THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF
THE FIBONACCI ASSOCIATION

VOLUME 14 NUMBER 1

CONTENTS

Some Operational Formulas . . . . *Humphrey Nash* 1
Minimal and Maximal Fibonacci Representations:
Boolean Generation . . . . *P. Monteiro and R. W. Newcomb* 9
Letter to the Editor 12
Some Remarks on Initial Digits . . . . *Basil Davis* 13
On Isomorphisms Between the Naturals and Integers . . . . *Samuel T. Stern* 15
A Fibonacci Curiosity . . . . *Leon Bankoff* 17
On Continued Fraction Expansions
Whose Elements are All Ones . . . . *Gregory Wulczyn* 18
On Congruences Modulo *A* Power of a Prime . . . . *M. G. Monzingo* 23
Fibonacci Ratio in Electric Wave Filters . . . . *L. G. Zuckerman* 25
On a Theorem of Kronecker . . . . *Michael Willett* 27
A Fibonacci Pleasantry . . . . *Leon Bankoff* 29
Column Generators for Coefficients of Fibonacci
And Fibonacci-Related Polynomials . . . . *Dean B. Priest and Stephen W. Smith* 30
A Summation Identity . . . . *Theodore J. Cullen* 35
Fibonacci Sequences
And Memory Management . . . . *T. G. Lewis, B. J. Smith and M. Z. Smith* 37
Interesting Properties of Laguerre Polynomials . . . . *N. C. Mohanty* 42
Numerator Polynomial Coefficient Array
For the Convolved Fibonacci Sequence . . . . . *G. E. Bergum and V. E. Hoggatt, Jr.* 43
Letter to the Editor 48
A Combinatorial Identity . . . . *Haim Hanani* 49
Some Remarks on the Periodicity of
The Sequence of Fibonacci Numbers . . . . *T. E. Stanley* 52
Parity Triangles of Pascal's Triangle . . . . *S. H. L. Kung* 54
The Saalschützian Theorems . . . . *L. Carlitz* 55
A Density Relationship
Between $ax + b$ and $[x/c]$ . . . . *Robert W. Stringall* 64
Generalized Bell Numbers . . . . *E. A. Enneking and J. C. Ahuja* 67
A Note on a Theorem of W. B. Ford . . . . *T. K. Puttaswamy* 74
A Method for the Evaluation of Certain
Sums Involving Binomial Coefficients . . . . *George H. Weiss and Menachem Dishon* 75
On the Order of Systems of Two Simultaneous
Linear Difference Equations in Two Variables . . . . *Robert Gordon Babb II* 78
Lucas Polynomials and Certain Circular Functions of Matrices . . . . . *J. E. Walton* 83
Advanced Problems and Solutions . . . . *Edited by Raymond E. Whitney* 88
Elementary Problems and Solutions . . . . *Edited by A. P. Hillman* 93

FEBRUARY 1976

THE FIBONACCI QUARTERLY

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

All subscription correspondence should be addressed to Professor Leonard Klosinski, Mathematics Department, University of Santa Clara, Santa Clara, California 95053. All checks ($12.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Two copies of manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State University, San Jose, California 95192. *All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscript sent to the editors.*

The Quarterly is entered as third-class mail at the St. Mary's College Post Office, Calif., as an official publication of the Fibonacci Association.

The Quarterly is published in February, April, October, and December each year.

Typeset by
HIGHLANDS COMPOSITION SERVICE
P.O. Box 760
Clearlake Highlands, Calif. 95422

SOME OPERATIONAL FORMULAS

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1. INTRODUCTION

In this paper we consider some simple variations of the derivative and the difference operator; deriving formulas for powers and factorials.

Let $s(n,k)$ denote the Stirling number of the first kind and $S(n,k)$ denote the Stirling number of the second kind. They are defined by:

(1.1) $$(x)_n = \sum_{k=1}^{n} s(n,k)x^k$$

(1.2) $$x^n = \sum_{k=1}^{n} S(n,k)(x)_k \,,$$

where

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

Substituting (1.1) in (1.2) or (1.2) in (1.1) shows that

$$a_n = \Sigma s(n,k)b_k \qquad \text{and} \qquad b_n = \Sigma S(n,k)a_k$$

are equivalent (inverse) relations.

Define

(1.3) $$A_n(x) = \sum_{k=1}^{n} s(n,k)x^k$$

(1.4) $$A^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)x^k$$

(1.5) $$B_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

(1.6) $$B^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)x^k \,.$$

Then $A_n(x) = (x)_n$, the falling factorial; $A^{(n)}(x) = x^{(n)}$, the rising factorial and $B_n(x)$ is the single variable Bell polynomial [3, p. 35]. We have $A_n(B(x)) = x^n = B_n(A(x))$, etc., where $(B(x))^k \equiv B_k(x)$, $(A(x))^k \equiv A_k(x)$.

We will employ the following special notation:

(1.7) $$[\theta\phi]^n = \theta^n\phi^n$$

and if

$$f_n(x) = \sum_{i=0}^{n} a_i x^i$$

then

$$f_n[\theta\phi] = \sum_{i=0}^{n} a_i [\theta\phi]^i = \sum_{i=0}^{n} a_i \theta^i \phi^i .$$

REMARK. When θ and ϕ commute or $n = 1$ then

$$[\theta\phi]^n = (\theta\phi)^n \qquad \text{and} \qquad f_n(\partial\phi) = f_n[\theta\phi] .$$

2. THE OPERATORS xD, Dx, $x\Delta$, Δx

Operators of the form $(xD)^n$, $D^n x^n$, $(\Delta x)^n$, etc., are often difficult to work with and we seek equivalent forms. First we note that

$$(2.1) \qquad (xD)_n = A_n(xD) = \sum_{k=1}^{n} S(n,k)(xD)^k = x^n D^n$$

follows by induction from

$$\begin{aligned}(xD)_{k+1} &= (xD)_k(xD-k) = x^k D^k(xD-k) = x^k(D^k x)D - kx^k D^k \\ &= x^k(xD^k + kD^{k-1})D - kx^k D^k = x^{k+1}D^{k+1} .\end{aligned}$$

But (2.1) admits the inverse

$$(2.2) \qquad (xD)^n = \Sigma S(n,k)x^k D^k = B_n[xD] .$$

Equation (2.2) can slo be shown directly using the recurrence for $S(n,k)$ [4, p. 218].

Similarly,

$$(2.3) \qquad (x\Delta)_n = A_n(x\Delta) = \sum_{k=0}^{n} a(n,k)(x\Delta)^k = x^{(n)}\Delta^n$$

follows by induction from

$$\begin{aligned}(x\Delta)_{k+1} &= (x\Delta - k)(x\Delta)_k = (x\Delta - k)x^{(k)}\Delta^k = \left\{ x\Delta x^{(k)} - kx^{(k)} \right\} \Delta^k \\ &= \left\{ xx^{(k)}\Delta + kx(x+1)^{(k-1)} + kx(x+1)^{(k-1)}\Delta - kx^{(k)} \right\} \Delta^k \\ &= \left\{ xx^{(k)}\Delta + kx(x+1)^{k-1}\Delta \right\} \Delta^k = (x+k)x^{(k)}\Delta\Delta^k = x^{(k+1)}\Delta^{k+1} .\end{aligned}$$

But (2.3) admits the inverse

$$(2.4) \qquad (x\Delta)^n = \Sigma S(n,k)x^{(k)}\Delta^k = B_n\left[x\Delta\right]$$

where $x^j \equiv x^{(j)}$.

Since

$$(Dx)^n = x^{-1}(xD)^{n+1}D^{-1} \qquad \text{and} \qquad (\Delta x)^n = x^{-1}(x\Delta)^{n+1}\Delta^{-1}$$

we have from (2.2) and (2.4), respectively,

$$(2.5) \qquad (Dx)^n = x^{-1}B_{n+1}[xD]D^{-1} = \sum_{k=1}^{n+1} S(n+1,k)x^{k-1}D^{k-1} ,$$

$$(\Delta x)^n = x^{-1}B_{n+1}[x\Delta]\Delta^{-1} = \sum_{k=1}^{n+1} S(n+1,k)(x+1)^{(k-1)}\Delta^{k-1}\,. \tag{2.6}$$

Using Leibnitz's formula for the derivative of a product we get; cf. [1. p.]

$$D^n x^n = \sum_{k=0}^{n} \binom{n}{k} (D^k x^n) D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (n)_k x^{n-k} D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} x^{n-k} D^{n-k}$$

Replacing $n-k$ by k we have

$$D^n x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} x^k D^k\,. \tag{2.7}$$

Using

$$D^{k+1}x^{k+1} = D^k\left\{x^{k+1}D + (k+1)x^k\right\} = D^k x^k\left\{xD + k + 1\right\}$$

we have by induction

$$D^n x^n = (xD+1)^{(n)} = (Dx)^{(n)} = A^{(n)}(Dx). \tag{2.8}$$

Since

$$(xD)^{(n)} = (xD)(xD+1)^{(n-1)} = (xD)(Dx)^{(n-1)} = xD\,D^{n-1}x^{n-1}$$

we have

$$(xD)^{(n)} = xD^n x^{n-1}\,.$$

Using the difference analogue of Leibnitz's formula [2, p. 96] we get cf. [1, p. 4],

$$\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}E^k x^{(n)}\Delta^k = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}(x+k)^{(n)}\Delta^k = \sum_{k=0}^{n} \binom{n}{k} (n)_{n-k}(x+n)^{(k)}\Delta^k\,.$$

Hence

$$\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!}(x+n)^{(k)}\Delta^k\,. \tag{2.9}$$

Using

$$\begin{aligned}\Delta^{k+1}(x)_{k+1} &= \Delta^k(\Delta(x)_{k+1}) = \Delta^k\left\{(x)_{k+1}\Delta + (k+1)(x)_k + (k+1)(x)_k\Delta\right\}\\ &= \Delta^k(x)_k\left\{(x-k)\Delta + (k+1) + (k+1)\Delta\right\}\\ &= \Delta^k(x)_k(x\Delta + \Delta + 1 + k) = \Delta^k(x)_k(\Delta x + k)\,,\end{aligned}$$

we have by induction

$$\Delta^n(x)_n = (\Delta x)^{(n)} = A^{(n)}(\Delta x). \tag{2.10}$$

But

$$\Delta^n x^{(n)} = \Delta^n(x+n-1)_n = (\Delta(x+n-1))^{(n)}\,;$$

hence using $\Delta x = x\Delta + \Delta + 1$ we have

$$\Delta^n x^{(n)} = ((x+n)\Delta + 1)^{(n)} = ((x+n)\Delta + n)_n\,. \tag{2.11}$$

Taking the inverse of (2.8) we have

$$(Dx)^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)D^k x^k = B^{(n)}[Dx]\,. \tag{2.12}$$

Taking the inverse of (2.10) we have

$$(\Delta x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) \Delta^k (x)_k = B^{(n)}[\Delta x] , \tag{2.13}$$

where $x^j \equiv (x)_j$.

Since

$$(xD)^{m+n} = (xD)^m (xD)^n \qquad \text{and} \qquad \left\{ (xD)^m \right\}^n = (xD)^{mn}$$

we have by (2.2)

$$B_{m+n}[xD] = B_m[xD] B_n[xD] , \qquad (B_m[xD])^n = B_{mn}[xD] . \tag{2.14}$$

Similarly (2.4) gives

$$B_{m+n}[x\Delta] = B_m[x\Delta] B_n[x\Delta], \qquad \left\{ B_m[x\Delta] \right\}^n = B_{mn}[x\Delta] . \tag{2.15}$$

Similar results also hold for $B^{(k)}[Dx]$ and $B^{(k)}[\Delta x]$.

3. THE OPERATORS $x(I + D)$, $x(I + \Delta)$, $(I + D)x$, $(I + \Delta)x$

Analogous to (2.1) is

$$(x(I + D))_n = A_n(x(I + D)) = x^n (I + D)^n = [x(I + D)]^n \tag{3.1}$$

which follows by induction from

$$\begin{aligned}(x(I + D))_{k+1} &= (x(I + D))_k (x(I + D) - k) = x^k (I + D)^k (x(I + D) - k) \\ &= x^k \left\{ x(I + D)^{k+1} + k(I + D)^k - k(I + D)^k \right\} = x^{k+1}(I + D)^{k+1} .\end{aligned}$$

But (3.1) admits the inverse

$$(x(I + D))^n = \sum_{k=1}^{n} S(n,k) x^k (I + D)^k = B_n[x(I + D)] . \tag{3.2}$$

Since

$$((I + D)x)^n = x^{-1}(x(I + D))^{n+1}(I + D)^{-1}$$

we have

$$((I + D)x)^n = \sum_{k=1}^{n+1} S(n + 1, k) x^{k-1} (I + D)^{k-1} . \tag{3.3}$$

Using

$$(I + D)^{n+1} x^{n+1} = (I + D)^n (I + D) x^{n+1} = (I + D)^n x^n (x + xD + n + 1) = (I + D)^n x^n ((I + D)x + n)$$

we have by induction

$$(I + D)^n x^n = ((I + D)x)^{(n)} = A^{(n)}((I + D)x) \tag{3.4}$$

which admits the inverse

$$((I + D)x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) (I + D)^k x^k = B^{(n)}[(I + D)x] . \tag{3.5}$$

By (3.4) and since $(I + D)x = (x + xD) + 1$,

$$(x(I + D))^{(n)} = (x + xD)^{(n)} = x(I + D)((I + D)x)^{n-1} = x(I + D)(I + D)^{n-1} x^{n-1} .$$

Hence

(3.6) $$(x(I+D))^{(n)} = x(I+D)^n x^{n-1} .$$

By (3.1) and since

$$(x+Dx)_n = (x+Dx)(x+xD)_n$$

we have

(3.7) $$((I+D)x)_n = (I+D)x^n(I+D)^{n-1} .$$

Using (1.4)

(3.8) $$(x(I+\Delta))^{(n)} = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)(x(I+\Delta))^k = A^{(n)}(x(I+\Delta)).$$

But,

(3.9) $$(x(I+\Delta))^n = x^{(n)}(I+\Delta)^n$$

follows by induction from

$$(x(I+\Delta))^{k+1} = (x(I+\Delta))(x(I+\Delta))^k = x(I+\Delta)x^{(k)}(I+\Delta)^k$$

$$= x\left\{x^{(k)} + x^{(k)}\Delta + k(x+1)^{(k-1)} + k(x+1)^{(k-1)}\Delta\right\}(I+\Delta)^k$$

$$= x\left\{x^{(k)} + k(x+1)^{(k-1)}\right\}(I+\Delta)^{k+1} = x(x+1)^{(k-1)}(x+k)(I+\Delta)^{k+1} = x^{(k+1)}(I+\Delta)^{k+1}.$$

Hence

(3.10) $$(x(I+\Delta))^{(n)} = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)x^{(k)}(I+\Delta)^k = A^{(n)}[x(I+\Delta)] ,$$

where $x^k \equiv x^{(k)}$.

Relation (3.8) admits the inverse

(3.11) $$(x(I+\Delta))^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)(x(I+\Delta))^{(k)} = B^{(n)}(x(I+\Delta)),$$

where $(x(I+\Delta))^k \equiv (x(I+\Delta))^{(k)}$.

Using (3.9), (3.11) may be rewritten

(3.12) $$(x)^{(n)}(I+\Delta)^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)(x(I+\Delta))^{(k)} .$$

Using (1.1)

(3.13) $$(x(I+\Delta))_n = \sum_{k=1}^{n} s(n,k)(x(I+\Delta))^k = A_n(x(I+\Delta))$$

and using (3.9)

(3.14) $$(x(I+\Delta))_n = \sum_{k=1}^{n} s(n,k)x^{(k)}(I+\Delta)^k = A_n[x(I+\Delta)] ,$$

where the inverses of (3.13 and (3.14 are, respectively,

(3.15) $$(x(I+\Delta))^n = \sum_{k=1}^{n} S(n,k)(x(I+\Delta))_k = B_n(x(I+\Delta))$$

and

(3.16) $$x^{(n)}(I+\Delta)^n = \sum_{k=1}^{n} S(n,k)(x(I+\Delta))_k = B_n(x(I+\Delta)).$$

Iterating $(I+\Delta)x = x + x\Delta + \Delta + I = (x+1)(I+\Delta)$ n times we have

(3.17) $$(I+\Delta)^n x = (x+n)(I+\Delta)^n .$$

More generally,

(3.18) $$(I+\Delta)^n x^{(n)} = (x+n)^{(n)}(I+\Delta)^n$$

as the following induction step shows:

$$(I+\Delta)^{n+1}x^{(n+1)} = (I+\Delta)^n(I+\Delta)x^{(n+1)} = (I+\Delta)^n(x+1)^{(n)}(x+n+1)(I+\Delta)$$
$$= (x+1+n)^{(n)}(I+\Delta)^n(x+n+1)(I+\Delta).$$

Using (3.17) we get

$$(x+1+n)^{(n)}(x+n+1+n)(I+\Delta)^n(I+\Delta) = (x+n+1)^{(n+1)}(I+\Delta)^{n+1} .$$

Replacing x by $x+1$ in (3.9) and using (3.17) for $n=1$ we have

(3.19) $$((I+\Delta)x)^n = (x+1)^{(n)}(I+\Delta)^n = (I+\Delta)^n(x)_n .$$

Similarly (3.10) becomes

(3.20) $$((I+\Delta)x)^{(n)} = A^{(n)}[(x+1)(I+\Delta)] = A^{(n)}[(I+\Delta)x],$$

where $(x+1)^k \equiv (x+1)^{(k)}$.

Equation (3.11) becomes

(3.21) $$((I+\Delta)x)^n = B^{(n)}((x+1)(I+\Delta)) = B^{(n)}[(I+\Delta)x].$$

Equation (3.14) becomes

(3.22) $$((I+\Delta)x)_n = A_n[(I+\Delta)x].$$

4. THE OPERATORS $xD^2x\ Dx^2D,\ x\Delta^2x-1,\ \Delta(x-1)^{(2)}\Delta$

We first note that xD and Dx commute, i.e.,

(4.1) $$xD^2x = xD\,Dx = x^2D^2 + 2xD = DxxD = Dx^2D$$

and we restrict our attention to xD^2x.

Since $xD^2x = xD\,Dx = xD(1+xD) = B_1[xD](1+B_1[xD])$,

$$(xD^2x)^n = \left\{B_1[xD](1+B_1[xD])\right\}^n .$$

By (2.14) this gives

(4.2) $$(xD^2x)^n = B_n[xD](1+B_1[xD])^n$$

or alternatively

(4.3) $$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[xD].$$

This becomes

(4.4) $$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k, j)x^jD^j$$

or utilizing (2.2),

(4.5) $$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} (xD)^{n+k} .$$

Since xD and Dx commute with each other,

$$(xD^2x)^n = (xD\,Dx)^n = (xD)^n(Dx)^n = [(xD)(Dx)]^n .$$

Using (2.2) and (2.12) this gives

(4.6) $$(xD^2x)^n = B_n[xD]B^{(n)}[Dx]$$

Comparison with (4.2) yields

(4.7) $$B^{(n)}[Dx] = \sum_{k=0}^{n} \binom{n}{k} B_k[xD] .$$

Since by (2.1) and (2.8),

$$x^nD^{2n}x^n = x^nD^nD^nx^n = (xD)_n(Dx)^{(n)}$$

and since

$$(xD-k)(Dx+k) = (xD-k)(xD+1+k) = xD^2x - k^{(2)}$$

we have, analogous to (2.1) and (2.8),

(4.8) $$x^nD^{2n}x^n = \prod_{k=0}^{n} (xD^2x - k^{(2)}).$$

Remark. $$D^nx^{2n}D^n = x^nD^{2n}x^n .$$

We note that $x\Delta$ and $\Delta(x-1)$ commute, i.e.,

(4.9) $$x\Delta^2(x-1) = x\Delta(1+x\Delta) = (1+x\Delta)x = (x-1)^{(2)}\Delta.$$

Writing

$$x\Delta^2(x-1) = x\Delta(1+x\Delta) = B_1[x\Delta](1+B_1[x\Delta])$$

we have using (2.14)

(4.10) $$(x\Delta^2(x-1))^n = B_n[x\Delta](1+B[x\Delta])^n$$

or

(4.11) $$(x\Delta^2(x-1))^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[x\Delta] = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k, j)x^jD^j$$

or using (2.4)

(4.12) $$(x\Delta^2(x-1))^n = \sum_{k=0}^{n} \binom{n}{k} (x\Delta)^{n+k} .$$

Since by (2.3) and (2.10)

$$x^{(n)}\Delta^n\Delta^n(x-1)_n = (x\Delta)_n(\Delta(x-1))^{(n)} = (x\Delta)_n(x\Delta+1)^{(n)}$$

and since

$$(x\Delta - k)(x\Delta + 1 + k) = (x\Delta^2(x-1) - k^{(2)})$$

we have, analogous to (4.8),

$$x^{(n)}\Delta^{2n}(x-1)_n = \prod_{k=0}^{n} (x\Delta^2(x-1) - k^{(2)}). \tag{4.13}$$

5. THE OPERATORS $x(I+D)^2x,\ x(I+\Delta)^2(x-1)$

The operators $x(I+D)$ and $(I+D)x$ commute, i.e.,

$$x(I+D)^2x = (I+D)x^2(I+D), \tag{5.1}$$

and we have using (3.2)

$$(x(I+D)^2x)^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[x(I+D)] = \sum_{k=0}^{n} \binom{n}{k} (x(I+D))^{n+k} \tag{5.2}$$

and

$$(x(I+D)^2x)^n = \sum_{n=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k,j)x^j(I+D)^j . \tag{5.3}$$

The operators $x(I+\Delta)$ and $(I+\Delta)(x-1)$ commute, i.e.,

$$x(I+\Delta)^2(x-1) = (I+\Delta)(x-1)^{(2)}(I+\Delta). \tag{5.4}$$

Using (3.18),

$$x(I+\Delta)^2(x-1) = x(I+\Delta)x(I+\Delta) = (x(I+\Delta))^2. \tag{5.5}$$

Hence by (3.9)

$$x(I+\Delta)^2(x-1))^n = (x(I+\Delta))^{2n} = x^{(2n)}(1+\Delta)^{2n} . \tag{5.6}$$

Since

$$\begin{aligned} x^{(n)}(I+\Delta)^n(I+\Delta)^n(x-1)_n &= x^{(n)}(1+\Delta)^n(1+\Delta)^n(x-n)^{(n)} \\ &= x^{(n)}(1+\Delta)^n x^{(n)}(1+\Delta)^n = x^{(n)}(x+n)^{(n)}(I+\Delta)^n(1+\Delta)^n \end{aligned}$$

we have

$$x^{(n)}(1+\Delta)^{2n}(x-1)_n = x^{(2n)}(1+\Delta)^{2n} \tag{5.7}$$

and comparing with (5.6)

$$(x(I+\Delta)^2(x-1))^n = x^{(n)}(I+\Delta)^{2n}(x-1)_n . \tag{5.8}$$

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MINIMAL AND MAXIMAL FIBONACCI REPRESENTATIONS: BOOLEAN GENERATION*

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"Then hear my message ere thou speed away." [1]

1. INTRODUCTION

Among the many important and interesting properties of Fibonacci numbers are those of yielding unique minimal and maximal representations of arbitrary nonnegative integers [2, p. 74]. Since these representations can be convenient for characterizing and calculating with integers, it is of interest to have an algorithmic process for obtaining the minimal and maximal representations. This we present here in terms of Boolean functions for which logic circuits are developed and from which hardware implementation can occur.

We take the j^{th} Fibonacci number as

(1a) $$F_j = F_{j-1} + F_{j-2}$$

(1b) $$F_0 = 0, \qquad F_1 = 1.$$

For concreteness we use the initial conditions of (1b) though as far as the algorithm to be developed is concerned others, such as $F_0 = 2, F_1 = 1$ for Lucas numbers, are equally satisfactory. Then it is known [3] that any nonnegative integer N can be represented as

(2a) $$N = \sum_{j=2}^{n} a_j F_j ; \qquad F_n \leqslant N < F_{n+1},$$

where each a_j is a binary number, that is either zero or unity. There are many such representations possible but that called the *minimal representation,* with [4] [5]

(2b) $$a_j a_{j+1} = 0, \qquad j = 2, 3, \cdots, n-1$$

and that called the *maximal representation,* with [6]

(2c) $$a_j + a_{j+1} \geqslant 1, \qquad j = 2, 3, \cdots, n-1$$

are unique. Indeed each of these two representations in itself uniquely characterizes the Fibonacci numbers [7]. In the following we shall represent N of (2a) by the coefficients, writing for convenience

(3) $$N = a_2 a_3 \cdots a_n .$$

Note that the least significant digits are to the left. Thus, for example

$$N = 24 = F_2 + F_3 + F_6 + F_7 = 1100110$$

has for its minimal and maximal forms $N = 0010001$ and $N = 1111010$, respectively.

"Before thee mountains rise and rivers flow." [1]

2. BOOLEAN EXPRESSIONS

We first set down the rules for obtaining the minimal and maximal representations from which the desired Boolean functions can be obtained. The rules follow by iteratively applying (1a), the iterations being indexed by time t in

*This work was supported by the Air Force Office of Scientific Research under Grant AFOSR 70-1910.

discrete increments t_k, $k = 0, 1, \cdots$. Thus, we assume that a configuration for N in the form of (3) is on hand at time t_k,

$$N = a_2(t_k)a_3(t_k)\cdots a_n(t_k).$$

The configuration is "changed" at the next instant of time t_{k+1} according to the following rules.

MINIMAL FORM RULE: If at time t_k any sequence 110 occurs in N replace it by time t_{k+1} by 001; repeat for all t_k until no changes occur.

As an example to illustrate the process consider the following:

$$\begin{aligned} t_0,\quad N = 61 &= 111011010 \\ t_1,\quad &= 100100110 \\ t_2,\quad &= 100100001\,. \end{aligned}$$

The maximal form rule is similar but uses a procedure which is the reverse of that for the minimal form rule.

MAXIMAL FORM RULE: If at time t_k any sequence 001 occurs in N replace it at time t_{k+1} by 110; repeat for all t_k until no changes occur.

This is illustrated by the following example:

$$\begin{aligned} t_0,\quad N = 13 &= 000001 \\ t_1,\quad &= 000110 \\ t_2,\quad &= 011010\,. \end{aligned}$$

Translation of the rules into Boolean expressions can occur through the use of a truth table [8, p. 50]. Alternately we can read off from the rules the Boolean conditions. For example, from the minimal form rule we have the following: A zero in the j^{th} position becomes changed to a one (that is, complemented) if a_{j-1} and a_{j-2} are both ones. If a_j is a one it remains a one if either $a_{j+1} = a_{j-1} = 0$ or $a_{j+1} = a_{j+2} = 1$; otherwise a_j becomes zero. Using standard Boolean symbols ($\cdot$ = and, + = or, $\bar{\ }$ = complement = not) we then have the following Boolean expression:

Minimal Form (assume $a_0(t) = a_1(t) = a_{n+2}(t) = 0$,

(4a) $$a_j(t_{k+1}) = [\bar{a}_j(t_k)\cdot a_{j-1}(t_k)\cdot a_{j-2}(t_k)] + \left\{ a_j(t_k)\cdot[(\bar{a}_{j-1}(t_k)\cdot\bar{a}_{j+1}(t_k)) + (a_{j+1}(t_k)\cdot a_{j+2}(t_k))]\right\}.$$

Using similar reasoning we have

Maximal Form (assume $a_1(t) \equiv 1,\ a_{n+1}(t) = a_{n+2}(t) \equiv 0$):

(4b) $$a_j(t_{k+1}) = [a_j(t_k)\cdot a_{j-1}(t_k)] + [\bar{a}_j(t_k)\cdot\bar{a}_{j+1}(t_k)\cdot a_{j+2}(t_k)] + [\bar{a}_{j-1}(t_k)\cdot\bar{a}_j(t_k)\cdot a_{j+1}(t_k)]\,.$$

As can be readily verified, beginning at t_0 with a given number N, the maximum time to reach either the minimal or maximal form using (4) is $t_{[n/2]}$, where $[\cdot]$ denotes the integer part; this maximum time is achieved when n is odd and the initial representation has all zeros or ones except for a_n.

Equations (4) can be implemented through logic circuits, these being shown in Figures 1 and 2 for one a_j. In the figures, at a given instant t_k, the binary values of the $a_j(t_k)$ are read into a register whose cells are so labelled. These values serve as inputs to the logic circuits shown. On being processed in the logic circuitry, designed according to (4), the result is fed back to the register cells to be clocked in at the next instant t_{k+1}. The minimum time difference, $t_{k+1} - t_k$, possible is seen to be the delay time for signals to traverse the logic circuits. After a time of at most $t_{[n/2]}$ the reading of the register will have settled to the required form. Of course, to completely implement (4) the end cells of the register, which have the assumed stationary values, remain constant between any two clock pulses.

The output of Fig. 2 can be derived from the circuit of Fig. 1 by complementing the initial input and the final output, and vice-versa.

It should be mentioned that the given rules are not the only ones available. For example, we could have used the alternate minimal form rule:

At time t_k proceed from higher to lower numbered indices replacing at time t_{k+1} the first zero followed by two ones by a one followed by two zeros; repeat for all t_k until no change occurs.

