A GENERALIZATION OF A SERIES OF DE MORGAN, WITH APPLICATIONS OF FIBONACCI TYPE

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Bromwich [1], p. 24, attributes the formula

(1) $$\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots+\frac{x^{2^{n-1}}}{1-x^{2^n}}=\frac{1}{1-x}-\frac{1}{1-x^{2^n}}$$

to Augustus de Morgan, together with the corresponding sums of the infinite series, namely $x(1-x)^{-1}$ if $|x|<1$, and $(1-x)^{-1}$ if $|x|>1$. As far as the authors know, the following generalization has not yet appeared in print.

(2) $$\sum_{n=0}^{N}\frac{(xy)^{m^n}[x^{m^n(m-1)}-y^{m^n(m-1)}]}{[x^{m^n}-y^{m^n}][x^{m^{n+1}}-y^{m^{n+1}}]}=\frac{y}{x-y}-\frac{y^{m^{N+1}}}{x^{m^{N+1}}-y^{m^{N+1}}}\qquad (x\neq y)$$

$$=\frac{x}{x-y}-\frac{x^{m^{N+1}}}{x^{m^{N+1}}-y^{m^{N+1}}}\qquad (m=2,3,\cdots).$$

To see this, note that the expression

$$[z^{m^n}+z^{2\cdot m^n}+\cdots+z^{(m-1)m^n}][1+z^{m^{n+1}}+z^{2m^{n+1}}+\cdots\text{ad inf}]=\frac{z^{m^n}(1-z^{m^n(m-1)})}{(1-z^{m^n})(1-z^{m^{n+1}})}\qquad (|z|<1)$$

is equal to the sum of all those powers of z where the powers are multiples of m^n but not multiples of m^{n+1}. Therefore

(3) $$\sum_{n=0}^{N}\frac{z^{m^n}(1-z^{m^n(m-1)})}{(1-z^{m^n})(1-z^{m^{n+1}})}=\frac{1}{1-z}-\frac{1}{1-z^{m^{N+1}}}\qquad (|z|<1).$$

On replacing z by y/x we obtain (2), and, on allowing N to tend to infinity we obtain, if $|x|\neq|y|$,

(4) $$\sum_{n=0}^{\infty}\frac{(xy)^{m^n}[x^{m^n(m-1)}-y^{m^n(m-1)}]}{(x^{m^n}-y^{m^n})(x^{m^{n+1}}-y^{m^{n+1}})}=\frac{\text{min (abs) }(x,y)}{x-y}\qquad (m=2,3,\cdots),$$

where min (abs) (x,y) signifies x or y, depending on whether $|x|<|y|$ or $|x|>|y|$, respectively.

To obtain examples, let a and b be positive integers and let u_n be the denominator of the $(n-1)^{th}$ convergent of the continued fraction

$$\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\cdots$$

so that

(5)
$$u_n = \frac{\xi^n - \eta^n}{\sqrt{(a^2+4b)}} ,$$

where

$$\xi = \frac{a+\sqrt{(a^2+4b)}}{2}, \qquad \eta = \frac{a-\sqrt{(a^2+4b)}}{2} .$$

Now put $x = \xi^k$ and $y = \eta^k$ in (2), where k is a positive integer, and we obtain

(6)
$$\sum_{n=0}^{N} \frac{(-b)^{km^n} u_{km^n(m-1)}}{u_{km^n} u_{km^{n+1}}} = \eta - \frac{\eta^{km^{N+1}}}{u_{km^{n+1}}} .$$

When $b = 1$ the formula simplifies somewhat. When $a = b = 1$, then $u_n = F_n$, the n^{th} Fibonacci number. Some special cases of these formulae are

(7)
$$\frac{2(2+1)}{2^3-1} + \frac{2^3(2^3+1)}{2^9-1} + \frac{2^9(2^9+1)}{2^{27}-1} + \cdots = 1 \quad (x=2, y=1, m=3 \text{ in } (4));$$

(8)
$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} + \cdots = \frac{7-\sqrt{5}}{2} \quad (a=b=k=1, m=2 \text{ in } (6), \text{ as } N \to \infty);$$

(9)
$$\frac{L_1}{F_3} + \frac{L_3}{F_9} + \frac{L_9}{F_{27}} + \cdots = \frac{\sqrt{5}-1}{2} \quad (a=b=k=1, m=3 \text{ in } (6), \text{ as } N \to \infty);$$

(where $L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, \cdots$ are the Lucas numbers),

(10)
$$\frac{F_1}{L_3} + \frac{F_3}{L_9} + \frac{F_9}{L_{27}} + \cdots = \frac{5-\sqrt{5}}{10} ,$$

(11)
$$\sum_{n=0}^{\infty} \frac{L_{k3^n}}{F_{k3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k F_k}, \qquad \sum_{n=0}^{\infty} \frac{F_{k3^n}}{L_{k3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k \sqrt{5} L_k} .$$

Further generalization. Formulae (2) and (4) can be further generalized to

(12)
$$\frac{y_0}{x_0 - y_0} = \sum_{n=0}^{m-1} \frac{x_{n+1}y_n - x_n y_{n+1}}{(x_n - y_n)(x_{n+1} - y_{n+1})} + \frac{y_m}{x_m - y_m} ,$$

where $\{x_n\}$ and $\{y_n\}$ are any sequences of real or complex numbers, with $x_n \neq y_n \ \forall n$, and

(13)
$$\sum_{n=0}^{\infty} \frac{x_{n+1}y_n - x_n y_{n+1}}{(x_n - y_n)(x_{n+1} - y_{n+1})} = \begin{cases} y_0/(x_0 - y_0) & \text{if } y_n/x_n \to 0 \\ x_0/(x_0 - y_0) & \text{if } x_n/y_n \to 0 \end{cases} .$$

Although (12) is more general than (2), its proof is obvious. A special case of (13), after a change of notation, is

(14)
$$\sum_{n=m}^{\infty} \frac{(xy)^{s(n)}[x^{s(n+1)-s(n)} - y^{s(n+1)-s(n)}]}{[x^{s(n)} - y^{s(n)}][x^{s(n+1)} - y^{s(n+1)}]} = \frac{y^{s(m)}}{x^{s(m)} - y^{s(m)}}$$

if $|x| > |y|$ and $s(n) \to \infty$ as $n \to \infty$.

Now put $x = \xi$, $y = \eta$ and we obtain

(15)
$$\sum_{n=m}^{\infty} \frac{(-b)^{s(n)} u_{s(n+1)-s(n)}}{u_{s(n)} u_{s(n+1)}} = \frac{\eta^{s(m)}}{u_{s(m)}}$$

and in particular

$$\sum_{n=0}^{\infty} \frac{(-1)^{s(n)} F_{s(n+1)-s(n)}}{F_{s(n)} F_{s(n+1)}} = \left(\frac{1-\sqrt{5}}{2}\right)^{s(0)} \Big/ F_{s(0)} . \tag{16}$$

For example, if $s(n) = F_{n+1}$,

$$\sum_{n=1}^{\infty} \frac{\epsilon_n F_{F_n}}{F_{F_{n+1}} F_{F_{n+2}}} = \frac{\sqrt{5}-1}{2}, \quad \text{where} \quad \epsilon_n = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 3 \\ -1 & \text{if } n \equiv 2 \pmod 3 \end{cases} \tag{17}$$

and if $s(n) = L_{n+1}$,

$$\sum_{n=0}^{\infty} \frac{\epsilon_n F_{L_n}}{F_{L_{n+1}} F_{L_{n+2}}} = \frac{\sqrt{5}-1}{2} . \tag{18}$$

Putting $s(n) = (n+1)k$ in (16) gives

$$\sum_{n=0}^{\infty} \frac{(-1)^{(n+1)k}}{F_{(n+1)k} F_{(n+2)k}} = \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k}{F_k^2} . \tag{19}$$

Putting $x_n = (n+2)^c$ and $y_n = 1$ in (13) gives, after a change of notation,

$$\sum_{n=2}^{\infty} \frac{(n+1)^c - n^c}{(n^c - 1)((n+1)^c - 1)} = \frac{1}{2^c - 1} \quad (c > 1). \tag{20}$$

Putting $x_n = e^{(n+1)t}$, $y_n = e^{-(n+1)t}$ in (13) gives

$$\sum_{n=1}^{\infty} \frac{1}{\cosh (2n+1)t - \cosh t} = \frac{e^{-|t|}}{2 \sinh^2 t} . \tag{21}$$

Historical note. The formula

$$\sqrt{3} = 2 - \frac{1}{4} - \frac{1}{4.14} - \frac{1}{4.14.194} - \cdots, \tag{22}$$

where $14 = 4^2 - 2$, $194 = 14^2 - 2, \cdots$, was drawn to the attention of I. J. Good by Dr. G. L. Camm in November 1947. (The sequence 4, 14, 194, ⋯ occurs also in tests for primality of the Mersenne numbers [4], p. 235.) The similar formula

$$\sqrt{r} = \frac{(r-1)\alpha_n}{4\beta_{n-1}} - \frac{r-1}{2}\left(\frac{1}{\beta_n} + \frac{1}{\beta_{n+1}} + \cdots\right), \tag{23}$$

where

$$r > 1, \quad \alpha_1 = 2\,\frac{m+1}{m-1}, \quad \alpha_{n+1} = \alpha_n^2 - 2, \quad \beta_0 = 1, \quad \beta_n = \alpha_1\alpha_2\cdots\alpha_n$$

was given in [3]; and formula (8) in [2] and [6]. Hoggatt [5] then noticed that

$$\sum_n \frac{1}{F_{k2^n}}$$

could similarly be summed. All these results follow from deMorgan's formula. I. J. Good noticed the generalization (4) in November 1947, but at that time did not see its application to the Fibonacci and similar sequences and therefore withheld its publication. P. S. Bruckman independently, and recently, noticed the more general formula (14). Alternate methods of proof appear in [7].

REFERENCES

1. T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series,* 2nd Ed., MacMillan, London, 1931.
2. I. J. Good, "A Reciprocal Series of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 4 (Dec. 1974), p. 346.
3. I. J. Good and T. N. Grover, "The Generalized Serial Test and the Binary Expansion of $\sqrt{2}$; Corrigendum and Addition," *J. Roy. Statist. Soc.,* A. Vol. 131 (1968), p. 434.
4. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Oxford, Clarendon Press, 1938.
5. V. E. Hoggatt, Jr., Private communication (14 December, 1974).
6. D. A. Millin, Advanced problem No. H-237, *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), p. 309.
7. V. E. Hoggatt, Jr. and Marjorie Bicknell, "A Primer for the Fibonacci Numbers, Part XV: Variations on Summing a Series of Reciprocals of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 14, No. 3, pp. 272–276.

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ON THE HARRIS MODIFICATION OF THE EUCLIDEAN ALGORITHM

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V. C. Harris[1] (see D. E. Knuth[2] also) modified the Euclidean algorithm (= algorithm by greatest integers) for finding the gcd of two odd integers $a > b > 1$. The conditions $a = bq + r$, $|r| < b$, $2|r$ define the integers q,r uniquely. In case $r = 0$, stop. In case $r \neq 0$, divide r by its highest power of 2 and obtain c (say); proceed with b, $|c|$ instead of a,b. Denote by $H(a,b)$ the number of steps in this Harris algorithm.

Example: $83 = 47\cdot 1 + 4\cdot 9, \quad 47 = 9\cdot 5 + 2\cdot 1, \quad 9 = 1\cdot 9; \quad H(83,47) = 3.$

Denote by $E(a,b)$ resp. $N(a,b)$ the number of steps in the algorithm by greatest resp. nearest integers for $a > b > 0$. According to Kronecker, $N(a,b) \leqslant E(a,b)$ always. In this note we prove that $H(a,b)$ is sometimes much larger than $E(a,b)$ and sometimes much smaller than $N(a,b)$.

Let

$$c_0 := 1, \quad c_{n+1} = 2c_n + 5 \quad (n \geqslant 0);$$

obviously

$$E(c_{n+1}, c_n) \leqslant 5 \quad (n \geqslant 0).$$

Furthermore, since

$$c_{n+2} = 3c_{n+1} - 2c_n, \quad 2 \nmid c_n \quad (n \geqslant 0),$$

the choice $a_k = c_k$, $b_k = c_{k-1}$ $(k > 0)$ gives

Theorem 1. For every integer $k > 0$ there exist odd integers $a_k > b_k > 0$ with

$$E(a_k, b_k) \leqslant 5, \quad H(a_k, b_k) = k.$$

Let

$$v_0 := 0, \quad v_1 := 1, \quad v_n := 2v_{n-1} + v_{n-2} \quad (n > 1);$$

then

$$(v_{n+1}, v_n) = 1, \quad v_n \leqslant 3^{n-1}, \quad 2|v_n \Leftrightarrow 2|n \quad (n \geqslant 0).$$

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[1] *The Fibonacci Quarterly,* Vol. 8, No. 1 (February, 1970), pp. 102–103.
[2] *The Art of Computer Programming,* Vol. 2, "Seminumerical Algorithms," Addison-Wesley Pub., 1969, pp. 300, 316

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POLYNOMIALS $P_{2n+1}(x)$ SATISFYING $P_{2n+1}(F_k) = F_{(2n+1)k}$

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Denote the polynomials defined indirectly and recursively by my Theorem [1] by $P_{2n+1}(x)$ so that

$$P_1(x) = x, \quad P_3(x) = 5x^3 + 3(-1)^k x, \quad P_5(x) = 25x^5 + 25(-1)^k x^3 + 5x,$$

$$P_7(x) = 125x^7 + 175(-1)^k x^5 + 70x^3 + 7(-1)^k x,$$

$$P_9(x) = 5^4 x^9 + 3^2 \cdot 5^3 (-1)^k x^7 + 3^3 \cdot 5^2 x^5 + 2 \cdot 3 \cdot 5^2 (-1)^k x^3 + 3^2 x,$$

$$P_{11}(x) = 5^5 x^{11} + 5^4 \cdot 11(-1)^k x^9 + 2^2 \cdot 5^3 \cdot 11 x^7 + 5^2 7 \cdot 11(-1)^k x^5 + 5^2 11 x^3 + 11(-1)^k x,$$

$$P_{13}(x) = 5^6 x^{13} + 5^5 13(-1)^k x^{11} + 5^5 13 x^9 + 2^2 \cdot 3 \cdot 5^3 13(-1)^k x^7 + 2 \cdot 5^2 7 \cdot 13 x^5 + 5 \cdot 7 \cdot 13(-1)^k x^3 + 13x,$$

$$P_{15}(x) = 5^7 x^{15} + 3 \cdot 5^7 (-1)^k x^{13} + 2 \cdot 3^2 5^6 x^{11} + 5^6 11(-1)^k x^9 + 2 \cdot 3^2 5^5 x^7 + 2 \cdot 3^3 5^2 7(-1)^k x^5$$
$$+ 2^2 5^2 7 x^3 + 3 \cdot 5(-1)^k x \quad .$$

Theorem [1] may be written as

Theorem 1.

$$(1) \qquad P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{s=1}^{n} \binom{2n+1}{n+1-s} [(-1)^{k+1}]^{n+1-s} P_{2s-1}(x) \, .$$

The following Theorem 2 gives an explicit expression for these polynomials:

Theorem 2.

$$(2) \qquad P_{2n+1}(x) = \sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!(2n+1-2r)!} x^{2n+1-2r} \quad .$$

Proof. The polynomials $P_3(x)$ obtained by substituting $n = 1$ into Eqs. (1) and (2) are easily shown to be identical. Using the second principle of mathematical induction, assume that (1) and (2) express identical polynomials $P_{2s+1}(x)$ for all $s < n$ [2]. Substituting the expression (2) into the right-hand side of Eq. (1), it will be shown that the resulting expression for $P_{2n+1}(x)$ is identical to that as determined by Eq. (2). Thus, Eq. (1) becomes

$$P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{s=1}^{n} \binom{2n+1}{n+1-s} [(-1)^{k+1}]^{n+1-s} P_{2s-1}(x),$$

where

$$P_{2s-1}(x) = \sum_{r=0}^{s-1} 5^{s-1-r} x^{2s-1-2r} (-1)^{kr} \frac{(2s-1)[(2s-2-r)!]}{r!(2s-2r-1)!} \quad .$$

Rearranging terms and changing the variable of summation by $t = s - r - 1$, eliminating r, and interchanging the order of summation on t and s obtain:

(3) $$P_{2n+1}(x) = 5^n x^{2n+1} - \sum_{t=0}^{n-1} 5^t(-1)^{kn-kt+n+1} \frac{(2n+1)!x^{2t+1}}{(2t+1)!} \cdot Q,$$

where

$$Q = \sum_{s=t+1}^{n} (-1)^s \frac{(2s-1)[(s+t-1)!]}{(s-t-1)!(n-s+1)!(n+s)!} .$$

Expression Q may be summed using the antidifference method as

$$Q = \frac{-(-1)^s(s+t-1)!}{(n-t)(n-s+1)!(n+s-1)!(s-t-2)!}\Bigg|_{s=t+1}^{s=n+1} = \frac{-(-1)^{n+1}(n+t)!}{(n-t)(2n)!(n-t-1)!} .$$

Substituting the latter for Q in (3) above and simplifying, obtain

(4) $$P_{2n+1}(x) = 5^n x^{2n+1} + \sum_{t=0}^{n-1} (-1)^{k(n-t)} \frac{5^t(2n+1)x^{2t+1}}{(2t+1)!(n-t)!} (n+t)!$$

Finally, changing the variable of summation to $r = n - t$, and noting the first term is represented with $r = 0$, Eq. (4) becomes (2) as desired.

Theorem 3. $$P_{(2m+1)(2n+1)}(x) = P_{2m+1}(P_{2n+1}(x)) .$$

Proof. Each of the polynomials is of degree $(2m+1)(2n+1)$, and since the same Fibonacci number, namely, $F_{(2m+1)(2n+1)k}$, is obtained for $x = F_k$, $k = 1, 2, 3, \cdots$, the polynomials have identical values for an infinite number of arguments, and thus by a well known property of polynomials, the polynomials in x are identical [3].

Theorem 4.

(5) $$P_{2n+5}(x) = [5x^2 + 2(-1)^k]P_{2n+3}(x) - P_{2n+1}(x) .$$

Proof. Substituting (1) into the right-hand member of Eq. (5) above, and multiplying by $[5x^2 + 2(-1)^k]$, one obtains three summations:

$$\sum_{r=0}^{n+1} 5^{n+2-r}(-1)^{kr} \frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+5-2r}$$

$$+\sum_{r=0}^{n+1} 5^{n+1-r}(-1)^{k(r+1)} \frac{2(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+3-2r}$$

$$-\sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!(2n+1-2r)!} x^{2n+1-2r} .$$

Replacing r by $r - 1$ and $r - 2$ in the second and third summations, respectively, each summation has the common factor $5^{n+2-r}(-1)^{kr}x^{2n+5-2r}$ with the range of summation overlapping for $r = 2$ to $n + 1$ as follows:

$$\sum_{r=0}^{n+1} 5^{n+2-r}(-1)^{kr} \frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!} x^{2n+5-2r}$$

$$+\sum_{r=1}^{n+2} 5^{n+2-r}(-1)^{kr} \frac{2(2n+3)[(2n+3-r)!]}{(r-1)!(2n+5-2r)!} x^{2n+5-2r} -$$

$$-\sum_{r=2}^{n+2} 5^{n+2-r}(-1)^{kr}\frac{(2n+1)[(2n+2-r)!]}{(r-2)!(2n+5-2r)!}x^{2n+5-2r}.$$

Collecting the overlapping portion of the summations in a single summation and simplifying the remaining individual terms, one obtains:

$$\sum_{r=2}^{n+1} 5^{n+2-r}(-1)^{kr}x^{2n+5-2r}\left\{\frac{(2n+3)[(2n+2-r)!]}{r!(2n+3-2r)!}+\frac{2(2n+3)[(2n+3-r)!]}{(r-1)!(2n+5-2r)!}-\frac{(2n+1)[(2n+2-r)!]}{(r-2)!(2n+5-2r)!}\right\}$$

(6) $$+5^{n+2}x^{2n+5}+5^{n+1}(-1)^k(2n+5)x^{2n+3}+(2n+5)(-1)^{kn}x.$$

The expression within the brace of Eq. (6) becomes

$$\frac{(2n+2-r)![(2n+3)(2n+5-2r)(2n+4-2r)+2r(2n+3)(2n+3-r)-r(r-1)(2n+1)]}{r!(2n+5-2r)!}$$

$$=\frac{(2n+2-r)!}{r!(2n+5-2r)!}[8n^3+48n^2+94n-8rn^2-34rn+2r^2n+5r^2-35r+60]$$

$$=\frac{(2n+2-r)!}{r!(2n+5-2r)!}[(2n+5)(2n+4-r)(2n+3-r)]=\frac{(2n+5)[(2n+4-r)!]}{r!(2n+5-2r)!}.$$

Substituting this simplified result for the brace in (6) and noting the three individual terms in (6) from the summation general summation term with $r = 0, 1,$ and $n+2$, respectively, Eq. (6) becomes $P_{2n+5}(x)$ as expressed by (2).

Theorem 5.

(7) $$[5x^2+4(-1)^k]P''_{2n+1}(x)+5xP'_{2n+1}(x)-5(2n+1)^2P_{2n+1}(x)=0.$$

Proof. Differentiating (2) and substituting into the left-hand member of (7), multiplying the binomial $[5x^2 + 4(-1)^k]$ appropriately in (7) to form two summations and changing the index of summation r to $r-1$ in the summation formed from $4(-1)^kP''_{2n+1}(x)$, one obtains four summations with like general terms in x with the range of summation overlapping from $r = 1$ to $n-1$. Factoring out the common factors, the left-hand member of (7) becomes

$$\sum_{r=1}^{n-1}\frac{5^{n-r+1}(2n+1)(-1)^{kr}(2n-r)!}{r!(2n-2r+1)!}\left\{(2n-2r+1)(2n-2r)+4r(2n-r+1)+(2n+1-2r)-(2n+1)^2\right\}x^{2n+1-2r}$$

$$+\frac{5^{n+1}(2n+1)[(2n)!]x^{2n+1}}{0!(2n-1)!}+\frac{5(-1)^{kn}(2n+1)[(n+1)!]4x}{(n-1)!1!}+\frac{5^{n+1}(2n+1)[(2n)!]x^{2n+1}}{0!(2n)!}+\frac{5(-1)^{kn}(2n+1)[n!]x}{n!0!}$$

$$-\frac{5^{n+1}(2n+1)^3[(2n)!]x^{2n+1}}{0!(2n+1)!}-\frac{5(-1)^{kn}(2n+1)^3(n!)x}{n!1!}.$$

The expression within the brace of the summation is easily shown to be zero, and the remaining six individual terms are easily shown to be zero also.

Theorem 6. The polynomials $P_{2n+1}(x)$ satisfy

$$P_{2n+1}(x)=\frac{2}{\sqrt5}T_{2n+1}\left(\frac{\sqrt5}{2}x\right)\quad\text{or}\quad(-1)^n\frac{2i}{\sqrt5}T_{2n+1}\left(-i\frac{\sqrt5}{2}x\right)$$

according to k odd or even, where $T_{2n+1}(x)$ is the Chebyshev polynomial of the first kind [4].

Proof. For k odd,

$$P'_{2n+1}(x)=\frac{2}{\sqrt5}T'_{2n+1}\left(\frac{\sqrt5}{2}x\right)\frac{\sqrt5}{2}\quad\text{and}\quad P''_{2n+1}(x)=\frac{2}{\sqrt5}T''_{2n+1}\left(\frac{\sqrt5x}{2}\right)\frac54$$

by applying the chain rule.

Substituting into (7) and changing the variable x to z by $x = (2/\sqrt{5})z$, obtain

$$(1-z^2)T''_{2n+1}(z) - z\cdot T'_{2n+1}(z) + (2n+1)^2 T_{2n+1}(z) = 0$$

defining the required polynomials [4: 22.6.9 p. 781]. The case for k even may be handled similarly.

REFERENCES

1. David G. Beverage, "A Polynomial Representation of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 9, No. 5 (Dec. 1971), pp. 541–544.
2. Nathan Jacobson, *Lectures in Abstract Algebra,* D. Van Nostrand, 1951, Vol. 1, p. 9.
3. L. E. Dickson, *New First Course in the Theory of Equations,* John Wiley & Sons, 1960, p. 15, Th. 4.
4. *Handbook of Mathematical Functions,* U.S. Dept. Commerce, National Bureau of Standards, Applied Math Series 55, pp. 773–795.

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Let $k > 0$, $2|k$, $K := 4k+3$; the conditions

$$v_{k+1}r_k + v_k r_{k+1} = 2^K, \quad 0 < r_{k+1} \leqslant 2v_{k+1}, \quad 2 \nmid r_{k+1}$$

define the integers r_{k+1}, r_k uniquely. Then $2r_{k+1} < r_k$. Let

$$r_j := 2r_{j+1} + r_{j+2} \quad (j = k-1, k-2, \cdots, 1);$$

then

$$0 < 2r_{j+1} < r_j, \quad 2 \nmid r_j \Leftrightarrow 2 \nmid j, \quad v_{j+1}r_j + v_j r_{j+1} = 2^K \quad (j = k-1, k-2, \cdots, 1);$$

$j = 1$ gives

$$2r_1 + r_2 = 2^K, \quad 0 < 2r_1 < 2^K.$$

Let $y_k := 2\cdot 2^K + r_1$, $x_k := 3y_k + 2^K$; then $2\cdot 2^K < y_k$, $2 \nmid y_k$, $2 \nmid x_k$. The defining equation for x_k gives $H(x_k, y_k) = 2$. The defining equations for x_k, y_k, r_j $(j = 1, 2, \cdots, k-1)$ are the beginning of an algorithm by greatest and by nearest integers for x_k, y_k and therefore $N(x_k, y_k) > k$. For an arbitrary integer $s > 0$, let $g_s := x_s$, $h_s := y_s$ in case $2|s$ and $g_s := x_{s+1}$, $h_s := y_{s+1}$ in case $2 \nmid s$. This proves

Theorem 2. For every integer $s > 0$ there exist odd integers $g_s > h_s > 0$ with $E(g_s, h_s) \geqslant N(g_s, h_s) > s$, $H(g_s, h_s) = 2$.

Nothing is known about the average size of $H(a,b)$.

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FIBONACCI NUMBERS AND UPPER TRIANGULAR GROUPS

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In this note we call attention to the curious fact that the Fibonacci numbers arise when we look at that familiar example from group theory, the $n \times n$ nonsingular upper triangular matrices. Once incidence subgroups are defined the result follows quite easily.

Let K be any field with more than two elements and let K^* denote the nonzero elements of K. We define T_n to be the group of all nonsingular $n \times n$ upper triangular matrices over K. That is $T_n = \left\{ (a_{ij}) \mid a_{ij} = 0 \text{ if } i > j,\ a_{ii} \in K^*,\ a_{ij} \in K \right\}$. The key definition is as follows.

Definition. A subgroup, H, of T_n is an *incidence subgroup* if

(a) The relations defining H can be given entirely by specifying the domain for each a_{ij}.

(b) The two possibilities for each a_{ii} are $a_{ii} = 1$ or $a_{ii} \in F^*$.

(c) The two possibilities for a_{ij} when $i < j$ are $a_{ij} = 0$ or $a_{ij} \in F$.

Since $H \subseteq T_n$ we automatically have $a_{ij} = 0$ whenever $i > j$. By way of example we have

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a,b \in K,\ \ c \in K^* \right\}$$

is an incidence subgroup of T_3.

$$\left\{ \begin{pmatrix} 1 & a & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a \in K \right\}$$

is a subgroup but not an incidence subgroup since the (1,2) and (1,3) entries are dependent.

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a,b \in K \right\}$$

is not a subgroup.

We let G' denote the commutator subgroup of G. Then it is easily shown that

$$T'_n = \left\{ (a_{ij}) \mid a_{ii} = 1,\ \ a_{ij} \in F \ \text{ if } \ i < j \right\}.$$

For instance

$$T'_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a,b,c \in F \right\}$$

which is an incidence subgroup.

Our result is the following:

Proposition 1. The number of incidence sugroups, S, of T_n such that $S' = T'_n$ is F_{n+2}, where

$$\left\{ F_n \right\}_1^\infty = \left\{ 1, 1, 2, 3, 5, 8, \cdots \right\}$$

is the sequence of Fibonacci numbers.

Proof. We must have $T_n \supseteq S \supseteq T_n'$ so that if $S = \{(a_{ij})\}$ we then have $a_{ij} = 0$ for $i > j$, $a_{ij} \in K$ for $i < j$, and for each i we must specify either $a_{ii} = 1$ or $a_{ii} \in K^*$.

Suppose we specify $1 = a_{ii} = a_{i+1,i+1}$. Note that the commutator

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & & & a_{1n} \\ 0 & a_{22} & & & & a_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & a_{i,i+1} & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & & & b_{1n} \\ 0 & b_{22} & & & & b_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & b_{i,i+1} & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & b_{nn} \end{pmatrix}$$

Using block multiplication and the above computation we have

$$A^{-1}B^{-1}AB = \begin{pmatrix} 1 & c_{12} & \cdots & & & c_{1n} \\ 0 & 1 & & & & c_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & 0 & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & 1 \end{pmatrix}$$

and such matrices will not yield all of T_n'.

Similarly

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1-a^{-1} \\ 0 & 1 \end{pmatrix}$$

and we can generate T_2' by choosing a appropriately.

Alternatively both

$$H_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in F^*,\ b \in F \right\} \quad \text{and} \quad H_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \middle| a \in F^*,\ b \in F \right\}$$

are nonabelian. If every 2×2 block,

$$\begin{pmatrix} a_{ii} & a_{i,i+1} \\ 0 & a_{i+1,i+1} \end{pmatrix},$$

along the main diagonal is either H_1, H_2 or T_2 then $a_{i,i+1} \in F$ is specified for each i. This yields $S' = T_n'$. Thus if no two consecutive entries on the main diagonal are specified as 1's then $S' = T_n'$.

To complete the proof we need the standard result (for instance see Niven [1]) that the number of sequences of n plus and minus signs with no two minus signs adjacent is F_{n+2}.

Incidence subgroups are themselves an interesting topic. The term comes from incidence algebra as used in the study of locally finite partially ordered sets. The following facts are known. If K is finite then most normal and all characteristic subgroups of T_n' are incidence subgroups (see Weir [2]). The center or commutator subgroup of any incidence subgroup is itself an incidence subgroup. The number of normal incidence subgroups of T_n' is given by the Catalan numbers.

If the number of incidence subgroups of T_n' were known it might be useful in determining the number of finite T_0 topologies. However this is an unsolved problem for n larger than nine.

REFERENCES

1. I. Niven, *Mathematics of Choice,* Random House, 1965, New York, pp. 52–53.
2. A. Weir, "Sylow p-Subgroups of the General Linear Groups Over Fields of Characteristic *p*," *Proc. A.M.S.,* 6 (1955), pp. 454–464.

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ON THE INFINITE MULTINOMIAL EXPANSION[1]

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Abel, [1], about 150 years ago gave the first proof of the Binomial Theorem for the case of an arbitrary complex exponent. From Abel's result one can deduce various versions of the Multinomial Expansion. In this note we shall derive one such form.

Let $n, a_1, a_r, \cdots, a_r$ be complex numbers with n not equal to a non-negative integer. *If the inequalities*

$$(1) \qquad |a_j| < |a_1 + a_2 + \cdots + a_{j-1}|,$$

for $j = 2, 3, \cdots, r$, *all hold, then the following Multinomial Expansion holds:*

$$(2) \qquad \left(\sum_{i=0}^{n} a_i \right)^n = \sum \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{r-1}+1)}{n_1!n_2!\cdots n_{r-1}!} a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\cdots-n_{r-1}},$$

where the summation is an iterated summation taken under all $n_i \geqslant 0$, *where* i *first takes on the value* $r-1$, *then* $r-2$, *and so on until the last value, 1, is taken on.*

We first establish the following *triple summation expansion:*

$$(3) \qquad \left(\sum_{i=1}^{r} a_i \right)^n = \sum_{j=2}^{r} \sum_{k=1}^{\infty} \binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell \right)^{n-k} + a_1^n$$

if the inequalities (1) all hold. Here we use the usual convention that $\binom{n}{k} = 0$ when n is a positive integer and $k > n$. Formula (3) is of interest in its own right. This author has found it, as well as Formula (7), to be of use in the representation of integers in specialized arithmetical systems, such as the binary system.

Indeed, let $z_1 = 0$ and

$$z_j = \sum_{\ell=1}^{j-1} a_\ell$$

for $j \geqslant 2$, so that the right side of (3) becomes, by (1),

$$\sum_{j=1}^{r} ((z_j + a_j)^n - z_j^n) = \sum_{j=1}^{r} (z_{j+1}^n - z_j^n) = z_{r+1}^n,$$

which is precisely the left side of (3).

Since $n - k \neq 0$, we can apply Formula (3) to the summation under ℓ on the right side of (3). This iterative process can be continued. After m iterations of Formula (3), $m \geqslant 0$ and not too large, we obtain

1 Received by the editors in July, 1973.

$$(4)\quad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2m}}{\ell_{2m+2}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2m+1}}^{\ell_{2m+2}} \left(\sum_{\ell=1}^{\ell_{2m+1}-1} a_\ell\right)^{n-\ell_2-\cdots-\ell_{2m+2}}$$

$$+\sum_{k=1}^{m}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n.$$

Here the indices vary over

$$(5)\quad \begin{cases} 2 \leqslant \ell_1 \leqslant r, \\ 2 \leqslant \ell_{2i+1} \leqslant \ell_{2i-1}-1, & \text{for } 1 \leqslant i \leqslant m, \\ 1 \leqslant \ell_{2i+2} < \infty, & \text{for } 0 \leqslant i \leqslant m. \end{cases}$$

The only restriction on m is that $m \leqslant r-2$, so that the first two inequalities in (5) are possible.

We let $m = r-2$, for $r \geqslant 2$. Then, by (5), $\ell_{2r-3} = 2$, so that Formula (4) becomes

$$(6)\quad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-4}}{\ell_{2r-2}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2r-3}}^{\ell_{2r-2}} a_1^{n-\ell_2-\cdots-\ell_{2r-2}}$$

$$+\sum_{k=1}^{r-2}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n$$

$$=\sum_{k=1}^{r-1}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n.$$

We now extend the range of ℓ_{2i}, for $1 \leqslant i \leqslant r-1$, to include 0. Then, the summation under k reduces to a single term $k = r-1$; and, by (5), the subscripts are uniquely determined:

$$\ell_1 = r, \qquad \ell_3 = r-1, \quad \cdots, \quad \ell_{2r-3} = 2.$$

It now follows from (6) that

$$\left(\sum_{i=1}^{r} a_i\right)^n = \sum \frac{n(n-1)\cdots(n-\ell_2-\cdots-\ell_{2r-2}+1)}{\ell_2!\,\ell_4!\cdots\ell_{2r-2}!} a_r^{\ell_2} a_{r-1}^{\ell_4}\cdots a_2^{\ell_{2r-2}} a_1^{n-\ell_2-\cdots-\ell_{2r-2}}$$

this result being valid for all $r \geqslant 1$. Here, we are employing the usual convention that the empty sum is 0 and the empty product is 1.

