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GENERAL IDENTITIES FOR FIBONACCI AND LUCAS NUMBERS WITH POLYNOMIAL SUBSCRIPTS IN SEVERAL VARIABLES

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Among the well known Fibonacci identities we have

$$F_{m+n} \equiv F_{m+1}F_n + F_mF_{n-1}$$

which may be written as

$$F_{m+1}F_n - F_1F_{m+n} \equiv F_mF_{n-1} .$$

In this form, we see a property which is common among Fibonacci and Lucas identities. Namely, that the sum of the subscripts of the first product $F_{m+1}F_n$ is identically equal to the sum of the subscripts of the second product F_1F_{m+n}.

What general identities do we have with this property? How does this property relate to the reducibility of a given form?

It is with these questions that we are principally concerned.

Definition 1. For every i, $1 \leqslant i \leqslant m$, let the domain of n_i be the set of integers. Then we let

$$P = \{ \text{polynomials in } n_1, n_2, \cdots, n_m \text{ with integral coefficients} \} .$$

For convenience in deriving general Fibonacci and Lucas identities for the forms

$$F_fF_g \pm F_hF_k, \qquad L_fL_g \pm L_hL_k, \qquad F_fL_g \pm F_hL_k ,$$

where $f, g, h, k \in P$, with the property that $f + g \equiv h + k$, we first express h and k in terms of f and g.

Lemma 1. If $f, g, h, k \in P$ such that $f + g \equiv h + k$, then there exists $f_1, f_2, g_1, g_2 \in P$ such that

$$f_1 + f_2 \equiv f, \qquad g_1 + g_2 \equiv g, \qquad f_1 + g_1 \equiv h, \quad \text{and} \quad f_2 + g_2 \equiv k.$$

Proof. Let

$$f_1 \equiv h, \qquad f_2 \equiv f - h, \qquad g_1 \equiv 0, \qquad g_2 \equiv g,$$

clearly,

$$f_1, f_2, g_1, g_2 \in P \qquad \text{and} \qquad f_1 + f_2 \equiv f, \qquad g_1 + g_2 \equiv g, \qquad f_1 + g_1 \equiv h .$$

$$f_2 + g_2 \equiv f - h + g$$

but, by hypothesis,

$$f + g \equiv h + k \Rightarrow f - h + g \equiv k \Rightarrow f_2 + g_2 \equiv k. \qquad \text{q.e.d.}$$

Theorem 1. Let $f, g, h, k \in P$ such that $f + g \equiv h + k$, then

$$F_fF_g - F_hF_k \equiv (-1)^{g+1}F_{f-h}F_{f-k} .$$

Proof. By hypothesis,

$$f + g \equiv h + k \qquad \text{and} \qquad f, g, h, k \in P$$

Hence, by Lemma 1, there exist $f_1, f_2, g_1, g_2 \in P$ such that

$$f_1 + f_2 \equiv f, \qquad g_1 + g_2 \equiv g, \qquad f_1 + g_1 \equiv h, \qquad f_2 + g_2 \equiv k .$$

Then, clearly,

$$F_fF_g - F_hF_k \equiv F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2} .$$

Using the Binet definition

$$\left(F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ where } n \in \text{[Integers]}, \qquad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \right)$$

we have

$$\begin{aligned}
F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2} &\equiv \left(\frac{\alpha^{f_1+f_2} - \beta^{f_1+f_2}}{\alpha-\beta}\right)\left(\frac{\alpha^{g_1+g_2} - \beta^{g_1+g_2}}{\alpha-\beta}\right) \\
&\quad - \left(\frac{\alpha^{f_1+g_1} - \beta^{f_1+g_1}}{\alpha-\beta}\right)\left(\frac{\alpha^{f_2+g_2} - \beta^{f_2+g_2}}{\alpha-\beta}\right) \\
&\equiv \frac{(\alpha^{f_1+f_2+g_1+g_2} - \beta^{f_1+f_2}\alpha^{g_1+g_2} - \alpha^{f_1+f_2}\beta^{g_1+g_2} + \beta^{f_1+f_2+g_1+g_2})}{(\alpha-\beta)^2} \\
&\quad - \frac{(\alpha^{f_1+f_2+g_1+g_2} - \beta^{f_1+g_1}\alpha^{f_2+g_2} - \alpha^{f_1+g_1}\beta^{f_2+g_2} + \beta^{f_1+g_1+f_2+g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{(-\beta^{f_1+f_2}\alpha^{g_1+g_2} + \alpha^{f_1+g_1}\beta^{f_2+g_2} - \alpha^{f_1+f_2}\beta^{g_1+g_2} + \beta^{f_1+g_1}\alpha^{f_2+g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{\beta^{f_2}\alpha^{g_1}(-\beta^{f_1}\alpha^{g_2} + \alpha^{f_1}\beta^{g_2}) + \alpha^{f_2}\beta^{g_1}(-\alpha^{f_1}\beta^{g_2} + \beta^{f_1}\alpha^{g_2})}{(\alpha-\beta)^2} \\
&\equiv \frac{(-\beta^{f_1}\alpha^{g_2} + \alpha^{f_1}\beta^{g_2})(\beta^{f_2}\alpha^{g_1} - \alpha^{f_2}\beta^{g_1})}{(\alpha-\beta)^2} \\
&\equiv \frac{(\alpha\beta)^{g_2}(-\beta^{f_1-g_2} + \alpha^{f_1-g_2})(\beta\alpha)^{g_1}(\beta^{f_2-g_1} - \alpha^{f_2-g_1})}{(\alpha-\beta)^2} \\
&\equiv \frac{(\alpha\beta)^{g_1+g_2+1}(\alpha^{f_1-g_2} - \beta^{f_1-g_2})(\alpha^{f_2-g_1} - \beta^{f_2-g_1})}{(\alpha-\beta)^2} \\
&\equiv (-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1} .
\end{aligned}$$

But

$g_1 + g_2 \equiv g$ and $f_1 - g_2 \equiv (f_1+f_2) - (f_2+g_2) \equiv f-k$ and $f_2 - f_1 \equiv (f_1 + f_2) - (f_1+g_1) \equiv f-h$.

Thus, by substituting

$$(-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1} \equiv (-1)^{g+1}F_{f-k}F_{f-h} \equiv (-1)^{g+1}F_{f-h}F_{f-k} \quad \text{q.e.d.}$$

Theorem 2. Let $f, g, h, k \in P$ such that $f+g \equiv h+k$, then

(a) $$L_fL_g - L_hL_k \equiv 5(-1)^gF_{f-h}F_{f-k}$$

and

(b) $$F_fL_g - F_hL_k \equiv (-1)^{g+1}F_{f-h}L_{f-k} .$$

Proof. The proof of 2(a) and 2(b) is virtually the same as that of Theorem 1 (where $L_n = \alpha^n + \beta^n$).

Corollary 1. Let $f, g, h, k \in P$ such that $f+g \equiv h+k$. Then

$$F_fF_g - F_hF_k \equiv -\frac{(L_fL_g - L_hL_k)}{5} .$$

Proof. Compare Theorems 2(a) and 1.

EXAMPLES AND APPLICATIONS

The degree of freedom offered by Theorems 1 and 2 together with the identity given in their hypothesis is large indeed. We will endeavor, with some examples, to indicate that degree of freedom.

EXAMPLE 1. By [1, p. 7], a general Turàn operator is defined by

$$Tf = T_xf(x) = f(x+u)f(x+v) - f(x)f(x+u+v).$$

"For the Fibonacci numbers it is a classic formula first discovered apparently by Tagiuri (Cf. Dickson [4, p. 404]) and later given as a problem in the *American Mathematical Monthly* (Problem 1396) that

$$T_nF_n = F_{n+u}F_{n+v} - F_nF_{n+u+v} = (-1)^nF_uF_v."$$

This is immediate from Theorem 1.

Let $f \equiv n+u$, $g \equiv n+v$, $h \equiv n$ and $k \equiv n+u+v$. Clearly,

$$f, g, h, k \in P \qquad \text{and} \qquad f+g \equiv h+k.$$

Thus, applying Theorem 1, we have

$$F_{n+u}F_{n+v} - F_nF_{n+u+v} \equiv (-1)^{n+v+1}F_{(n+u)-n}F_{(n+u)-(n+u+v)} \equiv (-1)^{n+v+1}F_uF_{-v}.$$

Now using the well known identity $(-1)^{m+1}F_m \equiv F_{-m}$ yields

$$(-1)^{n+v+1}F_uF_{-v} \equiv (-1)^{n+v+1}(-1)^{-v+1}F_uF_v \equiv (-1)^nF_uF_v,$$

the desired result.

EXAMPLE 2. By Theorem 2(a),

$$L_fL_g - L_hL_k \equiv (-1)^g5F_{f-h}F_{f-k}$$

if $f, g, h, k \in P$ and $f+g \equiv h+k$. Then too, $f-k \equiv h-g$ and $f-h \equiv k-g$.

Substituting, we obtain

$$L_fL_g - L_hL_k \equiv (-1)^g5F_{k-g}F_{h-g},$$

a trivial but equivalent form of Theorem 2(a).

Another equivalent form of Theorem 2(a) is

$$L_fL_g - 5F_hF_k \equiv (-1)^gL_{k-g}L_{h-g}.$$

To obtain this equivalent form, we write

$$f+(-g) \equiv (h-g)+(k-g).$$

Clearly,

$$f, (-g), (h-g), (k-g) \in P;$$

hence, Theorem 2(a) may be applied to these new polynomials, yielding,

$$L_fL_{(-g)} - L_{(h-g)}L_{(k-g)} \equiv (-1)^{-g}F_{(h-g)-(-\beta)}F_{(k-g)-(-g)}$$

then,

$$(-1)^gL_fL_g - L_{(h-g)}L_{(k-g)} \equiv (-1)^g5F_hF_k \Rightarrow L_fL_g - 5F_hF_k \equiv (-1)^gL_{k-g}L_{h-g}.$$

Similarly, Theorems 1 and 2 may be put into several other equivalent forms.

It would be natural to ask what $F_fF_g + F_hF_k$ would yield, subject to the condition

$$f, g, h, k \in P \qquad \text{and} \qquad f+g \equiv h+k,$$

with a proof analagous to that of Theorem 1. The result is, in at least one form,

$$F_fF_g + F_hF_k \equiv \frac{L_oL_{f+g}}{5} + (-1)^{g+1}\frac{L_{(f-h)}L_{(f-k)}}{5}.$$

However, this form is easily derived with the following method.

EXAMPLE 3.

$$f+g \equiv h+k \Rightarrow (0)+(f+g) \equiv h+k,$$

by Theorem 2(a),

$$\frac{L_{f+g}L_o}{5} - \frac{L_hL_k}{5} \equiv F_hF_k.$$

Now we use Theorem 2(a) to find an expression for F_fF_g and obtain

$$F_fF_g - \frac{L_hL_k}{5} \equiv \frac{(-1)^{g+1}L_{f-h}L_{f+k}}{5} \quad .$$

Adding these identities produces

$$F_fF_g + F_hF_k = \frac{L_oL_{f+g}}{5} + (-1)^{g+1}\,\frac{L_{f-h}L_{f-k}}{5} \quad .$$

Similarly, we find sums $L_fL_g + L_hL_k$ by using Theorem 2(b). Also, other sums with various equivalent forms may be found.

APPLICATION TO FIBONACCI AND LUCAS TRIPLES

Application of Theorems 1 and 2 to the Fibonacci and Lucas triples [2], generated by R. T. Hansen, allow Theorems 1 and 2 to be written in equivalent summation form for fixed integers.

Theorem 3. Let A, B be fixed integers; then

$$F_AF_B \equiv \sum_{K=0}^{B-1} (-1)^{B+1-K} F_{A-B+2K+1}$$

$$F_AL_B \equiv \sum_{K=0}^{A-1} (-1)^{B+K} L_{A-B-2K+1}$$

$$L_AL_B \equiv \sum_{K=0}^{A} (-1)^{B+K} L_{A-B-2(K+1)} + \sum_{K=0}^{A-2} (-1)^{B+K} L_{A-B-2K} \cdot$$

Proof. See [2] and directly apply Theorems 1 and 2.

Clearly, from these forms, the summation equivalents of Theorems 1 and 2, for fixed integer A, B, C, D such that $A + B = C + D$, may be obtained as immediate corollaries. We do not list these identities.

FURTHER APPLICATION OF THEOREMS 1 AND 2

We now apply Theorems 1 and 2 to find simple subscript properties between identically equal Fibonacci and Lucas products.

Lemma 2. Let $f, g \in P$ such that $f \neq 2$ and $g \neq 2$. If $F_f \equiv F_g$, then $|f| \equiv |g|$.

Proof.

$$F_f \equiv F_g \Rightarrow |F_f| \equiv |F_g| \Rightarrow F_{|f|} \equiv F_{|g|} \,.$$

Clearly,

$$\{F_N\}_{N=0}^{\infty}, \qquad N \neq 2, \qquad N \in [\text{Integers}],$$

is a strictly increasing sequence. Then $F_{|f|} \equiv F_{|g|}$ and $|f| \neq |g|$ is a contradiction to the fact that $\{F_N\}_{N=0}^{\infty}, N \neq 2,$ is strictly increasing. Thus,

$$F_f \equiv F_g \Rightarrow F_{|f|} \equiv F_{|g|} \Rightarrow |f| \equiv |g| . \qquad \text{Q.E.D.}$$

Theorem 4. Given $f, g, h, k \in P.$ If $F_fF_g \equiv F_hF_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, or $|f| \equiv |g|$ and $|g| \equiv |k|$ whenever

$$|f|, |g|, |h|, |k|, \notin \{0,2\}.$$

Proof. If $F_fF_g \equiv F_hF_k$, then

(1) $$|F_fF_g| \equiv |F_hF_k| \Rightarrow F_{|f|}F_{|g|} \equiv F_{|h|}F_{|k|}\ .$$

Since $f, g, h, k \in P$, they are functions of $n_1, n_2, \cdots,$ and n_m. Let n_i' for $1 \leqslant i \leqslant m$ be an arbitrary set of fixed values of n_i for $1 \leqslant i \leqslant m$, respectively. Then f_1, g_1, h_1, k_1 are the corresponding fixed integers. Assume W.L.O.G. that

$$|f_1| + |g_1| \geqslant |h_1| + |k_1|$$

and that $\{n_i'\}$ is such that

$$|f_1|, |g_1|, |h_1|, |k_1|$$

are not 2 or 0. Clearly, there exist K such that $K > 0$, $K \in$ [Integers] and

(2) $$|f_1| + |g_1| = |h_1| + |k_1| + K.$$

By Theorem 1,

$$F_{|f_1|}F_{|g_1|} - F_{|h_1|}F_{|k_1|+K} = (-1)^{|g_1|+1}F_{|f_1|-|h_1|}F_{|f_1|-(|k_1|+K)} = 0$$

if and only if

$$|f_1| - |h_1| = 0 \qquad \text{or} \qquad |f_1| - (|k_1| + K) = 0.$$

Without loss of generality, assume that

$$|f_1| - |h_1| = 0 \Rightarrow |f_1| = |h_1|\ .$$

Then by (2), $|g_1| = |k_1| + K$.

Suppose $K \neq 0$, then

$$F_{|f_1|}F_{|g_1|} = F_{|h_1|}F_{|k_1|+K} \neq F_{|h_1|}F_{|k_1|}$$

by Lemma 2.

Thus, if

$$F_{|f_1|}F_{|g_1|} = F_{|h_1|}F_{|k_1|}$$

it is required that $K = 0$. Thus,

$$|f_1| = |h_1| \qquad \text{and} \qquad |g_1| = |k_1|\ .$$

Further, since the selection of n_i' was arbitrary with the conditions of the theorems hypothesis, its conclusion holds. Q.E.D.

Note that the condition

$$|f|, |g|, |h|, |k| \notin \{2\}$$

is not really any restriction, practically speaking. That is $F_2 = F_1$, so if one agrees always to write F_2 as F_1 we could require only that $|f|, |g|, |h|, |k| \notin \{0\}$ in the hypothesis of Theorem 4.

Lemma 3. Let $f, g \in P$, if $L_f \equiv L_g$, then $|f| \equiv |g|$.

Proof. Construct an argument similar to Lemma 2.

Theorem 5. Let $f, g, h, k \in P$. If $L_fL_g \equiv L_hL_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, or $|f| \equiv |k|$ and $|g| \equiv |h|$.

Proof. Construct a proof analogous to Theorem 4 by using Lemma 3 and Theorem 2(a).

Theorem 6. Given $f, g, h, k \in P$. If $F_fL_g \equiv F_hL_k$, then $|f| \equiv |h|$ and $|g| \equiv |k|$, whenever $|f|, |h| \notin \{0,2\}$.

Proof. Construct a proof analogous to Theorem 4 by using Theorem 2(b). Informally speaking, Theorems 1, 2, 4, 5 and 6 seem to suggest that an algebraic structure for Fibonacci identities, based on the subscripts, can be formed. If the reader is interested in investigating this, he will be more successful in using the following form of Theorem 1:

$$F_{f_1+f_2}F_{g_1+g_2} - F_{f_1+g_1}F_{f_2+g_2} \equiv (-1)^{g_1+g_2+1}F_{f_1-g_2}F_{f_2-g_1}\ ,$$

where

$$f, g, h, k, f_1, f_2, g_1, g_2 \in P$$

and

$$f_1 + f_2 = f, \qquad g_1 + g_2 = g, \qquad f_1 + g_1 = h, \qquad f_2 + g_2 = k$$

and

$$f + g \equiv h + k.$$

Further, note that if we let

$$Q = \left\{ F_R F_S \mid R, S \in P \quad \text{and} \quad R + S \equiv f + g \right\}$$

then clearly

$$F_{f_1+f_2} F_{g_1+g_2},\ F_{f_1+g_1} F_{f_2+g_2} \in Q.$$

Also,

$$F_{f_1+f_2} F_{g_1+g_2} \equiv (-1)^{g_1+g_2+1} F_{f_1+f_2} F_{-g_1-g_2} \Rightarrow (-1)^{g_1+g_2+1} F_{f_1+f_2} F_{-g_1} F_{-g_2} \in Q$$

and then

$$(-1)^{g_1+g_2+1} F_{f_1-g_2} F_{f_2-g_1} \in Q.$$

The reader may enjoy investigating further in this or other directions.

SOME ADDITIONAL IDENTITIES

Theorem 7. Let $f, g, h \in P$ such that $f \equiv g + h$. Then,

(a) $$F_f - F_g L_h \equiv (-1)^g F_{h-g}$$

(b) $$L_f - L_g L_h \equiv (-1)^{g+1} L_{h-g}$$

(c) $$\frac{L_f}{5} - F_g F_h \equiv \frac{(-1)^g L_{h-g}}{5}.$$

Proof. By using the Binet definition we have

$$F_f - F_g L_h \equiv \frac{\alpha^f - \beta^f}{\alpha - \beta} - \frac{\alpha^g - \beta^g}{\alpha - \beta} \cdot \frac{\alpha^h + \beta^h}{1} \equiv \frac{(\alpha^f - \beta^f) - (\alpha^{g+h} - \beta^g \alpha^h + \alpha^g \beta^h - \beta^{g+h})}{\alpha - \beta}.$$

By hypothesis $f \equiv g + h$, hence by substituting $g + h$ for f in the above expression and simplifying we have

$$F_f - F_g L_h \equiv \frac{\beta^g \alpha^h - \alpha^g \beta^h}{\alpha - \beta}$$

$$\equiv (\alpha\beta)^g \frac{(\alpha^{h-g} - \beta^{h-g})}{\alpha - \beta} \equiv (-1)^g F_{h-g}.$$

The proofs of (b) and (c) are similar. Q.E.D.

Although not included, theorems corresponding to those in this paper may be developed for Fibonacci and Lucas triples as well. (The author did develop the $F_g F_h L_k - F_l F_m F_n$ form.) Clearly, the proofs for these, which are virtually the same as for Theorems 1 and 2, soon become cumbersome. We leave it to the reader to develop these to suit his needs.

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1. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 1, No. 1 (Feb. 1963), p. 8.
2. R. T. Hansen, "Generating Identities for Fibonacci and Lucas Triples," *The Fibonacci Quarterly,* Vol. 10, No. 5 (Dec. 1972), pp. 571–578.

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REPEATED BINOMIAL COEFFICIENTS AND FIBONACCI NUMBERS

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ABSTRACT

In this note, I show that there are infinitely many solutions to the equation

$$\binom{n+1}{k+1} = \binom{n}{k+2},$$

given by $n = F_{2i+2}F_{2i+3} - 1$, $k = F_{2i}F_{2i+3} - 1$, where F_n is the n^{th} Fibonacci number, beginning with $F_0 = 0$. This gives infinitely many binomial coefficients occurring at least 6 times. The method and results of a computer search for repeated binomial coefficients, up to 2^{48}, will be given.

1. INTRODUCTION

In [6], I have conjectured that the number of times an integer can occur as a binomial coefficient is bounded. A computer search up to 2^{48} has revealed only the following seven nontrivial repetitions:

$$120 = \binom{16}{2} = \binom{10}{3}; \quad 210 = \binom{21}{2} = \binom{10}{4}; \quad 1540 = \binom{56}{2} = \binom{22}{3};$$

$$7140 = \binom{120}{2} = \binom{36}{3}; \quad 11628 = \binom{153}{2} = \binom{19}{5}; \quad 24310 = \binom{221}{2} = \binom{17}{8};$$

and

$$3003 = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}.$$

In [2], it has been shown that the only numbers which are both triangular, i.e., $= \binom{n}{2}$ for some n, and tetrahedral, i.e., $= \binom{n}{3}$ for some n, are 1, 10, 120, 1540 and 7140. The first two are trivial and the last three were also found by the computer, giving a check on the search procedure.

The coefficient 3003 occurs in the following striking pattern in Pascal's triangle:

$$\begin{array}{ccccc} 1001 & & 2002 & & 3003 \\ & 3003 & & 5005 & \\ & & 8008 & & \end{array}$$

I had noticed this pattern some years ago when I discovered that it is the only solution to

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = 1:2:3,$$

and that there is at most one solution to this relation when the right-hand side is replaced by $a:b:c$. Hence I was led to consider determining solutions when the right-hand side was $a:b:a+b$, or, equivalently and more simply, solutions of

$$(1) \qquad \binom{n+1}{k+1} = \binom{n}{k+2}.$$

2. SOLUTION OF EQUATION (1)

From (1), we have $(n+1)(k+2) = (n-k)(n-k-1)$. Set $m = n+1$, $j = k+2$, thus obtaining $m^2 + (1-3j)m + j^2 - j = 0$. Solving for m gives

(2) $$m = [-1+3j \pm \sqrt{5j^2-2j+1}\,]/2 .$$

For this to make sense, we must have that $5j^2-2j+1$ is a perfect square, say v^2. We can rewrite this as

(3) $$(5j-1)^2 - 5v^2 = -4 .$$

Letting $u = 5j-1$, $C = -4$, we have the Pell-like equation

(4) $$u^2 - 5v^2 = C .$$

This can be completely solved by standard techniques [5, section 58, p. 204 ff]. The basic solutions are:

$$9 \pm 4\sqrt{5} \text{ when } C = 1; \quad 2 \pm \sqrt{5} \text{ when } C = -1; \quad \text{and} \quad 1 \pm \sqrt{5} \text{ and } 4 \pm 2\sqrt{5} \text{ when } C = -4 .$$

The class of solutions determined by $4+2\sqrt{5}$ is the same as the class determined by $4-2\sqrt{5}$, i.e., the class is ambiguous, in the terminology of [5]. Hence all solutions are given by

$$u_i + v_i\sqrt{5} = (-1+\sqrt{5})(9+4\sqrt{5})^i, \quad u_i + v_i\sqrt{5} = (1+\sqrt{5})(9+4\sqrt{5})^i, \quad u_i + v_i\sqrt{5} = (4+2\sqrt{5})(9+4\sqrt{5})^i ,$$

and their conjugates and negatives.

Let $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ define the Fibonacci numbers and let $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ define the Lucas numbers.

Lemma. $$(L_n + F_n\sqrt{5})(9+4\sqrt{5}) = L_{n+6} + F_{n+6}\sqrt{5} .$$

Proof. Let $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. By the Binet formulas, we have

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \qquad L_n = \alpha^n + \beta^n ,$$

and so $L_n + F_n\sqrt{5} = 2\alpha^n$. Hence the lemma reduces to showing $\alpha^6 = 9+4\sqrt{5}$, which is readily done.

Since the basic solutions $u_0 + v_0\sqrt{5}$ given above are respectively

$$L_{-1} + F_{-1}\sqrt{5}, \qquad L_1 + F_1\sqrt{5} \qquad \text{and} \qquad L_3 + F_3\sqrt{5} ,$$

the general solution of (4) can be written as

(5) $$L_{2i-1} + F_{2i-1}\sqrt{5}, \qquad i = 0, 1, \cdots$$

and we may now ignore the conjugates and negatives.

To solve (3), we must have $5j - 1 = L_{2i-1}$. From the Binet formula, one may obtain $L_i \equiv 2\cdot 3^i \pmod 5$ and hence $L_i \equiv -1 \pmod 5$ if and only if $i \equiv 3 \pmod 4$. Recalling that $j = k+2 \geq 2$, the solutions of (3) are thus

$$j = (L_{4i+3}+1)/5, \qquad v = F_{4i+3}, \qquad i = 1, 2, \cdots .$$

By standard manipulations, we obtain

(6) $$j = F_{2i}F_{2i+3}+1, \quad k = F_{2i}F_{2i+3}-1, \quad m = F_{2i+2}F_{2i+3} = (L_{4i+5}-1)/5, \quad n = F_{2i+2}F_{2i+3}-1.$$

Finally, observe that

$$\binom{n}{k} : \binom{n}{k+1} = (k+1):(n-k) = F_{2i} : F_{2i+1} ,$$

hence

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = F_{2i} : F_{2i+1} : F_{2i+2} .$$

The case $i = 1$ gives $n = 14$, $k = 4$ and

$$\binom{15}{5} = \binom{14}{6} = 3003 .$$

The case $i = 2$ gives $n = 103$, $k = 38$, $k+2 = 40$, and

$$\binom{104}{39} = \binom{103}{40} = 6121\ 81827\ 43304\ 70189\ 14314\ 82520 .$$

This number does not occur again as a binomial coefficient. The next values of (n,k) are (713, 271) and (4894, 1868).

Equation (1) has also been solved by Lind [4]. Hoggatt and Lind [3] have dealt with some related inequalities.

3. REMARKS

The coefficients

$$N = \binom{n+1}{k+1} = \binom{n}{k+1} = \binom{N}{1}$$

give us infinitely many binomial coefficients occurring at least six times. This has also been noted in [1, Theorem 3]. Since 3003 happens to be also a triangular number, one might hope that some more of these values might also be triangular. I first determined by calculation that

$$\binom{103}{40}$$

was not triangular and later I determined that it did not occur as any other binomial coefficient. These determinations are described below. I have not been able to discern any other patterns in the repetitions found.

One might try to extend the pattern of Eq. (1) and try to find

$$\binom{n}{k+4} = \binom{n+1}{k+3} = \binom{n+2}{k+2} .$$

This would require two solutions of (1) with consecutive values of n and inspection of (6) shows this is impossible.

The lemma is a special case of the general assertion that the solutions u_i, v_i of

$$u_i + v_i\sqrt{D} = (u_0 + v_0\sqrt{D})(a + b\sqrt{D})^i$$

both satisfy the same second-order recurrence relation:

$$u_{n+1} = 2au_n + (b^2D - a^2)u_{n-1} .$$

(In our particular case: $F_{n+6} = 18F_n - F_{n-6}$.) I do not see whether the fact that the three basic solutions happen to neatly fit together into a single linear recurrence is a happy accident or a general phenomenon. The converse problem of determining which pairs of recurrence relations give all solutions of a Pell-like equation seems interesting but I have not examined it.