This rule can be expressed in Boolean form by substituting

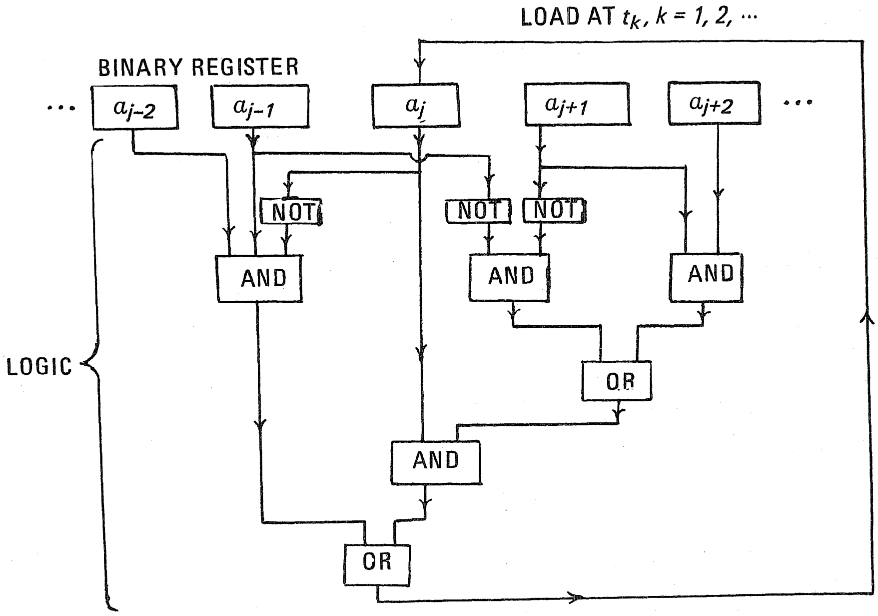

Figure 1

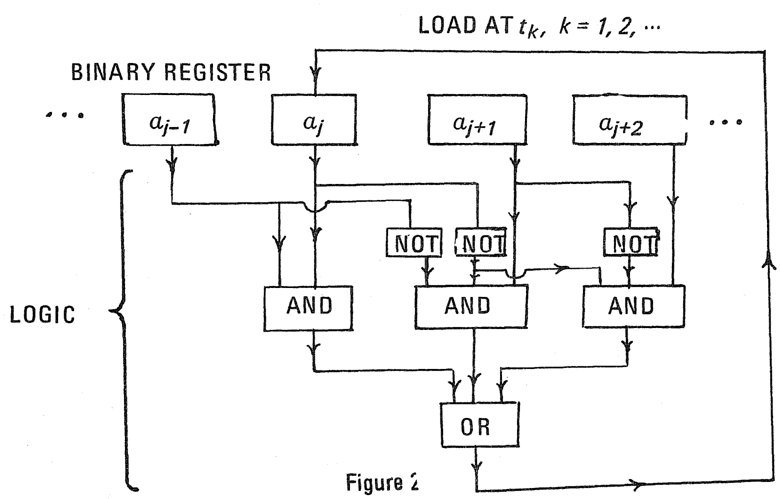

Figure 2

(5a) $$R_j = [a_{j-1}\cdot a_j\cdot \bar{a}_{j+1}\cdot C_{j+2}] + [a_j\cdot a_{j+1}\cdot \bar{a}_{j+2}\cdot C_{j+3}]$$

(5b) $$S_j = a_{j-2}\cdot a_{j-1}\cdot \bar{a}_j\cdot C_{j+1}$$

(5c) $$C_j = (\bar{a}_{j-2} + \bar{a}_{j-1} + a_j)\cdot C_{j+1}$$

(5d) $$C_{n+1} \equiv 1, \quad a_{n+1} = C_{n+2} = R_n \equiv 0$$

into

$$a_j(t_{k+1}) = [S_j(t_k) + \bar{R}_j(t_k)]\cdot a_j(t_k)$$

Equations (5) can be implemented by appropriate circuitry, as for (4), where R and S represent the reset and set inputs of an R-S flip-flop [8, p. 83] and C_j could be interpreted as a timing signal which signifies completion of changes (if any) in stage j. As before, a similar rule for the maximal form can be developed.

"When thou art weary, on the mountains stay,
And when exhausted, drink the rivers' driven spray." [1]

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LETTER TO THE EDITOR

December 2, 1975

Dear Dr. Hoggatt:

I showed Dr. James W. Follin, Jr., of the Applied Physics Laboratory the example in D. Shanks, "Incredible Identities," *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), pp. 271, 180. I think his generalization would be of interest.

Set $K^2 = m + n$. Then one has the identity

$$\sqrt{m} + \sqrt{2(K + \sqrt{m})} = \sqrt{K + \sqrt{n}} + \sqrt{K + m - \sqrt{n} + 2\sqrt{m(K - \sqrt{n})}}\ ,$$

which can be checked by squaring twice, while performing all simplifications, including substitution and observing a perfect square.

William G. Spohn, Jr.

SOME REMARKS ON INITIAL DIGITS

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It can be shown that the distribution of first digits among Fibonacci numbers is as follows: The probability that the first digit of a random Fibonacci number is n is given by

(1) $$P(n) = \log_{10}(1 + 1/n).$$

This property is true for any additive sequence of numbers, the m^{th} term of which is expressed as

(2) $$U_m = U_{m-1} + U_{m-2} \cdots U_{m-k}.$$

For Fibonacci and Lucas sequences $k = 2$. In the general case the ratio U_m/U_{m-1} tends to a limit, say R, as $m \to \infty$. R is related to k as follows:

(3) $$k = \frac{\log(2-R)^{-1}}{\log R}.$$

Hence an additive sequence tends towards a geometrical progression. As the reader may verify, the initial digits of any geometrical progression of real numbers will obey distribution (1), provided that the ratio is not a rational power of 10 (i.e., $10^{p/q}$, where p and q are integers).

The validity of the above law is tested below for the first 100 Fibonacci and Lucas numbers. The number of incidences of n as initial digit for Fibonacci numbers is given under A and that for Lucas numbers under B. The percentage calculated on the basis of distribution (1) is given under C.

n	A F nos.	B L nos.	C $100 \log_{10}(1 + 1/n)$
1	30	31	30.1
2	18	16	17.6
3	13	14	12.5
4	9	10	9.7
5	8	8	7.9
6	6	5	6.7
7	5	8	5.8
8	7	4	5.1
9	4	4	4.6

The close adherence to the law (1) is evident and the deviations can be explained as due to the finite number of the terms considered.

The distribution of first digits was the topic of a paper published by Benford [1] in 1938. It had been observed that the first few pages of logarithm books were consistently dirtier than the last few, indicating that the users had more occasion to look up numbers with smaller initial digits than larger ones. Benford collected a lot of numbers of the kind that users of logarithms were likely to deal with. They included surface areas of rivers, molecular weights of chemical compounds and such numbers as are found in scientific and statistical tables. Ignoring the decimal point and the magnitude of the numbers, he found that the first digits of these apparently random numbers followed very closely the following distribution:

The probability that the first digit of a random entry is n is given by $P(n) = \log_{10}(1 + 1/n)$. This is known as Benford's Law, and the distribution is identical with (1). Some of the various explanations put forward to explain this Law may be found in the references.

An elementary "explanation" may be provided as follows: Before computers were put into large-scale use (that is when logarithm books had to be used) it was difficult to deal with large and cumbersome numbers. To overcome this difficulty it was necessary that the units in which different quantities were measured were adjusted so as to render the measurements small (though greater than 1). This can be illustrated in the case of measurements on length. In atomic measurements Angstroms and other microscopic units were used to render the very small measurements close to unity. In everyday life units ranging from millimeters and inches to kilometers and miles are still used. In astronomy Astronomical Units, light years and *parsecs* are among the units employed. Similarly for mass, time, area, etc., the units are varied to suit the scale. Hence the numbers found in scientific and statistical tables would tend to be small in magnitude, except when the numbers are less than 1.

Thus one would expect the probability of the occurrence of a number of magnitude x to decrease monotonically with x when $x \geqslant 1$. Considering the simplest distribution $1/x$ as a trial function, one may write for $x \geqslant 1$,

$$f(x)dx = \frac{kdx}{x}, \tag{4}$$

where $f(x)dx$ is the probability of occurrence of a number in the range x to $x + dx$ and k is a constant of proportionality. The form of $f(x)$ in the region $0 \leqslant x < 1$ can be shown to be immaterial in obtaining the result below, provided $f(x)$ is finite throughout that interval.

If a number has initial digit n, it should lie between n and $n + 1$ or $10n$ and $10(n + 1)$ or $100n$ and $100(n + 1)$, etc. Hence the probability of occurrence of a number with initial digit n is

$$P(n) = \int_n^{n+1} \frac{kdx}{x} + \int_{10n}^{10(n+1)} \frac{kdx}{x} + \int_{100n}^{100(n+1)} \frac{kdx}{x} + \cdots .$$

Let there be m such integrals (thus putting an upper limit on x which will be removed by letting $m \to \propto$).

Then

$$P(n) = mk[\ln(n+1) - \ln(n)] = mk \ln(1 + 1/n).$$

Let $m \to \propto$ and $k \to 0$ so that mk remains finite. Then normalizing the probability,

$$mk = \frac{1}{\ln 10}.$$

Therefore,

$$P(n) = \log_{10}(1 + 1/n).$$

Thus, Benford's Law is obtained.

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ON ISOMORPHISMS BETWEEN THE NATURALS AND THE INTEGERS

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The mapping

$$g(m) = \left[\frac{m}{2}\right](-1)^m,$$

where $[x]$ denotes the greatest integer in x, from the set of naturals N onto the set of all integers I is one-to-one. This mapping fails to preserve natural order and the operations of ordinary addition and multiplication. For while $2 < 3$, $g(2) \nless g(3)$; also $g(2+3) \neq g(2) + g(3)$ and $g(2 \cdot 3) \neq g(2)g(3)$. However, it is possible to define an appropriate order relation $\wr$ and binary operations $(+)$ and $(\cdot)$ on I, while retaining natural order and ordinary addition and multiplication on N such that g will become an isomorphism of N to I, preserving order, addition, and multiplication as follows:

$$\text{(1)} \qquad x \wr y \text{ means } \begin{cases} |x| > |y| & \text{if } |x| \neq |y| \\ x < 0 \text{ and } y > 0 & \text{if } |x| = |y| \end{cases}$$

$$\text{(2)} \qquad x\,(+)\,y = \left[\frac{1 + |2x - \frac{1}{2}| + |2y - \frac{1}{2}|}{2}\right](-1)^{1+|2x-\frac{1}{2}|+|2y-\frac{1}{2}|}$$

$$\text{(3)} \qquad x\,(\cdot)\,y = \left[\frac{1 + |4x-1| + |4y-1| + |4x-1||4y-1|}{8}\right](-1)^{\frac{1+|4x-1|+|4y-1|+|4x-1||4y-1|}{4}}.$$

Noting that $[m/2]$ is equal to $m/2$ if m is even and $(m-1)/2$ if m is odd, it is easy to show that $m > n$ if and only if $g(m) \wr g(n)$. Furthermore,

$$g(m+n) = g(m)\,(+)\,g(n) \qquad \text{and} \qquad g(mn) = g(m)\,(\cdot)\,g(n).$$

An analogous treatment can be given the integers interpreted as equivalence classes of nonnegative integers. We let A be the set of all ordered pairs (a,b) of nonnegative integers and let $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. This defines an equivalence relation $\sim$ on A. Let B be the set of all equivalence classes of A with respect to this relation. Consider the mapping

$$\text{(4)} \qquad f(m) = K\left(\frac{m}{4}(1 + (-1)^m),\ \frac{m-1}{4}(1 + (-1)^{m-1})\right),$$

where $K(a,b)$ denotes the equivalence class of A which contains (a,b). f is one-to-one from N onto B. For let $K(a,b)$ represent an arbitrary element of B. If $a = b$ then $f(1) = K(a,b)$. If $a = b + k$, k a positive integer, then, $f(2k) = K(a,b)$. If $b = a + k$, k a positive integer, then $f(2k+1) = K(a,b)$. Furthermore if

$$K\left(\frac{m}{4}(1+(-1)^m),\ \frac{m-1}{4}(1+(-1)^{m-1})\right) = K\left(\frac{n}{4}(1+(-1)^n),\ \frac{n-1}{4}(1+(-1)^{n-1})\right)$$

then $(-1)^m(2m-1) = (-1)^n(2n-1)$. Hence m and n must be either both even or both odd, and it follows that $m = n$.

The absolute value of an element $K(a,b)$ of B, denoted by $|a,b|$ is defined as follows:

$$\text{(5)} \qquad |a,b| = \begin{cases} K(a,b) & \text{if } a > b \\ K(b,a) & \text{if } a \leqslant b \end{cases}.$$

The order relation Δ is defined on B as follows:

(6) $$K(a,b)\ \Delta\ K(c,d) \text{ if and only if } a+d > b+c.$$

The order relation ∇ is defined on B as follows:

(7) $$K(a,b)\ \nabla\ K(c,d) \text{ means } \begin{cases} |a,b|\ \Delta\ |c,d| & \text{if } |a,b| \neq |c,d| \\ a < b \text{ and } c > d & \text{if } |a,b| = |c,d|. \end{cases}$$

We show that with the relation of (7) on B and natural order on N the mapping (4) is an order isomorphism. For suppose that

$$K\left(\frac{m}{4}(1+(-1)^m), \frac{m-1}{4}(1+(-1)^{m-1})\right)\ \nabla\ K\left(\frac{n}{4}(1+(-1)^n), \frac{n-1}{4}(1+(-1)^{n-1})\right)$$

If these have the same absolute value, then by (7),

$$(-1)^{m+1}(2m-1) > 1 \quad \text{and} \quad (-1)^{n+1}(2n-1) < 1.$$

From the first of these inequalities we see that m is odd and since $2n-1$ is not zero $(-1)^{n+1}(2n-1)$ must be a negative integer, whence n is even. Thus

$$\left|0, \frac{m-1}{2}\right| = \left|\frac{n}{2}, 0\right|,$$

that is,

$$K\left(\frac{m-1}{2}, 0\right) = K\left(\frac{n}{2}, 0\right),$$

which implies $m > n$.

On the other hand if the two equivalence classes have different absolute values then

$$K\left(\frac{m}{2}, 0\right)\ \Delta\ K\left(\frac{n}{2}, 0\right)$$

if m and n are both even,

$$K\left(\frac{m-1}{2}, 0\right)\ \Delta\ K\left(\frac{n-1}{2}, 0\right)$$

if m and n are both odd, and

$$K\left(\frac{m-1}{2}, 0\right)\ \Delta\ K\left(\frac{n}{2}, 0\right)$$

if m is odd and n even. In each case we have $m > n$. If m is even and n odd then

$$K\left(\frac{m}{2}, 0\right)\ \Delta\ K\left(\frac{n-1}{2}, 0\right)$$

which implies $m \geqslant n$. But $m \neq n$. Hence $m > n$. Conversely, let $m > n$. Then if m and n are even,

$$\left|\frac{m}{2}, 0\right|\ \Delta\ \left|\frac{n}{2}, 0\right| \quad \text{and} \quad K\left(\frac{m}{2}, 0\right)\ \nabla\ K\left(\frac{n}{2}, 0\right).$$

If m and n are odd, then

$$\left|0, \frac{m-1}{2}\right|\ \Delta\ \left|0, \frac{n-1}{2}\right| \quad \text{and} \quad K\left(0, \frac{m-1}{2}\right)\ \nabla\ K\left(0, \frac{n-1}{2}\right).$$

If m is odd and n even and if also $m = n+1$, then

$$\left|0, \frac{m-1}{2}\right| = \left|\frac{n}{2}, 0\right|.$$

But if $m > n+1$, then

$$\left|0, \frac{m-1}{2}\right|\ \Delta\ \left|\frac{n}{2}, 0\right|.$$

Either way

$$K\left(0, \frac{m-1}{2}\right)\ \nabla\ K\left(\frac{n}{2}, 0\right).$$

If m is even and n odd, then

$$\left|\frac{m}{2},0\right| \vartriangle \left|0,\frac{n-1}{2}\right| \qquad \text{and} \qquad K\left(\frac{m}{2},0\right) \triangledown K\left(0,\frac{n-1}{2}\right).$$

Thus we have shown that $m > n$ if and only if $f(m) \triangledown f(n)$.

The operations $\oplus$, of addition and $\otimes$, of multiplication are defined on B as follows:

$$K(a,b) \oplus K(c,d) = \begin{cases} K(a+c,\, b+d) & \text{if } m,n \text{ are even} \\ K(b+d+1,\, a+c) & \text{if } m,n \text{ are odd} \\ K(b+c,\, a+d) & \text{if } m \text{ is even, } n \text{ odd} \\ K(a+d,\, b+c) & \text{if } m \text{ is odd, } n \text{ even} \end{cases} \tag{8}$$

$$K(a,b) \otimes K(c,d) = \begin{cases} K(2(a-b)(c-d),0) & \text{if } m,n \text{ are even} \\ K(c,d+2(a-b)(c-d)+b-a) & \text{if } m,n \text{ odd} \\ K(a+2(a-b)(d-c),b) & \text{if } m \text{ is even, } n \text{ odd} \\ K(c+2(a-b)(d-c),d) & \text{if } m \text{ is odd, } n \text{ even} \end{cases}$$

where m,n are the positive integers corresponding to (a,b) and (c,d), respectively in (4).

It is easy to show that

$$f(m+n) = f(m) \oplus f(n) \qquad \text{and} \qquad f(mn) = f(m) \otimes f(n).$$

A treatment similar to that above for arithmetic and geometric progressions can be found in [1].

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A FIBONACCI CURIOSITY

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In the Fibonacci sequence $F_0 = 0, F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$,

the sum of the digits of $F_0 = 0$
" " " " " " $F_1 = 1$
" " " " " " $F_5 = 5$
" " " " " " $F_{10} = 10$
" " " " " " $F_{31} = 31$
" " " " " " $F_{35} = 35$
" " " " " " $F_{62} = 62$
" " " " " " $F_{72} = 72$

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ON CONTINUED FRACTION EXPANSIONS WHOSE ELEMENTS ARE ALL ONES

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I. EVEN PERIOD EXPANSIONS

1. NUMBER THEORY REVIEW. Here is an example of an even continued fraction expansion of $\sqrt{D}$, D a non-square integer, with $D = 13$.

$$\sqrt{13} = 3 + \sqrt{13} - 3 = 3 + \frac{\sqrt{13}+3}{4}$$
$$\frac{\sqrt{13}+3}{4} = 1 + \frac{\sqrt{13}-1}{4} = 1 + \frac{\sqrt{13}+1}{3}$$
$$\frac{\sqrt{13}+1}{3} = 1 + \frac{\sqrt{13}-2}{3} = 1 + \frac{\sqrt{13}+2}{3}$$
$$\frac{\sqrt{13}+2}{3} = 1 + \frac{\sqrt{13}-1}{3} = 1 + \frac{\sqrt{13}+1}{4}$$
$$\frac{\sqrt{13}+1}{4} = 1 + \frac{\sqrt{13}-3}{4} = 1 + \frac{\sqrt{13}+1}{1}$$

Hence $\sqrt{13} = \langle 3, 1, 1, 1, 1, 6 \rangle$ and the solution of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

continued fraction elements c_i	3	1	1	1	1	6
signed denominators d_i	−4	3	−3	4	−1	
p convergents p_i	3	$\underline{4}$	$\underline{7}$	11	18	
q convergents q_i	1	$\underline{1}$	$\underline{2}$	3	5	

The q convergents are the Fibonacci numbers. The primitive solution of $x^2 - 13y^2 = -1$ is picked up from the half period. Thus

$$y = 1^2 + 2^2 = 5; \qquad x = 4 \times 1 + 7 \times 2 = 18.$$

In general for period $2r$,

$$y = q_r^2 + q_{r-1}^2 = q_{2r-1}; \qquad x = p_{r-1}q_{r-1} + p_r q_r = q_{2r-1}\,.$$

Also the representation of D as the sum of two squares can be found as

$$D = d_r^2 + (D - d_r^2) = d_r^2 + t^2\,,$$

where d_r is the middle denominator. Thus $13 = 3^2 + 2^2$. Finally for $D = 5$ (modulo 8), since a signed denominator is ± 4, the convergents under the -4 column are the coefficients of the cubic root of unity

$$\frac{3+\sqrt{13}}{2}$$

in the field $(1, \sqrt{13})$.

Since the period is even the x_0 of the quadratic congruence $x_0^2 \equiv -1 \pmod{13}$ is given by $x_0 \equiv x \equiv 18 \equiv 5$ (modulo 13).

2. FIBONACCI RELATIONS TO BE USED.

(a) $$(F_n, F_{n+1}) = 1.$$

(b) $$F_{2n}^2 + 1 = F_{2n-1}F_{2n+1}$$

(c) $$F_n^2 + F_{n+1}^2 = F_{2n+1}\,.$$

It may be noted that no odd Fibonacci number is ever divisible by a prime of the form $p = 4s + 3$ since from (b) $x^2 \equiv -1 \pmod{p}$ which is impossible.

3. EVEN VARIABLE DIFFERENCE TABLE: $D = m^2 + k$

$$\begin{array}{cccccc}
 & & & \overset{2n \text{ ones}}{} & \overset{n>1}{} & \\
m & 1 & 1 \cdots\cdots\cdots & 1 \cdots\cdots & 1 & 2m \\
-k & \multicolumn{3}{c}{\rule{8em}{0.4pt}} & -1 & \\
m & m+1 & 2m+1 & & mF_{2n+1} + F_{2n} & \\
1 & 1 & 2 & & F_{2n+1} &
\end{array}$$

The supposition $(mF_{2n+1} + F_{2n})^2 - F_{2n+1}^2(m^2 + k) = -1$ leads to

$$2mF_{2n}F_{2n+1} + F_{2n}^2 - kF_{2n+1}^2 = -1$$

$$2mF_{2n}F_{2n+1} - kF_{2n+1}^2 = -(F_{2n}^2 + 1) = -F_{2n-1}F_{2n+1}$$

$$2mF_{2n} - kF_{2n+1} = F_{2n-1}$$

Recalling that $(F_n, F_{n+1}) = 1$ and that F_{3n} is always even this linear diophantine equation will have an infinite number of positive integer solutions for m and k unless $2n + 1 \equiv 0 \pmod{3}$.

Example. $$D = m^2 + k, \qquad \sqrt{D} = \langle m, 1, 1, 1, 1, 1, 1, 2m \rangle$$

$$(13m + 8)^2 - 169(m^2 + k) = -1$$

$$16m - 13k = -5, \qquad k = m + \frac{3m+5}{13}$$

$$m = 7, \quad k = 7 + 2 = 9, \quad D = 58, \quad \sqrt{58} = \langle 7, 1, 1, 1, 1, 1, 1, 14 \rangle, \quad x^2 - 58y = -1$$

has primitive solution

$$x = 13m + 8 = 99, \quad y = 13.$$

$$m = 13 + 7 = 20, \quad k = 20 + 5 = 25, \quad D = 425, \quad \sqrt{425} = \langle 20, 1, 1, 1, 1, 1, 1, 40 \rangle, \quad x^2 - 425y^2 = -1$$

has primitive solution

$$x = 13m + 8 = 268, \quad y = 13.$$

In general if

$$D = 169m^2 - 140m + 29, \qquad \sqrt{D} = \langle 13m - 6, 1, 1, 1, 1, 1, 1, 26m - 12 \rangle$$

and the primitive solution of $x^2 - Dy^2 = -1$ is given by $x = 169m - 70$, $y = 13$.

II. ODD PERIOD EXPANSIONS

4. NUMBER THEORY REVIEW. Let $D = 135$

$$\sqrt{135} = 11 + \sqrt{135} - 11 = 11 + \frac{\sqrt{135} + 11}{14}$$

$$\frac{\sqrt{135} + 11}{14} = 1 + \frac{\sqrt{135} - 3}{14} = 1 + \frac{\sqrt{135} + 3}{9}$$

$$\frac{\sqrt{135} + 3}{9} = 1 + \frac{\sqrt{135} - 6}{9} = 1 + \frac{\sqrt{135} + 6}{11}$$

$$\frac{\sqrt{135} + 6}{11} = 1 + \frac{\sqrt{135} - 5}{11} = 1 + \frac{\sqrt{135} + 5}{10}$$

[continued on next page.]

$$\frac{\sqrt{135}+5}{10} = 1 + \frac{\sqrt{135}-5}{10} = 1 + \frac{\sqrt{135}+5}{11}$$

$$\frac{\sqrt{135}+5}{11} = 1 + \frac{\sqrt{135}-6}{11} = 1 + \frac{\sqrt{135}+6}{9}$$

$$\frac{\sqrt{135}+6}{9} = 1 + \frac{\sqrt{135}-3}{9} = 1 + \frac{\sqrt{135}+3}{14}$$

$$\frac{\sqrt{135}+3}{14} = 1 + \frac{\sqrt{135}-11}{14} = 1 + \sqrt{135}+11$$

$$\sqrt{135}+11 = 22$$

$$\sqrt{135} = \langle 11, 1, 1, 1, 1, 1, 1, 1, 22 \rangle \quad .$$

The solutions of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

c. f. elements	c_i	11	1	1	1	1	1	1	1	22
signed denominators	d_i	−14	9	−11	10	−11	9	−14	1	
p convergents	p_i	11	12	23	35	58	93	151	244	
q convergents	q_i	1	1	2	3	5	8	13	21	

The primitive solution of $x^2 - 135y^2 = 1$ is given by $x = p_8 = 244$, $y = q_8 = 21$. It can also be picked up from the half period. If the period is $2r+1$, $y = (q_r + q_{r-2})q_{r-1}$. Here

$$y = 3(2+5) = 21,$$

$$x = q_{r-1}p_{r-2} + q_r p_{r-1}.$$

Here $x = 3 \times 23 + 5 \times 35 = 244$.

5. FIBONACCI IDENTITIES TO BE USED.

(a) $$(F_{r-2}+F_r)F_{r-1} = F_{2r-2}$$

(b) $$F_{2n-1}^2 - 1 = F_{2n}F_{2n-2}$$

6. ODD VARIABLE DIFFERENCE TABLE : $D = m^2 + k$

$$\begin{array}{lllll} m & 1 & \overset{2r-1 \text{ ones}}{\cdots\cdots\cdots} & 1 & 2m \\ -k & & \text{———————} & 1 & \\ m & m+1 & 2m+1 & mF_{2r}+F_{2r-1} & \\ 1 & 1 & 2 & F_{2r} & \end{array}$$

The supposition $(mF_{2r}+F_{2r-1})^2 - F_{2r}^2(m+k) = 1$ leads to

$$2mF_{2r}F_{2r-1} + F_{2r-1}^2 - kF_{2r}^2 = 1$$

$$2mF_{2r}F_{2r-1} - F_{2r}^2 k = -(F_{2r-1}^2 - 1) = -F_{2r}F_{2r-2}$$

$$2mF_{2r-1} - kF_{2r} = -F_{2r-2}$$

Since $(F_{2r}, F_{2r-1}) = 1$, this linear diophantine equation will have an infinite number of positive integer solutions unless r is a multiple of 3. When $r = 3t$, F_{2r} is even, but F_{2r-2} is odd.

Example: $D = m^2 + k$, $\sqrt{D} = \langle m, 1, 1, 1, 2m \rangle$ $(3m+2)^2 - 9(m^2+k) = 1$

$$4m - 3k = -1, \quad k = m + \frac{m+1}{3}$$

$$m = 2, \quad k = 3, \quad D = 7, \quad \sqrt{7} = \langle 2, 1, 1, 1, 4 \rangle.$$

$x^2 - 7y^2 = 1$ has solution $x = 3 \times 2 + 2 = 8 \quad y = 3.$

Since $m = 2+3 = 5, \quad k = 5+2 = 7, \quad D = 32$ follows from $k = m + \dfrac{m+1}{3}$.

$x^2 - 32y^2 = 1$ has primitive solution $x = 3 \times 5 + 2 = 17, \quad y = 3.$ In general,

$$D = 9m^2 - 2m, \quad \sqrt{D} = \langle 3m-1, 1, 1, 1, 1, 6m-2 \rangle .$$

The primitive solution of $x^2 - Dy^2 = 1$ is tiven by $x = 9m - 1, \quad y = 3.$

7. $D = m^2 + k, \quad 2mF_r - kF_{r+1} = -F_{r-1}$

$$\sqrt{D} = m + \sqrt{D} - m = m + \frac{\sqrt{D}+m}{k}$$

$$\frac{\sqrt{D}+m}{k} = 1 + \frac{\sqrt{D}-(k-m)}{k} = 1 + \frac{\sqrt{D}+k-m}{2m+1-k}$$

$$\frac{\sqrt{D}+k-m}{2m+1-k} = 1 + \frac{\sqrt{D}-(3m+1-2k)}{2m+1-k} = 1 + \frac{\sqrt{D}+3m+1-2k}{4k-4m-1}$$

$$\frac{\sqrt{D}+3m+1-2k}{4k-4m-1} = 1 + \frac{\sqrt{D}-(6k-7m-2)}{4k-4m+1} = 1 + \frac{\sqrt{D}+6k-7m-2}{12m-9k+4}$$

$$\frac{\sqrt{D}+F_sF_{s-1}k - (1+2F_1F_2+\cdots+2F_{s-2}F_{s-1})m - (F_1^2+F_2^2+\cdots+F_{s-2}^2)}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

$$= 1 + \frac{\sqrt{D} - \overbrace{[(1+2F_1F_2+\cdots+2F_{s-1}F_s\, m) - F_sF_{s+1}k + (F_1^2F_2^2+\cdots+F_{s-1})]}^{\text{(A)}}}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

$$= 1 + \frac{D + \text{(A)}}{kF_{s+1}^2 - 2mF_sF_{s+1} - F_s^2} \quad .$$

For this last assumption to be valid,

$$(2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2)(kF_{s+1}^2 - 2mF_{s+1}F_s - F_s^2) \equiv m^2 + k - \text{(A)}^2 .$$

This identity will be proved by equating coefficients:

1. Coefficient of $-m^2$

$$4F_s^2F_{s-1}F_{s+1} = 4F_s^2[F_s^2+(-1)^s] = 4F_s^4 + 4(-1)^sF_s^2 = \frac{4}{25}(L_{4s}+L_{2s}-4) = [F_{s+2}F_s - F_{s+1}F_{s-2}]^2 - 1 .$$

2. Coefficient of $-k^2$

$$F_s^2\,F_{s+1}^2 = F_s^2F_{s+1}^2 .$$

3. Constant term:

$$-F_s^2F_{s-1}^2 = -(F_1^2+F_2^2+\cdots+F_{s-1}^2)^2 .$$

4. Coefficient of $2mk$

$$F_{s-1}F_sF_{s+1}^2 + F_s^3F_{s+1} = F_sF_{s+1}(F_{s-1}F_{s+1}+F_s^2) = [2L_{2s}+(-1)^s]F_sF_{s+1}$$

$$F_sF_{s+1}(1+2F_1F_2+\cdots+2F_{s-1}F_s = F_sF_{s+1}(F_{s+2}F_s - F_{s+1}F_{s-2}) = [2L_{2s}+(-1)^s].F_sF_{s+1}$$

5. Coefficient of k.

$$2F_sF_{s+1}(F_1^2+F_2^2+\cdots+F_{s-1}^2)+1 = 2F_s^2F_{s-1}F_{s+1}+1 = 1+2F_s^2[F_s^2+(-1)^s] = 2F_s^4+2F_s^2(-1)^s+1$$

$$F_{s-1}^2F_{s+1}^2 + F_s^4 = F_s^4 + [F_s^2+(-1)^s]^2 = 2F_s^4 + 2(-1)^sF_s^2 + 1 .$$

6. Coefficient of $-2m$

$$F_s^3F_{s-1}+F_{s-1}^2F_sF_{s+1} = F_{s-1}F_s[F_s^2+F_{s-1}F_{s+1}] = F_{s-1}F_s[F_s(F_{s+2}-F_{s+1})+F_{s-1}F_{s+1}]$$

$$= F_{s-1}F_s[F_sF_{s+2}-F_{s+1}(F_s-F_{s-1})] = F_{s-1}F_s(F_sF_{s+2}-F_{s+1}F_{s-2}).$$

$$(F_1^2+F_2^2+\cdots+F_{s-1}^2)(1+2F_1F_2+2F_sF_3+\cdots+2F_{s-1}F_s = F_{s-1}F_s[F_sF_{s+2}-F_{s+1}F_{s-2}]$$

In proving this identity the following Fibonacci identities were used:

(a) $$1+2F_1F_2+\cdots+2F_{s-1}F_s = F_sF_{s+2}-F_{s+1}F_{s-2}$$

(b) $$F_1^2+F_2^2+\cdots+F_s^2 = F_{s-1}F_s$$

(c) $$F_{s-1}F_{s+1} = F_s^2+(-1)^s\;.$$

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A MORE GENERAL FIBONACCI MULTIGRADE

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In a recent article I gave examples of multigrades based on Fibonacci series in which

$$F_{n+2} = F_{n+1}+F_n\;.$$

Here I first give a more general multigrade for series in which

$$F_{n+2} = yF_{n+1}+xF_n\,.$$

Consider

1 3 7 17 47 (where $x = 1, y = 2$).

