows:

$$(5)\qquad g(r,s,x) = x^s h(r,s,x^r)\,.$$

For the purposes of this paper, Eq. (2) is a more convenient definition.

$$(6)\qquad g(r,s,x) = \sum_{n=0}^{\infty} a_n \theta(n,r,s) x^n, \quad \text{where} \quad \theta(n,r,s) = \begin{cases} 1 & \text{if } n \equiv s \pmod r \\ 0 & \text{otherwise} \end{cases}$$

This is evident from the definition of $g(r,s,x)$ in (2). Another evident relation is:

$$(7)\qquad f(x) = \sum_{s=0}^{r-1} g(r,s,x)\,.$$

What is needed is an explicit expression for $\theta(n,r,s)$. Such an expression is conveniently provided by the following function:

$$\theta(n,r,s) = \frac{1}{r}\sum_{k=0}^{r-1} e^{(n-s)2k\pi i/r} = \frac{e^{(n-s)2k\pi i}-1}{r\left\{e^{(n-s)2k\pi i/r}-1\right\}} \quad \text{(provided } n \not\equiv s \ (mod\ r)\text{)}\ . \tag{8}$$

If $n = s + mr$, for some integer m, then $e^{(n-s)2k\pi i/r} = e^{2mk\pi i} = 1$. In this event, $\theta(n,r,s) = r/r = 1$. On the other hand, if $n \not\equiv s \ (mod\ r)$, the numerator of the second expression in (8) vanishes, but the denominator does not; i.e., $\theta(n,r,s) = 0$. Thus, $\theta(n,r,s)$ as defined in (8) has the desired properties we are seeking for this function. Accordingly,

$$g(r,s,x) = \sum_{n=0}^{\infty} a_n x^n \frac{1}{r}\sum_{k=0}^{r-1} e^{(n-s)2k\pi i/r}$$

$$= \frac{1}{r}\sum_{k=0}^{r-1} e^{-2sk\pi i/r} \sum_{n=0}^{\infty} a_n\left\{e^{2k\pi i/r}x\right\}^n = \frac{1}{r}\sum_{k=0}^{r-1} e^{-2sk\pi i/r} f(e^{2k\pi i/r}x)\,.$$

We may make a further simplification, by letting $w(r,k) = e^{2k\pi i/r}$, the $(k+1)^{th}$ r^{th} root of unity. We note that

$$w(r,k) = \left\{w(r,1)\right\}^k;$$

if we let w_r denote $w(r,1)$, then our relation takes the following form:

$$g(r,s,x) = \frac{1}{r}\sum_{k=0}^{r-1} w_r^{-sk} f(w_r^k x)\ . \tag{9}$$

This is the general expression we are seeking. Any further simplification will depend on the particular values of r and s, and on the specific form of $f(x)$. Indicated below are several special cases of (9) for the first few values of r and s, but for perfectly general $f(x)$:

$$g(1,0,x) = f(x), \quad g(2,0,x) = \tfrac{1}{2}\left\{f(x)+f(-x)\right\}, \quad g(2,1,x) = \tfrac{1}{2}\left\{f(x)-f(-x)\right\},$$

$$g(3,0,x) = \frac{1}{3}\left\{f(x)+f(ux)+f(u^2x)\right\} \text{ (where } u = \tfrac{1}{2}(-1+i\sqrt{3})\text{)}, \quad g(3,1,x) = \frac{1}{3}\left\{f(x)+u^2f(ux)+uf(u^2x)\right\},$$

$$g(3,2,x) = \frac{1}{3}\left\{f(x)+uf(ux)+u^2f(u^2x)\right\}, \quad g(4,0,x) = \frac{1}{4}\left\{f(x)+f(ix)+f(-x)+f(-ix)\right\}, \tag{10}$$

$$g(4,1,x) = \frac{1}{4}\left\{f(x)-if(ix)-f(-x)+if(-ix)\right\}, \quad g(4,2,x) = \frac{1}{4}\left\{f(x)-f(ix)+f(-x)-f(-ix)\right\},$$

$$g(4,3,x) = \frac{1}{4}\left\{f(x)+if(ix)-f(-x)-if(-ix)\right\}.$$

Note that the coefficients w_r^{-sk} are themselves r^{th} roots of unity, in permuted order (but with unity itself always first). If we sum $g(r,s,x)$ over s, keeping k fixed, the sum of these coefficients vanishes, except for $k = 0$, where it is unity. This is in accordance with our expected result in (7).

Many interesting special cases of (9) exist, and have been extensively studied, for specific functions $f(x)$. For example, if $f(x) = e^x$, Eq. (9) yields the following:

$$g(r,s,x) = \sum_{n=0}^{\infty} \frac{x^{nr+s}}{(nr+s)!} = \frac{1}{r}\sum_{k=0}^{r-1} w_r^{-sk} e^{w_r^k x}\ . \tag{11}$$

This may be further simplified and expressed as a strictly real function, involving trigonometric terms, but we will not do this here. It will suffice to say that the general form of (9) possesses an intrinsic symmetry which further manipulation tends to eliminate. For example, using identity (11),

$$g(3,0,x) = \frac{1}{3}\left\{e^x + e^{ux} + e^{u^2x}\right\},$$

where u is as defined in (10); however, we may also express $g(3,0,x)$ in real form:

$$g(3,0,x) = \frac{1}{3}\left\{e^x + 2e^{-\frac{1}{2}x}\cos\left(\tfrac{1}{2}x\sqrt{3}\right)\right\},$$

which is not as elegant a result as (11). Similarly, many special cases of (9) may be verified by the interested reader; it is the writer's opinion, nevertheless, that (9) possesses a special elegance just as it stands, limited though its practical usefulness may be.

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A FORMULA FOR $A_n^2(x)$

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This paper is a follow-up of [1], which dealt with certain combinatorial coefficients denoted by the symbol $A_n(x)$. We begin by recalling the definition of $A_n(x)$, which was given in [1]:

$$(1)\qquad (1-u)^{-1}(1+u)^x = \sum_{n=0}^{\infty} A_n(x)u^n\,; \quad \text{therefore,} \quad A_n(x) = \sum_{i=0}^{n}\binom{x}{i}\,,$$

which is a polynomial in x. In [1], the writer indicated that he had found the first few terms in the combinatorial expansion for $A_n^2(x)$, but was unable to obtain the general expansion. Formula (78) in [1] gave the first few terms of the expression, derived by direct expansion:

$$(2)\qquad A_n^2(x) = \binom{2n}{n}\left\{\binom{x}{2n} + \tfrac{1}{2}(n+2)\binom{x}{2n-1} + \left(\frac{n^3+2n^2+3n-4}{8n-4}\right)\binom{x}{2n-2} + \cdots\right\}.$$

The problem of obtaining the general term of the polynomial $A_n^2(x)$ has now been resolved. However, the expression is in the form of an iterated summation, which is indicated below:

$$(3)\qquad A_n^2(x) = \sum_{i=0}^{n} 3^i\binom{x}{i} + \sum_{i=n+1}^{2n}\binom{x}{i}\sum_{j=i-n}^{n}\binom{i}{j}\sum_{k=0}^{j+n-i}\binom{j}{k} \qquad (n = 1, 2, 3, \cdots)$$

Perhaps some interested reader can reduce this expression to a simpler one, involving only two (or possibly one) summation variables. If we denote the coefficient of $\binom{x}{i}$ as θ_i, relation (3) above yields the following values:

$$\theta_{2n} = \binom{2n}{n}; \quad \theta_{2n-1} = \frac{(2n-1)!}{n!n!}\,n(n+2); \quad \theta_{2n-2} = \frac{(2n-2)!}{n!(n-1)!}\,\tfrac{1}{2}(n^3+2n^2+3n-4)$$

(these last three values may be compared with those in (2));

$$\theta_{2n-3} = \frac{(2n-3)}{n!(n-2)!}\,\frac{1}{6}(n^4+n^3+8n^2+2n-24);$$

also, $\theta_{n+1} = 3^{n+1} - 2\cdot2^{n+1} + 1^{n+1}$; $\quad\theta_{n+2} = 3^{n+2} - 2\cdot2^{n+2} + 1^{n+2} - (n+2)(2^{n+2}-1) + (n+2)^2$.

In attempting to discover the law of formation of θ_i for $i > n$, it is clear that increasing difficulty is encountered as one recedes from either end of the second (iterated) summation in the right member of (3). Possibly, θ_i may be concisely expressed in terms of a finite difference operator, but this approach has not yet been fully explored.

A proof of (3) follows. The proof hinges on a formula due to Riordan, indicated as formula (6.44) in [2]. This formula is as follows:

$$(4)\qquad \sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{m-k}\binom{x}{m+n-k} = \binom{x}{m}\binom{x}{n}.$$

A slightly more convenient form of (4) is obtained by the substitution $i = m+n-k$, also observing that the upper limit in (4) need only equal min (m,n), since subsequent terms vanish. Then (4) takes the following form:

$$(5)\qquad \binom{x}{m}\binom{x}{n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{m}\binom{m}{i-n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{n}\binom{n}{i-m}.$$

Now

$$A_n^2(x) = \sum_{j=0}^{n}\binom{x}{j}\sum_{h=0}^{n}\binom{x}{h} = \sum_{j=0}^{n}\sum_{h=0}^{n}\sum_{i=\max(j,h)}^{j+h}\binom{x}{i}\binom{i}{j}\binom{j}{i-h},$$

(applying the result in (5)),

$$= \sum_{j=0}^{n}\sum_{i=j}^{j+n}\binom{x}{i}\binom{i}{j}\sum_{h=i-j}^{m}\binom{j}{i-h},$$

where $m = \min(i,n)$. Now let $h = i - j + k$. Then

$$A_n^2(x) = \sum_{j=0}^{n}\sum_{i=j}^{j+n}\binom{x}{i}\binom{i}{j}\sum_{k=0}^{m-i+j}\binom{j}{j-k} = \sum_{i=0}^{2n}\binom{x}{i}\sum_{j=i-m}^{m}\binom{i}{j}\sum_{k=0}^{m-i+j}\binom{j}{k}.$$

Distinguishing between the cases where $i \leqslant n$ and $i > n$, this expression may be simplified as follows:

$$\sum_{i=0}^{n}\binom{x}{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k} + \sum_{i=n+1}^{2n}\binom{x}{i}\sum_{j=i-n}^{n}\binom{i}{j}\sum_{k=0}^{n-i+j}\binom{j}{k}.$$

Comparing this with the right member of (3), we see that the only thing left to prove is that

$$3^i = \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}.$$

But this is an easy consequence of the binomial theorem, applied twice, since

$$\sum_{k=0}^{j}\binom{j}{k} = (1+1)^j = 2^j, \quad\text{and}\quad \sum_{j=0}^{i}\binom{i}{j}2^j = (1+2)^i = 3^i.$$

Hence (3) is proved. Obviously, the expression for θ_i given by (3), for $i > n$, is not unique. By various substitutions and/or translations, a wide variety of expressions for θ_i may be derived from the basic relationship in (3). For example, the following alternative formula is given, without proof:

$$\sum_{j=[\frac{1}{2}(1+i)]}^{n}\binom{i}{j}\sum_{k=0}^{2j-i}\binom{1+j}{k} = \sum_{j=2i-2n}^{i}\binom{i}{j}\sum_{k=i-n}^{j+n-i}\binom{j}{k} = \theta_i, \quad (i > n) \tag{6}$$

(where $[u]$ represents the integral part of u).

Attempts by the writer to obtain a generating function for the $A_n^2(x)$'s, in closed form, were unsuccessful. Can anyone help?

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THE GENERALIZED FIBONACCI NUMBER AND ITS RELATION TO WILSON'S THEOREM

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In this paper we consider the generalized Fibonacci second-order recurrence relation

(1) $$U_{k+2} = xU_{k+1} + yU_k\,,$$

with x and y variables. Then for certain x and y in (1) we introduce the following new theorems:

Theorem 1. If $U_{p-1} \equiv 0 \pmod{p^2}$, then $p > 3$ is *always* an odd prime.

Corollary 1. If $U_p + 1 \equiv 0 \pmod{p}$ then $p > 3$ is *always* an odd prime.

Corollary 2. If $U_p + 1 \equiv 0 \pmod{p^2}$ or $\pmod{p^3}$ then $(p-1)! + 1 \equiv 0$ respectively $\pmod{p^2}$ or $\pmod{p^3}$.

In the Addenda of this paper we also prove: If

$$F_n = k_1F_{n-1} + k_2F_{n-2}\,,$$

(where k_1 and k_2 are arbitrary constant numbers), then the following relation *always* holds

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^n k_2^n\,,$$

where

$$F_0 = 1, \quad F_1 = k_1, \quad F_2 = k_1^2 + k_2, \cdots .$$

NOTE. This paper was presented in person and in full at meeting No. 703 of The American Mathematical Society, New York, April 18–21, 1973. An abstract also appeared in the *Notices of the American Mathematical Society,* Vol. 20, No. 3, April 1973, issue No. 145, p. A-361, under 703-A22.

For clarity we write (1) as

(2) $$U_k = x_kU_{k-1} + y_kU_{k-2}\,,$$

where $k \geqslant 3$ is a positive integer, and the x_k, y_k are arbitrary variables.

$$U_k = x_kU_{k-1} + y_kU_{k-2}, \quad k \geqslant 2.$$

If $x_k = 2k-1$ and $y_k = -(k-1)^2$, then (2) becomes

(3) $$U_{k+1} = (2k+1)U_k - k^2U_{k-1}\,.$$

What we want to show next is that if in addition to (3) we let

(3b) $$U_k = kU_{k-1} + (k-1)!\,,$$

then

$$U_{k+1} = (k+1)U_k + k!\,.$$

To see this,

$$\begin{aligned} U_{k+1} &= (2k+1)(kU_{k-1} + (k-1)!) - k^2U_{k-1} = 2k^2U_{k-1} + kU_{k-1} - k^2U_{k-1} + (2k+1)(k-1)! \\ &= k^2U_{k-1} + kU_{k-1} + 2k! + (k-1)! = (k+1)(kU_{k-1} + (k-1)!) + k! = (k+1)U_k + k!\,, \end{aligned}$$

which is (3b) with k replaced by $k+1$. The proof is complete by induction. We then conclude that Eq. (3) may be written in the following two ways:

(4) $$U_{k+1} = (2k+1)U_k - k^2U_{k-1} = (k+1)U_k + k!,$$

where $k \geq 2,\ U_1 = 1,\ U_2 = 3,\ U_3 = 11, \cdots$.

H. Gupta has noticed that the sequence $1, 3, 11, 50, \cdots, U_{k+1} = (k+1)U_k + k!$ is really the second column of the array of STIRLING NUMBERS OF THE FIRST KIND. See Riordan [4], pp. 33 and 48. Of course, in the table the signs are alternating.

From page 33 of [4] we find

(A) $$s(k+1, n) = s(k, n-1) - ks(k, n)$$

so that we note that if $n = 2$, we get

$$s(k+1, 2) = s(k, 1) - ks(k, 2)$$

and, from the table on page 48 of [4], we note

$$s(k, 1) = (-1)^{k+1}(k-1)!.$$

Now let

$$V_k(-1)^{k+1} = s(k+1, 2),$$

then (A) becomes

$$V_k(-1)^{k+1} = (-1)^{k+1}(k-1)! - kV_{k-1}(-1)^k$$

or equivalently

$$V_{k+1} = kV_k + k!$$

which agrees with (4) for $k+1$. Q.E.D.

It is of course evident that

(5) $$m(m-2)!/m! = 1/(m-1),$$

and also that

(6) $$U = 2!(1) + 1!$$

(by (4)). Then, since $U_3 = 3U_2 + 2!$, we combine this equation with (5, with m = 3) and (6), which leads to $U_3 = 3!(1 + 1/2) + 2!$, and in the exact way we get

(7) $$U_4 = 4!(1 + 1/2 + 1/3) + 3!.$$

Then in the exact way we derived (7), step-by-step (with added induction we prove that

(8) $$U_k = k!(1 + 1/2 + 1/3 + \cdots + 1/(k-1)) + (k-1) = k!\left(\sum_{r=1}^{k} 1/r\right),$$

for $k = 1, 2, 3, \cdots$. (It may be interesting to emphasize the fact that we have found the explicit formula

$$\sum_{r=1}^{k} 1/r = U_k/k!.)$$

Now, using the well known fact that

(9) $$\phi(k-1) = \sum_{r=1}^{k} 1/r \equiv 0 \pmod{k^2},$$

if and only if $k > 3$ is an odd prime (see 1), we are in a position to prove the following theorems:

(10) *Theorem 1.* If $U_{p-1} \equiv 0 \pmod{p^2}$, then $p > 3$ is always an odd prime. The proof is immediate by combining (8, with $k = p-1$) with (9) which leads to the congruence $U_{p-1} = (p-1)!\phi(p-1) \equiv 0 \pmod{p^2}$.

(10a) *Corollary 1.* If $U_p + 1 \equiv 0 \pmod p$, then $p > 3$ is always an odd prime.

The proof of Corollary 1 is immediate by combining (3b, with k replaced by some odd prime number $p > 3$) with Wilson's theorem (Wilson's theorem: $(p-1)! + 1 \equiv 0 \pmod p$, if and only if p is a prime number), since

(10b) $$U_p + 1 \equiv pU_{p-1} + (p-1)! + 1 \equiv 0 \pmod p.$$

(10c) *Corollary 2.* If $U_p + 1 \equiv 0 \pmod{p^2}$ or $\pmod{p^3}$, then $(p-1)! + 1 \equiv 0$ respectively $\pmod{p^2}$ or $\pmod{p^3}$.

We easily prove (10c) by combining (10b) with (10). Since this leads to

(10d) $$U_p + 1 \equiv (p-1)! + 1 \pmod{p^3}.$$

ADDENDA

1. We write the following familiar congruence (see 2):

(11) If $p > 3$ is a prime then $(p-1)! \equiv pB_{p-1} - p \pmod{p^2}$,

where B is a Bernoulli number. Now, combining (11) with (10d) we have

(12) $$U_p \equiv pB_{p-1} - p \pmod{p^2}.$$

(13) 2. N. Neilsen (see 3) proved that: If $p = 2n+1$, $P = 1\cdot 3\cdot 5 \cdots (2n-1)$, and $p > 3$ is a prime, then

$$P \equiv (-1)^n 2^{3n} n! \pmod{p^3}.$$

Now, combining (10d) with the results in (13) leads to

(14) $$U_{2n+1} \equiv (-1)^n 2^{4n}(n!)^2 \pmod{p^3}, \quad \text{where} \quad 2n+1 = p \text{ is a prime } > 3.$$

It is easy to prove that

(15) $$((k-1)!)^2 = U_k^2 + U_{k-1}U_k - U_{k-1}U_{k+1} = F(k-1).$$

Proof. In (3c) we have $U_k - kU_{k-1} = (k-1)!$, we then put $(U_k - kU_{k-1})^2 = F(k-1)$, and this leads to

(15a) $$U_{k+1} = (2k+1) - k^2 U_{k-1},$$

where, since (15a) is identical with (4), we have proved that (15) holds. Now, in (15) we let $n = k-1$, so that

$$(n!)^2 = U_{n+1}^2 + U_nU_{n+1} - U_nU_{n+2} = F(n),$$

and combining this identity with (14), we have:

(16) $$U_{2n+1} \equiv (-1)^n 2^{4n}(F(n)) \pmod{p^3},$$

where $2n+1 = p$ is a prime > 3.

3. A generalized version of (4) may be derived in the following way: Put

(17) $$U_k = U_{k-1}x_k + (k-1)!$$

(where the x are arbitrary variables). Then, multiplying (17) through by k, we have

(17a) $$kU_k = kU_{k-1}x_k + k!,$$

but in (17) it is evident that

(17b) $$U_{k+1} = U_k x_{k+1} + k!,$$

and subtracting (17a) from this equation we get

(18) $$U_{k+1} = (k + x_{k+1})U_k - kx_kU_{k-1}.$$

Example of 3. We easily prove (4) with (17b) and (18), if we let

$$x_k = k, \quad x_{k+1} = k+1, \cdots, x_{k+j} = k+j \quad (j = 0, 1, 2, \cdots).$$

4. In conclusion, it may be interesting to note: If

(19) $$F_n = k_1F_{n-1} + k_2F_{n-2},$$

(where k_1 and k_2 are arbitrary constants) then the following relation always holds:

(19a) $$F_n^2 - F_{n+1}F_{n-1} = (-1)^n k_2^n,$$

where $F_0 = 1$, $F_1 = k_1$, $F_2 = k_1^2 + k_2, \cdots$.

Proof. In (19) we may write $F_{n+1} = k_1F_n + k_2F_{n-1}$, and combining this with (19a), we have

(20) $$k_1F_nF_{n-1} + k_2F_{n-1}^2 = F_n^2 + (-1)^{n+1}k_2.$$

Now, we multiply both sides of (20) by k_2 and then add

$$k_1^2F_n^2 + k_1k_2F_nF_{n-1}$$

to both sides of the result which leads to

$$k_1^2F_n^2 + 2k_1k_2F_nF_{n-1} + k_2^2F_{n-1}^2 = k_1^2F_n^2 + k_2F_n^2 + k_1k_2F_nF_{n-1} + (-1)^{n+1}k_2^{n+1} \quad \text{(20a)}$$

It is easily seen that

$$F_{n+2} = k_1F_{n+1} + k_2F_n = k_1^2F_n + k_1k_2F_{n-1} + k_2F_n ,$$

and combining this equation with (20a), we have

$$(k_1F_n + k_2F_{n-1})^2 = F_{n+1}^2 = F_{n+2}F_n + (-1)^{n+1}k_2^{n+1} . \quad \text{(20b)}$$

In the same way we found (20b), we proceed step-by-step (with added induction) and prove that the identities in (19) and (19a) are correct.

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PYTHAGOREAN TRIANGLES

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ABSTRACT

The first section of "Pythagorean Triangles" is primarily a portion of the history of pythagorean triangles and related problems. However, some new results and some new proofs of old results are presented in this section. For example, Fermat's Theorem is used to prove:

Levy's Theorem. If (x,y,z) is a pythagorean triangle such that $(7,x) = (7,y) = 1$, then 7 divides $x + y$ or $x - y$.

The historical discussion makes it reasonable to define pseudo-Sierpinski triangles as primitive pythagorean triangles with the property that $x = z - 1$, where z is the hypotenuse and x is the even leg. Whether the set of pseudo-Sierpinski triangles is finite or infinite is an open question. Some elementary, but new, results are presented in the discussion of this question.

An instructor of a course in Number Theory could use the material in the second section to present a coherent study of Fermat's Last Theorem and Fermat's method of infinite descent. These two results are used to prove the following familiar results.

(1A) *No pythagorean triangle has an area which is a perfect square.*

(2A) *No pythagorean triangle has both legs simultaneously equal to perfect squares.*

(3A) *It is impossible that any combination of two or more sides of a pythagorean triangle be simultaneously perfect squares.*

If 2 is viewed as a natural number for which Fermat's Last Theorem is true, then the following are obvious generalizations of 1A, 2A, and 3A.

(1B) *If* k *is an integer for which Fermat's Last Theorem holds, then there is no primitive pythagorean triangle whose area is a* k^{th} *power of some integer.*

(2B) *If* k *is some integer for which Fermat's Last Theorem is true, then there is no pythagorean triangle with the legs both equal to* k^{th} *powers of natural numbers.*

[Continued on Page 120.]

RELATIONS BETWEEN EULER AND LUCAS NUMBERS

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1. INTRODUCTION

In a previous article [1], the author presented a class of relations between Fibonacci-Lucas sequences and the generalized number sequences of Bernoulli. The same simple techniques can be used to obtain such identities involving other classical numbers.

The purpose of the present paper is to give explicit new relations and identities that involve *Lucas* numbers together with the famous numbers of *Euler.*

2. PRELIMINARIES

EULER NUMBERS

The *generalized Euler numbers* $E_n^{(m)}$ of the m^{th} *order* are defined by the generating function (see, for example [3]),

$$\frac{2^m}{(e^t+e^{-t})^m} = (\operatorname{sech} t)^m = \sum_{n=0}^{\infty} E_n^{(m)} \frac{t^n}{n!}, \qquad |t| < \pi/2 . \tag{1}$$

If $m = 1$, one writes $E_n^{(1)} \equiv E_n$, and has the more familiar Euler number sequence of the *first order:* 1, 0, –1, 0, 5, 0, –61, 0, 1385, 0, –50521, ⋯. The generalized numbers satisfy the *partial difference equation*

$$mE_n^{(m+1)} - E_{n+1}^{(m)} - mE_n^{(m)} = 0. \tag{2}$$

Moreover,

$$E_n^{(m)} = \frac{d^n}{dt^n}\,[(\operatorname{sech} t)^m]_{t=0}\,, \tag{3}$$

so one obtains the sequence

$$E_0^{(m)} = 1, \qquad E_1^{(m)} = 0, \qquad E_2^{(m)} = -m, \qquad E_3^{(m)} = 0, \qquad E_4^{(m)} = m(3m+2), \cdots, \tag{4}$$

with $E_{2k-1}^{(m)} = 0$ for $k \geqslant 1$.

If m is a *negative* integer, i.e., when $m = -p,\ p \geqslant 1$, the relation

$$E_n^{(-p)} = \frac{d^n}{dt^n}\,[(\cosh t)^p]_{t=0} \tag{5}$$

yields the explicit formula

$$E_{2k}^{(-p)} = \frac{1}{2^p}\sum_{j=0}^{p}\binom{p}{j}(p-2j)^{2k}, \qquad k \geqslant 0\,. \tag{6}$$

Euler and Bernoulli numbers of the first kind ($m = 1$) are related by the two equations

$$B_{2k} = \frac{-2k}{4^k(4^k-1)}\sum_{j=0}^{k-1}\binom{2k-1}{2j}E_{2j}, \qquad E_{2n} = 2 - \frac{1}{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k}(16)^k B_{2k}\,. \tag{7}$$

(See [2].)

LUCAS AND FIBONACCI NUMBERS

If

$$a = (1+\sqrt{5})/2 \quad \text{and} \quad b = (1-\sqrt{5})/2, \tag{8}$$

then the *Fibonacci* and *Lucas numbers* are defined respectively by the generating formulas

$$(9)\qquad \frac{e^{at}-e^{bt}}{\sqrt{5}} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}, \qquad e^{at}+e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

or explicitly by the equations

$$(10)\qquad F_n = \frac{a^n-b^n}{a-b}, \qquad L = a^n+b^n, \qquad n \geqslant 0$$

3. SOME IDENTITIES

With the above background preliminaries, we are in immediate position to obtain three identities. As in the previous article [1], we shall use inventive series manipulation as the fundamental method.

EXAMPLE 1

Note that

$$(11)\qquad e^{at}+e^{bt} = e^{t/2}(e^{ct}+e^{-ct}) = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

where the quantity c, which will occur frequently in subsequent equations, is

$$(12)\qquad c = \sqrt{5}/2.$$

We also have

$$\frac{e^{t/2}}{e^{at}+e^{bt}} = \frac{1}{e^{ct}+e^{-ct}} = ½ \sum_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}$$

or

$$(13)\qquad e^{at}+e^{bt} = \frac{2e^{t/2}}{\sum\limits_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}},$$

where we have made use of Eq. (1) with m = 1. Thus,

$$(14)\qquad \left[\sum_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}\right]\left[\sum_{s=0}^{\infty} L_s \frac{t^s}{s!}\right] = 2e^{t/2} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \frac{t^n}{n!}.$$

Application of Cauchy's rule for multiplying power series now yields

$$(15)\qquad \sum_{k=0}^{n} \binom{n}{k} c^k E_k L_{n-k} = 2^{1-n} \qquad n \geqslant 0.$$

Since $E_{2m-1} = 0$ for $m \geqslant 1$, and since $c = \sqrt{5}/2$, we have the identity*

$$(16)\qquad \sum_{k=0}^{[n/2]} \binom{n}{2k}\left(\frac{5}{4}\right)^k E_{2k} L_{n-2k} = 2^{1-n}$$

involving Euler numbers of the *first order* and the Lucas numbers. This identity holds for all $n \geqslant 0$.

EXAMPLE 2

Now

$$(e^{at}+e^{bt})^2 = e^t(e^{ct}+e^{-ct})^2,$$

or, in view of Eq. (1),

$$(17)\qquad \frac{e^t}{(e^{at}+e^{bt})^2} = \frac{1}{(e^{ct}+e^{-ct})^2} = ¼ \sum_{n=0}^{\infty} c^n E_n^{(2)} \frac{t^n}{n!},$$

*This particular identity is also found in [4].

where $E_n^{(2)}$ are Euler numbers of the *second order.* But it is also seen, using the second generating function in relations (9), that

$$(18)\qquad (e^{at}+e^{bt})^2 = [e^{2at}+e^{2bt}]+2e^t = \sum_{n=0}^{\infty}[2^nL_n+2]\frac{t^n}{n!} .$$

So, with (17) and (18), one has

$$\left[\sum_{n=0}^{\infty} c^nE_n^{(2)}\frac{t^n}{n!}\right]\left[\sum_{s=0}^{\infty}(2^sL_s+2)\frac{t^s}{s!}\right] = 4e^t = 4\sum_{n=0}^{\infty}\frac{t^n}{n!},$$

or the identity

$$\sum_{k=0}^{\infty}\binom{n}{k}c^kE_k^{(2)}[2^{n-k}L_{n-k}+2] = 4 .$$

Since the odd Euler numbers are zero, this can be written as

$$(19)\qquad \sum_{n=0}^{[n/2]}\binom{n}{2k}\left(\frac{5}{4}\right)^kE_{2k}^{(2)}[2^{n-2k}L_{n-2k}+2] = 4,\qquad n \geqslant 0.$$

EXAMPLE 3

Again, we have

$$(20)\qquad (e^{at}+e^{bt})^2 = 4e\,\frac{(e^{ct}+e^{-ct})^2}{4}$$

$$= 4\left[\sum_{s=0}^{\infty}\frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty}c^nE_n^{(-2)}\frac{t^n}{n!}\right],$$

where $E_n^{(-2)}$ are Euler numbers of *negative* second order. Once more we note that

$$(21)\qquad (e^{at}+e^{bt})^2 = \sum_{n=0}^{\infty}[2^nL_n+2]\frac{t^n}{n!},$$

and then equate this to the expression on the right in (20). Thus

$$[2^nL_n+2] = 4\sum_{k=0}^{n}\binom{n}{k}c^kE_k^{(-2)},$$

furnishing the identity

$$(22)\qquad L_n = 2^{1-n}\left[-1+2\sum_{k=0}^{[n/2]}\binom{n}{2k}\left(\frac{5}{4}\right)^kE_{2k}^{(-2)}\right],\qquad n \geqslant 0.$$

4. GENERALIZATION

The procedure just illustrated can easily be extended to furnish a whole new class of similar identities involving Lucas numbers and Euler numbers of higher order.

GENERAL CASE WHEN m IS AN ARBITRARY NEGATIVE INTEGER

We take $m = -p$, with p being a positive integer $\geqslant 1$. From Eq. (1) it is seen that

$$(23)\qquad \frac{(e^t+e^{-t})^p}{2} = (\cosh t)^p = \sum_{n=0}^{\infty}E_n^{(-p)}\frac{t^n}{n!},$$

where $E_n^{(-p)}$ are Euler numbers of the $(-p)^{th}$ order. We also note that

$$(24)\quad (e^{at}+e^{bt}) = [e^{pat}+e^{pbt}] + \sum_{r=1}^{p-1}\binom{p}{r} e^{[pa+(b-a)r]t} = \sum_{n=0}^{\infty}\left\{p^nL_n + \sum_{r=1}^{p-1}\binom{p}{r}[pa+(b-a)r]^n\right\}\frac{t^n}{n!},$$

and that

$$(25)\quad (e^{at}+e^{bt})^p = 2^pe^{pt/2}\,\frac{(e^{ct}+e^{-ct})^p}{2^p} = 2^p\left[\sum_{s=0}^{\infty}\left(\frac{p}{2}\right)^s\frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty}c^nE_n^{(-p)}\frac{t^n}{n!}\right].$$

Equating (24) and (25) now yields

$$(26)\quad p^nL_n + \sum_{r=1}^{p-1}\binom{p}{r}[pa+(b-a)r]^n = 2^p\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{2}\right)^{n-k}c^kE_k^{(-p)},$$

or the identity

$$(27)\quad L_n = p^{-n}\left\{-\sum_{r=1}^{p-1}\binom{p}{r}[pa+(b-a)r]^n + 2^p\sum_{k=0}^{[n/2]}\binom{n}{2k}\left(\frac{p}{2}\right)^{n-2k}\left(\frac{5}{4}\right)^kE_{2k}^{(-p)}\right\}.$$

This identity holds for each $p \geqslant 2$, and it furnishes an infinite number of identities. In the special case when $p = 1$, we have

$$(28)\quad L_n = 2^{1-n}\sum_{k=0}^{[n/2]}\binom{n}{2k}5^kE_{2k}^{(-1)},\qquad n \geqslant 0.$$

Equation (27) is remarkable in that it embodies explicit formulas for expressing any Lucas number in a finite sum involving any particular Euler sequence of negative order that one may choose.

GENERAL CASE WHEN m IS A POSITIVE INTEGER

Different types of identities are obtained when m is positive, but the technique of deriving them is the same. We present the result without showing the detailed development. It is as follows:

$$(29)\quad \sum_{k=0}^{[n/2]}\binom{n}{2k}\left(\frac{5}{4}\right)^kE_{2k}^{(m)}\left\{m^{n-2k}L_{n-2k} + \sum_{r=1}^{m-1}\binom{m}{r}[ma+(b-a)r]^n\right\} = 2^{m-n}m^n$$

which reduces to (16) when $m = 1$, and to (19) when $m = 2$. The identity (29) holds for all positive m, and represents a one-parameter family of identities that are valid for all $n \geqslant 0$.

5. REMARKS

By using the first equation given in (7), other identities, involving Fibonacci numbers and Euler numbers, can be found if B_{2k} in terms of Euler numbers is inserted in the identities obtained in [1].

Since

$$(30)\quad F_n = \frac{1}{5}[L_{n+1}+L_{n-1}],\qquad n \geqslant 1$$

Equation (27) can easily be used to explicitly express any Fibonacci number in terms that involve any Euler sequence of negative order.

It may interest the reader to extend our identities and to investigate how such relations may be applied. The author (as every Fibonacci-number enthusiast should do after recording his formulas) is turning his attention to the question of what might be *done* with them.

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SUMS AND PRODUCTS FOR RECURRING SEQUENCES

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In [1], we find many well known formulas which involve the sums of Fibonacci and Lucas numbers. For example, we have

(1) $$\sum_{i=1}^{n} F_i = F_{n+2} - 1, \qquad n \geq 1;$$

(2) $$\sum_{i=1}^{n} L_i = L_{n+2} - 3, \qquad n \geq 1;$$

(3) $$\sum_{i=1}^{n} F_{2i-1} = F_{2n}, \qquad n \geq 1;$$

(4) $$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2, \qquad n \geq 1.$$

Hence, it is natural to ask if there exist summation formulas for other lists of Fibonacci and Lucas numbers. If such formulas exist it is then natural to ask if the formulas can be extended to other recurring sequences. The purpose of this paper is to show that both of these questions can be answered in the affirmative. To do this, we first recall the following [1, p. 59]

(5) $$F_{n+k} + F_{n-k} = F_n L_k, \quad k \text{ even};$$

(6) $$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd};$$

(7) $$F_{n+k} - F_{n-k} = F_n L_k, \quad k \text{ odd};$$

(8) $$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}.$$

Using $L_n = \alpha^n + \beta^n$ where α and β are the roots of $x^2 - x - 1 = 0$ with $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$ it is easy to show that

(9) $$L_{n+k} + L_{n-k} = L_n L_k, \quad k \text{ even};$$

(10) $$L_{n+k} + L_{n-k} = 5F_n F_k, \quad k \text{ odd};$$

(11) $$L_{n+k} - L_{n-k} = L_n L_k, \quad k \text{ odd};$$

(12) $$L_{n+k} - L_{n-k} = 5F_n F_k, \quad k \text{ even}.$$

Observing that a sum involving 2^p terms, by combining pairs, reduces to a sum of 2^{p-1} terms, we were able to show

Theorem 1. If $k \geq 1$ then

(13) $$\sum_{i=0}^{2^j-1} F_{n+4ki} = F_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^i k}.$$

Proof. If $j = 1$ then

$$\sum_{i=0}^{1} F_{n+4ki} = F_n + F_{n+4k} = L_{2k}F_{n+2k} = F_{n+(2^1-1)2k}\prod_{i=1}^{1} L_{2^ik}$$

and the theorem is true.