4. THE COMPUTER SEARCH

Two separate computer searches were made. First an ALGOL program was used to search up to 2^{23} on the London Polytechnics' ICL 1905E. All the 4717 binomial coefficients $\binom{n}{k}$ with $k \geqslant 2$, $n \geqslant 2k$ and less than 2^{23} were formed by addition and stored in rows corresponding to the diagonals of Pascal's triangle. As each new coefficient was created, it was compared with the elements in the preceding rows. Since each row is in increasing order, a simple binary search was done in each preceding row and the process is quite quick. All the repeated values given in the Introduction were already determined in this search.

The second search was carried out using a FORTRAN program on the University of London Computer Centre's CDC 6600. Although the 6600 has a 60-bit word, it is difficult to use integers bigger than 2^{48} and overflow occurs with such integers. Consequently, I was only able to search up to 2^{48}. There are about 24×10^6 triangular numbers and about 12×10^4 tetrahedral numbers up to this limit. It is impractical to store all of these, so the program had to be modified. Fortunately, the results of [2], mentioned in the Introduction, implied that we did not have to compare these two sets. I wrote a subroutine to determine if an integer N was triangular or tetrahedral. This estimates the J such that $J(J+1)/2 = N$ by $J = [\sqrt{(2N)}] - 1$ and then computes the succeeding triangular numbers until they equal or exceed N. Two problems of overflow arose. Firstly: if N is large, the calculation of the first triangular number to be considered, i.e., $J(J+1)/2$, may cause an overflow when $J(J+1)$ is formed. This was resolved by examining J (mod 2) and computing either $(J/2)(J+1)$ or

$$J\left(\frac{J+1}{2}\right) .$$

Secondly: if N is larger than the largest triangular number less than 2^{48}, the calculation of the successive triangular numbers will produce an overflow before the comparison with N reveals that we have gone far enough. This was resolved by testing the index of the triangular numbers to see if overflow was about to occur. The test for tetrahedral numbers was similar, but requires testing J (mod 6).

The search then proceeded much as before. All coefficients $\binom{n}{k}$ with $k \geqslant 4$ and $n \geqslant 2k$ and less than 2^{48} were formed by addition and stored in rows. As each coefficient was formed, the subroutine was used to see if it was triangular or tetrahedral and binary search was used to see if it occurred in a preceding row.

I was rather startled that the second search produced no new results. The results 210, 11628, 24310 and 3003 were refound, which gave me some confidence in the process. I reran the program with output of the searching steps and this indicated that the program works correctly. So I am reasonably sure of the results, although still startled. I hope someone can extend this to higher limits, say 2^{59} and see if there are more repetitions.

The calculation of $N = \binom{103}{40}$ and the computational determination that it was not triangular were also complicated by overflow, since $N > 2^{48}$. First I attempted to compute only the 103rd row of the Pascal triangle by use of

$$\binom{103}{k} = \frac{104-k}{k}\binom{103}{k-1},$$

using double precision real arithmetic. However, this showed inaccuracies in the units place, beginning with $k = 33$. I then computed the entire triangle up to the 103rd row (mod 10^{14}) by addition. I could then overlap the two results to get N. The double precision calculation had been accurate to 27 of the 29 places.

I applied the idea of the subroutine to determine if N were triangular. This required some adjustments. Since $2N$ is bigger than 2^{96}, one cannot truncate $\sqrt{(2N)}$ to an integer. Instead $\sqrt{(N/2)}$ was calculated, truncated to an integer, converted to a double precision real and then doubled. Then the process of the subroutine was carried out, working in double precision real form. N was found to lie about halfway between two consecutive triangular numbers. These results for N were independently checked by Cecil Kaplinsky using multiprecision arithmetic on an IBM 360.

In a personal letter, D. H. Lehmer pointed out that one could determine that N was not triangular by noting its residue (mod 13). Following up on this suggestion, I computed the Pascal triangle (mod p) for small primes. Since $\binom{n}{k}$ (mod p) is periodic as a function of n [7, Theorem 38; 8; 9], one can deduce that $N \neq \binom{n}{k}$ for various k's by examination of N (mod p) and the possible values of $\binom{n}{k}$ (mod p). For example, $N \equiv 4$ (mod 13), but $\binom{n}{k} \not\equiv 4$ (mod 13) for $k = 2, 4, 6, 7, 8, 9, 10, 11, 12$. Using the primes 13, 19, 29, 31, 37, 53, 59 and 61, one can exclude all possibilities for k, other than 39 and 40 and hence N occurs exactly six times.

On the basis of the computer search and the scarcity of solutions of (1), I am tempted to make the following:

CONJECTURE. No binomial coefficient is repeated more than 10 times. (Perhaps the right number is 8 or 12?)

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A RECURSIVE METHOD FOR COUNTING INTEGERS NOT REPRESENTABLE IN CERTAIN EXPANSIONS

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1. INTRODUCTION

Let $\{P_i\}_1^\infty$ be a sequence of positive integers satisfying the inequality

$$P_{n+1} \geqslant 1 + \sum_1^n P_i \quad \text{for} \quad n \geqslant 1\,; \tag{1}$$

then it is well known ([1], Theorem 1; [2], Theorem 2; [3], Theorem 1) that any positive integer N possesses at most one representation as a sum of distinct terms from the sequence $\{P_i\}$. Such representations, when they exist, are thus unique, and we term a sequence $\{P_i\}$ of positive integers satisfying (1) a *sequence of uniqueness,* or briefly a *u-sequence.* Following Hoggatt and Peterson [1], we define $M(N)$ for each positive integer N as the number of positive integers less than N which are not representable as a sum of distinct terms from a given fixed u-sequence $\{P_i\}$. The principal result in [1] (Theorem 4) is that if N has a representation

$$N = \sum_1^n a_i P_i$$

with $\{a_i\}$ binary coefficients, then

$$M(N) = N - \sum_1^n a_i 2^{i-1}\,,$$

so that an explicit formula for $M(N)$ is available for *representable* positive integers. In general, a closed form expression for $M(N)$ as a function of N does not exist; our purpose in the present paper is to derive an expression from which $M(N)$ may be readily calculated for an arbitrary positive integer N.

2. DERIVATION

Throughout the following analysis, $\{P_i\}_1^\infty$ will denote a fixed u-sequence; we wish to find a recursive algorithm for determining $M(N)$.

First, we recall ([1], Theorem 2) that

$$M(P_n) = P_n - 2^{n-1} \quad \text{for} \quad n \geqslant 1,$$

so that only values of N not coinciding with terms of the u-sequence need be considered.

Theorem 1. Let N be an integer satisfying $P_n < N < P_{n+1}$ for some $n \geqslant 1$.

(i) If $P_n < N \leqslant \sum_1^n P_i$, then $M(N) = M(P_n) + M(N - P_n)$.

(ii) If $\sum_1^n P_i < N < P_{n+1}$, then $M(N) = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1 = N - 2^n$.

NOTE: Result (i) expresses $M(N)$ in terms of $M(P_n)$ and $M(N - P_n)$. But $N - P_n < P_n$ in case (i) since

$$P_n < N \leqslant \sum_1^n P_i \quad \text{implies} \quad 0 < N - P_n \leqslant \sum_1^{n-1} P_i < P_n,$$

the latter inequality following from the fact that $\{P_i\}$ is a u-sequence. Thus, if we consider the values $M(1)$, $M(2)$, $\cdots$, $M(P_n)$ as known, then $M(N)$ is determined from (i) whenever

$$P_n < N \leqslant \sum_1^n P_i,$$

while $M(N)$ is given explicitly by (ii) for the remaining values of N in (P_n, P_{n+1}).

Proof. Let N satisfy

$$P_n < N \leqslant \sum_1^n P_i.$$

Then $M(N)$ is equal to $M(P_n)$ plus the number of non-representable integers in the interval (P_n, N). But any integer K in (P_n, N) which is representable must have P_n in its representation (noting

$$\sum_1^{n-1} P_i < P_n),$$

and since $K = P_n + (K - P_n)$, we see $K - P_n$ must also be representable. Conversely if $K - P_n$ (which is less than P_n) is representable in terms of $P_1, \cdots, P_{n-1}$, then K is clearly representable. Thus the number of non-representable integers in (P_n, N) is equal to the number of non-representable integers less than $N - P_n$, or $M(N - P_n)$. Hence

$$M(N) = M(P_n) + M(N - P_n),$$

establishing (i).

For N satisfying

$$\sum_1^n P_i < N < P_{n+1},$$

it is obvious that N is not representable. Moreover

$$M\left(\sum_1^n P_i + 1\right) = M\left(\sum_1^n P_i\right), \qquad M\left(\sum_1^n P_i + 2\right) = M\left(\sum_1^n P_i\right) + 1$$

(assuming the arguments of the left-hand terms are $< P_{n+1}$) and in general (adding 1 to $M(N)$ each time N is increased by 1),

$$M(N) = M\left[\sum_1^n P_i + \left(N - \sum_1^n P_i\right)\right] = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1,$$

which is the first form of (ii). From Theorem 3 of [1],

$$M\left(\sum_1^n P_i\right) = \sum_1^n M(P_i) = \sum_1^n\left(P_i - 2^{i-1}\right) = \sum_1^n P_i - \sum_1^n 2^{i-1} = \sum_1^n P_i - (2^n - 1).$$

Then

$$M(N) = M\left(\sum_1^n P_i\right) + \left(N - \sum_1^n P_i\right) - 1 = \sum_1^n P_i - (2^n - 1) + \left(N - \sum_1^n P_i\right) - 1 = N - 2^n$$

as asserted.

Corollary 1. (Cf. [1], Theorem 4): If

$$N = \sum_1^n a_i P_i, \qquad \text{then} \qquad M(N) = \sum_1^n a_i M(P_i) = N - \sum_1^n a_i 2^{i-1}.$$

Proof. Let

$$N = \sum_1^n a_i P_i$$

with $a_n = 1$. Then

$$P_n \leqslant N \leqslant \sum_1^n P_i,$$

so that by (i) of Theorem 1, we have

$$M(N) = M(P_n) + M(N - P_n) = M(P_n) + M\left(\sum_1^K a_i P_i\right),$$

where $a_K = 1$ and $K < n$ (note K is simply the largest value of i less than n for which $a_i \neq 0$). Since

$$P_K \leqslant \sum_1^K a_i P_i \leqslant \sum_1^K P_i,$$

result (i) may be applied again and it is clear that successive iteration leads to

$$M(N) = \sum_1^n a_i M(P_i).$$

Using $M(P_i) = P_i - 2^{i-1}$, we have equivalently

$$M(N) = \sum_1^n a_i(P_i - 2^{i-1}) = \sum_1^n a_i 2^{i-1} = N - \sum_1^n a_i 2^{i-1}$$

as required.

Corollary 2. (Cf. [1], Theorem 3):

$$M\left(\sum_1^n P_i\right) = \sum_1^n M(P_i).$$

Proof. Immediate from Corollary 1 on taking all $a_i = 1$ for $i = 1, \cdots, n$.

3. EXAMPLE

Let $P_1 = 1, P_2 = 10, P_3 = 12, P_4 = 30, P_5 = 75, \cdots$ be the first 5 terms of a sequence which satisfies

$$P_{n+1} \geqslant 1 + \sum_{1}^{n} P_i$$

for all $n \geqslant 1$. Then, by direct enumeration

$M(1) = 0 = 1 - 2^0$	$M(13) = 8$
$M(2) = 0$	$M(14) = 8$
$M(3) = 1$	$M(15) = 9$
$M(4) = 2$	$M(16) = 10$
$M(5) = 3$	$M(17) = 11$
$M(6) = 4$	$M(18) = 12$
$M(7) = 5$	$M(19) = 13$
$M(8) = 6$	$M(20) = 14$
$M(9) = 7$	$M(21) = 15$
$M(10) = 8 = 10 - 2^1$	$M(22) = 16$
$M(11) = 8$	$M(23) = 16$
$M(12) = 8 = 12 - 2^2$	$M(24) = 16$
	$M(25) = 17$
	$M(26) = 18$
	$M(27) = 19$
	$M(28) = 20$
	$M(29) = 21$
	$M(30) = 22 = 30 - 2^3$

Now, note that all the values in the right-hand column may be calculated from those in the left-hand column; that is, if $12 < N < 30$, then we may apply Theorem 1 to see that

$$12 < N \leqslant 1 + 10 + 12 = 23 \rightarrow M(N) = M(12) + M(N-12)$$
$$23 < N < 30 \rightarrow M(N) = N - 2^3$$

Thus, for example, $N = 21$ is not representable but $M(21) = M(12) + M(9) = 8 + 7 = 15$, where we have assumed the values $M(1)$ through $M(12)$ are known. Similarly $N = 27$ is not representable but > 23, so $M(27) = 27 - 2^3 = 19$. Then, knowing $M(1)$ through $M(30)$, we may use Theorem 1 again to calculate $M(31)$ through $M(74)$. Note that for case (i) of Theorem 1, only one addition is needed, since $N - P_n$ always $< P_n$ in this case, while for case (ii), the result for $M(N)$ is explicitly given by $N - 2^n$.

4. CONCLUSION

A recursive scheme has been derived for calculating $M(N)$, the number of integers less than N not representable as a sum of distinct terms from a fixed u-sequence $\left\{P_i\right\}_1^{\infty}$. This approach has the advantage of not requiring any prior information concerning which positive integers are representable; however, if a representation for N is known, the result of Hoggatt and Peterson provides an explicit formula for $M(N)$, while in at least some of the remaining cases [(ii) of Theorem 1] an explicit formula is obtained from Theorem 1 of this paper. Other values of $M(N)$ for non-representable N are easily calculated via the recursion relation (i) of Theorem 1. In addition, Theorem 1 provides alternative somewhat simpler deviations of Theorems 3 and 4 in [1].

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A NOTE ON WEIGHTED SEQUENCES

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1. It is well known that the Catalan number

$$a(n) = \frac{1}{n+1}\binom{2n}{n} \tag{1.1}$$

satisfies the recurrence

$$a(n+1) = \sum_{j=0}^{n} a(j)a(n-j) \qquad (n = 0, 1, 2, \cdots). \tag{1.2}$$

Conversely if (1.2) is taken as definition together with the initial condition $a(0) = 1$ then one can prove (1.1). Thus (1.1) and (1.2) are equivalent definitions.

This suggests as possible q-analogs the following two definitions:

$$\bar{a}(n,q) = \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}, \tag{1.3}$$

where

$$[n+1] = \frac{1-q^{n+1}}{1-q}, \qquad \begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{(1-q^{2n})(1-q^{2n-1})\cdots(1-q^{n+1})}{(1-q)(1-q^2)\cdots(1-q^n)};$$

$$a(n+1,q) = \sum_{j=0}^{n} q^j a(j,q)a(n-j,q), \qquad a(0,q) = 1. \tag{1.4}$$

However (1.3) and (1.4) are not equivalent. Indeed

$$a(1,q) = 1, \qquad a(2,q) = 1+q, \qquad a(3,q) = (1+q)+q+q^2(1+q) = 1+2q+q^2+q^3,$$

$$\begin{aligned} a(4,q) &= (1+2q+q^2+q^3)+q(1+q)+q^2(1+q)+q^3(1+2q+q^2+q^3) \\ &= 1+3q+3q^2+3q^3+2q^4+q^5+q^6. \end{aligned}$$

On the other hand,

$$\bar{a}(1,q) = \frac{1}{[2]}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{1+q}\,\frac{1-q^2}{1-q} = 1,$$

$$\bar{a}(2,q) = \frac{1}{[3]}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{1+q+q^2}\,\frac{(1-q^4)(1-q^3)}{(1-q)(1-q^2)} = 1+q^2,$$

$$\begin{aligned} \bar{a}(3,q) = \frac{1}{[4]}\begin{bmatrix} 6 \\ 3 \end{bmatrix} &= \frac{1}{1+q+q^2+q^3}\,\frac{(1-q^6)(1-q^5)(1-q^4)}{(1-q)(1-q^2)(1-q^3)} \\ &= 1+q^2+q^3+q^4+q^6, \end{aligned}$$

$$\bar{a}(4,q) = \frac{1}{[5]}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}.$$

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Another well known definition of the Catalan number is the following. Let $f(n,k)$ denote the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = k \tag{1.5}$$

and

$$a_i \leqslant i \qquad (1 \leqslant i \leqslant n). \tag{1.6}$$

Then (see for example [1])

$$f(n,k) = \frac{n-k+1}{n}\binom{n+k-2}{n-1} \qquad (1 \leqslant k \leqslant n)$$

and in particular

$$f(n, n-1) = f(n,n) = \frac{1}{n}\binom{2n-2}{n-1} = a(n-1).$$

Next define $f(n, k, q)$ by means of [1]

$$f(n,k,q) = \sum q^{a_1+a_2+\cdots+a_n}, \tag{1.7}$$

where the summation is over all sequences $(a_1, a_2, \cdots, a_n)$ satisfying (1.5) and (1.6). It follows from this that the sum

$$f(n,q) = \sum_{k=1}^{n} f(n,k,q) \tag{1.8}$$

satisfies

$$f(n,q) = q^{-n} f(n+1, n, q) = q^{-n-1} f(n+1, n+1, q) \tag{1.9}$$

Moreover if we put

$$f(n+1, k+1, q) = q^{(k+1)(n+1)-\frac{1}{2}k(k+1)} b(n, k, q^{-1}) \tag{1.10}$$

then $b(n, k, q)$ satisfies

$$b(n,k,q) = q^{n-k} b(n, k-1, q) + b(n-1, k, q). \tag{1.11}$$

We shall show that

$$b(n,n,q) = a(n,q). \tag{1.12}$$

2. Returning to (1.4) we put

$$A(x,q) = \sum_{n=0}^{\infty} a(n,q)x^n. \tag{2.1}$$

Then

$$\begin{aligned} A(x,q) &= 1 + x \sum_{n=0}^{\infty} x^n \sum_{j=0}^{n} q^j a(j,q) a(n-j,q) \\ &= 1 + x \sum_{j=0}^{\infty} a(j,q) q^j x^j \sum_{n=0}^{\infty} a(n,q) x^n, \end{aligned}$$

so that

$$A(x,q) = 1 + xA(x,q)A(qx,q). \tag{2.2}$$

This gives

$$A(x,q) = \frac{1}{1 - xA(qx,q)},$$

which leads to the continued fraction

(2.3) $$A(x,q) = \frac{1}{1-}\,\frac{x}{1-}\,\frac{qx}{1-}\,\frac{q^2x}{1-}\,\cdots .$$

By a known result (see for example [3, p. 293])

$$\frac{1}{1-}\,\frac{x}{1-}\,\frac{qx}{1-}\,\frac{q^2x}{1-}\,\cdots = \frac{\Phi(qx,q)}{\Phi(x,q)},$$

where

(2.4) $$\Phi(x,q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}x^n}{(q)_n}$$

and

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

Therefore we get the identity

(2.5) $$A(x,q) = \frac{\Phi(qx,q)}{\Phi(x,q)} .$$

On the other hand it is proved in [1, (7.10)] that

(2.6) $$\sum_{k=0}^{\infty} b(n+k-1,k,q)x^k = \frac{\Phi(q^nx,q)}{\Phi(x,q)} \qquad (n > 0).$$

In particular, for $n = 1$, Eq. (2.6) reduces to

(2.7) $$\sum_{k=0}^{\infty} b(k,k,q)x^k = \frac{\Phi(qx,q)}{\Phi(x,q)} .$$

Comparing (2.7) with (2.5), we get

(2.8) $$b(k,k,q) = a(k,q).$$

3. For $x = -q$, Eq. (2.3) becomes

(3.1) $$A(-q,q) = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots .$$

It is known [3, p. 293] that the continued fraction

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\cdots = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} .$$

Thus (2.5) yields the identity

(3.2) $$\sum_{n=0}^{\infty} (-1)^n a(n,q)q^n = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} .$$

Another connection in which $a(n,q)$ occurs is the following. It can be shown that $a(n+1,q)$ is the number of *weighted* triangular arrays

(3.3) $$\begin{array}{ccccc} a_{11} & & a_{12} & \cdots & a_{1n} \\ & a_{21} & \cdots & a_{2,n-1} & \\ & & \cdots & & \\ & & a_{n1} & & \end{array} ,$$

where $a_{ij} = 0$ or 1 and

(3.4) $$a_{ij} \geqslant a_{i+1,j-1}, \qquad a_{ij} \geqslant a_{i+1,j} .$$

More precisely

$$a(n+1,\, q) \;=\; \sum q^{\Sigma a_{ij}}\,, \tag{3.5}$$

where the outer summation is over all (0.1) arrays (3.3) satisfying (3.4) and the sum Σa_{ij} is simply the number of ones in the array.

For example, for $n = 2$, we have the arrays

$$\begin{array}{cc} 0 & 0 \\ \multicolumn{2}{c}{0} \end{array} \qquad \begin{array}{cc} 1 & 0 \\ \multicolumn{2}{c}{0} \end{array} \qquad \begin{array}{cc} 0 & 1 \\ \multicolumn{2}{c}{0} \end{array} \qquad \begin{array}{cc} 1 & 1 \\ \multicolumn{2}{c}{0} \end{array} \qquad \begin{array}{cc} 1 & 1 \\ \multicolumn{2}{c}{1} \end{array}$$

This gives

$$1 + 2q + q^2 + q^3 \;=\; a(3, q).$$

For $n = 3$ we have

$$\begin{array}{ccc} 0 & 0 & 0 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|\; \begin{array}{ccc} 1 & 0 & 0 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 0 & 1 & 0 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 0 & 0 & 1 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|\; \begin{array}{ccc} 1 & 1 & 0 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 1 & 0 & 1 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 0 & 1 & 1 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|\; \begin{array}{ccc} 1 & 1 & 0 \\ \multicolumn{3}{c}{1\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 0 & 1 & 1 \\ \multicolumn{3}{c}{0\;1} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{0\;0} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|$$

$$\begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{1\;0} \\ \multicolumn{3}{c}{0} \end{array} \quad \begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{0\;1} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|\; \begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{1\;1} \\ \multicolumn{3}{c}{0} \end{array} \;\Bigg|\; \begin{array}{ccc} 1 & 1 & 1 \\ \multicolumn{3}{c}{1\;1} \\ \multicolumn{3}{c}{1} \end{array}$$

This gives

$$1 + 3q + 2q^2 + 3q^3 + 2q^6 + q^5 + q^6 \;=\; a(4, q).$$

Let $T_k(n)$ denote the number of solutions in non-negative integers a_{ij} of the equation

$$n = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} a_{ij}\,,$$

where the a_{ij} satisfy the inequalities

$$a_{ij} \geqslant a_{i+1,j}\,, \qquad a_{ij} \geqslant a_{i+1,j-1}\,.$$

It has been proved in [2] that

$$\sum_{n=1}^{\infty} T_k(n)x^n = \frac{1}{(1-x^{2k-1})(1-x^{2k-3})^2 \cdots (1-x^5)^{k-2}(1-x^3)^{k-1}(1-x)^k} \tag{3.6}$$

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AN APPLICATION OF SPECTRAL THEORY TO FIBONACCI NUMBERS

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It is well known that the n^{th} Fibonacci number, a_n

$$(a_0 = a_1 = 1, \quad a_n = a_{n-1} + a_{n-2}, \quad n \geq 2)$$

can be explicitly written in the form

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n = 0, 1, 2, \cdots, \tag{1}$$

where

$$\lambda_1 = \tfrac{1}{2}(1+\sqrt{5}), \quad \lambda_2 = \tfrac{1}{2}(1-\sqrt{5}).$$

The purpose of this note is to derive Binet's formula (1) from the spectral decomposition of the matrix A, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

First, note that for $n = 2, 3, 4, \cdots$, we have

$$A^n = \begin{pmatrix} a_n & a_{n-1} \\ a_{n-1} & a_{n-2} \end{pmatrix}. \tag{2}$$

Second, note that since A is a symmetric matrix, there is an orthogonal matrix, $P (P^TP = I)$, such that

$$P^{-1}AP = D = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \; ; \tag{3}$$

i.e., D is the diagonal matrix whose diagonal elements are the eigenvalues of A. These are the zeros of the characteristic equation $\lambda^2 - \lambda - 1 = 0$, of A. A short calcualtion reveals that

$$P = \frac{1}{d}\begin{pmatrix} 1 & \lambda_1 \\ -\lambda_1 & 1 \end{pmatrix},$$

where $d > 0$, and

$$d^2 = 1 + \lambda_1^2 = \sqrt{5}\lambda_1 \; ;$$

i.e.,

$$\frac{1}{d^2}\begin{pmatrix} 1 & -\lambda_1 \\ \lambda_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda_1 \\ -\lambda_1 & 1 \end{pmatrix} = D.$$

Raising the expression in (3) to the n^{th} power yields

$$(P^{-1}AP)^n = P^{-1}A^nP = D^n,$$

and, solving for A^n, we have

$$A^n = PD^nP^{-1}, \quad n \geq 2. \tag{4}$$

Equating the values in the upper left-hand corners of the two expressions in (4) gives formula (1) when $n \geq 2$. A simple check shows that (1) holds in fact, for all $n \geq 0$.

The above method for obtaining the explicit formula (1) is quite general; it can be used to obtain explicit formulae for terms in other linear recurrences. Unfortunately, it is not directly applicable to arithmetic sequences in prime number theory. In the case of the summatory function of $\Lambda(k)$ ($\Lambda(p^r) = \log p$ for any prime p, and

$$\Lambda(k) = 0$$

otherwise),

$$\Psi(X) = \sum_{k \leqslant x} \Lambda(k),$$

what seems to be needed is an operator, T, whose eigenvalues, ρ_k, are the zeros of the Riemann zeta-function. Then, given $x > 1$, we would have, on the one hand

$$\text{Trace}\left(\frac{x^T}{T}\right) = \sum_k \frac{x^{\rho_k}}{\rho_k},$$

and, on the other

$$\text{Trace}\left(\frac{x^T}{T}\right) = \sum_{j=0}^{\infty} \frac{\log^j x}{j!} (\text{Trace } T)^{j-1}.$$

An arithmetic interpretation of the right side of the last formula should yield an expression close to $x - \Psi(x)$.

