VARIATION IN THE NUMBER OF RAY- AND DISC-FLORETS IN FOUR SPECIES OF COMPOSITAE

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INTRODUCTION

The compositae is one of the largest families of the vascular plants, comprising about 1000 genera and about 30,000 species. The members of this family are distributed in almost all parts of the world and are readily recognized by their unique disc-shaped inflorescence, composed of numerous pentamerous florets packed on an involucrate head.

There is a good deal of variation in the numbers of ray-florets and disc-florets in many compositae. Moreover, beautiful phyllotactic configurations become visible due to the unique arrangement of florets/fruits in the head. Definite equiangular spirals appear on the head of a composite, which run either right-handed (counter-clockwise) or left-handed (clockwise). Another different set of spirals run opposite to the former spirals, and these intersect each other, such as: 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, ···. The numerators or the denominators of this series, when considered alone, form the successive stages of the famous Fibonacci Sequence.

In this note are presented the results of a study on the variation in the number of ray-florets in four species of compsoitae and also the variation in the number of disc-florets in one species.

PRESENTATION OF DATA

Variation in the number of ray-florets: Data on the variation in the number of ray-florets were obtained on:

1. *Tridax procumbens*
2. *Cosmos bipinnatus* (two varieties)
3. *Coreopsis tinctoria*
4. *Helianthus annuus.*

Tridax procumbens is an annual weed which usually grows on roadsides and wastelands. Its capitulum is small, the diameter of a head usually measuring 4.5 mm to 5.5 mm and raised on a long peduncle. *Cosmos bipinnatus* is a common flower which grows during winter and is available in white, pink and saffron colours. *Coreopsis tinctoria* and *Helianthus annuus* are also common in India. The head of *Coreopsis* is small, but that of *Helianthus* is quite large, with bright yellow ray-florets.

Data on *Tridax procumbens* were obtained from a locality near Andhra University, Waltair, Andhra Pradesh, India, during July 1972. In all, 4000 heads were observed.

Data on 300 heads of each of the two varieties of *Cosmos bipinnatus,* the saffron coloured variety and the white and pink coloured variety, were gathered from two localities at Calcutta. In both the localities, nearly 100 p.c. of the heads possessed eight rays each.

Data on 500 heads of *Coreopsis tinctoria* were collected from the gardens of Royal Agri-Horticultural Society, Calcutta.

One thousand and two heads of *Helianthus annuus* were observed in three different localities in Calcutta.

Variation in the number of disc-florets: Data on the variation in the number of disc-florets were collected on two varieties of *Cosmos bipinnatus,* the saffron coloured variety (Variety 1) and the white and pink coloured variety (Variety 2) from two localities in Calcutta.

Thirty heads of Variety 1 and 61 heads of Variety 2 were gathered from the gardens of the Indian Statistical Institute, Calcutta, and the number of disc-florets on each of the heads were counted.

DISCUSSION

From the data represented in Fig. 1, it is seen that though there is a great deal of variation in the numbers of ray-florets per head within the same species, the mode of each species invariably turns out to be a Fibonacci number. For *Tridax* the mode is at the fifth Fibonacci number; that is, at 5; for both varieties of *Cosmos* as well as for *Coreopsis* the mode is at the sixth Fibonacci number; that is, at 8; and for *Helianthus* the mode is at the eighth Fibonacci number; that is, at 21. Among the four species of compositae observed, the variation in the number of ray-florets is greatest for *Helianthus* and least for *Cosmos.*

Such variation surely has a genetic component and some (Ludwig, 1897) believed that (both within and between plants) it is largely a result of climatic factors and nutrition. These multimodal distributions are not totally new, and were demonstrated by Ludwig in the ray-florets of *Bellis perennis,* disc-florets in *Achilles millefolium* and flowers in the umbels of *Primula veris* as early as in 1890 (Briggs and Walters, 1969).

Such modal variation can be explained by the model suggested by Turing (1952). Turing considered a system of chemical substances, or "morphogens," reacting together and diffusing through a tissue. He showed that such a system, though originally homogeneous, may later develop a pattern or a structure due to instability of the homogeneous equilibrium. In the simple case of an isolated ring of cells, one form of instability gives rise to a standing wave of concentration of the morphogens. For any given set of values of the constants for the rates of reaction and diffusion there will be a "chemical wave-length" of β. If the circumference of the ring, s, is divided by β, the result will not usually be an integer. Yet the system necessarily forms an integral number of waves, typically the integer nearest to s/β.

This provides a simple model of the process whereby an integral number of discrete structures can arise from a homogeneous tissue. All individuals of a population will have n structures if:

$$n - (\tfrac{1}{2}) < s/\beta < n + (\tfrac{1}{2}).$$

Writing Var. (s), Var. (β), and Var. (s/β) for the variances of s, β, and s/β respectively, and Cov. (s,β) for the covariance between s and β, one can easily see that:

$$\text{Var.}\,(s/\beta) = \beta^{-4}[\beta^2\cdot\text{Var.}\,(s) + s^2\cdot\text{Var.}\,(\beta) - 2s\beta\cdot\text{Cov.}\,(s,\beta)]\,.$$

It is also interesting to know why the modes in the distribution of heads of ray-florets turn out to be Fibonacci numbers. An explanation which seems logical to us is the following:

The general formula for obtaining the Fibonacci numbers is:

$$F_n = F_{n-1} + F_{n-2},\ n = 3, 4, 5, \cdots; \qquad F_1 = F_2 = 1,$$

where F_n denotes the n^{th} Fibonacci number. When n is large, we can write:

$$F_n \doteq (1/\sqrt{5})\cdot[(1+\sqrt{5})/2]^n\,,$$

(where $\doteq$ means "approximately equal to"), and one can approximate it by the continuous curve:

$$\begin{aligned} y &= (1/\sqrt{5})\cdot[(1+\sqrt{5})/2]^x\,, \\ &= 0.4472 \times (1.6180)^x\,. \end{aligned}$$

For all practical purposes, the Fibonacci numbers lie on this curve in its higher stages, and moreover it represents perfect exponential growth; presumably tending to reduce the size of the florets to the optimum necessary for quick production of an adequate number of single seeded fruits. So the appearance of the Fibonacci numbers as the modes for the distribution of the ray-florets on the heads can be taken as an indication for perfect growth, as is usually the case. Also it is well known that Fibonacci phyllotaxis give optimum illumination to the photo-synthetic surfaces of plants (Davis, 1971).

The appearance of the Fibonacci numbers can also be explained on the consideration that the individual flowers emerge at a uniform speed at fixed intervals of time along a logarithmic spiral, $r = e^{a\alpha}$ with small a and with an initial angle α_1 = 137.5° (Mathai and Davis, 1974). They also show that the above logarithmic spiral may be a natural outcome of the supply of genetic material in the form of pulses at constant intervals of time and obeying the law of fluid flow.

A new theory which has been proposed by Leppik (1960) has emerged from studies of the behaviour of pollinating insects and their relationships with flowers. On the basis of numerous observations and behaviour tests, Leppik

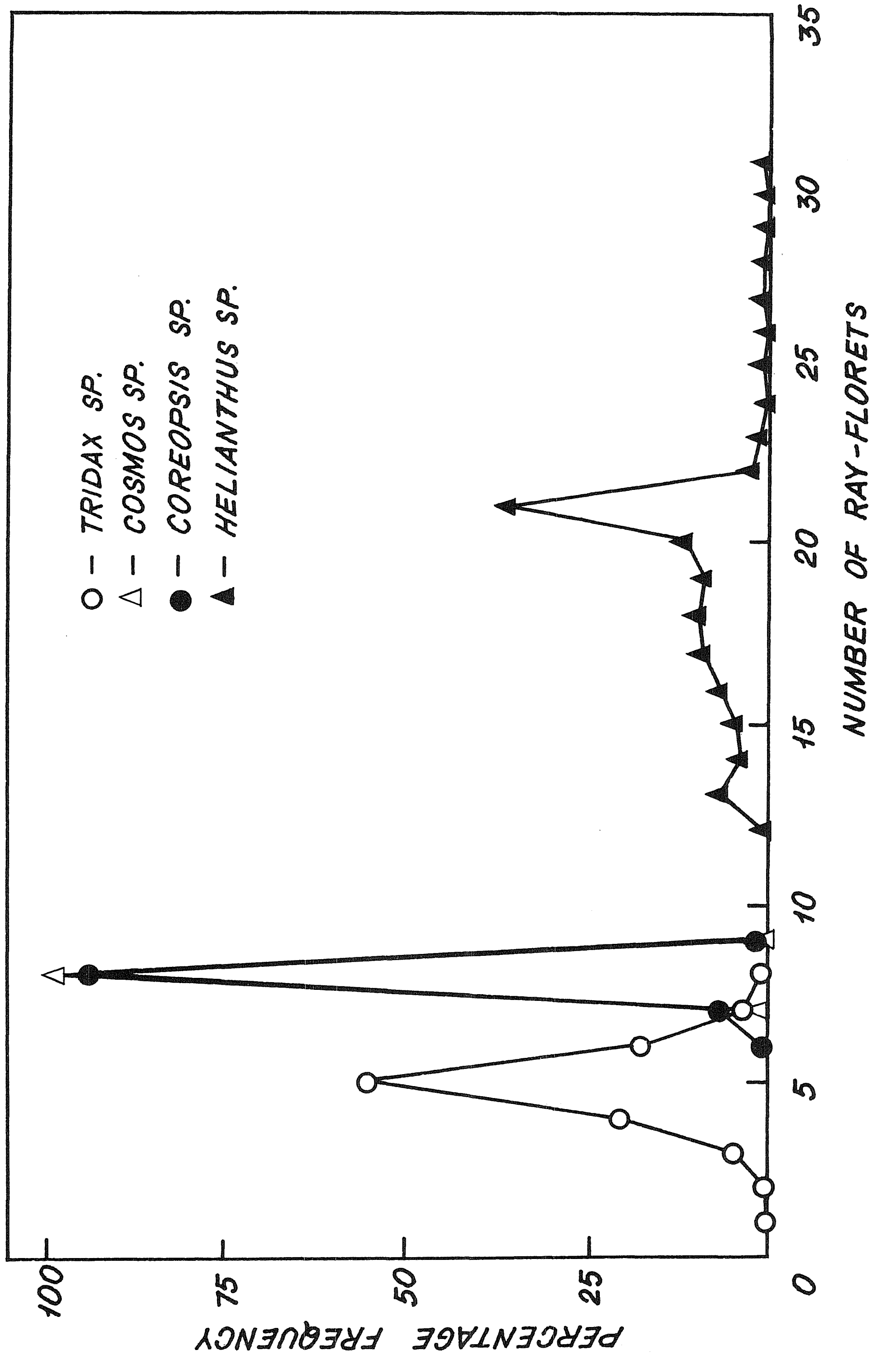
○ – TRIDAX SP.
△ – COSMOS SP.
● – COREOPSIS SP.
▲ – HELIANTHUS SP.
PERCENTAGE FREQUENCY
100
75
50
25
0
5
10
15
20
25
30
35
NUMBER OF RAY-FLORETS

Figure 1

has ascertained that most pollinating insects have the ability to distinguish floral characteristics—angular-form and radial-symmetry in particular. He has hypothesized that some numeral patterns, which include the Fibonacci numbers, are more symmetrically arranged than others. And hence, floral differentiation occurred and this has gradually led to the evolution of ecotypes with specific numeral patterns.

As we have already seen, in *Cosmos,* there is no (negligible) variation in the number of ray-florets and this turns out to be 8. But there is a great variation in the number of disc-florets. This shows that the correlation between the number of ray-florets and the number of disc-florets is almost zero. Also another interesting fact noticed is that there is no correlation between the size of a flower (when the head is looked at as a single flower) and the number of disc-florets present on the flower-head.

ACKNOWLEDGEMENT

We thank Prof. T. A. Davis for the numerous discussions on the subject and Mr. S.K. De for the drawing.

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*Indicates a relevant paper not discussed in the text.

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COMBINATORIAL NUMBERS IN $\mathbb{C}^n$

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1. INTRODUCTION

The use of linear algebra in combinatorial number theory was introduced in [4]. The present paper extends the notations and studies the general properties of product functions, i.e., combinatorial number systems in $\mathbb{C}^n$. Among the examples given are n-dimensional Bernoulli and Euler numbers which are useful in the expansion in series of functions in n variables. The methods and notations introduced here will be used in the study of functions and series in $\mathbb{C}^n$ that will be the subject of future investigations.

2. NOTATION

Let I be the set of positive integers, J the set of non-negative integers, and given $n \in I$, let $I(n) \subset I$, and $J(n) \subset J$ be such that if $k \in J(n)$, then $k \leqslant n$.

In order to avoid confusion we shall write Id for the identity operator or the identity matrix.

For $n \in I$, $k \in I(n)$, $X = [x_1, x_2, \cdots, x_n]$ is an n-dimensional vector and x_k are complex numbers, i.e., $x_k \in \mathbb{C}$, so that $X \in \mathbb{C}^n$.

Let

$$P = [p_1, p_2, \cdots, p_n], \qquad Q = [q_1, q_2, \cdots, q_n],$$

then $W(n) \subset \mathbb{C}^n$ be such that for $P \in W(n)$, $m \in I(n)$, $p_m \in J$, and for $P, Q \in W(n)$, $P \leqslant Q$, iff for all $m \in I(n)$, $p_m \leqslant q_m$.

We consider the following special vectors:

(2.1) $\qquad U \in W(n), \quad$ such that $\quad u_m = 1 \quad$ for all $\quad m \in I(n)$,

(2.2) $\qquad U(s) \in W(n), \quad$ such that $\quad u_m = \delta^s_m, \quad$ for all $\quad m \in I(n)$,

where δ^s_m is the Kronecker delta. It follows that

$$U = \sum_{s=1}^{n} U(s).$$

(2.3) $\qquad Z(s) \in W(n), \quad$ such that $\quad Z(s) = U - U(s), \quad$ i.e., $\quad z_m = 1 - \delta^s_m$.

(2.4) $\qquad Z(X,s) \in \mathbb{C}^n, \quad$ such that $\quad z_m = x_m(1 - \delta^s_m), \quad$ i.e., $\quad z_s = 0, \quad$ thus $\quad Z(U,s) = Z(s)$.

We next introduce for $X \in \mathbb{C}^n$

$$|X| = \sum_{m=1}^{n} x_m , \tag{2.5}$$

so that $|U| = n$, $|U(s)| = 1$, $|Z(s)| = n - 1$, and $|Z(X,s)| = |X| - x_s$.

We finally introduce the inner product in the usual way: If $X, Y \in \mathbb{C}^n$, then

$$X \cdot Y = \sum_{m=1}^{n} x_m \bar{y}_m , \tag{2.6}$$

where $\bar{y}_m$ is the complex conjugate of y_m. It follows that

$$\|X\| = (X \cdot X)^{1/2} = \left(\sum_{m=1}^{n} |x_m|^2 \right)^{1/2} . \tag{2.7}$$

If, however, $X, Y \in \mathcal{R}^n \subset \mathcal{C}^n$, where $\mathcal{R}^n$ is the space of real n vectors, then

$$X \cdot Y = \sum_{m=1}^{n} x_m y_m , \tag{2.8}$$

and

$$\|X\| = (X \cdot X)^{1/2} = \left[\sum_{m=1}^{n} x_m^2 \right]^{1/2} \tag{2.9}$$

3. FUNCTIONS OVER $\mathcal{C}^n$

We consider functions $\Phi: \mathcal{C}^n \to \mathcal{C}$.

A monomial in X can be written:

$$X^K = \prod_{m=1}^{n} x_m^{k_m} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} , \tag{3.1}$$

where $X \in \mathcal{C}^n$, $K \in W(n)$. In particular,

$$X^U = \prod_{m=1}^{n} x_m = x_1 x_2 \cdots x_n . \tag{3.2}$$

A polynomial in X, i.e., a polynomial in n variables, can be written

$$f(X,P) = \sum_{K=0}^{P} a(K) X^K , \tag{3.3}$$

where the summation is extended over all K such that $K \leqslant P$, $K,P \in W(n)$ and $a(K)$ are numbers. In the generally adopted polynomial sense the degree of $f(X,P)$ is clearly $p = |P|$.

More generally if $\varphi_k(x_k)$, $k \in I(n)$, is a sequence of functions, $\varphi_k : \mathcal{C} \to \mathcal{C}$, then with

$$\Phi = [\varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_n(x_n)],$$

$$\Phi^U = \prod_{k=1}^{n} \varphi_k(x_k) = \varphi(X), \tag{3.4}$$

is called a *product function* of the functions φ_k.

We study the following examples:

(i) If $\varphi_k = m_k$, $M = [m_1, m_2, \cdots, m_n] \in W(n)$, then with $k \in I(n)$

$$\Phi^U = M! = \prod_{k=1}^{n} m_k! , \tag{3.5}$$

(ii) If $M \in \mathcal{C}^n$ but $M \notin W(n)$, then we replace factorials by gamma functions thus if $\varphi_k = \Gamma(m_k + 1)$, then

$$\Phi^U = \Gamma(M + U) = \prod_{k=1}^{n} \Gamma(m_k + 1). \tag{3.6}$$

(iii) For $N, M \in W(n)$, $M \leqslant N$, and $k \in I(n)$, we have for

$$\varphi_k = \binom{n_k}{m_k} ,$$

$$\Phi^U = \prod_{k=1}^{n} \binom{n_k}{m_k} = \prod_{k=1}^{n} n_k!/m_k!(n_k - m_k)! = N!/M!(N-M)! = \binom{N}{M}. \tag{3.7}$$

It should be noted that $\binom{N}{M}$ is product function for binomial coefficients and *not* a multinomial coefficient. The corresponding multinomial coefficient would be (cf. [3])

$$\binom{|N|}{[M, N-M]} = \left[\sum_{k=1}^{n} n_k\right]!/M!(N-M)!,$$

where

$$[M, N-M] = [m_1, m_2, \cdots, m_n, n_1 - m_1, n_2 - m_2, \cdots, n_n - m_m] \in W(2n)$$

and clearly $|M| + |N - M| = |N|$.

(iv) For $N, M \in W(n)$, and $A, B \in \mathcal{C}^n$

$$(A+B)^N = \prod_{k=1}^{n} (a_k + b_k)^{n_k} = \prod_{k=1}^{n} \left[\sum_{m_k=0}^{n_k} \binom{n_k}{m_k} a_k^{m_k} b_k^{n_k - m_k}\right],$$

and by regrouping the terms we obtain

$$(A+B)^N = \sum_{M=0}^{N} \binom{N}{M} A^M B^{N-M}. \tag{3.8}$$

(v) For $X \in \mathcal{C}^n$, and with $eU = [e, e, \cdots, e]$, we define

$$e^X = (eU)^X = e^{|X|} = \prod_{k=1}^{n} e^{x_k} = \prod_{k=1}^{n} \left[\sum_{m_k=0} x_k^{m_k}/m_k!\right] = \sum_{M=0} X^M/M!, \tag{3.9}$$

and

$$e^{-X} = \sum_{M=0} (-1)^M X^M/M!,$$

where $(-1)^M = (-1)^{|M|}$.

It will be noted that whenever a summation goes to infinity the upper limit is left out.

4. UMBRAL CALCULUS

Umbral calculus consists in substituting indices for exponents. In [2] the following notation is used for the one dimensional case.

$$e^{ax} = \sum_{k=0} x^k a^k/k! \to [\exp ax, a^k = a_k] = \sum_{k=0} x^k a_k/k! \tag{4.1}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} \to [(a+b)^n, a^k = a_k, b^k = b_k] = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{4.2}$$

We shall change this notation and extend it to the n-dimensional case. The umbral expression corresponding to a vector exponent is clearly

$$A^K = \prod_{m=1}^{n} a_m^{k_m} \to A(K) = \prod_{m=1}^{n} a_m(k_m), \tag{4.3}$$

where instead of indices we write variables.

We now introduce the following convention: Whenever an element is to be written umbrally it will be underlined, thus

$$(4.4)\qquad (\underline{a}+\underline{b})^n = \sum_{m=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{m=0}^{n} \binom{n}{m} a(k)b(n-k)$$

and with $N,K \in W(n)$

$$(4.5)\qquad (\underline{A}+\underline{B})^N = \prod_{m=1}^{n} [\underline{a}(m)+\underline{b}(m)]^{n_m} = \sum_{K=0}^{N} \binom{N}{K} A(K)B(N-K),$$

but

$$(4.6)\qquad (A+\underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} A^K B(N-K) = \sum_{K=0}^{N} \binom{N}{K} A^{N-K} B(K),$$

and in particular

$$(4.7)\qquad (U+\underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} B(N-K) = \sum_{K=0}^{N} \binom{N}{K} B(K).$$

Similarly for the generalized exponential we have

$$(4.8)\qquad e^{\underline{A}X} = \sum_{K=0} X^K A(K)/K!\,.$$

It should be noted that the last umbral expression (4.8) is the exponential kind generating function for the numbers $A(K)$.

It should be noted that

$$e^{\underline{A}X}e^{\underline{B}X} = \Big[\sum_{S=0} X^S A(S)/S!\Big]\Big[\sum_{T=0} X^T B(T)/T!\Big] = \sum_{S=0}\sum_{T=0} X^{S+T} A(S)B(T)/S!T!\,.$$

Let $S+T=K$; observing that $\binom{K}{S} = K!/S!(K-S)!$, we have

$$e^{\underline{A}X}e^{\underline{B}X} = \sum_{K=0}\sum_{S=0} X^k A(S)B(K-S)/S!(K-S)! = \sum_{K=0} (X^K/K!) \sum_{S=0} \binom{K}{S} A(S)B(K-S)\quad,$$

but according to (4.5) the last sum is equal to $(\underline{A}+\underline{B})^K$, where the binomial coefficients for $S \geqslant K$ are all equal to zero. It follows that

$$(4.9)\qquad e^{\underline{A}X}e^{\underline{B}X} = \sum_{K=0} X^K(\underline{A}+\underline{B})^K/K! = e^{(\underline{A}+\underline{B})X}\quad,$$

i.e., the symbolic exponential follows the same law of addition as the ordinary exponential.

5. GENERATING FUNCTIONS

Let

$$\Phi(k) = [\varphi(k,1), \varphi(k,2), \cdots, \varphi(k,n)]$$

and using the notation of Section 2, we consider the product function

$$(5.1)\qquad \varphi(k) = [\Phi(k)]^U = \prod_{m=1}^{n} \varphi(k,m).$$

Let $v(t,m)$ be the generating function for the functions $\varphi(k,m)$, i.e.,

$$G\varphi(k,m) = \sum_{k=0} \varphi(k,m)t^k = v(t,m), \tag{5.2}$$

where $m \in I(n)$.

By taking the product

$$\prod_{m=1}^{n} \left[\sum_{k=0} \varphi(k,m)t_m^k \right] = \prod_{m=1}^{n} v(m,t_m) = [V(T)]^U = \Omega(T), \tag{5.3}$$

where $T = [t_1, t_2, \cdots, t_n] \in \mathbb{C}$, and

$$V(T) = [v(1,t_1), v(2,t_2), \cdots, v(n,t_n)] = \mathbb{C}^n$$

we thus obtain the generating function of the product function.

If $\varphi(k,m) = \varphi(k,x_m)$, then $v(m,t_m) = v(x_m, t_m)$ and (5.3) becomes

$$\prod_{m=1}^{n} v(x_m, t_m) = [V(X,T)]^U = \Omega(X,T). \tag{5.4}$$

We can state this result as follows:

PROPOSITION 1. The generating function of the product function of a set of functions is equal to the product of the generating functions of the set of functions.

6. INVERSION OF SERIES

Consider the series

$$A(N) = \sum_{K=0} f(N,K)B(K), \tag{6.1}$$

where the coefficients $f(N,K)$ are known. We say that (6.1) has an inverse if there exists a set of coefficients $g(N,K)$ such that

$$B(N) = \sum_{K=0} g(N,K)A(K), \tag{6.2}$$

both series being convergent.

PROPOSITION 2. If both series (6.1) and (6.2) are absolutely convergent they are inverses of each other if and only if f and g are quasi-orthogonal in the sense of [4] and [5].

PROOF:

$$A(N) = \sum_{K=0} f(N,K)B(K) = \sum_{K=0} f(N,K) \sum_{S=0} g(K,S)A(S) = \sum_{K=0}\sum_{S=0} f(N,K)g(K,S)A(S).$$

Since the series are absolutely convergent, their order can be deranged and the order of summation can be changed, thus

$$A(N) = \sum_{S=0} A(S) \left[\sum_{K=0} f(N,K)g(K,S) \right] = \sum_{S=0} A(S)\delta_N^S,$$

where δ_N^S is the Kronecker-Delta. It follows that

$$\sum_{K=0} f(N,K)g(K,S) = \delta_N^S,$$

which expresses quasi-orthogonality in the sense of [4] and [5].

PROPOSITION 3. $A(N) = (\underline{C} + \underline{B})^N$ and $B(N) = (\underline{G} + \underline{A})^N$ will be inverses of each other if $(\underline{C} + \underline{G})^T = \delta_0^T$.

PROOF. Since

$$A(N) = (\underline{C}+\underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} C(K)B(N-K) = \sum_{K=0}^{N} \binom{N}{K} B(K)C(N-K),$$

$$B(N) = (\underline{G}+\underline{A})^N = \sum_{K=0} \binom{N}{K} G(K)A(N-K) = \sum_{K=0} \binom{N}{K} A(K)G(N-K),$$

where both series involved are finite, i.e., present no problem of convergence, we apply the results of Proposition 2

$$\sum_{K=S}^{N} \binom{N}{K} C(N-K) \binom{K}{S} G(K-S) = \sum_{K=S}^{N} [N/K!(N-K)!][K!/S!(K-S)!]C(N-K)G(K-S)$$

$$= \binom{N}{S} \sum_{K=S}^{N} \binom{N-S}{N-K} C(N-K)G(K-S) = \delta_N^S.$$

Let $K - S = M$, i.e., $N - K = N - S - M$, so that

$$\binom{N-S}{N-K} = \binom{N-S}{N-S-N+K} = \binom{N-S}{M}.$$

The preceding quasi-orthogonality condition can thus be written

$$\binom{N}{S} \sum_{M=0}^{N} \binom{N-S}{M} G(M)C(N-S-M) = \binom{N}{S} (\underline{G}+\underline{C})^{N-S} = \delta_{N-S}^0,$$

or taking $N - S = T$,

$$(\underline{G}+\underline{C})^T = \delta_0^T. \tag{6.3}$$

It will be observed that (6.3) can be written for an arbitrary vector X in either form

$$e^{X(\underline{G}+\underline{C})} = 1, \tag{6.4}$$

or

$$e^{X\underline{G}} = 1/e^{X\underline{C}}. \tag{6.5}$$

7. OPERATORS IN $\mathbb{C}^n$

Let $D(m) = \partial/\partial x_m$, $m \in I(n)$, and $D = [D(1), D(2), \cdots, D(n)]$. We consider the *product operator*

$$\not{D} = D^U = \prod_{m=1}^{n} D(m) \tag{7.1}$$

and more generally $K = [k_1, k_2, \cdots, k_n] \in W(n)$

$$D^K = \prod_{m=1}^{n} [D(m)]^{k_m}. \tag{7.2}$$

Using this notation the n-dimensional Laplace operator can be written

$$\Delta_2 = \sum_{m=1}^{n} \partial^2/\partial x_m^2 = \sum_{m=1}^{n} D^{2U(m)}. \tag{7.3}$$

It is easily seen that for $k \in I(n)$

(7.4) $$C(k) = Z(X,k) + C_k U(k), \qquad c(k) = [C(k)]^U ,$$

$c(k)$ is such that $\not{D}c(k) = 0$. Considering now the vector

(7.5) $$C = [c(1), c(2), \cdots, c(n)]$$

it follows that $\not{D}C = 0$, and, if $\eta(X)$ is a function such that $\eta(0) = 0$, then $\eta(C)$ is the most general expression such that

(7.6) $$\not{D}\eta = 0,$$

where $\eta = \eta(C)$.

Similarly for difference operators we define $E(m)$ such that $E(m)f(x_k) = f(x_k + 1)\delta^k_m$, and

(7.7) $$E = [E(1), E(2), \cdots, E(n)]$$

(7.8) $$\not{E} = E^U = \prod_{m=1}^{n} E(m) .$$

We clearly have $E(m)\varphi(X) = \varphi[X + U(m)]$, and

(7.9) $$\not{E}X = E^U X = X + U, \qquad \not{E}\varphi(X) = \varphi(X + U).$$

The operator $\delta = \not{E} - Id$ is *not* a product operator of the form

$$\prod_{m=1}^{n} [E(m) - Id] .$$

We have however

(7.10) $$\delta X = U, \qquad \delta\varphi(X) = \varphi(X + U) - \varphi(X) .$$

The operator $\Delta(m) = E(m) - Id$ leads clearly to the

(7.11) $$\Delta = [\Delta(m_1), \Delta(m_2), \cdots, \Delta(m_n)]$$

(7.12) $$\not{\Delta} = \Delta^U = \prod_{k=1}^{n} \Delta(m_k).$$

It follows that $\not{\Delta}X = U$, but the general expression of $\not{\Delta}X^K$ is rather complicated.

The operator $M(m) = [E(m) + Id]/2$ leads similarly to

(7.13) $$M = [M(m_1), M(m_2), \cdots, M(m_n)]$$

and

(7.14) $$\not{M} = M^U = \prod_{k=1}^{n} M(m_k).$$

A more systematic study of the operators introduced here as well as the corresponding functional equations will be published in the future. We introduce here only what we need in view of the applications given.

8. RECURRENCE RELATIONS AND FUNCTIONAL EQUATIONS

Let $m \in I(n)$, and $a(m)$ be a one dimensional sequence of numbers satisfying a recurrence relation of the form

(8.1) $$\sum_{m=0}^{p} b(p,m)a(m) = 0, \qquad p \in J .$$

Let $k \in I(n)$, $m_k \in J$, $M = [m_1, m_2, \cdots, m_n]$, and

$$A(M) = [a(m_1), a(m_2), \cdots, a(m_n)]$$

and the associated product function be

(8.2) $$a(M) = [A(M)]^U = \prod_{k=1}^{n} a(m_k) .$$

By writing the product for (8.1) we obtain

$$\prod_{k=1}^{n}\left[\sum_{m_k=0}^{p} b(p,m_k)a(m_k)\right] = 0 .$$

Regrouping the terms we obtain

$$\sum_{M=0}^{pU} b(p,M)a(M) = 0, \tag{8.3}$$

where $B(p,M) = [b(p,m_1), b(p,m_2), \cdots, b(p,m_n)]$, and

$$b(p,M) = [B(p,M)]^U = \prod_{k=1}^{n} b(p,m_k).$$

Clearly $pU = [p, p, \cdots, p]$. We can state this result as follows.

PROPOSITION 4. If a sequence of numbers $a(m)$ satisfies a recurrence relation of the form (8.1) then the product function of the numbers $a(m)$, i.e., $a(M)$ satisfies a recurrence relation of the form (8.3).

If $\omega(m)$ is an operator such that

$$\omega(m)f(x_k) = \varphi(x_k)\delta_m^k , \tag{8.4}$$

where δ_m^k is the Kronecker delta.

Let $X \in \mathbb{C}$,

$$F(X) = [f(1,x_1), f(2,x_2), \cdots, f(n,x_n)] \in \mathbb{C}^n, \quad \Phi(X) = [\varphi(1,x_1), \varphi(2,x_2), \cdots, \varphi(n,x_\chi)] \in \mathbb{C}^n ,$$

$$\Omega = [\omega(1), \omega(2), \cdots, \omega(n)] \quad \text{and} \quad f(X) = [\Phi(X)]^U, \quad \varphi(X) = [\Phi(X)]^U, \quad \omega = \Omega^U,$$

then

$$\omega f(X) = \varphi(X). \tag{8.4}$$

9. EXAMPLES

(i) Consider the numbers $a_m = a(m)$ defined in [1] p. 231. They satisfy the relation

$$\sum_{m=0}^{n-1} a(m)(n-m)! = 0. \tag{9.1}$$

These numbers are the coefficients of the Bernoulli polynomials

$$\varphi_n(x) = \varphi(n,x) = \sum_{m=0}^{n} a(m)x^{n-m}/(n-m)! . \tag{9.2}$$

The numbers

$$B_m = B(m) = m!a(m) \tag{9.3}$$

are called Bernoulli numbers and satisfy the relation

$$(1+\underline{B})^n - B(n) = 0. \tag{9.4}$$

By using the Bernoulli numbers the polynomials of (9.2) can be written

$$\varphi(n,x) = (x+\underline{B})^n/n! . \tag{9.2a}$$

We introduce $M = [m_1, m_2, \cdots, m_n] \in W(n)$ and $A(M) = [a(m_1), a(m_2), \cdots, a(m_n)]$,

$$a(m) = [A(M)]^U = \prod_{k=1}^{n} a(m_k), \tag{9.5}$$

as well as

$$B(M) = [B(m_1), B(m_2), \cdots, B(m_n)],$$

(9.6) $$B(n,M) = [B(M)]^U = \prod_{k=1}^{n} B(m_k).$$

The numbers $B(n,M)$ are called the n-dimensional Bernoulli numbers. According to Section 8 we clearly have

(9.7) $$\sum_{M=0}^{P} a(M)/(P-M)! - a(P) = 0$$

and

(9.8) $$[U + \underline{B}(n)]^P - B(n,P) = 0.$$

(9.7) and (9.8) are the recurrence relations for the $a(M)$ and the n-dimensional Bernoulli numbers.

Consider next

$$P = [p_1, p_2, \cdots, p_n] \in W(n), \quad \Phi(P) = [\varphi(p_1, x_1), \varphi(p_2, x_2), \cdots, \varphi(p_n, x_n)]$$

and

(9.9) $$\varphi(P,X) = \prod_{k=1}^{n} \varphi(p_k, x_k) = \sum_{K=0}^{P} a(K)X^{P-K}/(P-K)! = \sum_{K=0}^{P} B(n,K)X^{P-K}/K!(P-K)! = [X + \underline{B}(n)]^P ,$$

from where it is easily seen that (cf. (7.1))

(9.10) $$\boldsymbol{D}\,\varphi(P,X) = \varphi(P-U, X).$$

On the other hand, according to [1], p. 231,

$$\Delta(k)\varphi(p_k, x_k) = x_k^{p_k-1}/(p_k-1)!,$$

so that by multiplication over k we obtain

(9.11) $$\boldsymbol{\Delta}\varphi(P,X) = X^{P-U}/(P-U)!\ .$$

According to Section 5 and [1], we obtain the generating function of the n-dimensional Bernoulli numbers as follows:

(9.12) $$\Omega(T) = T^U/(e^T - 1) = \sum_{M=0} B(n,M)T^M .$$

(ii) Consider the numbers $e(m)$ defined by the recurrence relation (cf. [1], p. 289)

(9.13) $$e(n) + \sum_{k=0}^{n} e(k)/(n-k)! = 0.$$

The numbers $e(m)$ are the coefficients of the Euler polynomials

(9.14) $$\eta(n,x) = \sum_{k=0}^{n} e(k)x^{n-k}/(n-k)!.$$

The numbers $t(n) = 2^n e(n)n!$ are called the tangent coefficients (cf. [1], p. 298) and the numbers

(9.15) $$\epsilon(n) = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t(k)$$

Euler numbers. According to [1], the tangent coefficients satisfy the recurrence relation

(9.16) $$(2+\underline{t})^n + t(n) = 0.$$

It is shown in [1] that (9.15) can be inverted to give $t(n) = [\underline{\epsilon} - 1]^n$. It follows that

(9.17) $$[\underline{\epsilon} + 1]^n + [\underline{\epsilon} - 1]^n = 0, \quad n > 0.$$

As before we introduce $M \in W(n)$ and $\eta(M) = [e(m_1), e(m_2), \cdots, e(m_n)]$, with

$$\eta(n,M) = [\eta(M)]^U = \prod_{m=1}^{n} e(m_k).$$

The n-dimensional tangent coefficients will be $T(M) = [t(m_1), t(m_2), \cdots, t(m_n)]$, so that

$$t(n,M) = [T(M)]^U = \prod_{k=1}^{n} t(m_k).$$

Finally let $\epsilon(M) = [\epsilon(m_1), \epsilon(m_2), \cdots, \epsilon(m_n)]$, so that

$$\epsilon(n,M) = [\epsilon(M)]^M = \prod_{k=1}^{n} \epsilon(m_k),$$

where the numbers $\epsilon(n,M)$ are called the n-dimensional Euler numbers. It is easily seen, like in the case of the Bernoulli numbers, that

(9.18) $$[\underline{\epsilon}(n) + 1]^P + [\underline{\epsilon}(n) - 1]^P = 0, \quad P > 0,$$

(9.19) $$t(n,P) + [2U + \underline{T}(n)]^P = 0, \quad P > 0,$$

(9.20) $$t(n,K) = K!2^K e(n,K),$$

(9.21) $$\epsilon(n,P) = [U + \underline{T}(n)]^P,$$

(9.22) $$t(n,M) = [\epsilon(n) - U]^M.$$

We introduce in the same way the n-dimensional Euler polynomials: Let

$$H(P) = [\eta(p_1, x_1), \eta(p_2, x_2), \cdots, \eta(p_n, x_n)],$$

where $P \in W(n)$. It follows that

(9.23) $$\eta(P,K) = \prod_{k=1}^{n} \eta(p_k, x_k) = \sum_{K=0}^{P} e(n,K)X^{P-K}/(P-K)!,$$

which defines the n-dimensional Euler polynomials.