By inspection we notice that

$$1^m+3^m+3^m+7^m = 0^m+4^m+4^m+6^m$$

$$3^m+7^m+7^m+17^m = 0^m+10^m+10^m+14^m\;,\text{ etc.}$$

(where $m = 1, 2$).

We can look at other series of a like kind:

1 3 10 33 109 (where $x = 1,\ y = 3$).

Here

$$1^m+3^m+3^m+3^m+10^m+10^m = 0^m+0^m+7^m+7^m+7^m+9^m$$

$$3^m+10^m+10^m+10^m+33^m+33^m = 0^m+0^m+23^m+23^m+23^m+30^m,\text{ etc.}$$

(where $m = 1, 2$)

1 3 11 39 139 (where $x = 2,\ y = 3$).

Here

$$1^m+1^m+3^m+3^m+3^m+11^m+11^m+11^m = 0^m+0^m+0^m+8^m+8^m+8^m+10^m+10^m$$

$$3^m+3^m+11^m+11^m+11^m+39^m+39^m+39^m = 0^m+0^m+0^m+28^m+28^m+28^m+36^m+36^m,\text{ etc.}$$

(where $m = 1, 2$)

The general series

$a \quad b \quad ax+by \quad bx+axy+by^2$

gives

$$x(a)^m+y(b)^m+(x+y-2)(ax+by)^m = (x+y-2)0^m+y(ax+by-b)^m+x(ax+by-a)^m$$

(where $m = 1, 2$).

Continued on page 66.

ON CONGRUENCE MODULO A POWER OF A PRIME

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A problem which appears in many textbooks in number theory, e.g. [1], is the following:
If $a^p \equiv b^p \pmod p$, then $a^p \equiv b^p \pmod{p^2}$.

In this paper this result will be generalized to higher powers of the prime p. Also, there will be a generalization to a composite modulus.

Lemma 1. If n is a positive integer for which $a^{p^n} \equiv b^{p^n} \pmod p$, then $a \equiv b \pmod p$.

Proof. Let $n = 1$; then, $a \equiv a^p \equiv b^p \equiv b \pmod p$ by Fermat's Theorem. Suppose that $a^{p^k} \equiv b^{p^k} \pmod p$ implies that $a \equiv b \pmod p$. If $a^{p^{k+1}} \equiv b^{p^{k+1}} \pmod p$, then $(a^{p^k})^p \equiv (b^{p^k})^p \pmod p$. Hence,

$$a^{p^k} \equiv (a^{p^k})^p \equiv (b^{p^k})^p \equiv b^{p^k} \pmod p$$

by Fermat's Theorem. By the induction hypothesis, $a \equiv b \pmod p$.

Lemma 2. If $a^{p^n} \equiv b^{p^n} \pmod{p^n}$, then

$$p^n \mid (a^{p^n-1} + a^{p^n-2}b + \cdots + b^{p^n-1}).$$

Proof. By Lemma 1, $p \mid (a-b)$, and, thus, $a = b + tp$ for some integer t. Then, with $d = p^n$,

$$\begin{aligned} a &= b + (tp) \\ a^2 &= b^2 + 2b(tp) + (tp)^2 \\ a^3 &= b^3 + \cdots + (tp)^3 \\ &\vdots \\ a^{d-1} &= b^{d-1} + (d-1)b^{d-2}(tp) + \cdots + (tp)^{d-1}. \end{aligned}$$

By multiplying the i^{th} row by b^{d-i-1}, we obtain:

$$\begin{aligned} b^{d-1} &= b^{d-1} \\ ab^{d-2} &= b^{d-1} + b^{d-2}(tp) \\ a^2b^{d-3} &= b^{d-1} + 2b^{d-2}(tp) + b^{d-3}(tp)^2 \\ &\vdots \\ a^{d-2}b &= b^{d-1} + (d-2)b^{d-2}(tp) + \cdots + b(tp)^{d-2} \\ a^{d-1} &= b^{d-1} + (d-1)b^{d-2}(tp) + \cdots + (tp)^{d-1}. \end{aligned}$$

The coefficient of $b^{d-k}(tp)^{k-1}$ in the expansion $a^{d-1} + a^{d-2}b + \cdots + b^{d-1}$ is

$$\sum_{i=k-1}^{d-1} \binom{i}{k-1}.$$

Using the identity

$$\binom{b+1}{a} = \binom{b}{a} + \binom{b}{a-1},$$

rewritten as

$$\binom{b}{a-1} = \binom{b+1}{a} - \binom{b}{a},$$

we have

$$\sum_{i=k-1}^{d-1} \binom{i}{k-1} = \sum_{i=k-1}^{d-1} \binom{i+1}{k} - \sum_{i=k-1}^{d-1} \binom{i}{k} = \binom{d}{k} + \sum_{i=k-1}^{d-2} \binom{i+1}{k} - \sum_{i=k}^{d-1} \binom{i}{k} - \binom{k-1}{k}$$

$$= \binom{d}{k} + \sum_{i=k}^{d-1} \binom{i}{k} - \sum_{i=k}^{d-1} \binom{i}{k} - 0 = \binom{d}{k}.$$

This implies that the k^{th} term of $a^{d-1} + \cdots + b^{d-1}$ expressed as a polynomial in (tp) is $(dr/k)b^{d-k}(tp)^{k-1}$, where

$$r = \binom{d-1}{k-1}.$$

If $(p,k) = 1$, then

$$p^n \mid (dr/k)b^{d-k}(tp)^{k-1}$$

since $p^n \mid d$. Suppose that $(p,k) \neq 1$; then, $k = p^m g$, where $m \neq 0$ and $p \nmid g$. To show that $m \leqslant k-1$, suppose to the contrary that $m > k-1$, i.e., $m \geqslant k$. Since $p > 1$, $p^m > m$. Hence, $p^m > m \geqslant k$, a contradiction. Thus, $m < k-1$, and

$$(dr/k)\,b^{d-k}(tp)^{k-1} = (dr/g)\,b^{d-k}t^{k-1}p^{k-m-1},$$

where p^{k-m-1} is integral. Since $p \nmid g$,

$$p^n \mid (dr/g)b^{d-k}t^{k-1}p^{k-m-1}.$$

Therefore, p^n divides each term of $a^{d-1} + \cdots + b^{d-1}$ expressed as a polynomial in (tp). The conclusion follows.

The next lemma is a generalization of the problem mentioned at the beginning of this paper.

Lemma 3. If $a^{p^n} \equiv b^{p^n} \pmod{p^n}$, then $a^{p^n} \equiv b^{p^n} \pmod{p^{n+1}}$.

Proof. Let $d = p^n$; then, by Lemma 1, $p \mid (a-b)$, and by Lemma 2, $p^n \mid (a^{d-1} + \cdots + b^{d-1})$. This implies that

$$p^{n+1} \mid (a-b)(a^{d-1} + \cdots + b^{d-1}),$$

i.e., $p^{n+1} \mid (a^{p^n} - b^{p^n})$.

Theorem. If $a^m \equiv b^m \pmod{m}$, then $a^m \equiv b^m \pmod{m \cdot p_1 p_2 \cdots p_r}$, where $p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ is the canonical factorization of m.

Proof. Let $q = m/p_i^{n_i}$; then

$$(a^q)^{p_i^{n_i}} \equiv a^m \equiv b^m \equiv (b^q)^{p_i^{n_i}} \pmod{p_i^{n_i}}.$$

By Lemma 3, $a^m \equiv b^m \pmod{p_i^{n_i+1}}$. The conclusion follows since the p_i are relatively prime.

The following example shows that in general the modulus in Lemma 3 and in the Theorem cannot be increased any more.

Example: $7^9 \equiv 1^9 \pmod{9}$ implies that $7^9 \equiv 1^9 \pmod{27}$, but $7^9 \not\equiv 1^9 \pmod{81}$.

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FIBONACCI RATIO IN ELECTRIC WAVE FILTERS

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In the classical theory of electric wave filters, a complete filter is composed of a series of sections in cascade or tandem, terminated at each end by a terminal half-section. Sections can be of T or π circuit configuration (sometimes called "mid-shunt" and "mid-series" sections).

Aside from the frequency selective properties of the filter, an important design requirement is that the image impedances at the input and output terminals shall be as nearly constant as possible throughout the greater part of the pass-band. For this reason the terminating half-sections are usually of the "m-derived" type, where m is a design parameter that can be chosen as anything in the interval between zero and unity, but is usually about 0.6 to satisfy the requirement of constant image impedance in the pass-band. This is the primary function of the terminating half-sections. Figure 1 shows a family of curves giving the variations in image impedance throughout the pass-band for various values of m. Notice that the curve for $m = 0.6$ results in a variation of the ordinate within $\pm 5\%$ over 75% of the pass-band, and provides a good approximation to constant image impedance over the useful range of frequency.

To demonstrate that the natural value for m is the inverse Fibonacci Ratio, $1/\phi$, where

$$\frac{1}{\phi} = \frac{\sqrt{5}-1}{2} = 0.618\,;$$

we use a low-pass T section as an example. Figure 2 shows a circuit diagram of such a mid-shunt section, together with a sketch of its frequency selective characteristic.

The salient features of the response function are the two frequencies f_c = cutoff and f_∞ = the frequency for infinite attenuation.

Letting the ratio $(f_c/f_\infty) = r$; the relation between m and r is the equation of a circle:

$$m^2 + r^2 = 1,$$

where m and r are both restricted to non-negative real values.

The design formulas are:

$$L_1 = mL_K \qquad\qquad L_K = \frac{R}{\pi f_c}$$

$$L_2 = \frac{1-m^2}{m}\,\frac{L_K}{4}$$

$$C_2 = mC_K \qquad\qquad C_K = \frac{1}{\pi R f_c}\,,$$

where R is the load resistance at the terminals and therefore the desired image impedance of the terminating half-section. The circuit diagram of the m-derived half-section is shown in Fig. 3.

We notice that the coefficient involving m of the midshunt inductance L_2 is: $(1 - m^2)/m$ and letting this coefficient equal unity gives:

$$1 - m^2 = m, \qquad m^2 + m - 1 = 0,$$

from which the positive real root is:

$$m = \frac{-1+\sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2} = 0.618 = \frac{1}{\phi}\,.$$

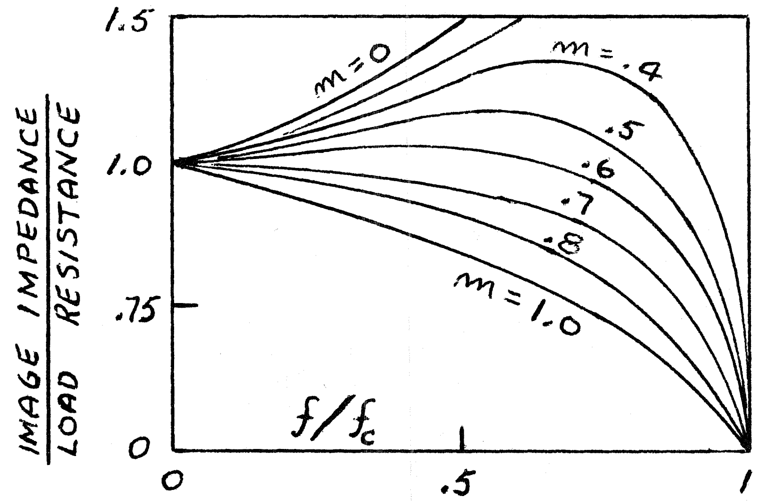

FIG. 1
IMAGE IMPEDANCE VS. FREQUENCY

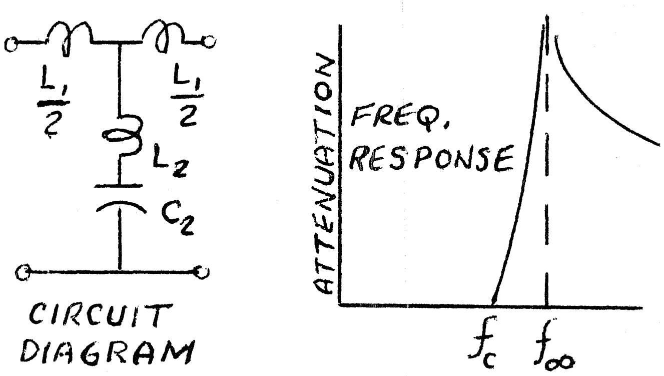

FIG 2: LOW-PASS T-SECTION

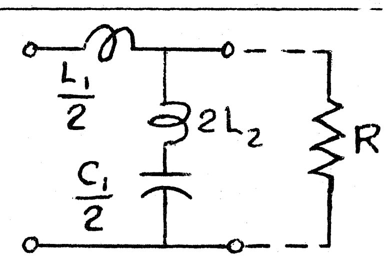

FIG. 3
CIRCUIT DIAGRAM OF m-DERIVED HALF-SECTION

Then:

$$r^2 = 1 - m^2 = 1 - \frac{1}{\phi^2} = 0.618 = \frac{1}{\phi} \quad , \quad \frac{1}{r} = \sqrt{\phi} = 1.272$$

This is substantially in agreement with a design rule that the frequency at infinite attenuation should be about 25% higher than the cutoff frequency.

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ON A THEOREM OF KRONECKER

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Consider the r^{th} order homogeneous linear recursion

(1) $$u_{n+r} = a_1u_{n+r-1} + \cdots + a_ru_n,\ a_r \neq 0,$$

over a field F of characteristic $p > 0$. Let

$$V = \left\{ v_n \right\}_o^\infty \subseteq F$$

be a non-trivial solution of the recursion (1) and let

$$f(x) \equiv x^r - a_1x^{r-1} - \cdots - a_r = \prod_{i=1}^{r} (x - r_i)$$

be factored completely in its splitting field K where $F \subseteq K$. The results which follow remain valid if F is also the complex field. The polynomial $f(x)$ is called a characteristic polynomial for the sequence V. If $\phi(n)$ is a sequence in K defined on the non-negative integers I then define the operator E by $E\phi(n) \equiv \phi(n+1)$ for $n \in I$. Recursion (1) may therefore be written as

(2) $$f(E)u_n \equiv 0.$$

The sequence V is said to satisfy a recursion of lower order if there exists a monic polynomial $g(x)$ over F such that $\deg g(x) < \deg f(x) = r$ and $g(E)v_n \equiv 0$. There exists a unique monic polynomial of lowest degree which is a characteristic polynomial for V, called the minimum polynomial for V [2]. The determination of the lowest order recursion that a given solution of (2) satisfies is an essential step in the study of the periodicity properties of such solutions. Define

(3) $$D(n) = \det \begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & v_{n+2} & & \\ \vdots & & \ddots & \vdots \\ v_{n+r-1} & \cdots & & v_{n+2r-2} \end{bmatrix},\quad n \in I.$$

The purpose of this note is to present a new proof of a classic theorem of Kronecker [1, p. 199] which does not depend on the notion of a fundamental solution set for (2). To this end Lemma 2 gives an explicit calculation of the values of $D(n)$.

Theorem 1. (Kronecker) The solution V of (2) satisfies a recursion of lower order if and only if $D(0) = 0$.

First define the polynomials

(4) $$f_k(x) = \prod_{i \neq k} (x - r_i), \qquad 1 \leq k \leq r.$$

We have

Lemma 1. $$f_k(E)v_n = r_k^n\,[f_k(E)v_o], \qquad n \in I.$$

Proof. Note that $f(x) = (x - r_k)f_k(x)$ and since polynomials in E commute as operators we have

(5) $$r_kf_k(E)v_n = E[f_k(E)v_n] = f_k(E)v_{n+1}.$$

The result follows from a repeated application of (5). Q.E.D.

Corollary 1. If $f_k(E)v_0 = 0$ then $f_k(E)v_n = 0,\ n \in I.$

The main result of this note is

Lemma 2. $$D(n) = (-1)^t \prod_{i=1}^{r} [f_i(E)v_n], \qquad t = r(r-1)/2, \quad n \in I.$$

Proof. (Induct on the order r of the recursion) If $r = 2$ then

$$D(n) = \det\begin{bmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2}\end{bmatrix} = v_nv_{n+2} - v_{n+1}^2 = -v_{n+1}^2 + v_n[(r_1+r_2)v_{n+1} - r_1r_2v_n]$$
$$= -(v_{n+1} - r_1v_n)(v_{n+1} - r_2v_n) = -[f_2(E)v_n][f_1(E)v_n].$$

Therefore the lemma is true for $r = 2$. Assume the lemma true for all recursions of order less than $r > 2$. Since

$$f_1(E)v_n = v_{n+r-1} + \sum_{i=2}^{r} c_iv_{n+r-i}$$

for some $c_i \in K$, we have that c_i times the $r+1-i$ row of $D(n)$ added to the r^{th} row for $2 \leqslant i \leqslant r$ gives

$$D(n) = \det\begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & & & \\ \vdots & \vdots & & \vdots \\ v_{n+r-2} & & & \\ f_1(E)v_n & f_1(E)v_{n+1} & \cdots & f_1(E)v_{n+r-1}\end{bmatrix}$$

which, by Lemma 1, gives

$$D(n) = [f_1(E)v_n]\det\begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & & & \\ \vdots & \vdots & & \vdots \\ v_{n+r-2} & & & \\ 1 & r_1 & & r_1^{r-1}\end{bmatrix}.$$

Multiplying column i by $-r_1$ and adding to column $i+1$ for $1 \leqslant i \leqslant r-1$, we have

$$D(n) = (-1)^{r-1}[f_1(E)v_n]\det\begin{bmatrix} w_n & \cdots & w_{n+r-2} \\ w_{n+1} & \ddots & \vdots \\ \vdots & & \\ w_{n+r-2} & \cdots & w_{n+2r-4}\end{bmatrix} \tag{6}$$

for

$$w_n \equiv v_{n+1} - r_1v_n, \qquad n \in I,$$

where the matrix appearing in (6) is $r-1$ square. Note that

$$f_1(E)w_n = f(E)v_n = 0, \qquad n \in I,$$

so that $f_1(x)$ is a characteristic polynomial for the sequence $\{w_n\}$. Let

$$g_k(x) \equiv \prod_{i \neq 1,k} (x - r_i), \qquad 2 \leqslant k \leqslant r.$$

Then, by the induction hypothesis, Eq. (6) becomes

$$D(n) = (-1)^{r-1}[f_1(E)v_n](-1)^{(r-1)(r-2)/2}\prod_{i=2}^{r}[g_i(E)w_n] = (-1)^{r(r-1)/2}[f_1(E)v_n]\prod_{i=2}^{r}[f_i(E)v_n].$$

Therefore mathematical induction yields the result. Q.E.D.

An immediate consequence of Corollary 1 and Lemma 2 is

Corollary 2. Either $D(n)$ is identically zero or never zero.

Zierler proves the following [2].

Lemma 3. Let $f(x)$ be a characteristic polynomial over the field F for the sequence

$$V = \left\{ v_n \right\} \subseteq F, \qquad V \neq 0,$$

and let $g(x)$ be the minimum polynomial for V. Then

(i) $$g(x) \mid f(x),$$

(ii) $h(x)g(x)$ is also a characteristic polynomial for V, where $h(x)$ is any monic polynomial over F.

To complete the proof of Theorem 1 we note that Lemma 3 implies that V satisfies a lower order recursion if and only if some $f_k(x)$ as defined in (4) is a characteristic polynomial for V. But then Lemma 2 and Corollary 2 imply that V satisfies a lower order recursion if and only if $D(0) = 0$.

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A FIBONACCI PLEASANTRY

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In the Fibonacci sequence $F_0 = 0, F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$, list the sums $F_n + n$ in ascending order of n and note the second differences. Do the same with $F_n - n$.

0 + 0 =	0			0 − 0 =	0		
		> 2				> 0	
1 + 1 =	2		> −1	1 − 1 =	0		> −1
		> 1				> −1	
1 + 2 =	3		> 1	1 − 2 =	−1		> 1
		> 2				> 0	
2 + 3 =	5		> 0	2 − 3 =	−1		> 0
		> 2				> 0	
3 + 4 =	7		> 1	3 − 4 =	−1		> 1
		> 3				> 1	
5 + 5 =	10		> 1	5 − 5 =	0		> 1
		> 4				> 2	
8 + 6 =	14		> 2	8 − 6 =	2		> 2
		> 6				> 4	
13 + 7 =	20		> 3	13 − 7 =	6		> 3
		> 9				> 7	
21 + 8 =	29		> 5	21 − 8 =	13		> 5
		> 14				> 12	
34 + 9 =	43		> 8	34 − 9 =	25		> 8
		> 22				> 20	
55 + 10 =	65		> 13	55 − 10 =	45		> 13

[Continued on page 41.]

COLUMN GENERATORS FOR COEFFICIENTS OF FIBONACCI AND FIBONACCI-RELATED POLYNOMIALS

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1. INTRODUCTION

Generating functions, row sums and rising diagonal sums for the Pascal triangle and types of Pascal triangles have been studied in [2] and [4]. Bicknell has pointed out in [1] that another Pascal-like array is observed if we consider the coefficients of the Fibonacci polynomials $F_n(t)$. These polynomials are such that

$$F_0(t) = 0, \quad F_1(t) = 1, \quad \text{and} \quad F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$$

for $n \geqslant 2$. The array is as follows:

Array 1

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8
0	0								
1	1								
2	0	1							
3	1	0	1						
4	0	2	0	1					
5	1	0	3	0	1				
6	0	3	0	4	0	1			
7	1	0	6	0	5	0	1		
8	0	4	0	10	0	6	0	1	
9	1	0	10	0	15	0	7	0	1

				...	
$\frac{x}{1-x^2}$	$\frac{x^2}{(1-x^2)^2}$	$\frac{x^3}{(1-x^2)^3}$	$\frac{x^4}{(1-x^2)^4}$		Column Generators
0^{th}	1^{st}	2^{nd}	3^{rd}		Column

Since the generating function for the zeroth column is $f(x) = x/(1-x^2)$ and since each nonzero a_{ij} has the Pascal-like property

$$a_{ij} = \sum_{k=0}^{i-1} a_{k,j-1}$$

for all i and j such that $i > j \geqslant 1$, then techniques similar to those in Theorem 1 of [4] can be used to show that the generating function for the k^{th} column ($k = 0, 1, 2, \cdots$) is

$$g_k(x) = f(x)[x/(1-x^2)]^k.$$

Moreover, the generating function for the row sums of this array is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x}{1-x^2} \right)^k = f(x) \frac{1-x^2}{1-x-x^2} = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n$$

as was to be expected. Again, employing results essentially the same as those in [2] and [4], the generating function for the rising diagonals of this array is

$$D(x) = \sum_{k=0}^{\infty} x^k g_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x^2}\right)^k = \left(\frac{x}{1-x^2}\right)\left(\frac{1-x^2}{1-2x^2}\right) = \frac{x}{1-2x^2} = \sum_{n=0}^{\infty} 2^n x^{2n+1}.$$

2. GENERATING FUNCTIONS FOR COEFFICIENTS OF $F_n'(t)$

Now we consider the array for $F_n'(t)$, the first derivative of each Fibonacci polynomial. It will be noted that this array is quite similar to the array suggested by Hoggatt in problem H-131 of this *Quarterly* [5]. In that problem it is required to show that sums, C_n, of the rising diagonals are given by $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j}F_j .$$

If we appropriately relabel the columns in that array, this is the same as showing

$$C_n = \sum_{j=0}^{n} F_{n-j}F_j$$

for $n = 0, 1, 2, \cdots$. Since the rising diagonal sums of that array are the same as the row sums of the array for the $F_n'(t)$ and since we can find the column generators for the array below, we can employ techniques similar to those used in the previous section to answer problem H-131. For consider:

Array 2

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7
0	0							
1	0							
2	1							
3	0	2						
4	2	0	3					
5	0	6	0	4				
6	3	0	12	0	5			
7	0	12	0	20	0	6		
8	4	0	30	0	30	0	7	
9	0	20	0	30	0	42	0	8
				...				
	$\frac{x^2}{(1-x^2)^2}$	$\frac{2x^3}{(1-x^2)^3}$	$\frac{3x^4}{(1-x^2)^4}$	$\frac{4x^5}{(1-x^2)^5}$		Column Generators		
	0^{th}	1^{st}	2^{nd}	3^{rd}		Column		

Denoting the generator of the zeroth column as $p(x)$, the column generator for the k^{th} column is given by

$$d_k(x) = p(x)(k+1)\left(\frac{x}{1-x^2}\right)^k$$

for $k = 0, 1, 2, \cdots$. The generating function for the row sums is given by

$$G(x) = \sum_{k=0}^{\infty} d_k(x) = p(x) \sum_{k=0}^{\infty} (k+1)\left(\frac{x}{1-x^2}\right)^k = p(x)\,\frac{1}{\left[1 - \frac{x}{1-x^2}\right]^2}$$

$$= \frac{x^2}{(1-x^2)} \cdot \frac{(1-x^2)^2}{(1-x-x^2)^2} = \left(\frac{x}{1-x-x^2}\right)^2$$

$$= \sum_{n=0}^{\infty} F_n(x) \cdot \sum_{n=0}^{\infty} F_n(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} F_{n-j}F_j \right) x^n \quad .$$

Since we have relabeled the zero$^{\text{th}}$ column, we have immediately that

$$C_n = \sum_{j=0}^{n} F_{n-j}F_j = F_n^{(1)}$$

for $n = 0, 1, 2, \cdots$, where $F_n^{(1)}$ represents the first Fibonacci convolution sequence [3].

3. GENERATING FUNCTIONS FOR THE COEFFICIENTS OF $I_n(t) = n \int_0^t F_n(t)dt$

The preceding suggests it would be in order to consider the array for

$$\int_0^t F_n(t)dt.$$

But this leads to an array containing fractions. To avoid this situation we consider the array for

$$I_n(t) = n \int_0^t F_n(t)dt$$

instead. This array now follows:

Array 3

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	0									
1	0	1								
2	0	0	1							
3	0	3	0	1						
4	0	0	4	0	1					
5	0	5	0	5	0	1				
6	0	0	9	0	6	0	1			
7	0	7	0	14	0	7	0	1		
8	0	0	16	0	20	0	8	0	1	
9	0	9	0	30	0	27	0	9	0	1

This array looks familiar since the array for the Lucas polynomials is Array 4 at the top of the next page.

It is easy to establish that

$$L_{2k-1}(t) = (2k-1)\int_0^t F_{2k-1}(t)dt = I_{2k-1}(t)$$

and

Array 4

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	2									
1	0	1								
2	2	0	1							
3	0	3	0	1						
4	2	0	4	0	1					
5	0	5	0	5	0	1				
6	2	0	9	0	6	0	1			
7	0	7	0	14	0	7	0	1		
8	2	0	16	0	20	0	8	0	1	
9	0	9	0	30	0	27	0	9	0	1

...

$$L_{2k}(t) = (2k)\int_0^t F_{2k}(t)dt + 2 = I_{2k}(t) + 2$$

for $k = 1, 2, 3, \cdots$; or if you prefer,

$$D_t[L_n(t)] = nF_n(t)$$

for $n = 1, 2, 3, \cdots$.

It is interesting to note that in each of the above arrays if we consider the left-most column as the zero$^{\text{th}}$ column, we do not obtain a Pascal-like triangle. However, if we consider the next column over as the zero$^{\text{th}}$ column, then we do have a Pascal-like array and the results of [4] are applicable.

Array 5

n	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	0								
1	1								
2	0	1							
3	3	0	1						
4	0	4	0	1					
5	5	0	5	0	1				
6	0	9	0	6	0	1			
7	7	0	14	0	7	0	1		
8	0	16	0	20	0	8	0	1	
9	9	0	30	0	27	0	9	0	1

...