Assume the proposition is true for j. Using (5), we have

$$\sum_{i=0}^{2^{j+1}-1} F_{n+4ki} = L_{2k}\sum_{i=0}^{2^j-1} F_{n+2k+8ki}$$

$$= L_{2k}F_{n+2k+(2^j-1)4k}\prod_{i=1}^{j} L_{2^{i+1}k}$$

$$= F_{n+(2^{j+1}-1)2k}\prod_{i=1}^{j+1} L_{2^ik}$$

and the theorem is proved.

Using (9) and an argument like that of Theorem 1, we have

$$\sum_{i=0}^{2^j-1} L_{n+4ki} = L_{n+(2^j-1)2k}\prod_{i=1}^{j} L_{2^ik}, \quad k \geqslant 1. \tag{14}$$

Using (8) and (14) with $j-1$ in place of j, $n+2k$ in place of n and $2k$ in place of k, one has

$$\sum_{i=0}^{2^j-1} (-1)^{i+1}F_{n+4ki} = F_{2k}L_{n+(2^j-1)2k}\prod_{i=2}^{j} L_{2^ik}, \quad k \geqslant 1. \tag{15}$$

Similarly, with the aid of (12) and Theorem 1, one obtains

$$\sum_{i=0}^{2^j-1} (-1)^{i+1}L_{n+4ki} = 5F_{2k}F_{n+(2^j-1)2k}\prod_{i=2}^{j} L_{2^ik}, \quad k \geqslant 1. \tag{16}$$

From (9) and (14), we have

$$\sum_{i=0}^{2^j-1} L_{n+(2i-1)k} = L_{n+(2^{j-1}-1)2k}\prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even} \tag{17}$$

while Theorem 1 with the aid of (12) gives

$$\sum_{i=0}^{2^j-1} (-1)^{i+1}L_{n+(2i-1)k} = 5F_kF_{n+(2^{j-1}-1)2k}\prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even}. \tag{18}$$

Theorem 1 together with (5) can be used to show

$$\sum_{i=0}^{2^j-1} F_{n+(2i-1)k} = F_{n+(2^{j-1}-1)2k}\prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even} \tag{19}$$

while (8) with (14) yields

$$\sum_{i=0}^{2^j-1} (-1)^{i+1}F_{n+(2i-1)k} = F_kL_{n+(2^{j-1}-1)2k}\prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even}. \tag{20}$$

Since we have used (5) and (8) as well as (9) and (12) on several occasions, it seems natural to ask if formulas exist using (6) and (7) as well as (10) and (11). With this in mind, we developed the next four formulas.

By use of (10) and (11), respectively with Theorem 1, we have

$$(21)\qquad \sum_{i=0}^{2^j-1} L_{n+(2i-1)k} = 5F_kF_{n+(2^{j-1}-1)2k}\prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd}$$

and

$$(22)\qquad \sum_{i=0}^{2^j-1} (-1)^{i+1}F_{n+(2i-1)k} = F_{n+(2^{j-1}-1)2k}\prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

Finally, if we apply (6) and (7) respectively with (14) we are able to show that

$$(23)\qquad \sum_{i=0}^{2^j-1} F_{n+(2i-1)k} = F_kL_{n+(2^{j-1}-1)2k}\prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd}$$

and

$$(24)\qquad \sum_{i=0}^{2^j-1} (-1)^{i+1}L_{n+(2i-1)k} = L_{n+(2^{j-1}-1)2k}\prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

To lift the results above to the generalized Fibonacci sequence which is defined recursively by

$$(25)\qquad H_0 = q, \quad H_1 = p, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geqslant 2$$

it is necessary and sufficient to examine formulas comparable to (5) through (12). To do this, we first define a generalized Lucas sequence by

$$(26)\qquad G_n = H_{n+1} + H_{n-1}.$$

In Horadam [3], it is shown that

$$(27)\qquad H_n = (r\alpha^n - s\beta^n)/2\sqrt{5},$$

where $r = 2(p - q\beta)$, $s = 2(p - q\alpha)$ and α, β are the usual roots of $x^2 - x - 1 = 0$. Furthermore, he shows that

$$(28)\qquad H_{n+k} = H_{n-1}F_k + H_nF_{k+1},$$

where the F_k are the Fibonacci numbers.

Using (27) and Binet's formula for F_k, a straightforward argument shows that

$$(29)\qquad H_nF_{k-1} - H_{n-1}F_k = (-1)^kH_{n-k}.$$

By (28) and (29) with the aid of $L_k = F_{k+1} + F_{k-1}$, we have

$$(30)\qquad H_{n+k} + H_{n-k} = H_nL_k, \quad k \text{ even}$$

and

$$(31)\qquad H_{n+k} - H_{n-k} = H_nL_k, \quad k \text{ odd}.$$

If we use (25), (28), and (29) together with the fact that $F_k = F_{k+1} - F_{k-1}$, we have

$$(32)\qquad H_{n+k} + H_{n-k} = G_nF_k, \quad k \text{ odd}$$

and

$$(33)\qquad H_{n+k} - H_{n-k} = G_nF_k, \quad k \text{ even}.$$

Replacing n by $n + k$ in (26) and using (28), we have

$$(34)\qquad G_{n+k} = H_{n-1}L_k + H_nL_{k+1}$$

while replacing n by $n - k$ in (26) and applying (29) gives

$$(35)\qquad G_{n-k} = (-1)^k(H_{n-1}L_k - H_nL_{k-1}).$$

Applying (34) and (35) as we did (28) and (29), we obtain

$$(36)\qquad G_{n+k} + G_{n-k} = G_nL_k, \quad k \text{ even};$$

$$(37)\qquad G_{n+k} + G_{n-k} = 5H_nF_k, \quad k \text{ odd};$$

(38) $$G_{n+k} - G_{n-k} = G_n L_k, \quad k \text{ odd};$$

(39) $$G_{n+k} - G_{n-k} = 5H_n F_k, \quad k \text{ even}.$$

Examining (30) through (33) and (36) through (39) with H replaced by F and G replaced by L, we obtain properties (5) thorugh (12). Hence, it is clear that identities (13) through (24) can be lifted to the generalized Fibonacci and Lucas sequences and in fact are

(40) $$\sum_{i=0}^{2^j-1} H_{n+4ki} = H_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \quad k \geq 1;$$

(41) $$\sum_{i=0}^{2^j-1} G_{n+4ki} = G_{n+(2^j-1)2k} \prod_{i=1}^{j} L_{2^ik}, \quad k \geq 1;$$

(42) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+4ki} = F_{2k} G_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geq 1;$$

(43) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+4ki} = 5F_{2k} H_{n+(2^j-1)2k} \prod_{i=2}^{j} L_{2^ik}, \quad k \geq 1;$$

(44) $$\sum_{i=0}^{2^j-1} G_{n+(2i-1)k} = G_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(45) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+(2i-1)k} = 5F_k H_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(46) $$\sum_{i=0}^{2^j-1} H_{n+(2i-1)k} = H_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(47) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+(2i-1)k} = F_k G_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ even};$$

(48) $$\sum_{i=0}^{2^j-1} G_{n+(2i-1)k} = 5F_k H_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(49) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} H_{n+(2i-1)k} = H_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(50) $$\sum_{i=0}^{2^j-1} H_{n+(2i-1)k} = F_k G_{n+(2^{j-1}-1)2k} \prod_{i=1}^{j-1} L_{2^ik}, \quad k \text{ odd};$$

(51) $$\sum_{i=0}^{2^j-1} (-1)^{i+1} G_{n+(2i-1)k} = G_{n+(2^{j-1}-1)2k} \prod_{i=0}^{j-1} L_{2^ik}, \quad k \text{ odd}.$$

The infinite sequence $\{x_n\}_{n=1}^{\infty}$ is called a recurring sequence if, from a certain point on, every term can be represented as a linear combination of the preceding terms of the sequence. Hence, the sequence $\{U_n(x,y)\}_{n=1}^{\infty}$

defined recursively by

(52) $$U_0(x,y) = 0, \quad U_1(x,y) = 1, \quad U_n(x,y) = xU_{n-1}(x,y) + yU_{n-2}(x,y), \quad n \geq 2.$$

where $U_n(x,y) \in F[x,y]$, F any field is a recurring sequence.

If we let λ_1 and λ_2 be the roots of the equation $\lambda^2 - x\lambda - y = 0$, where we assume $\lambda_1 = (x + \sqrt{x^2+4y})/2$, $y \neq 0$, and $x^2 + 4y$ is a nonperfect square different from zero, then it is easy to show that

(53) $$U_n(x,y) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Furthermore, if we let

(54) $$V_n(x,y) = \lambda_1^n + \lambda_2^n$$

then

(55) $$V_n(x,y) = yU_{n-1}(x,y + U_{n+1}(x,y).$$

Because of the y coefficient, the formulas (5) through (12) do not follow the same pattern for this recurring sequence. However, it can be shown using (53) through (55) together with the facts $\lambda_1\lambda_2 = -y$ and $\lambda_1 + \lambda_2 = x$ that

(56) $$U_{n+k}(x,y) + y^kU_{n-k}(x,y) = U_n(x,y)V_k(x,y), \quad k \text{ even};$$

(57) $$U_{n+k}(x,y) + y^kU_{n-k}(x,y) = V_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(58) $$U_{n+k}(x,y) - y^kU_{n-k}(x,y) = U_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(59) $$U_{n+k}(x,y) - y^kU_{n-k}(x,y) = V_n(x,y)U_k(x,y), \quad k \text{ even}$$

(60) $$V_{n+k}(x,y) + y^kV_{n-k}(x,y) = V_n(x,y)V_k(x,y), \quad k \text{ even}$$

(61) $$V_{n+k}(x,y) + y^kV_{n-k}(x,y) = (x^2+4y)U_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(62) $$V_{n+k}(x,y) - y^kV_{n-k}(x,y) = V_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(63) $$V_{n+k}(x,y) - y^kV_{n-k}(x,y) = (x^2+4y)U_n(x,y)U_k(x,y), \quad k \text{ even}.$$

Because of the y^k, it is quite obvious that formulas (13) through (24) do not have the same form for the recurring sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$. If we let the coefficients of $U_{n-2}(x,y)$ in (52) be $y = 1$ then the sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$ are sequences of polynomials in x. In fact, they are respectively the sequences of Fibonacci and Lucas polynomials. With $y = 1$, it is easy to see that formulas (56) through (63) are of the same nature as (5) through (12) with F in place of U and L in place of V. Hence, the formulas (13) through (24) can be lifted to the sequences $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$ if $y = 1$ by replacing F_n by $U_n(x,1)$ and L_n by $V_n(x,1)$. Of course, we have $x^2 + 4$ in palce of 5 in formulas (16), (18), and (21).

In conclusion, we will examine whiat happens if we consider the recurring sequence $\{H_n(x,y)\}_{n=1}^{\infty}$ where

(64) $$H_0(x,y) = f(x,y), \quad H_1(x,y) = g(x,y),$$
$$H_n(x,y) = xH_{n-1}(x,y) + yH_{n-2}(x,y), \quad n \geq 2.$$

By using properties of difference equations, it is easy to show that

(65) $$H_n(x,y) = (r(x,y)\lambda_1^n - s(x,y)\lambda_2^n)/2\sqrt{x^2+4y}$$

where λ_1 and λ_2 are as before, $r(x,y) = 2(g(x,y) - f(x,y)\lambda_2)$, and $s(x,y) = 2(g(x,y) - f(x,y)\lambda_1)$.

If we let

(66) $$G_n(x,y) = (r(x,y)\lambda_1^n + s(x,y)\lambda_2^n)/2$$

then

(67) $$G_n(x,y) = yH_{n-1}(x,y) + H_{n+1}(x,y).$$

Using (53) and (65), a direct calculation will show that

(68) $$H_n(x,y)U_{k+1}(x,y) + yH_{n-1}(x,y)U_k(x,y) = H_{n+k}(x,y)$$

and

(69) $$H_n(x,y)U_{k-1}(x,y) - H_{n-1}(x,y)U_k(x,y) = (-1)^ky^{k-1}H_{n-k}(x,y).$$

If we use (57) with (67) and (68) and remember that $U_1(x,y) = 1$, we obtain

(70) $$G_{n+k}(x,y) = yH_{n-1}(x,y)V_k(x,y) + H_n(x,y)V_{k+1}(x,y).$$

Using (55) with (69) and (67), it can be shown that

(71) $$H_{n-1}(x,y)V_k(x,y) - H_n(x,y)V_{k-1}(x,y) = (-1)^k y^{k-1} G_{n-k}(x,y).$$

Letting k be odd or even in (68) through (71), we have

(72) $$H_{n+k}(x,y) + y^k H_{n-k}(x,y) = H_n(x,y)V_k(x,y), \quad k \text{ even};$$

(73) $$H_{n+k}(x,y) + y^k H_{n-k}(x,y) = G_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(74) $$H_{n+k}(x,y) - y^k H_{n-k}(x,y) = H_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(75) $$H_{n+k}(x,y) - y^k H_{n-k}(x,y) = G_n(x,y)U_k(x,y), \quad k \text{ even};$$

(76) $$G_{n+k}(x,y) + y^k G_{n-k}(x,y) = G_n(x,y)V_k(x,y), \quad k \text{ even};$$

(77) $$G_{n+k}(x,y) + y^k G_{n-k}(x,y) = (x^2+4y)H_n(x,y)U_k(x,y), \quad k \text{ odd};$$

(78) $$G_{n+k}(x,y) - y^k G_{n-k}(x,y) = G_n(x,y)V_k(x,y), \quad k \text{ odd};$$

(79) $$G_{n+k}(x,y) - y^k G_{n-k}(x,y) = (x^2+4y)H_n(x,y)U_k(x,y), \quad k \text{ even}.$$

Observe that if we replace H by U and G by V then Eqs. (72) through (79) yield Eqs. (56) through (63).

If we let $y = 1$ in (64) then Eqs. (72) through (79) are those of (30) through (33) and (36) through (39) where we replace $V_n(x,y)$ by L_n, $H_n(x,y)$ by H_n, $G_n(x,y)$ by G_n, and $U_n(x,y)$ by F_n. The same substitutions in (40) through (51) will give us the summation-product relations relative to the sequences $\{H_n(x,y)\}$ and $\{G_n(x,y)\}$ if $y = 1$.

In conclusion, we observe several other results which are a direct consequence of the formulas of this paper [2; p. 19].

If we replace n by $k + 1$ in (5) through (8) we have F_k, L_k, F_{k+1}, and L_{k+1} are relatively prime to F_{2k+1} for $k \geq 1$. If we let $n = k + 2$ in (5) through (8), we have F_k, L_k, F_{k+2}, and L_{k+2} are all relatively prime to F_{2k+2} for $k \geq 1$. Letting $n = k + 1$ in (9) through (12), we see that F_k, L_k, F_{k+1}, and L_{k+1} are all relatively prime to L_{2k+1}.

If we let $n = k + 1$ in (56) through (59) with $y = 1$ we see that the Fibonacci polynomials $U_{2k+1}(x,1) \pm 1$ are factorable for $k \geq 2$. If $n = k$ with $y = 1$ in (56) through (59) then $U_{2k}(x,1)$ is factorable for $k \geq 2$.

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[Continued from Page 110.]

(3B) *If* k *is an integer for which Fermat's Last Theorem is true, then there is no pythagorean triangle with the hypotenuse and one of the legs equal to* k^{th} *powers of natural numbers.*

Proofs of 1B and 2B are provided in the complete text, but 3B remains an open question.

The authors have attempted to compile a complete bibliography related to pythagorean triangles. Included in the bibliography are 111 references to journal articles, 66 references to problems (with solutions) in *Amer. Math Monthly,* 17 references to notes in *Math. Gaz.,* and 12 references to notes in *Math. Mag.* Since it is impossible to compile such a bibliography without some omissions, the authors would appreciate receiving any references not already included in the bibliography.

The complete report of which this article is a summary consists of 23 pages. It may be obtained for $1.50 by writing the Managing Editor, Brother Alfred Frousseau, St. Mary's College, Moraga, California 94575.

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SOME IDENTITIES OF BRUCKMAN

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!. Bruckman [1] defined a sequence of numbers $\{A_n\}$ by means of

$$(1.1)\qquad (1-z)^{-1}(1+z)^{-1/2} = \sum_{n=0}^{\infty} A_n z^n ,$$

so that

$$(1.2)\qquad A_n = \sum_{k=0}^{n} (-1)^k 2^{-2k} \binom{2k}{k} .$$

He proved the striking result

$$(1.3)\qquad \sum_{n=0}^{\infty} 2^{2n} \frac{n!n!}{(2n+1)!} A_n^2 x^{2n+1} = \frac{\arctan x}{\sqrt{1-x^2}} ,$$

which is equivalent to

$$(1.4)\qquad A_n^2 = 2^{-2n}\binom{2n}{n} \sum_{k=0}^{n} (-1)^{n-k} 2^{-2k} \binom{2k}{k} \frac{2n+1}{2n-k+1} .$$

Gould [4] has discussed Bruckman's results in some detail and indicated their relationship to earlier results. He remarks that "a direct proof of (1.4) by squaring (1.2) is by no means trivial." However, he does not give a proof of the formula.

The purpose of this note is to show that (1.3) is a very special case of a much more general result involving hypergeometric polynomials. We also show how a generalized version of (1.3) can be obtained using a little calculus.

2. In the standard notation put

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} z^n ,$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1.$$

Weisner [6] has proved the formula

$$(2.1)\qquad \sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} F_n(-n,a;c;x)F(-n,b;c;y)$$
$$= (1-z)^{a+b-c}(1+(x-1)z)^{-a}(1+(y-1)z)^{-b} F(a,b;c;\zeta),$$

where

$$(2.2)\qquad \zeta = \frac{xyz}{(1+(x-1)z)(1+(y-1)z)} .$$

This result had indeed been proved earlier by Meixner [5]. For an elementary proof of (2.1) see [3].

Replacing x, y by $1-x$, $1-y$, respectively, Eq. (2.1) becomes

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(2.3)
$$\sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} F_n(-n, a; c; 1-x) F_n(-n, b; c; 1-y)$$
$$= (1-z)^{a+b-c}(1-xz)^{-a}(1-yz)^{-b} F(a,b;c;\bar{\zeta}),$$

where

(2.4)
$$\bar{\zeta} = \frac{(1-x)(1-y)z}{(1-xz)(1-yz)}.$$

In particular, for $c = a + b$, Eq. (2.3) reduces to

(2.5)
$$\sum_{n=0}^{\infty} \frac{(a+b)_n z^n}{n!} F_n(-n,a; a+b; 1-x) F(-n,b; a+b; 1-y)$$
$$= (1-xz)^{-a}(1-yz)^{-b} F(a,b;c;\bar{\zeta}).$$

Consider

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n,a;c;1-x) z^n = \sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{k!(c)_k} (1-x)^k$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!} (1-x)^k z^k \sum_{n=k}^{\infty} \frac{(c+k)_{n-k}}{(n-k)!} z^{n-k}$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!} (1-x)^k z^k (1-z)^{-c-k} = (1-z)^{-c} \left(1 + \frac{(1-x)z}{1-z} \right)^{-a},$$

where we have used

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n,a;c;1-x) z^n = (1-z)^{a-c}(1-xz)^{-a}.$$

Thus, for $c = a + b$, we have

(2.6)
$$\sum_{n=0}^{\infty} \frac{(a+b)_n}{n!} F(-n,a; a+b; 1-x) = (1-z)^{-b}(1-xz)^{-a}.$$

It follows that

(2.7)
$$F(-n, a; a+b; 1-x) = x^n F(-n, b; a+b; 1-x^{-1}).$$

3. We now specialize (2.5) by taking

(3.1)
$$a = \tfrac{1}{2}, \qquad b = 1, \qquad c = 3/2.$$

Then (2.5) becomes

(3.2)
$$\sum_{n=0}^{\infty} \frac{(3/2)_n z^n}{n!} F_n(-n,\tfrac{1}{2}; 3/2; 1-x) F(-n,1; 3/2; 1-y) = (1-xz)^{-\frac{1}{2}}(1-yz)^{-1} F(\tfrac{1}{2},1; 3/2; \quad).$$

In view of (2.7) this may be replaced by

(3.3)
$$\sum_{n=0}^{\infty} \frac{(3/2)_n y^n z^n}{n!} F_n(-n,\tfrac{1}{2}; 3/2; 1-x) F(-n,\tfrac{1}{2}; 3/2; 1-y^{-1}) = (1-xz)^{-\frac{1}{2}}(1-yz)^{-1} F(\tfrac{1}{2},1; 3/2; \quad).$$

We define the polynomial $A_n(x)$ by means of

$$\sum_{n=0}^{\infty} A_n(x)z^n = (1-z)^{-1}(1-xz)^{-1/2}. \tag{3.4}$$

This is equivalent to

$$A_n(x) = \sum_{k=0}^{n} 2^{-2k}\binom{2k}{k}x^k. \tag{3.5}$$

Comparing (3.4) with (1.1) or (3.5) with (1.2), it is evident that

$$A_n = A_n(-1). \tag{3.6}$$

It will also be convenient to define

$$\overline{A}_n(x) = x^n A_n(x^{-1}) = \sum_{k=0}^{n} 2^{-2k}\binom{2k}{k}x^{n-k}. \tag{3.7}$$

Comparing (3.4) with (2.6), we get

$$A_n(x) = \frac{(3/n)_n}{n!}F(-n, ½; 3/2; 1-x). \tag{3.8}$$

Thus (3.3) becomes

$$\sum_{n=0}^{\infty} \frac{n!z^n}{(3/2)_n} A_n(x)\overline{A}_n(y) = (1-xz)^{-1/2}(1-yz)^{-1}F(½,1;3/2;\quad). \tag{3.9}$$

Since

$$(3/2)_n = 2^{-2n}\frac{n!}{(2n+1)!}$$

and

$$zF(½,1;3/2;-z^2) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1} = \arctan z,$$

(3.9) may be replaced by

$$\sum_{n=0}^{\infty}(-1)^n 2^{2n}\frac{n!n!}{(2n+1)!}z^{2n+1}A_n(x)\overline{A}_n(y) \tag{3.10}$$

$$= \left\{(1-x)(1-y)(1+yz^2)\right\}^{-1/2}\arctan\left\{z\left(\frac{(1-x)(1-y)}{(1+xz^2)(1+yz^2)}\right)\right\}^{1/2}.$$

For $x = y = -1$, it is evident from (3.6) and (3.7) that

$$\sum_{n=0}^{\infty} 2^{2n}\frac{n!n!}{(2n+1)!}A_n^2 z^{2n+1} = ½(1-z^2)^{-1/2}\arctan\frac{2z}{1-z^2}. \tag{3.11}$$

For $y = x$, the right-hand side of (3.10) becomes

$$(1-x)^{-1}(1+xz^2)^{-1/2}\arctan\frac{(1-x)z}{1+xz^2} = \sum_{k=0}^{\infty}(-1)^k\frac{(1-x)^{2k}z^{2k+1}}{2k+1}(1+xz^2)^{-2k-(3/2)}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{(1-x)^{2k}z^{2k+1}}{2k+1}\sum_{j=0}^{\infty}(-1)^j\binom{2k+j+½}{j}x^jz^{2j}.$$

Comparing coefficients of z^{2n+1} we get

$$2^{2n}A_n(x)\overline{A}_n(x) = \frac{(2n+1)!}{n!n!}\sum_{j=0}^{n}\binom{2n-j+½}{j}\frac{x^j(1-x)^{2n-2j}}{2n-2j+1}. \tag{3.12}$$

The corresponding formula for $A_n(x)\overline{A}_n(y)$ is more complicated and will be omitted.

For $x = -1$, (3.12) reduces to

$$A_n^2 = \frac{(2n+1)!}{n!n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{2n-j+\frac{1}{2}}{j} \frac{2^{-2j}}{2n-2j+1}, \tag{3.13}$$

which may be compared with (1.4).

Formulas (3.11) and (1.3) are equivalent. This is a consequence of

$$\arctan \frac{2z}{1-z^2} = 2 \arctan z.$$

We remark that in a recent paper [2] Bruckman has considered a different generalization of A_n.

4. We can also prove (3.10) in the following way. To begin with, take

$$(1-z)^{-1}(1-xz)^{-\frac{1}{2}} = (1-z)^{-(3/2)} \left(1 + \frac{(1-x)z}{1-z} \right)^{\frac{1}{2}} = \sum_{k=0}^{\infty} (-1)^k 2^{-2k} \binom{2k}{k} \frac{(1-x)^k z^k}{(1-z)^{k+(3/2)}}$$

$$= \sum_{k=0}^{\infty} (-1)^k 2^{-2k} \binom{2k}{k} (1-x)^k z^k \sum_{j=0}^{\infty} \binom{k+j+\frac{1}{2}}{j} z^j .$$

It then follows from (3.4) that

$$A_n(x) = \sum_{k=0}^{n} (-1)^k 2^{-2k} \binom{2k}{k} \binom{n+\frac{1}{2}}{n-k} (1-x)^k = 2^{-2n} \frac{(2n+1)!}{n!n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1-x)^k}{2k+1} . \tag{4.1}$$

Since

$$\int_0^1 (1-(1-x)t^2)dt = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1-x)^k}{2k+1} ,$$

it follows that

$$A_n(x) = 2^{-2n} \frac{(2n+1)!}{n!\,n!} \int_0^1 (1-(1-x)t^2)^n dt \tag{4.2}$$

and

$$\bar{A}_n(x) = 2^{-2n} \frac{(2n+1)!}{n!n!} \int_0^1 (x+(1-x)t^2)^n dt. \tag{4.3}$$

Thus

$$\sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{n!n!}{(2n+1)!} A_n(x)\bar{A}_n(y) z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n A_n(x) z^{2n+1} \int_0^1 (y+(1-y)t^2)^n dt$$

$$= z \int_0^1 \left\{ 1+(y+(1-y)t^2)z^2 \right\}^{-1} \left\{ 1+x(y+(1-y)t^2)z^2 \right\}^{-\frac{1}{2}} dt.$$

We shall make use of the formula

$$\int \frac{dt}{(a'+b't^2)(a+bt^2)^{\frac{1}{2}}} = \frac{1}{(a'(ab'-a'b))^{\frac{1}{2}}} \arctan \left\{ x \left(\frac{ab'-a'b}{a'(a+bx^2)} \right) \right\}^{\frac{1}{2}} \tag{4.4}$$

where

$$\begin{cases} a = 1+xyz^2, & b = x(1-y)z^2 \\ a' = 1+yz^2, & b' = (1-y)z^2 \end{cases},$$

$$ab' = a'b = (1-x)(1-y)z^2, \qquad a+b = 1+xz^2 .$$

We therefore get

$$\sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{n!n!}{(2n+1)!} A_n(x)\bar{A}_n(y)z^{2n+1}$$

$$= \left\{(1-x)(1-y)(1+xz^2)\right\}^{-1/2} \arctan \left\{ z\left(\frac{(1-x)(1-y)}{(1+xz^2)(1+yz^2)}\right)\right\}^{1/2}$$

which is identical with (3.10).

APPENDIX

5. We shall prove the following identity:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} c_r x^r \sum_{s=0}^{n} (-1)^s \binom{n}{s} d_s y^s \tag{5.1}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(xyz)^n}{(1-z)^{2n}} C_n(\lambda, x, z) D_n(\lambda, y, z),$$

where $\{c_n\}$, $\{d_n\}$ are sequences of arbitrary complex numbers and

$$C_n(\lambda, x, z) = \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} c_{n+r} \left(\frac{-xz}{1-z}\right)^r, \qquad D_n(\lambda, y, z) = \sum_{s=0}^{\infty} \frac{(\lambda+n)_s}{s!} d_{n+s} \left(\frac{-yz}{1-z}\right)^s.$$

We may think of (4.1) as an identity between formal power series.

PROOF OF (5.1). The left-hand side of (4.1) is equal to

$$\sum_{r,s=0}^{\infty} (-1)^{r+s} c_r d_s x^r y^s \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \binom{n}{r}\binom{n}{s} z^n. \tag{5.2}$$

The right-hand side of (5.1) is equal to

$$(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (xyz)^n \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} c_{n+r}(-xz)^r \sum_{s=0}^{\infty} \frac{(\lambda+n)_s}{s!} d_{n+s}(-yz)^s (1-z)^{-2n-r-s}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(\lambda)_{n+r}(\lambda)_{n+s}}{n!\,r!\,s!\,(\lambda)_n} c_{n+r} d_{n+s} x^{n+r} y^{n+s} z^{n+r+s} (1-z)^{-2n-r-s}$$

$$= (1-z)^{-\lambda} \sum_{r,s=0}^{\infty} (-1)^{r+s} (\lambda)_r (\lambda)_s c_r d_s x^r y^s (1-z)^{-r-s} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n}.$$

Comparing this with (5.2), it is evident that it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \binom{n}{r}\binom{n}{s} z^n = (\lambda)_r(\lambda)_s (1-z)^{-\lambda-r-s} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n}. \tag{5.3}$$

If we multiply both sides of (5.3) by $x^r y^s$, and sum over r, s, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1+x)^n (1+y)^n z^n = (1-z)^{-\lambda} \sum_{r,s=0}^{\infty} (\lambda)_r (\lambda)_s \frac{x^r y^s}{(1-z)^{r+s}} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(xyz)^n}{(1-z)^{2n}} \sum_{r,s=0}^{\infty} \frac{(\lambda+n)_r (\lambda+n)_s}{r!s!} \frac{(xz)^r (yz)^s}{(1-z)^{r+s}} = (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}$$

$$\cdot \frac{(xyz)^n}{(1-z)^{2n}} \left(1 - \frac{xz}{1-z}\right)^{-\lambda-n} \left(1 - \frac{yz}{1-z}\right)^{-\lambda-n} = (1-z)^{\lambda}(1-(1+x)z)^{-\lambda}(1-(1+y)z)^{-\lambda}$$

$$\cdot \left[1 - \frac{xyz}{(1-(1+x)z)(1-(1+y)z)}\right]^{-\lambda} = (1-z)^{\lambda} \left\{[1-(1+x)z]\,[1-(1+y)z] - xyz\right\}^{-\lambda}.$$

Thus (5.3) is equivalent to

$$[1-(1+x)(1+y)z]^{-\lambda} = (1-z)^{\lambda}\left\{[1-(1+x)z][1-(1+y)z]-xyz\right\}^{-\lambda}$$

and so to

$$(1-z)[1-(1+x)(1+y)z] = [1-(1+x)z][1-(1+y)z]-xyz.$$

This equation is easily verified.

This completes the proof of (5.1).

The identity (5.1) contains numerous interesting special cases. In particular, taking

$$c_n = \frac{(a)_n}{(c)_n}, \qquad d_n = \frac{(b)_n}{(d)_n},$$

(5.1) becomes

$$(5.6)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(a)_r}{(c)_r}x^r\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(b)_s}{(d)_s}y^s$$

$$= (1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{(xyz)}{(1-z)^{2n}}C_n(x,z)D_n(y,z),$$

where now

$$(5.7)\qquad \begin{cases} C_n(x,z) = \displaystyle\sum_{r=0}^{\infty}\frac{(\lambda+n)_r(a)_{n+r}}{r!(c)_{n+r}}\left(\frac{-xz}{1-z}\right)^r \\ D_n(y,z) = \displaystyle\sum_{s=0}^{\infty}\frac{(\lambda+n)_s(b)_{n+s}}{s!(d)_{n+s}}\left(\frac{-yz}{1-z}\right)^s . \end{cases}$$

This result was proved in an entirely different way by Meixner [5].

We now specialize (5.6) further by taking $\lambda = c = d$. Thus (5.7) reduces to

$$C_n(x,z) = \sum_{r=0}^{\infty}\frac{(a)_{n+r}}{r!}\left(\frac{-xz}{1-z}\right)^r = (a)_n\left(1+\frac{xz}{1-z}\right)^{-a-n} = (a)_n(1-z)^{a+n}(1-(1-x)z)^{-a-n},$$

$$D_n(y,z) = (b)_n(1-z)^{b+n}(1-(1-y)z)^{-b-n}.$$

Therefore (5.6) reduces to

$$\sum_{n=0}^{\infty}\frac{(c)_n}{n!}z^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(a)_r}{(c)_r}x^r\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(b)_s}{(c)_s}y^s$$

$$= (1-z)^{a+b-c}(1-(1-x)z)^{-a}(1-(1-y)z)^{-b}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{xyz}{(1-(1-x)z)(1-(1-y)z)}\right)^n.$$

This is the same as (2.1).

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FORMAL PROOF OF EQUIVALENCE OF TWO SOLUTIONS OF THE GENERAL PASCAL RECURRENCE

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There have been numerous studies of the general Pascal recurrence relation

(1) $$f(x+1, y+1) - f(x,y+1) - f(x,y) = 0.$$

Defining

$$\Delta_x f(x,y) = f(x+1, y) - f(x,y), \qquad \Delta_y f(x,y) = f(x, y+1) - f(x,y),$$
$$E_x f(x,y) = f(x+1, y), \qquad E_y f(x,y) = f(x, y+1),$$

Milne-Thomson [8] notes that Eq. (1) may be recast in the form of the partial difference equation with constant coefficients

(2) $$E_y \Delta_x f(x,y) - f(x,y) = 0$$

for which one may write down the formal solution

(3) $$f(x,y) = (1 + E_y^{-1})^x \phi(y),$$

where $\phi(y)$ is an arbitrary function. Hence Milne-Thomson finds the classical formal solution (finite series when x is a positive integer)

(4) $$f(x,y) = \sum_{k=0}^{\infty} \binom{x}{k} \phi(y-k).$$

There is then also an alternative way to write such a formal series solution:

(5) $$f(x,y) = \sum_{k=0}^{\infty} \binom{x}{k} \phi(y-x+k).$$

These are old and well-known results, easily found in other treatises on the calculus of finite differences. The method of generating functions is used in [8] also and the results agree with the two possible series solutions we have quoted above.

As for getting a nice, elegant, explicit formula for the general solution to such partial difference equations (and of higher order), we would be remiss if we did not mention the two valuable papers of Carlitz [3] and [4]. Anyone working with arrays of numbers ought to consult these papers for a close-hand study of the interesting way Carlitz handles the equations. These papers deal with formulas for sums of powers of the natural numbers and the formulas involve Bernoulli and Stirling numbers as well as expansions of differential operators.

Most recently, Eq. (1) has arisen in some interesting new work on partitions [1], [5]. Carlitz's solution of a recurrence in [5] has now attracted Hansraj Gupta [7] who has announced the following result:

Theorem. Let $c(n+1, k) = c(n,k) + c(n, k-1)$, with $c(n,0) = a(n)$, $c(1,k) = b(k)$, $n,k \geqslant 1$, where $a(n)$ and $b(k)$ are arbitrary functions of n and k, respectively. Then, explicitly,

(6) $$c(n,k) = \sum_{r=k}^{n-1} \binom{r-1}{k-1} a(n-r) + \sum_{r=0}^{k-1} \binom{n-1}{r} b(k-r), \qquad k \geqslant 1.$$

This generalizes the solutions and formulas given in [1] and [5]. What we propose to do here is to show the equivalence of Gupta's formula (6) and the well-known formal series solution (4). We show that the one implies the other. A simple combinatorial identity listed in [6] equivalent to the Vandermonde convolution (addition formula) is used in the discussion.

We first need to reformulate Gupta's result in the notation of the present paper. In our notation, formula (6) becomes

(7) $$f(x,y) = \sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r, 0) + \sum_{r=0}^{y-1} \binom{x}{r} f(0, y-r),$$

for integers $x,y \geqslant 1$.