It can easily be checked that similarly to the one-dimensional case we have

(9.24) $$\not{D}\eta(P,X) = \eta(P-U, X)$$

and

(9.25) $$\not{M}\eta(P,X) = X^P/P!.$$

According to Section 8 we obtain the following generating function for the Euler numbers $\epsilon(n,K)$ and the numbers $e(n,K)$

(9.26) $$G\epsilon(n,P) = 2/[e^T + e^{-T}] = \sum_{K=0} \epsilon(n,K)T^K/K!$$

(9.27) $$Ge(n,P) = 2/[e^T + 1] = \sum_{K=0} e(n,K)T^K.$$

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CONSECUTIVE INTEGER PAIRS OF POWERFUL NUMBERS AND RELATED DIOPHANTINE EQUATIONS

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1. INTRODUCTION

S. W. Golomb [1] defined a *powerful number* to be a positive integer r, such that p^2 divides r whenever the prime p divides r, and discussed consecutive integer pairs of powerful numbers which fall into one or the other of two types. The types are TYPE I: pairs of consecutive powerful numbers one of which is a perfect square, and TYPE II: pairs of consecutive powerful numbers neither of which is a perfect square. He showed an infinity of cases of TYPE I by applying theory of the Pell equation.

The first purpose of this paper is to elaborate on Golomb's findings for TYPE I, through theory of the Pell equation, to give all cases of that type. Then, built on that theory, the second purpose is to formulate all pairs of consecutive powerful numbers of TYPE II, through certain solutions of another Diophantine equation.

2. CONSECUTIVE POWERFUL NUMBER PAIRS OF TYPE I

Consecutive powerful number pairs of TYPE I correspond to certain numbers satisfying the Pell equation

$$(1) \qquad X^2 - DY^2 = \pm 1 ,$$

where D is a given positive integer not a perfect square.

It is convenient to make the following definitions

Definition. The number $x + y\sqrt{D}$ is a solution of (1) if $x = X$ and $y = Y$ are integers satisfying (1).

Definition. A positive solution of (1) is a solution $x + y\sqrt{D}$ of (1) in which both integers x and y are positive.

Although at times we will consider solutions in which x or y may be negative, our main concern is with positive solutions. At all times our "integers" are assumed to be rational integers.

Definition. The positive solution $x + y\sqrt{D}$ of (1) in which X and Y have their least values is the fundamental solution of (1).

The fact that powers of the fundamental solution of (1) generate all positive solutions is well known [2] and is given here without proof in the following

Theorem 2.1. Solutions of the Pell equation (1) may be formulated by the following cases. (1) If equation (1) with the minus sign is not solvable, let $x + y\sqrt{D}$ be the fundamental solution of (1) with the plus sign. Then all positive solutions of the latter equation are given by

$$(2) \qquad x_i + y_i\sqrt{D} = (x + y\sqrt{D})^i$$

for positive integers i, and where $x_1, y_1 = x, y$. (2) If equation (1) with the minus sign is solvable and has fundamental solution $x + y\sqrt{D}$, then all its positive solutions are given by (2) for odd positive integers i. In this case the fundamental solution of (1) with the plus sign is $(x + y\sqrt{D})^2$, and all its positive solutions are given by $x_{2i} + y_{2i}\sqrt{D} = [(x + y\sqrt{D})^2]^i$ for positive integers i.

AMS 1970 *subject classifications.* Primary 10B05, 10B10; Secondary 12A25, 12A45.

Key words and phrases: powerful number, consecutive powerful number pair, positive solution, fundamental solution, smallest solution, property Q, least solution with property *Q*.

For purposes of this paper it is convenient to write solutions in the form $x_j + y_j\sqrt{D}$, where j is the exponent of the power of the fundamental solution and not necessarily the ordinal number of the solution in sequence. Without loss of generality we may assume that D is square-free.

In equation (1), for the two consecutive integers X^2 and DY^2, our desire that the latter be a powerful number motivates the following

Definition. A solution $u + v\sqrt{D}$ of equation (1) has property Q if for p a prime, $p \mid D$ implies $p \mid v$.

Definition. The least solution with property Q is the positive solution $u + v\sqrt{D}$ with property Q of (1), in which integers u and v have their least values.

Now if equation (1) with the minus sign has a solution with property Q, then D must be odd. For if $X^2 - D^2 = -1$ is solvable with D even, then no solution has property Q since $4 \nmid X^2 + 1 = DY^2$ for any integer X.

If equation (1) has the plus sign and if D is even, let $x + y\sqrt{D}$ be any solution. Then since x is odd

$$8 \mid (x-1)(x+1) = Dy^2\ ,$$

and $2 \mid y$ since D is square-free. So here $2 \mid D$ implies $2 \mid y$.

Now in the expaision of (2) we see that for any i

$$y_i = ix^{i-1}y + \binom{i}{3} x^{i-3}y^3D + \cdots, \tag{3}$$

where $y \mid y_i$, and D is a factor of every term on the right except the first. If the fundamental solution $x + y\sqrt{D}$ of (1) has property Q, then it is clear from (3) that all positive solutions have property Q. Otherwise, if we take $i = \Pi p_j$ the product of distinct odd primes p_j such that $p_j \mid D$ but $p_j \nmid y$, then we see from (3) that the solution $x_i + y_i\sqrt{D}$ has property Q. Moreover, this is the least solution with property Q since $(x,D) = 1$ and since for any $h < i$ there is at least one of the primes p_j such that $p_j \mid D$ but $p_j \nmid y_h$ in the solution $x_h + y_h\sqrt{D}$ of (1) as given by (2). We have proved the following

Theorem 2.2. For either choice of sign, let equation (1) have fundamental solution $x + y\sqrt{D}$. Then

(1). If (1) has the minus sign, and if D is even, no solutions have property Q. In all other cases (1) has a solution with property Q.

(2). If the fundamental solution has property Q, then all positive solutions have property Q.

(3). If the fundamental solution does not have property Q, then the least solution with property Q, when it exists, is the number $(x + y\sqrt{D})^i$, where the integer i is the product of the distinct odd primes dividing D but not dividing the integer y.

Before considering all solutions of (1) with property Q, we need two lemmas.

Lemma 2.3. Let $\alpha = x + y\sqrt{D}$ and $\beta = u + v\sqrt{D}$ be any solutions with property Q of equation (1) with either choice of sign for each solution. Then

(1). The product $\gamma = \alpha\beta = (xu + yvD) + (xv + yu)\sqrt{D}$ is a solution of (1) with the plus sign if α and β are solutions of (1) with the same sign, or γ is a solution of (1) with the minus sign for α and β solutions of (1) with opposite sign.

(2). The product solution γ has property Q.

Proof. The first conclusion follows from

$$(xu + yvD)^2 - D(xv + yu)^2 = (x^2 - Dy^2)(u^2 - Dv^2) = (\pm 1)(\pm 1) = \pm 1 .$$

Now by property Q for α and β if a prime $p \mid D$, then $p \mid y$ and $p \mid v$, so $p \mid (xv + yu)$ and γ has property Q.

Lemma 2.4. Let (1) with the minus sign be solvable with fundamental solution $x + y\sqrt{D}$ and have $x_i + y_i\sqrt{D}$ the least solution with property Q. Let $x_L + y_L\sqrt{D}$ be the least solution with property Q of (1) with the plus sign. Then $x_L + y_L\sqrt{D} = (x_i + y_i\sqrt{D})^2$.

Proof. By Theorem 2.1 the fundamental solution of (1) with the plus sign is

$$(x + y\sqrt{D})^2 = (x^2 + Dy^2) + 2xy\sqrt{D}\ .$$

By Theorem 2.2, D is odd, so $(2x,D) = 1$. Then the prime divisors of D not dividing $2xy$ are the same as those dividing D and not y for the least solution of (1) with the minus sign which has property Q. So that by Theorem 2.1 and by conclusion 3 of Theorem 2.2, we have

$$x_L + y_L\sqrt{D} = [(x+y\sqrt{D})^2]^i = [(x+y\sqrt{D})^i]^2 = (x_i+y_i\sqrt{D})^2 .$$

Theorem 2.5. Let $x_i + y_i\sqrt{D}$ be the least solution with property Q, when it exists, of equation (1). Then the formula

$$x_{ih} + y_{ih}\sqrt{D} = (x_i+y_i\sqrt{D})^h \tag{4}$$

gives all positive solutions with property Q of equation (1) with the plus sign for positive integers h, and gives all such solutions of (1) with the minus sign for odd positive integers h.

Proof. For either choice of sign in equation (1), since integers x_i and y_i are positive, by repeated applications of Lemma 2.3, we see that (4) always gives solutions as described in the statement of this theorem.

We now show that (4) gives all of the positive solutions with property Q of (1) for each choice of sign. Suppose that (1), for some choice of sign, has a positive solution $u + v\sqrt{D}$, which has property Q. and which is not given by formula (4). Then for some positive integer h, we have

$$(x_i+y_i\sqrt{D})^h < u+v\sqrt{D} < (x_i+y_i\sqrt{D})^{h+\epsilon} . \tag{5}$$

If equation (1) has the plus sign, then $\epsilon = 1$, or if equation (1) has the minus sign, then $\epsilon = 2$ with h odd; and respectively the number

$$x_i - y_i\sqrt{D} = \pm 1/(x_i+y_i\sqrt{D})$$

is positive or negative. For either case, multiplying inequalities (5) by $\pm(x_i - y_i\sqrt{D})^h$, whichever is positive, gives

$$1 < \alpha = (\pm u \pm v\sqrt{D})(x_i - y_i\sqrt{D})^h < \beta = (x_i+y_i\sqrt{D})^\epsilon .$$

For both of these cases, by Lemma 2.3, the number $\alpha = w+z\sqrt{D}$ is a solution with property Q of (1) with the plus sign, for some integer pair w,z. Substituting for α in the last inequalities, we get

$$1 < w+z\sqrt{D} < \beta = (x_i+y_i\sqrt{D})^\epsilon , \tag{6}$$

and the inequalities

$$0 < w-z\sqrt{D} = 1/(w+z\sqrt{D}) < 1 \tag{7}$$

imply that both integers w and z are positive. So $w+z\sqrt{D}$ is a positive solution of (1) with the plus sign. But in (6) we have

$$w+z\sqrt{D} < \beta = (x_i+y_i\sqrt{D})^\epsilon ,$$

where for $\epsilon = 1$ or for $\epsilon = 2$, by Lemma 2.4, the number β is the least solution with property Q of (1) with the plus sign, a contradiction. This completes the proof that (4) gives all positive solutions with property Q of equation (1).

Since equation (1) solutions $(x_i + y_i\sqrt{D})^h$ with property Q correspond to consecutive powerful numbers x_{ih}^2 Dy_{ih}^2 we have thus accounted for all consecutive powerful number pairs of TYPE I.

EXAMPLES. (1) The fundamental solution $24335 + 3588\sqrt{46}$ of $X^2 - 46Y^2 = 1$ has property Q, and hence all positive solutions have property Q. The corresponding powerful numbers are 592, 192, 225 and 592, 192, 224 or $(24335)^2$ and $(2^5)(3^2)(13^2)(23^3)$. (2) The fundamental solution of $X^2 - 6Y^2 = 1$ is $5 + 2\sqrt{6}$. The solution

$$(5+2\sqrt{6})^3 = 485 + 198\sqrt{6}$$

and all its powers have property Q. This solution leads to powerful numbers 235, 225 and 235, 224 or $(485)^2$ and $(2^3)(3^5)(11^2)$. (3) The fundamental solution of $X^2 - 5Y^2 = -1$ is $2+\sqrt{5}$. The solution

$$(2+\sqrt{5})^5 = 682 + 305\sqrt{5}$$

and its odd powers have property Q. This solution leads to the powerful numbers 465,124 and 465,125 or $(682)^2$ and $(5^3)(61^2)$.

3. CONSECUTIVE POWERFUL NUMBER PAIRS OF TYPE II

Consecutive powerful number pairs of this type correspond to certain numbers satisfying the Diophantine equation

$$mX^2 - nY^2 = \pm 1, \tag{8}$$

where m and n are given positive integers, and neither is a perfect square.

Our development of the theory of this equation proceeds along lines similar to that of the Pell equation (1). We begin by making the following definitions

Definition. The number $x\sqrt{m} + x\sqrt{n}$ is a solution of (8) if $x = X$ and $y = Y$ are integers satisfying (8).

Definition. A positive solution of (8) is a solution $x\sqrt{m} + y\sqrt{n}$ of (8) in which both integers x and y are positive.

As before, although some solutions under consideration may have negative x or y, our main concern is with positive solutions.

REMARK. If $x\sqrt{m} + y\sqrt{n}$ and $x'\sqrt{m} + y'\sqrt{n}$ are positive solutions of (8), then it is easily seen that the inequalities

$$x < x', \qquad y < y' \quad \text{and} \quad x\sqrt{m} + y\sqrt{n} < x'\sqrt{m} + y'\sqrt{n}$$

are equivalent. So among all the positive solutions, there is one solution in which both x and y have their least values.

Definition. The smallest solution of (8) is the positive solution $x\sqrt{m} + y\sqrt{n}$ in which both integers X and Y have their least values.

Analogous to Theorem 2.1 for solutions of the Pell equation is the theorem we now state without proof [3, Theorem 9] for solutions of equation (8)

Theorem 3.1. If equation (8) has smallest solution $x\sqrt{m} + y\sqrt{n}$, then all positive solutions of (8) are given by

$$x_i\sqrt{m} + y_i\sqrt{n} = (x\sqrt{m} + y\sqrt{n})^{2i+1} \tag{9}$$

for non-negative integers i, and where $x_0, y_0 = x,y$.

Without loss of generality we may assume that integers m and n are square-free. Moreover, our desire that consecutive integers mX^2 and nY^2 be a powerful number pair of TYPE II motivates the following

Definition. A solution $u\sqrt{m} + v\sqrt{n}$ of (8) has property Q if for p a prime, $p\,|\,mn$ implies $p\,|\,uv$.

Note that since $(mu,nv) = 1$, this definition is equivalent to saying the prime divisors of m divide u, and those dividing n divide v.

Definition. The least solution with property Q of equation (8) is the positive solution $u\sqrt{m} + v\sqrt{n}$ with property Q of (8), in which integers u and v have their least values.

Now from (9) we get x_i and y_i in the following expressions

$$\begin{aligned} x_i &= m^i\, x^{2i+1} + \dots + \binom{2i+1}{3} mn^{i-1}x^3y^{2i-2} + (2i+1)n^ixy^{2i}, \text{ and} \\ y_i &= (2i+1)m^ix^{2i}y + \binom{2i+1}{3} m^{i-1}nx^{2i-2}y^3 + \dots + n^iy^{2i+1} \quad . \end{aligned} \tag{10}$$

Note that x_i and y_i have m and n respectively as a factor of every term except one, the term in each case having the odd positive integer $2i + 1$ as a factor. Note also that $(mx,ny) = 1$, $x\,|\,x_i$, and $y\,|\,y_i$.

If one of m or n, say m, is even, we see in (10) that x_i is even for all i, if and only if x is even. Similarly, when n is even, y_i is even if and only if y is even. So for the possible prime divisor 2 of m or n, solutions with property Q of (8) depend solely on the parity of integers x or y respectively of the smallest solution $x\sqrt{m} + y\sqrt{n}$.

Now from (10) if the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8) has property Q, then all positive solutions have property Q since for p a prime, $p\,|\,mn$ implies $p\,|\,xy\,|\,x_iy_i$.

If the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8) does not have property Q, then for the odd integer $2i + 1$ in (9) and (10), take

$$2i + 1 = \Pi p_j \tag{11}$$

the product of distinct odd primes p_j such that $p_j\,|\,mn$ but $p_j \nmid xy$, and the solution $x_i\sqrt{m} + y_i\sqrt{n}$ as given in (9) and (10) has property Q. Moreover, this solution obtained in (11) is the least solution with property Q of (8). This is due to the fact that a positive solution $x_h\sqrt{m} + y_h\sqrt{n}$, with $h < i$, corresponds in (9) and (10) to an exponent $2h + 1$ such that $2h + 1$ and hence at least one of x_h or y_h are not divisible by some prime divisor p_j in (11) of mn.

We have proved the following

Theorem 3.2. Let equation (8) have smallest solution $x\sqrt{m} + y\sqrt{n}$. Then (1) If m (or n) is even and if x (or y) is odd, then no solutions have property Q. In all other cases (8) has a solution with property Q. (2) If the smallest solution has property Q, then all positive solutions have property Q. (3) If the smallest solution does not have property Q, then the least solution with property Q, when it exists, is the solution $x_i\sqrt{m} + y_i\sqrt{n}$ given by (9) for which the exponent $2i + 1$ is the product of the distinct odd primes dividing mn but not dividing xy.

Throughout the remainder of this discussion we will frequently be concerned with the Pell equation which we write in the form

$$R^2 - mnS^2 = 1, \tag{12}$$

and to which the same theory and definitions apply as to the Pell equation (1), since the product mn is square-free.

Before discussing all solutions with property Q of (8), we need the following two lemmas

Lemma 3.3. If $\alpha = u\sqrt{m} + v\sqrt{n}$ is a solution with property Q of (8), and $\beta = r + s\sqrt{(mn)}$ is a solution of (12) with property Q (as defined for that equation), then the product

$$\gamma = \alpha\beta = (ur + nvs)\sqrt{m} + (vr + mus)\sqrt{n}$$

is a solution with property Q of (8).

Proof. $(mu^2 - nv^2)(r^2 - mns^2) = (\pm 1)(1) = m(ur + nvs)^2 - n(vr + mus)^2 = \pm 1.$

So γ is a solution of (8). Now by the properties Q for α and β respectively, if p is a prime and $p \mid m$, then $p \mid (ur + nvs)$. Similarly, if a prime $q \mid n$, then $q \mid (vr + mus)$. So γ has property Q for equation (8).

Lemma 3.4. If

$$\alpha = u\sqrt{m} + v\sqrt{n} \qquad \text{and} \qquad \beta = u'\sqrt{m} + v'\sqrt{n}$$

are solutions with property Q of equation (8), then the product

$$\gamma = \alpha\beta = (muu' + nvv') + (uv' + u'v)\sqrt{(mn)}$$

is a solution with property Q of (12).

Proof. $(mu^2 - nv^2)(mu'^2 - nv'^2) = (\pm 1)(\pm 1) = (muu' + nvv')^2 - mn(uv' + u'v)^2 = 1,$
and γ is a solution of (12). Now by property Q for α and β solutions of (8), the prime divisors of m divide both u and u', and prime divisors of n divide both v and v'. So if a prime $p \mid mn$, then $p \mid (uv' + u'v)$, and γ has property Q for (12).

Theorem 3.5. Let $x_i\sqrt{m} + y_i\sqrt{n}$ be the least solution with property Q of (8), when it exists. Then all positive solutions of (8) with property Q are given by the formula

$$x_H\sqrt{m} + y_H\sqrt{n} = (x_i\sqrt{m} + y_i\sqrt{n})^{2h+1} \tag{13}$$

for non-negative integers h.

Proof. The subscript $H = 2ih + i + h$, by Theorem 3.1. Since $x_i\sqrt{m} + y_i\sqrt{n}$ is the least solution with property Q of (8), then by Lemmas 3.3 and 3.4 formula (13) gives a positive solution with property Q of (8) for every non-negative integer h.

Now suppose equation (8) has a positive solution $w\sqrt{m} + z\sqrt{n}$, with property Q, which is not given by (13). Then for some non-negative integer h, we have

$$(x_i\sqrt{m} + y_i\sqrt{n})^{2h+1} < w\sqrt{m} + z\sqrt{n} < (x_i\sqrt{m} + y_i\sqrt{n})^{2h+3}\,. \tag{14}$$

The number

$$x_i\sqrt{m} - y_i\sqrt{n} = \pm 1/(x_i\sqrt{m} + y_i\sqrt{n})$$

is positive or negative, respectively, according as equation (8) has the plus or minus sign. For either case, multiplying inequalities (14) by $\pm(x_i\sqrt{m} - y_i\sqrt{n})^{2h+1}$ whichever is positive, we get

$$1 < \alpha = (\pm w\sqrt{m} \pm z\sqrt{n})(x_i\sqrt{m} - y_i\sqrt{n})^{2h+1} < \beta = (x_i\sqrt{m} + y_i\sqrt{n})^2 \tag{15}$$

By Lemma 3.4, and since integers x_i and y_i are positive, the number $\beta = (x_i\sqrt{m} + y_i\sqrt{n})^2$ is a positive solution with property Q of (12). We will now show that β is the least solution with property Q of (12). Suppose on the contrary that the number $r_j + s_j\sqrt{(mn)}$ is the least solution with property Q of (12). By Theorem 2.1,

$$r_j + s_j\sqrt{(mn)} = [r + s\sqrt{(mn)}\,]^j,$$

where $r + s\sqrt{(mn)}$ is the fundamental solution of (12), for some positive integer j. By Theorem 2.2., the integer j is odd, and by [3, Theorem 6] the number

$$r + s\sqrt{(mn)} = (x\sqrt{m} + y\sqrt{n}\,)^2\ ,$$

where $x\sqrt{m} + y\sqrt{n}$ is the smallest solution of (8). A substitution gives

$$r_j + s_j\sqrt{(mn)} = [(x\sqrt{m} + y\sqrt{n}\,)^2]^j = [(x\sqrt{m} + y\sqrt{n})^j]^2\ .$$

By Theorem 3.1, and since $j = 2k + 1$ is odd,

$$(x\sqrt{m} + y\sqrt{n}\,)^j = (x\sqrt{m} + y\sqrt{n}\,)^{2k+1} = x_k\sqrt{m} + y_k\sqrt{n}\ ,$$

the k^{th} positive solution of (8). Then

$$r_j + s_j\sqrt{(mn)} = (x_k\sqrt{m} + y_k\sqrt{n}\,)^2 = mx_k^2 + ny_k^2 + 2x_ky_k\sqrt{(mn)}\ .$$

Now since $r_j + s_j\sqrt{(mn)}$ has property Q for (12) and since $s_j = 2x_ky_k$ it follows that if a prime $p\,|mn$, then $p|2x_ky_k$. In fact if a prime $p\,|mn$, then $p\,|x_ky_k$. This is obvious if p is an odd prime. Since equation (8) is assumed to have a solution with property Q, then by Theorem 3.2 if $2\,|mn$, then $2\,|xy\,|x_ky_k$ for all k, and where integers x,y are those of the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8). So the positive solution $x_k\sqrt{m} + y_k\sqrt{n}$ of (8) has property Q.

Then if

$$r_j + s_j\sqrt{(mn)} < \beta = (x_i\sqrt{m} + y_i\sqrt{n}\,)^2, \quad \text{or} \quad (x_k\sqrt{m} + y_k\sqrt{n}\,)^2 < (x_i\sqrt{m} + y_i\sqrt{n}\,)^2\ ,$$

it follows that

$$x_k\sqrt{m} + y_k\sqrt{n} < x_i\sqrt{m} + y_i\sqrt{n}$$

the least solution with property Q of (8), a contradiction. Thus we have shown the number $\beta = (x_i\sqrt{m} + y_i\sqrt{n})^2$ of inequalities (15) to be the least solution with property Q of (12).

Now consider the number α of inequalities (15). By Lemmas 3.3 and 3.4 the number $\alpha = u + v\sqrt{(mn)}$ is a solution with property Q of (12), for some pair of integers u and v. So that inequalities (15) become

$$1 < u + v\sqrt{(mn)} < (x_i\sqrt{m} + y_i\sqrt{n}\,)^2\ . \tag{16}$$

Then since $u + v\sqrt{(mn)}$ is a solution of (12), the inequalities

$$0 < u - v\sqrt{(mn)} = 1/[u + v\sqrt{(mn)}] < 1$$

imply that integers u and v are both positive, and $u + v\sqrt{(mn)}$ is a positive solution of (12).

We have shown that the existence of a positive solution $w\sqrt{m} + z\sqrt{n}$ with property Q of equation (8), which is not given by formula (13), implies the existence of a positive solution $u + v\sqrt{(mn)}$, with property Q, of equation (12), which by (16) is less than $(x_i\sqrt{m} + y_i\sqrt{n})^2$ the least solution with property Q of (12), a contradiction. This completes the proof that formula (13) gives all positive solutions with property Q of equation (8).

Since equation (8) solutions $(x_i\sqrt{m} + y_i\sqrt{n})^{2h+1}$ with property Q correspond to consecutive powerful numbers $mx_H^2\ \ ny_H^2$ we have thus accounted for all consecutive powerful number pairs of TYPE II and hence, with Section 2 of this paper, for all pairs of consecutive powerful numbers.

EXAMPLE. The equation $7X^2 - 3Y^2 = 1$ has smallest solution $2\sqrt{7} + 3\sqrt{3}$. The solution

$$(2\sqrt{7} + 3\sqrt{3}\,)^7 = 2{,}637{,}362\sqrt{7} + 4{,}028{,}637\sqrt{3}$$

and all its odd powers have property Q. This solution corresponds to the following consecutive powerful number pair of TYPE II,

$$48{,}689{,}748{,}233{,}308 = 7(2{,}637{,}362)^2 = (2^2)(7^3)(13^2)(43^2)(337^2)$$

and

$$48{,}689{,}748{,}233{,}307 = 3(4{,}028{,}637)^2 = (3^3)(139^2)(9661^2)\ .$$

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SUMS OF PARTITION SETS IN GENERALIZED PASCAL TRIANGLES I

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In the expansion

$$(1+x+x^2+\cdots+x^{k-1})^n = \sum_{i=0}^{(k-1)n} \begin{bmatrix} n \\ i \end{bmatrix}_k x^i, \qquad k \geqslant 2, \quad n \geqslant 0,$$

clearly

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_k = \begin{bmatrix} n \\ (k-1)n \end{bmatrix}_k = 1$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix}_k = \sum_{j=0}^{k-1} \begin{bmatrix} n-1 \\ r-j \end{bmatrix}_k, \begin{bmatrix} n \\ j \end{bmatrix}_k = 0, \qquad j<0, \quad j>(k-1)n.$$

For $k = 2$, these are the binomial coefficients and when dealing with these we shall use the usual notation:

$$\begin{bmatrix} n \\ r \end{bmatrix}_2 = \binom{n}{r}.$$

The problem of calculating sums of the following type for $k = 2$ was first treated by Cournot [2] and Ramus [5] and Ramus' method is outlined in [4]:

$$S(n,k,q,r) = \sum_{j=0}^{N} \begin{bmatrix} n \\ r+jq \end{bmatrix}_k,$$

where

$$N = \left[\frac{(k-1)n-r}{q} \right],$$

[·] denoting the greatest integer function. We wish here to investigate for certain fixed k and q the different values of these sums as r ranges from 0 to $q-1$ and, further, the differences between the sums.

1. THE METHOD OF RAMUS

Let ω be a primitive q^{th} root of unity then

$$\omega = \cos\frac{2\pi}{q} + i\sin\frac{2\pi}{q}.$$

Then

$$(1+\omega^0)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$(1+\omega)^n = \binom{n}{0} + \binom{n}{1}\omega + \binom{n}{2}\omega^2 + \binom{n}{3}\omega^3 + \cdots + \binom{n}{n}\omega^n$$

$$(1+\omega^2)^n = \binom{n}{0} + \binom{n}{1}\omega^2 + \binom{n}{2}\omega^4 + \binom{n}{3}\omega^6 + \cdots + \binom{n}{n}\omega^{2n}$$

$$\cdots$$

$$(1+\omega^{q-1})^n = \binom{n}{0} + \binom{n}{1}\omega^{q-1} + \binom{n}{2}\omega^{2(q-1)} + \binom{n}{3}\omega^{3(q-1)} + \cdots + \binom{n}{n}\omega^{n(q-1)} .$$

Multiplying each successive row by $\omega, \omega^{-r}, \omega^{-2r}, \cdots, \omega^{-(q-1)r}$, $0 \leqslant r \leqslant q-1$, and adding the products we get

$$q\left[\binom{n}{r} + \binom{n}{r+q} + \binom{n}{r+2q} + \cdots\right] = \sum_{\ell=0}^{q-1} (1+\omega^\ell)^n \omega^{-r\ell} = \sum_{\ell=0}^{q-1} (\omega^{\ell/2} + \omega^{-\ell/2})^n \omega^{-r\ell+\ell n/2}$$

$$= \sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n \omega^{\frac{\ell(n-2r)}{2}} = \sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n \left[\cos\frac{\ell(n-2r)2\pi}{2q} + i\sin\frac{\ell(n-2r)2\pi}{2q}\right] .$$

Since the left side is real, the coefficient of i on the right must be zero, hence

$$S(n,2,q,r) = \binom{n}{r} + \binom{n}{r+q} + \binom{n}{r+2q} + \cdots = \frac{1}{q}\sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n \cos\frac{\ell(n-2r)\pi}{q} .$$

Applying the same technique to the expansion $(1+x+x^2+\cdots+x^{k-1})^n$ one finds that

$$\mathrm{S}(\mathrm{n},\mathrm{k},\mathrm{q},\mathrm{r}) = \begin{bmatrix} n \\ r \end{bmatrix}_k + \begin{bmatrix} n \\ r+q \end{bmatrix}_k + \begin{bmatrix} n \\ r+2q \end{bmatrix}_k + \cdots = \frac{1}{q}\sum_{\ell=0}^{q-1} (1+\omega^\ell+\omega^{2\ell}+\cdots+\omega^{(k-1)\ell})^n \omega^{-r\ell}$$

$$= \begin{cases} \displaystyle \frac{1}{q}\sum_{\ell=0}^{q-1}\left[2\sum_{j=1}^{\frac{k-1}{2}}\cos\frac{\ell(k-2j+1)\pi}{q} + 1\right]^n \cos\frac{\ell(nk-n-2r)\pi}{q} & \text{for } k \text{ odd} \\ \displaystyle \frac{1}{q}\sum_{\ell=0}^{q-1}\left[2\sum_{j=1}^{k/2}\cos\frac{\ell(k-2j+1)\pi}{q}\right]^n \cos\frac{\ell(nk-n-2r)\pi}{q} & \text{for } k \text{ even} \end{cases}$$

2. THE CASES $k = 2,\ q = 3, 4$

This case is treated in [4] and more recently in [6]. From the formulas above one easily shows that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = \frac{1}{3}\left[2^n + \left(2\cos\frac{\pi}{3}\right)^n \cos\frac{n\pi}{3} + \left(2\cos\frac{2\pi}{3}\right)^n \cos\frac{2n\pi}{3}\right]$$

$$= \frac{1}{3}\left[2^n + \cos\frac{n\pi}{3} + (-1)^n \cos\frac{2n\pi}{3}\right] = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

$$\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{(n-2)\pi}{3}\right]$$

$$\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{(n-4)\pi}{3}\right] .$$

By examining the table for $\cos (n\pi)/3$ one sees that the three differences

$$\frac{2}{3}\left[\cos\frac{n\pi}{3} - \cos\frac{(n-2)\pi}{3}\right], \quad \frac{2}{3}\left[\cos\frac{(n-2)\pi}{3} - \cos\frac{(n-4)\pi}{3}\right] \quad \text{and} \quad \frac{2}{3}\left[\cos\frac{(n-4)\pi}{3} - \cos\frac{n\pi}{3}\right]$$

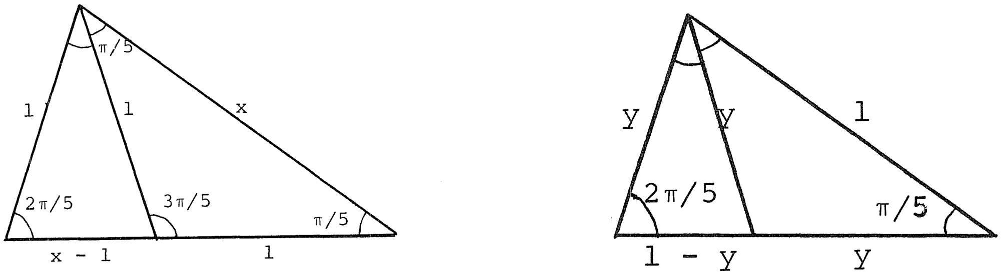
are 0, 1, −1. This problem appeared in the *American Mathematical Monthly* in May, 1938 as Problem E 300 (solution by Emma Lehmer) and again in February, 1956, as Problem E 1172. In slightly altered form it had appeared in the *Monthly* in 1932 as Problem 3497 (solution by Morgan Ward). It appeared as Problem B-6 in the 1974 William Lowell Putnam Contest.

The case of four sums ($q = 4$) yields in each case only two different or three different sums, depending on whether n is odd or even and the differences in the values are successive powers of 2, as the reader can verify. This appeared in *Mathematics Magazine* in November-December, 1952 as Problem 177 (solution by E. P. Starke).

3. THE CASE $k = 2, q = 5$

This case was treated tersely in the solution to a problem posed by E. P. Starke in the March, 1939, issue of the *National Mathematical Magazine,* where the differences were observed to be simple and predictable but the sums themselves were not seen to be reducible to simple form. We shall, therefore, treat this very interesting case at length, along with generalizations.

Consideration of the following two figures yields values of x and y:



(a) (b)

Figure 1

$$\frac{x}{1} = \frac{1}{x-1} \qquad \frac{1}{y} = \frac{y}{1-y} \qquad x = \frac{1+\sqrt{5}}{2} = \alpha \qquad y = -\frac{1-\sqrt{5}}{2} = -\beta,$$

where the signs are chosen so that x,y are positive. We note α is the golden ratio and the α,β are those of the Binet formulas for elements of Fibonacci and Lucas sequences, i.e., if

$F_1 = 1,\ \ F_2 = 1,\ \ F_n = F_{n-1} + F_{n-2},\ \ n > 2,$ and $L_1 = 1,\ \ L_2 = 3,\ \ L_n = L_{n-1} + L_{n-2},\ \ n > 2,$

then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n\ ,$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively [3].