$\frac{x(1+x^2)}{(1-x^2)^2}$	$\frac{x^2+x^4}{(1-x^2)^3}$	$\frac{x^3+x^5}{(1-x^2)^4}$	$\frac{x^4+x^6}{(1-x^2)^5}$	$\frac{x^5+x^7}{(1-x^2)^5}$	Column Generators
0^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}	Column

If we denote the generator of the 0^{th} column by $q(x)$, then the column generator for the k^{th} column $(k = 0,1,2,\cdots)$ is

$$h_k(x) = q(x)[x/(1-x^2)]^k .$$

The generating function for the row sums is

$$G(x) = \sum_{k=0}^{\infty} h_k(x) = q(x)\,\frac{1-x^2}{(1-x-x^2)} = (1+x^2)\left(\frac{x}{1-x^2}\right)\left(\frac{1}{1-x-x^2}\right) .$$

The generating function for the rising diagonals is

$$D(x) = \sum_{k=0}^{\infty} x^k h_k(x) = q(x) \cdot \frac{1-x^2}{1-2x^2} = (1+x^2)\left(\frac{x}{1-x^2}\right)\left(\frac{1}{1-2x^2}\right) \quad .$$

4. RELATIONSHIPS AMONG THE GENERATING FUNCTIONS

We now observe some relationships between the generating functions $g_k(x)$, $d_k(x)$ and $h_k(x)$. First

$$d_k(x) = (k+1)xg_k(x)$$

which was to have been anticipated in light of the connection between Array 2 and Problem H-131. However, the relationship between $h_k(x)$ and $g_k(x)$ is a little more surprising. Since Array 5 was obtained via an integration process, it might be felt that $h_k(x)$ should relate in some way to an integral of $g_k(x)$; but

$$h_k(x) = \frac{x}{(k+1)} g_k'(x)$$

which is easy to verify. This formula can be used to investigate some integral relationships however. Assuming each function is defined on $[o,x]$ and using an integration-by-parts formula we have

$$(k+1)\int_0^x h_k(x)dx + \int_o^x g_k(x)dx = xg_k(x).$$

Since $d_k(x) = (k+1)xg_k(x)$, we now have

$$d_k(x) = (k+1)^2 \int_o^x h_k(x)dx + (k+1)\int_o^x g_k(x)dx\,,$$

a formula involving all three generating functions for $k = 0, 1, 2, 3, \cdots$.

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A SUMMATION IDENTITY

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The purpose of this note is to generalize the following two well known formulas:

(1) $$\sum_{i=k}^{n}\sum_{j=k}^{n-i+k} = \sum_{j=k}^{n}\sum_{i=k}^{n-j+k}$$

(2) $$\sum_{i=k}^{n}\sum_{j=k}^{i} = \sum_{j=k}^{n}\sum_{i=j}^{n} \quad .$$

These are double summation operators, and the equality means that when either operator acts on an arbitrary doubly subscripted sequence, the same result is obtained.

To show how these formulas can be compounded, and to motivate the general result to follow, we offer the following example. Each equality is justified by (1), (2), or the fact that the operators may commute when the subscripts involved are independent of each other. Note that the use of the parenthesis is to indicate which pair of operators is being permuted, and is not to imply any sort of associative law.

$$\left(\sum_{i_1=k}^{n}\sum_{i_2=k}^{n-i_1+k}\right)\sum_{i_3=i_2}^{n}\sum_{i_4=k}^{i_3} = \sum_{i_2=k}^{n}\left(\sum_{i_1=k}^{n-i_2+k}\sum_{i_3=i_2}^{n}\right)\sum_{i_4=k}^{i_3} = \sum_{i_2=k}^{n}\sum_{i_3=i_2}^{n}\left(\sum_{i_1=k}^{n-i_2+k}\sum_{i_4=k}^{i_3}\right)$$

$$= \left(\sum_{i_2=k}^{n}\sum_{i_3=i_2}^{n}\right)\sum_{i_4=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} = \sum_{i_3=k}^{n}\left(\sum_{i_2=k}^{i_3}\sum_{i_4=k}^{i_3}\right)\sum_{i_1=k}^{n-i_2+k}$$

$$= \left(\sum_{i_3=k}^{n}\sum_{i_4=k}^{i_3}\right)\sum_{i_2=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} = \sum_{i_4=k}^{n}\sum_{i_3=i_4}^{n}\sum_{i_2=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} \quad .$$

If we examine the resulting equality of the first and last operators, we see that order is reversed with respect to the i_j's, i_4 replaces i_1 in the first factor, and the other factors are exchanged according to the schemes:

(3) $$\sum_{i=k}^{n-j+k} \longleftrightarrow \sum_{j=k}^{n-i+k}$$

(4) $$\sum_{i=j}^{n} \longleftrightarrow \sum_{j=k}^{i}$$

Theorem. Consider an operator of the form

$$\sum_{i_1=k}^{n}\prod_{j=2}^{t}\sum_{i_j=h_j}^{n_j} \; ,$$

where the pairs h_j, n_j are of the following types:

(5) $$h_j = k \quad \text{and} \quad n_j = n - i_{j-1} + k$$

(6) $$h_j = k \quad \text{and} \quad n_j = i_{j-1}$$

(7) $$h_j = i_{j-1} \quad \text{and} \quad n_j = n \;.$$

Then

$$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \sum_{i_t=k}^{n} \prod_{j=1}^{t-1} \sum_{i_{t-j}},$$

where each factor on the right (after the first) has been exchanged according to the scheme (3), (4).

Proof. Inductively, suppose that the theorem is true for $(t-1)$ factors. We have

(8) $$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \left(\sum_{i_1=k}^{n} \sum_{i_2=h_2}^{n_2} \right) \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} = \sum_{i_2=k}^{n} \sum_{i_1} \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} \quad ,$$

where the $\sum\limits_{i_1}$ factor has been transformed by (3) or (4). Now the factor $\sum\limits_{i_1}$ can commute with each $\sum\limits_{i_j=h_j}^{n_j}$, $t \geqslant j \geqslant 3$, since each of these h_j's and n_j's are independent of i_1. Hence

(9) $$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \left(\sum_{i_2=k}^{n} \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} \right) \sum_{i_1},$$

and the result follows from the induction hypothesis on the first $(t-1)$ factors.

Example.

$$\sum_{i_1=k}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} \binom{i_1}{k} = \sum_{i_t=k}^{n} \sum_{i_{t-1}=k}^{i_t} \cdots \sum_{i_1=k}^{i_2} \binom{i_1}{k} = \sum_{i_t=k}^{n} \sum_{i_{t-1}=k}^{i_t} \cdots \sum_{i_2=k}^{i_3} \binom{i_2+1}{k+1}$$

$$= \cdots = \sum_{i_t=k}^{n} \binom{i_t+t-1}{k+t-1} = \binom{n+t}{k+t}.$$

On the other hand, since

$$\sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} 1 = \binom{n-i_1+t-1}{t-1} \quad ,$$

$$\sum_{i_1=k}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} \binom{i_1}{k} = \sum_{i_1=k}^{n} \binom{i_1}{k} \sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} 1 = \sum_{i_1=k}^{n} \binom{i_1}{k} \binom{n-i_1+t-1}{t-1},$$

we have

$$\sum_{i_1=k}^{n} \binom{i_1}{k} \binom{n-i_1+t-1}{t-1} = \binom{n+t}{k+t}.$$

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FIBONACCI SEQUENCES AND MEMORY MANAGEMENT

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Fibonacci sequences have been studied from many points of view. We shall be concerned with sequences of integers which satisfy difference equations of the form

$$L_i = L_{i-1} + L_{i-k-1} . \tag{1}$$

For various values of k, we obtain generalized Fibonacci sequences as studied by Daykin [1] and Hoggatt [4].

In [4], many interesting properties of these sequences are derived by using generating functions and generalized diagonals of Pascal's Triangle. For our purposes, however, we need a formula which allows the direct calculation of any particular term in any one of the sequences determined by (1). The techniques are standard (see Miles [6] or Flores [2]) but will be developed here for completeness.

The advantages of closed-form formulas are important to many applications of Fibonacci sequences. In particular, the solution of (1) is useful to computer scientists in their study of algorithms. The polyphase sort algorithm, for instance, requires the use of Fibonacci numbers, and Fibonacci numbers arise naturally in the analysis of the algorithm to compute the greatest common divisor of two numbers. The application we investigate concerns the way Fibonacci numbers can be used to manage computer memory.

Consider the objective of keeping as many jobs in memory as possible. To implement this, the system must keep extensive tables of areas in memory and the size of each area. As jobs finish, the memory area becomes checkerboarded with vacant blocks of various sizes. Sometimes a new job is a little too big to fit in any of these areas, even though the total available area is sufficient to accomodate several new jobs. This checkerboarding is referred to as *external fragmentation* of memory. It can be alleviated by rearranging the jobs in memory so that all the vacant space is in one place. However, such operations require computer resources which could otherwise be used for user jobs in memory.

Some systems arbitrarily divide memory into blocks of fixed size, and force the requests for memory space to conform to these constraints. This makes it more economical for the system to manage the available blocks and their locations. On the other hand, this can be extravagant use of memory, because requests for space seldom fill the blocks to which they are assigned. The unused memory area toward the end of these blocks is called *internal fragmentation.*

In fact, there are memory management schemes which incorporate some of these features [7]. One such system is the Buddy System [5] and works as follows. The total memory size m is a power of 2, say $m = 2^n$. when the system notes a request for storage space, it tries to find the smallest block still a power of 2, which will hold the request. Larger blocks may be split in half if available, creating two smaller blocks, either of which might hold the request with less wasted space (internal fragmentation).

One feature of this system which reduces system overhead results from the fact that each block size (there are n distinct sizes) is twice the size of the next smaller block. That is the block sizes L_i satisfy the relation $L_i = 2L_{i-1}$. If it happens that two adjacent blocks of size L_{i-1} become free, they are recombined into one block of size L_i. This makes the tables and search procedures somewhat simpler.

Others ([3], [5]) have noticed that the Buddy System equation is a special case of the more general difference equation:

$$L_i = L_{i-1} + L_{i-k-1}, \qquad k = 0, 1, 2, \cdots . \tag{1}$$

For $k = 0$ (and appropriate initial values) we get the sequence 1, 2, 4, 8, 16, ⋯ . For $k = 1$, we can obtain the Fibonacci sequence 1, 1, 2, 3, 5, 6, 13, ⋯. For other values of k we will refer to the corresponding sequences as the k^{th} Fibonacci sequence (see Table 1).

Table 1
Generalized Fibonacci Sequences Giving Block Sizes 1 Through 250 (approx.)

i Level	L_i $k=0$	$k=1$	$k=2$	$k=3$	$k=4$
1	1	1	1	1	1
2	2	1	1	1	1
3	4	2	1	1	1
4	8	3	2	1	1
5	16	5	3	2	1
6	32	8	4	3	2
7	64	13	6	4	3
8	128	21	9	5	4
9	256	34	13	7	5
10		55	19	10	6
11		89	28	14	8
12		144	41	19	11
13		233	60	26	15
14			88	36	20
15			129	50	26
16			189	69	34
17			277	95	45
18				131	60
19				181	80
20				250	106
21					140
22					185
23					245

We can design a memory management scheme based on the k^{th} Fibonacci sequence, modeled after the Buddy System. Initially, memory is the size of an appropriate Fibonacci number, and requests for smaller pieces of memory are serviced by using Eq. (1) to split and reassemble blocks. The question is, does this improve utilization of memory? Table 1 shows there is a greater variety of block sizes as k increases. We conjecture that internal fragmentation decreases as k increases, but that system overhead increases.

For the moment we will disregard the overhead and examine the cost due to internal fragmentation. Let $\{L_i\}_{i=0}^{n}$ be the collection of block sizes, with $L_0 = 0$ and $L_n = m$ the memory size. If the system services a request for a certain number x of memory locations, it will allocate a block of size L_i, where $L_{i-1} < x \leqslant L_i$. The waste involved is $(L_i - x)$.

The requests for memory space are always for an integral number of locations, but for convenience let us assume that the request sizes are given by a continuous probability function $pdf(x)$. Then the expected average waste per request $\bar{w}$ is given by Hirschberg in [3]:

$$\bar{w} = \sum_{i=1}^{n} \int_{L_{i-1}}^{L_i} (L_i - x)pdf(x)dx.$$

Rewriting this, we obtain

$$\bar{w} = m - \bar{x} - \sum_{i=1}^{n} (d_i)cdf(L_{i-1}), \tag{2}$$

where:

$$m = \text{maximum memory size}$$

$$\bar{x} = \int_0^m x\,pdf(x) = \text{avg. request size}$$

$$n = \text{number of distinct block sizes}$$

$$d_i = L_i - L_{i-1}$$

$$cdf(z) = \int_0^z pdf(z)dx = \text{cumulative distribution function.}$$

The objective of memory management is to minimize $\bar{w}$ for a given $pdf(x)$. If we restrict our attention to Fibonacci type systems, we can gain some additional insight into minimizing $\bar{w}$.

Equation 1 gives rise to the characteristic polynomial, $x^{k+1} - x^k - 1 = 0$, of the k^{th} Fibonacci sequence. The polynomial (for fixed k) is known to have $(k+1)$ distinct roots which yield a closed-form expression for the n^{th} Fibonacci number. Note that $f(x) = x^{k+1} - x^k - 1$ has a real root between 1 and 2, since $f(1)$ is negative and $f(2)$ is positive. By Descartes' rule of signs, this is the only positive root, which will be denoted by a_1. Thus, $1 < a_1 \leqslant 2$ ($a_1 = 2$ for $k = 0$). Let the other roots of $f(x)$ be $a_2, a_3, \cdots, a_{k+1}$. It is easy to establish that a_1 is the root of largest modulus and, in fact, $|a_i| < 1$ for $i = 2, 3, \cdots, k+1$. (See, for example, [8].)

Evidently, any sequence of numbers $\{u_i\}$ satisfying

$$u_i = c_1a_1^i + c_2a_2^i + \cdots + c_{k+1}a_{k+1}^i$$

will be a k^{th} Fibonacci sequence satisfying Eq. (1). Specifying the initial $(k+1)$ terms of the sequence determines the constants $c_1, \cdots, c_{k+1}$, or specifying the constants determines the sequence. For the particular sequence $\{L_i\}$ in Table 1, we can write

$$L_i = c_1a_1^i + \cdots + c_{k+1}a_{k+1}^i\,.$$

Since $|a_i| < 1$ for $i = 2, 3, \cdots, k+1$, it follows that for sufficiently large i,

$$L_i \cong c_1a_1^i\;.$$

Some approximate values of c_1 and a_1 are given in Table 2. The initial segments in Table 1 can be obtained from the formula $L_i = c_1a_1^i$ (rounded to the nearest integer).

Table 2
Generators for Fibonacci Sequences

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$c(k)$	1	.44721	.41724	.39663	.38119
$a(k)$	2	1.61803	1.46557	1.38028	1.32472

Consider now the value of a_1 for different values of k. Let this root be denoted by $a(k)$. We have observed that $1 < a(k) \leqslant 2$. From Eq. (1) we see that

$$a(k) = 1 + \left(\frac{1}{a(k)}\right)^k\,,$$

and it follows that $a(k+1) < a(k)$ for every k. In fact, $\lim_{k\to\infty} a(k) = 1$

Let us apply the preceding observations to a particular example, in which the distribution of request sizes is given by the uniform distribution $pdf(x) = (1/m)$. Then $cdf(x) = (x/m)$, and $\bar{x} = (m/2)$. Let k be arbitrary but fixed. We write $a(k) = a$ and $c(k) = c$, so that $L_i \cong ca^i$, and $L_n = ca^n = m$. Then

(3)
$$\bar{w} = m - \bar{x} - \sum_{i=1}^{n} (d_i)cdf(L_{i-1}) = m - \frac{m}{2} - c\left(1 - \frac{1}{a}\right)\sum_{i=1}^{n} a^i cdf(ca^{i-1})$$

$$= \frac{m}{2} - c\left(1 - \frac{1}{a}\right)\sum_{i=1}^{n} a^i \frac{ca^i}{a^m} .$$

If we assume that $m >> 1$, then

$$a^2m^2 - 1 \cong a^2m^2, \qquad \text{and} \qquad \bar{w} \cong \frac{m}{2} - \frac{m}{a+1} .$$

Thus, $\bar{w}$ can be made as small as desired by increasing k, since a approaches 1 as k increases.

Intuitively, this is to be expected, since for any finite memory size m, if $k > m$, then the k^{th} Fibonacci sequence contains all the integers from 1 through m, and $\bar{w}$ should be zero. However, this leads to extreme overhead in memory management and places unreasonable demands on the search mechanism for allocation and release of area in memory.

The waste function $\bar{w}$ measures only the cost of internal fragmentation. Let us assume that the overhead associated with a memory system is given by a function of n, the number of distinct block sizes. Then a more complete cost function is

$$w = m - \bar{x} - \sum_{i=1}^{n} (d_i)cdf(L_{i-1}) + f(n).$$

This raises the possibility of optimizing the collection $\{L_i\}_{i=0}^{n}$ by considering the equations

$$\frac{\partial w}{\partial L_j} = 0 \quad \text{for} \quad j = 1, 2, \cdots, n-1$$

and the boundary conditions $L_0 = 0$, $L_n = m$. The solution is given by

(4)
$$L_{j+1} = L_j + \frac{cdf(L_j) - cdf(L_{j-1})}{cdf(L_j)} .$$

Continuing with the simple example of the uniform request distribution, let us assume conveniently that $f(n) = \beta \cdot n$, where $\beta > 0$ is a constant. We obtain

$$L_{j+1} = L_j + \frac{\frac{L_j}{m} - \frac{L_{j-1}}{m}}{\frac{1}{m}} = 2L_j - L_{j-1} .$$

The difference equation is not of the Fibonacci type, but does have a closed form solution

$$L_j = \frac{m}{n} j, \qquad j = 0, 1, \cdots, n .$$

So it is possible to optimize the collection $\{L_j\}$, which minimizes w, provided we know the nature of $f(n)$ in Eq. (4).

Unfortunately, other request distributions and other functions $f(n)$ do not lead to such nice solutions. Indeed the difference equations resulting from (4) are, in general, extremely difficult to solve analytically. For certain $pdf(x)$'s, however, solutions are of considerable importance to computer systems designers, and where closed-form solutions of the difference equations are not feasible, it is still important to apply numerical techniques to these problems.

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[Continued from Page 29.]

$$
\begin{array}{rcrcccc}
 & & & > & 35 & & \\
89 + 11 & = & 100 & & & > & 21 \\
 & & & > & 56 & & \\
144 + 12 & = & 156 & & & > & 34 \\
 & & & > & 90 & & \\
233 + 13 & = & 246 & & & > & 55 \\
 & & & > & 145 & & \\
377 + 14 & = & 391 & & & & \\
\text{etc.,} & \text{etc.,} & \text{etc.} & & & &
\end{array}
\qquad
\begin{array}{rcrcccc}
 & & & > & 33 & & \\
89 - 11 & = & 78 & & & > & 21 \\
 & & & > & 54 & & \\
144 - 12 & = & 132 & & & > & \\
 & & & > & 88 & & \\
233 - 13 & = & 220 & & & > & 55 \\
 & & & > & 143 & & \\
377 - 14 & = & 363 & & & & \\
\text{etc.,} & \text{etc.,} & \text{etc.} & & & &
\end{array}
$$

Now try it with the Lucas series 1, 3, 4, 7, 11, $\cdots$.

N.B—(In the reverse Fibonacci sequence, F_n is negative for even n).

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INTERESTING PROPERTIES OF LAGUERRE POLYNOMIALS

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Recent interest in optical communication has added to the importance of study of Laguerre polynomials [1] and distribution. We will establish two propositions which arise in studies of Laguerre distribution [2].

Definition.

$$L_n^{\alpha}(R) \triangleq \sum \binom{n+a}{n-i} \frac{(-1)^i}{i!} R^i ,$$

where

$$R^i \triangleq \int x^i p(x) dx .$$

Proposition 1:

$$\int L_n^{\alpha}(x) p(x) dx = L_n^{\alpha}(R) .$$

Proof.

$$\int L_n^{\alpha}(x) p(x) dx = \int \sum_{i=0}^{n} \binom{n+a}{n-i} \frac{(-1)^i}{i!} x^i p(x) dx = \sum_{i=0}^{w} \binom{n+a}{n-i} \frac{(-1)^i}{i!} \int x^i p(x) dx$$

$$= \sum_{i=0}^{n} \binom{n+a}{n-i} \frac{(-1)^i}{i!} R^i = L_n^{\alpha}(R).$$

Proposition 2. If $R^{i+j} = R^i R^j$, then

$$\int L_n^{\alpha}(x) L_m^{\beta}(x) p(x) dx = L_n^{\alpha}(R) L_m^{\beta}(R).$$

Proof.

$$\int L_n^{\alpha}(x) L_m^{\beta}(x) p(x) dx = \sum \sum \binom{m+\beta}{m-j} \binom{n+a}{n-i} \frac{(-1)^{i+j}}{i!\, j!} \int x^{i+j} p(x) dx$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n+a}{n-i} \binom{m+\beta}{m-j} \frac{(-1)^{i+j}}{i!\, j!} R^{i+j}$$

$$= \left\{ \sum \binom{n+a}{n-i} \frac{(-1)^i}{i!} R^i \right\} \left\{ \sum \binom{m+\beta}{m-j} \frac{(-1)^j}{j!} R^j \right\}$$

$$= L_n^{\alpha}(R) L_m^{\beta}(R) .$$

CONCLUSION

It is interesting to note that if $p(x) > 0$ and $\int p(x) dx = 1$ and $R^i < \infty \ \forall\, i$, then R^i are called moments of the random variable x. Expectation of Laguerre polynomials of random variables is Laguerre polynomials of moments.

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NUMERATOR POLYNOMIAL COEFFICIENT ARRAY FOR THE CONVOLVED FIBONACCI SEQUENCE

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1. INTRODUCTION

In [1], [2], and [3], Hoggatt and Bicknell discuss the numerator polynomial coefficient arrays associated with the row generating functions for the convolution arrays of the Catalan sequence and related sequences. In [4], Hoggatt and Bergum examine the irreducibility of the numerator polynomials associated with the row generating functions for the convolution arrays of the generalized Fibonacci sequence $\left\{ H_n \right\}_{n=1}^{\infty}$ defined recursively by

(1.1) $$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \qquad n \geq 3,$$

where the characteristic $P^2 - P - 1$ is a prime. The coefficient array of the numerator polynomials is also examined. The purpose of this paper is to examine the numerator polynomials and coefficient array related to the row generating functions for the convolution array of the Fibonacci sequence. That is, we let $P = 1$.

2. THE FIBONACCI ARRAY

We first note that many of the results of this section could be obtained from [4] by letting $P = 1$.

The convolution array, written in rectangular form, for the Fibonacci sequence is

Table 1
Convolution Array for the Fibonacci Sequence

1	1	1	1	1	1	1	1	...
1	2	3	4	5	6	7	8	...
2	5	9	14	20	27	35	44	...
3	10	22	40	65	98	140	192	...
5	20	51	105	190	315	490	726	...
8	38	111	256	511	924	1554	2472	...
13	71	233	594	1295	2534	4578	7776	...
21	130	474	1324	3130	6588	12,720	22,968	...
...	...	...	...	...	...	...	...	...

The generating function $C_m(x)$ for the m^{th} column of the convolution array is given by

(2.1) $$C_m(x) = (1 - x - x^2)^{-m}$$

and it is obvious that

(2.2) $$C_m(x) = (x + x^2)C_m(x) + C_{m-1}(x).$$

Hence, if $R_{n,m}$ is the element in the n^{th} row and m^{th} column of the convolution array then the rule of formation for the convolution array is

(2.3) $$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1}$$

which is representable pictorially by

$$\begin{array}{cc} & \boxed{w} \\ & \boxed{v} \\ \boxed{u} & \boxed{x} \end{array}$$

where

(2.4) $$x = u + v + w.$$

If $R_m(x)$ is the generating function for the m^{th} row of the convolution array then we see by (2.3) and induction that

(2.5) $$R_1(x) = \frac{1}{1-x}$$

(2.6) $$R_2(x) = \frac{1}{(1-x)^2}$$

and

(2.7) $$R_m(x) = \frac{N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \qquad m \geqslant 3$$

with $N_m(x)$ a polynomial of degree

$$\left[\frac{m-1}{2}\right],$$

where $[\;]$ is the greatest integer function.

The first few numerator polynomials are found to be

$$N_1(x) = 1$$
$$N_2(x) = 1$$
$$N_3(x) = 2 - x$$
$$N_4(x) = 3 - 2x$$
$$N_5(x) = 5 - 5x + x^2$$
$$N_6(x) = 8 - 10x + 3x^2$$
$$N_7(x) = 13 - 20x + 9x^2 - x^3$$
$$N_8(x) = 21 - 38x + 22x^2 - 4x^3.$$

Recording our results by writing the triangle of coefficients for these polynomials, we have

Table 2

Coefficients of Numerator Polynomials $N_m(x)$

1			
1			
2	−1		
3	−2		
5	−5	1	
8	−10	3	
13	−20	9	−1
21	−38	22	−4

Examining Tables 1 and 2, it appears as if there exists a relationship between the rows of Table 2 and the rising diagonals of Table 1. In fact, we shall now show that

$$N_m(x) = \sum_{n=1}^{k} R_{m-2n+2,n}(-x)^{n-1}, \qquad m \geqslant 2, \tag{2.8}$$

where

$$k = \left[\frac{m+1}{2}\right].$$

It is obvious from (2.5), (2.6), and (2.7) that the constant coefficient of $N_m(x)$ is F_m for all $m \geqslant 1$, where F_m is the m^{th} Fibonacci number. Furthermore, the rule of formation for the elements in Table 2 is given pictorially by

$$\begin{array}{|c|c|} \hline w & v \\ \hline \end{array}$$
$$\begin{array}{cc} & \boxed{u} \\ & \boxed{x} \end{array}$$

where

$$x = \pm(u + v - w) \tag{2.9}$$

with the sign chosen according as x is in an even or odd numbered column.

Letting $G_m(x)$ be the generating function for the m^{th} column and using (2.9) with induction, we see that

$$G_m(x) = \left(\frac{-1}{1-x-x^2}\right) G_{m-1}(x) = \frac{(-1)^{m-1}}{(1-x-x^2)^m} = (-1)^{m-1}C_m(x). \tag{2.10}$$

Equations (2.9) and (2.10) show that the columns of Table 2 are the columns of Table 1 shifted downward by the value of $2(m-1)$ and having the sign $(-1)^{m-1}$. Hence, Eq. (2.8) is proved.

Adding along rising diagonals of Table 2 is equivalent to

$$\sum_{k=0}^{\infty} x^{3k} G_{k+1}(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 + \frac{x^3}{1-x-x^2}\right) = (1-x-x^2+x^3)^{-1} \tag{2.11}$$

which is the generating function for the sequence defined by

$$S_n = \begin{cases} \left[\frac{n}{2}\right], & n \text{ even} \\ \left[\frac{n}{2}\right] + 1, & n \text{ odd} \end{cases}. \tag{2.12}$$

Letting

$$G_k^*(x) = (1-x-x^2)^{-k}, \tag{2.13}$$

we see that adding along rising diagonals with all signs positive is equivalent to

$$\sum_{k=0}^{\infty} x^{3k} G_{k+1}^*(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 - \frac{x^3}{1-x-x^2}\right) = \frac{1}{1-x-x^2-x^3} \tag{2.14}$$

which is the generating function for the sequence of Tribonacci numbers.

Since

$$\sum_{k=0}^{\infty} x^{2k} G_{k+1}(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 + \frac{x^2}{1-x-x^2}\right) = (1-x)^{-1}, \tag{2.15}$$

we know that the row sums of Table 2 are always one. This fact can also be shown in the following way. From (2.7), we determine that the generating function for the polynomials $N_m(x)$ is

$$\frac{1}{1-\lambda-(1-x)\lambda^2} = \sum_{k=0}^{\infty} N_{k+1}(x)\lambda^k. \tag{2.16}$$

Letting $x = 1$, we have

$$\frac{1}{1-\lambda} = \sum_{k=0}^{\infty} N_{k+1}(1)\lambda^k \tag{2.17}$$

so that $N_k(1) = 1$ for all $k \geq 1$. When $x = 0$, we obtain an alternate proof that the constant coefficient of $N_m(x)$ is F_m.

Row sums with all signs positive is given by

$$\sum_{k=0}^{\infty} x^{2k} G_k^*(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 - \frac{x^2}{1-x-x^2}\right) = (1-x-2x^2)^{-1} \tag{2.18}$$

which is the generating function for the sequence defined recursively by

$$T_1 = 1, \quad T_2 = 1, \quad T_n = T_{n-1} + 2T_{n-2}, \quad n \geq 3. \tag{2.19}$$

It is interesting to observe that by letting $x = -1$ in (2.16) we have $N_k(-1) = T_k$ for $k \geq 1$.

Adding along falling diagonals is equivalent to

$$\sum_{k=0}^{\infty} x^k G_{k+1}(x) = \frac{1}{1-x^2} \tag{2.20}$$

which is the generating function for the sequence defined by

$$S_n = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{2.21}$$

In conclusion, we note that the sum of falling diagonals with all elements positive is equivalent to

$$\sum_{k=0}^{\infty} x^k G_{k+1}^*(x) = \frac{1}{1-2x-x^2} \tag{2.22}$$

which is the generating function for the sequence of Pellian numbers defined recursively by

$$P_1 = 1, \quad P_2 = 2, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3. \tag{2.23}$$

3. PROPERTIES OF $\{N_m(x)\}_{m=1}^{\infty}$

The main purpose of this section is to show that if $m \geq 5$ then $N_m(x)$ is irreducible if and only if m is a prime. The irreducibility of $N_m(x)$ for $1 \leq m \leq 5$ is obvious.

By standard finite difference techniques, it can be shown that the auxiliary polynomial associated with

$$\{N_m(x)\}_{m=1}^{\infty}$$

is

$$\lambda^2 - \lambda - (1-x) = 0 \tag{3.1}$$

whose roots are

$$\lambda_1 = \frac{1+\sqrt{5-4x}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5-4x}}{2}. \tag{3.2}$$

Using (3.1) and induction, we have

$$N_m(x) = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2}, \quad m \geq 1. \tag{3.3}$$

Since $\lambda_1\lambda_2 = x - 1$, we can use (3.3) to show that

$$N_{m+n+1}(x) = N_{m+1}(x)N_{n+1}(x) + (1-x)N_m(x)N_n(x), \quad m \geq 1, \quad n \geq 1. \tag{3.4}$$

Following the arguments of Hoggatt and Long which can be found in [6], we obtain the following results.

$$(1-x, N_m(x)) = 1, \quad m \geq 1. \tag{3.5}$$

$$(N_m(x), N_{m+1}(x)) = 1, \quad m \geq 1. \tag{3.6}$$

(3.7) If $m \geqslant 3$ then $N_m(x)|N_n(x)$ if and only if $m|n$.

(3.8) Let $m \geqslant 5$. If $N_m(x)$ is irreducible then m is a prime.

(3.9) For $m \geqslant 1$, $n \geqslant 1$, $(N_m(x), N_n(x)) = N_{(m,n)}(x)$.