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MINIMUM SOLUTIONS TO $x^2 - Dy^2 = \pm 1$

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A solution pair (x_0, y_0) to $x^2 - Dy^2 = \pm 1$ shall be considered a minimum solution if y_0 is minimum. Throughout this article F_n stands for the n^{th} Fibonacci number of

$$\left\{1, 1, 2, 3, \cdots, F_{n+2} = F_{n+1} + F_n, \qquad F_1 = F_2 = 1\right\};$$

$$F_n = \frac{1}{\sqrt{5}}(a^n - b^n), \qquad a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2}, \qquad ab = -1.$$

I. CONTINUED FRACTION EXPANSION OF ODD PERIOD $2r+1$, $r \not\equiv 0 \pmod 3$, $r > 1$

Let $D = m^2 + k$, $m \leqslant k \leqslant 2m$, and assume a continued fraction expansion all of whose middle elements are ones, thus assuring minimum y.

$$y = F_{2r}, \qquad x = mF_{2r} + F_{2r-1}$$

$$(mF_{2r} + F_{2r-1})^2 - y^2(m^2 + k) = 1$$

upon using

$$F_{t+1}F_{t-1} - F_t^2 = (-1)^t$$

simplifies to

(1) $$2mF_{2r-1} - kF_{2r} = -F_{2r-2}.$$

This has integer solutions given by

$$m = sF_{2r} + \tfrac{1}{2}(F_{2r} + 1),$$

$$k = 2sF_{2r-1} + F_{2r-1} + 1, \qquad D = s^2F_{2r}^2 + s(F_{2r}^2 + 2F_{2r} + F_{2r-3}) + \tfrac{1}{4}(F_{2r} + 1)^2 + F_{2r-1} + 1.$$

$x^2 - Dy^2 = 1$ has integer solutions given by

$$x = sF_{2r}^2 + \tfrac{1}{2}F_{2r}(F_{2r} + 1) + F_{2r-1}, \qquad y = F_{2r}.$$

II. CONTINUED FRACTION EXPANSION OF ODD PERIOD $6r+1$

Let $D = m^2 + k$. Assume that the central integer in the continued fraction expansion is 2 and that all the other middle elements are ones. The half period expansion is:

		$(3r-1)$ ones			
m	1	1 ------------	1	1	2
$\frac{m}{1}$	$\frac{m+1}{1}$	$\frac{2m+1}{2}$	$\frac{mF_{3r-1}+F_{3r-2}}{F_{3r-1}}$	$\frac{mF_{3r}+F_{3r-1}}{F_{3r}}$	$\frac{\quad}{2F_{3r}+F_{3r-1}}$

$$y = F_{3r}[F_{3r-1} + 2F_{3r} + F_{3r-1}] = 2F_{3r}F_{3r+1}$$

$$x = 2mF_{3r}F_{3r+1} + F_{3r}^2 + F_{3r-1}F_{3r+1}$$

$$[2mF_{3r}F_{3r+1} + F_{3r}^2 + F_{3r-1}F_{3r+1}]^2 - 4F_{3r}^2F_{3r+1}^2(m^2 + k) = 1$$

simplifies to

$$m(F_{3r}^2 + F_{3r-1}F_{3r+1}) - F_{3r}F_{3r+1}k = -F_{3r}F_{3r-1}$$

(2) $$k = m + \frac{F_{3r-1}(mF_{3r-1} + F_{3r})}{F_{3r}F_{3r+1}}$$

(a) $$r = 2u + 1.$$

Equation (2) transformed has integer solutions for m and k given by

$$m = F_{6u+3}F_{6u+4}s + F^2_{6u+3}$$
$$k = (2F^2_{6u+3} - 1)s + F_{6u+1}F_{6u+2} + F^2_{6u+3}.$$

(b) $$r = 2u$$

Equation (2) transformed has integer solutions for m and k given by

$$m = F_{6u}F_{6u+1}s + F_{6u}F_{6u-1},$$
$$k = (2F^2_{6u} + 1)s + F_{6u-1}F_{6u-3} + F_{6u}F_{6u-1}$$

III. CONTINUED FRACTION EXPANSIONS OF EVEN PERIOD $2r + 2$, $r \not\equiv 1 \pmod 3$, $r \geq 1$, $D = m^2 + k$

Assume a continued fraction expansion all of whose middle elements are ones, thus assuring minimum y.

$$y = F_{2r+1}, \qquad x = mF_{2r+1} + F_{2r} \qquad \text{and} \qquad (mF_{2r+1} + F_{2r})^2 - F^2_{2r+1}(m^2 + k) = -1$$

simplifies to

(3) $$2mF_{2r} - kF_{2r+1} = -F_{2r-1}$$

Equation (3) has integer solutions given by

$$m = sF_{2r+1} + \tfrac{1}{2}(F_{2r+1} + 1), \qquad k = 2sF_{2r} + F_{2r} + 1,$$
$$D = s^2F^2_{2r+1} + s(F^2_{2r+1} + F_{2r+1} + 2F_{2r}) + \tfrac{1}{4}(F_{2r+1} + 1)^2 + F_{2r} + 1.$$

$x^2 - Dy^2 = -1$ has integer solutions given by

$$x = sF^2_{2r+1} + \tfrac{1}{2}F_{2r+1}(F_{2r+1} + 1) + F_{2r}, \qquad y = F_{2r+1}.$$

IV. CONTINUED FRACTION EXPANSIONS WITH EVEN PERIOD $6r - 2$, $r \geq 1$

Let $D = m^2 + k$ and assume that the two central elements are each two and the other middle elements are all ones. From the half period expansion:

$$\begin{array}{ccccc} & & \multicolumn{2}{c}{(3r-3)\text{ ones}} & \\ m & 1 & 1 \cdots\cdots & 1 & 2 \\ \hline m & m+1 & 2m+1 & mF_{3r-2} + F_{3r-3} & mF_{3r} + F_{3r-1} \\ 1 & 1 & 2 & F_{3r-2} & F_{3r} \end{array}$$

$$y = F^2_{3r-2} + F^2_{3r}$$
$$x = (F^2_{3r-2} + F^2_{3r})m + F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1}$$
$$(my + F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1})^2 - y^2(m^2 + k) = -1$$

simplifies to

$$2(F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1})m - (F^2_{3r} + F^2_{3r-2})k = -F^2_{3r-3} - F^2_{3r-1}$$

(4) $$k = m + \frac{m(F_{3r}F_{3r-3} + F_{3r-2}F_{3r-5}) + F^2_{3r-3} + F^2_{3r-1}}{F^2_{3r} + F^2_{3r-2}}$$

(a) $$r = 2u$$

$$k = m + \frac{m(F_{6u}F_{6u-3} + F_{6u-2}F_{6u-5}) + F^2_{6u-3} + F^2_{6u-1}}{F^2_{6u} + F^2_{6u-2}}$$

has integer solutions given by

$$m = \tfrac{1}{2}(F_{6u}F_{6u-3} + F_{6u-2}F_{6u-5} + 1),$$
$$k = m + \tfrac{1}{2}(F^2_{6u-3} + F^2_{6u-5} + 1)$$

(b)
$$r = 2u + 1$$
$$k = m + \frac{m(F_{6u}F_{6u+3} + F_{6u+1}F_{6u-2}) + F_{6u}^2 + F_{6u+2}^2}{F_{6u+3}^2 + F_{6u+1}^2}$$
has integer solutions given by
$$m = -\tfrac{1}{2}(F_{6u}F_{6u+3} + F_{6u+1}F_{6u-2} - 1), \qquad k = m - \tfrac{1}{2}(F_{6u}^2 + F_{6u-2}^2 - 1).$$

MINIMUM SOLUTION TABLE

period	D	x	y
2	$m^2 + 1 : 2$	$m : 1$	1
3	$m^2 + 2m : 3$	$m + 1 : 2$	1
4	$25s^2 + 64s + 41 : 41$	$25s + 32 : 32$	5
5	$9s^2 + 16s + 7 : 7$	$9s + 8 : 8$	3
6	13	18	5
7	21	55	12
8	58	99	13
9	135	244	21
10	113	776	73
11	819	1574	55
12	2081	4060	89
13	1650	8449	208
14	13834	27405	233
15	35955	71486	377
16	1370345	1551068	1325
17	244647	488188	987
18	639389	1276990	1597
19	1337765	4325751	3740
20	4374866	8745055	4181
21	11448871	22890176	6765
22	7877105	66688052	23761
23	78439683	156859562	17711
24	205337953	410643864	28657

A continued fraction expansion is unique; has the unique half-period relations
$$q_{2r} = q_r(q_{r-1} + q_{r+1}), \qquad p_{2r} = p_{r-1}q_r + p_rq_{r+1}$$
for period $2r + 1$,
$$q_{2r+1} = q_r^2 + q_{r+1}^2, \qquad p_{2r+1} = p_rq_r + p_{r+1}q_{r+1}$$
for period $2r + 2$; and furnish a minimum primitive solution to $x^2 - Dy^2 = \pm 1$. Using Fibonacci identities it can be shown that all the assumed continued fraction expansions obey the proper half-period relations, give a minimum primitive solution to $x^2 - Dy^2 = \pm 1$ and hence are the actual continued fraction expansions. The half-period relations are explicitly stated as the x and y values in II and IV. The other Fibonacci identities needed are

(1) $$F_{2r}^2 + 1 = F_{2r-1}F_{2r+1}$$

(2) $$F_r^2 + F_{r+1}^2 = F_{2r+1};$$

(3) $$F_{r-1}(F_{r-2} + F_r) = F_{2r-2};$$

(4) $$F_{2n-1}^2 - 1 = F_{2n}F_{2n-2}.$$

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DISTRIBUTION OF THE ZEROES OF ONE CLASS OF POLYNOMIALS

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INTRODUCTION

In the present paper we shall prove that the zeroes of the real polynomials

(1) $$f_0(x) = 0, \quad f_1(x) = s, \quad f_i(x) = x, \quad f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \quad n = 2, 3, \cdots$$

with $s \neq 0$ and $n \geqslant 2$ are simple, of the form $-2i \cos \theta$, where $i^2 = -1$. and if $2i \cos \theta_j^{(n+1)}, j = 1, \cdots, n$ are the zeroes of $f_{n+1}(x)$, then the points $\cos \theta_j^{(n+1)}, j = 1, \cdots, n$ are divided by $\cos \theta_j^{(n)}, j = 1, \cdots, n-1$ and for every interval between two successive points $-[\cos \theta_j^{(n+1)}, \cos \theta_{j+1}^{(n+1)}]$ one and only one of the following three possibilities holds:

(a) The interval contains one of $\cos \theta_j^{(n-k+i)}$, $1 \leqslant k \leqslant n-1, j = 1, \cdots, n-k$.

(b) It contains one of $\cos (j\pi/k)$, $j = 1, \cdots, k-1$ or

(c) One of the boundary points of it coincides with one of $\cos \theta_j^{(n-k+1)}$, and $\cos (j\pi/k)$ simultaneously.

When $s = 0$, then $f_n(x)$ becomes

$$f_0(x) = 0, \qquad f_n(x) = xu_{n-1}(x), \qquad n = 1, \cdots,$$

where $u_n(x)$ are derived from (1) for $s = 1$. $u_n(x)$ are Fibonacci polynomials.

1. ON THE ZEROES OF FIBONACCI POLYNOMIALS

From the well known formula:

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} 2^{n-2k} z^k = ((1+\sqrt{z+1})^{n+1} - (1-\sqrt{z+1})^{n+1})/2\sqrt{z+1}$$

and [2] it follows that:

(2) $$u_n(x) = (2^n\sqrt{x^2+4})^{-1}((x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n), \quad n = 0, 1, 2, \cdots.$$

Then for $x = 2i \cos \theta$ we get:

(3) $$u_n(2i \cos \theta) = -(i^{n+1} \sin n\theta)/\sin \theta\,.$$

So, the numbers $2i \cos (j\pi/n)$, where j is an integer and $\sin (j\pi/n) \neq 0$, are zeroes of $u_n(x)$, $n \geqslant 2$. But only $n-1$ of them are distinct. Indeed, if j gets values j_1 and j_i and $j_1 - j_2$ is a multiple of $2n$ then

$$\cos (j_1\pi/n) = \cos (j_2\pi/n).$$

Otherwise

$$\cos ((n+j)\pi/n) = \cos ((n-j)\pi/n) \quad \text{for} \quad 0 \leqslant j \leqslant n.$$

Therefore the numbers $2i \cos (j\pi/n)$, $j = 1, \cdots, n-1$ are $n-1$ different zeroes of (2). Since $u_n(x)$ is a polynomial of the $n-1^{th}$ degree they are all its zeroes.

2. DISTRIBUTION OF THE ZEROES OF $f_n(x)$, $n = 2, \cdots$, WHEN $s \neq 0$

By induction it may be proved that:

$$f_n(x) = u_n(x) + (s-1)u_{n-2}(x), \quad n \geqslant 2. \tag{4}$$

Owing to (3) and (4) we have:

$$f_n(2i\cos\theta) = i^{n-1}((\sin n\theta/\sin\theta) - (s-1)(\sin(n-2)\theta)/\sin\theta).$$

Functions

$$Q_n(\cos\theta) = \sin n\theta/\sin\theta, \quad n = 1, \cdots,$$

are Tchebishev's polynomials of second class. Let

$$Q_{-2}(\cos\theta) = -1, \quad Q_0(\cos\theta) = 0 \quad \text{and} \quad P_n(\cos\theta) = Q_n(\cos\theta) - (s-1)Q_{n-2}(\cos\theta), \quad n = 1, \cdots.$$

Then the following conditions are fulfilled:

$$P_0(\cos\theta) = s, \qquad P_2(\cos\theta) = 2\cos\theta,$$

$$P_{n+1}(\cos\theta) = 2\cos\theta P_n(\cos\theta) - P_{n-1}(\cos\theta), \qquad n = 1, 2, \cdots$$

and the polynomials

$$P_0(\cos\theta), \quad P_1(\cos\theta), \cdots, P_{n+1}(\cos\theta)$$

form a Sturm's row. From [1]—the zeroes of $P_{n+1}(\cos\theta)$ are real, distinct and the zeroes of $P_n(\cos\theta)$ divide those of $P_{n+1}(\cos\theta)$. So, $f_{n+1}(x)$ has n distinct zeroes—

$$2i\cos\theta_j^{(n+1)}, \qquad j = 1, 2, \cdots, \cdots, n$$

too and the points $\cos\theta_j^{(n+2)}$, $j = 1, \cdots, n$ are divided by $\cos\theta_j^{(n)}$, $j = 1, \cdots, n-1$.

The position of the zeroes of $P_{n-k}(\cos\theta)$ in relation to those of $P_n(\cos\theta)$ can be examined by the help of the lemmas:

Lemma 1.

$$P_n(\cos\theta) = Q_k(\cos\theta)P_{n-k}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k+1}(\cos\theta), \tag{4}$$

where n and k are positive integers and $n \geqslant 2$, $1 \leqslant k < n$.

This is proved by induction over n. It can be directly verified that it is valid for $n = 2$, $k = 1$ and for $n = 3$, $k = 1, 2$. If we assume that (4) is true for some $n - 1 > 3$, $k = 1, 2, \cdots, n-2$ and $n, k = 1, \cdots, n-1$, then

$$\begin{aligned} P_{n+1}(\cos\theta) &= 2\cos\theta P_n(\cos\theta) - P_{n-1}(\cos\theta) = 2\cos\theta(Q_k(\cos\theta)P_{n-k}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k-1}(\cos\theta)) \\ &= Q_k(\cos\theta)P_{n-k-1}(\cos\theta) + Q_{k-1}(\cos\theta)P_{n-k-2}(\cos\theta) \\ &= Q_k(\cos\theta)P_{n-k+1}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k}(\cos\theta) = Q_k(\cos\theta)P_{n-k+1}(\cos\theta), \end{aligned}$$

which is true for $k = 1, \cdots, n-2$. When $k = n-1$ and $k = n$, we have

$$P_{n+1}(\cos\theta) = 2\cos\theta\, Q_n(\cos\theta) - sQ_{n-1}(\cos\theta)$$

the validity of which is easily proved by induction over n.

Lemma 2.

$$P_{n-k}(\cos\theta_j^{(n+1)}) = Q_{k-1}(\cos\theta_j^{(n+1)})P_{n-1}(\cos\theta_j^{(n+1)}), \quad j = 1, 2, \cdots, n.$$

This can be proved by induction over k.

Owing to Lemma 1 and the results received above, the common zeroes of $P_n(\cos\theta)$ and $P_{n-k}(\cos\theta)$ are zeroes of $Q_{k-1}(\cos\theta)$. Moreover $P_n(\cos\theta)$ and $Q_{k-1}(\cos\theta)$ have no other common zeroes.

Let

$$(\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}), \quad 1 \leqslant j \leqslant n-1$$

be an interval between two successive zeroes of $P_n(\cos\theta)$ which doesn't contain any zeroes of $Q_{k-1}(\cos\theta)$.

Then

$$Q_{k-1}(\cos\theta_j^{(n+1)}), Q_{k-1}(\cos\theta_{j+1}^{(n+1)}) > 0$$

$$P_{n-1}(\cos\theta_j^{(n+1)}), P_{n-1}(\cos\theta_{j+1}^{(n+1)}) < 0$$

and by Lemma 2, we conclude that:

$$P_{n-k}(\cos\theta_j^{(n+1)}), \quad P_{n-k}(\cos\theta_{j+1}^{(n+1)}) < 0 .$$

This shows that $P_{n-k}(\cos\theta)$ has an odd number of zeroes in

$$[\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}] .$$

If $P_{n-k}(\cos\theta)$ has more than one zero in this interval, from Lemma 1 it will follow that $P_n(\cos\theta)$ has a zero in

$$(\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}) ,$$

which contradicts our assumption. Therefore every interval

$$[\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}]$$

which doesn't contain a zero of $Q_{k-1}(\cos\theta)$, contains only one zero of $P_{n-k}(\cos\theta)$. In a similar way it is proved that if in

$$[\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}]$$

there is no zero of $P_{n-k}(\cos\theta)$, it contains one zero of $Q_{k-1}(\cos\theta)$.

Thus we proved that in every interval between two successive points of

$$\cos\theta_j^{(n+1)}, \qquad j = 1, \cdots, n$$

there is either one and only one of

$$\cos\theta_j^{(n-k+1)}, \qquad j = 1, \cdots, n-k,$$

or one and only one of

$$\cos(j\Pi/k), \qquad j = 1, \cdots, k-1$$

or one of the boundary points of this interval coincides with one of

$$\cos\theta_j^{(n-k+1)}, \quad j = 1, \cdots, n-k \qquad \text{and of} \qquad \cos(j\Pi/k), \quad j = 1, \cdots, k-1.$$

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ON THE GENERALIZATION OF THE FIBONACCI NUMBERS

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I.

The Fibonacci numbers ($F_0 = F_1 = 1;\ F_n = F_{n-1} + F_{n-2}$, if $n \geq 2$) are very useful in describing the ladder-network of Fig. 1, if $r = R$ (cf. [1], [2], [3]). If the common value of the resistances R and r is chosen to be unity, the resistance Z_n of the ladder-network can be calculated on the following way:

$$Z_n = \frac{F_{2n}}{F_{2n-1}} \quad . \tag{1a}$$

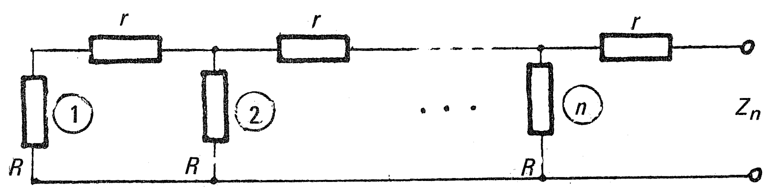

Figure 1

Let $R \neq r$. For the sake of convenient notation let $x = r/R$ and $z_n = Z_n/R$. Then

$$z_n = \frac{f_{2n}(x)}{f_{2n-1}(x)} \quad , \tag{1}$$

where $f_0(x) = f_1(x) = 1$; and for $n \geq 2$,

$$f_n(x) = \begin{cases} f_{n-1}(x) + f_{n-2}(x) & \text{if} \quad n \text{ is odd} \\ xf_{n-1}(x) + f_{n-2}(x) & \text{if} \quad n \text{ is even} \end{cases} \quad . \tag{*}$$

This fact gave us the idea to examine into the sequences, defined by a finite number of homogeneous linear recurrences which are to be used cyclically. We may assume without loss of generality that the length of the recurrences are equal and that this common length $\underline{m}$ equals the number of the recurrences:

$$f_n = \begin{cases} a_1^0 f_{n-1} + \cdots + a_m^0 f_{n-m} & \text{if } n \equiv 0 \pmod m \\ a_1^1 f_{n-1} + \cdots + a_m^1 f_{n-m} & \text{if } n \equiv 1 \pmod m \\ \vdots & \\ a_1^{m-1} f_{n-1} + \cdots + a_m^{m-1} f_{n-m} & \text{if } n \equiv m-1 \pmod m \end{cases} \quad .$$

It has been proved in [5] that the same sequence f_n can be generated by a certain unique recurrence too, which has length m^2 and "interspaces" of length $\underline{m}$, i.e.,

$$f_n = b_1 f_{n-m} + b_2 f_{n-2m} + \cdots + b_m f_{n-m^2} \quad .$$

Applying our results to (*) we have

$$f_n(x) = (2+x)\cdot f_{n-2}(x) - f_{n-4}(x),$$

or, after the calculation of the generating function and expanding it into Taylor-series,

$$(2)\qquad f_n(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} x^{[n/2]-i}.$$

This enables us to solve not only the problem of the lumped network mentioned above, but a special question of the theory of the distributed networks (e.g., transmission lines) can also be solved. If we want to describe the pair of transmission lines having resistance r_0 and shunt-admittance $1/R_0$ (see Fig. 2), then put $r = r_0/n$ and $R = R_0 \cdot n$. Applying (1) and (2) we have

$$Z_n^* = \frac{r_0}{n} \frac{g_n\left(\frac{R_0 n^2}{r_0}\right)}{h_n\left(\frac{R_0 n^2}{r_0}\right)},$$

where

$$g_n(x) = \sum_{j=1}^{n} \binom{2n-j}{j-1} \cdot x^j \qquad \text{and} \qquad h_n(x) = \sum_{j=0}^{n} \binom{2n-j}{j} \cdot x^j.$$

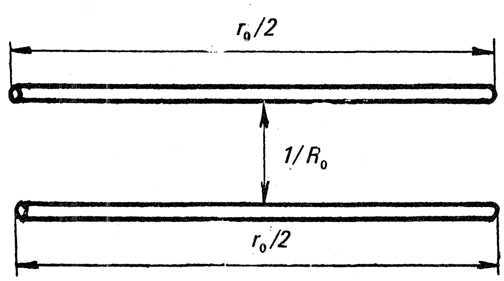

Figure 2

The following simultaneous system of recurrences can be found:

$$g_{n-1}(x) = h_n(x) - (1+x)h_{n-1}(x)$$

$$x^2 \cdot h_{n-2}(x) = g_n(x) - (1+x)g_{n-1}(x),$$

which enables us to give an explicit form to $g_n(x)$ and $h_n(x)$. At last

$$\lim Z_n^* = \sqrt{R_0 r_0} \cdot th\sqrt{r_0/R_0}, \quad \text{if} \quad n \to \infty,$$

where

$$th\, y = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

This is exactly the result, which can also be received from a system of partial differential equations (the telegraph-equations).

II.

On the other hand, (2) can be considered as a generalization of the Fibonacci sequence. Trivially $f_n(1) = F_n$; and

$$F_n = \sum_{i=0}^{[n/2]} \binom{n-i}{i} ,$$

a well-known result about the Fibonacci numbers. Similarly,

$$F_n = 3F_{n-2} - F_{n-4} ,$$

if $n \geqslant 4$, or

$$F_n = tF_{n-3} + (17-4t)F_{n-6} + (4-t)F_{n-9}$$

for any t, if $n \geqslant 9$ and an infinite number of longer recurrences (length m^2 and interspaces m for arbitrary $m = 2,3,\cdots$) could be similarly produced.

A possible further generalization of the Fibonacci numbers is

$$F_{n,p}(x) = \sum_{i=0}^{\left[\frac{n}{p+1}\right]} \binom{n-ip}{i} x^{\left[\frac{n}{p+1}\right]-i} ,$$

where p is an arbitrary non-negative integer.

This definition is the generalization of the $u(n;p,1)$ numbers of [4]. The following recurrence can be proved for the $F_{n,p}(x)$ polynomials:

$$xF_{n-p-1,p}(x) = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{n-(p+1)i,p}(x) . \tag{3}$$

Similarly, it can be easily proved that the generating function

$$\sum_{i=0}^{\infty} F_{i,p}(x) \cdot z^i$$

has the following denominator:

$$(1-z^{p+1})^{p+1} - x \cdot z^{p+1} .$$

As a last remark, it is to be mentioned that a further generalization of the functions $F_{n,p}(x)$ can be given (cf. [4]):

$$F_{n;p,q}(x) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} x^{\left[\frac{n}{p+q}\right]-i} ;$$

but this case is more difficult. A recurrence, similar to (3) can be found, which contains on the left side the higher powers of x, too. However, essentially new problems arise considering the case $q \geqslant 2$.

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THE GENERAL LAW OF QUADRATIC RECIPROCITY

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Extend the definition of the Jacobi Symbol to include values for negative second entry as follows:
If a is an integer and p is an odd prime, set

$$(a/p) \equiv a^{(p-1)/2} \pmod{p}$$

and

$$(a/p) = 0 \text{ or } \pm 1 .$$

Set

$$(a/1) = 1.$$

If b is an odd integer, set

$$(a/b_1b_2) = (a/b_1)(a/b_2).$$

Set

$$(0/-1) = 0.$$

Set

$$(-1/-1) = -1.$$

There is another way of defining negative second entry in the Jacobi Symbol, which is based upon

$$(-1/-1) = 1.$$

This method is given in [1, p. 38, Exercise IX, 5].

The Jacobi Symbol is only a definition and not a theorem; therefore it can be arbitrary as long as it satisfies two requirements: First, it must be consistent and, secondly, it must represent mathematical results clearly and elegantly. The definition given in this paper is superior from the second point of view. For example, with

$$(-1/-1) = 1$$

it is difficult to express the periodicity of the second entry. In fact, much of that periodicity is lost. But, with

$$(-1/-1) = -1,$$

the result is clearly stated in Corollary 2.

All of the known and proven properties of the Jacobi Symbol are retained in the extended definition (see [1, pp. 36–39] and [2, pp. 77–80]).

This refers in particular to the multiplicavity of the first entry, which is easily proved for negative second entry. Then

$$(a_1a_2/b) = (a_1/b)(a_2/b)$$

and

[Continued on page 321.]

NON-HYPOTENUSE NUMBERS

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The non-hypotenuse numbers $n = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 18, \cdots$ are those natural numbers for which there is *no* solution of

$$n^2 = u^2 + v^2 \qquad (u > v > 0). \tag{1}$$

Although they occur very frequently for small n they nonetheless have zero density—almost all natural numbers n do have solutions. Only 1/15.547 of the numbers around 10^{100} are NH numbers, and, around $2^{19937} - 1$, only 1/120.806.