From Fig. 1a, one sees that

$$\cos\frac{\pi}{5} = \frac{\alpha}{2} \qquad \text{and} \qquad \cos\frac{2\pi}{5} = \frac{\alpha - 1}{2}$$

and from $\alpha + \beta = 1$ one concludes that

$$\cos\frac{2\pi}{5} = \frac{-\beta}{2}\ .$$

From these one can construct a table of values for $\cos (n\pi)/5$. Then

$$S(n,2,5,r) = \binom{n}{r} + \binom{n}{r+5} + \binom{n}{r+10} + \cdots = \frac{1}{5}\sum_{\ell=0}^{4}\left(2\cos\frac{\ell\pi}{5}\right)^n \cos\frac{\ell(n-2r)\pi}{5}$$

$$= \frac{1}{5}\left[2^n + \alpha^n\cos\frac{(n-2r)\pi}{5} + (-\beta)^n\cos\frac{2(n-2r)\pi}{5} + (-\alpha)^n\cos\frac{4(n-2r)\pi}{5} + \beta^n\cos\frac{3(n-2r)\pi}{5}\right]$$

$$= \frac{1}{5}\left[2^n + 2\alpha^n\cos\frac{(n-2r)\pi}{5} + 2(-\beta)^n\cos\frac{2(n-2r)\pi}{5}\right] \text{ for } r = 0, 1, 2, 3, 4.$$

Let us examine, for example, $S(10m, 2, 5, 0)$:

$$\binom{10m}{0} + \binom{10m}{5} + \binom{10m}{10} + \cdots = \frac{1}{5}[2^{10m} + 2\alpha^{10m} + 2\beta^{10m}] = \frac{1}{5}[2^{10m} + 2L_{10m}] ,$$

where L_{10m} is a Lucas number. For $n = 10m + 1$,

$$S(10m+1,2,5,0) = \frac{1}{5}[2^{10m+1} + 2\alpha^{10m+1} \cdot (\alpha/2) - 2\beta^{10m+1}(\beta/2)] = \frac{1}{5}[2^{10m+1} + \alpha^{10m+2} + \beta^{10m+2}]$$

$$= \frac{1}{5}[2^{10m+1} + L_{10m+2}] .$$

We can continue to reduce these sums to the form $1/5[2^n + A]$, where A is a Lucas number or twice a Lucas number and can, in fact, form the following table for the values of A:

Table 1

n	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$10m$	$2L_{10m}$	L_{10m-1}	$-L_{10m+1}$	$-L_{10m+1}$	L_{10m-1}
$10m + 1$	L_{10m+2}	L_{10m+2}	$-L_{10m}$	$-2L_{10m+1}$	$-L_{10m}$
$10m + 2$	L_{10m+1}	$2L_{10m+2}$	L_{10m+1}	$-L_{10m+3}$	$-L_{10m+3}$
$10m + 3$	$-L_{10m+2}$	L_{10m+4}	L_{10m+4}	$-L_{10m+2}$	$-2L_{10m+3}$
$10m + 4$	$-L_{10m+5}$	L_{10m+3}	$2L_{10m+4}$	L_{10m+3}	$-L_{10m+5}$
$10m + 5$	$-2L_{10m+5}$	$-L_{10m+4}$	L_{10m+6}	L_{10m+6}	$-L_{10m+4}$
$10m + 6$	$-L_{10m+7}$	$-L_{10m+7}$	L_{10m+5}	$2L_{10m+6}$	L_{10m+5}
$10m + 7$	$-L_{10m+6}$	$-2L_{10m+7}$	$-L_{10m+6}$	L_{10m+8}	L_{10m+8}
$10m + 8$	L_{10m+7}	$-L_{10m+9}$	$-L_{10m+9}$	L_{10m+7}	$2L_{10m+8}$
$10m + 9$	L_{10m+10}	$-L_{10m+8}$	$-2L_{10m+9}$	$-L_{10m+8}$	L_{10m+10}

Thus we have formulas for all sums of the form

$$\sum_{t=0} \binom{n}{r+5t}, \qquad r = 0, 1, 2, 3, 4,$$

and since

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n ,$$

we note that the sum of the five elements on any row of the above table must be zero and, furthermore, it is clear from the method of generating Pascal's Triangle that each element of Table 1 must be the sum of the element above it and to the left of that. The following is the table of high and low values of the elements in Table 1:

Table 2

H	L
$2L_{10m}$	$-L_{10m+1}$
L_{10m+2}	$-2L_{10m+1}$
$2L_{10m+2}$	$-L_{10m+3}$
L_{10m+4}	$-2L_{10m+3}$
$2L_{10m+4}$	$-L_{10m+5}$
L_{10m+6}	$-2L_{10m+5}$
$2L_{10m+6}$	$-L_{10m+7}$
L_{10m+8}	$-2L_{10m+7}$
$2L_{10m+8}$	$-L_{10m+9}$
L_{10m+10}	$-2L_{10m+9}$

The differences between the highest value of the sums for given n and the lowest value is, therefore, always of the form

$$(2L_n + L_{n+1})/5.$$

That

$$(2L_n + L_{n+1})/5 = F_{n+1}$$

is proved easily by induction. We note that for each n there are only three different values for the five sums and that differences between the high and low values are Fibonacci numbers. Furthermore, the differences between the high and middle values, the middle and low values are again Fibonacci numbers. In fact, the three Fibonacci numbers have consecutive subscripts.

4. THE CASE $k = 3,\ q = 5$

In this case we are dealing with five sums of trinomial coefficients, and, for $r = 0, 1, 2, 3, 4,$

$$S(n,3,5,r) = \begin{bmatrix} n \\ r \end{bmatrix}_3 + \begin{bmatrix} n \\ r+5 \end{bmatrix}_3 + \begin{bmatrix} n \\ r+10 \end{bmatrix}_3 + \cdots = \frac{1}{5}\sum_{\ell=0}^{4}\left[2\cos\frac{2\ell\pi}{5} + 1\right]^n \cos\frac{2(n-r)\ell\pi}{5} .$$

But since

$$2\cos\frac{2\pi}{5} + 1 = 2\left(-\frac{\beta}{2}\right) + 1 = -\beta + 1 = \alpha = 2\cos\frac{8\pi}{5} + 1$$

and

$$2\cos\frac{4\pi}{5} + 1 = 2\left(-\frac{\alpha}{2}\right) + 1 = -\alpha + 1 = \beta = 2\cos\frac{6\pi}{5} + 1 ,$$

$$S(n,3,5,r) = \frac{1}{5}\left[3^n + \alpha^n\cos\frac{2(n-r)\pi}{5} + \beta^n\cos\frac{4(n-r)\pi}{5} + \beta^n\cos\frac{6(n-r)\pi}{5} + \alpha^n\cos\frac{8(n-r)\pi}{5}\right]$$

$$= \frac{1}{5}\left[3^n + 2\alpha^n\cos\frac{2(n-r)\pi}{5} + 2\beta^n\cos\frac{4(n-r)\pi}{5}\right] .$$

These sums reduce in each case to the form $1/5[3^n + B]$, where B is found in Table 3:

Table 3

n	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$10m$	$2L_{10m}$	L_{10m-1}	$-L_{10m+1}$	$-L_{10m+1}$	L_{10m-1}
$10m + 1$	L_{10m}	$2L_{10m+1}$	L_{10m}	$-L_{10m+2}$	$-L_{10m+2}$
$10m + 2$	$-L_{10m+3}$	L_{10m+1}	$2L_{10m+2}$	L_{10m+1}	$-L_{10m+3}$
$10m + 3$	$-L_{10m+4}$	$-L_{10m+4}$	L_{10m+2}	$2L_{10m+3}$	L_{10m+2}
$10m + 4$	L_{10m+3}	$-L_{10m+5}$	$-L_{10m+5}$	L_{10m+3}	$2L_{10m+4}$
$10m + 5$	$2L_{10m+5}$	L_{10m+4}	$-L_{10m+6}$	$-L_{10m+6}$	L_{10m+4}
$10m + 6$	L_{10m+5}	$2L_{10m+6}$	L_{10m+5}	$-L_{10m+7}$	$-L_{10m+7}$
$10m + 7$	$-L_{10m+8}$	L_{10m+6}	$2L_{10m+7}$	L_{10m+6}	$-L_{10m+8}$
$10m + 8$	$-L_{10m+9}$	$-L_{10m+9}$	L_{10m+7}	$2L_{10m+8}$	L_{10m+7}
$10m + 9$	L_{10m+8}	$-L_{10m+10}$	$-L_{10m+10}$	L_{10m+8}	$2L_{10m+9}$

Again, differences of the sums are Fibonacci numbers. If one examines cases for larger values of k and uses the fact that, for

$$q = 5, \qquad 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0,$$

one sees that the sums will be expressible in the form

$$\frac{1}{5}\left[k^n \pm C\right] ,$$

where C is a Lucas number or twice a Lucas number, and the differences will be consecutive Fibonacci numbers, in the cases where $k \equiv 2,3 \pmod 5$. In other cases, the sums take on a constant value or take on two values which differ by 1.

5. THE CASE OF $k=2,\ q=6$

Here

$$S(n,2,6,r) = \frac{1}{6}\sum_{\ell=0}^{5}\left(2\cos\frac{\ell\pi}{6}\right)^n \cos\frac{\ell(n-2r)\pi}{6} = \frac{1}{6}\left[2^n + 2(\sqrt{3})^n\cos\frac{(n-2r)\pi}{6} + 2\cos\frac{2(n-2r)\pi}{6}\right]$$

$r=0,\ 1,\ \cdots,\ 5,$ and the sums take the form $\frac{1}{6}[2^n+D]$, where, for $r = 0$, for example, D can be found in Table 4.

Table 4

n	D
$12m$	$2.3^{6m}+2$ (this breakd down for $m = 0$)
$12m+1$	$3^{6m+1}+1$
$12m+2$	$3^{6m+1}-1$
$12m+3$	-2
$12m+4$	$-3^{6m+2}-1$
$12m+5$	$-3^{6m+3}+1$
$12m+6$	$-2.3^{6m+3}+2$
$12m+7$	$-3^{6m+4}+1$
$12m+8$	$3^{6m+4}-1$
$12m+9$	-2
$12m+10$	$3^{6m+5}-1$
$12m+11$	$3^{6m+6}+1$

The other sums, for $r = 1,2,3,4,5$ can be computed easily and, not surprisingly, the largest and smallest sums differ by a power of 3 or twice a power of 3.

6. THE CASE OF $k=2,\ q=8$

The Pell numbers P_n are defined by the following:

$$P_1 = 1, \qquad P_2 = 2, \qquad P_n = 2P_{n-1}+P_{n-2}, \qquad n > 2,$$

and we shall define the Pell-Lucas sequence Q_n as satisfying the same recursion relation but $Q_1 = 2,\ Q_2 = 6$. The roots of the auxiliary equation $x^2-2x-1=0$ are, in this case,

$$\gamma = 1+\sqrt{2} \qquad \text{and} \qquad \delta = 1-\sqrt{2}$$

and the Binet-type formulas in this case are, analogously,

$$P_n = \frac{\gamma^n-\delta^n}{\gamma-\delta} \qquad \text{and} \qquad Q_n = \gamma^n+\delta^n .$$

For $q = 8$, the sums $S(n,2,8,r)$ for $r = 0,\ 1,\ 2,\ \cdots,\ 7$ can be written

$$S(n,2,8,r) = \binom{n}{r}+\binom{n}{r+8}+\binom{n}{r+16}+\cdots = \frac{1}{8}\sum_{\ell=0}^{7}\left(2\cos\frac{\ell\pi}{8}\right)^n\cos\frac{\ell(n-2r)\pi}{8} = \frac{1}{8}\left[2^n+\left(2\cos\frac{\pi}{8}\right)^n\right.$$

$$\cdot\cos\frac{(n-2r)\pi}{8}+\left(2\cos\frac{2\pi}{8}\right)^n\cos\frac{2(n-2r)\pi}{8}+\left(2\cos\frac{3\pi}{8}\right)^n\cos\frac{3(n-2r)\pi}{8}$$

$$\left.+\left(2\cos\frac{5\pi}{8}\right)^n\cos\frac{5(n-2r)\pi}{8}+\left(2\cos\frac{6\pi}{8}\right)^n\cos\frac{6(n-2r)\pi}{8}+\left(2\cos\frac{7\pi}{8}\right)^n\cos\frac{7(n-2r)\pi}{8}\right]$$

$$= \frac{1}{8}\left[2^n+2\cdot 2^{n/4}\gamma^{n/2}\cos\frac{(n-2r)\pi}{8}+2\cdot 2^{n/2}\cos\frac{2(n-2r)\pi}{8}+2\cdot 2^{n/4}(-\delta)^{n/2}\,\frac{3(n-2r)\pi}{8}\right] .$$

Table 5

	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$16m$	$2^{8m+1} + 2^{4m+1}Q_{8m}$	$2^{4m+2}P_{8m}$	-2^{8m+1}	$-2^{4m+2}P_{8m}$
$16m+1$	$2^{8m+1} + 2^{4m+2}P_{8m+1}$	$-2^{8m+1} + 2^{4m+2}P_{8m+1}$	$-2^{8m+1} + 2^{4m+2}P_{8m}$	$-2^{8m+1} - 2^{4m+2}P_{8m}$
$16m+2$	$2^{4m+1}Q_{8m+1}$	$2^{8m+2} + 2^{4m+3}P_{8m+1}$	$2^{4m+1}Q_{8m+1}$	-2^{8m+2}
$16m+3$	$-2^{8m+2} + 2^{4m+1}Q_{8m+1}$	$2^{8m+2} + 2^{4m+1}Q_{8m+2}$	$2^{8m+2} + 2^{4m+1}Q_{8m+2}$	$-2^{8m+2} + 2^{4m+1}Q_{8m+1}$
$16m+4$	-2^{8m+3}	$2^{4m+3}P_{8m+2}$	$2^{8m+3} + 2^{4m+2}Q_{8m+2}$	$2^{4m+3}P_{8m+2}$
$16m+5$	$-2^{8m+3} - 2^{4m+3}P_{8m+2}$	$-2^{8m+3} + 2^{4m+3}P_{8m+2}$	$2^{8m+3} + 2^{4m+3}P_{8m+3}$	$2^{8m+3} + 2^{4m+3}P_{8m+3}$
$16m+6$	$-2^{4m+2}Q_{8m+3}$	-2^{8m+4}	$2^{4m+2}Q_{8m+3}$	$2^{8m+4} + 2^{4m+4}P_{8m+3}$
$16m+7$	$2^{8m+4} - 2^{4m+2}Q_{8m+4}$	$-2^{8m+4} - 2^{4m+2}Q_{8m+3}$	$-2^{8m+4} + 2^{4m+2}Q_{8m+3}$	$2^{8m+4} + 2^{4m+2}Q_{8m+4}$
$16m+8$	$2^{8m+5} - 2^{4m+3}Q_{8m+4}$	$-2^{4m+4}P_{8m+4}$	-2^{8m+5}	$2^{4m+4}P_{8m+4}$
$16m+9$	$2^{8m+5} - 2^{4m+4}P_{8m+5}$	$2^{8m+5} - 2^{4m+4}P_{8m+5}$	$-2^{8m+5} - 2^{4m+4}P_{8m+4}$	$-2^{8m+5} + 2^{4m+4}P_{8m+4}$
$16m+10$	$-2^{4m+3}Q_{8m+5}$	$2^{8m+6} - 2^{4m+5}P_{8m+5}$	$-2^{4m+3}Q_{8m+5}$	-2^{8m+6}
$16m+11$	$-2^{8m+6} - 2^{4m+3}Q_{8m+5}$	$2^{8m+6} - 2^{4m+3}Q_{8m+6}$	$2^{8m+6} - 2^{4m+3}Q_{8m+6}$	$-2^{8m+6} - 2^{4m+3}Q_{8m+5}$
$16m+12$	-2^{8m+7}	$-2^{4m+5}P_{8m+6}$	$2^{8m+7} - 2^{4m+4}Q_{8m+6}$	$-2^{4m+5}P_{8m+6}$
$16m+13$	$-2^{8m+7} + 2^{4m+5}P_{8m+6}$	$-2^{8m+7} - 2^{4m+5}P_{8m+6}$	$2^{8m+7} - 2^{4m+5}P_{8m+7}$	$2^{8m+7} - 2^{4m+5}P_{8m+7}$
$16m+14$	$2^{4m+4}Q_{8m+7}$	-2^{8m+8}	$-2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+6}P_{8m+7}$
$16m+15$	$2^{8m+8} + 2^{4m+4}Q_{8m+8}$	$-2^{8m+8} + 2^{4m+4}Q_{8m+7}$	$-2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+4}Q_{8m+8}$

Continued, next page.

Table 5 (Cont'd)

	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$16m$	$2^{8m+1} - 2^{4m+1}Q_{8m}$	$-2^{4m+2}P_{8m}$	-2^{8m+1}	$2^{4m+2}P_{8m}$
$16m+1$	$2^{8m+1} - 2^{4m+2}P_{8m+1}$	$2^{8m+1} - 2^{4m+2}P_{8m+1}$	$-2^{8m+1} - 2^{4m+2}P_{8m}$	$-2^{8m+1} + 2^{4m+2}P_{8m}$
$16m+2$	$-2^{4m+1}Q_{8m+1}$	$2^{8m+2} - 2^{4m+3}P_{8m+1}$	$-2^{4m+1}Q_{8m+1}$	-2^{8m+2}
$16m+3$	$-2^{8m+2} - 2^{4m+1}Q_{8m+1}$	$2^{8m+2} - 2^{4m+1}Q_{8m+2}$	$2^{8m+2} - 2^{4m+1}Q_{8m+2}$	$-2^{8m+2} - 2^{4m+1}Q_{8m+1}$
$16m+4$	-2^{8m+3}	$-2^{4m+3}P_{8m+2}$	$2^{8m+3} - 2^{4m+2}Q_{8m+2}$	$-2^{4m+3}P_{8m+2}$
$16m+5$	$-2^{8m+3} + 2^{4m+3}P_{8m+2}$	$-2^{8m+3} - 2^{4m+3}P_{8m+2}$	$2^{8m+3} - 2^{4m+3}P_{8m+3}$	$2^{8m+3} - 2^{4m+3}P_{8m+3}$
$16m+6$	$2^{4m+2}Q_{8m+3}$	-2^{8m+4}	$-2^{4m+2}Q_{8m+3}$	$2^{8m+4} - 2^{4m+4}P_{8m+3}$
$16m+7$	$2^{8m+4} + 2^{4m+2}Q_{8m+4}$	$-2^{8m+4} + 2^{4m+2}Q_{8m+3}$	$-2^{8m+4} - 2^{4m+2}Q_{8m+3}$	$2^{8m+4} - 2^{4m+2}Q_{8m+4}$
$16m+8$	$2^{8m+5} + 2^{4m+3}Q_{8m+4}$	$2^{4m+4}P_{8m+4}$	-2^{8m+5}	$-2^{4m+4}P_{8m+4}$
$16m+9$	$2^{8m+5} + 2^{4m+4}P_{8m+5}$	$2^{8m+5} + 2^{4m+4}P_{8m+5}$	$-2^{8m+5} + 2^{4m+4}P_{8m+4}$	$-2^{8m+5} - 2^{4m+4}P_{8m+4}$
$16m+10$	$2^{4m+3}Q_{8m+5}$	$2^{8m+6} + 2^{4m+5}P_{8m+5}$	$2^{4m+3}Q_{8m+5}$	-2^{8m+6}
$16m+11$	$-2^{8m+6} + 2^{4m+3}Q_{8m+5}$	$2^{8m+6} + 2^{4m+3}Q_{8m+6}$	$2^{8m+6} + 2^{4m+3}Q_{8m+6}$	$-2^{8m+6} + 2^{4m+3}Q_{8m+5}$
$16m+12$	-2^{8m+7}	$2^{4m+5}P_{8m+6}$	$2^{8m+7} + 2^{4m+4}Q_{8m+6}$	$2^{4m+5}P_{8m+6}$
$16m+13$	$-2^{8m+7} - 2^{4m+5}P_{8m+6}$	$-2^{8m+7} + 2^{4m+5}P_{8m+6}$	$2^{8m+7} + 2^{4m+5}P_{8m+7}$	$2^{8m+7} + 2^{4m+5}P_{8m+7}$
$16m+14$	$-2^{4m+4}Q_{8m+7}$	-2^{8m+8}	$2^{4m+4}Q_{8m+7}$	$2^{8m+8} + 2^{4m+6}P_{8m+7}$
$16m+15$	$2^{8m+8} - 2^{4m+4}Q_{8m+8}$	$-2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} + 2^{4m+4}Q_{8m+8}$

One can reduce these sums to the form $\frac{1}{8}[2^n + E]$, where E is found in Table 5. $S(n,6,8,r)$ is similar.

Differences between the largest and smallest sums are, in this case, powers of 2 times Pell or Pell-Lucas numbers.

Further cases yield more differences which satisfy increasingly complicated linear recursion relations or combinations of such relations. Some of these, along with other techniques for handling such problems will appear in a later paper. Some generalizations to multinomial coefficients appear in [1].

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PER NØRGÅRD'S "CANON"

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Two preliminary facts must be stated to establish the relevance of what follows. And that which follows concerns a landmark in contemporary musical composition and publishing.

Per Nørgård is one of Denmark's leading composers. The Wilhelm Hansen Musik Forlag is one of Europe's most prestigious music publishers. These two forces have produced a musical composition that might well mark the beginning of a new era of music writing.

The composition under consideration for the moment is Per Nørgård's CANON for organ. It is based entirely and to the minutest detail on the Fibonacci numbers. These proportions are carried out with such precision that the composer found it necessary to invent a new system of notation. Conventional notation could not express the fluid rhythms generated by the Golden Mean. It is mainly this aspect of the composition that is being discussed herewith.

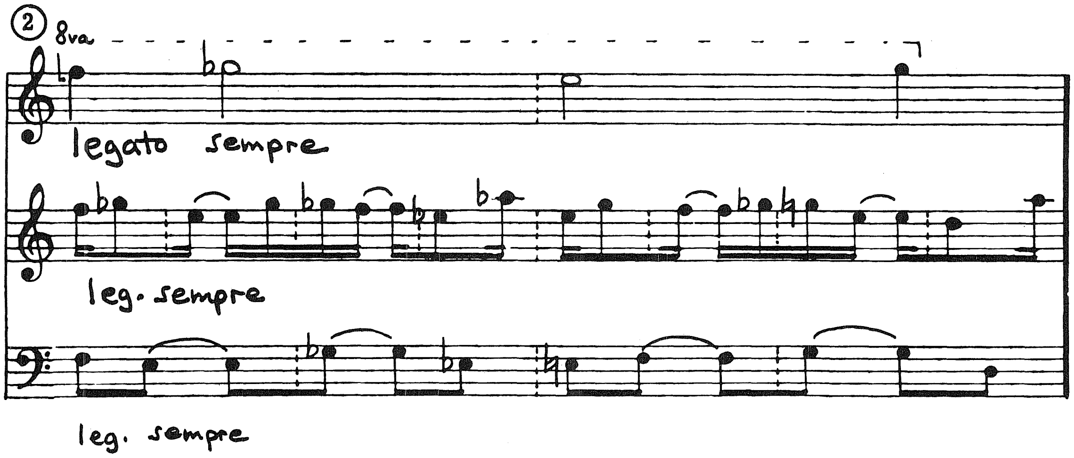
Actually CANON is not a canon in the usual sense of the term. Rather, it signifies music written according to "law." It is a series of truncated multiple augmentation canons in three-part texture. These fall into seven sections comprising 62 "stages," eight in each of the first six sections and fourteen in the last. Some of the "stages" are subdivided into smaller units in order to exploit further the proportions therein in varying time dimensions. The simple 1 : 1 and 1 : 2 time relationships occupy the two ends of the 25-minute composition with the higher ratios spiralling inward palindromically to 8 : 13 : 8 to form the peak of rhythmic complexity at the middle. The complete rhythmic scheme as it operates within the composition's seven principal sections has been tabulated by the composer as follows:

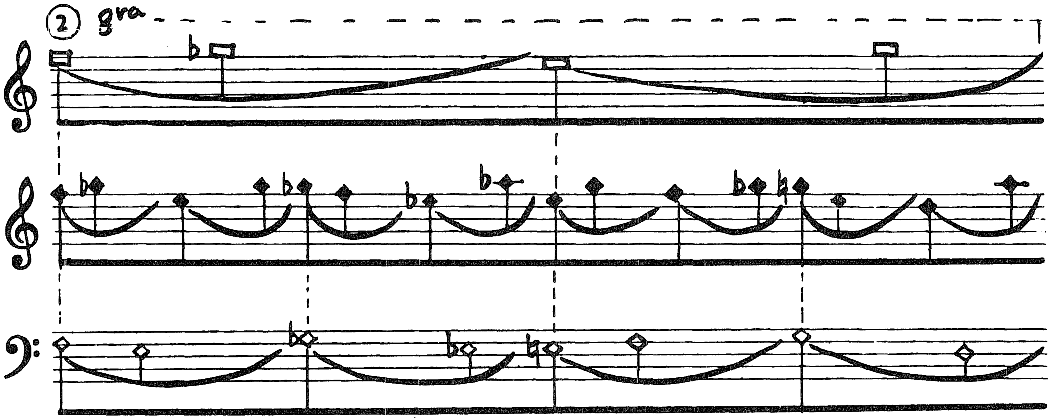
1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 in I,
1 : 2 / 2 : 1 / 1 : 2 / 2 : 1 in II,
2 : 3 : 5 : 3 / 3 : 5 : 3 : 2 in III,
3 : 5 : 8 : 5 : 8 : 13 : 8 : 5 in IV,
3 : 3 / 5 : 5 / 8 : 8 / 5 : 5 in V,
3 : 3 : 3 : 3 / 5 : 5 : 5 : 5 in VI, and
3 : 3 : 3 : 3 : 3 : 3 : 3 : 3
(ɔ : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1) in VII.

All of the above relationships indicate note values. For instance, 1 : 1 means notes of equal value, while 1 : 2 could mean a quarter-note followed by a half-note, and so on. As the augmentation ratios become higher, such as 5 : 8 and 8 : 13, it is at once obvious that the notation becomes cumbersome.

The notational problem encountered by Nørgård was to express accurately his augmentation proportions. In conventional notation absolute accuracy is not possible since this is fundamentally a duple system in which a whole-note is progressively divided into *two* half-notes, *four* quarter-notes, *eight* eighth-notes, *sixteen* sixteenth-notes, and so on. In terms of Fibonacci numbers it is possible. But, the composer required precision of 1 : 1.618 refinement. And this is absolutely impossible in conventional notation. So in order to express his intentions he invented a new kind of spatial notation wherein one cm. represents one second. In other words, the distance that comes *between* notes visually on the printed page is as important as the notes themselves. This presents a totally new and probably unwelcome problem for music engravers. In the case of the Wilhelm Hansen edition the new notation was not engraved, but merely a reproduction of the manuscript. The latter, however, is beautiful.

The following quotations show in both notations stage 2 of Section III, in which the 1 : 2, 2 : 1 proportions operate simultaneously in three dimensions.





The ingeniously contrived excerpt quoted above consists of an Augmentation Canon in *exact* Contrary Motion in the two lower parts, calculated in the theoretical key of E-flat major, while the uppermost part imitates the middle part at the octave above in quadruple augmentation and the bass part in double augmentation and in *exact* contrary motion. That is to say, the three strands of the contrapuntal fabric comprises three canons: bass + alto, bass + soprano, and alto + soprano. This is likewise true of the entire composition.

The complete proportion scheme of the three parts in relation to each other has been tabulated by the composer as shown on the following page.

Per Nørgård, in composing his CANON, and the Wilhelm Hansen Musik Forlag, in publishing it, have done something of far greater significance in the development of music than either of them may be presently aware. First, conventional notation has for a long time become increasingly inadequate to express accurately the musical thought of the contemporary composer. Experimentation is going on continuously in many quarters, but without much public notice. But, Nørgård's CANON is the first instance of a major music publisher investing heavily in a large serious work scored in a completely new notation that is likely to baffle the traditionally trained performer. It is to be hoped that still other publishers will become as venturesome. Would that the music industry in the United States would catch at least partially the pioneering spirit currently extant in Copenhagen!