In the steps below we need at one spot formula (3.4) from [6]:

$$\sum_{r=\alpha}^{\beta-\gamma} \binom{r}{\alpha}\binom{\beta-r}{\gamma} = \binom{\beta+1}{\alpha+\gamma+1} \quad . \tag{8}$$

We find then that assuming (4)

$$\sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r,0) = \sum_{r=y}^{x} \binom{r-1}{y-1} \sum_{j=0}^{\infty} \binom{x-r}{j} \phi(-j) = \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y}^{x} \binom{r-1}{y-1}\binom{x-r}{j}$$

$$= \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y-1}^{x-1} \binom{r}{y-1}\binom{x-1-r}{j} = \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y-1}^{x-1-j} \binom{r}{y-1}\binom{x-1-r}{j}$$

$$= \sum_{j=0}^{\infty} \phi(-j) \binom{x}{y+j} = \sum_{r=y}^{\infty} \binom{x}{r} \phi(y-r),$$

so that we have shown in fact

$$\sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r,0) = \sum_{r=y}^{\infty} \binom{x}{r} \phi(y-r). \tag{9}$$

Upon adding the trivial relation (clear from (4))

$$\sum_{r=0}^{y-1} \binom{x}{r} f(0,y-r) = \sum_{r=0}^{y-1} \binom{x}{r} \phi(y-r)$$

to both sides of (9), we find that we have proved (7). Conversely, it is easy to see how to follow the steps in reverse so that series (4) can be broken into two parts as specified in (7). Solving (1) in terms of an arbitrary function ϕ is equivalent to setting up the two sequences $f(x-r,0)$ and $f(0,y-r)$. We leave aside the discussion of convergence questions.

As a final observation, Cadogan [2] has shown how to solve the slight extension of (1): $f(k,n) = pf(k,n-1) + qf(k-1,n-1)$, where p,q are arbitrary fixed constants. He interprets the resulting arrays in terms of arithmetic and geometric sequences for certain choices of parameters. There is nothing new in this, but his paper is a worthwhile pedagogical survey written at an elementary level. Similarly, there is nothing "new" in the present paper, but we have spelled out the manipulations of our proof to show how one actually does the verification of equivalence. In a similar way the reader may write out the same argument using (5) instead of (4). Of course, the equivalence of these with Gupta's (6) has been shown here only when x,y are integers, and the reader must bear in mind that (4) and (5) are more general than (6) because they hold in cases where x,y are not integers. The series manipulations leading to (9) are easily justified because the series are really finite series, $\binom{x}{r} = 0$ for example, when $r > x$, x being a non-negative integer.

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NOTE ON SOME GENERATING FUNCTIONS

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1. In a recent paper in this Quarterly, Bruckman [2] defined a sequence of positive integers B_k by means of

$$(1.1) \qquad (1-x)^{-1}(1+x)^{-1/2} = \sum_{k=0}^{\infty} B_k \frac{x^k}{2^k \cdot k!} \quad .$$

This is equivalent to the recurrence

$$(1.2) \qquad B_k = B_{k-1} + (2k-1)(2k-2)B_{k-2} \qquad (k \geqslant 2), \qquad B_0 = B_1 = 1 \; .$$

Making use of (1.2) he showed that

$$(1.3) \qquad e^{x^2/2} \int_0^x e^{-u^2} du = \sum_{k=0}^{\infty} B_k \frac{x^{2k+1}}{(2k+1)!}$$

and

$$(1.4) \qquad (1-x^2)^{-1} \arctan x = \sum_{k=0}^{\infty} B_k^2 \frac{x^{2k+1}}{(2k+1)!} \quad .$$

Bateman [1] has discussed the polynomial $g_n(y,z)$ defined by

$$(1.5) \qquad (1+x)^{y+z}(1-x)^{-y} = \sum_{n=0}^{\infty} x^n g_n(y,z) \; ;$$

see also [3]. On the other hand the Jacobi polynomial [6, Ch. 16]

$$(1.6) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}$$

satisfies

$$(1.7) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)z^n = \left(1+\frac{x+1}{2}z\right)^{\alpha}\left(1+\frac{x-1}{2}z\right)^{\beta}$$

and in particular, for $x = 0$,

$$(1.8) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(0)z^n = (1+\tfrac{1}{2}z)^{\alpha}(1-\tfrac{1}{2}z)^{\beta} \; .$$

It follows from (1.1) and (1.8) that

$$(1.9) \qquad \frac{1}{k!} B_k = 2^{2k} P_k^{(-1/2-k,-1-k)}(0) = (-1)^k 2^{2k} P_k^{(-1-k,-1/2-k)}(0) \; .$$

We shall show that both (1.3) and (1.4) can be generalized considerably. We also obtain the following congruence for B_n :

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(1.10) $$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{r!\,m^r},$$

where m and r are arbitrary positive integers.

It would be of interest to find a combinatorial interpretation of B_k.

2. The writer [4] has obtained the following bilinear generating function:

(2.1) $$\sum_{n=0}^{\infty} \frac{n!}{(\gamma)_n}(x-1)^n(y-1)^n w^n P_n^{(\alpha-n,-\alpha-\gamma-n)}\left(\frac{x+1}{x-1}\right) P_n^{(\beta-n,-\beta-\gamma-n)}\left(\frac{y+1}{y-1}\right)$$
$$= (1-w)^{-\alpha-\beta-\gamma}(1-xw)^{\alpha}(1-yw)^{\beta} F\left[-\alpha, -\beta; \gamma; \frac{(x-1)(y-1)w}{(1-xw)(1-yw)}\right],$$

where as usual

$$F(z,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!\,(c)_n} z^n \qquad \text{and} \qquad (a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

In particular, for $x = y = -1$ and $\gamma = -\alpha-\beta$, Eq. (2.1) reduces to

(2.2) $$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} 4^n w^n P_n^{(\alpha-n,\beta-n)}(0) P_n^{(\beta-n,\alpha-n)}(0) = (1+w)^{\alpha+\beta} F\left[-\alpha, -\beta; -\alpha-\beta; \frac{4w}{(1+w)^2}\right].$$

It is convenient to replace α, β by $-\alpha, -\beta$, so that (2.2) becomes

(2.3) $$\sum_{n=0}^{\infty} \frac{n!}{(\alpha+\beta)_n} 4^n w^n P_n^{(-\alpha-n,-\beta-n)}(0) P_n^{(-\beta-n,-\alpha-n)}(0) = (1+w)^{-\alpha-\beta} F\left[\alpha, \beta; \alpha+\beta; \frac{4w}{(1+w)^2}\right].$$

Specializing further, we take $\beta = \alpha + \tfrac12$, so that

(2.4) $$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac12)_n} 4^n w^n P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0) P_n^{(-\alpha-\frac12-n,-\alpha-n)}(0)$$
$$= (1+w)^{-2\alpha-\frac12} F\left[\alpha, \alpha+\tfrac12; 2\alpha+\tfrac12, \frac{4w}{(1+w)^2}\right].$$

Next in formula (2) of [6, p. 66],

$$F\left[\tfrac12 a, \tfrac12 a + \tfrac12; a-b+1; \frac{4z}{(1+z)^2}\right] = (1+z)^a F[a,b; a-b+1; z]$$

take $a = 2\alpha$, $b = \tfrac12$. We get

(2.5) $$F\left[\alpha, \alpha+\tfrac12; 2\alpha+\tfrac12; \frac{4z}{(1+z)^2}\right] = (1+z)^2 F[2\alpha, \tfrac12; 2\alpha+\tfrac12; z].$$

Hence (2.4) becomes

(2.6) $$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac12)_n} 4^n w^n P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0) P_n^{(-\alpha-\frac12-n,-\alpha-n)}(0) = (1+w)^{-\frac12} F[2\alpha, \tfrac12; 2\alpha+\tfrac12; w].$$

Since

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

(2.6) may be written in the form

(2.7) $$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac12)_n} 4^n w^n \left\{P_n^{(-\alpha-n,-\alpha-\frac12-n)}(0)\right\}^2 = (1-w)^{-\frac12} F[2\alpha, \tfrac12; 2\alpha+\tfrac12; -w].$$

In particular, for $\alpha = \tfrac12$, it follows from (2.7) and (1.9) that

$$\sum_{n=0}^{\infty} \frac{2^{-2n} w^n}{n!(3/2)_n} B_n^2 = (1-w)^{-\frac12} F[1, \tfrac12; 3/2; -w].$$

Replacing w by z^2, this becomes

(2.8) $$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} B_n^2 = z(1-z^2)^{-1/2} F[1, 1/2; 3/2; -z^2] .$$

Since

$$zF[1, 1/2; 3/2; -z^2] = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)_n}{(3/2)_n} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} = \arctan z ,$$

it is evident that (2.8) is the same as (1.4).

3. In (2.1) take $x = -1,\ y = 0,\ \gamma = -\alpha - \beta$. Since, by (1.6),

$$P_n^{(\beta-n,\alpha-n)}(-1) = \binom{\alpha}{n} ,$$

it is clear that (2.1) reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} \binom{\alpha}{n} 2^n w^n P_n^{(\alpha-n,\beta-n)}(0) = (1+w)^{\alpha} F\left[-\alpha, -\beta; -\alpha-\beta; \frac{2w}{1+w}\right] .$$

Replacing α, β by $-\alpha, -\beta$, this becomes

(3.1) $$\sum_{n=0}^{\infty} (-1)^n \frac{(\alpha)_n}{(\alpha+\beta)_n} 2^n w^n P_n^{(-\alpha-n,-\beta-n)}(0) = (1+w)^{-\alpha} F\left[\alpha, \beta; \alpha+\beta; \frac{2w}{1+w}\right] .$$

In particular, for $\beta = 1/2$, we get

(3.2) $$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\alpha+1/2)_n} 2^n z^n P_n^{(-\alpha-n,-1/2-n)}(0) = (1-z)^{-\alpha} F\left[\alpha, 1/2; \alpha+1/2; \frac{-2z}{1-z}\right] .$$

For $\alpha = 1$, Eq. (3.2) becomes

(3.3) $$\sum_{n=0}^{\infty} \frac{z^n}{2^n (3/2)_n} B_k = (1-z)^{-1} F\left[1, 1/2; 3/2; -\frac{2z}{1-z}\right] .$$

This is not the same as (1.3).

The right-hand side of (3.2) is equal to

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r (1/2)_r}{r!(\alpha+1/2)_r} (-2z)^r (1-z)^{-\alpha-r} = \sum_{r=0}^{\infty} \frac{(\alpha)_r (1/2)_r}{r!(\alpha+1/2)_r} (-2z)^r \sum_{s=0}^{\infty} \frac{(\alpha+r)_s}{s!} z^s$$

$$= \sum_{n=0}^{\infty} (\alpha)_n z^n \sum_{r=0}^{n} (-2)^r \frac{(1/2)_r}{r!(n-r)!(\alpha+1/2)_r} .$$

Hence (3.2) implies

$$\sum_{n=0}^{\infty} \frac{2^n z^n}{(+1/2)_n} P_n^{(-\alpha-n,-1/2-n)}(0) = \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} (-2)^r \frac{(1/2)_r}{r!(n-r)!(\alpha+1/2)_r} = \sum_{r=0}^{\infty} \frac{(1/2)_r (-2z)^r}{r!(\alpha+1/2)_r} \sum_{n=r}^{\infty} \frac{z^{n-r}}{(n-r)!} ,$$

so that

(3.4) $$\sum_{n=0}^{\infty} \frac{z^n}{(\alpha+1/2)_n} P_n^{(-\alpha-n,-1/2-n)}(0) = e^{1/2 z} \sum_{r=0}^{\infty} \frac{(1/2)_r (-z)^r}{r!(\alpha+1/2)_r} .$$

For $\alpha = 1$, Eq. (3.4) reduces to (1.3) .

4. Put

(4.1) $$(1-x)^{\alpha}(1+x)^{\beta} = \sum_{n=0}^{\infty} c_n(\alpha,\beta) x^n .$$

Then

$$\sum_{m,n=0}^{\infty} c_m(a,\beta)c_n(a,\beta)x^m y^n = (1-x)^{\alpha}(1-y)^{\alpha}(1+x)^{\beta}(1+y)^{\beta} = (1+xy-x-y)^{\alpha}(1+xy+x+y)^{\beta}$$

$$= (1+xy)^{\alpha+\beta}\left(1-\frac{x+y}{1+xy}\right)^{\alpha}\left(1+\frac{x+y}{1+xy}\right)^{\beta}$$

$$= (1+xy)^{\alpha+\beta}\sum_{k=0}^{\infty} c_k(a,\beta)\left(\frac{x+y}{1+xy}\right)^k$$

$$= \sum_{k=0}^{\infty} c_k(a,\beta)\sum_{s=0}^{k}\binom{k}{s}x^s y^{k-s}\sum_{r=0}^{\infty}\binom{\alpha+\beta-k}{t}x^t y^t$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{\substack{s+t=m\\ k-s+t=n}}\binom{k}{s}\binom{\alpha+\beta-k}{t}c_k(a,\beta).$$

It follows that

$$c_m(a,\beta)c_n(a,\beta) = \sum_{t=0}^{\min(m,n)}\binom{m+n-2t}{m-t}\binom{\alpha+\beta-m-n+2t}{t}c_{m+n-2t}(a,\beta). \tag{4.2}$$

The proof follows Kaluza [6]; see also [3].

Comparing (4.1) with (1.1), we have

$$B_k = 2^k\cdot k!c_k(-1,-\tfrac{1}{2}). \tag{4.3}$$

Thus (4.2) implies

$$B_mB_n = \sum_{t=0}^{\min(m,n)}(-1)^t 2^t\binom{m}{t}\binom{n}{t}t!\prod_{j=0}^{t-1}(2m+2n-2t-2j+1)B_{m+n-2t}. \tag{4.4}$$

For $m = 1$, Eq. (4.4) reduces to (1.2). It is not difficult to prove (4.4) by induction.

The writer has proved the following result [5].

Let $f(n)$, $g(n)$ **denote polynomials in** n **with integral coefficients. Define** u_n **by means of**

$$u_{n+1} = f(n)u_n + g(n)u_{n-1} \qquad (n \geqslant 1), \tag{4.5}$$

where

$$u_0 = 1, \qquad u_1 = f(0), \qquad g(0) = 0. \tag{4.6}$$

Then u_n **satisfies the following congruence:**

$$\Delta^{2r}u_n \equiv \Delta^{2r-1}u_n \equiv 0 \pmod{m^r}, \tag{4.7}$$

for all $m \geqslant 1$, $n \geqslant 0$, $r \geqslant 1$, **where**

$$\Delta^r u_n = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}u_{n+sm}u_{(r-s)m}.$$

Comparing (4.5) with

$$B_{n+1} = B_n + 2n(2n+1)B_{n-1},$$

it is clear that (4.6) holds. We have therefore

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{m^{[(r+1)/2]}}. \tag{4.8}$$

However a better result can be obtained. By (4.4) we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm}B_{(r-s)m} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{t} (-1)^t 2^t \binom{n+sm}{t}\binom{(r-s)m}{t} t! \cdot \prod_{j=0}^{t-1}$$

$$\cdot (2n+2rm-2t-2j+1) \cdot B_{n+rm-2t} = \sum_{t} (-1)^t \frac{2^t}{t!} B_{n+rm-2t} \prod_{j=0}^{t-1} (2n+2rm-2t-2j+1)$$

$$\cdot \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s),$$

where

$$f(s) = (n+sm-t+1)_t((r-s)m-t+1)_t .$$

Clearly

$$f(s) = a_0 + a_1 sm + \cdots + a_{2t}(sm)^{2t} ,$$

where the a_i are integers. Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s) = \sum_{i=r}^{2t} a_i m^{2i} \Delta^r 0^i \equiv 0 \pmod{r!m^r}.$$

Since

$$\frac{2^t}{t!} \prod_{j=0}^{t-1} (2n+2rm-2t-2j+1)$$

is integral, it follows at once that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm}B_{(r-s)m} \equiv 0 \pmod{r!m^r}.$$

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A GENERALIZED PASCAL'S TRIANGLE

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1. INTRODUCTION

In the study of a combinatorial minimization problem related to multimodule computer memory organizations [5], a triangle of numbers is constructed, which enjoys many of the pleasant properties of Pascal's triangle [1,2]. These numbers originate from counting a set of points in the k-dimensional Euclidean space.

In this paper we only list some of the properties which are similar to those associated with Pascal's triangle. Other properties will be the subject of further investigation.

2. r-SPHERES IN RECTILINEAR METRIC

Let

$$U_r^{(k)} = \left\{ (x_1, x_2, \cdots, x_k) \,\middle|\, x_i \text{ integers, } i = 1, 2, \cdots, k, \text{ and } \sum_{i=1}^{k} |x_i| \le r \right\}.$$

The aim of this section is to obtain a formula for the cardinality $|U_r^{(k)}|$ of the set $U_r^{(k)}$.

Lemma 1. Let

$$S_j^{(k)} = \left\{ (x_1, x_2, \cdots, x_k) \,\middle|\, x_i \text{ integers, } i = 1, 2, \cdots, k \text{ and } \sum_{i=1}^{k} |x_i| = j \right\},$$

then

$$|S_j^{(k)}| = \begin{cases} 1, & j = 0, \\ \displaystyle\sum_{i=0}^{k-1} \binom{k}{i} \binom{j-1}{k-i-1} 2^{k-i}, & j \ge 1. \end{cases}$$

Proof. Note that the number of ways to place j nondistinct objects into k distinct cells is $\binom{k+j-1}{k-1}$. (See [3].) Consequently, the number of ways to place j nondistinct objects into k distinct cells such that none of them is empty is $\binom{j-1}{k-1}$. In $S_j^{(k)}$, if we group together all points $(x_1, x_2, \cdots, x_k)$ which have the same number of zero coordinates, the result follows.

Theorem 1.

$$|U_r^{(k)}| = \sum_{i=0}^{k} \binom{k}{i} \binom{r}{k-i} 2^{k-i}.$$

Proof. It follows from

$$|U_r^{(k)}| = 1 + \sum_{j=1}^{r} |S_j^{(k)}| \quad \text{and} \quad \sum_{i=0}^{n} \binom{i}{a} = \binom{n+1}{a+1}.$$

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The numbers $|S_j^{(k)}|$ and $|U_r^{(k)}|$ have the following geometric interpretation:

It suffices to mention the case $k = 2$. Partition the Euclidean plane into unit squares. Fix any square as the origin, which will be called the 0^{th} sphere. All squares which have at least one edge in common with the origin form the 1^{st} sphere. All those with at least one edge in common with a square in the 1^{st} sphere form the 2^{nd} sphere, and so on.

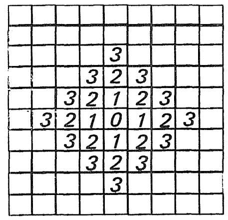

Figure 1

The numbers in Fig. 1 indicate what spheres the squares are in. $|S_j^{(2)}|$ is then the number of squares comprising the j^{th} sphere, i.e. its "surface area," and $|U_r^{(2)}|$ is the number of squares constituting the r^{th} sphere and its interior, i.e., its "volume."

The generalization to $k > 2$ is clear.

3. A GENERALIZED PASCAL'S TRIANGLE

For simplicity, let us write $M_{k,r}$ for $|U_r^{(k)}|$. We then have the following observations:

Theorem 2. (i)
$$M_{k,r} = M_{r,k}$$

(ii)
$$M_{k+1,r} = 2\sum_{j=0}^{r-1} M_{k,j} + M_{k,r}$$

(iii)
$$M_{k+1,r+1} = M_{k+1,r} + M_{k,r+1} + M_{k,r}$$

Proof. (i) If $k \geqslant r$, then

$$M_{k,r} = \sum_{i=k-r}^{k} \binom{k}{i}\binom{r}{k-i} 2^{k-i} = \sum_{j=0}^{r} \binom{k}{j+k-r}\binom{r}{r-j} 2^{r-j} = \sum_{j=0}^{r} \binom{r}{j}\binom{k}{r-j} 2^{r-j} = M_{r,k}.$$

Similarly for the case $k < r$.

(ii)
$$2\sum_{j=0}^{r-1} M_{k,j} = \sum_{i=0}^{k} \binom{k}{i}\binom{r}{k+1-i} 2^{k+1-i}$$

$$2\sum_{j=0}^{r-1} M_{k,j} + M_{k,r} = \binom{k}{0}\binom{r}{k+1} 2^{k+1} + \sum_{i=1}^{k}\binom{k}{i}\binom{r}{k+1-i} 2^{k+1-i} + \sum_{j=0}^{k-1}\binom{k}{j}\binom{r}{k-j} 2^{k-j} + \binom{k}{k}\binom{r}{0} 2^0$$

$$= \binom{k+1}{0}\binom{r}{k+1} 2^{k+1} + \sum_{i=1}^{k}\left[\binom{k}{i} + \binom{k}{i-1}\right]\binom{r}{k+1-i} 2^{k+1-i} + \binom{k+1}{k+1}\binom{r}{0} 2^0$$

$$= \sum_{i=0}^{k+1}\binom{k+1}{i}\binom{r}{k+1-i} 2^{k+1-i} = M_{k+1,r}.$$

(iii) It follows directly from (ii).

Theorem 3. For $k = 0, 1, 2, \cdots$, let

$$S_k = \begin{cases} \sum\limits_{i=0}^{(k-1)/2} M_{k-2i,i}, & \text{if } k \text{ is odd,} \\ \sum\limits_{i=0}^{k/2} M_{k-2i,i}, & \text{if } k \text{ is even.} \end{cases}$$

Then

$$S_{k+3} = S_k + S_{k+1} + S_{k+2}$$

for $k = 0, 1, 2, \cdots$.

In other words, they form a Tribonacci sequence.

Proof. It follows from (iii) of Theorem 2.

We can now construct a triangle of the numbers $\{M_{k,r}\}$, $k, r = 0, 1, 2, \cdots$. The n^{th} row consists of the numbers $\{M_{n-i,i}\}$ in the order of $i = 0, 1, 2, \cdots, n$ from left to right. The left diagonals thus consist of numbers with fixed r, and the right diagonals numbers with fixed k.

By (iii) of Theorem 2, each number in the n^{th} row is the sum of the three adjacent numbers in the $(n-1)^{st}$ and $(n-2)^{nd}$ rows. For example, the number 25 in the 5^{th} row is the sum of its three adjacent numbers 5, 7, 13 in the 3^{rd} and 4^{th} rows. Therefore, instead of using the formula in Theorem 1, we can fill in a row by adding appropriate numbers in the two preceding rows. Finally, by Theorem 3, the sums of the more gently sloping diagonals form the Tribonacci sequence, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, $\cdots$.

The first 10 rows of this generalized Pascal's triangle is displayed below.

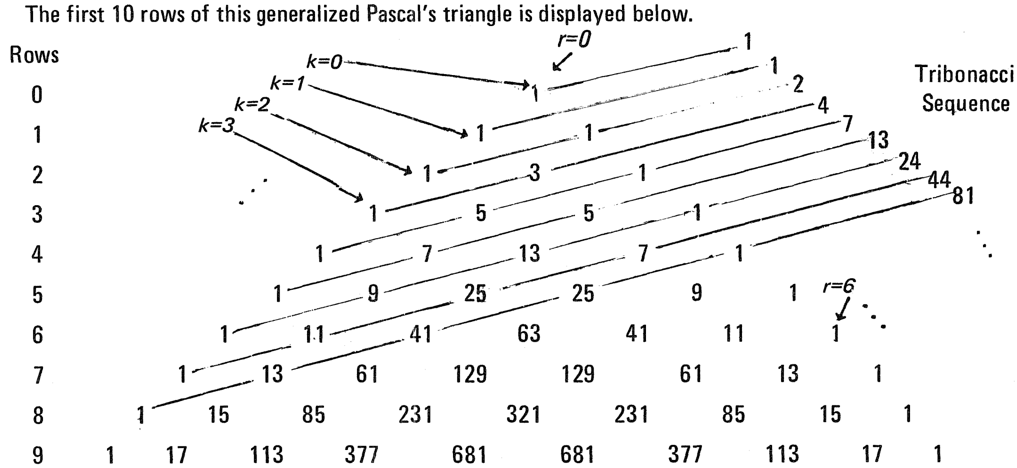

Figure 2

ACKNOWLEDGEMENT

The authors are grateful to Professor V.E. Hoggatt, Jr., for pointing out that in Professor Monte Boisen, Jr.'s article [4], the same triangle appeared but its generation was different.

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GENERALIZED FIBONACCI TILING

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1. INTRODUCTION

One way to easily establish the validity of

$$F_1^2+F_2^2+\cdots+F_n^2 = F_nF_{n+1}$$

is by use of a nice geometric argument as in Brother Alfred [1]. Thus by starting with two unit squares one can add a whirling array of squares (see Fig. 1) with Fibonacci number sides since the area as in Fig. 1 is

$$5\times 8 = F_1^2+F_2^2+F_3^2+F_4^2+F_5^2 = F_5F_6.$$

More generally, the rectangle has area F_nF_{n+1}.

This result is classic, but a new twist was added by H. L. Holden [2]. The centers of the outwardly spirally squares lie on two straight lines which are orthogonal. These two straight lines intersect in a point P, and the distances of the centers of the squares from P sequentially are proportional to the Lucas numbers. Holden also contains an extension to the generalized Fibonacci sequence with $H_1 = 1$ and $H_2 = p$ with $H_{n+2} = H_{n+1}+H_n$. This results in

$$H_1^2+H_2^2+H_3^2+\cdots+H_n^2 = H_nH_{n+1}-H_0.$$

In another paper, we will discuss the situation with inwinding spirals.

2. THE FIRST GENERALIZATION

Our method here is different than that used by Holden [2], but ours offers a neater way to get the centers of the squares, and we proceed principally by generating functions. (See Fig. 2.) We first discuss the geometry.

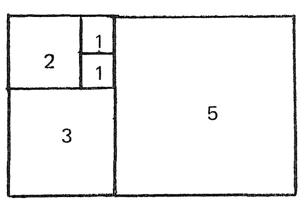


Figure 1

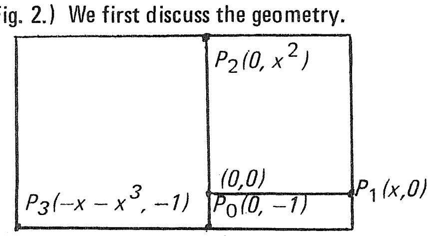


Figure 2

Start out with a unit square and make x copies. Then above that, make x copies of $x\cdot x$ squares. It is not difficult to see that the edges are $1, x, x^2+1, x^3+2x, \cdots, f_n(x)$, where $f_{n+2}(x) = xf_{n+1}(x)+f_n(x)$, which are the Fibonacci polynomials.

If we consider the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

as in Holden and $V_1 = (x,1)$, $V_2 = (-1,x)$, $V_3 = (-x,-1)$, $V_4 = (1,-x)$ with

$$V_{n+1} = V_n\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

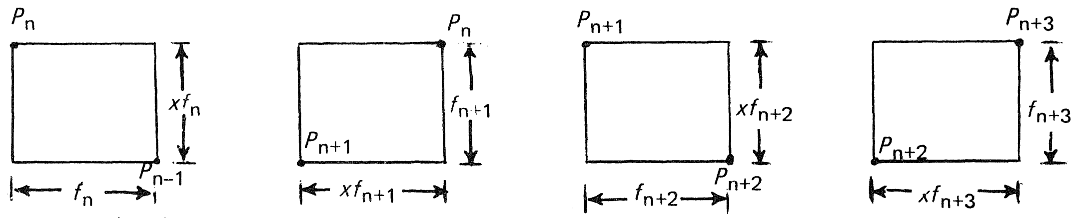
then

Theorem 1. $$P_n = P_{n-1}+V_nf_n(x).$$

*Fibonacci Scholar, Summer, 1974.

Proof. The proof proceeds in four parts.

Scheme 1: Scheme 2: Scheme 3: Scheme 4:



$P_n = P_{n-1} + (-1,x)f_n$ $\quad P_{n+1} = P_n + (-x,-1)f_{n+1}$ $\quad P_{n+2} = P_{n+1} + (1,-x)f_{n+2}$ $\quad P_{n+3} = P_{n+2} + (x,1)f_{n+3}$

These are the four critical turns in the sequence of expanding the outward spiralling of squares.

As a consequence of Theorem 1, one can prove

Theorem 2.

$$P_n = P_0 + \sum_{i=1}^{n} V_i f_i(x).$$

Proof.

$$P_1 = (0,-1) + (x,1)\cdot 1 = (x,0).$$

Assume

$$P_n = P_{n-1} + V_n f_n = P_0 + \sum_{i=1}^{n-1} V_i f_i + V_n f_n = P_0 + \sum_{i=1}^{n} V_i f_i.$$

We are now ready to get on with the general theorem by means of generating functions.

3. SOME NECESSARY IDENTITIES

Lemma 1.

$$\frac{\lambda x}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1}$$

Lemma 2.

$$\frac{(x^2+1)-\lambda^2}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n}$$

Lemma 3.

$$\frac{\lambda+\lambda^3}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n+1}(x)\lambda^{2n+1}$$

Since these are straightforward, the proofs will be omitted.

We may now give a generating function for the x-components of the corners, where $P_{n,x}$ denotes the x-coordinate of the point P_n.

Theorem 3.

$$\sum_{i=0}^{\infty} P_{i,x}\lambda^i = \frac{x(\lambda-\lambda^2+\lambda^3)}{1+\lambda^2(x^2+2)+\lambda^4}\cdot\frac{1}{1-\lambda}$$

Proof. From

$$P_n = (0,-1) + \sum_{i=1}^{n} V_i f_i(x),$$

$$\sum_{n=0}^{\infty} P_{n,x}\lambda^n = (0 + xf_1\lambda - f_2\lambda^2 - xf_3\lambda^3 + f_4\lambda^4 + xf_5\lambda^5 + \cdots)/(1-\lambda)$$

$$= [x(f_1\lambda - f_3\lambda^3 + f_5\lambda^5 - \cdots) - (f_2\lambda^2 - f_4\lambda^4 + f_6\lambda^6 - \cdots)]/(1-\lambda)$$

$$= \frac{\lambda x + x(\lambda + \lambda^3)}{1 + \lambda^2(x^2+2) + \lambda^4} \cdot \frac{1}{1-\lambda} .$$

Since P_n and P_{n-1} are opposite corners of square n, the x-coordinates of the centers C_n are given by

$$(P_{n,x} + P_{n-1,x})/2 = C_{n,x} .$$

$$\sum_{n=1}^{\infty} \lambda^n (P_{n,x} + P_{n-1,x})/2 = \frac{1+\lambda}{2(1-\lambda)} \cdot \frac{x(\lambda - \lambda^2 + \lambda^3)}{1 + \lambda^2(x^2+2) + \lambda^4} = \sum_{i=1}^{\infty} C_{i,x}\lambda^i$$

and

$$\sum_{n=1}^{\infty} (C_{n+2,x} - C_{n,x})\lambda^{n+1} = \frac{(1+\lambda)^2[x(\lambda - \lambda^2 + \lambda^3)]}{2(1 + \lambda^2(x^2+2) + \lambda^4)} + \frac{\lambda}{2} - \frac{\lambda^2 x}{2}$$

$$= \frac{-\lambda^3 x(x^2+2)/2 - \lambda^4 x(x^2+1+\lambda^2)/2}{1 + \lambda^2(x^2+2) + \lambda^4} ,$$

where

$$C_{1,x} = x/2 \qquad \text{and} \qquad C_{2,x} = x/2 .$$

There are two further things to do. These differences clearly alternate in sign. To convert to regular differences all the same sign, we change the minus sign to a plus in front of the even powered term and then replace λ^2 by $-\lambda^2$. This results in the following theorem:

Theorem 4.

$$G_x(x,\lambda) = \frac{-\lambda^3 x(x^2+2)/2 + \lambda^4 x(x^2+1-\lambda^2)/2}{1 - \lambda^2(x^2+2) + \lambda^4} .$$

Theorem 5. The generating function for the y-differences between alternate corners is

$$\frac{\lambda^3 x(x/2) + \lambda^4(x^2+1-\lambda^2)[(x^2+2)/2]}{1 - \lambda^2(x^2+2) + \lambda^4} .$$

Proof. For the y-differences one begins with

$$\sum_{i=0}^{\infty} P_{i,y}\lambda^i = \frac{1}{1-\lambda}\left[-1 + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right]$$

and

$$\sum_{i=1}^{\infty} P_{i,y}\lambda^i = \frac{1}{1-\lambda}\left[-\lambda + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right] .$$

$$\sum_{i=1}^{\infty} C_{i,y}\lambda^i = \sum_{i=1} \lambda^i (P_{i,y} + P_{i-1,y})/2 = \tfrac{1}{2}\left\{\frac{\lambda}{1-\lambda}\left[-1 + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right] + \frac{1}{1-\lambda}\left[-\lambda + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right]\right\}$$

$$= \frac{1}{2(1-\lambda)} \cdot \frac{-\lambda + \lambda^2(x^2+1) - \lambda^3(x^2+3) + \lambda^4 - 2\lambda^5}{1 + \lambda^2(x^2+2) + \lambda^4} .$$

Now to directly form the y-differences:

$$\sum_{i=1}^{\infty} (C_{i+2,y} - C_{i,y})\lambda^{i+2} = (1-\lambda^2)\sum_{i=1}^{\infty} C_{i,y}\lambda^i + C_{1,y}\lambda + C_{2,y}\lambda^2 .$$

But, $C_{1,y} = -\tfrac{1}{2}$ and $C_{2,y} = x^2/2$. Thus,

$$\sum_{i=1}^{\infty} (C_{i+2,y} - C_{i,y})\lambda^i = \frac{1+\lambda}{2} \cdot \frac{-\lambda + \lambda^2(x^2+1) - \lambda^3(x^2+3) + \lambda^4 - 2\lambda^5}{1+\lambda^2(x^2+2)+\lambda^4}$$

$$= \frac{\lambda^3 x^2 - \lambda^4(x^4+3x^2+2) - (x^2+2)\lambda^6}{2(1+\lambda^2(x^2+2)+\lambda^4)} .$$

From the diagrams it is clear that these differences are associated with odds and evens and on their respective lines they alternate in sign. We wish the initial values to be positive.

$$\sum_{i=1}^{\infty} |C_{i+2,y} - C_{i,y}|\lambda^i = -\left[\frac{(-\lambda^2)\lambda x^2 + (-\lambda)^2(x^4+3x^2+2) + (x^2+2)(-\lambda^2)^3}{1-\lambda^2(x^2+2)+\lambda^4}\right]$$

$$= \frac{\lambda^3 x(x/2) + \lambda^4(x^2+1-\lambda^2)[(x^2+2)/2]}{1-\lambda^2(x^2+2)+\lambda^4}$$

$$= \frac{x}{2}\sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1} + \frac{x^2+2}{2}\sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n} .$$

Recall that the x-differences

$$\sum_{i=1}^{\infty} (-1)^i |C_{i+2,x} - C_{i,x}|\lambda^i = -\frac{x^2+2}{2}\sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1} + \frac{x}{2}\sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n} .$$

Thus, uniformly we see that the slopes of the lines through the centers are

$$\frac{C_{n+2,y} - C_{n,y}}{C_{n+2,x} - C_{n,x}} = \frac{x/2}{-(x^2+2)/2} = \frac{-x}{x^2+2}, \quad \text{odd } n;$$

$$\frac{C_{n+2,y} - C_{n,y}}{C_{n+2,x} - C_{n,x}} = \frac{(x^2+2)/2}{(x/2)} = \frac{x^2+2}{x}, \quad \text{even } n .$$

Thus, the centers lie on two straight lines which are orthogonal since the product of their slopes is −1. This concludes the proof.

Theorem 6. The centers C_{2i+1} lie on a line with slope $-x/(x^2+2)$ and the centers C_{2i} lie on a line with slope $(x^2+2)/x$. These lines are orthogonal and intersect at the point (u,v), where

$$u = \frac{x}{x^2+4} \quad \text{and} \quad v = \frac{-2}{x^2+4} .$$

Proof. It is easy to show that the lines through C_1 and C_3, and through C_2 and C_4, respectively, do meet in the point (u,v) specified.