In a review of a table by A. H. Beiler [1], I had occasion to remark that if $NH(x)$ is the number of such $n \leqslant x$ then

$$NH(x) \sim Ax/\sqrt{\log x} \tag{2}$$

for some coefficient A. Recently, T. H. Southard wished to know this A because of an investigation [2] originating in a study of Jacobi theta functions. Inasmuch as most of the analysis and arithmetic has already been done in [3], one can be more precise and easily compute accurate values of A and C in the asymptotic expansion:

$$NH(x) = \frac{Ax}{\sqrt{\log x}}\left[1 + \frac{C}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]. \tag{3}$$

Landau's function $B(x)$ is the number of $n \leqslant x$ for which there *is* a solution of

$$n = u^2 + v^2 . \tag{4}$$

Note: n to the *first* power, and all u, v allowed. Then

$$B(x) = \frac{bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right] \tag{5}$$

and I evaluated

$$b = 0.764223654, \qquad c = 0.581948659 \tag{6}$$

in [3]. The n of (4) are those n divisible only by 2, by primes $p \equiv 1 \pmod 4$, and by even powers of primes $q \equiv 3 \pmod 4$. If $b_m = 1$ for any m = any such n, and $b_m = 0$ otherwise, one has the generating function

$$\sum_{m=1}^{\infty} \frac{b_m}{m^s} = f(s) = \frac{1}{1-2^{-s}} \prod_p \frac{1}{1-p^{-s}} \prod_q \frac{1}{1-q^{-2s}} . \tag{7}$$

In contrast, the NH numbers are those divisible by no prime p, and so they are generated by

$$g(s) = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} . \tag{8}$$

Since

(9) $$L(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots = \prod_p \frac{1}{1-p^{-s}} \prod_q \frac{1}{1+q^{-s}}$$

we can write

(10) $$g(s) = f(s)/L(s).$$

Landau [4] showed that $f(s)$ has a branch point at $s = 1$ and a convergent series

(11) $$f(s) = \frac{ais^2}{\sqrt{1-s}}\,[1 + a_1(1-s)/a + \cdots]$$

in its neighborhood for computable coefficients $a, a_1, \cdots$. In terms of these, one evaluates the coefficients of (5) as

(12) $$b = \frac{a\Gamma(\tfrac{1}{2})}{\pi}, \qquad c = (a_1 - a)/2a$$

with the usual method using Cauchy's theorem and integration around the branch point. But $L(s)$ is analytic at $s = 1$ and so we have, at once,

(13) $$d = \frac{a}{L(1)}, \qquad \frac{d_1}{d} = \frac{a_1}{a} + \frac{L'(1)}{L(1)}$$

for the new generator

(14) $$g(s) = \frac{dis^2}{\sqrt{1-s}}\,[1 + d_1(1-s)/d + \cdots].$$

Therefore

(15) $$A = b/L(1), \qquad C = c + L'(1)/2L(1)$$

give the wanted coefficients of (3). Of course, $L(1) = \pi/4$, and in [3] one has

(16) $$L'(1)/L(1) = \log\left[\left(\frac{\pi}{\tilde{\omega}}\right)^2 \frac{e^{\gamma}}{2}\right]$$

in terms of the Euler constant γ and the lemniscate constant $\tilde{\omega}$. So, from [3] one has

(17) $$A = \frac{2\sqrt{2}}{\pi}\prod_q (1-q^{-2})^{-1/2} = 0.97303978$$

and

(18) $$C = \frac{1}{2}\left[1 + \log\left(\frac{\pi}{\tilde{\omega}}\right) - \frac{1}{2}\frac{d}{ds}\log\prod_q \frac{1}{1-q^{-2s}}\bigg|_{s=1}\right] = 0.70475345.$$

In [2] Southard gives

$$NH(99999) - NH(99000) = 295,$$

while (3), (17) and (18) give

$$NH(99999) - NH(99000) = 289.36.$$

It is known that the third-order term in (3) is positive but it was not computed.

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[Continued from P. 318.]

$$(1/-1) = 1,$$
$$(-1/1) = 1,$$
$$(1/1) = 1.$$

The second entry of the Extended Jacobi Symbol is multiplicative by definition; it will be proved in the corollaries that both entries are also periodic.

The following results are easily derived:

Explicitly,

$$(0/1) = 1,$$
$$(0/b) = 0 \text{ if } b \neq 1,$$
$$(0/-b) = 0 \text{ if } -b \neq 1,$$
$$(2/\pm b) = (-1)^{(b^2-1)/8},$$
$$(-2/b) = (-1)^{(b^2+4b-5)/8},$$
$$(-2/-b) = (-1)^{(b^2-4b-5)/8}.$$

If $a \neq 0$, then

$$(-a^2/-1) = -1,$$
$$(-1/-b^2) = -1;$$
$$(-a/1) = 1,$$
$$(a/-1) = (a/-1) \text{ (see below)},$$
$$(-a/-1) = -(a/-1);$$

$$(1/b) = 1,$$
$$(-1/b) = (-1)^{(b-1)/2},$$
$$(1/-b) = 1,$$
$$(-1/-b) = (-1)^{(b+1)/2}.$$

[Continued on P. 324.]

THE FIBONACCI RATIOS F_{k+1}/F_k MODULO p

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It is well known that the ratios F_{k+1}/F_k converge to ϕ, the golden ratio. These fractions are alternately greater than and less than ϕ. However, interesting relationships also arise if we consider these ratios reduced modulo p, where p is an odd prime.

Before proceeding further, we will need a few definitions. Let R_k be the Fibonacci ratio F_{k+1}/F_k. Let $z(p)$ be the restricted period of the Fibonacci series reduced modulo p—that is, $F_{z(p)}$ is the first term $\equiv 0 \pmod p$ in the series. Let $\omega(p)$ be the period of the Fibonacci sequence modulo p and let

$$\beta(p) = \omega(p)/z(p).$$

If $z(p) \equiv 2 \pmod 4$, $\beta(p) = 1$; if $z(p) \equiv 4 \pmod 8$, $\beta(p) = 2$; and if $z(p) \equiv 1 \pmod 2$, $\beta(p) = 4$. See [1]. Further, let us agree to ignore all ratios modulo p which have 0 as a denominator.

Then, the Fibonacci ratios reduced modulo p repeat in periods of length $z(p) - 1$. This follows since the terms $F_{kz(p)+1}$ to $F_{(k+1)z(p)}$ are constant multiples of the first $z(p)$ terms. Furthermore, no two ratios within a period repeat, since this would imply that a term of the Fibonacci series preceding $F_{z(p)}$ was congruent to 0 modulo p.

Thus, if $z(p) = p + 1$, all the residues will be represented in the period of Fibonacci ratios reduced modulo p. The fact that none of these ratios repeat is an easy way of showing that $z(p) \leqslant p + 1$. A necessary but not sufficient condition for $z(p)$ to equal $p + 1$ is that $(5/p) = -1$ and $\beta(p) = 2$, which is equivalent to saying that $p \equiv 3$ or $7 \pmod{20}$. See [1]. For primes such as 3, 7, 23, and 43, $z(p)$ does in fact equal $p + 1$.

The Fibonacci ratios reduced modulo 7 are shown below:

k	F_k	R_{k-1}
1	1	
2	1	$1/1 \equiv 1 \pmod 7$
3	2	$2/1 \equiv 2 \pmod 7$
4	3	$3/2 \equiv 5 \pmod 7$
5	5	$5/3 \equiv 4 \pmod 7$
6	1	$1/5 \equiv 3 \pmod 7$
7	6	$6/1 \equiv 6 \pmod 7$
8	0	$0/6 \equiv 0 \pmod 7$

Theorem 1. $R_{z(p)-n} \equiv 1 - R_n \pmod p$ for $1 \leqslant n \leqslant z(p) - 1$.

Proof. This is true for $n = 1$, since $R_1 = 1/1 \equiv 1 \pmod p$ and $R_{z(p)-1} \equiv 0 \pmod p$.

Now assume that the hypothesis is true up to $n = k$. Let $R_k = r$. Then

$$\frac{F_{k+1}}{F_k} \equiv r \qquad \text{and} \qquad \frac{F_{z(p)-k+1}}{F_{z(p)-k}} \equiv 1 - r \pmod p$$

Thus, $F_{k+1} \equiv rF_k \pmod p$.

$$R_{k+1} = \frac{F_{k+2}}{F_{k+1}} = \frac{F_{k+1}+F_k}{F_{k+1}} \equiv \frac{(r+1)F_k}{rF_k} \equiv \frac{r+1}{r} \pmod p.$$

Also,

$$R_{z(p)-k-1} \equiv \frac{F_{z(p)-k}}{F_{z(p)-k-1}} \equiv \frac{F_{z(p)-k}}{F_{z(p)-k+1} - F_{z(p)-k}} \equiv \frac{F_{z(p)-k}}{(1-r)F_{z(p)-k} - F_{z(p)-k}}$$

$$\equiv \frac{F_{z(p)-k}}{-rF_{z(p)-k}} \equiv \frac{-1}{r} \pmod p.$$

But

$$\frac{r+1}{r} \equiv 1 - \left(\frac{-1}{r}\right) \pmod p$$

and we are done.

Theorem 2. $R_n \cdot R_{z(p)-n-1} \equiv -1 \pmod p$ for $1 \leqslant n \leqslant z(p) - 2$.

Proof. $R_1 \equiv 1$ and $R_2 \equiv 2 \pmod p$. By the previous theorem,

$$R_{z(p)-2} \equiv 1 - 2 \equiv -1 \equiv \frac{-1}{R_1} \pmod p.$$

Thus, the theorem holds for $n = 1$. The rest of the proof by induction is similar to the previous proof.

The remainder of this paper will be devoted to investigating what residues appear and do not appear among the Fibonacci ratios reduced modulo p. We will not consider such trivial residues as 2/1 or 3/2. By Theorem 1, if $z(p)$ is even then the ratio $R_{\frac{1}{2}z(p)}$ will be $\equiv \frac{1}{2} \pmod p$. If $z(p)$ is odd, then Theorem 1 implies that ½ will not appear among the Fibonacci ratios modulo p. Thus, if $\beta(p) = 1$ or 2, ½ appears among the Fibonacci ratios and if $\beta(p) = 4$, ½ will not be among the Fibonacci ratios modulo p.

By Theorem 2, if $z(p)$ is odd $R_{\frac{1}{2}(z(p)-1)}$ will be congruent to one of the square roots of $-1 \pmod p$. If $z(p)$ is even, no square roots of -1 will show up among the Fibonacci ratios reduced modulo p.

Combining theorems 1 and 2, we see that no solution of the congruence

$$1 - k \equiv \frac{-1}{k} \pmod p$$

will appear among the Fibonacci ratios modulo p. Solving for k, we see that

$$k \equiv \frac{1 \pm \sqrt{5}}{2} \pmod p$$

if $(5/p) = 0$ or 1. It turns out that for certain primes such as 11, 19, and 31, $z(p) = p - 1$, and every residue but

$$\frac{1 \pm \sqrt{5}}{2}$$

appears among the Fibonacci ratios modulo p. A necessary but not sufficient condition for this to occur is that $p \equiv$ 11 or 19 (mod 20).

We are now ready to summarize our results:

For all primes if the residue r appears among the Fibonacci ratios modulo p, then $1 - r$ and $-1/r$ will also appear.

$p = 5$: All residues except $\frac{1}{2} \equiv 3 \pmod 5$ will appear.

$p \equiv 3$ or $7 \pmod{20}$: All residues might appear since $z(p)$ might equal $p + 1$. In any case, the residue ½ will appear.

$p \equiv 11$ or $19 \pmod{20}$: The residue ½ appears. The residues

$$\frac{1 \pm \sqrt{5}}{2} \pmod p$$

do not appear. All other residues could appear since $z(p)$ might equal $p - 1$.

$p \equiv 13$ or $17 \pmod{20}$: The residue ½ does not appear. Exactly one square root of -1 appears.

$p \equiv 1$ or $9 \pmod{20}$ and $\beta(p) = 1$ or 2: The residue ½ appears. Both square roots of -1 and the residues

$$\frac{1 \pm \sqrt{5}}{2} \pmod{p}$$

do not appear.

$p \equiv 1$ or $9 \pmod{20}$ and $\beta(p) = 4$: The residues ½ and

$$\frac{1 \pm \sqrt{5}}{2} \pmod{p}$$

do not appear. Exactly one square root of $-1 \pmod{p}$ appears.

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[Continued from P. 321.]

If $(a,b) = 1$, then

$$(a^2/b^2) = 1,$$
$$(-a^2/b^2) = 1,$$
$$(a^2/-b^2) = 1,$$
$$(-a^2/-b^2) = -1;$$

$$(a/b^2) = 1,$$
$$(-a/b^2) = 1,$$
$$(a/-b^2) = (a/-1),$$
$$(-a/-b^2) = -(a/-1);$$

$$(a^2/b) = 1,$$
$$(-a^2/b) = (-1/b),$$
$$(a^2/-b) = 1,$$
$$(-a^2/-b) = -(-1/b);$$

$$(a/b) = (a/b),$$
$$(-a/b) = (a/b)(-1/b),$$
$$(a/-b) = (a/b)(a/-1),$$
$$(-a/-b) = -(a/b)(a/-1)(-1/b).$$

It remains to evaluate $(a/-1)$. Since $(-a^2/-1) = -1$, therefore $(a/-1) = -(-a/-1)$. This means that $(a/-1)$ cannot be defined in terms of an integer. Either $(a/-1) = 1$ if and only if a is positive or $(a/-1) = 1$ if and only if a is negative. The choice of alternative is dictated by the fact that $(1/-1) = 1$ and $(-1/-1) = -1$. Therefore, $(a/-1) = 1$ if and only if a is positive.

(See Tables 1 through 4.)

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ON ALTERNATING SUBSETS OF INTEGERS

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A finite set I of natural numbers is to be *alternating* [1] provided that there is an odd member of I between any two even members and an even member of I between any two odd members; equivalently, arranging the elements of I in increasing order yields a sequence in which consecutive elements have opposite parity. In this note we compute the number $a_{n,r}$ of alternating subsets of $\{1, 2, \cdots, n\}$ with exactly r elements, $0 \leqslant r \leqslant n$.

As a matter of notation we denote an alternating r-subset of $\{1, 2, \cdots, n\}$ by $(q_1, q_2, \cdots, q_r; n)$, where we assume $q_1 < q_2 < \cdots < q_r$.

Let $E_{n,r}$ (resp. $O_{n,r}$) be the number of alternating subsets of $\{1, 2, \cdots, n\}$ with r elements and with least element even (resp. odd). It follows that

(1) $$a_{n,r} = E_{n,r} + O_{n,r} \qquad (1 \leqslant r \leqslant n).$$

For reasons which will soon become evident we set $E_{n,0} = O_{n,0} = 1$; hence, $a_{n,0} = 2$ for $n > 0$. In addition, set $a_{0,0} = 1$.

Lemma. For any positive integer m,

$$E_{m+1,r} = O_{m,r}; \qquad 0 \leqslant r \leqslant m+1.$$

Proof. The case $r = 0$ is trivial. If $r = m + 1$, then

$$E_{m+1,m+1} = 0 = O_{m,m+1}.$$

For $1 \leqslant r \leqslant m$ consider the correspondence

$$(q_1, q_2, \cdots, q_r; m+1) \leftrightarrow (q_1 - 1, q_2 - 1, \cdots, q_r - 1; m).$$

If q_1 is even then it easily follows that the number of r-subsets of $\{1, 2, \cdots, m+1\}$ with least element even equals the number of r-subsets of $\{1, 2, \cdots, m\}$ with least element odd, q.e.d.

Proposition 1. For any positive integer m, and $1 \leqslant r \leqslant m+1$,

(2) $$a_{m+1,r} = a_{m,r-1} + a_{m-1,r}.$$

Proof. The case $m = 1$ is obvious, so assume $m \geqslant 2$. If $r = 1$ then

$$a_{m+1,1} = m+1 \qquad \text{while} \qquad a_{m,0} = 2, \qquad a_{m-1} = m-1;$$

hence (2) holds. For $r > 1$ we divide the r-subsets of $\{1, 2, \cdots, m+1\}$ (denoted as usual by $(q_1, q_2, \cdots, q_r; m+1)$) into two groups:

(i) $q_1 = 1$. Then $(q_2, \cdots, q_r; m+1)$ is an $(r-1)$-subset of $\{1, 2, \cdots, m+1\}$ which has an even least element, so there are $E_{m+1,r-1}$ such subsets.

(ii) $q_1 \geqslant 2$. Then the correspondence given in the previous lemma shows that the number of such r-subsets is $a_{m,r}$.

We thus conclude that

(3) $$a_{m+1,r} = E_{m+1,r-1} + a_{m,r}$$

whence it follows that

(4) $$a_{m+1,r} = E_{m+1,r-1} + E_{m,r-1} + a_{m-1,r}.$$

Applying the Lemma, Eq. (4) becomes

(5) $$a_{m+1,r} = O_{m,r-1} + E_{m,r-1} + a_{m-1,r}.$$

Substituting (1) in (5) yields (2), q.e.d.

We remark that (2) holds for $m = 0$ if we define $a_{n,r} = 0$ if $n < 0$ or $r < 0$.

The recurrence (2) can be solved using the standard technique of generating functions [2,3]. We first define

(6) $$A_n(x) = \sum_{k=0}^{\infty} a_{n,r}x^r .$$

Notice that $A_n(x)$ is a polynomial of degree n since $a_{n,r} = 0$ for $r > n$. Using (2) we deduce that for $n \geq 3$,

(7) $$A_n(x) = xA_{n-1}(x) + A_{n-2}(x),$$

while (6) and the boundary conditions on $a_{n,r}$ give

$$A_0(x) = a_{0,0} = 1$$

$$A_1(x) = a_{1,0} + a_{1,1}x = 2 + x$$

$$A_2(x) = a_{2,0} + a_{2,1}x + a_{2,2}x^2 = 2 + 2x + x^2 .$$

Set

$$A(y,x) = \sum_{n=0}^{\infty} A_n(x)y^n .$$

Then the above initial values together with (7) yield

(8) $$A(y,x) = \frac{(1+y)^2}{1 - xy - y^2} .$$

We now derive an explicit representation of $A_n(x)$. To begin, expand $1/(1 - xy - y^2)$ in a formal power series:

(9) $$\frac{1}{1 - xy - y^2} = \sum_{t=0}^{\infty} y^t(x+y)^t = \sum_{t=0}^{\infty} y^t \sum_{r=0}^{t} \binom{t}{r} x^{t-r}y^r = \sum_{t=0}^{\infty} \sum_{r=0}^{t} \binom{t}{r} x^{t-r}y^{t+r} .$$

Fix any integer $n \geq 0$. Then the coefficient of y^n in (9) is easily seen to be

(10) $$B_n(x) = \binom{n}{0} x^n + \binom{n-1}{1} x^{n-2} + \cdots + \binom{n-[n/2]}{[n/2]} x^{n-2[n/2]} .$$

It follows that $A_n(x)$, the coefficient of y^n in $A(y,x)$, is given by

(11) $$A_n(x) = \sum_{s=0}^{[n/2]} \binom{n-s}{s} x^{n-2s} + 2 \sum_{s=0}^{[(n-1)/2]} \binom{n-1-s}{s} x^{n-1-2s} + \sum_{s=0}^{[n/2]-1} \binom{n-2-s}{s} x^{n-2-2s}$$

$$= B_n(x) + 2B_{n-1}(x) + B_{n-2}(x) .$$

We now determine $a_{n,r}$, which, we recall is the coefficient of x^r in $A_n(x)$. We have two cases.

CASE 1. Assume $r \equiv n \pmod 2$. Then we can find $s \geq 0$ so that $n - r = 2s$, i.e., $s = \tfrac{1}{2}(n - r)$. Notice that $B_{n-1}(x)$ does not contain the term x^r. If $s = 0$, then $r = n$ and

$$a_{n,n} = \binom{n}{0} = 1;$$

otherwise we can rewrite r as $r = (n-2) - 2(s-1)$ and thus both $B_n(x)$ and $B_{n-2}(x)$ contain a term in x^r; hence

$$a_{n,r} = \binom{n - \frac{1}{2}(n-r)}{\frac{1}{2}(n-r)} + \binom{(n-2) - [\frac{1}{2}(n-r) - 1]}{\frac{1}{2}(n-r) - 1} \quad . \tag{12}$$

Simplifying (12) we have that for $r \equiv n \pmod 2$,

$$a_{n,r} = \binom{\frac{1}{2}(n+r)}{\frac{1}{2}(n-r)} + \binom{\frac{1}{2}(n+r) - 1}{\frac{1}{2}(n-r) - 1}. \tag{13}$$

CASE 2. Assume $r \not\equiv n \pmod 2$. Then the term x^r appears only in $B_{n-1}(x)$, so we obtain (in a fashion analogous to the one above) that

$$a_{n,r} = 2\binom{n - 1 - \frac{1}{2}(n-r-1)}{\frac{1}{2}(n-r-1)} \quad .$$

That is, for $r \not\equiv n \pmod 2$,

$$a_{n,r} = 2\binom{\frac{1}{2}(n+r-1)}{\frac{1}{2}(n-r-1)} . \tag{14}$$

We summarize these results in the following:

Proposition 2. Let $a_{n,r}$ be the number of alternating r-subsets of $\{1, 2, \cdots, n\}$.

(i) If $r \equiv n \pmod 2$,

$$a_{n,r} = \binom{\frac{1}{2}(n+r)}{\frac{1}{2}(n-r)} + \binom{\frac{1}{2}(n+r) - 1}{\frac{1}{2}(n-r) - 1} .$$

(ii) If $r \not\equiv n \pmod 2$,

$$a_{n,r} = 2\binom{\frac{1}{2}(n+r-1)}{\frac{1}{2}(n-r-1)} .$$

As a result of this development we obtain an interesting relation between the numbers $a_{n,r}$ and the Fibonacci numbers [3]:

Corollary. Let f_n be the Fibonacci sequence, i.e., $f_0 = f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Then we have

$$f_{n+2} = \sum_{r=0}^{n} a_{n,r}. \tag{15}$$

Proof. Recall (see [3], p. 89) that the ordinary generating function for the sequence f_n is

$$F(y) = \sum_{n=0}^{\infty} f_n y^n = \frac{1}{1 - y - y^2} \quad . \tag{16}$$

It follows from (8) that

$$A(y,1) = (1+y)^2 F(y) = \sum_{n=0}^{\infty} (f_n + 2f_{n-1} + f_{n-2})y^n ,$$

where $f_{-1} = f_{-2} = 0$. But from (7),

$$A(y,1) = \sum_{n=0}^{\infty} A_n(1)y^n ,$$

and

$$A_n(1) = \sum_{r=0}^{n} a_{n,r},$$

whence we conclude that

$$\sum_{r=0}^{n} a_{n,r} = f_n + 2f_{n-1} + f_{n-2}\ . \tag{17}$$

Using the recurrence

$$f_{n+1} = f_n + f_{n-1}\,,$$

the right-hand side of (17) simplifies to f_{n+2}, which is the desired result, q.e.d.

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[Continued from P. 324.]

TABLE 1
Jacobi Symbols: $b = 1$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
−7	1	1	−1	1
−5	1	1	−1	−1
−3	1	1	−1	1
−1	1	1	−1	−1
1	1	1	1	1
3	1	1	1	−1
5	1	1	1	1
7	1	1	1	−1

TABLE 2
Jacobi Symbols: $b = 3$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
−7	−1	−1	1	−1
−5	1	−1	−1	1
−3	0	0	0	0
−1	−1	1	1	−1
1	1	1	1	1
3	0	0	0	0
5	−1	−1	−1	−1
7	1	−1	1	1

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STRUCTURE OF THE REDUCED RESIDUE SYSTEM WITH COMPOSITE MODULUS

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In [1] a group-theoretical technique was employed to prove the following:

Theorem 1. Let

$$m = 2^e p_1^{e_1} \cdots p_k^{e_k}.$$

The congruence $x^2 \equiv 1 \pmod m$ has 2^k solutions if $e = 0, 1$, 2^{k+1} solutions if $e \geqslant 2$.

We extend this method to study the structure of the reduced residue system. Since the reduced residue system mod m is isomorphic to the automorphism group of cyclic group of order n, we need several lemmas on automorphism groups. Because of the existence of primitive root mod p^n, we have

Lemma 1. The automorphism group $A(C_{p^n})$ of the cyclic group of order p^n is cyclic, and its order is

$$\phi(p^n) = p^n - p^{n-1}.$$

Lemma 2. $A(C_{2^n})$ is cyclic if $n = 1, 2$. If $n > 2$,

$$A(C_{2^n}) = C_{2^{n-2}} \times C_2.$$

Proof. The first statement is obvious. For $n > 2$, the automorphism σ of C_{2^n} defined by $\sigma(a) = a^5$ has order 2^{n-2}; in fact if $n = 3$,

$$\sigma(a) = a^5, \qquad \sigma^2(a) = a,$$

so $|\sigma| = 2$. By induction on n,

$$\sigma^{2^{n-2}}(a) = a^{5^{2^{n-2}}} = a^{(5^{2^{n-3}})^2} = a^{(1+2^{n-1}+k2^n)^2} = a^{1+2^n} = a \text{ on } C_{2^n},$$

i.e., $\sigma^{2^{n-2}}$ = the identity automorphism on C_{2^n} but $\sigma^{2^{n-3}}$ is not, so $|\sigma| = 2^{n-2}$.

Next we show that every automorphism α on C_{2^n} is a product of a power of σ and an automorphism τ of order 2. Let α be defined by $\alpha(a) = a^t$, where t is odd, we have

$$\alpha(a) = a^{(-1)^{\frac{t-1}{2}} 5^i},$$

i.e., $\alpha(a) = \sigma^i \tau(a)$, where

$$\tau(a) = a^{(-1)^{\frac{t-1}{2}}}.$$

Theorem 2. Let

$$m = 2^e p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n},$$

where $e \geqslant 0$, $e_i \geqslant 1$. The reduced residue system mod m is generated by the powers of $n + k$ elements, with

$$k = \begin{cases} 0 & \text{if } e = 0 \text{ or } 1 \\ 1 & \text{if } e = 2 \\ 2 & \text{if } e > 2. \end{cases}$$

Proof.

$$C_m = C_{2^e} \times C_{p_1 e_1} \times \cdots \times C_{p_n e_n} A(C_m) = A(C_{2^e}) \times A(C_{p_1 e_1}) \times \cdots \times A(C_{p_n e_n})$$

$$A(C_{2^e}) = \begin{cases} (1) & \text{if } e = 0 \text{ or } 1 \\ C_2 & \text{if } e = 2 \\ C_{2^{e-2}} \times C_2 & \text{if } e \geq 3. \end{cases}$$

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1. H. S. Sun, "A Group-Theoretical Proof of a Theorem in Elementary Number Theory," *The Fibonacci Quarterly*, Vol. 11, No. 2 (April 1973), pp. 161–162.

[Continued from P. 328.]

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TABLE 3
Jacobi Symbols: $b = 5$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
−7	−1	−1	1	−1
−5	0	0	0	0
−3	−1	−1	1	−1
−1	1	1	−1	−1
1	1	1	1	1
3	−1	−1	−1	1
5	0	0	0	0
7	−1	−1	−1	1

TABLE 4
Jacobi Symbols: $b = 7$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
−7	0	0	0	0
−5	1	−1	−1	1
−3	1	1	−1	1
−1	−1	1	1	−1
1	1	1	1	1
3	−1	1	−1	−1
5	−1	−1	−1	−1
7	0	0	0	0

Then

$$\left(\frac{(a/-1)}{(b/-1)} \right) = 1$$

if and only if a is positive and/or b is positive; and

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ON AN INTERESTING PROPERTY OF 112359550561797752809

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In solving Problem 301 by J. A. Hunter in [1] an interesting Fibonacci property arose. The problem was to find the smallest positive integer with the property that when the digit 1 was appended to both ends, the new number was 99 times the old. If x is the original number then the problem can be restated by solutions x, k to

$$\frac{10^{k+2}+1}{89} = x \qquad \text{and} \qquad [\log_{10}x] = k,$$

where $[\cdots]$ is the greatest integer function. The problem can of course be generalized to other bases. In particular in the base g, $g-1$ plays the role of 9 in the base 10, so the original problem becomes

Generalized Problem: Find x, k if

$$g^{k+2} + gx + 1 = (g^2-1)x \;,$$

or equivalently

$$x = \frac{g^{k+2}+1}{g^2-g-1}\,, \qquad \text{and} \qquad k = [\log_g x] \;.$$

It is an easy inequality argument to show for a positive integer $g \geqslant 3$ that

$$g^k < \frac{g^{k+2}+1}{g^2-g-1} < g^{k+1} \;.$$

Thus the condition $[\log_g x] = k$ can be dropped for $g \geqslant 3$ and we will do so for the remainder.

By long division,

$$x = \frac{g^{k+2}+1}{g^2-g-1} = \left(\sum_{i=1}^{k+1} g^{k+1-i}F_i\right) + \frac{gF_{k+2}+F_{k+1}+1}{g^2-g-1} \;,$$

where F_i is the i^{th} Fibonacci number ($F_1 = F_2 = 1$, etc.). So all the solutions for a given g are found by finding the k's for which

$$\frac{gF_{k+2}+F_{k+1}+1}{g^{\ }-g-1}$$

is an integer.