The Multinomial Expansion (2), subject to the restrictions (1), now follows with a change of notation.

Another version of the Multinomial Theorem is

$$(7)\quad \left(\sum_{i=1}^{r} a_i\right)^n = (-1)^r \sum_{j=2}^{r} (-1)^j \left[\sum_{k=1}^{\infty} \binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell\right)^{n-k} + 2\left(\sum_{\ell=1}^{j-1} a_\ell\right)^n\right] + a_1^n;$$

valid under the conditions (1).

A good source for the Binomial Theorem and the Multinomial Theorem is Chrystal's *Algebra* [2], Volumes I and II. Our sequence of expository papers on the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [3], [4], [5], [6] and the present paper) will continue (Hilliker [7], [8]).

ADDENDUM. Here, as usual, z^n is defined to be that branch of the function $f(x) = e^{n \log z}$ defined over the complex z-plane with the nonpositive real axis included, and with $f(1) = 1$. That is, the logarithmic function is given by $\log z = \log |z| + i \arg z$ with $|\arg z| < \pi$. Our inequalities (1) imply that the quantities $a_1 + a_2 + \cdots + a_j$, for $1 \leqslant j \leqslant r$, are not 0. We shall need to assume that they are not negative real numbers. When n is a (negative) integer these restrictions which guarantee single-valuedness, may, naturally, be ignored. For more on this, and also for a development of the Binomial Theorem, that is, the Maclaurin expansion

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

for n and z complex and with $|z| < 1$, see Markushevich [9], I.

REFERENCES

1. Niels Henrick Abel, *Crelle's Journal,* I, 1826, p. 311. The proof for complex exponents also appears in Abel's *Oeuvres,* I, p. 219, Christiania, 1881.
2. G. Chrystal, *Textbook of Algebra,* Vols. I and II, Chelsea, New York, 1964. This is a reprint of works of 1886 and 1889. Also reprinted by Dover, New York, 1961.
3. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton, in regard to Newton's Discovery of the Binomial Theorem." First part, "Contributions of Pascal," *The Mathematics Student,* Vol. XL, No. 1 (1972), pp. 65–71.
4. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in regard to Newton's Discovery of the Binomial Theorem," second part, "Contributions to Archimedes," *The Mathematics Student,* Vol. XL, No. 4 (1973), 4 pp.
5. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in Regard to Newton's Discovery of the Binomial Theorem," third part, "Contributions of Cavalieri," *The Mathematics Student,* Vol. XLI, No. 2 (1974), 6 pp.
6. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in regard to Newton's Discovery of the Binomial Theorem," fourth part, "Contributions of Newton," *The Mathematics Student,* Vol. XLII, No. 1 (1974), 8 pp.
7. David Lee Hilliker, "On the infinite Multinomial Expansion," Second paper, *The Fibonacci Quarterly* (to appear).
8. David Lee Hilliker, "On the Multinomial Theorem," *The Fibonacci Quarterly* (to appear).
9. A. I. Markushevich, *Theory of Functions of a Complex Variable,* Volumes I, II, and III. Prentice-Hall, Englewood Cliffs, New Jersey, 1965. Translated from the Russian.

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THE SUM OF TWO POWERS IS A THIRD, SOMETIMES

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1. INTRODUCTION

We seek integer solutions to the Diophantine equation

$$x^n + y^m = z^k, \tag{1}$$

where n, m and k are positive integers. We have a general algorithm which sometimes augments primitive parameters to primitive solutions regardless of the choice of m, n, k. We classify the types of applications of this algorithm based on the greatest common divisor of the exponents. For some types all the primitive parameters augment to all the primitive solutions. For the type which includes the famous case $n = m = k > 2$, the finding of the primitive parameters which augment to primitive solutions is equivalent to the original problem. Without gain of generality (an expression of Professor DeHardt), we could extend this approach to a Diophantine equation with more powers on the left than two but only one power on the right.

2. HISTORY OF THE PROBLEM

In 1964 we obtained the computer solution $(1176)^2 + (49)^3 = (35)^4$ and from this one example we discovered our method of augmentation as well as a type of exponents for which we determined all primitive solutions. Subsequently that year Professor E. G. Strauss pointed out to us that this method could be applied successfully (i.e., yielding solutions) to another type. At this time we found that Basu [1] and others had found rational solutions for the first type mentioned above. Recently Beerenson [2] has found a similar method for finding integer solutions for this first type. At this later time we found that Teilhet [8] in 1903 used the method of augmentation for a special case $k = 3$, $m = n = 2$.

3. TRIVIAL SOLUTIONS

For completeness as well as for illustrating a simple case of primitive solutions, we now discuss the *trivial* solutions to (1), x_0, y_0, z_0, where $x_0y_0z_0 = 0$. Let us call the case $x = y = z = 0$, the zero case, and turn our attention elsewhere. Then exactly one of x_0, y_0, z_0 is zero, and the non-zero elements are both powers with common exponent the least common multiple of their corresponding exponents (x_0 corresponds to n, y_0 to m, z_0 to k). Thus for the *non-zero trivial* solutions with $x_0 = 0$, we say y_0, z_0 form a *primitive* solution if and only if there is no integer $d > 1$ such that $d^L | y_0$ and $d^L | z_0$ where $L = [m,k]$. Thus the possible candidates for a non-zero, trivial, primitive solution are: $y_0 = \pm 1$, $z_0 = \pm 1$.

4. PRIMITIVE SOLUTIONS AND THE CLASSIFICATION SCHEME

The computer example indicated to us that the usual definition of primitive solution x_0, y_0, z_0, namely, one where

$$(x_0, y_0) = (x_0, z_0) = (y_0, z_0) = 1,$$

was not adequate. Thus we give a new definition which reduces to the old when appropriate.

Definition: A solution u, v, w, $uvw \neq 0$, to (1) is called a (non-trivial) *primitive solution* if and only if there is no $t > 1$ such that $t^a | u$, $t^b | v$, $t^c | w$, where

$$a = L/n, \quad b = L/m, \quad c = L/k, \quad \text{and} \quad L = [n, m, k].$$

The case $n = m = k \geqslant 3$ is referred to as Fermat's Last Theorem (F.L.T.) wherein the conjecture states that there are no non-trivial solutions. This conjecture is true for $n < 25000$ [7]. If $(n,m,k) > 2$, then (1) for this type of exponents can be reduced to F.L.T.

The type $(n,m,k) = 2$ has not yet been completely resolved. If $n = 2h$, $m = 2i$, and $k = 2j$, and if $(h,i) = (h,j) = (i,j) = 1$, then (E.G. Strauss) all possible solutions can be obtained by augmentation. If $(h,i) \geqslant 2$, we can show there are no non-trivial solutions if F.L.T. holds. We conjecture the same holds for $(h,j) \geqslant 2$ and $(i,j) \geqslant 2$.

The type $(n,m,k) = 1$, but no one of n,m,k is relatively prime to the other two, is the only known type which sometimes yields a finite number of primitive solutions. In all other cases, as far as we know, if non-trivial solutions exist, there are an infinite number of primitive solutions.

We complete our classification scheme by mentioning the remaining type where one of n, m or k is relatively prime to the others. This is the "first type" referred to in Section 2.

5. THE METHOD OF AUGMENTATION

Let $D = [m,n]$ throughout this section.

Definition: Positive integers x_0 and y_0 are *primitive parameters* for (1) if and only if there is no $t > 1$ such that $t^d | x_0$ and $t^e | y_0$, where $d = D/n$ and $e = D/m$.

Definition: A primitive solution u,v,w, $uvw \neq 0$, to (1) is *an augmentation of primitive parameters* x_0, y_0 for (1) if and only if $u = x_0 z_0^d$, $v = y_0 z_0^e$, $z_0^{dn} = z_0^{em} = z_0^D$. If $z_0 > 1$, then we have a *proper augmentation.*

Theorem 1. If positive integers u,v,w form a primitive solution to (1) then there is a unique ordered pair x_0, y_0 which are primitive parameters so that u,v,w is an augmentation of x_0, y_0.

Proof. Let t be the largest positive integer for which $t^d | u$ and $t^e | v$ and $d = D/n$, $e = D/m$. Then $x_0 = u/t^d$ and $y_0 = v/t^e$ are primitive parameters, and u,v,w is an augmentation of x_0, y_0. Suppose x_1, y_1 are primitive parameters and

$$u = x_1 z_0^d, \quad v = y_1 z_0^e .$$

Let p be a prime such that $p^q \| t$ and $p^Q \| z_0$ and $q \neq Q$. Then $p^d | x_i$ and $p^e | y_i$, where $i = 0$ if $q < Q$ and $i = 1$ if $Q < q$. This contradicts the condition $x_i y_i$ are primitive parameters. Thus $t = z_0$ and $x_0 = x_1$ and $y_0 = y_1$.

Theorem 2. If x_0, y_0 are primitive parameters for (1) and $x_0^n + y_0^m$ is written as $a_k^k a_{k-1}^{k-1} \cdots a_2^2 a_1$, where each a_i, $i \neq k$, is squarefree and $(a_i, a_j) = 1$ for each $i < k$, $j < k$, $i \neq j$, then there is an augmentation to a positive primitive solution for (1) if and only if for each i, $1 \leqslant i < k$, either $a_i = 1$ or there is a solution g_i to $Dg_i \equiv -i \pmod{k}$ and g_i is the smallest such positive solution.

Proof. Suppose we have a primitive solution $u > 0$, $v > 0$, $w > 0$. Then

$$u = x_0 z_0^d, \quad v = y_0 z_0^e \quad \text{and} \quad w^k = a_k^k a_{k-1}^{k-1} \cdots a_2^2 a_1 z_0^D .$$

Hence

$$(w/a_k)^k = a_{k-1}^{k-1} \cdots a_2^2 a_1 z_0^D .$$

Suppose there is an i, $1 \leqslant i < k$, such that $a_i \neq 1$. Then for each prime p dividing a_i we have $p^i p^{gD} = p^{qk}$, where

$$p^g \| z_0 \quad \text{and} \quad p^q \| (w/a_k).$$

Thus $Dg \equiv -i \pmod{k}$. The smallest such positive solution is $< k/(D,k)$ [2, p. 51]. If $g > k/(D,k)$, then $gD > [D,k] = L = [n,m,k]$. Thus

$$p^L | u^n, \quad p^L | v^m, \quad p^L | w^k ,$$

and u,v,w is not a primitive solution.

Suppose the conditions hold, and we write b^c as $b \exp c$, then z_0^D is one if all $a_i = 1$ for $i \neq k$ and is the product of $a_i \exp D\, g_i$ for all i, $1 \leqslant i < k$ and $a_i \neq 1$, otherwise. Then

$$u = x_0 z_0^d, \quad v = y_0 z_0^e, \quad \text{and} \quad w = a_k \pi ,$$

where π is the product of the positive k^{th} roots of $a_i \exp f_i$, $f_i = Dg_i + i$, $a_i \neq 1$, or one. This is a solution to (1) but it may not be primitive when $z_0 \neq 1$.

If this is not a primitive solution, then there is a prime p such that

$$p^a|x_0z_0^d, \quad p^b|y_0z_0^e \quad \text{and} \quad p^c|w,$$

where $a = L/n$, $b = L/m$, $c = L/k$. If $p \nmid a_i$ for any i, $1 \leqslant i < k$, then $p \nmid z_0$ and $p^a|x_0$ and $p^b|y_0$. Since $L = DS$ for some integer S, $a = Sd$, $b = Se$, and x_0 and y_0 are not primitive parameters.

If $p|a_i$, then $p \exp g_i \| z_0$, and $g_i < k/(D,k)$. But

$$k/(D,k) = [D,k]/D = L/D = S.$$

Then $g_i \leqslant S - 1$, so

$$g_id \leqslant dS - d = a - d, \qquad g_ie \leqslant eS - e = b - e.$$

Since

$$p^a|x_0z_0^d, \qquad \text{and} \qquad p \exp g_id \,\|\, z_0^d,$$

then $p^d|x_0$; similarly $p^e|y_0$ and x_0, y_0 are not primitive parameters.

6. THE TYPE $(n,m,k) = 2$

Here $n = 2h$, $m = 2i$, $k = 2j$. For completeness we give a proof for a theorem in the literature [4] because it is easy and not too accessible.

V. A. Lebesque Theorem: If $x^{2t} + y^{2t} = z^2$ has a non-trivial solution then t is odd and $u^t + v^t = w^t$ has a non-trivial solution.

Proof. If t is even, we use the fact [3, p. 191] that $x^4 + y^4 = z^2$ has only trivial solutions. Then $x^t = 2rs$, $y^t = r^2 - s^2$ [3, p. 190]. But $(r + s, r - s) = 1$. (In this case the new and old definition of primitive are equivalent), hence $r + s = u^t$ and $r - s = v^t$, but either $2r = w^t$ or $2s = w^t$. In the former case, by adding $r + s$ to $r - s$, we obtain $u^t + v^t = w^t$. In the latter case subtract $r - s$ from $r + s$ and rename.

Lemma 1. If $n = 2$, $m = 2$, $k = 2t$, then all the primitive solutions to (1) are obtained by augmentation of primitive Pythagorean triples.

Proof. From a primitive Pythagorean triple [3, p. 190] x_0, y_0, z_0, we can use x_0, y_0 for primitive parameters, and if

$$x_0^2 + y_0^2 = a_{2t}^{2t}a_{2t-1}^{2t-1} \cdots a_2 a_1$$

under the conditions of Theorem 2, then $a_i = 1$ for all odd i and $2g_i \equiv -i \pmod{2t}$ can be solved for all needed even i. If $x_0^2 + y_0^2$ is not a square, then when written in the above form $a_i \neq 1$ form some odd i, and $2g_i \equiv -i \pmod{2t}$ has no solution, and there is no augmentation.

NOTE: Our method does not distinguish solutions 15, 20, 5, and 7, 24, 5 for $n = m = 2$, $k = 4$, except as proper or improper augmentations. For $n = k = 2$, $m = 4$ we use a general modification of Theorems 1 and 2 using

$$z_0^k - x_0^n = a_m^m a_{m-1}^{m-1} \cdots a_2^2 a_1.$$

Lemma 2. If $n = 2$, $m = 2s$, $k = 2t$, $(s,t) = 1$, then all primitive solutions to (1) are obtained from primitive solutions to (1) with $n = 2$, $k = 2$ by augmentation.

Proof. If x_0, y_0, z_0 is a primitive solution to (1) with $n = 2$, $m = 2s$, $k = 2$, then x_0, y_0 are primitive parameters for $n = 2$, $m = 2s$, $k = 2t$, $(s,t) = 1$, and the corresponding odd indexed $a_i = 1$, and $2sg_i \equiv -i \pmod{2t}$ can be solved for all even i. But if $y_0^2 + y_0^{2s}$ is not a square for x_0, y_0, primitive parameters then there is an odd i such that $a_i \neq 1$, and there is no solution to $2sg_i \equiv -i \pmod{2t}$.

Theorem 3. If

$$n = 2h, \quad m = 2i, \quad k = 2j, \quad (h,i) = (h,j) = (i,j) = 1$$

then there are an infinite number of primitive solutions to (1) obtained by none, one, or more augmentations of Pythagorean triples.

We do not give the proof since it repeats a third time essentially the proofs of the two lemmas.

7. THE TYPE $(m,n,k) = 1$ BUT NONE OF n,m,k IS RELATIVELY PRIME TO THE OTHER TWO

We know how to solve only $n = 2h$, $m = 3i$, $k = 6j$ for this type. We assume a result of Legendre [5], namely that $x^3 + y^3 = 2z^3$ implies $x = \pm y$. Using this hard to obtain result we give a proof of a theorem in the literature [6,9].

Thue-Lind Theorem. The only non-trivial primitive solutions to $x^2 + y^3 = z^6$ are $x = \pm 3$, $y = -2$, $z = \pm 1$.

Proof. First we note $(z^3 - x, z^3 + x) = 1$ or 2. In the former case, $z^3 - x = u^3$, $z^3 + x = v^3$, and $u^3 + v^3 = 2z^3$. By Legendre's result, $u = \pm v$. If $u = v$, $x = 0$, and if $u = -v$, then $z = 0$. Therefore for non-trivial solutions $(z^2 - x, z^3 + x) = 2$. Now

$$z^3 - x = 2y^3 \quad \text{and} \quad x^3 + x = 4v^3 \quad \text{or} \quad z^3 - x = 4u^3 \quad \text{and} \quad z^3 + x = 2v^3 .$$

One case can be obtained by the other by replacing x by $-x$, but x is a solution if and only if $-x$ is. Thus we consider the former case only. Then by adding, we obtain $z^3 + (-u)^3 = 2v^3$, and by Legendre's result, $z = u$ or $z = -u$. If $z = u$, then $v = 0$, and $y = 0$; thus for non-trivial solutions $z = -u$. Then $v = -u$, so, from $(u,v) = 1$, $u = \mp 1$, $v = \pm 1$; hence $y = -2$, $z = \pm 1$. Q.E.D.

Now any solution u, v, w to (1) for $n = 2h$, $m = 3i$, $k = 6j$ is a solution to the case $n = 2$, $m = 3$, $k = 6$ and hence

$$u^h = \pm 3a^3, \quad v^i = -2a^2, \quad w^j = \pm a.$$

If p is a prime greater than 3 and $p^d \| a$, then

$$h|3d, \quad i|2d, \quad \text{and} \quad j|d \quad \text{and} \quad [n,m,k]|6d$$

and this is not a primitive solution. Thus $a = 2^b 3^c$, and $j|b$ and $j|c$ and $h|3b$ and $i|2c$ and $h|1+3c$ and $i|1+2b$. Conversely if these conditions are met then there is a solution. Moreover, it can be shown there is a b and c if and only if $(h,i) = (h,j) = (i,j) = 1$. Note for $b = 4$ $c = 9$, we obtain 8, 9, 6 case as well as 8, 27, 6 case.

8. THE REMAINING CASE AND SUMMARY

The remaining case when one of n, m, k is relatively prime to the other two, then the conditions of Theorem 2 are always met and every set of primitive parameters augment, when the equation is written with the special exponent term being the only term of one side of the equation. For example, $n = 2$, $m = 3$, $k = 4$ then we write $z^4 - x^2 = y^3$, and, for example, 5, 24 being relatively prime are primitive parameters and $5^4 - 24^2 = 7^2$ from the Pythagorean triple, 7, 24, 25. Then we augment by 7^4 and obtain the solution we found on the computer.

Mr. Jim Grant, U.C.L.A. student, has also found an algorithm for obtaining rational solutions for this remaining case. He has made a real gain with his general approach because it not only applies to general Diophantine equations of this type but also applies to many other problems as well, including some differential equations.

REFERENCES

1. N. M. Basu, "On a Diophantine Equation," Ball, *Calcutta Math. Soc.* (1940), pp. 15–20.
2. R. G. Beerensson, "On the Equation $x^n \pm y^n = z^m$," *The Mathematical Gazette*, 54 (1970), pp. 138–139.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Oxford, Fourth Edition, 1960.
4. V. A. Lebesque (entry 31), L. E. Dickson, *History of the Theory of Numbers,* Vol. II, Chelsea, N.Y., 1952, p. 737.
5. A. M. Legendre (entry 184), L. E. Dickson, *ibid.,* p. 573.
6. Lind (entry 224), L. E. Dickson, *ibid,.,* p. 766.
7. J. L. Selfridge and B. W. Pollack, "Fermat's Theorem is True for any Exponent up to 25000," *Notices of AMS,* 11 (1964), p. 97.
8. P. F. Teilhet, "Cube somme de dans Carrés," *L'Intermediaire des Mathématicens,* Vol. 10, 1903, p. 210.
9. Axel Thue (entry 236), L. E. Dickson, *ibid.,* p. 580.

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THE RANK AND PERIOD OF A LINEAR RECURRENT SEQUENCE OVER A RING

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INTRODUCTION

Two problems from the theory of linear recurrent sequences are considered in this paper. The first is to establish the existence of the rank of the Lucas sequence over an arbitrary ring with an identity. In particular, a theorem of Wyler [10, Theorem 1], for second-order sequences over a commutative ring, is generalized to sequences of arbitrary order over an arbitrary (not necessarily commutative) ring. The second problem is to determine the period of a purely periodic Lucas sequence as a function of its rank. Solutions to this problem have previously been given in special cases: Vinson [7, Theorem 3] and Barner [1, Theorem 2] for the modular Fibonacci sequence; Ward [9] for modular integral sequences of arbitrary order in case the characteristic polynomial of the recurrence has distinct roots; and Wyler [10, Theorem 4] for second-order sequences over a commutative ring with odd prime power characteristic. A solution is given in the present paper for linear recurrent sequences of arbitrary order over an arbitrary commutative ring with an identity.

1. PERIODIC LINEAR RECURRENT SEQUENCES

Let R be an associative ring with an identity 1, and let $a_1, \cdots, a_k$ be elements of R. A sequence (w): $w_0, w_1, \cdots$ of elements in R that satisfy the recurrence

$$w_{n+k} = w_{n+k-1}a_1 + \cdots + w_n a_k$$

for $n \geqslant 0$ is said to be a (right) linear recurrent sequence associated with the list $(a_1, \cdots, a_k)$. Let $S(a_1, \cdots, a_k)$ be the collection of all linear recurrent sequences over R associated with $(a_1, \cdots, a_k)$ and let

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_k \\ 1 & 0 & \cdots & 0 & a_{k-1} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & a_1 \end{pmatrix} \in R^{k\times k}$$

be the companion matrix of $(a_1, \cdots, a_k)$.

The *Lucas sequence* of $S(a_1, \cdots, a_k)$ is the sequence (u): $u_0, u_1, \cdots$ associated with the list $(a_1, \cdots a_k)$ such that $u_0 = 0, \cdots, u_{k-2} = 0, u_{k-1} = 1$. (In case $k = 1$, then $u_0 = 1$). For n a non-negative integer and $\epsilon_k = (0, \cdots, 0, 1) \in R^k$, let $U_n \in R^{k\times k}$ be the matrix with $\epsilon_k A^{i-1+n}$ as its i^{th} row, $i = 1, \cdots, k$. Since the rows of U_0 are of the form $\epsilon_k = (0, \cdots, 0, 1)$, $\epsilon_k A = (0, \cdots, 1, *), \cdots, \epsilon_k A^{k-1} = (1, *, \cdots, *)$, then U_0 is invertible in $R^{k\times k}$.

Lemma 1. Let $(w) \in S(a_1, \cdots, a_k)$. Then for $n \geqslant 0$,

$$(w_n, \cdots, w_{n+k-1}) = (w_0, \cdots, w_{k-1})A^n = (w_0, \cdots, w_{k-1})U_0^{-1}U_n.$$

Proof. By finite induction on n, both the first equality and $U_n = U_0A^n$ are valid. Thus, since U_0 is invertible, then $A^n = U_0^{-1}U_n$, and the second equality holds.

Let $(w) \in S(a_1, \cdots, a_k)$. If there is a list of k consecutive elements of (w) that is equal to a preceding list of k consecutive elements of (w), then the sequence is said to be of finite period. Specifically, if

$$(w_{\alpha+\nu}, \cdots, w_{\alpha+\nu+k-1}) = (w_\alpha, \cdots, w_{\alpha+k-1}),$$

with $\alpha + \nu > \alpha \geqslant 0$, is the first such list, then $w_\alpha, w_{\alpha+1}, \cdots$ is periodic of period ν. In this case (w) is said to be periodic of index α and period ν. If the index $\alpha = 0$, then (w) is said to be purely periodic.

Similarly, the matrix A is said to be periodic if some term of the sequence $I, A, A^2, \cdots$ is equal to a preceding term.

If $A^{\alpha+\nu} = A^{\alpha}$, $\alpha+\nu > \alpha \geqslant 0$, is the first such term, then A is said to be periodic of index α and period ν.

Lemma 2. The following statements are equivalent:

(i) Every sequence of $S(a_1, \cdots, a_k)$ is periodic.

(ii) The Lucas sequence of $S(a_1, \cdots, a_k)$ is periodic.

(iii) A is periodic.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Let (u) be periodic. Then

$$U_{\alpha+\nu} = U_{\alpha}, \qquad \alpha+\nu > \alpha \geqslant 0.$$

Hence,

$$U_0 A^{\alpha+\nu} = U_0 A^{\alpha} \qquad \text{and} \qquad A^{\alpha+\nu} = A^{\alpha}.$$

That is, A is periodic.

(iii) $\Rightarrow$ (i). Let

$$A^{\alpha+\nu} = A^{\alpha}, \qquad \alpha+\nu > \alpha \geqslant 0.$$

Then

$$(w_{\alpha+\nu}, \cdots, w_{\alpha+\nu+k-1}) = (w_{\alpha}, \cdots, w_{\alpha+k-1}),$$

and (w) is periodic.

It is clear that the index of (w) is at most the index of A and that the period of (w) divides the period of A. Moreover, the index and period of the Lucas sequence are, respectively, the index and period of A.

Lemma 3. Let the Lucas sequence of $S(a_1, \cdots, a_k)$ be periodic. Then the following statements are equivalent:

(i) Every sequence of $S(a_1, \cdots, a_k)$ is purely periodic.

(ii) The Lucas sequence of $S(a_1, \cdots, a_k)$ is purely periodic.

(iii) a_k is right invertible in R.

(iv) a_k is not a right zero divisor in R.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial.

(ii) $\Rightarrow$ (iii). Let the Lucas sequence $(u) \in S(a_1, \cdots, a_k)$ be purely periodic. Then $U_\nu = U_0$ for $\nu > 0$. That is,

$$A^{\nu} = U_0^{-1} U_\nu = I.$$

If $[c_{ij}] = A^{\nu-1}$, then by direct calculation, $a_k c_{k,1} = 1$, and a_k is right invertible.

(iv) $\Rightarrow$ (i). Since (u) is periodic, then by Lemma 2, every $(w) \in S(a_1, \cdots, a_k)$ is periodic. Let

$$(w_0, \cdots, w_{k-1})A^{\alpha+\nu} = (w_0, \cdots, w_{k-1})A^{\alpha}, \qquad \alpha+\nu > \alpha \geqslant 0.$$

Also since a_k is not a right zero divisor, then A is right cancellable. Indeed, suppose $BA = 0$. Since

$$A^k = Ia_k + Aa_{k-1} + \cdots + A^{k-1}a_1,$$

then $Ba_k = 0$ and $B = 0$. Therefore,

$$(w_0, \cdots, w_{k-1})A^{\nu} = (w_0, \cdots, w_{k-1})$$

and (w) is purely periodic.

Reference is made at this point to DeCarli [4]; the main result given there follows immediately from Lemma 3.

2. THE RANK OF THE LUCAS SEQUENCE

A result of Wyler [8, Theorem 1], for second-order recurrences over a commutative ring, is now extended.

Theorem 1. Let $(u) \in S(a_1, \cdots, a_k)$ be the Lucas sequence, and suppose a_k is not a right zero divisor in R. Then there exists a unique non-negative integer ρ such that $u_n = 0, \cdots, u_{n+k-2} = 0$ if and only if n is a multiple of ρ. If $\rho = 0$, then (u) is not periodic. If $\rho > 0$, then (u) is periodic if and only if $u_{\rho+k-1}$ is of finite order in the unit group of R.

Proof. First, a matrix characterization of the condition $u_n = 0, \cdots, u_{n+k-2} = 0$ is provided. Specifically, suppose $u_n = 0, \cdots, u_{n+k-2} = 0$. Then

$$u_{n+k-1}\epsilon_k = (0, \cdots, 0, u_{n+k-1}) = \epsilon_k A^n, \quad \text{and} \quad u_{n+k-1}\epsilon_k A^i = \epsilon_k A^n A^i = \epsilon_k A^i A^n$$

for $i = 0, \cdots, k-1$. Therefore, $u_{n+k-1}U_0 = U_0A^n$. On the other hand, if $tU_0 = U_0A^n$ for some $t \in R$, then by checking the first row of this matrix, $u_n = 0, \cdots, u_{n+k-2} = 0$ and $u_{n+k-1} = t$. Consequently, $u_n = 0, \cdots, u_{n+k-2} = 0$ if and only if $A^n = U_0^{-1}tU_0$ for some $t \in R$; and in this case $t = u_{n+k-1}$.

Second, if $A^n = U_0^{-1}tU_0$, then t is not a right zero divisor in R. Indeed, suppose $vt = 0$ for $v \in R$. Then

$$vU_0A^n = vtU_0 = 0.$$

Since a_k is not a right zero divisor in R, then as in the proof of Lemma 3, A is right cancellable, $vU_0 = 0$ and $v = 0$.

The existence of ρ is now demonstrated. If $u_n = 0, \cdots, u_{n+k-2} = 0$ implies $n = 0$, then choose $\rho = 0$. In this case, by Lemma 3, the Lucas sequence (u) is not periodic. Thus, suppose $u_n = 0, \cdots, u_{n+k-2} = 0$ for some $n > 0$, and let ρ be the least such n. (If $k = 1$, then the condition is satisfied vacuously for every positive n and $\rho = 1$.) We show that every such n is a multiple of ρ. Indeed, let $A^\rho = U_0^{-1}sU_0$ with $s = u_{\rho+k-1}$. Then

$$A^{\rho q} = U_0^{-1}s^qU_0 \qquad \text{and} \qquad u_{\rho q} = 0, \cdots, u_{\rho q+k-2} = 0.$$

On the other hand, suppose

$$A^n = U_0^{-1}tU_0, \quad t \in R, \quad n = \rho q + \lambda, \quad 0 \leqslant \lambda < \rho.$$

Then

$$U_0^{-1}tU_0 = A^n = A^\lambda A^{\rho q} = A^\lambda U_0^{-1}s^qU_0,$$

where s^q is not a right zero divisor. Define

$$[d_{ij}] = D = U_0A^\lambda U_0^{-1}.$$

Then $Ds^q = It$. Since $d_{ij}s^q = 0$ for $i \neq j$, then $d_{ij} = 0$ for $i \neq j$. Also since

$$d_{ii}s^q = t = d_{11}s^q,$$

then $d_{ii} = d_{11} = d$, say, $i = 1, \cdots, k$. That is, $D = Id$ and $A^\lambda = U_0^{-1}dU_0$. Hence, by definition of ρ, it follows that $\lambda = 0$ and $n = \rho q$. That is, the desired ρ exists and is unique.

Finally, the last statement of the theorem is demonstrated. Indeed, if $s^q = 1$, then $A^{\rho q} = U_0^{-1}s^qU_0 = I$ and, by Lemma 2, (u) is periodic. Conversely, if (u) is periodic, then it is purely periodic and $A^\nu = I, \nu > 0$. Therefore, $A^\nu = U_0^{-1}1U_0$ and $\nu = \rho q$ for some q. Consequently,

$$I = A^{\rho q} = U_0^{-1}s^qU_0, \qquad Is^q = I,$$

$s^q = 1$, and $s = u_{\rho+k-1}$ is of finite order in the unit group of R.

The non-negative integer ρ of Theorem 1 is called the *rank* of the Lucas sequence associated with $(a_1, \cdots, a_k)$.

Corollary 1. Suppose a_k is not a right zero divisor in R. Let ρ be the rank of the Lucas sequence $(u) \in S(a_1, \cdots, a_k)$, and let $(w) \in S(a_1, \cdots, a_k)$. If $w_0 = 0$, then $w_\rho = 0$.

Proof. Let $\epsilon_1 = (1, 0, \cdots, 0) \in R^k$ and $\epsilon_k = (0, \cdots, 0, 1) \in R^k$. Since $\epsilon_k' = U_0\epsilon_1'$, where the prime denotes transpose, then $U_0^{-1}\epsilon_k' = \epsilon_1'$. Therefore,

$$A^\rho\epsilon_1' = U_0^{-1}u_{\rho+k-1}U_0\epsilon_1' = U_0^{-1}u_{\rho+k-1}\epsilon_k' = U_0^{-1}\epsilon_k'u_{\rho+k-1} = \epsilon_1'u_{\rho+k-1},$$

and

$$w_\rho = (w_\rho, \cdots, w_{\rho+k-1})\epsilon_1' = (w_0, \cdots, w_{k-1})A^\rho\epsilon_1' = (w_0, \cdots, w_{k-1})\epsilon_1'u_{\rho+k-1} = w_0u_{\rho+k-1}.$$

Consequently, if $w_0 = 0$, then $w_\rho = 0$. (Compare [3, Theorem 1].)

3. RELATIONS BETWEEN THE RANK AND PERIOD

In this section R is a *commutative* ring with identity 1. Also, (x,y) and $[x,y]$ denote the greatest common divisor and least common multiple of the positive integers x and y.

Theorem 2. Suppose a_k is not a zero divisor in R. Let the Lucas sequence $(u) \in S(a_1, \cdots, a_k)$ be of rank $\rho > 0$. Then (u) is periodic if and only if a_k is of finite order in the unit group of R. In this case, let ν be the period of (u), and let δ and β be the orders of $(-1)^{k-1}$ and $u_{\rho+k-1}$, respectively, in the unit group of R. Then

(i) $\nu = \rho\beta = (k, \beta)[\delta, \rho]$.
(ii) (k,β) is the order of $u_{\rho+k-1}^{[\delta,\rho]/\rho}$.

Proof. Since R is commutative, then

$$A^\rho = U_0^{-1}u_{\rho+k-1}U_0 = u_{\rho+k-1}I \qquad \text{and} \qquad ((-1)^{k-1}a_k)^\rho = \det A^\rho = (u_{\rho+k-1})^k .$$

Therefore, a_k is of finite order if and only if $u_{\rho+k-1}$ is of finite order. Consequently, by Theorem 1, (u) is periodic if and only if a_k is of finite order.

Now, suppose (u) is periodic of period ν, and let δ and β be the orders of $(-1)^{k-1}a_k$ and $u_{\rho+k-1}$, respectively, in the unit group of R. Since

$$I = A^\nu = (A^\rho)^{\nu/\rho} = (u_{\rho+k-1}I)^{\nu/\rho},$$

then $\beta|\nu/\rho$. On the other hand since

$$A^{\rho\beta} = (u_{\rho+k-1}I)^\beta = I,$$

then $\nu|\rho\beta$. Therefore, $\nu = \rho\beta$. Moreover, the order of

$$((-1)^{k-1}a_k)^\rho = (u_{\rho+k-1})^k$$

is $\delta/(\delta,\rho) = \beta/(k,\beta)$. Since $\delta/(\delta,\rho) = [\delta,\rho]/\rho$, then

$$\rho\beta = (k,\beta)[\delta,\rho].$$

Finally, since $\beta/(k,\beta) = [\delta,\rho]/\rho$, then (k,β) is the order of $u_{\rho+k-1}^{[\delta,\rho]/\rho}$.