Secondly, Nørgård has brought out in the open the fact that intelligent composers are deeply involved creatively with mathematically conceived structure. This is, of course, equally true of the whole gamut of music history. But, for some unexplainable reason, the academic world of music theory and music history has remained almost completely blind to it. The Fibonacci oriented underlay of a Palestrina mass, a Bach fugue or a Beethoven sonata-allegro is there for all to see. It is to be hoped that Nørgård's almost indecent exposure of his quite sophisticated working techniques will jolt the too often dreary "establishment" scholars and theorists into realizing that this sort of thing is new more in degree than in kind, and that it might prove rewarding to undertake a rather different type of approach.

```
I super   4     :     4     :     4     :     4
  medium  2  :  2  :  2  :  2  :  2  :  2  :  2  :  2
  sub     1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1

II        4   :  8                 :8        :        4
          2   :  4      :4    :    2  :2  :  4      :4    :    2
          1 : 2 : 2 : 1 : 1 : 2 : 2 : 1 : 1 : 2 : 2 : 1 : 1 : 2 : 2 : 1

III       8   :   12   :   20           :            12
          4     6    10         6     :6       10         6      4
          2 : 3 : 5 : 3 : 3 : 5 : 3 : 2 : 2 : 3 : 5 : 3 : 3 : 5 : 3 : 2

IV        16    :    26    :    42         :         26
          6 : 10 : 16  :  10:16  :  26     :    16 :: 10
          3 : 5 : 8 : 5 : 8 : 13 : 8 : 5 : 5 : 8 : 13 : 8 : 5 : 8 : 5 : 3

V     34          :         55         :         89          :         55
      13    :    21    :    34    :    21    :    34    :    55    :    34    :    21
       5 : 8 : 13 : 8 : 13 : 21 : 13 : 8 : 13 : 21 : 34 : 21 : 13 : 21 : 13 : 8

VI        34          :          34          :55          :          55
          13    :    13    :21    :    21    :34    :    34    :21    :    21
           5 : 5 : 8 : 8 : 13 : 13 : 8 : 8 : 13 : 13 : 21 : 21 : 13 : 13 : 8 : 8

          34          :          34          :      34      :      34
          13    :    13    :    13    :    13    :21    :    21    :    21    :    21
           5 : 5 : 5 : 5 : 8 : 8 : 8 : 8 : 13 : 13 : 13 : 13 : 8 : 8 : 8 : 8

VII       26          :          26          :      26      :      26
          13    :    13    :    13    :    13    :    13    :    13    :    13    :    13
           5 : 5 : 5 : 5 : 5 : 5 : 5 : 5 : 8 : 8 : 8 : 8 : 8 : 8 : 8 : 8
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BODE'S RULE AND FOLDED SEQUENCES

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THE THEORY

I have discovered a new and most interesting variation on recursions during my work on Bode's rule. Fibonacci-Lucas (F-L) sequences, by definition, satisfy $H_{k+1} = H_k + H_{k-1}$. A folded or crimped-in-upon-itself infinite sequence is N-cyclic and breaks the F-L rule only *once* per cycle, *i.e.,*

(1) $$G_{k+1} = G_k + G_{k-1} \quad \text{except that} \quad \{G_N\}_{jN} = \{G_N\}_0 \quad \text{for all } j,\ N.$$

As an example the $N = 3$ case is $\{G_3\} = \cdots 2, 0, 2, 2, 0, 2, 2, 0, \cdots$ in which $\{G_3\}_0 = \{G_3\}_3$. Application of (1) $N - 1$ times gives

(2) $$G_0F_{N-1} + G_1F_N = G_0\ ,$$

where $\{F\}$ is Fibonacci's sequence. This determines the sequences in Table 1. The partial sum of a F-L sequence is $(F_{n+2} - F_{m+1})$ for all $m,n,$ where F_m and F_n are the first and last terms. Using this the sum over one cycle of a folded F-L sequence gives $G_{N-1} + G_N - G_1$. Since (1) and (2) give $G_k = G_{jN+k}$ for all integers j,k we have

(3) $$\sum_{k=0}^{N-1} G_k = G_{-1} + G_0 - G_1\ .$$

An easy way to generate folded F-L sequences follows. From (2), $\{G_N\}_0$ must equal or be a multiple of F_N to avoid a fractional $\{G_N\}_1$. Let $\{G_N\}_0 = F_0 + F_N$ then (2) gives $\{G_N\}_1 = F_1 - F_{n-1}$. Thus every $\{G_N\}$ is simply the sum of a positive and negative Fibonacci sequence. We have

(4) $$\{G_N\}_k = F_k + (-1)^{N+1}F_{k-N}, \qquad 0 \leqslant k \leqslant N$$

and using the "skew symmetric" fact that

(5) $$F_{-k} = (-1)^{k+1}F_k$$

gives the simpler expression

(6) $$\{G_N\}_k = F_k + (-1)^kF_{N-k}, \qquad 0 \leqslant k \leqslant N.$$

One can also define negative folded F-L sequences which are finite and of length $N + 1$. Their definition is

(7) $$\{G_{-N}\}_k = F_k + (-1)^NF_{k-N}, \qquad 0 \leqslant k \leqslant N.$$

An example is $\{G_{-5}\} = -5, 4, -1, 3, 2, 5$. Substitution of (4) or (6) into (3) permits an explicit sum formula:

(8) $$\sum_{k=0}^{N-1} \{G_N\}_k = L_N + (-1)^{N+1} - 1.$$

When mod $(N,4) = 0$ then

$$-G_1/G_0 = (F_{N-1} - 1)/F_N = L_{m-1}/L_m\ ,$$

where $N = 2m$. The proof consists of crossmultiplying and inserting identity (7) of Hoggatt [1] which is true for all integers m. This reduces to a special case of identity (21) in the list [1]. Thus mod $(N,4) = 0$ gives folded Lucas sequences. Similarly when mod $(N,4) = 2$ then $-G_1/G_0 = F_{m-1}/F_m$ giving folded Fibonacci sequences. The proof is identical and ends with a special case of identity (23) in his list [1]. But the interesting cases are mod $(N,4) = 1$ or 3.

Table 1

Folded Fibonacci-Lucas Sequences

N	ONE CYCLE OF SEQUENCE													SUM
2						1	0							1
4					3	−1	2	1						5
6				8	−4	4	0	4	4					16
8			21	−12	9	−3	6	3	9	12				45
10		55	−33	22	−11	11	0	11	11	22	33			121
3						2	0	2						4
5					5	−2	3	1	4					11
7				13	−7	6	−1	5	4	9				29
9			34	−20	14	−6	8	2	10	12	22			76
11		89	−54	35	−19	16	−3	13	10	23	33	56		199
13	233	−143	90	−53	37	−16	21	5	26	31	57	88	145	521
15		···	−142	.92	−50	42	−8	34	26	···				1364
33					···	−1974;2584;610;3194;3804;6998;10802···								
∞					···	$-\sqrt{5}-1;\sqrt{5}+2;\ 1;\sqrt{5}+3;\sqrt{5}+4$ ···								

The reciprocal periods of planets and satellites are given by alternate members of an odd N-folded sequence. Their properties are studied best by placing the origin in the middle hence I define a half-integer subscript i given by $2k = 2i + N$.

Theorem: $\{G_N\}_{i+1}/\{G_N\}_i$ approaches a limiting value for all i as $N \to \infty$ for mod $(N,4) = 1$ and another for mod $(N,4) = 3$.

Proof: It is sufficient to prove this for one value of i whence it is true for all i by (1) aside from a constant factor which is of no interest. Write $h = \tfrac{1}{2}$ for typographical ease. I also define even integer $m = (N \pm 1)/2$ when mod $(N,4) = 3$ or 1, respectively. Then by (4) the middle pair for mod $(N,4) = 1$ is

$$\{G_N\}_h/\{G_N\}_{-h} = (F_{m+1}+F_{-m})/(F_m+F_{-m-1}) = F_{m-1}/F_{m+2} \to 1/\alpha^3 \quad \text{as} \quad N \to \infty$$

via Binet's expression since $\beta^N \to 0$ as $N \to \infty$, where $\alpha, \beta = (1 \pm \sqrt{5})/2$, respectively. For mod $(N;4) = 3$ we have:

$$\{G_N\}_h/\{G_N\}_{-h} = -F_{m+1}/F_{m-2} \to -\alpha^3 \quad \text{as} \quad N \to \infty$$

for the same reason. These initial ratios $\pm\alpha^{\mp 3}$ define $\{S\}$ and $\{S^*\}$ and apply to any star (planet) with an infinite number of planets (satellites). When mod $(N,4) = 1$ let $S_{-h} = 2+\sqrt{5}$ and $S_h = 1$ as in Table 1. Then (1) gives us

(9a) $$S_i = F_{i+h}S_h + F_{i-h}S_{-h}$$

for all positive or negative half-integers i. Similarly when mod $(N,4) = 3$ let $S^*_{-h} = -1$ and $S^*_h = 2+\sqrt{5}$ then

(9b) $$S^*_i = F_{i+h}S^*_h + F_{i-h}S^*_{-h}\ .$$

Substitution of (5) into (9a,b) proves the equivalence of S and S^* but for signs *i.e.,*

(10) $$S^*_i = (-1)^{i-h}S_{-i} \quad \text{or} \quad S^*_{-i} = (-1)^{i+h}S_i\ .$$

Use of $F_{i+h} + 2F_{i-h} = L_{i+h}$ in (9a,b) gives the elegant relations

(11) $$S_i = L_{i+h} + \sqrt{5}F_{i-h} \quad \text{and} \quad S^*_i = L_{i-h} + \sqrt{5}F_{i+h}$$

(11a) $$S_i = s_0(\sqrt{5}F_{i+h} - L_{i-h}) \quad \text{and} \quad S^*_i = s_0(L_{i+h} - \sqrt{5}F_{i-h})$$

which via Binet's theorem become

$$S_i = (a^i + (-1)^{i+h} a^{-i})\sqrt{s_0} \tag{12a}$$

$$S_i^* = (a^i + (-1)^{i-h} a^{-i})\sqrt{s_0} \tag{12b}$$

which immediately give (10) again and where $\sqrt{s_0} = (a^h + a^{-h}) = 2.058171 = 1/0.485868$.

The Lucas complement of any two-point sequence is defined by the two apart sum operator $\Sigma^\dagger$, namely

$$\Sigma^\dagger W_n = (W_{n+1} + W_{n-1})/d\ , \tag{13}$$

where d is the difference in the roots of the recursion's characteristic equation [2] and $d = \sqrt{5}$ for F-L sequences. It is known that $\Sigma^\dagger\Sigma^\dagger \equiv I$, the identity operator. We come now to the strongest property of $\{ S \}$ and $\{ S^* \}$. Aside from signs they are their *own* complements! The fact that this property is not true of the Fibonacci and Lucas sequences themselves indicates the greater importance of $\{ S \}$ and its approximation $\{ G_N \}$. After all $\{ G_N \}$ is a generalization of $\{ F \}$ and $\{ L \}$. Applying $\Sigma^\dagger$ to the elegant (11) immediately gives

$$\Sigma^\dagger S_i = S_i^* = (-1)^{i-h} S_{-i} \tag{14}$$

since

$$\sqrt{5}\,\Sigma^\dagger F_n = L_n \qquad \text{and} \qquad \Sigma^\dagger L_n = \sqrt{5}\,F_n .$$

Alternatively given (14) we can ask what the ratio S_h/S_{-h} in (9) must be. One obtains

$$S_{-h}^2 - 4S_{-h}S_h - S_h^2 = 0.$$

2. THE OBSERVATIONS

Several facts of satellites (planets) need to be explained. They can be remembered using the vowel mnemonic, *aei$\underline{\omega}\omega\epsilon$A.* They are: (i) rule(s) for the major semi-axes of the orbits, (ii) their near zero eccentricities, (iii) sin $i \approx 0$, *i.e.,* their orbital inclinations are nearly 0 or 180° for outer satellites, (iv) their spins are almost all counterclockwise (ccw) with a preference for $23° < \underline{\omega} < 29°$ where the sun and Jupiter are prominent exceptions, (v) their spins satisfy the narrow range $6 < P < 25$ hr unless tidally disturbed, (vi) the sun's obliquity $\epsilon = 7°$ hence the sun's equator does not lie in the invariable plane, (vii) the sun's Angular momentum is very small (it rotates in ≈ 30 day). I add (viii) that each satellite system has one or two satellites much more massive than the others. The massive satellites are called secondaries and all others are tertiaries. Thus Saturn's and Jupiter's secondaries are Titan+Hyperion and Galilei's quadruplet, respectively. The non-zero tilt of most of their axes suggests that the torque that each exerts on the other causing precession may be important. The ideal tilt is then 45°.

I envisage that the sun's family began with the sun and Jupiter (+ ?) Saturn from a contracting cloud and that all planetary and satellite systems start as binary systems, *i.e.,* a primary + secondary(ies). All other bodies, tertiaries, were subsequently formed by accretion. The sun's nebula would have dispersed early due to radiation pressure and infalling due to the Poynting-Robertson effect. Many planets and satellites should have formed from the nebula left around Jupiter. Binaries enable the capture of tertiary bodies. A single primary cannot capture a tertiary body whose orbit must *ab initio* be an ellipse or hyperbola. Outer satellites, those beyond the secondaries, act as if the secondaries were part of the primary. When the maximum elongation angle of the secondaries is very small they act as a point source. The number of major planets makes $N = 33$. Although N may be slightly different for the satellite systems $\{ G_N \} \to \{ S \}$ rapidly and for $N \geqslant 13$ the discrepancies are < one percent as Table 1 shows.

In Table 2, major bodies are capitalized. Also pons, faye, neujmin and hungaria refer to groups of comets at $61 < P < 77$ yr, $6.3 < P < 7.9$ yr, $P = 18$ yr, and a group of small planetoids $2.5 < P < 3.0$ yr named after the first discovered [3] 434 Hungaria ($P = 991$ da). There is a void in the planetoid distribution [4, p. 169] separating these from the normal asteroids indicated by a typical member Astrea. See also [10, 11]. Note that satellites of Saturn and Jupiter are included. The accuracy is very high. Discrepancies are never more than 2.5 percent except Jupiter (6%) and Galilei's quadruplet (10%) both of which are secondaries for which the rule is not intended. The observed Iapetus/Phoebe ratio (outer satellites of Saturn) is 6.938. The predicted ratio is very nearly $(76 + 21\sqrt{5})/(11 + 3\sqrt{5})$

= 75004/10802 ≈ 6.9435. This is amazing agreement, an error of 0.0008 parts! The observed and predicted Saturn/Uranus ratios are 2.852 and 10802/3804 = 2.840, an error of 0.004. The Direct/Retrograde satellites of Jupiter give 2.835. Again excellent agreement. The agreement for Venus/Earth is similar. The data in [5] give JXII/Retrograde = 1.18 compared with a predicted 1.20. Planet X was predicted [6] from perturbations of Halley's comet. I find its period to be 521 to 524 yr. Bailey [7] proposed that the moon was once between Venus and Mercury.

Table 2

Reciprocal Periods

Sun	pons	Nep		X		Uran		Satur	neujm	Jup	faye
3524578, ···	4558,	−1974,	2584,	610,	3194,	3804,	6998,	10802,	17800,	28602,	46402
	JXII			*		Retr		Direct Phoebe			

Astrea	Hungar	Mars	earth	Venus	luna?	Mercury		Sun	
075004,	121406,	196410,	317816,	514226,	832042,	1346268,	2178310,	3524578,	(5702888)
lapetus			T+H		Galilei	Rhea	dione	Tethys	Almathea

The predicted effective solar rotation period is 32.8 day. If all planetary (and satellite) systems have about the same number of bodies and if these are tied to the primary's rotation then stars rotating much faster than the sun will have their planets too close to permit life. This would be true of white stars (earlier than type F5) whose rotation period is about 0.01 of the sun's. The theory predicts Mercury's period to be 86 ± 0.2 day. Hence some mechanism decreased its orbital energy and increased its orbital angular momentum. Furthermore the planetary rotations seem to be quantized near 1.14, 0.70, 0.43, 0.27 (asteroids), 0.17, 0.10 (solar grazing body) day. Although Folded sequences have made excellent predictions far more accurate than any previous work the sequences for Jupiter's and Saturn's satellites can be modified to include the sun's motion around the planet. For Saturn's satellites the sequence: 0.093, 0.636, 1.815, 4.809, 12.612 ··· is equally good. Similarly a Jovian sequence: 0.23, 1.35, 3.82, 10.11 ··· is a good predictor. The units are $(\text{kiloday})^{-1}$. The first term of each sequence is the motion of planet and sun around each other and is already determined by the sequence for the planets [8, 11].

Alternate F-L members approach the limit $\xi = (3+\sqrt{5})/2$. The limiting distance ratio, d, is given by Kepler's III law: $a^3 = p^2$. Hence $d = 1.899547627$. Planets and satellites were accreted from grain orbits of maximum eccentricity e thus $(1-e)/(1+e) = 1/d$. This function occurs often in science and deserves its own name. I define

$$oin(x,p) = (1-x)/(1+px)$$

because it is its own inverse, *i.e.,* if $y = oin(x,p)$ then $x = oin(y,p)$, where p is a parameter. This gives $e = 0.3102$.

3. INNER SATELLITES

Once again the god of time Chronos or Saturn holds the secret. Table 3 gives the reciprocal periods, Ω,

Table 3

Inner Saturnian Satellites

Rhea		Dione		Tethys		M+2E		Janus		Rotn?			Cassin	Ω_g?
221.4		365.4		529.7		838.0		1310.6		2091.5		3345.0		5379.4
	144.0		164.3		308.3		472.6		780.9		1253.5		2034.4	
123.7		20.3		144.0		164.3		308.3		472.6		780.9		1253.5

in $(\text{kiloday})^{-1}$. The errors for Rhea, Dione and Tethys are each 0.01 percent! The first differences are the synodic frequencies and they are a F-L sequence! Cassini's division falls on one of these values. Slightly different sequences occur in [9]. I must define two new operators. They are a forward knight operator $K \equiv \Delta + \Delta^2$ and a backward knight operator $N \equiv \nabla - \nabla^2$ by analogy with the chess piece. More generally:

$$K^p \equiv \Delta^p \sum_{x=0}^{p} \binom{p}{x} \Delta^x \tag{15}$$

For any F-L sequence $K^p F_n = F_n$ for all n and all integers $p \geqslant 0$. Carried leftwards Table 3 predicts the period of a grazing satellite to be 0.155 day. Table 3 is a shifted F-L sequence. It satisfies

$$K\Omega_n + \Omega^{\circ} = \Omega_n , \tag{16}$$

where Ω° = 0.0571 inverse days. But Ω° is very nearly the mean reciprocal period of massive Titan and Hyperion! For the terrestrial planets the situation is almost as good. Errors are negligible but for Mars. Here we have (16) with

Table 4

Terrestrial Planets in (Kiloday)$^{-1}$

Hungar		(Mars)		Earth		Venus		luna?		Mercury		(Vulcan)		
1.026		1.851		2.738		4.450		7.049		11.360		18.270		29.49
	0.825		0.887		1.712		2.599		4.311		6.910		11.221	
		0.062		0.825		0.887		1.712		2.599		4.311		

Ω° = 0.000 139 invday. This compares well with the synodic frequency between Jupiter and Saturn, 0.000 137 9. Hence the frequencies of inner planets are increased by the frequency of Jupiter and Saturn conjunctions. Carried leftwards Table 4 suggests a solar rotation of 25 day. The Martian error can [9] be removed by writing Ω° = –.000 256 at a small expense to Mercury. The inner Uranian triplet satisfies

$$\delta^2\Omega = \nabla\Omega \qquad \text{and} \qquad \Delta\Omega_n + \Omega^{\circ} = \Omega_n ,$$

where Ω° = 0.0858 invday and δ is the central difference operator.

COLOPHON

Johannes Kepler's Zeroth Law appeared in the year F_{17}. It was the first cosmological attempt and states that planets orbit in spheres which in- and circum-scribe the F_5 perfect solids arranged in the order 2, 8, –8, 0, –2 of faces minus vertices—all but one, members of $\{F\}$. It is to his faith in pure mathematics that I am indebted. F_{14} years later I found that the universal answer is $(\delta^2 - I)\Omega \to 0$. Another genius, J. C. Maxwell, also began his life [11, p. 93] studying the perfect solids and provided us with an elegant derivation of kinetic theory to which I am also indebted. Pussy willow leaves [1] and Houseleek petals display the 5/13 arrangement. And in haiku let us say:

Nature numbers hides
In shells, petals, moons to find
Is to hear with her

and in tanka style:

Each conjunction that I see
So real that
'junctions too must fit the rule
For are not 'junctions real
And earth's motion their conjunctions.

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DIOPHANTINE REPRESENTATION OF THE LUCAS NUMBERS

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The Lucas numbers, 1, 3, 4, 7, 11, 18, 29, ⋯, are defined recursively by the equations

$$L_1 = 1, \qquad L_2 = 3 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n .$$

We shall show that the Lucas numbers may be defined by a particularly simple Diophantine equation and thus exhibit them as the positive numbers in the range of a very simple polynomial of the 9th degree.

Our results are based upon the following identity

$$L_{n+1}^2 - L_{n+1}L_n - L_n^2 = 5(-1)^{n+1} . \tag{1}$$

This identity (cf. [1] p. 2 No. 6) actually *defines* the Lucas numbers in the following sense.

Theorem 1. For any positive integer y, in order that y be a Lucas number, it is necessary and sufficient that there exist a positive number x such that

$$y^2 - yx - x^2 = \pm 5 . \tag{2}$$

Proof. The Proof is virtually identical to that of the analogous result for Fibonacci numbers proved in [2].

Theorem 2. The set of all Lucas numbers is identical with the position values of the polynomial

$$y(1 - ((y^2 - yx - x^2)^2 - 25)^2) \tag{3}$$

as the variables x and y range over the positive integers.

Proof. We have only to observe that the right factor of (3) cannot be positive unless equation (2) holds. Here we are using an idea of Putnam [3].

It will be seen that the polynomial (3) also gives certain negative values. This is unavoidable. It is easy to prove that a polynomial which takes *only* Lucas number values must be constant (cf. [2] Theorem 3).

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PASCAL, CATALAN, AND GENERAL SEQUENCE CONVOLUTION ARRAYS IN A MATRIX

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The Catalan numbers $\{1, 1, 2, 5, 14, 42, \cdots\}$ are the first sequence in a sequence of sequences S_i which arise in the first column of matrix inverses of matrices containing certain columns of Pascal's triangle, and which also can be obtained from certain diagonals of Pascal's triangle [1], [2]. These sequences S_i are also the solutions for certain ballot problems, which counting process also yields their convolution arrays. The convolution triangles for the sequences S_i contain determinants with special values and occur in matrix products yielding Pascal's triangle. Surprisingly enough, we can also find determinant properties which hold for any convolution array.

1. ON THE CATALAN NUMBERS AND BALLOT PROBLEMS

When the central elements of the even rows of Pascal's triangle are divided sequentially by 1, 2, 3, 4, ⋯, to obtain 1/1 = 1, 2/2 = 1, 6/3 = 2, 20/4 = 5, 70/5 = 14, 252/6 = 42, ⋯, the Catalan sequence $\{C_n\}$ results,

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \cdots. \tag{1.1}$$

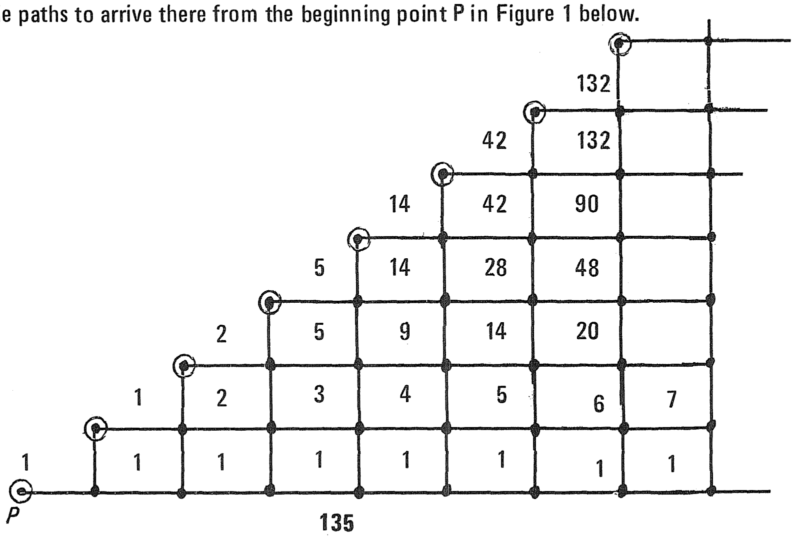
The Catalan sequence has the generating function [4]

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \tag{1.2}$$

and appears in several ways in Pascal's triangle.

The Catalan numbers also arise as the solution to a counting problem, being the number of paths possible to travel from a point to points lying along a rising diagonal, where one is allowed to travel from point to point within the array by making one move to the right horizontally or one move vertically. Each point in the array is marked with the number of possible paths to arrive there from the beginning point P in Figure 1 below.

Figure 1



Given two sequences $\{a_n\}$ and $\{b_n\}$, there is a sequence $\{c_n\}$ called the *convolution* of the two sequences,

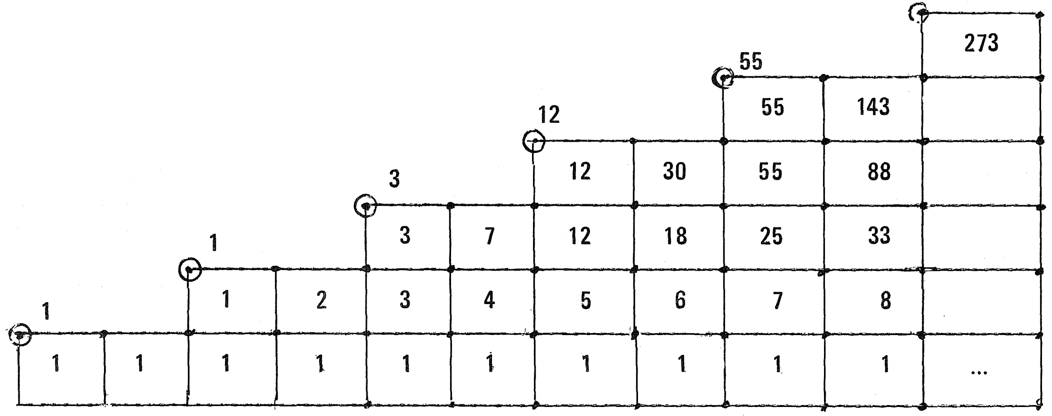
$$c_n = \sum_{k=0}^{n} b_{n-k}a_k \;.$$

If the sequences have generating functions $A(x)$, $B(x)$, and $C(x)$, respectively, then $C(x) = A(x)B(x)$. The successive convolutions of the Catalan sequence with itself appear as successive columns in the *convolution triangle*

1							
1	1						
2	2	1					
5	5	3	1				
14	14	9	4	1			
42	42	28	14	5	1		
132	132	90	48	20	6	1	
...	...	...	...	...	...	...	...

Notice that these same sequences appear on successive diagonals in Figure 1.

Call the Catalan sequence S_1, the first of a sequence of sequences S_i which arise as the solutions to similar counting problems where one changes the array of points. The counting problem related to S_2 we illustrate in Figure 2 below. The circled vertices yield $S2 = \{1,\ 1,\ 3,\ 12,\ 55,\ 273,\ \cdots\}$; under this is the first convolution $\{1, 2, 7, 30, 143, \cdots\}$, which can be computed from the definition of convolution. Successive diagonals continue to give successive convolutions of S_2.



										273
								55	143	
						12	30	55	88	
				3	7	12	18	25	33	
		1	2	3	4	5	6	7	8	
1	1	1	1	1	1	1	1	1	1	...

Figure 2

Similarly for $S_3 = \{1, 1, 4, 22, 140, \cdots\}$, the circled vertices are the sequence S_3, and under this appears the first convolution, and so on, as shown in Figure 3.

The sequences S_i and their convolution triangles are the solution to such counting problems, where one counts the number of paths possible to arrive at each point in the array from a beginning point from which one is allowed to travel from point to point within the array by making one move to the right horizontally or one move vertically. For the sequence S_i, the points in the grid are arranged so that the successive circled points are i to the right and one above their predecessors. By the rule of formation as compared to the rule of formation of the convolution array for S_i as found in [1], one sees that we have the same sequences S_i in both cases. Here, we go on to relate these convolution arrays to Pascal's triangle as matrix products.

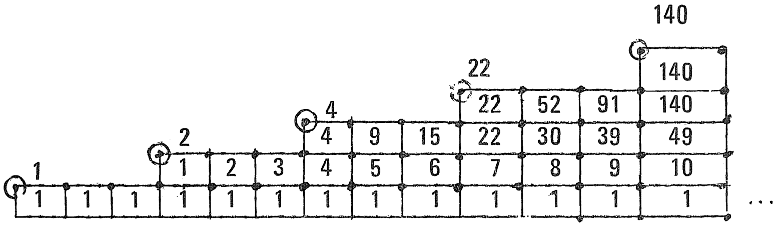


Figure 3

2. THE CATALAN CONVOLUTION TRIANGLE IN A MATRIX

Write a matrix A which contains the rows of Pascal's triangle from Fig. 1 written on and below the main diagonal with alternating signs. Write a matrix B containing the Catalan convolution triangle on and below its main diagonal, augmented by the first column of the identity matrix on the left. Then B is the matrix inverse of A, so that $AB = I$, the identity matrix, where, of course, all matrices have the same order. That is, for order 7,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & 6 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 \end{bmatrix} = I .$$

As proof, the columns of A are generated by $[x(1-x)]^{j-1}$ while those of B are generated by $[(1-\sqrt{1-4x})/2]^{j-1}$. The columns of AB, then, are the composition

$$[(1-\sqrt{1-4(1-x)x})/2]^{j-1} = x^{j-1} ,$$

the column generators for the identity matrix. Notice that the row sums of the absolute values of the elements of A are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, ⋯, while the row sums of B are the Catalan numbers.

Now, if the Catalan convolution triangle is written as a square array and used to form a matrix C, and if Pascal's triangle is written as a square array to form matrix P, then P is the matrix product AC. First, we illustrate for 5 × 5 matrices A, C, and P:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 9 & 14 & 20 \\ 5 & 14 & 28 & 48 & 75 \\ 14 & 42 & 90 & 165 & 275 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} .$$

Again considering the column generators and finding their composition, we prove that $AC = P$. The column generators of C are $[(1-\sqrt{1-4x})/2x]^{j-1}$, making the column generators of the matrix product AC to be

$$[(1-\sqrt{1-4x(1-x)})/2x(1-x)]^{j-1} = [1/(1-x)]^{j-1} ,$$

the generating functions of the columns of Pascal's triangle written in the form of P.

Fortunately, the finite $n \times n$ lower left matrices A have determinants whose values are determined by an $n \times n$ determinant within the infinite one. For infinite matrices A, B, and C, if we know that $AB = C$ by generating functions, then it must follow that $AB = C$ for $n \times n$ matrices, A, B, and C, because each $n \times n$ matrix is the same as the $n \times n$ block in the upper left in the respective infinite matrix. That is, adding rows and columns to the $n \times n$ matrices A and B does not alter the minor determinants we had, and similarly, the $n \times n$ matrix C agrees with the infinite matrix C in its $n \times n$ upper left corner. We write the Lemma,

Lemma. Let A be an infinite matrix such that all of its non-zero elements appear on and below its main diagonal, and let $A_{n\times n}$ be the $n \times n$ matrix formed from the upper left corner of A. Let B and C be infinite matrices with $B_{n\times n}$ and $C_{n\times n}$ the $n \times n$ matrices formed from their respective upper left corners. If $AB = C$, then $A_{n\times n}B_{n\times n} = C_{n\times n}$.

We will frequently consider $n \times n$ submatrices of infinite matrices in this paper, but we will not describe the details above in each instance. We can apply earlier results [2], [3] to state the following theorems for the Catalan convolution array, since each submatrix of C in Theorem 2.1 and 2.2 is multiplied by a submatrix of A which has a unit determinant to form the similarly placed submatrix within Pascal's triangle written in rectangular form.

Theorem 2.1. The determinant of any $n \times n$ array taken with its first row along the row of ones in the Catalan convolution array written in rectangular form is one.

Theorem 2.2. The determinant of any $k \times k$ array taken from the Catalan convolution array written in rectangular form with its first row along the second row of the Catalan convolution array and its first column the j^{th} column of the array has its value given by the binomial coefficient

$$\binom{k+j-1}{k}.$$

On the other hand, taking alternate columns of Pascal's triangle with alternating signs to form matrix Q and alternate columns of the Catalan convolution triangle to form matrix R as indicated below produces a pair of matrix inverses, where the row sums of absolute values of the elements of Q are the alternate Fibonacci numbers 1, 2, 5, 13, 34, ⋯, F_{2k+1}, ⋯, while the row sums of R are

$$1, 2, 6, 20, 70, \cdots, \binom{2n}{n}, \cdots,$$

the central column of Pascal's triangle. For 6 × 6 matrices Q and R,

$$QR = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ -1 & 6 & -5 & 1 & 0 & 0 \\ 1 & -10 & 15 & -7 & 1 & 0 \\ -1 & 15 & -35 & 28 & -9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 \end{bmatrix} = I.$$

Here, the j^{th} column of Q is generated by

$$Q(x) = \frac{1}{1+x} \cdot \left(\frac{x}{(1+x)^2}\right)^{j-1}$$

while the j^{th} column of R is generated by

$$R(x) = \frac{1-\sqrt{1-4x}}{2x} \cdot \left(\frac{(1-\sqrt{1-4x})^2}{4x}\right)^{j-1}$$

so that the j^{th} column of QR is generated by

$$\left(\frac{1-\left(1-\frac{4x}{(1+x)^2}\right)^{2}}{\frac{4x}{(1+x)^2}}\right)^{j-1} = \left(\frac{\left(1-\frac{1-x}{1+x}\right)}{\frac{4x}{(1+x)^2}}\right)^{j-1} = \left(\frac{\frac{4x^2}{(1+x)^2}}{\frac{4x}{(1+x)^2}}\right)^{j-1} = x^{j-1}$$