Substituting (3.2) into (3.3) and expanding by the binomial theorem, we obtain the following.

(3.10) $N_{2n+1}(x)$ is a monic polynomial of degree n.

(3.11) $$4^n N_{2n+1}(x) \equiv 2n+1 \pmod{5-4x}.$$

Let p be an odd prime, say $p = 2n+1$. By expanding (3.3) and collecting like powers of x, we obtain

(3.12) $$N_p(x) = \frac{1}{2^{p-1}} \sum_{m=0}^{n} \sum_{j=m}^{n} \binom{p}{2j+1}\binom{j}{m} 5^{j-m}(-4x)^m \equiv \sum_{m=0}^{n} \binom{n}{m} 5^{n-m}(-4x)^m \pmod p$$
$$\equiv (5-4x)^{\frac{p-1}{2}} \pmod p.$$

In order to prove the converse of (3.8), we present the following argument.

Suppose that for some prime p, $p > 5$, $N_p(x)$ is reducible. Then, by (3.10), there exist two monic polynomials such that

$$N_p(x) = f(x)g(x)$$

or

$$N_p(x^2) = f(x^2)g(x^2).$$

Since all the powers of $f(x^2)$ and $g(x^2)$ are even, we can use the division algorithm to obtain

$$4^t f(x^2) = \ell_1(x)(5-4x^2)+h$$

and

$$4^q g(x^2) = \ell_2(x)(5-4x^2)+g\ ,$$

where t and q are respectively the degrees of $f(x)$ and $g(x)$ and h and g are integers.

By (3.11), we see that

(3.13) $$4^{\frac{p-1}{2}} f(x^2)g(x^2) \equiv p \equiv hg \pmod{5-4x^2}.$$

Hence, we assume without loss of generality that $h = \pm p$ and $g = \pm 1$.

If $p \equiv \pm 2 \pmod 5$ then 5 is a quadratic nonresidue so that $5-4x^2$ is irreducible in the unique factorization domain $Z_p[x]$. Hence, by (3.12), we conclude that $g(x) \equiv (5-4x^2)^k \pmod p$ for some integer k. If $p \equiv \pm 1 \pmod 5$ then 5 is a quadratic residue so that

$$(5-4x^2) = (a-2x)(a+2x) \text{ in } Z_p[x] \text{ with } a^2 \equiv 5 \pmod p.$$

Therefore, by (3.12),

$$g(x^2) \equiv (a-2x)^{k_1}(a+2x)^{k_2} \pmod p$$

for some integers k_1 and k_2. However, $g(x^2)$ is even so that $k_1 = k_2$. In both cases, there exists an integer k such that

(3.14) $$g(x^2) = \ell_3(x)p + (5-4x^2)^k.$$

Since $\ell_3(x)$ is obviously even, we know that

(3.15) $$4^q \ell_3(x) \equiv c \pmod{5-4x^2}$$

for some integer c so that

(3.16) $$4^q g(x^2) \equiv \pm 1 \equiv pc \pmod{5-4x^2}$$

which is impossible. Hence, $N_p(x)$ is irreducible.

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LETTER TO THE EDITOR

October 13, 1975

Dear Professor Hoggatt:

It was with some surprise that I read Miss Ada Booth's article "Idiot's Roulette Revisited" in the April 1975 issue of *The Fibonacci Quarterly.* The problem she discusses—given N places circularly arranged and successively casting out the C^{th} place, determine which will be the last remaining place—is quite old and commonly referred to as the Josephus problem. The name alludes to a passage in the writings of Flavius Josephus [7], a Jewish historian who relates how after the fall of Jotapata, he and forty other Jews took refuge in a nearby cave, only to be discovered by the Romans. In order to avoid capture, everyone in the group, save Josephus, resolved on mass suicide. At Josephus' suggestion, lots were drawn, and as each man's lot came up, he was killed. By means not made clear in the passage, Josephus ensured that the lots of himself and one other were the last to come up, at which point he persuaded the other man that they should surrender to Vespasian.

Bachet [2], in one of the earliest works on recreational mathematics, proposed a definite mechanism by which this could have been accomplished: all forty-one people are placed in a circle, Josephus placing himself and the other man at the 16^{th} and 31^{st} places; every third person is then counted off and killed. This is, of course, a special case of the question Miss Booth considers.

Miss Booth's iterative solution to the general problem was apparently first discovered by Euler [5] in 1771 and then rediscovered by P. G. Tait [9], the English physicist and mathematician, in 1898. Tait points out that the method enables one to calculate the last r places to be left, not merely the last as in Miss Booth's article. Although Euler and Tait content themselves with demonstrating how the iterative solution works and do not actually derive the formula for Miss Booth's sequence of "subtraction numbers," in the 1890's Schubert and Busche [8, 4] derived a formula for this sequence (slightly modified) via a wholly different attack on the problem ("Oberreihen"). (Ahrens [1] has an excellent description of this work, as well as a comprehensive review of the history of the problem. Ball and Coxeter [3] briefly touch on the problem but omit any mention of the work of Schubert and Busche.)

[Continued on Page 51.]

A COMBINATORIAL IDENTITY

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Let $q < p < k$ and v be positive integers, n be a nonnegative integer, $\ell_0 = 1$ and $\{\ell_1, \ell_2 \cdots\}$ be a sequence of marks. Further let $T_{k,j}$ be the Stirling numbers of the first kind defined as the coefficients of

(1) $$f(x) = \sum_{j=1}^{k} T_{k,j}x^j = x(x-1)(x-2)\cdots(x-k+1)$$

and let

(2) $$L(v,p,q) = \sum r_1 r_2 \cdots r_v \ell_{d_1}\ell_{d_2}\cdots\ell_{d_v},$$

where the summation is over all the sequences of integers $r_1, r_2, \cdots, r_v$ satisfying

$$p = r_0 \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_v = p - q, \quad \text{and} \quad d_i = r_{i-1} - r_i .$$

In connection with integration of differential equations of a group, A Ran proved in his thesis [1], using analytical methods, that

(3) $$\sum_{j=1}^{k} T_{k,j}L(j+n,p,q) \equiv 0$$

identically, i.e., that on the left side of (3) the coefficient of every product $\Pi \ell_i^{\alpha_i}$ equals zero.

Here the proof of (3) is given by combinatorial methods. To begin we write (2) in the form

(4) $$L(v,p,q) = \sum{}^{*} R(v,p,q,a,\pi\ell_i^{\alpha_i}) \prod_{i\geqslant 1} \ell_i^{\alpha_i},$$

where the summation Σ^* is over all sequences of nonnegative integers $\alpha_1, \alpha_2, \cdots, \alpha_q$ satisfying $\Sigma i\alpha_i = q$, and

(5) $$a = \sum \alpha_i,$$

and prove the following

Lemma.

(6) $$R(v,p,q,a,\pi\ell_i^{\alpha_i}) = \sum_{h=0}^{q} c_h(p-h)^v,$$

where the coefficients c_h do not depend on v (but may depend on p,q,a and $\pi\ell_i^{\alpha_i}$) and are such that

(7) $$\sum_{h=0}^{q} c_h(p-h)^t = 0, \qquad t = 0, 1, \cdots, a-1.$$

Proof. The proof is given by induction on a. For $a = 1$ we have

$$R(v,p,q,1,\ell_q) = (p-q)\sum_{i=0}^{v-1} p^i(p-q)^{v-i-1} = \frac{p-q}{q}(p^v - (p-q)^v),$$

which satisfies both (6) and (7).

Suppose now that (6) and (7) are satisfied for $a = b - 1$. It is easily seen that

$$R(v,p,q,b,\pi\ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \sum_{\beta=0}^{v-b} p^{\beta} R(v-\beta-1, p-\eta, q-\eta, b-1, \pi\ell_i^{\alpha_i}/\ell_\eta),$$

where η obtains the values of i for which $a_i \geqslant 1$. We make use of (6) with $a = b - 1$ and in order to stress that the coefficients c_h depend on η we write them in the form $c_{\eta,h}$. We have

$$R(v,p,q,b,\pi\ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \sum_{\beta=o}^{v-b} p^{\beta} \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1}$$

$$= \sum_{\eta} (p-\eta) \left[\sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^v - (p-h)^v) - \sum_{\beta=v-b+1}^{v-1} p^{\beta} \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1} \right].$$

By (7) follows that

$$\sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1} = 0$$

for every η and for $0 \leqslant v - \beta - 1 \leqslant b - 2$, i.e., for $v - b + 1 \leqslant \beta \leqslant v - 1$ and consequently

$$R(v,p,q,b,\pi\ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \cdot \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^v - (p-h)^v) \tag{8}$$

which proves (6) for $a = b$.

To prove (7) let us denote for every η

$$D_\eta(t) = \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^t - (p-h)^t). \tag{9}$$

Evidently $D_\eta(0) = 0$. For $t \geqslant 1$ we have

$$D_\eta(t) = \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} \cdot h \sum_{i=0}^{t-1} p^i (p-h)^{t-i-1} = \sum_{i=0}^{t-1} p^i \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{t-i-1}.$$

By (7) with $a = b - 1$,

$$\sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{t-i-1} = 0$$

for $t = 1, 2, \cdots, b - 1$ and $0 \leqslant i \leqslant t - 1$ and consequently $D_\eta(t) = 0$ for $0 \leqslant t \leqslant b - 1$. By (6), (8) and (9),

$$\sum_{h=o}^{q} c_h(p-h)^t = R(t,p,q,b,\pi\ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) D_\eta(t) = 0, \qquad t = 0, 1, \cdots, b-1$$

which proves (7) with $a = b$.

Theorem.

$$\sum_{j=1}^{k} T_{k,j} L(j+n,p,q) \equiv 0.$$

Proof. By (4), (6) and (1) we have

$$\sum_{j=1}^{k} T_{k,j} L(j+n, p, q) = \sum_{j=1}^{k} T_{k,j} \sum \prod_{i=1}^{q} \ell_i^{\alpha_i} \sum_{h=o}^{q} c_h (p-h)^{j+n}$$

$$= \sum{}^{*} \prod_{i=1}^{q} \ell_i^{\alpha_i} \sum_{h=0}^{q} c_h (p-h)^n \sum_{j=1}^{k} T_{k,j} (p-h)^j = \sum{}^{*} \prod_{i=1}^{q} \ell_i^{\alpha_i} \sum_{h=o}^{q} c_h (p-h)^n f(p-h).$$

By definition $p-h$ is an integer satisfying $1 \leqslant p-h \leqslant p \leqslant k-1$ and consequently by (1), $f(p-h)=0$ which proves the theorem.

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[Continued from Page 48.]

Much more recently (1973), Jacobczyk [6] has given new iterative procedures for determining answers to both:

(a) for each k, $1 \leqslant k \leqslant N$, which will be the k^{th} place to be cast out?

(b) for each k, $1 \leqslant k \leqslant N$, when will the k^{th} place be cast out?

(The "Oberreihen" methods described by Ahrens also provide answers to both questions.)

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Sandy L. Zabell

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SOME REMARKS ON THE PERIODICITY OF THE SEQUENCE OF FIBONACCI NUMBERS

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In the work of Wall [2], a function ϕ was defined by "$\phi(m)$ is the length of the period of the sequence of Fibonacci numbers reduced to least non-negative residues modulo m, for $m > 2$." Thus, the domain of ϕ is the set of positive integers greater than 2, and the range was shown to be a subset of the set of all even integers. Below, I determine the range of ϕ exactly. In [1] I proved the following

Theorem A. If m is an integer greater than 3 then $\phi(F_m) = 2m$ if m is even and $\phi(F_m) = 4m$ if m is odd. Here, F_m is the m^{th} Fibonacci number, where

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1} \quad (n \geqslant 1).$$

Theorem 2 of [2] shows that the values of ϕ are completely known provided its values at all prime powers are known. But, as the table of values included in [2] shows, the values that ϕ takes at primes do not seem to follow any simple pattern. In an attempt to find more of the values of ϕ I will prove the following

Theorem B. If $m \geqslant 2$ then $\phi(F_{m-1} + F_{m+1}) = 4m$ if m is even and $\phi(F_{m-1} + F_{m+1}) = 2m$ if m is odd.

Theorems A and B have the following

Corollary. The range of ϕ is the set of all even integers greater than 4.

Proof. It is clear that we cannot have an integer n for which $\phi(n) = 2$ or $\phi(n) = 4$. Suppose that r is an even integer other than 2 or 4. If r is a multiple of 4, say $r = 4s$, then $\phi(F_{s-1} + F_{s+1}) = r$ if s is even, while $\phi(F_s) = r$ if s is odd and $s > 3$. Also $\phi(F_6) = 12$. If r is not a multiple of 4, say $r = 2s$, where s is odd and $s > 1$, then

$$\phi(F_{s-1} + F_{s+1}) = r.$$

A subsidiary result is required to prove Theorem B. In the following, the symbol $\equiv$ denotes congruence modulo $(F_{m-1} + F_{m+1})$.

Lemma. For $1 \leqslant r \leqslant m$ let $G_r = F_{m-1} + F_{m+1} - F_r$. Then

(i)
$$F_{m+r} \equiv \begin{cases} F_{m-r} & \text{if } 0 \leqslant r \leqslant m \text{ and } r \text{ is even} \\ G_{m-r} & \text{if } 1 \leqslant r \leqslant m-1 \text{ and } r \text{ is odd.} \end{cases}$$

If m is a positive even integer then

(ii)
$$F_{2m+r} \equiv G_r \quad \text{if } 0 \leqslant r \leqslant m.$$

(iii)
$$F_{3m+r} \equiv \begin{cases} G_{m-r} & \text{if } 0 \leqslant r \leqslant m \text{ and } r \text{ is even} \\ F_{m-r} & \text{if } 1 \leqslant r \leqslant m-1 \text{ and } r \text{ is odd.} \end{cases}$$

Proof. We prove these results by induction on r.

(i) The assertion here is trivially true if $r = 0$ or $r = 1$. Suppose the result is true for $r - 1$ and r. If $r + 1$ is odd then

$$\begin{aligned} F_{m+r+1} &= F_{m+r} + F_{m+r-1} \equiv F_{m-r} + G_{m-r+1} \quad \text{by hypothesis} \\ &= F_{m-1} + F_{m+1} + F_{m-r} - F_{m-r+1} \\ &= F_{m-1} + F_{m+1} - F_{m-(r+1)} = G_{m-(r+1)}. \end{aligned}$$

If $r + 1$ is even then

$$\begin{aligned} F_{m+r+1} &= F_{m+r} + F_{m+r-1} \\ &\equiv G_{m-r} + F_{m-(r-1)} \quad \text{by hypothesis} \\ &= F_{m-1} + F_{m+1} + F_{m-(r+1)} \\ &\equiv F_{m-(r+1)} \quad . \end{aligned}$$

(ii) The case in which $r = 0$ follows directly from (i) with $r = m$. The result is also true for $r = 1$ because

$$\begin{aligned} F_{2m+1} &= F_{2m} + F_{2m-1} \\ &\equiv F_0 + G_{m-(m-1)} \quad \text{by (i)} \\ &= G_1 \quad . \end{aligned}$$

Suppose the result is true for $r - 1$ and r. Then

$$\begin{aligned} F_{2m+r+1} &= F_{2m+r} + F_{2m+r-1} \\ &\equiv G_r + G_{r-1} \quad \text{by hypothesis} \\ &\equiv F_{m-1} + F_{m+1} - F_{r+1} \\ &= G_{r+1} \quad . \end{aligned}$$

(iii) The case in which $r = 0$ follows directly from (ii) with $r = m$. When $r = 1$ we have

$$\begin{aligned} F_{3m+1} &= F_{3m} + F_{3m-1} \\ &\equiv G_m + G_{m-1} \quad \text{by (ii)} \\ &= F_{m-1} + 2F_{m+1} - F_m \\ &\equiv F_{m-1} \quad , \end{aligned}$$

so that the result is true for $r = 1$. Suppose it is true for $r - 1$ and r. If $r + 1$ is odd then

$$\begin{aligned} F_{3m+r+1} &= F_{3m+r} + F_{3m+r-1} \\ &\equiv G_{m-r} + F_{m-r+1} \quad \text{by hypothesis} \\ &\equiv F_{m-(r+1)} \quad , \end{aligned}$$

while if $r + 1$ is even we have

$$\begin{aligned} F_{3m+r+1} &= F_{3m+r} + F_{3m+r-1} \\ &\equiv F_{m-r} + G_{m-r+1} \\ &= G_{m-(r+1)} \quad . \end{aligned}$$

This finishes the proof of the Lemma.

We may now prove Theorem B by noticing that if m is even then the sequence of Fibonacci numbers reduced modulo $(F_{m-1} + F_{m+1})$ consists of repetitions of the numbers

$$F_0, F_1, \cdots, F_m, F_{m+1}, F_{m-2}, G_{m-3}, F_{m-4}, G_{m-5}, \cdots, F_2, G_1, 0,$$

$$G_1, G_2, \cdots, G_{m-1}, G_m, F_{m-1}, G_{m-2}, F_{m-3}, G_{m-4}, \cdots, G_2, F_1,$$

while if m is odd we obtain

$$F_0, F_1, \cdots, F_m, F_{m+1}, F_{m-2}, G_{m-3}, F_{m-4}, G_{m-5}, \cdots, G_2, F_1 \ .$$

Thus, counting, and noticing that $G_1 \neq F_1$, we obtain the required results.

Using Theorem A, it may be shown that if $m > 4$ then

$$\phi(F_{m-1} + F_{m+1}) = \tfrac{1}{2}(\phi(F_{m-1}) + \phi(F_{m+1})).$$

I conclude by conjecturing that if k is a positive integer with $m - k > 3$ then

$$\phi(F_{m-k} + F_{m+k}) = \frac{k}{2}\,(\phi(F_{m-k}) + \phi(F_{m+k})).$$

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PARITY TRIANGLES OF PASCAL'S TRIANGLE

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In the Pascal's triangle of binomial coefficients, $\binom{n}{r}$, let every odd number be represented by an asterisk, "*," and every even number by a cross, "†." Then we discover another diagram which is quite interesting.

Every nine (odd) numbers form a triangle having exactly one (odd) even number in its interior (odd!). Thus we shall designate it as an Odd-triangle.

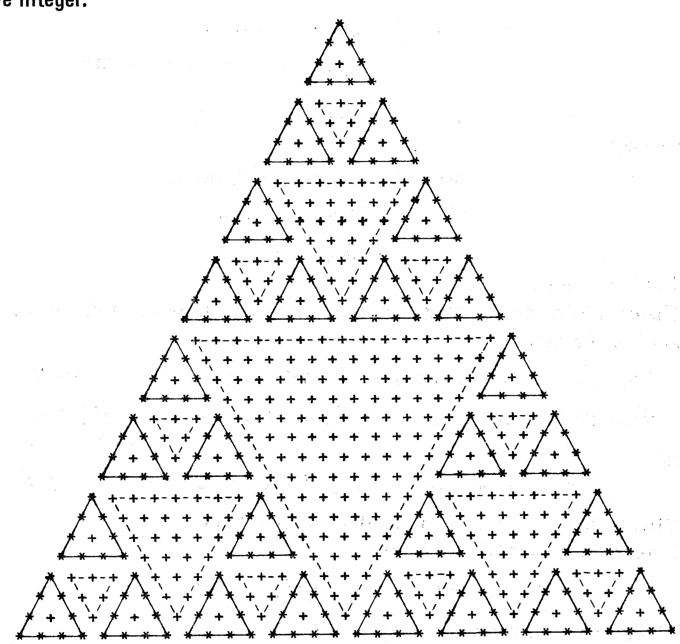
The even numbers also form triangles whose sizes vary but each of these triangles contains an even number of crosses. This set of triangles is called Even-triangles.

The present diagram $(n = 31)$ can be easily extended along the outermost apex of Pascal's triangle. Some partial observations are:

(a) If $n = 2^i - 1$ and $0 \leqslant r \leqslant 2^i - 1$, then $\binom{n}{r}$ is odd,

(b) If $n = 2^i$ and $1 \leqslant r \leqslant 2^i - 1$, then $\binom{n}{r}$ is even,

where i is a nonnegative integer.



Parity Triangles of $\binom{n}{r}$

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THE SAALSCHÜTZIAN THEOREMS

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1. Saalschütz's theorem reads

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!(c)_k (d)_k} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \tag{1.1}$$

where

$$c + d = -n + a + b + 1 \tag{1.2}$$

and

$$(a)_k = a(a+1)\cdots(a+k-1), \qquad (a)_0 = 1.$$

The theorem has many applications. For example, making use of (1.1), one can prove [3, §6], [7, p. 41]

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{2j \leqslant n} \frac{(n+j)!}{(j!)^3 (n-j)!} x^j (1+x)^{n-2j}. \tag{1.3}$$

In particular, for $x = 1$, (1.3) reduces to

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{2j \leqslant n} \frac{(n+1)!}{(j!)^3 (n-j)!} 2^{n-2j}, \tag{1.4}$$

a result due to MacMahon. For $x = -1$, (1.3) yields Dixon's theorem:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3}. \tag{1.5}$$

Saalschütz's theorem is usually proved (see for example [2, p. 9], [6, p. 86], [8, p. 48]) by showing that it is a corollary of Euler's theorem for the hypergeometric function:

$$F(a,b;c;x) = (1-x)^{c-a-b} F(c-a, c-b; c; x), \tag{1.6}$$

where as usual

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n.$$

As for (1.6), the usual method of proof is by making use of the hypergeometric differential equation.

The writer [3, §6] has given an inductive proof of (1.1). We shall now show how to prove the theorem by using only Vandermonde's theorem

$$F(-n,a;c;1) = \frac{(c-a)_n}{(c)_n}. \tag{1.7}$$

Supported in part by NSF grant GP-37924.

We then show how the q-analog of (1.1) can be proved in an analogous manner (for statement of the q-analog see §5 below). Finally, in §6, we prove a q-analog of (1.5).

2. To begin with, we note that (1.4) is implied by the familiar formula

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}, \tag{2.1}$$

where a,b are non-negative integers. Since each side of (2.1) is a polynomial in $a,b,$ it follows that (2.1) holds for arbitrary $a,b.$ Replacing a by $-a$ and b by $c+n-1,$ (2.1) becomes

$$\sum_{k=0}^{n} \binom{-a}{k}\binom{c+n-1}{n-k} = \binom{c-a+n-1}{n},$$

that is,

$$\sum_{k=0}^{n} (-1)^k \frac{(a)_k(c+k)_{n-k}}{k!(n-k)!} = \frac{(c-a)_n}{n!}.$$

This is the same as

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(c)_k} = \frac{(c-a)_n}{(c)_n}, \tag{2.2}$$

so that we have proved (1.4).

Now, by (2.2),

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(b)_n}{n!(c)_n} x^n = \sum_{n=0}^{\infty} \frac{(b)_n}{n!} x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{(c)_k} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(b)_k}{k!(c)_k} x^k \sum_{n=0}^{\infty} \frac{(b+k)_n}{n!} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(b)_k}{k!\,(c)_k} x^k(1-x)^{-b-k},$$

so that

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(b)_n}{n!(c)_n} x^n = (1-x)^{-b} \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} \left(\frac{x}{x-1}\right)^k. \tag{2.3}$$

We have accordingly proved the well-known formula

$$F\left(a,b;c;\frac{x}{x-1}\right) = (1-x)^b F(c-a,b;c;x). \tag{2.4}$$

In the next place, by (2.3),

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(c-b)_n}{n!(c)_n} x^n = (1-x)^{b-c} \sum_{k=0}^{\infty} \frac{(a)_k(c-b)_k}{k!(c)_k} \left(\frac{x}{x-1}\right)^k = (1-x)^{b-c} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \left(\frac{x}{x-1}\right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(b)_j}{(c)_j}$$

$$= (1-x)^{b-c} \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!(c)_j} \left(\frac{x}{1-x}\right)^j \sum_{k=0}^{\infty} \frac{(a+j)_k}{k!} \left(\frac{-x}{1-x}\right)^k =$$

$$= (1-x)^{b-c}\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!(c)_j}\left(\frac{x}{1-x}\right)^j\left(1+\frac{x}{1-x}\right)^{-a-j} = (1-x)^{a+b-c}\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!\,(c)_j}x^j.$$

This evidently proves (1.6).

To see that (1.1) and (1.6) are equivalent, consider

$$(1-x)^{a+b-c}F(a,b;c;x) = \sum_{j=0}^{\infty}\frac{(c-a-b)_j}{j!}x^j\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}x^k$$

$$= \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\frac{(a)_k(b)_k(c-a-b)_{n-k}}{k!(c)_k(n-k)!}\;.$$

Since

$$(a)_{n-k} = \frac{(a)_n}{(a+n-k)\cdots(a+n-1)} = (-1)^k\frac{(a)_n}{(-a-n+1)_n}\,,$$

it follows that

$$(1-x)^{a+b-c}F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(c-a-b)_n}{n!}x^n\sum_{k=0}^{n}\frac{(-n)_k\,(a)_k(b)_k}{k!(c)_k(a+b-c-n+1)_k}$$

Hence (1.6) is equivalent to

$$\frac{(c-a-b)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(a)_k(b)_k}{k!(c)_k(a+b-c-n+1)_k} = \frac{(c-a)_n(c-b)_n}{n!(c)_n}\,,$$

which is itself equivalent to (1.1).

3. It may be of interest to remark that (2.4) is a special case of the following identity:

$$\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\lambda_r x^r = (1-x)^{-a}\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\mu_r\left(\frac{x}{x-1}\right)^r\,, \tag{3.1}$$

where

$$\mu_r = \sum_{s=0}^{r}(-1)^s\binom{r}{s}\lambda_s\,,\qquad \lambda_r = \sum_{s=0}^{r}(-1)^s\binom{r}{s}\mu_s\,. \tag{3.2}$$

Indeed

$$(1-x)^{-a}\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\mu_r\left(\frac{x}{x-1}\right)^r = \sum_{r=0}^{\infty}(-1)^r\frac{(a)_r}{r!}\mu_r x^r(1-x)^{-a-r}$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{(a)_r}{r!}\mu_r x^r\sum_{s=0}^{\infty}\frac{(a+r)_s}{s!}x^s$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mu_r$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n}{n!}\lambda_n x^n\;.$$

For $\lambda_r = (b)_r/(c)_r$, (3.1) reduces to an identity equivalent to (2.4). For

$$\lambda_r = \frac{1}{c+r}, \qquad \mu_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{1}{c+s} = \frac{r!}{(c)_{r+1}}$$

and (3.1) becomes

$$\sum_{r=0}^{\infty} \frac{(a)_r}{r!} \frac{x^r}{c+r} = (1-x)^{-a} \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_{r+1}} \left(\frac{x}{x-1}\right)^r . \tag{3.3}$$

4. We turn next to the q-analog of Saalshütz's theorem. We shall use the following notation. Put

$$(a)_n = (a)_{n,q} = (1-a)(1-qa)\cdots(1-q^{n-1}a), \qquad (a)_0 = 1; \tag{4.1}$$

in particular

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} ;$$

it occurs in the q-binomial theorem

$$(x)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} x^k .$$

We also put

$$e(x) = e(x,q) = \prod_{n=0}^{\infty} (1-q^n x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} ,$$

where $|q| < 1$, $|x| < 1$. A more general result used below is

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{e(x)}{e(ax)} . \tag{4.2}$$

We shall also use the identity

$$\frac{1}{e(x)} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{x^n}{(q)_n} .$$

For completeness we sketch the proof of (4.2). Put

$$f(x) = \frac{e(x)}{e(ax)} = \sum_{n=0}^{\infty} A_n x^n .$$

Since $e(qx) = (1-x)e(x)$, it follows that

$$f(qx) = \frac{1-x}{1-ax} f(x),$$

so that

$$(1-x) \sum_{n=0}^{\infty} A_n x^n = (1-ax) \sum_{n=0}^{\infty} A_n q^n x^n .$$

This gives

$$A_n - A_{n-1} = q^n A_n - q^{n-1} a A_{n-1}, \qquad (1-q^n)A_n = (1-q^{n-1}a)A_{n-1}.$$

Since $A_0 = 1$, we get

$$A_n = \frac{1-q^{n-1}a}{1-q^n} A_{n-1} = \frac{(a)_n}{(q)_n},$$

thus proving (4.2).

We shall also require the following formulas:

$$\sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k-1)} (c/a)^k = \frac{(c/a)_n}{(c)_n}, \tag{4.3}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k+1)-nk} = \frac{(c/a)_n}{(c)_n} a^n, \tag{4.4}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(a)_n q^{\frac{1}{2}n(n-1)}}{(q)_n (c)_n} (c/a)^n = \frac{e(c)}{e(c/a)}. \tag{4.5}$$

To prove (4.3), we note first that it follows from (4.2) and the evident identity

$$\frac{e(x)}{e(ax)} \frac{e(ax)}{e(abx)} = \frac{e(x)}{e(abx)}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b)_{n-k} a^{n-k} = (ab)_n. \tag{4.6}$$

Replacing by $= q^{-n+1}/c$, this becomes

$$\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (q^{-n+1}/c)_{n-k} a^{n-k} = (q^{-n+1}a/c)_n. \tag{4.7}$$

Now

$$\begin{aligned}(q^{-n+1}a/c)_n &= \left(1 - \frac{q^{-n+1}a}{c}\right)\left(1 - \frac{q^{-n+2}a}{c}\right) \cdots \left(1 - \frac{a}{c}\right) \\ &= (-1)^n q^{-\frac{1}{2}n(n-1)} (a/c)^n (c/a)_n \ ;\end{aligned}$$

similarly

$$\begin{aligned}(q^{-n+1}/c)_{n-k} &= \left(1 - \frac{q^{-n+1}}{c}\right)\left(1 - \frac{q^{-n+2}}{c}\right) \cdots \left(1 - \frac{q^{-k}}{c}\right) \\ &= (-1)^{n-k} q^{-\frac{1}{2}n(n-1)+\frac{1}{2}k(k-1)} c^{-n+k} (q^k c)_{n-k} \\ &= (-1)^{n-k} q^{-\frac{1}{2}n(n-1)+\frac{1}{2}k(k-1)} c^{-n+k} (c)_n/(c)_k.\end{aligned}$$

Hence (4.7) becomes

$$\sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} (c/a)^k = \frac{(c/a)_n}{(c)_n},$$

so that we have proved (3.3).