The first part of (i) in Theorem 2 is due to Carmichael [2]. The second part of (i) is an extension of a result of Ward [9] for modular integral sequences. (See also Robinson [6].)

Corollary 2. Let the conditions be as in Theorem 2. Then

(i) $\delta|\nu$.
(ii) $\beta|k\delta$.
(iii) $\beta|k$ if and only if $\delta|\rho$.

This corollary includes several facts that have been previously observed for some special sequences. For example, let 0, 1, 1, 2, 3, 5, ⋯ be the sequence of Fibonacci numbers reduced modulo $m > 2$. In this case, $k = 2$, $a_1 = a_2 = 1$, and $\delta = 2$. In particular $2|\nu$. (See for example Wall [8, Theorem 4].) Also, $\beta|4$, and $\beta|2$ if and only if $2|\rho$. In other words, $\beta|2$ if $2|\rho$ and $\beta = 4$ if $2\nmid\rho$. (See Vinson [7, Theorem 3].)

Corollary 3. Let the conditions be as in Theorem 2, and suppose k is a prime. Then

(i) $\nu = k[\delta,\rho]$ if $u_{\rho+k-1}^{[\delta,\rho]/\rho} \neq 1$.

(ii) $\nu = [\delta,\rho]$ if $u_{\rho+k-1}^{[\delta,\rho]/\rho} = 1$.

In particular, the relation between the rank and period of the Fibonacci sequence modulo a prime may now be given. (See Barner [1, Theorem 2] or Herrick [5, Theorem 3].)

Corollary 4. Let the Fibonacci sequence reduced modulo an odd prime be of rank ρ and period ν. Then

(i) $\nu = 4\rho$ if $2\nmid\rho$.
(ii) $\nu = 2\rho$ if $2|\rho$, $2|\rho/2$.
(iii) $\nu = \rho$ if $2|\rho$, $2\nmid\rho/2$.

Proof. Let R be the ring of integers modulo an odd prime; in particular, $k = 2$ and $\delta = 2$. If ρ is odd, then $\beta = 4$ and, by Theorem 2(i), $\nu = 4\rho$. Thus, suppose ρ is even and let A be the companion matrix associated with $a_1 = 1$, $a_2 = 1$. Clearly

$$(-1)^{\rho/2}A^{\rho/2} = (\det A^{\rho/2})A^{\rho/2} = ((\operatorname{adj} A^{\rho/2})A^{\rho/2})A^{\rho/2} = (\operatorname{adj} A^{\rho/2})A^\rho = (\operatorname{adj} A^{\rho/2})u_{\rho+1}.$$

Since the off diagonal elements of $A^{\rho/2}$ are not zero and are the negatives of the off diagonal elements of adj $A^{\rho/2}$, then it follows that $u_{\rho+1} = -(-1)^{\rho/2}$. Therefore, since $[2,\rho]/\rho = 1$,

$$u_{\rho+1}^{[2,\rho]/\rho} = u_{\rho+1} = -(-1)^{\rho/2} = \begin{cases} -1 & \text{if } 2|\rho/2 \\ 1 & \text{if } 2\nmid\rho/2 . \end{cases}$$

Consequently, by Corollary 3, $\nu = 2\rho$ if $2|\rho/2$ and $\nu = \rho$ if $2\nmid\rho/2$.

A slight extension of the foregoing argument provides another proof of the main theorem of Wyler [10]. In fact, Wyler [10, Theorem 4] is valid for every purely periodic second-order Lucas sequence over a commutative ring with 1 satisfying the following two properties: $1 + 1$ is not a zero divisor, and $u^2 = 1$ implies either $u = 1$ or $u = -1$.

REFERENCES

1. K. Barner, "Zur Fibonacci-Folge modulo p," *Monatsh. Math.,* 69 (1965), pp. 97–104.
2. R. D. Carmichael, "On Sequences of Integers Defined by Recurrence Relations," *Quart. J. Math.,* 48 (1920), pp. 343–372.
3. E. C. Dade, D. W. Robinson, O. Taussky, and M. Ward, "Divisors of Recurrent Sequences," *J. Reine Angew. Math.,* 214/215 (1964), pp. 180–183.
4. R. J. DeCarli, "Periodicity Over the Ring of Matrices," *The Fibonacci Quarterly,* Vol. 11, No. 5 (Dec. 1973), pp. 466–468.
5. D. L. Herrick, "On the Periodicity of the Terminal Digits in the Fibonacci Sequence," *The Fibonacci Quarterly,* Vol. 11, No. 5 (Dec. 1973), pp. 535–538.
6. D. W. Robinson, "The Fibonacci Matrix Modulo m," *The Fibonacci Quarterly,* Vol. 1, No. 1 (Feb. 1963), pp. 29–36.
7. J. Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," *The Fibonacci Quarterly,* Vol. 1, No. 1 (Feb. 1963), pp. 37–45.
8. D. D. Wall, "Fibonacci Series Modulo m," *Amer. Math. Monthly,* 67 (1960), pp. 525–532.
9. M. Ward, "The Modular Period of a Linear Integral Recurrence," unpublished manuscript dated Europe; July–October, 1962.
10. O. Wyler, "On Second-Order Recurrences," *Amer. Math. Monthly,* 72 (1965), pp. 500–506.

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LETTER TO THE EDITOR

GENERALIZED FIBONACCI NUMBERS AND UNIFORM DISTRIBUTION MOD 1

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In the following I want to comment on a paper by William Webb concerning the distribution of the first digits of Fibonacci numbers [1] and to give a partial answer to some questions raised by the author. In fact, restriction to Fibonacci-related sequences makes it possible to obtain a number of results. (F_n) or 1, 1, 2, 3, 5, ⋯ stands for the sequence of Fibonacci numbers.

Theorem 1. Let k be an integer different from 0. Then the sequence $(\log F_n^{1/k})$ is uniformly distributed mod 1 (abbreviated u.d. mod 1).

Proof. We apply a classic result of J. G. van der Corput: Let (u_n) be a sequence of real numbers. If

$$\lim_{n\to\infty} (u_{n+1} - u_n)$$

exists and is irrational, then the sequence (u_n) is u.d. mod 1. See [2], p. 28.

Now set $u_n = \log F_n^{1/k}$. Then

$$u_{n+1} - u_n = \log F_{n+1}^{1/k} - \log F_n^{1/k} = \frac{1}{k}\log\frac{F_{n+1}}{F_n},$$

which tends to

[Continued on page 253.]

THE SUMS OF CERTAIN SERIES CONTAINING HYPERBOLIC FUNCTIONS

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1. INTRODUCTION

In this paper we are concerned with the summation of a number of series. They are

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^{4p-1}\sinh r\pi}, \quad \sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}}, \quad \sum_{r=0}^{\infty} \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}, \quad \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^{4p-3}\cosh (2r+1)\frac{\pi}{2}}$$

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}r^{4p-3}}{\sinh r\pi}, \quad \sum_{r=0}^{\infty} \frac{(-1)^r(2r+1)^{4p-1}}{\cosh (2r+1)\frac{\pi}{2}}$$

and

$$\sum_{r=1}^{\infty} \frac{\left\{2^{4p}\coth r\frac{\pi}{2} - \coth 2r\pi\right\}}{r^{4p+1}},$$

where $p = 1, 2, 3, \cdots$.

Certain of the above series have been extensively discussed in the past. Results for particular values of p are given by Ramanujan in [4], while Phillips, Sandham and Watson in [3, 5, 6] have determined, by varying methods, sums for general p. The last series of the group, however, seems to have received less attention. It differs from the others in that it contains the inverse powers of $4p + 1$. Further, it is closely related to the Riemann Zeta function $\zeta(4p + 1)$. As this paper shows, the sums of the series, where they are not identically zero, satisfy recursive relations containing binomial coefficients.

Thus if we write

$$T_{4p-1} = \frac{(-1)^p(4p)!}{\pi^{4p-1}2^{2p-2}} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{4p-1}\sinh r\pi}$$

then

$$\sum_{p=1}^{n} \binom{4n+2}{4p} T_{4p-1} = 1 \qquad n = 1, 2, \cdots.$$

The recursive relations are themselves of interest and can be inverted. Their inversion, which leads to the sums of the various series, involves the Bernoulli and the lesser known Euler numbers.

All results are obtained by considering the Neumann problem for the rectangle. Although this problem is of an elementary nature and is in fact discussed in both contemporary and established literature on Laplace's equation, a complete solution to it does not seem to be available. Kantorovich and Krylov in [2] proposed a method of solution but the suggested method contains, as we shall show, an error of principle. Once this error is removed the method can be applied to solve the problem. Initially, therefore, we state and solve the Neumann problem for the rectangle and then subsequently in Section 3 make appropriate use of the solution to obtain the various results.

2. THE NEUMANN PROBLEM FOR THE RECTANGLE

This problem requires the determination of a function $\phi(x,y)$ satisfying

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for} \quad 0 < x < a, \quad 0 < y < b \tag{2.1}$$

$$\phi_y(x,0) = f(x), \quad \phi_y(x,b) = g(x) \quad \text{for} \quad 0 \leqslant x \leqslant a \tag{2.2}$$

$$\phi_x(0,y) = F(y), \quad \phi_x(a,y) = G(y) \quad \text{for} \quad 0 \leqslant y \leqslant b, \tag{2.3}$$

where $f(x)$, $g(x)$, $F(y)$ and $G(y)$ are known functions and the subscripts x and y are used to denote partial differentiation.

It is necessary for a solution that

$$\int_c \frac{\partial \phi}{\partial n}\, ds = 0, \tag{2.4}$$

where c is the boundary of the rectangle, $\partial/\partial n$ denotes differentiation with respect to the outward normal to c and s refers to arc length. The condition (2.4) is equivalent to

$$\int_0^a (f-g)dx + \int_0^b (F-G)dy = 0. \tag{2.5}$$

We now briefly describe the method used by Kantorovich and Krylov in [2]. We put $\Phi = U + V$, where U and V are functions of x and y. We choose the function U so that it satisfies (2.1), (2.2) and $U_x(0,y) = U_x(a,y) = 0$ for $0 \leqslant y \leqslant b$, while V satisfies (2.1), (2.3) and $V_y(x,0) = V_y(x,b) = 0$ for $0 \leqslant x \leqslant a$.

Thus, the original Neumann problem is replaced by two other Neumann problems, one for U and the other for V. It is evident that if we can find U and V we shall fulfill the conditions imposed on ϕ by (2.1) to (2.3). By virtue of (2.4) the existence of U requires

$$\int_0^a (f-g)dx = 0.$$

Likewise, the existence of V requires

$$\int_0^b (F-G)dy = 0.$$

However, given functions f, g, F and G satisfying (2.5), it does not necessarily follow that the integrals

$$\int_0^a (f-g)dx \quad \text{and} \quad \int_0^b (F-G)dy$$

are each zero, and therefore the functions U and V may not exist. Yet the difficulty is readily overcome. We write

$$\phi = A(x^2 - y^2) + U + V,$$

where A is some constant to be found, while the functions U and V each satisfy (2.1) and the further conditions:

$$U_x(0,y) = U_x(a,y) = V_y(x,0) = V_y(x,b) = 0$$

$$U_y(x,0) = f(x), \quad U_y(x,b) = g(x) + 2Av \quad \text{for} \quad 0 \leqslant x \leqslant a$$

$$V_x(0,y) = F(y), \quad V_x(a,y) = G(y) - 2Aa \quad \text{for} \quad 0 \leqslant y \leqslant b.$$

Using (2.4) we require for the existence of U and V

$$\int_0^a \left\{ g(x) + 2Ab - f(x) \right\} dx = 0, \quad \text{i.e.,} \quad 2abA = \int_0^a (f-g)dx$$

and

$$\int_0^b \left\{ G(y) - 2Aa - F(y) \right\} dy = 0 \quad \text{or} \quad 2abA = \int_0^b (G-F)dy.$$

Equation (2.5) shows that these two expressions for A are consistent. Having found A, we can now follow the procedure given in [2] to determine U and V. In fact, it can be verified directly that to within an arbitrary constant Φ is given by

$$\phi = A(x^2 - y^2) + \tfrac{1}{2}f_0 y + \tfrac{1}{2}F_0 x + \sum_{r=1}^{\infty} \frac{a\left\{ g_r \cosh \frac{r\pi y}{a} - f_r \cosh \frac{r\pi}{a}(b-y)\right\}}{r\pi \sinh \frac{r\pi b}{a}} \cos \frac{r\pi x}{a} \tag{2.6}$$

$$+ \sum_{r=1}^{\infty} \frac{b\left\{ G_r \cosh \frac{r\pi x}{b} - F_r \cosh \frac{r\pi}{b}(a-x)\right\}}{r\pi \sinh \frac{r\pi a}{b}} \cos \frac{r\pi y}{b},$$

where f_r, g_r $(r = 0, 1, 2, \cdots)$ are the Fourier cosine coefficients for $f(x)$ and $g(x)$, respectively, over the range $0 \leqslant x \leqslant a$ and F_r, G_r $(r = 0, 1, 2, \cdots)$ are the Fourier cosine coefficients of $F(y)$ and $G(y)$ over $0 \leqslant y \leqslant b$.

3. APPLICATION OF THE SOLUTION TO THE NEUMANN PROBLEM

We put $a = b = \pi$ and define functions $\phi(x,y,4n)$, where $n = 1, 2, 3, \cdots$, by

$$2\phi(x,y,4n) = (x+iy)^{4n} + (x-iy)^{4n}. \tag{3.1}$$

It is readily verified that these functions satisfy (2.1). Further, using (2.2) and (2.3), we deduce for them that $f(x)$ and $F(y)$ are both identically zero. In addition

$$g(x) = 2n\left\{(\pi+ix)^{4n-1} + (\pi-ix)^{4n-1}\right\} \quad \text{and} \quad G(y) = 2n\left\{(\pi+iy)^{4n-1} + (\pi-iy)^{4n-1}\right\}.$$

Thus, the Fourier coefficients f_r and F_r are all zero, while $g_r = G_r = I_r(n)$ $(r = 1, 2, \cdots)$, where

$$I_r(n) = Re\,\frac{4n}{\pi}\int_0^{\pi} [(\pi+ix)^{4n-1} + (\pi-ix)^{4n-1}]e^{irx}dx \tag{3.2}$$

using the result

$$2abA = \int_0^a (f-g)dx$$

we find that the constant A vanishes and hence with the help of (2.6) we can write

$$\phi(x,y,4n) = c_{4n} + \sum_{r=1}^{\infty} I_r(n)\,\frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r \sinh r\pi} \tag{3.3}$$

the c_{4n} $(n = 1, 2, \cdots)$ being constants which have yet to be determined. Successive integration by parts of (3.2) leads to the result

$$I_r(n) = \frac{(-1)^{n+r}}{r^2}\pi^{4n-3}2^{2n}(4n)(4n-1) + \frac{4n}{r^4}(4n-1)(4n-2)(4n-3)I_r(n-1). \tag{3.4}$$

In particular

$$I_r(1) = (-1)^{r+1}\frac{48\pi}{r^2}$$

so that putting $n = 1$ in (3.1) and (3.3) we find

$$x^4 - 6x^2y^2 + y^4 = c_4 + 48\pi \sum_{r=1}^{\infty} (-1)^{r+1}\frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r^3 \sinh r\pi}. \tag{3.5}$$

Repeated application of (3.4) yields

$$I_r(n) = (-1)^r \left\{ \frac{a_2(n)}{r^2} + \frac{a_6(n)}{r^6} + \frac{a_{10}(n)}{r^{10}} + \frac{a_{4n-2}(n)}{r^{4n-2}} \right\}, \tag{3.6}$$

where, for example,

$$a_2(n) = (-1)^n \pi^{4n-3} 2^{2n} 4n(4n-1) \tag{3.7}$$

and more generally

$$a_{4p-2}(n) = (-1)^{n-p+1} \pi^{4n-4p+1} 2^{2n-2p+2} (4p-2)! \binom{4n}{4p-2}, \quad p = 1, 2, \cdots, n. \tag{3.8}$$

Using this last result, it follows

$$a_{4p+2}(n+1) = (4n+4)(4n+3)(4n+2)(4n+1) a_{4p-2}(n)$$

and hence from (3.6) that

$$(4n+4)(4n+3)(4n+2)(4n+1) \frac{I_r(n)}{r^4} = I_r(n+1) + \frac{(-1)^{r+1}}{r^2} a_2(n+1). \tag{3.9}$$

We now proceed to find the constants c_{4n} occurring in (3.3). We integrate Eq. (3.3) twice with respect to x and twice with respect to y. These integrations will introduce arbitrary functions of x and y. We have, therefore,

$$\frac{-\phi(x, y, 4n+4)}{(4n+1)(4n+2)(4n+3)(4n+4)} + xP_n(y) + Q_n(y) + yp_n(x) + q_n(x)$$

$$= c_{4n} \frac{x^2y^2}{4} - \sum_{r=1}^{\infty} I_r(n) \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r^5 \sinh r\pi},$$

where $p_n(x)$, $q_n(x)$, $P_n(y)$ and $Q_n(y)$ are arbitrary functions which may depend on n. Noting the result contained in (3.9) we can write this equation in the alternative form

$$\phi(x, y, 4n+4) = \sum_{r=1}^{\infty} \left\{ I_r(n+1) + \frac{(-1)^{r+1}}{r^2} a_2(n+1) \right\} \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r \sinh r\pi}$$

$$+ (4n+1)(4n+2)(4n+3)(4n+4) \left\{ xP_n(y) + Q_n(y) + yp_n(x) + q_n(x) - c_{4n} \frac{x^2y^2}{4} \right\}.$$

This reduces with the help of (3.3) and (3.5) to

$$0 = -c_{4n+4} + \frac{a_2(n+1)}{48\pi} (x^4 - 6x^3y^2 + y^4 - c_4)$$

$$+ (4n+1)(4n+2)(4n+3)(4n+4) \left\{ xP_n(y) + Q_n(y) + yp_n(x) + q_n(x) - c_{4n} \frac{x^2y^2}{4} \right\}.$$

This is an identity. Hence equating to zero the coefficient of x^2y^2 we deduce with the aid of (3.7)

$$c_{4n} = \frac{(-1)^n \pi^{4n} 2^{2n}}{(2n+1)(4n+1)}. \tag{3.10}$$

Thus we have

$$(x+iy)^{4n} + (x-iy)^{4n} = 2c_{4n} + 2 \sum_{r=1}^{\infty} I_r(n) \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r \sinh r\pi}, \tag{3.11}$$

where c_{4n} is given by (3.10) and $I_r(n)$ by results (3.6) and (3.8). Putting $x = y = 0$ in (3.11) and simplifying we obtain

$$0 = \frac{1}{(4n+1)(4n+2)} + \sum_{r=1}^{\infty} \frac{(-1)^r}{r \sinh r\pi} \left\{ \sum_{p=1}^{n} \frac{(-1)^{p-1}(4p-2)! \binom{4n}{4p-2}}{\pi^{4p-1} 2^{2p-2} r^{4p-2}} \right\} \quad n = 1, 2, \cdots.$$

Thus if we write

$$T_{4p-1} = \frac{(-1)^p(4p)!}{\pi^{4p-1}2^{2p-2}} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{4p-1}\sinh r\pi}, \quad p = 1, 2, \cdots \tag{3.12}$$

then it follows T_{4p-1} satisfies the recursive relation

$$1 = \sum_{p=1}^{n} \binom{4n+2}{4p} T_{4p-1} \quad n = 1, 2, \cdots . \tag{3.13}$$

This is the first of our results. We now show how this recursive relation can be inverted to give T_{4p-1} in terms of the Bernoulli numbers. To do this, we observe that (3.13) can be put in the alternative form

$$\frac{1}{(4n+2)!} = \sum_{p=1}^{n} \frac{T_{4p-1}}{(4p)!(4n+2-4p)!} .$$

Multiplying both sides of this equation by x^{4n+2} and summing from $n = 1$ to ∞ yields

$$\sum_{n=1}^{\infty} \frac{x^{4n+2}}{(4n+2)!} = \sum_{n=1}^{\infty} \sum_{p=1}^{n} \frac{T_{4p-1}x^{4n+2}}{(4p)!(4n+2-4p)!} = \left\{ \sum_{p=1}^{\infty} \frac{T_{4p-1}x^{4p}}{(4p)!} \right\} \left\{ \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} \right\} .$$

After some manipulation we obtain

$$\sum_{p=1}^{\infty} T_{4p-1} \frac{x^{4p}}{(4p)!} = 1 - \frac{x^2}{\cosh x - \cos x} = 1 + \frac{x^2}{2} \operatorname{cosech} ax \operatorname{cosech} iax, \tag{3.14}$$

where $2a = 1 + i$.

Using the expansion of cosech x given in [1] Eq. (3.15) leads after some simplification to

$$T_{4p-1} = \frac{(-1)^{p+1}}{2^{2p-2}} \sum_{q=0}^{2p} (-1)^2(2^{2q-1}-1)(2^{4p-2q-1}-1) \binom{4p}{2q} B_q B_{2p-q} .$$

It should be noted that B_0 is taken as -1 while the Bernoulli numbers are defined here by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{p=1}^{\infty} (-1)^{p+1} B_p \frac{x^{4p}}{(2p)!} .$$

With the help of (3.12) we deduce

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^{4p-1}\sinh r\pi} = \pi^{4p-1} \sum_{q=0}^{2p} \frac{(-1)^q(2^{2q-1}-1)(2^{4p-2q-1}-1)}{(2q)!\,(4p-2q)!} B_q B_{2p-q} .$$

In a similar manner if we put $x = y = \pi$ in (3.11) and define s_{4p-1} by

$$s_{4p-1} = (-1)^{p-1}\pi^{1-4p}2^{-2p+2}(4p)! \sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}}$$

then

$$\sum_{p=1}^{n} \binom{4n+2}{4p} s_{4p-1} = 2n(4n+3) \tag{3.16}$$

s_{4p-1} can also be expressed in terms of the Bernoulli numbers. By writing (3.16) in the form

$$\sum_{p=1}^{n} \frac{s_{4p-1}}{(4n-4p+2)!(4p)!} = \frac{1}{2}\left\{\frac{1}{(4n)!} - \frac{2}{(4n+2)!}\right\}$$

and following a procedure similar to that for T_{4p-1} we find

$$\sum_{p=1}^{\infty} s_{4p-1} \frac{x^{4p}}{(4p)!} = -\frac{x^2}{2} \coth ax \coth iax - 1.$$

Since (see [1]),

$$x \coth x = \sum_{p=0}^{\infty} (-1)^{p+1} B_p 2^{2p} \frac{x^{2p}}{(2p)!}$$

we have

$$s_{4p-1} = 2^{2p} \sum_{q=0}^{2p} (-1)^{p+q} \binom{4p}{2q} B_q B_{2p-q}$$

giving

(3.17) $$\sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}} = 2^{4p-2}\pi^{4p-1} \sum_{q=0}^{2p} \frac{(-1)^{q+1} B_q B_{2p-q}}{(2q)!(4p-2q)!}.$$

We next put $x = 0$, $y = \pi$ in (3.11) and subtract from twice the result the expressions obtained by putting $x = y = 0$ and $x = y = \pi$. This leads to

(3.18) $$\pi^{4n}[1+(-1)^{n+1}2^{2n-1}] = \sum_{r=0}^{\infty} 2I_{2r+1}(n) \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}.$$

Writing

(3.19) $$Q_{4p-1} = (-1)^{p-1}\pi^{-4p+1}2^{-2p+4}(4p)! \sum_{r=0}^{\infty} \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}$$

(3.18) gives with the aid of (3.6) and (3.19)

(3.20) $$\sum_{p=1}^{n} \binom{4n+2}{4p} Q_{4p-1} = (4n+1)(4n+2)\left\{1+(-1)^{n+1}2^{1-2n}\right\}.$$

This is the third of the recursive relations and may be compared directly in form with (3.13) and (3.16). Q_{4p-1} can also be expressed in terms of the Bernoulli numbers.

From (3.20) we deduce

$$x^2 \sum_{n=1}^{\infty} \left\{1+(-1)^{n+1}2^{1-2n}\right\} \frac{x^{4n}}{(4n)!} = \left\{\sum_{p=1}^{\infty} Q_{4p-1} \frac{x^{4p}}{(4p)!}\right\}\left\{\sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}\right\}$$

or, after some manipulation,

$$\sum_{p=1}^{\infty} Q_{4p-1} \frac{x^{4p}}{(4p)!} = x^2 \frac{\left\{\cosh x + \cos x - 2\cosh ax - 2\cos ax + 2\right\}}{\cosh x - \cos x},$$

where as before $2a = 1+i$.

The right-hand side of (3.21) can be expressed as

$$\frac{x^2}{2}\left\{\coth\frac{ax}{2}\coth\frac{iax}{2} - 2\coth ax\coth iax - 2\operatorname{cosec} ax \operatorname{cosech} iax - \tanh\frac{ax}{2}\tanh\frac{iax}{2}\right\}.$$

Recalling the expansions for $\coth x$, $\operatorname{cosech} x$ already used and noting that in [1] for $\tanh x$ we obtain after some manipulation

$$Q_{4p-1} = \frac{(-1)^p(4p)!}{2^{2p-3}}\sum_{q=0}^{2p}(-1)^q\frac{(2^{4p-2q}-1)(2^{2q}-1)}{(2q)!(4p-2q)!}B_qB_{2p-q}$$

and hence by (3.19)

$$(3.22)\qquad \sum_{r=1}^{\infty}\frac{\tanh(2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}} = \frac{\pi^{4p-1}}{2}\sum_{q=1}^{2p-1}(-1)^{q+1}\frac{(2^{4p-2q}-1)(2^{2q}-1)}{(2q)!(4p-2q)!}B_qB_{2p-q}.$$

The expression in (3.11) can be differentiated as many times as we wish with respect to x and y at points within the rectangle.

Differentiating once with respect to x and once with respect to y gives

$$(3.23)\qquad (2n)(4n-1)i[(x+iy)^{4n-2}-(x-iy)^{4n-2}] = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}r}{\sinh r\pi}[\sinh ry\sin rx + \sinh rx \sin ry]$$

$$\times\left\{\sum_{p=1}^{n}(-1)^{-p+n+1}\pi^{4n-4p+1}\frac{2^{2n-2p+2}(4p-2)!}{r^{4p-2}}\binom{4n}{4p-2}\right\}.$$

Putting $x = y = \pi/2$ in (3.23) and defining R_{4p-3} by

$$(3.24)\qquad R_{4p-3} = \frac{(-1)^{p-1}}{\pi^{4p-3}2^{2p-1}}(4p-2)!\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^{4p-3}\cosh(2r+1)\frac{\pi}{2}}$$

yields

$$(3.25)\qquad \frac{(4n)(4n-1)}{2^{4n}} = \sum_{p=1}^{n}\binom{4n}{4p-2}R_{4p-3}.$$

The quantities R_{4p-3} can be expressed in terms of the Euler numbers (see [1]).

Following a procedure similar to earlier ones, we can deduce from (3.25) that

$$\sum_{p=0}^{\infty}R_{4p+1}\frac{x^{4p}}{(4p+2)!} = \frac{1}{2^4}\sec\frac{ax}{2}\sec\frac{iax}{2}.$$

Since

$$\sec x = \sum_{q=0}^{\infty}E_q\frac{x^{2q}}{(2q)!},$$

where $E_1, E_2, \cdots$, are the Euler numbers and E_0 is taken as unity, we obtain

$$(3.26)\qquad R_{4p+1} = (4p+2)!\frac{(-1)^p}{2^{6p+4}}\sum_{q=0}^{2p}(-1)^q\frac{E_qE_{2p-q}}{(2q)!(4p-2q)!}$$

and hence

$$(3.27)\qquad \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^{4p+1}\cosh (2r+1)\frac{\pi}{2}} = \pi^{4p+1}2^{-4p-3}\sum_{q=0}^{2p} \frac{(-1)^q E_q E_{2p-q}}{(2q)!(4p-2q)!}, \quad p = 0, 1, 2, \cdots.$$

Putting $n = 1$ in (3.23) yields

$$(3.28)\qquad xy = 2\pi \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r\sinh r\pi} \left\{ \sinh ry \sin rx + \sinh rx \sin ry \right\}.$$

Hence differentiating once with respect to x and then y we have, on putting $x = y = \pi/2$

$$1 = 4\pi \sum_{r=1}^{\infty} \frac{r(-1)^{r+1}}{\sinh r\pi}.$$

If we differentiate (3.28) $(2p+1)$ times with respect to x and $(2p+1)$ times with respect to y then for $x = y = \pi/2$ we find

$$\sum_{r=1}^{\infty} \frac{r^{4p+1}(-1)^r}{\sinh r\pi} = 0, \quad p = 1, 2, \cdots.$$

Likewise differentiating (3.28) $(2p)$ times with respect to x and $2p$ times with respect to y leads to

$$\sum_{r=0}^{\infty} \frac{(2r+1)^{4p-1}(-1)^r}{\cosh (2r+1)\frac{\pi}{2}} = 0, \quad p = 1, 2, \cdots.$$

We now proceed to find the sum of the last of the series referred to in the Introduction. Using the results of Section 2, it can be shown for $n = 1, 2, \cdots$

$$(3.29)\qquad \frac{1}{2}\left\{ (x+iy)^{4n+2} + (x-iy)^{4n+2} \right\} = (-1)^n \pi^{4n} 2^{2n}(x^2-y^2) + \sum_{r=1}^{\infty} (-1)^r \frac{\left\{\cosh rx \cos ry - \cosh ry \cos rx\right\}}{r\sinh r\pi}$$

$$\times \left\{ \sum_{p=1}^{n} (-1)^{n+1-p} \frac{\pi^{4n-4p+1}}{r^{4p}} 2^{2n-2p+2}(4p)! \binom{4n+2}{4p} \right\}.$$

The constant appearing in the Neumann solution is determined here to be zero by observing that each side of (3.29) vanishes when $x = y = 0$.

Putting $x = \pi$, $y = 0$ in (3.29) and defining M_{4p+1} by

$$(3.30)\qquad M_{4p+1} = (-1)^{p+1}\pi^{-4p-1}2^{-2p+2}(4p)! \sum_{r=1}^{\infty} \frac{1+(-1)^{r+1}\cosh r\pi}{r^{4p+1}\sinh r\pi}, \quad p = 1, 2, \cdots$$

leads to

$$(3.31)\qquad \sum_{p=1}^{n} M_{4p+1}\binom{4n+2}{4p} = 1 + (-1)^{n+1}2^{-2n}.$$

From the recurrence relation (3.31) we deduce

$$\sum_{p=1}^{\infty} M_{4p+1}\frac{x^{4p}}{(4p)!} = 1 + \frac{i}{2}\cot\frac{\alpha x}{2}\tan\frac{i\alpha x}{2} - \frac{i}{2}\tan\frac{\alpha x}{2}\cot\frac{i\alpha x}{2}$$

and hence

$$M_{4p+1} = (-1)^p \frac{(4p)!}{2^{2p-2}} \sum_{s=0}^{2p} (-1)^s \frac{(2^{4p-2s+2}-1)}{(2s)!(4p-2s+2)!} B_s B_{2p+1-s} .$$

Since

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\cosh r\pi + 1}{r^{4p+1}\sinh r\pi} = \sum_{r=1}^{\infty} \left\{ \frac{\coth \frac{r\pi}{2} - 2^{-4p}\coth 2r\pi}{r^{4p+1}} \right\}$$

we can, noting (3.30), obtain the required sum. It also follows for $p \geqslant 3$ we can obtain a good approximation to

$$\sum_{r=1}^{\infty} \frac{\coth \frac{r\pi}{2}}{r^{4p+1}}$$

in terms of the Bernoulli numbers.

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REFERENCES

1. R. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series,* MacMillan & Co. Ltd., Second Edition, (1926), pp. 297–299.
2. Kantorovich and Krylov, *Approximate Methods of Higher Analysis* (translated by Curtis D. Benster), P. Boordhoff Ltd., Gronigen, Netherlands (1958), pp. 7–9.
3. E. G. Phillips, "Note on Summation of Series," *Journal London Math. Society,* 4 (1929), pp. 114–116.
4. S. Ramanujan, *Collected Papers,* Edited by G. H. Hardy, et al, Cambridge University Press, 1927.
5. H. F. Sandham, "Some Infinite Series," *Proc. Amer. Math. Soc.,* 5 (1954), pp. 430–436.
6. G. N. Watson, "Theorems Stated by Ramanujan (II)," *J. London Math. Society,* 3 (1928), pp. 216–225.

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SEQUENCES OF MATRIX INVERSES FROM PASCAL, CATALAN, AND RELATED CONVOLUTION ARRAYS

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A sequence of sequences S_i arising from the first column of matrix inverses of matrices containing certain columns of Pascal's triangle provided a fruitful study in [1]. Here, we use convolution arrays of the sequences S_i to form a sequence of matrix inverses, leading to inter-relationships between the sequences S_i. The proofs involve generating functions for the columns of infinite matrices, and have diverse applications.

1. SEQUENCES OF MATRIX INVERSES

In this paper, we return to the sequences S_i arising from the first column of P_i^{-1} as in [1]. We form a series of $n \times n$ matrices $P_{i,j}$ by placing every j^{th} column of the convolution triangle for the sequence S_i on and below the main diagonal, and zeroes elsewhere. Then, to relate to the matrix P_i from [1] which was formed by writing the $(i+1)^{st}$ columns of Pascal's triangle on and below the main diagonal, in the new notation, $P_{i-1} = P_{0,i}$, or, every i^{th} column of the convolution array for the sequence $S_0 = \{1, 1, 1, \cdots\}$, which is Pascal's triangle. As a second example, the matrix $P_{1,3}$ would contain every third column of the convolution array for the Catalan sequence S_1 written in triangular form.

We call the inverse of $P_{i,j}$ the matrix $P_{i,j}^{-1}$ and record these inverses in the tables that follow.

Now, let us analyze the results. First, we look at the form of the elements of each matrix inverse, disregarding signs, for $P_{1,j}$. For $j = 1$, the rows of Pascal's triangle appear on and below the main diagonal; these columns are also the columns of the convolution triangle for the sequence $S_{-1} = \{1, 1, 0, 0, 0, \cdots\}$. The column generators, alternating signs included, are $(1-x)^{k-1}$, which are the reciprocals of the column generators for Pascal's triangle, where we do not adjust for the triangular form. For $j = 2$, we have alternate columns of Pascal's triangle, or alternate columns of the convolution triangle for S_0. (In fact, notice that each array contains columns of the convolution array for its left-most column.) For $j = 3$, we have every third column of the convolution triangle for S_1, while $j = 4$ gives fourth columns for S_2.