Solving the equation

$$\frac{g^{k+2}+1}{g^{\ }-g-1} = x$$

for x and k is equivalent to solving the congruence $g^t \equiv -1 \pmod{g^2-g-1}$ for $t \geqslant 2$. As a matter of fact, since $10^{22} \equiv -1 \pmod{89}$ we see that for $g = 10$, all solutions x are given by

$$x = \frac{10^{22+44j}+1}{89} \;.$$

The first such x is 112359550561797752809.

In the remainder of this paper we will always use p to denote an odd prime. It is easy to show that $g^t \equiv -1 \pmod{p^a}$ has a solution t if and only if $\mathrm{ord}_p g$ is even, where $\mathrm{ord}_p g$ means the order of g in the multiplicative group of integers modulo p. In this case t is an odd number times ½ $\mathrm{ord}_p g$. Then using the Chinese Remainder Theorem, the fact that $\mathrm{ord}_m g = \text{l.c.m.} \left\{ \mathrm{ord}_{p^a} g ; p^a \| m \right\}$, and the fact that $\mathrm{ord}_{p^a} g$ is a power of p times $\mathrm{ord}_p g$, it is an elementary argument to show for m odd and $(g,m) = 1$ that $g^t \equiv -1 \pmod m$ has a solution t if and only if there is an $x \geqslant 1$ such that $2^x \| \mathrm{ord}_p g$ for each $p | m$, in which case t is an odd number times ½ $\mathrm{ord}_m g$. Compiling this result with our earlier discussion and the fact that $g^2 - g - 1$ is always odd leads to the following theorem.

Theorem 1. Let $g \geqslant 3$ be an integer. Then the following statements are equivalent.

(a) The Generalized Problem has a solution x, k.

(b) There is an integer k such that

$$\frac{gF_{k+2} + F_{k+1} + 1}{g^2 - g - 1}$$

is an integer.

(c) There is an integer $x \geqslant 1$ such that $2^x \| \mathrm{ord}_p g$ for every prime $p | g^2 - g - 1$.

If these statements hold, then $k + 2$ is any odd number times ½ $\mathrm{ord}_{g^2-g-1} g$.

The question naturally arises as to how many bases g are there for which the Generalized Problem has a solution. Towards this end let A denote the set of those $g \geqslant 3$ for which the Generalized Problem has a solution and let

$$B = \left\{ g \geqslant 3 : g \notin A \right\}.$$

Let p be a prime of the form 3 (mod 4) which divides $h^2 - h - 1$ some h. Then p also divides $(-h+1)^2 - (-h+1) - 1$. Furthermore

$$\left(\frac{h}{p}\right)\left(\frac{-h+1}{p}\right) = \left(\frac{-1}{p}\right) = -1,$$

where $(\cdots/p)$ is the Legendre symbol. So either

$$\left(\frac{h}{p}\right) = 1 \qquad \text{or} \qquad \left(\frac{-h+1}{p}\right) = 1.$$

Let a_p stand for h or $-h + 1$ according as to which Legendre symbol is 1. Then if $g \equiv a_p \pmod p$ we have that $p | g^2 - g - 1$ and that $\mathrm{ord}_p g$ is odd (since

$$g^{(p-1)/2} \equiv a_p^{(p-1)/2} \equiv \left(\frac{a_p}{p}\right) \equiv 1 \pmod p \quad \text{and} \quad \frac{p-1}{2}$$

is odd). On the other hand if p is any prime of the form 1 or 4 (mod 5) then $p | h^2 - h - 1$ for

$$h = \frac{1}{2}(1+b)(1+p),$$

where $b^2 \equiv 5 \pmod p$. (Note that

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = 1,$$

so b exists.) Therefore if $p \equiv 1$ or 4 (mod 5) and in addition $p \equiv 3 \pmod 4$, i.e., $p \equiv 11$ or 19 (mod 20), then there is an a_p such that for every $g \equiv a_p \pmod p$ we have $\mathrm{ord}_p g$ is odd and $p | g^2 - g - 1$. Let $P = \left\{ p : p \text{ is a prime of the form 11 or 19 (mod 20)} \right\}$ and let $C = \left\{ g \geqslant 3 : g \equiv a_p \pmod p \text{ for some } p \in P \right\}$. Then Theorem 1 implies $C \subset B$. Furthermore, Dirichlet's theorem on primes in arithmetical progressions implies

$$\sum_{p \in P} \frac{1}{p} = \infty .$$

It then follows that the asymptotic density of C, and hence B, is 1. We have thus proved the following theorem.

Theorem 2. The probability of a random choice of a base $g \geq 3$ not yielding a solution to the Generalized Problem is 1.

In light of this theorem it seems that the choice of the base 10 in the problem as originally stated was a wise choice! We leave as an entertaining problem for the reader the question of the identity of the bases g less than 100 for which there is a solution.

We have shown that in some sense A has far fewer elements than B. But is A finite or infinite? If $g \equiv 3 \pmod 4$ is a prime and $p = g^2 - g - 1$ is also a prime, then $p \equiv 1 \pmod 4$ and

$$\left(\frac{g}{p}\right) = \left(\frac{p}{g}\right) = \left(\frac{-1}{g}\right) = -1.$$

Hence $g^t \equiv -1 \pmod p$ has a solution and $g \in A$. We note that Schinzel's Conjecture H [2] implies there are infinitely many primes $g \equiv 3 \pmod 4$ for which $g^2 - g - 1$ is also prime. Hence if this famous conjecture is true it follows that our set A is infinite.

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[Continued from P. 330.]

$$\left(\frac{(-1/a)}{(-1/b)}\right) = (-1)^{(a-1)(b-1)/4} = 1$$

if and only if $a \equiv 1 \pmod 4$ and/or $b \equiv 1 \pmod 4$.

If $A = \pm 1$ and $B = \pm 1$ are logical variables, then the sixteen functions of those variables are given by $\pm 1, \pm A, \pm B, \pm AB$ and $\pm(\pm A/\pm B)$. This is a result that cannot be obtained with the definition $(-1/-1) = 1$. If $A = (-1/b)$ and $B = (-2/b)$, then the logical functions of A and B give the congruence of b modulo 8. For example,

$$(A/B) = (-1)^{(b^3-b^2+7b-7)/16} = 1$$

if and only if $b \equiv 1$, 3 or 5 (mod 8). The function -1 is a null function which cannot occur.

If $b = \pm p_1 p_2 \cdots p_k$ with p_i not necessarily distinct, and n is the number of p_i for which $(a/p) = -1$, then

$$(ab) = \left(\frac{(a/-1)}{(b/-1)}\right)(-1)^n .$$

Theorem. If $ab \equiv 1 \pmod 2$ and $(a,b) = 1$, then

$$(a/b)(b/a) = \left(\frac{(a/-1)}{(b/-1)}\right)\left(\frac{(-1/a)}{(-1/b)}\right) .$$

In other words,

$$(a/b)(b/a) = 1$$

if and only if ((a is positive and/or b is positive) and ($a \equiv 1 \pmod 4$ and/or $b \equiv 1 \pmod 4$)) or (a is negative and b is negative and $a \equiv -1 \pmod 4$ and $b \equiv -1 \pmod 4$)).

Proof.

$$((-1/a)/(-1/b)) = -1$$

if and only if

$$(-1/a) = (-1/b) = -1;$$

$$((-1/-a)/(-1/b)) = -1$$

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DISTRIBUTION OF THE FIRST DIGITS OF FIBONACCI NUMBERS

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In a recent paper [1], J. L. Brown, Jr., and R. L. Duncan showed that the sequence $\{\ell n F_n\}$ is uniformly distributed modulo 1 (u.d. mod 1), where ℓn denotes the natural logarithm and F_n is the n^{th} Fibonacci number. In this paper we show that some modifications of these ideas have some interesting consequences concerning the distribution of the first digits of the Fibonacci numbers. This also answers a question raised in Problem H-125.

It has been noticed, and proved in the probabilitic or measure theoretic sense, that the proportion of physical constants whose first significant digit is less than or equal to a given digit a (in base 10), is $\log_{10}(1+a)$. See [2], [4]. We will show that a wide class of sequences, including the Fibonacci numbers, have a natural density satisfying a similar distribution. Hence, roughly speaking, a large percentage of the Fibonacci numbers have a small first digit.

Let b be a given positive integer. All of our numbers will now be written in base b. Let $\{a_n\}$ be a given sequence of positive numbers. For any digit d in base b, let x_d = number of $n \leqslant x$ such that the first digit of a_n is $\leqslant d$. More generally, if

$$a = a_0 b^k + a_1 b^{k-1} + \cdots, \qquad a_0 \neq 0,$$

define

$$a^* = ab^{-k} \quad ;$$

so that $1 \leqslant a^* < b$ and a and a^* have the same digits. Then if λ is any number $1 \leqslant \lambda \leqslant b$, define x_λ = the number of $n \leqslant x$ such that $a_n^* \leqslant \lambda$. Also, let $x_\lambda(k)$ = the number of $n \leqslant x$ such that $b^k \leqslant a_n \leqslant \lambda b^k$. Hence

$$x_\lambda = \sum x_\lambda(k).$$

We will say that a sequence $\{a_n\}$ is logarithmicly distributed (LD) if $x_\lambda \sim x \log \lambda$, where log means $\log_b$. The connection between this type of distribution of first digits and uniform distribution mod 1 is given by:

Theorem 1. $\{a_n\}$ is LD in base b if and only if $\{\log a_n\}$ is u.d. mod 1.

Proof. $1 \leqslant a_n^* \leqslant \lambda$, if and only if $b^k \leqslant a_n \leqslant \lambda b^k$ for some integer k, if and only if $k \leqslant \log a_n \leqslant k + \log \lambda$ for some integer k, if and only if $(\log a_n) \leqslant \log \lambda$, where (m) denotes the fractional part of m. Hence x_λ = number of $n \leqslant x$ such that $(\log a_n) \leqslant \log \lambda$, and so $x_\lambda \sim x \log \lambda$ if and only if $\{\log a_n\}$ is u.d. mod 1.

Corollary 1. $\{a^n\}$ is LD if and only if a is not a rational power of b.

Proof. This follows immediately from the fact that $\{n \log a\}$ is u.d. mod 1 if and only if $\log a$ is irrational [3]. This last result follows from Weyl's theorem that $\{\beta_j\}$ is u.d. mod 1 if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h \beta_j} = 0$$

for all integers $h > 0$ [3].

Using Weyl's theorem and results concerning trigonometric sums, we can show that sequences such as $\{a^{p_n}\}$ and $\{n^n\}$ are LD where p_n denotes the n^{th} prime.

The following results can be proved using Weyl's theorem, but they can also be obtained directly from the definition of x_λ without recourse to any considerations of uniform distribution.

Theorem 2. If $\{a_n\}$ is LD then

(i) $\{ca_n\}$ is LD for all constants $c > 0$,

(ii) $\{a_n^k\}$ is LD for all positive integers k

(iii) $\{1/a_n\}$ is LD

(iv) $\{\beta_n\}$ is LD if $\beta_n \sim a_n$.

Proof. We illustrate the methods used by proving (iii).

Let $S = \{a_n\}$ be LD and let $S' = \{1/a_n\}$. Let x_λ refer to S, x'_λ refer to S', etc. Then

$$b^k \leqslant \frac{1}{a_n} \leqslant \lambda b^k$$

if and only if

$$\frac{1}{\lambda} b^{-k} \leqslant a_n \leqslant b^{-k} \;;$$

hence

$$x'_\lambda(k) = x_b(-k-1) - x_{b/\lambda}(-k-1)$$

which implies

$$x'_\lambda = \sum x'_\lambda(k) = \sum x_b(-k-1) - x_{b/\lambda}(-k-1)$$
$$= x_b - x_{b/\lambda} \sim x - x \log(b/\lambda) \sim x \log \lambda.$$

We are now ready to show:

Theorem 3. $\{F_n\}$ is LD.

Proof.

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} .$$

Since

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0,$$

$$F_n \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} .$$

Now

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

is LD by Corollary 1,

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

is LD by Theorem 2-(i), and so F_n is LD by Theorem 2-(iv).

Theorem 3 is easily extended to other recurrence sequences.

It should also be noted that examples can be constructed which show that

$$\{\alpha_n\} \quad \text{and} \quad \{\beta_n\}$$

LD does not imply that any of

$$\{\alpha_n^{1/k}\}, \qquad \{\alpha_n\beta_n\}, \qquad \text{or} \qquad \{\alpha_n+\beta_n\}$$

are LD. It might be interesting to obtain necessary and/or sufficient conditions for these implications to hold.

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[Continued from P. 333.]

if and only if

$$(-1/a) \neq (-1/b) = -1\,;$$

$$((-1/a)/(-1/-b)) = -1$$

if and only if

$$(-1/a) \neq (-1/b) = 1\,;$$

$$((-1/-a)/(-1/-b)) = -1$$

if and only if

$$(-1/a) = (-1/b) = 1\,.$$

Now stipulate that

$$(a/-1) = (b/-1) = 1\,.$$

Then, by the classic Law of Quadratic Reciprocity,

$$(a/b)(b/a) = ((-1/a)/(-1/b)). \tag{1}$$

But

$$(-a/b) = (a/b)(-1/b)$$

and

$$(b/-a) = (b/a)(b/-1)\,.$$

Since $(b/-1) = 1$, therefore

[Continued on P. 339.]

GENERALIZATIONS OF EULER'S RECURRENCE FORMULA FOR PARTITIONS

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INTRODUCTION

In 1954, H. L. Alder [1] showed that, as a generalization of the Rogers-Ramanujan identities, there exist polynomials $G_{k,n}(x)$ such that

$$(1)\qquad \prod_{\substack{n=1\\ n\not\equiv 0,\pm k \,(\mathrm{mod}\ 2k+1)}}^{\infty} (1-x^n)^{-1} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)}{(1-x)(1-x^2)\cdots(1-x^n)},$$

and

$$(2)\qquad \prod_{\substack{n=1\\ n\not\equiv 0,\pm 1 \,(\mathrm{mod}\ 2k+1)}}^{\infty} (1-x^n)^{-1} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)x^n}{(1-x)(1-x^2)\cdots(1-x^n)},$$

where k is a positive integer and the left-hand side of (1) is the generating function for the number of partitions into parts $\not\equiv 0, \pm k \pmod{2k+1}$, while the left-hand side of (2) is the generating function for the number of partitions into parts $\not\equiv 0, \pm 1 \pmod{2k+1}$. As Alder remarks, when $k = 2$, identities (1) and (2) reduce to the Rogers-Ramanujan identities for which $G_{2,n}(x) = x^{n^2}$.

Alder showed that identities similar to (1) and (2) exist for the generating function for the number of partitions into parts $\not\equiv 0, \pm(k-r) \pmod{2k+1}$ for all r with $0 \leqslant r \leqslant k-1$, so that, for a given modulus $2k+1$, there exist k such identities.

We shall show in this paper that a similar generalization is possible for recursion formulae for the number of unrestricted or restricted partitions of n. The best known of these is the Euler identity for the number of unrestricted partitions of n:

$$(3)\qquad p(n) = \sum_{j} (-1)^{j+1} p\left(n - \frac{3j^2 \pm j}{2}\right),$$

where the sum extends over all positive integers j for which the arguments of the partition function are non-negative. Another recursion formula was obtained by Hickerson [2], who showed that $q(n)$, the number of partitions of n into distinct parts, is given by

$$(4)\qquad q(n) = \sum_{j=-\infty}^{\infty} (-1)^j p(n-(3j^2+j)),$$

where the sum extends over all integers j for which the arguments of the partition function are non-negative.

We shall show here that these and other recursion formulas are special cases of the following

Theorem. If we denote the number of partitions of n into parts $\neq 0, \pm(k-r) \pmod{2k+a}$ by $p'(0, k-r, 2k+a; n)$, then for $0 \leqslant r \leqslant k-1$,

$$p'(0, k-r, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + (2r+a)j}{2}\right), \tag{5}$$

where the sum extends over all integers j for which the arguments of the partition function are non-negative.

Proof. Using Jacobi's triple product identity

$$\prod_{n=0}^{\infty} (1-y^{2n+2})(1+y^{2n+1}z)(1+y^{2n+1}z^{-1}) = \sum_{j=-\infty}^{\infty} y^{j^2} z^j .$$

with

$$y = x^{(2k+a)/2}, \qquad z = -x^{(2r+a)/2},$$

we obtain

$$\prod_{n=0}^{\infty} (1-x^{(2k+a)n+(2k+a)})(1-x^{(2k+a)n+k+r+a})(1-x^{(2k+a)n+k-r}) = \sum_{j=-\infty}^{\infty} (-1)^j x^{\frac{(2k+a)j^2+(2r+a)j}{2}} .$$

Dividing both sides by

$$\prod_{s=1}^{\infty} (1-x^s),$$

the left-hand side becomes the generating function for the number of partitions of n into parts $\neq 0, \pm(k-r) \pmod{2k+a}$. Equating coefficients of x^n in the resulting equation yields the theorem.

Corollary 1. For $r = 0$, we obtain the following recursion formula

$$p'(0, k, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + aj}{2}\right), \tag{6}$$

where it shall be understood here and henceforth

$$\sum_j$$

denotes a sum over all integers for which the arguments of the partition function are non-negative.

Corollary 2. If in (6), we let $k = a = 1$, then $p'(0,1,3;n) = 0$ and

$$\sum_j (-1)^j p\left(n - \frac{3j^2+j}{2}\right) = 0$$

or

$$p(n) = \sum_{j \neq 0} (-1)^{j+1} p\left(n - \frac{3j^2+j}{2}\right),$$

which is the Euler identity (3).

Corollary 3. If in (6), we let $k = 2$, $a = 1$, we obtain a recursion formula for $p'(0,2,5;n)$, which by the first Rogers-Ramanujan identity is equal to the number of partitions of n into parts differing by at least 2, or $q_2(n)$. Therefore we have

$$q_2(n) = \sum_j (-1)^j p\left(n - \frac{5j^2+j}{2}\right). \tag{7}$$

Corollary 4. If in (5), we let $r = k - a$, we obtain

$$p'(0, a, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + (2k-a)j}{2}\right) . \tag{8}$$

Corollary 5. If in (8), we let $k = a = 2$, we obtain a recursion formula for $p'(0,2,6;n)$, which is equal to $q(n)$, the number of partitions of n into odd parts, so that we have

$$q(n) = \sum_j (-1)^j p(n - (3j^2 + j)),$$

which is (4).

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[Continued from P. 336.]

$$\begin{aligned}(-a/b)(b/-a) &= (a/b)(b/a)(-1/b)\\ &= ((-1/a)/(-1/b))(-1/b)\\ &= -1\end{aligned}$$

if and only if

$$(-1/a) \ne (-1/b) = -1.$$

Therefore,

$$(-a/b)(b/-a) = ((-1/-a)/(-1/b)). \tag{2}$$

Also,

$$(a/-b) = (a/b)(a/-1)'$$

and

$$(-b/a) = (b/a)(-1/a).$$

Since $(a/-1) = 1$, therefore

$$\begin{aligned}(a/-b)(-b/a) &= (a/b)(b/a)(-1/a)\\ &= ((-1/a)/(-1/b))(-1/a)\\ &= -1\end{aligned}$$

if and only if

$$(-1/a) \ne (-1/b) = 1.$$

Therefore,

$$(a/-b)(-b/a) = ((-1/a)/(-1/-b)). \tag{3}$$

Finally,

$$(-a/-b) = -(a/b)(a/-1)(-1/b)$$

and

$$(-b/-a) = -(b/a)(b/-1)(-1/a).$$

[Continued on P. 342.]

ON LUCAS NUMBERS WHICH ARE ONE MORE THAN A SQUARE

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Let F_n be the n^{th} term in the Fibonacci sequence, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n,$$

and let L_m be the n^{th} term in the Lucas sequence, defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

In a previous paper [4], the author proved that the only numbers in the Fibonacci sequence of the form $y^2 + 1$ are

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2 \quad \text{and} \quad F_5 = 5.$$

The purpose of the present paper is to prove the corresponding result for Lucas numbers. In particular, we prove the following:

Theorem. The only numbers in the Lucas sequence of the form

$$y^2 + 1, \quad y \in z, \quad y \geqslant 0$$

are $L_0 = 2$ and $L_1 = 1$.

In the course of our investigations, we shall require the following results, some of which were proved by Cohn [1], [2], [3].

(1) $$L_{2n} = L_n^2 + 2(-1)^{n-1}.$$

(2) $$(F_{3n}, L_{3n}) = 2 \quad \text{and} \quad (F_n, L_n) = 1 \text{ if } 3 \nmid n.$$

(3) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

(4) $$\text{If} \quad F_{2n} = x^2, \; n > 0, \quad \text{then} \quad 2n = 0, 2 \text{ or } 12.$$

(5) The only non-negative solutions of the equation $x^2 - 5y^4 = 4$ are

$$[x,y] = [2,0], [3,1] \text{ and } [322,12].$$

(6) L_n is never divisible by 5 for any n.

(7) $$\text{If} \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \quad \text{then} \quad F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

(8) $$F_{2n} = F_n L_n.$$

(9) $$\text{If} \quad L_n = x^2, \; n > 0, \quad \text{then} \quad n = 1 \text{ or } 3.$$

(10) $$\text{If} \quad L_n = 2x^2, \; n > 0, \quad \text{then} \quad n = 0 \text{ or } 6.$$

We now return to the proof of our theorem, and consider two cases,

CASE I. n even: If $L_{2n} = y^2 + 1$, then by (1), either

$$y^2 + 1 = L_n^2 + 2 \quad \text{or} \quad y^2 + 1 = L_n^2 - 2.$$

The first case yields

$$L_n^2 - y^2 = -1, \quad L_n = 0, \quad y = 1,$$

which is impossible. The second case yields

$$L_n^2 - y^2 = 3,$$

and it is easily proved that the only integer solution of this equation is

$$L_n = 2, \quad y = 1.$$

CASE II. n odd: First, we prove the following Lemmas:

Lemma 1. If $F_{2n} = 5x^2$ then $n = 0$.

Proof. By (8), we have $F_n L_n = 5x^2$ and, by (2), either

$$(F_n, L_n) = 1 \qquad \text{or} \qquad (F_n, L_n) = 2.$$

If $(F_n, L_n) = 1$, then, by (6),

$$F_n = 5s^2, \qquad L_n = t^2.$$

But then $n = 1$ or 3 and $F \neq 5s^2$. If $(F_n, L_n) = 2$, then we conclude that

$$F_n = 10s^2, \qquad L_n = 2t^2.$$

By (10), $n = 0$ or 6. But $F_n = 10s^2$ only for $n = 0$.

Lemma 2. The only integer solution of the equation $u^2 - 125v^4 = 4$ is

$$u = \pm 2, \quad v = 0.$$

Proof. If $u^2 - 125v^4 = 4$, then u and $5v^2$ are a set of solutions of

$$p^2 - 5q^2 = 4$$

thus

$$u + 5v^2\sqrt{5} = 2\left(\frac{3+\sqrt{5}}{2}\right)^n = 2\alpha^{2n}, \qquad u - 5v^2\sqrt{5} = 2\beta^{2n}.$$

so $F_{2n} = 5v^2$ and thus $v = 0$.

Now let us use (3) with n odd and $L_n = y^2 + 1$. We get

(11) $$(y^2 + 1)^2 + 4 = 5x^2,$$

and we wish to show that the only integer solution of this equation is $y = 0, x = 1$. Note first that if y is odd the equation is impossible mod 16.

On factorizing (11) over the Gaussian integers, we set

$$(y^2 + 1 + 2i)(y^2 + 1 - 2i) = 5x^2.$$

Since y is even, the two factors on the left-hand side of this equation are relatively prime. Thus we conclude

$$y^2 + 1 + 2i = (1 + 2i)(a + bi)^2.$$

This yields

$$a^2 + ab - b^2 = 1, \qquad a^2 - 4ab - b^2 = y^2 + 1,$$

i.e.,

(12) $$a^2 + ab - b^2 = 1$$

and

$$5ab = -y^2.$$

The first equation of (12) yields $(a,b) = 1$, and it may be written

(13) $$(2a + b)^2 - 5b^2 = 4.$$

Since $(a,b) = 1$ the second equation of (12) yields either

(14) $$b = \pm t^2, \qquad a = \mp 5a^2$$

or

(15) $$b = \pm 5t^2, \qquad a = \mp s^2 .$$

Equations (13) and (14) yield

$$(\mp 10s^2 \pm t^2)^2 - 5t^4 = 4.$$

By (5), the only integer solutions of this equation occur for $t = 0$, 1 or 12. But none of these values of t yield a value for s. Equations (13) and (15) yield

$$(\mp 2s^2 \pm 5t^2)^2 - 125t^4 = 4.$$

By Lemma 2, $t = 0, s = 1, a = \pm 1, b = 0, L_n = 1.$ The proof is complete.

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[Continued from P. 339.]

Since

$$(a/-1) = (b/-1) = 1,$$

therefore

$$\begin{aligned}(-a/-b)(-b/-a) &= (a/b)(b/a)(-1/a)(-1/b)\\ &= ((-1/a)/(-1/b))(-1/a)(-1/b)\\ &= 1\end{aligned}$$

if and only if

$$(-1/a) = (-1/b) = 1.$$

Therefore,

(4) $$(-a/-b)(-b/-a) = -((-1/-a)/(-1/-b)).$$

From (1), (2), (3) and (4), it can be seen that the theorem is true for all sixteen combinations of

$$(a/-1) = \pm 1, \qquad (b/-1) = \pm 1, \qquad (-1/a) = \pm 1 \quad \text{and} \quad (-1/b) = \pm 1.$$

Corollary 1. If $a \equiv 0$ or 1 (mod 2), $b \equiv 1$ (mod 2) and $(a,b) = 1$, and if $a_1 \equiv a_2$ (mod b), then

$$(a_1a_2/b) = \left(\frac{(a_1a_2/-1)}{(b/-1)}\right) .$$

In other words, $(a_1a_2/b) = 1$ if and only if a_1a_2 is positive and/or b is positive.

