THE ALGEBRA OF FIBONACCI REPRESENTATIONS

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1. INTRODUCTION AND SUMMARY

A Fibonacci representation has been defined [1, 2, 3, 5, 8] as a finite sequence of ones and zeroes (in effect) read positionally from right to left, in which a one in position i signifies the presence of the Fibonacci number f_i, where we take $f_1 = 1$, $f_2 = 1$. The integer thus represented is the sum of the Fibonacci numbers whose presence is indicated by the ones appearing in the representation.

Our purpose in this paper is to generalize the notion of Fibonacci representations in such a way as to provide for a natural algebraic and geometric setting for their analysis. In this way many known results are unified and simplified and new results are obtained. Some of the results extend to Fibonacci representations of higher order, but we do not present these because we have been unable to extend the theory as a whole and because of the length of the paper.

The first step is to extend the Fibonacci numbers through all negative indices using the defining recursion $f_{n+2} = f_n + f_{n+1}$, as has been done by Klarner [14]. The second step is to introduce arbitrary integer coefficients. Thus an *extended* Fibonacci representation is a finite sequence of integers, together with a point which sets off the position of f_0. Positions are numbered as is customary for positional notation, and an integer k_i in position i signifies $k_i f_i$. The integer thereby represented is $\sum k_i f_i$, the summation extending over those finitely many i for which $k_i \neq 0$.

Let τ denote the golden ratio taken greater than one. Then $\tau^2 = 1 + \tau$ and $1/\tau^2 = 1 - (1/\tau)$. The ring I of quadratic integers in the quadratic extension field $Q[\tau]$ of the rationals consists of those elements of the form $m + n\tau$ (or $(m/\tau) + n$) in which m and n are ordinary integers.

Each Fibonacci representation $\sum k_i f_i$ determines another integer by taking its *left shift;* this gives $\sum k_{i-1} f_i$. For each Fibonacci representation $\sum k_i f_i$ we define a quadratic integer in I said to be *determined* by the representation $\sum k_i f_i$; it is

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i \,.$$

This quadratic integer is equal to the sum $\sum k_i \tau^i$, which is a pseudo-polynomial in τ. Because of this the usual arithmetical algorithms for addition, subtraction and multiplication, when applied to the Fibonacci representations, yield results which interpret in terms of the ring structure in I. For example, 12.1 represents 2 and 121. represents 3 so that 12.1 determines the quadratic integer $(2/\tau) + 3$. Similarly, 1.1 determines $(1/\tau) + 1$. Since

$$\left(\frac{1}{\tau} + 1\right)\left(\frac{2}{\tau} + 3\right) = \frac{3}{\tau} + 5\,,$$

we predict that the usual multiplication algorithm when applied to 12.1 and 1.1 will produce a representation of 3 whose left shift represents 5, and indeed this is true of the result, which is 13.31.

A Fibonacci representation is *canonical* if either all of the non-zero k_i are +1 or else all of the non-zero k_i are -1, and no two non-zero k_i are consecutive. A basic theorem in this paper is that each quadratic integer in I is determined by exactly one canonical representation. A *resolution algorithm* is introduced which is shown to reduce any Fibonacci representation to the unique canonical representation which determines the same quadratic integer. As a result, the canonical Fibonacci representations in the usual arithmetical algorithms plus the resolution algorithm form a ring isomorphic to the ring I under the correspondence

$$\sum k_i f_i \to \sum k_i \tau^i, \quad \text{or the same,} \quad \sum k_i f_i \to \frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i .$$

Clearly the subring of representations of zero will be isomorphic to the integers under the left shift $\sum k_i f_i \to \sum k_{i-1} f_i$, since in the case of zero representations this amounts to $\sum k_i f_i \to \sum k_i \tau^i$. The reader is referred to the text for sample calculations with the resolution algorithm.

One consequence of the foregoing remarks is that for every pair of integers m and n there is exactly one canonical Fibonacci representation of m whose left shift represents n (canonically). (This appears in [14] for natural m and n and in [13] for the general case.) This representation can be determined from the resulution algorithm by starting with $n.m$ which represents $\frac{m}{\tau} + n$. This of course provides an infinity of canonical representations for each integer m, one corresponding to each choice of n.

By identifying the quadratic integer $\frac{m}{\tau} + n$ with the point (m,n) in the plane, which in the present context we refer to as the *Fibonacci plane*, we are able to arrive at simple geometric characterizations of those choices of n (for a given m) which will result in the standard Fibonacci representations in the literature, and some new ones in addition. Formulas giving n as a function of m for these **representations** are an immediate consequence of the geometry.

It is shown in Section 2 that the space of integer sequences $\{x_n\}$ satisfying $x_{n+2} = x_n + x_{n+1}$ is naturally isomorphic to the ring I. Consequently the results on canonical Fibonacci representations interpret for these sequences, which we call *Fibonacci sequences.* Namely, given a Fibonacci sequence with zeroth and first terms x_0 and x_1, respectively, let $\sum k_i f_i$ be the canonical Fibonacci representation which determines the quadratic integer $\frac{x_0}{\tau} + x_1$. Then the sequence $\{x_n\}$ is uniquely expressible as a signed sum of distinct, non-consecutive shifts of the sequence $\{f_n\}$ of Fibonacci numbers, and the sum is exactly that which is determined by the canonical representation $\sum k_i f_i$, wherein position i is associated with the i^{th} left shift of $\{f_n\}$. Moreover, the canonical representation $\sum k_i f_i$ represents the term x_0 and its various left and right shifts represent the corresponding terms of the sequence $\{x_n\}$. (Again see [14] for special cases and see [8, 13] for generalizations.) That is, every Fibonacci sequence of integers appears canonically "in Fibonacci" as a sequence of shifts of a fixed, signed block of zeroes and ones.

Consider for example the Fibonacci sequence having $x_0 = 5$, $x_1 = 7$. By the resolution algorithm 7.5 reduces to 10100.1. This means that $\{x_n\}$ is the sum of the fourth and second left shifts and the first right shift of $\{f_n\}$. Moreover, the sequence $\cdots$, 5, 7, 12, $\cdots$ appears "in Fibonacci" as $\cdots$, 10100.1, 101001., 1010010., $\cdots$.

Various other results appear in the paper, such as other canonical Fibonacci representations obtained by geometric means, hyperbolic flows and number theoretic properties of flow constants.

It is a pleasure to acknowledge the kind assistance of many of our colleagues and acquaintances, particularly that of Professors J. Luh and L. Carlitz.

2. THE RINGS $\underline{X}$, $\underline{F}$, M AND I AND THE FIBONACCI PLANE P

2.1 Introduction. The analysis of Fibonacci representations presented here rests on a natural ring structure for the space of integer Fibonacci sequences. The purpose of Section 2 is to introduce the space of integer Fibonacci sequences, to show how its natural ring structure arises, and to introduce other isomorphic rings of interest in investigating and interpreting Fibonacci representations.

2.2. The Spaces X and $\underline{X}$. Let X denote the collection of integer sequences $\{x_n\}_0^\infty$ satisfying the linear, second-order recursion

(2.1) $$x_{n+2} = x_n + x_{n+1} .$$

These *Fibonacci sequences* form a module over the ring Z of integers under termwise operations.

Let $f = \{f_n\}_0^\infty$ be the solution of Eq. (2.1) such that $f_0 = 0$ and $f_1 = 1$; the terms of f are customarily called the *Fibonacci numbers.*

Let $\sigma: X \to X$ be the *left shift* on X, defined by

$$\sigma(\{x_n\}_0^\infty) = \{x_{n+1}\}_0^\infty. \tag{2.2}$$

By Eq. (2.1),

$$\sigma^2 - \sigma - \mathbb{1} = \mathbb{0}, \tag{2.3}$$

where $\mathbb{1}$ and $\mathbb{0}$ denote the identity and zero operators on X, respectively. σ is an automorphism of the Z-module X, and its inverse σ^{-1} is the right shift on X, with the understanding that the zeroth term of $\sigma^{-1}(\{x_n\}_0^\infty)$ is to be $x_1 - x_0$.

X is a two-dimensional Z-module; one basis for X is $\{f, \sigma(f)\}$, in which

$$x = (x_1 - x_0)f + x_0\sigma(f), \qquad x \in X. \tag{2.4}$$

Each sequence $x = \{x_n\}_0^\infty$ in X can be extended by Eq. (2.1) in just one way to a double-ended sequence $\underline{x} = \{x_n\}_{-\infty}^\infty$; the collection of double-ended solutions of Eq. (2.1) is also a Z-module under termwise operations, and is isomorphic to X under the correspondence $\alpha(x) = \underline{x}$. Members of $\underline{X}$ shall also be called Fibonacci sequences, but referred to as *extended* when it is necessary to distinguish them from the sequences in X. In particular, the terms of $\underline{f} = \alpha(f)$ are called *extended Fibonacci numbers.* For these numbers it is readily verified that

$$f_{-n} = (-1)^{n+1}f_n, \qquad n \in Z. \tag{2.5}$$

No confusion will arise from using $\sigma: \underline{X} \to \underline{X}$ to denote the left shift on $\underline{X}$ as well as the left shift on X, and Eq. (2.3) is valid in either case. Since α is an isomorphism, $\{\underline{f}, \sigma(\underline{f})\}$ is a basis for $\underline{X}$ in which

$$\underline{x} = (x_1 - x_0)\underline{f} + x_0\sigma(\underline{f}), \qquad \underline{x} \in \underline{X}. \tag{2.6}$$

The inverse of σ on $\underline{X}$, σ^{-1}, is the right shift on $\underline{X}$.

2.3. Two Theorems. This section consists of the statement and proof of two theorems. The first of these shows that a certain class of quotient rings can be characterized as a certain class of modules. Because X belongs to the latter class (in this connection Eq. (2.4) is critical) the theorem provides a ring structure for X. The results apply equally to $\underline{X}$.

The second theorem shows how the members of this class of quotient rings can be realized as matrix rings; this results in a representation of X and $\underline{X}$ as a certain collection of 2×2 matrices in the usual operations. This theorem appears in MacDuffee [15] for algebras.

Theorem 2.1. Let R be a commutative ring with unity 1 and let $p(\lambda)$ be a monic polynomial in $R[\lambda]$ of degree n. Let S be the quotient ring of polynomials modulo $p(\lambda)$ and define $\Lambda: S \to S$ by

$$\Lambda([q(\lambda)]) = [\lambda q(\lambda)]$$

for each equivalence class $[q(\lambda)]$ in S. If S is considered to be an R-module in the operations

$$[q_1(\lambda)] + [q_2(\lambda)] = [q_1(\lambda) + q_2(\lambda)], \qquad r[q(\lambda)] = [rq(\lambda)],$$

then $\Lambda \in \mathrm{Hom}_R(S,S)$ and S is n-dimensional over R with basis

$$\{[1], \Lambda([1]), \cdots, \Lambda^{n-1}([1])\} = \{[1], [\lambda], \cdots, [\lambda^{n-1}]\}.$$

Furthermore $p(\Lambda) = 0$ and $p(\lambda)$ is the polynomial in $R[\lambda]$ of least degree which is monic and which annihilates Λ.

Conversely, let S be an n-dimensional R-module over a commutative ring R with unity 1, let $\Lambda \in \mathrm{Hom}_R(S,S)$ and suppose there exists $s \in S$ such that $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ is a basis for S over R. Then there exists $p(\lambda)$ in $R[\lambda]$ which is monic and of degree n, such that $p(\Lambda) = 0$ and such that S is isomorphic to the quotient ring $R[\lambda]/(p(\lambda))$ considered as an R-module. One isomorphism is the mapping ϕ which sends

$$\sum_{i=0}^{n-1} r_i\Lambda^i(s) \quad \text{to} \quad \left[\sum_{i=0}^{n-1} r_i\lambda^i\right].$$

This isomorphism induces a multiplication on S by

$$s_1 * s_2 = \phi^{-1}(\phi(s_1)\phi(s_2)),$$

and this induced multiplication makes S into a ring with unity s. ϕ is then a ring isomorphism under which the action of Λ in S corresponds to multiplication by the element $[\lambda]$ in the quotient ring $R[\lambda]/(p(\lambda))$.

Proof. Let $p(\lambda) = \lambda^n - r_{n-1}\lambda^{n-1} - \cdots - r_0$. Note that for $q_1(\lambda), q_2(\lambda) \in R[\lambda]$, we have $q_1(\Lambda), q_2(\Lambda) \in \mathrm{Hom}_R(S,S)$, and

(2.7) $$q_1(\Lambda)([q_2(\lambda)]) = [q_1(\lambda)q_2(\lambda)] = q_2(\Lambda)([q_1(\lambda)]).$$

Now $\{[1], \Lambda([1]), \cdots, \Lambda^{n-1}([1])\}$ is the same as $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$, and the latter is a basis for S over R because p is monic. Moreover, by Eq. (2.7)

$$p(\Lambda)([\lambda^i]) = [\lambda^i p(\lambda)] = [0]$$

so that $p(\Lambda)$ vanishes on a basis and therefore is zero. If $q(\lambda) \in R[\lambda]$ is monic and $q(\Lambda) = 0$, then by Eq. (2.7)

$$q(\Lambda)([1]) = [q(\lambda)] = [0]$$

so that $q(\lambda)$ is a multiple of $p(\lambda)$. Since both $q(\lambda)$ and $p(\lambda)$ are monic, $q(\lambda)$ cannot have lesser degree than $p(\lambda)$ has. Thus $p(\lambda)$ is the polynomial in $R[\lambda]$ of least degree which is monic and annihilates Λ.

Now suppose S is an n-dimensional R-module over a commutative ring R with unity 1, and let $\Lambda \in \mathrm{Hom}_R(S,S)$ and $s \in S$ such that $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ is a basis for S over R. Since $\Lambda^n(s) \in S$, there exist unique elements $r_0, r_1, \cdots, r_{n-1}$ of R such that

$$\Lambda^n(s) = \sum_{i=0}^{n-1} r_i\Lambda^i(s).$$

Define $p(\lambda) \in R[\lambda]$ by

$$p(\lambda) = \lambda^n - r_{n-1}\lambda^{n-1} - \cdots - r_0,$$

so that

$$p(\Lambda)(s) = \Lambda^n(s) - \sum_{i=0}^{n-1} r_i\Lambda^i(s) = 0.$$

But then

$$p(\Lambda)(\Lambda^i(s)) = \Lambda^i(p(\Lambda)(s)) = 0$$

for each natural number i, and hence $p(\Lambda)$ vanishes on a basis and is therefore zero.

Define ϕ as in the statement of the theorem; that is

$$\phi\left(\sum_{i=0}^{n-1} r_i\Lambda^i(s)\right) = \left[\sum_{i=0}^{n-1} r_i\lambda^i\right].$$

It is clear that ϕ is a module homomorphism, and must indeed be an isomorphism because it sends the basis $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ onto the basis $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$. The rest of the theorem now follows readily from the manner in which the multiplication $*$ is induced on S.

Theorem 2.2. Let R be a commutative ring with a unity and let $p(\lambda) \in R[\lambda]$ be monic with degree n. Let S be the quotient ring $R[\lambda]/(p(\lambda))$. Then each congruence class in S contains exactly one polynomial (possibly zero) of degree less than n. Given $q(\lambda) \in R[\lambda]$ let

$$\left.\begin{aligned}
q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i\lambda^i\,, \\
\lambda q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i'\lambda^i\,, \\
\lambda^2 q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i''\lambda^i\,, \\
&\ \ \vdots \\
\lambda^{n-1} q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i^{(n-1)}\lambda^i\,,
\end{aligned}\right\} \text{modulo } p(\lambda)$$

the right-hand sides being uniquely determined by the choice of $[q(\lambda)]$. Define a mapping γ which sends the congruence class $[q(\lambda)]$ in S to the $n \times n$ matrix

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ r_0' & r_1' & \cdots & r_{n-1}' \\ r_0'' & r_1'' & \cdots & r_{n-1}'' \\ \vdots & \vdots & & \vdots \\ r_0^{(n-1)} & r_1^{(n-1)} & \cdots & r_{n-1}^{(n-1)} \end{pmatrix}.$$

Then γ is a ring isomorphism from S onto a subring of the ring of $n \times n$ matrices over R in the usual operations.

Proof. We have seen in the previous theorem that S is an R-module with basis $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$. In this basis, multiplication by $[q(\lambda)]$ in S is a module endomorphism on S which is represented by the foregoing matrix. The mapping which sends $[q(\lambda)]$ in S to the endomorphism induced by multiplication by $[q(\lambda)]$ is a ring isomorphism of S onto a subring of the endomorphism ring of S. Since representation of these endomorphisms by matrices in a given basis is also a ring isomorphism, the theorem follows.

2.4. Application to X. We now apply Theorem 2.1 to X, taking for Λ the left shift σ and for s the sequence f of Fibonacci numbers. Equation (2.3) gives $p(\lambda) = \lambda^2 - \lambda - 1$, so we let F denote the quotient ring $Z[\lambda]/(\lambda^2 - \lambda - 1)$. If $x \in X$, by Eq. (2.4) $x = (x_1 - x_0)f + x_0\sigma(f)$, so we define $\phi : X \to F$ by $\phi(x) = (x_1 - x_0) + x_0\lambda$, which can be written $\phi(x) = x_{-1} + x_0\lambda$ if we introduce (for X) the abbreviation $x_{-1} = x_1 - x_0$. For $y \in X$, $\phi(y) = y_{-1} + y_0\lambda$, so

$$x * y = \phi^{-1}([x_{-1} + x_0\lambda])([y_{-1} + y_0\lambda])$$

which works out to

$$x * y = (x_{-1}y_{-1} + x_0y_0)f + (x_{-1}y_0 + x_0y_0 + x_0y_{-1})\sigma(f). \tag{2.8}$$

This equation defines the multiplication in X which makes X into a ring with unity f. Moreover, the left shift σ in X corresponds under ϕ to multiplication by $[\lambda]$ in F, which means that the left shift on X can be realized by multiplication in X by $\sigma(f)$; thus

$$x * \sigma(f) = (x_{-1}f + x_0\sigma(f)) * \sigma(f) = x_0f + (x_{-1} + x_0)\sigma(f) = x_0f + x_1\sigma(f) = \sigma(x). \tag{2.9}$$

If $q(\lambda)$ is equivalent to $m + n\lambda$ modulo $\lambda^2 - \lambda - 1$, then $\lambda q(\lambda)$ is equivalent to $n + (m+n)\lambda$ modulo $\lambda^2 - \lambda - 1$. It follows from Theorem 2.2 that X is isomorphic to the ring M of 2×2 matrices of the form

$$\begin{pmatrix} m & n \\ n & m+n \end{pmatrix}, \quad m,n \in Z,$$

under the transformation $\psi : X \to M$ defined by

$$\psi(x) = \begin{pmatrix} x_{-1} & x_0 \\ x_0 & x_1 \end{pmatrix} . \tag{2.10}$$

It is clear that the remarks of this section apply as well to $\underline{X}$, in view of Eq. (2.6); Eqs. (2.8), (2.9), (2.10) are valid with identical right-hand sides when x and y are replaced by $\underline{x}$ and $\underline{y}$ on the left-hand sides.

2.5. Extension of F to $\underline{F}$. By a *pseudo-polynomial* over Z is meant a finite sum of the form

$$\sum_i k_i \lambda^i$$

in which each $i \in Z$ and each $k_i \in Z$. The collection $Z\langle\lambda\rangle$ of all pseudo-polynomials over Z in the indeterminant λ is a ring in the obvious way, in which $p(\lambda) = \lambda^2 - \lambda - 1$ generates an ideal $(\lambda^2 - \lambda - 1)$. Let $\underline{F}$ denote the quotient ring $Z\langle\lambda\rangle/(\lambda^2 - \lambda - 1)$.

Since $\lambda^{-1}(\lambda^2 - \lambda - 1) = \lambda - 1 - \lambda^{-1} \in (\lambda^2 - \lambda - 1)$ in $Z\langle\lambda\rangle$, we see that in $Z\langle\lambda\rangle$

$$\lambda^{-1} \equiv \lambda - 1 \mod (\lambda^2 - \lambda - 1). \tag{2.11}$$

It follows by taking powers on each side of this congruence that every pseudo-polynomial is equivalent modulo $\lambda^2 - \lambda - 1$ in $Z\langle\lambda\rangle$ to a polynomial. Since polynomials are equivalent modulo $\lambda^2 - \lambda - 1$ in $Z\langle\lambda\rangle$ if and only if they are equivalent modulo $\lambda^2 - \lambda - 1$ in $Z[\lambda]$, it is possible to map each equivalence class in F unambiguously onto the equivalence class in $\underline{F}$ containing the same polynomials, and this mapping $\beta : F \to \underline{F}$ is an onto ring isomorphism.

Define a mapping $\underline{\phi} : \underline{X} \to \underline{F}$ by

$$\underline{\phi}(\underline{x}) = [x_{-1} + x_0\lambda] . \tag{2.12}$$

$\underline{\phi}$ is a ring isomorphism if the multiplication in $\underline{X}$ is defined by Eq. (2.8). In fact, $\underline{\phi} = \beta \circ \phi \circ \alpha^{-1}$, making the extension of F to $\underline{F}$ the exact counterpart of the extension of X to $\underline{X}$ in the sense that the following diagram commutes:

$$\begin{tikzcd} \underline{X} \arrow[r, "\underline{\phi}"] & \underline{F} \\ X \arrow[u, "\alpha"] \arrow[r, "\phi"] & F \arrow[u, "\beta"'] \end{tikzcd} \tag{2.13}$$

Under the isomorphism $\underline{\phi}$, the left shift in $\underline{X}$ corresponds to multiplication by $[\lambda]$ in $\underline{F}$ and the right shift in $\underline{X}$ corresponds to multiplication by $[\lambda^{-1}] = [\lambda - 1]$ in $\underline{F}$. It follows that the left and right shifts commute with the multiplication in $\underline{X}$ (and in X) in the sense that

$$\sigma^n(\underline{x} * \underline{y}) = \sigma^n(\underline{x}) * \underline{y} = \underline{x} * \sigma^n(\underline{y}) \tag{2.14}$$

for all integers n. Taking $\underline{y} = \underline{f}$, the unity in $\underline{X}$, and $n = \pm 1$ gives two analogues in $\underline{X}$ of Eq. (2.9):

$$\sigma(\underline{x}) = \underline{x} * \sigma(\underline{f}), \tag{2.15}$$

$$\sigma^{-1}(\underline{x}) = \underline{x} * \sigma^{-1}(\underline{f}). \tag{2.16}$$

It now follows that any endomorphism of the Z-module $\underline{X}$ which is a pseudo-polynomial in the shift σ can be achieved by multiplication in $\underline{X}$ by that element of $\underline{X}$ which is the value of the corresponding pseudo-polynomial in $\underline{f}$.

2.6. The Ring I and the Fibonacci Plane P. Conjugation and Flows. Let τ be the positive root of $\lambda^2 - \lambda - 1$; then $\tau = \frac{1}{2}(1 + \sqrt{5})$ which is the famous *golden ratio,* taken greater than 1. Let Q denote the field of rational numbers. The quadratic extension field $Q[\tau]$ is isomorphic to $Q[\lambda]/(\lambda^2 - \lambda - 1)$ in the standard way; each equivalence class in $Q[\lambda]/(\lambda^2 - \lambda - 1)$ corresponds to the number in $Q[\tau]$ which is the common value assumed by all members of the equivalence class under the evaluation $\lambda \to \tau$. It is well known that the ring I of quadratic integers in $Q[\tau]$ consists precisely of those members of the form $m + n\tau$, $m, n \in Z$. Thus we define an isomorphism $\zeta : F \to I$ by

(2.17) $$\zeta([p(\lambda)]) = p(\tau).$$

This same formula, in which $p(\lambda)$ can be an arbitrary pseudo-polynomial in $Z\langle\lambda\rangle$ serves to define an isomorphism $\underline{\zeta} : \underline{F} \to I$. Diagram (2.13) then becomes

(2.18)
$$\begin{tikzcd}
\underline{X} \arrow[r, "\underline{\phi}"] & \underline{F} \arrow[dr, "\underline{\zeta}"] & \\
& & I \\
X \arrow[uu, "\alpha"] \arrow[r, "\phi"] & F \arrow[uu, "\beta"] \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

which still commutes. We note that under the identification $\underline{\zeta} \circ \underline{\phi}$ (resp. $\zeta \circ \phi$) each integral power of σ on $\underline{X}$ (resp. X) corresponds to multiplication by that power of τ in I, and similarly for pseudo-polynomials in σ. The identification $\underline{\zeta} \circ \underline{\phi}$ maps $\underline{x} \in \underline{X}$ to $x_{-1} + x_0\tau = (x_0/\tau) + x_1 \in I$.

The other root of $\lambda^2 - \lambda - 1$ is $-(1/\tau) = 1 - \tau$, and the automorphism of $Q[\tau]$ which fixes Q and sends τ to $-(1/\tau)$ is of course called *conjugation*. Denoting the conjugate of $p + q\tau$ by $\overline{p + q\tau}$,

$$\overline{p + q\tau} = p + q - q\tau,$$

or, alternatively,

(2.19) $$\overline{\frac{p}{\tau} + q} = -\frac{p}{\tau} + q - p\,.$$

Conjugation is involutory and therefore has a fixed point space and an involuted space. This is best considered geometrically, and for this and other purposes we introduce the Fibonacci Plane P. In analogy with complex numbers, we associate to each number $(p/\tau) + q$ in $Q[\tau]$ the point (p,q) in the *Fibonacci plane.* Even though the points of the rational plane suffice to represent $Q[\tau]$, we include all real number pairs into the Fibonacci plane. Equation (2.19) is then extended to the Fibonacci plane so as to send each point $(u,v) \in P$ to $(-u, v - u)$ $\in P$. This is a linear transformation over the reals and is involutory. It consists of a non-orthogonal reflection of each point P in the V-axis $(u = 0)$ along the line $K : v = \frac{1}{2}u$. This is illustrated in Figure 2.1.

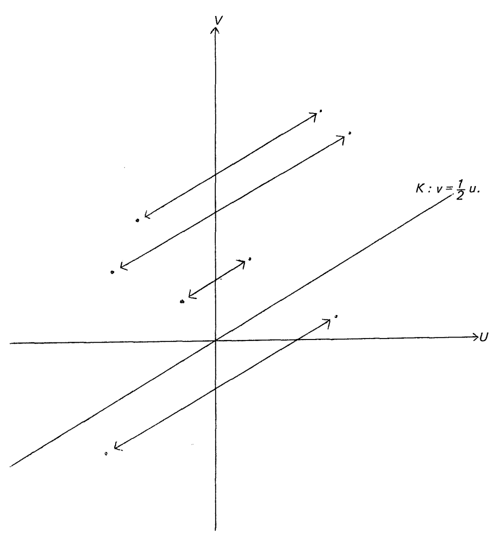

Fig. 2.1 Conjugation in P

Given $(p/\tau) + q \in Q[\tau]$, one readily verifies that

$$p = \frac{\left(\frac{p}{\tau} + q\right) - \overline{\left(\frac{p}{\tau} + q\right)}}{2\tau - 1}, \tag{2.20}$$

and

$$q = \frac{(p + q\tau) - \overline{(p + q\tau)}}{2\tau - 1} \tag{2.21}$$

These formulas are analogous to those for the real and imaginary parts of a complex number.