These results continue for $P_{2,j}^{-1}$ in Table 1.2. When $j = 1$, disregarding the alternating signs of the array, we have every column of the convolution triangle for the sequence $S_{-2} = \{1, 1, -1, 2, -5, 14, -42, \cdots\}$ which contains the Catalan numbers or S_1, taken with alternating signs, following the initial term. If the generating function of S_1 is $C(x)$, then the generating function for S_{-2} is $1/C(x)$. Then, for $j = 2$, we have every second column of the array for S_{-1}; for $j = 3$, every third column of the array for S_0, or, every third column of Pascal's triangle. These results continue, so that when $j = 4$, we have every fourth column of the convolution array for the Catalan numbers, or S_1; $j = 5$, the fifth columns of the array for S_2; $j = 6$, the sixth columns of the array for S_3; and for $j = 7$, the seventh columns of the convolution array for S_4.

Inspecting Table 1.3 for the form of $P_{3,j}^{-1}$ verifies that these results continue. When $j = 1$, every column of the convolution array for the sequence $S_{-3} = \{1, 1, -2, 7, -30, \cdots\}$ appears. Notice that S_{-3} contains the elements of the first convolution of S_2, or of S_2^2, taken with alternating signs and with one additional term preceding the sequence. If the generating function for S_2 is $D(x)$, then the generating function for S_{-3} is $1/D^2(x)$. For $j = 2$, we have every second column of the convolution array for S_{-2}; $j = 3$, every third column of the array for S_{-1}; $j = 4$, every fourth column of the array for S_0; $\cdots$, and for $j = 8$, we have every eighth column of the array for S_4.

Table 1.0
Non-Zero Elements of the Matrices $P_{0,j}^{-1}$ and $P_{0,j}$

	$P_{0,j}^{-1}$						$P_{0,j}$					
	1						1					
	−1	1					1	1				
$j = 1$	1	−2	1				1	2	1			
	−1	3	−3	1			1	3	3	1		
	1	−4	6	−4	1		1	4	6	4	1	
	−1	5	−10	10	−5	1	1	5	10	10	5	1
	...						...					
	1						1					
	−1	1					1	1				
$j = 2$	2	−3	1				1	3	1			
	−5	9	−5	1			1	6	5	1		
	14	−28	20	−7	1		1	10	15	7	1	
	...						...					
	1						1					
	−1	1					1	1				
$j = 3$	3	−4	1				1	4	1			
	−12	18	−7	1			1	10	7	1		
	55	−88	42	−10	1		1	20	28	10	1	
	...						...					
	1						1					
	−1	1					1	1				
$j = 4$	4	−5	1				1	5	1			
	−22	30	−9	1			1	15	9	1		
	140	−200	72	−13	1		1	35	45	13	1	
	...						...					

Table 1.1
Non-Zero Elements of $P_{1,j}^{-1}$ and $P_{1,j}$

	$P_{1,j}^{-1}$						$P_{1,j}$					
	1						1					
	−1	1					1	1				
$j = 1$	0	−2	1				2	2	1			
	0	1	−3	1			5	5	3	1		
	0	0	3	−4	1		14	14	9	4	1	
	...						...					
	1						1					
	−1	1					1	1				
$j = 2$	1	−3	1				2	3	1			
	−1	6	−5	1			5	9	5	1		
	1	−10	15	−7	1		14	28	20	7	1	
	...						...					
	1						1					
	−1	1					1	1				
$j = 3$	2	−4	1				2	4	1			
	−5	14	−7	1			5	14	7	1		
	14	−48	35	−10	1		14	48	35	10	1	
	...						...					
	1						1					
	−1	1					1	1				
$j = 4$	3	−5	1				2	5	1			
	−12	25	−9	1			5	20	9	1		
	55	−130	63	−13	1		14	75	54	13	1	
	−273	700	−408	117	−17	1	42	275	273	104	17	1
	...						...					

Table 1.2

Non-Zero Elements of $P_{2,j}^{-1}$ and $P_{2,j}$

	$P_{2,j}^{-1}$					$P_{2,j}$				
	1					1				
	−1	1				1	1			
$j = 1$	−1	−2	1			3	2	1		
	−2	−1	−3	1		12	7	3	1	
	−5	−2	0	−4	1	55	30	12	4	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 2$	0	−3	1			3	3	1		
	0	3	−5	1		12	12	5	1	
	0	−1	10	−7	1	55	55	25	7	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 3$	1	−4	1			3	4	1		
	−1	10	−7	1		12	18	7	1	
	1	−20	28	−10	1	55	88	42	10	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 4$	2	−5	1			3	5	1		
	−5	20	−9	1		12	25	9	1	
	14	−75	54	−13	1	55	130	63	13	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 5$	3	−6	1			3	6	1		
	−12	33	−11	1		12	33	11	1	
	55	−182	88	−16	1	55	182	88	16	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 6$	4	−7	1			3	7	1		
	−22	49	−13	1		12	42	13	1	
	140	−357	130	−19	1	55	245	117	19	1
	…					…				
	1					1				
	−1	1				1	1			
$j = 7$	5	−8	1			3	8	1		
	−35	68	−15	1		12	52	15	1	
	285	−606	270	−22	1	55	320	150	22	1
	…					…				

Table 1.3
Non-Zero Elements of $P^{-1}_{3,j}$ and $P_{3,j}$

	$P^{-1}_{3,j}$					$P_{3,j}$				
	1					1				
	−1	1				1	1			
j = 1	−2	−2	1			4	2	1		
	−7	−3	−3	1		22	9	3	1	
	−30	−10	−3	−4	1	140	52	15	4	1
	1					1				
	−1	1				1	1			
j = 2	−1	−3	1			4	3	1		
	−2	0	−5	1		22	15	5	1	
	−5	−1	5	−7	1	140	91	30	7	1
	1					1				
	−1	1				1	1			
j = 3	0	−4	1			4	4	1		
	0	6	−7	1		22	22	7	1	
	0	−4	21	−10	1	140	140	49	10	1
	1					1				
	−1	1				1	1			
j = 4	1	−5	1			4	5	1		
	−1	15	−9	1		22	30	9	1	
	1	−35	45	−13	1	140	200	72	13	1
	1					1				
	−1	1				1	1			
j = 5	2	−6	1			4	6	1		
	−5	27	−11	1		22	39	11	1	
	14	−110	77	−16	1	140	272	99	16	1
	1					1				
	−1	1				1	1			
j = 6	3	−7	1			4	7	1		
	−12	42	−13	1		22	49	13	1	
	55	−245	117	−19	1	140	357	130	19	1
	1					1				
	−1	1				1	1			
j = 7	4	−8	1			4	8	1		
	−22	60	−15	1		22	60	15	1	
	140	−456	165	−22	1	140	456	165	22	1
	1					1				
	−1	1				1	1			
j = 8	5	−9	1			4	9	1		
	−35	81	−17	1		22	72	17	1	
	285	−759	221	−25	1	140	570	204	25	1

To generalize, $P^{-1}_{i,j}$ contains the sequence S_{j-i-1} along its first column and the j^{th} columns of the convolution triangle for the sequence S_{j-i-1}, taken with alternating signs, on and below its main diagonal, with, of course, zeroes everywhere above its main diagonal. The sequences S_i, $i \geqslant 0$, were explored in [1]. The sequences S_{-i}, $i \geqslant 2$, are all related to the sequences S_i by

$$S_{-i} - 1 = S_{i-1}^{i-1}$$

so that, if the initial one is deleted, the sequence S_{-1} is identical to the $(i-2)$nd convolution of the sequence S_{i-1}, $i \geq 2$. Also, if the generating function for S_i is $G(x)$, then the generating function for S_{-i-1} is $1/G^i(x)$, $i \geq 1$, and S_{-1} has a generating function which is the reciprocal of that for S_0.

There are other patterns which occur for the matrices $P_{i,j}^{-1}$. Except for the alternating signs, $P_{i,j}^{-1}$ is identical to $P_{i,j}$ for $j = 2i+1$. Furthermore, this property still holds if we form $P_{i,j}^*$ from *any* set of j^{th} columns of the convolution triangle for S_i, $j = 2i+1$. For example $P_{1,3}^{-1}$ contained the same elements as $P_{1,3}$ except for the alternating signs, where $P_{1,3}$ contained the zeroeth, third, sixth, ⋯ , columns of the convolution triangle for S_1 so that its k^{th} column was the $(3k)^{th}$ column of the convolution array. Form $P_{1,3}^*$ to contain every third column of the convolution array for S_1, but beginning from the first convolution, so that the k^{th} column of $P_{1,3}^*$ is the $(3k+1)^{st}$ column of the array, and $P_{1,3}^{*-1}$ has the same elements as $P_{1,3}^*$, taken with alternating signs. Similarly, if we form the matrix $P^{**}_{1,3}$ from the $(3k+2)^{nd}$ columns of the convolution array for S_1, $P_{1,3}^{**-1}$ has the same elements as $P_{1,3}^{**}$ taken with alternating signs. For example, using 5 × 5 matrices,

$$P_{1,3}^{*-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 \\ 14 & 20 & 8 & 1 & 0 \\ 42 & 75 & 44 & 11 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 5 & -5 & 1 & 0 & 0 \\ -14 & 20 & -8 & 1 & 0 \\ 42 & -75 & 44 & -11 & 1 \end{bmatrix}$$

$$P_{1,3}^{**-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 9 & 6 & 1 & 0 & 0 \\ 28 & 27 & 9 & 1 & 0 \\ 90 & 110 & 54 & 12 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 9 & -6 & 1 & 0 & 0 \\ -28 & 27 & -9 & 1 & 0 \\ 90 & -110 & 54 & -12 & 1 \end{bmatrix}$$

Notice that we can consider $n \times n$ submatrices of the infinite matrices of this paper, since for infinite matrices A, B, and C, if we know that $AB = C$ by generating functions, then it must follow that $AB = C$ for $n \times n$ matrices A, B, and C, because each $n \times n$ matrix is the same as the $n \times n$ block in the upper left in the respective infinite matrix. We write the Lemma,

Lemma. Let A be an infinite matrix such that all of its non-zero elements appear on and below its main diagonal, and let $A_{n\times n}$ be the $n \times n$ matrix formed from the upper left corner of A. Let B and C be infinite matrices with $B_{n\times n}$ and $C_{n\times n}$ the $n \times n$ matrices formed from their respective upper left corners. If $AB = C$, then $A_{n\times n}B_{n\times n} = C_{n\times n}$.

Returning for a moment to Tables 1.1, 1.2, and 1.3, notice that the row sums of $P_{1,1}$ are $\{1, 2, 5, 14, 42, \cdots\}$, or S_1^2; the row sums of $P_{2,2}$ are $\{1, 2, 7, 30, 143, \cdots\}$, or S_2^2; and the row sums of $P_{3,3}$ are $\{1, 2, 9, 52, 320, \cdots\}$, or S_3^2. We easily prove that

Theorem. The successive row sums of $P_{i,i}$ are S_i^2.

Proof. Let $S_i(x)$ be the generating function for the sequence S_i. Then the row sums are

$$R(x) = S_i(x) + xS_i^{i+1}(x) + x^2S_i^{2i+1} + \cdots = S_i(x)/[1 - xS_i^i(x)]$$

by summing the infinite geometric series. But, by [1],

$$1 = S_i(x) - xS_i^{i+1}(x),$$

so that $R(x) = S_i^2(x)$ upon simplication.

2. PROOF OF RESULTS AND FURTHER APPLICATIONS

Now, we establish firmly the matrix inverse results of this paper. Let $S_i(x)$ denote the generating function for the sequence S_i, and let $S_k \downarrow S_{k-1}$ mean that S_{k-1} is the solution to

$$S_k(x/s(x)) = S(x),$$

with $S(0) = 1$. It will follow that

$$S_2 \downarrow S_1 \downarrow S_0 \downarrow S_{-1} \downarrow S_{-2} \cdots .$$

Now, it turns out that

$$S_{-1}(x) = \frac{1}{S_0(-x)} = \frac{1}{\dfrac{1}{1-(-x)}} = 1+x.$$

From this we can show

$$\frac{1}{S_0(-x)} \downarrow \frac{1}{S_1(-x)} \downarrow \frac{1}{S_2(-x)} \cdots$$

as follows:

$$S_1(xS_2(x)) = S_2(x)$$

is trivial, but this continues as

$$\frac{1}{S_1(-x/1/S_2(-x))} = \frac{1}{S_2(-x)},$$

and thus we can generally say

$$\frac{1}{S_m(-x)} \downarrow \frac{1}{S_{m+1}(-x)} .$$

Notice that, if $S_0(x) = 1/(1-x)$, then $S_n(x)$ satisfies

$$\frac{1}{1-xS_n^n(x)} = S_n(x) \quad \text{or} \quad 1 = S_n(x) - xS_n^{n+1}(x).$$

Now, let us look at our general (Pascal) problem. (We denote each matrix by giving successive column generators.) The two infinite matrices

$$(f^m(x), f^{m+k}(x), f^{m+2k}(x), \cdots) \quad \text{and} \quad (A^m(x), A^{m+k}(x), A^{m+2k}(x), \cdots)$$

are matrix inverses if

$$A(x)f(xA^k(x)) = 1 \quad \text{or} \quad f(x/[1/A^k(x)]) = 1/A(x).$$

That is, $1/A(x)$ is k steps down the descending chain of sequences from $f(x)$. Let us examine the two together.

$$(S_2(x), S_2^6(x), S_2^{11}(x), \cdots)^{-1}$$

is given by

$$(A(x), A^6(x), A^{11}(x), \cdots),$$

where

$$S_2(x/1/A^5(x)) = 1/A(x)$$

so we go down five sequences from $S_2(x)$:

$$S_2(x), S_1(x), S_0(x), S_{-1}(x) = 1/S_0(-x), S_{-2}(x) = 1/S_1(-x), S_{-3}(x) = 1/S_2(-x)$$

so that

$$1/A(x) = S_2(-x).$$

This verifies that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 & 0 \\ 12 & 33 & 11 & 1 & 0 \\ 55 & 182 & 88 & 16 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 3 & -6 & 1 & 0 & 0 \\ -12 & 33 & -11 & 1 & 0 \\ -55 & -182 & 88 & -16 & 1 \end{bmatrix}$$

Lemma. Two infinite matrices

$$(f(x), xf(x)A(x), x^2f(x)A^2(x), x^3f(x)A^3(x), \cdots) \quad \text{and} \quad (g(x), xg(x)B(x), x^2g(x)B^2(x), x^3g(x)B^3(x), \cdots)$$

are inverses if

$$g(x)B(x)A(xB(x))f(xB(x)) = 1.$$

There are several interesting applications. Consider the central column of Pascal's triangle, $\{1, 2, 6, 20, \cdots\}$ which, upon proper processing, originally gave us the Catalan sequence. Let $f(x)$ be the generating function for the central column of Pascal's triangle, and take

$$f(x) = 1/\sqrt{1-4x}, \quad A(x) = (1-\sqrt{1-4x})/2x, \quad g(x) = 1-2x, \quad B(x) = 1-x,$$

$$A(x(B(x)) = \frac{1-\sqrt{1-4x(1-x)}}{2x(1-x)} = \frac{x}{1-x} = \frac{1}{B(x)}$$

so that $B(x)A(x(B(x)) = 1$. Now

$$f(x(B(x)) = \frac{1}{\sqrt{1-4x(1-x)}} = \frac{1}{1-2x} = \frac{1}{g(x)}$$

so that $g(x)f(xB(x)) = 1$ also. This matrix uses elements from the central column of Pascal's triangle and the columns parallel to it, and has inverse whose columns have the coefficients from the generating functions. For example, for the 5×5 case.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 10 & 10 & 4 & 1 & 0 \\ 20 & 35 & 56 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 2 & -4 & 1 & 0 \\ 0 & 0 & 5 & -5 & 1 \end{pmatrix}$$

If we now go to the m^{th} columns, then

$$g(x)B^m(x)A^m(x(B(x))f(xB(x)) = 1$$

so it seems to naturally break into two separate parts:

$$(1) \qquad B(x)A(x(B(x)) = 1$$

$$(2) \qquad g(x)f(xB(x)) = 1,$$

where we already know how to solve (1), but (2) is something new when combined with (1) since the above has to hold for all $m \geqslant 0$.

Let us consider $S_2^* = \{1, 3, 15, 84, \cdots\}$, the diagonal of Pascal's triangle, which, upon proper processing, lead to our sequence $S_2 = \{1, 1, 3, 12, 55, 273, \cdots\}$. Let $S_2^*(x)$ be the generating function for S_2^*, and take

$$f(x) = S_2^*(x), \qquad A(x) = S_2(x),$$

and let $S_1(x) = C(x)$ be the generating function for the sequence $S_1 = \{1, 1, 2, 5, 14, 70, \cdots\}$, the Catalan sequence. Then,

$$g(x) = 1 - 3xC(x) \qquad B(x) = 1 - xC(x)$$

$$= \frac{3}{C(x)} - 2 \qquad = \frac{1}{C(x)}$$

$$= \frac{3}{S_1(x)} - 2 \qquad = \frac{1}{S_1(x)}$$

Now, let $S_3^* = \{1, 4, 28, 220, \cdots\}$, the diagonal of Pascal's triangle which led to the sequence S_3. Here, we use

$$1 = S_3(x) - xS_3^3(x)$$

and we can write

$$f(x) = S_3^*(x), \quad A(x) = S_3(x), \quad B(x) = 1/S_3(x), \quad \text{and} \quad g(x) = 1 - 4xS_3^2(x) = 4/S_3(x) - 3.$$

Generally speaking, we take

$$f(x) = S_k^*(x), \quad A(x) = S_k(x), \quad B(x) = 1/S_k(x), \quad g(x) = 1 - (k+1)xS_k^{k-1}(x) = \frac{k+1}{S_k(x)} - k.$$

Lemma. The two infinite matrices

$$(f(x)A^m(x),\, xf(x)A^{m+k}(x),\, x^2f(x)A^{m+2k}(x),\, \cdots) \quad \text{and} \quad (g(xB^m(x),\, xg(x)B^{m+k}(x),\, x^2g(x)B^{m+2k}(x),\, \cdots)$$

are inverses if $f(xB^k(x))g(x) = 1$ and $A(xB^k(x))B(x) = 1$, simultaneously.

The Lemma is the same as considering the two infinite matrices

$$(F(x),\, xF(x)A^k(x),\, x^2F(x)A^{2k}(x),\, \cdots) \quad \text{and} \quad (G(x),\, xG(x)B^k(x),\, x^2G(x)B^{2k}(x),\, \cdots),$$

where

$$F(x) = f(x)A^m(x); \quad G(x) = g(x)B^m(x); \quad A^k(xB^k(x))f(xB^k(x))A^m(xB^k(x))B^m(x)g(x) = 1$$

or

$$[A(xB^k(x))B(x)]^m = 1 \quad \text{and} \quad f(xB^k(x))g(x) = 1, \quad A(0) = B(0) = 1.$$

With application to the sequences S_i of this paper, we can take

$$f(x) = D_0(x), \quad A(x) = S_k(x), \quad g(x) = 1-(k+1)S_k^{k-1}(x), \quad \text{and} \quad B(x) - 1/S_{k-1}(x).$$

The above lemma can also be illustrated by taking

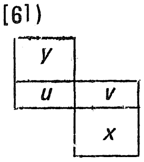
$$f(x) = 1/(1-x), \quad A(x) = (1+x)/(1-x), \quad g(x) = (3+x-\sqrt{1+6x+x^2})/2,$$

and

$$B(x) = [-(1+x)+\sqrt{1+6x+x^2}]/2x.$$

This arises from the triangular matrix (from a paper by Alladi [6])

$$\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 3 & 1 & & \\
1 & 5 & 5 & 1 & \\
1 & 7 & 13 & 7 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$



$x = u+v+y$

where the column generators are successively given by

$$\frac{1}{1-x},\quad \frac{x(1+x)}{(1-x)^2},\quad \cdots,\quad \frac{x^n(1+x)^n}{(1-x)^{n+1}},\quad \cdots$$

The lemmas of this section also apply to some other interesting sequences. Suppose we take the sequence $\{1, 1, 2, 4, 8, 16, \cdots\}$ which is generated by

$$f(x) = \frac{1-x}{1-2x} = 1+\frac{x}{1-2x} = 1+\sum_{n=0}^{\infty} 2^n x^n .$$

Let $Hf(x) = S(x)$, where $S(0) = 1$ and $S(x)$ satisfies $f(xS(x)) = S(x)$. Then $H^2f(x) = S(x)$ means that $f(xS^2(x)) = S(x)$, $S(0) = 1$.

$$H\left(\frac{1-x}{1-2x}\right) = g(x)$$

which is the generating function for $\{1, 1, 3, 11, 45, 197, 903, \cdots,\}$. (See Riordan [2], p. 168), while

$$H^2\left(\frac{1-x}{1-2x}\right) = H(g(x)) = h(x),$$

which is the generating function for the sequence $\{1. 1, 4, 21, 126, 818, 5594, \cdots\}$ given by Carlitz [4]. There is another sequence from the same article by Carlitz, but first we note $B = \{1, 1, 3, 11, 45, \cdots, b_n, \cdots\}$ obeys

$$(n+1)b_n - 3(2n-1)b_{n-1} + (n-2)b_{n-2} = 0.$$

We solve the quadratic

$$2xS^2 - (x+1)S + 1 = 0$$

$$S(x) = \frac{1 + x \pm \sqrt{1 + 2x + x^2 - 8x}}{4x}$$

$$S(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x} = \sum_{n=0}^{\infty} b_n x^n \ .$$

From this, we should be able to establish the recurrence. We also note that, where $C(x) = (1 - \sqrt{1-4x})/2x$ is the generator for the Catalan sequence,

$$\frac{1}{1+x}\, C\left(\frac{2x}{(1+x)^2}\right) = S(x)$$

which comes from Riordan [2], p. 168.

There is another application. Let $f(x)$ generate the odd numbers. Then the solution to $f(x(S(x)) = S(x)$, $S(0) = 1$, is the sequence $\{1, 3, 14, 79, 494, 3294, \cdots\}$ given by Carlitz [4], which has generating functions

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \cdots$$

$$\frac{1 + xS(x)}{(1 - xS(x))^2} = S(x)$$

$$1 + xS(x) = S(x) - 2xS^2(x) + x^2S^3(x)$$

$$0 = x^2S^3(x) - 2xS^2(x) + (1-x)S(x) - 1,$$

where $S(x)$ generates $\{1, 3, 14, 79, 494, 3294, \cdots\}$. As verification,

$$S = \{1, 3, 14, 79, 494, 3294, \cdots\}, \quad S^2 = \{1, 6, 37, 242, 1658, \cdots\}$$

$$S^3 = \{1, 9, 69, 516, \cdots\}, \quad -1S^0 = \{-1, 0, 0, 0, 0, 0, \cdots\}$$

$$S = \{1, 3, 14, 79, 494, 3294, \cdots\}, \quad -xS = \{0, -1, -3, -14, -79, -494, \cdots\}$$

$$-2xS^2 = \{0, -2, -12, -74, -484, -3316, \cdots\}, \quad x^2S^3 = \{0, 0, 1, 9, 69, 516, \cdots\}$$

with all vertical sums equalling zero.

With the methods of this section, it is easily shown that, if $P^*_{i,j}$ is the matrix formed by moving each column of $P_{i,j}$ up to form a rectangular array, $P^{-1}_{i,j}P^*_{i,j}$ contains Pascal's triangle written in rectangular form, which is such a prolific result that it is the content of another paper [3]. Paul Bruckman has proved the matrix theorems used in this section in [5]. See [7] also.

REFERENCES

1. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices," *The Fibonacci Quarterly,* Vol. 14, No. 4 (Dec. 1976), to appear.
2. John Riordan, *Combinatorial Analysis,* John Wiley and Sons, 1968.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix," *The Fibonacci Quarterly,* Vol. 14, No. 2 (April, 1976), pp. 135–142.
4. L. Carlitz, "Enumeration of Two-Line Arrays," *The Fibonacci Quarterly,* Vol. 11, No. 2 (April, 1973), pp. 113–130.
5. Paul Bruckman, Private Communication
6. Krishnaswami Alladi, Private Communication.
7. V. E. Hoggatt, Jr., and Paul S. Bruckman, "The H-Convolution Transform," *The Fibonacci Quarterly,* Vol. 13, No. 4 (December 1975), pp. 357–367.

NOTE: In the paper, "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix," the publication dates of references 1 and 2 were inadvertently interchanged. Reference 4 should have read: Michael Rondeau, "The Generating Functions for the Vertical Columns of $(N+1)$-Nomial Triangles," unpublished Master's Thesis, San Jose State University, December, 1973.

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ON A GENERALIZATION OF THE FIBONACCI NUMBERS USEFUL IN MEMORY ALLOCATION SCHEMA; OR ALL ABOUT THE ZEROES OF $Z^k - Z^{k-1} - 1,\ k > 0$

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ABSTRACT

A generalization of the Fibonacci numbers arises in the theory of dynamic storage allocation schema. The associated linear recurrence relation involves the polynomial $Z^k - Z^{k-1} - 1,\ k \geqslant 1$. A theorem is proven showing that all the zeroes of this polynomial lie in the intersection of two annuli.

Complete information about the sequence then follows, e.g., expressing the elements in terms of certain sums of binomial coefficients and sums of powers of roots, limits of quotients of terms, and limits of roots. Tables useful for storage design are included.

A certain linear recurrence relation arises in the theory of memory allocation schema which generalizes the linear recurrence defining the Fibonacci numbers. The generalized numbers may be expressed as the coefficients of a rational generating function where the denominator of the rational function involves the trinomial $Z^k - Z^{k-1} - 1$. From this fact follows two expressions for the numbers themselves, one in terms of linear combinations of the powers of the roots of the trinomial, and another expression giving the numbers as sums of binomial coefficients which lie on a line of rational slope falling across Pascal's triangle. The former expression gives complete information on the limit of successive quotients. This latter data depends upon the location of the roots of this trinomial: all complex zeroes lie in the intersection of two annuli in the complex plane. See Table 1 and Figure 1 for explicit numbers and visulization of the following central theorems.

Theorem A. Let $k \geqslant 1$. All of the k zeroes of $Z^k - Z^{k-1} - 1$ are distinct and lie in the intersection of the two annuli

$$\lambda_0 \leqslant |Z| \leqslant \lambda_1 \qquad \text{and} \qquad \lambda_1 - 1 \leqslant |Z-1| \leqslant 1 + \lambda_0\,,$$

where $\lambda_\epsilon = \lambda_\epsilon(k)$ is the largest (positive) real solution of

$$r^k + (-1)^\epsilon r^{k-1} - 1 = 0, \quad \epsilon = 0, 1, \quad 0 < \lambda_0 < 1 < \lambda_1 < 2.$$

Table 2 gives approximate values of these $\lambda_\epsilon = \lambda_\epsilon(k),\ k = 1, 2, \cdots, 20, 100.$

Theorem B. Let $k \geqslant 1$. Define $f_{k,n} = f_{k,n-1} + f_{k,n-k}\,;\ f_{k,j} = 0,\ j < k;\ f_{k,k} = 1.$ Then

$$\lim_{n\to\infty} \frac{f_{k,n+1}}{f_{k,n}} = \lambda_1(k) \quad \text{and} \quad \lim_{n\to\infty} \lambda_1(k) = 1\,.$$

The proofs of these theorems depend upon two sequences of lemmas, those bearing more directly upon Theorem A or B; we number the lemmas accordingly.

Lemma A1. Let $p(Z) = Z^k - Z^{k-1} - 1,\ k \geqslant 1$. None of the zeroes of $p(Z)$ are rational; all of the zeroes of $p^{(1)}(Z)$ are rational.

Proof. Since

$$p^{(1)}(Z) = kZ^{k-2}\left(Z - \frac{k-1}{k}\right)$$

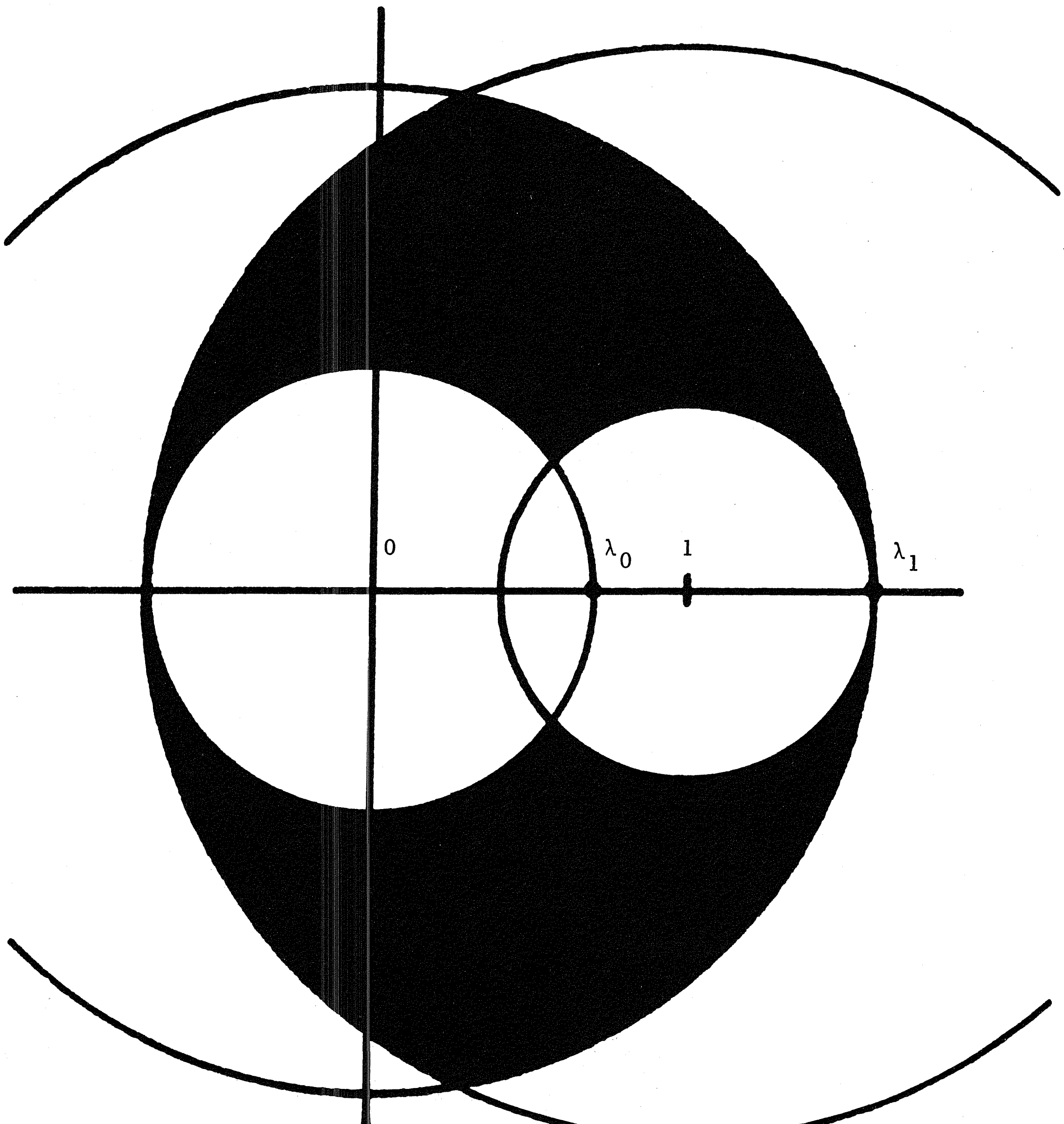

Figure 1. The Two Annuli Theorem
(The shaded region represents the region in which all of the complex zeroes of $Z^k - Z^{k-1} - 1$ must lie.)

Table 1
The Sequences $f_{k,n} = f_{k,n-1} + f_{k,n-k}$
With $f_{k,j} = 0, j < k; f_{k,k} = 1, k \geqslant 1$

n^k	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	0
3	4	1	1	0	0	0	0	0	0	0	0
4	8	2	1	1	0	0	0	0	0	0	0
5	16	3	1	1	1	0	0	0	0	0	0
6	32	5	2	1	1	1	0	0	0	0	0
7	64	8	3	1	1	1	1	0	0	0	0
8	128	13	4	2	1	1	1	1	0	0	0
9	256	21	6	3	1	1	1	1	1	0	0
10	512	34	9	4	2	1	1	1	1	1	0
11	1024	55	13	5	3	1	1	1	1	1	1
12	2048	89	19	7	4	2	1	1	1	1	1
13	4096	144	28	10	5	3	1	1	1	1	1
14	8192	233	41	14	6	4	2	1	1	1	1
15	16384	377	60	19	8	5	3	1	1	1	1
16	32768	610	88	26	11	6	4	2	1	1	1
17	65536	987	129	36	15	7	5	3	1	1	1
18	131072	1597	189	50	20	9	6	4	2	1	1
19	262144	2584	277	69	26	12	7	5	3	1	1
20	524288	4181	406	95	34	16	8	6	4	2	1
21	1048576	6765	595	131	45	21	10	7	5	3	1
22	2097152	10946	872	181	60	27	13	8	6	4	2
23	4194304	17711	1278	250	80	34	17	9	7	5	3
24	8388608	28657	1873	345	106	43	22	11	8	6	4
25	16777216	46368	2745	476	140	55	28	14	9	7	5
26	33554432	75025	4023	657	185	71	35	18	10	8	6
27	67108864	121393	5896	907	245	92	43	23	12	9	7
28	134217728	196418	8641	1252	325	119	53	29	15	10	8
29	268435456	317811	12664	1728	431	153	66	36	19	11	9
30	536870912	514299	18560	2385	571	196	83	44	24	13	10
31	1073741824	832040	27201	3292	756	251	105	53	30	16	11

Table 2
$\lambda_\epsilon = \lambda_\epsilon(k)$, $\epsilon = 0, 1$ is the Largest Positive Real Root of $r^k + (-1)^\epsilon r^{k-1} - 1$.
The roots are truncated to 25 decimal places; see [3].

k	$\lambda_1(k)$	$\lambda_0(k)$
1	2.0000000000000000000000000	0.0000000000
2	1.6180339887498948482045868	0.6180339887498948482045868
3	1.4655712318767680266567312	0.7548776662466927600495088
4	1.3802775690976141156733016	0.8191725133961644396995711
5	1.3247179572447460259609088	0.8566748838545028748523248
6	1.2851990332453493679072604	0.8812714616335695944076491
7	1.2554228710768465432050014	0.8986537126286992932608757
8	1.2320546314285722959319676	0.9115923534820549186286736
9	1.2131497230596399145540815	0.9215993196339830062994303
10	1.1974914335516807746915412	0.9295701282320228642044130
11	1.1842763223508938723515139	0.9360691110777583783971914
12	1.1729507500239802071448788	0.9414696173216352043780467
13	1.1631197906692044180088153	0.9460285282856136156355381
14	1.1544935507090564328867379	0.9499283999636198830314051
15	1.1468540421995067272864110	0.9533025374016641591079826
16	1.1400339374770049101652704	0.9562505576379890668254960
17	1.1339024903348373489121350	0.9588484010075613716652026
18	1.1283559396916029856471042	0.9611549719964985735216646
19	1.1233108062463267587889592	0.9632166633389015467989664
20	1.1186991080522260494554442	0.9650705109167162350928078
100	1.034	0.9930

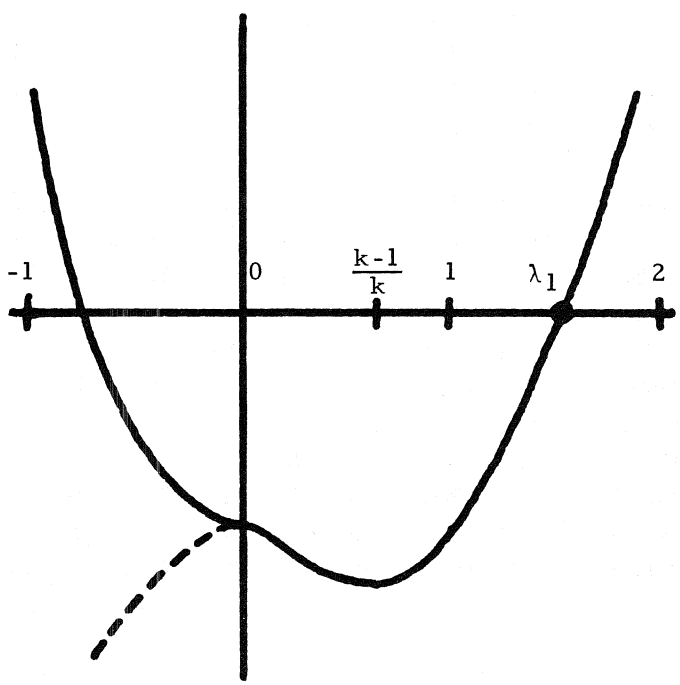

Figure 2. Combined graph of $x^k - x^{k-1} - 1 = y$ for k even and odd. There is a local minimum at $x = \frac{k-1}{k}$.

we see that the roots of $p^{(1)}(Z)$ are 0 with multiplicity $k-2$ and $(k-1)/k$ with multiplicity 1, both rational. Since $p(Z)$ is monic with integer coefficients any rational root must be a gaussian integer. From the relation $Z^{k-1}(Z-1)=1$ it is easy to infer that Z cannot be integral.