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SUMS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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The purpose of this note is to announce the following formulae, where H_0 and H_1 are chosen arbitrarily and $H_n = H_{n-1} + H_{n-2}$ for $n > 1$:

$$(*)\qquad \sum_{k=0}^{n} H_k H_{k+2m+1} = \begin{cases} H_{m+n+1}^2 - H_{m+1}^2 + H_0 H_{2m+1}, & \text{if } n \text{ is even} \\ H_{m+n+1}^2 - H_m^2, & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{k=0}^{n} H_k H_{k+2m} = \begin{cases} H_{m+n} H_{m+n+1} - H_m H_{m+1} + H_0 H_{2m}, & \text{if } n \text{ is even} \\ H_{m+n} H_{m+n+1} - H_{m-1} H_m, & \text{if } n \text{ is odd}. \end{cases}$$

These results may be established by first proving the corresponding formulas for Fibonacci numbers and then expanding the expressions on the left side of (*) by using the well-known relation

$$H_n = F_{n-1} H_0 + F_n H_1 .$$

To prove (*) for Fibonacci numbers the method of generating functions is utilized. Using Binet's formulae for Fibonacci and Lucas numbers, one finds that

$$\sum_{n=0}^{\infty} F_{n+m}^2 x^n = \frac{F_m^2 + [F_{m-1}F_m + (-1)^m]x - F_{m-1}^2 x^2}{(1+x)(1-3x+x^2)} \quad \text{and} \quad \sum_{n=0}^{\infty} F_n F_{n+m} x^n = \frac{F_{m+1}x - F_{m-1}x^2}{(1+x)(1-3x+x^2)} .$$

Moreover,

$$\sum_{n=0}^{\infty} \left(\sum_{n=0}^{n} F_k F_{k+m} \right) x^n = \left(\sum_{n=0}^{\infty} x^n \right) \left(\sum_{n=0}^{\infty} F_n F_{n+m} x^n \right) = \frac{F_{m+1}x - F_{m-1}x^2}{(1-x)(1+x)(1-3x+x^2)} ,$$

and with the methods of Gould [1] one can derive the bisection generating functions

$$\sum_{n=0}^{\infty} F_{2n+m}^2 x^n = \frac{F_m^2 + [(-1)^m - 3F_{m-2}F_m]x + F_{m-2}^2 x^2}{(1-x)(1-7x+x^2)} ,$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n} F_k F_{k+m} \right) x^n = \frac{F_{m+3}x - F_{m-1}x^2}{(1-x)(1-7x+x^2)} ,$$

and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n+1} F_k F_{k+m} \right) x^n = \frac{F_{m+1} - F_{m-3}x}{(1-x)(1-7x+x^2)} .$$

The proof of (*) for Fibonacci numbers is then completed by observing the relationships among these generating functions. For example,

$$\sum_{n=0}^{\infty}\left(F^2_{2n+m+2}-F^2_m\right)x^n = \sum_{n=0}^{\infty} F^2_{2n+m+2}x^n - F^2_m\sum_{n=0}^{\infty}x^n$$

$$= \frac{F^2_{m+2}+[(-1)^{m+2}-3F_mF_{m+2}]x+F^2_mx^2}{(1-x)(1-7x+x^2)} - \frac{F^2_m}{1-x}$$

$$= \frac{(F^2_{m+2}-F^2_m)+[(-1)^m-3F_mF_{m+2}+7F^2_m]x}{(1-x)(1-7x+x^2)}$$

$$= \frac{F_{2m+2}-F_{2m-2}x}{(1-x)(1-7x+x^2)}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2n+1} F_kF_{k+2m+1}\right)x^n\ ,$$

and hence,

$$\sum_{k=0}^{2n+1} F_kF_{k+2n+1} = F^2_{2n+m+2}-F^2_m\ .$$

The other three cases are similar.

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[Continued from P. 342.]

Proof. The corollary is known to be true for $(b/-1)=1$. Then the following results can be calculated:

If

$$(a_1a_2/-1) = 1,$$

then

$$(a_1a_2/b) = 1,$$
$$(-a_1a_2/b) = (-1/b),$$
$$(a_1a_2/-b) = 1\,,$$
$$(-a_1a_2/-b) = -(-1/b);$$

If $(a_1a_2/-1) = -1$, then

$$(a_1a_2/b) = 1,$$
$$(-a_1a_2/b) = (-1/b),$$
$$(a_1a_2/-b) = -1\,,$$
$$(-a_1a_2/-b) = (-1/b).$$

[Continued on P. 349.]

A PRIMER ON THE PELL SEQUENCE AND RELATED SEQUENCES

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1. INTRODUCTION

Regular readers of this journal are well acquainted with basic properties and identities relating to the Fibonacci sequence and its associated sequence, the Lucas sequence, but may be unaware that the Pell sequence is one of many other sequences which share a large number of the same basic properties. The reader should supply the analogous Fibonacci identities, verify formulas numerically, and provide proofs for formulae given here. The proofs are very similar to those for the Fibonacci case.

2. THE PELL SEQUENCE

By observation of the sequence $\{1, 2, 5, 12, 29, 70, 169, \cdots, P_n, \cdots\}$ it is easily seen that each term is given by

(1) $$P_n = 2P_{n-1} + P_{n-2}, \qquad P_1 = 1, \qquad P_2 = 2.$$

The sequence can be extended to include

$$P_0 = 0, \qquad P_{-1} = 1, \qquad P_{-2} = -2, \qquad P_{-3} = 5, \quad \cdots \quad P_{-n} = (-1)^{n+1}P_n\,.$$

The associated sequence $\{R_n\}$, where $R_n = P_{n-1} + P_{n+1}$, has

(2) $$R_n = 2R_{n-1} + R_{n-2}, \qquad R_1 = 2, \qquad R_2 = 6,$$

with first few members given by 2, 6, 14, 34, 82, 198, $\cdots$, and can be extended to include

$$R_0 = 2, \qquad R_{-1} = -2, \qquad \cdots, \qquad R_{-n} = (-1)^n R_n.$$

The Pell numbers enjoy a Binet form. If we take the equation

$$y^2 = 2y + 1$$

which has roots

$$\alpha = (2 + \sqrt{8})/2 \qquad \text{and} \qquad \beta = (2 - \sqrt{8})/2,$$

then it can be proved by mathematical induction that

(3) $$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad R_n = \alpha^n + \beta^n, \qquad \alpha^n = \frac{R_n + P_n\sqrt{8}}{2}\,.$$

Using the Binet form, one can prove that P_{nk} is evenly divisible by P_k, $k \neq 0$, so that the Pell sequence also shares many divisibility properties of the Fibonacci numbers.

Geometrically, the Fibonacci numbers are related to the Golden Rectangle, which, of course, has the property that upon removing one square with edge equal to the width of the rectangle, the rectangle remaining is again a Golden Rectangle. The equation related to the Pell numbers arises from the ratio of length to width in a "silver rectangle" of length y and width 1 such that, when two squares with side equal to the width are removed, the remaining rectangle has the same ratio of length to width as did the original rectangle, or such that

$$\frac{y}{1} = \frac{1}{y-2}, \qquad y^2 - 2y - 1 = 0,$$

so that $y = \alpha$, the positive root given above.

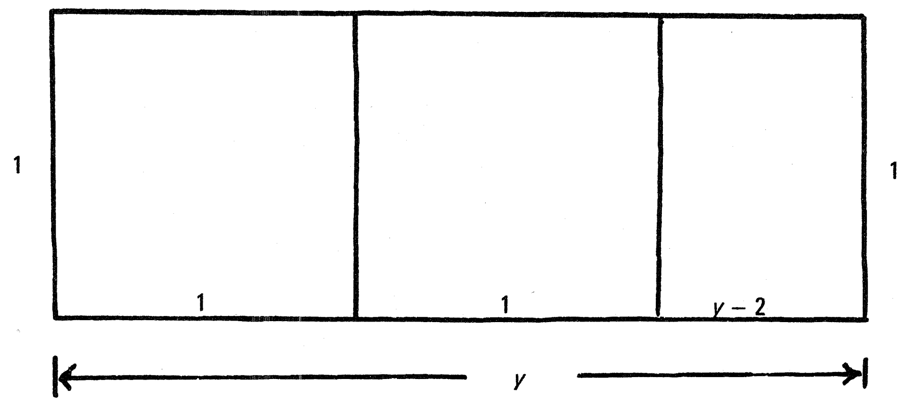

Figure 1

Some simple identities for Pell numbers follow. No attempt was made for completeness; these identities merely indicate some directions that can be explored in finding identities. (Most of these identities can be found in Serkland [1] and Horadam [2].)

(4) $$P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

(5) $$R_{n+1}R_{n-1} - R_n^2 = 8(-1)^{n+1}$$

(6) $$R_n = P_{n+1} + P_{n-1}$$

(7) $$8P_n = R_{n+1} + R_{n-1}$$

(8) $$P_{2n+1} = P_{n+1}^2 + p_n^2$$

(9) $$2P_{2n} = P_{n+1}^2 - P_{n-1}^2$$

(10) $$P_{-n} = (-1)^{n+1}P_n$$

(11) $$R_{-n} = (-1)^n R_n$$

(12) $$P_{2n} = P_n R_n$$

(13) $$R_n + R_{n+1} = 4P_{n+1}$$

(14) $$R_0 + R_1 + R_2 + \cdots + R_n = 2P_{n+1}$$

(15) $$P_{n+p+1} = P_{n+1}P_{p+1} + P_n P_p$$

(16) $$R_n^2 - 8P_n^2 = 4(-1)^n$$

(17) $$P_1^2 + P_2^2 + P_3^2 + \cdots + P_n^2 = (P_n P_{n+1})/2$$

(18) $$\frac{x}{1 - 2x - x^2} = \sum_{i=1}^{\infty} P_n x^n$$

(19) $$\sum_{k=0}^{n} \binom{n}{k} 2^k P_{2k} = P_{2n}$$

(20) $$\sum_{k=0}^{n} \binom{n}{k} P_k P_{n-k} = 2^n P_n$$

The Fibonacci numbers were generated by a matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which satisfied the equation whose roots provide the Binet form for the Fibonacci numbers. The Pell numbers are also generated by a matrix M,

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$$

which can be proved by mathematical induction. The matrix M provides some identities immediately. For example,

$$\det M^n = (\det M)^n = (-1)^n = P_{n+1}P_{n-1} - P_n^2$$

and expanding $M^{n+p} = M^n M^p$ gives

$$P_{n+p+1} = P_{n+1}P_{p+1} + P_n P_p$$

upon equating elements in the upper left. The matrix M also satisfies the equation related to the Pell sequence, $M^2 = 2M + 1$.

3. THE GENERAL SEQUENCE

Since the Fibonacci sequence and the Pell sequence share so many basic properties, and since they have the same starting values but different, though related, recurrence relations, it seems reasonable to ask what properties the sequence $\{U_n\}$,

(21) $$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+1} = bU_n + U_{n-1},$$

which includes both the Fibonacci sequence ($b = 1$) and the Pell sequence ($b = 2$) as special cases, will have.

The first few values of $\{U_n\}$ are:

$$\begin{aligned} U_0 &= 0 \\ U_1 &= 1 \\ U_2 &= b \\ U_3 &= b^2 + 1 \\ U_4 &= b^3 + 2b \\ U_5 &= b^4 + 3b^2 + 1 \\ U_6 &= b^5 + 4b^3 + 3b \\ U_7 &= b^6 + 5b^4 + 6b^2 + 1 \\ U_8 &= b^7 + 6b^5 + 10b^3 + 4b \\ \cdots & \quad \cdots \end{aligned}$$

These are just the Fibonacci polynomials $F_n(x)$ (see [3]) given by

$$F_0(x) = 0, \qquad F_1(x) = 1, \qquad F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$$

evaluated at $x = b$. That is,

$$F_n(1) = F_n, \qquad F_n(2) = P_n, \quad \text{and} \quad F_n(b) = U_n.$$

Thus, any known identities for Fibonacci polynomials establish the same identities for $\{F_n\}$, $\{P_n\}$, and $\{U_n\}$. $\{U_n\}$ has an associated sequence $\{V_n\}$, $V_n = U_{n-1} + U_{n+1}$, where

(22) $$V_0 = 2, \qquad V_1 = b, \qquad V_{n+2} = bV_{n+1} + V_n.$$

Using identities for Fibonacci polynomials given in [2], we have

(23) $$U_m = V_k U_{m-k} + (-1)^{k+1} U_{m-2k}$$

(24) $$U_{-n} = (-1)^{n+1} U_n$$

(25) $$V_{-n} = (-1)^n V_n$$

(26) $$V_n = bU_n + 2U_{n-1}$$

(27) $$bV_n = U_{n+2} - U_{n-2}$$

(28) $$U_{2n} = U_n V_n$$

(29) $$U_{m+n} + (-1)^n U_{m-n} = U_m V_n$$

Also from [2], we can also state that U_{nk} is always divided evenly by U_n, $n \neq 0$.

Now, if we explore the related equation

$$y^2 = by + 1$$

with roots

$$\alpha = \frac{b + \sqrt{b^2+4}}{2} \qquad \text{and} \qquad \beta = \frac{b - \sqrt{b^2+4}}{2},$$

it can be shown by mathematical induction that

(30) $$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n,$$

(31) $$\alpha^n = \frac{V_n + U_n\sqrt{b^2+4}}{2}.$$

Geometrically, U_n and V_n are related to "silver rectangles." (See Raab [4].) If a rectangle of length y and width 1 has dimensions such that, when b squares with side equal to the width are removed, the rectangle remaining has the same ratio of length to width as the original, then the ratio of length to width is $\alpha = (b + \sqrt{b^2+4})/2$, as seen by the following:

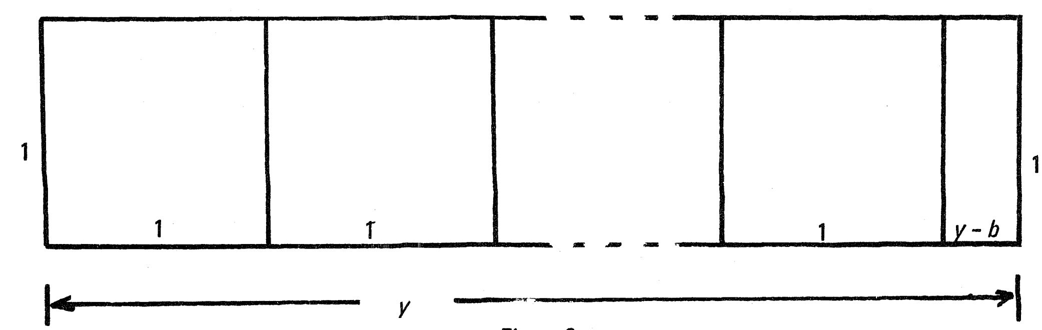

Figure 2

$$\frac{y}{1} = \frac{1}{y-b}, \qquad y^2 - by - 1 = 0, \qquad y = \frac{b + \sqrt{b^2+4}}{2}.$$

Further, it can be proved that

(32) $$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{b + \sqrt{b^2+4}}{2} = \alpha,$$

Serkland [1] and Horadam [2] establish that the generating function for $\{U_n\}$ is

(33) $$\frac{x}{1 - bx - x^2} = \sum_{i=0}^{\infty} U_n x^n .$$

Now, it is well known that the Fibonacci polynomials are generated by a matrix. (See [1], [3], for example.) That the matrix Q below generates $\{U_n\}$ can be established by mathematical induction:

$$Q = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q^n = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} .$$

Since $\det Q^n = (\det Q)^n = (-1)^n$, we have

(34) $$U_{n+1}U_{n-1} - U_n^2 = (-1)^n .$$

Using $Q^{m+n} = Q^m Q^n$ and equating elements in the upper left gives us

(35) $$U_{m+n+1} = U_{m+1}U_{n+1} + U_m U_n$$

(36) $$U_{2n+1} = U_{n+1}^2 + U_n^2 .$$

Many other identities can be found in the same way. Note that the characteristic polynomial of Q is $x^2 - bx - 1 = 0$.

Summation identities can also be generalized [1], [2], as, for example,

(37) $$U_0 + U_1 + U_2 + \cdots + U_n = (U_n + U_{n+1} - 1)/b$$

(38) $$V_0 + V_1 + V_2 + \cdots + V_n = (V_n + V_{n+1} + b - 2)/b$$

(39) $$U_0^2 + U_1^2 + U_2^2 + \cdots + U_n^2 = (U_n U_{n+1})/b .$$

The reader is left to see what other identities he can find which hold for the general sequence.

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1. Carl E. Serkland, *The Pell Sequence and Some Generalizations,* Unpublished Master's Thesis, San Jose State University, San Jose, California, August, 1972.
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4. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," *The Fibonacci Quarterly,* Vol. 1, No. 3 (Oct. 1963), pp. 21–31.

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[Continued from P. 344.]

Corollary 2. If $ab \equiv 1 \pmod 2$ and $(a,b) = 1$, and if $b_1 \equiv b_2 \pmod{2a}$, then

$$(a/b_1 b_2) = \left(\frac{(-1/a)}{(-1/b_1 b_2)} \right) .$$

In other words,

$$(a/b_1 b_2) = 1$$

if and only if $a \equiv 1 \pmod 4$ and/or $b_1 b_2 \equiv 1 \pmod 4$.

Proof. From $(b_1 b_2/a)$, $(-b_1 b_2/a)$, $(b_1 b_2/-a)$ and $(-b_1 b_2/-a)$, the following results can be obtained by quadratic reciprocity:

[Continued on P. 384.]

PALINDROMIC COMPOSITIONS

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In this paper, we discuss palindromic compositions of integers n using members of general sequences of positive integers as summands. A palindromic composition of n is a composition that reads the same forward as backward, as $5 = 1 + 3 + 1$, but not $5 = 3 + 1 + 1$. We derive formulas for the number of palindromic representations of any integer n as well as for the compositions of n. The specialized results lead to generalized Fibonacci sequences, interleaved Fibonacci sequences 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, ⋯, and rising diagonal sums of Pascal's triangle.

1. GENERATING FUNCTIONS

Let

$$\{a_k\}_{k=0}^{\infty}$$

be any increasing sequence of positive integers from which the compositions of a non-negative integer n are made. Then let

$$F(x) = x^{a_0} + x^{a_1} + \cdots + x^{a_k} + \cdots,$$

which will allow us to write generating functions for the number of palindromic compositions P_n as well as the number of compositions C_n made from the sequence

$$\{a_k\}_{k=0}^{\infty}.$$

Theorem 1.1. The number of compositions C_n of a non-negative integer n is given by

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-F(x)}.$$

Proof. Now $C_0 = 1$ and $C_1 = C_2 = \cdots = C_{a_0-1} = 0$ because the numbers $1, 2, 3, \cdots, a_0 - 1$ have no compositions, while the number 0 has a vacuuous composition using no summands from the given sequence. Next,

$$C_n = C_{n-a_0} + C_{n-a_1} + \cdots + C_{n-a_s} + \cdots,$$

where $C_j = 0$ if $j < 0$. Thus,

$$\sum_{n=0}^{\infty} C_n x^n = (x^{a_0} + x^{a_1} + x^{a_2} + \cdots) \sum_{n=0}^{\infty} C_n x^n + 1$$

from which Theorem 1.1 follows immediately.

Theorem 1.2. The number of palindromic compositions P_n of a non-negative integer n is given by

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1+F(x)}{1-F(x^2)}$$

or

$$\sum_{n=1}^{\infty} P_n x^n = \frac{F(x)+F(x^2)}{1-F(x^2)} .$$

Proof. First, we can make a palindromic composition by adding an a_k to each side of an existing palindromic composition. Thus

$$P_n = P_{n-2a_0} + P_{n-2a_1} + \cdots + P_{n-2a_s} + \cdots,$$

where $P_j = 0$ if $j < 0$. Thus

$$\sum_{n=0}^{\infty} P_n x^n = x^{2a_0}(P_0 + P_1 x + P_2 x^2 + \cdots) + x^{2a_1}(P_0 + P_1 x + P_2 x^2 + \cdots)$$
$$+ x^{2a_2}(P_0 + P_1 x + P_2 x^2 + \cdots) + \cdots + (x^{a_0} + x^{a_1} + x^{a_2} + \cdots),$$

where the terms $x^{a_0} + x^{a_1} + x^{a_2} + \cdots$ account for the single palindromic compositions not achievable in the first form. Theorem 1.2 is immediate.

We note that the function

$$F(x) = x^{a_0} + x^{a_1} + \cdots + x^{a_s} + \cdots$$

is such that

$$F^i(x) = \sum_{n=0}^{\infty} R(n)x^n,$$

where $R(n)$ is the i-part composition of n;

$$F^i(x^2) = \sum_{n=0}^{\infty} R^*(n)x^n,$$

where $R^*(n)$ is the $2i$-part palindromic composition of n; and

$$F(x)F^i(x^2) = \sum_{n=0}^{\infty} R^{**}(n)x^n,$$

where $R^{**}(n)$ is the $(2i+1)$-part palindromic composition of n.

Next, we find the number of occurrences of a_k in the compositions and in the palindromic compositions of n.

Theorem 1.3. Let A_n be the number of times a_k is used in the compositions of n. Then

$$\sum_{n=0}^{\infty} A_n x^n = \frac{x^{a_k}}{[1-F(x)]^2} .$$

Proof. It is easy to see that

$$A_n = A_{n-a_0} + A_{n-a_1} + \cdots + A_{n-a_k} + C_{n-a_k} + \cdots,$$

where C_j and $A_j = 0$ if $j < 0$.

$$\sum_{n=0}^{\infty} A_n x^n = (x^{a_0} + x^{a_1} + \cdots + x^{a_s} + \cdots) \sum_{n=0}^{\infty} A_n x^n + x^{a_k} \sum_{n=0}^{\infty} C_n x^n$$

from which Theorem 1.3 follows after applying Theorem 1.1.

It follows from Theorem 1.3 that the total use of all a_k is given by all integer counts in the expansion of

$$\frac{F(x)}{[1-F(x)]^2} .$$

Since the number of plus signs occurring is given by the total number of integers used minus the total number of compositions less the one for zero, the number of plus signs has generating function given by

$$\frac{F(x)}{[1-F(x)]^2} - \frac{F(x)}{1-F(x)} = \frac{F^2(x)}{[1-F(x)]^2} .$$

Theorem 1.4. The number of occurrences of a_k in the palindromic compositions of n, denoted by U_n, is given by the generating function

$$\frac{x^{a_k}}{1-F(x^2)} + \frac{2x^{2a_k}(1+F(x))}{[1-F(x^2)]^2} = \sum_{n=0}^{\infty} U_n x^n .$$

Proof. To count the occurrences of a_k in the palindromic compositions of n,

$$U_n = U_{n-2a_0} + U_{n-2a_1} + \cdots + (U_{n-2a_k} + 2P_{n-2a_k}) + \delta = \begin{cases} 1 \text{ if } n = a_k \\ 0 \text{ if } n \neq a_k \end{cases}$$

the one being for the single palindrome a_k, and U_j and $P_j = 0$ for $j < 0$.

$$\sum_{n=0}^{\infty} U_n x^n = x^{2a_0}(U_0 + U_1 x + U_2 x^2 + \cdots) + x^{2a_1}(U_0 + U_1 x + U_2 x^2 + \cdots)$$
$$+ \cdots + x^{2a_s}(U_0 + U_1 x + U_2 x^2 + \cdots) + \cdots + x^{a_k}$$
$$+ 2x^{2a_k} \sum_{n=0}^{\infty} P_n x^n .$$

Therefore, applying Theorem 1.2 and simplifying yields Theorem 1.4.

As before, from Theorem 1.4 we can write the total number of integers in all palindromic compositions displayed in the form of the generating function

$$\frac{F(x)}{1-F(x^2)} + \frac{2F(x^2)(1+F(x))}{[1-F(x^2)]^2} .$$

Now, in getting all the plus signs counted we need only subtract the generating function for the palindromic compositions of all n except zero. Thus

$$\frac{F(x)}{1-F(x^2)} + \frac{2F(x^2)(1+F(x))}{[1-F(x^2)]^2} - \frac{F(x^2)+F(x)}{1-F(x^2)} = \frac{F(x^2)[1+2F(x)+F(x^2)]}{[1-F(x^2)]^2} .$$

2. APPLICATIONS AND SPECIAL CASES

The results of Section 1 are of particular interest in several special cases.

When the summands are 1 and 2, $F(x) = x + x^2$ gives the result of [1] that the number of compositions of n is F_{n+1}, the $(n+1)^{st}$ Fibonacci number, since by Theorem 1.1,

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-(x+x^2)} = \sum_{n=0}^{\infty} F_{n+1} x^n , \tag{2.1}$$

where we recognize the generating function for the Fibonacci sequence. Theorem 1.2 gives the number of palindromic compositions as

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1+x+x^2}{1-(x^2+x^4)} \tag{2.2}$$

which is the generating function for the interleaved Fibonacci sequence 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, ⋯ .

When the summands are 1, 2, and 3, $F(x) = x + x^2 + x^3$ in Theorem 1.1 gives the generating function for the Tribonacci numbers 1, 1, 2, 4, 7, ⋯, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} T_{n+1}x^n \tag{2.3}$$

while the number of palindromic compositions from Theorem 1.2 becomes

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1+x+x^2+x^3}{1-x^2-x^4-x^6} \tag{2.4}$$

which generates the interleaved generalized Tribonacci sequence 1, 1, 2, 2, 3, 3, 6, 6, 11, 11, 20, 20,⋯.

When the summands are 1, 2, 3, ⋯, k, then $F(x) = x + x^2 + \cdots + x^k$ in Theorem 1.1 gives the generating function for a sequence of generalized Fibonacci numbers $\{F_n^*\}$ defined by

$$F_{n+k}^* = F_{n+k-1}^* + F_{n+k-2}^* + \cdots + F_n^*, \quad F_1^* = 1, \quad F_n^* = 2^{n-1}, \quad n = 2, 3, 4, \cdots, k,$$

so that $C_n = F_{n+1}^*$.

When the summands are the positive integers, $F(x) = x + x^2 + x^3 + \cdots = x/(1-x)$ in Theorem 1.1 gives the number of compositions of n as 2^{n-1}, $n \geq 1$, since

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x}{1-x}} = \frac{1-x}{1-2x} \tag{2.5}$$

which generates 1, 1, 2, 4, 8, 16, 32, ⋯ . Applying Theorem 1.2 to find the number of palindromic compositions gives the generating function for the sequence 1, 1, 2, 2, 4, 4, 8, 8, ⋯, or, $P_n = 2^{[n/2]}$, $n = 0, 1, 2, \cdots$, where $[x]$ is the greatest integer function.

Taking odd summands 1, 3, 5, 7, ⋯, and using $F(x) = x + x^3 + x^5 + x^7 + \cdots = x/(1-x^2)$ in Theorem 1.2 to find the number of palindromic compositions of n again gives the generating function for the interleaved Fibonacci sequence 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, ⋯, while Theorem 1.1 gives the number of compositions of n as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = \sum_{n=0}^{\infty} (F_{n+1} - F_{n-1})x^n \tag{2.6}$$

so that $C_n = F_n$.

If we use the sequence 1, 2, 4, 5, 7, 8, ⋯, the integers omitting all multiples of 3, then

$$F(x) = (x+x^2) + (x^4+x^5) + (x^7+x^8) + \cdots = (x+x^2)/(1-x^3)$$

yields the number of compositions of n as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x+x^2}{1-x^3}} = \frac{1-x^3}{1-x-x^2-x^3} \tag{2.7}$$

so that, returning to Eq. (2.3), $C_n = T_{n+1} - T_{n-2}$, where T_n is the n^{th} Tribonacci number.

If we take

$$F(x) = x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x^2}{1-x},$$

the number of compositions of n using the sequence of integers greater than 1 is given by

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x^2}{1-x}} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n-1}x^n \tag{2.8}$$

so that $C_n = F_{n-1}$. Applying Theorem 1.2 we again find the number of palindromic compositions to be the interleaved Fibonacci sequence, but with the subscripts shifted down from before, as 1, 0, 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, ⋯. (Note: Zero is represented vacuously; one not at all.)

The sequence of multiples of k used for summands leads to

$$F(x) = x^k + x^{2k} + x^{3k} + \cdots = x^k/(1-x^k),$$

which in Theorem 1.1 gives us

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-x^k}{1-2x^k} = 1 + \sum_{m=1}^{\infty} 2^{m-1}x^{km} \tag{2.9}$$

so that the number of compositions of n is 2^{m-1} if $n = km$ or 0 if $n \neq km$ for an integer m.

3. SEQUENCES WHICH CONTAIN REPEATED ONE'S

Compositions formed from sequences which contain repeated one's also lead to certain generalized Fibonacci numbers. We think of labelling the one's in each case so that they can be distinguished. These are weighted compositions.

First, 1, 1, and 2 used as summands gives $F(x) = x + x + x^2$ so that

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} P_{n+1}x^n \tag{3.1}$$

so that $C_n = p_{n+1}$ where p_n is the n^{th} Pell number defined by $p_1 = 1, p_2 = 2, p_{n+2} = 2p_{n+1} + p_n$. Applying Theorem 1.2, we find that we have the generating function for the sequence 1, 2, 3, 4, 7, 10, 17, 24, 41, ⋯, which is a sequence formed from interleaved generalized Pell sequences, having the same recursion relation as the Pell sequence but different starting values.