To prove (4.4), rewrite (4.6) in the form

(4.8)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b)_{n-k} b^k = (ab)_n .$$

Then exactly as above

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (q^{-n+1}/c)_{n-k} (q^{-n+1}/c)^k = (q^{-n+1}a/c)_n ,$$

which reduces to

$$\sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k+1)-nk} (a)_k (q^k c)_{n-k} = (c/a)_n a^n .$$

As for (4.5), we take

$$\frac{e(a)}{e(c)} \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n q^{\frac{1}{2}n(n-1)}}{(q)_n (c)_n} (c/a)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(c/a)^n}{(q)_n} \frac{e(q^n a)}{e(q^n c)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(c/a)^n}{(q)_n} \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} (q^n a)^k$$

$$= \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} a^k \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(q^k c/a)^n}{(q)_n}$$

$$= \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} a^k \frac{1}{e(q^k c/a)}$$

$$= \frac{1}{e(c/a)} \sum_{k=0}^{\infty} \frac{a^k}{(q)_k} = \frac{e(a)}{e(c/a)} .$$

This evidently proves (4.5).

5. The q-analog of Saalschütz's theorem reads

(5.1)
$$\sum_{k=0}^{n} \frac{(q^{-n})_k (a)_k (b)_k}{(q)_k (c)_k (d)_k} q^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} ,$$

where

(5.2)
$$cd = q^{-n+1} ab .$$

The theorem is usually proved (see for example [2, p. 68], [8, p. 96]) as a special case of a much more elaborate result for generalized basic hypergeometric series. We shall give a proof analogous to the proof in §2 of the ordinary Saalschütz theorem.

Making use of (5.2), we may rewrite (5.1) as follows.

$$\sum_{k=0}^{n} \frac{(q^{-n})_k (a)_k (b)_k}{(q)_k (c)_k (q^{-n+1}ab/c)_k} q^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} . \tag{5.3}$$

Since

$$(q^{-n})_k = (-1)^k q^{\frac{1}{2}k(k-1)-nk} (q)_n / (q)_{n-k} ,$$

$$(q^{-n+1}ab/c)_k = (-1)^k q^{\frac{1}{2}k(k+1)-nk} (ab/c)^k (c/ab)_n (c/ab)_{n-k} ,$$

(5.3) becomes

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k (b)_k}{(c)_k (c/ab)_n} (c/ab)_{n-k} \left(\frac{c}{ab}\right)^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} .$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (c)_k} \left(\frac{cx}{ab}\right)^k \sum_{n=0}^{\infty} \frac{(c/ab)_n}{(q)_n} x^n .$$

Hence, by (4.2), we have

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \frac{e(x)}{e(cx/ab)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (c)_k} \left(\frac{cx}{ab}\right)^k , \tag{5.4}$$

an identity due to Heine. Clearly (5.3) and (5.4) are equivalent, so it will suffice to prove (5.4).

By (4.3),

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n &= \sum_{n=0}^{\infty} \frac{(c/b)_n}{(q)_n} x^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k-1)} (c/a)^k \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k \sum_{n=0}^{\infty} \frac{(q^k c/b)_n}{(q)_n} x^n \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k \frac{e(x)}{e(q^k cx/b)} \\
&= \frac{e(x)}{e(cx/b)} \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k . \qquad (5.5)
\end{aligned}$$

Next, using (4.4) we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k q^{\frac{1}{2}k(k-1)}}{(q)_k (cx/b)_k} (cx/ab)^k \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(b)_j}{(c)_j} q^{\frac{1}{2}j(j+1)-jk} \\
&= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j (cx/b)_j} (cx/ab)^j \sum_{k=0}^{\infty} (-1)^k \frac{(q^j a)_k}{(q)_k (q^j cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/ab)^k .
\end{aligned}$$

By (4.5) the inner sum is equal to

$$\frac{e(q^j cx/b)}{e(cx/ab)} = \frac{e(cx/b)}{e(cx/ab)} (cx/b)_j .$$

Hence we have

$$(5.6) \qquad \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/ab)^k$$

$$= \frac{e(cx/b)}{e(cx/ab)} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j} (cx/ab)^j .$$

Combining (5.5) and (5.6), we get

$$(5.7) \qquad \sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \frac{e(x)}{e(cx/ab)} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j} (cx/ab)^j .$$

Thus we have proved (5.4) and so have proved (5.1).

6. We now give an application of (5.1). Making some changes in notation, (5.1) can be written in the following form.

$$(6.1) \qquad \sum_{j=0}^{k} \frac{(q^{-k})_j (q^k a)_j (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^j = \frac{(a/b)_k (a/c)_k}{(qb)_k (qc)_k} \left(\frac{qbc}{a}\right)^k .$$

It follows that

$$\sum_{k=0}^{\infty} \frac{(a)_k (a/b)_k (a/c)_k}{(q)_k (qb)_k (qc)_k} \left(\frac{qbcx}{a}\right)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k \sum_{j=0}^{k} \frac{(q^{-k})_j (q^k a)_j (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^j$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{(q)_k} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a)_{j+k} (qbc/a)_j}{(qb)_j (qc)_j} q^{\frac{1}{2}j(j+1)-jk}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j} (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^{-\frac{1}{2}j(j-1)} x^j \sum_{x=0}^{\infty} \frac{(q^{2j} a)_k}{(q)_k} (q^{-j} x)^k .$$

We now take $a = q^{-2m}$ and replace x by $q^m x$, where m is a non-negative integer. The above identity becomes

$$(6.2) \qquad \sum_{k=0}^{2m} \frac{(q^{-2m})_k (q^{-2m}/b)_k (q^{-2m}/c)_k}{(q)_k (qb)_k (qc)_k} (q^{3m+1} bcx)^k$$

$$= \sum_{j=0}^{m} (-1)^j \frac{(q^{-2m})_{2j} (q^{2m+1} bc)_j}{(q)_j (qb)_j (qc)_j} q^{mj - \frac{1}{2}j(j-1)} x^j \sum_{k=0}^{2m-2j} \frac{(q^{-2m+2j})_k}{(q)_k} (q^{m-j} x)^k .$$

The inner sum on the right is equal to

$$\sum_{k=0}^{2m-2j} (-1)^k \begin{bmatrix} 2m-2j \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (q^{-m+j} x)^k = (q^{-m+j} x)_{2m-2j} .$$

We have therefore proved the identity

$$(6.3) \qquad \sum_{k=0}^{2m} \frac{(q^{-2m})_k(q^{-2m}/b)_k(q^{-2m}/c)_k}{(q)_k(qb)_k(qc)_k}(q^{3m+1}bcx)^k$$

$$= \sum_{j=0}^{m} (-1)^j \frac{(q^{-2m})_{2j}(q^{-2m+1}/bc)_j}{(q)_j(qb)_j(qc)_j} q^{mj-\frac{1}{2}j(j-1)}x^j(q^{-m+j}x)_{2m-2j} .$$

For $x = 1$, (6.3) becomes

$$(6.4) \qquad \sum_{k=0}^{2m} \frac{(q^{-2m})_k(q^{-2m}/b)_k(q^{-2m}/c)_k}{(q)_k(qb)_k(qc)_k}(q^{3m+1}bc)^k$$

$$= (-1)^m \frac{(q)_{2m}(q^{-2m+1}/bc)_m}{(q)_m(qb)_m(qc)_m} q^{-\frac{1}{2}m(3m+1)} .$$

In particular, for $b = c = 1$, (6.4) reduces to

$$(6.5) \qquad \sum_{k=0}^{2m} (-1)^k \begin{bmatrix} 2m \\ k \end{bmatrix}^3 q^{\frac{3}{2}(m-k)^2+\frac{1}{2}(m-k)} = (-1)^m \frac{(q)_{3m}}{((q)_m)^3} ,$$

a result due to Jackson [5] and Bailey [1]. Jackson's more general results can also be proved [4] using the q-analog of Saalschütz's theorem.

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A DENSITY RELATIONSHIP BETWEEN $ax + b$ AND $[x/c]$

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This note is motivated by the following problem originating in combinatorial logic. Let f and g be the functions on the set of positive integers defined by $f(x) = 3x$ and $g(x) = [x/2]$, where $[r]$ denotes the greatest integer less than or equal to the real number r. Let Γ denote the collection of all composite functions formed by repeated applications of f and g. For which positive integers k does there exist $h \in \Gamma$ such that $h(1) = k$? For example, if f, g and Γ are defined as above, then

$$f(1) = 3, \qquad f^2(1) = 9, \qquad f^3(1) = 27, \qquad gf^3(1) = 13, \qquad fgf^3(1) = 39 \quad \text{and} \quad gfgf^3(1) = 19.$$

Thus, given any number from the collection $\{3, 9, 27, 13, 39, 39\}$ there exists an $h \in \Gamma$ such that $h(1)$ is the given number. The following theorem verifies that every positive integer can be obtained in this manner.

Before stating the theorem, the following conventions are adopted. The set of non-negative integers, the set of positive integers and the set of positive real numbers are denoted by N, N^+ and R^+, respectively. If f and g are functions on N to N, then the composite function $g \cdot f$ is defined by $g \cdot f(x) = g(f(x))$ and the functions obtained by repeated applications of f, n-times, will be denoted by f^n. If r is a real number then the greatest integer less than or equal to r is denoted by $[r]$. Finally, two integers a and c are said to be power related provided there exist $m,n \in N^+$ such that $a^m = c^n$.

Theorem 1. Let $a \neq 1$, $c \neq 1$ be positive integers. Let $b \in N$ and let f and g be the functions on N to N defined by $f(x) = ax + b$ and $g(x) = [x/c]$. If a and c are not power related and if $u,v \in N^+$, then there exist $m,n \in N^+$ such that $g^m \cdot f^n(u) = v$.

Using this theorem with $a = 3$, $b = 0$ and $c = 2$ and noting that 2 and 3 are not power related leads to the previously mentioned result.

A related theorem will be proved from which Theorem 1 will follow. Three lemmas will be employed. Indications of proof will be provided for all three.

Lemma 1. Let $a,c \in N^+$, $a \neq 1$, $c \neq 1$. The collection $\{a^n/c^m : m,n \in N\}$ is dense in R^+ if and only if a and c are not power related.

Proof. This result is well known and is generally considered to be folklore; a guide to its proof is given.

Using the continuity of the logarithm and results found on pages 71–75 of [1], the following statements can be shown to be equivalent.

(a) The collection $\{a^n/c^m : n,m \in N\}$ is dense in R^+.

(b) The collection $\{n - m(\log c/\log a) : n,m \in N\} \cap R^+$ is dense in R^+.

(c) The quotient $(\log c/\log a)$ is irrational.

(d) The numbers a abd c are not power related.

Lemma 2. Let a and b be positive integers with the additional property that the collection $\{a^n/c^m : n,m \in N\}$ is a dense subset of R^+. Then if $n_0 \in N^+$, the collection $\{a^n/c^m : n > n_0; n,m \in N\}$ is also a dense subset of R^+.

Proof. The subset $\{(a^n/c^m)^{n_0} : n,m \in N\} \subseteq \{a^n/c^m : n > n_0; n,m \in N\}$ is dense in R^+.

Lemma 3. Let $a,b \in N$, where $a \neq 0$ and $a \neq 1$. If f is defined on N by $f(x) = ax + b$, then

$$f^n(x) = a^n x + \frac{a^n - 1}{a-1}\, b = a^n \left(\frac{(a-1)x + b(1-a^{-n})}{a-1} \right)$$

for all $n \in N^+$.

Proof. A straightforward induction argument establishes the lemma.

Theorem 2. Let a and c be positive integers neither of which is 1. Let $b \in N$. Let f denote the function on N defined by $f(x) = ax + b$. If a and c are not power related, then for all $u \in N^+$, the collection

$$A(u) = \left\{ \frac{f^n(u)}{c^m} : m,n \in N \right\}$$

is dense in R^+.

Proof. Let $r \in R^+$ and let $\epsilon > 0$ be given. The quotient

$$\frac{r(a-1)}{(a-1)u + b(1-a^{-n})}$$

decreases as n increases and has limiting value

$$\frac{r(a-1)}{(a-1)u+b},$$

as $n \to \infty$. Choose n_0 such that $n > n_0$ implies

$$\frac{r(a-1)}{(a-1)u+b} + \frac{\frac{\epsilon}{2}(a-1)}{(a-1)u+b} > \frac{r(a-1)}{(a-1)u+b(1-a^{-n})} .$$

Then for $n > n_0$,

$$\frac{r(a-1)}{(a-1)u+b(1-a^{-n})} < \frac{(r+(\epsilon/2))(a-1)}{(a-1)u+b} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b} \leqslant \frac{(r+\epsilon)(a-1)}{(a-1)u+b(1-a^{-n})} .$$

Since a and c are not power related, Lemma 1 yields the fact that $\left\{ a^n/c^m : m,n \in N \right\}$ is a dense subset of R^+. By Lemma 2, it is possible to choose m_1, n_1 such that $n_1 > n_0$ and

$$\frac{(r+(\epsilon/2))(a-1)}{(a-1)u+b} < \frac{a^{n_1}}{c^{m_1}} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b} .$$

It follows that

$$\frac{r(a-1)}{(a-1)u+b(1-a^{-n_1})} < \frac{a^{n_1}}{c^{m_1}} < \frac{(r+\epsilon)(a-1)}{(a-1)u+b(1-a^{-n_1})}$$

and

$$r < \frac{a^{n_1}}{c^{m_1}} \, \frac{(a-1)u+b(1-a^{-n_1})}{a-1} < r+\epsilon .$$

By Lemma 3,

$$r < \frac{f^{n_1}(u)}{c^{m_1}} < r+\epsilon .$$

Hence $A(u)$ is dense in R^+.

An additional lemma will expedite the proof of Theorem 1.

Lemma 4. Let $c \in N^+$. Let g be defined on R^+ by $g(x) = [x/c]$. If $v \in N^+$ and if r is a real number such that $vc^n \leqslant r < (v+1)c^n$, then $g^n(r) = v$.

Proof. The proof is by induction on n. If $n = 1$, then $vc \leqslant r \leqslant (v+1)c$ implies $r = vc + s$, where $s \in R^+$ or $s = 0$ and $0 \leqslant s < c$. It follows that

$$g(r) = \left[\frac{vc+s}{c} \right] = \left[v + \frac{s}{c} \right] \quad \text{and} \quad \frac{s}{c} < 1 .$$

Hence $g(r) = v$. Suppose $g^k(r) = v$ whenever $vc^k \leqslant r < (v+1)c^k$. Suppose, in addition, that $vc^{k+1} \leqslant r_0 < (v+1)c^{k+1}$. Then

$$g^{k+1}(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) \quad \text{and} \quad vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

It follows that

$$vc^k \leqslant \frac{r_0}{c} < (v+1)c^k\ .$$

Hence by the induction hypothesis

$$g^{k+1}(r_0) = g^k \cdot g(r_0) = g^k\left(\left[\frac{r_0}{c}\right]\right) = v\ .$$

To prove Theorem 1, employ Theorem 2 to obtain positive integers n and m such that

$$v < \frac{f^n(u)}{c^m} < v+1$$

and apply Lemma 4.

REFERENCE

1. Ivan Niven, "Irrational Numbers," *The Carus Mathematical Monographs,* No. 11, published by The Mathematical Association of America.

Continued from page 22. *******

We can add any quantity B to each term:

$$x(a+B)^m + y(b+B)^m + (x+y-2)(ax+by+B)^m = (x+y-2)B^m + y(ax+by+B-b)^m + x(ax+by+B-a)^m$$

(where $m = 1, 2$).

A special case of a Fibonacci-type series is

$$1^m \quad 2^m \quad 3^m \quad \cdots \quad n^m\ .$$

Consider the series when $m = 2$:

(1) $\quad 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad \cdots \quad \cdots$

where

$$F_n = 3(F_{n-1} - F_{n-2}) + F_{n-3}$$

[we obtain our coefficients from Pascal's Triangle], i.e.,

$$(x+3)^2 = 3[(x+2)^2 - (x+1)^2] + x^2\ .$$

I have found by conjecture that

$$1^m - 4^m - 4^m - 4^m + 9^m + 9^m + 9^m - 16^m = -0^m - 12^m - 12^m - 12^m + 7^m + 7^m + 7^m + 15^m$$

(where $m = 1, 2$).

[I hope the reader will accept the strange -0^m for the time being.]

If we express the series (1) above in the form

$$a \quad b \quad 3(c-b)+a \quad \text{etc.,}$$

our multigrade appears as follows

$$a^m - 3b^m + 3c^m - [3(c-b)+a]^m = -0^m - 3(3c-4b+a)^m + 3(2c-3b+a)^m + [3(c-b)]^m$$

(where $m = 1, 2$).

We could, of course, write the above as

$$(x^2)^m - 3[(x+1)^2]^m + 3[(x+2)^2]^m - [3[(x+2)^2 - (x+1)^2] + x^2]^m$$
$$= -0^m - 3[x^2 - 4(x+1)^2 + 3(x+2)^2]^m + 3[x^2 - 3(x+1)^2 - 4(x+2)^2]^m + [3[(x+2)^2 - (x+1)^2]^m$$

(where $m = 1, 2$).

Continued on page 82.

GENERALIZED BELL NUMBERS

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1. INTRODUCTION

In the notation of Riordan [2], the Stirling numbers of the second kind, $S(n,k)$, with arguments n and k are defined by the relation

$$(1.1)\qquad t^n = \sum_{k=0}^{n} S(n,k)(t)_k, \qquad n > 0,$$

where $(t)_n = t(t-1)\cdots(t-n+1)$ is the factorial power function. They have been utilized by Tate and Goen [4] in obtaining the distribution of the sum of zero-truncated Poisson random variables where

$$(1.2)\qquad (e^t - 1)^k/k! = \sum_{n=k}^{\infty} S(n,k)t^n/n!\ .$$

The Bell numbers or exponential numbers B_n can be expressed as

$$(1.3)\qquad B_n = \sum_{k=0}^{n} S(n,k), \qquad n \geqslant 0,$$

with $B_0 \equiv 1$. They have been investigated by many authors: see [1] and [3] for lists of references. Uppuluri and Carpenter [7] have recently studied the moment properties of the probability distribution defined by

$$(1.4)\qquad p(k) = S(n,k)/B_n, \qquad k = 1, 2, \cdots, n,$$

and give

$$(1.5)\qquad \sum_{k=1}^{n} k^r S(n,k) = \sum_{i=1}^{r} \binom{r}{i} C_i B_{n+r-i}\ ,$$

where the sequence $\{C_n, n = 0, 1, \cdots\}$ is defined by

$$(1.6)\qquad \sum_{k=0}^{\infty} C_k x^k/k! = \exp(1 - e^x)\ .$$

Tate and Goen [4] have also derived the n-fold convolution of independent random variables having the Poisson distribution truncated on the left at 'c' in terms of the generalized Stirling numbers of the second kind, $d_c(n,k)$ given by

$$(1.7)\qquad (e^t - 1 - t - \cdots - t^c/c!)^k/k! = \sum_{n=k(c+1)}^{\infty} d_c(n,k)t^n/n!\ ,$$

where $d_c(n,k) = 0$ for $n < k(c+1)$. They give an explicit representation for $d_c(n,k)$ too complicated to reproduce here. The $d_c(n,k)$ can be shown to satisfy the recurrence formula

(1.8)
$$d_c(n+1,k) = kd_c(n,k) + \binom{n}{c} d_c(n-c, k-1),$$
where $d_c(0,0) = 1$ for all c.

Definition 1. We define the numbers $B_c(n)$ given by

(1.9)
$$B_c(n) = \sum_{k=0}^{n} d_c(n,k),$$

for $c \geqslant 1$ and $n \geqslant 0$ as generalized Bell numbers. It may be noted that $B_0(n) = B_n$.

Definition 2. A random variable X is said to have the generalized Bell distribution (GBD) if its probability function is given by

(1.10)
$$p_c(k) = d_c(n,k)/B_c(n), \qquad k = 0, 1, \cdots, n.$$

It may also be noted that when $c = 0$ and $n > 0$ (1.10) reduces to (1.4) as then $d_0(n,0) = 0$.

In this paper we investigate some properties of the numbers $B_c(n)$ and provide recurrence relations for the ordinary and factorial moments of the GBD. It is shown that the related results obtained by Uppuluri and Carpenter [7] follow as special cases for $c = 0$.

2. PROPERTIES OF $B_c(n)$

Property 1.

(2.1)
$$\sum_{n=0}^{\infty} B_c(n)t^n/n! = \exp\,(e^t - 1 - t - \cdots - t^c/c!).$$

This is immediately evident upon expansion of the right-hand side making use of (1.7).

Lemma 1.

(2.2)
$$d_c(n+1,k) = \sum_{m=0}^{n-c} \binom{n}{m} d_c(m, k-1).$$

Proof. Differentiating both sides of (1.7) with respect to t and expanding in powers of t we obtain

$$\sum_{r=c}^{\infty} \sum_{m=0}^{\infty} \binom{r+m}{m} d_c(m, k-1)t^{r+m}/(r+m)! = \sum_{n=0}^{\infty} d_c(n,k)t^{n-1}/(n-1)!\,.$$

Interchanging sums on the left-hand side and equating coefficients of t^n we are led to Lemma 1.

Property 2.

(2.3)
$$B_c(n+1) = \sum_{m=0}^{n-c} \binom{n}{m} B_c(m)\,.$$

This is now immediate from Definition 1 and Lemma 1. We note that when $c = 0$ (2.3) reduces to the known relation

$$B_{n+1} = \sum_{m=0}^{n} \binom{n}{m} B_m$$

for Bell numbers.

In attempting to find a recurrence relation in c for $B_c(n)$ we first need

Lemma 2.

(2.4)
$$d_c(n,k) = \sum_{i=0}^{k} [(-1)^i n!/i!(c!)^i(n-ci)!]\,d_{c-1}(n-ci, k-i),$$

for $c \geq 1$.

Proof. See Riordan [2], p. 102.

Using Lemma 2 we can now write

$$B_c(n) = \sum_{i=0}^{n} [(-1)^i \binom{n}{i} (n-i)!/(c!)^i(n-ci)!] \sum_{k=i}^{n} d_{c-1}(n-ci, k-i).$$

It follows directly from the above that we now have

Property 3.

$$B_c(n) = \sum_{i=0}^{n} [(-1)^i \binom{n}{i} (n-i)!/(c!)^i(n-ci)!] B_{c-1}(n-ci), \qquad c \geq 1. \tag{2.5}$$

The well-known Dobinski formula for Bell numbers has the form

$$B_{n+1} = e^{-1}(1^n + 2^n/1! + 3^n/2! + \cdots). \tag{2.6}$$

When $c = 1$ Property 1 gives us a formula similar to that of Dobinski.

Property 4.

$$B_1(n) = e^{-1}((-1)^n/1! + 1^n/2! + 2^n/3! + \cdots). \tag{2.7}$$

Property 3 suggests that we may write the generalized Bell numbers as a linear combination of the Bell numbers. Write the right-hand side of (2.1) in the form

$$\exp(e^t - 1 - t - t^2/2! - \cdots - t^c/c!) = \exp(e^t - 1)H(t), \tag{2.8}$$

where

$$H(t) = \sum_{r=0}^{\infty} b_c(r)t^r/r!, \qquad c \geq 1. \tag{2.9}$$

Property 5.

$$B_c(n) = \sum_{j=0}^{n} \binom{n}{j} b_c(j)B_{n-j}, \qquad c \geq 0. \tag{2.10}$$

Proof. Expand the right-hand side of (2.8) in powers of t. Property 5 now follows from (2.1), with $c = 0$, and (2.9).

For the purposes of enumeration the recurrence relation for $b_c(r)$,

$$b_c(r+1) = -\sum_{i=0}^{c-1} \binom{r}{i} b_c(r-i), \qquad c \geq 1, \tag{2.11}$$

with $b_0(j) = 0$ for all $j \geq 0$ and $b_c(0) = 1$, can be obtained by differentiating both sides of (2.8) with respect to t, using (2.9), and equating coefficients. With $b_1(j) = (-1)^j$ we alternately have Property 4 from Property 5.

Making use of the above properties, the first few values of $B_c(n)$ are as follows:

Table 1

Table for $B_c(n)$

$c \backslash n$	0	1	2	3	4	5	6	7
0	1	1	2	5	15	52	203	877
1	1	0	1	1	4	11	41	162
2	1	0	0	1	1	1	11	36

3. RECURRENCE RELATIONS FOR MOMENTS OF THE GBD

Let X be a random variable having the generalized Bell distribution defined by (1.10). The r^{th} ordinary moment of X is given by

$$\mu_c(x^r) = \sum_{k=0}^{n} k^r d_c(n,k)/B_c(n). \tag{3.1}$$

Let

$$B_c(n,r) = \sum_{k=0}^{n} k^r d_c(n,k). \tag{3.2}$$

Property 6.

$$B_c(n, r+1) = B_c(n+1, r) - \binom{n}{c} \sum_{j=0}^{r} \binom{r}{j} B_c(n-c, j). \tag{3.3}$$

Proof. Multiply both sides of (1.8) by k^r and sum over k. We have for every choice of c

$$B_c(n+1, r) = B_c(n, r+1) + \binom{n}{c} \sum_{k=0}^{n} k^r d_c(n-c, k-1)$$

$$= B_c(n, r+1) + \binom{n}{c} \sum_{j=0}^{r} \binom{r}{j} B_c(n-c, j).$$

Property 6 follows immediately. When $c = 0$, $B_0(n,r)$ becomes $B_n^{(r)}$ in [7] with Property 6 replaced by

Property 7.

$$B_n^{(r+1)} = B_{n+1}^{(r)} - \sum_{j=0}^{r} \binom{r}{j} B_n^{(j)} . \tag{3.4}$$

Property 7 is not given however by Uppuluri and Carpenter.

In attempting to express $B_c(n,r)$ as a linear combination of the generalized Bell numbers we are led after expanding (3.3) for the first few values of r to the following:

Property 8.

$$B_c(n,r) = \sum_{i=0}^{r} \sum_{j=0}^{i} a_{i,j}(n,r,c) B_c(n+r-i-jc), \tag{3.5}$$

where $a_{i,j}(n,r,c)$ satisfies the recurrence relation

$$a_{i,j}(n, r+1, c) = a_{i,j}(n+1, r, c)$$

$$- \binom{n}{c} \sum_{s=r-i+j}^{r} \binom{r}{s} a_{i+s-r-1,j-1}(n-c, s, c), \tag{3.6}$$

with $a_{0,0}(n,r,c) = 1$ and $a_{i,j}(n,r,c) = 0$ if $i > r$, $j > i$, or $j = 0$ and $i > 0$.

The proof consists of substituting (3.5) into (3.3) and equating appropriate coefficients.

Comparing (3.5) with (1.5) when $c = 0$ we must have

$$\sum_{j=0}^{i} a_{i,j}(n,r,0) = \binom{r}{i} C_i , \tag{3.7}$$

independent of n for $i = 1, 2, \cdots, r$. By starting with (3.6) and summing out j one can show that

$$C_{k+1} = -\sum_{i=0}^{k} \binom{k}{i} C_i \tag{3.8}$$

which agrees with Proposition 3 in [7]. We note also when $c = 0$

$$a_{i,j}(n,r,0) = (-1)^j \binom{r}{i} S(i,j), \tag{3.9}$$

independent of n, as (3.6) is then equivalent to

$$S(i,j) = \sum_{k=0}^{i-1} \binom{i-1}{k} S(k, j-1), \tag{3.10}$$

a property of Stirling numbers of the second kind.

Now let

$$W_c(n,r) = \sum_{j=0}^{n} (j)_r d_c(n,j). \tag{3.11}$$

Then the factorial moments of the generalized Bell distribution are given by

$$\nu_c((x)_r) = W_c(n,r)/B_c(n). \tag{3.12}$$

We now seek a recurrence formula for $W_c(n,r)$ and investigate the special case $c = 0$.

Property 9.

$$W_c(n, r+1) = W_c(n+1, r) - rW_c(n,r) - \binom{n}{c} [W_c(n-c, r) + rW_c(n-c, r-1)]. \tag{3.13}$$

Proof. From (3.11)

$$W_c(n, r+1) = \sum_{j=0}^{n} (j)_{r+1} d_c(n,j) = \sum_{j=0}^{n} j(j)_r d_c(n,j) - rW_c(n,r).$$

Hence

$$\sum_{j=0}^{n} j(j)_r d_c(n,j) = W_c(n, r+1) + rW_c(n,r). \tag{3.14}$$

Using (1.8) we can write, with $c \geqslant 1$,

$$W_c(n, r+1) = \sum_{j=0}^{n} (j)_r [d_c(n+1,j) - \binom{n}{c} d_c(n-c, j-1)] - rW_c(n,r)$$

$$= W_c(n+1, r) - rW_c(n,r) - \binom{n}{c} \sum_{j=0}^{n-1} (j+1)_r d_c(n-c,j).$$

Now with (3.14) and the fact that

$$(j+1)_r = j(j)_{r-1} + (j)_{r-1}$$

we have the desired recurrence relation stated in Property 9. One can verify directly that when $c = 0$ we have

Property 10.

$$W_0(n, r+1) = W_0(n+1, r) - (r+1)W_0(n,r) - rW_0(n, r-1), \tag{3.15}$$

so that (3.13) is true for all c.