Corollary A1. Define the collection of zeroes of $p(Z)$ to be

$$Z_k = \left\{ Z \in \mathbb{C} : p(Z) = 0 \right\} = \left\{ \lambda_{k,j} ; 1 \leqslant j \leqslant k \right\}.$$

Then $[Z_k] = k$, i.e., the roots are distinct, and we can order them

$$|\lambda_{k,j}| \leqslant |\lambda_{k,j+1}|, \qquad j = 1, 2, \cdots, k-1$$

with equality iff $\lambda_{k,j}$ is the complex conjugate of $\lambda_{k,j+1}$.

Proof. From Lemma A1 we have proven that $p(Z)$ and $p^{(1)}(Z)$ are relatively prime ($\mathbb{C}$ is algebraically closed) which is sufficient for the roots to be distinct. We note that in addition to nonreal complex zeroes occurring in conjugate pairs, exactly two roots are real if k is even and exactly one is real if k is odd.

Lemma A2. There exist numbers, $0 < \lambda_0 < 1 < \lambda_1 < 2$ dependent only upon k, $k > 1$, such that all of the zeroes of $p(Z) = Z^k - Z^{k-1} - 1$ lie in an annulus $\lambda_0 \leqslant |Z| \leqslant \lambda_1$ centered at 0 and in an annulus $\lambda_1 - 1 \leqslant |Z-1| \leqslant 1 + \lambda_0$ centered at 1.

Proof. Since $p(0) \neq 0$, any complex zero Z of $p(Z)$ has norm $|Z| = r > 0$ and $p(Z) = 0$ gives $|Z-1| = r^{1-k}$. Thus any zero lies on the intersection of the two circles $|Z| = r$ and $|Z-1| = r^{1-k}$ with fixed centers. There are two cases of empty intersection: one circle lying wholly inside the other. Comparing radii of these circles there will be a non-vacuous intersection if $r \leqslant 1 + r^{1-k}$ or if $r \leqslant \lambda_1$ where λ_1 is the largest positive root of $p(Z)$. (!). The second case of $|Z| = r$ lying inside $|Z-1| = r^{1-k}$ yields $0 \leqslant r^k + r^{k-1} - 1$ or $r \geqslant \lambda_0$ where λ_0 is the largest positive root of $q(Z) = Z^k + Z^{k-1} - 1$. Locating these roots gives the inequalities above and noting that $\lambda_0^{1-k} = 1 + \lambda_0$ $\lambda_1^{1-k} = \lambda_1 - 1$ bounds the radius r^{1-k}

Corollary A2. Set $\lambda_{k,k} = \lambda_1(k) = \lambda_1$. Then $\lambda_1(k)$ is real and $|\lambda_{k,j}| < \lambda_1(k)$ for $1 \leqslant j \leqslant k$.

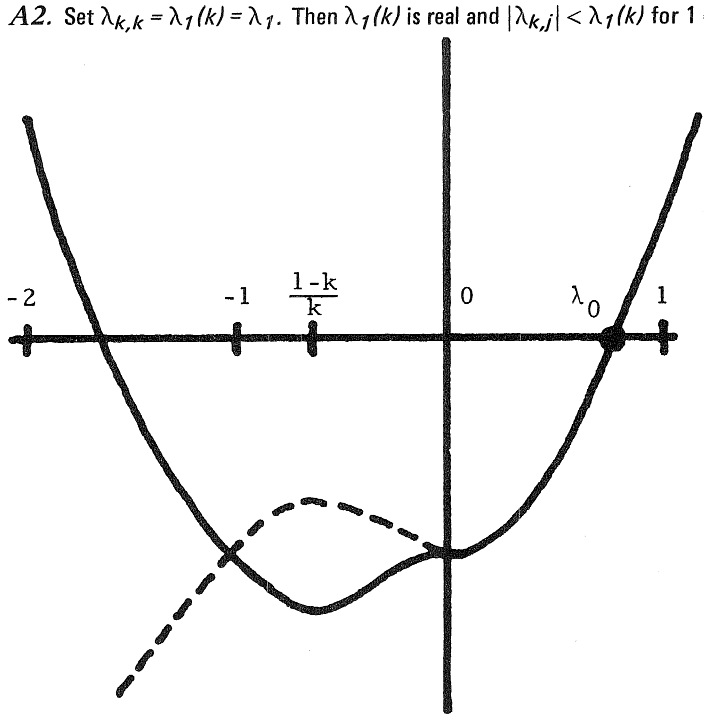

Figure 3. Combined Graph of $x^k + x^{k-1} - 1 = y$ for k Even and Odd. There is a local maximum and minimum at $x = (1-k)/k$.

Proof. λ_1 is, from the proof of Lemma A2 the largest possible real root of $p(Z)$. Note that if k is even that $-\lambda_0$ is the smallest real root of $p(Z)$.

Lemma A3. Let

$$\sum_{1 \leqslant j \leqslant k} C_j \lambda_{k,j}^n$$

be any (complex) linear combination of the n^{th} powers of the zeroes of $p(Z)$. Then, for

$$A = \sum_{1 \leqslant j \leqslant k} |C_j| \leqslant k \max_{1 \leqslant j \leqslant k} |C_j|, \qquad \left| \sum_{1 \leqslant j \leqslant k} C_j \lambda_{k,j}^n \right| \leqslant A \lambda_1^n .$$

Proof. This follows directly from Corollary A2 and the usual absolute value inequalities. This Lemma gives information on the rate of growth of the integers $f_{k,n}$.

Lemma A4. For $p_k(x) = x^k - x^{k-1} - 1$,

$$1 + \sum_{1 \leqslant j \leqslant k} p_j(x) = x^k - k.$$

Proof. The sum telescopes. The purpose of this simple Lemma is to motivate the next Lemma; the largest positive real zero of the sum is $k^{1/k}$.

Lemma A5. Let $k > 3$. Then $1 < \lambda_1(k) < k^{1/k}$.

Proof. Since $p(1) = -1$ we need only show that $p(k^{1/k}) > 0$. For $k > 3$ it is clear that

$$1 + \frac{1}{2(k-1)} < \ln k.$$

But

$$1 + \frac{1}{2(k-1)} = \frac{1}{2k} + \frac{1}{2k^2} + \frac{1}{2k^3} + \cdots > 1 + \frac{1}{2k} + \frac{1}{3k^2} + \frac{1}{4k^3} + \cdots = -k \ln\left(1 - \frac{1}{k}\right)$$

so that

$$-k \ln\left(1 - \frac{1}{k}\right) < \ln k.$$

Rewriting, we have

$$\ln\left(\frac{k-1}{k}\right) > \ln k^{-1/k},$$

exp is order preserving so that

$$1 - \frac{1}{k} > k^{-1/k}.$$

Then

$$0 < \frac{1}{k} + \frac{1}{k^{1/k}} < 1.$$

But this gives

$$0 < k - k^{1-(1/k)} - 1 = p(k^{1/k}).$$

Lemma A6. Let $k > 2$. Then $k^{-1/k} < \lambda_0(k) < 1$.

Proof. For

$$q(z) = x^k + x^{k-1} - 1, \quad q(1) = 1$$

it is sufficient to show that $q(k^{-1/k}) < 0$. It is clear that $k^{1/k} < k - 1$ for k an integer larger than two. But then $1 + k^{1/k} < k$ gives

$$0 > \frac{1}{k} + \frac{k^{1/k}}{k} - 1 = q(k^{1/k}).$$

Lemma A7. $\lim_{k \to \infty} k^{1/k} = 1.$

Proof. This follows from $\lim_{k \to \infty} (\ln k)/k = 0.$

The development concludes the proof of Theorem A and the second limit of Theorem B. We now proceed to the rationality of the generating function, the two closed form expressions for its coefficients and the limit of successive ratios.

Lemma B1. Let $f_{k,n}$ be defined as in Theorem B. For $k \geqslant 1$, the generating function for $f_{k,n}$, viz.,

$$G_k(t) = \sum_{n \geqslant 0} f_{k,n} t^n \tag{1}$$

is a rational function of t. In fact,

$$G_k(t) = \frac{t^k}{1 - t - t^k} \tag{2}$$

$$= t^k \sum_{1 \leqslant j \leqslant k} \frac{A_{k,j}}{1 - \lambda_{k,j} t}, \tag{3}$$

where the $\lambda_{k,j}$ are as in Corollary A1 and

$$A_{k,j} = B_{k,j} \lambda_{k,j}^k \tag{4}$$

with

$$B_{k,j} = \frac{\lambda_{k,j} - 1}{k\lambda_{k,j} - (k-1)} . \tag{5}$$

Proof. Given equations (2) and (3), we have

$$1 = \sum_{1 \leqslant j \leqslant k} A_{k,j} \frac{1 - t - t^k}{1 - \lambda_{k,j} t} . \tag{3'}$$

From Lemma A1, and letting $t \to \lambda_{k,j}^{-1}$ we have

$$A_{k,j} = \frac{\lambda_{k,j}^k}{k - \lambda_{k,j}^{k-1}} \tag{4'}$$

which, with $\lambda_{k,j}^{1-k} = \lambda_{k,j} - 1$ yields (5).

From the initial conditions, $f_{k,j} = 0, j < k, f_{k,k} = 1$ we have $f_{k,k+j} = 1, 0 \leqslant j < k$ by referring to the relation

$$f_{k,n} = f_{k,n-1} + f_{k,n-k} . \tag{6}$$

Then

$$G_k(t) = \sum_{k \leqslant n < 2k} f_n t^n + \sum_{n \geqslant 2k} f_n t^n \tag{7}$$

and

$$tG_k(t) = \sum_{k < n < 2k} f_{n-1} t^n + \sum_{n \geqslant 2k} f_{n-1} t^n \tag{8}$$

$$t^k G_k(t) = 0 + \sum_{n \geqslant 2k} f_{n-k} t^n . \tag{9}$$

From the relation (6) we have the equation

$$tG_k(t) - \sum_{k<n<2k} f_{n-1}t^n + t^kG_k(t) = G_k(t) - \sum_{k\leqslant n<2k} f_nt^n . \tag{10}$$

Isolating $G_k(t)$ and noting that

$$t^k = \sum_{k\leqslant n<2k} f_nt^n - \sum_{k<n<2k} f_{n-1}t^n \tag{11}$$

we have

$$G_k(t) = \frac{t^k}{1-t-t^k} \tag{12}$$

$$= \frac{t^k}{\prod\limits_{1\leqslant j\leqslant k} (1-\lambda_{k,j}t)} , \tag{13}$$

where $\lambda_{k,j}$ are the solutions of $Z^k - Z^{k-1} = 0$. Clearly,

$$\sum_{1\leqslant j\leqslant k} \lambda_{k,j} = 1, \qquad \prod_{1\leqslant j\leqslant k} \lambda_{k,j} = (-1)^{k-1} . \tag{14}$$

Since, by Lemma A1 the $\lambda_{k,j}$ are all distinct we have the partial fractions decomposition stated in the Lemma, Eq. (3).

Lemma B2. Let $k \geqslant 1$.

$$f_{k,n} = \sum_{0\leqslant m\leqslant (n-k)/k} \binom{n-k-(k-1)m}{m} . \tag{15}$$

Proof. From Eq. (2) in Lemma B1 we have

$$G_k(t) = \frac{t^k}{1-(t+t^k)} \tag{16}$$

$$= t^k \sum_{s\geqslant 0} t^s(1+t^{k-1})^s \tag{17}$$

$$= \sum_{s\geqslant 0} t^{s+k} \sum_{0\leqslant m\leqslant s} \binom{s}{m} t^{(k-1)m} \tag{18}$$

$$= \sum_{s\geqslant 0}\sum_{0\leqslant m\leqslant s} \binom{s}{m} t^{s+k+(k-1)m} \tag{19}$$

$$= \sum_{n\geqslant 0} t^n \sum_{0\leqslant m\leqslant (n-k)/k} \binom{n-k-(k-1)m}{m} . \tag{20}$$

Thus (15) follows from the definition of $G_k(t)$. Note that if $k = 1$,

$$f_{1,n} = \sum_{0 \le m \le n-1} \binom{n-1}{m} = 2^{n-1}, \tag{21}$$

corresponding to summing Pascal's triangle horizontally. If $k = 2$, the case of Fibonacci numbers yields the familiar

$$f_{2,n} = \sum_{0 \le m \le (n-2)/2} \binom{n-2-m}{m}, \tag{22}$$

corresponding to summing the binomial coefficients lying upon lines of slope 1 through Pascal's triangle. In general one sums along lines of slope $k-1$. See Figure 4.

	1							
	1	1						
	1	2	1					
	1	3	3	1				
$f_{1,5} = 16$	1	4	6	4	1			
	1	5	10	10	5	1		
$f_{2,7} = 8$	1	6	15	20	15	6	1	
	1	7	21	35	35	21	7	1
	1	8	28	56	70	56	28	
$f_{3,10} = 9$	1	9	36	84	126	126		
	1	10	45	120	210			
	1	11	55	165				
	1	12	66					
$f_{4,14} = 14$	1	13						
	1							

Figure 4. The Numbers $f_{k,n}$ as Sums of Binomial Coefficients Lying Upon Lines of Slope $k-1$ through Pascal's Triangle. (See Lemma B2.)

Lemma B3. Let $k \ge 1$. Then

$$f_{k,n} = \sum_{1 \le j \le k} \frac{(\lambda_{k,j} - 1)}{k\lambda_{k,j} - (k-1)} \lambda_{k,j}^n, \tag{23}$$

where the $\lambda_{k,j}$ are the zeroes of

$$z^k - z^{k-1} - 1.$$

Proof. From Eq. (3),

$$G_k(t) = t^k \sum_{1 \le j \le k} A_{k,j} \sum_{n \ge 0} \lambda_{k,j}^n t^n, \tag{24}$$

$$= \sum_{n \ge 0} t^{n+k} \sum_{1 \le j \le k} A_{k,j} \lambda_{k,j}^n, \tag{25}$$

$$= \sum_{n \ge k} t^n \sum_{1 \le j \le k} B_{k,j} \lambda_{k,j}^n. \tag{26}$$

Table 3

Real and Complex Zeroes Rounded to Five Places, $\lambda_{k,j}, j = 1, 2, \cdots, k$, of the Polynomial $Z^k - Z^{k-1} - 1$ for $k = 1, 2, \cdots, 10$ (The zeroes are listed in decreasing order of modulus. A more complete table of these roots, $k = 1, 2, \cdots, 20$ to 28 significant digits is available upon request .)

k	$\lambda_{k,k}$					
1	2.00000					
2	1.61803	−0.61803				
3	1.46557	−0.23279 ± i0.79255				
4	1.38028	0.21945 ± i0.91447	−0.81917			
5	1.32472	0.50000 ± i0.86603	−0.66236 ± i0.56228			
6	1.28520	0.67137 ± i0.78485	−0.37333 ± i0.82964	−0.88127		
7	1.25542	0.78019 ± i0.70533	−0.10935 ± i0.93358	−0.79855 ± i0.42110		
8	1.23205	0.85224 ± i0.63526	0.10331 ± i0.95648	−0.61578 ± i0.68720	−0.91159	
9	1.21315	0.90173 ± i0.57531	0.26935 ± i0.94058	−0.41683 ± i0.84192	−0.86082 ± i0.33435	
10	1.19749	0.93677 ± i0.52431	0.39863 ± i0.90691	−0.23216 ± i0.92442	−0.73720 ± i0.57522	−0.92957

Lemma B4. Fix $k \geqslant 1$. Then

$$\lim_{n \to \infty} \frac{f_{k,n+1}}{f_{k,n}} = \lambda_{k,\max}, \tag{27}$$

where $\lambda_{k,\max}$ is the largest positive real root of $Z^k - Z^{k-1} - 1$. In fact, $\lambda_{k,\max} = \lambda_{k,k}$.

Proof. From Lemma B3,

$$\frac{f_{k,n+1}}{f_{k,n}} = \frac{\sum\limits_{1 \leqslant j \leqslant k} B_{k,j}\lambda_{k,j}^{n+1}}{\sum\limits_{1 \leqslant j \leqslant k} B_{k,j}\lambda_{k,j}^{n}}. \tag{28}$$

Define $\lambda_{k,\max}$ to be the zero of $Z^k - Z^{k-1} - 1$ with largest absolute value. Then

$$\frac{f_{k,n+1}}{f_{k,n}} = \lambda_{k,\max} \frac{\sum\limits_{1 \leqslant j \leqslant k} B_{k,j}\left(\frac{\lambda_{k,j}}{\lambda_{k,\max}}\right)^{n+1}}{\sum\limits_{1 \leqslant j \leqslant k} B_{k,j}\left(\frac{\lambda_{k,j}}{\lambda_{k,\max}}\right)^{n}}. \tag{29}$$

Letting $n \to \infty$, each sum in the quotient has one or two terms depending upon whether $\lambda_{k,\max}$ is real or complex and in the latter case the limit need not exist. But from the proof of Lemma A2, $\lambda_{k,\max}$ is real and is equal to $\lambda_{k,k}$. (Each nonreal complex root has absolute value r such that $1 + r^{1-k} > r$ or $r < \lambda_{k,k} = \lambda_1(k)$.) Since

$$\lim_{n \to \infty} (\lambda_{k,j}/\lambda_{k,k})^n = \delta_k^j,$$

the Lemma follows.

Lemma A8. Let $k > 1$. Then

$$\lambda_\epsilon(k) = \lim_{n \to \infty} \mu_{\epsilon,n} \tag{30}$$

where

$$\mu_{0,n+1} = (1 + \mu_{0,n})^{1/(1-k)}, \qquad \mu_{0,0} = 1 \qquad \text{and} \qquad \mu_{1,n+1} = 1 + \mu_{1,n}^{1-k}, \qquad \mu_{1,0} = 1.$$

Proof. Clear

Lemma A9. For $k \geqslant 0$

(31) $$\lambda_1(k) > \lambda_1(k+1).$$

In other words $\lambda_1(k)$ converges monotonically to 1 as $k \to \infty$.

Proof. [2]. Let $r = \lambda_1(k)$, $s = \lambda_1(k+1)$. Then $r > 1$, $s > 1$, $r \neq s$, and

$$r(r^k - r^{k-1} - 1) = 0, \qquad s^{k+1} - s^k - 1 = 0.$$

Subtracting the second equation from the first and dividing through by $r - s$ we have

(32) $$\frac{(r^{k+1} - s^{k+1})}{r-s} - \frac{(r^k - s^k)}{r-s} = \frac{r-1}{r-s}.$$

But the left-hand side is positive because it equals

(33) $$r^k + (s-1)(r^{k-1} + r^{k-2}s + \cdots + rs^{k-2} + s^{k-1}).$$

Thus $r - s > 0$.

REFERENCES

1. T. Norman, B. Hays, H. R. P. Ferguson, *An Analysis of Dynamic Storage Allocation Schemes Based on Generalized Fibonacci Sequences,* 1974, to appear.
2. D. W. Robinson, private communication.

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A NOTE ON SOME ARITHMETIC FUNCTIONS CONNECTED WITH THE FIBONACCI NUMBERS

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1. INTRODUCTION AND PRELIMINARIES

The Fibonacci numbers are defined as usual by

(1.1) $$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 2)$$

and the Lucas numbers are defined by

(1.2) $$L_0 = 1, \quad L_1 = 3, \quad L_n = L_{n-1} + L_{n-2} \quad (n \geqslant 2).$$

Recall that if x is any real number, $[x]$ is defined to be the greatest integer less than or equal to x, and $\{x\} = x - [x]$ is called the fractional part of x. Thus we have $0 \leqslant \{x\} < 1$.

In [1] and [2], L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville have introduced and studied the arithmetic functions a and b which are defined by

(1.3) $$a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n], \quad \text{where} \quad \alpha = ½(1+\sqrt{5}), \text{ and } n > 0.$$

The functions a and b satisfy many relations which follow from (1.3), e.g.,

(1.4) $$b(n) = a(n) + n = a^2(n) + 1 \quad (n \geqslant 1)$$

(1.5) $$ab(n) = a(n) + b(n) = ba(n) + 1 \quad (n \geqslant 1).$$

Here, and throughout this paper, juxtaposition of functions indicates composition.

The equalities (1.4) and (1.5) are given in [1], along with many other properties of a and b.

In the present paper we show that the function a has the following property: Let $j > 0$ and let n be an integer with $n > F_{2j}$. If $a(n) \not\equiv a(n - F_{2j}) \pmod{F_{2j+1}}$ then

$$a(n) \equiv a(n + kF_{2j}) \pmod{F_{2j+1}} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2.$$

In fact, we have the stronger result, that

$$a(n + kF_{2j}) = a(n) + kF_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2.$$

In addition, if $a(n) \not\equiv a(n - F_{2j}) \pmod{F_{2j+1}}$, then

$$a(n + L_{2j}F_{2j}) = a(n) + L_{2j}F_{2j} - 1.$$

We give conditions on n for deciding whether or not $a(n) \equiv a(n + (L_{2j} - 1)F_{2j}) \pmod{F_{2j+1}}$.

Finally, we have similar results for the Fibonacci numbers of odd index, and for the Lucas numbers.

2. VALUES OF THE FUNCTION α WHICH ARE CONGRUENT MODULO A FIXED FIBONACCI NUMBER

We shall require a few facts about the Fibonacci and Lucas numbers, which may be found in V.E. Hoggatt, Jr. [4].

If $\alpha = ½(1+\sqrt{5})$ and $\beta = ½(1-\sqrt{5})$ (i.e., α and β are the roots of the equation $x^2 - x - 1 = 0$), then the Fibonacci numbers, defined by (1.1), are also given by

(2.1) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n = 0, 1, 2, \cdots)$$

and the Lucas numbers defined by (1.2) are given by

(2.2) $$L_n = \alpha^n + \beta^n \quad (n = 0, 1, 2, \cdots).$$

Using (2.1) and (2.2), it is easy to check that

(2.3) $$\alpha F_n = F_{n+1} - \beta^n \quad (n = 0, 1, \cdots)$$

and

(2.4) $$\alpha L_n = L_{n+1} - \beta^{n-1}(1+\beta^2) \quad (n = 0, 1, \cdots).$$

The main results of the present paper are given in the next four theorems.

2.5 *Theorem.* Let n be a positive integer. Write $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > F_{2j}$ $(j > 0)$ and that $a(n - F_{2j}) \neq a(n) - F_{2j+1}$. Then

(i) $$\epsilon > 1 - \beta^{2j}$$

(ii) $$a(n + kF_{2j}) = a(n) + kF_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2$$

(iii) If $\epsilon \geqslant 1 - \beta^{2j} + \beta^{4j}$, then

$$a(n + (L_{2j} - 1)F_{2j}) = a(n) + (L_{2j} - 1)F_{2j+1}$$

(iv) If $\epsilon < 1 - \beta^{2j} + \beta^{4j}$, then

$$a(n + (L_{2j} - 1)F_{2j}) = a(n) + (L_{2j} - 1)F_{2j+1} - 1$$

(v) $$a(n + L_{2j}F_{2j}) = a(n) + L_{2j}F_{2j+1} - 1.$$

2.6 *Theorem.* Let n be a positive integer, and set $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > F_{2j+1}$ and

$$a(n - F_{2j+1}) \neq a(n) - F_{2j+2}.$$

Then

(i) $$\epsilon < |\beta|^{2j+1}$$

(ii) $$a(n + kF_{2j+1}) = a(n) + kF_{2j+2} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j+1} - 1$$

(iii) If $\epsilon < \beta^{4j+2}$, then we have

$$a(n + L_{2j+1}F_{2j+1}) = a(n) + L_{2j+1}F_{2j+2}$$

(iv) If $\epsilon \geqslant \beta^{4j+2}$, then

$$a(n + L_{2j+1}F_{2j+1}) = a(n) + L_{2j+1}F_{2j+2} + 1$$

(v) $$a(n + (L_{2j+1} + 1)F_{2j+1}) = a(n) + (L_{2j+1} + 1)F_{2j+2} + 1.$$

2.7 *Theorem.* Let n be a positive integer, and set $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > L_{2j}$ $(j > 0)$ and that $a(n - L_{2j}) \neq a(n) - L_{2j+1}$. Then

(i) $$\epsilon < |\beta|^{2j-1}(1+\beta^2)$$

(ii) $$a(n + kL_{2j}) = a(n) + kL_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, F_{2j} - 1.$$

(iii) If $\epsilon < \beta^{4j}$, then

$$a(n + F_{2j}L_{2j}) = a(n) + F_{2j}L_{2j+1}$$

(iv) If $\epsilon \geqslant \beta^{4j}$, then

$$a(n + F_{2j}L_{2j}) = a(n) + F_{2j}L_{2j+1} + 1$$

(v) $$a(n + (F_{2j} + 1)L_{2j}) = a(n) + (F_{2j} + 1)L_{2j+1} + 1.$$

2.8 *Theorem.* Let n be a positive integer, with $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > L_{2j+1}$ $(j > 0)$ and that $a(n - L_{2j+1}) \neq a(n) - L_{2j+2}$. Then

(i) $$\epsilon > 1 - \beta^{2j}(1 + \beta^2)$$

(ii) $$a(n + kL_{2j+1}) = a(n) + kL_{2j+2} \quad \text{for} \quad k = 0, 1, \cdots, F_{2j+1} - 2$$

(iii) If $\epsilon > 1-\beta^{2j}(1+\beta^2)+\beta^{4j+2}$, then

$$a(n+(F_{2j+1}-1)L_{2j+1}) = a(n)+(F_{2j+1}-1)L_{2j+2}$$

(iv) If $\epsilon \leqslant 1-\beta^{2j}(1+\beta^2)+\beta^{4j+2}$, then

$$a(n+(F_{2j+1}-1)L_{2j+1}) = a(n)+(F_{2j+1}-1)L_{2j+2}-1$$

(v) $$a(n+F_{2j+1}L_{2j+1}) = a(n)+F_{2j+1}L_{2j+2}-1.$$

The proofs of Theorems 2.5–2.8 are given in §3.

It is natural to ask about the values of $a(kF_m)$ and $a(kL_m)$, and in fact, we have the following theorem (which is not quite a direct corollary of the preceding results).

2.9 *Theorem.* Let $j > 0$ be any integer. Then

(a) $$a(kF_{2j}) = kF_{2j+1}-1 \quad \text{for} \quad k = 1, 2, \cdots, L_{2j}-1$$

and $$a(L_{2j}F_{2j}) = L_{2j}F_{2j+1}-2.$$

(b) $$a(kF_{2j+1}) = kF_{2j+2} \quad \text{for} \quad k = 1, 2, \cdots, L_{2j+1}$$

and $$a((L_{2j+1}+1)F_{2j+1}) = (L_{2j+1}+1)F_{2j+1}+1.$$

(c) $$a(kL_{2j}) = kL_{2j+1} \quad \text{for} \quad k = 1, 2, \cdots, F_{2j}$$

and $$a((F_{2j}+1)L_{2j}) = (F_{2j}+1)L_{2j+1}+1$$

(d) $$a(kL_{2j+1}) = kL_{2j+2}-1 \quad \text{for} \quad k = 1, 2, \cdots, F_{2j+1}-1$$

and $$a(F_{2j+1}L_{2j+1}) = F_{2j+1}L_{2j+2}-2.$$

Proof. The proofs of all four parts are very similar, and we prove only (a). By (1.3) we have

$$a(kF_{2j}) = [k\alpha F_{2j}] = [k(F_{2j+1}-\beta^{2j})],$$

where the last equality follows from (2.3). It is easy to check, using (2.2), that

$$L_{2j}\beta^{2j} = 1+\beta^{4j} > 1,$$

while

$$(L_{2j}-1)\beta^{2j} = 1+\beta^{4j}-\beta^{2j}$$

and since $|\beta| < 1$, we have $\beta^{2j} > \beta^{4j}$, so that

$$(L_{2j}-1)\beta^{2j} < 1.$$

Then for all $k = 1, 2, \cdots, L_{2j}-1$, we have $k\beta^{2j} < 1$, while $L_{2j}\beta^{2j} > 1$. This proves (a).

3. PROOFS

We prove in detail only Theorems 2.5 and 2.7. It is then obvious how to prove Theorems 2.6 and 2.8.

Proof of Theorem 2.5. From the definition (1.3) of the function a, we have $a(n) = m$, and

$$\begin{aligned} a(n-F_{2j}) &= [\alpha(n-F_{2j})] = [\alpha n-\alpha F_{2j}] \\ &= [m+\epsilon-(F_{2j+1}-\beta^{2j})] \qquad \text{(by (2.3))} \\ &= [m-F_{2j+1}+(\epsilon+\beta^{2j})]. \end{aligned}$$

Now the assumption

$$a(n-F_{2j}) \neq a(n)-F_{2j+1}$$

implies that

$$\epsilon+\beta^{2j} > 1$$

and this proves part (i).

To see (ii), suppose that $k > 0$ is any integer. Then

$$\begin{aligned} a(n+kF_{2j}) &= [\alpha n + k(\alpha F_{2j})] \\ &= [m+\epsilon+k(F_{2j+1}-\beta^{2j})] \quad \text{(by (2.3))} \\ &= [m+kF_{2j+1}+\epsilon-k\beta^{2j}] . \end{aligned}$$

To prove (ii), we need only show that for all k satisfying $0 \leqslant k \leqslant L_{2j}-2$, we have

$$0 < \epsilon - k\beta^{2j} < 1. \tag{3.1}$$

By (i), we have $\epsilon > 1-\beta^{2j}$. It suffices to show

$$1-\beta^{2j} \geqslant k\beta^{2j} > 0 \quad (k = 0, 1, \cdots, L_{2j}-2) \tag{3.2}$$

or equivalently,

$$(k+1)\beta^{2j} \leqslant 1 \quad (k = 0, 1, \cdots, L_{2j}-2). \tag{3.3}$$

Clearly, if we can show

$$(L_{2j}-1)\beta^{2j} \leqslant 1, \tag{3.4}$$

the inequality (3.3) will follow. By (2.2), we have

$$L_{2j}\beta^{2j} = (\alpha^{2j}+\beta^{2j})\beta^{2j} = 1+\beta^{4j} \tag{3.5}$$

and so

$$(L_{2j}-1)\beta^{2j} = 1+\beta^{4j}-\beta^{2j}.$$

Since $|\beta| < 1$, we have $\beta^{4j} < \beta^{2j}$ for all $j > 0$, and this proves (3.4).

To see (iii) and (iv), we have

$$a(n+(L_{2j}-1)F_{2j}) = [m+\epsilon+(L_{2j}-1)F_{2j+1}-(L_{2j}-1)\beta^{2j}] .$$

If $0 \leqslant \epsilon-(L_{2j}-1)\beta^{2j} < 1$, then we have

$$a(n+(L_{2j}-1)F_{2j}) = a(n)+(L_{2j}-1)F_{2j+1}.$$

But since

$$(L_{2j}-1)\beta^{2j} = 1-\beta^{2j}+\beta^{4j},$$

then

$$0 \leqslant \epsilon-(L_{2j}-1)\beta^{2j} < 1$$

is equivalent to

$$0 \leqslant \epsilon-(1-\beta^{2j}+\beta^{4j}) < 1 \tag{3.6}$$

or equivalently,

$$0 < 1-\beta^{2j}+\beta^{4j} \leqslant \epsilon < 1 \tag{3.7}$$

(since we always have $0 < \epsilon < 1$). This proves (iii).

It is clear that if (3.6) (and hence (3.7)) does not hold, then we must have

$$\epsilon-(L_{2j}-1)\beta^{2j} < 0 \tag{3.8}$$

since $0 < \epsilon < 1$ and $(L_{2j}-1)\beta^{2j} > 0$. It is evident that if (3.8) holds, then

$$a(n+(L_{2j}-1)F_{2j}) = a(n)+(L_{2j}-1)F_{2j+1}-1.$$

This proves (iv).