Theorem 7. If Q is the point

$$\left(\frac{x}{x^2+4}, \frac{-2}{x^2+4}\right),$$

then the center C_n is D_n units from Q, where

$$D_n = \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)}$$

and $\mathcal{L}_n(x)$ is the n^{th} Lucas polynomial, $\mathcal{L}_1(x) = x$, $\mathcal{L}_2(x) = x^2+2$, and $\mathcal{L}_{n+1}(x) = x\mathcal{L}_n(x) + \mathcal{L}_{n-1}(x)$.

Proof. Given that $C_1 = (x/2, -1/2)$ and $C_2 = (x/2, x^2/2)$, one can compute

$$D_1^2 = \left(\frac{x}{x^2+4} - \frac{x}{2}\right)^2 + \left(\frac{-2}{x^2+4} + \frac{1}{2}\right)^2 = \frac{x^2(x^2+2)^2 + x^4}{[2(x^2+4)]^2} = \frac{x^6+5x^4+4x^2}{[2(x^2+4)]^2}$$

$$D_1 = \frac{x\sqrt{x^4+5x^2+4}}{2(x^2+4)} \quad .$$

It is also easy to verify that

$$D_2 = \frac{(x^2+2)\sqrt{x^4+5x^2+4}}{2(x^2+4)} \quad .$$

Now consider the centers C_{n+2} and C_n. The points lie on a line through

$$\left(\frac{x}{x^2+4}, \frac{-2}{x^2+4}\right)$$

which separates them. The x and y differences from C_{n+2} and C_n are

$$-\frac{x^2+2}{2} f_n(x) \quad \text{and} \quad \frac{x}{2} f_n(x),$$

respectively, for one line or

$$\frac{x}{2} f_n(x) \quad \text{and} \quad \frac{x^2+2}{2} f_n(x),$$

respectively, for the second line. Thus the distance

$$|C_{n+2} - C_n| = \sqrt{x^4+5x^2+4}\, f_{n+1}(x)/2$$

in any case.

There is an identity for Lucas polynomials (see [3], p. 82)

$$\mathcal{L}_{n+1}(x) + \mathcal{L}_{n-1}(x) = (x^2+4)f_n(x).$$

Now, suppose

$$D_n = \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} \quad .$$

Then

$$|C_{n+2} - C_n| - D_n = D_{n+2}$$

$$\frac{f_{n+1}(x)\sqrt{x^4+5x^2+4}}{2} - \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} = \frac{\mathcal{L}_{n+2}(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} \quad .$$

This concludes the proof of Theorem 7.

3. THE SECOND GENERALIZATION

In the last section we considered the rectangle whose edges were $f_n(x)$ and $xf_n(x)$, where

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

with

$$f_0(x) = 0 \quad \text{and} \quad f_1(x) = 1;$$

that is, the Fibonacci polynomials. Here we consider the sequence of polynomials such that

$$U_1(x) = 1, \qquad U_2(x) = P, \qquad \text{and} \qquad U_{n+2}(x) = xU_{n+1}(x) + U_n(x),$$

the generalized Fibonacci polynomials. We shall prove the following theorem.

Theorem 8. If one starts with a $1 \times p$ rectangle and adds counter-clockwise rectangles p by $px, \cdots, U_n(x)$ by $xU_n(x)$, then such squares in the whirling array have their centers on two straight lines with slopes $-(x^2+2)/x$ and $x/(x^2+2)$, which are orthogonal.

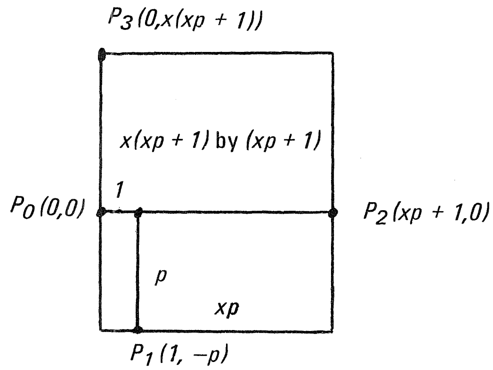


Figure 3

Proof. To establish that the centers lie on two perpendicular straight lines we shall have to find the coordinates of the vertices P_n. As before, we consider the rotation matrix

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the sequence of vectors

$$V_1^* = (1, -x), \qquad V_2^* = (x, 1), \qquad V_3^* = (-1, x), \qquad V_4^* = (-x, -1),$$

where $V_n^* = V_m^*$ when $n \equiv m \pmod 4$.

We shall also need the following identities for Fibonacci and Lucas polynomials (see [3]):

$$f_{n+2k}(x) - f_{n-2k}(x) = \mathcal{L}_n(x) f_{2k}(x)$$

$$f_{n+2k}(x) + f_{n-2k}(x) = f_n(x) \mathcal{L}_{2k}(x)$$

$$f_{n+2k+1}(x) + f_{n-2k-1}(x) = \mathcal{L}_n(x) f_{2k+1}(x)$$

$$f_{n+2k+1}(x) - f_{n-2k-1}(x) = f_n(x) \mathcal{L}_{2k+1}(x).$$

It is then easy to establish, on lines similar to Theorem 1, that

(1) $$P_n = P_{n-1} + V_n U_n(x),$$

where the $\{U_n\}$ is the sequence of polynomials

(2) $$U_1(x) = 1, \qquad U_2(x) = p, \qquad U_n(x) = xU_{n-1}(x) + U_{n-2}(x).$$

One also recognizes from (2) that the U_n obey

(3) $$U_{n+1}(x) = pf_n(x) + f_{n-1}(x),$$

where the $f_n(x)$ are the standard Fibonacci polynomials.

If $P_{n,x}$ and $P_{n,y}$ denote the x- and y-coordinates of P_n, one can establish from (1) that

$$\begin{aligned} P_{2n+1,x} &= (U_1 - U_3 + U_5 - \cdots + (-1)^n U_{2n+1}) + (-x)(-U_2 + U_4 - \cdots + (-1)^n U_{2n}) \\ &= (-1)^n f_{n+1}(x) U_{n+1}(x) + (-1)^{n+1} x f_n(x) U_{n+1}(x) = (-1)^n U_{n+1}(x) f_{n-1}(x) \end{aligned}$$

from which one can easily deduce that

$$P_{2n+3,x} = (-1)^{n+1} U_{n+2}(x) f_n(x), \qquad P_{2n-1,x} = (-1)^{n-1} U_n(x) f_{n-2}(x).$$

We also have

$$\begin{aligned} P_{2n,x} &= [U_1(x) - U_3(x) + U_5(x) - \cdots + (-1)^{n-1}U_{2n-1}(x)] \\ &\quad -x[-U_2(x) + U_4(x) - \cdots + (-1)^nU_{2n}(x)] \\ &= (-1)^{n-1}f_n(x)U_n(x) + (-1)^{n+1}xf_n(x)U_{n+1}(x) \\ &= (-1)^{n-1}f_n(x)U_{n+2}(x) \end{aligned}$$

which implies that

$$\begin{aligned} P_{2n+2,x} &= (-1)^nf_{n+1}(x)U_{n+3}(x), \\ P_{2n-2,x} &= (-1)^nf_{n-1}(x)U_{n+1}(x). \end{aligned}$$

Now, because the C_n are the centers of the squares, we get

$$C_n = (P_n + P_{n-1})/2$$

and so we get

$$\begin{aligned} C_{2n+3,x} - C_{2n+1,x} &= (P_{2n+3,x} + P_{2n+2,x} - P_{2n+1,x} - P_{2n,x})/2 \\ &= \tfrac{1}{2}[(-1)^{n+1}U_{n+2}(x)f_n(x) + (-1)^nf_{n+1}(x)U_{n+3}(x) \\ &\quad + (-1)^{n+1}U_{n+1}(x)f_{n-1}(x) + (-1)^nf_n(x)U_{n+2}(x)] \\ &= \frac{(-1)^n}{2}\,[(pf_{n+2}(x) + f_{n+1}(x))f_{n+1}(x) - (pf_n(x) + f_{n-1}(x))f_{n-1}(x)] \\ &= \frac{(-1)^n}{2}\,[p(f_{n+2}(x)f_{n+1}(x) - f_n(x)f_{n-1}(x)) + f_{n+1}^2(x) - f_{n-1}^2(x)] \\ &= \frac{(-1)^n}{2}\,[pxf_{2n+1}(x) + xf_{2n}(x)] = \frac{(-1)^n}{2}\,xU_{2n+2}(x). \end{aligned}$$

We have also

$$\begin{aligned} C_{2n+2,x} - C_{2n,x} &= (P_{2n+2,x} + P_{2n+1,x} - P_{2n,x} - P_{2n-1,x})/2 \\ &= \tfrac{1}{2}[(-1)^nf_{n+1}(x)U_{n+3}(x) + (-1)^nU_{n+1}(x)f_{n-1}(x) \\ &\quad + (-1)^nf_n(x)U_{n+2}(x) + (-1)^nU_n(x)f_{n-2}(x)] \\ &= \frac{(-1)^n}{2}\,[(pf_{n+2}(x) + f_{n+1}(x))f_{n+1}(x) + (pf_n(x) + f_{n-1}(x))f_{n-1}(x) \\ &\quad + (pf_{n+1}(x) + f_n(x))f_n(x) + (pf_{n-1}(x) + f_{n-2}(x))f_{n-2}(x)] \\ &= \frac{(-1)^n}{2}\,[pf_{n+1}(x)(f_{n+2}(x) + f_n(x)) + pf_{n-1}(x)(f_n(x) + f_{n-2}(x)) \\ &\quad + f_{n-2}^2(x) + f_{n-1}^2(x) + f_n^2(x) + f_{n+1}^2(x)] \\ &= \frac{(-1)^n}{2}\,[pf_{2n+2}(x) + pf_{2n-2}(x) + f_{2n+1}(x) + f_{2n-3}(x)] \\ &= \frac{(-1)^n}{2}\,[p\mathcal{L}_2(x)f_{2n}(x) + \mathcal{L}_2(x)f_{2n-1}(x)] = \frac{(-1)^n}{2}\,\mathcal{L}_2(x)U_{2n+1}(x). \end{aligned}$$

Now we shift our attention to the y-coordinates. From (1) we get

$$\begin{aligned} P_{2n+1,y} &= (-x)(U_1 - U_3 + \cdots + (-1)^nU_{2n+1}) \\ &\quad - (-U_2 + U_4 - U_6 + \cdots + (-1)^nU_{2n}) \\ &= (-1)^{n+1}xf_{n+1}(x)U_{n+1}(x) + (-1)^{n+1}f_n(x)U_{n+1}(x) \\ &= (-1)^{n+1}U_{n+1}(x)f_{n+2}(x). \end{aligned}$$

and

$$\begin{aligned} P_{2n+3,y} &= (-1)^nU_{n+2}(x)f_{n+3}(x), \\ P_{2n-1,y} &= (-1)^nU_n(x)f_{n+1}(x). \end{aligned}$$

We also have

$$\begin{aligned} P_{2n,y} &= (-x)[U_1(x) - U_3(x) + \cdots + (-1)^{n-1}U_{2n-1}(x)] \\ &\quad - [-U_2(x) + U_4(x) - \cdots + (-1)^n U_{2n}(x)] \\ &= (-1)^n x f_n(x) U_n(x) + (-1)^{n+1} f_n(x) U_{n+1}(x) \\ &= (-1)^{n+1} f_n(x) U_{n-1}(x) \end{aligned}$$

and

$$\begin{aligned} P_{2n+2,y} &= (-1)^n f_{n+1}(x) U_n(x), \\ P_{2n-2,y} &= (-1)^n f_{n-1}(x) U_{n-2}(x). \end{aligned}$$

From the above calculations, we find

$$C_{2n+3,y} - C_{2n+1,y} = (P_{2n+3,y} + P_{2n+2,y} - P_{2n+1,y} - P_{2n,y})/2$$

$$\begin{aligned} C_{2n+3,y} - C_{2n+1,y} &= \tfrac{1}{2}[(-1)^n U_{n+2}(x) f_{n+3}(x) + (-1)^n f_{n+1}(x) U_n(x) \\ &\quad + (-1)^n U_{n+1}(x) f_{n+2}(x) + (-1)^n f_n(x) U_{n-1}(x)] \\ &= \frac{(-1)^n}{2}[(pf_{n+1}(x) + f_n(x)) f_{n+3}(x) \\ &\quad + (pf_{n-1}(x) + f_{n-2}(x)) f_{n+1}(x) + (pf_n(x) + f_{n-1}(x)) f_{n+2}(x) \\ &\quad + (pf_{n-2}(x) + f_{n-3}(x)) f_n(x)] \\ &= \frac{(-1)^n}{2}[pf_{n+1}(x)(f_{n+3}(x) + f_{n-1}(x)) \\ &\quad + pf_n(x)(f_{n+2}(x) + f_{n-2}(x)) + f_n(x)(f_{n+3}(x) + f_{n-3}(x)) \\ &\quad + f_{n-2}(x) f_{n+1}(x) + f_{n-1}(x) f_{n+2}(x)] \\ &= \frac{(-1)^n}{2}[pf_{n+1}^2(x)\mathcal{L}_2(x) + pf_n^2(x)\mathcal{L}_2(x) \\ &\quad + f_n(x)\mathcal{L}_n(x) f_3(x) + f_{2n}(x)] \\ &= \frac{(-1)^n}{2}\mathcal{L}_2(x) U_{2n+2}(x). \end{aligned}$$

Our final step is to find

$$\begin{aligned} C_{2n+2,y} - C_{2n,y} &= (P_{2n+2,y} + P_{2n+1,y} - P_{2n,y} - P_{2n-1,y})/2 = \tfrac{1}{2}[(-1)^n f_{n+1}(x) U_n(x) \\ &\quad + (-1)^{n+1} U_{n+1}(x) f_{n+2}(x) + (-1)^n f_n(x) U_{n-1}(x) + (-1)^{n+1} U_n(x) f_{n+1}(x)] \\ &= \frac{(-1)^{n+1}}{2}[(pf_n(x) + f_{n-1}(x)) f_{n+2}(x) - (pf_{n-2}(x) + f_{n-2}(x)) f_n(x)] \\ &= \frac{(-1)^{n+1}}{2}[pf_n(x)(f_{n+2}(x) - f_{n-2}(x)) + f_{n+2}(x) f_{n-1}(x) - f_n(x) f_{n-3}(x)] \\ &= \frac{(-1)^{n+1}}{2}[pf_n(x)\mathcal{L}_n(x) f_2(x) + x f_{2n-1}(x)] = \frac{(-1)^{n+1}}{2} x U_{2n+1}(x). \end{aligned}$$

So, from the above results, we have

$$\frac{C_{2n+3,y} - C_{2n+1,y}}{C_{2n+3,x} - C_{2n+1,x}} = \frac{(-1)^n \mathcal{L}_2(x) U_{2n+2}(x)\cdot 2}{2\cdot U_{2n+1}(x)\cdot x(-1)^n} = \frac{\mathcal{L}_2(x)}{x}$$

which tells us that the C_n for odd n lie on a line with slope $(x^2+2)/2$. We also find

$$\frac{C_{2n+2,y} - C_{2n,y}}{C_{2n+2,x} - C_{2n,x}} = \frac{(-1)^{n+1} x U_{2n+1}(x)\cdot 2}{2\cdot(-1)^n \mathcal{L}_2(x) U_{2n+1}(x)} = -\frac{x}{\mathcal{L}_2(x)}$$

which tells us that the C_n for even n lie on a line with slope $-x/(x^2+2)$. Further, since the products of the slopes is -1, these lines are perpendicular. This proves Theorem 8.

From the above result it follows almost trivially that

Theorem 9. If D_n is the distance of C_n from the point of intersection of the two lines of centers, then

$$D_n = \frac{\mathcal{L}_n^*(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)},$$

where the $\mathcal{L}_n^*(x)$ are the generalized Lucas polynomials

$$\mathcal{L}_1^* = p, \qquad \mathcal{L}_2^* = xp+2, \qquad \text{and} \qquad \mathcal{L}_{n+2}^*(x) = x\mathcal{L}_{n+1}^*(x) + \mathcal{L}_n^*(x).$$

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A LEAST INTEGER SEQUENCE INVESTIGATION

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In the fall semester of 1964, four students, Robert Lera, Ron Staszkow, Rod Arriaga, and Robert Martel began an investigation along with their teacher, Brother Alfred Brousseau, of a problem that arose in connection with a Putnam examination question. The problem was to prove that if

$$P_{n+1} = [p_n + p_{n-1} + p_{n-2}]/p_{n-3}$$

produced an endless sequence of integers while the quantities p_i remained less in absolute value than an upper bound A, then the sequence must be periodic. The divergent idea that led to the research was this: How can one insure an infinite sequence of integers from such a recursion formula? One quick answer was to use the greatest integer function.

Initially an investigation was begun on:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}}\right],$$

where the square brackets mean: "take the greatest integer less than or equal to the quantity enclosed within the brackets." Very quickly, zero entered into the sequence with the result that there were mathematical complications once it arrived at the denominator.

To avoid this problem, it was decided to try using "the least integer function" instead of the greatest integer function. The notation adopted was:

$$[x]^* = n,$$

where n is the least integer greater than or equal to x. With this approach starting with three positive integers the function:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}}\right]^*$$

gives terms that are always ≥ 1.

The problem was enlarged by introducing two parameters, p and q, defining:

$$a_{n+1} = [(pa_n + qa_{n-1})/a_{n-2}]^*.$$

For any given pair (p,q) we have a set of sequences determined by assigning any three initial positive integers (a_1, a_2, a_3). We shall speak of these as the sequences belonging to (p,q).

Our least integer function representation is actually equivalent to two inequality relations. Thus if

$$[A]^* = B$$

the equivalent inequality statements are:

$$A \leqslant B \quad \text{and} \quad A > B - 1,$$

where B is an integer.

A GENERAL PERSPECTIVE

Early in the work with sequences, when examining such cases as (1,1), (1,2), (2,1), (2,2), (2,3), (3,4), it was noted that when $p \geqslant q$, only periodic sequences are found, whereas when $p < q$, there were also non-periodic sequences as well. This became one of the general topics of the research. One theorem along these lines can be stated immediately.

Theorem. Every set of sequences (p,q) has at least one periodic sequence.

Proof. Consider the sequence determined by the three quantities $p + q$, $p + q$, $p + q$. The fourth term calculates out as $p + q$. Thus this sequence will continue indefinitely with the single quantity $p + q$.

SEQUENCES OF TYPE (1,1)

The general recursion relation for this type of sequence is:

$$a_{n+1} = [(a_n + a_{n-1})/a_{n-2}]^*.$$

To explore this case it was found convenient to set up a table of the quantities a_1, a_2, a_3 for all values $a_i \leqslant 6$, $(i = 1,2,3)$. Starting with 1,1,1, for example, one obtains: 1,1,1,2,3,5,4,3,2,2,2,···. This was called a B sequence, the period being of length one and consisting of the single number 2.

Four distinct periods were found:

A ··· 4,2,2,1,2,2,4,3 (Length 8)	B ··· 2 (Length 1)
C ··· 1,3,1,4,2,6,2,4 (Length 8)	D ··· 2,3 (Length 2) .

It was demonstrated that these are the only four periods among sequences of type (1,1), the argument following along these lines.

(1) Only a finite number of sequences all of whose elements are less than or equal to 6 lead to D.

(2) Any sequence involving an element 1 leads to one of the periods A, B, or C.

(3) For a sequence determined by three elements none of which is 1 and at least one of which is greater than 6, an element 1 will eventually appear in the sequence.

SEQUENCES OF TYPE (3,1)

Using starting numbers up to six, five periods were found:

A: 1,1,1,4,13,43,36,12,2,1,1,2,7,23,38,20,5. (Length 17)
B: 2,2,2,4,7,13,12,7,3,2,2,3,6,11,13,9,4. (Length 17)
C: 3,3,3,4,5,7,7,6,4. (Length 9)
D: 5,4. (Length 2)
E: 4. (Length 1).

We arrived at no proof that these were all the periods in this case.

SEQUENCES OF PERIOD TWO IN (p,p), $(p, p + 1)$, and $(p + 1, p)$

An extensive investigation was made of the number of sequences of period two for these sets of sequences (21 pages). In all these cases it was possible to arrive at a formula of some complexity for determining the number of such sequences.

IMPOSSIBILITY OF VARIOUS PERIODS

Regardless of the values of p and q, it was shown by a detailed consideration of inequalities that it was impossible to have a period of length three. Similarly, it was proved that there are no periods of length four or five. With six, the work became quite complicated and as a result there was no demonstration of the impossibility of this case.

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IDENTITIES RELATING THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS

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1. INTRODUCTION

If $i \geqslant 0$ and $n \geqslant 1$, let $q_i^e(n)$ denote the number of partitions of n into an even number of parts, where each part occurs at most i times and let $q_i^o(n)$ denote the number of partitions of n into an odd number of parts, where each part occurs at most i times. If $i \geqslant 0$, let $q_i^e(0) = 1$ and $q_i^o(0) = 0$. For $i \geqslant 0$ and $n \geqslant 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

For $i = 1$, it is well known [1] that

$$\Delta_1(n) = \begin{cases} (-1)^j & \text{if } n = \tfrac{1}{2}(3j^2 \pm j) \text{ for some } j = 0, 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 3$, Dean R. Hickerson [2] has proved that

$$\Delta_3(n) = \begin{cases} (-1)^n & \text{if } n = \tfrac{1}{2}(j^2 + j) \text{ for some } j = 0, 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

For i an even number, Hickerson [2] has proved that

$$\Delta_i(n) = (-1)^n p_i^d(n),$$

where $p_i^d(n)$ is the number of partitions of n into distinct odd parts which are not divisible by $i + 1$ and $p_i^d(0) = 1$.

In this paper, we obtain formulae for $\Delta_i(n)$ for $i = 5$ and 7 in terms of the number of partitions into distinct parts taken from certain sets. These formulae, like those above, will allow rapid calculation of $\Delta_i(n)$ even for large values of n without the need to determine either $q_i^e(n)$ or $q_i^o(n)$. They will also allow verification of a conjecture by Hickerson [3] that, for $i = 5$ and 7, $\Delta_i(n)$ is nonnegative if n is even and nonpositive if n is odd.

2. THEOREMS

Theorem 1.

$$\Delta_5(n) = (-1)^n \sum_{j=0}^{\infty} q_{3,6}^d(n - (3j^2 \pm 2j)),$$

where $q_{3,6}^d(n)$ denotes the number of partitions of n into distinct parts each of which is congruent to 3 (modulo 6), $q_{3,6}^d(0) = 1$, and where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Proof. The generating function for Δ_i is given by

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n = (1 - x + x^2 - \cdots + (-1)^i x^i)(1 - x^2 + x^4 - \cdots + (-1)^i x^{2i})(1 - x^3 + x^6 + \cdots + (-1)^i x^{3i}) \cdots$$

(1)
$$= \prod_{j=1}^{\infty} (1 - x^j + x^{2j} - \cdots + (-1)^i x^{ij}) = \prod_{j=1}^{\infty} \frac{1 + (-1)^i x^{(i+1)j}}{1 + x^j}.$$

Therefore,

(2) $$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \prod_{j=1}^{\infty} \frac{1-x^{6j}}{1+x^j} = \prod_{j=1}^{\infty} \frac{(1-x^{6j})(1-x^j)}{1-x^{2j}} = \prod_{j=1}^{\infty} (1-x^{6j})(1-x^{2j-1})$$

$$= \prod_{j=0}^{\infty} (1-x^{6j+1})(1-x^{6j+5})(1-x^{6j+6}) \prod_{j=0}^{\infty} (1-x^{6j+3}).$$

Applying Jacobi's identity

(3) $$\prod_{j=0}^{\infty} (1-x^{2kj+k-\ell})(1-x^{2kj+k+\ell})(1-x^{2kj+2k}) = \sum_{j=-\infty}^{\infty} (-1)^j x^{kj^2+\ell j}$$

with $k = 3,\ \ell = 2,$ to the triple product in (2), we obtain

(4) $$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \sum_{j=-\infty}^{\infty} (-1)^j x^{3j^2+2j} \prod_{j=0}^{\infty} (1-x^{6j+3}).$$

Since

$$\prod_{j=0}^{\infty} (1-x^{6j+3}) = \sum_{k=0}^{\infty} (-1)^k q_{3,6}^d(k)x^k ,$$

we can write (3) as

$$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \left(\sum_{j=0}^{\infty} (-1)^j x^{3j^2\pm 2j}\right)\cdot\left(\sum_{k=0}^{\infty} (-1)^k q_{3,6}^d(k)x^k\right)$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty} (-1)^j(-1)^{n-(3j^2\pm 2j)} q_{3,6}^d(n-(3j^2\pm 2j))\right\}x^n$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty} (-1)^{n-(3j^2-j\pm 2j)} q_{3,6}^d(n-(3j^2\pm 2j))\right\}x^n .$$

But $3j^2 - j \pm 2j \equiv 0 \pmod 2$. Hence

$$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty} (-1)^n q_{3,6}^d(n-(3j^2\pm 2j))\right\}x^n .$$

Equating coefficients on both sides, we obtain the theorem.

To illustrate that Theorem 1 allows very rapid calculation of $\Delta_5(n)$, we consider the case $n = 20$, for which we have

$$\Delta_5(20) = \left(\sum_{j=0}^{\infty} q_{3,6}^d(20-(3j^2\pm 2j)\right) = q_{3,6}^d(15)+q_{3,6}^d(12) = 2 ,$$

all other terms in the sum being 0. This checks with

$$q_5^e(20) - q_5^o(20) = 236-234 = 2 ,$$

obtained by computer.

Theorem 2.

$$\Delta_7(n) = (-1)^n \sum_{j=0}^{\infty} q_4^d(n-(2j^2\pm j)) ,$$

where $q_4^d(n)$ denotes the number of partitions of n into distinct parts, each of which is divisible by 4, $q_4^d(0) = 1$, and where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Proof. Using (1), we have

$$\sum_{n=0}^{\infty} \Delta_7(n)x^n = \prod_{j=1}^{\infty} \frac{1-x^{8j}}{1+x^j} = \prod_{j=1}^{\infty} \frac{1-x^{4j}}{1+x^j}(1+x^{4j})$$

(5)

$$= \prod_{j=0}^{\infty} (1-x^{4j+1})(1-x^{4j+3})(1-x^{4j+4}) \prod_{j=0}^{\infty} (1+x^{4j+4}) .$$

Applying Jacobi's identity (3) with $k = 2,\ \ell = 1,$ to the triple product in (5), we obtain

$$\sum_{n=0}^{\infty} \Delta_7(n)x^n = \sum_{j=-\infty}^{\infty} (-1)^j x^{2j^2+j} \prod_{j=0}^{\infty} (1+x^{4j+4}) = \left(\sum_{j=0}^{\infty} (-1)^j x^{2j^2 \pm j}\right)\left(\sum_{k=0}^{\infty} q_4^d(k)x^k\right)$$

(6)

$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (-1)^j q_4^d(n-(2j^2 \pm j)) \right\} x^n .$$

Equating coefficients on both sides, we obtain

$$\Delta_7(n) = \sum_{j=0}^{\infty} (-1)^j q_4^d(n-(2j^2 \pm j)) .$$

Now for $n \equiv a \pmod 4,\ 0 \leqslant a \leqslant 3,$ and observing that $q_4^d(n) = 0$ unless n is divisible by 4, we have

$$\Delta_7(n) = \sum_{\substack{j \leqslant 0 \\ 2j^2 \pm j \equiv a \pmod 4}} (-1)^j q_4^d(n-(2j^2 \pm j))$$

$$= (-1)^a \sum_{\substack{j \geqslant 0 \\ 2j^2 \pm j \equiv a \pmod 4}} q_4^d(n-(2j^2 \pm j)) = (-1)^n \sum_{j=0}^{\infty} q_4^d(n-(2j^2 \pm j)) .$$

The formulae of Theorems 1 and 2 show that $\Delta_i(n)$ for i = 5 and 7 is nonnegative if n is even and nonpositive if n is odd.

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FIBONACCI AND RELATED SEQUENCES IN PERIODIC TRIDIAGONAL MATRICES

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1. INTRODUCTION

Tridiagonal matrices are matrices like

$$(1)\qquad \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{bmatrix}$$

and are made up of three diagonal sequences $\{a_i\}, \{b_i\}, \{c_i\}$ of real or complex numbers. They are of much use in the numerical analysis of matrices. They also have interesting arithmetical properties being connected with the theories of continued fractions, recurring sequences of the second order, and, in special cases, permutations, graph theory, and partitions. We shall be considering two functions of such matrices, the determinant and the permanent.

By the permanent of the matrix

$$A = \{a_{ij}\}_{n\times n}$$

is meant the sum

$$\operatorname{per} A = \sum_{(\pi)} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$$

extending over all permutations

$$\pi : \begin{pmatrix} 1, & 2, \cdots, & n \\ \pi(1), & \pi(2), \cdots, & \pi(n) \end{pmatrix} .$$

Thus the definition of the permanent is simpler than the corresponding definition of the determinant in that no distinction is made between odd and even permutations. In spite of this apparent simplicity, permanents are usually much more difficult than determinants in their computation and manipulation. For tridiagonal matrices, however, determinants and permanents are not very different. In fact we see that

$$\operatorname{per}\begin{bmatrix} b_1 & c_1 \\ a_2 & b_2 \end{bmatrix} = b_1 b_2 + a_2 c_1$$

and

$$\operatorname{per}\begin{bmatrix} b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 \end{bmatrix} = b_1 b_2 b_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

and, in general, the permanent of the tridiagonal matrix based on $\{a_i\}, \{b_i\}, \{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}, \{b_i\}, \{c_i\}$. Thus it is sufficient and simpler to consider the permanent function of tridiagonal matrices. In fact we shall need only the method of expansion by minors in developing what follows.

2. STANDARDIZATION OF TRIDIAGONAL MATRICES

For our present purposes we make the assumption that the elements b on the main diagonal are all different from zero. It is therefore possible to divide the elements in each row by its main diagonal element. Thus we obtain a matrix of the form

(2)
$$\begin{bmatrix} 1 & C_1 & 0 & 0 & 0 & 0 \\ A_2 & 1 & C_2 & 0 & 0 & 0 \\ 0 & A_3 & 1 & C_3 & 0 & 0 \\ 0 & 0 & A_4 & 1 & C_4 & 0 \\ 0 & 0 & 0 & A_5 & 1 & C_5 \\ 0 & 0 & 0 & 0 & A_6 & 1 \end{bmatrix}$$

whose permanent (or determinant) is related to that of the original matrix (1) by the factor $b_1b_2 \cdots b_6$. Our next step towards standardization is to observe that the permanent of (2) is not a function of A_2 *and* C_1 but only of their product A_2C_1. To see this, we expand the permanent by minors in the first column obtaining

$$\text{per}\begin{bmatrix} 1 & C_2 & 0 & 0 & 0 \\ A_3 & 1 & C_3 & 0 & 0 \\ 0 & A_4 & 1 & C_4 & 0 \\ 0 & 0 & A_5 & 1 & C_5 \\ 0 & 0 & 0 & A_6 & 1 \end{bmatrix} + A_2C_1\,\text{per}\begin{bmatrix} 1 & C_3 & 0 & 0 \\ A_4 & 1 & C_4 & 0 \\ 0 & A_5 & 1 & C_5 \\ 0 & 0 & A_6 & 1 \end{bmatrix}$$

which is a function of A_2C_1. By induction, therefore, the permanent of such a matrix as (2) will depend only on

$$A_2C_1,\ A_3C_2,\ \cdots,\ A_nC_{n-1}\ .$$

Hence, without loss of generality, we may assume that the C's are all equal to 1 and by an obvious change in notation define the standard tridiagonal matrix by

$$M = M_n = M_n(a_1, a_2, \cdots, a_{n-1}) = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \end{bmatrix}.$$

We denote the permanent of this matrix M by

$$\Delta = \Delta_n = \Delta_n(a_1, a_2, \cdots, a_{n-1}) = \text{per}\, M_n(a_1, a_2, \cdots, a_{n-1}).$$

We also adopt the conventions

(3) $$\Delta_0 = 1 \quad \text{and} \quad \Delta_{-1} = 0\ .$$

3. BASIC PROPERTIES

We begin with the basic recurrence for Δ_n.

Theorem 1. If $n \geqslant 1$,

$$\Delta_n(a_1, \cdots, a_{n-1}) = \Delta_{n-1}(a_1, \cdots, a_{n-2}) + a_{n-1}\Delta_{n-2}(a_1, \cdots, a_{n-3}).$$

Proof. This follows at once by expanding Δ_n by minors of the elements of the last column of $M_n(a_1, \cdots, a_{n-1})$. This recurrence is an efficient way of calculating successive Δ's when the a's are given. It is clear from (4) that Δ_n is linear in each of its independent variables $a_1, \cdots, a_{n-1}$. For future use we give Table 1 of Δ_n. We observe from this table that Δ_n is unaltered when its arguments are reversed. In general we have

Theorem 2. $$\Delta_n(a_1, a_2, \cdots, a_{n-1}) = \Delta_n(a_{n-1}, a_{n-2}, \cdots, a_1).$$

Proof. The theorem holds trivially for $n = 0, 1, 2$. If true for $n - 1$ amd $n - 2$, (4) becomes

$$\Delta_n(a_1, a_2, \cdots, a_{n-1}) = \Delta_{n-1}(a_{n-2}, \cdots, a_1) + a_{n-1}\Delta_{n-2}(a_{n-3}, \cdots, a_1).$$

But the right-hand side is the result of expanding the permanent of $M_n(a_{n-1}, a_{n-2}, \cdots, a_1)$ by minors of elements of its first row. Hence the theorem is true for n and the induction is complete.

Table 1

n	$\Delta_n(a_1, a_2, \cdots, a_{n-1})$
-1	0
0	1
1	1
2	$1 + a_1$
3	$1 + a_1 + a_2$
4	$1 + a_1 + a_2 + a_3 + a_1a_3$
5	$1 + a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4 + a_1a_4$
6	$1 + \sum_{i=1}^{5} a_i + a_1a_3 + a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_5 + a_1a_3a_5$

Since Δ_n is linear in each variable a_j one can ask what are the functions G_j and H_j in

(5) $$\Delta_n(a_1, \cdots, a_{n-1}) = G_j + H_ja_j \qquad (1 \leqslant j < n).$$

It is clear from (4) that when $j = n - 1$

$$G_{n-1} = \Delta_{n-1}(a_1, \cdots, a_{n-2}), \qquad H_{n-1} = \Delta_{n-2}(a_1, \cdots, a_{n-3}).$$

The general theorem is

Theorem 3. In (5),

$$G_j = \Delta_{n-j}(a_{j+1}, \cdots, a_{n-1})\Delta_j(a_1, \cdots, a_{j-1})$$
$$H_j = \Delta_{n-j-1}(a_{j+1}, \cdots, a_{n-1})\Delta_{j-1}(a_1, \cdots, a_{j-2}) .$$

Proof. This can be proved by expanding Δ_n by minors of the elements of its j^{th} column and using Laplacian development of these minors. However, a simpler proof is afforded by the introduction of the following generalized permanents $\Delta_{K,r}$ defined for $K \leqslant r$ by

(6) $$\Delta_{K,r} = \Delta_{K,r}(a_1, a_2, \cdots) = \Delta_K(a_{r-K+1}, a_{r-K+2}, \cdots, a_{r-1}) = \Delta_K(a_{r-1}, a_{r-2}, \cdots, a_{r-K+1}).$$

In particular we have

$$\Delta_{K,K} = \Delta_K(a_1, a_2, \cdots, a_{K-1}).$$

Theorem 1 applied to these two equivalent definitions gives us the following useful relations.