Let $\underline{x} \in \underline{X}$. In view of the remarks immediately following diagram (2.18), we have

$$\tau^n \left(\frac{x_0}{\tau} + x_1\right) = \frac{x_n}{\tau} + x_{n+1} \tag{2.22}$$

for every integer n. By taking $\underline{x} = \underline{f} \in \underline{X}$, we obtain the well known identity

$$\tau^n = \frac{f_n}{\tau} + f_{n+1} = f_{n-1} + f_n\tau, \quad n \in Z. \tag{2.23}$$

The use of Eq. (2.20) in conjunction with Eq. (2.22) enables one to solve for the general term of sequences in $\underline{X}$ in terms of terms number 0 and 1:

$$x_n = \frac{\tau^n\left(\frac{x_0}{\tau} + x_1\right) - \overline{\tau^n\left(\frac{x_0}{\tau} + x_1\right)}}{2\tau - 1}, \quad n \in Z. \tag{2.24}$$

As a special case of Eq. (2.24) we obtain the classical Binet formula; taking $\underline{x} = \underline{f}$ gives

$$f_n = \frac{\tau^n - \overline{\tau}^n}{2\tau - 1} = \frac{\tau^n - (-\tau)^{-n}}{\sqrt{5}} \tag{2.25}$$

We introduce two *principal axes* L_1 and L_2 into the Fibonacci plane by

$$L_1 : v = \tau u, \qquad L_2 : v = -(1/\tau)u.$$

These two axes are perpendicular and divide the Fibonacci plane into four regions $\mathcal{L}_I$, $\mathcal{L}_{II}$, $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$, as illustrated in Fig. 2.2, in which for later reference also appear the V-axis and the line K.

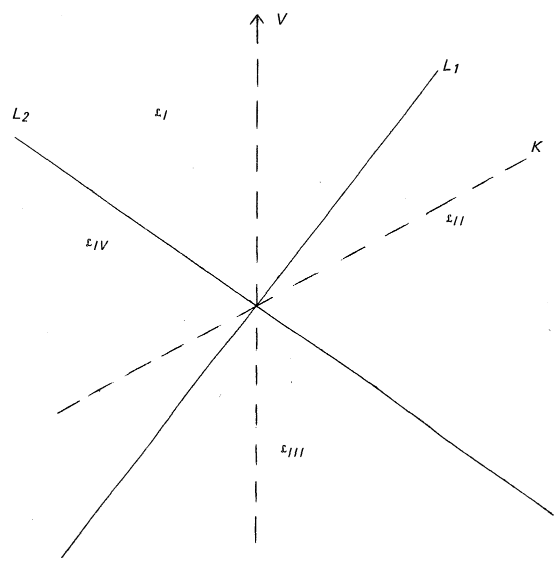

Fig. 2.2 The Principal Axes with V and K

To each point (u,v) in the Fibonacci plane we associate a pair of distances d_1 and d_2 as follows: d_i is the vertical distance (that is, distance parallel to the V-axis) from the line L_i to the point (u,v), measured positively upward, $i = 1,2$. This is illustrated in Fig. 2.3, from which the following equations follow readily:

$$d_1(u,v) = \overline{\frac{u}{\tau} + v}, \tag{2.26}$$

$$d_2(u,v) = \frac{u}{\tau} + v. \tag{2.27}$$

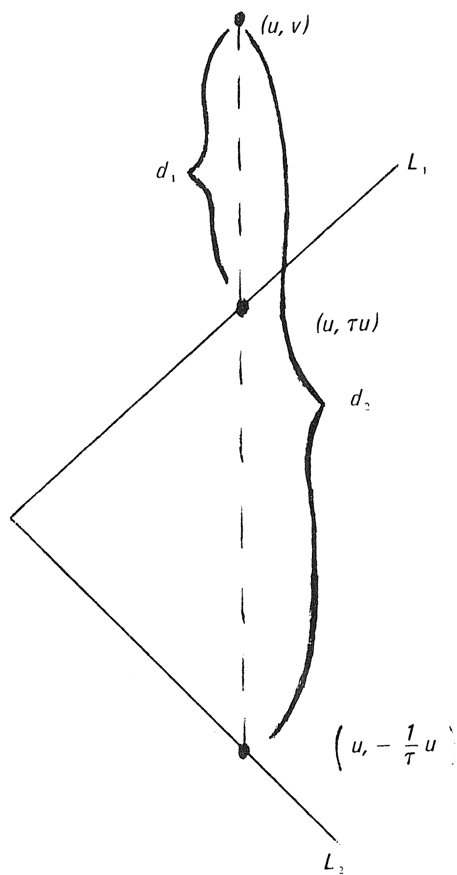

Fig. 2.3 The Distances d_1 and d_2

It is clear that each of the regions $\mathcal{L}_I$, $\mathcal{L}_{II}$, $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$ has a characteristic pair of signs for d_1 and d_2.
Each element $(p/\tau) + q \in Q[\tau]$ has a *norm* given by

$$\nu\left(\frac{p}{\tau} + q\right) = \left(\frac{p}{\tau} + q\right)\left(\overline{\frac{p}{\tau} + q}\right) = q^2 - pq - p^2. \tag{2.28}$$

We see that this norm is a determinant

(2.29)
$$\nu\left(\frac{p}{\tau}+q\right) = \begin{vmatrix} q & p \\ p+q & q \end{vmatrix} .$$

In case
$$\frac{p}{\tau}+q = \underline{\zeta} \circ \underline{\phi}(\underline{x}) = \frac{x_0}{\tau}+x_1 ,$$

$$\nu\left(\frac{x_0}{\tau}+x_1\right) = \begin{vmatrix} x_1 & x_0 \\ x_0+x_1 & x_1 \end{vmatrix} = \begin{vmatrix} x_{-1} & x_0 \\ x_0 & x_1 \end{vmatrix},$$

which is the determinant of the matrix $\psi(\underline{x})$ given by Eq. (2.10).

Equation (2.28) can be extended to the entire Fibonacci plane, giving a quantity

(2.30)
$$\nu(u,v) = v^2 - uv - u^2$$

at each point. $\nu(u,v)$ is an indefinite quadratic form which vanishes precisely on L_1 and L_2. For each non-zero real number ν_0, the graph in the Fibonacci plane of the equation

(2.31)
$$\nu(u,v) = \nu_0$$

is called a *Fibonacci flow,* and ν_0 is the *flow constant.* The flows are rectangular hyperbolas in the Fibonacci plane having L_1 and L_2 for asymptotes. The flows with positive constants lie in the regions $\mathcal{L}_I$ and $\mathcal{L}_{III}$ and the flows with negative constants lie in the regions $\mathcal{L}_{II}$ and $\mathcal{L}_{IV}$.

It is readily verified that each point in the Fibonacci plane lies on the same flow as its conjugate. Figure 2.1 shows that the line K divides the plane into two halves, each of which is set-wise invariant under conjugation. Since the flows with positive constants lie in regions $\mathcal{L}_I$ and $\mathcal{L}_{III}$, it can be seen in Fig. 2.2 that the two branches of these flows always lie on opposite sides of K. It follows that on each such flow, every point and its conjugate lie on the same branch of the flow. Similarly, the V-axis divides the Fibonacci plane into two halves such that every point of either half is sent to the other half by conjugation. Since the two branches of any flow with negative constant always lie on opposite sides of the V-axis, on each of these flows, every point and its conjugate lie on opposite branches of the flow.

Let $\underline{x} \in \underline{X}$. By Eq. (2.22),
$$\tau^n\left(\frac{x_0}{\tau}+x_1\right) = \frac{x_n}{\tau}+x_{n+1} .$$

Taking norms on both sides gives

(2.32)
$$\nu(x_n, x_{n+1}) = (-1)^n \nu(x_0, x_1).$$

It follows that

(2.33)
$$\nu(x_{2n}, x_{2n+1}) = \nu(x_0, x_1)$$

and

(2.34)
$$\nu(x_{2n-1}, x_{2n}) = \nu(x_{-1}, x_0) = -\nu(x_0, x_1)$$

for all integers n. Corresponding to each sequence $\underline{x} \in \underline{X}$ we define a sequence $\xi(\underline{x}) = \left\{\xi_n\right\}_{-\infty}^{\infty}$ in P by

(2.35)
$$\xi_n = (x_{2n}, x_{2n+1}), \quad n \in Z .$$

Then every point of the sequence $\xi(\underline{x})$ lies on the flow

(2.36)
$$\nu(u,v) = \nu(x_0, x_1),$$

and the sequence $\xi(\underline{x})$ is called the *embedding of $\underline{x}$ in the flow (2.36).* The sequence $\underline{x}$ is said to be of type I, II, III or IV according as to whether the point $\xi_0 = (x_0, x_1)$ is in region $\mathcal{L}_I$, $\mathcal{L}_{II}$, $\mathcal{L}_{III}$ or $\mathcal{L}_{IV}$. According to Eqs. (2.26) and (2.27), this depends only on the signs of $(x_0/\tau)+x_1$ and $x_0/\tau + x_1$.

Equations (2.26) and (2.27), in conjunction with Eq. (2.22) give

$$d_1(\xi_n) = d_1(x_{2n}, x_{2n+1}) = \frac{1}{\tau^{2n}}\left(\overline{\frac{x_0}{\tau}+x_1}\right), \tag{2.37}$$

and

$$d_2(\xi_n) = d_2(x_{2n}, x_{2n+1}) = \tau^{2n}\left(\frac{x_0}{\tau}+x_1\right). \tag{2.38}$$

From these equations it is obvious that all points ξ_n of a given embedding lie in the same one of the four regions $\mathfrak{L}_I$, $\mathfrak{L}_{II}$, $\mathfrak{L}_{III}$ and $\mathfrak{L}_{IV}$, and thus on the same branch of a flow. It is also clear that $\xi(\underline{x})$ and $\xi(-\underline{x}) = -\xi(\underline{x})$ lie on opposite branches of the same flow. The strict monotonicity of the right sides of Eqs. (2.37) and (2.38) as functions of n shows that, for sequences in $\underline{X}$ of a given type, all embeddings possess an identical orientation, progressing always from one end of its branch to the other as n increases. This naturally orients the flows to conform with the orientation of the embeddings, as shown in Fig. 2.4. Notice that the flows are so oriented that $\lim (v/u) = \tau$ along the positive sense on every flow.

For certain purposes the study of Fibonacci sequences of type I is sufficient, because a sequence of any other type can be made type I by either negation, shifting, or both.

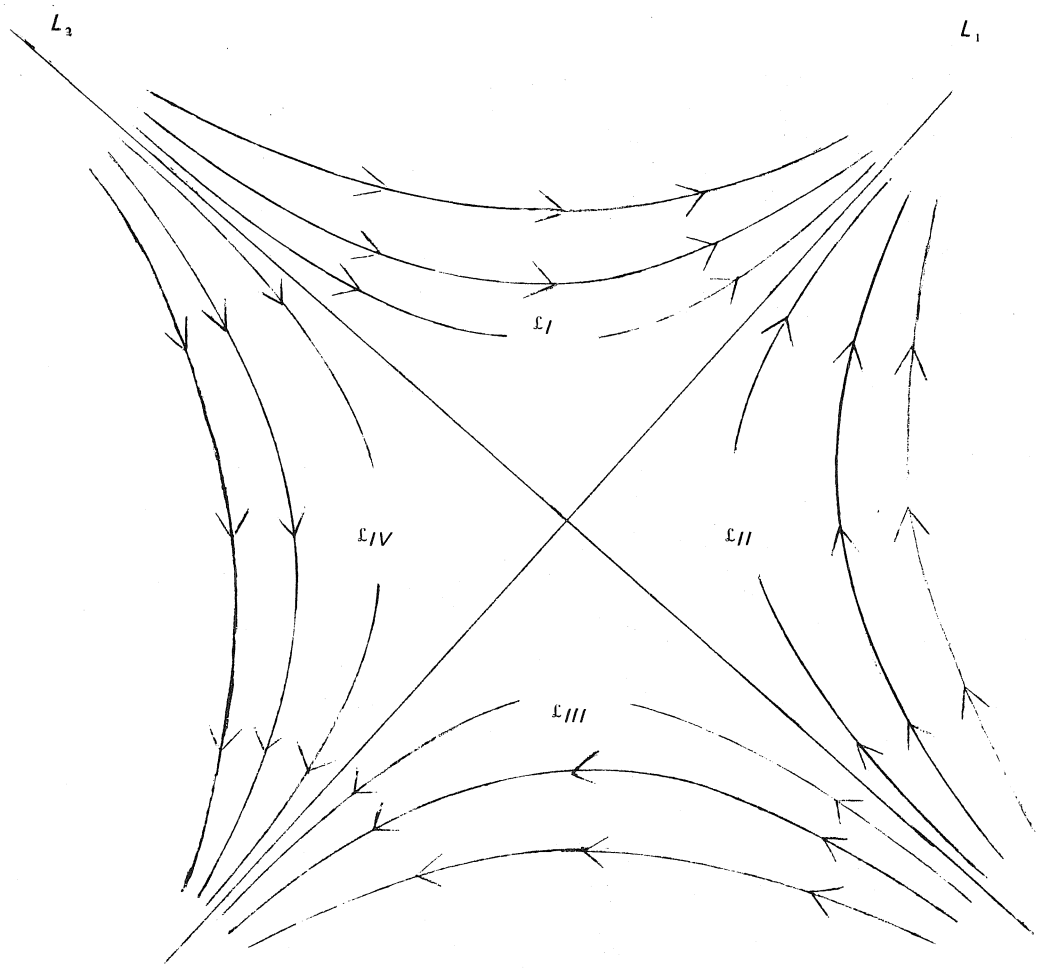

Figure 2.4 Orientation of the Fibonacci Flows

3. FIBONACCI REPRESENTATIONS

3.1. Introduction. In Section 3 we introduce the general and the canonical Fibonacci representations, together with a positional notational system for performing arithmetic with the representations. Basic existence and uniqueness theorems are presented, and the results are interpreted in the contexts of some of the rings discussed in Section 2. An algorithm is given for determining canonical representations, and consideration is given to the relationship of previous results on Fibonacci representations to those given here.

3.2. General Fibonacci Representations, Positional Notation and Arithmetic

Definition 3.1. Given $m \in Z$, a pseudo-polynomial $p(\lambda) = \sum k_i\lambda^i \in Z\langle\lambda\rangle$ is said to *represent* m if $\sum k_if_i = m$. The sum $\sum k_if_i$ is called a *Fibonacci representation of* m, corresponding to the pseudo-polynomial $p(\lambda)$.

Suppose $p(\lambda) \in Z\langle\lambda\rangle$ represents m. By Eq. (2.23) we have

$$p(\tau) = \sum k_i\tau^i = \frac{\sum k_if_i}{\tau} + \sum k_{i-1}f_i. \tag{3.1}$$

This gives the following theorem.

Theorem 3.1. The pseudo-polynomial $p(\lambda)$ represents $m \in Z$ if and only if $p(\tau)$ is of the form $(m/\tau) + n$, $n \in Z$. The integer n is represented by $\lambda p(\lambda)$.

In this way we associate to each Fibonacci representation an element $(m/\tau) + n \in I$, a point $(m,n) \in P$ and an ordered pair (m,n), all said to be *determined by the representation* $\sum k_if_i$.* We note from Eq. (3.1) that $\sum k_{i-1}f_i$ is a representation for n. The representation $\sum k_{i-1}f_i$ is called the *left shift* of the representation $\sum k_if_i$.

We introduce a positional notation for Fibonacci representations by listing the coefficients k_i in the conventional manner from left to right, with a point appearing between the positions corresponding to k_0 and k_{-1}. Because the k_i themselves can contain multiple digits and even minus signs, any coefficient consisting of more than a single digit must be enclosed in parentheses. A minus sign preceding the entire listing is understood to apply to every term in the listing. Thus, for example, 2(11)0.5 represents $2f_2 + 11f_1 + 5f_{-1} = 18$. The left shift of this is 2(11)05. (where the coefficients are left shifted, not the point) which represents $2f_3 + 11f_2 + 5f_0 = 15$. Thus the associated pair determined by this representation is (18, 15).

Since the usual algorithms for addition and multiplication follow from the interpretation of a positional representation as a pseudo-polynomial in a base, and since Theorem 3.1 relates Fibonacci representations to pseudo-polynomials in τ, the standard algorithms, when applied to Fibonacci representations, interpret in terms of the operations on the associated quadratic integers in I determined by the given Fibonacci representations.

For example, what will result from applying the standard multiplication algorithm to 201.1 and 1(−1).01? Formally, we obtain

$$\begin{array}{rrrrrrr}
 & & & 2 & 0 & 1. & 1 \\
 & & & 1 & (-1). & 0 & 1 \\
\hline
 & & & 2 & 0 & 1 & 1 \\
 & (-2) & 0 & (-1) & (-1) & & \\
2 & 0 & 1 & 1 & & & \\
\hline
2 & (-2) & 1 & 2. & (-1) & 1 & 1
\end{array}$$

*It is critical in the sequel to distinguish between that which is *determined* by Σk_if_i and that which is *represented* by Σk_if_i.

On the other hand, the representation 201.1 determines the quadratic integer $(3/\tau) + 5$ and the representation $1(-1).01$ determines the quadratic integer $(0/\tau) + 1 = 1$. The product of these in I is $(3/\tau) + 5$, which is the same as the quadratic integer determined by the result of the foregoing calculation.

Of course, since I is a ring, subtraction of Fibonacci representations is also possible, and again interprets in terms of subtraction of the corresponding quadratic integers.

It is therefore clear that for some purposes Fibonacci representations are best considered as representations of quadratic integers in I, rather than ordinary integers in Z. The usual attitude towards these representations, as reflected in Definition 3.1, does not allow for a full understanding of the arithmetic of the representations.

In the general class of representations under discussion, no restrictions have been placed on the coefficients k_i, other than that they be integers. Consequently, no necessity for carrying and/or borrowing arises in performing the arithmetic. However, in working with canonical representations the necessity does arise, and can be treated by the exchange of an integer k in any particular position for an integer k in each of the two positions immediately to its right. This is justified by the identities

$$kf_i = kf_{i-1} + kf_{i-2} \qquad \text{and} \qquad k\tau^i = k\tau^{i-1} + k\tau^{i-2}.$$

Thus, for example, 21.0 = 3.2 = .53, etc., or, going the other direction, 21.0 = 110.0 = 1000.0, and the equalities here apply not only to the represented integers, but to the associated quadratic integers as well.

Let $\mathcal{Z}$ denote the collection of Fibonacci representations of zero. Members of $\mathcal{Z}$ determine natural integers in I, since by Theorem 3.1 the quadratic integers determined by members of $\mathcal{Z}$ will be of the form $(m/\tau) + n$ with $m = 0$. Therefore, under the usual arithmetical algorithms, *the collection $\mathcal{Z}$ of representations of zero forms a ring on which the left shift is a homomorphism onto the ring* Z *of integers.* Thus, for example, 100.01 is a representation of 0 which determines the quadratic integer $(0/\tau) + 3$, and $-1.$ is a representation of 0 which determines the quadratic integer $(0/\tau) - 1$. The sum and product of these representations will determine $(0/\tau) + 2$ and $(0/\tau) - 3$, respectively.

3.3 Canonical Fibonacci Representations

Definition 3.2. A Fibonacci representation $\sum k_i f_i$ is *positive canonical* if (1) $k_i \neq 0 \Rightarrow k_i = 1$ and (2) $k_i k_{i+1} = 0$ for all i. A Fibonacci representation is *negative canonical* if its negative is positive canonical. A Fibonacci representation is *canonical* if it is either positive canonical or negative canonical.

We agree to write every negative canonical representation (other than all zeroes) with a prefixed minus sign, so that every canonical Fibonacci representation consists of a possible minus sign followed by a finite sequence of ones and zeroes with a point (which may be omitted if it immediately follows the last significant digit) in which no two ones occur consecutively. The representation consisting of all zeroes can be written 0 or .0 and is the only canonical representation which is both positive canonical and negative canonical.

In any canonical Fibonacci representation other than 0, the positions (indices) of the left-most and right-most ones appearing in the representation shall be called the *upper degree* and *lower degree* of the representation, respectively. The upper and lower degree of the representation 0 are defined to be $-\infty$ and $+\infty$, respectively.

Theorem 3.2. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation with sum n. Suppose the upper degree of the representation is negative and let r denote the lower degree. Then the sum n is positive if and only if r is odd, in which case r is the unique negative index j such that $-f_{j+1} < n \leq -f_{j-1}$. Similarly, n is negative if and only if r is even, in which case r is the unique negative index j such that $-f_{j-1} < n \leq -f_{j+1}$.

Proof. From Eq. (2.5) we have that $f_i > 0$ if i is odd and $f_i < 0$ if i is even and not zero. Thus if r is odd, the representation $\sum k_i f_i$ can sum to at most

$$\sum_{\substack{j=r \\ j \text{ odd}}}^{-1} f_j,$$

this being the sum of all the positive terms that could appear in the representation, and none of the negative. But

$$\sum_{\substack{j=r \\ j \text{ odd}}}^{-1} f_j = -f_{r-1}$$

by Eq. (2.5) and standard identities for the Fibonacci numbers. Still assuming that r is odd, the smallest even index that can appear in $\sum k_i f_i$ is $r+3$, so that the representation must sum to at least

$$f_r + \sum_{\substack{j=r+3 \\ j \text{ even}}}^{-2} f_j,$$

since this sum includes all of the negative terms that might appear and excludes all of the positive terms that might not. But

$$\sum_{\substack{j=r+3 \\ j \text{ even}}}^{-2} f_j = -f_{r+2} + 1, \qquad \text{and} \qquad f_r + (-f_{r+2} + 1) = 1 - f_{r+1} > -f_{r+1}$$

and so

(3.2) $$-f_{r+1} < \sum k_i f_i \leqslant -f_{r-1}, \qquad r \text{ odd}.$$

By entirely similar reasoning one shows that

(3.3) $$-f_{r-1} < \sum k_i f_i \leqslant -f_{r+1}, \qquad r \text{ even}.$$

In view of inequalities 3.2 and 3.3, we define, for each negative integer j, an interval I_j by

$$I_j = (-f_{j+1}, -f_{j-1}] \quad \text{for } j \text{ odd},$$
$$I_j = (-f_{j-1}, -f_{j+1}] \quad \text{for } j \text{ even}.$$

Some of these intervals are shown in Fig. 3.1.

j odd		j even	
j	$I_j = (-f_{j+1}, -f_{j-1}]$	j	$I_j = (-f_{j-1}, -f_{j+1}]$
−1	(0,1]	−2	(−2,−1]
−3	(1,3]	−4	(−5,−2]
−5	(3,8]	−6	(−13,−5]
−7	(8,21]	−8	(−34,−13]
−9	(21,55]	−10	(−89,−34]

Fig. 3.1 Some of the Intervals I_j Determined by the Inequalities of Theorem 3.2

As is clear from Fig 3.1, the intervals I_j are pairwise disjoint and their union contains the set of all non-zero integers. Therefore the sum n of $\sum k_i f_i$ falls in the interval I_j if and only if $j = r$, and the theorem is proved.

Alternative proof. The conjugate of $\sum k_i \tau^i$ is $\sum (-1)^i k_i \tau^{-i}$. Using the assumptions of the theorem, we see that

$$\sum k_i\tau^i - \overline{\sum k_i\tau^i}$$

is of the form

$$k_{-1}\left(\frac{1}{\tau}+\tau\right)+k_{-2}\left(\frac{1}{\tau^2}-\tau^2\right)+k_{-3}\left(\frac{1}{\tau^3}+\tau^3\right)+\cdots+k_r\left(\frac{1}{\tau^{-r}}-(-1)^r\tau^{-r}\right),$$

where $k_r = 1$. If r is odd, this is at most

$$\left(\frac{1}{\tau}+\tau\right)+\left(\frac{1}{\tau^3}+\tau^3\right)+\cdots+\left(\frac{1}{\tau^{-r}}+\tau^{-r}\right)$$

and is at least

$$\left(\frac{1}{\tau^2}-\tau^2\right)+\left(\frac{1}{\tau^4}-\tau^4\right)+\cdots+\left(\frac{1}{\tau^{3-r}}-\tau^{3-r}\right)+\left(\frac{1}{\tau^{-r}}+\tau^{-r}\right).$$

By performing the obvious summations and simplifications, and by employing a similar argument when r is even, we get

$$\tau^{-r-1}-\frac{1}{\tau^{-r-1}}+\tau+\frac{1}{\tau} \leqslant \sum k_if_i - \overline{\sum k_if_i} \leqslant \tau^{-r+1}-\frac{1}{\tau^{-r+1}},\quad r \text{ odd},$$

and

$$-\tau^{-r+1}-\frac{1}{\tau^{-r+1}}+\tau+\frac{1}{\tau} \leqslant \sum k_if_i - \overline{\sum k_if_i} \leqslant -\tau^{-r-1}-\frac{1}{\tau^{-r-1}},\quad r \text{ even}.$$

Inequalities (3.2) and (3.3) now follow from these by using Eqs. (2.23) and (2.20).

For each non-zero integer n, define $r(n)$ to be the unique negative index j such that $n \in I_j$.

Theorem 3.3. If n is a non-zero integer then either $n = f_{r(n)}$ or else $r(n - f_{r(n)}) \geqslant r(n)+2$.

Proof. Suppose that $n \neq f_{r(n)}$, so that $r(n) < -2$. For $r(n)$ odd, it follows from the definition of $r(n)$ that

$$-f_{r(n)+1} < n \leqslant -f_{r(n)-1}.$$

Subtracting $f_{r(n)}$ throughout gives

$$-f_{r(n)+2} < n - f_{r(n)} \leqslant -f_{r(n)+1}.$$

As can be seen from Fig. 3.1, if k is an odd, negative index, then

$$(-f_{k+2}, -f_{k+1}] = (-f_{k+2}, -1] \cup (-1, 0] \cup (0, -f_{k+1}] = \left(\bigcup_{\substack{j=k+3\\ j \text{ even}}}^{-2} I_j\right) \cup (-1, 0] \cup \left(\bigcup_{\substack{j=k+2\\ j \text{ odd}}}^{-1} I_j\right).$$

From the definition of r, necessarily

$$r(n - f_{r(n)}) \geqslant r(n)+2.$$

Similar reasoning applies in case $r(n)$ is even.