the generating function for the identity matrix.

3. MATRICES FORMED FROM CONVOLUTION TRIANGLES OF THE SEQUENCES S_i

Now we generalize, applying similar thinking to the sequences S_i. We use the notation of [2], letting $P_{i,j}$ be the infinite matrix formed by placing every j^{th} column (beginning with the zeroth column which contains the sequence S_i^1) of the convolution triangle for the sequence S_i on and below its main diagonal, and zeroes elsewhere. Then $P_{0,j}$ contains every j^{th} column of the convolution array for $S_0 = \{1, 1, 1, \cdots\}$, which is Pascal's triangle. Let $P'_{i,j}$ denote the matrix formed as $P_{i,j}$ from every j^{th} column of the convolution array for S_i but beginning with the column which contains S_i°, so that the n^{th} column of the matrix contains the sequence $S_i^{(n-1)j}$. Let $P^*_{i,j}$ be formed as $P_{i,j}$ except that the columns are written in a rectangular display, so that the first row is a row of ones. Then, referring to Section 2,

$$A = P'^{-1}_{1,1}\,, \qquad B = P'_{1,1}\,, \qquad C = P^*_{1,1}\,, \qquad P = P^*_{0,1}\,, \qquad R = P_{1,2}\,,$$

so that $AC = P$ becomes

$$P'^{-1}_{1,1}P^*_{1,1} = P^*_{0,1}\,. \tag{3.1}$$

One also finds that

$$P^{-1}_{1,1}P^*_{1,1} = P'^*_{0,1}\,. \tag{3.2}$$

We extend these results to S_2, where we will illustrate first $P^{-1}_{2,2}P_{2,2} = I$ for 5 × 5 submatrices [see 2]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 1 & 0 \\ 0 & -1 & 10 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 12 & 12 & 5 & 1 & 0 \\ 55 & 55 & 25 & 7 & 1 \end{bmatrix} = I$$

Here, the row sums of $P_{2,2}$ are 1, 2, 7, 30, 143, ···, which we recognize as S_2^2, the first convolution of S_2. Notice that $P^{-1}_{2,2}$ contains the odd rows of Pascal's triangle as its columns.

If we form $P'^{-1}_{2,2}$ using the even rows of Pascal's triangle taken with alternate signs on and below the main diagonal, then

$$P'^{-1}_{2,2}P^*_{2,1} = P^*_{0,1}\,, \tag{3.3}$$

which we illustrate for 5 × 5 matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 6 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 7 & 12 & 18 & 25 \\ 12 & 30 & 55 & 88 & 130 \\ 55 & 143 & 273 & 455 & 700 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}.$$

Of course, this means that the results of Theorems 2.1 and 2.2 also apply for the sequence S_2.

Now, if the matrix $P^*_{2,2}$ is formed from every other column of the S_2 convolution array written in rectangular form, the matrix product $P'^{-1}_{2,2}P^*_{2,2}$ becomes the matrix containing every other column of Pascal's triangle written in rectangular form. For example, for the 5 × 5 case,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 6 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 \\ 3 & 12 & 25 & 42 & 63 \\ 12 & 55 & 130 & 245 & 408 \\ 55 & 273 & 700 & 1428 & 2565 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 \\ 1 & 6 & 15 & 28 & 45 \\ 1 & 10 & 35 & 84 & 165 \\ 1 & 15 & 70 & 210 & 495 \end{bmatrix}.$$

This also means that, using earlier results [3], if we take the determinant of any square submatrix of $P^*_{2,2}$ with its first row taken along the first row of $P^*_{2,2}$ the determinant value will be $2^{[k(k-1)/2]}$ if the submatrix taken has order k.

If we shift the columns of $P^*_{2,1}$ one to the left so that the new matrix begins with S_2^2 in its first column, we find that, for 5 × 5 submatrices,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 1 & 0 \\ 0 & -1 & 10 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 7 & 12 & 18 & 25 & 33 \\ 30 & 55 & 88 & 130 & 182 \\ 143 & 273 & 455 & 700 & 1020 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

We shall show that this is also true for the infinite matrices indicated, which means, in light of our previous results, that the results of Theorems 2.1 and 2.2 also apply to the rectangular convolution array for S_2 if we truncate its zero$^{\text{th}}$ column.

Using every other column, we can make some interesting shifts. We already observed that $P'^{-1}_{2,2}P^*_{2,2} = P^*_{0,2}$. We also can write $P^{-1}_{2,2}P^*_{2,2}$, which provides every other column of Pascal's triangle, beginning with the column of integers. We also can write two matrix products relating the matrix containing the odd columns of the convolution matrix for S_2 to matrices containing every other column of Pascal's triangle, each of which is illustrated below for 4×4 submatrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 7 & 18 & 33 & 52 \\ 30 & 88 & 182 & 320 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 6 & 15 & 28 \\ 1 & 10 & 35 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 7 & 18 & 33 & 52 \\ 30 & 88 & 182 & 320 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 10 & 21 & 36 \\ 4 & 20 & 56 & 120 \end{bmatrix}.$$

Since we can establish that the corresponding infinite matrices do have the product indicated, if we form a rectangular array from the convolution array for S_2 using every other column, whether we take the odd columns only, or the even columns only, the determinant of any $k \times k$ submatrix of either array which has its first row taken along the first row of the array will have determinant value given by $2^{[k(k-1)/2]}$.

Next, form $P_{3,3}$ containing every third column of the convolution triangle for S_3. Then, from [2], $P^{-1}_{3,3}$ contains every third row of Pascal's triangle taken with alternate signs on and below the main diagonal with zeroes elsewhere, as illustrated for 5 × 5 submatrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & 6 & -7 & 1 & 0 \\ 0 & -4 & 28 & -10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 \\ 22 & 22 & 7 & 1 & 0 \\ 140 & 140 & 49 & 10 & 1 \end{bmatrix} = I,$$

Notice that the row sums of $P_{3,3}$ are 1, 2, 9, 52, 340, ⋯, or S_3^2, the first convolution of S_3. As before, we find that

$$P'^{-1}_{3,3}P^*_{3,1} = P^*_{0.1}, \tag{3.4}$$

which allows us to again extend Theorems 2.1 and 2.2. We also find

$$P'^{-1}_{3,3}P^*_{3,3} = P^*_{0,3} \tag{3.5}$$

which is illustrated for 5 × 5 submatrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -6 & 1 & 0 \\ 0 & -1 & 15 & -9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 7 & 10 & 13 \\ 4 & 22 & 49 & 85 & 130 \\ 22 & 140 & 357 & 700 & 1196 \\ 140 & 969 & 2695 & 5740 & 10647 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 7 & 10 & 13 \\ 1 & 10 & 28 & 55 & 91 \\ 1 & 20 & 84 & 220 & 455 \\ 1 & 35 & 210 & 715 & 1820 \end{bmatrix}$$

Using earlier results [3], this means that, if we take any $k \times k$ submatrix of $P^*_{3,3}$ which has its first row along the first row of $P^*_{3,3}$, the value of its determinant is $3^{[k(k-1)/2]}$. However, we have the same result if we take every third column to form the array, whether we take columns of the convolution array for S_3 of the form $3k$, $3k+1$, or $3k+2$.

Next, we summarize our results. First, the matrix $P^{-1}_{i,i}$ always contains the i^{th} rows of Pascal's triangle written on and below the main diagonal with alternating signs, beginning with the first row, and zeroes elsewhere. That is, the j^{th} column of $P^{-1}_{i,i}$ contains the coefficients of $(1-x)^{1+i(j-1)}$, $j = 1, 2, \cdots$, on and below the main diagonal, and zeroes above the main diagonal. Inspecting $P_{i,i}$ gave the row sums as S^2_i, the first convolution of S_i. Both of these results were proved in [2].

If we form the matrix $P'^{-1}_{i,i}$ using the i^{th} rows of Pascal's triangle taken with alternate signs on and below the main diagonal, but beginning with the zeroth row, so that the j^{th} column contains the coefficients of $(1-x)^{i(j-1)}$, $j = 1, 2, \cdots$, and form the matrix $P^*_{i,1}$ so that its elements are the convolution triangle for S_i written in rectangular form, then

$$P'^{-1}_{i,i} P^*_{i,1} = P^*_{0,1}, \tag{3.6}$$

the matrix containing Pascal's triangle written in rectangular form.

If we form an infinite matrix $P^*_{i,i}$ from every i^{th} column of the convolution array for the sequence S_i, then the matrix product

$$P'^{-1}_{i,i} P^*_{i,i} = P^*_{0,i} \tag{3.7}$$

the matrix formed from every i^{th} column of Pascal's triangle written in rectangular form. Further, (3.7) is only one of $2i$ similar matrix products which we could write. By adjusting the columns of $P^{-1}_{i,i}$ to write modified matrices which are formed using the i^{th} rows of Pascal's triangle as before but taking the first column of the new matrix $(P^{-1}_{i,i})_r$ to contain the r^{th} row, so that its j^{th} column contains the $(r + (j-1)i)$ row of Pascal's triangle or the coefficients of $(1-x)^{r+i(j-1)}$, $j = 1, 2, \cdots$, on and below its main diagonal, we can write

$$(P^{-1}_{i,i})_r P^*_{i,i} = (P^*_{0,i})_r, \qquad r = 0, 1, \cdots, i-1, \tag{3.8}$$

where $(P^*_{0,i})_r$ contains every i^{th} column of Pascal's triangle written in rectangular form beginning with its r^{th} column. Notice that $r = 0$ in (3.8) gives (3.7), and that

$$P'^{-1}_{i,i} = (P^{-1}_{i,i})_0 \qquad \text{while} \qquad P^{-1}_{i,i} = (P^{-1}_{i,i})_1 .$$

We can also write

$$P^{-1}_{i,i}(P^*_{i,i})_r = (P^*_{0,i})_{r-1}, \qquad r = 0, 1, \cdots, i-1, \tag{3.9}$$

where $(P^*_{i,i})_r$ contains the i^{th} columns of the convolution array for S_i beginning with the r^{th} column. Also,

$$(P^{-1}_{i,i})_r (P^*_{i,i})_r = (P^*_{0,i})_0 = P^*_{0,i}, \qquad r = 0, 1, \cdots, i-1 \tag{3.10}$$

$$(P^{-1}_{i,i})_j (P^*_{i,i})_r = (P^*_{0,i})_{r-j} . \tag{3.11}$$

The matrix identities of this section are proved next.

4. PROOF OF THE MATRIX IDENTITIES GIVEN IN SECTION 3

The proof of (3.3) follows from [2] but is a little subtle since we do not have explicit formulas for the generating functions for S_i, $i \geqslant 2$. However, we do have the following from [2]: If $S_i(x)$ is the generating function for S_i and if $S_0(x) = f(x)$, then

$$f(xS_1(x)) = S_1(x); \qquad f(S_2^2(x)) = S_2(x); \cdots; \qquad f(xS_k^k(x)) = S_k(x).$$

And further, $f(1/S_{-1}(x)) = S_1(x)$, etc. That means

$$S_2(x(1-x)^2) = S_2(x/[1/(1-x)^2]) = S_0(x) = \frac{1}{1-x}$$

which generates Pascal's triangle.

The general case given in (3.6) follows easily by replacing 2 with i in the above discussion. We prove (3.7) by taking

$$S_i(x[(1-x)^i]^i = \left(\frac{1}{1-x}\right)^i$$

the generating function for the matrix containing the i^{th} columns of Pascal's triangle. Equation (3.8) merely starts the matrices with shifted first columns, but it is the constant difference of the columns, or the power of $(1-x)$ which is the ratio of two successive column generators, which is used in the relationships shown above.

All of this raises a very interesting situation. Clearly, if we can obtain Pascal's triangle from the convolution array for S_i by matrix multiplication, then we can get the convolution array for any S_k by multiplying the convolution array for S_i by a suitable matrix. The possibilities are endless. Also, one can factor Pascal's triangle matrix when written in its rectangular form into several factors.

Now, in all of these special matrix multiplications, when $AB = C$, the column arrangement of B determines the column configuration of C. Whatever appears in A for i^{th} columns of a convolution array for S_i will appear as the rectangular convolution array for S_i (every column) if the proper middle matrix is used. Starting with, say, $S_2^2(x)$ as the first column of A and then $xS_2^5(x)$, $x^2S_2^8(x)$, $\cdots$, one can use as the middle matrix the one with column generators $(1+x)$, $(1+x)^2$, $(1+x)^3$, $\cdots$, where $S_{-1}(x) = (1+x)$. Now $S_{-1}(xS_2^3(x)) = S_2(x)$, etc. Thus the columns of the right-most matrix are S_2^3, S_2^6, S_2^9, $\cdots$, as is to be expected.

5. DETERMINANT IDENTITIES IN CONVOLUTION ARRAYS

Since, in Section 3, we found several ways that $P_{i,1}^*$ and $P_{i,i}^*$, when multiplied by matrices having unit determinants, yield matrices containing columns of Pascal's triangle, and since the $n \times n$ submatrices taken in the upper left corners have the same multiplication properties as the infinite matrices from which they are taken in these cases, we have several theorems we can write by applying earlier results concerning determinant values found within Pascal's triangle [3]. Specifically, (3.5) and (3.7) allow us to write the very general theorem,

Theorem 5.1. Write the convolution array in rectangular form for any of the sequences S_i. Any $n \times n$ submatrix of the array which has its first row taken along the row of ones of the array has a determinant with value one. Any $n \times n$ submatrix of the array such that its first column lies in the j^{th} column of the array and its first row is taken along the row of integers of the array has determinant value given by the binomial coefficient $\binom{n+j-1}{n}$. Any $n \times n$ matrix formed such that its columns are every r^{th} column of the convolution array beginning with the j^{th} column, $j = 0, 1, \cdots, r-1$, has a determinant value of $r^{n(n-1)/2}$.

However, the surprising thing about Theorem 5.1 is that so much of it can be stated for the convolution array of *any* sequence whatever! Hoggatt and Bergum [5] have found that if S is any sequence with first term 1, then the rows of its convolution array written in rectangular form are arithmetic progressions of order 0, 1, 2, 3, $\cdots$ with constants 1, s_2, s_2^2, s_2^3, $\cdots$, where s_2 is the second term of sequence S. Applying Eves' Theorem [3],

Theorem 5.2. Let S be a sequence with first term one. If any $n \times n$ array is taken from successive rows and columns of the rectangular convolution array for S such that the first row includes the row of ones, then the determinant has value one if the second term of the sequence is one and value $s_2^{n(n-1)/2}$ if the second term of S is s_2.

Theorem 5.3. Let S be a sequence with first and second term both one. If any $n \times n$ array is formed from successive rows and columns of the rectangular convolution array for S such that the first row includes the row of integers and the first column includes S^{j-1}, $j = 1, 2, \cdots$, then the determinant of the array is given by the binomial coefficient $\binom{n+j-1}{n}$.

Conjecture. Let S be a sequence with first term one and second term s_2. If any $n \times n$ array is formed using the successive rows and columns of the rectangular convolution array for S such that the first row includes the row $1u_2$, $2u_2$, $3u_2$, $4u_2$, $\cdots$, and the first column includes S^{j-1}, $j = 1, 2, \cdots$, then the determinant of the array is given by

$$s_2^{n(n-1)/2}\binom{n+j-1}{n}.$$

For further interesting relationships, see Hoggatt and Bruckman [1].

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LETTER TO THE EDITOR

April 21, 1975

Dear Editor:

Following are some remarks on some formulas of Trumper [5].

Trumper has proved seven formulas of which the following is entirely characteristic

$$F_nF_m - F_xF_{n+m-x} = (-1)^{m+1}F_{x-m}F_{n-x}\ . \tag{1}$$

He actually gives 13 formulas, but the duplicity arises from the trivial replacement of x by $-x$ in all but the seventh formula.

It is of interest to note that the formulas are not really new in the sense that they can all be gotten from the single formula

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^nF_nF_b \tag{2}$$

by use of the negative transformation

$$F_{-n} = (-1)^{n+1}F_n\,. \tag{3}$$

For example, in (1) replace n by $n+x$ and m by $m+x,$ and we have

$$F_{x+n}F_{x+m} - F_xF_{x+n+m} = (-1)^{m+x+1}F_{-m}F_n = (-1)^xF_mF_n\,,$$

the last step following by (3). But the formula is then simply a restatement of (2) with n replaced by $x,$ a by $n,$ and b by $m.$ Similarly, for his formula (4), which we may rewrite as

$$F_{n+x}F_m - F_nF_{m+x} = (-1)^{m+1}F_{n-m}F_x\,,$$

we have only to set $x=a,$ $m=n+b$ and use (3) again to get (2), and all steps are reversible. The reader may similarly derive the other formulas.

For reference to the history of (2), see [1, p. 404], [2], [3]. Formula (2) was posed as a problem [6]. Tagiuri is the oldest reference [4] of which I know. Formula (2) is the unifying theme behind all the formulas in [5].

[Continued on page 146.]

IN-WINDING SPIRALS

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1. INTRODUCTION

In [1] Holden discusses the system of outwinding squares met in the geometric proof of

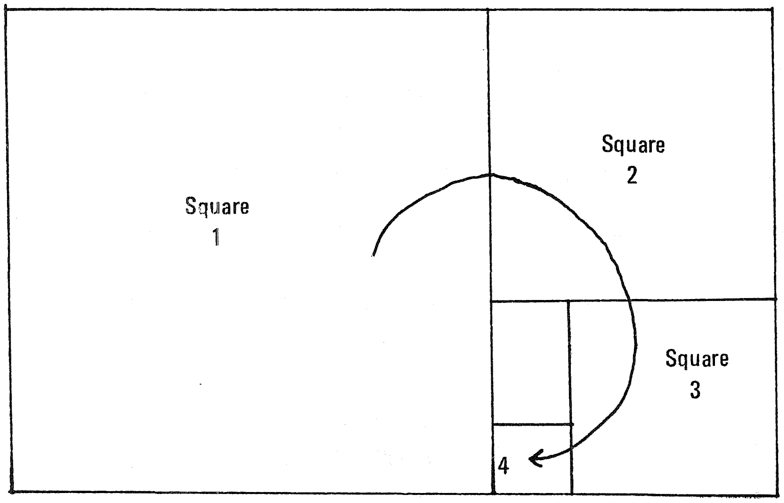
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1} .$$

In fact, it is shown there that the centers of the squares lie on two orthogonal lines with slopes $-1/3$ and 3, respectively. This was the original problem. Further, if one lets $u_1 = 1$, $u_2 = p$, and $u_{n+2} = u_{n+1} + u_n$, one obtains the generalized Fibonacci sequence. Here the tiling is not made up of squares but contains one 1 by $(p - 1)$ rectangle but the whirling squares still have their centers on two orthogonal straight lines.

It is the purpose of this paper to extend the results to in-winding systems of squares and rectangles. For background and generalizations, see Hoggatt and Alladi [2].

2. THE CLASSIC EXAMPLE

The Golden Section Rectangle yields one beautiful example of in-winding spirals of squares. We start with a rectangle such that if one cuts a square from it, then the remaining rectangle is similar to the original one. The ratio of length to width of this rectangle is $\alpha = (1 + \sqrt{5})/2$.



We now repeat the cutting off of a square from the second rectangle, then a square from the third rectangle, and so on for n steps. This will leave some 1 by $(p - 1)$ rectangle in the middle of the system of squares. One immediately notices that if the rectangle is $1 \times (p - 1)$, then the n^{th} square was $p \times p$ and reversing the construction you are indeed adding squares on such that the sides form a generalized Fibonacci sequence. That is, the resulting squares have their centers on two mutually perpendicular straight lines. Now, since the out-winding squares from the $1 \times (p - 1)$

rectangles have centers on two mutually orthogonal lines for one n the same pair of lines hold for all n. In other words, the nested set of rectangles converges onto a point. In the case of the Golden Section rectangle the sequence of corners of the rectangles lie on two mutually orthogonal lines and further the common point of intersection of this pair of lines coincides with that of the pair of lines determined by the centers of the in-winding squares.

Suppose we let $f_0 = p$ and $f_1 = 1$, $f_2 = f_0 - f_1 = p - 1$, $f_3 = 1 - (p-1) = 2 - p$, $f_4 = (p-1) - (2-p) = 2p - 3$, $f_5 = (2-p) - (2p-3) = 5 - 3p, \cdots$, so that in general one gets

$$f_n = (F_{n+1} - pF_n)(-1)^n\ ,$$

where F_n is the n^{th} Fibonacci number. In the event that $p = \alpha$, then

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} F_n\left(\frac{F_{n+1}}{F_n} - \alpha\right) = 0.$$

This suggests that although we know

$$\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n} - \alpha\right) = 0$$

in reality

$$\lim_{n\to\infty} F_n\left(\frac{F_{n+1}}{F_n} - \alpha\right) = 0.$$

To see this, we look at

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} - \alpha = \frac{\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} + \alpha\beta^n}{\alpha^n - \beta^n} = \frac{-\beta^n(\alpha - \beta)}{\alpha^n - \beta^n}\ .$$

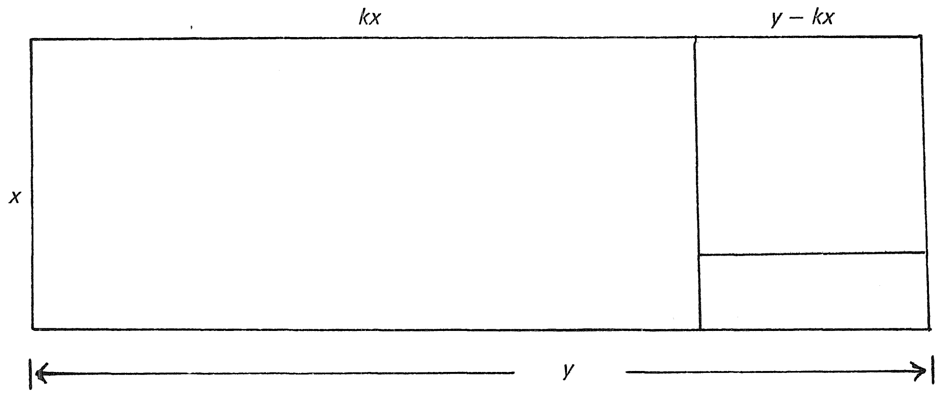
Thus,

$$F_n\left(\frac{F_{n+1}}{F_n} - \alpha\right) = -\beta^n \to 0.$$

This seems to indicate that unless $\alpha = p$, the process will not converge for squares.

3. A GENERALIZATION: THE SILVER RECTANGLE

Suppose we cut off k squares from the rectangle and then want the remaining rectangle to be similar to the original.



$$\frac{y}{x} = \frac{x}{y - kx}$$

$$y^2 - kxy - x^2 = 0$$

$$\left(\frac{y}{x}\right)^2 - k\left(\frac{y}{x}\right) - 1 = 0$$

$$\frac{y}{x} = \frac{k \pm \sqrt{k^2 + 4}}{2}$$

Since we wish $y/x > 0$, then $\alpha = (k + \sqrt{k^2+4})/2$ is selected. In reality, this leads naturally to the Fibonacci polynomials. Suppose again we start out with $f_0 = p$ and $f_1 = 1$, $f_2 = p - k$,

$$f_3 = 1 - k(p-k) = k^2 - kp + 1 = (k^2+1) - pk$$

$$f_4 = (p-k) - k(k^2 - kp + 1) = (-k^3 - 2k) + p(k^2+1) = -u_4(k) + pu_3(k)$$

$$f_n = (-1)^n [u_{n+1}(k) - pu_n(k)]\,,$$

where $u_n(k)$ is the n^{th} Fibonacci polynomial. Once again $\lim\limits_{n\to\infty} f_n$ does not exist unless

$$p = (k + \sqrt{k^2+4})/2\,;$$

then

$$f_n = (-1)^n u_n(k) \left(\frac{u_{n+1}(k)}{u_n(k)} - p \right).$$

$$\lim_{n\to\infty} f_n = 0$$

as before. When $k = 1(un(1) = F_n)$ so that unless $p = \alpha$, then

$$f_n = (-1)^n [u_{n+1}(k) - \alpha u_n(k) - (p-\alpha)u_n(k)] = (-1)\cdot 1 + (-1)^n(\alpha - p)u_n(k)$$

which diverges since $\lim\limits_{n\to\infty} u_n(k) \to \infty$ for each $k > 0$.

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[Continued from Page 143.]

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* * *

ARITHMETIC SEQUENCES OF HIGHER ORDER

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I

Definition 1. Given a sequence of numbers

$$a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n \quad \cdots \tag{1}$$

we call first differences of (1) the numbers of the sequence

$$D_0^1 \quad D_1^1 \quad D_2^1 \quad \cdots \quad D_n^1 \quad \cdots$$

with

$$D_n^1 = a_{n+1} - a_n .$$

By recurrence we define the differences of order k of (1) as the first differences of the sequence of differences of order $k - 1$ of (1), namely the numbers of the sequence

$$D_0^k \quad D_1^k \quad D_2^k \quad \cdots \quad D_n^k \quad \cdots \tag{2}$$

with

$$D_n^k = D_{n+1}^{k-1} - D_n^{k-1} . \tag{3}$$

Observe that (3) is also valid for $k = 1$ if we rename $a_n = D_n^0$.

Definition 2. The sequence (1) is arithmetic of order k if the differences of order k are equal, whereas the differences of order $k - 1$ are not equal. It follows that the differences of order higher than k are null.

Proposition 1. Given a sequence (1), if there exists a polynomial $p(x)$ of degree k with leading coefficient c such that $a_n = p(n)$ for $n = 0, 1, 2, \cdots$ then the sequence is arithmetic of order k and the differences of order k are equal to $k!c$.

Proof. Let $p(x) = cx^k + bx^{k-1} + \cdots$ (the terms omitted are always of less degree than those written). Then

$$a_n = cn^k + bn^{k-1} + \cdots$$

hence

$$D_n^1 = a_{n+1} - a_n = c[(n+1)^k - n^k] + b[(n+1)^{k-1} - n^{k-1}] + \cdots = ckn^{k-1} + \cdots$$

therefore, for the first differences we have a polynomial $p_1(x) = kcx^{k-1} + \cdots$ of degree $k - 1$ and leading coefficient kc such that $D_n^1 = p_1(n)$. Repeating the same process k times we come to the conclusion that $D_n^k = p_k(n)$ for a polynomial $p_k(x)$ of degree zero and leading coefficient $k!c$; hence $D_n^k = k!c$ for $n = 0, 1, 2, \cdots$.

EXAMPLE. The sequence

$$0 \quad 1 \quad 2^k \quad 3^k \quad \cdots \quad n^k \quad \cdots \tag{4}$$

for k a positive integer is arithmetic of order k and $D_n^k = k!$.

Proposition 2. For any sequence (1), arithmetic or not, we have

$$D_n^k = \binom{k}{0} a_{n+k} - \binom{k}{1} a_{n+k-1} + \binom{k}{2} a_{n+k-2} + \cdots \pm \binom{k}{k} a_n .$$

The proof is straightforward using induction on k with the help of (3).

In particular for the sequence (4) we have

(5) $$D_n^k = \binom{k}{0}(n+k)^k - \binom{k}{1}(n+k-1)^k + \binom{k}{2}(n+k-2)^k - \cdots \pm \binom{k}{k} n^k,$$

where the coefficient of n^{k-i} $(i = 0, 1, 2, \cdots, k)$ is

$$\binom{k}{0}\binom{k}{i} k^i - \binom{k}{1}\binom{k}{i}(k-1)^i + \binom{k}{2}\binom{k}{i}(k-2)^i - \cdots \mp \binom{k}{k-1}\binom{k}{i} 1^i \pm \binom{k}{k}\binom{k}{i} 0^i$$

(we assume that $0^i = 0$ for $i = 1, 2, \cdots, k$ and $0^0 = 1$). Hence the coefficient of n^{k-i} $(i = 1, 2, \cdots, k)$ in (5) is

$$\binom{k}{i}\left[\binom{k}{0} k^i - \binom{k}{1}(k-1)^i + \binom{k}{2}(k-2)^i - \cdots \pm \binom{k}{k-1} 1^i\right]$$

and the coefficient of n^k

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots \pm \binom{k}{k}.$$

Since we know that $D_n^k = k!$ we have the remarkable equalities:

(i) $$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots \pm \binom{k}{k} = 0$$

(which is a very well known fact since it is the development of $(1-1)^k$).

(6) (ii) $$\binom{k}{0} k^i - \binom{k}{1}(k-1)^i + \binom{k}{2}(k-2)^i - \cdots \pm \binom{k}{k-1} 1^i = 0$$

for $i = 1, 2, \cdots, k-1$

(7) (iii) $$\binom{k}{0} k^k - \binom{k}{1}(k-1)^k + \binom{k}{2}(k-2)^k - \cdots \pm \binom{k}{k-1} 1^k = k!$$

A fourth identity can be obtained from (5) with $n = 0$ and (21), namely

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}(k-1-j)^k = (k-1)!\binom{k}{2},$$

which can also be written in the form

(iv) $$\binom{k}{0} k^{k+1} - \binom{k}{1}(k-1)^{k+1} + \binom{k}{2}(k-2)^{k+1} - \cdots \pm \binom{k}{k-1} 1^{k+1} = k!\binom{k+1}{2}.$$

II

Starting with $k+1$ numbers $A_0, A_1, \cdots, A_k$ we form the "generalized" triangle of Pascal

$$\begin{array}{c}
A_0 \qquad A_1 \qquad A_2 \quad \cdots \quad A_k \\
A_0 \qquad A_0 + A_1 \qquad A_1 + A_2 \quad \cdots \quad A_k \\
A_0 \qquad 2A_0 + A_1 \qquad A_0 + 2A_1 + A_2 \qquad A_1 + 2A_2 + A_3 \quad \cdots \quad A_k \\
A_0 \qquad 3A_0 + A_1 \qquad 3A_0 + 3A_1 + A_2 \qquad \cdots
\end{array}$$

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where each number is the sum of the two above. We observe that the coefficient of A_0 in the h^{th} entry of the n^{th} row is $\binom{n-1}{h-1}$; the coefficient of A_1 is $\binom{n-1}{h-2}$ $\cdots$ and the coefficient of A_k is $\binom{n-1}{h-k-1}$. (We set $\binom{n}{j}$ = 0 whenever $j > n$ or $j < 0$.) Therefore the h^{th} entry of the n^{th} row is

(8) $$\binom{n-1}{h-1} A_0 + \binom{n-1}{h-2} A_1 + \cdots + \binom{n-1}{h-k-1} A_k$$

In particular, for the triangle over the $k+1$ differences $a_0, D_0^1, D_0^2, \cdots, D_0^k$ of the sequence (1) assumed to be arithmetic of order k, in view of (3) and taking into account that $D_0^k = D_1^k = \cdots$ we have

$$\begin{array}{ccccccccc}
 & & & & a_0 \quad D_0^1 \quad D_0^2 \cdots D_0^k & & & & \\
 & & & a_0 \quad a_1 \quad D_1^1 \quad D_1^2 \cdots D_1^k & & & & & \\
 & & a_0 \quad S_1^1 \quad a_2 \quad D_2^1 \quad D_2^2 \cdots D_2^k & & & & & & \\
 & a_0 \quad S_1^2 \quad S_2^1 \quad a_3 \quad D_3^1 \quad D_3^2 \cdots D_3^k & & & & & & & \\
a_0 \quad S_1^3 \quad S_2^2 \quad S_3^1 \quad a_4 \quad D_4^1 \quad D_4^2 \cdots D_4^k & & & & & & & & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & &
\end{array}$$

where

$$S_0^1 = a_0 \quad S_i^1 = S_{i-1}^1 + a_i \text{ and } S_n^k = S_{n-1}^k + S_n^{k-1} .$$

Since in this triangle a_n is the $(n+1)^{th}$ entry of the $(n+1)^{th}$ row, we have

$$a_n = \binom{n}{n} a_0 + \binom{n}{n-1} D_0^1 + \binom{n}{n-2} D_0^2 + \cdots + \binom{n}{n-k} D_0^k \tag{9}$$

or, equivalently,

$$a_n = a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_0^2 + \cdots + \binom{n}{k} D_0^k . \tag{10}$$

Observe that if the sequence (1) is not arithmetic we still can construct a "generalized" triangle of Pascal starting with an infinity of entries in the first row.

$$\begin{array}{c} a_0 \quad D_0^1 \quad D_0^2 \quad \cdots \quad D_0^n \quad \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

and then instead of (10) we would have

$$a_n = a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_0^2 + \cdots + \binom{n}{n} D_0^n .$$

Proposition 2. If (1) is an arithmetic sequence of order k, we can find a polynomial $p(x)$ of degree k such that $a_n = p(n)$.

Proof.

$$p(x) = a_0 + \binom{x}{1} D_0^1 + \binom{x}{2} D_0^2 + \cdots + \binom{x}{k} D_0^k$$

with

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

is obviously a polynomial of degree k and in view of (10), $a_n = p(n)$.

For the partial sum $S_n^1 = a_0 + a_1 + \cdots + a_n$ we have a formula similar to (10). In fact, observing that S_n^1 is the $(n+1)^{th}$ entry of the $(n+2)^{th}$ row in the "generalized" triangle of Pascal, we have

$$S_n^1 = \binom{n+1}{n} a_0 + \binom{n+1}{n-1} D_0^1 + \cdots + \binom{n+1}{n-k} D_0^k$$

or, equivalently,

$$S_n^1 = \binom{n+1}{1} a_0 + \binom{n+1}{2} D_0^1 + \cdots + \binom{n+1}{k+1} D_0^k . \tag{11}$$

Therefore $S_n^1 = q(n)$, where $q(x)$ is a polynomial of degree $k+1$. This was to be expected, since obviously the sequence $S_0^1, S_1^1, \cdots, S_n^1, \cdots$ is arithmetic of order $k+1$.

EXAMPLES. If we apply (11) to the sequences of type (4) with $k = 1, 2, 3, 4$ we obtain the well known formulas

1. $$0 + 1 + 2 + \cdots + n = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 = \frac{n^2+n}{2}$$

2. $$0 + 1^2 + 2^2 + \cdots + n^2 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$0 + 1^3 + 2^3 + \cdots + n^3 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 6 + \binom{n+1}{4} 6 = \frac{n^4+2n^3+n^2}{4}$$

4. $$0 + 1^4 + 2^4 + \cdots + n^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

III

We now know that the sum

$$S_k(n) = 0 + 1^k + 2^k + \cdots + n^k$$

is given by a polynomial in n of degree $k + 1$. The question arises, how to find out the coefficients of this polynomial?