(7) $$\Delta_{K,r} = \Delta_{K-1,r} + a_{r-K+1}\Delta_{K-2,r}$$

(8) $$\Delta_{K,r} = \Delta_{K-1,r-1} + a_{r-1}\Delta_{K-2,r-2}.$$

We claim now that for $0 \leqslant K < n$

(9) $$\Delta_n = \Delta_{K,n}\Delta_{n-K} + a_{n-K}\Delta_{K-1,n}\Delta_{n-K-1} .$$

In fact this is trivial when $K = 0$ by (3) and (6) and when $K = 1$ it is a restatement of Theorem 1. To proceed inductively for K to $K + 1$ we note that

$$\Delta_{n-K} = \Delta_{n-(K+1)} + a_{n-(K+1)}\Delta_{n-1-(K+1)}$$

by Theorem 1. Substituting this into our induction hypothesis (9) we obtain

$$\Delta_n = \Delta_{n-(K+1)}\left\{\Delta_{K,n} + a_{n-K}\Delta_{K-1,n}\right\} + a_{n-(K+1)}\Delta_{K,n}\Delta_{n-1-(K+1)} .$$

But by (7) the quantity in the braces in $\Delta_{K+1,n}$. Hence our induction is complete. If now we put $K = n - j$ and $r = n$ in (6) and (9) the theorem follows.

As a corollary we have

$$\frac{\partial \Delta_n(a_1, a_2, \cdots, a_{n-1})}{\partial a_j} = \Delta_{j-1}(a_1, a_2, \cdots, a_{j-2})\Delta_{n-j-1}(a_{j+2}, \cdots, a_{n-1}).$$

4. CONNECTION WITH CONTINUED FRACTIONS

The ratio of two Δ's is the convergent of a continued fraction. More precisely we have

Theorem 4.

$$1 + \frac{a_1|}{|1} + \frac{a_2|}{|1} + \cdots + \frac{a_{n-1}|}{|1} = \frac{\Delta_n(a_1, a_2, \cdots, a_{n-1})}{\Delta_{n-1}(a_2, a_3, \cdots, a_{n-1})} .$$

Proof. By Theorems 2 and 1 we may write

$$\frac{\Delta_n(a_1, \cdots, a_{n-1})}{\Delta_{n-1}(a_2, \cdots, a_{n-1})} = \frac{\Delta_n(a_{n-1}, \cdots, a_1)}{\Delta_{n-1}(a_{n-1}, \cdots, a_2)}$$

$$= \frac{\Delta_{n-1}(a_{n-1}, \cdots, a_2) + a_1\Delta_{n-2}(a_{n-1}, \cdots, a_3)}{\Delta_{n-1}(a_{n-1}, \cdots, a_2)} = 1 + \frac{a_1}{\Delta_{n-1}/\Delta_{n-2}} .$$

Iterating this identity until we reach $\Delta_1/\Delta_0 = 1$, we obtain the theorem.

As an example, in case all the a's are equal to 1 we get the Fibonacci irrational

$$\theta = \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{1|}{|1} + \frac{1|}{|1} + \cdots$$

whose successive convergents

$$1, 2/1, 3/2, 5/3, 8/5, \cdots$$

are the ratios of consecutive Fibonacci numbers F_{n+1}/F_n. Hence

(10) $$\Delta_n(1, 1, 1, \cdots, 1) = F_{n+1}$$

a fact which follows at once from (4). Conversely as soon as we have developed other formulas like (10) we can evaluate other continued fractions of Ramanujan type given in Theorem 4.

5. PERMANENTS WITH PERIODIC ELEMENTS

We are now prepared to consider the case in which the elements a of Δ are periodic of period p so that $a_{j+p} = a_j$. We shall find that the permanents

$$\Delta_s, \Delta_{s+p}, \Delta_{s+2p}, \cdots$$

constitute in this case a recurring series of the second order with constant coefficients depending only on p and the values of $a_1, a_2, \cdots, a_p$ but not depending on s. From this it will follow that Δ_n is a linear combination of two Lucas functions U_h and U_{h+1}, where $h = [n/p]$ whose coefficients now depend on $s = n - hp$. More precisely

$$U_h = U_h(P,Q) = (a^h - b^h)/(a-b),$$

where

$$P = a+b, \quad Q = ab$$

and

(11) $$U_0 = 0, \quad U_1 = 1, \quad U_2 = P$$

and

(12) $$U_h = PU_{h-1} - QU_h .$$

We denote the $n \times n$ permanent based on the periodic a's by

$$\Delta_n(\dot{a}_1, a_2, \cdots, \dot{a}_p)$$

so that (10) becomes

$$\Delta_n(\dot{1}) = F_{n+1} .$$

6. THE CASE $p = 1$

In this simple case we have

Theorem 5.

(13) $$\Delta_n(\dot{a}_1) = U_{n+1}(1, -a_1).$$

Proof. By (12),

$$U_{n+1}(1, -a_1) = U_n(1, -a_1) + a_1 U_{n-1}(1, -a_1).$$

But by (4),

$$\Delta_n(\dot{a}_1) = \Delta_{n-1}(\dot{a}_1) + a_1 \Delta_{n-2}(\dot{a}_1)$$

since $a_{n-1} = a_1$ for all n.

Hence both Δ_n and U_{n+1} satisfy the same recurrence. They also have the same starting values for $n = -1$ and $n = 0$. Hence the two functions coincide.

Corollary. $\Delta_{n-1}(\dot{a}) = \left\{(1+\sqrt{1+4a})^n - (1-\sqrt{1+4a})^n\right\} / (2^n\sqrt{1+4a})$.

Proof. Referring to (13) we see that a and b are roots of $x^2 - x - a = 0$. Examples of the Corollary are

$$\Delta_{n-1}(\dot{0}) = 1$$

$$\Delta_{n-1}(-\dot{1}) = \frac{\sqrt{12}}{3} \sin(\pi n/3)$$

$$\Delta_{n-1}(\dot{2}) = \left\{2^n - (-1)^n\right\}/3.$$

This last example leads, via Theorem 4, to

$$1 + \frac{2|}{|1} + \frac{2|}{|1} + \frac{2|}{|1} + \cdots = 2$$

as is easily verified.

7. THE CASE $p = 2$

This case is also relatively simple. We have

Theorem 6. $\Delta_n(\dot{a}_1, \dot{a}_2) = (1 + a_1 + a_2)\Delta_{n-2}(\dot{a}_1, \dot{a}_2) - a_1 a_2 \Delta_{n-4}(\dot{a}_1, \dot{a}_2)$.

Proof. First suppose n is odd so that $a_{n-1} = a_2$. Then Theorem 1 gives

$$\Delta_n = \Delta_{n-1} + a_2\Delta_{n-2} = \Delta_{n-2} + a_1\Delta_{n-3} + a_2\Delta_{n-2}.$$

But $$\Delta_{n-3} = \Delta_{n-2} - a_2\Delta_{n-4}.$$

Elimination of Δ_{n-3} gives the theorem for n odd. If n is even, we simply interchange the roles of a_1 and a_2.

The counterpart of Theorem 5 for $p = 2$ is

Theorem 7. $$\Delta_{2n}(\dot{a}_1, \dot{a}_2) = U_{n+1}(1 + a_1 + a_2, a_1 a_2) - a_2 U_n(1 + a_1 + a_2, a_1 a_2)$$
$$\Delta_{2n+1}(\dot{a}_1 \dot{a}_2) = U_{n+1}(1 + a_1 + a_2, a_1 a_2).$$

Proof. Let $W_n = \Delta_{2n}(\dot{a}_1, \dot{a}_2)$. By Theorem 6

$$W_n = (1 + a_1 + a_2)W_{n-1} - a_1 a_2 W_{n-2}$$

with

$$W_0 = 1, \quad W_1 = \Delta_2(a_1, a_2) = 1 + a_1.$$

But

$$U_{n+1}(1 + a_1 + a_2, a_1 a_2) - a_2 U_n(1 + a_1 + a_2, a_1 a_2)$$

enjoys the same recurrence and the same initial conditions. This proves the first part of the Theorem. The second part is proved in the same way.

We note that, unlike $\Delta_{2n}(a_1, a_2)$, the function $\Delta_{2n+1}(\dot{a}_1, \dot{a}_2)$ is symmetric in a_1 and a_2.

Examples of Theorem 7 are

$$\Delta_{2n+1}(\dot{0}, \dot{1}) = 2^n, \quad \Delta_{2n}(\dot{0}, \dot{1}) = 2^{n-1}, \quad \Delta_{2n}(\dot{1}, \dot{0}) = 2^n$$
$$\Delta_{2n+1}(\dot{1}, -\dot{1}) = F_{n+1}, \quad \Delta_{2n}(\dot{1}, -\dot{1}) = F_{n+2}, \quad \Delta_{2n}(-\dot{1}, \dot{1}) = F_{n-1}$$
$$\Delta_{2n+1}(\dot{\omega}, \dot{\omega}^2) = \tfrac{1}{2}i^n(1+(-1)^n), \quad \Delta_{4n}(\dot{\omega}, \dot{\omega}^2) = (-1)^n$$
$$\Delta_{2n+1}(-\dot{\omega}, \dot{\omega}^2) = n+1, \quad \Delta_{2n}(-\dot{\omega}, -\dot{\omega}^2) = 1 - n\omega$$
$$\Delta_{2n-1}(i, -i) = \frac{\sqrt{12}}{3}\sin(\pi n/3)$$
$$\Delta_{2n-1}(\dot{1}, \dot{2}) = \frac{(2+\sqrt{2})^n - (2-\sqrt{2})^n}{2\sqrt{2}}, \quad \Delta_{2n}(\dot{1}, \dot{2}) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2}$$

Here

$$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{3}\,i}{2} \; .$$

The last two results easily lead to

$$1 + \frac{1|}{|1} + \frac{2|}{|1} + \frac{1|}{|1} + \frac{2|}{|1} + \cdots = \sqrt{2}\;.$$

Inspection of the above examples shows them to behave exponentially, linearly or periodically as $n \to \infty$. This is a general fact, true of periodic a's of any period length p.

8. THE GENERAL PERIODIC CASE

We now take up the complicated general case of $p \geqslant 3$, although the theorems we are about to obtain hold for $p = 1$ and 2. For this purpose we enlarge the definition (6) of $\Delta_{K,r}$ to include the cases $K > r$. That is, we define for the periodic case

$$\Delta_{K,r}(\dot{a}_1, a_2, \cdots, \dot{a}_p) = \Delta_K(a_{r-K+1}, a_{r-K+2}, \cdots, a_{r-1}),$$

where the subscripts of the a's are to be interpreted modulo p. Thus if $p = 4$,

$$\Delta_{5,2}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_5(a_{-2}, a_{-1}, a_0, a_1) = \Delta_5(a_2, a_3, a_4, a_1)$$
$$\Delta_{4,1}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_4(a_{-2}, a_{-1}, a_0) = \Delta_4(a_2, a_3, a_4)$$
$$\Delta_{3,0}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_3(a_2, a_3)\;.$$

It is easily verified that

$$\Delta_{5,2}(\dot{a}_1, a_2, a_3, \dot{a}_4) = \Delta_{4,1} + a_1\Delta_{3,0}$$

which for $K = 5$ and $r = 2$ is a particular case of (7). Formulas (7) and (8) are still true in general by Theorem 1.

Theorem 8. For $0 \leqslant s < p$ let

$$A(p,s) = \Delta_{p,s} + a_s\Delta_{p-2,s-1}$$
$$B(p,s) = a_s(\Delta_{p,s}\Delta_{p-2s-1} - \Delta_{p-1,s}\Delta_{p-1,s-1}).$$

Then if $n \equiv s \pmod p$,

$$\Delta_{n+p} = A(p,s)\Delta_n - B(p,s)\Delta_{n-p}\;,$$

where the argument in all the Δ's is $(\dot{a}_1, a_2, \cdots, \dot{a}_p)$.

Proof. Let $n = ph + s$. If in (9) we set $K = p$ and use the fact that $a_{n+i} = a_{s+i}$ we get

(14) $$\Delta_{ph+s} = \Delta_{p,s}\Delta_{p(h-1)+s} + a_s\Delta_{p-1,s}\Delta_{p(h-1)+s-1}\;.$$

In the same way replacing n by $n - p$ and setting $K = p - 1$ we have

(15) $$\Delta_{p(h-1)+s-1} = \Delta_{p-1,s-1}\Delta_{p(h-2)+s} + a_s\Delta_{p-2,s-1}\Delta_{p(h-2)+s-1}\;.$$

Beginning with (14) and continually applying (15) gives the following for Δ_n

$$\Delta_{ph+s} = \Delta_{p,s}\Delta_{p(h-1)+s} + \Delta_{p-1,s}\Delta_{p-1,s-1}\sum_{\mu=1}^{h-1} a_s^{\mu}\left\{\Delta_{p-2,s-1}\right\}^{\mu-1}\Delta_{p(h-\mu-1)+s} + \Delta_{p-1,s}a_s^h\left\{\Delta_{p-2,s-1}\right\}^{h-1}\Delta_{s-1}\,.$$

(16) $$\Delta_{p-1,s}\Delta_{p-1,s-1}\sum_{\mu=1}^{h-1} a^{\mu}\left\{\Delta_{p-2,s-1}\right\}^{\mu-1}\Delta_{p(h-\mu-1)+s}$$
$$= \Delta_{ph+s} - \Delta_{ps}\Delta_{p(h-1)+s} - a_s^h\Delta_{p-1,s}\Delta_{s-1}\left\{\Delta_{p-2,s-1}\right\}^{h-1}.$$

Next we multiply both sides of (16) by $a_s\Delta_{p-2,s-1}$ and add

$$a_s\Delta_{p-1,s}\Delta_{p-1,s-1}\Delta_{p(h-1)+s}$$

to both sides. If we subtract this result from (16) when h is replaced by $h+1$ we get

$$\Delta_{p(h+1)+s} - \Delta_{p,s}\Delta_{ph+s} = a_s\left\{\Delta_{p-2,s-1}\Delta_{ph+s} - \Delta_{p,s}\Delta_{p-2,s-1}\Delta_{p(h-1)+s} + \Delta_{p-1,s-1}\Delta_{p-1,s}\Delta_{p(h-1)+s}\right\}.$$

Collecting the coefficients of Δ_{ph+s} and $\Delta_{p(h-1)+s}$ gives us the theorem.

Our next goal is to show that $A(p,s)$ and $B(p,s)$ depend on p but not on s.

Theorem 9. $$B(p,s) = (-1)^p a_1a_2\cdots a_p.$$

Proof. It will suffice to show that

(17) $$\Delta_{p,s}\Delta_{p-2,s-1} - \Delta_{p-1}\Delta_{p-1,s-1} = (-1)^p a_{s-1}a_{s-2}\cdots a_{s-p+1},$$

where the subscripts on the a's are to be taken modulo p, because then, by definition of $B(p,s)$ we

$$B(p,s) = (-1)^p a_sa_{s-1}\cdots a_{s-p+1} = (-1)^p a_1a_2\cdots a_p.$$

To prove (17) we note that it holds for $p = 1$ since the left member is -1 and the product of a's is vacuous. Assuming the result holds for p and noting that (7) gives

$$\Delta_{p+1,3} = \Delta_{p,s} + a_{s-p}\Delta_{p-1,s}$$

and

$$\Delta_{p,s-1} - \Delta_{p-1,s-1} = a_{s-p}\Delta_{p,s}\Delta_{p-2,s-1}.$$

We have

$$\begin{aligned}\Delta_{p+1,s}\Delta_{p-1,s-1} - \Delta_{p,s}\Delta_{p,s-1} &= -\Delta_{p,s}[\Delta_{p,s-1} - \Delta_{p-1,s-1}] \\ &\quad + a_{s-p}\Delta_{p-1,s}\Delta_{p-1,s-1} \\ &= -a_{s-p}[\Delta_{p,s}\Delta_{p-2,s-1} - \Delta_{p-1,s}\Delta_{p-1,s-1}] \\ &= (-1)^{p+1}a_{s-1}a_{s-2}\cdots a_{s-p+1}a_{s-p}.\end{aligned}$$

Hence (17) holds for $p+1$ and the induction is complete.

Theorem 10. $A(p,s)$ is not a function of s.

Proof. Using both (7) and (8) with $k = p$ and $r = s$ and $s = 1$ we have

$$\begin{aligned}A(p,s) &= \Delta_{p,s} + a_s\Delta_{p-2,s-1} = a_s\Delta_{p-2,s-1} + \Delta_{p-1,s-1} + a_{s-1}\Delta_{p-2,s-2} \\ &= a_{s-1}\Delta_{p-2,s-2} + \Delta_{p,s-1} = A(p,s-1).\end{aligned}$$

Hence $A(p,s)$ does not depend on s.

We can write

(18) $$A(p,s) = A(p,p) = P_p = P = \Delta_p(a_1,\cdots,a_{p-1}) + a_p\Delta_{p-2}(a_2,\cdots,a_{p-2})$$

and

(19) $$Q_p = Q = (-1)^p a_1a_2\cdots a_p$$

and restate Theorem as follows

Theorem 11. $$\Delta_{n+p} = P\Delta_n - Q\Delta_{n-p}.$$

Armed with this information we can at once evaluate $\Delta_n(a_1,\cdots,a_p)$ as a linear combination of two consecutive members of the Lucas sequence $\left\{U_m(P,Q)\right\}$ as follows.

Theorem 12.

(20) $$\Delta_{hp+s} = \Delta_sU_{h+1}(P,Q) + (\Delta_{p+s} - P\Delta_s)U_h(P,Q).$$

Proof. This relation holds for $h = 0$ and, since $U_2(P,Q) = P$ for $h = 1$. By Theorems 11 and 12 both sides enjoy the same recurrence. Hence they coincide.

9. MORE ON THE FUNCTION P

The function

$$P = P_p(a_1, a_2, \cdots, a_p)$$

defined by (18) is not as simple as Q. We already know that

$$P_1 = 1 \quad \text{and} \quad P_2 = 1 + a_1 + a_2 .$$

We can tabulate P_p as follows

Table 2

p	$P_p(a_1, a_2, \cdots, a_p)$
1	1
2	$1 + a_1 + a_2$
3	$1 + a_1 + a_2 + a_3$
4	$1 + a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4$
5	$1 + \sum_{i=1}^{5} a_i + \sum_{i=1}^{3} a_ia_{i+2} + \sum_{i=1}^{2} a_ia_{i+3}$
6	$1 + \sum_{i=1}^{6} a_i + \sum_{i<j\leqslant 6} a_ia_j - \sum_{i=1}^{5} a_ia_{i+1} + \sum_{i=1}^{2} a_ia_{i+2}a_{i+4}$

Further entries in this table are left to the curiosity of the reader. It will be observed that the entries cease to be symmetric functions of the a's with $p = 4$.

10. FIBONACCI-TYPE Δ'S

The permanent of a tridiagonal matrix with periodic a's will depend on Fibonacci numbers if we can make $P = 1$ and $Q = -1$ since

$$U_m(1, -1) = F_m .$$

For $p = 3$ this requires

$$P_3 = 1 + a_1 + a_2 + a_3 = 1, \qquad -Q_3 = a_1a_2a_3 = 1 .$$

This means that the three a's are the roots any cubic equation of the form

$$x^3 + cx - 1 = 0. \tag{21}$$

The simplest example is $c = 0$ for which

$$a_1 = 1, \quad a_2 = \omega, \quad a_3 = \omega^2$$

or some other permutation of these. For this case Theorem 12 gives the examples

$$\Delta_{3h}(\dot{1}, \omega, \dot{\omega}^2) = F_{h+1} - \omega^2 F_n$$

$$\Delta_{3h+1}(\dot{1}, \omega, \dot{\omega}^2) = F_{h+1} + \omega^2 F_n$$

$$\Delta_{3h+2}(\dot{1}, \omega, \dot{\omega}^2) = 2F_{h+1} .$$

Another special case is that of $c = -2$ in which the roots of (21) are -1 and the two Fibonacci irrationals, for example

$$a_1 = \theta, \qquad a_2 = \overline{\theta}, \qquad a_3 = -1.$$

For this choice we get

$$\Delta_{3h}(\dot{\theta}, \bar{\theta}, -\dot{1}) = F_{h+2}$$
$$\Delta_{3h+1}(\dot{\theta}, \bar{\theta}, -\dot{1}) = F_{h+1} - \theta F_h$$
$$\Delta_{3h+2}(\dot{\theta}, \bar{\theta}, -\dot{1}) = (1+\theta)F_{h+1}\,.$$

The reader may wish to write such formulas for other permutations of $\theta, \bar{\theta}, -1$.

For $p = 4$ our requirement becomes

(22) $$a_1 + a_2 + a_3 + a_4 + a_1a_3 + a_2a_4 = 0, \qquad a_1a_2a_3a_4 = -1.$$

Examples are

$$a_1 = \mathrm{i}, \quad a_2 = -1, \quad a_3 = -i, \quad a_4 = 1, \quad a_1 = \omega, \quad a_2 = \theta, \quad a_3 = \omega^2, \quad a_4 = \bar{\theta}\,.$$

More general examples are

$$a_1 = \tfrac{1}{2}(-t + \sqrt{t^2 - 4\epsilon}), \qquad a_2 = \tfrac{1}{2}(t + \sqrt{t^2 + 4\epsilon}),$$
$$a_3 = \tfrac{1}{2}(-t - \sqrt{t^2 - 4\epsilon}), \qquad a_4 = \tfrac{1}{2}(t - \sqrt{t^2 + 4\epsilon}),$$

where t is any real or complex parameter and $\epsilon^2 = 1$. In any case there are eight permutations of the four a's that maintain (22). These are, in cycle notation

$$(1)\,(2)\,(3)\,(4), \quad (1)\,(3)\,(24), \quad (13)\,(2)\,(4), \quad (13)\,(24)\,(12)\,(34), \quad (14)\,(23), \quad (1234), \quad (1432).$$

With any one of these choices we have for $\Delta_n = \Delta_n(\dot{a}_1, a_2, a_3, \dot{a}_4)$

$$\Delta_{4h} = F_{h+1} - a_4(1+a_2)F_h$$
$$\Delta_{4h+1} = F_{h+1} - a_1a_4F_h$$
$$\Delta_{4h+2} = (1+a_1)F_{h+1} - a_1a_2a_4F_h$$
$$\Delta_{4h+3} = (1+a_1+a_2)F_{h+1}\,.$$

Instead of forcing Δ_n to involve the Fibonacci numbers we can make it a linear function of n by choosing $P = 2$ and $Q = 1$ because $U_n(2,1) = n$.

For $p = 3$ the conditions become

(23) $$a_1 + a_2 + a_3 = 1, \qquad a_1a_2a_3 = -1.$$

One obvious solution is to choose two of the a's equal to 1 and the third -1. Thus we find

$$\Delta_{3h}(\dot{1}, 1, -\dot{1}) = 2h+1, \quad \Delta_{3h+1}(\dot{1}, 1, -\dot{1}) = 1, \quad \Delta_{3h+2}(\dot{1}, 1, -\dot{1}) = 2h+2, \quad \Delta_{3h}(\dot{1}, -1, \dot{1}) = 1,$$
$$\Delta_{3h+1}(\dot{1}, -1, \dot{1}) = 2h+1, \quad \Delta_{3h+2}(\dot{1}, -1, \dot{1}) = 2h+2, \quad \Delta_{3h}(-\dot{1}, 1, \dot{1}) = 1, \quad \Delta_{3h+1}(-\dot{1}, 1, \dot{1}) = 1, \quad \Delta_{3h+2}(-\dot{1}, 1, \dot{1}) = 0.$$

Another choice of a's satisfying (23) is any permutation of

$$-2\cos(2\pi/7), \quad -2\cos(4\pi/7), \quad -2\cos(6\pi/7).$$

The most general solutions of (23) are of course the roots of

$$x^3 - x^2 + cx + 1 = 0$$

and this leads to the linear function

$$\Delta_{3h+s} = \Delta_s + (\Delta_{s+3} - \Delta_s)h\,.$$

The reader may have observed in the above that, of all the formulas for Δ_{hp+s}, the simplest is that for $s = p-1$. The reason for this phenomenon is to be seen by substituting $s = -1$ in Theorem 12. We obtain simply

$$\Delta_{hp-1} = \Delta_{p-1}U_p(P,Q)\,.$$

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A MAXIMUM VALUE FOR THE RANK OF APPARITION OF INTEGERS IN RECURSIVE SEQUENCES

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We define the sequence $R_0, R_1, R_2, \cdots$ by the recursive relation

$$R_{n+1} = aR_n + bR_{n-1}$$

in which $b = 1$ or -1; a and the discriminant $\Delta = a^2 + 4b$ are positive integers. In addition, we have the initial conditions $R_0 = 0$ and R_1 may be any positive integer. We now state the following:

Theorem. The rank of apparition of an integer M in the sequence $R_0, R_1, R_2, \cdots$ does not exceed $2M$.

Proof. First we observe that R_1 divides all terms of the sequence. If the theorem holds for the sequence

$$0 = \frac{R_0}{R_1}, \quad 1 = \frac{R_1}{R_1}, \frac{R_2}{R_1}, \cdots$$

then it apparently holds for the sequence $R_0, R_1, R_2, \cdots$. Therefore we may suppose in what follows, that $R_1 = 1$.

Let M be a positive integer

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad .$$

Here $p_1, p_2, \cdots, p_k$ denote the different primes of M and $\alpha_1, \alpha_2, \cdots, \alpha_k$ their powers. To each p_i $(i = 1, 2, \cdots, k)$ we assign a number s_i:

$$s_i = p_i \pm 1 \text{ if } p_i \text{ is odd and } p_i \nmid \Delta;$$

the minus sign is to be taken if Δ is a quadratic residue of p_i and plus sign if it is a nonresidue

$$s_i = p_i \text{ if } p_i \text{ is odd and } p_i \mid \Delta.$$

$$s_i = 3 \text{ if } p_i = 2 \text{ and } \Delta \text{ odd.}$$

$$s_i = 2 \text{ if } p_i = 2 \text{ and } \Delta \text{ even.}$$

Let m be any common multiple of the numbers $s_1 p_1^{\alpha_1 - 1}, s_2 p_2^{\alpha_2 - 1}, \cdots, s_k p_k^{\alpha_k - 1}$ then $M \mid R_m$. In the case that m constitutes the least common multiple of the mentioned numbers, the proof can be found in Carmichael [1]. From the known property $R_q \mid R_{nq}$, n and q denote positive integers, it appears that m may be any common multiple (the property $R_q \mid R_{nq}$ can be found in Bachman [2]).

Now suppose that M contains only odd primes $p_1, p_2, \cdots, p_k$ with $p_1 \nmid \Delta, p_2 \nmid \Delta, \cdots, p_k \nmid \Delta$, then it is not difficult to verify that the product

$$m = 2\,\frac{s_1 p_1^{\alpha_1 - 1}}{2}\,\frac{s_2 p_2^{\alpha_2 - 1}}{2} \cdots \frac{s_k p_k^{\alpha_k - 1}}{2} \tag{1}$$

is a common multiple of the numbers $s_1 p_1^{\alpha_1 - 1}, \cdots, s_k p_k^{\alpha_k - 1}$ and therefore $M \mid R_m$. It is easy to verify that

$$\frac{m}{M} \leqslant \frac{4}{3} \quad .$$

The extension is easily made to the case where M contains also odd primes $q_1, q_2, \cdots, q_\ell$ with $q_1 \mid \Delta, \cdots, q_\ell \mid \Delta$ and /or to the case where M is even.

In the first case we form a common multiple by multiplying (1) with $q_1^{\beta_1} q_2^{\beta_2} \cdots q_\ell^{\beta_\ell}$ (the numbers $\beta_1, \cdots, \beta_\ell$ constitute the powers of $q_1, \cdots, q_\ell$ in M).

In the second case we multiply (1) with 2^γ if Δ is even and with $3.2^{\gamma-1}$ if Δ is odd (γ is the power of 2 which is contained in M). We now obtain

$$m \leqslant \frac{4}{3}M \quad \text{if} \quad \Delta \quad \text{is even}$$
$$m \leqslant 2M \quad \text{if} \quad \Delta \quad \text{is odd.}$$

This completes the proof.

SOME EXAMPLES

1. The Fibonacci sequence: $a = b = 1 \quad \Delta = 5 \quad R_1 = F_1 = 1.$
   If $M = 21$ then $p_1 = 3 \quad p_2 = 7$ so $s_1 = 4 \quad s_2 = 8$ and $m = 2 \cdot \frac{4}{2} \cdot \frac{8}{2} = 16.$
   Therefore $21 | F_{16}$ (in fact $21 | F_8$).
   If $M = 110 = 2 \cdot 5 \cdot 11$ then $m = 3 \cdot 5 \cdot 2 \cdot \frac{10}{2} = 150$ so $110 | F_{150}$.
   The only numbers having a rank of apparition equal to $2M$ are $6, 30, 150, 750, \cdots$ so $6 | F_{12}, 30 | F_{60}, 150 | F_{300}$, etc.
2. The Pell numbers: $0, 1, 2, 5, 12, 29, 70, \cdots \; a = 2 \quad b = 1 \quad \Delta = 8.$
   The numbers $3, 9, 27, \cdots$ constitute the only numbers having a rank of apparition equal to $\frac{4}{3}M$. So $3 | R_4, 9 | R_{12}$, etc.

In the special case $b = -1$ the theorem can be strengthened. We use the same notation as before. First we prove the following

Lemma. Let $b = -1$. If p_i is an odd prime and $p_i \nmid \Delta$ then

$$p_i \mid R_{s_i/2}$$

Proof. We suppose again $R_1 = 1$. Next we introduce the auxiliary sequence $T_0, T_1, T_2, \cdots$ with $T_{n+1} = aT_n - T_{n-1}$ and the initial conditions $T_0 = 2 \quad T_1 = a$. The following properties apply: (Proof in Bachmann [2])

$$\text{I.} \quad p_i \mid R_{s_i}$$
$$\text{II.} \quad p_i \mid T_{s_i} - 2$$
$$\text{III.} \quad R_{2n} = R_n T_n \quad (n \text{ is a positive integer})$$
$$\text{IV.} \quad T_{2n} = T_n^2 - 2 \quad (n \text{ is a positive integer}).$$

Take $n = s_i/2$ in III and IV. From II and IV it follows

$$p_i \nmid T_{s_i/2}.$$

From I and III it then follows $p_i \mid R_{s_i/2}$. This proves the lemma. Now let M be again an integer

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Further let m be the product of the numbers

$$(s_i p_i^{\alpha_i - 1})/2$$

respectively $s_i p_i^{\alpha_i - 1}$ $(i = 1, 2, \cdots, k)$, where we have to choose the first number if p_i is an odd prime and $p_i \nmid \Delta$; the second number if $p_i \mid \Delta$ or $p_i = 2$. By Carmichael's method it can be proved that again $M \mid R_m$.

It is easy to verify that $m \leqslant M$ if Δ is even and that $m \leqslant \frac{3}{2}M$ if Δ is odd. So we have found:

The rank of apparition does not exceed M if $b = -1$ and Δ is even.
The rank of apparition does not exceed $\frac{3}{2}M$ if $b = -1$ and Δ is odd.

EXAMPLES

PREAMBLE: The equation $X^2 - NY^2 = 1$ in which N constitutes a positive integer, not a square, and X and Y are integers, is called Pell's equation. For given N, an infinite number of pairs X and Y exist, which satisfy the equation. If X_1 and Y_1 constitute the smallest positive solution, all solutions can be found from the recursive relations

$$X_{n+1} = 2X_1 X_n - X_{n-1} \qquad Y_{n+1} = 2X_1 Y_n - Y_{n-1}$$

with initial conditions $X_0 = 1, \; Y_0 = 0.$

The sequence $Y_0, Y_1, Y_2, \cdots$ does satisfy the conditions of the strengthened theorem.

EXAMPLE 1. Let $N = 3$, so $X^2 - 3Y^2 = 1$ then $X_1 = 2, \; Y_1 = 1, \Delta = 12.$ The sequence $Y_0, Y_1, Y_2, \cdots$ consists of the numbers $0, 1, 4, 15, 56, 209, \cdots$. If $M = 110 = 2\cdot 5\cdot 11$ then $m = 2\cdot\frac{6}{2}\cdot\frac{10}{2} = 30$ so $110 \mid Y_{30}$. If $M = 18 = 2\cdot 3^2$ then $m = 2\cdot 3^2 = 18$ so $18 \mid Y_{18}$.

EXAMPLE 2. $X^2 - 2Y^2 = 1$ then $X_1 = 3,\ Y_1 = 2,\ \Delta = 32$.

The sequence $Y_0, Y_1, Y_2, \cdots$ consists of the numbers $0, 2, 12, 70, \cdots$ (which are Pell numbers with even subscript). The rank of apparition of any number M is less than M.

REMARK

If $b \neq \pm 1$ the theorem will generally not be valid; e.g., on taking $a = 4, b = 6, R_1 = 1$ any number M containing the factor 3 will not divide a member of the sequence.

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FIBONACCI AND LUCAS SUMS IN THE r-NOMIAL TRIANGLE

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ABSTRACT

Closed-form expressions not involving $c_n(p,r)$ are derived for

(1) $$\sum_{n=0}^{p(r-1)} c_n(p,r) f^m_{bn+j}(x)$$

(2) $$\sum_{n=0}^{p(r-1)} c_n(p.r) \ell^m_{bn+j}(x)$$

(3) $$\sum_{n=0}^{p(r-1)} c_n(p,r)(-1)^n f^m_{bn+j}(x)$$

(4) $$\sum_{n=0}^{p(r-1)} c_n(p,r)(-1)^n \ell^m_{bn+j}(x),$$

where $c_n(p,r)$ is the coefficient of y^n in the expansion of the r-nomial

$$(1 + y + y^2 + \cdots + y^{r-1})^p, \qquad r = 2, 3, 4, \cdots, \qquad p = 0, 1, 2, \cdots,$$

and $f_n(x)$ and $\ell_n(x)$ are the Fibonacci and Lucas polynomials defined by

$$f_1(x) = 1, \qquad f_2(x) = x, \qquad f_n(x) = xf_{n-1}(x) + f_{n-2}(x);$$
$$\ell_1(x) = x, \qquad \ell_2(x) = x^2 + 2, \qquad \ell_n(x) = x\ell_{n-1}(x) + \ell_{n-2}(x).$$

Fifty-four identities are derived which solve the problem for all cases except when both b amd m are odd; some special cases are given for that last possible case. Since $f_n(1) = F_n$ and $\ell_n(1) = L_n$, the n^{th} Fibonacci and Lucas numbers respectively, all of the identities derived here automatically hold for Fibonacci and Lucas numbers. Also, $f_n(2) = P_n$, the n^{th} Pell number. These results may also be extended to apply to Chebychev polynomials of the first and second kinds.

The entire text of this 51-page paper is available for \$2.50 by writing the Managing Editor, Brother Alfred Brousseau, St. Mary's College, Moraga, California 94575.

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EXPONENTIAL MODULAR IDENTITY ELEMENTS AND THE GENERALIZED LAST DIGIT PROBLEM

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INTRODUCTION

Led intuitively by the fact that the last digit of the positive integral powers of the non-negative integers repeat every fourth power, we proceed to an analogous general result for the last z digits (for z a positive integer). To do this we need first to define and build up some theory and properties for the orders and complete classes of Exponential Modular Identity Elements (EMIE). The last section then applies these..