Theorem 3.4. Every integer n, positive, negative or zero has a unique positive canonical Fibonacci representation with negative upper degree. For $n = 0$ the representation is .0. For $n \neq 0$ the representation is $f_{j_s} + f_{j_{s-1}} + \cdots + f_{j_1}$ in which $j_1 = r(n)$,

$$j_i = r\left(n - \sum_{p=1}^{i-1} f_{j_p}\right)$$

for $1 < i \leqslant s$ and s is the first positive integer such that

$$n - \sum_{p=1}^{s} f_{j_p} = 0.$$

Proof. Certainly .0 represents the number zero canonically, and has negative upper degree $-\infty$. According to Theorem 3.2, any other positive canonical Fibonacci representation with negative upper degree cannot represent zero. If $n \neq 0$, define, as in the statement of the theorem, $j_1 = r(n)$ and

$$j_i = r\left(n - \sum_{p=1}^{i-1} f_{j_p}\right)$$

whenever $i > 1$ and

$$n - \sum_{p=1}^{i-1} f_{j_p} \neq 0.$$

According to Theorem 3.3, $j_i \geqslant j_{i-1} + 2$ for each i, and since all j_i are negative, the process must terminate after finitely many steps. The only way for this to happen is for

$$n - \sum_{p=1}^{s} f_{j_p} = 0$$

for some s.

This establishes the existence of the representation. If

$$\sum k_i f_i \quad \text{and} \quad \sum k_i' f_i$$

are two positive canonical representations for n both having negative upper degree, Theorem 3.2 states that they have equal lower degree. If the f_i of least index is subtracted from both representations, the results are still positive canonical, still of negative upper degree and still equal, so Theorem 3.2 applies again. Continued application of this argument proves the representations to be identical.

We note that the assumption of negative upper degree is essential to Theorem 3.4. For example, the sum $f_n + f_{-n}$ is a positive canonical representation of 0 for every even integer n.

Define a strip S in the Fibonacci plane to consist of all of those points (u,v) for which $0 \leqslant d_2 < 1$. Based on simple geometry, one readily concludes that for a quadratic integer $(m/\tau) + n \in I$,

(3.4) $$(m,n) \in S \quad \text{if and only if} \quad n = -\left[\left|\frac{m}{\tau}\right|\right],$$

where $[|\ |]$ denotes the greatest integer function.

Theorem 3.5. A quadratic integer $(m/\tau) + n \in I$ is determined by a positive canonical Fibonacci representation with negative upper degree if and only if $(m,n) \in S$, i.e., if and only if $(m/\tau) + n \in [0,1)$. In this case, the positive canonical representation with negative upper degree is unique.

Proof. Suppose $(m/\tau) + n$ is so determined. Then there exists a positive canonical Fibonacci representation $\sum k_i f_i$ of negative upper degree such that $(m/\tau) + n = \sum k_i \tau^i$. Clearly

$$0 \leqslant \sum k_i \tau^i < \tau^{-1} + \tau^{-3} + \tau^{-5} + \cdots = 1, \quad \text{so } (m,n) \in S.$$

Suppose $(m,n) \in S$. By Theorem 3.4 there exists for m a positive canonical Fibonacci representation $\sum k_i f_i$ with negative upper degree. By the first half of this proof, if $(m/\tau) + n'$ is the quadratic integer determined by this representation, then $(m, n') \in S$. In view of condition 3.4,

$$n' = n = -\left[\left|\frac{m}{\tau}\right|\right]$$

so that $\sum k_i f_i$ determines $(m/\tau) + n$.

The uniqueness follows from the uniqueness of the representation of m as asserted in Theorem 3.4.

Theorem 3.6. For each quadratic integer $(m/\tau) + n \in I$ there is one and only one canonical Fibonacci representation $\sum k_i f_i$ which determines $(m/\tau) + n$. Points in regions $\mathcal{L}_I$ and $\mathcal{L}_{II}$ correspond to positive canonical representations and points in $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$ correspond to negative canonical representations.

Proof. Since the negative of any point (m,n) in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ is in $\mathcal{L}_{III}$ or $\mathcal{L}_{IV}$ and *vice versa,* it is sufficient to prove existence and uniqueness of canonical representations determined by 0 and by points in L_I and L_{II}, showing that the latter are necessarily positive canonical.

Let $\sum k_i f_i$ be a canonical Fibonacci representation other than 0. Theorem 3.5 assures that $\sum k_i f_i$ cannot determine 0 if the representation is positive canonical with negative upper degree. But with the proper choice of sign and exponent p, $\pm \sum k_i f_{i-p}$ is positive canonical with negative upper degree and determines $\pm \dfrac{1}{\tau^p}\sum k_i \tau^i \neq 0$, so that $\sum k_i \tau^i \neq 0$. Thus the only canonical Fibonacci representation determining zero is 0.

Let $(m/\tau) + n$ be a point in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ (in the sense that (m,n) is in $\mathcal{L}_I$ or $\mathcal{L}_{II}$) other than zero. Then $d_2(m,n) > 0$ and

$$\lim_{s} d_2\left(\frac{1}{\tau^s}\left(\frac{m}{\tau}+n\right)\right) = 0.$$

Thus for some sufficiently large s, say $s = s_0$,

$$\frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right) \in S \quad \text{and so} \quad \frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right)$$

is determined by a positive canonical Fibonacci representation $\sum k_i f_i$ with negative upper degree. Thus

$$\frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right) = \sum k_i \tau^i, \quad \text{so that} \quad \frac{m}{\tau}+n = \sum k_i \tau^{i+s_0},$$

showing that $(m/\tau) + n$ is determined by the positive canonical representation $\sum k_i f_{i+s_0}$. If $\sum k_i f_i$ and $\sum k_i' f_i$ are positive canonical representations determining $(m/\tau) + n$, then for some integer t_0, $\sum k_i f_{i-t_0}$ and $\sum k_i' f_{i-t_0}$ both have negative upper degree and both determine

$$\frac{1}{\tau^{t_0}}\left(\frac{m}{\tau}+n\right)$$

so by Theorem 3.5 are identical. Hence $\sum k_i f_i$ and $\sum k_i' f_i$ are also identical. Finally, no point in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ other than (0,0) can be determined by a negative canonical representation, since for such a representation

$$d_2 = \sum k_i \tau^i < 0.$$

For each positive real number u, define $s(u)$ to be the largest integer exponent i such that $\tau^i \leqslant u$. Given $u > 0$, let $i_1 = s(u)$, $i_2 = s(u - \tau^{i_1})$ and in general

$$i_j = s\left(u - \sum_{p=1}^{j-1} \tau^{i_p}\right) \quad \text{so long as} \quad u - \sum_{p=1}^{j-1} \tau^{i_p} \neq 0.$$

The sequence $i_1, i_2, \cdots$ terminates at any j such that

$$u - \sum_{p=1}^{j} \tau^{i_p} = 0;$$

otherwise it continues indefinitely.

Definition 3.3. For a given positive real number u, let $i_1, i_2, \cdots$ be the sequence determined above. The sum

$$\sum_p \tau^{i_p}$$

is called the *τ-expansion* of u. If u is negative, the τ-expansion of u is that of $-u$, preceded by a minus sign. The τ-expansion of zero is simply 0.

Theorem 3.7. Let u be a real number with τ-expansion

$$\sum_p \tau^{i_p}.$$

Then $\{i_p\}$ is a (decreasing) sequence in which $i_{p+1} \leqslant i_p - 2$ for each p, and the expansion $\sum_p \tau^{i_p}$ sums to u. Conversely, let u be a real number such that

$$|u| = \sum_j \tau^{i_j}$$

in which $\{i_j\}$ is a (decreasing) sequence in which $i_{j+1} \leqslant i_j - 2$ for each j. If $\{i_j\}$ is not ultimately regular of the form $\cdots, J, J-2, J-4, J-6, \cdots$ then

$$\sum_j \tau^{i_j}$$

is the τ-expansion of $|u|$. If $\{i_j\}$ is ultimately regular of the form $\cdots, J, J-2, J-4, J-6, \cdots$ let j_0 be the index such that $i_{j_0} = J$, $i_{j_0+1} = J-2$, etc., and $i_{j_0-1} > J+2$. Then

$$\sum_j \tau^{i_j} = \sum_{j=1}^{j_0-1} \tau^{i_j} + \tau^{J+1},$$

and the right-hand side of this equation is the τ-expansion of $|u|$.

Proof. The sequence $\{i_p\}$ is decreasing by construction. If two integers i_p are consecutive, say $i_{p_0} = n$ and $i_{p_0+1} = n-1$, then

$$s\left(u - \sum_{p=1}^{p_0} \tau^{i_p}\right) = n-1 \quad \text{and} \quad \tau^{n-1} \leqslant u - \sum_{p=1}^{p_0} \tau^{i_p}.$$

Adding $\tau^{i_{p_0}} = \tau^n$ to both sides gives

$$\tau^{n-1} + \tau^n \leqslant u - \sum_{p=1}^{p_0-1} \tau^{i_p}.$$

Because $\tau^{n-1}+\tau^n=\tau^{n+1}$, this gives

$$\tau^{n+1} \leqslant u-\sum_{p=1}^{p_0-1} \tau^{i_p}$$

contradicting the definition of

$$n = i_{p_0} = s\left(u-\sum_{p=1}^{p_0-1} \tau^{i_p}\right) .$$

Therefore, no two i_p can be consecutive, so that $i_{p+1} \leqslant i_p - 2$ for each p. Thus for each j,

$$0 \leqslant u-\sum_{p=1}^{j-1} \tau^{i_p} \leqslant \tau^{i_j}.$$

Either

$$u = \sum_{p=1}^{j-1} \tau^{i_p}$$

for some j, or else the sequence $\left\{i_j\right\}$ decreases to $-\infty$; in either case $\sum_p \tau^{i_p} = u$.

Now suppose that

$$|u| = \sum_j \tau^{i_j}$$

in which $i_{j+1} \leqslant i_j - 2$ for each j. Then for each n,

$$\sum_{j>n} \tau^{i_j} \leqslant \tau^{i_n-2}+\tau^{i_n-4}+\tau^{i_n-6}+\cdots$$

and the latter is a geometric series with sum τ^{i_n-1}. Thus if the sequence $\left\{i_j\right\}$ is not ultimately regular as stated in the theorem, for each n,

$$\sum_{j>n} \tau^{i_j} < \tau^{i_n-1}. \quad \text{Therefore} \quad \sum_{j\geqslant n} \tau^{i_j} < \tau^{i_n}+\tau^{i_n-1} = \tau^{i_n+1} \quad \text{so} \quad s\left(\sum_{j\geqslant n}\tau^{i_j}\right) = i_n .$$

It now follows by induction that

$$\sum_j \tau^{i_j}$$

is the τ-expansion of $|u|$.

If

$$\sum_j \tau^{i_j}$$

is ultimately regular as stated in the theorem, because $\tau^J+\tau^{J-2}+\tau^{J-4}+\cdots$ is a geometric sum equal to τ^{J+1}, the sum

$$\sum_j \tau^{i_j} \quad \text{is equal to} \quad \sum_{j=1}^{j_0-1} \tau^{i_j}+\tau^{J+1},$$

and because $i_{j_0-1} > J+2$, this latter sum must be the τ-expansion of $|u|$, by the part of the theorem already proved.

Theorem 3.8. The τ-expansion of a real number u is finite if and only if u is a quadratic integer in I. In this case the τ-expansion of u is identical to the pseudo-polynomial in τ determined by the canonical Fibonacci representation of Theorem 3.6. (A generalization of this result appears in [8].)

Proof. On the one hand, Eq. (2.23) assures that any finite τ-expansion sums to a quadratic integer in I. On the other hand, the pseudo-polynomial in τ determined by the Fibonacci representation of Theorem 3.6 satisfies the conditions of Theorem 3.7 and must therefore be the τ-expansion of u.

Theorem 3.9. The usual ordering on the real numbers is identical to the lexicographic ordering on their τ-expansions.

Corollary. The lexicographic ordering on the canonical Fibonacci representations coincides with the usual real ordering on the quadratic integers they determine.

We omit the proof of Theorem 3.9 because the proof is straightforward and the theorem is of a standard type.

Canonical Fibonacci representations with negative upper degree are of interest because of their existence and uniqueness properties (Theorem 3.4) and because their consideration leads to a general existence and uniqueness theorem for canonical Fibonacci representations (Theorem 3.6). Further study of the significance of the upper and lower degrees of canonical Fibonacci representations leads to additional existence and uniqueness theorems, and relates to the Fibonacci representations in the literature.

Theorem 3.10. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau) + n$. Then $\sum k_i f_i$ has lower degree r if and only if $\overline{(m/\tau)+n} \in J_r$, where

$$J_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right)$$

if r is an odd integer and

$$J_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right)$$

if r is an even integer.

Proof. Let r be the lower degree of $\sum k_i f_i$, so that

$$\overline{\frac{m}{\tau}+n} = (-1)^r\left(\frac{1}{\tau^r} - \frac{k_{r+1}}{\tau^{r+1}} + \frac{k_{r+2}}{\tau^{r+2}} - \cdots\right).$$

If r is odd, this expression is strictly greater than

$$-\left(\frac{1}{\tau^r} + \frac{1}{\tau^{r+2}} + \frac{1}{\tau^{r+4}} + \cdots\right) = -\frac{1}{\tau^{r-1}}$$

and is strictly less than

$$-\frac{1}{\tau^r} + \frac{1}{\tau^{r+3}} + \frac{1}{\tau^{r+5}} + \cdots = -\frac{1}{\tau^r} + \frac{1}{\tau^{r+2}} = -\frac{1}{\tau^{r+1}}.$$

If r is even, the limits are similarly found to be

$$\frac{1}{\tau^{r+1}} \quad \text{and} \quad \frac{1}{\tau^{r-1}}.$$

Therefore, for each integer r define J_r to be

$$\left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right) \text{ if } r \text{ is odd,} \quad \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right) \text{ if } r \text{ is even.}$$

Some of these intervals are shown in Fig. 3.2.

As can be seen in Fig. 3.2, the intervals J_r are pairwise disjoint and cover all real numbers except 0 and those of the form $-\tau^n$, n even and τ^n, n odd. We note that none of these numbers can be the conjugate of a positive canonical Fibonacci representation different from 0 since their conjugates are negative canonical.

r	r odd: $J_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right)$	r	r even: $J_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right)$
-7	$(-\tau^8, -\tau^6)$	-6	(τ^5, τ^7)
-5	$(-\tau^6, -\tau^4)$	-4	(τ^3, τ^5)
-3	$(-\tau^4, -\tau^2)$	-2	(τ, τ^3)
-1	$(-\tau^2, -1)$	0	$\left(\frac{1}{\tau}, \tau\right)$
1	$\left(-1, -\frac{1}{\tau^2}\right)$	2	$\left(\frac{1}{\tau^3}, \frac{1}{\tau}\right)$
3	$\left(-\frac{1}{\tau^2}, -\frac{1}{\tau^4}\right)$	4	$\left(\frac{1}{\tau^5}, \frac{1}{\tau^3}\right)$
5	$\left(-\frac{1}{\tau^4}, -\frac{1}{\tau^6}\right)$	6	$\left(\frac{1}{\tau^7}, \frac{1}{\tau^5}\right)$

Fig. 3.2 Some of the Intervals J_r Determined by Theorem 3.10

Thus we can see that if the lower degree of the representation is r then $\overline{(m/\tau) + n} \in J_r$. If, on the other hand, $\overline{(m/\tau) + n} \in J_r$, the lower degree cannot be other than r because $J_s \cap J_r = \phi$ for $s \neq r$.

The previous theorem has a companion theorem whose proof is omitted for obvious reasons.

Theorem 3.11. Let $\sum k_i f_i$ be a negative canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau) + n$. Then $\sum k_i f_i$ has lower degree r if and only if $\overline{(m/\tau) + n} \in J'_r$, where

$$J'_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right) \quad \text{for } r \text{ odd}, \quad J'_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right) \quad \text{for } r \text{ even}.$$

Here are two more theorems whose straightforward proofs are omitted.

Theorem 3.12. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau) + n$. Then $\sum k_i f_i$ has upper degree p if and only if $(m/\tau) + n \in K_p$, where $K_p = [\tau^p, \tau^{p+1})$ for each integer p.

Theorem 3.13. Let $\sum k_i f_i$ be a negative canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau) + n$. Then $\sum k_i f_i$ has upper degree p if and only if $(m/\tau) + n \in K'_p$, where $K'_p = (-\tau^{p+1}, -\tau^p]$ for each integer p.

According to Theorem 3.6, the canonical Fibonacci representation which determines $(m/\tau) + n$ is positive canonical if and only if $(m,n) \in \pounds_I \cup \pounds_{II}$ and is negative canonical if and only if $(m,n) \in \pounds_{III} \cup \pounds_{IV}$. In consideration of the case $r = 0$ in Theorems 3.10 and 3.11, we define subsets T_1 and T_2 of the Fibonacci plane as follows:

$$T_1 = \left\{(u,v) \in P : \frac{1}{\tau} < d_1(u,v) < \tau\right\} \cap (\pounds_I \cup \pounds_{II}),$$

and

$$T_2 = \left\{(u,v) \in P : -\tau < d_1(u,v) < -\frac{1}{\tau}\right\} \cap (\pounds_{III} \cup \pounds_{IV}).$$

T_1 and T_2 are shown in Fig. 3.3, and each is seen to be a half-strip with vertical thickness one. Using this fact and the fact that neither the boundary of T_1 nor that of T_2 can contain a point (m,n) with both m and n integers, one can conclude that for every integer $m \neq 0$ there exists a unique integer n such that $(m,n) \in T_1 \cup T_2$.

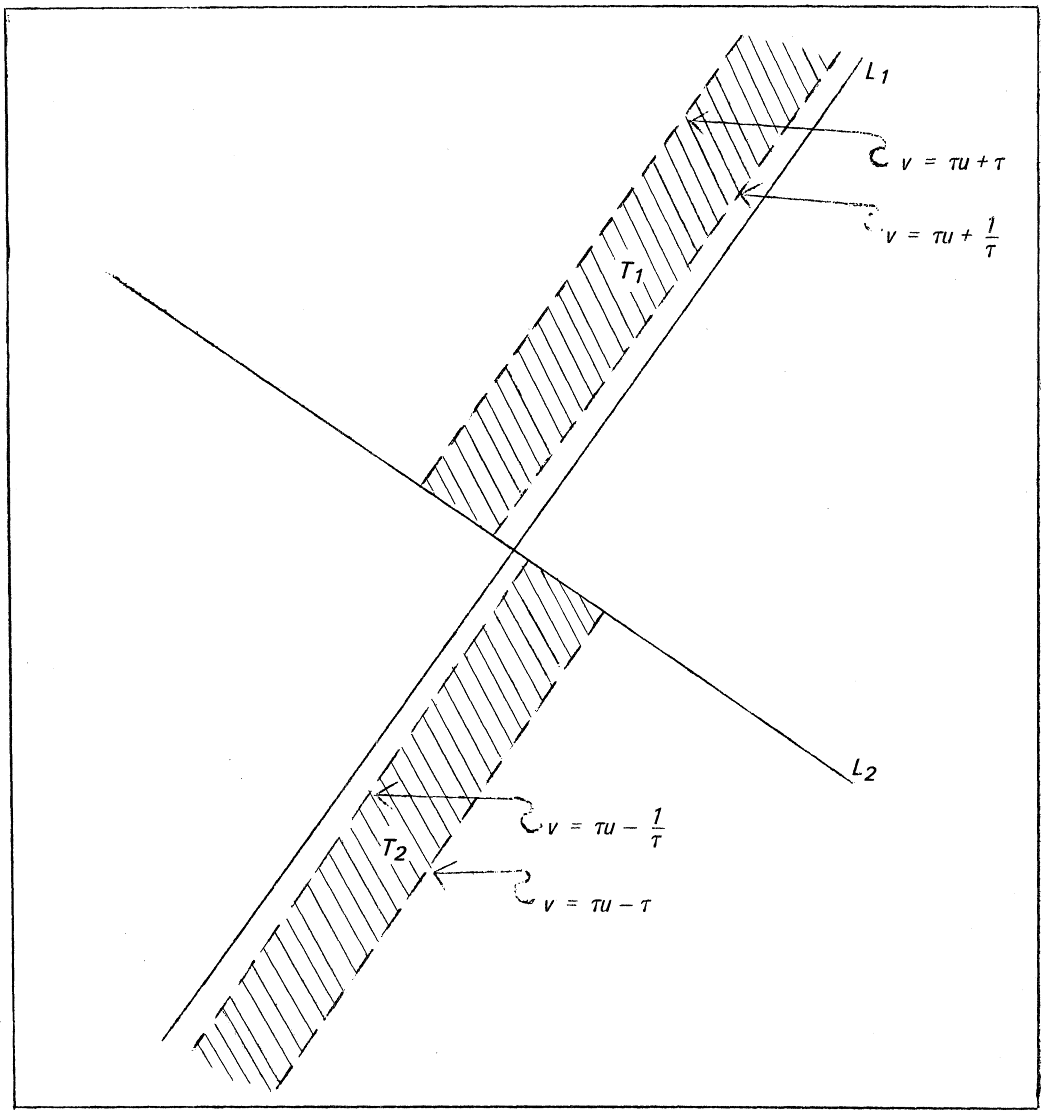

Fig. 3.3 The Regions T and T_2

More precisely, if $(m/\tau) + n \in I$, then

$$(m,n) \in T_1 \quad \text{if and only if} \quad m \geqslant 0 \quad \text{and} \quad n = [\![(1+m)\tau]\!], \tag{3.5}$$

and

$$(m,n) \in T_2 \quad \text{if and only if} \quad m \leqslant 0 \quad \text{and} \quad n = -[\![(1-m)\tau]\!] . \tag{3.6}$$

We thus obtain the following theorem.

Theorem 3.14. Every non-zero integer m has a unique canonical Fibonacci representation with lower degree equal to 0. If $m > 0$ this representation is positive canonical, if $m < 0$ this representation is negative canonical. The integer 0 has exactly two canonical Fibonacci representations with lower degree 0; they are $\pm f_0$.

As an illustration of Theorem 3.14 we note that if $m = -6$ condition 3.6 gives $n = -11$. The canonical Fibonacci representation which determines $(-6/\tau) - 11$ is -100101. (Section 3.4 describes an algorithm by which one can determine the canonical representation -100101. from the quadratic integer $-(6/\tau) - 11$.)

The critical point in the proof of Theorem 3.14 is the selection of the set $T_1 \cup T_2$ in such a way that for each m there is one and only one n for which $(m,n) \in T_1 \cup T_2$ (which, incidentally, failed for $m = 0$). This depended on the fact that the width of the interval J_0 was one, so Theorem 3.14 would fail for other choices of r. If $r < 0$ the intervals are too wide, so that one could prove existence but not uniqueness, whereas if $r > 0$ the intervals are too narrow, so that one could prove uniqueness but not existence.

However, anytime a set such as $T_1 \cup T_2$ can be found, having the property that for each integer m there is exactly one integer n such that (m,n) is in the set, a new theorem like Theorem 3.14 or Theorem 3.4 results. We shall dignify this observation by a definition after which we shall show how the usual Fibonacci representations in the literature result as special cases.

Definition 3.4. Let S be a non-empty subset of Z. A subset U of the Fibonacci plane is said to be *selective on S* if for each $m \in S$ there exists one and only one $n \in Z$ such that $(m,n) \in U$. For each $m \in S$ the canonical Fibonacci representation which determines $(m/\tau) + n$, $(m,n) \in U$, shall be called the *U-representation of m.*

For example, let

$$J = \bigcup_{r=2}^{\infty} J_r, \quad J' = \bigcup_{r=2}^{\infty} J'_r, \quad U_1 = \left\{ (u,v) \in P : \overline{\frac{u}{\tau} + v} \in J \right\} \cap (\mathcal{L}_I \cup \mathcal{L}_{II}),$$

and

$$U_2 = \left\{ (u,v) \in P : \frac{u}{\tau} + v \in J' \right\} \cap (\mathcal{L}_{III} \cup \mathcal{L}_{IV}).$$

Theorem 3.15. $U_1 \cup U_2$ is selective on the set Z^* of non-zero integers.

Proof. We note that each of U_1 and U_2 is a strip on either side of L_1 with a sequence of lines removed, since U_1 consists of those points (u,v) in $\mathcal{L}_I \cup \mathcal{L}_{II}$ for which $-(1/\tau^2) < d_1(u,v) < (1/\tau)$ but $d_1(u,v) \neq (1/\tau^k)$ for k odd positive, $d_1(u,v) \neq -(1/\tau^k)$ for k even positive and $d_1(u,v) \neq 0$, and U_2 is of similar structure. The vertical thickness of each of the strips U_1 and U_2 is 1. Moreover, none of the missing half-lines can contain (m,n) in $Z^* \times Z$, as can be seen from the following type of argument. If m and n are integers and $d_1(m,n) = (1/\tau^k)$, k odd and positive, then $\overline{(m/\tau) + n} = (1/\tau^k)$ so that $(m/\tau) + n = -\tau^k = -(f_k/\tau) - f_{k+1}$ by Eq. (2.23). Thus $(m,n) = (-f_k, -f_{k+1})$ which is not in $\mathcal{L}_I \cup \mathcal{L}_{II}$. Other cases follow similarly.

It is therefore clear that U_1 is selective on the positive integers and U_2 is selective on the negative integers. We leave it to the reader to show that there is no possibility of the union $U_1 \cup U_2$ failing to be selective near the origin as a result of vertical overlap.

It is not difficult to show that if $(m/\tau) + n \in I$, then for $m > 0$

$$(3.7) \qquad (m,n) \in U_1 \cup U_2 \quad \text{if and only if} \quad n = [|(1+m)\tau|] - 1$$

and for $m < 0$,

$$(3.8) \qquad (m,n) \in U_1 \cup U_2 \quad \text{if and only if} \quad n = -[|(1-m)\tau|] + 1.$$

Theorem 3.15 has the following corollary.

Corollary. Every non-zero integer has a unique canonical Fibonacci representation with lower degree greater than 1. For 0, no such representation exists.

Of course these are the well known *Zeckendorff representations* which have been extensively treated in various contexts [1, 2, 3, 4, 5, 6, 11]. Because their properties are well known we shall not discuss them further at this point. We note in passing that this corollary follows immediately from Theorem 3.14 by the removal of the f_0 term in each of the canonical representations of lower degree zero.