Obviously the coefficient of n^0 is zero, since $S_k(0) = 0$, and the coefficient of n^{k+1} is $1/(k + 1)$ as we can see from (11). Hence the polynomial form for $S_k(n)$ is

(12) $$S_k(n) = 1/(k+1)n^{k+1} + h_0 n^k + h_1 n^{k-1} + \cdots + h_{k-1}n$$

for some coefficients $h_0, h_1, \cdots, h_{k-1}$. Since $S_k(n) - S_k(n - 1) = n^k$, we have

$$\frac{1}{k+1}[n^{k+1} - (n-1)^{k+1}] + h_0[n^k - (n-1)^k] + h_1[n^{k-1} - (n-1)^{k-1}] + \cdots$$
$$+ h_i[n^{k-i} - (n-1)^{k-i}] + \cdots + h_{k-1} = n^k$$

and taking coefficients of the different powers of n, we have the following equations: (the first is an identity, the rest form a linear system of k equations in k unknowns, which permits to compute recursively $h_0, h_1, \cdots, h_{k+1}$).

(13) $$\left\{\begin{array}{l} \frac{1}{k+1}\binom{k+1}{1} = 1 \\ \frac{1}{k+1}\binom{k+1}{2} - \binom{k}{1} h_0 = 0 \\ \frac{1}{k+1}\binom{k+1}{3} - \binom{k}{2} h_0 + \binom{k-1}{1} h_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{k+1}\binom{k+1}{i+1} - \binom{k}{i} h_0 + \binom{k-1}{i-1} h_i - \cdots \pm \binom{k-i+1}{1} h_{i-1} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{k+1}\binom{k+1}{k+1} - \binom{k}{k} h_0 + \binom{k-1}{k-1} h_1 - \cdots \pm \binom{1}{1} h_{k-1} = 0. \end{array}\right.$$

From the second equation we obtain $h_0 = ½$, independent of k. If we set

(14) $$h_1 = \binom{k}{1} b_1 \qquad h_2 = \binom{k}{2} b_2 \cdots h_{k-1} = \binom{k}{k-1} b_{k-1}$$

and observe that

$$\binom{k-j}{i-j} h_j = \binom{k-j}{i-j}\binom{k}{j} b_j = \binom{k}{i}\binom{i}{j} b_j$$

we can write the i^{th} equation in (13) in the form

$$\frac{1}{i+1}\binom{k}{i} - ½\binom{k}{i} + \binom{k}{i}\binom{i}{1} b_1 - \binom{k}{i}\binom{i}{2} b_2 + \cdots \pm \binom{k}{i}\binom{i}{j} b_j + \cdots \pm \binom{k}{i}\binom{i}{i-1} b_{i-1} = 0$$

or, equivalently:

$$\frac{1}{i+1} - \frac{1}{2} + \binom{i}{1} b_1 - \binom{i}{2} b_2 + \cdots \pm \binom{i}{j} b_j + \cdots \pm \binom{i}{i-1} b_{i-1} = 0.$$

Hence the system (13), after omitting the first two identities reduces to:

$$(15)\qquad \left\{\begin{array}{c}
\frac{1}{3}-\frac{1}{2}+\binom{2}{1}b_1=0\\
\frac{1}{4}-\frac{1}{2}+\binom{3}{1}b_1-\binom{3}{2}b_2=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{1}{i+1}-\frac{1}{2}+\binom{i}{1}b_1-\binom{i}{2}b_2+\cdots\pm\binom{i}{j}b_j+\cdots\pm\binom{i}{i-1}b_{i-1}=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{1}{k+1}-\frac{1}{2}+\binom{k}{1}b_i-\binom{k}{2}b_2+\cdots\pm\binom{k}{k-1}b_{k-1}=0\,.
\end{array}\right.$$

We will call Bernoulli numbers the numbers $b_1, b_2, \cdots$. The Bernoulli numbers have over the numbers $h_1, h_2, \cdots$ the advantage that they do not depend on k, as we can see from system (15). Equation (14) permits to calculate for each k the h's in terms of the b's.

Proposition. The even Bernoulli numbers are null.

Proof: Writing $n = 1$ in (12) we have

$$\frac{1}{k+1}+\frac{1}{2}+h_1+h_2+\cdots+h_{k-1}=1\,.$$

On the other hand, the last equation in (13) is

$$\frac{1}{k+1}-\frac{1}{2}+h_1-h_2+\cdots\pm h_{k-1}=0.$$

Adding and subtracting these two equations, we obtain:

$$(16)\qquad \left\{\begin{array}{l}
h_1+h_3+\cdots=\frac{1}{2}-\frac{1}{k+1}\\
h_2+h_4+\cdots=0
\end{array}\right.$$

The second equation in (16) can be written

$$\binom{k}{2}b_2+\binom{k}{4}b_4+\cdots=0,$$

where the sum is extended to all the subscripts less than or equal to $k-1$. For $k=3$ we get $b_2=0$; for $k=5$, $b_4=0$, etc., which proves the proposition.

The first equation in (16) for $k = 3, 5, 7, \cdots$ yields the infinite system of equations:

$$(17)\qquad \left\{\begin{array}{c}
\binom{3}{1}b_1=\frac{1}{2}-\frac{1}{4}\\
\binom{5}{1}b_1+\binom{5}{3}b_3=\frac{1}{2}-\frac{1}{6}\\
\binom{7}{1}b_1+\binom{7}{3}b_3+\binom{7}{5}b_5=\frac{1}{2}-\frac{1}{8}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right.$$

and for $k = 2, 4, 6, \cdots$ the system

$$(18)\qquad \left\{\begin{array}{c}
\binom{2}{1}b_1=\frac{1}{2}-\frac{1}{3}\\
\binom{4}{1}b_1+\binom{4}{3}b_3=\frac{1}{2}-\frac{1}{5}\\
\binom{6}{1}b_1+\binom{6}{3}b_3+\binom{6}{5}b_5=\frac{1}{2}-\frac{1}{7}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right.$$

Subtracting the equations in (18) from those in (17), we have

$$(19)\qquad \left\{\begin{array}{c} \binom{2}{0} b_1 = \frac{1}{3\cdot 4} \\ \binom{4}{0} b_1 + \binom{4}{2} b_3 = \frac{1}{5\cdot 6} \\ \binom{6}{0} b_1 + \binom{6}{2} b_3 + \binom{6}{4} b_5 = \frac{1}{7\cdot 8} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right.$$

Any of the infinite systems (17), (18) or (19) permits to find recursively the Bernoulli numbers with odd subscripts.

Substituting in (12) the Bernoulli numbers, we express

$$S_k(n) = 0 + 1^k + \cdots + n^k$$

in the form

$$(20)\qquad S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + b_1 \binom{k}{1} n^{k-1} + b_3 \binom{k}{3} n^{k-3} + \cdots ,$$

where the coefficients of the different powers of n are products of a combinatorial number of k and a number which does not depend on k.

NOTE. If we compute the coefficient of the k^{th} power of n in (11) we have

$$-\frac{(k+1)(k-2)}{2(k+1)!} D_0^k + \frac{1}{k!} D_0^{k-1} .$$

On the other hand for the sequence $0, 1^k, 2^k, \cdots$ that coefficient is ½, and $D_0^k = k!$. Hence, for this particular sequence we have

$$(21)\qquad 2D_0^{k-1} = (k-1)k! .$$

EXAMPLES. From (10) we obtain:

$$b_1 = \frac{1}{12} \qquad b_3 = -\frac{1}{120} \qquad b_5 = \frac{1}{252} \qquad b_7 = -\frac{1}{240} \qquad b_9 = \frac{1}{132}$$

which, substituted in (20) for $k = 1, 2, \cdots, 11$ yields the formulas:

$$1 + 2 + \cdots + n = \frac{1}{2} n^2 + \frac{1}{2} n$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$$

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 - \frac{1}{30} n$$

$$1^5 + 2^5 + \cdots + n^5 = \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 - \frac{1}{12} n^2$$

$$1^6 + 2^6 + \cdots + n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 - \frac{1}{6} n^3 + \frac{1}{42} n$$

$$1^7 + 2^7 + \cdots + n^7 = \frac{1}{8} n^8 + \frac{1}{2} n^7 + \frac{7}{12} n^6 - \frac{7}{24} n^4 + \frac{1}{12} n^2$$

$$1^8 + 2^8 + \cdots + n^8 = \frac{1}{9} n^9 + \frac{1}{2} n^8 + \frac{2}{3} n^7 - \frac{7}{15} n^5 + \frac{2}{9} n^3 - \frac{1}{30} n$$

$$1^9 + 2^9 + \cdots + n^9 = \frac{1}{10} n^{10} + \frac{1}{2} n^9 + \frac{3}{4} n^8 - \frac{7}{10} n^6 + \frac{1}{2} n^4 - \frac{3}{20} n^2$$

$$1^{10} + 2^{10} + \cdots + n^{10} = \frac{1}{11} n^{11} + \frac{1}{2} n^{10} + \frac{5}{6} n^9 - n^7 + n^5 - \frac{1}{2} n^3 + \frac{5}{66} n$$

$$1^{11} + 2^{11} + \cdots + n^{11} = \frac{1}{12} n^{12} + \frac{1}{2} n^{11} + \frac{11}{12} n^{10} - \frac{11}{8} n^8 + \frac{11}{6} n^6 - \frac{11}{8} n^4 + \frac{5}{12} n^2$$

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DIVISIBILITY PROPERTIES OF CERTAIN RECURRING SEQUENCES

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We shall consider the sequences, $\{w_n(r,s;a,c)\}$, defined by $w_0 = r$, $w_1 = s$ and $w_n = aw_{n-1} + cw_{n-2}$ for $n \geqslant 2$; henceforth denoted by $\{w_n\}$ where no ambiguity may result. We shall confine our attention to those sequences for which r, s, a, and c are integers with $(a,c) = 1$, $(r,s) = 1$, $(s,c) = 1$, $ac \neq 0$ and $w_n \neq 0$ for $n \geqslant 1$. The major result of this paper will be a complete classification of all sequences $\{w_n\}$ for which $w_k \mid w_{2k}$ for all integers $k \geqslant 1$.

If $w_0 = 0$ and $w_1 = 1$, we have a well known sequence which we shall denote, following Carmichael [1], by $\{D_n(a,c)\}$, or $\{D_n\}$ if no ambiguity may result, and concerning which we shall assume the following facts to be known (cf. [1], [2]):

F1: $(D_n, c) = 1$ for all $n \geqslant 1$,

F2: $(D_n, D_{n+1}) = 1$ for all n.

F3: If c is even, then D_n is odd for all n.

If c is odd and a is even, then $D_n \equiv n \pmod 2$ for all n.
If both a and c are odd, then D_n is even if and only if $n \equiv 0 \pmod 3$.

F4: Let $b = a^2 + 4c$ and let p be an odd prime.

$$\text{Let} \quad (b/p) = \begin{cases} (b/p) & \text{if } (b,p) = 1 \\ 0 & \text{if } p \mid b\,. \end{cases}$$

If $(p,c) = 1$, then $p \mid D_p - (b/p)$.

F5: $D_{m+n} = cD_mD_{n-1} + D_{m+1}D_n$ for all $m \geqslant 0$ and $n \geqslant 1$.

F6: If $m \mid n$, then $D_m \mid D_n$.

If $w_0 = 2$ and $w_1 = a$, we have a well known sequence which we shall denote, following Carmichael [1], by $\{S_n(a,c)\}$,* or by $\{S_n\}$ if no ambiguity may result, and concerning which we assume the following fact to be known:

F7: $D_{2n} = D_nS_n$ for all n.

Theorem 1: $w_n(r,s;a,c) = sD_n(a,c) + rcD_{n-1}(a,c)$ for all $n \geqslant 1$.

The proof is by complete mathematical induction on n:

1. $sD_1 + rcD_0 = s = w_1$.
2. $sD_2 + rcD_1 = as + rc = w_2$
3. Suppose the theorem is true for all n less than some fixed integer $k \geqslant 3$. Then $w_{k-1} = sD_{k-1} + rcD_{k-2}$ and

$$w_{k-2} = sD_{k-2} + rcD_{k-3}.$$

So

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*We differ from Carmichael in requiring that $(a,2) = 1$. If $(a,2) = 2$, $w_n(1, (a/2); a,c) = \frac{1}{2}S_n(a,c)$ for all n, and hence the former sequence has essentially the same divisibility properties as the latter.

$$w_k = a(sD_{k-1} + rcD_{k-2}) + c(sD_{k-2} + rcD_{k-3}) = s(aD_{k-1} + cD_{k-2}) + rc(aD_{k-2} + cD_{k-3}) = sD_k + rcD_{k-1}.$$

Using (F1), (F2) and the fact that $(r,s) = 1$, we have:

Corollary: $(w_n,D_n) = (r, D_n) = (r, w_n)$, $(w_n, D_{n-1}) = (s, D_{n-1}) = (s, w_n)$.

Theorem 2: $(w_n, w_{n+1}) = 1$ for all $n \geqslant 0$.

The proof is by induction on n:

1. $(w_0, w_1) = (r, s) = 1.$
2. $(w_1, w_2) = (s, as + cr) = (s, cr) = 1.$
3. Suppose $(w_{k-1}, w_k) = 1$ for some fixed integer $k \geqslant 2$. Let $(w_k, w_{k+1}) = d$. Since $w_{k+1} = aw_k + cw_{k-1}$, $d|cw_{k-1}$, whence $d|c$. Now $w_k = aw_{k-1} + cw_{k-2}$, whence $d|n$. Hence $d = 1$.

Theorem 3: $(w_n, c) = 1$ for all $n \geqslant 1$.

Proof:

1. $(w_1, c) = (s, c) = 1.$
2. Suppose $n \geqslant 2$. Then $w_n = aw_{n-1} + cw_{n-2}$. Let $d = (w_n, c)$. Then $d|aw_{n-1}$. Hence, by Theorem 2, $d = 1$.

Theorem 4. (a) If c is even, then w_n is odd for all $n \geqslant 1$.

(b) If a is even and c is odd, then
 (i) If n is odd, then $w_n \equiv s \pmod 2$.
 (ii) If n is even, then $w_n \equiv r \pmod 2$.

(c) If a and c are both odd, then
 (i) If $n \equiv 0 \pmod 3$, then $w_n \equiv r \pmod 2$.
 (ii) If $n \equiv 1 \pmod 3$, then $w_n \equiv s \pmod 2$.
 (iii) If $n \equiv 2 \pmod 3$, then $w_n \equiv r + s \pmod 2$.

Proof: Part (a) is immediate from Theorem 3.
Parts (b) and (c) follow from (F3) and Theorem 1.

Corollary: If r is even, then $w_n \equiv D_n \pmod 2$ for all n.

Theorem 5: Let p be any odd prime.

(a) If $p|c$, then $(p, w_n) = 1$ for all $n \geqslant 1$.
(b) If $(p,c) = 1$, then $p|w_{p-(b/p)}$ if and only if $p|r$.

Proof: Part (a) is immediate from Theorem 3.
Part (b) follows from (F4) and Theorem 1.

REMARK: The only recurring sequences for which $p|w_{p-(b/p)}$ for more than a finite number of primes p are $\pm\ D_n(a,c)$.

Theorem 6: $w_{m+n} = cD_{n-1}w_m + D_nw_{m+1}$ for all $m \geqslant 0$ and $n \geqslant 1$.

Proof:

$$\begin{aligned} w_{m+n} &= sD_{m+n} + rcD_{m+n-1} \quad \text{(by Theorem 1);} \\ &= s(cD_mD_{n-1} + D_{m+1}D_n) + rc(cD_{m-1}D_{n-1} + D_mD_n) \quad \text{(by F5);} \\ &= cD_{n-1}(sD_m + rcD_{m-1}) + D_n(sD_{m+1} + rcD_m) \\ &= cD_{n-1}w_m + D_nw_{m+1} \quad \text{(by Theorem 1).} \end{aligned}$$

Corollary 1: $(w_n, w_k) = (w_n, D_{n-k}) = (w_k, D_{n-k})$, where $n \geqslant k \geqslant 0$.

Proof: This corollary is immediate if $n = k$. Suppose $n \geqslant k \geqslant 0$. Then

$$w_n = w_{k+(n-k)} = cD_{n-k-1}w_k + D_{n-k}w_{k+1}.$$

Hence if $d|w_n$ and $d|w_k$, then $d|D_{n-k}w_{k+1}$. By Theorem 2, $(w_k, w_{k+1}) = 1$. Hence $d|D_{n-k}$.
Similarly, if $d|w_n$ and $d|D_{b-k}$, then $d|cD_{n-k-1}w_k$. But $(D_{n-k}, dD_{n-k-1}) = 1$. So $d|w_k$.
Finally, if $d|w_k$ and $d|D_{n-k}$, then $d|w_n$.

Corollary 2: $w_k|w_n$ if and only if $w_k|D_{n-k}$, where $n \geqslant k \geqslant 1$.

Corollary 2: $w_k | w_n$ if and only if $w_k | D_{n-k}$, where $n \geqslant k \geqslant 1$.

Corollary 3: (a) $w_k | w_{mk}$ if and only if $w_k | D_{(m-1)k}$ for $n \geqslant 1$.

(b) If $w_k | D_{tk}$, then $w_k | w_{mk}$ whenever $m \equiv 1 \pmod{t}$.

Proof: Part (a) is immediate from Corollary 2 with $n = mk$.

Part (b). By (F6), $D_{tk} | D_{ntk}$ for all positive integers n. Then $w_k | D_{ntk}$, whence $w_k | w_{(nt+1)k}$ for all non-negative integers n.

Corollary 4: (a) $w_k | w_{2k}$ if and only if $w_k | r$.

(b) $w_k | w_{3k}$ if and only if $w_k | rS_k$.

(c) $w_k | w_{3k}$ for all $k \geqslant 1$ if and only if $w_k | r(2s - ar)$ for all $k \geqslant 1$.

Proof: Part (a) follows from Corollary 3(a) and the corollary to Theorem 1.

Part (b) follows from (F7), Corollary 3(a) and the corollary to Theorem 1.

Part (c): Suppose that $w_k | w_{3k}$ for all $k \geqslant 1$. By Part (b), $w_k | rS_k$ for all $k \geqslant 1$. In particular, $w_1 | rS_1$, i.e., $s | ra$. Since $(r,s) = 1$, sa. Let $a = sd$. We shall prove by complete mathematical induction on k that

$$S_k(a,c) = dw_k(r,s;a,c) + c(2 - rd)D_{k-1}(a,c) \quad \text{for all } k \geqslant 1.$$

1. $$dw_1 + c(2 - rd)D_0 = ds + 0 = a = S_1.$$

2. $$dw_2 + c(2 - rd)D_1 = d(as + cr) + c(2 - rd) = a^2 + 3c = S_2.$$

3. Suppose that the theorem is true for all integers k less than some fixed integer $t \geqslant 3$.

$$\begin{aligned} S_t &= aS_{t-1} + cS_{t-2} = a[dw_{t-1} + c(2 - rd)D_{t-2}] + c[dw_{t-2} + c(2 - rd)D_{t-3}] \\ &= adw_{t-1} + c(2 - rd)(D_{t-1} - cD_{t-3}) + cdw_{t-2} + c^2(2 - rd)D_{t-3} \\ &= adw_{t-1} + c(2 - rd)D_{t-1} - c^2(2 - rd)D_{t-3} + cdw_{t-2} + c^2(2 - rd)D_{t-3} \\ &= d(aw_{t-1} + cw_{t-2}) + c(2 - rd)D_{t-1} = dw_t + c(2 - rd)D_{t-1}. \end{aligned}$$

Hence if $p | w_n$ and $p | S_n$, then $p | c(2 - rd)D_{n-1}$. So by Theorem 3 and the corollary to Theorem 1, $p | (2 - rd)s$. Thus, by Part (b), if $w_k | w_{3k}$ for all $k \geqslant 1$, then $w_k | r(2s - ar)$ for all $k \geqslant 1$.

Conversely, suppose $w_k | r(2s - ar)$ for all $k \geqslant 1$. Since $w_1 | r(2s - ar)$ and $(r,s) = 1$, $s | a$. Then, letting $a = sd$, it follows from the first half of the proof that $(S_k, w_k) = (2s - ar, w_k)$ for all $k \geqslant 1$. Hence, by Part (b) and the corollary to Theorem 1, if $w_k | r(2s - ar)$ for all $k \geqslant 1$, then $w_k | w_{3k}$ for all $k \geqslant 1$.

Lemma 1: $w_k | w_{2k}$ for all $k \geqslant 1$ if and only if $w_k w_{k+1} | r$ for all $k \geqslant 1$.

Proof: The "if" part is immediate by Corollary 4, Part (a).

Suppose that $w_k | w_{2k}$ for all $k \geqslant 1$. By Corollary 4 (a), $w_k | r$ and $w_{k+1} | r$. But by Theorem 2, $(w_k, w_{k+1}) = 1$. Hence $w_k w_{k+1} | r$.

Lemma 2: If $r \neq 0$ and $(a,r) = 1$, then $w_k | w_{2k}$ for all k only in the following cases:

(a) $r = s = \pm 1$, $a + c = 1$; in which cases $\{w_n\} = \pm\{1, 1, \cdots\}$.

(b) $r = \pm 1$, $s = \mp 1$, $-a + c = 1$; in which cases $\{w_n\} = \pm\{1, -1, 1, -1, \cdots\}$.

(c) $r = \pm 2$, $s = \mp 1$, $a = c = -1$; in which cases $\{w_n\} = \pm\{2, -1, -1, 2, -1, -1, \cdots\}$.

(d) $r = \pm 2$, $s = \pm 1$, $a = 1$, $c = -1$; in which cases $\{w_n\} = \pm\{2, 1, -1, -2, -1, 1, 2, 1, -1, -2, -1, 1, \cdots\}$.

Proof: Suppose $w_n(r,s;a,c)$ is a sequence for which $w_k | w_{2k}$ for all k. Then, by Corollary 4 (a), $w_k | w_{2k}$ for all k. Since $(s,r) = 1$, $s = w_1$ and $w_1 | r$, we may conclude that $s = \pm 1$. Now $w_n(r,1;a,c) = -w_n(-r,-1;a,c)$ for all n. So it suffices to consider the case where $s = 1$.

Since $w_2 | r$ and $(w_2, r) = (a + cr, r) = (a,r) = 1$, $w_2 = \pm 1$. We shall prove by complete mathematical induction on n that $w_n(r,s;a,c) = (-1)^{n+1} w_n(-r,s;-a,c)$ for all $n \geqslant 0$:

(1) $w_0(r,s;a,c) = r = (-1)^1(-r) = (-1)^1 w_0(-r,s;-a,c)$.

(2) $w_1(r,s;a,c) = s = (-1)^2(s) = (-1)^2 w_1(-r,s;-a,c)$.

(3) Suppose that the theorem is true for all integers n less than some fixed integer $k \geqslant 3$.

$$\begin{aligned} w_k(r,s;a,c) &= aw_{k-1}(r,s;a,c) + cw_{k-2}(r,s;a,c) = (-1)^k aw_{k-1}(-r,s;-a,c) + (-1)^{k-1} cw_{k-2}(-r,s;-a,c) \\ &= (-1)^{k+1}[(-a)w_{k-1}(-r,s;-a,c) + cw_{k-2}(-r,s;-a,c)] = (-1)^{k+1} w_k(-r,s;-a,c). \end{aligned}$$

Hence it suffices to consider the case where $w_2 = 1$.

CASE I: Suppose that $a \geqslant 1$.

Then $c \leqslant -1$. For were $c \geqslant 1$, we would have $w_{i+1} > w_i > 1$ for $i \geqslant 3$, contradicting the fact that $w_i \leqslant |r|$ for all i. Also since $r = (1-a)/c$, $1 - a \leqslant c \leqslant -1$. So $a + c \geqslant 1$.

(a) If $a + c = 1$, it is easily seen that the sequence reduces to $\{w_0, w_1, \cdots\} = \{1, 1, \cdots\}$.

(b) Suppose that $a + c > 1$. We shall prove by induction on i that $w_i > w_{i-1}$ for $i \geqslant 3$.

(1) By hypothesis it is true for $i = 3$.

(2) Suppose it to be true for i equal to some fixed integer $n \geqslant 3$. Then $w_{n+1} = aw_n + cw_{n-1} > w_n(a+c) > w_n$.

But this means that the w_i's form an unbounded sequence, which is impossible since $w_i \leqslant |r|$ for all i.

CASE II: Suppose that $a \leqslant -1$.

Since $a + c \mid a - 1$, either $c = -1$ or $0 < c < -2a + 1$.

(a) Suppose $c = -1$. Then $w_4 = a^2 - a - 1$ and, since $w_4 \mid r$, $a^2 - a - 1 \leqslant 1 - a$. Hence $a^2 \leqslant 2$, i.e., $a = -1$.

Then $r = -2$ and this yields the sequence $\{-2, 1, 1, -2, 1, 1, \cdots\}$.

(b) Suppose $c > 0$. Now $r = (1-a)/c$ and $a + c \mid r$. So $ac + c^2 \mid a - 1$.

$$\therefore\ ac + c^2 \leqslant 1 - a$$

$$\therefore\ a(c+1) \leqslant 1 - c^2$$

$$\therefore\ a \leqslant \frac{1 - c^2}{c+1} = 1 - c\,.$$

Also $ac + c^2 \geqslant a - 1$, whence $a(c-1) \geqslant -c^2 - 1$. Hence either $c = 1$ or

$$c - 1 \leqslant -a \leqslant \frac{c^2 + 1}{c - 1} = (c+1) + \frac{2}{c-1}\,.$$

Thus case (b) reduces to the following four subcases:

(i) $c = 1$. Now $w_3 \mid D_3$, i.e., $a + 1 \mid a^2 + 1$. Since $a^2 + 1 = (a+1)(a-1) + 2$, $a + 1 \mid 2$. So $a = -2$ or $a = -3$.

1. If $c = 1$ and $a = -2$, then $r = 3$ but $w_5 = -7$.
2. If $c = 1$ and $a = -3$, then $r = 4$ but $w_4 = 7$.

(ii) $a = -c - 1$. Then $w_4 = 2c + 1$, $r = (c+2)/c$ and $w_4 \mid r$. Hence $2c^2 + c \leqslant c + 2$. So $c = 1$, a case already considered.

(iii) $a = -c + 1$. But then $a + c = 1$, a case already considered.

(iv) $c = 2$ and $a = -5$. Then $r = 5$ but $w_4 = 17$.

This exhausts all of the possible cases. The other six sequences mentioned in the theorem are precisely those obtained from the sequences $\{1, 1, \cdots\}$ and $\{-2, 1, 1, -2, 1, 1, \cdots\}$ by the permutations of sign outlined at the beginning of the proof.

Theorem 7. If $r \neq 0$, then $w_k \mid w_{2k}$ for all k only in the cases listed in Lemma 2.

Proof: We shall prove that if $r \neq 0$ and $(a,r) = d > 1$, then w_k fails to divide w_{2k} for some k. The theorem will then follow by Lemma 2. Suppose the contrary, i.e., suppose there exists a sequence $w_n(r,s;a,c)$ such that $w_k \mid w_{2k}$ for all k. As in Lemma 2, $s = \pm 1$ and, moreover, we need only consider the case where $s = 1$.

Then $w_2 \mid r$ and $w_2 \mid D_2$, where $D_2 = a$. So $w_2 \mid d$. But $d \mid w_2$, since $w_2 = as + cr$. Thus $w_2 = \pm d$ and, as in the lemma, we need only consider the case where $w_2 = d$.

Suppose $a > 0$ and $d > 0$, $c < 0$ for otherwise the w_i's would become unboundedly large.

Now $d(ad + c) \mid r$ by Lemma 1 and $r = (d - a)/c \neq 0$. Hence $c(ad + c) \mid 1 - (a/d)$ and $1 - (a/d) < 0$.

Since $c \mid 1 - (a/d)$, $1 - (a/d) \leqslant c < 0$. Since $ad + c \mid 1 - (a/d)$, $ad + 1 - (a/d) \leqslant ad + c \leqslant (a/d) - 1$.

$$\therefore\ ad \leq \frac{2a}{d} - 2.$$

$$d^2 \leq 2 - \frac{2d}{a} < 2,$$

which is impossible since $d \geq 2$. Hence $a < 0$.

Since $cd(ad+c)\,|\,a-d$, $a - d \leq cd(ad+c) \leq d - a$.

Suppose $c < 0$. Now $acd^2 + c^2d \leq d - a$.

$$\therefore\ a(cd^2+1) \leq d(1-c^2).$$

$$\therefore\ a \geq \frac{d(1-c^2)}{cd^2+1} \geq 0,$$

contradicting the fact that $a < 0$. So $a < 0$ and $c > 0$.

Now $acd^2 + c^2d > a - d$.

$$\therefore\ a(cd^2-1) \geq -d(c^2+1).$$

$$\therefore\ a \geq -\frac{d(c^2+1)}{cd^2-1}.$$

Since $a \leq -1$,

$$\frac{c^2d+d}{cd^2-1} \geq 1.$$

$$\therefore\ c^2d + d \geq cd^2 - 1.$$

$$\therefore\ d[c(c-d)+1] \geq -1.$$

Since $d > 1$, $d[c(c-d)+1] \geq 0$, whence $c(c-d) \geq -1$. Then, since $c \neq 0$ and $(c,d) = 1$, either $c > d$ or $c = 1$ and $d = 2$. But in the latter case, the inequalities

$$-\frac{d(c^2+1)}{cd^2-1} \leq a \leq -1$$

imply that $a = -1$, contradicting the fact that $d\,|\,a$.

Now, since $cd\,|\,a-d$, $c \leq 1 - (a/d) < 1 - a$. So $0 < d < c \leq 1 - (a/d) < 1 - a$.

Suppose that $a = -d$. Then $a + cr = -a$, i.e., $cr = -2a$ and $a\,|\,r$.

CASE I: $r = -a$ and $c = 2$.

Then $ad + c = 2 - a^2$ and $ad + c\,|\,-a$. Hence either $a = -1$ or $a = -2$.

But both possibilities are inadmissible since $d = -a > 1$ and $(a,c) = 1$.

CASE II: $r = -2a$ and $c = 1$.

Then $ad + c = 1 - a^2$ and $ad + c\,|\,-2a$. But this requires that $1 - a^2$ must divide 2, since $(a, 1-a^2) = 1$, and this is not satisfied by any integer a. Hence $a \leq -2d$.

Suppose that $d > 2$. By Lemma 1, $w_3w_4\,|\,r$. It follows that $(ad+c)(a^2d + ac + cd) \geq a - d$.

$$\therefore\ a - d \leq a^3d^2 + 2a^2cd + acd^2 + ac^2 + c^2d \leq a^3d^2 + 2a^2cd + acd^2 \leq d^2a^3 + 2a^2(1-a)d + ad^3.$$

$$\therefore\ 0 < d^2a^3 + 2a^2(1-a)d + ad^3 - a + d = (d^2-2d)a^3 + 2da^2 + (d^3-1)a + d < (d^2-2d)a^3 + 2da^2$$
$$< a^3 + 2da^2 = a^2(a+2d) \leq 0,$$

a contradiction. Hence $d = 2$. Then

$$ad + c = 2a + c \quad \text{and} \quad r = \frac{2-a}{c}$$

By Lemma 1, $d(ad+c)\,|\,r$. So $4a + 2c > a - 2$.

$$\therefore\ c \geq -\frac{3}{2}a - 1 > -a - 1.$$

Hence $-a - 1 < c < -a + 1$, i.e., $c = -a$. But this contradicts the facts that $(a,c) = 1$ and $a < -1$.

Thus we have verified that there is no sequence $w_n(r,s;a,c)$ for which $r \neq 0$, $(a,r) > 1$ and $w_k\,|\,w_{2k}$ for all k.

CONCLUDING REMARKS

This theorem completes the identification of those sequences for which $w_k | w_{2k}$ for all $k \geqslant 1$; those sequences being

$$\pm \left\{ D_n(a,c) \right\} ; \qquad \pm \left\{ w_n(1,1;a,c) \right\} ,$$

where

$$a + c = 1; \qquad \pm \left\{ w_n(1,-1;a,c) \right\} ,$$

where

$$-a + c = 1; \qquad \pm \left\{ w_n(2,-1;-1,-1) \right\} \quad \text{and} \quad \pm \left\{ w_n(2,1;1,-1) \right\} .$$

These sequences, it is clear are precisely those for which $w_k | w_{mk}$ for all integers $k \geqslant 1$ and $m \geqslant 0$. In fact, an inspection of the proofs of Lemma 2 and Theorem 7 discloses that these are the only sequences for which $w_k | w_{2k}$ for $1 \leqslant k \leqslant 5$ and $\left\{ |w_k| \,\middle|\, k - 1, 2, \cdots \right\}$ is bounded.

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RECURSION RELATIONS OF PRODUCTS OF LINEAR RECURSION SEQUENCES

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Given two sequences S_i and T_i governed respectively by linear recursion relations

$$S_n = \sum_{i=1}^{p} a_i S_{n-i} \tag{1}$$

of order p and

$$T_n = \sum_{i=1}^{q} b_i T_{n-1} \tag{2}$$

of order q. Required to find the recursion relation of the term-by-term product of the two sequences $Z_i = S_i T_i$.

Initially we shall assume that the roots of the auxiliary equations corresponding to the above recursion relations are distinct so that:

$$S_n = \sum_{i=1}^{p} A_i s_i^n \,, \tag{3}$$

where s_i $(i = 1,p)$ are the roots of $x^p - a_1 x^{p-1} - a_2 x^{p-2} \cdots - a_p = 0$. Similarly,

$$T_n = \sum_{i=1}^{q} B_i t_i^n \,, \tag{4}$$

where t_i $(i = 1,q)$ are the roots of $x^q - b_1 x^{q-1} - b_2 x^{q-2} \cdots - b^q = 0$.

No universal formulation applying to all orders has been arrived at so that the results will be given as a series of algorithms applying to particular cases. The method employed is to find the products of the general terms (3) and (4) and then note the new set of roots for the recursion relation of the product. By finding the symmetric functions of these roots one can arrive at the recursion relation of the term-by-term product.

1. GEOMETRIC PROGRESSION BY ANOTHER SEQUENCE

A geometric progression is a linear recursion relation of the first order:

$$S_n = rS_{n-1}$$

whose general term can be taken as $S_n = Ar^n$. If such a term be multiplied by (4) one has:

$$S_n T_n = \sum_{i=1}^{q} B_i (rt_i)^n \tag{5}$$

Thus these terms behave as belonging to an auxiliary equation whose roots are rt_i $(i = 1,q)$. Consequently by finding the symmetric functions of these quantities one arrives at the linear recursion relation governing the terms $Z_n = S_n T_n$. It is not difficult to verify that this leads to:

(6)
$$Z_n = \sum_{i=1}^{q} B_i r^i Z_{n-i}$$

2. TWO RELATIONS OF THE SECOND ORDER

Let the auxiliary equations corresponding to two linear recursion relations of the second order be:

$$x^2 + a_1 x + b_1 = 0$$
$$x^2 + a_2 x + b_2 = 0$$

Let the terms of the sequence governed by the first relation be:

$$S_n = Ar^n + Bs^n$$

and the terms governed by the second sequence be:

$$T_n = Cu^n + Dv^n \ .$$

Then

$$Z_n = S_n T_n = AC(ru)^n + AD(rv)^n + BC(su)^n + BD(sv)^n \ .$$

The roots of the auxiliary equation for Z_n are ru, rv, su, sv. To obtain the coefficients of this equation we calculate the symmetric functions of these roots.

$$S_{4,1} = (r+s)(u+v) = (-a_1)(-a_2) = a_1 a_2$$
$$S_{4,2} = r^2uv + rsu^2 + rsuv + rsuv + rsv^2 + s^2uv = uv(r^2+s^2) + rs(u^2+v^2) + 2rsuv$$
$$= b_1(a_2^2 - 2b_2^2) + b_2(a_1^2 - 2b_1) + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$
$$S_{4,3} = r^2su^2v + r^2suv^2 + rs^2u^2v + rs^2uv^2 = rsuv(r+s)(u+v) = b_1b_2a_1a_2$$
$$S_{4,4} = r^2s^2u^2v^2 = b_1^2b_2^2 \ .$$

The recursion relation for the product of two sequences of the second order is thus

$$x^4 - a_1a_2x^3 + (b_1a_2^2 + b_2a_1^2 - 2b_1b_2)x^2 - a_1a_2b_1b_2x + b_1^2b_2^2 = 0 \ .$$

EXAMPLE. The sequence 1, 4, 17, 72, 305, ⋯ is governed by $T_{n+1} = 4T_n + T_{n-1}$ while 1, –5, 26, –135, 701, ⋯ is governed by $T_{n+1} = -5T_n + T_{n-1}$. The product sequence is 1, –20, 442, –9720, 213805, ⋯ . In terms of the above formulation, $a_1 = -4$, $b_1 = -1$, $a_2 = 5$, $b_2 = -1$. The auxiliary equation for the product sequence is given by:

$$x^4 + 20x^3 - 43x^2 + 20x + 1 = 0 .$$

$$(-9720)(-20) + 442*43 + (-20)(-20) - 1 = 213805 .$$

SECOND- AND THIRD-ORDER RECURSION RELATIONS

Given two sequences S_n, T_n governed respectively by the relations

(7)
$$x^2 + a_1x + b_1 = 0$$

(8)
$$x^3 + a_2x^2 + b_2x + c_2 = 0$$

with roots r_1, s_1 and r_2, s_2, t_2, respectively. The recursion relation of the product S_nT_n will have for roots r_1r_2, r_1s_2, r_1t_2, s_1r_2, s_1s_2, s_1t_2. The symmetric functions of these roots are as follows.

(9)
$$S_{6,1} = (r_1+s_1)(r_2+s_2+t_2) = a_1a_2$$

(10)
$$S_{6,2} = r_1s_1(s_2^2 + t_2^2 + r_2^2) + (r_1^2 + s_1^2)(r_2s_2 + r_2t_2 + s_2t_2) + 2r_1s_1(r_2s_2 + r_2t_2 + s_2t_2)$$
$$= b_1(a_2^2 - 2b_2) + (a_1^2 - 2b_1)b_2 + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$

(11)
$$S_{6,3} = (r_1^3 + s_1^3)(r_2s_2t_2) + r_1s_1(r_1+s_1)(r_2+s_2+t_2)(r_2s_2 + r_2t_2 + s_2t_2)$$
$$= (a_1^3 - 3a_1b_1)c_2 + a_1b_1a_2b_2$$

(12) $$S_{6,4} = r_1s_1(r_1^2+s_1^2)r_2s_2t_2 \quad + s_2 + t_2) + r_1^2s_1^2(r_2s_2+r_2t_2+s_2t_2)^2$$
$$= b_1a_1^2a_2c_2 + b_1^2b_2^2 - 2b_1^2a_2c_2$$

(13) $$S_{6,5} = r_1^2s_1^2(r_1+s_1)r_2s_2t_2(r_2s_2+r_2t_2+s_2t_2) = b_1^2a_1b_2c_2$$

(14) $$S_{6,6} = r_1^3s_1^3r_2^2s_2^2t_2^2 = b_1^3c_2^2$$

EXAMPLE

$x^2-4x+3 = 0$	$x^3-3x^2+6x-3 = 0$	
2	3	6
7	−21	-147
22	−60	-1320
67	−45	-3015
202	162	32724
607	576	349632
1822	621	1131462
5467	−1107	−6051969
16402	−5319	−87242238

$$a_1 = -4, \qquad b_1 = 3, \qquad a_2 = -3, \qquad b_2 = 6, \qquad c_2 = -3$$

$$S_{6,1} = 12$$

$$S_{6,2} = 3*9+6*16-2*18 = 87$$

$$S_{6,3} = (-64+36)(-3)+216 = 300$$

$$S_{6,4} = 3*16*9+9*36-2*9*9 = 594$$

$$S_{6,5} = 9(-4)*6(-3) = 648$$

$$S_{6,6} = 27*9 = 243\ .$$

The recursion relation corresponds to:

$$x^6 - x^5 - 12x^5 + 87x^4 - 300x^3 + 594x^2 - 648x + 243 = 0.$$

CHECK

12(−6051969) − 87(1131462) + 300(349632) − 594(32724) + 648(−3015) − 243(−1320) = −87242238.

TWO THIRD-ORDER RELATIONS

For two sequences governed by the relations:

$$x^3 + a_1x^2 + b_1x + c_1 = 0 \qquad \text{and} \qquad x^3 + a_2x^2 + b_2x + c_2 = 0$$

The coefficients of the recursion relation of the product are found to be:

x^9	1
x^8	$-a_1a_2$
x^7	$a_1^2b_2 + a_2^2b_1 - 2b_1b_2$
x^6	$-a_1^3c_2 - a_2^3c_1 - 3c_1c_2 + 3a_1b_1c_2 - 3a_2b_2c_1 - a_1a_2b_1b_2$
x^5	$b_1^2b_2^2 - 2a_2b_1^2c_2 - 2a_1b_2^2c_1 + a_1^2a_2b_1c_2 + a_1a_2^2b_2c_1 - a_1a_2c_1c_2$
x^4	$-a_1b_1^2b_2c_2 - a_2b_1b_2^2c_1 + 2a_1^2\ {}_2c_1c_2 + 2a_2^2b_1c_1c_2 - a_1^2a_2^2c_1c_2 + b_1b_2c_1c_2$
x^3	$b_1^3c_2^2 + b_2^3c_1^2 - 3a_1b_1c_1c_2^2 - 3a_2b_2c_1^2c_2 + 3c_1^2c_2^2 + a_1a_2b_1b_2c_1c_2$
x^2	$-a_2b_1^2c_1c_2^2 - a_1b_2^2c_1^2c_2 + 2a_1a_2c_1^2c_2^2$
x	$b_1b_2c_1^2c_2^2$
1	$-c_1^3c_2^3$

EXAMPLE $x^3 - 3x^2 + 5x - 2 = 0$ $x^3 + 4x^2 - 7x - 3 = 0$

1	0	0
−2	1	−2
−9	2	−18
−15	−1	15
−4	21	−84
45	−85	−3825
125	484	60500
142	−2468	−350456
−109	13005	−1417545
−787	−67844	53393228
−1532	355007	−543870724
−879	−1855921	1631354559