Finally, to see (v), we have from (3.5) that $L_{2j}\beta^{2j} = 1+\beta^{4j}$. Then

$$a(n+L_{2j}F_{2j}) = [m+\epsilon+L_{2j}(F_{2j+1}-\beta^{2j})] = [m+L_{2j}F_{2j+1}-1+\epsilon-\beta^{4j}]$$

We must show that

$$0 < \epsilon-\beta^{4j} < 1. \tag{3.9}$$

It is easy to compute that

(3.10) $$.6 < |\beta| < .7$$

so that $\beta^2 < \frac{1}{2}$ and $\beta^4 < \frac{1}{4}$. By (i) we know $\epsilon > 1 - \beta^{2j}$, and since $j > 0$, this gives $\epsilon > \frac{1}{2}$. But also, $\beta^{4j} \leqslant \beta^4 < \frac{1}{4}$, and (3.9) follows. This proves (v) and completes the proof of Theorem 2.5.

Proof of Theorem 2.7. As before, we have $a(n) = m$, and

$$a(n - L_{2j}) = [m + \epsilon - \alpha L_{2j}] = [m + \epsilon - (L_{2j+1} - \beta^{2j-1}(1+\beta^2))]$$

(by (2.4)). Then the assumption $a(n - L_{2j}) \neq a(n) - L_{2j+1}$ is equivalent to

(3.11) $$\epsilon + \beta^{2j-1}(1+\beta^2) < 0,$$

since $\beta < 0$. Clearly (3.11) is the same as

(3.12) $$\epsilon < |\beta|^{2j-1}(1+\beta^2)$$

and this proves (i).

To see (ii), we first have, for any integer $k > 0$,

$$a(n + kL_{2j}) = [m + \epsilon + k(L_{2j+1} - \beta^{2j-1}(1+\beta^2))].$$

As in the proof of Theorem 2.5, we need to show that

(3.13) $$0 < \epsilon + (F_{2j} - 1)|\beta|^{2j-1}(1+\beta^2) < 1.$$

We first note that, since $\alpha\beta = -1$,

(3.14) $$1 + \beta^2 = 1 - \frac{\beta}{\alpha} = \frac{\alpha - \beta}{\alpha}.$$

Then we have

$$F_{2j}|\beta|^{2j-1}(1+\beta^2) = \frac{\alpha^{2j} - \beta^{2j}}{\alpha - \beta} \cdot |\beta|^{2j-1} \cdot \frac{\alpha-\beta}{\alpha} = 1 - \frac{|\beta|^{4j-1}}{\alpha} = 1 - \beta^{4j}.$$

Then, using (i), we have (since $j > 0$)

$$0 < \epsilon + (F_{2j} - 1)|\beta|^{2j-1}(1+\beta^2)$$
$$< |\beta|^{2j-1}(1+\beta^2) + (1 - \beta^{4j}) - |\beta|^{2j-1}(1+\beta^2) = 1 - \beta^{4j} < 1.$$

It follows that if $0 \leqslant k \leqslant F_{2j} - 1$, we have

(3.16) $$0 < \epsilon + k|\beta|^{2j-1}(1+\beta^2) < 1$$

and (ii) is proved.

It is clear from (3.15) that if $0 < \epsilon < \beta^{4j}$, then

(3.17) $$0 < \epsilon + F_{2j}|\beta|^{2j-1}(1+\beta^2) = \epsilon + (1 - \beta^{4j}) < 1$$

and (iii) follows. On the other hand, if $\epsilon \geqslant \beta^{4j}$, then

$$\epsilon + F_{2j}|\beta|^{2j-1}(1+\beta^2) = \epsilon + (1 - \beta^{4j}) \geqslant 1,$$

and this proves (iv).

To see (v), we have

(3.18) $$(F_{2j} + 1)|\beta|^{2j-1}(1+\beta^2) = (1 - \beta^{4j}) + |\beta|^{2j-1}(1+\beta^2) > 1$$

and it follows that (v) holds.

This completes the proof of Theorem 2.7.

In view of (1.3), it is clear that Theorems 2.5 – 2.8 all remain true if we substitute the function b for the function a wherever it appears.

REFERENCES

1. L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville, "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 1–28.
2. ________, "Lucas Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 29–42.
3. L. Carlitz, R. Scoville, and T. P. Vaughan, "Some Arithmetic Functions Connected with the Fibonacci Numbers," to appear.
4. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton Mifflin Co., Boston, 1969.

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SOME BINOMIAL SUMS

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1. Put

$$A(n) = \sum_{k=0}^{n+1} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^3 , \tag{1.1}$$

where it is understood that

$$\binom{n}{-1} = \binom{n}{n+1} = 0 \qquad (n \geq 0).$$

Consideration of this sum was suggested by the following problem proposed by H. W. Gould [1]. Let

$$A_p(n) = \sum_{0 \leq 2k \leq n} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^p .$$

Then

$$A_2(2m+1) = (2m+1)A_1(2m+1).$$

It is noted that this result does not hold for even n.

Since

$$A(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{n-k+1} - \binom{n}{n-k} \right\}^3 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{k-1} - \binom{n}{k} \right\}^3 ,$$

so that

$$A(n) = (-1)^n A(n), \tag{1.2}$$

therefore

$$A(2m+1) = 0. \tag{1.3}$$

However (1.2) gives no information about $A(2m)$. By (1.1) we have

$$A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2$$

$$- \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}^3 = 2 \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 .$$

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Thus if we put

$$S_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3, \quad S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1},$$

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n}{k-1}^2,$$

it is clear that

(1.4) $$A(n) = 2S_0(n) - 3S_1(n) + 3S_2(n).$$

In the next place, we have

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n}{k-1}^2 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1}\binom{n}{n-k}^2$$

$$= \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{k-1}\binom{n}{k}^2,$$

so that

(1.5) $$S_2(n) = (-1)^{n+1} S_1(n)$$

and (1.4) becomes

(1.6) $$A(n) = 2S_0(n) - 3\left\{1 + (-1)^n\right\} S_1(n).$$

In particular we have

(1.7) $$\begin{cases} A(2m) = 2S_0(2m) - 6S_1(2m) \\ A(2m+1) = 2S_0(2m+1). \end{cases}$$

It is well known (see for example [2, p. 13], [3, p. 243]) that $S_0(2m+1) = 0$, while

(1.8) $$S_0(2m) = (-1)^m \frac{(3m)!}{(m!)^3}.$$

However $S_1(n)$ does not seem to be known.

2. In order to evaluate $S_1(2m)$ we proceed as follows. We have

$$S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \left\{\binom{n+1}{k} - \binom{n}{k}\right\} = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n+1}{k} - S_0(n)$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n+1}{k}\left\{\binom{n+1}{k} - \binom{n}{k-1}\right\} - S_0(n)$$

so that

(2.1) $$S_1(n) = T_0(n) - T_1(n) - S_0(n),$$

where

$$T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+1}{k}^2, \qquad T_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n+1}{k}\binom{n}{k-1},$$

Now

$$T_1(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1} \binom{n+1}{n-k+1} \binom{n}{n-k}$$

$$= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} \binom{n+1}{k} \binom{n}{k} ,$$

that is,

(2.2) $$T_1(n) = (-1)^{n+1} T_1(n).$$

Therefore $T_1(2m) = 0$ and (2.1) yields

(2.3) $$S_1(2m) = T_0(2m) - S_0(2m).$$

In the next place

$$T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \binom{n+1}{n-k}^2$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1}^2 = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k} \right\}$$

$$= -(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k+1} \right\}$$

$$= -(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2$$

$$- (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+1}{k} + \binom{n+1}{k+1} \right\}$$

$$= -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n+1}{k+1} - (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2$$

$$+ (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2 ,$$

so that

(2.4) $$\left\{ 1 + (-1)^n \right\} T_0(n) = -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n+1}{k+1}$$

$$+ (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2 .$$

For $n = 2m + 1$, (2.4) gives no information about $T_0(2m + 1)$; indeed each sum on the right vanishes. For $n = 2m$, however, (2.4) becomes

(2.5)
$$2T_0(2m) = -2\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+1}{k}\binom{2m+1}{k+1} + \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+2}{k+1}^2 .$$

It is known [3, p. 243] that

(2.6)
$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+1}{k}\binom{2m+1}{k+1} = (-1)^m\frac{(3m+1)!}{m!m!(m+1)!}$$

and

(2.7)
$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+2}{k+1}^2 = (-1)^m\frac{2(3m+2)!}{m!m!(m+1)!(2m+1)} .$$

Substituting from (2.6) and (2.7) in (2.5), we get

(2.8)
$$T_0(2m) = (-1)^m\frac{(3m+1)!}{(m!)^3(2m+1)} .$$

Therefore by (2.3) and (1.8)

(2.9)
$$S_1(2m) = (-1)^m\frac{(3m)!}{m!m!(m-1)!(2m+1)} .$$

Finally, by (1.6) and (2.9),

(2.10)
$$A(2m) = -2(-1)^m\frac{(3m)!(m-1)}{(m!)^3(2m+1)} .$$

This completes the evaluation of the sum $A(2m)$. Note that we have not evaluated $S_1(2m+1)$.

3. For completeness we give a simple proof of (1.8), (2.6) and (2.7). We assume Saalschütz's theorem [2, p. 9]:

(3.1)
$$\sum_{k=0}^{n}\frac{(-n)_k(a)_k(b)_k}{k!(c)_k(d)_k} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} ,$$

where

$$(a)_k = a(a+1)\cdots(a+k-1), \quad (a)_0 = 1$$

and

(3.2)
$$c+d = -n+a+b+1.$$

We rewrite (3.1) in the following way:

(3.3)
$$\sum_{r=0}^{j}\frac{(-j)_r(a+j)_r(b+c-a+1)_r}{r!(b+1)_r(c+1)_r} = \frac{(a-b)_j(a-c)_j}{(b+1)_j(c+1)_j} ;$$

the condition (3.2) is automatically satisfied. Multiplying both sides of (3.3) by $(a)_jx^j/j!$ and summing over j, it follows that

$$\sum_{j=0}^{\infty}\frac{(a)_j(a-b)_j(a-c)_j}{j!(b+1)_j(c+1)_j}x^j = \sum_{j=0}^{\infty}\frac{(a)_j}{j!}x^j\sum_{r=0}^{\infty}\frac{(-j)_r(a+j)_r(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{(a)_{2r}(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}x^r\sum_{j=0}^{\infty}\frac{(a+2r)_j}{j!}x^j = \sum_{r=0}^{\infty}(-1)^r\frac{(a)_{2r}(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}x^r(1-x)^{-a-2r} .$$

Now take $a = -n$ and we get

$$\sum_{j=0}^{\infty} \frac{(-n)_j(-n-b)_j(-n-c)_j}{j!(b+1)_j(c+1)_j} x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(-n)_{2r}(b+c+n-1)_r}{r!(b+1)_r(c+1)_r} x^r(1-x)^{n-2r}. \tag{3.4}$$

For $n = 2m$ and $x = 1$, (3.4) reduces to

$$\sum_{j=0}^{\infty} \frac{(-2m)_j(-2m-b)_j(-2m-c)_j}{j!(b+1)_j(c+1)_j} = (-1)^m \frac{(2m)!(b+c+2m+1)_m}{m!(b+1)_m(c+1)_m}. \tag{3.5}$$

Now let b,c be non-negative integers. Then (3.5) yields

$$\sum_{j=0}^{2m} (-1)^m \binom{2m}{j}\binom{2m+b+c}{j+b}\binom{2m+b+c}{j+c} \tag{3.6}$$

$$= (-1)^m \frac{(2m)!(3m+b+c)!(2m+b+c)!}{m!(m+b)(m+c)!(2m+b)!(2m+c)!}.$$

For $b = c = 0$ we get (1.8); for $b = 0$, $c = 1$ we get (2.6); for $b = c = 1$ we get (2.7).

REFERENCES

1. E 2395, *Amer. Math. Monthly,* 80 (1973), p. 75; solution, 80 (1973), p. 1146.
2. W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge, 1935.
3. L. J. Slater, *Generalized Hypergeometric Functions,* Cambridge, 1966.

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[Continued from Page 214.]

$$\frac{1}{k} \log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$. Since this limiting value is an irrational number, the sequence (u_n) is u.d. mod 1.

REMARK. Let p and q be non-negative integers. Then the sequence

$$p,\ q,\ p+q,\ p+2q,\ 2p+3q,\ \cdots$$

or (H_n), $n = 1, 2, \cdots$ with

$$H_n = qF_{n-1} + pF_{n-2} \quad (n \geqslant 3), \quad H_1 = p, \quad H_2 = q$$

possesses the property shown in Theorem 1. For if $v_n = \log H_n^{1/k}$, we have

$$v_{n+1} - v_n \to \frac{1}{k} \log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$.

Theorem 2. Let p, q, p^* and q^* be non-negative integers. Let (H_n) be the sequence

$$p,\ q,\ p+q,\ p+2q,\ 2p+3q,\ \cdots$$

and (H_n^*) the sequence

$$p^*,\ q^*,\ p^*+q^*,\ p^*+2q^*,\ 2p^*+3q^*,\ \cdots.$$

[Continued on Page 276.]

RESTRICTED COMPOSITIONS

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1. INTRODUCTION

A *composition* of the integer n into k parts is defined [1, p. 107] as the number of ordered sets of non-negative integers $(a_1, a_2, \cdots, a_k)$ such that

(1.1) $$a_1 + a_2 + \cdots + a_k = n.$$

It is well known and easy to prove that the number of such compositions is equal to the binomial coefficient

$$\binom{n+k-1}{k-1} \quad .$$

If we require that the a_i be strictly positive then of course the number of solutions of (1.1) is equal to

$$\binom{n-1}{k-1}.$$

In the present paper we consider the problem of determining the number of solutions of (1.1) when we require that

(1.2) $$a_i \neq a_{i+1} \qquad (i = 1, 2, \cdots, k-1).$$

Let $c(n,k)$ denote the number of solutions of (1.1) and (1.2) in *positive* a_i and let $\bar{c}(n,k)$ denote the number of solutions of (1.1) and (1.2) in *non-negative* a_i. Then clearly

(1.3) $$c(n,k) = \bar{c}(n-k,k).$$

Also it is evident from the definition that

(1.4) $$\bar{c}(n,k) = 0 \qquad (k > 2n+1).$$

We shall show that

(1.5) $$\sum_{n,k=0}^{\infty} c(n,k)x^n z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \dfrac{x^j z^j}{1-x^j}} \quad .$$

For $z = 1$, this reduces to

(1.6) $$\sum_{n=0}^{\infty} c(n)x^n = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \dfrac{x^j}{1-x^j}} \quad ,$$

where

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$$(1.7)\qquad c(n) = \sum_{k=1}^{n} c(n,k), \qquad c(0) = 1.$$

Thus $c(n)$ is the number of solutions of (1.1) and (1.2) with $a_i > 0$ when the number of parts is unrestricted.

It follows from (1.3) and (1.6) that

$$(1.8)\qquad 1 + \sum_{n,k=1}^{\infty} \bar{c}(n,k)x^n z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \dfrac{z^j}{1-x^j}}.$$

This is also proved independently.

The generating function for

$$(1.9)\qquad \bar{c}(n) = \sum_{k} \bar{c}(n,k), \qquad \bar{c}(0) = 1$$

is less immediate. It is proved that

$$(1.10)\qquad \sum_{0}^{\infty} \bar{c}(n)x^n = \frac{1}{1-(1-x)\sum_{1}^{\infty} \dfrac{x^{2j-1}}{(1-x^{2j-1})(1-x^{2j})}}.$$

It is of some interest to determine the radius of convergence of the series

$$(1.11)\qquad \sum_{0}^{\infty} c(n)x^n, \qquad \sum_{0}^{\infty} \bar{c}(n)x^n.$$

We show that the radius of convergence of the first is at least ½; the radius of convergence of the second is also probably > ½ but this is not proved.

2. GENERATING FUNCTIONS FOR $c(n,k)$ AND $c_a(n,k)$

It is convenient to define the following refinements of $c(n,k)$ and $\bar{c}(n,k)$. Let $c_a(n,k)$ denote the number of solutions of (1.1) and (1.2) in positive integers a_i with $a_i = a$; $\bar{c}_a(n,k)$ is defined as the corresponding number when the $a_i \geqslant 0$. Clearly

$$(2.1)\qquad c(n,k) = \sum_{a=1}^{n} c_a(n,k), \qquad \bar{c}(n,k) = \sum_{a=0}^{n} \bar{c}_a(n,k).$$

The enumerant $c_a(n,k)$ satisfies the recurrence

$$(2.2)\qquad c_a(n,k) = \sum_{b \neq a} c_b(n-a, k-1) \qquad (k > 1).$$

If we put, for $k \geqslant 1$,

$$F_a(x,k) = \sum_{n=1}^{\infty} c_a(n,k)x^n, \qquad \Phi_k(x,y) = \sum_{a=1}^{\infty} F_a(x,k)y^a,$$

it follows from (2.2) that

$$F_a(x,k) = x^a \sum_{b \neq a} F_b(x,k-1) \qquad (k > 1).$$

Then

$$\Phi_k(x,y) = \sum_{a=1}^{\infty} (xy)^a \sum_{b \neq a} F_b(x,k-1) = \sum_{b=1}^{\infty} F_b(x,k-1) \sum_{a \neq b} (xy)^a = \sum_{b=1}^{\infty} F_b(x,k-1) \left(\frac{xy}{1-xy} - (xy)^b \right),$$

so that

$$\Phi_k(x,y) = \frac{xy}{1-xy} \Phi_{k-1}(x,1) - \Phi_{k-1}(x,xy) \qquad (k > 1). \tag{2.3}$$

Iterating (2.3), we get

$$\Phi_k(x,y) = \frac{xy}{1-xy} \Phi_{k-1}(x,1) - \frac{x^2y}{1-x^2y} \Phi_{k-2}(x,1) + \Phi_{k-2}(x,x^2y) \qquad (k > 2)$$

and generally

$$\Phi_k(x,y) = \sum_{j=1}^{s} (-1)^{j-1} \frac{x^jy}{1-x^jy} \Phi_{k-j}(x,1) + (-1)^s \Phi_{k-s}(x,x^sy) \qquad (k > s).$$

In particular, for $s = k-1$, this becomes

$$\Phi_k(x,y) = \sum_{j=1}^{k-1} (-1)^{j-1} \frac{x^jy}{1-x^jy} \Phi_{k-j}(x,1) + (-1)^{k-1} \Phi_1(x,x^{k-1},y) \qquad (k > 1). \tag{2.4}$$

Since

$$\Phi_1(x,y) = \sum_{a=1}^{\infty} (xy)^a = \frac{xy}{1-xy},$$

it is clear that (2.4) may be replaced by

$$\Phi_k(x,y) = \sum_{j=1}^{k} (-1)^{j-1} \frac{x^jy}{1-x^jy} F_{k-1}(x,1) \qquad (k \geq 1), \tag{2.5}$$

where it is understood that

$$\Phi_0(x,y) = 1. \tag{2.6}$$

For $y = 1$, (2.5) reduces to

$$\Phi_k(x,1) + \sum_{j=1}^{k} (-1)^j \frac{x^j}{1-x^j} \Phi_{k-j}(x,1) = \delta_{k,1}, \tag{2.7}$$

where $\delta_{k,1}$ is the Kronecker delta.

Using (2.6), this gives

$$\sum_{k=0}^{\infty} z^k \left\{ \Phi_k(x,1) + \sum_{j=1}^{k} (-1)^j \frac{x^j}{1-x^j} \Phi_{k-j}(x,1) \right\} = 1$$

and therefore

$$\sum_{k=0}^{\infty} \Phi_k(x,1)z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \frac{x^jz^j}{1-x^j}}. \tag{2.8}$$

In view of (2.1), (2.8) can be written in the more explicit form

$$(2.9)\qquad \sum_{n,k=0}^{\infty} c(n,k)x^n z^k = \frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{x^j z^j}{1-x^j}} .$$

We now put

$$(2.10)\qquad \frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{x^j z^j}{1-x^j}} = \sum_{k=0}^{\infty} \frac{P_k(x)}{(x)_k} z^k ,$$

where

$$(x)_k = (1-x)(1-x^2)\cdots(1-x^k), \qquad (x)_0 = 1.$$

Clearly

$$(2.11)\qquad \Phi_k(x,1) = \frac{P_k(x)}{(x)_k} .$$

The $P_k(x)$ are polynomials in x that satisfy

$$(2.12)\qquad P_k(x) = \sum_{j=1}^{k} (-1)^{j-1} \begin{bmatrix} k \\ j \end{bmatrix} (x)_{j-1} x^j P_{k-j}(x) \qquad (k \geq 1),$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(x)_k}{(x)_j (x)_{k-j}} .$$

The first few values of $P_k(x)$ are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 2x^3, \quad P_3(x) = x^4 + x^5 + 4x^6 .$$

In the next place, by (2.5),

$$\sum_{k=1}^{\infty} \Phi_k(x,y)z^k = \sum_{k=1}^{\infty} z^k \sum_{j=1}^{k} (-1)^{j-1} \frac{x^j y}{1-x^j y} \Phi_{k-j}(x,1) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j y z^j}{1-x^j y} \sum_{k=0}^{\infty} \Phi_k(x,1)z^k .$$

Hence, by (2.8),

$$(2.13)\qquad \sum_{k=1}^{\infty} \Phi_k(x,y)z^k = \frac{\sum_{j=1}^{\infty} (-1)^{j-1} \dfrac{x^j y z^j}{1-x^j y}}{1+\sum_{j=1}^{\infty} (-1)^j \dfrac{x^j z^j}{1-x^j}} .$$

This evidently reduces to (2.8) when $y = 1$.

Note that the LHS of (2.13) is equal to

$$(2.14)\qquad \sum_{n=1}^{\infty} \sum_{a,k} c_a(n,k)x^n y^a z^k .$$

3. GENERATING FUNCTION FOR $c(n)$ AND RELATED FUNCTIONS

For $z = 1$, (2.8) reduces to

(3.1)
$$\sum_{k=0}^{\infty} \Phi_k(x,1) = \frac{1}{1-\sum_{j=1}^{\infty} (-1)^{j-1} \dfrac{x^j}{1-x^j}}.$$

We have

$$\sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{1-x^j} = \sum_{j,k=1}^{\infty} (-1)^{j-1} x^{jk} = \sum_{n=1}^{\infty} x^n \sum_{j|n} (-1)^{j-1}.$$

Put

(3.2)
$$d'(n) = \sum_{j|n} (-1)^{j-1};$$

thus $d'(n)$ is the number of odd divisors of n less the number of even divisors.

For $n = 2^r m$, where m is odd, and $r \geq 0$,

$$d'(n) = \sum_{s=0}^{r} \sum_{j|m} (-1)^{2^s j-1} = (1-r) \sum_{j|m} 1,$$

so that

(3.3)
$$d'(n) = -(r-1)d(m),$$

where $d(n)$ is the number of divisors of n.

Thus we may replace (3.1) by

(3.4)
$$\sum_{k=0}^{\infty} \Phi_k(x,1) = \frac{1}{1-\sum_{1}^{\infty} d'(n)x^n}.$$

Since

$$\sum_{k=0}^{\infty} \Phi_k(x,1) = 1 + \sum_{n,k,a=1}^{\infty} c_a(n,k)x^n = 1 + \sum_{n=1}^{\infty} c(n)x^n,$$

we have therefore

(3.5)
$$1 + \sum_{n=1}^{\infty} c(n)x^n = \frac{1}{1-\sum_{j=1}^{\infty} (-1)^{j-1} \dfrac{x^j}{1-x^j}} = \frac{1}{1-\sum_{n=1}^{\infty} d'(n)x^n}.$$

It follows that $c(n)$ satisfies the recurrence

(3.6)
$$c(n) = \sum_{j=1}^{n} d'(j)c(n-j) \qquad (n \geq 1),$$

where $c(0) = 1$.

It is also of some interest to take $z = -1$ in (2.8). We get

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,1) = \frac{1}{1+\sum_{1}^{\infty} \dfrac{x^j}{1-x^j}} = \frac{1}{1+\sum_{1}^{\infty} d(n)x^n}.$$

Since

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,1) = 1 + \sum_{n,k,a=1}^{\infty} (-1)^k c_a(n,k)x^n = 1 + \sum_{n=1}^{\infty} c^*(n)x^n ,$$

where

$$c^*(n) = \sum_{k,a=1}^{n} (-1)^k c_a(n,k), \tag{3.7}$$

we get

$$1 + \sum_{1}^{\infty} c^*(n)x^n = \frac{1}{1 + \sum_{1}^{\infty} d(n)x^n} . \tag{3.8}$$

This yields the recurrence

$$c^*(n) + \sum_{j=1}^{n} d(j)c^*(n-j) = 0 \qquad (n \geqslant 1), \tag{3.9}$$

where $c^*(0) = 1$.

The first few values of $c^*(n)$ are

$$c^*(1) = -1, \quad c^*(2) = -1, \quad c^*(3) = 1, \quad c^*(4) = 0, \quad c^*(5) = 1, \quad c^*(6) = -2 .$$

It is also of interest to take $y = -1$ in (2.13). For $y = -1$, $z = 1$ we get

$$\sum_{k=1}^{\infty} \Phi_k(x,-1) = \frac{\sum_{1}^{\infty} (-1)^j \frac{x^j}{1+x^j}}{1 + \sum_{1}^{\infty} (-1)^j \frac{x^j}{1-x^j}} ,$$

so that

$$\sum_{k=0}^{\infty} \Phi_k(x,-1) = \frac{1 + 2\sum_{1}^{\infty} (-1)^j \frac{x^j}{1-x^{2j}}}{1 + \sum_{1}^{\infty} (-1)^j \frac{x^j}{1-x^j}} . \tag{3.10}$$

If we take $y = z = -1$ in (2.13) we get

$$\sum_{k=1}^{\infty} (-1)^k \Phi_k(x,-1) = \frac{\sum_{1}^{\infty} \frac{x^j}{1+x^j}}{1 + \sum_{1}^{\infty} \frac{x^j}{1-x^j}} ,$$

so that

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,-1) = \frac{1 + 2\sum_{1}^{\infty} \frac{x^j}{1-x^{2j}}}{1 + \sum_{1}^{\infty} \frac{x^j}{1-x^j}} = \frac{1 + 2\sum_{1}^{\infty} d_0(n)x^n}{1 + \sum_{1}^{\infty} d(n)x^n} , \tag{3.11}$$

where $d_o(n)$ denotes the number of *odd* divisors of n. Note that the LHS of (3.11) is equal to

$$1+\sum_{n=1}^{\infty} x^n \sum_{a,k} (-1)^{a+k} c_a(n,k). \tag{3.12}$$

4. GENERATING FUNCTION FOR $\bar{c}(n,k)$ AND $\bar{c}_a(n,k)$

While generating functions for $\bar{c}(n,k)$ and $\bar{c}_a(n,k)$ can be obtained from those for $c(n,k)$ and $c_a(n,k)$ by using (1.3), it is of some interest to derive them independently. Put

$$\bar{F}_a(x,k) = \sum_{n=0}^{\infty} \bar{c}_a(n,k)x^n, \qquad \bar{\Phi}_k(x,y) = \sum_{a=0}^{\infty} \bar{F}_a(x,k)y^a .$$

Then, exactly as in Section 2,

$$\bar{c}_a(n,k) = \sum_{b\neq a} c_b(n-a,k),$$

so that

$$\bar{F}_a(n,k) = x^a \sum_{b\neq a} \bar{F}_b(x,k-1)$$

and

$$\bar{\Phi}_k(x,y) = \sum_{a=0}^{\infty} (xy)^a \sum_{b\neq a} \bar{F}_b(x,k-1) = \sum_{b=0}^{\infty} \bar{F}_b(x,k-1)\left(\frac{1}{1-xy} - (xy)^b\right).$$

Thus

$$\bar{\Phi}_k(x,y) = \frac{1}{1-xy}\Phi_{k-1}(x,1) - \bar{\Phi}_{k-1}(x,xy) \qquad (k>1). \tag{4.1}$$

As above, iteration yields

$$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{1-x^jy}\bar{\Phi}_{k-j}(x,1) + (-1)^{k-1}\bar{\Phi}_1(x,x^{k-1}y) \qquad (k>1).$$

Since

$$\bar{\Phi}_1(x,y) = \sum_{a=0}^{\infty} (xy)^a = \frac{1}{1-xy},$$

we get

$$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{1-x^jy}\bar{\Phi}_{k-j}(x,1) \qquad (k \geqslant 1), \tag{4.2}$$

where

$$\bar{\Phi}_0(x,y) = 1. \tag{4.3}$$

For $y = 1$, (4.2) reduces to

$$\bar{\Phi}_k(x,1) + \sum_{j=1}^{k} \frac{(-1)^j}{1-x^j}\bar{\Phi}_{k-j}(x,1) = \delta_{k,0}. \tag{4.4}$$

It follows that

(4.5)
$$\sum_{k=0}^{\infty} \bar{\Phi}_k(x,1)z^k = \frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{z^j}{1-x^j}} .$$

Now put

$$\frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{z^j}{1-x^j}} = \sum_{k=0}^{\infty} \frac{\bar{P}_k(x)}{(x)_k} z^k ,$$

so that

(4.6)
$$\bar{\Phi}_k(x,1) = \frac{\bar{P}_k(x)}{(x)_k} .$$

The $\bar{P}_k(x)$ are polynomials in x that satisfy the recurrence

(4.7)
$$\bar{P}_k(x) = \sum_{j=1}^{k} (-1)^{j-1} \begin{bmatrix} k \\ j \end{bmatrix} (x)_{j-1}\bar{P}_{k-j}(x) \qquad (k \geqslant 1);$$

also it is clear from the definition that

(4.8)
$$P_k(x) = x^k \bar{P}_k(x).$$

For $x = 1$, (4.7) reduces to

$$\bar{P}_k(1) = k\bar{P}_{k-1}(1),$$

so that

(4.9)
$$\bar{P}_k(1) = k!.$$

Also it is easy to show by induction that

$$\deg \bar{P}_j(x) \leqslant \tfrac{1}{2}j(j-1).$$

Indeed, assuming that this holds for $j < k$, it follows that the degree of the j^{th} term on the right of (4.7)

$$\leqslant j(k-j) + \tfrac{1}{2}j(j-1) + \tfrac{1}{2}(k-j)(k-j-1) = \tfrac{1}{2}k(k-1).$$

Let γ_k denote the coefficient of $x^{\frac{1}{2}k(k-1)}$ in $\bar{P}_k(x)$. Then we have

$$\gamma_k = \sum_{j=1}^{k} \gamma_{k-j} = \sum_{j=0}^{k-1} \gamma_j \qquad (k \geqslant 1).$$

This gives

$$\sum_{k=0}^{\infty} \gamma_k x^k \left(1 - \sum_{j=1}^{\infty} x^j \right) = 1,$$

so that

$$\sum_{k=0}^{\infty} \gamma_k x^k = \frac{1-x}{1-2x} .$$

Thus $\gamma_k = 2^{k-1}$, $k \geqslant 1$, and so

(4.10)
$$\deg \bar{P}_k(x) = \tfrac{1}{2}k(k-1).$$

Since, by (2.4),

$$\bar{c}(n,k) = 0 \qquad (k > 2n+1),$$

it follows that $\bar{P}_k(x)$ begins with a term in $x^{[k/2]}$; moreover the coefficient of this term is 1 for k odd and 2 for k even and positive.

It is clear from the recurrence (4.7) that all the coefficients are integers. It would be interesting to know if they are positive.

If we put

$$\bar{P}_k(x) = \sum_j \gamma(k,j)x^j \quad \text{and} \quad \frac{1}{(x)_k} = \sum_{n=0}^{\infty} p(n,k)x^n ,$$

so that $p(n,k)$ is the number of partitions (in the usual sense) of n into parts $\leqslant k$, it follows from (4.6) that

$$\bar{c}(n,k) = \sum_j p(n-j,\, k)\gamma(k,j). \tag{4.11}$$

Returning to (4.2), we have

$$\sum_{k=1}^{\infty} \bar{\Phi}_k(x,y)z^k = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy} \sum_{k=0}^{\infty} \bar{\Phi}_k(x,1)z^k .$$

This gives

$$\sum_{k=1}^{\infty} \bar{\Phi}_k(x,y)z^k = \frac{\displaystyle\sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy}}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}} . \tag{4.12}$$

We may rewrite (4.5) and (4.12) as

$$1+\sum_{n,k=1}^{\infty} \bar{c}(n,k)x^nz^k = \frac{1}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}} , \tag{4.13}$$

$$1+\sum_{n=1}^{\infty}\sum_{a,k} \bar{c}_a(n,k)x^ny^az^k = \frac{\displaystyle\sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy}}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}} . \tag{4.14}$$

By (1.3) we have

$$\bar{c}(n,k) = c(n+k,\, k). \tag{4.15}$$

Hence, replacing z by xz in (4.5), we have

$$1+\sum_{n,k=1}^{\infty} c(n+k,\, k)x^{n+k}z^k = \frac{1}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{x^jz^j}{1-x^j}} . \tag{4.16}$$

This is of course equivalent to (2.9).

Since

$$\bar{c}_a(n,k) = c_{a+1}(n+k,\, k) \qquad (k > 0),$$

the equivalence of (4.16) and (2.9) follows easily.

Note that it follows from (4.6) and (4.12) that

$$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{1-x^jy} \frac{\bar{P}_{k-j}(x)}{(x)_{k-j}} . \tag{4.17}$$

In addition to (4.15) another relation expressing $\bar{c}(n,k)$ in terms of $c(n,k)$ can be obtained by considering the possible location of zero elements. There may be a zero on the extreme left or the extreme right; also there may be one or more zeros on the inside. Thus we get relations such as the following.

$$\bar{c}(0,0) = \bar{c}(0,1) = 1, \qquad \bar{c}(n,1) = c(n,1) = 1 \quad (n \geqslant 1),$$
$$\bar{c}(n,2) = c(n,2) + 2c(n,1) \qquad (n \geqslant 2),$$

$$\bar{c}(n,3) = c(n,3) + 2c(n,2) + x(n,1) + \sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 1),$$

$$\bar{c}(n,4) = c(n,4) + 2c(n,3) + c(n,2) + 2 \sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 1) + 2 \sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 2).$$

It follows that

$$\sum_0^\infty \bar{\Phi}_k(x,1)z^k = 1 + z + (1+z)^2 \sum_1^\infty \Phi_k(x,1)z^k + (1+z)^2 z \left\{ \sum_1^\infty \Phi_k(x,1)z^k \right\}^2 + (1+z)^3 z \left\{ \sum_1^\infty \Phi_k(x,1)z^k \right\}^3 + \cdots$$

$$= 1 + z + \frac{(1+z)^2 \sum_1^\infty \Phi_k(x,1)z^k}{1 - z \sum_1^\infty \Phi_k(x,1)z^k} .$$

It is easily verified that this is in agreement with (2.8) and (4.5).

5. GENERATING FUNCTIONS FOR $c(n)$ AND $\bar{c}(n)$

We may not put $z = 1$ in (4.5) since the right-hand side then becomes meaningless. We can get around this difficulty in the following way.