In general, if we use the sequence 1, 1, 1, ⋯, 1, 2 (k one's) as summands, $F(x) = x + x + x + \cdots + x + x^2 = kx + x^2$ in Theorem 1.1 gives

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-kx-x^2} = \sum_{n=0}^{\infty} p^*_{n+1}x^n\,, \tag{3.2}$$

where

$$p^*_1 = 1,\quad p^*_2 = k,\quad p^*_{n+2} = kp^*_{n+1} + p^*_n.$$

Thus, the number of compositions of n formed from this sequence is $C_n = p^*_{n+1}$. The number of palindromic compositions is again a sequence formed from two interleaved generalized Pell sequences, having the same recursion relation as p^*_n but different starting values. The starting values for one sequence are 1 and $k+1$; for the second, k and k^2. Thus, the interleaved sequence begins

$$1,\ k,\ k+1,\ k^2,\ k^2+k+1,\ k^3+k,\ k^3+k^2+2k+1,\ k^4+2k^2,\ \cdots.$$

One other special case using repeated ones is interesting. When the sequence 1, 1, 1, 1, 2 is used as summands,

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-4x-x^2} = \sum_{n=0}^{\infty} \frac{F_{3(n+1)}}{2}\,x^n \tag{3.3}$$

using the known generating function [2], where L_k is the k^{th} Lucas number,

(3.4)
$$\frac{F_k x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn} x^n .$$

Actually, as a bonus, this gives us two simple results; F_{3k} is always divisible by 2, since C_n is an integer, and, from the recursion relation $C_{n+2} = 4C_{n+1} + C_n$, we have

$$F_{3(n+2)} = 4F_{3(n+1)} + F_{3n} .$$

But, we can go further. Equation (3.4) combined with Theorem 1.1 for odd k gives us

(3.5)
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - L_k x - x^2} = \sum_{n=0}^{\infty} (F_{k(n+1)}/F_k) x^n, \quad k \text{ odd},$$

so that

$$C_n = F_{k(n+1)}/F_k$$

when L_k repeated ones and a 2 are used for the sequence from which the compositions of n are made, k odd. Since C_n is an integer, we prove in yet another way that F_k divides F_{kn} [3], as well as write the formula

(3.6)
$$F_{k(n+2)} = L_k F_{k(n+1)} + F_{kn}, \quad k \text{ odd},$$

4. APPLICATIONS TO RISING DIAGONAL SUMS IN PASCAL'S TRIANGLE

The generalized Fibonacci numbers of Harris and Styles [4], [5] are the numbers $u(n; p,q)$ which are found by taking the sum of elements appearing along diagonals of Pascal's triangle written in left-justified form. The number $u(n; p,q)$ is the sum of the elements found by beginning with the left-most element in the n^{th} row and taking steps of p units up and q units right throughout the array. We recall that

(4.1)
$$\frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n; p,q) x^n .$$

Note that $p = q = 1$ yields the Fibonacci numbers, or, $F_{n+1} = u(n; 1,1)$. Now, Eq. (4.1) combined with Theorem 1.1 gives us the number of compositions of n from the sequence $\{1, p+1\}$ as

(4.2)
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - x - x^{p+1}} = \sum_{n=0}^{\infty} u(n; p,1) x^n$$

so that $C_n = u(n; p,1)$, the sequence of diagonal sums found in Pascal's triangle by taking steps of p units up and 1 unit right throughout the array. Note again that $p = 1$ gives us the Fibonacci sequence.

Suppose that the compositions are made from the sequence of integers greater than or equal to $p + 1$. Then

$$F(x) = x^{p+1} + x^{p+2} + x^{p+3} + \cdots = x^{p+1}/(1-x),$$

so that Theorem 1.1 gives

(4.3)
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - \dfrac{x^{p+1}}{1-x}} = \frac{1-x}{1 - x - x^{p+1}} = \sum_{n=0}^{\infty} [u(n; p,1) - u(n-1; p,1)] x^n$$

and the number of compositions of n becomes

$$C_n = u(n; p,1) - u(n-1; p,1) .$$

Again the special case $p = 1$ yields Fibonacci numbers, with $C_n = F_{n-1}$.

Now, if the compositions are made from the sequence $1, p+2, 2p+3, \cdots$ or the sequence formed by taking every $(p+1)^{st}$ integer,

$$F(x) = x + x^{p+2} + x^{2p+3} + x^{3p+4} + \cdots = x/(1 - x^{p+1})$$

in Theorem 1.1 gives

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - \dfrac{x}{1 - x^{p+1}}} = \frac{1 - x^{p+1}}{1 - x - x^{p+1}} \tag{4.4}$$

so that

$$C_n = u(n;p,1) - u(n-p-1;p,1).$$

Again, $p = 1$ yields Fibonacci numbers, being the case of the sequence of odd integers, where $C_n = F_n$, as in (2.6).

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A NOTE ON TOPOLOGIES ON FINITE SETS

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In an article [1] by D. Stephen, it was shown that an upper bound for the number of elements in a non-discrete topology on a finite set with n elements is $3(2^{n-2})$ and moreover, that this upper bound is attainable. The following example and theorem furnish a much easier proof of these results.

Example. Let b, c be distinct elements of a finite set X with $n (n \geq 2)$ elements. Define

$$\Gamma = \{ A \subset X \mid b \in A \text{ or } c \notin A \}.$$

Now Γ is a topology on X and since there are 2^{n-1} subsets of X containing b and 2^{n-2} subsets of X which do not intersect $\{b, c\}$ we have

$$2^{n-1} + 2^{n-2} = 3(2^{n-2})$$

elements in Γ.

Theorem. If Σ is a non-discrete topology on a finite set X, then Σ is contained in a topology of the type defined in the example.

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THE H-CONVOLUTION TRANSFORM

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1. INTRODUCTION

We form the complete convolution array for a sequence whose generating function is

$$f(x) = \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} a_{i,0} x^i \tag{1.1}$$

with $f(0) = f_0 = a_{00} \neq 0$, and let

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij} x^i, \qquad j = 0, \pm 1, \pm 2, \pm 3, \cdots; \tag{1.2}$$

note that

$$a_{i,-1} = \delta_{i,0} = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}.$$

This convolution array is the source of an infinite number of sequences which are intimately related to the coefficients of $f(x)$

Form a new sequence whose generating function $S_1(x)$ is given by

$$Hf(x) = S_1(x) \tag{1.3}$$

and

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i = \sum_{i=0}^{\infty} a_i x^i . \tag{1.4}$$

We call the sequence $\{s_i\}_{i=0}^{\infty}$ the H-convolution transform of the sequence $\{f_i\}_{i=0}^{\infty}$, but it is easier to express this relationship between the generating functions. That is, $H\{f_i\}_{i=0}^{\infty} = \{s_i\}_{i=0}^{\infty}$ is expressed $Hf(x) = S_1(x)$.

In the next section we shall prove that, if $Hf(x) = S_1(x)$, then $f(xS_1(x)) = S_1(x)$ with $f(0) = S_1(0) \neq 0$. It is well known that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.5}$$

defines the Catalan numbers, whose generating function is $C(x) = [1 - \sqrt{1-4x}]/2x$. Let $f(x) = 1/(1-x)$. The Catalan generating function satisfies $1 + xC^2(x) = C(x)$. This implies that $1/[1 - xC(x)] = C(x)$. That is, if

$$f(x) = 1/(1-x),$$

then

$$f(xC(x)) = 1/[1 - xC(x)] = C(x),$$

so that from Pascal's triangle generator we get the Catalan number generator; $H(1/(1-x)) = C(x)$, $C(0) = 1$.

2. LAGRANGE'S THEOREM

Lagrange's Theorem: (As in Polya and Szegö [1])

Let $f(z)$ and $\varphi(z)$ be regular about $z = 0$ and $f(0) \neq 0$, $\varphi(0) \neq 0$, and $z = \omega\varphi(z)$. Then

$$\frac{f(z)}{1-\omega\varphi'(z)} = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \left.\frac{d^n(f(x)\varphi^n(x))}{dx^n}\right|_{x=0} .$$

Since

$$\varphi(0) \neq 0, \quad \text{and} \quad \omega = z/\varphi(z) = g(z),$$

if $f(z) = 1$, then we are dealing only with reversal of power series [2].

We now use Lagrange's theorem to prove our major result.

Theorem 1. Let $f(x)$ be analytic about $x = 0$, with $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i ,$$

and let

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i ;$$

then $f(xS_1(x)) = S_1(x)$, and $S_1(0) = f(0) \neq 0$.

Proof of Theorem 1. Let

$$xS_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^{i+1} ;$$

then

$$\frac{d}{dx}(xS_1(x)) = \sum_{i=0}^{\infty} a_{ii}x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!} \left.\frac{d^i}{dx^i}(f^{i+1}(x))\right|_{x=0}$$

which can be visualized for Lagrange's theorem as

$$\frac{d}{dx}(xS_1(x)) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \left.\frac{d^i}{dx^i}(f(x)f^i(x))\right|_{x=0},$$

or

$$\frac{d}{dx}(xS_1(x)) = \frac{f(z)}{1-xf'(z)} ,$$

with $\omega = x$ and $\varphi(z) = f(z)$. From $z = xf(z)$, $x \neq 0$,

$$\text{(2.1)} \qquad \frac{df}{dz} = f'(z) = \frac{x - z\dfrac{dx}{dz}}{x^2} ,$$

and so

$$\text{(2.2)} \qquad 1 - xf'(z) = 1 - \frac{x - z\dfrac{dx}{dz}}{x} = \frac{z}{x}\frac{dx}{dz} .$$

Thus,

$$\frac{d}{dx}(xS_1(x)) = \frac{dz}{dx} ,$$

which implies that $xS_1(x) = z + c$; but $xS_1(x) \to 0$ and $z \to 0$ as $x \to 0$. Thus, $c = 0$ and $xS_1(x) = z$. Thus,

$$xS_1(x) = xf(xS_1(x))$$

or

$$S_1(x) = f(xS_1(x)) = f(z).$$

From $z = xf(z)$ to $z = xS_1(x)$ is a reversal of power series, and a necessary and sufficient condition for $S_1(x)$ to be regular about $x = 0$ is that $x = z/[f(z)] = g(z)$ be such that $g'(0) \neq 0$. Clearly, this is guaranteed by $f(0) \neq 0$, since

$$g'(z) = [f(z) - zf'(z)]/f^2(z) \qquad \text{and} \qquad g'(0) = 1/f(0) \neq 0.$$

See Copson [2].

We thus see that if $f(x)$ is regular about $x = 0$ and $f(0) \neq 0$, then $Hf(x) = S_1(x)$ is a function such that $f(0) = S_1(0) \neq 0$ and $f(xS_1(x)) = S_1(x)$, and $S_1(x)$ is regular about $x = 0$.

Corollary. $\varphi(z) = S(x)$, where $\varphi(xS(x)) = S(x)$.

We now proceed to another important

Theorem 2. Let $f(x)$ be regular about $x = 0$ and $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i, \qquad j = 0, 1, 2, \cdots,$$

and

$$G_j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j}\, a_{i,i+j-1}x^i .$$

Then $G_j(x) = S_1^j(x)$ for $j = 1, 2, 3, \cdots$.

Proof of Theorem 2.

$$x^jG_j(x) = \sum_{i=0}^{\infty} \frac{jx^{i+j}}{(i+j)i!} \left.\frac{d^i(f^j(x)f^i(x))}{dx^i}\right|_{x=0},$$

or

$$\frac{d}{dx}\,(x^jG_j(x)) = jx^{j-1} \sum_{i=0}^{\infty} \frac{x^i}{i!} \left.\frac{d^i(f^j(x)f^i(x)}{dx^i}\right|_{x=0}$$

$$= \frac{jx^{j-1}f^j(z)}{1-xf'(z)} = jx^{j-1}f^{j-1}(z)\,\frac{dz}{dx},$$

with $\omega = x$ and $f(z)$ replaced by $f^j(z)$; the last step follows from (2.2), the result in the proof of Theorem 1. Thus,

$$\frac{d}{dx}\,(x^jG_j(x)) = jz^{j-1}\,\frac{dz}{dx},$$

which implies that $x^jG_j(x) = z^j + c$. Since $x^jG_j(x)$ and $z^j \to 0$ as $x \to 0$, then $c = 0$, so that

$$G_j(x) = z^j/x^j = f^j(z) = S_1^j(x),$$

since the same hypotheses of Theorem 1 are used in Theorem 2, and there $f(z) = S_1(x)$. Thus,

$$S_1^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j}\,\frac{x^i}{i!} \left.\frac{d^i(f^j(x)f^i(x))}{dx^i}\right|_{x=0}$$

$$= \sum_{i=0}^{\infty} \frac{j}{i+j}\, a_{i,i+j-1}x^i , \qquad j = 1, 2, 3, \cdots.$$

The next theorem is harder to prove.

Theorem 3. Let $f(x)$ be regular about $x = 0$, $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i, \qquad j = 0, 1, 2, \cdots.$$

Let

$$G_{-j}(x) = \sum_{i=0}^{\infty}{}' \frac{-j}{i-j}\,\frac{x^i}{i!}\,\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\Bigg|_{x=0},$$

where the prime indicates $i \neq j$. Then

$$G_{-j}(x) + \frac{x^j}{j!}\,\frac{d^j}{dx^j}\,(S_1^{-j}(x))\Big|_{x=0} = S_1^{-j}(x).$$

Proof of Theorem 3. Clearly the missing term is indeterminate since

$$\frac{d^j}{dx^j}\,(f^0(x))\Big|_{x=0} = \begin{cases} 0, \text{ if } j \neq 0\,; \\ 1, \text{ if } j = 0\,; \end{cases}$$

in either case, the missing term is 0/0. Now

$$x^{-j}G_{-j}(x) = \sum_{i=0}^{\infty}{}' \frac{-j}{i-j}\,\frac{x^{i-j}}{i!}\,\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\Bigg|_{x=0}$$

so that

$$\frac{d}{dx}\,(x^{-j}G_{-j}(x)) = -jx^{-j-1}\sum_{i=0}^{\infty} \frac{x^i}{i!}\,\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\Bigg|_{x=0}.$$

Thus, by Lagrange's theorem, with $\omega = x$, $\varphi(z) = f(z)$, and $f(z)$ replaced by $(f(z))^{-j}$, and by the result (2.2) in the proof of Theorem 1,

$$\frac{d}{dx}\,(x^{-j}G_{-j}(x)) = -jx^{-j-1}f^{-j-1}(z)\,\frac{dz}{dx} = -jz^{-j-1}\,\frac{dz}{dx},$$

since $z = xf(z)$, so that

$$x^{-j}G_{-j}(x) = z^{-j} + c,$$

and

$$G_{-j}(x) = f^{-j}(z) + cx^j = S_1^{-j}(x) + cx^j.$$

Recall that $G_{-j}(x)$ has a zero coefficient for x^j. Thus, we can get equality if and only if

$$c = -\frac{1}{j!}\,\frac{d^j}{dx^j}\,(S_1^{-j}(x))\Big|_{x=0},$$

which concludes the proof of Theorem 3.

3. APPLICATIONS OF THESE THEOREMS

The three theorems we have proved now give us an explicit set of instructions on how to convert the entire convolution array generated by the powers of $f(x)$ into the entire convolution array for $S_1(x)$.

The central falling diagonal is converted into $S_1(x)$, and the diagonals parallel to this are explicitly converted into $S_1^j(x)$ for all integral j, where $f(0) = S_1(0)$ and $f(xS_1(x)) = S_1(x)$. We have in reality explicitly derived series expansions for all $S_1^j(x)$ in terms of the entries of the convolution array for $f(x)$. This is

$$S_1^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j} a_{i,i+j-1} x^i , \tag{3.1}$$

where

$$a_{i,i+j-1} = \frac{1}{i!} \left. \frac{d^i(f^j(x)f^i(x))}{dx^i} \right|_{x=0} ,$$

for all integral j, with special attention given when $i+j=0$, as earlier discussed. This, of course, can now be repeated any number of times.

A particularly pleasing special case of sequences of convolution arrays arises upon taking $f(x) = 1/(1-x)$, giving rise to the generating functions for the columns of Pascal's triangle. This paper proves and generalizes the results found when considering Catalan and related sequences which arose from inverses of matrices containing certain columns of Pascal's triangle [3], [4], [5], [6].

4. FURTHER GENERALIZATIONS

We can, of course, apply the convolution transform H to $f(x)$ several times. $Hf(x) = S_1(x)$ means $f(xS_1(x)) = S_1(x)$, and $H^2f(x) = S_2(x)$ means that $HS_1(x) = S_2(x)$, where $S_1(xS_2(x)) = S_2(x)$. Further, we can show $f(xS_2^2(x)) = S_2(x)$ as follows:

$$f(xS_1(x)) = S_1(x);$$

replace x by $xS_2(x)$ to obtain

$$f(xS_2(x)S_1(xS_2(x))) = f(xS_2^2(x)) = S_1(xS_2(x)) = S_2(x).$$

In general, one can show that, if

$$S_k(xS_{k+1}(x)) = S_{k+1}(x),$$

then

$$H^k f(x) = S_k(x) \qquad \text{and} \qquad f(xS_k^k(x)) = S_k(x). \tag{4.1}$$

Thus, one can secure an infinite sequence of generating functions from one generating function, $f(x)$.

We can now discuss the inverse convolution transform, H^{-1}. From $f(xS_1(x)) = S_1(x)$, we look at $S_1(x/f(x))$, replace x by $xS_1(x)$, so that

$$S_1(xS_1(x)/f(xS_1(x))) = S_1(x) = f(xS_1(x));$$

thus

$$S_1(x/f(x)) = f(x).$$

$H^{-1}S_1(x) = f(x)$ means $S_1(x/f(x)) = f(x)$. If we designate $f(x) = S_0(x)$, then

$$H^1S_0(x) = S_1(x),$$

and, in general,

$$H^kS_0(x) = S_k(x) \qquad \text{and} \qquad H^{-k}S_0(x) = S_{-k}(x), \tag{4.2}$$

generating a doubly infinite sequence of generating functions from the convolution array for $f(x) = S_0(x)$.

We now derive the explicit formulas for these.

Theorem 4.

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} a_{i,ki+j-1} x^i , \qquad k = 0, 1, 2, \cdots .$$

Proof of Theorem 4.

We consider the elements a_{ij} of the convolution array for $f(x)$ such that $f(0) \neq 0$ and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i \,,$$

j an integer. We proceed first for j positive.

For $S_1(x)$, the elements processed are a_{ii}; for $S_2(x)$, the elements processed are $a_{i,2i}$; and for $S_k(x)$, the elements processed are $a_{i,ki}$. This is, of course, done sequentially. Consider the element $a_{i,ki+j-1}$. We now find the sequential factors to convert it into the coefficient of x^i in $S_k^j(x)$.

First, we consider the diagonals parallel to the principal falling diagonal a_{ii}; the diagonal $S_1^{(k-1)i+j}(x)$ contains $a_{i,ki+j-1}$ and was multiplied by

$$\frac{(k-1)i+j}{ki+j}\,.$$

In the diagonals parallel to the $a_{i,2i}$, the diagonal $S_2^{(k-2)i+j}(x)$ contains $a_{i,ki+j-1}$ and was multiplied by an additional factor of

$$\frac{(k-2)i+j}{(k-1)i+j}\,,$$

and so on. In the diagonals parallel to $a_{i,ki}$, $S_k^j(x)$ picked up a factor of $j/(i+j)$. Thus, for the terms of $S_k^j(x)$

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j}\cdot\frac{i+j}{2i+j}\cdot\ \cdots\ \cdot\frac{(k-1)i+j}{ki+j}\, a_{i,ki+j-1}x^i$$

$$= \sum_{i=0}^{\infty} \frac{j}{ki+j}\, a_{i,ki+j-1}x^i \,.$$

This can also be established by induction. Look at $a_{i,(k+1)i+j-1}$. Each factor we used before has its right subscript of a_{ij} advanced by i so that

$$S_{k+1}^j(x) = \sum_{i=0}^{\infty} \frac{j}{(k+1)i+j}\, a_{i,(k+1)i+j-1}x^i \,.$$

This holds for $j = 1, 2, 3, \cdots$, and concludes the proof of Theorem 4, for j positive. For $j = 0$, $S_k^0(x) = 1$. For $j \leqslant 0$, there are special problems to surmount.

Theorem 5. If $f^{-1}(xS^k(x)) = S(x)$, with $S(0) = f^{-1}(0) \neq 0$, then $S(x) = S_{-k}^{-1}(x)$.

Proof. The function $f^{-1}(x)$ induces a two-sided sequence of generating functions. From $f^{-1}(xS^k(x)) = S(x)$, we imply

$$S(x/(f^{-1}(x))^k) = f^{-1}(x)$$
$$S(xf^k(x)) = f^{-1}(x)$$
$$S^{-1}(xf^k(x)) = f(x)\,.$$

But $S_{-k}(xf^k(x)) = f(x)$, so that $S(x) = S_{-k}^{-1}(x)$.

Theorem 6. For $j > 0$, $k > 0$,

$$S_{-k}^{-j}(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,\frac{x^i}{i!}\,\frac{d^i(f^{-j}(x)f^{-ki}(x))}{dx^i}\Bigg|_{x=0}\,.$$

Proof. Apply Theorem 4 to the function $F(x) = f^{-1}(x)$. Thus for $j > 0$ and $k > 0$

$$S_{-k}^{-j}(x) = \sum_{i=0}^{\infty} \frac{-j}{-ki-j}\,\frac{x^i}{i!}\,\frac{d^i(f^{-1}(x))^j(f^{-k}(x))^i}{dx^i}\Bigg|_{x=0}$$

This is equivalent to the theorem.

SUMMARY:

(4.3)
$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} \frac{x^i}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0}$$

now holds for $j \geq 1$, $k \geq 1$, or $j \leq -1$, $k \leq -1$. The case $j = 0$, $k \neq 0$ is routine and $k = 0$ for any j is routine.

We note that in the proof sequence of Theorem 4, there are no zero factors except when $j = 0$.

Theorem 7 (The Completion of Theorem 4).

If $f(z)$ is regular about $z = 0$ and $f(0) \neq 0$, then, for $k \neq 0$,

(i)
$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} \frac{x^i}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0}$$

when $-j/k \neq m$, a positive integer.

The prime below indicates $i \neq m$,

(ii)
$$S_k^j(x) = \sum_{i=0}^{\infty}{}' \frac{j}{ki+j} \frac{x^i}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0} + \frac{x^m}{m!} \frac{d^m(f^j(x)f^{ki}(x))}{dx^m}\bigg|_{x=0}$$

when $-j/k = m$, a positive integer.

Proof of Theorem 7. Let

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} \frac{x^i}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0}$$

for $j \neq 0$, and $g_0(x) = 1$.

Case (i).

$$x^{j/k}g_j(x) = \sum_{i=0}^{\infty} \frac{j/k}{i+j/k} \frac{x^{i+j/k}}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0} .$$

Taking the derivative,

(4.4)
$$\frac{d}{dx}(x^{j/k}g_j(x)) = \left(\frac{j}{k}x^{j/k-1}\right)\sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\bigg|_{x=0}$$
$$= \frac{j}{k}x^{j/k-1}f^j(z)\frac{x}{z}\frac{dz}{dx} .$$

But
$$z = x\varphi(z) = xf^k(z).$$

From the corollary to Theorem 1, $f^k(z) = S(x)$, where $f^k(xS(x)) = S(x)$. To identify $S(x)$, recall that
$$f(xG^k(x)) = G(x)$$
implies that
$$G(x) = S_k(x)$$

as defined for $f(x)$; hence,

$$S(x) = G^k(x) = S_k^k(x)$$

so that $f(z) = S_k(x)$.

Returning now to (4.4),

$$\frac{d}{dx}\,(x^{j/k} g_j(x)) = \frac{j}{k}\, x^{j/k-1} f^{j-k}(z)\, \frac{dz}{dx}\,. \,.$$

From $z/x = f^k(z)$, then $(z/k)^{1/k} = f(z)$, so that

$$f^{j-k}(z) = (z/x)^{(j-k)/k} \qquad \text{and} \qquad x^{j/k-1} f^{j-k}(z) = z^{j/k-1}\,.$$

Therefore,

$$\frac{d}{dx}\,(x^{j/k} g_j(x)) = \frac{j}{k}\, z^{j/k-1}\, \frac{dz}{dx}\,,$$

so that

$$x^{j/k} g_j(x) = z^{j/k} + C$$

$$g_j(x) = \frac{z^{j/k}}{x^{j/k}} + Cx^{-j/k}\,.$$

Thus,

$$g_j(x) = f^j(z) + Cx^{-j/k} = S_k^j(x) + Cx^{-j/k}\,.$$

From the definition of

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ik+j}\, \frac{x^i}{i!}\, \left.\frac{d^i(f^j(x) f^{ki}(x))}{dx^i}\right|_{x=0}\,,$$

where $-j/k \ne m$, a positive integer, we see that $g_j(x)$ has a Maclaurin power series. Further, $S_k(x)$ is regular about $x = 0$, $S_k(0) \ne 0$, and hence $S_k^j(x)$ is regular about $x = 0$ and $S_k^j(0) \ne 0$; thus $S_k^j(x)$ also has a power series expansion. Their difference is a power series so that if $-j/k \ne m$, a positive integer, then $C = 0$, and the proof of part (i) is complete. Since $S_k^0(x) = 1$, then Theorem 7, part (i), is valid for all integral j and $S_0(x) = f(x)$ does not need such a form.

Case (ii). If $-j/k = m$, a positive integer, then

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\, \frac{x^i}{i!}\, \left.\frac{d^i(f^j(x) f^{ki}(x))}{dx^i}\right|_{x=0}$$

when written as above has an indeterminate term; thus, as in the form in part (ii), it should be primed. Thus, $g_j(x)$ has no term when $ki + j = 0$, so it is necessary and sufficient that in

$$g_j(x) = S_k^j(x) + Cx^{-j/k}\,,$$

$$C = -\frac{1}{m!}\, \frac{d^m}{dx^m}\, \left.(S_k^{-mk}(x))\right|_{x=0}\,.$$

This completes the proof of part (ii).

Theorem 8. When $-j/k \ne m$, m a positive integer,

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\, a_{i,ki+j-1} x^i\,.$$

When $-j/k = m$, m a positive integer,

$$S_k^j(x) = \sum_{i=0}^{\infty}{}' \frac{j}{ki+j} a_{i,ki+j-1}x^i + \frac{x^m}{m!}\frac{d^m}{dx^m}(S_k^{-mk}(x))\Bigg|_{x=0} .$$

Theorem 8 is simply a collection of results in terms of

$$f^{j+1}(x) = \sum_{i=0}^{\infty} a_{ij}x^i .$$

Theorem 9. Let

$$f(xS_k^k(x)) = S_k(x);$$

then

$$\sum_{m=0}^{\infty} \frac{x^m}{m!}\frac{d^m}{dx^m}(S_k^{-k}(x))^m\Bigg|_{x=0} = f^k(x)\frac{d}{dx}(xf^{-k}(x)) .$$

Proof. Let

$$f(z) = 1, \quad z = xS_k^{-k}(z);$$

then

$$\sum_{m=0}^{\infty} \frac{x^m}{m!}\frac{d^m}{dx^m}(S_k^{-k}(x))^m\Bigg|_{x=0} = \frac{x}{z}\frac{dz}{dx} ,$$

where

$$z = xS(x) \quad \text{and} \quad S_k^{-k}(xS(x)) = S(x),$$

but

$$S_k(xf^{-k}(x)) = f(x),$$

so that

$$S_k^{-k}(xf^{-k}(x)) = f^{-k}(x).$$

That is, $S(x) = f^{-k}(x)$. Further,

$$x/z = S_k^{-k}(z) = S^{-1}(x) = f^k(x),$$

so that

$$\sum_{m=0}^{\infty} \frac{x^m}{m!}\frac{d^m}{dx^m}(S_k^{-k}(x))^m\Bigg|_{x=0} = f^k(x)\frac{d}{dx}(xf^{-k}(x)) . \tag{4.5}$$

Since $f^{j+1}(x)$ is implicit in our problem, we can express Eq. (4.5) in a better form.