3449	9705201	33473238249

$$a_1 = -3, \quad b_1 = 5, \quad c_1 = -2, \quad a_2 = 4, \quad b_2 = -7, \quad c_2 = -3.$$

The recursion relation for the product is:

$$x^9 + 12x^8 + 87x^7 - 88x^6 + 97x^5 + 2665x^4 + 563x^3 - 828x^2 - 1260x - 216 = 0.$$

CHECK

$$-12*1631354559 - 87(-543870724) + 88*53393228 - 97(-1417545) - 2665(-350456)$$
$$- 563*60500 + 828(-3825) + 1260(-84) + 216*15 = 33473238249.$$

SECOND AND FOURTH ORDERS

Given two sequences governed by the following relations, respectively:

$$x^2 + a_1x + b_1 = 0$$

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0.$$

The coefficients of the product recursion relation are:

x^8	1
x^7	$-a_1a_2$
x^6	$a_1^2b_2 + a_2^2b_1 - 2b_1b_2$
x^5	$-a_1^3c_2 - a_1a_2b_1b_2 + 3a_1b_1c_2$
x^4	$a_1^4d_2 - 4a_1^2b_1d_2 + 2b_1^2d_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2$
x^3	$-a_1^3a_2b_1d_2 - a_1b_1^2b_2c_2 + 3a_1a_2b_1^2d_2$
x^2	$a_1^2b_1^2b_2d_2 + b_1^3c_2^2 - 2b_1^3b_2d_2$
x	$-a_1b_1^3c_2d_2$
1	$b_1^4d_2^2$

EXAMPLE $x^2 - 3x + 2 = 0$ $x^4 + 2x^3 - 3x^2 + x - 3 = 0$

10	−8	−80
22	34	748
46	−97	−4462
94	319	29986
190	−987	−187530
382	3130	1195660
766	−9831	−7530546
1534	30996	47547864
3070	−97576	−299558320
6142	307361	1887811262
12286	−967939	−11892098554

CHECK

The recursion relation of the product corresponds to:

$$x^8 + 6x^7 - 7x^6 - 27x^5 + 5x^4 - 144x^3 + 188x^2 - 72x + 144 = 0$$

$$-6(1887811262) + 7(-299558320) + 27{*}47547864 + (-5)(-7530546) + 144{*}1195660 - 188(-187530) + 72{*}29986 - 144(-4462) = -11892098554\,.$$

THIRD- AND FOURTH-ORDER SEQUENCES

For two sequences governed respectively by the relations corresponding to:

$$x^3 + a_1x^2 + b_1x + c_1 = 0$$

and

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0$$

the coefficients for the auxiliary equation of the product are given by

x^{12} $\quad 1$

x^{11} $\quad -a_1a_2$

x^{10} $\quad a_1^2b_2 + a_2^2b_1 - 2b_1b_2$

x^9 $\quad -a_1^3c_2 - 3c_1c_2 + 3a_1b_1c_2 + 3a_2b_2c_1 - a_1a_2b_1b_2 - a_2^3c_1$

x^8 $\quad a_1^4d_2 - 4a_1^2b_1d_2 - a_1a_2c_1c_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2 - 2a_1b_2^2c_1 + a_1a_2^2b_2c_1 + 2b_1^2d_2 + 4a_1c_1d_2$

x^7 $\quad -a_1^3a_2b_1d_2 - 5a_2b_1c_1d_2 + 3a_1a_2b_1^2d_2 + a_1^2a_2c_1d_2 - a_1b_1^2b_2c_2 + 2a_1^2b_2c_1c_2 + b_1b_2c_1c_2 - a_1^2a_2^2c_1c_2 + 2a_2^2b_1c_1c_2 - a_2b_1b_2^2c_1$

x^6 $\quad a_1^2b_1^2b_2d_2 - 2a_1^3b_2c_1d_2 - 2b_1^3b_2d_2 + 4a_1b_1b_2c_1d_2 - 3b_2c_1^2d_2 + a_1^3a_2^2c_1d_2 + 3a_2^2c_1^2d_2 - 3a_1a_2^2b_1c_1d_2 + b_1^3c_2^2 - 3a_1b_1c_1c_2^2 + 3c_1^2c_2^2 + b_2^3c_1^2 - 3a_2b_2c_1^2c_2 + a_1a_2b_1b_2c_1c_2$

x^5 $\quad -a_1b_1^3c_2d_2 + 3a_1^2b_1c_1c_2d_2 + b_1^2c_1c_2d_2 - 5a_1c_1^2c_2d_2 + a_1a_2b_2c_1^2d_2 - a_1^2a_2b_1b_2c_1d_2 + 2a_2b_1^2b_2c_1d_2 - a_2b_1^2c_1c_2^2 + 2a_1a_2c_1^2c_2^2 - a_1b_2^2c_1^2c_2$

x^4 $\quad b_1^4d_2^2 - 4a_1b_1^2c_1d_2^2 + 2a_1^2c_1^2d_2^2 + 4b_1c_1^2d_2^2 + a_1a_2b_1^3c_1c_2d_2 - 2a_1^2a_2c_1^2c_2d_2 - a_2b_1c_1^2c_2d_2 + a_1^2b_2^2c_1^2d_2 - 2b_1b_2^2c_1^2d_2 + b_1b_2c_1^2c_2^2$

x^3 $\quad -a_2b_1^3c_1d_2^2 + 3a_1a_2b_1c_1^2d_2^2 - 3a_2c_1^3d_2^2 + 3b_2c_1^3c_2d_2 - a_1b_1b_2c_1^2c_2d_2 - c_1^3c_2^3$

x^2 $\quad b_1^2b_2c_1^2d_2^2 - 2a_1b_2c_1^3d_2^2 + a_1c_1^3c_2^2d_2$

x $\quad -b_1c_2c_1^3d_2^2$

1 $\quad c_1^4d_2^3$

EXAMPLE

$x^3 - x^2 - x - 1 = 0$	$x^4 - x^3 - x^2 - x - 1 = 0$	PRODUCT
6	21	126
11	39	429
20	76	1520
37	147	5439
68	283	19244
125	545	68125
230	1051	241730
423	2026	856998
778	3905	3038090
1431	7527	10771137
2632	14509	38187688
4841	27967	135388247
8904	53908	479996832
16377	103911	1701750447
30122	200295	6033285990

The recursion relation for the product corresponds to the equation:

$$x^{12} - x^{11} - 4x^{10} - 12x^9 - 17x^8 - 12x^7 - 5x^6 + 10x^5 - 7x^4 - 2x^3 + 0x^2 + x - 1 = 0.$$

CHECK

$$479996832 + 4*135388247 + 12*38187688 + 17*10771137 + 12*3038090 + 5*856998$$
$$- 10*241730 + 7*68125 + 2*19244 - 1520 + 429 = 1701750447 .$$

REPEATED ROOTS

The case of [illegible] d roots can be handled in the same way but with some modifications in the procedure for finding the symr [illegible] ıctions. This discussion will be limited to the important case in which one of the sequences has a general term given by a polynomial function. The recursion relation for such a polynomial function of the n^{th} degree has its coefficients determined by the expansion of

$$(x-1)^{n+1} = 0 .$$

In other words there are $n+1$ roots all equal to unity.

QUADRATIC POLYNOMIAL SEQUENCE

The general procedure can be illustrated by the case of a sequence whose terms are given by a quadratic polynomial function. To keep the resulting formulas reasonably simple, let the other sequence be limited to order five and be in the form:

$$x^5 - a_1x^4 + a_2x^3 - a_3x^2 + a_4x - a_5 = 0$$

so that the quantities a_i are the symmetric functions of the roots. If the roots are given by r_i, the general term of the sequence would be:

$$T_n = \sum A_i r_i^n .$$

If the polynomial function is $f(n) = B_1n^2 + B_2n + B_3$, the product of the terms of the two sequences is:

$$Z_n = T_n f(n) = B_1 n^2 \sum A_i r_i^n + B_2 n \sum A_i r_i^n + B_3 \sum A_i r_i^n .$$

This shows that the equation for the product has the roots r_i taken three times. The problem then is to find the symmetric functions for three such sets of roots taken together. Suppose we wish to find $S_{15,9}$, the symmetric function of these fifteen roots taken nine at a time. The various cases can be found by taking the partitions of 9 into three or less parts, the largest being five (since this limitation was set on the order of the recursion relation). These partitions would be: 54, 531, 522, 441, 432, 333. Hence

$$S_{15,9} = 6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3$$

the coefficients being determined by the multinomial coefficient corresponding to the number of ways the various groups of roots can be selected.

For the quadratic polynomial function and linear recursion relations up to the fifth order the coefficients of the product recursion relation are as follows:

1
$-3a_1$
$3a_2 + 3a_1^2$
$-[3a_3 + 6a_1a_2 + a_1^3]$
$3a_4 + 6a_1a_3 + 3a_2^2 + 3a_2a_1^2$
$-[3a_5 + 6a_4a_1 + 6a_3a_2 + 3a_3a_1^2 + 3a_2^2a_1]$
$6a_5a_1 + 6a_4a_2 + 3a_3^2 + 3a_4a_1^2 + 6a_3a_2a_1 + a_2^3$
$-[6a_5a_2 + 6a_4a_3 + 3a_5a_1^2 + 6a_4a_2a_1 + 3a_3^2a_1 + 3a_3a_2^2]$
$6a_5a_3 + 3a_4^2 + 6a_5a_2a_1 + 6a_4a_3a_1 + 3a_4a_2^2 + 3a_3^2a_2$
$-[6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3]$
$3a_5^2 + 6a_5a_4a_1 + 6a_5a_3a_2 + 3a_4^2a_2 + 3a_4a_3^2$
$-[3a_5^2a_1 + 6a_5a_4a_2 + 3a_5a_3^2 + 3a_4^2a_3]$
$3a_5^2a_2 + 6a_5a_4a_3 + a_4^3$
$-[3a_5^2a_3 + 3a_5a_4^2]$
$3a_5^2a_4$
$-a_5^3$

EXAMPLE $x^5 - 2x^4 + x^3 + x^2 - 3x - 2 = 0 \quad f(n) = n^2$

		PRODUCT
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
11	36	396
26	49	1274
54	64	3456
94	81	7614
151	100	15100
254	121	30734
477	144	68688
939	169	158691
1788	196	350448
3224	225	725400
5660	256	1448960
10079	289	2912831
18516	324	5999184

The recursion relation of the product corresponds to the equation:

$$x^{15} - 6x^{14} + 15x^{13} - 17x^{12} - 6x^{11} + 42x^{10} - 38x^9 - 21x^8 + 69x^7 - 17x^6 - 54x^5$$
$$+ 33x^4 + 21x^3 - 42x^2 - 36x - 8 = 0.$$

CHECK

6*725400 – 15*350448 + 17*158691 + 6*68688 – 42*30734 + 38*15100 + 21*7614 – 69*3456
+ 17*1274 + 54*396 – 33*125 – 21*64 + 42*27 + 36*8 + 8*1 = 1448960.

ARITHMETIC PROGRESSION

An arithmetic progression is given by a polynomial function of the first degree so that its recursion relation corresponds to $(x - 1)^2 = 0$ with the root 1 taken twice. For a fifth order linear recursion relation such as given under the quadratic polynomial the coefficients of the equation corresponding to the linear recursion relation for the product are as follows.

$$\begin{gathered}
1 \\
-2a_1 \\
2a_2 + a_1^2 \\
-[2a_3 + 2a_2a_1] \\
2a_4 + 2a_3a_1 + a_2^2 \\
-[2a_5 + 2a_4a_1 + 2a_3a_2] \\
2a_5a_1 + 2a_4a_2 + a_3^2 \\
-[2a_5a_2 + 2a_4a_3] \\
2a_5a_3 + a_4^2 \\
-2a_5a_4 \\
a_5^2
\end{gathered}$$

POLYNOMIAL OF THE THIRD DEGREE

For a third-degree polynomial and a recursion relation up to the third order the coefficients of the equation corresponding to the linear recursion relation of the product are given by the following.

$$1$$
$$-4a_1$$
$$4a_2 + 6a_1^2$$
$$-[4a_3 + 12a_2a_1 + 4a_1^3]$$
$$12a_3a_1 + 6a_2^2 + 12a_2a_1^2 + a_1^4$$
$$-[12a_3a_2 + 12a_3a_1^2 + 12a_2^2a_1 + 4a_2a_1^3]$$
$$6a_3^2 + 24a_3a_2a_1 + 4a_3a_1^3 + 4a_2^3 + 6a_2^2a_1^2$$
$$-[12a_3^2a_1 + 12a_3a_2^2 + 12a_3a_2a_1^2 + 4a_2^3a_1]$$
$$12a_3^2a_2 + 6a_3^2a_1^2 + 12a_3a_2^2a_1 + a_2^4$$
$$-[4a_3^3 + 12a_3^2a_2a_1 + 4a_3a_2^3]$$
$$4a_3^3a_1 + 6a_3^2a_2^2$$
$$-4a_3^3a_2$$
$$a_3^4$$

EXAMPLE

$$x^3 - 3x^2 + x - 2 = 0 \quad \text{and} \quad f(n) = n^3$$

1	1	1
2	8	16
3	27	81
9	64	576
28	125	3500
81	216	17496
233	343	79919
674	512	345088
1951	729	1422279
5645	1000	5645000
16332	1331	21737892
47253	1728	81653184
136717	2197	300367249
395562	2744	1085422128
1144475	3375	3862603125

The recursion relation of the product corresponds to the relation:

$$x^{12} - 12x^{11} + 58x^{10} - 152x^9 + 267x^8 - 384x^7 + 442x^6 - 396x^5 + 337x^4 - 184x^3 + 120x^2 - 32x + 16 = 0.$$

CHECK

12*81653184 − 58*21737892 + 152*5645000 − 267*1422279 + 384*345088 − 442*79919 + 396*17496 − 337*3500 + 184*576 − 120*81 + 32*16 − 16 = 300367249

REPEATED ROOTS IN GENERAL

Given a sequence whose recursion relation has p repeated roots and another whose recursion relation has q repeated roots. We would have:

$$S_n = r^n(a_0 + a_1n + \cdots + a_{p-1}n^{p-1})$$
$$T_n = s^n(b_0 + b_1n + \cdots + b_{q-1}n^{q-1})$$
$$Z_n = S_nT_n = (rs)^n(c_0 + c_1n + \cdots + c_{p+q-2}n^{p+q-2})$$

so that the recursion relation of the product is of order $p + q - 1$.

If the first recursion relation has m distinct roots r_i and a repeated root r of multiplicity p, while the second has n distinct roots s_j and a repeated root of multiplicity q, the product recursion relation has the following number of roots: $mn + pn + qm + p + q - 1 = (m + p)(n + q) - (p - 1)(q - 1)$. The symmetric functions of these roots will give the coefficients of a recursion relation of this order. Similar considerations can be applied when there are a number of repeated roots of various multiplicities.

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TREBLY-MAGIC SYSTEMS IN A LATIN 3-CUBE OF ORDER EIGHT

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The Tarry-Escott problem requires that for each positive integer t the least integer $N(t)$ be found such that there exist two distinct sets of integers $\left\{a_i\right\}, \left\{b_i\right\}, i = 1 \cdots N(t)$ such that $a_i^m = b_i^m$ for $m = 1 \cdots t$. It is easily shown that for each t, $N(t) \geqslant t + 1$ and that for small values of t equality holds. For example $N(2) = 3$ since the sets $\left\{1,8,9\right\}$ and $\left\{3,4,11\right\}$ satisfy the equations $1 + 8 + 9 = 3 + 4 + 11$ and $1^2 + 8^2 + 9^2 = 3^2 + 4^2 + 11^2$. A complete solution to the problem is unknown.

We call a system $L = \left\{S_i\right\}_{i=1}^{n}$ of sets of integers t-magic if the numbers

$$\sum_{s \in S_t} s^m$$

are independent of the choice of S_i for $m = 1 \cdots t$. Thus a solution to the Tarry-Escott problem is a t-magic system of two sets of cardinality $N(t)$.

It has been shown [1] that for appropriate choices of n and k, orthogonal systems of magic Latin k-cubes of order n can be constructed. In this paper we exhibit a Latin 3-cube of order 8 in which are embedded subcubes possessing hypermagic properties.

The cube (Fig. 1) comprises 8^3 ordered triples with entries 0,1,2,3,4,5,6,7. It is orthogonal, viz., each of the triples from 000 to 777 appears exactly once. In the diagram we show the cube as a set of eight squares which are to be placed one above the other to form the complete 3-dimensional array. After each of the entries is attached one of the letters *a, b, c, d.* Each of the rows in each square is labeled with one of the symbols R_{00}, R_{01}, R_{11}, R_{20}, R_{21}, R_{30}, R_{31} and each of the columns is labeled with one of $K_{00}, K_{01}, \cdots, K_{31}$. Thus the totality of entries R_{ij} represents a set of rows parallel to one of the horizontal edges of the cube. A similar statement can be made about all entries labeled K_{ij}.

The two subcubes that we consider are designated as A and B. They are constructed as follows. Cube A is obtained by deleting the second entry in each cell of the original cube and regarding the remaining pair as a two-digit number in base eight. So that each of the first 64 positive integers may appear in each subsquare of the cube we add 1 to each of the two-digit numbers. Thus the first row of the first square of cube A is: *20a 33b 76c 51d 44a 67b 22c 05d* R_{00}. Cube B is constructed exactly the same way, deleting the first entry in each cell. For convenience in computation we convert the entries to base ten.

We denote by A_k the k^{th} (horizontal) square of cube A and by B_k the k^{th} square of cube B. Then a_{ijk} is the entry in the i^{th} row, k^{th} column of A_k and b_{ijk} the corresponding entry in B_k.

We now observe that for fixed j,k

$$\sum_i a_{ijk} = \sum_i b_{ijk} = 260$$

	K_{00}	K_{01}	K_{10}	K_{11}	K_{20}	K_{21}	K_{30}	K_{31}	
1	117a	332b	775c	550d	443a	666b	221c	004d	R_{00}
2	532a	717b	350c	175d	066a	243b	604c	421d	R_{10}
3	050b	275a	632d	417c	504b	721a	366d	143c	R_{11}
4	475b	650a	217d	032c	121b	304a	743d	566c	R_{30}
5	643c	466d	021a	204b	317c	132d	575a	750d	R_{20}
6	266c	043d	404a	621b	732c	517d	150a	375b	R_{01}
7	704d	521c	166b	343a	250d	075c	432b	617a	R_{31}
8	321d	104c	543b	766a	675d	450c	017b	232a	R_{21}

Square 1

	K_{00}	K_{01}	K_{10}	K_{11}	K_{20}	K_{21}	K_{30}	K_{31}	
1	425b	600a	247d	062c	171b	354a	713d	536c	R_{30}
2	000b	225a	662d	447c	554b	771a	336d	113c	R_{10}
3	562a	747b	300c	125d	036a	213b	654c	471d	R_{11}
4	147a	362b	725c	500d	413a	636b	271c	054d	R_{00}
5	371d	154c	513b	736a	625d	400c	047b	262a	R_{20}
6	754d	571c	136b	313a	200d	025c	462b	647a	R_{31}
7	236c	031d	454a	671b	762c	547d	100a	325b	R_{01}
8	613c	436d	071a	254b	347c	162d	525a	700b	R_{21}

Square 2

1	361d	144c	503b	726a	635d	410c	057b	272a	R_{20}
2	744d	561c	126b	303a	210d	035c	472b	657a	R_{30}
3	226c	003d	444a	661b	772c	557d	110a	335b	R_{00}
4	603c	426d	061a	244b	357c	172d	535a	710b	R_{21}
5	435b	610a	257d	072c	161b	344a	703d	526c	R_{31}
6	010b	235a	672d	457c	544b	761a	326d	103c	R_{10}
7	572a	757b	310c	135d	026a	203b	644c	461d	R_{11}
8	157a	372b	735c	510d	403a	626b	261c	044d	R_{01}

Square 3

1	653c	476d	031a	214b	307c	122d	565a	740b	R_{20}
2	276c	053d	414a	631b	722c	507d	140a	365b	R_{00}
3	714d	531c	176b	353a	240d	065c	422b	670a	R_{30}
4	331d	114c	553b	776a	665d	440c	007b	222a	R_{21}
5	107a	322b	765c	540d	453a	676b	231c	014d	R_{01}
6	522a	707b	340c	165d	076a	253b	614c	431d	R_{10}
7	040b	265a	622d	407c	514b	731a	376d	153c	R_{11}
8	465b	640a	207d	022c	131b	314a	753d	576c	R_{31}

Square 4

1	206c	023d	464a	641b	752c	577d	130a	315b	R_{00}
2	623c	406d	041a	264b	377c	152d	515a	730b	R_{20}
3	341d	164c	523b	706a	615d	430c	077b	252a	R_{21}
4	764a	541c	106b	323a	230d	015c	452b	677a	R_{30}
5	522a	777b	330c	115d	006a	223b	664c	441d	R_{10}
6	177a	352b	715c	530d	423a	606b	241c	064d	R_{01}
7	415b	630a	277d	052c	141b	364a	723d	506c	R_{31}
8	030b	215a	652d	477c	564b	741a	306d	123c	R_{11}

Square 5

1	734d	511c	156b	373a	260d	045c	402b	627a	R_{30}
2	311d	134c	573b	756a	645d	460c	027b	202a	R_{20}
3	673c	456d	011a	234b	327c	102d	545a	760b	R_{21}
4	256c	073d	434a	611b	702c	527d	160a	345b	R_{00}
5	060b	245a	602d	427c	534b	711a	356d	173c	R_{10}
6	445b	660a	227d	002c	111b	334a	773d	556c	R_{31}
7	127a	302b	745c	560d	473a	656b	211c	034d	R_{01}
8	502a	727b	360c	145d	056a	273b	634c	411d	R_{11}

Square 6

1	070b	255a	612d	437c	524b	701a	346d	163c	R_{10}
2	455b	670a	237d	012c	101b	324a	763d	546c	R_{30}
3	137a	312b	755c	570d	463a	646b	201c	024d	R_{00}
4	512a	737b	370c	155d	046a	263b	624c	401d	R_{11}
5	724d	501c	146b	363a	270d	055c	412b	637a	R_{31}
6	301d	124c	563b	746a	655d	470c	037b	212a	R_{20}
7	663c	446d	001a	224b	337c	112d	555a	770b	R_{21}
8	246c	063d	424a	601b	712c	537d	170a	355b	R_{01}

Square 7

1	542a	767b	320c	105d	016a	233b	674c	451d	R_{10}
2	167a	342b	705c	520d	433a	616b	251c	074d	R_{00}
3	405b	620a	267d	042c	151b	374a	733d	516c	R_{30}
4	020b	205a	642d	467c	574b	751a	316d	133c	R_{11}
5	216c	033d	474a	651b	742c	537d	120a	305b	R_{01}
6	633c	416d	051a	274b	367c	142d	505a	720b	R_{20}
7	351d	174c	533b	716a	605d	420c	067b	242a	R_{21}
8	774d	551c	116b	333a	220d	005c	442b	667a	R_{31}

Square 8

Figure 1

and for fixed i,k

$$\sum_j a_{ijk} = \sum_j b_{ijk} = 260.$$

Similarly

$$\sum_i a_{ijk}^2 = \sum_i b_{ijk}^2 = \sum_j a_{ijk}^2 = \sum_j b_{ijk}^2 = 11180.$$

Thus in a natural way, we have exhibited a system of 256 sets of eight integers that is 2-magic.

We now define a system of 196 sets of 16 integers that is 3-magic. This system has the pleasant property that it includes the principal diagonals as well as the rows and columns of cubes A and B.

Let A_{ka} be the set of 16 numbers in A_k that are followed by the letter *a*. Let $A_{kb}, A_{kc}, A_{kd}, B_{ka}, B_{kb}, B_{kc}, B_{kd}$ be similarly defined. (This defines 64 sets.)

Let AR_{ki} (resp. B_{ki}) be the set of 16 numbers in A_k (resp. B_k) that lie in rows R_{i0} or R_{i1}, $i = 0, 1, 2, 3$. (This defines 64 sets.) Let AK_{ki} (resp. B_{ki}) be the set of 16 numbers in A_k (resp. B_k) that lie in columns K_{i0} or K_{i1}, $i = 0, 1, 2, 3$. (This defines 64 sets.)

Let AD_a (resp. BD_a) be the set of numbers in the two main diagonals of cube A (resp. B) of the form a_{iii} or $a_{8-i,8-i,i}$ (resp. b_{iii}, $b_{8-i,8-i,i}$). It will be observed that each of these entries is labeled by the letter *a*. Similarly let AD_d (resp. BD_d) be the set in the other two main diagonals

$$\{a_{i,8-i,i}\}, \quad \{a_{8-i,i,i}\}, \quad \{b_{i,8-i,i}\}, \quad \{b_{8-i,i,i}\}.$$

(This defines 4 sets.)

Now let L be the system of 196 sets defined above. It can be verified that L is a 3-magic system. Explicitly, if $S \in L$ then

$$\sum_{s\in S} s = 520, \qquad \sum_{s\in S} s^2 = 22360 \quad \text{and} \quad \sum_{s\in S} s^3 = 1081600.$$

We remark in conclusion that we have by no means exhausted the hypermagic systems that can be extracted from the cubes. To this end we append the following constructions.

HYPERMAGIC CONSTRUCTIONS

In what follows, when it is mentioned that sets of numbers (in this case each set contains 16 two-digit numbers) are equal in sum, this will mean that they have the same sum of k^{th} powers for $k = 1, 2$ and 3.

We also point out that each row in every one of the eight squares has two numbers that end in *a*, two numbers that end in *b*, two numbers that end in *c*, and two numbers that end in *d*.

I	1	2	3	4	5	6	7	8	
II	2	1	4	3	6	5	8	7	
III	2	7	4	5	6	3	8	1	
IV	3	4	1	2	7	8	5	6	
V	4	5	2	7	8	1	6	3	
VI	4	3	2	1	8	7	6	5	
VII	5	6	7	8	1	2	3	4	
VIII	6	3	8	1	2	7	4	5	
IX	6	5	8	7	2	1	4	3	
X	7	8	5	6	3	4	1	2	
XI	8	7	6	5	4	3	2	1	
XII	8	1	6	3	4	5	2	7	
									Z
	1	2	3	4	5	6	7	8	= Square Number

Figure 2 Chart

How to Read the Figure 2 Chart

The numbers on the bottom of the Chart (below line *Z*) each denotes the number of some square in the cube. The number in the column above the number denoting a square denotes a row number (counting from top to bottom) in the particular square listed on the bottom of the column. For example: Cell (VII,6) = 2 denotes the 8 numbers on row 2 to Square 6. *Each of the 6 numbers on a row in the Chart represents a magic system.* For example: We write the numbers on row VII to get row 5 in square 1, row 6 in square 2, ··· row 4 in square 8. We now arrange the (resulting) 64 3-digit numbers so that the 16 numbers that end in *a* are in (say) column 1, the 16 numbers that end in *b* are in column 2, and the 16 numbers that end in *c* are in column 3, and the 16 numbers that end in *d* are (say) in column 4.

We first consider the first and third digit of each and every number in the 4 columns (that is cube A) and after adding 1 to each pair of digits we express the 64 2-digit numbers in the scale of 10.

We now add (in cube A) the 16 numbers in column 1 to get the sum *s*1,
" " " " " " " " " " 2 " " " " *s*2,
" " " " " " " " " " 3 " " " " *s*3,
" " " " " " " " " " 4 " " " " *s*4.

Then for the sum of the k^{th} powers (for k = 1, 2 and 3) we have $s1 \overset{3}{=} s2 \overset{3}{=} s3 \overset{3}{=} s4$ (in cube A).

The exact relationship between the numbers in cube A also holds true for cube B (in the 2nd and 3rd digits).

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* * *

SOME FACTORABLE DETERMINANTS

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A number of computer programs for evaluating determinants of large order are available, however, these programs are quite cumbersome if the determinants are non-symmetric and their order is large. It is rather difficult to test out these computer programs on account of the presence of round-off errors. In many situations, where a researcher is more interested in error assessment, the problem becomes exasperating.

To ease this problem Bowman and Shenton [1] have recently quoted a non-symmetric determinant of order $(s + 1)$, given by Painvin [2], which is factorable and have used an ingenious method to show that two other determinants can be reduced to the s^{th} power of a number n, which occurs in the determinant. Since there is only one number n, in each of the determinants, which can be changed arbitrarily the use of these results becomes highly restricted.

We quote below more general forms, containing two arbitrary numbers n and s, of these two factorable determinants. Their proofs are not being given as they are exactly similar to the one given by Bowman and Shenton.

$$(1)\qquad \begin{vmatrix} n-2as & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\ a(2s-1) & n-2a(s-1) & \cdots & \cdot & 0 & 0 & \binom{s}{1}(-2)^{s-1} \\ a(1-s)/2 & a(2s-3) & \cdots & \cdot & \cdot & \cdot & \binom{s}{2}(-2)^{s-2} \\ 0 & a(2-s)/2 & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & n-6a & 0 & 0 & \cdot \\ \cdot & \cdot & \cdots & 5a & n-4a & 0 & \cdot \\ \cdot & \cdot & \cdots & -2a/2 & 3a & n-2a & \binom{s}{s-1}(-2)^1 \\ 0 & 0 & \cdots & 0 & -a/2 & a & \binom{s}{s}(-2)^0 \end{vmatrix} = n^s$$

It may be noted that there is no "a" in the last column. Each value of a, positive or negative or zero, gives a different determinant, however, the value of the determinant remains unaltered by a and is equal to n^s.

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*This work was done by the author while he was spending part of his sabbatical leave at the Computer Center, University of Georgia, Athens, U.S.A., whose help is gratefully acknowledged.

$$(2)\quad \begin{vmatrix} n-2as & 0 & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\ 2a(s-1) & n-2a(s-1) & 0 & \cdots & \cdot & \cdot & \cdot & \binom{s}{1}(-2)^{s-1} \\ a(1-s)/2 & 2a(s-2) & n-2a(s-2) & \cdots & \cdot & \cdot & \cdot & \binom{s}{2}(-2)^{s-2} \\ 0 & a(2-s)/2 & 2a(s-3) & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & a(3-s)/2 & \cdots & \cdot\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdots & \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & n-6a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 4a & n-4a & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdots & -2a/2 & 2a & n-2a & \cdot \\ 0 & 0 & 0 & \cdots & 0 & -a/2 & 0 & \binom{s}{s}(-2)^0 \end{vmatrix} = (n-2a)^s$$

The value of the above factorable determinant depends upon the value of *a*. When *n* is replaced by $n+2$ and $a=1$, the above result becomes identical with Bowman and Shenton's result.

We also give here a more general form of Painvin's factorable determinant. For all values of *n* and *a*, taking either the upper sign or the lower sign at all places, the value of the determinant is $(n+as/2)^{s+1}$.

$$(3)\quad \begin{vmatrix} n & \pm a/2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ +as/2 & n+a & \pm 2a/2 & 0 & \cdots & \cdot & \cdot & 0 \\ 0 & \mp a(s-1)/2 & n+2a & \pm 3a/2 & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \mp a(s-2)/2 & n+3a & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \mp a(s-3)/2 & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & n+(s-2)a & \pm(s-1)a/2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \mp 2a/2 & n+(s-1)a & \pm sa/2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mp a/2 & n+sa \end{vmatrix} = \left(n+\frac{as}{2}\right)^{s+1}$$

Evidently, when $a=-1$, and we take the lower sign, the above reduces to Painvin's result.

Proof. Let *r* denote the number of the row. If the respective rows are multiplied by $(-1)^{r-1}$, $r=1,2,\cdots,s+1$ and added into the first row, then $(n+as/2)$ comes out as a common factor leaving $1,-1,\cdots,(-1)^{s-1}$ as the elements. The order of the determinant can be now reduced by unity by multiplying the new first row by $(\mp as/2)$ and subtracting it from the second row.

In the second operation the respective rows are multiplied by $(-1)^{r-1}\binom{r}{1}$, $r=1,2,\cdots,s$ and added to the first row to give another $(n+as/2)$ as a common factor. The order of the determinant can again be reduced by unity by multiplying the new first row by $\mp a(s-1)/2$ and subtracting it from the second row.

In the third operation the respective rows are multiplied by $(-1)^{r-1}\binom{r+1}{2}$, $r=1,2,\cdots,s-1$ and added together to give another factor $(n+as)/2)$ and then reduction of the order follows the above procedure.

Repeating these operations $(s-4)$ times more, one can easily find that the given determinant reduces to

$$(n+as/2)^{s-1}\begin{vmatrix} n & \pm s^2a/2 \\ \pm a/2 & n+sa \end{vmatrix}$$

which gives our result.

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NOTE: The author offers a reward of $25 for non-trivial generalizations of the three results in (1), (2) and (3).

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FIBONACCI TRIANGLE

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1. DEFINITION

The Fibonacci sequence $\{f_n\}$ is defined by the recursive relation

$$f_n = f_{n-1} + f_{n-2} \tag{1.1}$$

with

$$f_0 = f_1 = 1.^* \tag{1.2}$$

Let us define the set of integers $\{f_{m,n}\}$ with two suffices

$$f_{m,n} = f_{m-1,n} + f_{m-2,n} \tag{1.3}$$

$$f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2} \tag{1.4}$$

$$(m \geqslant 2,\ m \geqslant n \geqslant 0)$$

with

$$f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1. \tag{1.5}$$

These numbers can be arranged triangularly as in Fig. 1,

									1									
								1		1								
							2		1		2							
						3		2		2		3						
					5		3		4		3		5					
				8		5		6		6		5		8				
			13		8		10		9		10		8		13			
		21		13		16		15		15		16		13		21		
	34		21		26		24		25		24		26		21		34	
55		34		42		39		40		40		39		42		34		55

Figure 1. Fibonacci Triangle

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*Another set of the initial values can be chosen as

$$f_0 = 0, \qquad f_1 = 1. \tag{1.2'}$$

where the entries in a row have the same m and line up according to the value of n $(0\text{-}m)$ from left to right.

Let us call them the Fibonacci Triangle.* A number of interesting relations were found, part of which will be given in this paper.

2. RELATION WITH THE FIBONACCI NUMBERS

As evident from the definitions (1.3)–(1.5) we have four Fibonacci sequences in the Triangle;

(2.1) $$f_{m,0} = f_{m,m} = f_m$$

(2.2) $$f_{m,1} = f_{m,m-1} = f_{m-1}\,.$$

Successive application of (1.3) to itself gives the following relation

(2.3) $$f_{m,n} = f_k \cdot f_{m-k,n} + f_{k-1} \cdot f_{m-k-1,n} \qquad (1 \leqslant k \leqslant m-n-1).$$

By putting $k = m-n-1$ into (2.3) one gets

(2.4) $$f_{m,n} = f_{m-n-1} \cdot f_{n+1,n} + f_{m-n-2} \cdot f_{n,n}$$

(from (2.1) and (2.2))

$$= f_n (f_{m-n-1} + f_{m-n-2}).$$

It follows then that

(2.5) $$f_{m,n} = f_{m-n} \cdot f_n \qquad (m \geqslant n \geqslant 0).$$

It means that the Fibonacci Triangle is constructed by the self-multiplication of the Fibonacci sequence, or symbolically,

(2.6) $$\{f_{m,n}\} = \{f_m\} \times \{f_n\}.$$

In other words the Fibonacci Triangle is the 2-dimensional Fibonacci sequence. Then extension to the k-dimensional Fibonacci sequence

(2.7) $$\{f_{m_1,m_2,\cdots,m_k}\} = \{f_{m_1}\} \times \{f_{m_2}\} \times \cdots \times \{f_{m_k}\}$$

is straightforward, but we will not duscuss them further.

It is proved from (2.5) or (2.6) that the Fibonacci sequences multiplied by the Fibonacci numbers are seen in the Triangle alongside of the "roof." That the Triangle is symmetric,

(2.8) $$f_{m,n} = f_{m,m-n}\,,$$

is directly proved from (2.5). On the center line of the Triangle the squares of $\{f_n\}$ are lined up,

(2.9) $$f_{m,m/2} = (f_{m/2})^2 \qquad (m = \text{even}).$$

Application of the Binet's formula

(2.10) $$f_n = (\alpha^{n+1} - \beta^{n+1})/\sqrt{5}$$
$$\alpha = (1+\sqrt{5})/2, \qquad \beta = (1-\sqrt{5})/2$$

to (2.5) gives

(2.11) $$f_{m,n} = \{(\alpha^{m+2} + \beta^{m+2}) + (-1)^n(\alpha^{m-2n} + \beta^{m-2n})\}/5.$$

The Lucas sequence $\{g_n\}$, which is defined by

(2.12) $$g_n = g_{n-1} + g_{n-2}$$

with

(2.13) $$g_0 = 2, \qquad g_1 = 1,$$

is expressed as

(2.14) $$g_n = \alpha^n + \beta^n.$$