Other choices of selective sets produce other interesting classes of representations. We describe some of them by way of the following theorems whose proofs are omitted in the interest of brevity.

Theorem 3.16. Every non-zero integer has a unique canonical Fibonacci representation with positive, odd lower degree. For each non-zero integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.9) $$n = [|m\tau|] \quad \text{if } m > 0,$$

(3.10) $$n = -[|-m\tau|] \quad \text{if } m < 0.$$

These are the so-called *second-canonical representations* appearing in [5].

Theorem 3.17. Every integer m has a unique positive canonical Fibonacci representation with upper degree 1. For each integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.11) $$n = -\left[\left|\frac{m-1}{\tau}\right|\right] + 1.$$

As examples, we note that for $m = -7$, $n = 6$ and $(-7/\tau) + 6$ is determined by 10.000001; for $m = 0$, $n = 2$ and $(0/\tau) + 2$ is determined by 10.01.

Theorem 3.17 has an obvious counterpart in terms of negative canonical Fibonacci representations which we shall omit.

Theorem 3.18. Every integer m has a unique positive canonical Fibonacci representation with upper degree either 0 or -1. For each integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.12) $$n = -\left[\left|\frac{m-1}{\tau}\right|\right].$$

Once again the theorem has a counterpart in terms of negative canonical representations.

The theorems we have listed here give the consequences of some of the most obvious choices of the selective sets. It is clear that many other possibilities exist and that the general selective set does not necessarily relate to upper and lower representational degrees.

We conclude this section with an interesting decomposition theorem which is an immediate consequence of the foregoing results on canonical Fibonacci representations. The theorem is stated only for the half-plane $\mathfrak{L}_I \cup \mathfrak{L}_{II}$, but has at least one obvious extension to the entire Fibonacci plane.

Given any lattice point $(m,n) \in \mathfrak{L}_I \cup \mathfrak{L}_{II}$, the quadratic integer $(m/\tau) + n$ is determined by precisely one positive canonical Fibonacci representation $\sum k_i f_i$. This representation naturally decomposes into the sum of terms with nonnegative indices and the sum of terms with negative indices; in positional notation this corresponds to the portion to the left of the point and the portion to the right of the point.

With only one restriction, any positive canonical Fibonacci representation with nonnegative lower degree can be added to any positive canonical Fibonacci representation with negative upper degree to yield a positive canonical Fibonacci representation; the exception is of course the case of zero lower degree and -1 upper degree.

If we consult Theorems 3.10 and 3.12, we find that the positive canonical Fibonacci representation which determines a lattice point $(m,n) \in \mathfrak{L}_I \cup \mathfrak{L}_{II}$:

has nonnegative lower degree if and only if $-1 < d_1(m,n) < \tau$,

has lower degree zero if and only if $(1/\tau) < d_1(m,n) < \tau$,

has negative upper degree if and only if $0 < d_2(m,n) < 1$,

and

has upper degree $= -1$ if and only if $(1/\tau) \leq d_2(m,n) < 1$.

Therefore, let U_1 denote the semi-strip

$$U_1 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : -1 < d_1(u,v) < \tau \right\},$$

let U'_1 denote the sub-semi-strip

$$U'_1 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : (1/\tau) < d_1(u,v) < \tau \right\},$$

let U_2 denote the semi-strip

$$U_2 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : 0 < d_2(u,v) < 1 \right\},$$

and let U_2' denote the sub-semi-strip

$$U_2' = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : (1/\tau) \leqslant d_2(u,v) < 1 \right\} .$$

Then we have the following theorem.

Theorem 3.19. Every integer pair $(m,n) \in \mathfrak{L}_I \cup \mathfrak{L}_{II}$ can be decomposed into the sum of an integer pair in U_1 and an integer pair in U_2. This decomposition continues to exist and becomes unique in the presence of the restriction that the summands not lie one in U_1' and the other in U_2'.

3.4 The Resolution Algorithm

We have seen that every Fibonacci representation determines a quadratic integer which in turn is determined by a unique canonical Fibonacci representation. In this section we present an algorithm for passing from any Fibonacci representation to the canonical representation determining the same quadratic integer; we call it the *resolution algorithm.*

Let W be the class of Fibonacci representations $\sum k_i f_i$ in which $k_i \geqslant 0$ for all i. We begin by defining the algorithm and proving its convergence on W.

Given a Fibonacci representation $\sum k_i f_i$, a pair (k_i, k_{i-1}) of consecutive coefficients shall be called a *significant pair* if it is not of the form (1,0) or $(0,n)$. It is clear that a Fibonacci representation in W fails to be canonical if and only if it contains a significant pair. In any non-canonical representation the significant pair (k_i, k_{i-1}) with largest index i is called the *first significant pair.*

On the class W the resolution algorithm consists of the repetition of the following operation Ω on the first significant pair: (i) if both members of the pair (k_i, k_{i-1}) are positive, replace k_{i+1} by $k_{i+1}+k$, replace k_i by $k_i - k$ and replace k_{i-1} by $k_{i-1}-k$, where k is any integer satisfying $0 < k \leqslant \min \left\{ k_i, k_{i-1} \right\}$; (ii) if one member of the pair is zero (it must be k_{i-1} since the pair is significant) replace k_i by $k_i - j$, k_{i-1} by j and k_{i-2} by $k_{i-2}+j$ where j is any integer satisfying $0 < j < k_i$, and then immediately apply (i) to the new first significant pair $(k_i - j, j)$ obtaining $k_{i+1}+k$, $k_i - j - k$, $j - k$ and $k_{i-2}+j$ as the final replacements for k_{i+1}, k_i, k_{i-1} and k_{i-2}, respectively, where, as required in (i), k is any integer satisfying $0 < k \leqslant \min \left\{ k_i - j, j \right\}$.

As explained in Section 3.2, operations (i) and (ii) will not alter the quadratic integer determined by the representation. A convenient choice for k in operation (i) is the largest, i.e., $k = \min \left\{ k_i, k_{i-1} \right\}$ and a convenient choice for j in operation (ii) is the smallest, i.e., $j = 1$, which changes k_{i+1}, k_i, k_{i-1} and k_{i-2} to $k_{i+1}+1$, $k_i - 2$, 0 and $k_{i-1}+1$, respectively. The reader will discover that these convenient choices are not necessarily the most efficient, but we shall not be concerned with that problem at this time. We establish the convergence of this algorithm after looking at two brief examples.

Example 1. Find the canonical Fibonacci representation which determines the quadratic integer $(3/\tau) + 2$. As a pseudo-polynomial in τ this has positional notation 2.3. Applying operations (i) and (ii) as required and using the choices for j and k suggested as convenient, we obtain

$$2.3 = 20.1 = 100.2 = 101.001.$$

Example 2. Determine the canonical representation of 6 given by Theorem 3.16. Since $[|6\tau|] = 9$, we form $(6/\tau) + 9$ and obtain

$$9.6 = 63. = 330. = 3000. = 11010. = 100010.$$

Consider a Fibonacci representation $\sum k_i f_i$ in W and let W_0 be the subset of those representations in W which determine the same quadratic integer as does $\sum k_i f_i$. Order W_0 lexicographically on the positional notations of its members (with points aligned) and observe that the operator Ω sends any non-canonical member of W_0 to another element of W_0 which is strictly greater in the lexicographic ordering.

Now for any integer r the number of representations in W_0 having lower degree greater than or equal to r is finite. For if K is an integer such that $\tau^K > \sum k_i \tau^i$ and if N is an integer such that $N\tau^r > \sum k_i \tau^i$, then every representation in W_0 must have upper degree less than K and every coefficient less than N making the total number of possibilities less than or equal to $N(K-r)$.

Thus if we begin with a representation $\sum k_i f_i$ in W_0 and apply the operation Ω repeatedly, after finitely many steps we must arrive a first time at a representation $\sum k_i^* f_i$ which is either canonical or else has the property that the application of Ω to $\sum k_i^* f_i$ necessarily produces a representation with lower degree less than that of $\sum k_i f_i$. A finite sequence of representations in W_0 produced by starting with $\sum k_i f_i$ and repeatedly applying Ω until arriving at such a representation $\sum k_i^* f_i$ shall be called an Ω*-cycle.*

Theorem 3.20. Let $\sum k_i f_i$ and $\sum k_i^* f_i$ be the first and last representations of an Ω-cycle, and let r be the lower degree of $\sum k_i f_i$. Then if $\sum k_i^* f_i$ is not canonical, it either has first significant pair $(k_r^*, k_{r-1}^*) = (n,0)$ with $n > 1$ or first significant pair $(k_{r+1}^*, k_r^*) = (n,0)$ with $n > 1$.

Proof. If $\sum k_i^* f_i$ is not canonical, then the application of Ω must necessarily lessen the lower degree to less than r, and the lower degree of $\sum k_i^* f_i$ must still be greater than or equal to r since, by the definition of an Ω-cycle, $\sum k_i^* f_i$ must be the first representation encountered for which the application of Ω produces lower degrees less than r. Now of operations (i) and (ii), only (ii) can lessen the lower degree, and when (ii) is applied to a first significant pair (k_i, k_{i-1}) it alters only k_{i+1}, k_i, k_{i-1}, and k_{i-2}. It follows that the first significant pair of $\sum k_i^* f_i$ must be of the form $(n,0)$ with $n > 1$, since otherwise (i) would apply instead of (ii), and that the position of this first significant pair must either be (k_r^*, k_{r-1}^*) or (k_{r+1}^*, k_r^*), since the application of (ii) to pairs positioned further to the left cannot alter k_i^* for $i < r$, and pairs further to the right cannot be significant.

Intuitively, Theorem 3.20 says that the last representation in an Ω-cycle is canonical except possibly for having an integer greater than 1 in the right-most non-zero position.

Theorem 3.21. Let $\sum k_i f_i$ and $\sum k_i^* f_i$ be the first and last representations of an Ω-cycle. Then $\sum k_i^* f_i$ is independent of the various possible choices for k and j in alternatives (i) and (ii) for Ω. That is, all Ω-cycles beginning with $\sum k_i f_i$ terminate with $\sum k_i^* f_i$.

Proof. Consider two Ω-cycles with first representation $\sum k_i f_i$. Let them have last representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$. In accordance with Theorem 3.20, we distinguish six cases: (a) (k_r^*, k_{r-1}^*) and (k_r^{**}, k_{r-1}^{**}) are both first significant pairs, (b) (k_{r+1}^*, k_r^*) and (k_{r+1}^{**}, k_r^{**}) are both first significant pairs, (c) (k_r^*, k_{r-1}^*) and (k_{r+1}^{**}, k_r^{**}) are both first significant pairs, (d) (k_{r+1}^*, k_r^*) and (k_r^{**}, k_{r-1}^{**}) are both first significant pairs, (e) precisely one of the two representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ is canonical and finally (f) both of the representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ are canonical.

In case (a) let $(k_r^*, k_{r-1}^*) = (n^*, 0)$ and $(k_r^{**}, k_{r-1}^{**}) = (n^{**}, 0)$ wherein n^* and n^{**} are integers greater than 1. Then we write

$$\sum k_i^* \tau^i = \sum_{i \geq r+2} k_i^* \tau^i + n^* \tau^r$$

and similarly

$$\sum k_i^{**} \tau^i = \sum_{i \geq r+2} k_i^{**} \tau^i + n^{**} \tau^r .$$

Since $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ are both in W_0, the sums $\sum k_i^* \tau^i$ and $\sum k_i^{**} \tau^i$ are equal. We therefore have

$$\sum_{i \geqslant r+2} k_i^{**}\tau^i - \sum_{i \geqslant r+2} k_i^{*}\tau^i = (n^* - n^{**})\tau^r,$$

and taking conjugates on both sides,

$$\overline{\sum_{i \geqslant r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geqslant r+2} k_i^{*}\tau^i} = (-1)^r(n^* - n^{**})\tau^{-r} .$$

Now

$$\sum_{i \geqslant r+2} k_i^{*}f_i \quad \text{and} \quad \sum_{i \geqslant r+2} k_i^{**}f_i$$

are positive canonical Fibonacci representations with lower degree $\geqslant r+2$. Referring to Theorem 3.10, we deduce that

$$\overline{\sum_{i \geqslant r+2} k_i^{*}f_i} \quad \text{and} \quad \overline{\sum_{i \geqslant r+2} k_i^{**}f_i}$$

must both lie in the interval

$$\left(-\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r+2}} \right)$$

if r is odd and in the interval

$$\left(-\frac{1}{\tau^{r+2}}, \frac{1}{\tau^{r+1}} \right)$$

if r is even. In either case,

$$\overline{\sum_{i \geqslant r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geqslant r+2} k_i^{*}\tau^i}$$

must lie in the interval $\left(-\frac{1}{\tau^r}, \frac{1}{\tau^r} \right)$, so that

$$-\tau^{-r} < (-1)^r(n^* - n^{**})\tau^{-r} < \tau^{-r}.$$

This clearly gives $n^* = n^{**}$, making

$$\sum_{i \geqslant r+2} k_i^{*}\tau^i = \sum_{i \geqslant r+2} k_i^{**}\tau^i .$$

By the uniqueness of canonical representations,

$$\sum_{i \geqslant r+2} k_i^{*}f_i \quad \text{and} \quad \sum_{i \geqslant r+2} k_i^{**}f_i$$

must be identical and hence the theorem is proved for case (a).

Case (b) is clearly equivalent to case (a) by a shift.

In case (c) let $(k_r^*, k_{r-1}^*) = (n^*, 0)$ and $(k_{r+1}^{**}, k_r^{**}) = (n^{**}, 0)$. Then by similar reasoning we get successively

$$\sum_{i \geqslant r+2} k_i^{**}\tau^i - \sum_{i \geqslant r+2} k_i^{*}\tau^i = -n^{**}\tau^{r+1} + n^*\tau^r ,$$

$$\overline{\sum_{i \geqslant r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geqslant r+2} k_i^{*}\tau^i} = (-1)^r(n^{**}\tau^{-r-1} + n^*\tau^{-r}),$$

$$-\frac{1}{\tau^r} < (-1)^r \frac{n^{**}\tau^{-1} + n^*}{\tau^r} < \frac{1}{\tau^r}, \quad \frac{n^{**}}{\tau} + n^* < 1.$$

Since n^{**} and n^* must both be greater than 1, this is impossible, so that case (c) cannot occur.

Case (d) is clearly equivalent to case (c) by an interchange of symbols.

The calculations for cases (a) and (c) can be applied as well as when either $n^* = 1$ or 0 or $n^{**} = 1$ or 0 to show that if one of the two representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ is canonical, so is the other. The uniqueness theorem for canonical representations then takes care of case (e) and case (f), as well.

Corollary. Let $\sum k_i f_i$ and $\sum k_i' f_i$ be two Fibonacci representations which determine the same quadratic integer and which have the same lower degree. Then any Ω-cycle which begins with $\sum k_i f_i$ must end with the same representation as does any Ω-cycle which begins with $\sum k_i' f_i$.

Proof. The proof of Theorem 3.21 uses only the properties that the last representation of the Ω-cycle is in W_0 and has the same lower degree as the first; the actual values of the k_i are immaterial.

Corollary. Let $\sum k_i^* f_i$ be the last representation of some Ω-cycle, and suppose that $\sum k_i^* f_i$ is not canonical. Apply Ω to $\sum k_i^* f_i$ to obtain a new representation of lesser lower degree. This new representation begins a new Ω-cycle whose last representation is independent of the choice of k and j in (ii) when reducing the lower degree of $\sum k_i^* f_i$.

Proof. Any choice for k and j in (ii) will send $\sum k_i^* f_i$ to a representation of lower degree exactly two smaller than that of $\sum k_i^* f_i$. Moreover, since all such representations continue to represent the same quadratic integer, the preceding corollary assures that all consequent Ω-cycles must terminate in the same representation.

Theorem 3.22. Let $\sum k_i^* f_i$ be the last representation of some Ω-cycle. Suppose $\sum k_i^* f_i$ is non-canonical and let it have first significant pair $(n^*, 0)$. Apply Ω to $\sum k_i^* f_i$ to generate a new Ω-cycle with last representation $\sum k_i^{**} f_i$. Let $(n^{**}, 0)$ be the first significant pair in $\sum k_i^{**} f_i$ if the latter is non-canonical. Then $n^{**} \leqslant \frac{1}{2} n^*$.

Proof. Since $\sum k_i^* f_i$ ends an Ω-cycle, the last non-zero pair of consecutive integers in the positional notation for $\sum k_i^* f_i$ must be $0, n^*$. Applying (ii) with $j = 1$, $k = 1$, these two integers and the two following on the right become $1, n^* - 2, 0, 1$. At this point everything to the left of these four positions is canonical in the sense that it contains no significant pairs. If the position immediately to the left of these four contains a 1, then it, together with the 1 to its right form the new first significant pair and these two ones are replaced on the next step by a new 1 in the first position to the left of the pair. If this 1 is adjacent to another on its left, this pair is now the first significant pair and is replaced by a new 1 in the first position to its left, and so forth. This process continues until the new 1 stands alone, in which case the resultant representation ends in $0, n^* - 2, 0, 1$ with no significant pairs to the left of these four positions. On the other hand, if for the last four significant positions $1, n^* - 2, 0, 1$ no 1 appears immediately to the left of these four positions, $(1, n^* - 2)$ becomes the first significant pair so the last four significant positions become, upon the next application of Ω with $k = 1$, $0, n^* - 3, 0, 1$. The new 1 which now appears immediately to the left of these four positions behaves as just described, moving to the left each time it pairs with another 1 immediately to its left, the process terminating when the new 1 finally stands alone. At this point the representation terminates with $0, n^* - 3, 0, 1$ with no significant pairs appearing to the left of these four positions.

In either case, the next application of Ω calls for operation (ii), for which we once again select j and $k = 1$. By an exact repetition of the arguments just presented, we see that after finitely many applications of Ω we arrive at a representation of the form $0, n^* - k_2, 0, 2$ with no significant pairs to the left of these four positions, and having $k_2 \geqslant 4$. By induction we arrive after m steps at a representation ending in $0, n^* - k_m, 0, m$

with $k_m \geqslant 2m$ and no significant pairs to the left of these four positions. When finally $n^* - k_m = 0$ or 1 the end of the Ω-cycle has been reached, and the representation ends in either $0, 1, 0, m$ or $0, 0, 0, m$. Also $n^* - k_m \leqslant 1$ and $k_m \geqslant 2m$. This gives $m \leqslant \frac{1}{2}k_m$ with $k_m = n^*$ or $k_m = n^* - 1$ and the theorem is proved because the last representation of the Ω-cycle does not depend on our particular choices for j and k in the various applications of Ω.

Corollary. For each Fibonacci representation $\sum k_i f_i \in W$ the resolution algorithm converges in finitely many steps to the canonical representation determining the same quadratic integer.

To extend the resolution algorithm to the general case, we show how every other case can be reduced to the case of representations in W.

Let $\sum k_i f_i$ be a Fibonacci representation with upper degree p and lower degree r. If $p - r \geqslant 2$, eliminate k_p by adding it to each of k_{p-1} and k_{p-2}. This does not alter r but reduces p by at least one and thus reduces $p - r$ by at least one. Clearly by repeating this process finitely many times the Fibonacci representation can be reduced to one containing at most two non-zero coefficients, and these will be adjacent. If these two numbers are both nonnegative, or if at any point prior to arriving at this pair all of the coefficients become nonnegative, one should revert to the resolution algorithm as defined for representations in W. If these two numbers are both non-positive, or if at any point prior to arriving at this pair all of the coefficients become non-positive, one should factor a minus sign to the front of the entire representation and then treat as in the case of W, the minus remaining in place during the remaining operations.

Therefore we may assume that we have arrived at a Fibonacci representation containing exactly two non-zero coefficients which are adjacent and such that one is positive and the other is negative. Let this pair of coefficients be a, b. If we continue the operation of eliminating the first member of the pair by adding it to each of the two positions immediately to its right we obtain successively the pairs

$$(a,b), \quad (a+b, a), \quad (2a+b, a+b), \quad (3a+2b, 2a+b), \quad (5a+3b, 3a+2b), \cdots .$$

The pairs (a,b), $(a+b, 2a+b)$, $(3a+2b, 5a+3b)$ belong to the embedding $\xi((b-a)\underline{f} + a\sigma(\underline{f}))$ defined by Eq. (2.35). Because of the orientation of the flows as seen in Fig. 2.4, the ratio of the second to the first term in each pair must eventually become positive and remain so (approaching τ) and therefore we must eventually arrive at a pair in which both members have the same sign. At this point we may proceed as indicated previously.

Thus the entire resolution algorithm is seen to proceed in the following phases in the general case:

<u>Phase I</u>: Reduce the representation to a pair.
<u>Phase II:</u> Continue reduction until like signs are obtained.
<u>Phase III</u>: Factor our minus signs and apply Ω repeatedly.

It is perhaps worthwhile to present one worked-out example.

$$\begin{aligned} 20(-7).046 &= 2(-5).046 = (-3).246 = .(-1)16 = .005 = .01301 \\ &= .10201 = .11002 = 1.00002 = 1.0001001 . \end{aligned}$$

The reader can verify that the first and last representations (and all those in between) determine the quadratic integer $(10/\tau) - 5$.

This algorithm now makes possible an arithmetic for the canonical Fibonacci representations. One performs the standard algorithms and interprets them as in Section 3.2 and then resolves the results to make them once again canonical. In this way we obtain our final theorem in this section.

Theorem 3.23. The canonical Fibonacci representations form a ring in the usual arithmetical algorithms, followed by resolution. This ring is isomorphic to the ring I of quadratic integers in $Q[\tau]$ under the mapping which sends each canonical representation $\sum k_i f_i$ to the quadratic integer which it determines, namely

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i .$$

The canonical representations of zero form a subring which is isomorphic to the ring Z of ordinary integers under the left shift

$$\sum k_i f_i \to \sum k_{i-1} f_i \,.$$

This ring is actually a linear algebra over Z in the obvious way; hence the title of our paper.

3.5 Applications in $\underline{X}$ and $\underline{F}$.

As explained in Section 2.6, each sequence $\underline{x} \in \underline{X}$ goes over to the quadratic integer $(x_0/\tau) + x_1$ under the isomorphism $\underline{\zeta} \circ \underline{\phi}$, and moreover, the left shift in $\underline{X}$ corresponds to multiplication by τ in I. In positional notation $(x_0/\tau) + x_1$ is denoted by $x_1 . x_0$ which represents x_0, and multiplication by τ produces $x_1 x_0 .$ which represents x_1, etc. Thus we see that successive terms in $\underline{x}$ are represented by successive shifts of each Fibonacci representation for x_0 which determines $(x_0/\tau) + x_1$. (For the particular representation $x_1 . x_0$ this amounts to a restatement of Eq. (2.6).) There is for the quadratic integer $(x_0/\tau) + x_1$ a unique canonical Fibonacci representation in the sense of Theorem 3.6. This representation signifies a finite sum of multiples of 1.0 by non-consecutive powers of τ, which corresponds in $\underline{X}$ to a finite sum of non-consecutive shifts of $\underline{f}$, since $\underline{\zeta} \circ \underline{\phi}$ sends $\underline{f}$ to 1. Hence we obtain the following theorem.

Theorem 3.24. Every Fibonacci sequence $\underline{x} \in \underline{X}$ is uniquely expressible as a signed finite sum of non-consecutive shifts of the sequence $\underline{f}$ of Fibonacci numbers. The appropriate sum is precisely that indicated by the sign and shifts of 1. appearing in the canonical Fibonacci representation which determines the quadratic integer $\underline{\zeta} \circ \underline{\phi}(\underline{x}) = (x_0/\tau) + x_1$. Furthermore, the canonical representation $\underline{\zeta} \circ \underline{\phi}(\underline{x})$ represents x_0 and successive left or right shifts of this representation yield representations for the terms of $\underline{x}$ given by the corresponding shifts in $\underline{X}$.

An example of this theorem appears in the principal introduction (Section 1). As pointed out in the introduction, the second statement in this theorem appears in [14] for nonnegative Fibonacci sequences and in [8] and [13] for more general sequences.

Thus each Fibonacci sequence, when appropriately represented "in Fibonacci" consists of consecutive shifts of a basic block of ones and zeroes, and two sequences are made up of shifts of the same basic block if and only if each sequence is a shift of the other, in which case we say that the two sequences are *equivalent.* It is clear that this equivalence is a true equivalence relation in which each equivalence class determines a signed basic block of zeroes and ones, and vice-versa.

Theorem 3.14 provides a ready-made enumeration of these equivalence classes if we agree to distinguish between -0 and 0 for purposes of listing. For every basic block can be so shifted as to have lower degree zero, so every Fibonacci sequence in $\underline{X}$ is equivalent to exactly one which under $\underline{\zeta} \circ \underline{\phi}$ is determined by a canonical representation of lower degree zero, and by Theorem 3.14, there is an exact correspondence between the set $\cdots -2, -1, -0, 0, 1, 2, 3, \cdots$ and the canonical representation of lower degree zero. Thus for each $m = \pm 0, \pm 1, \pm 2, \cdots$ we can refer to the m^{th} *equivalence class* in $\underline{X}$.

In Fig. 3.4 we list for several values of m the pair (m,n) with $n = [|(m+1)\tau|]$, the canonical Fibonacci representation of lower degree zero which determines the pair (m,n) and some of subsequent terms of the embedding of the Fibonacci sequence in the flow passing through the point (m,n). Flow constants are also given for later reference.

The reader will perhaps notice that the canonical representations increase in strict lexicographic order in the sense that they increase with no omissions within the class of canonical representations of lower degree 0. This can be proved easily from Eq. (3.5), the corollary to Theorem 3.9 and Theorem 3.14; however it is also an immediate consequence of the known properties of the Zeckendorff representations and their simple connection with the canonical representations with lower degree zero.