The $W_0(n,r)$ may also be expressed as a linear combination of the Bell numbers. In fact using the same substitution procedure as before for Property 8 one can prove

Property 11.

$$W_0(n,r) = \sum_{i=0}^{r} a(r,i)B_{n+r-i}, \tag{3.16}$$

where $a(r,i)$ satisfies the recurrence relation

$$a(r+1,i) = a(r,i) - (r+1)a(r,i-1) - ra(r-1,i-2), \tag{3.17}$$

with $a(r,0) = 1$, $a(r,i) = 0$ if $i > r$, and $a(r,r) = (-1)^r$. A table of the $a(n,k)$ is as follows:

Table 2

Table for $a(n,k)$

(3.18)

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	−1					
2	1	−3	1				
3	1	−6	8	−1			
4	1	−10	29	−24	1		
5	1	−15	75	−145	89	−1	
6	1	−21	160	−545	814	−415	1

We note that the $a(n,k)$ are the coefficients of a special case of the Poisson-Charlier polynomials (cf. Szegö [6], p. 34). Touchard [5] gives formulas for the first seven polynomials corresponding to the coefficients in the table above. The polynomials take the form

$$h_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x)_{n-i}. \tag{3.19}$$

If we write

$$(x)_{n-i} = \sum_{k=0}^{n-i} s(n-i,k)x^k, \qquad n-i > 0, \tag{3.20}$$

where the $s(n,k)$ are the Stirling numbers of the first kind (see Riordan [2] p. 33), then

$$h_n(x) = \sum_{k=0}^{n} \left[\sum_{i=0}^{n-k} (-1)^i \binom{n}{i} s(n-i,k) \right] x^k. \tag{3.21}$$

Hence $a(n,k)$ has the representation

$$a(n,k) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} s(n-i,n-k). \tag{3.22}$$

Investigating the general case using similar procedures as before one can easily prove

Property 12.

$$W_c(n,r) = \sum_{i=0}^{r} \sum_{j=0}^{i} b_{i,j}(n,r,c)B_c(n+r-i-jc), \tag{3.23}$$

where $b_{i,j}(n,r,c)$ satisfies the recurrence relation

$$\begin{aligned} b_{i,j}(n, r+1, c) &= b_{i,j}(n+1, r, c) - rb_{i-1,j}(n,r,c) \\ &\quad - \binom{n}{c}\left[b_{i-1,j-1}(n-c, r, c) - rb_{i-2,j-1}(n-c, r-1, c)\right], \end{aligned} \tag{3.24}$$

with $b_{r,j}(n,r,c) = 0$, for $j = 0, 1, \cdots, r-1$, $b_{0,0}(n,r,c) = 1$, and $b_{r,r}(n,r,c) = (-1)^r n!/(c!)^n (n-rc)!$.

Comparing (3.16) and (3.23) when $c = 0$, we have

$$a(r,i) = \sum_{j=0}^{i} b_{i,j}(n,r,0). \tag{3.25}$$

Hence in view of (3.22)

$$b_{i,j}(n,r,0) = (-1)^j \binom{r}{j} s(r-j, r-i) \tag{3.26}$$

independent of n.

Recurrence relations for the ordinary and factorial moments are readily obtained from (3.3), (3.4), (3.13), and (3.15).

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A NOTE ON A THEOREM OF W. B. FORD

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W. B. Ford's theorem as stated in [1] on page 205 is incorrect. We observe that in Ford's proof, he claims

$$\lim_{n \to \infty} D_n = 0$$

on page 207 in [1]. But his hypotheses do not guarantee at all that $D_n \to 0$ as $n \to \infty$, when

$$\max_{n \to \infty} |g(2n + \tfrac{1}{2} + iy)| = \infty$$

for small values of y. Ford's proof holds, if we make an accurate restatement of Ford's theorem with appropriate generality, as follows:

If the coefficient $g(n)$ of the power series

$$f(z) = \sum_{n=0}^{\infty} g(n) z^n \tag{1}$$

radius of convergence > 0 may be considered as a function $g(s)$ of the complex variable $s = x + iy$ and as such satisfies the following two conditions, when considered throughout each right half plane $x > x_0$, where x_0 is any arbitrary large negative number.

(a) The function $g(s)$ is single valued and analytic except for a finite number of poles situated at the points $s = s_1, s_2, \cdots, s_p$ which lies within a Band B:

$$|Im\, s_i| < c, \qquad Re\, s_i < c,$$

where c is a fixed positive constant and $i = 1, 2, \cdots, p$. Furthermore, none of the s_i is a negative integer and p may increase as x_0 is decreased.

(b) For any point $s = x + iy$ to the right of the line $x = x_0$ and outside the Band B,

$$|g(x + iy)| < ke^{(\gamma+\epsilon)|y|}, \tag{2}$$

where γ is some fixed value such that $0 < \gamma < \pi$ and ϵ is any positive number. The value of k depends upon x_0 and ϵ.

Then the function $g(s)$ as defined by (1) will be analytic in a sector $S: \gamma < \arg z < 2\pi - \gamma$ and for z's of large modulus in Sector S, $f(z)$ may be developed asymptotically

$$f(z) \approx \sum_{n=1}^{\infty} r_n - \sum_{n=1}^{\infty} \frac{g(-n)}{z^n}, \tag{3}$$

where r_n represents the residue of the function

$$\frac{\pi g(x)(-z)^s}{\sin \pi s}$$

at the point $s = s_n$, $n = 1, 2, \cdots, p$.

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A METHOD FOR THE EVALUATION OF CERTAIN SUMS INVOLVING BINOMIAL COEFFICIENTS

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Recently T. V. Narayana presented two verifications of the sum

$$(1)\qquad S = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}\, u^r v^s = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} C_{r,s} u^r v^s$$

$$= \tfrac{1}{2}\left(1-u-v-\sqrt{1-2(u+v)+(u-v)^2}\,\right)$$

first derived by him in [1], and by Kreweras in [2], [3]. No direct proof of this formula seems to have been given. It is the purpose of this note to present an analytic derivation of Eq. (1) and to suggest a method more generally applicable to summing series with binomial coefficients. The method involves the introduction of an integral representation for at least one of the binomial coefficients.

To begin with let us transform the series of Eq. (1) by using the integral representation

$$(2)\qquad \frac{1}{r+s-1} = \int_1^{\infty} \frac{dt}{t^{r+s}}$$

and interchange the orders of summation and integration (a step that can be justified in detail for values of u and v for which the original series converges). Then we can write

$$(3)\qquad S = \int_1^{\infty} dt \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \binom{r+s-1}{r}\binom{r+s-1}{s}\left(\frac{u}{t}\right)^r\left(\frac{v}{t}\right)^s$$

so that we need only find the sum of the simpler series

$$(4)\qquad F(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \binom{r+s-1}{r}\binom{r+s-1}{s} x^r y^s$$

with $x = u/t,\ y = v/t$. At this point we introduce the integral representation

$$(5)\qquad \binom{r+s-1}{s} = \frac{1}{2\pi i}\oint \frac{(1+z)^{r+s-1}}{z^{s+1}}\, dz\,,$$

where the contour will be chosen as the unit circle. We can again interchange orders of summation and integration to find

$$(6)\qquad F(x,y) = \frac{1}{2\pi i}\oint \sum_{r=1}^{\infty} x^r \sum_{s=1}^{\infty} \binom{r+s-1}{r}\frac{(1+z)^{r+s-1}}{z^{s+1}}\, y^s dz\,.$$

But the summation over s can be effected explicitly using the formula

(7) $$\sum_{j=0}^{\infty} \binom{r+j}{j} a^j = \frac{1}{(1-a)^{r+1}}$$

valid for $|a| < 1$. In this way we find

$$F(x,y) = \frac{y}{2\pi i} \oint \frac{dz}{z^2} \sum_{r=1}^{\infty} x^r(1+z)^r \frac{z^{r+1}}{[z-(1+z)y]^{r+1}}$$

(8) $$= -\frac{xy}{2\pi i} \oint \frac{dz(1+z)}{[z(1-y)-y][xz^2+z(x+y-1)+y]}$$

$$= \frac{-y}{2\pi i(1-y)} \oint \frac{dz(1+z)}{\left(z-\frac{y}{1-y}\right)\left(z^2+z\left(\frac{x+y-1}{x}\right)+\frac{y}{x}\right)} .$$

The quadratic form in z can be factored in the form

(9) $$z^2 + z\left(\frac{x+y-1}{x}\right) + \frac{y}{x} = (z-z_+)(z-z_-) ,$$

where

(10) $$z_{\pm} = \frac{1}{2x}\left(1-x-y \pm \sqrt{(1-x-y)^2-4xy}\right) .$$

It is easily verified that the only root of Eq. (8) that lies in the unit circle as x or y tends to zero is z_-, hence in the evaluation of the contour integral in Eq. (8), we need only be concerned about the poles at $z = y/(1-y)$ and at $z = z_-$. The residue of the integrand at $z = y/(1-y)$ is found to be $(1-y)/y$ and the residue at $z = z_-$ is

(11) $$\frac{1+z_-}{(z_- - z_+)\left(z_- - \frac{y}{1-y}\right)} = \frac{-(1-y)(1-x-y+\sqrt{(1-x-y)^2-4xy})}{2y\sqrt{(1-x-y)^2-4xy}} .$$

If we add the contributions from the two poles we find

(12) $$F(x,y) = \frac{(1-x-y+\sqrt{(1-x-y)^2-4xy})}{2\sqrt{(1-x-y)^2-4xy}} - 1 = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2\sqrt{(1-x-y)^2-4xy}} .$$

If we now return to the integral over t, we find that S can be expressed as

(13) $$S = \int_1^{\infty} F(u/t, v/t)dt = \int_1^{\infty} \frac{t-u-v-\sqrt{(t-u-v)^2-4uv}}{2\sqrt{(t-u-v)^2-4uv}}\, dt .$$

Letting $t-u-v = \zeta$, we can transform this last integral to

(14) $$S = \int_{1-u-v}^{\infty} \frac{\zeta - \sqrt{\zeta^2-4uv}}{2\sqrt{\zeta^2-4uv}}\, d\zeta .$$

Finally, the substitution $\zeta = 2\sqrt{uv}\cosh\theta$ allows us to express S as

$$S = \sqrt{uv} \int_{\cosh^{-1}\left(\frac{1-u-v}{2\sqrt{uv}}\right)}^{\infty} e^{-\theta} d\theta = \sqrt{uv}\, \exp\left[-\cosh^{-1}\left(\frac{1-u-v}{2\sqrt{uv}}\right)\right]$$

$$= \tfrac{1}{2}\left(1-u-v-\sqrt{(1-u-v)^2-4uv}\right)$$

as found in the earlier references.

Another set of identities that has been the subject of several recent notes, [4]–[6], is the following

$$A = \sum_{n=0}^{N} (-1)^n \binom{n+\epsilon-1}{n}\binom{\epsilon}{N-n} = 0$$

(16)

$$B = \sum_{n=0}^{N} (-1)^n \binom{n+\epsilon-1}{N-1}\binom{N}{n} = 0$$

These can both be derived in the same way as the identity of Eq. (1). In the expression for A we note that the upper limit of the sum can be chosen to be ∞ if we use the convention that

$$\binom{a}{-j} = 0$$

for j any positive integer. If we then use an integral representation for $\binom{\epsilon}{N-n}$ we find

$$A = \frac{1}{2\pi i}\sum_{n=0}^{\infty} (-1)^n \binom{n+\epsilon-1}{n} \oint \frac{(1+z)^{\epsilon}}{z^{N+1-n}}\,dz = \frac{1}{2\pi i}\oint \frac{dz}{z^{N+1}} = 0 \quad . \tag{17}$$

Similarly the series of B can be expressed as

$$B = \frac{1}{2\pi i}\sum_{n=0}^{N} (-1)^n \binom{N}{n} \oint \frac{(1+z)^{n+\epsilon-1}}{z^{N}}\,dz = \frac{(-1)^N}{2\pi i}\oint (1+z)^{\epsilon-1}dz = 0\,. \tag{18}$$

where the contour can be suitably modified when a branch cut must be made.

The preceeding analysis is of interest not for its derivation of known results but because it gives a method that can be tried on many similar problems. In cases where a summation in closed form is not possible, the integral representation can sometimes lead to asymptotic results.

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ON THE ORDER OF SYSTEMS OF TWO SIMULTANEOUS LINEAR DIFFERENCE EQUATIONS IN TWO VARIABLES

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1. INTRODUCTION

Several techniques are known for solving general linear difference equations [1, 2, 3]. We are not concerned here with specific techniques for actually solving difference equations (whether numerically or symbolically). Rather, the main problem dealt with is that of determining the *order* (number of initial conditions) of systems of two simultaneous linear difference equations in two variables. Since the order for non-homogeneous equations is that of the associated homogeneous equations, we content ourselves with the homogeneous case. This paper examines the definition of two-dimensional sequences by a system of two simultaneous linear difference equations in two variables and the initial value problem is solved algorithmically.

While it is relatively simple in the one-dimensional case to specify suitable initial conditions, the same problem in two dimensions is considerably more complicated. The traditional algebraic approach relies upon the representation of the elements of two-dimensional sequences as the matrix product of two geometric progressions, one considered as a row matrix, the other as a column matrix.

In the author's algorithmic approach a suitably defined finite subset of elements of the sequence is selected. Using constraints determined by the difference equations, certain elements of the subset are chosen whose values are determined by the values of the remaining elements of the subset that in turn are determined by initial values. Induction is used to prove that the entire sequence is determined by the initial values. The algorithm has been programmed in FORTRAN.

2. THE DEFINITION OF TWO-DIMENSIONAL FIBONACCI SEQUENCES

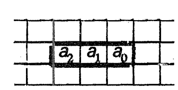
Any linear difference equation in one variable can be written in the following form:

$$c_0 f(i+m_0) = \sum_{k=1}^{n} c_k f(i+m_k) \ , \tag{1}$$

where f is a function on the integers, i.e., a sequence, $M = (m_k)$ is a vector with $n + 1$ integer components, and the c's are non-zero coefficients and are distinct. For some purposes, it is more convenient to express linear difference equations diagrammatically rather than strictly algebraically as in (1). For example, the diagram, or *pattern* as we will call it, for the Fibonacci recursion relation is shown in Fig. 1. The two variable equation corresponding to (1) is

$$c_0 f(i+m_0, j+n_0) = \sum_{k=1}^{n} c_k f(i+m_k, j+n_k), \tag{2}$$

where f is a function of two integer variables, m_k corresponds to the column index for the k^{th} term, and n_k corresponds to the row index. $M = (m_k)$ and $N = (n_k)$ are vectors with $n + 1$ integer components with $(m_i, n_i) \neq (m_j, n_j)$ if $i \neq j$. The c's are non-zero coefficients.



$$a_0 = a_1 + a_2$$

Figure 1 Pattern for the Fibonacci Recursion Relation

One of the simplest non-trivial two dimensional sequences is the "Fibonacci Multiplication Table," derived from the simultaneous equations

$$f(i+2, j) = f(i+1, j) + f(i, j) \tag{3}$$

$$f(i, j+2) = f(i, j+1) + f(i, j). \tag{4}$$

The pattern for Eq. (3) is

$$\begin{array}{|c|c|c|}\hline a_2 & a_1 & a_0 \\ \hline\end{array} \qquad a_0 = a_1 + a_2$$

and the pattern for Eq. (4) is

$$\begin{array}{|c|}\hline b_2 \\ \hline b_1 \\ \hline b_0 \\ \hline\end{array} \qquad b_0 = b_1 + b_2 \ .$$

Equations (3) and (4) with initial conditions

$$f(0,0) = 0, \qquad f(0,1) = 0,$$
$$f(1,0) = 0, \qquad f(1,1) = 1,$$

lead to a sequence with the property that

$$f(1, j) = f(i, 1)f(1, j). \tag{5}$$

Since row 1 and column 1 contain ordinary Fibonacci sequences, the sequence may be looked at as a multiplication table for the Fibonacci numbers.

3. INITIAL CONDITIONS FOR LINEAR DIFFERENCE EQUATIONS

In the one-dimensional case, the order is easily determined by inspection (see [4]). If the equation is written in the form of (1), the order, which we will call N_g is

$$N_g = \max_{i,j} |m_i - m_j| . \tag{6}$$

This number N_g is also one less than the width in grid squares of the pattern for the equation.

The set $G = \quad g_k$ giving a possibility for the relative positions of the initial values will be diagrammed on a grid in analogy to the way patterns are diagrammed. For example, the pattern

$$\begin{array}{|c|c|c|c|c|}\hline a_2 & & a_1 & & a_0 \\ \hline\end{array} \qquad a_0 = a_1 + a_2$$

requires four initial values, since the width is 5. One valid g-pattern for these initial values is

$$\begin{array}{|c|c|c|c|c|c|}\hline g_1 & & g_2 & g_2 & & g_4 \\ \hline\end{array}$$

In the traditional algebraic approach to initial conditions in two dimensions, we represent the solution by a matrix product of two geometric progressions. If we use R for the horizontal ratio and S for the vertical ratio, then the analog of Eq. (2) is

$$c_0 R^{m_0} S^{n_0} = \sum_{k=1}^{n} c_k R^{m_k} S^{n_k} . \tag{7}$$

If we form the Eqs. (7) for two patterns, and let the first pattern have degree d_1 in R and e_1 in S, and the equation for the second pattern have degree d_2 in R and e_2 in S, then solving the equations simultaneously using the resultant (see [5]), we find there are at most M_g initial conditions required, where

$$M_g = (d_1 + d_2) \max(e_1, e_2) \min(d_1, d_2). \tag{8}$$

This method may require much tedious algebraic manipulation. Also, the theory does not provide in general even one valid g-pattern. The algorithm described in the following section solves the initial value problem without relying on geometric progressions. Also, it has the advantage of yielding a family of valid g-patterns.

4. AN EFFICIENT ALGORITHM FOR DETERMINING SETS OF INITIAL CONDITIONS IN TWO DIMENSIONS

Given two patterns for two linear difference equations in two unknowns, the algorithm described in this section first constructs a special set of adjacent grid squares (corresponding to the elements of a two-dimensional sequence) called a *starting set.* Then the number of initial conditions necessary and sufficient to determine the values for all of the elements in the starting set is calculated by matrix operations on the coefficients of equations implied by the difference equations. A form of two-dimensional induction is attempted to check whether the values for the entire sequence can be determined from the equations already solved and the values for the elements of the starting set. If the induction step fails, either the equations were not independent, or the starting set was not large enough. Assuming the latter, the starting set is enlarged and the procedure is repeated until either the induction step succeeds, or too many initial conditions are required for the equations to have been independent.

ALGORITHM FOR THE TWO-DIMENSIONAL INITIAL VALUE PROBLEM

Given two patterns with elements labelled $a_0, a_1, \cdots, a_n$ and $b_0, b_1, \cdots, b_m$, representing two linear difference equations in two variables, find N_g, the number of initial values necessary and sufficient to define a complete two-dimensional sequence and find at least one valid g-pattern if N_g is finite.

STEP 1. (Initialize the first starting set.) Fix the position of the pattern whose squares are labelled with a's on a grid. Let S be the set of all grid squares necessary and sufficient to represent

$$a_i = b_0 = \sum_{k=1}^{m} b_k$$

for $i = 0, 1, \cdots, n.$ (Note that the squares representing $a_0, a_1, \cdots, a_n$ are in S.)

STEP 2. (Check for horizontal gaps.) If there is no element of $\bar{S}$ between two elements of S in the same row of the grid, then go to Step 4.

STEP 3. (Augment S to reduce a horizontal gap.) Replace S by $S \cup R$, where R is the set of all grid squares one grid square to the right of a grid square in S. Go to Step 2.

STEP 4. (Check for vertical connectedness.) If each row containing an element of S (except the bottom-most) contains an element of S that is vertically adjacent to an element of S in the next row down, then go to Step 6.

STEP 5. (Augment S to reduce a vertical gap.) Replace S by $S \cup B$, where B is the set of all grid squares one grid square below an element of S. Go to Step 2.

STEP 6. (Set up equations.) Associate the i^{th} grid square in S with the variable x_i. Form M' as a coefficient matrix with columns representing the variables x_i and whose rows are the coefficients of all possible equations determined by the two patterns and involving only elements of the starting set S.

STEP 7. (Echelonize M'.) Put M' into echelon form M.

STEP 8. (Count free variables.) Label the distinguished column variables of M with x_i's, and the free-variable columns with g_i's. Let n_g = the number of g_i's. (Note that n_g is also the difference between the number of columns and the number of non-zero rows of M.)

STEP 9. (Check for dependent equations.) If $n_g > M_g$ from Eq. (8) then stop. $N_g = \infty$.

STEP 10. (Check horizontal induction.) Check whether M is row-equivalent to a matrix G, in echelon form, all of whose free variables correspond to grid squares of S that have a grid square corresponding to a distinguished variable on the right. (In forming G, columns of M may be interchanged if a non-zero value appears in both columns for any one row.) If so, go to Step 12.

STEP 11. (Augment S.) Replace S by $S \cup T$, where T is the set of all elements that are one grid square left of, right of, above, or below an element of S. Go to Step 6.

STEP 12. (Check vertical induction.) Check whether M is row equivalent to an echelon matrix H all of whose free variables correspond to grid squares that have a grid square corresponding to a distinguished variable one grid square below. If not, go to Step 11. Otherwise, the algorithm terminates, N_g is equal to the n_g calculated in Step 8. The grid squares correspond to the g_k's for the matrices M, G, or H, form valid g-patterns. □

The goal of Steps 1 through 5 is to find a starting set with the following properties:

(1) The set should be connected, that is, it should be possible to go from each element to every other element remaining within the set and using only moves of one square horizontally or vertically.

(2) Every element of S should appear in at least one equation formed in Step 6.

The reason behind requirement (1) is that it has been found empirically that when it is satisfied the algorithm never needs to execute Step 11 and repeat Step 6 and the following steps. This has not been proved, however. The reason for requirement (2) is to avoid introducing extraneous free variables into the starting set. If an element appears in no equations for a particular starting set, that element will always appear to be a free variable, even though it would not necessarily be free if a larger starting set were used that allowed it to appear in an equation.

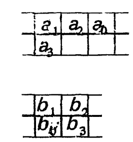
We now give a proof that the number n_g calculated in Step 8 is always a lower bound on the number of initial conditions necessary to define a complete sequence.

Proof. If Step 12 is reached and is successful, then n_g is a sufficient number of initial conditions, and all the values for elements of the sequence outside the starting set are derivable from the values of the starting set given n_g initial conditions in the positions of the free variables (the g's). Including equations involving elements outside the starting set would not add any new information to the system. If either the horizontal or the vertical induction fails, and all bordering values are not deriveable, then, since each g_i is necessary because at least one x_j depends on it, n_g is a lower bound on the number of initial conditions required. □

The procedure must terminate (i.e., is an algorithm) because, if a finite number of initial conditions exists, the starting set must eventually include at least one possible set of locations for those initial conditions, since all of the elements in the sequence are eventually included in the starting set.

The claim for efficiency in the title of this section is based on the observation that, for all cases tried, the number of zero rows in the matrix M, the echelon form of M', is $(N3 + 1)(N5 + 1)$, where $N3$ is the number of times Step 3 was executed in constructing S, and $N5$ is the number of times Step 5 was executed. This means that, if deriving values for the elements of a two-dimensional sequence is the object of discovering the number N_g and a valid g-pattern, most of the rows of M, with the exception of a limited number of zero rows, are useful for back-substitution in M given values for the g_i's. Also, using the two-dimensional induction technique, the value for any element in the sequence can be determined using only repeated back-substitution in M.

As a specific example of the algorithm, we give the results for the two patterns shown in Fig. 2. The number of initial conditions n_g for this case is 3, and a valid g-pattern is shown in Fig. 3. A portion of the two-dimensional sequence determined by $g_1 = 0$, $g_2 = 1$, $g_3 = 2$ is shown in Fig. 4. More detail on the operation of the algorithm, as well as the results for many other cases, are given in [6].



$$a_0 = a_1 + a_2 + a_3$$

$$b_0 = b_1 + b_2 + b_3$$

Fig. 2 Patterns for $f(m + 2, n + 1) = f(m, n + 3) + f(m + 1, n + 1) + f(m,n)$ and $f(m,n) = f(m, n + 3) + f(m + 1, n + 1) + f(m + 1, n)$

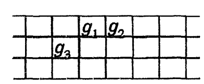


Fig. 3 A valid g-pattern for the patterns in Fig. 2

			43	−30	21	−14	11	-6	5	-6	-5
−23	16	−11	8	-5	4	-3	0	−5	−4	-3	8
−4	3	−2	1	−2	−1	−2	1	6	15	22	17
−1	0	−1	0	1	4	7	8	1	−20		
0	1	2	3	2	−3	−14	−29	−38	−19		
1	0	−3	-8	−13	−12	5	48				
−4	−5	−2	9	30	55						
7	16	23	16	−23							
0	−23										

Fig. 4 A portion of a two-dimensional sequence satisfying the patterns shown in Fig. 2. The initial values are circled.

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Continued from page 66. *******

If we add any quantity B to each term, the above becomes

$$(x^2+B)^m - 3[(x+1)^2+B]^m + 3[(x+2)^2+B]^m - [3[(x+2)^2-(x+1)^2]+x^2+B]^m$$
$$= -B^m - 3[x^2-4(x+1)^2+3(x+2)^2+B]^m + 3[x^2-3(x+1)^2-4(x+2)^2+B]^m + [3[(x+2)^2-(x+1)^2]+B]^m$$

(where $m = 1, 2$).

Finally, take the series in which

$$F_n = A_{n-1}F_{n-1} + A_{n-2}F_{n-2} \cdots A_2F_2 + A_1F_1 .$$

We conjecture that

$$A_1F_1^m + A_2F_2^m + A_3F_3^m \cdots A_{n-2}F_{n-2}^m + A_{n-1}F_{n-1}^m + \left(\sum_1^{n-1} A - 2\right)F_n^m$$

(2)

$$= A_1(F_n-F_1)^m + A_2(F_n-F_2)^m + A_3(F_n-F_3)^m \cdots A_{n-2}(F_n-F_{n-2})^m + A_{n-1}(F_n-F_{n-1})^m + \left(\sum_1^{n-1} A-2\right)0^m$$

(where $m = 1, 2$).

Proof: When $m = 1$,

$$\text{L.H.S.} = \left(\sum_1^{n-1} A - 1\right)F_n .$$

When $m = 1$,

$$\text{R.H.S.} = (A_1+A_2+A_3\cdots A_{n-2}+A_{n-1})F_n - (A_1F_1+A_2F_2+A_3F_3\cdots A_{n-2}F_{n-2}+A_{n-1}F_{n-1}) = \left(\sum_1^{n-1} A-1\right)F_n .$$

$$\therefore\ \text{L.H.S.} = \text{R.H.S.}$$

When $m = 2$,

$$\text{L.H.S.} = A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-2}F_{n-2}^2 + A_{n-1}F_{n-1}^2 + \left(\sum_1^{n-1} A-2\right)F_n^2 .$$

When $m = 2$,

$$\begin{aligned}\text{R.H.S.} &= A_1F_n^2 - 2A_1F_1F_n + A_1F_1^2 + A_2F_n^2 - 2A_2F_2F_n + A_2F_2^2 \\ &\quad + A_3F_n^2 - 2A_3F_3F_n + A_3F_3^2 + \cdots \\ &\quad + A_{n-1}F_n^2 - 2A_{n-1}F_{n-1}F_n + A_{n-1}F_{n-1}^2 \\ &= \sum_1^{n-1} AF_n^2 - 2F_n \cdot F_n + A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-1}F_{n-1}^2 \\ &= \left[\sum_1^{n-1} A-2\right]F_n^2 + A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-1}F_{n-1}^2 = \text{L.H.S.}\end{aligned}$$

If we add any quantity B to each term, we get

$$A_1(F_1+B)^m + A_2(F_2+B)^m + A_3(F_3+B)^m \cdots A_{n-2}(F_{n-2}+B)^m + A_{n-1}(F_{n-1}+B)^m + \left(\sum_1^{n-1} A-2\right)(F_n+B)^m$$

$$= A_1(F_n-F_1+B)^m + A_2(F_n-F_2+B)^m + A_3(F_n-F_3+B)^m \cdots A_{n-2}(F_n-F_{n-2}+B)^m + A_{n-1}(F_n-F_{n-1}+B)^m$$
$$+\left(\sum_1^{n-1} A-2\right)B^m \quad (\text{where } m = 1, 2).$$

Continued on page 92.

LUCAS POLYNOMIALS AND CERTAIN CIRCULAR FUNCTIONS OF MATRICES

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INTRODUCTION

1. The *fundamental function* $U_n(p,q)$ as defined by Lucas [4] uses the second-order recurrence relation

(1) $$U_{n+2} = pU_{n+1} - qU_n \qquad (n \geqslant 0)$$

with initial values $U_0 = 0$ and $U_1 = 1$. For example, we find by calculation, that

(1') $$\begin{cases} U_2 = p & U_3 = p^2 - q \\ U_4 = p^3 - 2pq & U_5 = p^4 - 3p^2q + q^2 \cdots \end{cases}$$

so that, by induction

(2) $$U_n = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} p^{n-2r} q^r$$

As the sequence $\{U_n\}$ has only been defined for $n \geqslant 0$, and as we often require negative-valued subscripts, we find, by calculation of the U's that

(3) $$U_{-n} = -q^{-n} U_n$$

to allow unrestricted values of n.

2. In addition, Lucas [4] also defined the *primordial function* $V_n(p,q)$ by

(4) $$V_{n+2} = pV_{n+1} - qV_n \qquad (n \geqslant 0)$$

with $V_0 = 2$ and $V_1 = p$. For example,

(4') $$\begin{cases} V_2 = p^2 - 2q & V_3 = p^3 - 3pq \\ V_4 = p^4 - 4p^2q + 2q^2 & V_5 = p^5 - 5p^3q + 5pq^2 \end{cases}$$
$$\cdots$$

As in Lucas [4], it can easily be verified that

(5) $$V_{2n+1} = pU_{2n+1} - 2qU_{2n}$$

and

(6) $$V_{2n+1} = 2U_{2n+2} - pU_{2n+1}\,.$$

3 In [1], Barakat considered the matrix exponential e^X for the 2×2 matrix

(7) $$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where he took

(8) $$trX = p \qquad \text{and} \qquad \det X = q\,.$$

By showing that we could express X^n in terms of the U_n for unrestricted values of n, viz:

$$X^n = U_nX - qU_{n-1}I \qquad \text{and} \qquad X^{-n} = -qU_{-n}X^{-1} + U_{-n+1}I \tag{9}$$

(where I is the unit matrix of order 2).

Barakat [1] was then able to obtain various summation formulas for the Lucas polynomials by the use of the matrix exponential function, where

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{and} \qquad e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} X^{-n} \tag{10}$$

4. It is the purpose of this paper to extend the work of Barakat [1] by considering the matrix sine and cosine for 2×2 matrices, and their corresponding connections with the sequences $\{U_n\}$ and $\{V_n\}$. As special cases, we will then examine the relationships between the Lucas polynomials and the Chebychev polynomials. We commence with an investigation of the sine of a matrix. For every square matrix X, the sine of X is defined by the power series

$$\sin X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n+1)!} \tag{11}$$

We then give a set of parallel results for the cosine function, where we define the cosine of every square matrix X by the power series

$$\cos X = \sum_{n=0}^{\infty} (-1)^n \frac{X^{2n}}{(2n)!} \,. \tag{12}$$

Expansions (11) and (12) are perfectly valid since, as the functions $\sin z$ and $\cos z$ converge for all z, the eigenvalues of X lie within the circle of convergence of radius $R = \infty$.