To begin with, we shall get crude upper bounds for $c(n)$ and $\bar{c}(n)$. Let $\nu(n,k)$ denote the number of solutions in positive integers of

$$n = a_1 + a_2 + \cdots + a_k$$

and let $\bar{\nu}(n,k)$ denote the number of solutions in non-negative integers. Then

$$\nu(n,k) = \binom{n-1}{k-1}, \qquad \bar{\nu}(n,k) = \binom{n+k-1}{k-1} .$$

Clearly

$$c(n,k) \leqslant \nu(n,k), \qquad \bar{c}(n,k) \leqslant \bar{\nu}(n,k).$$

It follows that

$$c(n) \leqslant 2^{n-1} \qquad (n \geqslant 1), \tag{5.1}$$

so that the radius of convergence of

$$\sum_0^\infty c(n)x^n \tag{5.2}$$

is at least ½.

As for $\bar{c}(n)$, since

$$\bar{c}(n,k) = 0 \qquad (k > 2n+1),$$

we get

$$\bar{c}(n) \leqslant \sum_{k=1}^{2n+1} \binom{n+k-1}{k-1} = \sum_{k=0}^{2n} \binom{n+k}{k} \leqslant \sum_{k=0}^{2n} \binom{3n}{k},$$

so that

(5.3)
$$\bar{c}(n) \leqslant 2^{3n}.$$

Hence the radius of convergence of

(5.4)
$$\sum_0^\infty \bar{c}(n)x^n$$

is at least 1/8;

Presumably these bounds are by no means best possible. It seems likely that the radius of convergence of (5.4) is about ½.

Next consider

$$\sum_{j=1}^{2k} (-1)^{j-1} \frac{z^j}{1-x^j} = \sum_{j=1}^{k} \left(\frac{z^{2j-1}}{1-x^{2j-1}} - \frac{z^{2j}}{1-x^{2j}} \right) = \sum_{j=1}^{\infty} \frac{1-z+x^{2j-1}(z-x)}{(1-x^{2j-1})(1-x^{2j})} z^{2j-1}.$$

Thus (4.5) becomes

(5.5)
$$\sum_0^\infty \Phi_k(x,1)z^k = \frac{1}{1-\sum_1^\infty \dfrac{1-z+x^{2j-1}(z-x)}{(1-x^{2j-1})(1-x^{2j})}}.$$

It is now permissible to let $z \to 1$. We get

(5.6)
$$\sum_0^\infty \bar{c}(n)x^n = \frac{1}{1-(1-x)\sum_1^\infty \dfrac{x^{2j-1}}{(1-x^{2j-1})(1-x^{2j})}}.$$

For $x = ½$ we get

$$\frac{1/2}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)} + \frac{1/8}{\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)} + \frac{1/32}{\left(1-\frac{1}{32}\right)\left(1-\frac{1}{64}\right)} = \frac{4}{3} + \frac{16}{105} + \frac{32}{31.63} < 1.$$

Thus the radius of convergence of (5.4) is probably somewhat greater than ½.

REFERENCE

1. John Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.

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SUMS OF COMBINATION PRODUCTS

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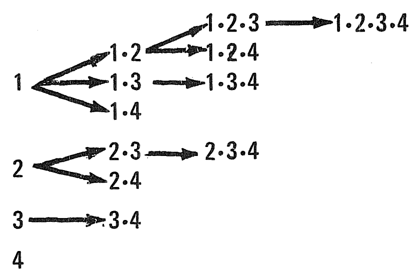
INTRODUCTION

The combinations of the integers 1, 2, 3, 4 can be represented by the following diagram:



We will be interested in developing methods for evaluating sums of the form

$$1\cdot 2+1\cdot 3+1\cdot 4+2\cdot 3+2\cdot 4+3\cdot 4 \qquad\text{and}\qquad 1\cdot 2\cdot 3+1\cdot 2\cdot 4+1\cdot 3\cdot 4+2\cdot 3\cdot 4\,.$$

We let

$$\sum_{\substack{x_1<\cdots<x_r\\ M_n}} (x_1\cdot x_2\cdot \cdots \cdot x_r)$$

denote the sum of all products of the form $x_1\cdot x_2\cdot \cdots \cdot x_r$, where

$$x_1<x_2<\cdots<x_r;\quad x_1,x_2,\cdots,x_r\in\{1,2,\cdots,n\},\quad\text{and}\quad n\geqslant r\geqslant 2.$$

For example,

$$\sum_{\substack{x_1<x_2\\ M_4}} x_1x_2 = 1\cdot 2+1\cdot 3+1\cdot 4+2\cdot 3+2\cdot 4+3\cdot 4 \quad\text{and}\quad \sum_{\substack{x_1<x_2\\ M_3}} x_1x_2 = 1\cdot 2+1\cdot 3+2\cdot 3\,.$$

We define

$$A_r^n = \sum_{\substack{x_1<\cdots<x_r\\ M_n}} (x_1\cdot x_2\cdot\cdots\cdot x_r),\qquad r\geqslant 2,\quad\text{and}\quad A_1^n=\sum_{i=1}^n i\,.$$

In this paper we develop formulas for A_2^n, A_3^n, A_4^n. We also provide a general approach for finding A_r^n when $n \geqslant r \geqslant 5$.

A. We now develop a formula for A_2^n. Consider

$$\left(\sum_{i=1}^n i\right)\left(\sum_{j=1}^n j\right)=\sum_{i=1}^n\sum_{j=1}^n ij=\sum_{i\neq j} ij+\sum_{i=1}^n i^2\,.$$

Thus,

$$(1) \qquad \sum_{i \neq j} ij = \left(\sum_{i=1}^{n} i \right)^2 - \sum_{i=1}^{n} i^2 .$$

Now,

$$\sum_{i \neq j} ij = 2 \sum_{\substack{i<j \\ M_n}} ij .$$

Thus,

$$2 \sum_{\substack{i<j \\ M_n}} ij = \left[\frac{n(n+1)}{2} \right]^2 - \frac{n(n+1)(2n+1)}{6} .$$

Thus, we have

Theorem 1. Say $n \geqslant 2$. Then

$$2 \sum_{\substack{i<j \\ M_n}} ij = \left(\sum_{i=1}^{n} i \right)^2 - \sum_{i=1}^{n} i^2 = \frac{3(n^4 - n^2) + 2(n^3 - n)}{4(3)} .$$

For example, with $n = 3$,

$$2(1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3) = \frac{3(3^4 - 3^2) + 2(3^3 - 3)}{4(3)} .$$

We could also find

$$\sum_{\substack{i<j \\ M_n}} ij$$

by using the method of undetermined coefficients. We begin by assuming that

$$\sum_{\substack{x_1 < \cdots < x_r \\ M_n}} (x_1 \cdot \cdots \cdot x_r)$$

is a polynomial of degree $2r$ in n (we assume that the coefficient of n^0 is zero):

$$2 \sum_{\substack{i<j \\ M_n}} ij = An^4 + Bn^3 + Cn^2 + Dn .$$

Now,

$$\sum_{\substack{i<j \\ M_2}} ij = 1 \cdot 2 = 2, \quad \sum_{\substack{i<j \\ M_3}} ij = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11 ,$$

$$\sum_{\substack{i<j \\ M_4}} ij = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35 ,$$

$$\sum_{\substack{i<j \\ M_5}} ij = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 1 \cdot 5 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 3 \cdot 4 + 3 \cdot 5 + 4 \cdot 5 = 85 .$$

Thus,

$$2(2) = A\cdot 2^4 + B\cdot 2^3 + C\cdot 2^2 + D\cdot 2, \quad 2(11) = A\cdot 3^4 + B\cdot 3^3 + C\cdot 3^2 + D\cdot 3,$$
$$2(35) = A\cdot 4^4 + B\cdot 4^3 + C\cdot 4^2 + D\cdot 4, \quad 2(85) = A\cdot 5^4 + B\cdot 5^3 + C\cdot 5^2 + D\cdot 5.$$

Solving this system for A,B,C,D should provide the required answer. Generalizing Theorem 1, we have

Theorem 2. Say $a_i, a_j \in \{a_1, a_2, \cdots, a_n\}$ and $n \geq 2$. Then

$$2\sum_{i<j} a_i a_j = \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2 .$$

For example, letting $a_i = i^2$,

$$2\sum_{i<j} (ij)^2 = \left(\sum_{i=1}^{n} i^2\right)^2 - \sum_{i=1}^{n} i^4 .$$

Similarly, letting $a_i = 1/i$,

$$2\sum_{i<j} \frac{1}{ij} = \left(\sum_{i=1}^{n} \frac{1}{i}\right)^2 - \sum_{i=1}^{n} \frac{1}{i^2} .$$

For example,

$$2\left(\frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3}\right) = \left(1 + \frac{1}{2} + \frac{1}{3}\right)^2 - \left(1 + \frac{1}{4} + \frac{1}{9}\right).$$

Now, say $x^3 + Bx^2 + Cx + D = 0$. Then, by Theorem 2, letting a_i equal the i^{th} root of the above equation,

$$2C = B^2 - (a_1^2 + a_2^2 + a_3^2).$$

Say $B = C = 0$. Then $a_1^2 + a_2^2 + a_3^2 = 0$. Thus, we have

Theorem 3. Say $r_1, r_2, \cdots, r_n$ are the roots of $x^n = -D$, and $n \geq 3$. Then $r_1^2 + r_2^2 + \cdots + r_n^2 = 0$.

B. We now develop a formula for A_3^n. Consider:

$$\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j\right)\left(\sum_{k=1}^{n} k\right) = \left(\sum_{i=1}^{n} i\right)^3 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk.$$

We consider

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk$$

to be a sum of products having three factors. Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk = \sum_{\substack{\text{all factors}\\ \text{equal}}} ijk + \sum_{\substack{\text{all factors}\\ \text{different}}} ijk + \sum_{\substack{\text{two factors}\\ \text{equal}}} ijk .$$

Now, if the product $1\cdot 2\cdot 4$ appears in the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk,$$

the following products also appear:

$$1\cdot 4\cdot 2, \quad 2\cdot 1\cdot 4, \quad 2\cdot 4\cdot 1, \quad 4\cdot 1\cdot 2, \quad 4\cdot 2\cdot 1 .$$

These may be considered as rearrangements of $1\cdot 2\cdot 4$. We note that the number of permutations of three distinct objects taken three at a time is six.

If the product $1\cdot 1\cdot 4$ appears in the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk,$$

the following products also appear:

$$1\cdot4\cdot1, \qquad 4\cdot1\cdot1.$$

We note that the number of permutations of three objects, two of which are of one kind, is three. Thus,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk = \sum_{i=1}^{n} i^3 + 6\sum_{\substack{i<j<k\\M_n}} ijk + \left(3\sum_{i=1}^{n} 1^2 i - 3\cdot1^3\right) + \left(3\sum_{i=1}^{n} 2^2 i - 3\cdot2^3\right)$$

$$+\cdots+\left(3\sum_{i=1}^{n} n^2 i - 3n^3\right) = \sum_{i=1}^{n} i^3 + 6\sum_{\substack{i<j<k\\M_n}} ijk + 3\sum_{i=1}^{n} i(1^2+2^2+\cdots+n^2) - 3(1^3+2^3+\cdots+n^3)$$

$$= \sum_{i=1}^{n} i^3 + 6\sum_{\substack{i<j<k\\M_n}} ijk + 3\left(\sum_{i=1}^{n} i\right)\left(\sum_{i=1}^{n} i^2\right) - 3\sum_{i=1}^{n} i^3.$$

Thus, we have

Theorem 4. Say $n \geqslant 3$. Then

$$6\sum_{\substack{i<j<k\\M_n}} ijk = \left(\sum_{i=1}^{n} i\right)^3 + 2\sum_{i=1}^{n} i^3 - 3\left(\sum_{i=1}^{n} i^2\right)\left(\sum_{i=1}^{n} i\right) = \frac{n^6}{8} - \frac{n^5}{8} - \frac{3n^4}{8} + \frac{n^3}{8} + \frac{n^2}{4}.$$

For example, with $n = 4$,

$$6(1\cdot2\cdot3 + 1\cdot2\cdot4 + 1\cdot3\cdot4 + 2\cdot3\cdot4) = \left(\sum_{i=1}^{4} i\right)^3 + 2\sum_{i=1}^{4} i^3 - \left(\sum_{i=1}^{4} i^2\right)\left(\sum_{i=1}^{4} i\right).$$

We now give an alternate derivation of the formula for $\sum_{\substack{i<j<k\\M_n}} ijk$

Consider: $3(1\cdot2) + 4(1\cdot2 + 1\cdot3 + 2\cdot3) + 5(1\cdot2 + 1\cdot3 + 1\cdot4 + 2\cdot3 + 2\cdot4 + 3\cdot4) = (1\cdot2\cdot3) + (1\cdot2\cdot4 + 1\cdot3\cdot4 + 2\cdot3\cdot4) + (1\cdot2\cdot5 + 1\cdot3\cdot5 + 1\cdot4\cdot5 + 2\cdot3\cdot5 + 2\cdot4\cdot5 + 3\cdot4\cdot5)$. This suggests that

$$\sum_{\substack{i<j<k\\M_n}} ijk = 3\sum_{\substack{i<j\\M_2}} ij + 4\sum_{\substack{i<j\\M_3}} ij + \cdots + n\sum_{\substack{i<j\\M_{n-1}}} ij.$$

Thus, we conjecture,

$$\sum_{\substack{i<j<k\\M_n}} ijk = \sum_{w=2}^{n-1} (w+1)\sum_{\substack{i<j\\M_w}} ij. \tag{2}$$

Thus, by Theorem 1, we conjecture

$$\sum_{\substack{i<j<k\\M_n}} ijk = \sum_{w=2}^{n-1} (w+1)\left[\frac{3(w^4-w^2)+2(w^3-w)}{24}\right]$$

and we have

Theorem 5. Say $n \geqslant 3$. Then

$$24 \sum_{\substack{i<j<k \\ M_n}} ijk = \sum_{i=1}^{n} (3i^5 + 5i^4 - i^3 - 5i^2 - 2i) - 3n^5 - 5n^4 + n^3 + 5n^2 + 2n.$$

We can prove Theorem 5 by using Theorem 4 and the following formulas:

$$2\sum_{i=1}^{n} i = n^2 + n, \quad 3\sum_{i=1}^{n} i^2 = n^3 + \frac{3n^2}{2} + \frac{1n}{2}, \quad 4\sum_{i=1}^{n} i^3 = n^4 + 2n^3 + n^2,$$

$$5\sum_{i=1}^{n} i^4 = n^5 + \frac{5n^4}{2} + \frac{5n^3}{3} - \frac{1n}{6}, \qquad 6\sum_{i=1}^{n} i^5 = n^6 + 3n^5 + \frac{5n^4}{2} - \frac{1n^2}{2}.$$

C. We now develop a formula for A_4^n. Consider:

$$4(1\cdot2\cdot3) + 5(1\cdot2\cdot3 + 1\cdot2\cdot4 + 1\cdot3\cdot4 + 2\cdot3\cdot4) = (1\cdot2\cdot3\cdot4) + (1\cdot2\cdot3\cdot5 + 1\cdot2\cdot4\cdot5 + 1\cdot3\cdot4\cdot5 + 2\cdot3\cdot4\cdot5).$$

This suggests that

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = 4\sum_{\substack{i<j<k \\ M_3}} ijk + 5\sum_{\substack{i<j<k \\ M_4}} ijk + \cdots + n\sum_{\substack{i<j<k \\ M_{n-1}}} ijk.$$

Thus, we conjecture,

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{w=3}^{n-1} (w+1) \sum_{\substack{i<j<k \\ M_w}} ijk. \tag{3}$$

Thus, by Theorem 4, we conjecture,

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{w=3}^{n-1} \frac{(w+1)}{24}\left(\frac{w^6}{2} - \frac{w^5}{2} - \frac{3w^4}{2} + \frac{w^3}{2} + w^2\right)$$

and we have

Conjecture 1. Say $n \geq 4$. Then

$$24\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{i=1}^{n}\left(\frac{1i^7}{2} - 2i^5 - i^4 + \frac{3i^3}{2} + i^2\right) - \frac{n^7}{2} + 2n^5 + n^4 - \frac{3n^3}{2} - n^2.$$

Comparing (2) and (3) we have

Conjecture 2. Say $n \geq r \geq 3$. Then

$$\sum_{\substack{x_1<\cdots<x_r \\ M_n}} \prod_{i=1}^{r} x_i = \sum_{w=r-1}^{n-1} (w+1) \sum_{\substack{x_1<\cdots<x_{r-1} \\ M_w}} \prod_{i=1}^{r-1} x_i.$$

Thus, we have

Conjecture 3. Conjecture 2 and Theorem 1 provide a recursive method for determining $A_3^n, A_4^n, A_5^n, \cdots$.

D. Theorem 6. Say $n \geq 2$. Then

$$(n-1)! = n^{n-1} + \sum_{i=1}^{n-1} (-1)^i A_i^{n-1} n^{n-(i+1)}.$$

Proof.

$$(n-1)! = (n-1)(n-2)\cdots[n-(n-1)] = n^{n-1} + (-1)^1 A_1^{n-1} n^{n-2} + (-1)^2 A_2^{n-1} n^{n-3} + (-1)^3 A_3^{n-1} n^{n-4} + \cdots + (-1)^{n-1} A_{n-1}^{n-1} n^{n-n}.$$

E. Theorem 7. The A_i^n can be solved for by Cramer's rule. Also,

$$\sum_{i=1}^{n} A_i^n = (n+1)! - 1.$$

Proof. Let $f(x) = (x+1)(x+2)\cdots(x+n) = (x+n)!/x!$. Then $f(x) = x^n + A_1^n x^{n-1} + A_2^n x^{n-2} + \cdots + A_{n-1}^n x + A_n^n$. Thus,

$$A_1^n 1^{n-1} + A_2^n 1^{n-2} + \cdots + A_{n-1}^n 1^1 + A_n^n = f(1) - 1^n$$
$$A_1^n 2^{n-1} + A_2^n 2^{n-2} + \cdots + A_{n-1}^n 2^1 + A_n^n = f(2) - 2^n$$
$$\vdots$$
$$A_1^n n^{n-1} + A_2^n n^{n-2} + \cdots + A_{n-1}^n n^1 + A_n^n = f(n) - n^n,$$

where the A_i^n can be solved for by Cramer's rule.

F. Theorem 8. Say $n \geqslant r \geqslant 1$ and $f(x) = (x+n)!/x!$. Then

$$A_r^n = \frac{f^{[n-r]}(0)}{(n-r)!},$$

where $f^{[n-r]}(0)$ denotes the $n-r$ derivative evaluated at zero.

Proof. Say $f(x) = (x+1)(x+2)\cdots(x+n) = (x+n)!/x!$. Then $f(x) = x^n + A_1^n x^{n-1} + \cdots + A_{n-1}^n x + A_n^n$. Now $f(x)$ is a polynomial of degree n. Thus, by Taylor's formula,

$$f(x) = f(0) + f^{[1]}(0)x + \frac{f^{[2]}(0)x^2}{2!} + \cdots + \frac{f^{[n]}(0)x^n}{n!}.$$

Thus, comparing the coefficients of the above two equations, the theorem is proved.

G. A Curiosity. Let

$$T_Q = -\sum_{x_1=1}^{Q} x_1 + \sum_{x_1=2}^{Q} x_1 \sum_{x_2=1}^{x_1-1} x_2 - \sum_{x_1=3}^{Q} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3 + \sum_{x_1=4}^{Q} x_1 \sum_{x_2=3}^{x_1-1} x_2 \sum_{x_3=2}^{x_2-1} x_3 \sum_{x_4=1}^{x_3-1} x_4 - \cdots + w(Q,Q), \tag{4}$$

where

$$w(v,Q) = (-1)^v \sum_{x_1=v}^{Q} x_1 \left[\sum_{x_2=v-1}^{x_1-1} x_2 \sum_{x_3=v-2}^{x_2-1} x_3 \cdots \sum_{x_v=1}^{x_{v-1}-1} x_v\right].$$

$$\therefore\ T_Q = -\sum_{x_1=1}^{Q} x_1 + \sum_{v=2}^{Q} w(v,Q).$$

Thus,

$$T_1 = -[1] \tag{5}$$

$$T_2 = -[1+2] + [2(1)] \tag{6}$$

$$T_3 = -[1+2+3] + [2(1) + 3(1+2)] - \left\{3[2(1)]\right\} = -[1+2+3] + [(2\cdot1) + (3\cdot1) + (3\cdot2)] - [(3\cdot2\cdot1)]. \tag{7}$$

This suggests

Conjecture 4.

$$A_2^n = \sum_{x_1=2}^{n} x_1 \sum_{x_2=1}^{x_1-1} x_2, \quad A_3^n = \sum_{x_1=3}^{n} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3$$

$$\cdots A_{n-1}^n = \sum_{x_1=n-1}^{n} x_1 \sum_{x_2=n-2}^{x_1-1} x_2 \sum_{x_3=n-3}^{x_2-1} x_3 \cdot \cdots \cdot \sum_{x_{n-1}=1}^{x_{n-2}-1} x_{n-1}$$

and

Conjecture 5.

$$T_n = \sum_{i=1}^{n} (-1)^i A_i^n .$$

We note that from Conjecture 4,

$$A_2^n \stackrel{?}{=} \sum_{x_1=2}^{n} x_1 \sum_{x_2=1}^{x_1-1} x_2 = \sum_{i=2}^{n}\sum_{j=1}^{i-1} ij = \sum_{i=2}^{n} \frac{i(i-1)i}{2} = \frac{1}{2}\sum_{i=2}^{n} (i^3 - i^2)$$

$$= \frac{1}{2}\left\{\left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6}\right\}$$

which agrees with Theorem 1.

Similarly,

$$A_3^n \stackrel{?}{=} \sum_{x_1=3}^{n} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3 = \sum_{i=3}^{n}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} ijk .$$

We note that $T_3 = T_2 - 3 + 3[1+2] - 3[2(1)] = T_2 - 3 - 3T_2$ and $T_4 = T_3 - 4 + 4[1+2+3] - 4[2(1) + 3(1+2)] + 4\left\{3[2(1)]\right\} = T_3 - 4 - 4T_3$. Thus, $T_3 = -2T_2 - 3$ and $T_4 = -3T_3 - 4$. This suggests

Theorem 9. Say $Q \geq 1$. Then

(8) $$T_Q = -(Q-1)T_{Q-1} - Q.$$

We leave the proof to the reader.

We might hope that the T_Q represent a new species of number. Let's see; i.e., from (5), (6), (7) we have

$$T_1 = -1, \qquad T_2 = -3 + 2 = -1, \qquad T_3 = -6 + 11 - 6 = -1.$$

This suggests

Theorem 10. Say $Q \geq 1$. Then $T_Q = -1$.

Proof (induction). By (5) we know that $T_1 = -1$. Say k is a fixed integer greater than or equal to two and $T_{k-1} = -1$. Then, by (8), $T_k = -1$ and the theorem is proved.

Hence, from Conjecture 5 and the above theorem, we have

Conjecture 6.

$$\sum_{i=1}^{n} (-1)^i A_i^n = -1 .$$

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A PRIMER FOR THE FIBONACCI NUMBERS, PART XV
VARIATIONS ON SUMMING A SERIES OF RECIPROCALS OF FIBONACCI NUMBERS

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It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci numbers such that the subscripts are terms of geometric progressions. However, in [1] Good shows that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2} \tag{1}$$

a problem proposed by Millin [2]. This particular series can be summed in several different ways.

Method I. Write out the first few terms of (1),

$$1, \quad 1+1, \quad 1+1+\frac{1}{3} = \frac{7}{3}, \quad 1+1+\frac{1}{3}+\frac{1}{21} = \frac{50}{21}, \quad \cdots .$$

Now,

$$\frac{50}{21} = 1+\frac{29}{21} = 1+\frac{L_7}{F_8},$$

which suggests that

$$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\cdots+\frac{1}{F_{2^n}} = 1+\frac{L_{2^n-1}}{F_{2^n}}. \tag{2}$$

From [3], we write

$$L_m L_{m+1} - L_{2m+1} = (-1)^m \tag{3}$$

from which it follows that

$$1+\frac{L_{2^n-1}}{F_{2^n}}\cdot\frac{L_{2^n}}{L_{2^n}}+\frac{1}{F_{2^{n+1}}} = 1+\frac{L_{2^{n+1}-1}}{F_{2^{n+1}}}$$

since $F_m L_m = F_{2m}$. Thus, we can prove (2) by mathematical induction. If we compute the limit as $n \to \infty$ for (2), then we have the infinite sum of (1), for (see [3])

$$\lim_{n\to\infty}\left(1+\frac{L_{2^n-1}}{F_{2^n}}\right) = \lim_{n\to\infty}\left(1+\frac{L_{2^n}}{F_{2^n}}\cdot\frac{L_{2^n-1}}{L_{2^n}}\right) = 1+\sqrt{5}\cdot\frac{1}{\alpha},$$

where $\alpha = (1+\sqrt{5})/2$, which simplifies to $(7-\sqrt{5})/2$.

The limits used above can be easily derived from the well-known

$$F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}, \qquad L_n = \alpha^n+\beta^n,$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$ are the roots of $x^2-x-1=0$.

$$\lim_{n\to\infty}\frac{L_n}{F_n} = \lim_{n\to\infty}(\alpha-\beta)\frac{\alpha^n+\beta^n}{\alpha^n-\beta^n} = \lim_{n\to\infty}(\alpha-\beta)\frac{1+(\beta/\alpha)^n}{1-(\beta/\alpha)^n} = \sqrt{5}$$

since $(\alpha-\beta) = \sqrt{5}$ and $\beta/\alpha < 1$. In an entirely similar manner, we could show that

$$\lim_{n\to\infty} L_{n+r}/L_n = \alpha^r, \qquad \lim_{n\to\infty} F_{n+r}/F_n = \alpha^r.$$

Method II. Returning to the first few terms of (1),

$$\frac{50}{21} = 2 + \frac{8}{21} = 2 + \frac{F_6}{F_8},$$

which suggests

(4) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 2 + \frac{F_{2^n-2}}{F_{2^n}}.$$

If we take the limit as $n \to \infty$ of the right-hand side of (4), we obtain $2 + 1/\alpha^2 = (7 - \sqrt{5})/2$. We can prove (4) by induction, since

$$2 + \frac{F_{2^n-2}}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} = 2 + \frac{(F_{2^{n+1}})(F_{2^n-2})/F_{2^n} + 1}{F_{2^{n+1}}} = 2 + \frac{L_{2^n}F_{2^n-2} + 1}{F_{2^{n+1}}}.$$

We need to establish that

$$F_{2^n-2}L_{2^n} + 1 = F_{2^{n+1}-2}$$

which follows from (see [3], [4])

(5) $$F_{m+p} - F_{m-p} = F_pL_m, \quad p \text{ even},$$

where $m + p = 2^{n+1} - 2,\ m - p = 2,\ m = 2^n,\ p = 2^n - 2$, so that

$$F_{2^{n+1}-2} - F_2 = F_{2^n-2}L_{2^n}.$$

Method III. Examining the first terms of (1) yet again,

$$\frac{50}{21} = 3 - \frac{13}{21} = 3 - \frac{F_7}{F_8}$$

suggests

(6) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 3 - \frac{F_{2^n-1}}{F_{2^n}},$$

used by Good [1], where the limit as $n \to \infty$ of the right side is $3 - 1/\alpha = (7 - \sqrt{5})/2$. Establishing (6) by induction involves showing that

$$3 - \frac{F_{2^n-1}}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} = 3 - \frac{L_{2^n}F_{2^n-1} - 1}{F_{2^{n+1}}} = 3 - \frac{F_{2^{n+1}-1}}{F_{2^n}},$$

where we need

$$L_{2^n}F_{2^n-1} = F_{2^{n+1}-1} + F_1$$

which follows from [3], [4]

$$F_{m+p} + F_{m-p} = L_mF_p, \quad p \text{ odd},$$

where $m + p = 2^{n+1} - 1,\ m - p = 1,\ m = 2^n,\ p = 2^n - 1$.

Method IV. Proceeding in a similar manner, we notice that

$$\frac{50}{21} = 4 - \frac{34}{21} = 4 - \frac{F_9}{F_8}$$

and

$$\lim_{n \to \infty}\left(4 - \frac{F_{2^n+1}}{F_{2^n}}\right) = 4 - \alpha = 4 - \frac{1+\sqrt{5}}{2} = \frac{7-\sqrt{5}}{2},$$

if indeed

(7) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 4 - \frac{F_{2^n+1}}{F_{2^n}}.$$

Thus one expects

$$L_{2^n}F_{2^n+1} - 1 = F_{2^{n+1}+1}$$

which follows from [3], [4]

(8) $$F_{m+p} + F_{m-p} = L_p F_m, \quad p \text{ even},$$

where $m+p = 2^{n+1}+1,\ m-p = 1,\ m = 2^n+1,\ p = 2^n$.

Method V. Again looking at the early terms of (1),

$$\frac{50}{21} = 5 - \frac{F_{10}}{F_8}$$

suggests

(9) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 5 - \frac{F_{2^n-2}}{F_{2^n}},$$

where the limit of the right-hand side as $n \to \infty$ is $5 - \alpha^2 = 5 - (\alpha + 1) = 4 - \alpha$ again. From the form of (9) and earlier experience, one expects

$$F_{2^n+2}L_{2^n} - 1 = F_{2^{n+1}+2}$$

which follows from (8), where $m+p = 2^{n+1}+2,\ m-p = 2,\ m = 2^n+2$ and $p = 2^n$.

Method VI. One last time, we inspect the early terms of (1) to observe

$$\frac{50}{21} = 6 - \frac{76}{21} = 6 - \frac{L_9}{F_8}$$

which has the form of

(10) $$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_{2^n}} = 6 - \frac{L_{2^n+1}}{F_{2^n}}.$$

The proof of (10) by induction depends upon the identity

$$L_{2^n+1}L_{2^n} - 1 = L_{2^{n+1}+1}$$

which follows readily from (3). The limit as $n \to \infty$ of the right-hand side of (10) follows from

$$\lim_{n\to\infty} \frac{L_{2^n+1}}{F_{2^n}} = \lim_{n\to\infty} \frac{L_{2^n}}{F_{2^n}} \cdot \frac{L_{2^n+1}}{L_{2^n}} = \sqrt{5}\cdot\alpha,$$

becoming $6 - \sqrt{5}\cdot\alpha$, which simplifies to $(7 - \sqrt{5})/2$.

Method VII. We again return to the early terms of (1), but we proceed in a different manner.

$$2 + \frac{1}{3} + \frac{1}{21} = 2 + \frac{7+1}{21} = 2 + \frac{L_4+1}{F_8}$$

$$2 + \frac{L_4+1}{F_8} + \frac{1}{F_{16}} = 2 + \frac{L_8L_4 + L_8 + 1}{F_{16}} = 2 + \frac{L_{12} + L_8 + L_4 + 1}{F_{16}}.$$

Assume that

(11) $$\sum_{j=0}^{n} 1/F_{2^j} = 2 + \frac{L_{2^n-4} + L_{2^n-8} + L_{2^n-12} + \cdots + L_4 + 1}{F_{2^n}}.$$

Since

$$\lim_{n\to\infty} \frac{L_{m-r}}{F_m} = \sqrt{5}\cdot\alpha^{-r},$$

the limit as $n \to \infty$ of the right-hand side of (11) becomes

$$2 + \sqrt{5}(\alpha^{-4} + \alpha^{-8} + \alpha^{-12} + \cdots) + 0 = 2 + \sqrt{5}\cdot\alpha^{-4}[1/(1-\alpha^{-4})] = 2 + \sqrt{5}[1/(\alpha^4 - 1)]$$
$$= 2 + \sqrt{5}[1/(\alpha^2+1)(\alpha^2-1)] = 2 + \sqrt{5}[1/(\sqrt{5}\,\alpha)(\alpha)] = 2 + 1/\alpha^2$$

since

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \alpha^2 + 1 = \alpha + 2 = \frac{1+\sqrt{5}}{2} + 2 = \frac{5+\sqrt{5}}{2} = \sqrt{5}\cdot\alpha.$$

Also, since $\alpha^n = (L_n + F_n\sqrt{5})/2$, $\alpha^2 = (3+\sqrt{5})/2$, and the above becomes

$$2 + 1/\alpha^2 = 2 + (3-\sqrt{5})/2 = (7-\sqrt{5})/2.$$

Here, (11) can be proved by induction if the identity

$$(12) \qquad L_{2^n}(L_{(2^n-4)} + L_{(2^n-8)} + \cdots + L_4 + 1) = L_{2^{n+1}-4} + L_{2^{n+1}-8} + \cdots + L_4$$

is known. (See [5]).

We could also have used

$$(13) \qquad \sum_{j=1}^{n} L_{2kj} = \frac{L_{2k(n+1)} - L_{2kn} - L_{2k} - 2}{L_{2k} - 2}$$

to sum the numerator of (11), and proceeded as in [6].

Method VIII. Starting with the first few partial sums,

$$\frac{1}{F_1} + \frac{1}{F_2} = 1 + \frac{L_2}{F_4}, \qquad \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} = 1 + \frac{L_2+1}{F_4}, \qquad \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} = 1 + \frac{L_6+L_4+L_2+1}{F_8}$$

Generally,

$$(14) \qquad \sum_{j=0}^{n} 1/F_{2^j} = 1 + \frac{L_{2^n-2} + L_{2^n-4} + \cdots + L_2 + 1}{F_{2^n}},$$

but

$$L_{2m} + L_{2m-2} + \cdots + L_2 = L_{2m+1} - 1.$$

Thus

$$(15) \qquad \sum_{j=0}^{n} 1/F_{2^j} = 1 + \frac{L_{2^n-1}}{F_{2^n}} = A$$

so that

$$\lim_{n\to\infty} A = 1 + \sqrt{5}/\alpha = (7-\sqrt{5})/2.$$

Method IX. I. J. Good [7] uses the identity

$$\sum_{n=1}^{\infty} (xy)^{2^{n-1}}/(x^{2^n} - y^{2^n}) = \frac{\text{min abs}\,(x,y)}{x-y},$$

where $x = (1+\sqrt{5})/2$ and $y = (1-\sqrt{5})/2$. This is not quite complete by itself.

Method X. On the other hand, L. Carlitz [8] uses

$$\sum_{n=0}^{\infty} 1/F_{2^n} = \sum_{i=0}^{\infty} \frac{\alpha-\beta}{\alpha^{2^i} - \beta^{2^i}} = 1 + \sum_{i=1}^{\infty} \frac{\alpha-\beta}{\alpha^{2^i} - \beta^{2^i}} = (\alpha-\beta)\sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} \beta^{j2^i}/\alpha^{(j+1)2^i}\right) + 1,$$

but $(\alpha\beta)^2 = 1$, so that this is

$$(\alpha-\beta)\sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} \alpha^{-(2j+1)2^i}\right) + 1$$

but clearly, every even number greater than zero can be written as $(2j+1)2^i$. Thus, this is

$$1 + (\alpha-\beta)\sum_{n=1}^{\infty} \alpha^{-2n} = 1 + \frac{\alpha^{-2}(\alpha-\beta)}{1-\alpha^{-2}} = 1 + \frac{\alpha-\beta}{\alpha^2-1} = 1 + \sqrt{5}/\alpha = \frac{7-\sqrt{5}}{2}.$$

Method XI. For yet another method see A. G. Shannon's solution in the April 1976 Advanced Problem Section solution to H-237.