At this point we can see that the pairs appearing in the right-hand column of Fib. 3.4 are the *Wythoff pairs* as defined in [17] and discussed in [5, 17, 18, 19]. For given any pair (a,b) in this column, let $b - a = k$. Then $(k.a)$ is determined by an odd shift of the canonical Fibonacci representation appearing in the same row as (a,b), so (k,a) is determined by a canonical Fibonacci representation of positive, odd lower degree. By Eq. (3.9) we have $a = [|k\tau|]$, so that $b = a + k = [|k\tau + k|] = [|k\tau^2|]$. Since the left shifts of the canonical representations of

(m,n)	Flow Constant	Canonical Representation	Subsequent Points of Embedding in Fibonacci Flow
(0,1)	1	1.	(1,2), (3,5), (8,13) ···
(1,3)	5	101.	(4,7), (11,18), (29,47), ···
(2,4)	4	1001.	(6,10), (16,26), (42,68) ···
(3,6)	9	10001.	(9,15), (24,39), (63,102) ···
(4,8)	16	10101.	(12,20), (32,52), (84,136) ···
(5,9)	11	100001.	(14,23), (37,60), (97,157) ···
(6,11)	19	100101.	(17,28), (45,73), (118,191) ···
(7,12)	11	101001.	(19,31), (50,81), (131,212) ···
(8,14)	20	1000001.	(22,36), (58,94), (152, 246) ···
(9,16)	31	1000101.	(25,41), (66,101), (167,268) ···
(10,17)	19	1001001.	(27,44), (71,115), (186,301) ···
(11,19)	31	1010001.	(30,49), (79,128), (207,335) ···
(12,21)	45	1010101.	(33,54), (87,141), (228,369) ···
(13,22)	29	10000001.	(35,57), (92,149), (241,390) ···
(14,24)	44	10000101.	(38,62), (100,162), (262,424) ···
(15,25)	25	10001001.	(40,65), (105,170), (275,445) ···

Fig. 3.4 Some Data on the Equivalence Classes in $\underline{X}$

lower degree zero must represent positive integers, and since $([|k\tau|], [|k\tau^2|])$ is known to be the k^{th} Wythoff pair for each k, the right column contains only Wythoff pairs. But by the corollary to Theorem 3.15 and the fact that all possible basic blocks occur in the table, every positive integer must occur somewhere in the right column and therefore all Wythoff pairs must be present.

It now follows from the discussion in [17] that the first pairs appearing in the right column are the *primitive* Wythoff pairs (defined in [17]). If we throw in the negatives of the primitive Wythoff pairs and the pair (0, 0) and refer to this larger collection as the primitive Wythoff pairs, we obtain the following generalization of the results in [17].

Theorem 3.25. Every Fibonacci sequence in $\underline{X}$ is from some point forward identical to the sequence initiated by a primitive Wythoff pair, and for non-equivalent sequences these primitive Wythoff pairs are distinct.

Thus the primitive Wythoff pairs furnish a system of representatives for the equivalence classes in $\underline{X}$ just as do the pairs in the first column of Fig. 3.4.

Our last theorem in this section is the only application of canonical representations to $\underline{F}$. While it is too obvious at this point to require proof, it is of sufficient interest to be stated formally.

Theorem 3.26. Every equivalence class in the quotient ring $\underline{F}$ contains a unique pseudo-polynomial $\sum k_i\lambda^i$ for which either $k_i \neq 0$ implies $k_i = 1$ or else $k_i \neq 0$ implies $k_i = -1$ and for which $k_ik_{i+1} = 0$ for every i. For each equivalence class this pseudo-polynomial is precisely the one associated with the canonical Fibonacci representation which determines the image of the equivalence class under the isomorphism $\underline{\zeta}$.

For example, the equivalence class $[2\lambda^2 + \lambda^{-1} - \lambda^{-3}]$ in $\underline{F}$ goes to $2\tau^2 + \tau^{-1} - \tau^{-3}$ under $\underline{\zeta}$. By the resolution algorithm (and a shortcut)

$$200.10(-1) = 200.01 = 1001.01$$

so that $[2\lambda^2 + \lambda^{-1} - \lambda^{-3}] = [\lambda^3 + 1 + \lambda^{-2}]$.

4. THE FIBONACCI FLOWS

4.1 Introduction

In this section we consider properties of the Fibonacci flows such as which integers are flow numbers, how many non-equivalent sequences are embedded on a given flow and how the embeddings situate with respect to one another on a flow. Much of this material comes out in a standard way from the analysis of representations by quadratic forms, such as can be found in [16]. In these cases we simply provide a statement of the results.

4.2 The Flow Constants

Suppose $\nu_0 \neq 0$ is a flow constant for some $\underline{x} \in \underline{X}$ so that by Eqs. (2.28) and (2.36)

$$\nu_0 = x_1^2 - x_1x_0 - x_0^2 \tag{4.1}$$

for the pair (x_0, x_1) of integers. Thus the flow constants ν for sequences in $\underline{X}$ are precisely those integers which are represented by the indefinite quadratic form $x_1^2 - x_1x_0 - x_0^2$. This form has discriminant 5, and since all forms of discriminant 5 are equivalent under unimodular transformations, by the standard reduction of the problem of representation to that of equivalence we find that the integers ν having primitive representations by the form $x_1^2 - x_1x_0 - x_0^2$ are precisely those for which the quadratic congruence

$$\mu^2 \equiv 5 \bmod 4|\nu| \tag{4.2}$$

is solvable. Using quadratic reciprocity we obtain the following theorem.

Theorem 4.1. The positive integers having primitive representations by the indefinite form $x_1^2 - x_0x_1 - x_0^2$ are those of the form

$$\nu = 5^\beta p_1^{\gamma_1} \cdots p_k^{\gamma_k} \tag{4.3}$$

in which $\beta = 0$ or 1 and $p_1, \cdots, p_k$ are distinct primes, each of which is congruent either to 1 or 9 modulo 10. Those integers which are flow constants for sequences in $\underline{X}$ are therefore all numbers of the form $\pm k^2\nu$, wherein ν is given by Eq. (4.3) and k is an arbitrary integer.

We deliberately allow the case $k = 0$ in Theorem 4.1 to account for the zero sequence in $\underline{X}$. The reader is referred to Fig. 3.4 for some data on the flow constants.

4.3 The Embeddings

Having established the form of the flow constants it is natural to inquire as to how many and which embeddings occur on a given flow. For each given flow constant all of the embeddings on that flow can be computed by the method of reduction of quadratic forms. In this connection we point out that the automorphs of the form $x_1^2 - x_1x_0 - x_0^2$ are the linear transformations given by matrices of the form

$$\pm\begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}, \pm\begin{pmatrix} f_{2n-1} & -f_{2n-2} \\ f_{2n} & -f_{2n-1} \end{pmatrix} \tag{4.4}$$

in which n is an arbitrary integer. The reader will have little difficulty in showing that if (x_0, x_1) is a representation for some number ν by the form $x_1^2 - x_1x_0 - x_0^2$, then the other representations of ν generated from (x_0, x_1) by these automorphs are precisely the other points of the embedding of $(x_1 - x_0)\underline{f} + x_0\sigma(\underline{f})$, their negatives, their conjugates, and the negatives of their conjugates.

Thus in any given case one determines the solutions for μ of the congruence 4.2 such that $0 \leqslant \mu < 2|\nu|$, and then determines for each solution, by reducing the corresponding equivalent form, a primitive representation (x_1, x_0) which then generates by 4.4 its embedding, the negative of its embedding and the conjugates of these. In addition, non-primitive representations will arise if the prime power factorization of $|\nu|$ contains exponents greater than or equal to 2. For by factoring p^2 from ν and determining primitive representations for ν/p^2, say (u_0, u_1), we obtain the non-primitive representations (pu_0, pu_1) for ν. Since the automorphs 4.4 generate only primitive representations from primitive representations, the non-primitive representation (pu_0, pu_1) and all the other representations determined from it by the automorphs 4.4 are necessarily distinct from all of the primitive representations. An extension of this argument shows that if the squares of two distinct primes occur as factors of $|\nu|$ then the corresponding non-primitive representations are distinct. It follows that there is no upper limit to the number of non-equivalent embeddings that can lie on the same flow.

As an example consider the flow constant 121. The solutions of $\mu^2 \equiv 5 \bmod 484$ with $0 \leqslant \mu < 282$ are $\mu \equiv 73, 169 \bmod 484$. These determine the equivalent forms $121x_1^2 + 73x_1x_0 + 11x_0^2$ and $121x_1^2 + 169x_1x_0 + 59x_0^2$. By reducing these two forms we determine the primitive representations $(-3, 10)$ and $(-7, 10)$, respectively.

The only square factor of 121 is 121 itself, so the non-primitive representations of 121 will be given by multiplying by 11 the primitive representations of 1. The solutions of $\mu^2 \equiv 5 \bmod 4$ with $0 \leqslant \mu < 2$ are $\mu = 1$, only, giving the equivalent form $x_1^2 + x_1x_0 - x_0^2$. This form is already reduced and determines the primitive representation (0,1), which gives the non-primitive representation (0, 11) of 121. If we note that the embedding containing (–7, 10) has for its next point (3, 13) which in turn has as its conjugate (–3, 10), we can state that the only embeddings on flow 121 are, up to equivalence (i.e., shifts) those of the sequence $17\underline{f} - 7\sigma(\underline{f})$, its negative, its conjugate, the negative of its conjugate and similarly for $11\underline{f}$. However $11\underline{f}$ is self-conjugate so the total number of non-equivalent sequences embedded on flow number 121 is 6.

The next theorem shows that the number of non-equivalent sequences on a given flow can be simply computed from the ordinary prime factorization of the flow constant, without actually determining the embedded sequences. Let ν_0 be an arbitrary non-zero flow constant and let the prime factorization of ν_0 have the form

$$\nu_0 = \pm 5^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_j^{m_j} \tag{4.5}$$

in which $p_1, p_2, \cdots, p_k$ are the prime factors of ν_0 which are congruent to 1 or 9 modulo 10, and $q_1, q_2, \cdots, q_j$ are the other prime factors of ν_0 different from 5. Note that by Theorem 4.1 each of the exponents $m_1, m_2, \cdots, m_j$ is even, say $m_i = 2r_i$, $i = 1, 2, \cdots, j$.

Theorem 4.2. Let ν_0 be a non-zero flow constant which factors as in 4.5. Then the number of non-equivalent Fibonacci sequences in $\underline{X}$ embedded on the flow $\nu(\underline{x}) = \nu_0$ is equal to

$$2(1+n_1)(1+n_2)\cdots(1+n_k).$$

Proof. We shall use the following known facts from elementary number theory: (1) $Z[\tau]$ is a unique factorization domain, (2) if n is a positive integer which is not a square then $\sqrt{n} \in Z[\tau]$ if and only if the square-free part of n is 5 and (3) the units in $I = Z[\tau]$ are exactly the elements $\pm\tau^n$, $n = 0, \pm 1, \pm 2, \cdots$.

The first step of the proof is to establish that the natural integers which are prime in I are precisely the natural primes which are not 5 and are not congruent to 1 or 9 mod 10. Clearly Theorem 4.1 precludes any other possibilities for the primes in I. On the other hand, we now show that if p is any natural prime which factors non-trivially in I, then the factorization of p is necessarily of the form $p = \pm\alpha\bar{\alpha}$, with α and $\bar{\alpha}$ prime in I. From this it will follow, again by Theorem 4.1, that each natural prime not 5 or congruent to 1 or 9 mod 10 is prime in I.

Suppose p is a natural prime which factors in I as $p = \alpha\beta$, when neither α nor β is a unit in I. Then also $p = \bar{\alpha}\bar{\beta}$ so $p^2 = \alpha\bar{\alpha}\beta\bar{\beta}$, and thus $\alpha\bar{\alpha} = \pm\beta\bar{\beta} = \pm p$. Hence α and β have prime norm and therefore are prime in I. Since I is a unique factorization domain, either $\beta = u\alpha$ where u is a unit in I or $\beta = u\bar{\alpha}$ where u is a unit in I. If $\beta = u\bar{\alpha}$ then we have $p = \alpha\beta = u\alpha\bar{\alpha}$. Since $\alpha\bar{\alpha}$ is an integer, necessarily $u = \pm 1$ giving the desired result: $p = \pm\alpha\bar{\alpha}$. If $\beta = u\alpha$ then $p = u\alpha^2 = \bar{u}\bar{\alpha}^2$ so that $\alpha = v\bar{\alpha}$ for some unit v since I is a unique factorization domain. This gives $p = uv\alpha\bar{\alpha}$, and since uv is a unit in I, we are back to the previous case.

The next step of the proof is to show that if a natural prime p is a non-prime in I with prime factorization $p = \pm\alpha\bar{\alpha}$ in I, then $\bar{\alpha}$ is an associate of α if and only if $p = 5$. For if $p = 5$ we have $5(2\tau-1)^2$ and $\overline{2\tau-1} = -(2\tau-1)$. On the other hand, if $\bar{\alpha} = u\alpha$ where u is a unit in I, then $p = \pm\alpha\bar{\alpha} = \pm u\alpha^2 = \pm\bar{u}\bar{\alpha}^2$ and so $p^2 = u\bar{u}(\alpha\bar{\alpha})^2$, whence $u\bar{u} = 1$. Now a unit u is of the form $\pm\tau^n$, so $\bar{u}$ is $\pm(-1)^n\tau^{-n}$. Therefore $u\bar{u} = (-1)^n$ showing that n is necessarily even, say $n = 2k$. This gives

$$u = \pm\tau^{2k}, \quad p = \pm\tau^{2k}\alpha^2 = \pm(\tau^k\alpha)^2.$$

Since all members of I are real we have $p = (\tau^k\alpha)^2$ whence $\sqrt{p} \in I$. But p is prime, so $p = 5$.

Having dispensed with these technicalities we can now complete the proof. Let $\nu(\underline{x}) = \nu(x_0, x_1) = \nu_0$, so that

$$\nu_0 = \left(\frac{x_0}{\tau} + x_1\right)\left(\overline{\frac{x_0}{\tau} + x_1}\right).$$

Let the prime factorization of $(x_0/\tau) + x_1$ in I be

$$\frac{x_0}{\tau} + x_1 = \alpha_1\alpha_2\cdots\alpha_s,$$

so

$$\overline{\frac{x_0}{\tau} + x_1} = \bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_s,$$

and

$$\nu_0 = (a_1\bar{a}_1)(a_2\bar{a}_2)\cdots(a_s\bar{a}_s) \tag{4.6}$$

the latter necessarily being the prime factorization of ν_0 into natural primes.

On the other hand, by Eq. (4.5)

$$\nu_0 = \pm 5^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{2r_1} q_2^{2r_2} \cdots q_j^{2r_j}.$$

Each q_i is prime in I and each p_i factors by Theorem 4.1 as $p_i = \gamma_i\bar{\gamma}_i$, each factor being prime in I. We have seen that 5 factors as $-(2\tau - 1)(\overline{2\tau - 1}) = (2\tau - 1)^2$. Thus

$$\nu_0 = \pm(2\tau - 1)^{2n_0} \gamma_1^{n_1}\bar{\gamma}_1^{n_1}\gamma_2^{n_2}\bar{\gamma}_2^{n_2}\cdots\gamma_k^{n_k}\bar{\gamma}_k^{n_k} q_1^{2r_1} q_2^{2r_2}\cdots q_j^{2r_j} \tag{4.7}$$

is the prime factorization of ν_0 in I.

By comparing Eqs. (4.6) and (4.7) we see that, up to units $a_1, a_2, \cdots, a_s$ must consist of the following: r_1 occurrences of q_1, r_2 occurrences of q_2, and so on, up to r_j occurrences of q_j, n_0 occurrences of $2\tau - 1$ (we note that up to this point there is no choice in the assignment of the a_i except order and units) and n_1 occurrences of either γ_1 or $\bar{\gamma}_1$, n_2 occurrences of either γ_2 or $\bar{\gamma}_2$, and so on, up to n_k occurrences of either γ_1 or $\bar{\gamma}_k$. In this last listing — the occurrences of the γ_i and $\bar{\gamma}_i$ — there are $(1 + n_1)(1 + n_2) \cdots (1 + n_k)$ possible choices of the corresponding a_i and distinct choices must yield distinct values for $(x_0/\tau) + x_1$ since I is a unique factorization domain and since no γ_i and $\bar{\gamma}_i$ can be associates.

The introduction of units into the above assignments of $a_1, a_2, \cdots, a_s$ can only produce a multiplication on the resulting value of $(x_0/\tau) + x_1$ by a factor of the form $\pm\tau^n$. Since under $\underline{\zeta} \circ \underline{\phi}$ the shifts in $\underline{X}$ correspond to multiplication by powers of τ in I, the effect of the introduction of these units on the sequence in $\underline{X}$ which is embedded in the flow $\nu = \nu_0$ is to either produce an equivalent sequence or the negative of an equivalent sequence. These negatives always occur since, for example, $1 - \tau$ is a unit with norm -1. Thus we must double our previous count, thereby obtaining the final result.

$$2(1 + n_1)(1 + n_2) \cdots (1 + n_k).$$

As an illustration of this theorem, we note that since $121 = 11^2$ and $11 \equiv 1 \bmod 10$, the number of non-equivalent embeddings on the flow $\nu = 121$ is $2(1 + 2) = 6$, in agreement with our earlier calculations.

Our next task is to establish a separation theorem for distinct embeddings in the same branch of a flow. To this end we define functions $C_T(t)$ and $S_T(t)$ for each real number t by

$$C_T(t) = \frac{\tau^t + \tau^{-t}}{2}, \tag{4.8}$$

$$S_T(t) = \frac{\tau^t - \tau^{-t}}{2}. \tag{4.9}$$

The resemblance to the hyperbolic sine and cosine is evident and in fact

$$C_T(t) = \cosh(t \ln \tau), \tag{4.10}$$

$$S_T(t) = \sinh(t \ln \tau) \tag{4.11}$$

for each real number t. Based on these relations one readily verifies that

$$u(t) = x_0 C_T(t) + \frac{2\tau - 1}{5}(2x_1 - x_0)S_T(t), \tag{4.12}$$

$$v(t) = x_1 C_T(t) + \frac{2\tau - 1}{5}(2x_0 + x_1)S_T(t) \tag{4.13}$$

are parametric equations for the branch of the flows passing through the point (x_0, x_1). For each real t the point $(u(t), v(t))$ can be thought of as representing

$$\frac{u(t)}{\tau} + v(t),$$

and one finds readily from Eqs. (4.12) and (4.13) that

$$\frac{u(t)}{\tau} + v(t) = \left(\frac{x_0}{\tau} + x_1\right)\tau^t . \tag{4.14}$$

Of course this relation is not unanticipated and would serve well in the place of Eqs. (4.12) and (4.13), except that the representation $(u/\tau) + v$ is not unique when u and v are irrational. In any case Eq. (4.14) assures that the orientation of the flows induced by the parameterization in Eqs. (4.12) and (4.13) agrees with the orientation induced by the embeddings, and that given two points (u_1, v_1) and (u_2, v_2) on the same branch of a type I or type II flow, (u_2, v_2) follows (u_1, v_1) in the orientation of Eqs. (4.12) and (4.13) if and only if $(u_2/\tau) + v_2$ is greater than $(u_1/\tau) + v_1$ as a real number, while the opposite holds for type III or type IV flows. This observation provides a proof of our next theorem.

Theorem 4.3. Every pair of distinct embeddings on a single branch of a Fibonacci flow perfectly separate one another in the sense that between every pair of consecutive points of either embedding there occurs exactly one point of the other.

Proof. Assume the flow to be of type I or type II; an obvious parallel argument applies in the other cases. Suppose (m_1, n_1) and (m_2, n_2) are consecutive points of one embedding so that

$$\frac{m_2}{\tau} + n_2 = \left(\frac{m_1}{\tau} + n_1\right)\tau^2 .$$

The points (x_{2n}, x_{2n+1}) of the other embedding will all follow the relation

$$\frac{x_{2n}}{\tau} + x_{2n+1} = \left(\frac{x_0}{\tau} + x_1\right)\tau^{2n}$$

for the appropriate values of x_0 and x_1, and since the sequences are of type I or type II, we have

$$\frac{x_0}{\tau} + x_1 > 0.$$

If now $2k$ is the smallest positive integer such that

$$\left(\frac{x_0}{\tau} + x_1\right)\tau^{2k} > \frac{m_1}{\tau} + n_1$$

it follows readily that

$$\left(\frac{x_0}{\tau} + x_1\right)\tau^{2k-2} < \frac{m_1}{\tau} + n_1 < \left(\frac{x_0}{\tau} + x_1\right)\tau^{2k} < \frac{m_2}{\tau} + n_2 < \left(\frac{x_0}{\tau} + x_1\right)\tau^{2k+2}$$

and the theorem is proved.

4.4. A Final Theorem

Our last theorem does not concern embeddings but nevertheless fits in conveniently at this point of the paper. We have earlier been concerned with the various ways in which the natural integers can be represented canonically by the sequence $\underline{f}$. We consider now the canonical representations by an arbitrary sequence $\underline{x} \in \underline{X}$ with initial terms x_0 and x_1 which are relatively prime. (The case in which x_0 and x_1 are not relatively prime is a simple extension of this case.) We want to know which natural integers have canonical representations by $\underline{x}$ — meaning sums of the form $\sum k_i x_i$ in which all but finitely many of the k_i are zero, no two consecutive k_i are non-zero, and either all non-zero k_i are 1 or else all non-zero k_i are -1 — we want to determine all such canonical representations when they exist.

In view of the analysis in 3 it is natural to associate to each canonical representation $\sum k_i x_i$ the quadratic integer

$$\frac{\sum k_i x_i}{\tau} + \sum k_{i-1} x_i .$$

From foregoing results we have

$$\frac{\sum k_i x_i}{\tau} + \sum k_{i-1} x_i = \sum k_i \left(\frac{x_i}{\tau} + x_{i+1}\right) = \sum k_i \tau^i \left(\frac{x_0}{\tau} + x_1\right)$$
$$(4.15) \qquad = \left(\frac{x_0}{\tau} + x_1\right) \sum k_i \tau^i = \left(\frac{x_0}{\tau} + x_1\right) \left(\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i\right) .$$

The factor

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i$$

we know from Section 3 can be equal to any quadratic integer in I so we see that a natural integer m has a canonical representation by $\underline{x}$ if and only if there exists a natural integer n such that $(m/\tau) + n$ belongs to the principal ideal in I generated by $(x_0/\tau) + x_1$. In this case, the representation $(m/\tau) + n$ by $\underline{x}$ agrees with that of

$$\left(\frac{m}{\tau} + n\right) \left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

by $\underline{f}$. Our last theorem shows that such n exist for each m and characterizes all such.

Theorem 4.4. Let m be an arbitrary natural integer. Then the canonical representations of m by $\underline{x}$ have the same coefficient sets as the canonical representations of the quadratic integers

$$\left(\frac{m}{\tau} + n\right) \left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

by $\underline{f}$ where n is any natural integer such that

$$nx_0 \equiv mx_1 \bmod (x_1^2 - x_1 x_0 - x_0^2).$$

Moreover the foregoing congruence is solvable for n because x_0 is prime to $x_1^2 - x_1x_0 - x_0^2$. For each solution n, the resulting canonical representation of m by $\underline{x}$ has for its left shift a canonical representation for n by $\underline{x}$.

Proof. In view of the remarks preceding the statement of the theorem, we need only show that the condition

$$nx_0 \equiv mx_1 \bmod (x_1^2 - x_1 x_0 - x_0^2)$$

is necessary and sufficient for $(m/\tau) + n$ to belong to the principal ideal in I generated by $(x_0/\tau) + x_1$. Now given m and n, there exist a and b in Z such that

$$\frac{m}{\tau} + n = \left(\frac{x_0}{\tau} + x_1\right) \left(\frac{a}{\tau} + b\right)$$

if and only if

$$\left(\frac{m}{\tau} + n\right) \left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

is in I. But

$$\left(\frac{x_0}{\tau} + x_1\right)^{-1} = \frac{1}{\nu(x_0, x_1)} \overline{\left(\frac{x_0}{\tau} + x_1\right)} ,$$

so

$$\left(\frac{m}{\tau} + n\right) \left(\frac{x_0}{\tau} + x_1\right)^{-1} = \frac{1}{\nu(x_0,x_1)} \left(\frac{m}{\tau} + n\right) (-x_0\tau + x_1)$$

$$= \frac{1}{\nu(x_0,x_1)} \left(\frac{mx_1 - nx_0}{\tau} + nx_1 - mx_0 - nx_0\right) .$$

Thus the necessary and sufficient conditions for $(m/\tau) + n$ to be in the principal ideal generated by $(x_0/\tau) + x_1$ are that

$$mx_1 - nx_0 \equiv 0 \bmod \nu(x_0, x_1), \quad \text{and} \quad nx_1 - mx_0 - nx_0 \equiv 0 \bmod \nu(x_0, x_1).$$

The second of these two congruences is a consequence of the first, as follows. Since x_0 and x_1 are relatively prime it follows simply that x_0 and $\nu(x_0, x_1)$ and that x_1 and $\nu(x_0, x_1)$ are relatively prime. Let x_1^{-1} denote the inverse of x_1 mod $\nu(x_0, x_1)$, so from the first congruence we have

$$m \equiv x_1^{-1} n x_0 \bmod \nu(x_0, x_1)$$

whence

$$nx_1 - mx_0 - nx_0 \equiv nx_1 - x_1^{-1} n x_0^2 - nx_0 \equiv nx_1^{-1}(x_1^2 - x_1 x_0 - x_0^2) \equiv 0 \bmod \nu(x_0, x_1).$$

Thus we have shown that the condition

$$mx_1 - nx_0 \equiv 0 \bmod \nu(x_0, x_1)$$

is necessary and sufficient for $(m/\tau) + n$ to belong to the ideal generated by $(x_0/\tau) + x_1$, and the theorem follows.

5. CONCLUSION

We conclude with a number of comments concerning the foregoing material and possible extensions thereof. First of all, the necessity of distinguishing between the integer "represented" by $\sum k_i f_i$ and the quadratic integer "determined" by $\sum k_i \tau^i$ is unsatisfactory, since in view of all that has been shown it is clearly more natural to "represent" the quadratic integer $\sum k_i \tau^i$ then the ordinary integer $\sum k_i f_i$. The necessity for this distinction exists because in the special case that $\sum k_i \tau^i$ is a natural integer it does not coincide with the natural integer $\sum k_i f_i$. This in turn traces to Eq. (3.1) in which $\sum k_i f_i$ is the coefficient of $\frac{1}{\tau}$ rather than the τ-free part of the expression. All of this can be corrected by defining $g_n = f_{n-1}$ for every n and then defining Fibonacci representations to have the form $\sum k_i g_i$ instead of $\sum k_i f_i$. In this case Eq. (3.1) becomes

$$k_i \tau^i = \sum k_i g_i + \left(\sum k_i g_{i+1} \right) \tau .$$

Furthermore one may take s to be the sequence g in Section 2.4 and many notational asymmetries are eliminated. For example we find that ϕ maps $\underline{x}$ to $x_0 + x_1 \tau$ rather than $(x_0/\tau) + x_1$. Also, Theorem 3.23 then states that the canonical representations which determine (or now we can say represent) natural integers from a ring isomorphic to the integers under the correspondence $\sum k_i g_i \to \sum k_i \tau^i$. All of this is an argument in favor of defining the Fibonacci numbers by the sequence g instead of f. We have not done this because we do not wish to conflict with the definitions already present in the literature, and moreover, this would have the effect increasing the disparity between the positional notation we use, which includes a position for f_0, and that currently in use for Zeckendorff representations, which terminates with the f_1 term. Additional indication for the indexing of the Fibonacci numbers by g instead of by f appears in [19].