Thus one gets

---

*The author noticed that the term "Fibonacci Triangle" is used for quite a different array of integers. [1].

(2.15) $$f_{m,n} = \left\{ g_{m+2} + (-1)^n g_{m-2n} \right\} /5$$

(2.16) $$= \left\{ g_{m+2} + (-1)^{m-n} q_{2n-m} \right\} /5.$$

In deriving (2.16) the following relation

(2.17) $$g_{-n} = (-1)^n g_n$$

was used.

3. MAGIC DIAMOND

As all the entries in the Triangle are generated from four one's (1.5) forming a diamond, any quartet $(f_{m,n}, f_{m-1,n}, f_{m-1,n-1}, f_{m-2,n-1})$ in the Triangle generates the nearest neighbors to the four corners and will be called as a "magic diamond."

Application of (1.3) into (1.4) gives "downward generation"

(3.1) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} + f_{m-2,n-1} = f_{m+2,n+1}\ ;$$

as illustrated in Fig. 2a. Similarly one gets "upward generation"

(3.2) $$f_{m,n} - f_{m-1,n} - f_{m-1,n-1} + f_{m-2,n-1} = f_{m-4,n-2}$$

as in Fig. 2b.

"Leftward generation" and "rightward generation" (Figs. 2c,d) are obtained respectively as

(3.3) $$f_{m,n} + f_{m-1,n} - f_{m-1,n-1} - f_{m-2,n-1} = f_{m-1,n-2}$$

and

(3.4) $$f_{m,n} - f_{m-1,n} + f_{m-1,n-1} - f_{m-2,n-1} = f_{m-1,n+1}\,.$$

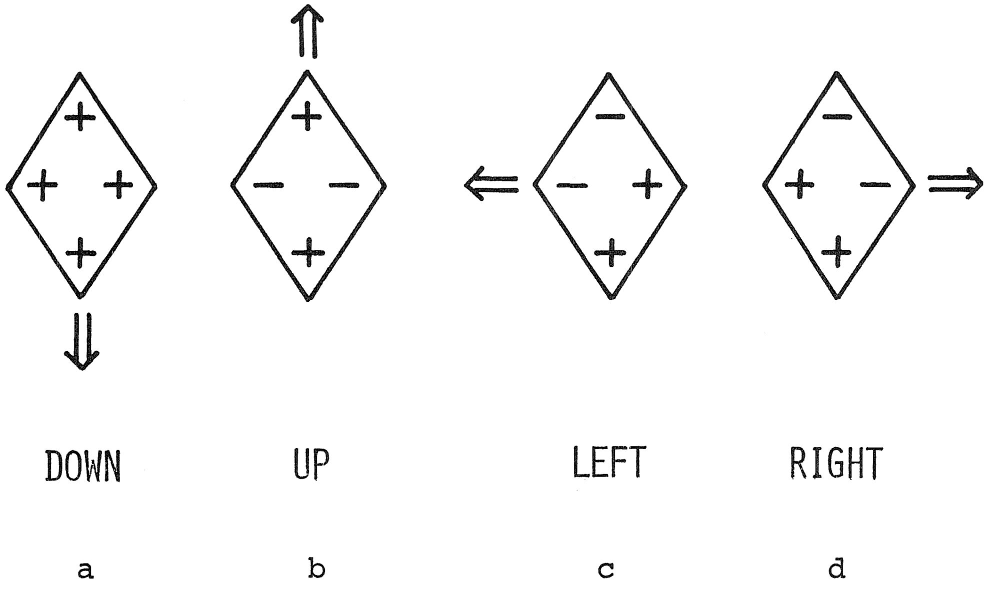

Figure 2. Magic Diamond

From (2.5) one gets

(3.5) $$f_{m,n} \cdot f_{m-2,n-1} = f_{m-1,n-1} \cdot f_{m-1,n}$$

or

(3.6) $$f_{m,n} \div f_{m-1,n} \div f_{m-1,n-1} \times f_{m-2,n-1} = 1,$$

which shows the stability of the "magic diamond" (see Fig. 3).

It is verified from (2.5) that the four numbers at the corners in any parallelogram are stable like an "Amoeba."

(3.7) $$f_{m,n} \cdot f_{m-k-\ell,n-k} = f_{m-k,n-k} \cdot f_{m-\ell,n} \,.$$

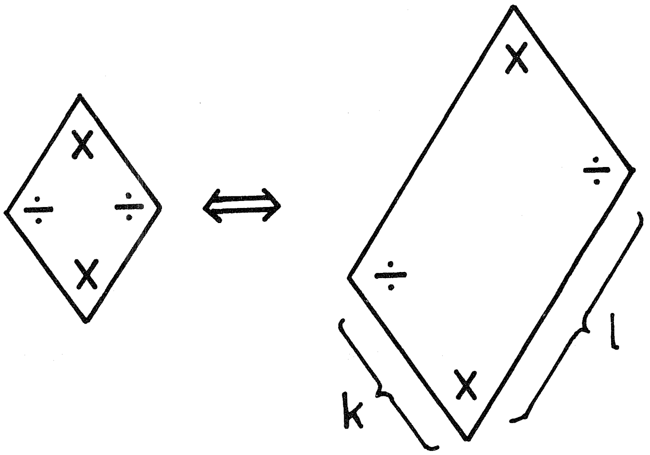

Figure 3 Amoeba

4. CRAWLING CRAB

The sum of the three entries in any downward triangle $(f_{m,n}, f_{m-1,n}, f_{m-1,n-1})$ or a "Crab" is kept constant as long as the Crab crawls sideways (see Fig. 4a),

(4.1) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m,\ell} + f_{m-1,\ell} + f_{m-1,\ell-1} \qquad (m-1 \geqslant n,\ \ell \geqslant 1).$$

Proof. From (1.3) and (1.4) one gets

(4.2) $$f_{m+1,n} = f_{m,n} + f_{m-1,n} = f_{m,n-1} + f_{m-1,n-2}$$

and

(4.3) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m,n-1} + f_{m-1,n-2} + f_{m-1,n-1} \,.$$

This relation is transmitted along a given row m and yields (4.1).

It is easy to derive from (4.1) the following relation

(4.4) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m+1} \cdot$$

Application of (1.3) to (4.4) gives

(4.5) $$f_{m+1,n} + f_{m-1,n-1} = f_{m+1} \,.$$

Combination of (4.4) and (4.5) with proper shift of suffices one gets the transmission property pertinent to an upward triangle (Fig. 4b),

(4.6) $$f_{m,n} + f_{m,n-1} - f_{m-1,n-1} = f_m \,.$$

5. ROLLING DUMBBELL

The relation (4.5) means that the sum of any two vertical neighbors in the Fibonacci Triangle is kept constant for a horizontal movement. Add up the both sides of the two equations derived from (4.5) by substituting $(m = m + 2, n = n + 1)$ and $(m = m - 2, n = n - 1)$, subtract (4.5) from the sum, and one gets

(5.1) $$f_{m+3,n+1} + f_{m-3,n-2} = f_{m+2} + f_{m-1} = 2f_{m+1} \,.$$

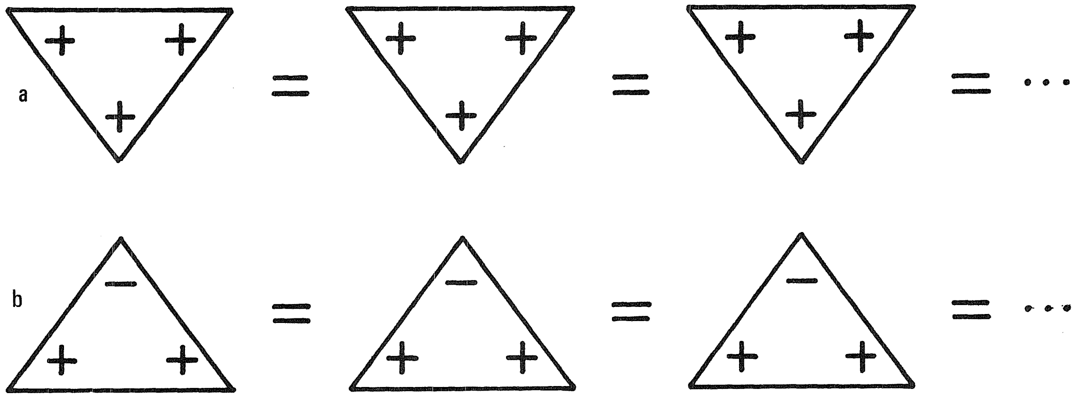

Figure 4. Crawling Crab

More generally one gets the "Rolling Dumbbell" relation (Fig. 5a)

$$f_{m+2k+1,n+k} + f_{m-2k-1,n-k-1} = f_{2k}\cdot f_{m+1} = f_{m+2k+1,m+1} \tag{5.2}$$

(from (2.5)).

By putting $m = m - 2$ and $n = n - 1$ into (4.5) one gets

$$f_{m-1,n-1} + f_{m-3,n-2} = f_{m-1}\,. \tag{5.3}$$

Subtraction of (5.3) from (4.5) followed by substitution $m = m + 1$ and $n = n + 1$ gives

$$f_{m+2,n+1} - f_{m-2,n-1} = f_{m+1}\,. \tag{5.4}$$

This is extended to the expression

$$f_{m+2k,n+k} - f_{m-2k,n-k} = f_{2k-1}\cdot f_{m+1} = f_{m+2k,m+1}\,, \tag{5.5}$$

which is illustrated in Fig. 5b.

6. RELATION WITH THE TOPOLOGICAL INDEX

The present author has defined the topological index Z_G for characterizing the topological nature of a non-directed graph G [2]. A non-adjacent number $p(G,k)$ for graph G is defined as the number of ways in which k disconnected lines are chosen from G; $p(G,0)$ being defined as unity for all the cases. The topological index Z_G is the sum of the $p(G,k)$ numbers.

It is shown [2] that the non-adjacent numbers and the topological index of a path progression S_N* are given by

$$p(S_N, k) = \binom{N-k}{k} \tag{6.1}$$

$$Z_{S_N} = \sum_{k=0}^{[N/2]} \binom{N-k}{k} = f_N\,. \tag{6.2}$$

The topological index of a series of path progressions recurses,

$$Z_{S_N} = Z_{S_{N-1}} + Z_{S_{N-2}}\,, \tag{6.3}$$

as the Fibonacci sequence (1.1). This is a special case of the composition principle (a recursion formula) of Z_G,

$$Z_G = Z_{G-\ell} + Z_{G\ominus\ell} \tag{6.4}$$

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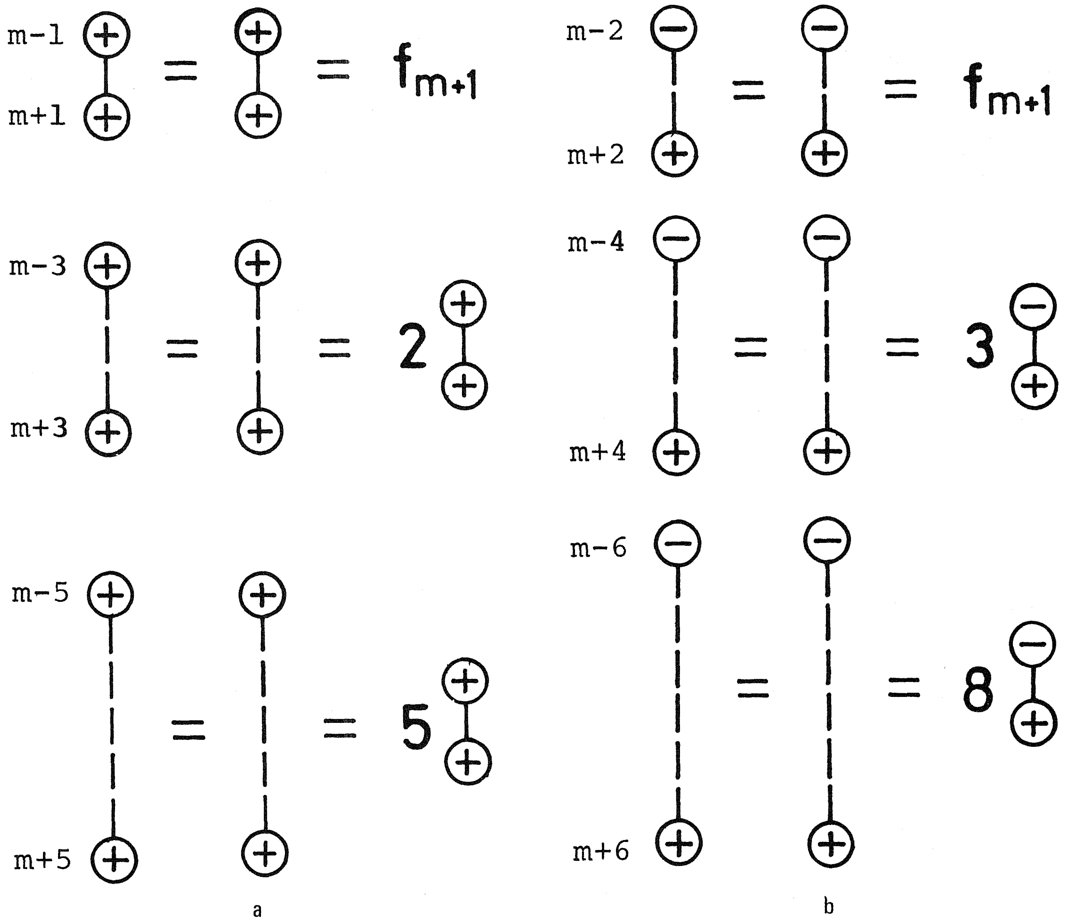
*A path progression S_N is a graph composed of linearly connected N points. A point is S_1 and a line joining a pair of given points is S_2 and so on.



a b

Figure 5. Rolling Dumbbell

where $G - \ell$ is a subgraph of G derived from G by deleting line ℓ, and $G \ominus \ell$ is further derived from $G - \ell$ by deleting all the lines which were adjacent to line ℓ in G.

In Fig. 6 a graphical equivalent of the Fibonacci Triangle is given, where the "roofs" are omitted owing to their redundancy. Note, however, that in this case (1.3)–(1.5) should read

$$(1.3') \qquad f_{m,n} = f_{m-1,n} + f_{m-2,n}$$

$$(1.4') \qquad f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2} \qquad (m \geqslant 3,\ m-1 \geqslant n \geqslant 1)$$

$$(1.5') \qquad f_{1,1} = 1, \qquad f_{2,1} = f_{1,2} = 2, \qquad f_{3,2} = 4.$$

Except for the difference in the boundary conditions all the relations pertinent to the Fibonacci Triangle hold for the topological indices of the trangle array of the graphs in Fig. 6.

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2. H. Hosoya, "Topological Index and Fibonacci Numbers with Relation to Chemistry," *The Fibonacci Quarterly,* Vol. 11, No. 3 (Oct. 1973), pp. 255–266.

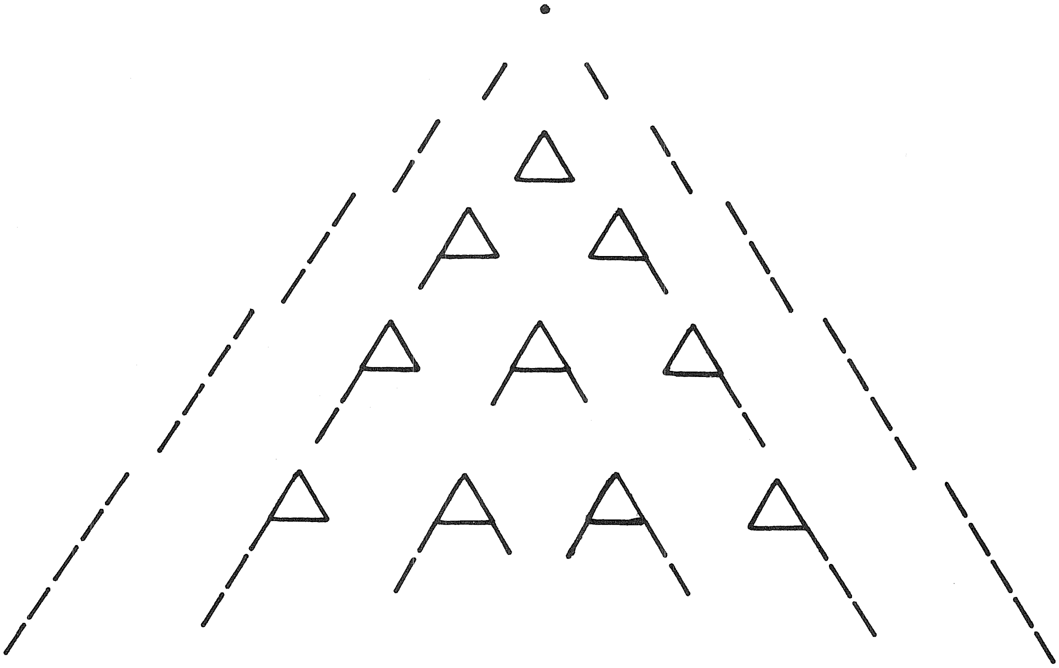

Figure 6. Graphical Equivalent of Fibonacci Triangle

ACKNOWLEDGEMENT

The author would like to thank the editor, Dr. V. E. Hoggatt, Jr., for his suggestion of the name "Amoeba" for Figure 3.

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A CONJECTURE RELATING QUARTIC RECIPROCITY AND QUARTIC RESIDUACITY TO PRIMITIVE PYTHAGOREAN TRIPLES

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CONJECTURE

(a) If

$$p = a^2 + b^2 \equiv 1 \pmod 4$$

is prime, $q \equiv 1 \pmod 8$ is prime with $(p/q) = 1$, and (x,y,z) is a primitive Pythagorean triple, then either $a^2 \equiv x^2$ with $b^2 \equiv y^2 \pmod q$ for some (x,y) and $a^2 \equiv -x^2$ with $b^2 \equiv -y^2 \pmod q$ for other (x,y) or $a^2 \not\equiv \pm x^2$ with $b^2 \equiv \pm y^2 \pmod q$ for any (x,y);

$$(\sqrt{p}/q)(\sqrt{q}/p) = 1$$

if and only if the first alternative is true, in which case

$$(z/q)(\sqrt{q}/p) = 1.$$

(b) If $q \equiv 5 \pmod 8$, then either $a^2 \equiv x^2$ with $b^2 \equiv y^2 \pmod{2q}$ for some (x,y) and $a^2 \equiv q - x^2$ with $b^2 \equiv q - y^2 \pmod{2q}$ for other (x,y) or $a^2 \equiv -x^2$ with $b^2 \equiv -y^2 \pmod{2q}$ for some (x,y) and $a^2 \equiv q + x^2$ with $b^2 \equiv q + y^2 \pmod{2q}$ for other (x,y);

$$(\sqrt{p}/q)(\sqrt{q}/p) = 1$$

if and only if the first alternative is true, and

$$(z/q)(\sqrt{q}/p) = 1$$

if and only if $a \equiv x \pmod 2$.

(c) If $q \equiv 3 \pmod 8$, then $a^2 \equiv x^2$ with $b^2 \equiv y^2 \pmod{2q}$ for some (x,y) and $a^2 \equiv q + x^2$ with $b^2 \equiv q + y^2 \pmod{2q}$ for other (x,y);

$$(z/q)(\sqrt{-q}/p) = 1$$

in the first case and

$$(-z/q)(\sqrt{-q}/p) = 1$$

in the second case.

(d) If $q \equiv 7 \pmod 8$, then $a^2 \equiv x^2$ with $b^2 \equiv y^2 \pmod q$ for some (x,y) and

$$(z/q)(\sqrt{-q}/p) = 1.$$

In the following examples, (x,y,z) is the smallest primitive Pythagorean triple that satisfies the congruence:

$p = a^2 + b^2$	(x, y, z)	q or $2q$
5 = 1 + 4,	(21, 20, 29)	
	(12, 35, 37)	(mod 22);
	(77, 36, 85)	
	(20, 21, 29)	(mod 38);
	(57, 176, 185)	
	(12, 5, 13)	(mod 58);
	(435, 308, 533)	(mod 31);
	(—, —, —)	
	(—, —, —)	(mod 41);
29 = 25 + 4,	(5, 12, 13)	
	(20, 21, 29)	
41 = 25 + 16,	(5, 12, 13)	
	(20, 21, 29)	
101 = 1 + 100,	(21, 20, 29)	
	(12, 5, 13)	
109 = 9 + 100,	(21, 20, 29)	
	(12, 5, 13)	(mod 10);
13 = 9 + 4,	(3, 4, 5)	
	(12, 5, 13)	(mod 6);
	(—, —, —)	
	(—, —, —)	(mod 17);
	(20, 21, 29)	(mod 23);
	(7, 24, 25)	
	(84, 437, 445)	(mod 58);
17 = 1 + 16,	(21, 20, 29)	
	(12, 35, 37)	
29 = 25 + 4,	(77, 36, 85)	
	(8, 15, 17)	(mod 26);
17 = 1 + 16,	(39, 80, 89)	
	(20, 99, 101)	(mod 38);
53 = 49 + 4,	(112, 15, 113)	
	(40, 9, 41)	
149 = 49 + 100,	(7, 24, 25)	
	(45, 28, 53)	(mod 17);
41 = 25 + 16,	(615, 728, 953)	
	(116, 837, 845)	(mod 122);
61 = 25 + 36,	(87, 416, 425)	
	(45, 28, 53)	(mod 41).

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-261 Proposed by A. J. W. Hilton, University of Reading, Reading, England.

It is known that, given k a positive integer, each positive integer n has a unique representation in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t} \quad ,$$

where $t = t(n,k)$, $a_i = a_i(n,k)$, $(i = t, \cdots, k)$, $t \geqslant 1$ and, if $k > t$, $a_k > a_{k-1} > \cdots > a_t$. Call such a representation the *k-binomial representation* of n.

Show that, if $k \geqslant 2$, $n = r + s$, where $r \geqslant 1$, $s \geqslant 1$ and if the k-binomial representations of r and s are

$$r = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_u}{u}, \qquad s = \binom{c_k}{k} + \binom{c_{k-1}}{k-1} + \cdots + \binom{c_v}{v}$$

then

$$\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1} \leqslant \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{b-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}$$

H-262 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$L_{p^2} \equiv 1 \pmod{p^2} \rightleftharpoons L_p \equiv 1 \pmod{p^2}.$$

H-263 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $L^2_{2mn} \equiv 4 \pmod{L^2_m}$ for every $n,m = 1, 2, 3, \cdots$.

SOLUTIONS

WFFLE!

H-234 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Suppose an alphabet, $A = \{x_1, x_2, x_3, \cdots\}$, is given along with a binary connective, P (in prefix form). Define a well formed formula (wff) as follows: a wff is

(1) x_i for $i = 1, 2, 3, \cdots$, or
(2) If A_1, A_2 are wff's, then PA_1A_2 is a wff and
(3) The only wff's are of the above two types.

A wff of order n is a wff in which the only alphabet symbols are $x_1, x_2, \cdots, x_n$ in that order with each letter occurring exactly once. There is one wff of order 1, namely x_1. There is one wff of order 2, namely Px_1x_2. There are two wff's of order 3, namely $Px_1Px_2x_3$ and $PPx_1x_2x_3$, and there are five wff's of order 4, etc.

Define a sequence $\left\{G_i\right\}_{i=1}^{\infty}$ as follows:

G_i is the number of distinct wff's of order i.

(a). Find a recurrence relation for $\left\{G_i\right\}_{i=1}^{\infty}$ and

(b). Find a generating function for $\left\{G_i\right\}_{i=1}^{\infty}$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Let F_k denote any arbitrary wff of order k. In order to form F_n $(n = 2, 3, \cdots)$, we need to apply P to all possible distinct "products" of the form $F_k F_{n-k}$, $(k = 1, 2, \cdots, n-1)$. Hence, we obtain the recursion:

$$\text{(a)} \qquad G_n = \sum_{k=1}^{n-1} G_k G_{n-k}, \qquad n = 2, 3, 4, \cdots, \text{ with } G_1 = 1 .$$

The above recursion is the well known relation which yields the Catalan numbers, defined by:

$$\text{(b)} \qquad G_{n+1} = \frac{\binom{2n}{n}}{n+1}; \qquad \text{thus,} \qquad (G_n) = (1, 1, 2, 5, 14, 42, 132, 429, \cdots) .$$

We shall give a brief derivation of (b) from (a), using generating functions. Let us define the generating function for the G_n's:

$$\text{(1)} \qquad y = \sum_{n=0}^{\infty} G_{n+1} x^n = \sum_{n=1}^{\infty} G_n x^{n-1} ;$$

then

$$y^2 = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} G_{k+1} G_{n-k+1} = \sum_{n=0}^{\infty} x^n \sum_{k=1}^{n+1} G_k G_{n+2-k} = \sum_{n=2}^{\infty} x^{n-2} \sum_{k=1}^{n-1} G_n G_{n-k}$$

$$= \sum_{n=2}^{\infty} G_n x^{n-2} ,$$

using (a). Hence,

$$xy^2 = \sum_{n=2}^{\infty} G_n x^{n-1} = y - G_1 = y - 1$$

(using (1)). Thus, y is a solution of the quadratic equation

$$\text{(2)} \qquad xy^2 - y + 1 = 0; \quad \text{we note that} \quad y(0) = G_1 = 1.$$

Solving the quadratic, we obtain two solutions:

$$y = \frac{1 \pm \sqrt{1-4x}}{2x} .$$

The positive sign must be rejected, since this solution is not defined at $x = 0$. Thus, $y = [1 - (1-4x)^{1/2}]/2x$; it is easy to verify, by L'Hospital's rule, that $\lim_{x \to 0} y = 1$. Expanding this expression by the binomial theorem, or otherwise, we find

$$y = \frac{1}{2x} - \frac{1}{2x} \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = -\frac{1}{2x} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n = 2 \sum_{n=0}^{\infty} \binom{1/2}{n+1} (-4x)^n .$$

Comparing coefficients with (1), we have:

$$G_{n+1} = 2\binom{1/2}{n+1}(-4)^n = 2\cdot\tfrac{1}{2}\cdot\frac{\binom{-1/2}{n}}{n+1}\cdot(-4)^n = \frac{\binom{2n}{n}}{n+1}\ ,$$

which establishes (b).

Also solved by A. Shannon, R. L. Goodstein, and the Proposer.

SUM DIFFERENTIAL EQUATION!

H-235 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

a. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} F_{n+1}x^n\ .$$

b. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} L_{n+1}x^n\ .$$

Solution by A. G. Shannon, University of New England, Armidale, N.S.W.

Consider $\{H_n\} : H_n = H_{n-1} + H_{n-2}$

and $\{H_n\} = F_n$ when $H_1 = H_2 = 1$

and $\{H_n\} = L_n$ when $H_1 = 1,\ H_2 = 3.$

Let $Y = y(x).$

Then

$$y = \sum_{n=0}^{\infty} H_{n+1}x^n \quad\text{and}\quad y' = \sum_{n=0}^{\infty} (n+1)H_{n+2}x^n \quad\text{and}\quad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)H_{n+3}x^n\ .$$

Thus

$$(x^2 + x - 1)y'' + 2(2x+1)y' + 2y = \sum_{n=2}^{\infty} n(n-1)H_{n+1}x^n + \sum_{n=1}^{\infty} n(n+1)H_{n+2}x^n - \sum_{n=0}^{\infty} (n+1)(n+2)H_{n+3}x^n$$

$$+ 4\sum_{n=1}^{\infty} nH_{n+1}x^n + 2\sum_{n=0}^{\infty} (n+1)H_{n+2}x^n + 2\sum_{n=0}^{\infty} H_{n+1}x^n$$

$$= \sum_{n=2}^{\infty} (n^2 + 3n + 2)(H_{n+1} + H_{n+2} - H_{n+3})x^n\ .$$

Thus

$$(x^2 + x - 1)y'' + 2(2x+1)y' + 2y = 0 \quad \text{for } all \quad \{H_n\}.$$

Also solved by P. Bruckman, F. D. Parker, and the Proposer.

SUM PRODUCT!

H-236 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1) \qquad \sum_{n=0}^{\infty} (-1)^n x^{n^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{k=1}^{\infty} (1-x^k),$$

$$(2) \qquad \sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{k=1}^{\infty} (1-x^k),$$

where $(x)_k = (1-x)(1-x^2)\cdots(1-x^k),\ (x)_0 = 1.$

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

We begin with the well known Jacobi "triple-product" formula:

$$(1) \qquad \prod_{r=1}^{\infty} (1+x^{2r-1}w)(1+x^{2r-1}w^{-1})(1-x^{2r}) = \sum_{n=-\infty}^{\infty} x^{n^2}w^n = 1+\sum_{n=1}^{\infty} x^{n^2}(w^n+w^{-n});$$

the following treatment is formal, and avoids questions of convergence, but it may be shown that the manipulations are valid in the unit disk $|x| < 1$. Setting $w = -1$ in (1), the left-hand side becomes:

$$\prod_{r=1}^{\infty} (1-x^{2r-1})^2(1-x^{2r}) = \prod_{r=1}^{\infty} (1-x^{2r-1})(1-x^r) = \prod_{r=1}^{\infty} \frac{(1-x^{2r-1})(1-x^{2r})}{(1+x^r)}$$

$$= \prod_{r=1}^{\infty} \frac{(1-x^r)}{(1+x^r)} \ .$$

Hence, we obtain the identity

$$(2) \qquad \prod_{r=1}^{\infty} \left(\frac{1-x^r}{1+x^r}\right) = -1+2\sum_{n=0}^{\infty} (-1)^n x^{n^2} = 1-2\sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2}$$

Next, we will establish the following identity:

$$(3) \qquad \prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} \frac{x^n w^n}{(x)_n} \quad ,$$

where

$$(x)_0 = 1, \qquad (x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad n = 1,2,3,\cdots.$$

Letting

$$f(w,x) = \prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} A_n(x)w^n \ ,$$

we first note that $f(0,x) = 1 = A_0(x)$; also, we observe that

$$f(wx,x) = \prod_{r=1}^{\infty} (1-x^{r+1}w)^{-1} = \prod_{r=2}^{\infty} (1-x^r w)^{-1} = (1-xw)f(w,x).$$

Hence, by substituting into the series form, we obtain the recursion:

$$x^n A_n(x) = A_n(x) - xA_{n-1}(x), \qquad n = 1,2,3,\cdots, \qquad A_0(x) = 1,$$

i.e.,

$$A_n(x) = x/(1-x^n)A_{n-1}(x).$$

By an easy induction, we establish that $A_n(x) = x^n/(x)_n$ for all n, where $(x)_n$ is defined in (3). This establishes (3).

If, in (3), we replace w by $-w$, we obtain:

(4) $$\prod_{r=1}^{\infty} (1+x^r w)^{-1} = \sum_{n=0}^{\infty} \frac{x^n(-1)^n w^n}{(x)_n} .$$

Adding, and then subtracting, both sides of (3) and (4), we obtain:

(5) $$\prod_{r=1}^{\infty} (1-x^r w)^{-1} + \prod_{r=1}^{\infty} (1+x^r w)^{-1} = 2\sum_{n=0}^{\infty} \frac{x^{2n}w^{2n}}{(x)_{2n}} ,$$

and

(6) $$\prod_{r=1}^{\infty} (1-x^r w)^{-1} - \prod_{r=1}^{\infty} (1+x^r w)^{-1} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}w^{2n+1}}{(x)_{2n+1}} .$$

If, in (5) and (6), we set $w = 1$, and multiply throughout by

$$\prod_{r=1}^{\infty} (1-x^r),$$

we obtain:

(7) $$1+\prod_{r=1}^{\infty}\left(\frac{1-x^r}{1+x^r}\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{r=1}^{\infty} (1-x^r) ,$$

and

(8) $$1-\prod_{r=1}^{\infty}\left(\frac{1-x^r}{1+x^r}\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{r=1}^{\infty} (1-x^r) .$$

Now if, in (7) and (8), we substitute the expression obtained in (2) for the infinite product on the left-hand side, simplify and divide by 2, we obtain the desired results:

(9) $$\sum_{n=0}^{\infty} (-1)^n x^{n^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{r=1}^{\infty} (1-x^r),$$

and

(10) $$\sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{r=1}^{\infty} (1-x^r) .$$

Also solved by P. Tracy and the Proposer.

SUM RECIPROCAL!

H-237 Proposed by D. A. Miller, High School Student, Annville, Pennsylvania.

Prove

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2} .$$

Editorial Note: A solution to this problem appears in a note (accepted Feb. 27, 1973) appearing in the Dec. '74 issue o the *Quarterly*, p. 346.

Solution by A. G. Shannon, University of New England, Armidale, N.S.W.

Let

$$F(x) = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{F_{2^k}} .$$

Then

$$F(\alpha x) = \sum_{k=1}^{\infty} \frac{\alpha^{2^{k-1}} x^{2^{k-1}}}{F_{2^k}}$$

and so

$$F(\alpha x) + F(\beta x) = \sum_{k=0}^{\infty} \frac{\alpha^{2^{k-1}} + \beta^{2^{k-1}}}{F_{2^k}} x^{2^{k-1}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{F_{2^{k-1}}} = \sum_{k=0}^{\infty} \frac{x^{2^k}}{F_{2^k}} = x + F(x^2).$$

So

$$x + F(x^2) = F(\alpha x) + F(\beta x).$$

Put $x = -\beta$:

$$-\beta + F(\beta^2) = F(-\beta^2) + F(-\alpha\beta)$$

or

$$F(1) = -\beta + 2\beta^2$$

since

$$\alpha\beta = -1$$

and

$$F(\beta^2) = F(-\beta^2) + 2\beta^2 .$$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = 1 + F(1)$$

$$= 1 - \beta + 2\beta^2$$

$$= 2 - (1 + \beta - \beta^2) + \beta^2$$

$$= 2 + \frac{(3 - \sqrt{5})}{2} = \frac{7 - \sqrt{5}}{2} .$$

Also solved by I. J. Good (see note), J. Shallit, W. Brady, P. Bruckman, F. Higgins, L. Carlitz, and the Proposer.

Editorial Note. Kurt Mahler reports

$$\sum_{n=0}^{\infty} \frac{1}{L_{2^n}}$$

is transcendental.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S. E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 0, \qquad L_1 = 1.$$

Also a and b designate the roots of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-328 Proposed by Walter Hansell, Mill Valley, California, and V. E. Hoggatt, Jr., San Jose, California.

Show that

$$6(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

is always a sum

$$m^2 + (m^2 + 1) + (m^2 + 2) + \cdots + (m^2 + r)$$

of consecutive integers, of which the first is a perfect square.

B-329 Proposed by Herta T. Freitag, Roanoke, Virginia.

Find r, s, and t as linear functions of n such that $2F_r^2 - F_sF_t$ is an integral divisor of $L_{n+2} + L_n$ for $n = 1, 2, \cdots$.

B-330 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let
$$G_n = F_n + 29F_{n+4} + F_{n+8}\ .$$

Find the greatest common divisor of the infinite set of integers $\{G_0, G_1, G_2, \cdots\}$.

B-331 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that
$$F_{6n+1}^2 \equiv 1 \pmod{24}.$$

B-332 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $a(n)$ be the number of ordered pairs of integers (r,s) with both $0 \leqslant r \leqslant s$ and $2r + s = n$. Find the generating function
$$A(x) = a(0) + xa(1) + x^2a(2) + \cdots.$$

B-333 Proposed by Phil Mana, Albuquerque, New Mexico.

Let S_n be the set of ordered pairs of integers (a,b) with both $0 < a < b$ and $a + b \leqslant n$. Let T_n be the set of ordered pairs of integers (c,d) with both $0 < c < d < n$ and $c + d > n$. For $n \geqslant 3$, establish at least one bijection (i.e., 1-to-1 correspondence) between S_n and T_{n+1}.

SOLUTIONS

SO BEE IT

B-304 Proposed by Sidney Kravitz, Dover, New Jersey.

According to W. Hope-Jones "The Bee and the Pentagon," *The Mathematical Gazette,* Vol. X, No. 150, 1921 (Reprinted Vol. LV, No. 392, March 1971, Page 220) the female bee has two parents but the male bee has a mother only. Prove that if we go back n generations for a female bee she will have F_n male ancestors in that generation and F_{n+1} female ancestors, making a total of F_{n+2} ancestors.

Solution by Sister Marion Beiter, Rosary Hill College, Buffalo, New York.

The proof is by induction. Let $P(n)$ be the statement of the problem. $P(n)$ holds for $n = 1$.

One generation back a female bee will have $F_1 = 1$ male ancestor and $F_{1+1} = 1$ female ancestor, a total of $F_{1+2} = 2$. If $P(n)$ holds for $n = k$, it holds for $n = k + 1$:

If we go back k generations for a female bee she will have F_k male ancestors in that generation and F_{k+1} female ancestors, making a total of F_{k+2} ancestors.

Then if we go back $k + 1$ generations, she will have F_{k+1} male ancestors (from the F_{k+1} females in the k^{th} generation), and F_{k+2} female ancestors (from the total F_{k+2} ancestors in the k^{th} generation). This makes a total of $F_{k+1} + F_{k+2} = F_{k+3}$. Hence, $P(n)$ holds for all natural numbers n.

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Graham Lord, A. G. Shannon, and the Proposer.

A TELESCOPING SUM

B-305 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n} = L_{2n} \sum_{k=1}^{n} L_{2n+4k-2} .$$

Solution by Graham Lord, Universite Laval, Quebec, Canada.

The following steps use standard identities:

$$\begin{aligned} L_{2n} \sum_{k=1}^{n} L_{2n+4k-2} &= L_{2n} \left(\sum_{k=1}^{n} \left\{ F_{2n+4k-2+2} - F_{2n+4k-2-2} \right\} \right) \\ &= L_{2n}(F_{6n} - F_{2n}) = L_{2n} \cdot F_{2n} \cdot L_{4n} \\ &= F_{4n} \cdot L_{4n} = F_{8n} . \end{aligned}$$

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, and the Proposer.

SOMETHING SPECIAL

B-306 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n+1} - 1 = L_{2n} \sum_{k=1}^{n} L_{2n+4k-1} .$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

For generalized Fibonacci numbers defined by letting H_0 and H_1 be arbitrary integers and $H_n = H_{n-1} + H_{n-2}$ for $n \geqslant 2$, it is known that

$$\sum_{k=1}^{n} H_{4k-1} = F_{2n} H_{2n+1} .$$

(See, for example, Identity (9) in Iyer's article, *FQ*, 7 (1969), pp. 66–72.) More generally,

$$\sum_{k=1}^{n} H_{2n+4k-1} = F_{2n} H_{4n+1} .$$

Specializing this identity to Lucas numbers and using (I_7) and (I_{24}) of Hoggatt's *Fibonacci and Lucas Numbers,* one obtains

$$L_{2n} \sum_{k=1}^{n} L_{2n+4k-1} = L_{2n}F_{2n}L_{4n+1} = F_{4n}L_{4n+1} = F_{8n+1} - 1.$$

Also solved be Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, and the Proposer.

MODULARLY MOVING MAVERICK

B-307 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$(1+x+x^2)^n = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots,$$

(where, of course, $a_{n,k} = 0$ for $k > 2n$). Also let

$$A_n = \sum_{j=0}^{\infty} a_{n,4j}, \qquad B_n = \sum_{j=0}^{\infty} a_{n,4j+1}, \qquad C_n = \sum_{j=0}^{\infty} a_{n,4j+2}, \qquad D_n = \sum_{j=0}^{\infty} a_{n,4j+3}.$$

Find and prove the relationship of A_n, B_n, C_n and D_n to each other. In particular, show the relationship among these four sums for $n = 333$.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

One may easily prove by induction on n that the trinomial coefficients $a_{n,k}$ satisfy the recursion formula

$$a_{n,k} = a_{n-1,k-2} + a_{n-1,k-1} + a_{n-1,k}$$

for $n > 0$ with initial values

$$a_{0,k} = \begin{cases} 1 \text{ if } k = 0 \\ 0, \text{otherwise} \end{cases}.$$

By letting $x = 1$ in the defining equation one may also deduce that

$$\sum_{k=0}^{\infty} a_{n,k} = 3^n.$$

This last fact and two applications of the recurrence relationship readily yield the following identities for $n \geqslant 2$:

$$A_n = 2\cdot 3^{n-2} + C_{n-2}, \qquad C_n = 2\cdot 3^{n-2} + A_{n-2},$$
$$B_n = 2\cdot 3^{n-2} + D_{n-2}, \qquad D_n = 2\cdot 3^{n-2} + B_{n-2}.$$

Iteration on n, upon summation of the resulting geometric series, yields the following formula for each

$$X \in \{A, B, C, D\}, \qquad m \in \{0, 1, 2, 3,\}, \qquad n = 0, 1, 2, \cdots:$$

$$X_{4n+m} = \tfrac{1}{4}(3^{4n+m} - 3^m) + X_m.$$

Less compactly, but more in the spirit of comparison one finds

$$B_{4n} = C_{4n} = D_{4n} = A_{4n} - 1, \qquad B_{4n+1} = C_{4n+1} = A_{4n+1} = D_{4n+1} + 1,$$
$$B_{4n+2} = D_{4n+2} = A_{4n+2} = C_{4n+2} - 1, \qquad C_{4n+3} = D_{4n+3} = A_{4n+3} = B_{4n+3} + 1,$$

for each $n = 0, 1, 2, \cdots$. In particular,

$$A_{333} = B_{333} = C_{333} = D_{333} + 1 = \tfrac{1}{4}(3^{333} + 1).$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

A GARBLED HINT

B-308 Proposed by Phil Mana, Albuquerque, New Mexico.

(b) Let r be a real number such that $\cos(r\pi) = p/q$, with p and q relatively prime positive integers and q not in $\{1, 2, 4, 8, \cdots\}$. Prove that r is not rational.

[The (a) part has been deleted due to an error in it.]

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Ill.

(b) We first recall the multiple-angle formula from trigonometry:

$$(1) \qquad \cos n\theta = \tfrac{1}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2\cos\theta)^{n-2k}, \qquad n = 1, 2, 3, \cdots.$$

We also recall, or we may easily show, that this is a polynomial with integer coefficients, in $\cos\theta$.

Suppose now that $r = u/v$ is rational (with u and v relatively prime natural numbers), and satisfies:

$$(2) \qquad \cos(r\pi) = p/q,$$

where p and q are relatively prime natural numbers and $(q,2) = 1$ (i.e., q is odd), except $q \neq 1$. Note that this restricts q more than in the original problem, but we will deal with the remaining values of q later. Letting $\theta = r\pi$ and $n = v$ in (1), we get:

$$(-1)^u = \cos u\pi = \tfrac{1}{2} \sum_{k=0}^{[v/2]} (-1)^k \frac{v}{v-k} \binom{v-k}{k} \left(\frac{2p}{q}\right)^{v-2k} = \frac{2^{v-1}p^v}{q^v} - \frac{v2^{v-3}p^{v-2}}{q^{v-2}} + \cdots$$

$$= \frac{2^{v-1}p^v + q^2M}{q^v},$$

where M is some integer.

Since $(2,q) = (p,q) = 1$, it follows that $(2^{v-1}p^v + q^2M, q^v) = 1$; but then q^v cannot divide $(2^{v-1}p^v + q^2M)$, and their ratio cannot be $(-1)^u = \pm 1$. This contradiction shows that r cannot be rational, when q is as stated above.

Suppose now, as before, that (2) holds for some rational r, where $q = 2^s t$, $(2,p) = (p,t) = (2,t) = 1$, s and t are natural numbers, $t \geq 3$. As before,

$$(-1)^u = (2^{v-1}p^v + q^2M)/q^v = (2^{v-1}p^v + 2^{2s}t^2M)/2^{sv}t^v.$$

Since $(2,t) = (p,t) = 1$, the indicated ratio cannot be an integer, and we have again reached a contradiction. Hence, we have proved that the only possible values of q satisfying (2) are $q = 1$ and $q = 2$; this, in turn, implies that $\cos(r\pi) = 0, \pm\tfrac{1}{2}, \pm 1$ are the only possible values, corresponding to $r = n + \tfrac{1}{2}$, $n \pm 1/3$, and n, respectively, where n is an arbitrary integer. This is a stronger result that originally sought.

Also solved by the Proposer. The error in Part (a) was pointed out by Paul S. Bruckman and Herta T. Freitag.

AN ANALOGUE OF $a^n = aF_n + F_{n-1}$

B-309 Corrected version of B-284.

Let $z^2 = xz + y$ and let k, m, and n be nonnegative integers. Prove that

(a) $z^n = p_n(x,y)z + Q_n(x,y)$, where p_n and Q_n are polynomials in x and y with integer coefficients and p_n has degree $n - 1$ *in* x *for* n > 0.

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0.$$

Composite of solutions by David Zeitlin, Minneaspolis, Minnesota, and the Proposer.

(a) Let $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = xU_{n+1} + yU_n$ for $n = 0, 1, \cdots$. Then one easily proves by induction that $z^n = zU_n + U_{n-1}$ using $z^{k+1} = xz^k + yz^{k-1}$.

(b) z satisfies a quadratic equation over the field $F = Q(x,y)$ of polynomials in x and y with rational coefficients. Hence $F[z]$ is a vector space of degree 2 over F. Thus any three powers of z are linearly dependent over F. Clearing denominators, gives the desired result.

Also solved by Paul S. Bruckman and Herta T. Freitag.

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