REFERENCES

1. I. J. Good, "A Reciprocal Series of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 4 (Dec. 1974), p. 346.
2. D. A. Millin, Problem H-237, *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), p. 309.
3. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton-Mifflin, Boston, 1969.
4. I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," *The Fibonacci Quarterly,* Vol. 1, No. 2 (April 1963), pp. 75–80.
5. Ken Siler, "Fibonacci Summations," *The Fibonacci Quarterly,* Vol. 1, No. 3 (Oct. 1963), pp. 67–69.
6. V. E. Hoggatt, Jr., and Marjorie Bicknell, "A Reciprocal Series of Fibonacci Numbers with subscripts $k2^n$," *The Fibonacci Quarterly,* to appear.
7. I. J. Good and P. S. Bruckman, "A Generalization of a Series of De Morgan with Applications of Fibonacci Type," *The Fibonacci Quarterly,* Vol. 14, No. 3 (Oct. 1976), pp. 193–196.
8. L. Carlitz, private communication.

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[Continued from Page 253.]

Then the sequence

$$(w_n) = (\log H_n H_n^*)$$

is u.d. mod 1.

Proof. We have

$$w_{n+1} - w_n = \log \frac{H_{n+1}}{H_n} + \log \frac{H_{n+1}^*}{H_n^*} ,$$

which tends to

$$2 \log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$ for

$$\frac{H_{n+1}}{H_n} = \frac{qF_n + pF_{n-1}}{qF_{n-1} + pF_{n-2}} = \frac{q(F_n/F_{n-1}) + p}{q(F_{n-1}/F_{n-2}) + p} \cdot \frac{F_{n-1}}{F_{n-2}}$$

goes to

$$\frac{1+\sqrt{5}}{2}$$

as $n \to \infty$

Theorem 3. Let p, q, p^*, q^*, H_n and H_n^* have the same meaning as in Theorem 2. Then the sequence

$$(x_n) = (\log (H_n + H_n^*))$$

is u.d. mod 1.

Proof. By the definitions of H_n and H_n^* we have

$$H_n + H_n^* = (q+q^*)F_{n-1} + (p+p^*)F_{n-2} \quad (n \geq 3)$$

and so we see that

$$x_{n+1} - x_n = \log ((H_{n+1} + H_{n+1}^*)/(H_n + H_n^*)) = \log \frac{(q+q^*)F_n + (p+p^*)F_{n-1}}{(q+q^*)F_{n-1} + (p+p^*)F_{n-2}} ,$$

[Continued on Page 281.]

A MODEL FOR POPULATION GROWTH

DAVID A. KLARNER
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In 1969, Parberry [1] posed and solved an interesting problem in population growth analogous to the rabbit problem considered by Fibonacci. In this note we describe how one might treat a generalization of these problems. First, we state the problems of Fibonacci and Parberry and note what they have in common.

The situation considered by Fibonacci involves two types of rabbit which will be denoted B and F (for baby and female, respectively). Starting with one individual of type B, a sequence of generations of rabbits is formed as follows: Each individual of type B in the n^{th} generation matures to become an individual of type F in the $(n+1)^{st}$ generation. Also, each individual of type F in the n^{th} generation gives birth to an individual of type B in the $(n+1)^{st}$ generation, and survives to become an individual of type F in the $(n+1)^{st}$ generation. A family tree may be drawn which represents this process; see Figure 1.

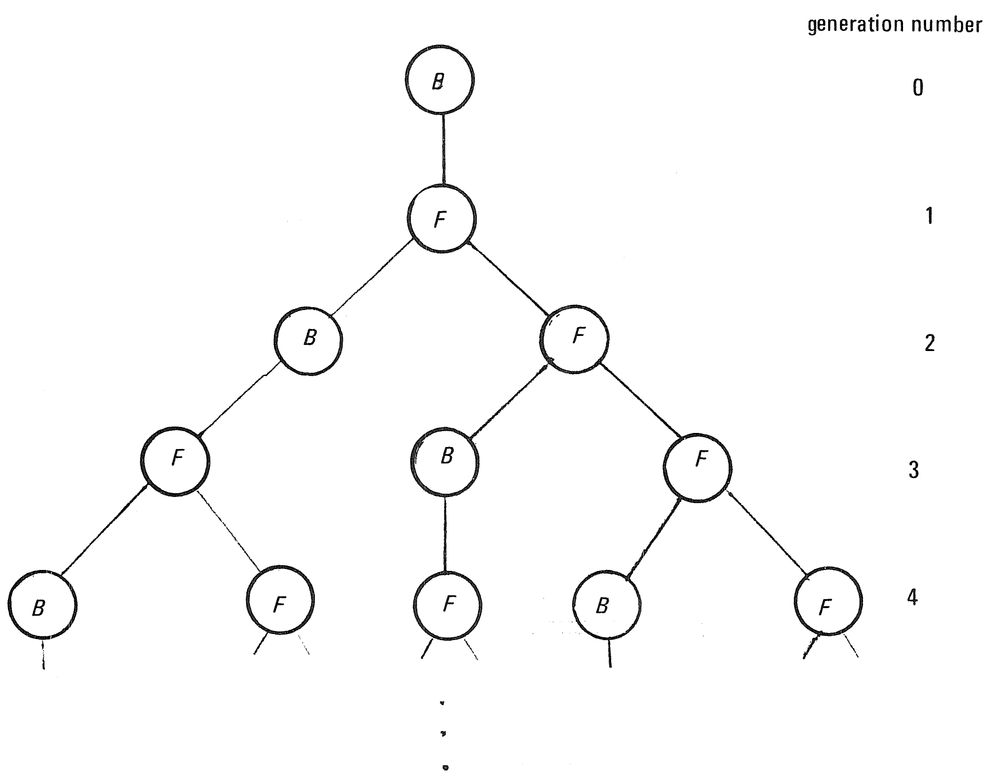

Figure 1. Family Tree of Fibonacci's Rabbits

Parberry considered populations of diatoms, one-celled algae whose reproductive capabilities can be classified according to size and maturity. Changes in classification along with reproduction are assumed to take place at regular intervals which will be called generations. Let m and n denote natural numbers, and let

$$S_1, \cdots, S_m, S_{m+1}, \cdots, S_{m+n}$$

denote a classification of the diatoms. Diatoms of type S_i for $i = 1, \cdots, m$ split to form two new diatoms, one of type S_i and the other of type S_{i+1}, but diatoms of type S_{m+i} for $i = 1, \cdots, n$ can only mature to become diatoms of type S_{m+i+1}, where a diatom of type S_{m+n+1} is defined to be of type S_1. For example, when $m = 2$, $n = 1$, the family tree of diatoms descending from one individual of type S_1 is shown in Figure 2.

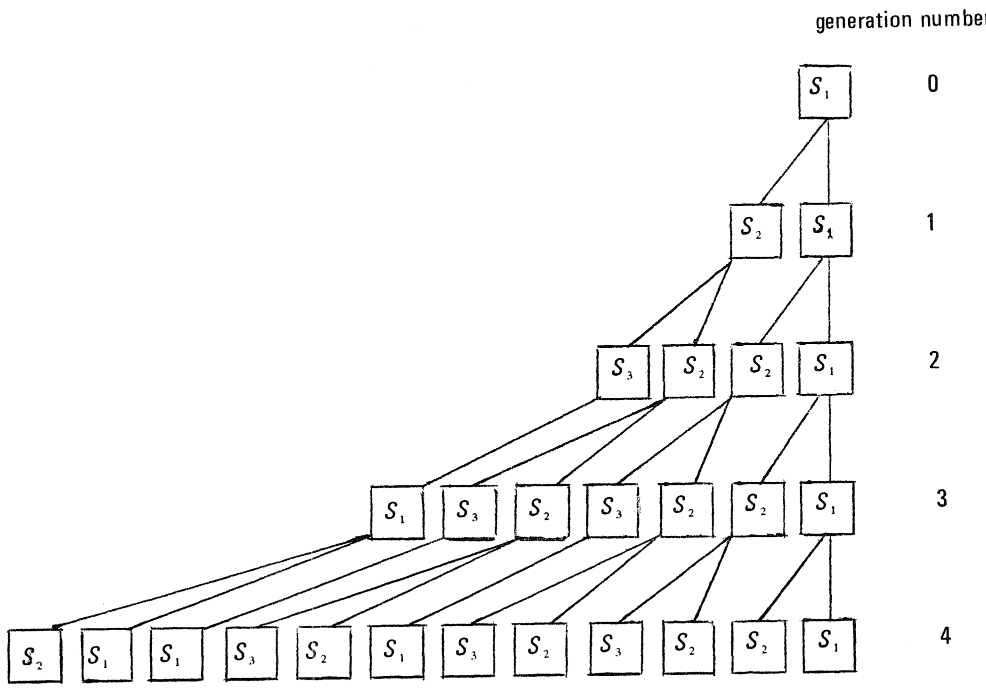

Figure 2. Family Tree of Diatoms

The problems of Fibonacci and Parberry have common features which are embodied in the following generalization. There is a finite set $T = \{1, \cdots, t\}$ of types of individuals, and each individual of type i in the n^{th} generation gives rise to f_{ij} individuals of type j in the $(n+1)^{st}$ generation $(1 \leq i, j \leq t)$ for $n = 0, 1, \cdots$. Also, there is an initial population containing f_i individuals of type i. Let $f_i(n)$ denote the number of individuals of type i in the n^{th} generation. (Thus, $f_i = f_2(0)$), and put

$$f(n) = f_1(n) + \cdots + f_t(n).$$

The sequences

$$(f_i(n) : n = 0, 1, \cdots)$$

are of interest: How are they related, how should they be calculated, what is their rate of growth, and so on. There is a very simple theory which explains these things.

Each of the $f_j(n-1)$ individuals of type j in the $(n-1)^{st}$ generation gives rise to f_{jk} individuals of type k in the n^{th} generation; hence, summing on j we have

(1) $$f_k(n) = f_1(n)f_{1k} + \cdots + f_t(n)f_{tk}.$$

This may be expressed in terms of matrices as

$$[f_1(n) \cdots f_t(n)] = [f_1(n-1) \cdots f_t(n-1)] \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1t} \\ f_{21} & f_{22} & \cdots & f_{2t} \\ \vdots & \vdots & & \\ f_{t1} & f_{t2} & \cdots & f_{tt} \end{bmatrix},$$

or, with an obvious notational convention, this may be succinctly expressed as

(2) $$\bar{f}(n) = \bar{f}(n-1)F.$$

Using (2), an easy induction argument gives

(3) $$\bar{f}(n) = \bar{f}(0)F^n.$$

Now (3) can be used to show that each of the sequences $(f_i(n) : n = 0, 1, \cdots)$ satisfies a certain difference equation, hence, the sequence $(f(n) : n = 0, 1, \cdots)$ also satisfies this difference equation. Recall the Cayley-Hamilton Theorem: Every square matrix M satisfies its characteristic equation. Thus, if we form the polynomial $c_F(x) = \det (xI - F)$, where I denotes the $t \times t$ identity matrix, then $c_F(F)$ is the all-zero matrix. Hence,

(4) $$F^n c_F(F) = 0$$

for $n = 0, 1, \cdots$. Let

$$c_F(x) = x^t - a_1x^{t-1} - \cdots - a_t,$$

and let $f_{ij}(n)$ denote the $(i,j)^{th}$ entry of F^n for $n = 0, 1, \cdots$. Then (4) implies

(5) $$f_{ij}(n+t) - a_1f_{ij}(n+t-1) - \cdots - a_tf_{ij}(n) = 0$$

for $n = 0, 1, \cdots$ and $1 \leqslant i, j \leqslant t$. Since each of the sequences

$$(f_{ij}(n) : n = 0, 1, \cdots)$$

satisfies the same difference equation given in (5), any linear combination of these sequences also satisfies this difference equation. In particular, this implies

(6) $$f_i(n+t) = a_1f_i(n+t-1) + \cdots + a_tf_i(n)$$

for $n = 0, 1, \cdots$; also, the sequence

$$(f(n) : n = 0, 1, \cdots)$$

satisfies this difference equation.

The matrix F may satisfy a polynomial equation with degree less than t; if so, this polynomial may be used in place of $c_F(x)$ to obtain a lower order difference equation. It is well known and easy to prove that there is a polynomial, unique up to a constant factor, having minimal degree such that F satisfies the corresponding polynomial equation. This polynomial, called the *minimal polynomial* of F, is a factor of $c_F(x)$.

Returning to Fibonacci's problem, the matrix involved is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}:$$

The minimal polynomial of this matrix is $x^2 - x - 1$: Hence, $f(n)$, the number of rabbits in the n^{th} generation satisfies

(7) $$f(n+2) - f(n+1) - f(n) = 0$$

for $n = 0, 1, \cdots$; also, we have $f(0) = f(1) = 1$, so this gives Fibonacci's sequence 1, 1, 2, 3, 5, 8, ⋯.

A more realistic model of a rabbit population would reflect the fecundity of the female depending on her age. For example, type 1 matures to become type 2; type 2 has a litter of 3 type 1's and matures to become type 3; type 3 has a litter of 4 type 1's and matures to become type 4; type 4 has a litter of 2 type 1's and dies. The matrix involved is

(8) $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic equation is

$$x^4 - 3x^2 - 4x - 2 = (x+1)(x^3 - x^2 - 2x - 2).$$

Suppose the initial population consists of one rabbit of type 1, then the family tree shown in Figure 3 results.

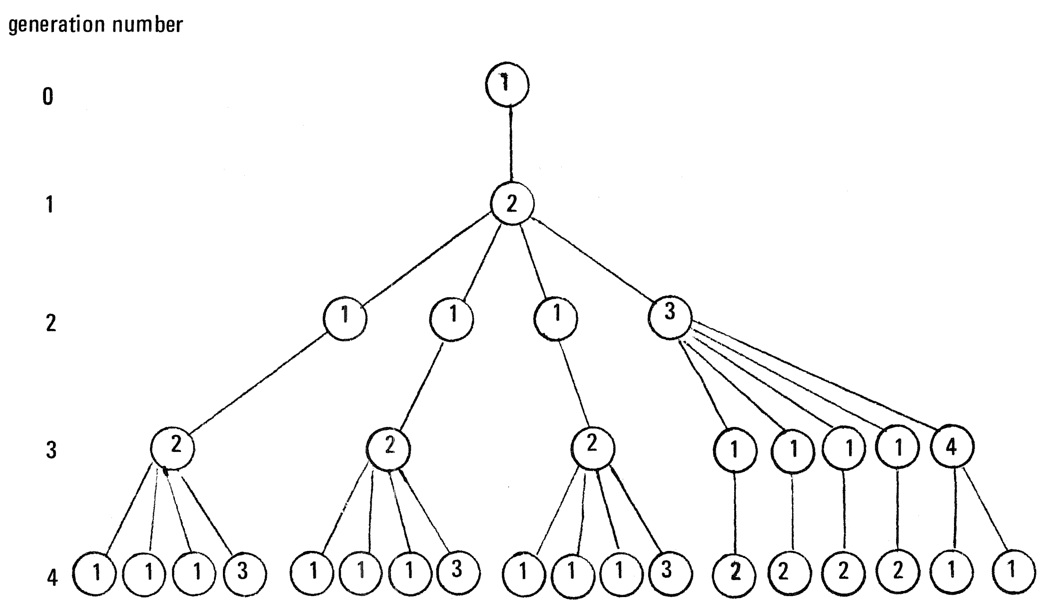

Figure 3

The number of rabbits in the n^{th} generation satisfies

(9) $$f(n+4) = 3f(n+2) + 4f(n+1) + 2f(n),$$

but it is easy to check that the initial conditions

$$f(0) = f(1) = 1, \quad f(2) = 4, \quad \text{and} \quad f(3) = 8$$

give rise to a sequence 1, 7, 4, 8, 18, ⋯ which satisfies the lower order difference equation

(10) $$f(n+3) = f(n+2) + 2f(n+1) + 2f(n).$$

This relation arose because

$$x^3 - x^2 - 2x - 2$$

is a factor of

$$x^4 - 3x^2 - 4x - 2.$$

Since

$$x^3 - x^2 - 2x - 2$$

has a real zero θ between 2.2 and 2.3, it follows that

$$f(n) > (2.2)^n$$

for all sufficiently large n.

REFERENCE

1. Edward A. Parberry, "A Recursion Relation for Populations of Diatoms," *The Fibonacci Quarterly*, Vol. 7, No. 4 (Dec. 1969), pp. 449–456.

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[Continued from Page 276.]

which tends to

$$\log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$ and this completes the proof.

In addition we want to mention another interesting property possessed by the sequences of the previous theorems. This property can be shown by applying a result of Vanden Eynden (see [2] p. 307): Let (C_n) be a sequence of real numbers such that the sequence (C_n/m) is u.d. mod 1 for all integers $m \geqslant 2$. Then the sequence $([C_n])$ of integral parts is u.d. in the ring of integers $\mathbb{Z}$.

Theorem 4. The sequences

$$([\log F_n^{1/k}]), \quad ([\log H_n H_n^*]) \quad \text{and} \quad ([\log (H_n + H_n^*)])$$

are u.d. in $\mathbb{Z}$.

Proof. It is easily seen that for all non-zero integers m the expressions

$$\frac{1}{m}\log F_n^{1/k}, \quad \frac{1}{m}\log (H_n H_n^*) \quad \text{and} \quad \frac{1}{m}\log (H_n + H_n^*)$$

satisfy the condition in van der Corput's Theorem.

REFERENCES

1. William Webb, "Distribution of the First Digits of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 13, No. 4 (Dec. 1975), pp. 334–336.
2. L. Kuipers and H. Niederreiter, "Uniform Distribution of Sequences," 1974.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-264 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-s-i+1}{n-s} = \sum_{i=0}^{n-s} \binom{r+i}{i}\binom{m+n-r-i+1}{m-r} .$$

H-265 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that $F_{2^3 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}$, where $k \geqslant 1$.

H-266 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Find all identities of the form

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = s^n F_{tn}$$

with positive integral r, s and t.

SOLUTIONS

TRIPLE PLAY

H-238 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} x^m y^n z^p ,$$

where the summation is restricted to m,n,p such that

$$m \leqslant n+p, \quad n \leqslant p+m, \quad p \leqslant m+n.$$

Solution by D. Russell, Digital Systems Lab, Stanford, California.

If $m+n+p$ is even, then either (1) exactly one of m, n, or p is even, or (2) all of m, n, and p are even. In either case, $m+n-p$ is also even. Let $a = \frac{1}{2}(m+n-p)$, and similarly let $b = \frac{1}{2}(n+p-m)$ and $c = \frac{1}{2}(p+m-n)$; because of the restrictions, all of a, b, c are non-negative. Then $m = a+c$, $n = a+b$, and $p = b+c$, and

$$x^m y^n z^p = x^{a+c} y^{a+b} z^{b+c} = (xy)^a (yz)^b (xz)^c .$$

This is a general term of the generating function

$$T_{\text{even}} = \frac{1}{(1-xy)(1-yz)(1-xz)}$$

and it is easily seen that *all* terms $x^m y^n z^p$ of T_{even} satisfy the restrictions and that $m+n+p$ is even.

Consider the terms where $m+n+p$ is odd and $m+n+p \geqslant 3$ (no terms exist with $m+n+p=1$). Either (1) exactly one of m, n, or p is odd, or (2) all of m, n, and p are odd. In either case, in the restriction $m \leqslant n+p$, equality may not hold, since then one side of the relation would be even and the other would be odd. But if $m < n+p$, then $m-1 \leqslant n+p-2$. Let $m' = m-1$, $n' = n-1$, $p' = p-1$. Then $x^{m'} y^{n'} z^{p'}$ satisfies the restrictions and $m'+n'+p'$ is even. The terms

$$x^m y^n z^p = (xyz) x^{m'} y^{n'} z^{p'}$$

with $m+n+p$ odd are thus terms of the generating function $T_{\text{odd}} = xyzT_{\text{even}}$ and *all* terms of T_{odd} are easily seen to satisfy the restrictions with $m+n+p$ odd.

Since $m+n+p$ is either odd or even, the sum S is given by

$$S = \frac{1+xyz}{(1-xy)(1-yz)(1-xz)} .$$

Also solved by P. Bruckman, W. Brady, M. Klamkin, O. P. Lossers, A. Shannon, and the Proposer.

FERMAT' INEQUALITY

H-239 Proposed by D. Finkel, Brooklyn, New York.

If a Fermat number $2^{2^n}+1$ is a product of precisely two primes, then it is well known that each prime is of the form $4m+1$ and each has a unique expression as the sum of two integer squares. Let the smaller prime be a^2+b^2, $a > b$; and the larger prime be c^2+d^2, $c > d$. Prove that

$$\left| \frac{c}{a} - \frac{d}{b} \right| \leqslant \frac{1}{100} .$$

Also, given that $2^{2^6}+1 = (274{,}177)(67{,}280{,}421{,}310{,}721)$ and that $274{,}177 = 516^2 + 89^2$, express the 14-digit prime as a sum of two squares.

Solution by the Proposer.

It is well known that

(1) $$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 = (ac-bd)^2 + (ad+bc)^2 .$$

Let $c/a = r$ and $d/b = r'$. Then

(2) $$(a^2+b^2)(c^2+d^2) = (a^2r+b^2r')^2 + (abr'-abr)^2 = (a^2r-b^2r')^2 + (abr'+abr)^2 .$$

One of the four squares on the right-hand side of (2) must be 1^2. Taking a, b, c, and d as positive, it is obviously not the first one on the top line or the last one on the bottom line. Clearly $ac > bd$ and thus $(a^2r - b^2r')^2 > 1$. Hence,

$$(abr'-abr)^2 = 1^2 \quad \text{or} \quad |r'-r| = \frac{1}{ab} .$$

The smallest Fermat number which is a product of exactly two primes is $2^{2^5}+1 = (641)(6{,}700{,}417)$. Here $641 = 25^2 + 4^2$ and $ab = 100$. No other Fermat number can have an ab product as low as 100.* Hence the result,

$$\left| \frac{c}{a} - \frac{d}{b} \right| \leqslant \frac{1}{100} ,$$

follows. For the last part of the problem, let the smaller prime be $p_1 = a^2+b^2$ and the larger prime be

$$p_2 = c^2+d^2 = a^2r^2 + b^2(r')^2 .$$

Now r and r' are approximately equal and $p_2/p_1 \sim r^2$. Since $c = ar$ and $d = br'$, a simple calculation leads to

$$p_2 = 8083111^2 + 1394180^2 .$$

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*Beiler, *Recreations in the Theory of Numbers,* pp. 143, 175.

E-GAD

H-240 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Let

$$S(m,n,p) = (q)_n (q)_p \sum_{i=0}^{\min(n,p)} \frac{q^{mi+(n-i)(p-i)}}{(q)_i (q)_{n-i} (q)_{p-i}},$$

where

$$(q)_j = (1-q)(1-q^2)\cdots(1-q^j), \qquad (q)_0 = 1.$$

Show that $S(m,n,p)$ is symmetric in m,n,p.

Solution by the Proposer.

Put

$$e(x) = \prod_{n=0}^{\infty} (1-q^n x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n}.$$

It is well known and easy to show that

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{e(x)}{e(ax)},$$

where

$$(a)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a), \qquad (a)_0 = 1.$$

It follows that

$$\sum_{r,s=0}^{\infty} \frac{q^{rs}x^r y^s}{(q)_r (q)_s} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{s=0}^{\infty} \frac{(q^r y)^s}{(q)_s} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} e(q^r y) = e(y) \sum_{r=0}^{\infty} \frac{(y)_r}{(q)_r} x^r,$$

so that

$$\sum_{r,s=0}^{\infty} \frac{q^{rs} x^r y^s}{(q)_r (q)_s} = \frac{e(x)e(y)}{e(xy)}.$$

Then

$$\frac{e(x)e(y)e(z)}{e(xyz)} = \frac{e(x)e(yz)}{e(xyz)} \frac{e(y)e(z)}{e(yz)}$$

$$= \sum_{m,i=0}^{\infty} \frac{q^{mi}x^m (yz)^i}{(q)_m (q)_i} \sum_{j,k=0}^{\infty} \frac{q^{jk} y^j z^k}{(q)_j (q)_k} = \sum_{m,n,p=0}^{\infty} \frac{x^m y^n z^p}{(q)_m} \sum_{\substack{i+j=n \\ i+k=p}} \frac{q^{mi+jk}}{(q)_i (q)_j (q)_k}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{x^m y^n z^p}{(q)_m} \sum_{i=0}^{\min(n,p)} \frac{q^{mi+(n-i)(p-i)}}{(q)_i (q)_{n-i} (q)_{p-i}},$$

so that

$$(*) \qquad \frac{e(x)e(y)e(z)}{e(xyz)} = \sum_{m,n,p=0}^{\infty} \frac{x^m y^n z^p}{(q)_m (q)_n (q)_p} S(m,n,p).$$

The stated result follows at once.

REMARK. Since

$$\frac{1}{e(z)} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}z^n}{(q)_n},$$

it follows from (*) that

$$\sum_{m,n,p=0}^{\infty} \frac{x^m y^n z^n}{(q)_m (q)_n (q)_p} S(m,n,p) = \sum_{i,j,k=0}^{\infty} \frac{x^i y^j z^k}{(q)_i (q)_j (q)_k} \sum_{s=0}^{\infty} (-1)^s q^{\frac{1}{2}s(s-1)} \frac{(xyz)^s}{(q)_s},$$

so that

$$S(m,n,p) = \sum_{s=0}^{\min(m,n,p)} (-1)^s q^{\frac{1}{2}s(s-1)} \frac{(q)_m (q)_n (q)_p}{(q)_s (q)_{m-s} (q)_{n-s} (q)_{p-s}}.$$

HARMONIC

H-241 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\frac{1}{1-x^n} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1 - xe^{\frac{2k\pi}{n}i}}.$$

Solution by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Using partial fractions, we have

$$\frac{1}{1-x^n} = -\frac{1}{x^n - 1} = -\sum_{k=0}^{n-1} \frac{A_k}{x - e^{\frac{2\pi k}{n}i}}$$

$$A_k = \frac{1}{\phi'\left(e^{\frac{2\pi k}{n}i}\right)}; \quad \phi(x) \equiv x^n - 1$$

[e.g., see Edwards, *Integral Calculus,* Vol. 1, p. 145.]

$$\frac{1}{1-x^n} = -\sum_{k=0}^{n-1} \frac{1}{ne^{\frac{2\pi k(n-1)}{n}i}\left(x - e^{\frac{2\pi k}{n}i}\right)} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1 - xe^{\frac{2\pi k(n-1)}{n}}} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1 - xe^{\frac{2\pi k}{n}i}}$$

since $(n, n-1) = 1$.

Also solved by C. Bridger, P. Smith, and the Proposer.

PELL-MELL

H-243 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that for each triangular number $t_n = \frac{n}{2}(n+1)$ there exist an infinite number of nonsquare positive integers D such that $t_{n+r}^2 - t_n^2 D = 1$.

Solution by the Proposer.

In the Pellian equation $x^2 - Dy^2 = 1$, let $x = t_{mt_n^2+1}$, $y = t_n$, $mn \neq 0$.

$$[(mt_n^2+1)(mt_n^2+2)+2]\,[(mt_n^2+1)(mt_n^2+2)-2] = 4t_n^2 D.$$

$$[m^2t_n^4 + 3mt_n^2 + 4](m^2t_n^2 + 3m) = 4D.$$

The left-hand side is congruent to zero modulo 4 for the conditions (1) m an even integer, (2) m odd, t_n odd, (3) m odd, t_n even. Hence D is an integer, and not an integer square since the difference of these two integer squares is never one.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S. E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-334 Proposed by Phil Mana, Albuquerque, New Mexico.

Are all the terms prime in the sequence 11, 17, 29, 53, ⋯ defined by $u_0 = 11$ and $u_{n+1} = 2u_n - 5$ for $n \geqslant 0$?

B-335 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain a closed form for

$$\sum_{i=0}^{n-k} (F_{i+k}L_i + F_iL_{i+k}).$$

B-336 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $Q_0 = 1 = Q_1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$. Show that $2(Q_{2n}^2 - 1)$ is a perfect square for n = 1, 2, 3, ⋯.

B-337 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Show that there are infinitely many points with both x and y rational on the ellipse $25x^2 + 16y^2 = 82$.

B-338 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let k and n be positive integers. Let $p = 4k + 1$ and let h be the largest integer with $2h + 1 \leqslant n$. Show that

$$\sum_{j=0}^{h} p^j \binom{n}{2j+1}$$

is an integral multiple of 2^{n-1}.

B-339 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the validity of E. Cesàro's symbolic Fibonacci-Lucas identity $(2u + 1)^n = u^{3n}$; after the binomial expansion has been performed, the powers of u are used as either Fibonacci or Lucas subscripts. (For example, when n = 2 one has both

$$4F_2 + 4F_1 + F_0 = F_6 \qquad \text{and} \qquad 4L_2 + 4L_1 + L_0 = L_6.)$$

SOLUTIONS

SPECIAL BINOMIAL COEFFICIENTS

B-310 Proposed by Daniel Finkel, Brooklyn, New York.

Find some positive integers n and r such that the binomial coefficient $\binom{n}{r}$ is divisible by $n+1$.

Solution by David Singmaster, Polytechnic of the South Bank, London, England.

For $n \leqslant 100$, I find the following solutions for $(n+1) \mid \binom{n}{r}$, with $2r \leqslant n$;

$$n = 29, \quad r = 6, 7, 14;$$
$$n = 59, \quad r = 12, 13, 14, 15;$$
$$n = 69, \quad r = 21, 22, 23, 24;$$
$$n = 83, \quad r = 36, 37, 38, 39, 40, 41;$$
$$n = 89, \quad r = 15, 18, 19, 20, 21, 22, 23, 40, 41, 42, 44.$$

One can show that $n+1$ must have at least three prime factors.

Also solved by the Proposer.

A NONHOMOGENEOUS RECURRENCE

B-311 Proposed by Jeffrey Shallit, Wynnewood, Pennsylvania.

Let k be a constant and let $\{a_n\}$ be defined by

$$a_n = a_{n-1} + a_{n-2} + k, \qquad a_0 = 0, \qquad a_1 = 1.$$

Find

$$\lim_{n \to \infty} (a_n / F_n).$$

Solution by Graham Lord, Université Laval, Québec.

With $a_0 = 0$ and $a_1 = 1$ then $a_n = F_n + (F_{n+1} - 1)k$ (use induction) and so the limit is $1 + \alpha k$.

Also solved by George Berzsenyi, Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Frank Higgins, Harvey J. Hindin, Mike Hoffman, John W. Milsom, C.B.A. Peck, A. G. Shannon, Martin C. Weiss, Gregory Wulczyn, David Zeitlin, Larry Zimmerman, and the Proposer.

DOUBLY-TRUE FIBONACCI ALPHAMETIC

B-312 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Solve the doubly-true alphametic

```
  ONE
  ONE
  ONE
  TWO
THREE
-----
EIGHT
```

Unity is not normally considered so, but here our ONE is prime!

Solution by Charles W. Trigg, San Diego, California.

$O \neq$ zero, $T + 1 = E$, and ONE is prime, so $E = 3$, 7, or 9. Then $4E + O = T + 10k$.

If $E = 3$, then O = zero, which is not acceptable.

If $E = 9$, then $T = 8$, $O = 2$, $H = 7$, and the sum of the digits in the hundreds' column is < 30. Hence, $E \neq 9$.

If $E = 7$, then $T = 6$, and $O = 8$, whereupon $N = 2$ or 5, since ONE is prime. But if $N = 2$, then $I = 2$. Consequently, $N = 5$, $H = 9$, $I = 2$, $W = 4$, $R = 1$, and $G = 3$.

The unique reconstruction of the addition is:

$$857 + 857 + 857 + 648 + 69177 = 72396.$$

Also solved by Hai Vo Ba, Richard Blazej, Paul S. Bruckman, Madeleine Hatzenbuehler and George Berzsenyi (jointly), John W. Milsom, C.B.A. Peck, A. G. Shannon, Martin C. Weiss, and the Proposer.

EXPONENTIATING LUCAS INTO FIBONACCI

B-313 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$M(x) = L_1x + (L_2/2)x^2 + (L_3/3)x^3 + \cdots.$$

Show that the Maclaurin series expansion for $e^{M(x)}$ is

$$F_1 + F_2x + F_3x^2 + \cdots.$$

1. Solution by Graham Lord, Universite Laval, Québec.

If L_n is replaced by $\alpha^n + \beta^n$ then $M(x)$ becomes

$$-\ln(1-\alpha x)(1-\beta x)$$

which is the same as

$$-\ln(1-x-x^2).$$

Hence $e^{M(x)}$ is $1/(1-x-x^2)$, that is

$$F_1 + F_2x + F_3x^2 + \cdots.$$

2. Solution by Martin C. Weiss, San Jose, California.

$$M'(x) = L_1 + L_2x + L_3x^2 + \cdots = (1+2x)/(1-x-x^2).$$

Integrating, $M(x) = -\ln(1-x-x^2)$. Hence,

$$e^{M(x)} = 1/(1-x-x^2) = F_1 + F_2x + F_3x^2 + \cdots.$$

Also solved by Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Harvey J. Hindin, Mike Hoffman, A. G. Shannon, Sahib Singh, David Zeitlin, and the Proposer.

LUCAS NUMBERS ENDING IN THREE

B-314 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2p^k} \equiv 3 \pmod{10}$ for all primes $p \geqslant 5$.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

For all primes $p \geqslant 5$, $p \equiv \pm 1 \pmod 6$. Hence, for all natural k, $p^k \equiv \pm 1 \pmod 6$, which implies $2p^k \equiv \pm 2 \pmod{12}$.

If we now write down the Lucas sequence (mod 10), we readily find that the cycle has length 12 (i.e. $L_{m+12} \equiv L_m \pmod{10}$, $\forall m$); it is also easy to observe that

$$L_n \equiv 3 \pmod{10} \quad \text{iff} \quad m \equiv \pm 2 \pmod{12}.$$

Combining this with the first result above, it follows that $L_{2p^k} \equiv 3 \pmod{10}$, for all prime $p \geqslant 5$.

Also solved by Frank Higgins, Graham Lord, A. G. Shannon, Martin C. Weiss, Gregory Wulczyn, David Zeitlin, and the Proposer.

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