We mention that the convergence proof of the resolution algorithm is really a second proof of the existence of canonical Fibonacci representations corresponding to the quadratic integers in I. We could have formulated and proved Theorem 3.26 and then the earlier theorems could be derived therefrom. This has a certain appeal because it is more intrinsically algebraic, but it was felt that the information contained in the statements and proofs of Theorems 3.2, 3.3 and 3.4 warranted their inclusion.

A number of likely extensions and applications of the material in this paper suggest themselves. In references [7, 8, 9, 10] one finds investigations of other types of representations: Lucas representations, Pellian representations and so forth. The theorems of Section 2.3 have been stated with deliberate generality in anticipation of other applications, and it would be of interest to determine for what general class of representations the algebraic approach we have taken could succeed. In addition, there is a possibility that other of the results in the foregoing references could be interpreted and possibly extended in the light of these investigations.

Theorem 3.8 clearly suggests a Fibonacci representatiön for rational numbers. These representations will in general be infinite and divergent, but possibly converge in some generalized sense to the rational numbers they represent.

The resolution algorithm in conjunction with Eqs. (3.7) or (3.9) offers a method of computing first and second canonical representations of positive integers in the sense of [5]. However, Eqs. (3.7) and (3.9) involve the irrationality τ. It would be of interest to determine an algorithm for generating these representations which does not involve irrationalities and also does not involve tables of Fibonacci numbers (as do the extant algorithms).

Finally, the pleasant properties of the Wythoff pairs in terms of Fibonacci representations as evidenced by Fig. 3.4 and pointed out in [18], together with the connection of Fibonacci representations with the ring I as explored in this paper suggests that their role in Wythoff's game might be derivable from formal algebraic arguments, in the spirit of what has been done by Gleason for the game of nim [12].

REFERENCES

1. J. L. Brown, Jr., "Zeckendorff's Theorem and Some Applications," *The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 162–168.
2. L. Carlitz, "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 6, No. 2 (April 1968), pp. 193–220.
3. L. Carlitz, "Fibonacci Representations II," *The Fibonacci Quarterly,* Vol. 8, No. 2 (April 1970), pp. 113–134.
4. L. Carlitz, R. Scoville and T. Vaughn, "Some Arithmetic Functions Related to Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 11, No. 3 (Oct. 1973), pp. 337–386.
5. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 1–28.
6. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Addendum to the Paper 'Fibonacci Representations'," *The Fibonacci Quarterly,* Vol. 10, No. 5 (Dec. 1972), pp. 527–530.
7. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 43–70.
8. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order—II," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 71–80.
9. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Lucas Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 29–42.
10. L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville, "Pellian Representations," *The Fibonacci Quarterly,* Vol. 10, No. 4 (Dec. 1972), pp. 449–488.
11. L. Carlitz and R. Scoville, "Partially Ordered Sets Associated with Fibonacci Representations," *Duke Math. Journal,* 40 (1973), pp. 511–524.
12. A. Gleason, *Nim and Other Oriented Graph Games,* MAA Film.
13. V. E. Hoggatt, Jr., and M. Bicknell, "Generalized Fibonacci Polynomials and Zeckendorff's Theorem," *The Fibonacci Quarterly,* Vol. 11, No. 4 (Dec. 1973), pp. 399–419.
14. D. Klarner, "Partitions of N into Distinct Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 6, No. 2 (Feb. 1968), pp. 235–244.
15. C. C. MacDuffee, *An Introduction to Abstract Algebra,* New York, John Wiley and Sons, 1940.
16. G. E. Mathews, *Theory of Numbers,* New York, Chelsea Publishing Co.
17. R. Silber, "A Fibonacci Property of Wythoff Pairs," *The Fibonacci Quarterly,* Vol. 14, No. 4 (Nov. 1976), pp. 380–384.
18. R. Silber, "Wythoff's Nim and Fibonacci Representations," *The Fibonacci Quarterly,* to appear, Feb. 1977.
19. R. Silber, "On the N Canonical Fibonacci Representation of Order N," *The Fibonacci Quarterly,* to appear, February, 1977.
20. W.A. Wythoff, "A Modification of the Game of Nim," *Nieuw Archief voor Wiskunde,* 2nd Series, 7(1907), pp. 199–202.

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SET PARTITIONS

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1. Let Z_n denote the set $\{1, 2, \cdots, n\}$. Let $S(n,k)$ denote the number of partitions of Z_n into k non-empty subsets $B_1, \cdots, B_k$. The B_k are called *blocks* of the partition. Put

$$n_j = |B_j| \qquad (j = 1, 2, \cdots, k),$$

so that

$$n_1 + n_2 + \cdots + n_k = n. \tag{1.1}$$

It is convenient to introduce a slightly different notation. Put

$$n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n, \tag{1.2}$$

where

$$k_j \geqslant 0 \qquad (j = 1, 2, \cdots, n)$$

and

$$k_1 + k_2 + \cdots + k_n = k. \tag{1.3}$$

We call (1.2) a number partition of the integer n; the condition (1.3) indicates that the partition is into k parts, not necessarily distinct. For brevity (1.2) is often written in the form

$$n = 1^{k_1} 2^{k_2} \cdots n^{k_n}. \tag{1.4}$$

Corresponding to the partition (1.2) we have

$$\frac{n!}{(1!)^{k_1}(2!)^{k_2} \cdots (n!)^{k_n}} \, \frac{1}{k_1!\, k_2! \cdots k_n!} \tag{1.5}$$

set partitions. Hence

$$S(n,k) = \sum \frac{1}{(1!)^{k_1}(2!)^{k_2} \cdots (n!)^{k_n}} \, \frac{1}{k_1!\, k_2! \cdots k_n!}, \tag{1.6}$$

where the summation is over all nonnegative $k_1, k_2, \cdots, k_n$ satisfying (1.2) and (1.13). Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} S(n,k) z^k &= \sum_{k_1, k_2, \cdots = 0}^{\infty} \left(\frac{x}{1!}\right)^{k_1} \left(\frac{x^2}{2!}\right)^{k_2} \cdots \frac{z^{k_1}}{k_1!} \frac{z^{k_2}}{k_2!} \cdots \\
&= \sum_{k_1, k_2, \cdots = 0}^{\infty} \frac{1}{k_1!} \left(\frac{xz}{1!}\right)^{k_1} \frac{1}{k_2!} \left(\frac{x^2 z}{2!}\right)^{k_2} \cdots \\
&= \exp\left(xz + \frac{x^2 z}{2!} + \frac{x^3 z}{3!} + \cdots\right)
\end{aligned}$$

and we get the well known formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} S(n,k) \frac{x^n}{n!} z^k = \exp(z(e^x - 1)). \tag{1.7}$$

*Supported in part by NSF grant GP-17031.

It is clear from (1.7) that

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k, \tag{1.8}$$

which implies

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n, \tag{1.9}$$

the familiar formula for a Stirling number of the second kind.

Next put

$$A_n(z) = \sum_{k=0}^{n} S(n,k)z^k \tag{1.10}$$

and in particular

$$A_n = A_n(1) = \sum_{k=0}^{n} S(n,k). \tag{1.11}$$

The polynomial $A_n(z)$ is called a single-variable Bell polynomial. The number A_n is evidently the total number of set partitions of Z_n.

From (1.7) and (1.10) we have

$$\sum_{n=0}^{\infty} A_n(z)\frac{x^n}{n!} = \exp(z(e^x - 1)). \tag{1.12}$$

Differentiation with respect to x gives

$$A_{n+1}(z) = z\sum_{r=0}^{\infty}\binom{n}{r} A_r(z) \tag{1.13}$$

while differentiation with respect to z gives

$$A_n'(z) = \sum_{r=0}^{n-1}\binom{n}{r} A_r(z). \tag{1.14}$$

Hence

$$A_{n+1}(z) = zA_n(z) + zA_n'(z). \tag{1.15}$$

By (1.10), (1.15) is equivalent to the familiar recurrence

$$S(n+1, k) = S(n, k-1) + kS(n, k).$$

If we take $z = 1$ in (1.13) we get

$$A_{n+1} = \sum_{r=0}^{n}\binom{n}{r} A_r \qquad (A_0 = 1). \tag{1.16}$$

This recurrence can be proved directly in the following way. Consider a partition of Z_{n+1} into k blocks $B_1, B_2, \cdots, B_k$. Assume that the element $n+1$ is in B_k and let B_k contain r additional elements, $r \geq 0$. Keeping these r elements fixed it is clear that $B_1, \cdots, B_{k-1}$ furnishes a partition of Z_{n-r} into $k-1$ blocks. Since the r elements in B_k can be chosen in $\binom{n}{r}$ ways we get

$$A_{n+1} = \sum_{r=0}^{n}\binom{n}{r} A_{n-r} = \sum_{r=0}^{n}\binom{n}{r} A_r.$$

For a detailed discussion of the numbers A_n see [5]. The polynomial $A_n(z)$ is discussed in [1].

We now define (compare [4, Ch. 4])

$$S_1(n,k) = \sum \frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}} \frac{1}{k_1!\,k_2!\cdots k_n!}\,, \tag{1.17}$$

where again the summation is over all nonnegative k_1, k_2, k_n satisfying (1.2) and (1.3). This definition should be compared with (1.6). It follows from (1.17) that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} S_1(n,k)z^k = \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{1}{k_1!}\left(\frac{xz}{1}\right)^{k_1} \frac{1}{k_2!}\left(\frac{x^2z}{2}\right)^{k_2} \cdots$$

$$= exp\left(xz + \frac{x^2z}{2} + \frac{x^3z}{3} + \cdots\right)$$

$$= exp\left(z\log\frac{1}{1-x}\right)\,,$$

so that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} S_1(n,k)\frac{x^n}{n!}z^k = (1-x)^{-z}\,. \tag{1.18}$$

It follows that

$$\sum_{k=0}^{n} S_1(n,k)z^k = z(z+1)\cdots(z+n-1)\,, \tag{1.19}$$

and therefore $S_1(n,k)$ is a Stirling number of the first kind.

We may restate (1.17) in the following way. Let

$$B_1, B_2, \cdots, B_k \tag{1.20}$$

denote a typical partition of Z_n into k blocks with $n_j = |B|_j$. Then

$$S_1(n,k) = \quad (n_1-1)!(n_2-1)!\cdots(n_k-1)!\,, \tag{1.21}$$

where the summation is over all partitions (1.20) such that

$$n_1+n_2+\cdots+n_k = n.$$

2. We again consider the number partition

$$n = k_1{\cdot}1 + k_2{\cdot}2 + \cdots + k_n{\cdot}n \qquad (k_1+\cdots+k_n = k). \tag{2.1}$$

This may be replaced by

$$n = n_1+n_2+\cdots+n_k\,, \tag{2.2}$$

where

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k\,. \tag{2.3}$$

If there are no other conditions the partition is said to be *unrestricted.* We may, on the other hand, assume that

$$n_1 > n_2 > \cdots > n_k\,, \tag{2.4}$$

in which case we speak of partitions into unequal parts. Alternatively we may assume that in (2.2) the parts n_j are odd. If $q(n)$ denotes the number of partitions into distinct parts and $r(n)$ the number of partitions into odd parts, it is well known that [3, Ch. 19]

$$q(n) = r(n). \tag{2.5}$$

This discussion suggests the following two problems for set partitions.

1. Determine the number of set partitions into k blocks of unequal length.
2. Determine the number of set partitions into k blocks, the number of elements in each block being odd.

We shall first discuss Problem 2. The results are similar to those of § 1 above. Let $U(n,k)$ denote the number of set partitions of Z_n into k blocks

$$B_1, B_2, \cdots, B_k \tag{2.6}$$

with

$$n_j = |B_j| \equiv 1 \pmod 2 \qquad (j = 1, 2, \cdots, k). \tag{2.7}$$

In addition we define $V(n,k)$ as the number of set partitions of Z_n into k blocks (2.6) with

$$n_j = |B_j| \equiv 0 \pmod 2 \qquad (j = 1, 2, \cdots, k). \tag{2.8}$$

(In the case of number partitions, the number of partitions

$$n = n_1 + n_2 + \cdots + n_k,$$

where

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k, \qquad n_j \equiv 0 \pmod 2,$$

is of course equal to the number of unrestricted partitions of $n/2$.)

Exactly as in (1.6) we have

$$U(n,k) = \sum \frac{n!}{(1!)^{k_1}(3!)^{k_2}} \frac{1}{k_1!\,k_2!\cdots}, \tag{2.9}$$

where the summation is over all nonnegative $k_1, k_2, \cdots$ such that

$$\begin{cases} n = k_1 \cdot 1 + k_2 \cdot 3 + k_3 \cdot 5 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases} \tag{2.10}$$

Similarly we have

$$V(n,k) = \sum \frac{n!}{(2!)^{k_1}(4!)^{k_2}\cdots} \frac{1}{k_1!\,k_2!\cdots}, \tag{2.11}$$

where now the summation is over all nonnegative $k_1, k_2, \cdots$ such that

$$\begin{cases} n = k_1 \cdot 2 + k_2 \cdot 4 + k_3 \cdot 6 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases}. \tag{2.12}$$

It follows from (2.9) and (2.10) that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} U(n,k) z^k = \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{1}{k_1!}\left(\frac{xz}{1!}\right)^{k_1} \frac{1}{k_2!}\left(\frac{x^3 z}{3!}\right)^{k_2} \cdots$$

$$= \exp\left\{ z\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right\},$$

so that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} U(n,k) \frac{x^n}{n!} z^k = \exp(z \sinh x). \tag{2.13}$$

The corresponding generating function for $V(n,k)$ is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} V(n,k) \frac{x^n}{n!} z^k = \exp(z(\cosh x - 1)). \tag{2.14}$$

It is evident from the definitions that

$$U(n,k) = 0 \quad (n \equiv k+1 \pmod 2), \qquad V(n,k) = 0 \quad (n \equiv 1 \pmod 2).$$

Corresponding to the polynomial $A_n(z)$ and the number A_n we define

$$(2.15)\qquad \begin{cases} U_n(z) = \sum_{k=0}^{n} U(n,k)z^k \\ U_n = U_n(1) = \sum_{k=0}^{n} U(n,k) \end{cases}$$

and

$$(2.16)\qquad \begin{cases} V_n(z) = \sum_{k=0}^{n} V(n,k)z^k \\ V_n = V_n(1) = \sum_{k=0}^{n} V(n,k). \end{cases}$$

Clearly U_n is the total number of set partitions satisfying (2.6) and (2.7), while V_n is the total number of set partitions satisfying (2.6) and (2.8).

By (2.13) and (2.15) we have

$$(2.17)\qquad \sum_{n=0}^{\infty} U_n(z)\frac{x^n}{n!} = \exp(z \sinh x)$$

and by (2.14) and (2.15)

$$(2.18)\qquad \sum_{n=0}^{\infty} V_n(z)\frac{x^n}{n!} = \exp(z(\cosh x - 1)).$$

Differentiating (2.17) with respect to x we get

$$\sum_{n=0}^{\infty} U_{n+1}(z)\frac{x^n}{n!} = z \cosh x \exp(z \sinh x).$$

This implies

$$(2.19)\qquad U_{n+1}(z) = z \sum_{2r\leqslant n} \binom{n}{2r} U_{n-2r}(z).$$

Differentiation of (2.17) with respect to z gives

$$\sum_{n=0}^{\infty} U_n'(z)\frac{x^n}{n!} = \sinh x \exp(z \sinh x)$$

so that

$$(2.20)\qquad U_n'(z) = \sum_{2r<n} \binom{n}{2r+1} U_{n-2r-1}(z).$$

Put $F(x,z) = \exp(z \sinh x)$. Since

$$\frac{\partial^2}{\partial z^2}F(x,z) = \sinh^2 x F(x,z),$$

$$\frac{\partial^2}{\partial x^2}F(x,z) = \frac{\partial}{\partial x}(z\cosh x)F(x,z) = (z^2\cosh^2 x + z\sinh x)F(x,z),$$

it follows that

$$\frac{\partial^2}{\partial x^2}F(x,z) = z^2F(x,z) + z\frac{\partial}{\partial z}F(x,z) + z^2\frac{\partial^2}{\partial z^2}F(x,z).$$

This implies

(2.21) $$U_{n+2}(z) = z^2U_n(z) + zU'_n(z) + z^2U''_n(z) = z^2U_n(z) + (zD_z)^2U_n(z)$$

and therefore

(2.22) $$U(n+2,k) = U(n,k-2) + k^2U(n,k).$$

This splits into the following pair of recurrences

(2.23) $$\begin{cases} U(2n+2,2k) = U(2n,2k-2) + 4k^2U(2n,2k) \\ U(2n+1,2k+1) = U(2n-1,2k-1) + (2k+1)^2U(2n-1,2k+1). \end{cases}$$

To get explicit formulas for $U(n,k)$ we return to (2.13). We have

$$\exp(z\sinh x) = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}(e^x - e^{-x})^k = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}e^{(k-2j)x}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}(k-2j)^n,$$

which yields

(2.24) $$U(n,k) = \frac{1}{2^kk!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}(k-2j)^n.$$

Similarly, since $\cosh x - 1 = 2\sinh^2 \tfrac{1}{2}x$,

$$\exp(z(\cosh x - 1)) = \exp(2\sinh^2 \tfrac{1}{2}x) = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})^{2k}$$

$$= \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}e^{(k-j)x}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{2k\leqslant n}\frac{(z/2)^k}{k!}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(k-j)^n,$$

we get

(2.25) $$V(n,k) = \frac{1}{2^kk!}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(k-j)^n.$$

Comparing (2.25) with (2.24), we get

(2.26) $$V(2n,k) = \frac{(2k)!}{2^{2n-k}k!}U(2n,2k).$$

Thus the first of (2.23) gives

(2.27) $$V(2n+2,k) = (2k-1)V(2n,k-1) + k^2V(2n,k).$$

If we put

(2.28) $$V(2n,k) = \frac{(2k)!}{2^kk!}V'(n,k),$$

(2.27) becomes

(2.29) $$V'(n+1,k) = V'(n,k-1) + k^2V'(n,k).$$

Returning to (2.18) we have

$$\sum_{n=0}^{\infty} V_{n+1}(z)\frac{x^n}{n!} = z \sinh x \exp (z(\cosh x - 1)).$$

This implies

(2.30) $$V_{n+1}(z) = z \sum_{2r<n} \binom{n}{2r+1} V_{n-2r-1}(z).$$

Differentiation of (2.18) with respect to z gives

$$\sum_{n=0}^{\infty} V'_n(z)\frac{x^n}{n!} = (\cosh x - 1) \exp (z \cosh x - 1)$$

which implies

(2.31) $$V'_n(z) = \sum_{0<r\leqslant 2n} \binom{n}{2r} V_{n-2r}(z).$$

It is evident from (2.15) and (2.19) that

(2.32) $$U_{n+1} = \sum_{2r\leqslant n} \binom{n}{2r} U_{n-2r}.$$

Similarly from (2.30) and (2.16) we have

(2.33) $$V_{n+1} = \sum_{2r<n} \binom{n}{2r+1} V_{n-2r-1}.$$

Since $V_n = 0$ unless n is even, we may replace (2.33) by

(2.34) $$V_{2n+2} = \sum_{r=0}^{n} \binom{2n+1}{2r+1} V_{2n-2r}.$$

It is easy to prove (2.32) and (2.34) directly by a combinatorial argument, exactly like the combinatorial proof of (1.16).

The first few values of U_n, V_{2n} follow.

$$U_0 = U_1 = U_2 = 1, \quad U_3 = 2, \quad U_4 = 5, \quad U_5 = 12, \quad U_6 = 36,$$
$$V_0 = V_2 = 1, \quad V_4 = 4, \quad V_6 = 31, \quad V_8 = 379.$$

The following values of $U(2n, 2k)$, $V'(n,k)$, $V(2n+1, 2k+1)$ are computed by means of (2.23) and (2.29).

$U(2n, 2k)$

$n \backslash k$	1	2	3	4
1	1			
2	4	1		
3	16	20	1	
4	64	336	56	1

$U(2n+1, 2k+1)$

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	10	1		
3	1	91	35	1	
4	1	820	966	84	1

$V'(n, k)$

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	5	1		
3	1	21	14	1	
4	1	85	147	30	1

For additional properties of $U(n,k)$ see [2].

3. Put

$$(3.1)\qquad P_n(z) = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2).$$

Then, by the second of (2.23),

$$z^2P_n(z) = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2)[z^2-(2k+1)^2-(2k+1)^2]$$

$$= \sum_{k=0}^{n} [U(2n-1, 2k-1)+(2k+1)^2U(2n-1, 2k+1)]z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2)$$

$$= \sum_{n=0}^{n} U(2n+1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2),$$

so that

$$z^2P_n(z) = P_{n+1}(z).$$

Since $P_1(z) = z$, it follows that $P_n(z) = z^{2n-1}$ and (3.1) becomes

$$(3.2)\qquad z^{2n-1} = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2).$$

Similarly it follows from the first of (2.23) that

$$(3.3)\qquad z^{2n-1} = \sum_{k=0}^{n-1} U(2n, 2k)z(z^2-2^2)(z^2-4^2)\cdots(z^2-(2k-2)^2).$$

By (2.26), (2.28) and (3.2) we have also

(3.4) $$z^{2n-1} = \sum_{k=0}^{n-1} V_n'(n,k)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2).$$

Formula (1.17) for $S_1(n,k)$ suggests the following definitions.

(3.5) $$U_1(n,k) = \sum \frac{n!}{1^{k_1}3^{k_2}5^{k_3}} \frac{1}{k_1!\,k_2!\,k_3!\cdots},$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$, such that

$$\begin{cases} n = k_1\cdot 1 + k_2\cdot 3 + k_3\cdot 5 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases};$$

(3.6) $$V_1(n,k) = \sum \frac{n!}{2^{k_1}4^{k_2}6^{k_3}\cdots} \frac{1}{k_1!\,k_2!\,k_3!},$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$ such that

$$\begin{cases} n = k_1\cdot 2 + k_2\cdot 4 + k_3\cdot 6 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases}.$$

We observe that $U_1(n,k)$ is the number of permutations of Z_n with k cycles each of odd length while $V_1(n,k)$ is the number of permutations of Z_n with k cycles each of even length.

It follows from (3.5) that

(3.7) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} U_1(n,k) \frac{x^n}{n!} z^k = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z}.$$

Similarly, by (3.6),

(3.8) $$\sum_{n=0}^{\infty} \sum_{2k\leqslant n} V_1(n,k) \frac{x^n}{n!} z^k = (1-x^2)^{-\frac{1}{2}z},$$

so that

(3.9) $$V_1(n,k) = \frac{(2n)!}{2^k n!} S_1(n,k).$$

This is also clear if we compare (3.6) with (1.17).

It is easily verified that

$$(1-x^2)\frac{\partial}{\partial x}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z} = z\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z}.$$

If we put

$$U_{1,n}(z) = \sum_k U_1(n,k)z^k$$

it follows from (3.7) that

(3.10) $$U_{1,n+1}(z) - n(n-1)U_{1,n-1}(z) = zU_{1,n}(z).$$

This is equivalent to

(3.11) $$U_1(n+1,k) = U_1(n,k-1) + n(n-1)U_1(n-1,k).$$

Notice that this recurrence is somewhat different in form from the familiar recurrence for $S_1(n,k)$.

By expanding the right member of (3.7) we get

(3.12) $$U_{1,n}(z) = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \binom{\frac{1}{2}z}{r} \qquad (n \geq 1).$$

To verify directly that (3.12) implies (3.10) we take

$$zU_{1,n}(z) = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \left\{ 2(r+1) \binom{\frac{1}{2}z}{r+1} + 2r \binom{\frac{1}{2}z}{r} \right\}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ 2r \binom{n-1}{r-1} + r \binom{n-1}{r-2} \right\}.$$

On the other hand

$$U_{1,n+1}(z) - n(n-1)U_{1,n-1}(z) = (n+1)! \sum_{r=1}^{n+1} 2^r \binom{n}{r-1} \binom{\frac{1}{2}z}{r} - (n-1)n! \sum_{r=1}^{n-1} 2^r \binom{n-2}{r-1} \binom{\frac{1}{2}z}{r}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ (n+1) \binom{n}{r-1} - (n-1) \binom{n-2}{r-1} \right\}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ 2r \binom{n-1}{r-1} + r \binom{n-1}{r-2} \right\}.$$

It is evident from (3.5) that

(3.13) $$U_1(n,k) = 0 \qquad (n \equiv k+1 \pmod 2).$$

This is also clear from either (3.10) or (3.11).