1. EXPONENTIAL MODULAR IDENTITY ELEMENTS

Let n be a positive integer and let a be any z digit positive integer. Define:

$$A(z) = \left\{ a: a^n \equiv a \pmod{10^z}, \text{ for all } n \right\}.$$

Less formally, $A(z)$ is the set of all z digit non-negative integers, each of which, when raised to any positive integral powers, will end in itself. Term elements of $A(z)$ Exponential Modular Identity Elements (EMIE) of order z. Let

$$A = \cup A(z),$$

where the union is over all $z \in I^+ = \{$ positive integers $\}$. A subclass of A all of whose elements have the same last digit is termed a class. There are a countable infinity of orders but only four complete classes. (Complete, here, means the class contains elements of every order.) The first ten orders and the four complete classes are:

z	(Order)	c	(Complete Class)
1	$\{$ 0, 1, 5, 6 $\}$	1	$\{$ 0, 00, 000, 0000, $\cdots$ $\}$
2	$\{$ 00, 01, 25, 76 $\}$	2	$\{$ 1, 01, 001, 0001, $\cdots$ $\}$
3	$\{$ 000, 001, 625, 376 $\}$	3	$\{$ 5, 25, 625, 0625, $\cdots$ $\}$
4	$\{$ 0000, 0001, 0625, 9376 $\}$	4	$\{$ 6, 76, 376, 9376, $\cdots$ $\}$
5	$\{$ 00000, 00001, 90625, 09376 $\}$		
6	$\{$ 000000, 000001, 890625, 109376 $\}$		
7	$\{$ 0000000, 0000001, 2890625, 7109376 $\}$		
8	$\{$ 00000000, 00000001, 12890625, 87109376 $\}$		
9	$\{$ 000000000, 000000001, 212890625, 787109376 $\}$		
10	$\{$ 0000000000, 0000000001, 8212890625, 1787109376 $\}$		

Note that order and complete class uniquely determine an EMIE. Classes 1 and 2 are totally specified. Elements of Class 2 are universal identity elements because any element of class 2 of the z^{th} order when multiplied by any positive integer is congruent to the last z digits of that positive integer modulo 10^z. Elements of all other classes are existential identities. Define $\sim$ and τ to be binary relations satisfying: $a \sim b$ iff a and b are elements of the same complete class; $a \; \tau \; b$ iff a and b are elements of the same order. Since $\sim$ and τ satisfy the reflexive, symmetric and transitive properties, they are equivalence relations and the orders (complete classes) have union A and partition A into countably infinite (4) mutually disjoint equivalence classes of cardinality 4 (aleph null). A is neither closed under addition nor multiplication. Complete Class 1 is trivially closed under addition and multiplication and complete Class 2 under multiplication only. The other complete classes and the orders are closed under neither operation. But since the closure property is necessary even for a semi-group, group theory doesn't seem to be of any use here. Our integral specifications designate us to number theory. From elementary number theory:

$$a^2 \equiv a \pmod{10^z} \Rightarrow a^n \equiv a \pmod{10^z}$$

which obviously is useful since it allows us to deal only with squares, but is still quite insufficient. After introducing notation, I present the most useful of the properties I have developed.

Notation: $A(z,c,n)$ is the n^{th} power of the EMIE of z^{th} order and complete class c. $L(a,b,n)$ is the last a digits of the n^{th} power of b. If $n = 1$, we may omit the n. Of course z, c, n, a, b are positive integers. b can equal 0. $a^N = a{*}{*}N$. $I, I_1, I_2, \cdots$ represent arbitrary positive integers.

Property 1. $$L(z-1, A(z,c)) = A(z-1, c).$$

Proof. If this were not so then the last $z-1$ digits of $A(z,c,2)$ would not equal the last $z-1$ digits of $A(z,c)$ and so the last z digits of $A(z,c,2)$ would not equal the last z digits of $A(z,c)$. But this contradicts $A(z,c)$ being an element of the z^{th} order so the property must be true.

Property 2. (a) $$L(z+k, A(z,3,(2\ell+1)10^k)) = A(z+k,3)$$

(b) $$L(z+k+1, A(z,3,(2\ell)10^k)) = A(z+k+1,3),$$

where $z, \ell, k \in I^+$, $z \geq k$ and in (a) ℓ can be 0.

Proof. $A(z+k,3)$ is EMIE, so

$$\begin{aligned} A(z+k,3) &\equiv A(z+k,3,j10^k) = (10^z x + A(z,3)){*}{*}j10^k \equiv 0+0+\cdots+0+A(z,3,j10^k) \\ &\equiv A(z,3,(2\ell+1)10^k) \equiv L(z+k, A(z,3,(2\ell+1)10^k)) \pmod{10^{z+k}}, \end{aligned}$$

where $j = 2\ell+1$ and x is the appropriate nonnegative integer. (Note: Though x is unique for given z and k, it does not make any difference whether we know what it is or not as far as this particular result goes.)

Also,

$$\begin{aligned} A(z+k+1,3) &\equiv (10^z y + A(z,3)){*}{*}m10^k \equiv 10^{z+k}(ym)(\cdots 5) + A(z,3,m10^k) \\ &\equiv A(z,3(2\ell)10^k) \equiv L(z+k+1, A(z,3,(2\ell)10^k)) \pmod{10^{z+k+1}} \end{aligned}$$

using the fact that $m = 2\ell$ is even and y is the appropriate nonnegative integer.

Therefore

$$A(z+k,3) \equiv L(z+k, A(z,3,(2\ell+1)10^k)) \pmod{10^{z+k}} \quad \text{and} \quad A(z+k+1,3) \equiv L(z+k+1, A(z,3,(2\ell)10^k)),$$

but the first pair are both $z+k$ digit numbers and so are equal. Likewise the second pair are both $z+k+1$ digit numbers and so are equal.

Property 3. (a) $$L(z+k, A(z,4,j10^k)) = A(z+k,4)$$

(b) $$L(z+k+1, A(z,4,(5\ell)10^k)) = A(z+k+1,4),$$

where $z \geq k$.

Proof. $$\begin{aligned} A(z+k,4) &\equiv A(z+k,4,j10^k) = (10^z x + A(z,4)){*}{*}j10^k \equiv 0+0+\cdots+0+A(z,4,j10^k) \\ &\equiv L(z+k, A(z,4,j10^k)) \pmod{10^{z+k}} \end{aligned}$$

so

$$A(z+k,4) = L(z+k, A(z,4,j10^k))$$

because they are both $z+k$ digit figures.

Also,

$$\begin{aligned} A(z+k+1,4) &\equiv (10^z x + A(z,4)){*}{*}(5\ell)10^k \equiv 0+0+\cdots+0+(5\ell)10^k\,10^z x(\cdots 6) + A(z,4,(5\ell)10^k) \\ &\equiv L(z+k+1), A(z,4,(5\ell)10^k)). \end{aligned}$$

Thus

$$A(z+k+1,4) = L(z+k+1, A(z,4,(5\ell)10^k)).$$

Property 4. $2^j A(n,3,b) \equiv 5^j A(n,4,b) \equiv 0 \pmod{10^i}$, where $1 \leq i \leq \min(j,n) = m$.

Proof. $2^j A(n,3,b) \equiv 2^j A(n,3) = 2^j L(n,A(1,3,10^{n-1}(2\ell+1)))$.

(Using property 2(a) with $z = 1$, $k = n-1$). But let $b' = (2\ell+1)10^{n-1}$ then this is congruent to

$$L(n,2^j A(1,3,b')) = L(n,10^m(2^{j-m}5^{b'-m})) \equiv L(m,10^m I) = 0 \pmod{10^m} \equiv 0 \pmod{10^i},$$

where $1 \leqslant i \leqslant m$ and $I = 2^{j-m}5^{b'-m}$ is a positive integer.

Also
$$5^j A(n,4,b) \equiv 5^j A(n,4) = 5^j L(n,A(1,4,k10^{n-1})) \equiv L(n,5^j A(1,4,k10^{(n-1)}))$$
$$= L(n,30^m 5^{j-m}6^{b''}) \equiv L(m,30^m I') = L(m,10^m I'') = 0 \pmod{10^m} \equiv 0 \pmod{10^i},$$

where $1 \leqslant i \leqslant m$; $b'' = k1U^{n-1} - m$, $I'' = 3^m I'$; $I' = 5^{j-m}6^{b''}$.

$$\therefore\ 2^j A(n,3,b) \equiv 5^j A(n,4,b) \equiv 0 \pmod{10^i}.$$

Property 5. (a) $L(z+k+j, A(z,3,d)) = A(z+k+j,3)$

(b) $L(z+k+j, A(z,4,d')) = A(z+k+j,4)$,

where $d = \ell 2^j 10^k$, $d' = \ell 5^j 10^k$, $1 \leqslant j,\ k \leqslant z, \ell, j, k, z \in I^+$.

Proof.

$$A(z+k+j,3) \equiv A(z+k+j,3,d) = (10^2x + A(z,3))**d \equiv \binom{d}{d-2} 10^{2z}x^2 A(z,3,d-2)$$
$$+ \binom{d}{d-1} 10^z x A(z,3,d-1) + A(z,3,d) \equiv 10^k 2^{j-1}\ell(d-1)10^{2z}x^2 A(z,3,d-2)$$
$$+ 10^k 2^j \ell 10^z x A(z,3,d-1) + A(z,3,d) = 10^{2z+k} I_1 + 10^{z+k}(2^j A(z,3,d-1))I_2 + A(z,3,d),$$

since $2z+k \geqslant z+k+j$ and min $(j,z) = j$ so by Property 4, $2^j A(z,3,d-1) \equiv 0 \pmod{10^j}$ therefore, $2^j A(z,3,d-1) = 10^j I$. Hence,

$$A(z+k+j,3) \equiv 0 + 0 + A(z,3,d) \equiv L(z+k+j,A(z,3,d)) \pmod{10^{z+k+j}}.$$

Thus, $A(z+k+j,3) = L(z+k+j,A(z,3,d))$.

Also,

$$A(z+k+j,4) \equiv A(z+k+j,4,d') = (10^z x + A(z,4))**d' \equiv 0 + 0 + \cdots + 0 + \frac{10^k 5^j \ell(d'-1)}{2} 10^{2z}x^2 A(z,4,d'-2)$$
$$+ 10^k 5^j \ell 10^z x A(z,4,d'-1) + A(z,4,d') \equiv 10^{2z+k} I + 10^{z+k}(5^j A(z,4,d'-1))I_1 + A(z,4,d')$$

and by using Property 4 and $2z+k \geqslant z+k+j$ get

$$A(z+k+j,4) \equiv A(z,4,d') \equiv L(z+k+j, A(z,4,d')) \pmod{10^{z+k+j}}.$$

Thus, $A(z+k+j,4) = L(z+k+j, A(z,4,d'))$.

Note that by placing $j = 0,1$ in each of these yields Properties 2(a) and 3(a). Property 6 is thus an extension of the (a) parts of 2 and 3 made possible by using 4. [For the first part of 2 you must restrict further replacing all positive integers ℓ by only the odd integers $2\ell + 1$.]

Notation. $T(a,b)$ is the a^{th} digit from the end of the nonnegative integer b, $F(b)$ is the first digit of b.

Property 6. $L(1,2nx + T(z+1,A(z,4,2n)) = x$, where $x = F(A(z+1,4))$ and $n,z \in I^+ = T(z+1,A(z+1,4))$.

Proof. $A(z+1,4) \equiv A(z+1,4,2n) = (10^z x + A(z,4))**2n \equiv 0 + 0 + \cdots + 0 + 2n10^z xA(z,4,2n-1) + A(z,4,2n)$

since $2z \geqslant z+1 \equiv 10^z xn2(\cdots 6) + A(z,4,2n) \equiv 2xn10^z + A(z,4,2n) \equiv L(z+1,2xn10^z + A(z,4,2n))$

$$= 10^z T(z+1,2xn10^z + A(z,4,2n)) + L(z,2xn10^z + A(z,4,2n)) = 10^z T(z+1,2xn10^z + A(z,4,2n)$$
$$+ L(z,A(z,4,2n)) = 10^z T(z+1,2xn10^z + A(z,4,2n)) + L(z,A(z,4))$$
$$= 10^z T(z+1,2xn10^z + A(z,4,2n)) + A(z,4)$$

$$\therefore\ x = F(A(z+1,4)) = T(z+1,A(z+1,4)) = T(z+1,10^z T(z+1,2xn10^z + A(z,4,2n) + A(z,4)))$$
$$= T(z+1,10^z T(z+1,2xn10^z + A(z,4,2n)) + T(z+1,A(z,4))) = T(z+1,10^z T(z+1,2xn10^z$$
$$+ A(z,4,2n))) = T(z+1,2xn10^z + A(z,4,2n)) = T(z+1,10^z 2nT(z+1,A(z+1,4)) + A(z,4,2n)$$
$$= T(z+1,T(z+1,A(z+1,4))10^z 2n + T(z+1,A(z,4,2n))) = L(2nT(z+1,A(z+1,4)) + T(z+1,A(z,4,2n))$$
$$= L(2nF(A(z+1,4)) + T(z+1,A(z,4,2n))) = F(A(z+1,4))$$

replacing k for 2 in the above argument:

Property 6 (extended).

$$L(L(6L(k))nx + T(z+1, A(z,4,kn))) = x,$$

where $x = F(A(z+1,4))$ and

$$L(6L(k)) = \begin{cases} i & \text{if } k \equiv i \pmod 5,\ i = 0, 2, 4 \text{ (even)} \\ 5+i & \text{if } k \equiv i \pmod 5,\ i = 1, 3 \text{ (odd)} \end{cases}.$$

Note: $L(k) = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9, $\quad L(6L(k)) = 0, 6, 2, 8, 4, 0, 6, 2, 8,$ or 4

using k from 1 to 9 consecutively.

It is easy to see further that:

Property 6 (extended further). $\quad L(L(6L(k)L(n))x + T(z+1, A(z,4,kn)) = x,$

where $x = F(A(z+1,4))$ and

$$L(6L(k)L(n)) = L(6L(kn)) = \begin{cases} i & \text{if } kn \equiv i \pmod 5,\ i = 0, 2, 4 \\ 5+i & \text{if } kn \equiv i \pmod 5,\ i = 1, 3 \end{cases}.$$

Property 6 (final).

$$L(ax + T(z+1, A(z,4,m)) = x,$$

where $x = F(A(z+1,4))$ and

$$a = L(6L(a_1)L(a_2)\cdots L(a_k)) = L(6L(m)) \quad \text{for} \quad m = a_1a_2a_3\cdots a_k$$

$$a = \begin{cases} i & \text{if } m \equiv i \pmod 5 \ i = 0, 2, 4 \\ 5+i & \text{if } m \equiv i \pmod 5 \ i = 1, 3 \end{cases}.$$

Property 7.

$$L(L(5L(k))nx + T(z+1, A(z,3,kn))) = x,$$

where $x = F(A(z+1,3))$ and $k,n,z \in I^+$ and

$$L(5L(k)) = 5i,$$

where $k \equiv i \pmod 2$ and $i = 0$ or 1.

Proof.

$$\begin{aligned} A(z+1,3) &\equiv A(z+1,3,kn) = (10^z x + A(z,3))^{**}kn \equiv kn(10^z x)A(z,3,kn-1) + A(z,3,kn) \equiv knx10^z(\cdots 5) \\ &\quad + A(z,3,kn) \equiv 5L(k)nx10^z + A(z,3,kn) \equiv L(5L(k))nx10^z + A(z,3,kn) \\ &\equiv L(z+1, L(5L(k))nx10^z + A(z,3,kn)) = 10^z T(z+1, L(5L(k))nx10^z + A(z,3,kn)) \\ &\quad + L(z, L(5L(k))nx10^z + A(z,3,kn)). \end{aligned}$$

Let $a = L(5L(k))$. Then $L(z, anx10^z + A(z,3,kn)) = L(z, A(z,3,kn)) = A(z,3)$ so

$$A(z+1,3) \equiv 10^z T(z+1, anx10^z + A(z,3,kn)) + A(z,3).$$

Therefore

$$\begin{aligned} x &= F(A(z+1,3)) = T(z+1, A(z+1,3)) = T(z+1, 10^z T(z+1, anx10^z + A(z,3,kn)) + A(z,3)) \\ &= T(z+1, 10^z T(z+1, anx10^z + A(z,3,kn)) = T(z+1, anx10^z + A(z,3,kn)) = T(z+1, a10^z nF(A(z+1,3)) + A(z,3,kn)) \\ &= T(z+1, 10^z anT(z+1, A(z+1,3)) + A(z,3,kn)) = T(z+1, 10^z anT(z+1, A(z+1,3)) + T(z+1, A(z,3,kn))) \\ &= L(anT(z+1, A(z+1,3)) + T(z+1, A(z,3,kn)) = L(anF(A(z+1,3)) + T(z+1, A(z,3,kn)) \\ &= L(anx + T(z+1, A(z,3,kn))) = L(L(5L(k))nx + T(z+1, A(z,3,kn))) \end{aligned}$$

[all congruences are modulo 10^{z+1}] and

$L(k) = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9, $\quad L(5L(k)) = 0, 5, 0, 5, 0, 5, 0, 5, 0,$ or 5.

$$\therefore\ L(5L(k)) = 5i, \text{ where } k \equiv \pmod 2 \text{ and } i = 0 \text{ or } 1$$

Clearly, essentially repeating all steps for the generalized constant a we have

Property 7 (extended).

$$L(ax + T(z+1, A(z,3,m))) = x = F(A(z+1,3)),$$

where

$$a,m,z \in I^+, \quad m = a_1a_2\cdots a_k, \quad a = L(5L(a_1)L(a_2)\cdots L(a_k)) = L(5L(m)),$$

and

$$a = \begin{cases} 0 & \text{if } m \text{ is even} \\ 5 & \text{if } m \text{ is odd} \end{cases} = \begin{cases} 0 & \text{if every } a_i \ (1 \leqslant i \leqslant k) \text{ is odd} \\ 5 & \text{if at least one } a_i \ (1 \leqslant i \leqslant k) \text{ is even} \end{cases}$$

Property 8 $A(n,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j}$, $A(n,4) \equiv A(i,4,5^{j-i}m) \pmod{10^j}$,

$$A(j,3) \equiv A(i,3,2^{n-i}m) \pmod{10^j}, \quad A(j,4) \equiv A(i,4,5^{n-i}m) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Proof. Let $z = i$, $j = n - i$, $k = 0$, $\ell = m$ in Property 5(a); then $L(n,A(i,3,2^{n-i}m) = A(n,3)$. So

$$A(n,3) \equiv A(i,3,2^{n-i}m) \pmod{10^n}$$

for all n. In particular, $A(j,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j}$, but $A(n,3) \equiv A(j,3) \pmod{10^j}$. Thus

$$A(n,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j} \quad \text{and} \quad A(j,3) \equiv A(i,3,2^{n-i}m) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Likewise, using Property 5(b) we get $A(n,4) = L(n,A(i,4,5^{n-i}m)$. So $A(n,4) \equiv A(i,4,5^{n-i}m) \pmod{10^n}$. Thus

$$A(n,4) \equiv A(i,3,5^{j-i}m) \pmod{10^j} \quad \text{and} \quad A(i,4,5^{n-i}m) \equiv A(j,4) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Property 9. (a) $$T(z+1,A(z+1,3)) + T(z+1,A(z+1,4)) = 9$$

(b) $$\sum_{i=1}^{4} A(z,i) = 10^z + 2$$

(c) $$A(z,3) + A(z,4) = 10^z + 1$$

(d) $$A(z,3) + A(z,4) = 10^z + A(z,1) + A(z,2)$$

(e) $$A(z,3) + A(z,4) \equiv A(z,1) + A(z,2) \equiv 1 \pmod{10^z}.$$

Uncompleted Proof. *IF* we assume for the moment that 9(a) is true then it is easy to show the rest. (I know 9(a) is true at least for $z = 1, 2, \cdots, 11$ because of direct calculation but can't prove it in general. Can the reader?) For we know that $L(1,A(z,3)) + L(1,A(z,4)) = 5 + 6 = 11$ and that $A(z,1) = 0$ and $A(z,2) = 1$ for all z. So for $z = 1$ we have

$$A(z,3) + A(z,4) = A(1,3) + A(1,4) = 5 + 6 = 11 = 10^1 + 1 = 10^z + 1.$$

So (c) is true at least for $z = 1$. Now, assume (c) true for $k - 1$; then

$$\begin{aligned} A(z,3) + A(z,4) &= 10^{z-1}T(z,A(z,3)) + L(z-1,A(z,3)) + 10^{z-1}T(z,A(z,4)) + L(z-1,A(z,4)) \\ &= 10^{z-1}(T(z,A(z,3)) + T(z,A(z,4))) + L(z-1,A(z,3)) + L(z-1,A(z,4)) \\ &= 10^{z-1}(9) + A(z-1,3) + A(z-1,4) = 10^{z-1}(9) + 10^{z-1} + 1 = 10^z + 1 \end{aligned}$$

so if (c) is true for $z - 1$ then it is true for z and so by induction we get (c): $A(z,3) + A(z,4) = 10^z + 1$ $z \in I^+$ but, $A(z,1) = 0$ and $A(z,2) = 1$ so

$$\sum_{i=1}^{4} A(z,i) = A(z,1) + A(z,2) + A(z,3) + A(z,4) = 0 + 1 + 10^z + 1 = 10^z + 2,$$

which is (b). Also since $A(z,1) + A(z,2) = 1$, $A(z,3) + A(z,4) = 10^z + 1 = 10^z + A(z,1) + A(z,2)$ so $A(z,3) + A(z,4) \equiv A(z,1) + A(z,2) \equiv 1 \mod (10^z)$, which are (d) and (e).

The largest order I've calculated is:

$$A_{12} = \{000000000000, 000000000001, 918212890625, 081787109376\}.$$

Note that:

$$A(12,1) + A(12,2) + A(12,3) + A(12,4) = 0 + 1 + 918212890625 + 81787109376 = 10^{12} + 2$$

and

$$T(12,A(12,3)) + T(12,A(12,4)) = 9 + 0 = 9 \quad \text{and} \quad T(i+1,A(12,3)) + T(i+1,A(12,4)) = 9 \quad i = 1,2,\cdots,11$$

which means Property 9 is true for at least order 12. (Concluding Property 9 true for the 12^{th} order concludes it true for all lower orders.)

For minimum effort in finding further orders use:

$$L(z+1, A(z,3,2)) = A(z+1,3) \quad \text{and} \quad L(2x + T(z+1, A(z,4,2)) = x = T(z+1, A(z+1,4)) = F(A(z+1,4)).$$

These are restrictions of Property 5 and 6, respectively. If I could prove Property 9, I could cut the work in half calculating only the first of these. Each succeeding calculation of higher orders checks the lower ones. Further casting out of nine's and casting out of eleven's are enormously timesaving checks which can be used on both the total product and the partial products. Calculate only one of Classes 3 and 4 (Classes 1 and 2 are completely determined) then use Property 9 and obtain easily the assumed, but unproved, value of the other. If the assumed value is true to the appropriate of the two given equations, then all lower orders are found and proved true *PLUS* you at the same time find and prove the next order of that class. You can now keep raising the order as long as you like and then repeat the above process saving more time the longer you wait to repeat. (That is, as long as Property 9 does continue to hold true—a high probability—you save. At any rate, you haven't lost anything if it doesn't work but you will have practically halved the time if it does—and for large digits, believe me, it helps!!!) This method to a large extent, but not quite, makes up for the lack of a solid proof of Property 9 for the particular problem of building up orders.

2. APPLICATIONS OF EMIE

Observe from the table below the repetitive sequence (listed to the left) of the last digits of a finite subset of the set of nonnegative integers to all positive integral powers. The bar means "repeated."

	x	x^2	x^3	x^4	x^5	x^6
$\overline{1}$	1	1	1	1	1	1
$\overline{2,4,8,6}$	2	4	8	16	32	64
$\overline{3,9,7,1}$	3	9	27	81	243	729
$\overline{4,6}$	4	16	64	256	1024	4096
$\overline{5}$	5	25	125	625	3125	15625
$\overline{6}$	6	36	216	1296	7776	46656
$\overline{7,9,3,1}$	7	49	343	2401	16807	117649
$\overline{8,4,2,6}$	8	64	512	4096	32768	262144
$\overline{9,1}$	9	81	729	6561	59049	531441
$\overline{0}$	10	100	1000	10000	100000	1000000
$\overline{1}$	11	121	1331	14641	161051	1771561
$\overline{2,4,8,6}$	12	144	1728	20736	248831	2985984
$\overline{3,9,7,1}$	13	169	2197	28561	371293	4826809

Obviously, by knowing recursively the last digit for all x^n, where $x \in \{0,1,2,3,4,5,6,7,8,9\}$ you can determine all the last digits of all y^n, where $y \in I^+ \cup \{0\}$ and $n \in I^+$. Noting that column 5 repeats 1, 6 repeats 2, and so on, it is logical to induce that the last digit of the positive integral powers of the nonnegative integers repeat every 4 powers. 0,1,5, and 6 repeat every time with themselves because they are EMIE of order one. 4 and 9 repeat every two times on EMIE's of 6 and 1, respectively. 2,3,7 and 8 repeat every four times on EMIE's of 6,1,1,6, respectively. I shall now state and prove this induction aided by the EMIE background. Let $L(1,a) = L(a)$.

Last Digit Property (LDP). $\quad x^{4n+m} \equiv L(y^m) \pmod{10},$

where

$$x = (10a+y); \quad m \in \{1, 2, 3, 4\}; \quad a, x, 4n+m \in I^+ \quad \text{and} \quad y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Proof: $\quad x^{5n+m} = x^{4n}x^m = (10a+y)^{4n}(10a+y)^m \equiv y^{4n}y^m$

but $y = 0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9, so $y \equiv 0, \pm 1, \pm 2, \pm 4, 5$ so $y^4 \equiv 0, 1, 6, 1, 6$ or $s \in A(1)$. Therefore,

$$y^{4n}y^m \equiv y^4y^m \equiv yL(y^4y^{m-1}) \equiv yL(y^4)L(y^{m-1}).$$

But

$$yL(y^4) \equiv 0\cdot 0, 1\cdot 1, 2\cdot 6, 3\cdot 1, 4\cdot 6, 5\cdot 5, 6\cdot 6, 7\cdot 1, 8\cdot 6, 9\cdot 1 \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 = y.$$

So continuing from above,

$$x^{5n+m} \equiv yL(y^{m-1}) \equiv L(y)L(y^{m-1}) \equiv L(y^m) \pmod{10}.$$

Having proved LDP, it is only natural that one wonder whether there exists a similar theorem for the last z digits, where z is a positive integer greater than or equal to two. Consider first the case of the last two digits and the number 2. (We shall use LzDPa to mean the Last z Digit Property of powers of a.)

L2DP2
$$2^{20n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \\ 52 & \text{if } n \geqslant 1 \text{ and } m = 1 \pmod{10^2} \\ L(2,2^m) & \text{otherwise,} \end{cases}$$

where $m \in \{1,2,\cdots,20\}$, $n \in I^+$.

Proof. $2^{20} = (2^{10})^2 = (1024)^2 \equiv 24^2 \equiv 76 = A(2,4) \pmod{10^2}$ so for $n \geqslant 1$,

$$2^{20n+m} = (2^{20})^n 2^m \equiv 76^n 2^m \equiv 76\cdot 2^m = 75\cdot 2^m + 2^m \equiv L(2,2^m) \pmod{10^2},$$

where $m \neq 0,1$; if $m = 0$ and $n \geqslant 1$, $2^{20n+m} \equiv 75\cdot 2^0 + 2^0 = 76$; if $m = 1$ and $n \geqslant 1$, $2^{20n+m} \equiv 75\cdot 2^1 + 2^1 \equiv 52$; if $n = 0$, $2^{20n+m} = 2^m \equiv L(2,2^m) \pmod{10^2}$.

L2DP3
$$3^{20n+m} \equiv L(2,3^m) \pmod{10^2}.$$

Proof.

$$3^{20} = (3^3)^6 9 = (27)^6 9 \equiv (29)^3 9 \equiv (41)(29)(9) \equiv (41)(61) \equiv (41)(-39) = -(40^2-1) = -40^2+1 \equiv 01 \pmod{10^2};$$

$$\therefore\ 3^{20n+m} = (3^{20})^n (3^m) \equiv (01)^n (3^m) = 3^m \equiv L(2,3^m) \pmod{10^2}$$

if $n \geqslant 1$; obvious if $n = 0$.

L2DP4
$$4^{10n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \pmod{10^2} \\ L(2,4^m) & \text{otherwise.} \end{cases}$$

Proof
$$4^{10} = 2^{20} \equiv 76 \pmod{10^2}.$$

Therefore, if $n \geqslant 1$, $m \neq 0$,

$$4^{10n+m} \equiv 76^n 4^m \equiv 76\cdot 4^m = 75\cdot 4^m + 4^m \equiv L(2,4^m) \pmod{10^2};$$

if $n = 0$,

$$4^{10n+m} = 4^m \equiv L(2,4^m) \pmod{10^2};$$

if $n \geqslant 1$ and $m = 0$,

$$4^{10n+m} = (4^{10})^n \equiv 76^n \equiv 76 \pmod{10^2}.$$

L2DP5
$$5^n \equiv \begin{cases} 5 & n = 1 \\ 25 & n \geqslant 2 \end{cases} \pmod{10^2}.$$

Proof. $5^2 = 25$; if $n \geqslant 2$,

$$5^n = 5^{n-2}5^2 = (25\cdot 5)5^{n-3} \equiv 25\cdot 5^{n-3} \equiv \cdots \equiv 25\cdot 5^{n-(n-1)} \equiv 25 \pmod{10^2};$$

if $n = 1$, $5^n = 5^1 \equiv 5 \pmod{10^2}$.

Another way: $5^{2n} = (5^2)^n = (25)^n \equiv 25$; $5^{2n+1} \equiv 5\cdot 25 \equiv 25$ for $n \in I^+$; $5^1 \equiv 5 \pmod{10^2}$.

L2DP6
$$6^{5n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \\ 56 & \text{if } n \geqslant 1 \text{ and } m = 1 \pmod{10^2} \\ L(2,6^m) & \text{otherwise} \end{cases}$$

Proof.
$$6^5 \equiv (16)(36) = 26^2 - 10^2 \equiv 76;$$

if $m \neq 0,1$ and $n \geqslant 1$,

$$6^{5n+m} \equiv 76^n 6^m \equiv 76\cdot 6^m = 75\cdot 6^m + 6^m \equiv L(2,6^m);$$

if $m = 0$ and $n \geqslant 1$,

$$6^{5n+m} \equiv (76)^n \equiv 76;$$

if $m = 1$, $n \geqslant 1$,

$$6^{5n+m} \equiv 76^n\cdot 6 \equiv 76\cdot 6 \equiv 56;$$

if $n = 0$, $6^{5n+m} \equiv L(2,6^m)$.

Since the proofs that follow immediately hereafter are completely analogous to the preceding ones, I will leave them to the reader and merely state the results for reference. (I present them here even though I am also going to discuss a general last two digit property because we can in general get much more information about specific bases than we can about all bases. Also, it is illustrative in getting a good grasp to look back to the analogous occurrence in LDP and the material just preceding.)

L2DP7 $$7^{4n+m} \equiv L(2,7^m) \pmod{10^2}.$$

L2DP8 $$8^{20n+m} \equiv \begin{cases} 76 & \text{if } m=0, n \geq 1 \\ L(2,8^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP9 $$9^{10n+m} \equiv L(2,9^m) \pmod{10^2}.$$

L2DP10 $$10^n \equiv \begin{cases} 10 & \text{if } n=1 \\ 00 & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP11 $$11^{10n+m} \equiv L(2,11^m) \pmod{10^2}.$$

L2DP12 $$12^{20n+m} \equiv \begin{cases} 76 & \text{if } n \geq 1 \text{ and } m=0 \\ L(2,12^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP13 $$13^{20n+m} \equiv L(2,13^m) \pmod{10^2}.$$

L2DP14 $$14^{10n+m} \equiv \begin{cases} 76 & \text{if } m=0, n \geq 1 \\ 64 & \text{if } m=1, n \geq 1 \\ L(2,14^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

I now hazard my best guesses as to the general L2P and LzP. These guesses come from knowledge of the above stated results when the base is known and from the fact that having studied a moderately sized table I have found no contradictions as yet. I have found much affirmation at least for the concepts which lie at the heart of the property (that in L2P we see repetition every 20 powers and in LzP we see it every $4\cdot 5^{z-1}$ powers). The particular side conditions are more questionable. I present my guesses as an aid to those who want to research my guess and perhaps find a solution. I present incomplete proofs in order to illustrate where in the proof I make assumptions I cannot prove. Even so, I hope you will find them stimulating if only in providing the direction your approach should or could take.

L2P $$x^{20n+m} = x^{4\cdot 5^1 n+m} \equiv \begin{cases} \text{side conditions} \\ L(2,y^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

where plausible side conditions might be:

$$\begin{cases} 76 & \text{if } 2|x,\ m=0,\ n \geq 1 \\ 50+y & \text{if } 2|x,\ 4\nmid x,\ m=1,\ n \geq 1 \end{cases}$$

and $x = (100a+y)$.

$$m \in \{1,2,3,\cdots,20\}, \quad a, x, 20n+m \in I^+, \quad y \in \{0,1,2,\cdots,99\} = H.$$

Incomplete Proof. *IF* we ignore side conditions and *IF* we assume y^{20} is EMIE of order 2 for all $y \in \{0,1,\cdots,99\}$. (We know this is true for $y \leq 14$. Anyone for computing the last 85 so we can discard this assumption? If you take this approach, you can get L2D but try using it for L3D where y takes on 1000 values and so on. Eventually you will have to stop. You will have gained some ground, but *hopefully* there is an easier way. I think so.) Now

$$x^{20n+m} = (100a+y)^{20n+m} \equiv y^{20n+m} = (y^{20})^n y^m \equiv (y^{20})y^m \equiv L(2,y^{20})y^m \equiv L(2,y^m) \pmod{10^2}.$$

The last step can be made since we know what EMIE's of order 2 are and what they do when multiplied by any of all possible last 2 digits configurations. This is an exercise in computation that I will not present here.

The following property is presented on an even less sound basis than the previous one (L2P):

LzP $$x^{4\cdot 5^{z-1}n+m} \equiv \begin{cases} \text{side conditions} \\ L(z,y^m) & \text{otherwise} \end{cases} \pmod{10^z}$$

where plausible side conditions might be

$$\begin{cases} A(z,4) & \text{if } m=0, n \geq 1 & 2|x \\ 5\cdot 10^{z-1}+y & \text{if } m=1, n \geq 1 & 2|x,\ 4\nmid x \end{cases}$$

and $x = (10^z a + y)$

$$m = \{1,2,3,\cdots,4\cdot 5^{z-1}\}, \quad a, x, 4\cdot 5^{z-1}n+m \in I^+, \quad y \in \{0,1,2,3,\cdots,10^z-1\} = H'.$$

Incomplete Proof. *IF* we ignore side conditions, and *IF* we assume $y^{4\cdot 5^{z-1}}$ is EMIE of order z for all $y \in H'$, then

$$x^{4\cdot 5^{n+m}} = (10^z a+y)**4\cdot 5^{z-1}n+m \equiv y^{4\cdot 5^{z-1}n+m} = (y^{4\cdot 5^{z-1}})^n y^m \equiv L(z,y^{4\cdot 5^{z-1}})y^m \equiv L(z,y^m) \pmod{10^z}.$$

(The last step would have to also be shown. For any particular value of z, we can do a lot of computation as noted in L2P above. However, I hope there is an easier way.)

I leave you at this open-ended point. I feel there is a lot of room for more research in both theory and applications of EMIE. I append some numerical examples.

APPENDIX

EXAMPLES

$$2^{28} = 2^{6\cdot 4+4} \equiv 2^4 \equiv 6 \pmod{10}$$

$$12^{101} \equiv 2^{25\cdot 4+1} \equiv 2^1 = 2 \pmod{10}$$

$$36{,}487{,}697^{36{,}766{,}542} \equiv 7^{9191635(4)+2} \equiv 7^2 \equiv 9 \pmod{10}$$

$$2485^{137653} \equiv 5^{137653} \equiv 5 \pmod{10}$$

$$19^{21} \equiv 9^{5\cdot 4+1} \equiv 9^1 = 9 \pmod{10}$$

$$2^{148} \equiv 2^{36\cdot 4+4} \equiv 2^4 \equiv 6 \pmod{10}$$

$$3^{1081} \equiv 3^{20(54)+1} \equiv 3^1 \equiv 03 \pmod{10^2}$$

$$485^{1085} \equiv 85^{100(10)+85} \equiv 85^{85} \equiv 225^{42}\cdot 85 \equiv 625^{21}\cdot 85 \equiv 625\cdot 85 \equiv 125 \pmod{10^3}$$

$$\begin{aligned} 2^{10^{10^{10^{10}}}} &\equiv 376 \pmod{10^3} \\ &\equiv 081787109376 \pmod{10^{12}} \\ &\equiv A(10^{10^{10}}+1,4) \pmod{10^{(10^{10^{10}}+1)}} \end{aligned}$$

$$\begin{aligned} 5^{4^{5^{6^7}}} &\equiv 0625 \pmod{10^4} \\ &\equiv 918212890625 \pmod{10^{12}} \\ &\equiv A(2(5^{6^7})-1,3) \pmod{10^{2(5^{6^7}-1)}}. \end{aligned}$$

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LETTER TO THE EDITOR

February 15, 1974

Dear Dr. Hoggatt:

I have discovered the theorem below and was advised to forward it to you as being the most suitable publisher, should it turn out to be original.