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i$$

$$f^k(x) = \sum_{i=0}^{\infty} a_{i,k-1}x^i$$

$$f^{-k}(x) = \sum_{i=0}^{\infty} a_{i,-k-1}x^i$$

$$xf^{-k}(x) = \sum_{i=0}^{\infty} a_{i,-k-1}x^{i+1}$$

$$\frac{d}{dx}(xf^{-k}(x)) = \sum_{i=0}^{\infty} (i+1)a_{i,-k-1}x^i = \sum_{i=0}^{\infty} b_{i,-k-1}x^i .$$

Let

$$f^k(x)\frac{d}{dx}(xf^{-k}(x)) = \sum_{m=0}^{\infty} A_m x^m ;$$

then

$$(4.6) \qquad A_m = \sum_{t=0}^{m} a_{t,k-1}b_{m-t,-k-1} = \sum_{t=0}^{m} (m+1-t)a_{t,k-1}a_{m-t,-k-1}, \qquad k \neq 0 .$$

Comment: For each $S_k^j(x)$, there is one term (when $-j/k = m$, m a positive integer) that is not easily specified by the convolution array for $f(x)$. With Theorem 9, we now know how to get that missing term in terms of the convolution array coefficients for $f(x)$ as given in Eq. (4.6).

5. FURTHER GENERALIZED IDENTITIES

The following is a consequence of Theorem 8 for $-j/k \neq m$, a positive integer.

Theorem 10 (A Generalized Identity)

Let

$$G_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} a_{i,ki+j-1}x^i = S_k^j(x),$$

$$G_s(x) = \sum_{i=0}^{\infty} \frac{s}{ki+s} a_{i,ki+s-1}x^i = S_k^s(x);$$

then

$$G_{s+j}(x) = \sum_{i=0}^{\infty} \frac{s+j}{ki+(s+j)} a_{i,ki+s+j-1}x^i = S_k^{s+j}(x).$$

Thus, by convolution it is true that

$$(5.1) \qquad \frac{s+j}{kn+s+j} a_{n,kn+s+j-1} = \sum_{t=0}^{n} \frac{j}{kt+j} a_{t,kt+j-1} \frac{s}{k(n-t)+s} a_{n-t,k(n-t)+s-1} .$$

Corollary 3 (Abel Convolution Formula)

Let $f(x) = e^x$ and $k = 1$ in Theorem 10; then by exponential convolution,

$$\frac{s+j}{n+s+j}(n+s+j)^n = \sum_{t=0}^{n}\binom{n}{t}\frac{j}{t+j}(t+j)^t\,\frac{s}{n-t+s}(n-t+s)^{n-t}\ .$$

Corollary 2 (Generalized Abel Convolution Formula)

Use Theorem 10 with $f(x) = e^x$ and k a positive integer; then

$$\frac{s+j}{kn+j+s}[(n+1)k+s+j-1]^n$$

$$= \sum_{t=0}^{n}\binom{n}{t}\frac{j}{kt+j}[(t+1)k+j-1]^t\,\frac{s}{k(n-t)+s}[(n-t+1)k+s-1]^{n-t}.$$

See Raney [14], who conjectured this form.

Corollary 3 (Hagen-Rothe Identity)

Let $f(x) = (1+x)^a$, $k = 1$, in Theorem 10; then

$$\frac{s+j}{n+s+j}\binom{a(n+s+j)}{n} = \sum_{t=0}^{n}\frac{s}{t+s}\binom{a(t+s)}{t}\frac{j}{n-t+j}\binom{a(n-t+j)}{n-t}\ .$$

Corollary 4 (Generalized Hagen-Rothe Identity)

Let $f(x) = (1+x)^a$ and k be a positive integer in Theorem 10; then

$$\frac{s+j}{kn+s+j}\binom{a[k(n)+s+j}{n}$$

$$= \sum_{t=0}^{n}\frac{j}{kt+j}\binom{a[(k)t+j}{t}\frac{s}{k(n-t)+s}\binom{a[k(n-t)+s}{n-t}$$

6. FINAL REMARKS

I. Schur in [8] has done much in this area. Schur [8] and Carlitz [7] give derivations of Lagrange's theorem. H. W. Gould in [13] has summarized much of what has been done earlier. There is still much that can be done for specialized functions $f(z)$.

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[Continued from Page 356.]

Proof. Since Σ is a non-discrete topology on X there exists $c \in X$ with $\{c\} \notin \Sigma$. Let Δ be the topology on X generated by

$$\Sigma \cup \{\{x\} \mid x \in X \setminus \{c\}\}$$

and notice Δ is non-discrete since $\{c\} \notin \Delta$.

Consider

$$S = \cap \{A \in \Delta \mid c \in A\}.$$

Since Δ is finite if $S = \{c\}$ then $\{c\} \in \Delta$. Thus, choose $b \in S \setminus \{c\}$. Let

$$\Gamma = \{B \subset X \mid b \in B \text{ or } c \notin B\}.$$

Let $T \in \Delta$. If $c \in T$ then $S \subset T$ and so $b \in T$ which implies $T \in \Gamma$. If $c \notin T$ then $T \in \Gamma$ by definition of Γ. Hence

$$\Sigma \subset \Delta \subset \Gamma.$$

Corollary. Every non-discrete topology on a finite set with n elements is contained in a non-discrete topology with $3(2^{n-2})$ elements.

REFERENCE

1. D. Stephen, "Topology in Finite Sets," *American Mathematican Monthly,* 75 (1968), pp. 739–741.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-255 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n}{j+k}\binom{2m+2n}{2m-j+k} = (-1)^{m+n}\frac{(3m+3n)!(2m)!(2n)!}{m!n!(m+n)!(2m+n)!(m+2n)!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

H-256 Proposed by E. Karst, Tucson, Arizona.

Find all solutions of

(i) $$x+y+z = 2^{2n+1}-1$$

and

(ii) $$x^3+y^3+z^3 = 2^{6n+1}-1,$$

simultaneously for $n < 5$, given that

(a) x, y, z are positive rationals
(b) $2^{2n+1}-1,\ 2^{6n+1}-1$ are integers
(c) $n = \log_2\sqrt{t}$, where t is a positive integer.

H-257 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Consider the array, D, indicated below in which F_{2n+1} $(n = 0, 1, 2, \cdots)$ is written in staggered columns

$$D:\quad \begin{array}{cccccc}
1 \\
2 & 1 \\
5 & 2 & 1 \\
13 & 5 & 2 & 1 \\
34 & 13 & 5 & 2 & 1 \\
89 & 34 & 13 & 5 & 2 & 1 \\
\cdots
\end{array}$$

i) Show that the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.
(ii) Show that the rising diagonal sums are $F_{n+1}F_{n+2}$ $(n = 0, 1, 2, \cdots)$.
(iii) Show that if the columns are multiplied by 1, 2, 3, $\cdots$ sequentially to the right, then the row sums are $F_{2n+3} - 1$ $(n = 0, 1, 2, \cdots)$

READER COMMENTS

Paul Bruckman noted that H-241 is identical to H-206.

Charles Wall noted that H-188 is a weaker version of B-141.

H-239 Correction

The given inequality should read

$$\left|\frac{c}{a} - \frac{d}{b}\right| \leqslant \frac{1}{100} \qquad \text{not} \qquad \left|\frac{c}{d} - \frac{d}{b}\right| \leqslant \frac{1}{100} \quad .$$

SOLUTIONS

A NEST OF SUBSETS

H-223 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

Let S be a set of k elements. Find the number of sequences $(A_1, A_2, \cdots, A_n)$ where each A_i is a subset of S, and where $A_1 \subseteq A_2$, $A_2 \supseteq A_3$, $A_3 \subseteq A_4$, $A_4 \supseteq A_5$, etc.

Solution by the Proposers.

Let ϕ_1 be the characteristic function of A_1, ϕ_2 the characteristic function of A_2', ϕ_3 of A_3', ϕ_4 of A_4', etc. The condition on the A_i's is equivalent to

$$(1)' \qquad \phi_i(j) = 1 \Rightarrow \phi_{i+1}(j) = 0, \qquad \forall_{i,j} .$$

For instance, suppose $A_i \subseteq A_{i+1}$. Then $i+1$ is even. If $\phi_i(j) = 1$, then $j \in A_i$, $j \in A_{i+1}$, $j \notin A'_{i+1}$ and $\phi_{i+1}(j) = 0$.

The matrix $(\phi_i(j))$ has k columns each of which is a sequence of 0's and 1's of length n in which no 1's occur consecutively. Since there are F_{n+2} such sequences, there are F^k_{n+2} matrices satisfying (1)'.

SUM LEGENDRE

H-227 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} (aj+ck)^m (bj+dk)^n = m!n! \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r} d^{n-r} (bc)^r .$$

In particular, show that the Legendre polynomial $P_n(x)$ satisfies

$$(n!)^2 P_n(x) = \sum_{j,k=0}^{n} (-1)^{j+k} \binom{n}{j}\binom{n}{k} (aj+ck)^n (bj+dk)^n ,$$

where $ad = \tfrac{1}{2}(x+1)$, $bc = \tfrac{1}{2}(x-1)$.

Solution by the Proposer. We have

$$\sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} (aj+ck)^m (bj+dk)^n$$

$$= \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} a^{m-r} c^r b^{n-s} d^s j^{m+n-r-s} k^{r+s}$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} a^{m-r} c^r b^{n-s} d^s S_{m,n} ,$$

where

$$S_{m,n} = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^{m+n-r-s} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{r+s} .$$

Since

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^t = \begin{cases} m! & (t = m) \\ 0 & (t < m), \end{cases}$$

it follows that $S_{m,n} = 0$ unless

$$\begin{cases} m+n-r-s \geq m \\ r+s \geq n \end{cases},$$

that is, $r+s=s$. Hence

$$\sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} (aj+ck)^m (bj+dk)^n = m!n! \sum_{r+s=n} \binom{m}{r}\binom{n}{s} a^{m-r} c^r b^{n-s} d^s$$

$$= m!n! \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r} d^{n-r} (bc)^r .$$

Since (see for example G. Szegö's *Orthogonal Polynomials,* p. 67)

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k},$$

the second assertion follows at once.

A TRIANGULAR ARRAY

H-229 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array $A(n,k)$ $(0 \leq k \leq n)$ is defined by means of

$$(*) \qquad \begin{cases} A(n+1, 2k) = A(n, 2k-1) + aA(n, 2k) \\ A(n+1, 2k+1) = A(n, 2k) + bA(n, 2k+1) \end{cases}$$

together with

$$A(0,0) = 1, \qquad A(0,k) = 0 \qquad (k \neq 0)$$

Find $A(n,k)$ and show that

$$\sum_k A(n, 2k)(ab)^k = a(a+b)^{n-1},$$

$$\sum_k A(n, 2k+1)(ab)^k = (a+b)^{n-1} .$$

Solution by the Propsoer.

It follows from the definition that

$$A(n,0) = a^n \qquad (n = 0, 1, 2, \cdots).$$

Then

$$A(n,1) = a^{n-1} + bA(n-1, 1)$$

so that

$$A(n,1) = \frac{a^n - b^n}{a-b}.$$

Put

$$A_k(x) = \sum_{n=k}^{\infty} A(n,k)x^n.$$

Then by (*)

$$A_{2k}(x) = \sum_{n=k}^{\infty} (A(n-1,2k-1) + aA(n-1,2k))x^n = xA_{2k-1}(x) + axA_{2k}(x),$$

so that

$$(1-ax)A_{2k} = xA_{2k-1}(x).$$

Similarly

$$(1-bx)A_{2k+1}(x) = xA_{2k}(x).$$

It follows that

$$(**)\qquad \begin{cases} A_{2k+1}(x) = x^{2k+1}(1-ax)^{-k-1}(1-bx)^{-k-1} \\ A_{2k}(x) = x^{2k}(1-ax)^{-k-1}(1-bx)^{-k} \end{cases}.$$

Since

$$(1-ax)^{-k-1} = \sum_{r=0}^{\infty} \binom{k+r}{k} a^r x^r,$$

we get

$$\begin{cases} A(n,2k+1) = \displaystyle\sum_{r=0}^{n-2k-1} \binom{k+r}{k}\binom{n-k-r-1}{k} a^r b^{n-2k-r-1} \\ A(n,2k) = \displaystyle\sum_{r=0}^{n-2k} \binom{k+r}{k}\binom{n-k-r-1}{k-1} a^r b^{n-2k-r} \end{cases}.$$

It follows from (**) that

$$(***)\qquad \begin{cases} \displaystyle\sum_{k=0}^{\infty} A_{2k}(x)y^{2k} = \frac{1-bx}{(1-ax)(1-bx)-x^2y^2} \\ \displaystyle\sum_{k=0}^{\infty} A_{2k+1}(x)y^{2k+1} = \frac{xy}{(1-ax)(1-bx)-x^2y^2} \end{cases}.$$

Hence

$$\sum_{k=0}^{\infty} A_k(x)y^k = \frac{1-bx+xy}{(1-ax)(1-bx)-x^2y^2}.$$

For $a = b$ this reduces to

$$\frac{1}{1-ax-xy}$$

which is correct.

Finally, taking $y^2 = ab$ in (***), we get

$$\sum_k A(n,2k)(ab)^k = a(a+b)^{n-1}, \qquad \sum_k A(n,2k+1)(ab)^k = (a+b)^{n-1}.$$

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems to Professor A.P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-316 Proposed by J.A.H. Hunter, Fun with Figures, Toronto, Ont., Canada.

Solve the alphametic:

```
  TWO
THREE
THREE
-----
EIGHT
```

Believe it or not, there must be no 8 in this!

B-317 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that L_{2n-1} is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, \cdots$.

B-318 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, \cdots$.

B-319 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove or disprove:

$$\frac{1}{L_2} + \frac{1}{L_6} + \frac{1}{L_{10}} + \cdots = \frac{1}{\sqrt{5}}\left(\frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots\right) .$$

B-320 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m} .$$

B-321 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m+1}.$$

SOLUTIONS

A COMBINATORIAL PROBLEM

B-292 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain and prove a formula for the number $S(n,t)$ of terms in $(x_1 + x_2 + \cdots + x_n)^t$, where n and t are integers with $n > 0$, $t \geq 0$.

I. Solution by Graham Lord, Secane, Pennsylvania.

$S(n,t)$ is the number of unordered selections of size t and a set of n elements, that is:

$$S(n,t) = \binom{n+t-1}{t}.$$

This is a well known result. See for example H.H. Ryser, "Combinatorial Mathematics," *Carus Monograph,* American Math Association, 1963.

II. Solution by Frank Higgins, Naperville, Illinois.

$$S(n,t) = \binom{n+t-1}{t}.$$

For $n = 1$, the formula clearly holds for all integers $t \geq 0$. Suppose the formula holds for some integer $n \geq 1$ and all integers $t \geq 0$. Now, for any integer $t \geq 0$, we have that

$$(x_1 + x_2 + \cdots + x_n + x_{n+1})^t = [(x_1 + x_2 + \cdots + x_n) + x_{n+1}]^t = \sum_{k=0}^{t} \binom{t}{k} (x_1 + x_2 + \cdots + x_n)^{t-k} x_{n+1}^k$$

and hence, by the induction hypothesis, that

$$S(n+1,t) = \sum_{k=0}^{t} \binom{n+t-k-1}{t-k} = \binom{n+t}{t}$$

which completes the proof.

Also solved by Paul S. Bruckman, Jeffrey Shallit, A.C. Shannon, Gregory Wulczyn, and the Proposer.

THE FIRST SIX FIBONACCI TERMS

B-293 Proposed by Harold Don Allen, Nova Scotia Teachers College, N.S., Canada.

Identify T, W, H, R, E, F, I, V and G as distinct digits in $\{1, 2, \cdots, 9\}$ such that we have the following sum (in which 1 and 0 are the digits 1 and 0):

```
    1
    1
  TWO
THREE
 FIVE
-----
EIGHT
```

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

It is easy to see that the digit carried from the thousands column must be 1; consequently, $T + 1 = E$. Applying this fact to the ones column yields the congruence $2T + 4 \equiv T \pmod{10}$ whose only solution is $T = 6$. Therefore, $E = 7$ follows. On the basis of the thousands column one can also easily deduce that $I \leq 5$. Furthermore, it is evident that the values of V and W are interchangeable. The value of H determines the possible values for V and W, resulting in the following ten cases:

(1) $H = 1; V, W \in \{5,8\}$:	(6) $H = 4; V, W \in \{1,5\}$:
(2) $H = 1; V, W \in \{4,9\}$;	(7) $H = 5; V, W \in \{3,4\}$;
(3) $H = 2; V, W \in \{1,3\}$;	(8) $H = 5; V, W \in \{8,9\}$;
(4) $H = 2; V, W \in \{5,9\}$;	(9) $H = 8; V, W \in \{1,9\}$;
(5) $H = 3; V, W \in \{1,4\}$;	(10) $H = 9; V, W \in \{3,8\}$.

All but two of these lead to contradictions. Case (4) yields one solution, from Case (9) two solutions are obtained; they are given below.

1	1	1
1	1	1
690	610	610
62477	68577	68277
8157	3297	5497
71326	72486	74386

As remarked earlier, upon interchanging the values of V and W, three additional solutions may be given. It may be of interest to note that the number of essentially different solutions, the possible values of F (commonly used to denote the Fibonacci numbers), as well as the possible values of H (often used to denote generalized Fibonacci numbers) are all Fibonacci numbers.

Also (partially) solved by Paul S. Bruckman, Warren Cheves, J.A.H. Hunter, John W. Milsom, Carl Moore, Jim Pope, A.C. Shannon, and the Proposer.

A FORMULA SYMMETRIC IN k AND n

B-294 Proposed by Richard Blazej, Queens Village, New York.

Show that

$$F_n L_k + F_k L_n = 2F_{n+k}.$$

Solution by Frank Higgins, Naperville, Illinois.

Using the Binet formulas we have

$$F_n L_k + F_k L_n = \left(\frac{a^n - b^n}{\sqrt{5}}\right)(a^k + b^k) + \left(\frac{a^k - b^k}{\sqrt{5}}\right)(a^n + b^n) = 2\left(\frac{a^{n+k} - b^{n+k}}{\sqrt{5}}\right) = 2F_{n+k}\ .$$

Also solved by George Berzsenyi, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Mike Hoffman, Peter A. Lindstrom, Graham Lord, John W. Milsom, Carl Moore, F.D. Parker, Jeffrey Shallit, A.C. Shannon, Paul Smith, Gregory Wulczyn, and the Proposer.

CONVOLUTION OR DOUBLE SUM

B-295 Proposed by V.E. Hoggatt, Jr., California State University, San Jose, California.

Find a closed form for

$$\sum_{n=1}^{n} (n + 1 - k)F_{2k} = nF_2 + (n-1)F_4 + \cdots + F_{2n}\ .$$

Solution by Graham Lord, Secane, Pennsylvania.

The sum of the first k odd indexed Fibonacci numbers is F_{2k} and that of the first k even indexed ones is $F_{2k+1} - 1$, where $k \geq 1$.

Therefore,

$$\sum_{k=1}^{n} (n+1-k)F_{2k} = \sum_{j=1}^{n}\sum_{i=1}^{i} F_{2i} = \sum_{j=1}^{n} (F_{2j+1} - 1)$$
$$= F_{2(n+1)} - n - 1\,.$$

NOTE: Compare B-290.

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Carl Moore, Jeffrey Shallit, A.C. Shannon, Paul Smith, and the Proposer.

A MOST CHALLENGING PROBLEM

B-296 Proposed by Gary Ford, Vancouver, B.C., Canada.

Find constants a and b and a transcendental function G such that

$$G(y_{n+3}) + G(y_n) + G(y_{n+2})G(y_{n+1})$$

whenever y_n satisfies $y_{n+2} = ay_{n+1} + by_n$.

I. Solution by Carl F. Moore, Tacoma, Washington.

Two solutions are given by:

(1) $\quad a = b = 1 \quad$ and $\quad G(u) = 2\cos u\,,$

(2) $\quad a = b = 1 \quad$ and $\quad G(u) = c^u + c^{-u} \quad (c \neq 1)\,.$

[Notice $G(u) = 2\cosh u$ is a pleasing special case.]

To show (1),

$$G(y_{n+3}) + G(y_n) = 2\cos(y_{n+3}) + 2\cos(y_n) = 2(\cos(y_{n+3}) + \cos(y_n))$$
$$= 2\left(2\cos\frac{y_{n+3}+y_n}{2}\cdot\cos\frac{y_{n+3}-y_n}{2}\right)$$
$$= 2\left(2\cos\frac{2y_{n+2}}{2}\cdot\cos\frac{2y_{n+1}}{2}\right)$$
$$= (2\cos(y_{n+2}))\cdot(2\cos(y_{n+1})) = G(y_{n+2})\cdot G(y_{n+1})\,.$$

To show (2),

$$G(y_{n+3}) + G(y_n) = (c^{y_{n+3}} + c^{-y_{n+3}}) + (c^{y_n} + c^{-y_n}) = c^{y_{n+2}+y_{n+1}} + c^{-y_{n+2}-y_{n+1}} + c^{y_{n+2}-y_{n+1}}$$
$$+ c^{y_{n+1}-y_{n+2}} = c^{y_{n+2}}\cdot c^{y_{n+1}} + c^{-y_{n+2}}\cdot c^{-y_{n+1}} + c^{y_{n+2}}\cdot c^{-y_{n+1}} + c^{y_{n+1}}\cdot c^{-y_{n+2}}$$
$$= (c^{y_{n+2}} + c^{-y_{n+2}})\cdot(c^{y_{n+1}} + c^{-y_{n+1}}) = G(y_{n+2})\cdot G(y_{n+1})\,.$$

II. Solution by the Proposer.

Let $G(x) = c^x + c^{-x}$, with c any (complex) constant and let $\{y_n\}$ be a generalized Fibonacci sequence (satisfying $y_{n+2} = y_{n+1} + y_n$ and having any initial conditions).

There were no other solvers.

PARTIAL FRACTIONS

B-297 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Obtain a recursion formula and a closed form in terms of Fibonacci and Lucas numbers for the sequence (G_n) defined by the generating function:

$$(1-3x-x^2+5x^3+x^4-x^5)^{-1} = G_0+G_1x+G_2x^2+\cdots+G_nx^n+\cdots.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

We note that

$$G_{n+5}-3G_{n+4}-G_{n+3}+5G_{n+2}+G_{n+1}-G_n = 0.$$

Since

$$(1-3x-x^2+5x^3+x^4-x^5) = (1-3x+x^2)(1-x-x^2)(1+x),$$

we obtain, using partial fractions,

$$\frac{10}{1-3x-x^2+5x^3+x^4-x^5} = \frac{18-7x}{1-3x+x^2} - \frac{5(2+x)}{1-x-x^2} + \frac{2}{1+x}.$$

If $W_{n+2}=aW_{n+1}+bW_n$, then

$$\sum_{n=0}^{\infty} W_nx^n = \frac{W_0+(W_1-aW_0)x}{1-ax-bx^2}.$$

Thus,

$$\frac{18-7x}{1-3x+x^2} = \sum_{n=0}^{\infty} L_{2n+6}x^n; \qquad \frac{2+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+3}x^n; \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^nx^n.$$

Thus,

$$G_n = \frac{1}{10}(L_{2n+6}-5F_{n+3}+2(-1)^n).$$

Also solved by Frank Higgins, Carl F. Moore, A.C. Shannon, Gregory Wulczyn, and the Proposer.

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VOLUME INDEX

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THE GENERAL LAW OF QUADRATIC RECIPROCITY

If $(-1/b_1b_2) = 1$, then

$$(a/b_1b_2) = 1,$$
$$(-a/b_1b_2) = 1,$$
$$(a/-b_1b_2) = (a/-1),$$
$$(-a/-b_1b_2) = -(a/-1);$$

If $(-1/b_1b_2) = -1$, then

$$(a/b_1b_2) = (-1/a),$$
$$(-a/b_1b_2) = -(-1/a),$$
$$(a/-b_1b_2) = (a/-1)(-1/a),$$
$$(-a/-b_1b_2) = (a/-1)(-1/a).$$

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2. William Judson LeVeque, *Topics in Number Theory,* Vol. 1, Addison-Wesley, Reading, Mass., 1956.

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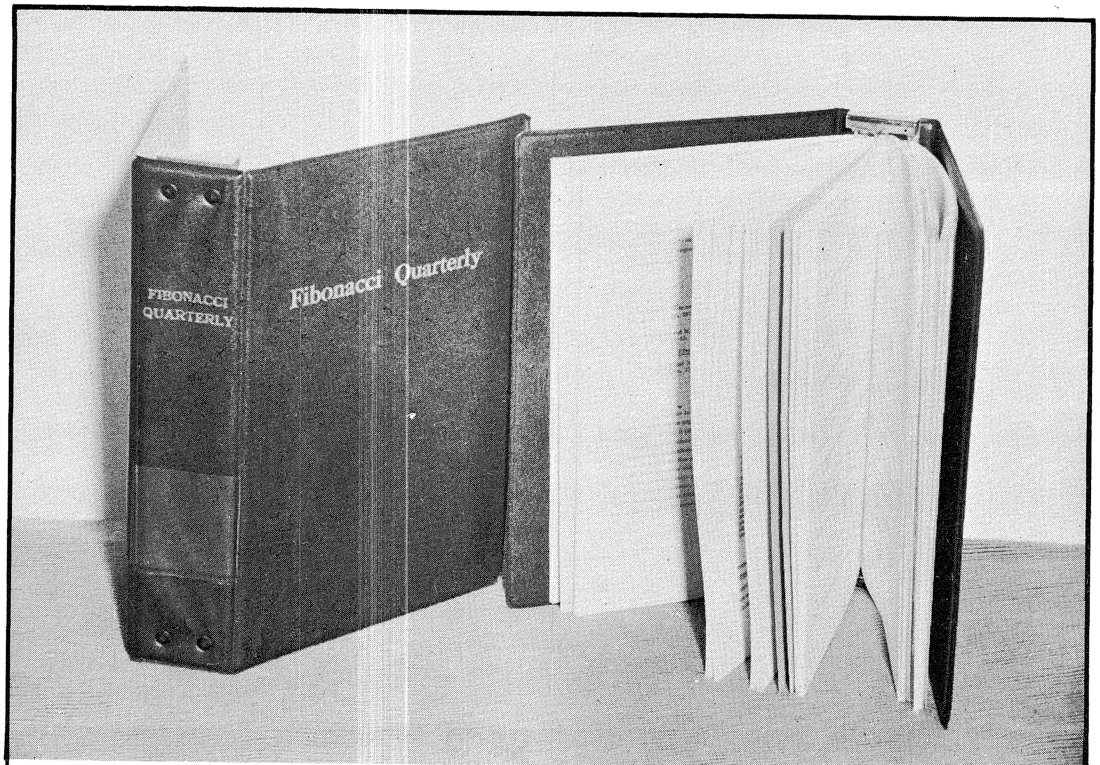

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