By means of (3.10) we get

$$U_{1,1}(z) = z, \quad U_{1,2}(z) = z^2, \quad U_{1,3}(z) = 2z + z^3,$$
$$U_{1,4}(z) = 8z^2 + z^4, \quad U_{1,5}(z) = 24z + 20z^3 + z^5.$$

The number

(3.14) $$U_{1,n} = U_{1,n}(1) = \sum_k U_1(n,k)$$

evidently denotes the total number of permutations of Z_n into cycles of odd length. By (3.12) we have

(3.15) $$U_{1,n} = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \binom{\frac{1}{2}}{r} \qquad (n \geq 1).$$

Alternatively, by (3.7) and (3.17),

$$\sum_{n=0}^{\infty} U_{1,n} \frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = (1+x)(1-x^2)^{-\frac{1}{2}} = (1+x) \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{2}\right)^{2n},$$

which yields

(3.16) $$U_{1,2n} = (2n)! \binom{2n}{n} 2^{-2n} = (1.3.5 \cdots (2n-1))^2,$$

(3.17) $$U_{1,2n+1} = (2n+1)! \binom{2n}{n} 2^{-2n} = (2n+1)U_{1,2n}.$$

4. To obtain an array orthogonal to $U(n,k)$ we consider the expansion

$$(\sqrt{1+x^2}-x)^{-z} = \sum_{n=0}^{\infty} C_n(z)\frac{x^n}{n!} \tag{4.1}$$

If we denote the left member of (4.1) by F, we have

$$\frac{\partial F}{\partial x} = \frac{z}{\sqrt{1+x^2}}F, \qquad \frac{\partial^2 F}{\partial x^2} = \left(\frac{z^2}{1+x^2} - \frac{xz}{(1+x^2)^{3/2}}\right)F,$$

which gives

$$(1+x^2)\frac{\partial^2 F}{\partial x^2} + x\frac{\partial F}{\partial x} = z^2F. \tag{4.2}$$

Substituting from (4.1) in (4.2) we get

$$C_{n+2}(z) + n(n-1)C_n(z) + nC_n(z) = z^2C_n(z),$$

so that

$$C_{n+2}(z) = (z^2-n^2)C_n(z). \tag{4.3}$$

Since $C_0(z) = 1$, $C_1(z) = z$, it follows that

$$\begin{cases} C_{2n}(z) = z^2(z^2-2^2)(z^2-4^2)\cdots(z^2-(2n-2)^2) \\ C_{2n+1}(z) = z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2n-1)^2). \end{cases} \tag{4.4}$$

Therefore (4.1) becomes

$$(\sqrt{1+x^2}-x)^{-z} = \sum_{n=0}^{\infty} \frac{z^2(z^2-2^2)\cdots(z^2(2n-2)^2)}{(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{z(z^2-1^2)\cdots(z^2-(2n-1)^2)}{(2n+1)!}x^{2n+1}. \tag{4.5}$$

If we differentiate both sides of (4.5) with respect to z and then put $z = 0$, we get

$$\log(\sqrt{1+x^2}-x) = -\sum_{n=0}^{\infty}(-1)^n\frac{1^2\cdot 3^2\cdots(2n-1)^2}{(2n+1)!}x^{2n+1}.$$

Thus (4.5) becomes

$$\exp\left\{z\sum_{n=0}^{\infty}(-1)^n\frac{1^2\cdot 3^2\cdots(2n-1)^2}{(2n+1)!}x^{2n+1}\right\} \tag{4.6}$$

$$= \sum_{n=0}^{\infty}\frac{z^2(z^2-2^2)\cdots(z^2-(2n-2)^2)}{(2n)!}x^{2n}$$

$$+ \sum_{n=0}^{\infty}\frac{z(z^2-1^2)\cdots(z^2-(2n-1)^2)}{(2n+1)!}x^{2n+1}.$$

Now replace x by ix and z by $-iz$ and we get

$$(4.7)\quad \exp\left\{z\sum_{n=0}^{\infty} 1^2\cdot 3^2\cdots(2n-1)^2\frac{x^{2n+1}}{(2n+1)!}\right\}=\sum_{n=0}^{\infty}\frac{z^2(z^2+2^2)\cdots(z^2+(2n-2)^2)}{(2n)!}x^{2n}$$

$$+\sum_{n=0}^{\infty}\frac{z(z^2+1^2)(z^2+3^2)\cdots(z^2+(2n-1)^2)}{(2n+1)!}x^{2n+1}.$$

We now define $W(n,k)$ by means of

$$(4.8)\quad \begin{cases} z^2(z^2+2^2)(z^2+4^2)\cdots(z^2+(2n-2)^2)=\sum_{k=0}^{n} W(2n,2k)z^{2k} \\ z(z^2+1^2)(z^2+3^2)\cdots(z^2+(2n-1)^2)=\sum_{k=0}^{n} W(2n+1,2k+1)z^{2k+1}. \end{cases}$$

It follows at once from (3.2), (3.3) and (4.8) that

$$(4.9)\quad \sum_{j=k}^{n}(-1)^{n-j}W(2n,2j)U(2j,2k)=\sum_{j=k}^{n}(-1)^{j-k}U(2n,2j)W(2j,2k)=\delta_{n,k},$$

$$(4.10)\quad \sum_{j=k}^{n}(-1)^{n-j}W(2n+1,2j+1)U(2j+1,2k+1)$$

$$=\sum_{j=k}^{n}(-1)^{j-k}U(2n+1,2j+1)W(2j+1,2k+1)=\delta_{n,k}.$$

By means of (4.7) we can exhibit $W(n,k)$ in a form similar to (2.9) and (2.11). Indeed it is evident from (4.7) and (4.8) that

$$(4.11)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} W(n,k)\frac{x^n}{n!}z^k=\exp\left\{z\sum_{n=0}^{\infty} f(n)\frac{x^{2n+1}}{(2n+1)!}\right\},$$

where for brevity we put

$$f(n)=1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2.$$

It follows from (4.11) that

$$(4.12)\quad W(n,k)=\sum\frac{n!}{(1!)^{k_1}(3!)^{k_2}(5!)^{k_3}\cdots}\;\frac{(f(1))^{k_1}(f(2))^{k_2}(f(3))^{k_3}\cdots}{k_1!\,k_2!\,k_3!\cdots}$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$ such that

$$(4.13)\quad n=k_1\cdot 1+k_2\cdot 3+k_3\cdot 5+\cdots,\qquad k=k_1+k_2+k_3+\cdots.$$

Moreover, in view of the definition of $U(n,k)$, we have the following combinatorial interpretation of $W(n,k)$: $W(n,k)$ is the number of *weighted* number partitions (4.13): to each partition we assign the weight

$$\frac{n!}{(1!)^{k_1}(3!)^{k_2}(5!)^{k_3}\cdots}\;\frac{(f(1))^{k_1}(f(2))^{k_2}(f(3))^{k_3}\cdots}{k_1!\,k_2!\,k_3!}.$$

A different interpretation is suggested by (4.8).

5. We now return to Problem 1 as stated in the beginning of §2.

Let $T(n,k)$ denote the number of set partitions of Z_n into k blocks

$$B_1, B_2, \cdots B_k$$

of unequal length. Then it is evident that we have the generating function

$$\sum_{n=0}^{\infty} \sum_{k} T(n,k) \frac{x^n}{n!} z^k = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n!}\right) . \tag{5.1}$$

This is equivalent to

$$T(n,k) = \sum \frac{n!}{n_1! n_2! \cdots n_k!} , \tag{5.2}$$

where the summation is over all $n_1, n_2, \cdots, n_k$ such that

$$n = n_1 + n_2 + \cdots + n_k, \qquad n_1 > n_2 > \cdots > n_k > 0. \tag{5.3}$$

In other words, $T(n,k)$ can be thought of as a weighted number partition: to each partition (5.3) we assign the weight

$$\frac{n!}{n_1! n_2! \cdots n_k!} ;$$

this weight is of course the number of admissible set partitions corresponding to the given number partition.

We can define a function that includes $T(n,k)$, $U(n,k)$, $V(n,k)$ as special cases. Let

$$\underline{r} = (r_1, r_2, r_3, \cdots) \tag{5.4}$$

be a sequence in which r_j is either a nonnegative integer or infinity. Let $S(n,k|\underline{r})$ denote the number of set partitions of Z_n into k blocks $B_1, B_2, \cdots, B_k$ with the requirement that, for each j, there are at most r_j blocks of length j. Thus, for example, we have

$$S(n,k|\underline{r}) = \begin{cases} S(n,k) & \underline{r} = (\infty, \infty, \infty, \cdots) \\ U(n,k) & \underline{r} = (\infty, 0, \infty, 0, \cdots) \\ V(n,k) & \underline{r} = (0, \infty, 0, \infty, \cdots) \\ T(n,k) & \underline{r} = (1, 1, 1, \cdots) \end{cases} . \tag{5.5}$$

For an arbitrary sequence (5.4) we have the generating function

$$\sum_{n=0}^{\infty} \sum_{k} S(n,k|\underline{r}) \frac{x}{n!} z^k = \prod_{j=1}^{\infty} \left\{ \sum_{k=0}^{r_j} \frac{1}{k!} \left(\frac{x^j z}{j!}\right)^k \right\} . \tag{5.6}$$

Clearly (5.6) reduces to a known result in each of the cases (5.5).

We shall now obtain some more explicit results for the enumerant $T(n,k)$. It is convenient to define

$$T_n(z) = \sum_{k} T(n,k) z^k \tag{5.7}$$

and

$$T_n = T_n(1) = \sum_{k} T(n,k). \tag{5.8}$$

Then, by (5.1),

$$\sum_{n=0}^{\infty} T_n(z) \frac{x^n}{n!} = \prod_{n=1}^{\infty} \left(1 - \frac{x^n z}{n!}\right) . \tag{5.9}$$

Put

$$F = F(x,z) = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n!}\right) .$$

Then it is easily verified that

(5.10) $$\log F(x,z) = \sum_{n=1}^{\infty} F_n(z) \frac{x^n}{n!} ,$$

where

(5.11) $$F_n(z) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{s(r!)^s} z^s .$$

Differentiating (5.10) with respect to x, we get

$$\frac{F_x(x,z)}{F(x,z)} = \sum_{n=0}^{\infty} F_{n+1}(z) \frac{x^n}{n!} .$$

This implies the recurrence

(5.12) $$T_{n+1}(z) = \sum_{r=0}^{n} \binom{n}{r} F_{r+1}(z) T_{n-r}(z) .$$

Differentiating (5.10) with respect to z, we get

$$\frac{F_z(x,z)}{F(x,z)} = \sum_{n=1}^{\infty} F'_n(z) \frac{x^n}{n!}$$

and therefore

(5.13) $$T'_n(z) = \sum_{r=1}^{n} \binom{n}{r} F'_n(z) T_{n-r}(z).$$

Written at length, (5.13) becomes

(5.14) $$\sum_k kT(n,k)z^k = \sum_{r=1}^{n} \binom{n}{r} T(n-r,j) \sum_{st=r} (-1)^{s-1} \frac{r!}{(t!)^s} z^s .$$

This gives

(5.15) $$kT(n,k) = \sum_{\substack{0<st\leqslant n \\ s\leqslant t}} (-1)^{s-1} \binom{n}{st} \frac{(st)!}{(t!)^s} T(n-st, k-s) .$$

It is obvious that

(5.16) $$T(n,1) = 1 \qquad (n \geqslant 1).$$

Using (5.14) we get

(5.17) $$T(n,2) = \tfrac{1}{2}(2^n - 2) - \tfrac{1}{2} \binom{n}{n/2} = S(n,2) - \tfrac{1}{2} \binom{n}{n/2} .$$

If we put

(5.18) $$G_k(x) = \sum_n T(n,k) \frac{x^n}{n!}$$

and

(5.19) $$H_j(x) = \sum_{t=1}^{\infty} \frac{x^{jt}}{(t!)^j} ,$$

then by (5.14)

$$kG_k(x) = \sum_{s=1}^{\infty} (-1)^{s-1} H_s(x) G_{k-s}(x). \tag{5.20}$$

Thus for example

$G_1(x) = H_1(x) = e^x - 1, \quad 2!G_2(x) = H_1^2(x) - H_2(x), \quad 3!G_3(x) = H_1^3(x) - 3H_1(x)H_2(x) + 2H_3(x)$

and so on.

If we take $z = 1$ in (5.12) we get the recurrence

$$T_{n+1} = \sum_{r=0}^{\infty} \binom{n}{r} F_{r+1}(1) T_{n-r}. \tag{5.21}$$

Unfortunately the numbers

$$F_n(1) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{s(r!)^s}$$

are not simple. We note that

$$\sum_{n=1}^{\infty} F_n(1) \frac{x^n}{n!} = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} H_s(x). \tag{5.22}$$

Analogous to (5.2) we may define

$$T_1(n,k) = \sum \frac{n!}{n_1 n_2 \cdots n_k}, \tag{5.23}$$

where again the summation is over all $n_1, n_2, \cdots, n_k$ such that

$$n = n_1 + n_2 + \cdots + n_k, \qquad n_1 > n_2 > \cdots > n_k > 0.$$

Then $T_1(n,k)$ denotes the number of permutations of Z_n with k cycles of unequal length. From (5.23) we obtain the generating function

$$\sum_{n=0}^{\infty} \sum_{k} T_1(n,k) \frac{x^n}{n!} z^k = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n}\right). \tag{5.24}$$

As above we define

$$T_{1,n}(z) = \sum_k T_1(n,k) z^k, \qquad T_{1,n} = T_{1,n}(1) = \sum_k T_1(n,k).$$

We can obtain recurrences for $T_1(n,k)$ and $T_{1,n}$ similar to those for $T(n,k)$ and T_n. In particular we have

$$T_{1,n+1} = \sum_{r=0}^{n} \binom{n}{r} F_{1,r+1}(1) T_{1,n-r}, \tag{5.25}$$

where

$$F_{1,n}(1) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{sr^s}.$$

We remark that $T_{1,n}$ is the total number of permutations of Z_n with cycles of unequal length. Note that

$$\sum_{n=1}^{\infty} T_{1,n} \frac{x^n}{n!} = \prod_{n=1}^{\infty} \left(1 + \frac{x^n}{n}\right). \tag{5.26}$$

Finally, as in (5.4), let

$$\underline{r} = (r_1, r_2, r_3, \cdots) \tag{5.27}$$

be a sequence in which each r_j is either a nonnegative integer or infinity. Let $S_1(n,k|\underline{r})$ denote the number of permutations π in Z_n with the requirement that, for each i, the number of cycles of length i in π is at most r_i. Then

$$S_1(n,k|\underline{r}) = \begin{cases} S_1(n,k) & \underline{r} = (\infty, \infty, \infty, \cdots) \\ U_1(n,k) & \underline{r} = (\infty, 0, \infty, 0, \cdots) \\ V_1(n,k) & \underline{r} = (0, \infty, 0, \infty, \cdots) \\ T_1(n,k) & \underline{r} = (1, 1, 1, \cdots). \end{cases}$$

For an arbitrary sequence (5.27) we have the generating function

$$\sum_{n=0}^{\infty} \sum_{k} S_1(n,k|\underline{r}) \frac{x^n}{n!} z^k = \prod_{j=1}^{\infty} \left\{ \sum_{k=0}^{r_j} \frac{1}{k!} \left(\frac{x^j z}{j}\right)^k \right\}. \tag{5.28}$$

The following question is of some interest. For what sequences (5.27) will the orthogonality relations

$$\sum_{j=k}^{n} (-1)^{n-j} S_1(n,j|\underline{r}) S(j,k|\underline{r}) \tag{5.29}$$

$$= \sum_{j=k}^{n} (-1)^{j-k} S(n,j|\underline{r}) S_1(j,k|\underline{r}) = \delta_{n,k}$$

be satisfied?

Alternatively we may ask for what pairs of sequences $\underline{r}$, $\underline{s}$ will the orthogonality relations

$$\sum_{j=k}^{n} (-1)^{n-j} S_1(n,j|\underline{r}) S(j,k|\underline{s}) = \sum_{j=k}^{n} (-1)^{j-k} S(n,j|\underline{s}) S_1(j,k|\underline{r}) = \delta_{n,k} \tag{5.30}$$

be satisfied?

REFERENCES

1. L. Carlitz, "Single Variable Bell Polynomials," *Collectanea Mathematica,* Vol. 14 (1962), pp. 13–25.
2. L. Carlitz and John Riordan, "The Divided Central Differences of Zero," *Canadian Journal of Mathematics,* Vol. 15 (1963), pp. 94–100.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Clarendon Press, Oxford, 1938.
4. John Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.
5. Gian-Carlo Rota, "The Number of Partitions of a Set," *American Math. Monthly,* Vol. 71 (1964), pp. 498–504.

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PRIMITIVE PERIODS OF GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

In this paper we are concerned with the primitive periodicity of Fibonacci-type sequences; where the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$; and the generalized Fibonacci sequence $\{H_n\}_{n=0}^{\infty}$ has any two relatively prime starting values with the rule, $H_{n+2} = H_{n+1} + H_n$. The Lucas sequence $\{L_n\}_{n=0}^{\infty}$ is defined with $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$; and the generalized Lucas sequence $\{G_n\}_{n=0}^{\infty}$ is defined recursively by $G_n = H_{n+1} + H_{n-1}$. We will see that in one case, that of modulo 3^n, all generalized Fibonacci sequences have the same primitive periodicity. Then we will observe that the primitive periods of $\{F_n\}$ and $\{L_n\}$ are the same, modulus p^m, where p is a prime, $p \neq 5$.

Prior to examination of the Fibonacci case mod 3^n we will prove the following theorem:

Theorem. If $n \mid F_m$, then $n^k \mid F_{mn^{k-1}}$. We use the fact that

$$\alpha^m = F_m\alpha + F_{m-1} \qquad \beta^m = F_m\beta + F_{m-1} \qquad (\alpha^{mn^{k-2}})^n = \alpha^{mn^{k-1}}$$

$$\alpha^{mn^{k-2}} = \alpha F_{mn^{k-2}} + F_{mn^{k-2}-1} \qquad \beta^{mn^{k-2}} = \beta F_{mn^{k-2}} + F_{mn^{k-2}-1} \quad .$$

By definition,

$$F_{mn^{k-1}} = \sum_{j=0}^{n} \binom{n}{j} (F_{mn^{k-2}})^j (F_{mn^{k-2}-1})^{n-j} F_j$$

$$F_{mn^{k-1}} = 0 + nF_{mn^{k-2}}(F_{mn^{k-2}-1})^{n-1}F_1 + \binom{n}{2} (F_{mn^{k-2}})^2 F_{mn^{k-2}-1} F_2 \;.$$

By induction, $n^{k-1} \mid F_{mn^{k-2}}$. Clearly, n^{k-1} also divides all successive terms as j is increasing. Our proof is complete.

2. THE FIBONACCI CASE MOD 3^n

Theorem 1. The period (not necessarily primitive) of the Fibonacci sequence modulo 3^n is $2^3 \cdot 3^{n-1}$. We will prove that: (A) $F_{2^3\cdot 3^{n-1}} \equiv F_0 \pmod{3^n}$ and (B) $F_{2^3\cdot 3^{n-1}+1} \equiv F_1 \pmod{3^n}$.

A. The proof is direct.

$3 \mid F_{2^2}$, thus $3^k \mid F_{2^2\cdot 3^{k-1}}$, using the theorem; If $m \mid F_n$, then $m^k \mid F_{nm^{k-1}}$.

It follows that $3^k \mid F_{2^3\cdot 3^{k-1}}$, thus $F_{2^3\cdot 3^{k-1}} \equiv 0 \pmod{3^k}$.

Hence Part A is proved.

B. (1) First, $$F_{2^3\cdot 3^{n-1}+1} = (F_{2^2\cdot 3^{n-1}+1})^2 + (F_{2^2\cdot 3^{n-1}})^2$$
using the identity

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n \,.$$

Now, since $(F_{2^2\cdot 3^{n-1}})^2 \equiv 0 \pmod{3^n}$ From Part A, it follows that

(2) $(F_{2^3\cdot 3^{n-1}})^2 \equiv 1 \pmod{3^n}$ from the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

(3) Now, substituting into (1) we have $F_{2^3\cdot 3^{n-1}+1} \equiv 1 + 0 \pmod{3^n}$. Hence Part B is proved.

Theorem 2. The primitive period of the Fibonacci sequence modulo 3^n is $2^3\cdot 3^{n-1}$. Secondly, $2^2\cdot 3^{n-1}$ is the entry point of 3^n. Let 3^k be the highest power of 3 dividing F_n; the notation is $3^k \| F_n$.

(4) We now prove that $3^n \| F_{2^2\cdot 3^{n-1}}$. The proof is by induction. We will have to consider three cases,

CASE 1. $n = 1$. $3^1 \| F_{2^2\cdot 3^{1-1}}$; $3 \| F_4 = 3$ and, $3^{1+1} \nmid F_{2^3\cdot 3^{1-1}}$ $9 \nmid F_8 = 21$.

CASE 2. $n = 2$. $3^2 \| F_{2^2\cdot 3^{2-1}}$; $9 \| F_{12} = 144$ and, $3^{2+1} \nmid F_{2^3\cdot 3^{2-1}}$ $27 \nmid F_{24} = 46368$.

CASE 3. $n > 2$. Assume $3^k \| F_{2^2\cdot 3^{k-1}}$; then we claim $3^{k+1} \nmid F_{2^3\cdot 3^{k-1}}$.

$$F_{2^3\cdot 3^{n-1}} = (F_{2^2\cdot 3^{n-1}})(L_{2^2\cdot 3^{n-1}})$$

using the identity $F_{2n} = F_nL_n$. Now, given that $3^n \| F_{2^2\cdot 3^{n-1}}$ and since (F_n, L_n) is 1 or 2, then

$$3^{n+1} \nmid F_{2^3\cdot 3^{n-1}}.$$

(5) If $3^{k+1} \nmid F_{2^2\cdot 3^k}$ then 3^{k+1} divides a smaller F_m whose subscript is a multiple of the first F_m that is divisible by 3^k. It must be of the form, $p(2^2\cdot 3^{k-1})$. Clearly, $p \neq 1$, for that contradicts our assumption that $3^k \| F_{2^2\cdot 3^{k-1}}$. And $p \neq 2$, for $3^{k+1} \nmid F_{2^3\cdot 3^{k-1}}$. We conclude that $p = 3$, hence the first F_m divisible by 3^{k+1} is $F_{2^2\cdot 3^k}$. Furthermore,

$$F_{2^2\cdot 3^k} = F_{2^2\cdot 3^{k-1}}(5(F_{2^2\cdot 3^{k-1}})^2 + 3)$$

implies $3^{k+1} \| F_{2^2\cdot 3^k}$ as it clearly shows $3^{k+2} \nmid F_{2^2\cdot 3^k}$. Our claim in (4) is true; our proof is complete by induction.

(6) Now that we have found the first F_m divisible by 3^k, we can write the primitive period modulo 3^k as a multiple of that subscript. The primitive period is of the form $s(2^2\cdot 3^{k-1})$. We have shown that when $s = 2$ we have a period, not necessarily primitive. We must examine $s < 2$, that is, $s = 1$. If the primitive period were to be $1(2^2\cdot 3^{k-1})$, then we would need

$$F_{2^2\cdot 3^{k-1}} \equiv F_0 \quad\text{and}\quad F_{2^2\cdot 3^{k-1}+1} \equiv F_1 \pmod{3^k}.$$

We claim that the latter is false.

(7) We assert that $F_{2^2\cdot 3^{k-1}} \not\equiv F_1 \pmod{3^k}$, but that

$$F_{2^2\cdot 3^{k-1}+1} \equiv (-F_1) \pmod{3^k}.$$

This follows by induction.

(8) Case 1. $k = 1$. $F_{2^2\cdot 3^{1-1}+1} = F_5 = 2 \equiv -1 \pmod{3}$.

Case 2. $k = 2$. $F_{2^2\cdot 3^{2-1}+1} = F_{13} = 233 \equiv -1 \pmod{3^2}$.

Case 3. $k > 2$. Assume that $F_{2^2\cdot 3^{k-1}+1} \equiv -1 \pmod{3^k}$.

(9) Recall from Theorem 1, that $F_{2^2\cdot 3^{k-1}+1} \equiv 1 \pmod{3^k}$ and that $F_{2^2\cdot 3^{k-1}} \equiv 0 \pmod{3^k}$.

(10) Observe that

$$F_{2^2\cdot 3^k+1} = (F_{2^2\cdot 3^{k-1}+1})(F_{2^3\cdot 3^{k-1}+1}) + (F_{2^2\cdot 3^{k-1}})(F_{2^3\cdot 3^{k-1}}),$$

using the identity $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

(11) Now substituting (9) into (10) and using our inductive assumption in (8) we have

$$F_{2^2\cdot 3^k+1} \equiv (-1)(1) + (0)(0) \pmod{3^{k+1}}.$$

That is, $F_{2^2\cdot 3^k+1} \equiv (-F_1) \pmod{3^{k+1}}$ and our proof is complete.

(12) We conclude that $s \not< 2$, thus $s = 2$ provides the primitive period and Theorem 2 is proved.

3. THE GENERAL FIBONACCI CASE MOD 3^n

Theorem 3A. The period (not necessarily primitive) of any generalized Fibonacci sequence modulo 3^n is $2^3\cdot 3^{n-1}$. We will prove that: (A) $H_{2^3\cdot 3^{n-1}+1} \equiv H_1 \pmod{3^n}$ and (B) $H_{2^3\cdot 3^{n+1}+2} \equiv H_2 \pmod{3^n}$.

A. We will have to consider three cases.

Case 1. $n = 1$. $$H_{2^3 \cdot 3^{1-1}+1} = H_9 = 21H_2 + 13H_1 \equiv H_1 \pmod{3^n}.$$

Case 2. $n = 2$. $$H_{2^3 \cdot 3^{2-1}+1} = H_{25} = 46368H_2 + 28657H_1 \equiv H_1 \pmod{3^2}.$$

Case 3. $n > 2$.

(13) First, $$H_{2^3 \cdot 3^{n-1}+1} = H_1 F_{2^3 \cdot 3^{n-1}-1} + H_2 F_{2^3 \cdot 3^{n-1}}$$

from the identity $$H_{n+1} = H_1 F_{n-1} + H_2 F_n.$$

(14) But since

$$F_{2^3 \cdot 3^{n-1}} \equiv 0 \pmod{3^n}, \quad \text{and} \quad F_{2^3 \cdot 3^{n-1}-1} = F_{2^3 \cdot 3^{n-1}+1} - F_{2^3 \cdot 3^{n-1}} = 1 - 0 = 1$$

from the recursion rule that $F_{m-1} = F_{m+1} - F_m$; we substitute (14) into (13) to obtain that

(15) $$H_{2^3 \cdot 3^{n-1}+1} \equiv H_1(1) + H_2(9) \pmod{3^n}$$

and Part A is proved.

B. First, $$H_{2^3 \cdot 3^{n-1}+2} = H_1 F_{2^3 \cdot 3^{n-1}} + H_2 F_{2^3 \cdot 3^{n-1}+1}$$

from the identity $$H_{n+2} = H_1 F_n + H_2 F_{n+1}.$$

Since $F_{2^3 \cdot 3^{n-1}} \equiv 0 \pmod{3^n}$ from 1-A, and

$$F_{2^3 \cdot 3^{n-1}+1} \equiv 1 \pmod{3^n}$$

from 1-B, Part B follows immediately.

Theorem 3B. The primitive period of any generalized Fibonacci sequence modulo 3^n is $2^3 \cdot 3^{n-1}$.

In Theorem 3A we proved that the period is at most $2^3 \cdot 3^{n-1}$. It remains to show that the primitive period is no smaller.

Consider the generalized Fibonacci sequence $\{H_n\}$, $(H_1, H_2) = 1$. Adding alternate terms we derive another generalized sequence $\{D_n\}$. We observe: $H_2 + H_0 = kD_1$ where k is an integer, $H_3 + H_1 = kD_2$, and so on.

We need to examine the possible values for k. We rewrite the equations above:

$$2H_2 - H_1 = kD_1 \qquad H_2 + 2H_1 = kD_2.$$

We solve for H_1 and H_2:

$$H_2 = \frac{k}{5}(2D_1 + D_2) = \frac{k}{5}(D_3 + D_1) \qquad H_1 = \frac{k}{5}(2D_2 - D_1) = \frac{k}{5}(D_2 + D_0).$$

If $k = 5$, then $\{H_n\}$ is a generalized Lucas sequence. If $5 \nmid k$, then $k = 1$ because $(H_1, H_2) = 1$, and 5 must divide $(D_3 + D_1)$ and $(D_2 + D_0)$. Thus $k = 1$ implies that $\{D_n\}$ is a generalized Lucas sequence.