Consider the function

$$F_x(n) = 1 + \sum_{i=1}^{i=\frac{2n+(-1)^{(n+1)}-3}{4}} \left\{ \left(\frac{x^i}{i!}\right) \prod_{j=i+1}^{j=2i} (n-j) \right\} .$$

We make the convention that $F_x(1) = 1$ for all x.

It is easily established that for all ℓ the coefficient of $x^{(\lambda-1)}$ in $F_x(n)$ added to the coefficient of x^λ in $F_x(n+1)$ gives the coefficient of x^λ in $F_x(n+2)$, and thus we have:

$$xF_x(n) + F_x(n+1) = F_x(n+2) .$$

$F_1(n)$ is the Fibonacci series.

Theorem. Any prime factor of $F_x(P)$, where P is prime, is congruent to ± 1 or 0 (mod P). (We assume $P \neq 2$ since if $P = 2$ the theorem is trivial.)

Lemma 1. For any ℓ,

$$(\ell+1)(\ell+2)\cdots(2\ell) = (2)(6)\cdots(4\ell-2) .$$

This is easily proved by induction.

Lemma 2. The coefficient of x^ℓ in $F_x(p)$ is congruent to the coefficient of x^ℓ in the binomial expansion of

$$\left[x + \left(\frac{p+1}{4}\right)\right]^{\left(\frac{p-1}{2}\right)} \pmod p,$$

where p is prime, and $p \neq 2$.

Since $p \neq 2$, p is odd and $F_x(p)$ is of order

$$\frac{2p+(-1)^{p+1}-3}{4} = \left(\frac{p-1}{2}\right) \quad \text{in } x.$$

From Lemma 1 we have

$$\frac{(\ell+1)(\ell+2)\cdots(2\ell)}{\ell!} = \frac{(2)(6)\cdots(4\ell-2)}{\ell!} .$$

Thus

$$\frac{(p-(\ell+1))(p-(\ell+2))\cdots(p-2\ell)}{\ell!} \equiv \frac{(2p-2)(2p-6)\cdots(2p-(4\ell-2))}{\ell!} \equiv 4^\ell \frac{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}-1\right)\cdots\left(\frac{p-1}{2}-(\ell-1)\right)}{\ell!}$$

(mod p). But

$$4^\ell \equiv \left(\frac{p+1}{4}\right)^{(-\ell)} \pmod p$$

and by Fermat's Theorem

$$\left(\frac{p+1}{4}\right)^{(p-1)} \equiv 1 \pmod p,$$

moreover

$$\left(\frac{p+1}{4}\right)^{\left(\frac{p-1}{2}\right)} \equiv 1 \pmod p$$

since

$$\left(\frac{p+1}{4}\right)^{\left(\frac{p-1}{2}\right)} \equiv -1 \pmod p$$

would imply

$$\left(\frac{1}{4}\right)^{\left(\frac{p-1}{2}\right)} = 4^{\left(\frac{1-p}{2}\right)} \equiv -1 \pmod p$$

or

$$4^{\left(p-1-\left(\frac{1-p}{2}\right)\right)} \equiv -1 \pmod p,$$

applying Fermat's theorem again, and this gives

$$2^{(p-1)} \equiv -1 \pmod p$$

which is absurd since $p \neq 2$. Thus

$$4^{\lambda} \equiv \left(\frac{p+1}{4}\right)^{\left(\frac{p-1}{2}-\ell\right)} \pmod p,$$

and so:

$$\frac{(p-(\ell+1))(p-(\ell+2))\cdots(p-2\ell)}{\ell!} \equiv \left(\frac{p+1}{4}\right)^{\left(\frac{p-1}{2}-\ell\right)} \frac{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}-\ell\right)\cdots\left(\frac{p-1}{2}-(\ell-1)\right)}{\ell!}$$

$\pmod p$ which is equivalent to the lemma.

Lemma 3. $F_x(p) \equiv \pm 1$ or $0 \pmod p$, where p is prime and $p \neq 2$.

From Lemma 2, it follows that

$$F_x(p) \equiv \left(x+\frac{p+1}{4}\right)^{\left(\frac{p-1}{2}\right)} \pmod p.$$

Thus by Fermat's theorem, either

$$x \equiv -\left(\frac{p+1}{4}\right) \mod p$$

in which case $F_x(p) \equiv 0 \pmod p$, or

$$\left\{F_x(p)\right\}^2 - 1 \equiv 0 \pmod p$$

in which case $F_x(p) \equiv \pm 1 \pmod p$.

Lemma 4. $\left\{F_x(n)\right\}^2 - \left\{F_x(n-1)\right\}\left\{F_x(n+1)\right\} = -x^{(n-1)}$ for all n.

This is easily proved by induction on n using the relationship

$$xF_x(n) + F_x(n+1) = F_x(n+2).$$

Lemma 5. When $x \not\equiv 0 \pmod p$, at least one of $F_x(p)$, $F_x(p-1)$, $F_x(p+1)$ is congruent to $0 \pmod p$, where p is prime and $p \neq 2$.

It follows from Lemma 4, using Fermat's theorem, that

$$\left\{F_x(p)\right\}^2 - \left\{F_x(p-1)\right\} \cdot \left\{F_x(p+1)\right\} \equiv 1 \pmod p.$$

Thus if $F_x(p) \not\equiv 0 \pmod p$, by Lemma 3,

$$\left\{F_x(p)\right\}^2 \equiv 1 \pmod p$$

in which case

$$\left\{F_x(p-1)\right\}\left\{F_x(p+1)\right\} \equiv 0 \pmod p,$$

and the lemma follows.

Now if $x \equiv 0 \pmod p$, $F_x(n) \equiv 1 \pmod p$ for all n, by the definition of $F_x(n)$.

If $x \not\equiv 0 \pmod p$, from Lemma 5 there exists a number α such that $F_x(\alpha) \equiv 0 \pmod p$, we assume that α is the least such number, and $\alpha > 1$ since $F_x(1) = 1$ for all x. It can be shown inductively that $F_x(n+\alpha) \equiv sF_x(n) \pmod p$ for all n, where $s \equiv F_x(\alpha+1) \pmod p$, and $s \not\equiv 0$ since $s \equiv 0$ would imply $F_x(\alpha-1) \equiv 0 \pmod p$. Then if $F_x(r) \equiv 0 \pmod p$, there exists r' such that

$$r' \equiv r \pmod \alpha, \quad 0 < r' \leqslant \alpha, \quad \text{and} \quad F_x(r') \equiv 0 \pmod p.$$

By the definition of α, $r' < \alpha$ is absurd, therefore $r' = \alpha$.

Let P be prime and p a prime factor of $F_x(P)$. Then

$$F_x(P) \equiv 0 \pmod p \quad \text{and} \quad x \not\equiv 0 \pmod p$$

since, if $x \equiv 0 \pmod p$, $F_x(n) \equiv 1 \pmod p$ for all n.

Thus $P \equiv 0 \pmod \alpha$ and since P is prime, $P = \alpha$. Let p' be either $p, p-1$, or $p+1$, such that

$$F_x(p') \equiv 0 \pmod p$$

(from Lemma 3). Then p' is an integral multiple of P and the theorem follows.

I mentioned this result to Dr. P.M. Lee of York University and he has pointed out to me that Lemma 3 can be derived from H. Siebeck's work on recurring series (L.E. Dickson, *History of the Theory of Numbers,* p. 394f). A colleague of his has also discovered a non-elementary proof of the above theorem.

I am myself only an amateur mathematician, so I would ask you to excuse any resulting awkwardnesses in my presentation of this theorem and proof.

Yours faithfully,
Alexander G. Abercrombie

[Continued from Page 146.]

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There is room for considerable work regarding possible lengths of periods. For various values of p and q we found periods of lengths: 1, 2, 8, 9, 17, 25, 33, 35, 42, 43, 61, 69.

GENERALIZED PERIODS

For various sequence types, it is possible to arrive at generalized periods. Some examples are the following.

$(p, p-1)$: $2p-2, 2p-3, 2p-3, 2p-2, 2p, 2p+2, 2p+3, 2p+2, 2p$, where p is large enough to make all quantities positive.

(p,p): $2p, 2p+2, 2p, 2p+1, 2p-1, 2p, 2p-1, 2p+1$, where $p \geqslant 2$.

$2p-1, 2p+1, 2p-1, 2p+2, 2p, 2p+3, 2p, 2p+2$, where $p \geqslant 2$, and many others.

$(p+1, p)$: $2p-1, 2p, 2p+2, 2p+4, 2p+5, 2p+4, 2p+2, 2p, 2p-1$ fpr $p \geqslant 3$. (Period of length 9)

$2p, 2p+1, 2p+5, 2p+5, 2p+5, 2p+1, 2p, 2p-3, 2p-1, 2p-1, 2p+4, 2p+4, 2p+7, 2p+3,$
$2p+2, 2p-3, 2p-2, 2p-3, 2p+2, 2p+3, 2p+8, 2p+7, 2p+4, 2p+4, 2p-1, 2p-1, 2p-3,$
for $p \geqslant 24$ (Period of length 26), and many others.

A schematic method was used which made the work of arriving at these results somewhat less laborious.

NON-PERIODIC SEQUENCES

In studying the sequences (3,4), non-periodic sequences of a quasi-periodic type were found. They have the peculiar property that alternate terms form a regular pattern in groups of four, while the intermediate terms between these pattern terms become unbounded. This situation arises in sequences (p,q) for which q is greater than p.

As an example of such a non-periodic sequence in the case (4,7) the sequence beginning with 1,3,4, follows:

1, 3, 4, 37, 59, 124, 25, 17, 2, 6, 3, 27, 22, 93, 20, 34, 3, 13, 3, 35, 13, 99, 14, 58, 4, 31, 3, 58, 9, 148, 12, 121, 4, 72, 3, 129, 8, 312, 11, 279, 4, 179, 3, 317, 8, 751, 10, 663, 4, 466, 3, 819, 8, 1922, 10, 1687, 4, 1183, 3, 2074, 8, 4850, 10, 4249, 4, 2976, 3, 5211, 8, 12170, 10, ⋯.

Note the regular periodicidity of 3,8,10,4 with the sets of intermediate terms increasing as the sequence progresses.

The various types of non-periodic sequence for (4,7) are:

[Continued on Page 184.]

SIGNED b-ADIC PARTITIONS

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INTRODUCTION

The common type of partition problem can be stated as follows: let $S \subseteq N$, given $n \in N$, how many ways can we write $n = s_1 + s_2 + \cdots + s_k$, $s_i \in S$? For instance, S might be the squares or the cubes, k might be fixed or not.

This paper considers the question: given b, how many ways can we write $n = a_0 + a_1 b + a_2 b^2 + \cdots + a_m b^m$, $a_i \in \{0, 1, -1, 2, -2, \cdots, b-1, 1-b\}$? An algorithm is derived to answer this question. This algorithm produces for each n a tree, for which questions of height and width are answered.

1. THE DECOMPOSITION ALGORITHM

1.1 Definition. Let $b > 1$ be fixed. A *k-decomposition* of n, $k > 0$, is a partition of n of the form $n = a_0 + a_1 b + a_2 b^2 + \cdots + a_m b^m$, where each $a_i \in \{0, 1, -1, 2, -2, \cdots, b-1, 1-b\}$ and $a_i \neq 0$ for exactly k values of i. A *decomposition* of n is a k-decomposition of n for some (unspecified) k.

The number of k-decompositions of n will be denoted $R_k(n)$. Clearly $R_k(-n) = R_k(n)$, so WLOG we shall assume that $n \geq 0$.

1.2 Theorem.

(a)
$$R_k(bn) = R_k(n)$$

(b) If $n \equiv a \pmod b$, $a \neq 0$, and if $k > 1$, then
$$R_k(n) = R_{k-1}(n-a) + R_{k-1}(n-a+b)$$

(c)
$$R_1(n) = \begin{cases} 1 & \text{if } n = ab^j \text{ for some } j \geq 1, \text{ some } 0 < a < b \\ 0 & \text{if } n \neq ab^j \text{ for any } j, \text{ any } a \end{cases}$$

(d)
$$R_k(0) = 0 \quad \text{for all } k$$

(e)
$$\text{If } 0 < a < b, \text{ then } R_k(a) = 1 \quad \text{for all } k.$$

Proof.

(a) Given any k-decomposition of n, multiplying the expression by b produces a k-decomposition of bn. So $R_k(bn) \geq R_k(n)$. Given any k-decomposition of bn, $bn = a_0 + a_1 b + a_2 b^2 + \cdots + a_m b^m$, clearly $b \mid a_0$, so $a_0 = 0$. Dividing the expression by b produces a k-decomposition of n. So $R_k(n) \geq R_k(bn)$.

(b) Let $n \equiv a \pmod b$. Consider any k-decomposition of n, $n = a_0 + a_1 b + \cdots + a_m b^m$. $n \equiv a_0 \pmod b$; hence $a \equiv a_0 \pmod b$. Thus either $a = a_0$ or $a = a_0 + b$. That is, the first term of the decomposition is either a or $a - b$. The remaining $k - 1$ terms then are a $(k-1)$-decomposition of $n - a$ or of $n - (a - b)$, respectively.

(c) Immediate from the definition.

(d) Assume false. Then for some k there is at least one k-decomposition of 0, $0 = a_0 + a_1 b + \cdots + a_m b^m$. Place the terms with $a_i < 0$ on the left side of the expression. Then some integer has two distinct representations in base b—contradiction.

(e)
$$\begin{aligned} R_k(a) &= R_{k-1}(a-a) + R_{k-1}(a-a+b) \quad \text{by part (b).} \\ &= 0 + R_{k-1}(1) \quad \text{by parts (d) and (a)} \\ &= R_{k-2}(1-1) + R_{k-2}(1-1+b) = 0 + R_{k-2}(1) \\ &= \cdots = R_1(1) \\ &= 1 \quad \text{by part (c).} \end{aligned}$$

This theorem enables us quickly to find $R_k(n)$. Moreover, unwinding the algorithm, we can find the k-decompositions.

Example 1. Let $b = 4$.

$$R_5(3) = R_4(0)+R_4(4) = 0+R_4(1) = R_3(0)+R_3(4) = R_3(1) = R_2(0)+R_2(4) = R_2(1) = R_1(0)+R_1(4) = 1,$$

a result we know already. Unwinding the algorithm,

$$4 = 4, \quad 1 = -3+4, \quad 4 = -12+16, \quad 1 = -3-12+16, \quad 4 = -12-48+64,$$
$$1 = -3-12-48+64, \quad 4 = -12-48-192+256,$$
$$3 = -1-12-48-192+256 = -1-3\cdot 4-3\cdot 4^2-3\cdot 4^3+1\cdot 4^4 .$$

The pattern is clear, so from now on we shall use part (e) of the theorem and stop the algorithm whenever the argument n is less than b. Moreover, because of part (a), we shall consider only n such that b does not divide n.

Example 2. Let $b = 3$.

$$R_4(17) = R_3(15)+R_3(18) = R_3(5)+R_3(2) = R_2(3)+R_2(6)+R_3(2) = R_2(1)+R_2(2)+R_3(2) = 1+1+1 = 3.$$

Unwinding,

$1 = -2+3$	$2 = -1+3$	$2 = -1-6+9$
$3 = -6+9$	$6 = -3+9$	$18 = -9-54+81$
$5 = 2-6+9$	$5 = -1-3+9$	$17 = -1-9-54+81$
$15 = 6-18+27$	$15 = -3-9+27$	
$17 = 2+6-18+27$	$17 = 2-3-9+27$	
$= 2+2\cdot 3-2\cdot 3^2+1\cdot 3^3$		

Example 3. Let $b = 2$.

$$R_3(11) = R_2(10)+R_2(12) = R_2(5)+R_2(3) = R_1(4)+R_1(6)+R_1(2)+R_1(4) = 1+0+1+1 = 3.$$

Unwinding,

$4 = 4$	$2 = 2$	$4 = 4$
$5 = 1+4$	$3 = 1+2$	$3 = -1+4$
$10 = 2+8$	$12 = 4+8$	$12 = -4+16$
$11 = 1+2+8$	$11 = -1+4+8$	$11 = -1-4+16$

1.3. Each time k decreases by one, each term $R_k(\cdot)$ splits into at most two terms $R_{k-1}(\cdot)$. In completing the algorithm, there are $k-1$ such steps. Hence $R_k(n) \leqslant 2^{k-1} < 2^k$ for all n. We have the well known result

Theorem. $\{b^i : i = 0, 1, 2, \cdots\}$ is a Sidon set. (See [2], pp. 124, 127.)

1.4 Lemma. If $n = a_0 + a_1 b + a_2 b^2 + \cdots + a_m b^m$ is any decomposition of n, $a_m \neq 0$, then $a_m > 0$.

Proof. If $a_m < 0$, then

$$n = \sum_{i=0}^{m-1} a_i b^i + a_m b^m \leqslant \sum_{i=0}^{m-1} (b-1)b^i - b^m = b^m - 1 - b^m = -1$$

—a contradiction.

1.5 Definition. A k-decomposition of n is *basic* if (a) $a_m > 1$, or if (b) $a_{m-1} \geqslant 0$ (or both).

Theorem. Let $b^{h-1} < n < b^h$. Then for any basic decomposition of n,

(a) $$i > h \Rightarrow a_i = 0$$

(b) $$0 \leqslant a_h \leqslant 1$$

(c) If $a_h = 0$, then $a_{h-1} > 0$

(d) If $a_h = 1$, then $a_{h-1} = 0$; and if $a_j b^j$ is the last non-zero term before $a_h b^h$, then $a_j < 0$.

Proof. (a) By the lemma above, if $a_m b^m$ is the last non-zero term, $a_m > 0$. Assume $m > h$.

Case 1. $a_m > 1$. Then

$$n = \sum_{i=0}^{m} a_i b^i \geqslant \sum_{i=0}^{m-1} (1-b)b^i + 2b^m = b^m + 1 > b^h$$

—a contradiction.

Case 2. $a_m = 1$ and $a_{m-1} \geqslant 0$. Then

$$n \geqslant \sum_{i=0}^{m-2} (1-b)b^i + 0b^{m-1} + b^m = 1 + b^{m-1}(b-1) \geqslant 1 + b^{m-1} \geqslant 1 + b^h$$

—a contradiction.

(b) By part (a), there are no terms in the decomposition after $a_h b^h$, so $a_h \geqslant 0$. Assume $a_h > 1$. Then

$$n \geqslant \sum_{i=0}^{h-1} (1-b)b^i + 2b^h = 1 + b^h$$

—a contradiction.

(c) If $a_h = 0$, then there are no terms after $a_{h-1}b^{h-1}$, so $a_{h-1} \geqslant 0$. Assume $a_{h-1} = 0$. Then

$$n \leqslant \sum_{i=0}^{h-2} (b-1)b^i = b^{h-1} - 1$$

—a contradiction.

(d) If $a_h = 1$, then by the definition of a basic decomposition $a_{h-1} \geqslant 0$. Assume $a_{h-1} > 0$. Then

$$n \geqslant \sum_{i=0}^{h-2} (1-b)b^i + 1b^{h-1} + 1b^h = 1 + b^h$$

—a contradiction. The same reasoning shows that if the next to last non-zero coefficient is a_j, $j < h$, then $a_j < 0$.

Corollary. Let $b^{h-1} < n < b^h$, and let $k > h$. Then no k-decomposition of n is basic.

Proof. Every basic decomposition of n ends with $a_{h-1}b^{h-1}$ or with $a_{h-2}b^{h-2} + 0 \cdot b^{h-1} + 1 \cdot b^h$. In either case there are at most h non-zero terms in the sum.

1.6 Theorem. Starting with $R_k(a)$, $0 < a < b$, the unwinding of the algorithm produces a basic decomposition of n iff $k = 1$.

Proof. Start with a k-decomposition of a.

Case 1. $k = 1$. The reverse algorithm starts: $x_1 = a$; then $x_2 = ab^p$, $p \geqslant 1$; then $x_3 = ab^p + a'$.

Case 1a. $a > 1$ or $p > 1$. Then a' can be any integer such that $0 < |a'| < b$.

Case 1b. $a = p = 1$. Then $x_3 = b + a'$. If $a' < 0$, $x_3 < b$. But the forward algorithm stops as soon as the argument is less than b. So $a' > 0$. In either case there is a basic 2-decomposition of x_3. The next step is to multiply by b^q for some $q \geqslant 1$. Clearly the resulting 2-decomposition is basic. Then add a''; the new 3-decomposition is still basic. Continue until a basic decomposition of n is reached.

Case 2. $k > 1$. By the corollary above, since $a < b^1$, no k-decomposition of a is basic. That is, the reverse algorithm starts

$$a = a_0 + a_1 b + \cdots + a_{m-1}b^{m-1} + b^m,$$

with $a_{m-1} < 0$. Multiplying by b^p produces a non-basic k-decomposition. Then adding a' gives a non-basic $(k+1)$-decomposition. Continue, ending with a non-basic decomposition of n.

1.7 Definition. Let $B_k(n)$ be the number of basic k-decompositions of n. Let

$$B(n) = \sum_{k=1}^{\infty} B_k(n).$$

Remark. Since $n < b^h$, $k > h \Rightarrow B_k(n) = 0$ (corollary above), the sum is only finite.

Theorem. If $b^{h-1} < n < b^h$, $k > h$, then $R_k(n) = R_h(n) = B(n)$; and $B(n) \leqslant 2^{h-1}$.

Proof. If $k > h$, no k-decomposition of n is basic. Thus the algorithm goes all the way: every end term is of the form $R_s(a)$, $0 < a < b$, $s > 1$. Once all the $a < b$ appear, no more decompositions can appear. Each basic decomposition occurs from unwinding each $R_1(a)$, choosing $k \leqslant h$ so that $s = 1$ when the a first appears. The inequality is from 1.3.

2. THE CASE $b = 2$

From the algorithm, we see that if neither n nor $n+1$ is divisible by b, then their k-decompositions differ only in the first term. Therefore, for simplification we shall assume that $b = 2$, unless specifically stated otherwise. Of course, we restrict n to be odd.

2.1 By the algorithm, $R_k(n) = R_{k-1}(n-1) + R_{k-1}(n+1)$. Let $n - 1 = 2^p x$ and $n + 1 = 2^q y$, x and y odd. Note that $\min(p,q) = 1$ and that $\max(p,q) \geqslant 2$, as $n-1$ and $n+1$ are consecutive even integers.

Definition. Given x, y odd, if there exists an (odd) n such that $R_k(n) = R_{k-1}(x) + R_{k-1}(y)$, write $x * y = n$. If no such n exists, then $x * y$ is undefined.

Remark. By the uniqueness of the algorithm,

(a) $\quad x * y = y * x$ if either exists, and

(b) $\quad x * y = u * v \Rightarrow \{x,y\} = \{u,v\}$.

2.2 Theorem. Let y be given. If $x \geqslant y$, then $x * y$ exists iff $x = 2^i y + 1$ or $x = 2^i y - 1$ for some $i \geqslant 1$. If so, then $x * y = 2^{i+1}y + 1$ or $x * y = 2^{i+1}y - 1$, respectively.

Proof. By the algorithm, if $x * y$ is to exist, there must exist p, $q \geqslant 1$ such that $2^p x - 2^q y = \pm 2$. By the note above, $p = 1$ and $q \geqslant 2$. So $x = 2^{q-1}y \pm 1$. Let $i = q - 1 \geqslant 1$. $x * y$ is the odd integer between $2x$ and $2^q y$. So

$$x * y = (½)[2x + 2^q y] = (½)[2(2^i y \pm 1) + 2^{i+1}y] = 2^{i+1} \pm 1.$$

Corollary. If GCD $(x,y) > 1$, then $x * y$ does not exist. In particular, if $y > 1$, then $y * y$ does not exist.

2.3 Theorem. $3 * 1 = \{5,7\}$. In all other cases, $x * y$ is unique.

Proof. WLOG $x \geqslant y$. If $x * y$ exists, $x = 2^i y \pm 1$. If $x * y$ is not unique, then x must be expressible in two ways, i.e.,

$$x = 2^p y + 1 = 2^q y - 1$$

for some p, $q \geqslant 1$. Then

$$2^q y - 2^p y = 2, \qquad 2^{q-1}y - 2^{p-1}y = 1.$$

Since y divides the left side, $y = 1$. Then $p = 1$ and $q = 2$. So

$$x = 2^1 \cdot 1 + 1 = 3 = 2^2 \cdot 1 - 1, \quad \text{and} \quad x * y = 2^2 \cdot 1 + 1 = 5, \qquad x * y = 2^3 \cdot 1 - 1 = 7.$$

2.4 Theorem. Given $x > 3$, there exist two y, $y < x$, such that $x * y$ exists.

Proof. $x = 2^i y \pm 1$, so $y = (x-1)/2^p$ and $y = (x+1)/2^q$, y odd. These numbers are distinct unless $(x-1)/2^p = (x+1)/2^q$. If so, then since $x - 1$, $x + 1$ are consecutive even numbers, both divisible by some power of 2, $x = 3$.

Corollary. If $a * b$ exists, then the integers y, $y < a * b$, such that $(a * b) * y$ exists are $y = a$ and $y = b$.

Proof. If $a * b$ exists, WLOG $a \geqslant b$. Then $a = 2^i b \pm 1$. By the theorem, if $(a * b) * y$ exists, then

$$y = \frac{(a*b) \mp 1}{2^p} = \frac{(2^{i+1}b \pm 1) \mp 1}{2^p} = \left\{ \frac{2^{i+1}b}{2^p}, \frac{2^{i+1}b \pm 2}{2^q} \right\}$$

$$= \{b, 2^i b \pm 1\} = \{b, a\}.$$

Remark. If $a = b$, by the Corollary of 2.2, $a = b = 1$, and so $y = 1$.

2.5 Theorem. If $x * y$ exists, then exactly one of $\{x, y, x * y\}$ is divisible by 3.

Proof. $1 * 1 = 3$. Assume now WLOG that $x > y$. So $x = 2^i y \pm 1$, and $x * y = 2^{i+1}y \pm 1$.

Case 1. Clearly if $3 | y$, 3 divides neither x nor $x * y$.

Case 2. If $3 | x$, 3 cannot divide y. Assume $3 | x * y$. Then $3 | (x * y - x)$, so

$$3 | (2^{i+1}y - 2^i y), \; 3 | 2^i y \text{ — a contradiction.}$$

Case 3. Assume that 3 divides neither x nor y. To show $3 | x * y$.

Case 3a. $y \equiv 1 \pmod 3$. Since $2^i \equiv (-1)^i \pmod 3$,

$$x = 2^i y \pm 1 \equiv (-1)^i \pm 1 \pmod 3.$$

Since $x \not\equiv 0 \pmod 3$, if i is even, we must use the $+1$, and if i is odd, we must use the -1. Then

$$x * y = 2^{i+1}y \pm 1 \equiv (-1)^{i+1} \pm 1 \pmod 3, \qquad x * y \equiv 0 \pmod 3$$

whether i is even or odd.

Case 3b. $y \equiv -1 \pmod 3$. Then

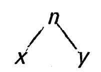
$$x \equiv (-1)^{i+1} \pm 1 \pmod 3.$$

If i is even, we use the -1; if i is odd, the $+1$. So

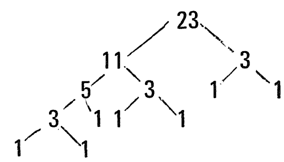
$$x * y \equiv (-1)^{i+2} \pm 1 \pmod 3 \equiv 0 \pmod 3$$

in both cases.

2.6. The expression $n = x * y$ can conveniently be expressed visually as



If x or $y > 1$, it in turn can be written as a $*$-product. Each n has in this manner associated with it a tree. For example, for $n = 23$, the tree is as in Fig. 1.



$$H(23) = 5, \quad W(23) = 7$$

Figure 1

Remark. Since $x,y < x * y$, the numbers decrease down the tree, and every chain ends with 1. The tree associated with n, without integers at the nodes, with the longer chain always to the left at every node, will be denoted $T(n)$.

2.7 Definition. If the length of the longest chain in the tree is ℓ, then the *height* of the tree, denoted $H(n)$, is defined by $H(n) = \ell + 1$. The number of branches of the tree (= number of times 1 appears) is the *width* of the tree, denoted by $W(n)$.

Lemma. Let $n = x * y$, $x \geqslant y$. Then

(a) $$H(n) = 1 + H(x)$$

(b) $$W(n) = W(x) + W(y).$$

Proof. Obvious from the definition of $T(n)$.

Theorem. Let $2^{h-1} < n < 2^h$. Then

(a) $$H(n) = h$$

(b) $$W(n) = B(n), \text{ the number of basic decompositions}$$

(c) $$h \leqslant W(n) \leqslant 2^{h-1}$$

Proof. (a) If $h = 1$, $H(1) = 1$; if $h = 2$, $H(3) = 2$. Assume that for all $n < 2^k$, the statement is true. Let $2^k < n < 2^{k+1}$. The algorithm starts: $R_s(n) = R_{s-1}(n-1) + R_{s-1}(n+1)$.

Case 1. $n - 1$ is divisible by 4. Then $n + 1$ is not divisible by 4, so $2^k < n + 1 < 2^{k+1}$. $2^{k-1} < (n+1)/2 < 2^k$. By the inductive hypothesis, $H((n+1)/2) = k$. By the lemma, $H(n) = k + 1$.

Case 2. $n + 1$ is divisible by 4. Then $2^k < n - 1 < 2^{k+1}$; $2^{k-1} < (n-1)/2 < 2^k$. So $H((n-1)/2) = k$; $H(n) = k + 1$.

(b) The algorithm produces the numbers at the nodes of the tree. As soon as a 1 appears, the branch stops. Starting with $R_1(1)$, following each chain upwards produces each of the basic decompositions.

(c) The second inequality is the Theorem of 1.7. The first is obvious for $n = 1, 3$. Assume the first inequality is true for all $n < 2^k$. Let $2^k < n < 2^{k+1}$. $n = x * y$ for some $x > y$, $2^{k-1} < x < 2^k$. By the inductive hypothesis, $W(x) \geqslant k$. So $W(n) = W(x) + W(y) \geqslant k + 1$.

2.8 Lemma. Let $0 < t < 2^{h-1}$, t odd. Then $T(2^{h-1} + t) = T(2^h - t)$.

Proof. If $h = 2$, then $t = 1$. $2^{2-1} + 1 = 3 = 2^2 - 1$; the result is automatically true. If $h = 3$, then $t = 1$ or 3. $2^{3-1} + 1 = 5$ and $2^3 - 1 = 7$; while $2^{3-1} + 3 = 7$ and $2^3 - 3 = 5$. We know $T(5) = T(7)$.

Assume that the statement is true for all $k \leqslant h$. Let t be any odd number such that $0 < t < 2^k$. If $2^k + t = 2^{k+1} - t$, then $t = 2^{k-1}$; since t is odd, $t = k = 1$.

Case 1. $t+1$ is divisible by 4. Then

$$2^k+t = \frac{2^k+t+1}{2^p} * \frac{2^k+t-1}{2},$$

where 2^p is the highest power of 2 that divides $t+1$, $2 \leqslant p \leqslant k$.

$$= \left(2^{k-p}+\frac{t+1}{2^p}\right) * \left(2^{k-1}+\frac{t-1}{2}\right);$$

and

$$2^{k+1}-t = \frac{2^{k+1}-(t+1)}{2^p} * \frac{2^{k+1}-(t-1)}{2} = \left(2^{k-p+1}-\frac{t+1}{2^p}\right) * \left(2^k-\frac{t-1}{2}\right).$$

By the inductive hypothesis,

$$T\left(2^{k-p}+\frac{t+1}{2^p}\right) = T\left(2^{k-p+1}-\frac{t+1}{2^p}\right)$$

and

$$T\left(2^{k-1}+\frac{t-1}{2}\right) = T\left(2^k-\frac{t-1}{2}\right) .$$

Thus $T(2^k+t)$ and $T(2^{k+1}-t)$ have the same right branch, the same left branch, and therefore are equal.

Case 2. $t-1$ is divisible by 4. Interchange $t-1$, $t+1$ in the above proof.

Theorem. If $h \geqslant 3$, there are 2^{h-3} different trees of height h associated with the odd integers.

Proof. For $h=3$, $T(5)=T(7)$, so there is one tree of height 3. Let $k \geqslant 3$. To each x, $2^{k-1} < x < 2^k$ there exist $y_1 \neq y_2$, $y_i < x$, such that $x * y_i$ exists. Since $H(y_1) \neq H(y_2)$, $T(x*y_1) \neq T(x*y_2)$. Therefore the number of trees of height $k+1$ is at least twice the number of trees of height k. Hence the number of trees of height h is at least 2^{h-3}.

Between 2^{h-1} and 2^h there are 2^{h-2} odd integers. By the lemma, each tree of height h is associated with at least two integers. Hence the number of trees of height h is at most 2^{h-3}.

2.9 Theorem. $W(2^{h-1}+1) = W(2^h-1) = h$; the minimum possible width of a tree of height h is attained.

Proof. If $h=3$, $W(2^{3-1}+1) = W(5) = 3$. Assume that $W(2^{k-1}+1) = k$.

$$2^k+1 = (2^{k-1}+1)*1.$$

It follows that

$$W(2^k+1) = W(2^{k-1}+1)+W(1) = k+1.$$

Since $W(n) \geqslant h$ if $2^{h-1} < n < 2^h$, the minimum width is attained. Lastly, by the lemma above, $W(2^h-1) = h$.

Theorem. (a) The maximum width of any tree of height h is F_{h+1}, where F_i is the i^{th} Fibonacci number.

(b) This width is attained for

$$n = (2^{h+1}+(-1)^h)/3, \quad h \geqslant 1,$$

and for

$$n = (5\cdot 2^{h-1}+(-1)^{h-1})/3, \quad h \geqslant 2.$$

Proof. For $h=1$, $W(1)=1$. For $h=2$, $W(3)=2$. For $h=3$, $W(5) = W(7) = 3$.

(a) For each k, the maximum width is attained by at least two values of n. Call the smallest of these values n_k, i.e., $\{n_k\} = \{1, 3, 5, 11, \cdots\}$. Assume:

(1) $$W(n_i) = F_{i+1}, \qquad i = 1, 2, \cdots, k$$

(2) $n_k = n_{k-1} * n_{k-2}$. The two inductive hypotheses are true for $k=3$. By the Corollary of 2.4, $n_k * n_{k-1} = n$ exists; so

$$W(n) = W(n_k)+W(n_{k-1}) = F_{k+1}+F_k = F_{k+2}.$$

$T(n)$ has as its left branch the widest tree of height k, as its right branch the widest tree of height $k-1$. Hence $T(n)$ is the widest tree of height $k+1$, and there is only one such tree. Since n is the smaller integer whose tree has this width, $n = n_{k+1}$.

(b) Claim: $n_h = 2n_{h-1}+(-1)^h$. Statement is true for $h=2$. Assume it is true for $h=k$. Then $2n_k = 4n_{k-1}+2(-1)^k$. Using the algorithm, we can calculate $n_{k+1} = n_k * n_{k-1}$. Since $2n_k$ and $4n_{k-1}$ differ by 2,

$$n_{k+1} = (½)[2n_k + 4n_{k-1}] = (½)[2n_k + 2n_k - 2(-1)^k] = 2n_k + (-1)^{k+1}.$$

Claim proved. Assume

$$n_k = \frac{2^{k+1} + (-1)^k}{3}.$$

By the claim,

$$n_{k+1} = 2\left(\frac{2^{k+1}+(-1)^k}{3}\right) + (-1)^{k+1} = \frac{2^{k+2}+(-1)^{k+1}}{3}.$$

Lastly, if m_h is the larger number such that $W(m_h) = F_{h+1}$, by the Lemma of 2.8, $m_h + n_h = 2^{h-1} + 2^h$. So

$$m_h = 3\cdot 2^{h-1} - n_h = \frac{5\cdot 2^{h-1} + (-1)^{h-1}}{3}.$$

Theorem. If the base is $b > 2$, then $W((b^h - 1)/(b-1)) = 2^{h-1}$; that is, the maximum width attained is the maximum possible.