Summation Formulas – The Sine

5. If we substitute (9) into (11), then

$$\sin X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (U_{2n+1}X - qU_{2n}I)$$

Thus, we have

$$\sin X = X \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+1} - Iq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n} \tag{13}$$

6 By using Sylvester's matrix interpolation formula, viz. Bellman [2]:

If $f(t)$ is a polynomial of degree $\leqslant N-1$, and if $\lambda_1, \lambda_2, \cdots \lambda_N$ are the N distinct eigenvalues of X, then

$$f(X) = \sum_{i=1}^{N} f(\lambda_i) \prod_{\substack{1\leqslant j\leqslant N\\ j\neq i}} \left[\frac{X - \lambda_j I}{\lambda_i - \lambda_j}\right] \,, \tag{14}$$

we can show that if λ_1 and λ_2 are the eigenvalues of our 2×2 matrix X defined in (7), then

$$f(X) = \sum_{i=1}^{2} f(\lambda_i) \prod_{\substack{1\leqslant j\leqslant N\\ j\neq 1}} \left[\frac{X - \lambda_j I}{\lambda_i - \lambda_j}\right] = f(\lambda_1) \prod_{\substack{1\leqslant j\leqslant N\\ j\neq 1}} \left[\frac{X - \lambda_1 I}{\lambda_1 - \lambda_j}\right] + f(\lambda_2) \prod_{\substack{1\leqslant j\leqslant N\\ j\neq 2}} \left[\frac{X - \lambda_2 I}{\lambda_2 - \lambda_j}\right] =$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_1 I)f(\lambda_1)\right\} - \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_2 I)f(\lambda_2)\right\}$$

Hence, we have

$$\sin X = \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_2 I)\sin\lambda_1 - (X - \lambda_2 I)\sin\lambda_2\right\}$$

so that

(15) $$\sin X = \frac{1}{\lambda_1 - \lambda_2}\left[(\sin\lambda_1 - \sin\lambda_2)X - (\lambda_1\sin\lambda_1 - \lambda_2\sin\lambda_2)I\right]$$

7. Now, the characteristic equation of X is

$$|X - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$
$$=: a_{11}a_{22} - \lambda(a_{11} + a_{22}) + \lambda^2 - a_{12}a_{21}$$
$$= \lambda^2 - p\lambda + q = 0\,.$$

Thus, as in Barakat [1], the eigenvalues λ_1 and λ_2 satisfy the quadratic equation

(16) $$\lambda^2 - p\lambda + q = 0$$

so that

(17) $$\lambda_1 = \frac{p+\delta}{2} \quad \text{and} \quad \lambda_2 = \frac{p-\delta}{2} \quad \text{(say)}$$

(18) $$\delta = \Delta^{1/2} = (p^2 - 4q)^{1/2}\,.$$

8. Substituting these values for λ_1 and λ_2 in (15) eventually gives

(19) $$\sin X = \left[2\delta^{-1}\sin\frac{\delta}{2}\cos\frac{p}{2}\right]X - \left[\delta^{-1}p\sin\frac{\delta}{2}\cos\frac{p}{2} + \sin\frac{p}{2}\cos\frac{\delta}{2}\right]I\,.$$

Thus, on comparing Eqs. (13) and (19), we see that

(20) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1} = 2\delta^{-1}\sin\frac{\delta}{2}\cos\frac{p}{2}$$

and

(21) $$q\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n} = \delta^{-1}p\sin\frac{\delta}{2}\cos\frac{p}{2} + \sin\frac{p}{2}\cos\frac{\delta}{2}\,.$$

9. If we rewrite (5) in the form

(22) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1} - 2q\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n}$$

we have, on using (20) and (21), that

(23) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = -2\sin\frac{p}{2}\cos\frac{\delta}{2}\,.$$

Re-writing (6) as

(24) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+2} - p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1}$$

gives

(25) $$2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} + p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1}$$

Using (20) and (24) in (25) yields, on calculation,

(26) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+2} = \delta^{-1} p \sin\frac{\delta}{2}\cos\frac{p}{2} - \sin\frac{p}{2}\cos\frac{\delta}{2}.$$

Summation Formulas – The Cosine

10. If we parallel the work in paragraphs 5 to 9 for the cosine of the matrix X as defined in (12), we also have the following results:

(27) $$\cos X = I - X\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} + Iq\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1}$$

so that

(28) $$\cos X = -X\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} + Iq\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1}$$

since, when $n = 0$,

$$-X(-1)U_0 = 0$$

on using (1) and

$$IqU_{-1} = Iq\cdot q^{-1} = I$$

on using (3).

(29) $$\cos X = \left[-2\delta^{-1}\sin\frac{p}{2}\sin\frac{\delta}{2}\right]X - \left[\cos\frac{p}{2}\cos\frac{\delta}{2} - \delta^{-1}p\sin\frac{p}{2}\sin\frac{\delta}{2}\right]I$$

(30) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} = 2\delta^{-1}\sin\frac{p}{2}\sin\frac{\delta}{2}$$

(31) $$q\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1} = \delta^{-1}p\sin\frac{p}{2}\sin\frac{\delta}{2} - \cos\frac{p}{2}\cos\frac{\delta}{2}$$

(32) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} V_{2n} = 2\cos\frac{p}{2}\cos\frac{\delta}{2}$$

(33) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n+1} = \delta^{-1}p\sin\frac{p}{2}\sin\frac{\delta}{2} + \cos\frac{p}{2}\cos\frac{\delta}{2}$$

Chebychev Polynomials

11. As in Horadam [3], which deals among other things with Chebychev polynomials in relation to a certain generalized recurrence sequence, write

(34) $$x = \cos\theta \quad \text{with} \quad p = 2x \quad \text{and} \quad q = 1.$$

Then the U_n are precisely the Chebychev polynomials of the first kind, $S_n(x)$. Thus

(35) $$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \qquad (n \geqslant 0),$$

where

(36) $$S_{n+2} = 2xS_{n+1} - S_n \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = 1.$$

Likewise, the V_n are the Chebychev polynomials of the second kind, $t_n(x) = 2T_n(x)$, where

(37) $$T_{n+2} = 2xT_{n+1} - T_n \quad \text{with} \quad T_0 = 1 \quad \text{and} \quad T_1 = x$$

so that

(38) $$t_0 = 2 \qquad \text{and} \qquad t_1 = 2x\,(=p).$$

Thus

$$V_n(2x, 1) = 2T_n(x) = 2\cos n\theta \quad (n \geqslant 0). \tag{39}$$

Putting $q = 1$ in (20) and using (35) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} S_{2n+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)!\sin x}$$

$$= \frac{1}{\sin X} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left\{ \frac{e^{i(2n+1)x} - e^{-i(2n+1)x}}{2i} \right\}$$

$$= \frac{1}{\sin X} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (e^{ix})^{2n+1}}{(2n+1)!} - \frac{(-1)^n (e^{-ix})^{2n+1}}{(2n+1)!} \right\}$$

$$= \frac{1}{2i\sin x} \left\{ \sin e^{ix} - \sin e^{-ix} \right\}$$

$$= \frac{1}{2i\sin x}\, 2\cos\frac{e^{ix}+e^{-ix}}{2}\, \sin\frac{e^{ix}-e^{-ix}}{2}$$

$$= \frac{1}{i\sin x} \cos(\cos x)\sin(i\sin x)$$

$$= \frac{1}{i\sin x} \cos(\cos x) i\sinh(\sin x)$$

$$= \frac{\cos(\cos x)\sinh(\sin x)}{\sin x}.$$

Thus, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)!\sin x} = \frac{\cos(\cos x)\sinh(\sin x)}{\sin x}. \tag{40}$$

Similarly, from (21), (30) and (32) and using (35) and (37), it can be shown that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cos(2n+1)x = \sin(\cos x)\cosh(\sin x) \tag{41}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\sin 2nx}{(2n)!\sin x} = \frac{\sin(\cos x)\sinh(\sin x)}{\sin x} \tag{42}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\cos 2nx}{(2n)!} = -\cos(\cos x)\cosh(\sin x) \tag{43}$$

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-258 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum x^a y^b z^c t^d,$$

where the summation is over all non-negative a, b, c, d such that

$$\begin{cases} 2a \leqslant b+c+d \\ 2b \leqslant a+c+d \\ 2c \leqslant a+b+d \\ 2d \leqslant a+b+c. \end{cases}$$

H-259 Proposed by R. Finkelstein, Tempe, Arizona.

Let p be an odd prime and m an odd integer such that $m \not\equiv 0 \pmod p$. Let $F_{mp} = F_p \cdot Q$. Can $(F_p, Q) > 1$?

H-260 Proposed by H. Edgar, San Jose State University, San Jose, California.

Are there infinitely many subscripts, n, for which F_n or L_n are prime?

Editorial Note: Good luck on this one!

SOLUTIONS

CORRECTION

H-179 Proposed by D. Singmaster, Bedford College, University of London, England.

Let k numbers $p_1, p_2, \cdots, p_k$ be given. Set $a_n = 0$ for $n < 0$; $a_0 = 1$ and define a_n by the recursion

$$a_n = \sum_{i=1}^{k} p_i a_{n-i} \quad \text{for} \quad n > 0.$$

1. Find simple necessary and sufficient conditions on the p_i for $\lim_{n \to \infty} a_n$ to exist and be (a) finite and non-zero, (b) zero, (c) infinite.
2. Are the conditions: $p_i \geqslant 0$ for $i = 1, 2, \cdots, p_1 > 0$ and

$$\sum_{i=1}^{k} p_i = 1$$

sufficient for $\lim_{n \to \infty} a_n$ to exist, be finite and be non-zero?

SOME SQUARE

H-230 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

(a) If 5 is a quadratic nonresidue of a prime p ($p \neq 5$), then $p \mid F_{k(p+1)}$, k a positive integer.
(b) If 5 is a quadratic residue of a prime p, then $p \mid F_{k(p-1)}$, k a positive integer.

Solution by J. L. Hunsucker, University of Georgia, Athens, Georgia.

In problem H-221 of this Journal (Vol. 2, No. 3), L. Carlitz gave the theorem:

Let p be an odd prime, $p \neq 5$. If $p \equiv 1 \pmod 4$ then $(F_{p-1}/2) \equiv 0 \pmod p$ for $(5/p) = 1$ and $(F_{p+1}/2) \equiv 0 \pmod p$ for $(5/p) = -1$; if $p \equiv 3 \pmod 4$ then $(L_{p-1}/2) \equiv 0 \pmod p$ for $(5/p) = 1$ and $(L_{p+1}/2) \equiv 0 \pmod p$ for $(5/p) = -1$.

Using the theorem that $F_n \mid F_{kn}$ in the case $p \equiv 1 \pmod 4$ and for the case $p \equiv 3 \pmod 4$, using in addition to $F_n \mid F_{kn}$, the theorem that $L_n \mid F_m$ if and only if $m = 2kn$ we see that H-230 follows immediately from H-221.

Also solved by P. Tracy and the Proposer.

RECURRENT THEME

H-231 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

1. Let $A_0 = 0$, $A_1 = 1$, and

$$\begin{cases} A_{2k+1} = A_{2k} + A_{2k-1} \,, \\ A_{2k+2} = A_{2k+1} - A_{2k} \,. \end{cases}$$

Find A_n.

2. Let $B_0 = 2$, $B_1 = 3$, and

$$\begin{cases} B_{2k+1} = B_{2k} + B_{2k-1} \,, \\ B_{2k+2} = B_{2k+1} - B_{2k} \,. \end{cases}$$

Find B_n.

Solution by Robert M. Guili, San Jose State University, San Jose, California.

1.
$$\{A_i \mid i = 0, 1, 2, \cdots\} = \{0, 1, 1, 2, 1, 3, 2, 5, 3, 8, \cdots\}$$
$$\begin{array}{ccccc} & (F_2) & (F_3) & (F_4) & (F_5) \quad (F_6) \cdots \\ (F_0) & (F_1) & (F_2) & (F_3) & (F_4) \cdots \end{array}$$

$$A_{2k+1} = F_{k+2}, \qquad A_{2k+2} = F_{k+1} \quad \text{for } k = 0, 1, 2, \cdots.$$

2.
$$\{B_i \mid i = 0, 1, 2, \cdots\} = \{2, 3, 1, 4, 3, 7, 4, 11, 7, 18, \cdots\}$$
$$\begin{array}{ccccc} & (L_2) & (L_3) & (L_4) & (L_5) \quad (L_6) \cdots \\ (L_0) & (L_1) & (L_2) & (L_3) & (L_4) \cdots \end{array}$$

$$B_{2k+1} = F_{k+2}, \qquad B_{2k+2} = F_{k+1} \quad \text{for } k = 0, 1, 2, \cdots.$$

To derive these two solutions note that by combining the two equations

$$\begin{cases} H_{2k+1} = H_{2k} + H_{2k-1} \\ H_{2k+2} = H_{2k+1} - H_{2k} \,, \end{cases}$$

we get $H_{2k+2} = H_{2k-1}$. Using this relation to replace H_{2k} in the first equation, and H_{2k+1} in the second, we get

$$\begin{cases} H_{2k+1} = H_{2k-3} + H_{2k-1} \\ H_{2k+2} = H_{2k+4} - H_{2k-2} \,. \end{cases}$$

Now let $m = 2k - 1$, and $n = 2k + 2$ for $k = 0, 1, 2, \cdots$, which yields

$$\begin{cases} H_{m+1} = H_{m-1} + H_m \\ H_{n+1} = H_{n-1} + H_n \,. \end{cases}$$

These we recognize as the generalized Fibonacci recursive relation. By applying the starting values (A_0, A_1, A_2) and (B_0, B_1, B_2) in problems 1 and 2, respectively, we get the desired result.

Also solved by P. Tracy, A. Shannon, V. E. Hoggatt, Jr., P. Bruckman, and the Proposer.

USING YOUR GENERATOR

H-232 Proposed by R. Garfield, the College of Insurance, New York, New York.

Define a sequence of polynomials $\quad G_k(x) \;{}_{k=0}^{\infty}$ as follows:

$$\frac{1}{1-(x^2+1)t^2-xt^3} = \sum_{k=0}^{\infty} G_k(x)t^k .$$

1. Find a recursion formula for $G_k(x)$.
2. Find $G_k(1)$ in terms of the Fibonacci numbers.
3. Show that when $x = 1$, the sum of any 4 consecutive G numbers is a Lucas number.

Solution by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

SOLUTION 1.

$$\begin{aligned}\frac{1}{1-(x^2+1)t^2-xt^3} &= 1+t^2(x^2+1)+t^3x+t^4(x^2+1)^2+t^5\left[\binom{2}{1}x\right]+t^6[(x+1)^3+x^2] \\ &\quad +\cdots+t^{2k}\left[\binom{k}{0}(x^2+1)^k+\binom{k-1}{2}(x^2+1)^{k-3}x^2+\binom{k-2}{4}(x^2+1)^{k-6}x^4+\cdots\right] \\ &\quad +t^{2k+1}\left[\binom{k}{1}(x^2+1)^{k-1}x+\binom{k-1}{3}(x^2+1)^{k-4}x^3+\binom{k-2}{5}(x^2+1)^{k-7}x^5+\cdots\right].\end{aligned}$$

SOLUTION 2.

$$\begin{aligned}\frac{1}{1-2t^2-t^3} &= \frac{1}{(t+1)(1-t-t^2)} = \frac{1}{t+1}+\frac{t}{1-t+t^2} = 1-t+t^2-t^3+\cdots+F_nt^{n+1} \\ &= t^{n+1}[F_n+(-1)^{n+1}].\end{aligned}$$

SOLUTION 3.

$$\begin{aligned}&F_n+(-1)^{n+1}+F_{n+1}+(-1)^{n+2}+F_{n+2}+(-1)^{n+3}+F_{n+3}+(-1)^{n+4} \\ &\qquad = \frac{1}{\sqrt5}[a^n(1+a+a+a)-b^n(1+b+b+b)] = \frac{1}{\sqrt5}\left[a^n\left(\frac{4+2\sqrt5}{2}\right)+b^n\left(\frac{4-2\sqrt5}{2}\right)\right]\sqrt5 \\ &\qquad = a^{n+3}+b^{n+3} = L_{n+3}.\end{aligned}$$

Also solved by C. Chouteau, P. Bruckman, A. Shannon, and the Proposer.

GENERAL–IZE

H-233 Proposed by A. G. Shannon, NSW Institute of Technology, Broadway, and The University of New England, Armidale, Australia.

The notation of Carlitz* suggests the following generalization of Fibonacci numbers. Define

$$f_n^{(r)} = (a^{nk+k}-b^{nk+k})/(a^k-b^k),$$

where $k = r-1$, and a,b are the zeros of x^2-x-1, the auxiliary polynomial of the ordinary Fibonacci numbers, $f_n^{(2)}$.
Show that

(a) $$\sum_{n=0}^{\infty} f_n^{(r)} x^n = 1/(1-(a^k+b^k)x+(a^kb^k)x^2)\,.$$

Let $f_k = (a^{k+1}-b^{k+1})/(a-b)$, and prove that

(b) $$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}\,.$$

(Note that when $r=2$ (and so $k=1$), $f_k = f_{k-1} = 1$, $f_{k-2} = 0$, and (b) reduces to the well known

$$f_n^{(2)} = \sum_{0 \le 2m \le n} \binom{n-m}{m}\,.)$$

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

We form the series

$$\sum_{n=0}^{\infty} f_n^{(r)} x^n = \sum_{n=0}^{\infty} \left(\frac{a^{nk+k}-b^{nk+k}}{a^k-b^k}\right) x^n = \frac{a^k}{a^k-b^k}\sum_{n=0}^{\infty}(a^kx)^n - \frac{b^k}{a^k-b^k}\sum_{n=0}^{\infty}(b^kx)^n$$

$$= \frac{a^k}{a^k-b^k}\cdot\frac{1}{1-a^kx} - \frac{b^k}{a^k-b^k}\cdot\frac{1}{1-b^kx} = \left\{(1-a^kx)(1-b^kx)\right\}^{-1}$$

$$= \left\{1-xL_k+(-1)^kx^2\right\}^{-1} = \left\{1-(a^k+b^k)x+(ab)^kx^2\right\}^{-1},$$

which is the result of part (a). Now consider the series $S(x)$ defined as follows:

$$S(x) = \sum_{n=0}^{\infty} x^n \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}\,;$$

then

$$S(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{n-m} x^n \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=m}^{n} x^n \binom{m}{s-m}\binom{n-m}{s-m} f_{k-1}^{2s-2m} f_{k-2}^{2m-s} f_k^{n-s}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=m}^{n} \theta(n,m,s) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{n} \theta(n,m,s)$$

$$= \sum_{m=0}^{\infty}\sum_{s=m}^{\infty}\sum_{n=s}^{\infty} \theta(n,m,s)$$

$$= \sum_{m,s,n=0}^{\infty} x^{n+m+s} \binom{m}{s}\binom{n+s}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n}$$

$$= \sum_{m,s=0}^{\infty} x^{m+s}\binom{m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} \sum_{n=0}^{\infty}\binom{n+s}{n}(xf_k)^n =$$

$$= \sum_{m,s=0}^{\infty} x^{m+s} \binom{m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} (1 - xf_k)^{-s-1}$$

$$= (1 - xf_k)^{-1} \sum_{m=0}^{\infty} x^m f_{k-2}^m \sum_{s=0}^{\infty} \binom{m}{s} \left\{ \frac{xf_{k-1}^2}{f_{k-2}(1 - xf_k)} \right\}^s$$

$$= (1 - xf_k)^{-1} \sum_{m=0}^{\infty} (xf_{k-2})^m \left\{ 1 + \frac{xf_{k-1}^2}{f_{k-2}(1 - xf_k)} \right\}^m$$

$$= (1 - xf_k)^{-1} \left\{ 1 - xf_{k-2} - \frac{x^2 f_{k-1}^2}{1 - xf_k} \right\}^{-1}$$

$$= \left\{ (1 - xf_k)(1 - xf_{k-2}) - x^2 f_{k-1}^2 \right\}^{-1}$$

$$= \left\{ 1 - x(f_k + f_{k-2}) + x^2 (f_k f_{k-2} - f_{k-1}^2) \right\}^{-1}$$

$$= \left\{ 1 - x(F_{k+1} + F_{k-1}) + x^2 (F_{k+1}F_{k-1} - F_k^2) \right\}^{-1} \quad (F_k \text{ is the } k^{th} \text{ Fibonacci number})$$

$$= \left\{ 1 - L_k x + (-1)^k x^2 \right\}^{-1} = \sum_{n=0}^{\infty} f_n^{(r)} x^n, \text{ by part (a).}$$

Comparing coefficients of the power series, this establishes part (b). N.B. $F_{(n+1)k}/F_k = f_n^{(r)}$.

Also solved by the Proposer.

Editorial Note: Dale Miller's name appeared incorrectly in H-237.

Continued from page 82. *******

Returning to (2) above, we can generate multigrades of higher orders. (For the standard method employed, see below.[1]) I give now, as an example, a third-order multigrade:

$$A_1 F_1^m + A_2 F_2^m \cdots A_{n-1} F_{n-1}^m + \left(\sum_1^{n-1} A - 2 \right) F_n^m + A_1(2F_n - F_1)^m + A_2(2F_n - F_2)^m \cdots A_{n-1}(2F_n - F_{n-1})^m$$

$$+ \left(\sum_1^{n-1} A - 2 \right) F_n^m = A_1(F_n - F_1)^m + A_2(F_n - F_2)^m \cdots A_{n-1}(F_n - F_{n-1})^m + \left(\sum_1^{n-1} A - 2 \right) 0^m$$

$$+ A_1(F_n + F_1)^m + A_2(F_n + F_2)^m \cdots A_{n-1}(F_n + F_{n-1}) + \left(\sum_1^{n-1} A - 2 \right) (2F_n)^m$$

(where $m = 1,2,3$).

[I have added F_n to each term in (2), and added the L.H.S. totals to the original R.H.S. and *vice versa.*] Expressed more tidily, the above becomes

$$A_1[(F_1)^m + (2F_n - F_1)^m] + A_2[(F_2)^m + (2F_n - F_2)^m] \cdots A_{n-1}[(F_{n-1})^m - (2F_n - F_{n-1})^m] + 2\left[\sum_1^{n-1} A - 2 \right] F_n^m$$

$$= A_1[(F_n - F_1)^m + (F_n + F_1)^m] + A_2[(F_n - F_2)^m + (F_n + F_2)^m] \cdots A_{n-1}[(F_n - F_{n-1})^m + (F_n + F_{n-1})^m]$$

$$+ \left(\sum_1^{n-1} A - 2 \right) [(2F_n)^m + 0^m] \qquad \text{(where } m = 1, 2, 3\text{)}.$$

Again, if we add any quantity B to each term, the final 0^m terms each become B^m.

REFERENCE

1. M. Kraitchik, *Mathematical Recreations,* George Allen & Unwin, London, 1960, page 79.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-322 Proposed by Sidney Kravitz, Dover, New Jersey.

Solve the following alphametic in which no 6 appears:

```
   A R K I N
   A L D E R
   S A L L E
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 A L L A D I
```

(All the names are taken from the front cover of the April 1975 *Fibonacci Quarterly*.)

B-323 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Prove that
$$F_{n+r}^2 - (-1)^r F_n^2 = F_r F_{2n+r}\,.$$

B-324 Proposed by Herta T. Freitag, Roanoke, Virginia.

Determine a constant k such that, for all positive integers n,

$$F_{3n+2} \equiv k^n F_{n-1} \pmod 5\,.$$

B-325 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Prove that there does not exist an *even* single-valued function G such that

$$x + G(x^2) = G(ax) + G(bx) \quad \text{on} \quad -a \leqslant x \leqslant a.$$

B-326 Based on the solution to B-303 by David Zeitlin, Minneapolis, Minnesota.

For positive integers n, let $\sigma(n)$ be the sum of the positive integral divisors of n. Prove that

$$\sigma(mn) \geqslant 2\sqrt{\sigma(m)\sigma(n)} \quad \text{for} \quad m > 1 \quad \text{and} \quad n > 1.$$

B-327 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find all integral values of r and s for which the equality

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{ri} = s^n L_n$$

holds for all positive integers n.

SOLUTIONS

A CORRECTION

Jeffrey Shallit points out that the second solution $\pi^2/2^2$ to problem B-274 is incorrect and that a correct solution using π, $/$, and 2 is $\pi^2/2^{2^2}$.

AN APPLICATION OF THE BINET FORMULAS

B-298 Proposed by Richard Blazej, Queens Village, New York.

Show that
$$5F_{2n+3}F_{2n-3} = L_{4n} + 18.$$

Solution by Gerald E. Bergum, South Dakota State University, Brookings, South Dakota.

Using
$$F_n = (a^n - b^n)/\sqrt{5}, \qquad L_n = a^n + b^n,$$
and the fact that

$$ab = -1, \quad 5F_{2n+3}F_{2n-3} = (a^{2n+3} - b^{2n+3})(a^{2n-3} - b^{2n-3}) = a^{4n} + b^{4n} - (a^6 + b^6)(ab)^{2n-3}$$
$$= L_{4n} - L_6(-1)^{2n-3} = L_{4n} + L_6 = L_{4n} + 18.$$

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Ralph Garfield, Frank Higgins, Graham Lord, John W. Milsom, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, David Zeitlin, and the Proposer.

A CONVOLUTION FORMULA

B-299 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple closed form for

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k}.$$

Solution by Frank Higgins, Naperville, Illinois.

Using Problem B-295, we have

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k} = F_{2n+3} - \sum_{k=1}^{n} (n+1-k)F_{2k} - \sum_{k=1}^{n} F_{2k}$$
$$= F_{2n+3} - (F_{2n+2} - (n-1)) - (F_{2n+1} - 1) = n+2.$$

Also solved by Gerald E. Bergum, George Berzsenyi, Wray C. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, and the Proposer.

ANOTHER CONVOLUTION

B-300 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple closed form for

$$L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k}.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Utilizing well known formulae and the solution to Problem B-299, one finds that

$$\begin{aligned} L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k} &= L_{2n+2} - \sum_{k=1}^{n} (n+2-k)F_{2k} - \sum_{k=1}^{n} F_{2k} \\ &= L_{2n+2} - (F_{2n+3} - n - 2) - (F_{2n+1} - 1 \\ &= L_{2n+2} + n + 3 - (F_{2n+1} + F_{2n+3}) \\ &= L_{2n+2} + n + 3 - L_{2n+2} = n + 3. \end{aligned}$$

Also solved by Gerald E. Bergum, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Frank Higgins, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Jeffrey Shallit, A. C. Shannon, and the Proposer.

GREATEST INTEGER IDENTITY

B-301 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $[x]$ denote the greatest integer in x, i.e., the integer m with $m \leqslant x < m + 1$. Also let

$$A(n) = (n^2 + 6n + 12)/12 \quad \text{and} \quad B(n) = (n^2 + 7n + 12)/6.$$

Does $[A(n)] + [A(n+1)] = [B(n)]$ for all integers n? Explain.

Solution by Graham Lord, Secane, Pennsylvania.

The identity is correct, as can be seen upon placing $n = 6m, 6m + 1, \cdots, 6m + 5$ successively. For example, with $n = 6m + 4$:

$$\begin{aligned} [A(n)] + [A(n+1)] &= [A(6m+4)] + [A(6m+5)] = (3m^2 + 7m + 4) + (3m^2 + 8m + 5) \\ &= [6m^2 + 15m + (56/6)] = B(6m+4) = B(n). \end{aligned}$$

Also solved by Paul S. Bruckman, David Zeitlin, and the Proposer.

COMPOSITE FIBONACCI NEIGHBORS

B-302 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Prove that $F_n - 1$ is a composite integer for $n \geqslant 7$ and that $F_n + 1$ is composite for $n \geqslant 4$.

Solution by John Ivie, Student, University of California, Berkeley, California.

Using the Binet Formulas, the following identities can be established:

$$\begin{aligned} F_{4k} - 1 &= L_{2k-1}F_{2k+1}; \quad & F_{4k} + 1 &= L_{2k+1}F_{2k-1}. \\ F_{4k+1} - 1 &= L_{2k+1}F_{2k}; & F_{4k+1} + 1 &= L_{2k}F_{2k+1}. \\ F_{4k+2} - 1 &= L_{2k+2}F_{2k}; & F_{4k+2} + 1 &= L_{2k}F_{2k+2}. \\ F_{4k+3} - 1 &= L_{2k+1}F_{2k+2}; & F_{4k+3} + 1 &= L_{2k+2}F_{2k+1}. \end{aligned}$$

Since $F_n > 1$ for $n > 2$ and $L_n > 1$ for $n > 1$, one thus sees that $F_n - 1$ is composite for $n \geqslant 7$ and $F_n + 1$ is composite for $n \geqslant 4$.

Also solved by Gerald E. Bergum, George Berzsenyi, Paul S. Bruckman, Graham Lord, A. C. Shannon, David Zeitlin, and the Proposer.

A SIGMA FUNCTION INEQUALITY

B-303 Proposed by David Singmaster, Polytechnic of the South Bank, London, England.

In B-260, it was shown that $\sigma(mn) > \sigma(m) + \sigma(n)$, where $\sigma(n)$ is the sum of the positive integral divisors of n. What relation holds between $\sigma(mn)$ and $\sigma(m)\sigma(n)$?

Solution by Frank Higgins, Naperville, Illinois.

Let

$$m = \prod_{i=1}^{k} p_i^{\alpha_i} \qquad \text{and} \qquad n = \prod_{i=1}^{k} p_i^{\beta_i},$$

where each α_i and β_i is non-negative and where $p_1, p_2, \cdots, p_k$ are distinct prime numbers. Since

$$\frac{p_i^{\alpha_i+\beta_i+1} - 1}{p_i - 1} \leqslant \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \cdot \frac{p_i^{\beta_i+1} - 1}{p_i - 1},$$

where equality holds iff $\alpha_i\beta_i = 0$, it follows that $\sigma(mn) \leqslant \sigma(m)\sigma(n)$, where equality holds iff $(m,n) = 1$.

Also solved by Paul S. Bruckman, Graham Lord, C. B. A. Peck, David Zeitlin, and the Proposer.

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