Proof. It is clear that $W(b+1) = W(b+2) = 2$. Assume that $W(m) = W(m+1) = 2^{k-1}$ where $m = (b^k - 1)/(b-1)$.

$$m * (m+1) = \{bm+1, bm+2, \cdots, bm+b-1\}$$

(from the obvious definition of $x * y$, $\{x * y\}$ has at least $b - 1$ elements.) So

$$W(bm+1) = W(bm+2) = W(m) + W(m+1) = 2^k \quad \text{and} \quad bm+1 = b\left(\frac{b^k-1}{b-1}\right) + 1 = \frac{b^{k+1}-1}{b-1}.$$

Remark. Comparison of the preceding two theorems shows why the special case $b = 2$ is more interesting than the general case. The trees for $b = 2$ are of special type: at any node the two sub-trees are always of unequal heights.

3. THE PROBLEM OF WIDTHS

3.1 Theorem. $2|W(n)$ iff $3|n$.

Proof. $W(1) = 1$ and $W(3) = 2$. Assume the statement is true for all $n \leqslant k$. Consider $W(k+1)$. Let $k+1 = x * y$.

Case 1. $k+1$ is divisible by 3. By the Theorem of 2.5, neither x nor y is divisible by 3. By the inductive hypothesis $W(x)$ and $W(y)$ are odd. Hence $W(k+1) = W(x) + W(y)$ is even.

Case 2. $k+1$ is not divisible by 3. Then one of x,y is. So $W(k+1)$ = even + odd = odd.

3.2. An interesting but unsolved question is the following: given w, find all (odd) n such that $W(n) = w$.

If $n > 2^w$, then $H(n) > w$, so $W(n) > w$ (Theorem of 2.7). Thus all solutions n satisfy $n < 2^w$. At least one pair of solutions always exists, because

$$W(2^{w-1} + 1) = W(2^w - 1) = w$$

(first Theorem of 2.9). From the theorem above it appears that there should be fewer solutions for w even than for w odd. An examination of a short table of solutions, found by the algorithm, shows little regularity.

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2. Walter Rudin, *Fourier Analysis on Groups,* Interscience, New York, 1962.

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IDIOT'S ROULETTE REVISITED

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The winter 1973 issue of the California Mathematics Council *Bulletin* carried an article under the title "Idiot's Roulette." It discussed a counting-out puzzle, in which N people stand in a circle surrounding an executioner, who goes around and around the circle, shooting every second person as he counts. The problem is to determine the "safe" position, X, as a function of N. That is—which will be the last person left, according to the original numbering? An intuitive solution was presented, developed by looking for patterns, and the author asked for further comments on possible proofs.

The problem is a special case of a more general counting-out problem I had been playing with the previous fall, although in a somewhat less bloodthirsty fashion, —and the analysis which provides an iterative solution for the general case incidentally yields a closed-form solution for the special case where the countoff spacing = 2.

The general problem: Given N places around the circle and a countoff spacing = C, such that every C^{th} place drops out, the count continuing around the circle until only one place is left, —which of the numbers 1 to N will be the last place L ?

Assume the count "1" starts with place number 1. A different starting point simply rotates the problem around the circle, changing nothing essential. This seemingly trivial observation, however, provides a key to the analysis and solution of the problem. So let us consider what happens if we start the count at some other number, say at $J + 1$ instead of 1. This is equivalent to rotating the problem J places around the circle, so the game would end at $L + J$ instead of L, unless $L + J > N$, in which case the modular nature of our numbering makes the last place $L + J - N$.

Now return to the original problem. The count starts at place number 1, with countoff spacing C and N people. Call the solution for the last place winner L_N. (For simplicity in the following discussion, we shall restrict ourselves to the case where $N \geqslant C - 2$. See footnote 1 for more complete analysis.) L_N is a function of C and N. Now consider the problem for the same countoff spacing C, but with one more person in the circle. After our first loser is counted out at place C, this reduces to a circle of N places in which the count starts at $C + 1$. So $L_{N+1} = L_N + C$, unless $L_N + C > N + 1$, in which case we have

$$L_{N+1} = L_N + C - (N + 1).$$

The table on the following page shows the situation, for example, for $C = 2$, and several values of N.

I shall now introduce some terminology which will help us develop an iterative solution for the general problem. Noting that, for a given C, each time we add a place to the circle we add C to the old solution, write the solution in the form $L_N = CN - I_N$, since some integer I_N certainly must exist which will make the statement true. (Example: In the table, where $C = 2$ and $N = 4$, $L_4 = 2(4) - 7$, and $I_4 = 7$. For $N = 5, 6$, and 7 also, $I_N = 7$. For $N = 8$, however, this is no longer true. $I_8 = 15$.)

---

[1] If $C = 1$, the problem is trivial, with $L_N = N$ for all N. If $C > 1$, the general statement becomes:

$$L_{N+1} = L_N + C - T_N(N + 1) \quad \text{and} \quad I_{N+1} = I_N + T_N(N + 1), \quad \text{where} \quad T_N = \left[C - \frac{I_N + 1}{N + 1} \right].$$

For $N \geqslant C - 2$, T_N must be either 0 or 1, and the analysis in the article holds completely. For small N, however, some of the S values generated may not actually be used, with the general statement being: If $I_N = S_K$,

$$I_{N+1} = S_{k+T_N}.$$

For example, if $C = 4$, $S = \{3, 5, 7, 10, 14, 19, 26, \cdots\}$. $I_1 = S_1 = 3$; $I_2 = S_3 = 7$, since

$$T_N = \left[4 - \frac{3 + 1}{2} \right] = 2.$$

Similarly, for $C = 7$, $S = \{6, 8, 10, 12, 15, 18, 22, 26, 31, 37, \cdots\}$ $I_1 = S_1 = 6$; $I_2 = S_4 = 12$; $I_3 = S_6 = 18$; $I_4 = S_8 = 26$; $I_5 = S_9 = 31$.

N	L_N	L_{N+1}	$(C = 2)$
4	1	3	
5	3	5	
6	5	7	
7	7	1	$(7 + 2 - 8)$
8	1	3	
9	3	5	

The problem now is to find the appropriate I_N such that $L_N = CN - I_N$, where $1 \leqslant L_N \leqslant N$.

We can restate the condition that $CN - I_N \leqslant N$ to obtain

(1) $$N \leqslant \frac{I_N}{C-1} .$$

Next look at the statements about $L_{N+1} = C(N+1) - I_{N+1}$: If

(2) $$L_N + C \leqslant N+1, \qquad L_{N+1} = L_N + C = CN - I_N + C = C(N+1) - I_N .$$

Thus, if $L_N + C \leqslant N+1$, $I_{N+1} = I_N$, while if

(3) $$L_N + C > N+1, \qquad L_{N+1} = L_N + C - (N+1) = C(N+1) - (I_N + N + 1) \quad \text{and} \quad I_{N+1} = I_N + N + 1 .$$

Call S the set of *distinct* subtraction integers, where $I_N \neq I_{N+1}$, and let M be the set of $(N+1)$ values at which this occurs. Then we can restate, from (3), $S_{k+1} = S_k + M_k$; and also rewrite the inequality

$$L_N + C = CN - S_k + C = C(N+1) - S_k = CM_k - S_k > M_K ,$$

from which we obtain:

(4) $$M_k > \frac{S_k}{C-1} .$$

Similarly, rewriting (1) we have

$$M_k - 1 \leqslant \frac{S_k}{C-1}, \qquad \text{or} \qquad M_k \leqslant \frac{S_k}{C-1} + 1 .$$

Combining this statement with (4), we obtain

(5) $$\frac{S_k}{C-1} < M_k \leqslant \frac{S_k}{C-1} + 1 ,$$

which can be solved in terms of the greatest integer function:

(6) $$M_k = \left[\frac{S_k}{C-1} \right] + 1,$$

(where $[x]$ is defined as the greatest integer $\leqslant x$).

For a circle where $N = M_k$ places, then, our last place winner

$$L = CM_k - (S_k + M_k).$$

Since $S_{k+1} = S_k + M_k$, we have the following iterative formula for subtraction integers:

(7) $$S_{k+1} = S_k + \left[\frac{S_k}{C-1} \right] + 1 = \left[\frac{C}{C-1} S_k \right] + 1 .$$

To obtain a starting point for the set of S values, we note that, for $N = 1$, $L_N = 1$, whatever the value of C. Hence $1 = C - S_1$, and $S_1 = C - 1$. Given a particular C, we can generate a set of subtraction integers[1]. For example, for $C = 3$:

$$S_1 = 2; \quad S_{k+1} = \left[\frac{3}{2} S_k \right] + 1,$$

and the set of S values is

$$\{ 2, 4, 7, 11, 17, 26, 40, 61, \cdots \}$$

To apply the formula $L_N = CN - S_k$, we simply choose the proper S_k so that[2]

$$1 \leqslant L_N \leqslant N .$$

(Uniqueness of S_k can be shown readily from the equivalent condition that $(C-1)N \leqslant S_k \leqslant CN.$)

For the very special case of $C = 2$, the solution reduces neatly to a closed form, because

$$\frac{C}{C-1} = 2 ,$$

an integer. We can show by mathematical induction that for $C = 2$,

$$S_k = 2^k - 1 ,$$

since

$$S_1 = C - 1 = 2^1 - 1 ,$$

and

$$S_{k+1} = 2S_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1 .$$

Therefore we can write: If

(8) $$C = 2, \qquad L = 2N - (2^k - 1) \quad \text{and} \quad 1 \leqslant 2N - (2^k - 1) \leqslant N .$$

By rewriting the inequality in (8) we can obtain an explicit solution for k in terms of N. We have

$$2^k - 1 + 1 \leqslant 2N ;$$

hence $2^k \leqslant 2N$, and $k \leqslant 1 + \log_2 N$. We Also have

$$2N \leqslant N + 2^k - 1 ;$$

therefore $N \leqslant 2^k - 1$, and $N < 2^k$. Thus $\log_2 N < k$. Combining the inequalities:

(9) $$\log_2 N < k \leqslant 1 + \log_2 N, \quad \text{so} \quad k = 1 + [\log_2 N] .$$

An explicit formula can therefore be written for L.

(10) $$L = 2N - (2^{1+[\log_2 N]} - 1) = 1 + 2(N - 2^{[\log_2 N]})$$

and the roulette player can avoid the executioner if he quickly counts how many share his possible fate and uses his fingers to calculate powers of 2! [3]

---

[2] I tried a number of computer runs to obtain M_k and S_k sets for various values of C. The resulting sequences of num-bers looked hauntingly familiar, as though they ought to be expressible in some more elegant form. It might be interesting to follow up on this.

[3] This paper also provides a solution for the Population Explosion problem of Brother Alfred Brousseau, *The Fibonacci Quarterly,* Vol. 6, No. 1 (February 1968), pp. 58–59.

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[Continued from Page 173.]

1-1-29-29	2-2-15-15	3-3-10-10	4-4-8-8
5-5-6-6	1-2-29-15	1-15-29-2	1-3-29-10
1-10-29-3	1-4-29-8	1-8-29-4	1-5-29-6
1-6-29-5	2-3-15-10	2-10-15-3	2-4-15-8
2-8-15-4	2-5-15-6	2-6-15-5	3-4-10-8
3-8-10-4	3-5-10-6	3-6-10-5	4-5-8-6
4-6-8-5			

For any given values of p and q, it is not difficult to determine all such non-periodic sequence types.

A MODIFIED TYPE OF SEQUENCE

The students created another type of sequence in which the multipliers interchange their position from one step to the next. Thus for $T_1(2,1)$ where the multipliers are 2 and 1 and then 1 and 2, starting with 2,5,7, the next term is $[(2 * 7 + 5)/2]^* = 10$; the following term is $[(1 * 10 + 2 * 7)/5]^* = 5$, etc. The periods for $T_1(2,1)$ were found to be:

A: 3,3,2,3,3,5,4,5	*E:* 2,4,2,4	*I:* 4,2,4,3,6,4,6,3
B: 3,3,3,3,···	*F:* 4,3,4,3	*J:* 5,1,6,3,15,6,12,2
C: 3,2,3,3,5,5,5,3	*G:* 2,5,2,5	*K:* 5,3,5,3
D: 2,3,2,4,3,5,3,4	*H:* 3,1,3,3,9,7,9,3	*L:* 6,1,5,2,12,6,15,3
		M: 3,4,3,4

It should be noted that a period 3,4,3,4 in this setup is not the same as a period 4,3,4,3. Evidently this opens up another broad area for investigation.

CONCLUSION

The purpose of reporting this research is in the first instance to offer a model of cooperative effort where teacher and students work on a problem of unknown potential. Secondly, we feel that we have just scratched the surface and wish to open up the many possibilities to interested parties, especially people who have access to computer time.

Just a few of the points for investigation may be indicated.

(1) A major conjecture to be proved: For sequences of type (p,q), if $p \geqslant q$, all sequences are periodic; for sequences with $p < q$, some sequences are periodic and some non-periodic of the type mentioned in this summary.

(2) Additional work on possible and non-possible period lengths.

(3) Determining the lengths of periods for given values of p and q.

(4) In the case of periodic sequences, finding upper bounds for the values of terms in the periods.

(5) Arriving at additional generalized sequences for other values of p and q than (p,p), $(p+1,p)$, $(p, p+1)$.

(6) Modifying the work to include more terms in the numerator with a corresponding number of multipliers.

(7) Studying the least integer functions which involve non-linear combinations of the previous terms.

NOTE

The complete report of which this article is a summary consists of 54 pages. It may be obtained for $2.50 by writing the Managing Editor, Brother Alfred Brousseau, St. Mary's College, Moraga, California 94575.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within to months after publication of the problem.

H-249 Proposed by F. D. Parker, St. Lawrence University, Canton, New York.

Find an explicit formula for the coefficients of the Maclaurin series for

$$\frac{b_o + b_1x + \cdots + b_kx^k}{1 + \alpha x + \beta x^2} .$$

H-250 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that if

$$A(n)F_{n+1} + B(n)F_n = C(n) \qquad (n = 0, 1, 2, \cdots),$$

where the F_n are the Fibonacci numbers and $A(n)$, $B(n)$, $C(n)$ are polynomials, then

$$A(n) \equiv B(n) \equiv C(n) \equiv 0.$$

H-251 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Prove the identity:

$$\sum_{n=0}^{\infty} \frac{x^{n^2}}{[(x)_n]^2} = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} ,$$

where

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \quad (x)_0 = 1 .$$

SOLUTIONS

SOME SUM

H-219 Proposed by Paul Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity

$$(-1)^n \binom{x}{n} \sum_{i=0}^{n} \binom{n}{i} (-2)^i \cdot \frac{x-n}{x-i} = \sum_{i=0}^{n} \binom{x}{i} ,$$

where

$$\binom{x}{i} = \frac{x(x-1)(x-2)\cdots(x-i+1)}{i!}$$

(x) not necessarily an integer.

Solution and generalization by H. Gould, West Virginia University, Morgantown, West Virginia.

We shall obtain the slightly more general formula

$$(1) \qquad (-1)^n \binom{x}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+t)^k \frac{x-n}{x-k} = \sum_{k=0}^{n} \binom{x}{k} t^k .$$

Examination of Bruckman's formula suggests that the formula can be found from the partial fraction expansion

$$(2) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x}{x+k} = \binom{x+n}{n}^{-1} ,$$

which is formula (1.41) in my book, *Combinatorial Identities* (a standardized set of tables listing 500 binomial coefficient summations, revised edition, published by the author, Morgantown, W. Va., 1972). This is a familiar and well-known formula. Besides (2) we shall need below the formula

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k} ,$$

the binomial theorem, and simple operations on series.

We make a straightforward attack on the left-hand side of (1) and find

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+t)^k \frac{x-n}{x-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x-n}{x-k} \sum_{j=0}^{k} \binom{k}{j} t^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} t^j \sum_{k=j}^{n} (-1)^k \binom{n-j}{k-j} \frac{x-n}{x-k} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^j \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \frac{x-n}{x-j-k}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^j \frac{n-x}{j-x} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \frac{-x+j}{-x+j+k}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^j \frac{n-x}{j-x} \binom{n-x}{n-j}^{-1} , \text{ by (2)},$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^j \binom{n-x-1}{n-j}^{-1} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} t^{n-j} \binom{n-x-1}{j}^{-1}$$

Therefore,

$$(-1)^n \binom{x}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+t)^k \frac{x-n}{x-k} = \binom{x}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^{n-j} \binom{n-x-1}{j}^{-1}$$

$$= \binom{x}{n} \binom{n-x-1}{n}^{-1} \sum_{j=0}^{n} (-1)^j \binom{n-x-1-j}{n-j} t^{n-j} = \binom{x}{n} (-1)^n \binom{x}{n}^{-1} \sum_{j=0}^{n} (-1)^j (-1)^{n-j} \binom{x}{n-j} t^{n-j}$$

$$= \sum_{j=0}^{n} \binom{x}{n-j} t^{n-j} = \sum_{j=0}^{n} \binom{x}{j} t^j ,$$

as desired to show. Bruckman's formula (1) occurs when $t = 1$, and formula (2) occurs when $t = 0$. Thus (2) is not only used to prove (1) but is a special case of it.

We may rewrite (1) in the form

$$(3) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1+t)^k}{x-k} = (-1)^n \frac{1}{x-n} \binom{x}{n}^{-1} \sum_{j=0}^{n} \binom{x}{j} t^j .$$

Recall the simple, well known inversion pair

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k),$$

and we see that (3) inverts to give

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x-n}{x-k} \binom{x}{k}^{-1} \sum_{j=0}^{k} \binom{x}{j} t^j = (1+t)^n . \tag{4}$$

Now, however, the power series expansion of $(1+t)^n$ is unique, so that the coefficient of t^j in (4) must be precisely $\binom{n}{j}$, so that we have evidently proved

$$\binom{x}{j} \sum_{k=j}^{n} \binom{n}{k} \binom{x}{k}^{-1} \frac{x-n}{x-k} = \binom{n}{j} \tag{5}$$

for all real x. This formula is actually just a special case of (4.1) in *Combinatorial Identities* which occurs when we set $z = n$ there and replace x by $x - 1$. However (5) is an interesting way to express this case.

Many other interesting sums can be found from (1). Thus by taking r^{th} derivatives we have at once the identity

$$\binom{x}{n} \sum_{k=r}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{r} (1+t)^{k-r} \frac{x-n}{x-k} = \sum_{k=r}^{n} \binom{x}{k} \binom{k}{r} t^{k-r}, \tag{6}$$

which will express other relations in *Combinatorial Identities* in different ways. For $t = 0$, Eq. (6) yields nothing more than a variant of (2) again.

[See also Paul Bruckman, Problem H-219, *The Fibonacci Quarterly*, Vol. 11, No. 2 (April 1973), p. 185.]

Also solved by G. Lord, P. Tracy, L. Carlitz, and the Proposer.

ON Q

H-220 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}},$$

where

$$(z)_n = (1-z)(1-qz)\cdots(1-q^{n-1}z), \qquad (z)_0 = 1 .$$

Solution by the Proposer.

It is well known that

$$\frac{1}{(z)_{k+1}} = \sum_{r=0}^{\infty} \begin{bmatrix} k+r \\ r \end{bmatrix} z^r ,$$

where

$$\begin{bmatrix} k+r \\ r \end{bmatrix} = \frac{(q)_{k+r}}{(q)_k (q)_r} = \begin{bmatrix} k+r \\ k \end{bmatrix}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{k=0}^{\infty} a^k z^k \sum_{r=0}^{\infty} \begin{bmatrix} k+r \\ r \end{bmatrix} z^r = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k .$$

On the other hand,

$$\sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}} = \sum_{r=0}^{\infty} a^r q^{r^2} z^{2r} \sum_{s=0}^{\infty} \begin{bmatrix} r+s \\ s \end{bmatrix} z^s \sum_{j=0}^{\infty} \begin{bmatrix} r+j \\ j \end{bmatrix} a^j z^j$$

$$= \sum_{n=0}^{\infty} z^n \sum_{2r+s+j=n} \begin{bmatrix} r+s \\ s \end{bmatrix}\begin{bmatrix} r+j \\ j \end{bmatrix} a^{r+j} q^{r^2} = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} a^k \sum_{r=0}^{k} \begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n-k \\ r \end{bmatrix} q^{r^2} .$$

Hence it remains to show that

$$\sum_{r=0}^{k} \begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n-k \\ r \end{bmatrix} q^{r^2} = \begin{bmatrix} n \\ k \end{bmatrix}$$

or what is the same thing

$$(*) \qquad \sum_{r=0}^{k} \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ r \end{bmatrix} q^{r^2} = \begin{bmatrix} m+n \\ m \end{bmatrix} .$$

This can be proved rapidly as follows. We recall that

$$(1+z)(1+qz)\cdots(1+q^{n-1}z) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} z^r .$$

Then

$$\sum_{k=0}^{m+n} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} z^k = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} (q^n z)^r \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} z^s .$$

The coefficient of z^n on the right is equal to

$$\sum_{r+s=n} \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+nr+\frac{1}{2}s(s-1)} = \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)+nr+\frac{1}{2}(n-r)(n-r-1)}$$

$$= q^{\frac{1}{2}n(n-1)} \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ r \end{bmatrix} q^{r^2} .$$

This proves (*).

CONGRUENCE FOR F_n AND L_n

H-221 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let $p = 2m + 1$ be an odd prime, $p \neq 5$. Show that if m is even then

$$F_m \equiv 0 \pmod p \qquad \left(\left(\frac{5}{p}\right) = +1\right)$$

$$F_{m+1} \equiv 0 \pmod p \qquad \left(\left(\frac{5}{p}\right) = -1\right) \quad ;$$

If m is odd then

$$L_m \equiv 0 \pmod p \qquad \left(\left(\frac{5}{p}\right) = +1\right)$$

$$L_{m+1} \equiv 0 \pmod p \qquad \left(\left(\frac{5}{p}\right) = -1\right) \quad ,$$

where $\left(\frac{5}{p}\right)$ is the Legendre symbol.

Solution by the Proposer.

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

where $\alpha + \beta = 1$, $\alpha\beta = -1$.

Recall the identities

$$(*) \qquad L_{2m+1} - 1 = \begin{cases} 5F_mF_{m+1} & (m \text{ even}) \\ L_mL_{m+1} & (m \text{ odd}) \end{cases}.$$

Since $L_p \equiv 1 \pmod p$, it follows that

$$\begin{cases} F_mF_{m+1} \equiv 0 \pmod p & (m \text{ even}) \\ L_mL_{m+1} \equiv 0 \pmod p & (m \text{ odd}) \end{cases}.$$

1. Let m be even. Since $(F_m, F_{m+1}) = 1$, it follows that either F_m or $F_{m+1} \equiv 0 \pmod p$ but not both. Since

$$F_m \equiv 0 \pmod p \rightleftarrows \alpha^{2m} \equiv 1 \pmod p$$

and

$$F_{m+1} \equiv 0 \pmod p \rightleftarrows \alpha^{2m+2} \equiv -1 \pmod p,$$

we must show that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv -1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

Now when $\left(\frac{5}{p}\right) = +1$, $p = \pi\pi'$, where π, π' are primes in the quadratic field $Q(\sqrt{5})$. Since

$$N(\pi) = N(\pi') = p$$

and α is a unit of the field we have

$$\alpha^{p-1} \equiv 1(\pi), \qquad \alpha^{p-1} \equiv 1(\pi')$$

and therefore $\alpha^{p-1} \equiv 1 \pmod p$.

On the other hand if $\left(\frac{5}{p}\right) = -1$, p remains a prime in $Q(\sqrt{5})$. Since

$$\alpha^p = \left(\frac{1+\sqrt{5}}{2}\right)^p \equiv \frac{1 + 5^{\frac{1}{2}(p-1)}\sqrt{p}}{2} \equiv \frac{1-\sqrt{p}}{2} \pmod p,$$

it is clear that $\alpha^p \equiv \beta \pmod p$, so that $\alpha^{p+1} \equiv \alpha\beta \equiv -1 \pmod p$.

2. Now let m be odd. Since $(L_m, L_{m+1}) = 1$, it follows from (*) that either L_m or $L_{m+1} \equiv 0 \pmod p$ but not both. Since

$$L_m \equiv 0 \pmod p \rightleftarrows \alpha^{2m} \equiv 1 \pmod p$$

and

$$L_{m+1} \equiv 0 \pmod p \rightleftarrows \alpha^{2m+2} \equiv -1 \pmod p,$$

it suffices to show that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv -1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

However the proof of these congruences for m even applies also when m is odd.

This completes the proof.

REMARK. We have incidentally proved that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

The first of these congruences is immediate but the second is less obvious.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems to Professor A.P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-304 Proposed by Sidney Kravitz, Dover, New Jersey.

According to W. Hope-Jones, "The Bee and the Pentagon," *The Mathematical Gazette,* Vol. X, No. 150, 1921 (Reprinted Vol. LV, No. 392, March 1971, Page 220), the female bee has two parents but the male bee has a mother only. Prove that if we go back n generations for a female bee she will have F_n male ancestors in that generation and F_{n+1} female ancestors, making a total of F_{n+2} ancestors.

B-305 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n} = L_{2n} \sum_{k=1}^{n} L_{2n+4k-2}\ .$$

B-306 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n+1} - 1 = L_{2n} \sum_{k=1}^{n} L_{2n+4k-1}\ .$$

B-307 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$(1 + x + x^2)^n = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots,$$

(where, of course, $a_{n,k} = 0$ for $k > 2n$). Also let

$$A_n = \sum_{j=0}^{\infty} a_{n,4j}, \quad B_n = \sum_{j=0}^{\infty} a_{n,4j+1}, \quad C_n = \sum_{j=0}^{\infty} a_{n,4j+2}, \quad D_n = \sum_{j=0}^{\infty} a_{n,4j+3}\ .$$

Find and prove the relationship of A_n, B_n, C_n, and D_n to each other. In particular, show the relationships among these four sums for $n = 333$.

B-308 Proposed by Phil Mana, Albuquerque, New Mexico.

(a) Let $c_n = \cos(n\theta)$ and find the integers a and b such that $c_n = ac_{n-1} + bc_{n-2}$ for $n = 2, 3, \cdots$.

(b) Let r be a real number such that $\cos(r\pi) = p/q$, with p and q relatively prime positive integers and q not in 1, 2, 4, 8, ⋯ . Prove that r is not rational.

B-309 Corrected Version of B-284.

Let $z^2 = xz + y$ and let k, m, and n be nonnegative integers. Prove that:

(a) $z^n = p_n(x,y)z + Q_n(x,y)$, where p_n and q_n are polynomials in x and y with integer coefficients and p_n has degree $n - 1$ in x for $n > 0$.

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0 .$$

SOLUTIONS

THE EDITOR'S DIGITS

B-280 Proposed by Maxey Brooke, Sweeney, Texas.

Identify $A, E, G, H, J, N, O, R, T, V$ as the ten distinct digits such that the following holds with the dots denoting some seven-digit number and ϕ representing zero:

$$\begin{array}{r} VERNER \\ \times\quad \underline{\qquad\quad E} \\ \cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot \\ -\underline{R\phi\phi\phi\phi JR} \\ HOGGATT \end{array}$$

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle Campus.

The unique solution to the problem is the following:

$$\begin{array}{r} 971471 \\ \times\quad \underline{\qquad 7} \\ 6800297 \\ -\underline{1000031} \\ 5800266 \end{array}$$

i.e., we have:

$$\begin{array}{c} AEGHJNORTV \\ 2705348169 \end{array}$$

Proof. Let the product $VERNER \times E$ be denoted by P in this discussion, and let the first digit of P be denoted by Y. Since P is a 7-digit number, and $VERNER$ is a 6-digit number, then $E \geqslant 2$. Since R and H are both at least 1, their total must be at least 3 (since $R \neq H$); hence, $E \geqslant 4$ and $Y \geqslant 3$.

Since $R + T \equiv ER \pmod{10}$, we initially obtain 39 possibilities for E, T, R with $E \geqslant 4$. Taking into account the possible values of J, we are left with 26 possibilities for E, T, R, J.

Now $Y \leqslant E - 1$ (since $V \leqslant 9$); moreover, since $H \geqslant 1$, we must have $R \leqslant E - 2$. Taking this requirement into account, we further reduce the list to only 13 possibilities. By a slightly tedious but manageable process of elimination, we conclude the result indicated above.

Also solved by John W. Milsom, C. B. A. Peck, Richard D. Plotz, and the Proposer.

ONES FOR TEE

B-281 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $T_n = n(n+1)/2$. Find a positive integer b such that for all positive integers m, $T_{11\cdots1} = 11\cdots1$, where the subscript on the left side has m 1's as the digits in base b and the right side has m 1's as the digits in base b^2.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

More will be shown to be true. Suppose the base on the right side is the positive integer c, instead of b^2. The equality for $m = 1$ is automatically satisfied and for $m = 2$ is $(1+b)(2+b) = 2(1+c)$, i.e., $3b + b^2 = 2c$. For $m = 3$ the resulting equation is

$$(1+b+b^2)(2+b+b^2) = 2(1+c+c^2).$$

These last two equations in b and c force $b^2 = 2b+3$ and hence $b = 3$ (since it is a positive integer), and $c = b^2 = 9$. Finally as $(3^m - 1)(3^m + 1) = (3^{2m} - 1)$ then $T_{11\cdots 1}$, in base 3, equals $11 \cdots 1$, in base 9, for all positive integers greater than 2.

Also solved by Paul S. Bruckman, Herta T. Freitag, C.B.A. Peck, Bob Prielipp, Paul Smith, Gregory Wulczyn, and the Proposer.

LUCAS RIGHT TRIANGLES

B-282 Proposed by Herta T. Freitag, Roanoke, Virginia.

Characterize geometrically the triangles that have

$$L_{n+2}L_{n-1}, \quad 2L_{n+1}L_n, \quad \text{and} \quad 2L_{2n}+L_{2n+1}$$

as the lengths of the three sides.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

Since

$$[2L_{2n}+L_{2n+1}]^2 = [L_{2n}+L_{2n+2}]^2 = [L_{n-1}L_{n+1}+3(-1)^n + L_nL_{n+2}+3(-1)^{n+1}]^2$$

(see the Solution to Problem B-256, p. 221, *The Fibonacci Quarterly,* April 1974)

$$= [L_{n-1}L_{n+1}+L_nL_{n+2}]^2 = [(L_{n-1}+L_n)L_{n+1}+L_n^2]^2 = [L_{n+1}^2+L_n^2]^2 = [2L_{n+1}L_n]^2 + [L_{n+1}^2 - L_n^2]^2$$

$$= [2L_{n+1}L_n]^2 + [(L_{n+1}+L_n)(L_{n+1}-L_n)]^2 = [2L_{n+1}L_n]^2 + [L_{n+2}L_{n-1}]^2,$$

the triangles are right triangles.

Also solved by Richard Blazej, Paul S. Bruckman, Wray G. Brady, C.B.A. Peck, Gregory Wulczyn, and the Proposer.

RATIONAL APPROXIMATION OF COS $\pi/6$ AND SIN $\pi/6$

B-283 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the ordered triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$, a odd, $c < 1000$, and c/a as close to 2 as possible. [This approximates the sides of a 30°, 60°, 90° triangle with a Pythagorean triple.]

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

It is clearly sufficient to find a triple of the form $(u^2 - v^2, 2uv, u^2 + v^2)$, with u, v of opposite parity. We must then find the minimum value for $u^2 + v^2 < 1000$ of

$$\left|2 - \frac{u^2+v^2}{u^2-v^2}\right| = \left|\frac{u^2-3v^2}{u^2-v^2}\right| .$$

If $|u^2 - 3v^2| = 2$ then u, v are of the same parity and a is even. Hence, if $|u^2 - 3v^2| > 1$,

$$\left|\frac{u^2-3v^2}{u^2-v^2}\right| > \left|\frac{u^2-3v^2}{u^2+v^2}\right| \geqslant \frac{3}{1000} .$$

For $u^2 + v^2 < 1000$ the Pellian equation $|u^2 - 3v^2| = 1$ has solutions $(u,v) = (2,1), (7,4), (26,15)$. The solution $(26, 15)$ yields the triple $(451, 780, 901)$ which is best possible, since

$$\left|2 - \frac{901}{451}\right| = \frac{1}{451} < \frac{3}{1000} .$$

Also solved by Paul S. Bruckman, Gregory Wulczyn, and the Proposer.

CORRECTED AND REINSERTED

Problem B-284 has been corrected and reinserted as B-309 above.

VERY SLIGHT VARIATION ON A PREVIOUS PROBLEM

B-285 Proposed by Barry Wolk, University of Manitoba, Winnipeg, Manitoba, Canada.

Show that

$$F_{k(n+1)}/F_k = \sum_{k=0}^{[n/2]} (-1)^{r(k-1)} \binom{n-r}{r} L_k^{n-2r} .$$

Solution by C.B.A. Peck, State College, Pennsylvania.

This was H-135, Part II and was proved by induction on n in *The Fibonacci Quarterly,* Vol. 7, No. 5, p. 519. (The exponent of -1 in that problem has + instead of −, but $(-1)^{2r} = 1$.)

Also solved by P.S. Bruckman and the Proposer.

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SUSTAINING MEMBERS

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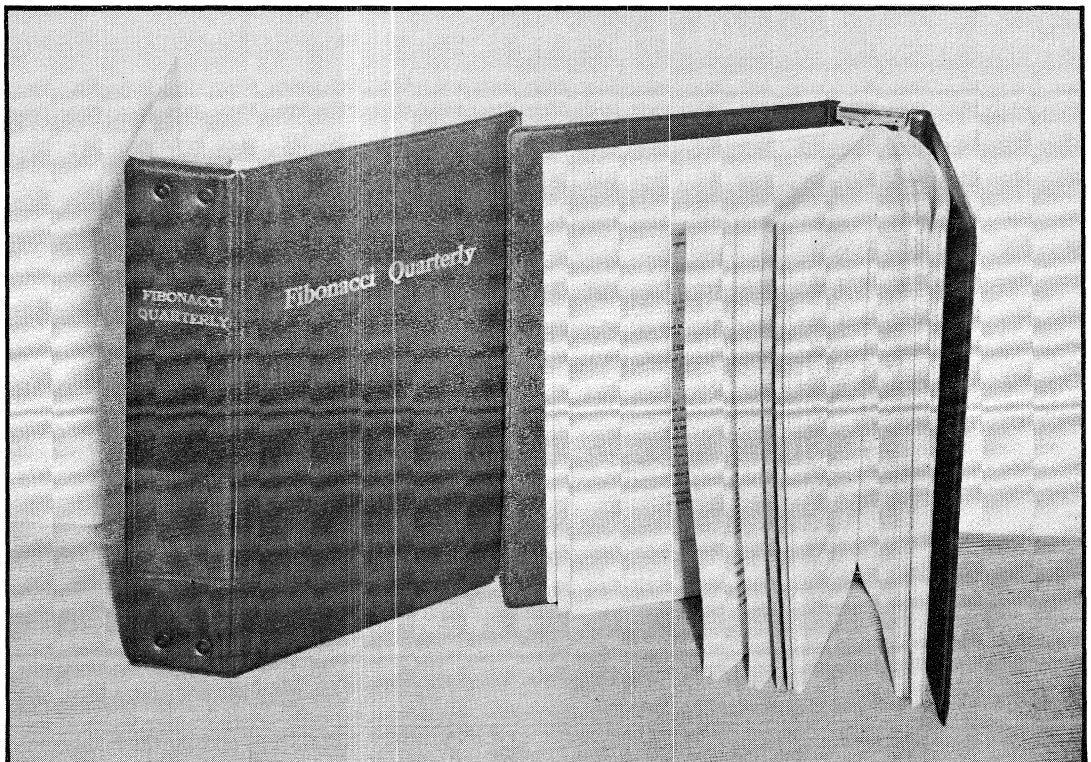

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