We conclude that modulo 5^n is the only prime modulus in which the primitive period of a generalized Fibonacci sequence will be smaller than in the Fibonacci case. We note that it will be smaller by a factor of five. Hence, our proof of Theorem 3B is complete.

Example. The period modulo 5^n of the Fibonacci sequence is $4 \cdot 5^n$ while the period mod 5^n of the Lucas sequence is $4 \cdot 5^{n-1}$.

4. THE FIBONACCI AND LUCAS CASES MOD p^m

Lemma 1. A prime p, does not divide $\{L_n\}$ if and only if the entry point of p in $\{F_n\}$, (EP_F), is odd. We will examine two cases in the proof.

Case 1: Given $p \nmid \{L_n\}$.

(16) Assume EP_F is even, that is, $EP_F = F_{2k}$, we write $p \,\|\, F_{2k}$.

(17) $p \,\|\, F_{2k}$ implies $p \nmid F_k$.

Recall the identity $F_{2k} = F_k L_k$. Therefore, $p \mid L_k$. This contradicts that $p \nmid \{L_n\}$.

(18) Hence our assumption in (16) is not true, so EP_F is odd. We conclude that $p \nmid \{L_n\}$ implies EP_F is odd.

Case 2: Given EP_F is odd.

(19) Assume $p \mid \{L_n\}$. Then there exists k such that $p \,\|\, L_k$.

(20) Recall that the greatest common divisor of (F_n, L_n) is 1 or 2. Hence $p \nmid F_k$.

(21) $p \| L_k$ implies $p \| F_{2k}$ from the identity $F_{2k} = F_k L_k$. This contradicts that EP_F is odd.

(22) Therefore $p \nmid \{L_n\}$. We conclude that EP_F is odd implies that $p \nmid \{L_n\}$ and our proof of Lemma 1 is complete.

Lemma 2. A prime p divides $\{L_n\}$ if and only if EP_F is either of the form 2 (odd) or 2^m (odd), $m \geqslant 2$. This follows immediately from Lemma 1 and the identity $F_{2^n k} = F_{2^{n-1}k} \cdot L_{2^{n-1}k}$.

Theorem 4. The primitive periods of $\{F_n\}$ and $\{L_n\}$ are of the same length, modulus p, for p a prime, $p \neq 5$.

Case 1. The primitive period for $\{L_n\}$ is no longer than for $\{F_n\}$.

(23) We have $L_{n+k} - L_{n-k} = L_n L_k$, k odd.

(24) $L_{n+k} - L_{n-k} = 5F_n F_k$, k even.

Now, let $2k$ denote the length of the period of $\{F_n\}$. Thus k denotes half the period of $\{F_n\}$. When EP_F of p is odd then the period, $2k$, is $4(EP_F)$. Thus $k = 2(EP_F)$ so k is even. Likewise, when EP_F of p is of the form 2^m (odd) for $m \geqslant 2$, then the period, $2k$, is $2(EP_F)$. Thus $k = EP_F = 2^m$ (odd) so k is even.

Note, above that either $k = 2EP_F$ or $k = EP_F$, thus $F_k \equiv 0$, mod p. Hence, $L_{n+k} - L_{n-k} \equiv 0$, mod p. It follows that the period of $\{L_n\}$ is $2k$ which is the period of $\{F_n\}$.

Now we consider the special case when EP_F is of the form 2 (odd). Then the period, $2k$, is EP_F, and $k = ½2$ (odd) so k is odd. We will use Eq. (23). We recall that $F_{2k} F_k L_k$ implies $p | L_k$ since EP_F of p is F_{2k} implies $p \nmid F_k$. Hence $p | L_k$ means $L_k \equiv 0$, mod p. Therefore $L_{n+k} - L_{n-k} \equiv 0$, mod p. It follows that the period of $\{L_n\}$ is $2k$, again the same as the period of $\{F_n\}$.

Case 2. The primitive period for $\{L_n\}$ is no shorter than for $\{F_n\}$.

A. First we will consider the situation in which EP_F is odd. Then the period is $4(EP_F)$ and $k = 2(EP_F)$. By Lemma 1, $p \nmid \{L_n\}$.

(25) Assume the primitive period for $\{L_n\}$ is shorter than for $\{F_n\}$, that is, the primitive period for $\{L_n\}$ is half the period of $\{F_n\}$. Then the period for $\{L_n\}$ is $2(EP_F)$. We use Eq. (23) since

(26) EP_F is odd. We have $L_{n+EP_F} - L_{n-EP_F} = L_n L_{EP_F}$. But $p \nmid \{L_n\}$ thus $p \nmid L_{EP_F}$ so Eq. (26) is not con-

(27) gruent to zero. Therefore, the period cannot be $2(EP_F)$. Our assumption in (25) is false, so when EP_F is odd the period of $\{L_n\}$ is no shorter than for $\{F_n\}$.

B. Now we consider the situation in which EP_F **is** of the form $2d$, where d is odd. Then the period for $\{F_n\}$ is EP_F.

(28) Assume the primitive period for $\{L_n\}$ is shorter than for $\{F_n\}$. We note that $L_d \equiv 0$ since EP_F is F_{2d} and the fact that $F_{2d} = F_d L_d$. Now, assuming the primitive period for $\{L_n\}$ is smaller means that there exists c where $c < d$ such that $L_{n+c} - L_{n-c} = L_n L_c$. This would meet the requirement since the period $2c < 2d$. However, $L_c = 0$ implies that $F_{2c} \equiv 0$ mod p which contradicts that EP_F of p is F_{2d}.

(29) Our assumption in (28) is false, so when EP_F is of the form $2d$ where d is odd, then the period for $\{L_n\}$ is no shorter than for $\{F_n\}$.

C. Lastly, we consider the situation in which EP_F is of the form $2^m d$, where d is odd and $m \geqslant 2$. Then the

(30) period for F_n, is $2EP_F$. Assuming the primitive period for $\{L_n\}$ is smaller, then it too must be even since the period for $\{F_n\}$ is even. There exists b where $b < EP_F$ such that $L_{n+b} - L_{n-b} = 5F_n F_b$.

But if $2b$ is to be the period for $\{L_n\}$ then $5F_nF_b \equiv 0 \bmod p$. But $F_b \not\equiv 0 \bmod p$ since $b < EP_F$. Our

(31) assumption in (30) must be false. We conclude that if EP_F is of the form 2^md, where d is odd, $m \geq 2$, then the primitive period for $\{L_n\}$ is no shorter than for $\{F_n\}$.

Our conclusions in (27), (29), and (31) prove that Case 2 is true. Thus our proof of Theorem 4 is complete.

Examples of Theorem 4

Example 1. EP_F of p is odd.

Take $p = 13$. The $EP_F = 7$. We see the length of the primitive period of $\{F_n\}$ is 28.

Period of $\{F_n\}$ mod 13 = 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0.

Period of $\{L_n\}$ mod 13 = 1, 3, 4, 7, 11, 5, 3, 8, 11, 6, 4, 10, 1, 11, 12, 10, 9, 6, 2, 8, 10, 5, 2, 7, 9, 3, 12, 2.

We see that the primitive period of $\{F_n\}$ is exactly the same length as the primitive period of $\{L_n\}$.

We also observe that Lemma 1 is demonstrated as $p \nmid \{L_n\}$.

Example 2. EP_F of p is of the form 2 (odd).

Take $p = 29$. The EP_F = 14 = 2(7). The length of the primitive period of $\{F_n\}$ is 14.

Period of $\{F_n\}$ mod 29 = 1, 1, 2, 3, 5, 8, 13, 21, 5, 26, 2, 28, 1, 0.

Period of $\{L_n\}$ mod 29 = 1, 3, 4, 7, 11, 18, 0, 18, 18, 7, 25, 3, 28, 2.

We see that the primitive period of $\{F_n\}$ is exactly the same length as of $\{L_n\}$.

Also note that the $EP_F = 2EP_L$. We see Lemma 2 demonstrated.

Example 3. EP_F of p is of the form 2^m (odd), $m \geq 2$.

Take p = 47. The EP_F = 16 = $2^4(1)$. The length of the primitive period of $\{F_n\}$ is 32.

Period of $\{F_n\}$ mod 47 = 1, 1, 2, 3, 5, 8, 13, 21, 34, 8, 42, 3, 45, 1, 46, 0, 46, 46, 45, 44, 42, 39, 34, 26, 13, 39, 5, 44, 2, 46, 1, 0.

Period of $\{L_n\}$ mod 47 = 1, 3, 4, 7, 11, 18, 29, 0, 29, 29, 11, 40, 36, 29, 18, 0, 18, 18, 36, 7, 43, 3, 46, 2.

Again we see that the primitive period of $\{F_n\}$ is exactly the same as for $\{L_n\}$.

We notice that the $EP_F = 2EP_L$, and we see Lemma 2 demonstrated.

Comment. In this study we came across an unanswered problem that was discovered by D. D. Wall in 1960. It concerns the hypothesis that "Period mod $p^2 \neq$ Period mod p." He ran a test on a digital computer that verified the hypothesis was true for all p less than 10,000. Until this day no one as yet has proven that the Period mod p^2 = Period mod p is impossible.

We give an example to show that the above hypothesis does not hold for composite numbers. Period mod 12^2 = Period mod 12 = 24.

Period mod 12 of $\{F_n\}$ = 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, 5, 10, 3, 1, 4, 5, 9, 2, 11, 1, 0.

Period mod 12^2 of $\{F_n\}$ = 1, 1, 2, 3, 5, 8, 13, 21, 55, 89, 0, 89, 89, 34, 123, 13, 136, 5, 141, 2, 143, 1, 0.

We note that EP_F of 12 = EP_F of 12^2.

REFERENCE

D. D. Wall, "Fibonacci Series Modulo *m*," *The Amer. Math. Monthly,* Vol. 67, No. 6 (June-July 1960), pp. 525–532.

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ON THE PRIME FACTORS OF $\binom{n}{k}$

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A well known theorem of Sylvester and Schur (see [5]) states that for $n \geqslant 2k$, the binomial coefficient $\binom{n}{k}$ always has a prime factor exceeding k. This can be considered as a generalization of the theorem of Chebyshev: There is always a prime between m and $2m$. Set

$$\binom{n}{k} = u_n(k)v_n(k)$$

with

$$u_n(k) = \prod_{\substack{p^\alpha \| \binom{n}{k} \\ p<k}} p^\alpha, \qquad v_n(k) = \prod_{\substack{p^\alpha \| \binom{n}{k} \\ p \geq k}} p^\alpha .$$

In [4] it is proved that $v_n(k) > u_n(k)$ for all but a finite number of cases (which are tabulated there).

In this note, we continue the investigation of $u_n(k)$ and $v_n(k)$. We first consider $v_n(k)$, the product of the large prime divisors of $\binom{n}{k}$.

Theorem.

$$\max_{1 \leqslant k \leqslant n} v_n(k) = e^{\frac{n}{2}(1+o(1))} .$$

Proof. For $k < \epsilon n$ the result is immediate since in this case $\binom{n}{k}$ itself is less than $e^{n/2}$. Also, it is clear that the maximum of $v_n(k)$ is not achieved for $k > n/2$. Hence, we may assume $\epsilon n \leqslant k \leqslant n/2$. Now, for any prime

$$p \in \left(\frac{n-k}{r}, \frac{n}{r} \right]$$

with $p \geqslant k$ and $r \geqslant 1$, we have $p \mid v_n(k)$. Also, if $k^2 > n$ then $p^2 \nmid v_n(k)$ so that in this case the contribution to $v_n(k)$ of the primes

$$p \in \left(\frac{n-k}{r}, \frac{n}{r} \right]$$

is (by the Prime Number Theorem (PNT)) just $e^{\frac{k}{r}(1+o(1))}$. Thus, letting $\frac{n}{t+1} < k \leqslant \frac{n}{t}$, we obtain

$$v_n(k) = \exp\left[\left(\sum_{r=1}^{t-1} \frac{k}{r} + \left(\frac{n}{t} - k \right) \right) \right] (1+o(1)) = \exp\left[\left(\frac{n}{t} \sum_{r=1}^{t-1} \frac{1}{r} \right) (1+o(1)) \right]$$

$$\leqslant e^{\frac{n}{2}(1+o(1))}$$

and the theorem is proved.

It is interesting to note that since

$$\frac{n}{t} \sum_{r=1}^{t-1} \frac{1}{r} = \frac{1}{2}$$

for both $t = 2$ and $t = 3$ then

$$\lim_{n} v_n(k)^{1/n} = e^{1/2}$$

for any $k \in \left(\frac{n}{3}, \frac{n}{2}\right)$.

In Table 1, we tabulate the least value $k^*(n)$ of k for which $v_n(k)$ achieves its maximum value for selected values of $n \leqslant 200$. It seems likely that infinitely often $k^*(n) = \frac{n}{2}$ but we are at present far from being able to prove this.

Table 1

n	$k^*(n)$	n	$k^*(n)$	n	$k^*(n)$
2	1	10	2	18	8
3	1	11	3	19	9
4	2	12	6	20	10
5	2	13	4	50	22
6	2	14	4	100	42
7	3	15	5	200	100
8	4	16	6		
9	2	17	7		

Note that

$$v_7(0) < v_7(1) < v_7(2) < v_7(3).$$

It is easy to see that for $n > 7$, the $v_n(k)$ cannot increase monotonically for $0 \leqslant k \leqslant \frac{n}{2}$.

Next, we mention several results concerning $u_n(k)$. To begin with, note that while $u_7(k) = 1$ for $0 \leqslant k \leqslant \frac{n}{2} = \frac{7}{2}$, this behavior is no longer possible for $n > 7$. In fact, we have the following more precise statement.

Theorem. For some $k \leqslant (2 + o(1)) \log n$, we have $u_n(k) > 1$.

Proof. Suppose $u_n(k) = 1$ for all $k \leqslant (2 + \epsilon) \log n$. Choose a prime $p < (1 + \epsilon) \log n$ which does not divide $n + 1$. Such a prime clearly exists (for large n) by the PNT. Since $p \nmid n + 1$ then for some k with $p < k < 2p$,

$$p^2 \mid n(n-1) \cdots (n-k+1), \quad p^2 \nmid k!$$

Thus, $p \mid u_n(k)$ and since

$$k < 2p < (2 + 2\epsilon) \log n,$$

the theorem is proved.

In the other direction we have the following result.

Fact. There exist infinitely many n so that for all $k \leqslant (1/2 + o(1)) \log n$, $u_n(k) = 1$.

Proof. Choose $n + 1 = [\ell.\text{c.m.} \{1, 2, \cdots, t\}]^2$. By the PNT, $n = e^{(2+o(1))t}$. Clearly, if $m \leqslant t$ then $m \nmid \binom{n}{t}$. Thus,

$$u_n(k) = 1 \quad \text{for} \quad k \leqslant \left(\frac{1}{2} + o(1)\right) \log n$$

as claimed.

In Table 2 we list the least value $n^*(k)$ of n such that $u_n(i) = 1$ for $1 \leqslant i \leqslant k$

Table 2

k	$n^*(k)$
1	1
2	2
3	3
4	7
5	23
6	71

Of course, for $k \leqslant 2$, $u_n(k) = 1$ is automatic. By a theorem of Mahler [11], it follows that

$$u_n(k) < n^{1+\epsilon}$$

for $k \geqslant 3$ and large n. It is well known that if $p^\alpha \mid \binom{n}{k}$ then $p^\alpha \leqslant n$. Consequently,

$$u_n(k) \leqslant n^{\pi(k)},$$

where $\pi(k)$ denotes the number of primes not exceeding k. It seems likely that the following stronger estimate holds:

$$(*) \qquad u_n(k) < n^{(1+o(1))(1-\gamma)\pi(k)}, \quad k \geqslant 5,$$

where γ denotes Euler's constant. It is easy to prove (*) for certain ranges of k. For example, suppose k is relatively large compared to n, say, $k = n/t$ for a large fixed t. Of course, any prime $p \in (n - n/t, n)$ divides $v_n(k)$ and by the PNT

$$\prod_{n(1-1/t)<p<n} p = e^{(1+o(1))n/t}.$$

More generally, if $rp\epsilon(n - n/t, n)$ with $r < t$ then $p \geqslant k$ and $p \mid v_n(k)$ so that again by the PNT

$$\prod_{\frac{n}{r}\left(1-\frac{1}{t}\right)<p<\frac{n}{r}} p = e^{(1+o(1))n/rt}.$$

Thus

$$v_n(k) \geqslant \prod_{1\leqslant r<t}\ \prod_{\frac{n}{t}\left(1-\frac{1}{t}\right)<p<\frac{n}{r}} p = \exp\left((1+o(1)) \sum_{1\leqslant r<t} \frac{1}{r}\right)\frac{n}{t}$$

$$= \exp\left((1+o(1))(\log t+\gamma)\right)\frac{n}{t}.$$

But by Stirling's formula we have

$$\binom{n}{n/t} = e^{\frac{n}{t}\log t+\frac{n}{t}+o\left(\frac{n}{t}\right)}.$$

Thus,

$$u_n(k) = \binom{n}{k} / v_n(k) \leqslant e^{\frac{n}{t}\log t+\frac{n}{t}+o\left(\frac{n}{t}\right)-(1+o(1))(\log t+\gamma)\frac{n}{t}}$$

$$= e^{(1+o(1))(1-\gamma)\frac{n}{t}} = n^{(1+o(1))(1-\gamma)\pi(k)}$$

which is just (*).

In contrast to the situation for $v_n(k)$, the maximum value of $u_n(k)$ clearly occurs for $k \geqslant \frac{n}{2}$. Specifically, we have the following result.

Theorem. The value $\hat{k}(n)$ of k for which $u_n(k)$ assumes its maximum value satisfies

$$\hat{k}(n) = (1+o(1))\left(\frac{e}{e+1}\right) n.$$

Proof. Let $k = (1-c)n$. For $c \leqslant \frac{1}{2}$,

$$v_n(k) = \prod_{n-k<p\leqslant n} p = e^{(1+o(1))cn}.$$

Since

$$\binom{n}{k} = \binom{n}{cn} = e^{-(c\log c+(1-c)\log(1-c))(1+o(1))n}$$

then

$$u_n(k) = \binom{n}{k} / v_n(k) = e^{-(1+o(1))(c+\log c^c(1-c)^{1-c})n}.$$

A simple calculation shows that the exponent is maximized by taking $c = \frac{1}{e+1} = 0.2689\cdots$.

Concluding remarks. We mention here several related problems which were not able to settle or did not have time to investigate. One of the authors [8] previously conjectured that $\binom{2n}{n}$ is never squarefree for $n > 4$ (at present this is still open). Of course, more generally, we expect that for all a, $\binom{2n}{n}$ is always divisible by an a^{th} power of a prime $> k$ if $n > n_0(a,k)$. We can show the much weaker result that $n = 23$ is the largest value of n for which *all* $\binom{n}{k}$ are squarefree for $0 \leqslant k \leqslant n$. This follows from the observation that if p is prime and $p^{\alpha} \nmid \binom{n}{k}$ for any k then $p^{\beta} | n + 1$, where

$$p^{\beta} \geqslant \frac{n+1}{p^{\alpha}-1}\ .$$

Thus, $2^2 \nmid \binom{n}{k}$ for any k implies $2^{\beta} | n + 1$ where $2^{\beta} \geqslant \frac{n+1}{3}$. Also, $3^2 \nmid \binom{n}{k}$ for any k implies $3^{\gamma} | n + 1$ where $3^{\gamma} \geqslant \frac{n+1}{8}$. Together these imply that $d = 2^{\beta}3^{\gamma} | n + 1$ where $d \geqslant (n + 1)^2/24$. Since d cannot exceed $n + 1$ then $n + 1 \leqslant 24$ is forced, and the desired result follows.

For given n let $f(n)$ denote the largest integer such that for *some* k, $\binom{n}{k}$ is divisible by the $f(n)^{th}$ power of a prime. We can prove that $f(n) \to \infty$ as $n \to \infty$ (this is not hard) and very likely $f(n) > c \log n$ but we are very far from being able to prove this. Similarly, if $F(n)$ denotes the largest integer so that for *all* k, $1 \leqslant k < n$, $\binom{n}{k}$ is divisible by the $F(n)^{th}$ power of some prime, then it is quite likely that $\overline{\lim}\ F(n) = \infty$, but we have not proved this.

Let $P(x)$ and $p(x)$ denote the greatest and least prime factors of x, respectively. Probably

$$P\left(\binom{n}{k}\right) > \max\,(n - k, k^{1+\epsilon})$$

but this seems very deep (for related results see the papers of Ramachandra and others [11], [12]).

J. L. Selfridge and P. Erdös conjectured and Ecklund [1] proved that $p\left(\binom{n}{k}\right) < \frac{n}{2}$ for $k > 1$, with the unique exception of $p\left(\binom{7}{3}\right) = 5$. Selfridge and Erdös [9] proved that

$$p\left(\binom{n}{k}\right) < \frac{c_1 n}{k^{c_2}}$$

and they conjecture

$$p\left(\binom{n}{k}\right) < \frac{n}{k} \quad \text{for} \quad n > k^2\ .$$

Finally, let $d\left(\binom{n}{k}\right)$ denote the greatest divisor of $\binom{n}{k}$ not exceeding n. Erdös originally conjectured that $d\left(\binom{n}{k}\right) > n - k$ but this was disproved by Schinzel and Erdös [13]. Perhaps it is true however, that $d_n > cn$ for a suitable constant c.

For problems and results of a similar nature the reader may consult [2], [3], [6], [7] or [10].

REFERENCES

1. E. Ecklund, Jr., "On Prime Divisors of the Binomial Coefficients," *Pac. J. of Math.*, 29 (1969), pp. 267–270.
2. E. Ecklund, Jr., and R. Eggleton, "Prime Factors of Consecutive Integers," *Amer. Math. Monthly*, 79 (1972) pp. 1082–1089.
3. E. Ecklund, Jr., P. Erdös and J. L. Selfridge, "A New Function Associated with the Prime Factors of $\binom{n}{k}$, *Math. Comp.*, 28 (1974), pp. 647–649.
4. E. Ecklund, Jr., R. Eggleton, P. Erdös and J. L. Selfridge, "On the Prime Factorization of Binomial Coefficients," to appear.
5. P. Erdös, "On a Theorem of Sylvester and Schur," *J. London Math. Soc.*, 9 (1934), pp. 282–288.
6. P. Erdös, "Some Problems in Number Theory," *Computers in Number Theory, Proc. Atlas Symp.*, Oxford, 1969, Acad. Press (1971), pp. 405–414.

7. P. Erdös, "Über die Anzahl der Primzahlen-von $\binom{n}{k}$," *Archiv der Math.,* 24 (1973).
8. P. Erdös, "Problems and Results on Number Theoretic Properties of Consecutive Integers and Related Questions," *Proc. Fifth Manitoba Conf. on Numerical Mathematics* (1975), pp. 25–44.
9. P. Erdös and J. L. Selfridge, "Some Problems on the Prime Factors of Consecutive Integers," *Pullman Conf. on Number Theory,* (1971).
10. P. Erdös, R. L. Graham, I. Z. Ruzsa and E. G. Straus, "On the Prime Factors of $\binom{2n}{n}$," *Math Comp.,* 29 (1975), pp. 83–92.
11. K. Ramachandra, "A Note on Numbers with a Large Prime Factor II," *J. Indian Math. Soc.,* 34 (1970), pp. 39–48.
12. K. Ramachandra, "A Note on Numbers with a Large Prime Factor III," *Acta Arith.,* 19 (1971), pp. 49–62.
13. A. Schinzel, "Sur un Probleme de P. Erdös," *Coll. Math.,* 5 (1952), pp. 198–204.

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MIXED NEAREST NEIGHBOR DEGENERACY FOR PARTICLES ON A ONE-DIMENSIONAL LATTICE SPACE

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1. INTRODUCTION

In a recent article [1] expressions were presented which describe exactly the number of independent ways of arranging q indistinguishable particles on a one-dimensional lattice space of N equivalent compartments, in such a way as to create

1. n_{11} occupied nearest neighbor pairs
2. n_{00} vacant nearest neighbor pairs.

The present paper is concerned with the degeneracy associated with n_{01}, the number of mixed (one compartment empty, one occupied) nearest neighbor pairs.

Ising [2] has developed relationships which describe approximately the degeneracy associated with mixed nearest neighbor pairs. The purpose of the present paper is to develop an expression which describes exactly the degeneracy of arrangements containing a prescribed number of mixed nearest neighbor pairs.

2. CALCULATION

To determine $A(n_{01}, q, N)$, the number of independent arrangements arising when q particles are placed on a one-dimensional lattice space of N equivalent compartments in such a way as to create n_{01} mixed nearest neighfor pairs, we must consider the situations when n_{01} is odd and when it is even.

1. n_{01} odd

When n_{01} is odd, one and only one end compartment must be occupied. (See Fig. 1.) If the occupied end compartment is on the left-hand side we construct "units" consisting of a particle or of a continguous group of particles *and* the adjacent vacancy just to the right then we observe that there are $\frac{n_{01}-1}{2}$ permutable "units." We initially regard these "units" as identical regardless of the number of particles (≥ 1) of which they are composed (see cross-hatched "units" in Fig. 1). These "units" can be permuted to form other independent arrangements having the same n_{01}.

There are $N - q$ vacancies but not all of these vacancies can be permuted to form independent arrangements; there are $\frac{n_0 - 1}{2} + 1$ vacancies which form mixed nearest neighbor pairs. Thus there are $N - q - \left(\frac{n_0-1}{2}\right) - 1$ permutable vacancies and a total of $N - q - 1$ objects which can be permuted. These can be arranged in

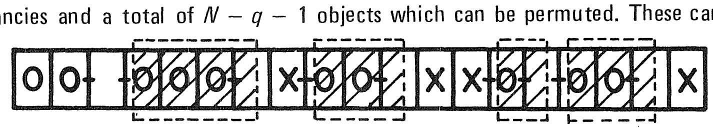

Figure 1. Shown is an arrangement of $q = 10$ particles on a linear array of $N = 19$ equivalent compartments which creates $n_{01} = 9$ mixed nearest neighbor pairs. There are $\left(\frac{n_{01}-1}{2}\right) = 4$ permutable "units" (cross-hatched) and $N - q - 1 - \left(\frac{n_{01}-1}{2}\right) = 4$ permutable vacancies (marked with x's). Thus there are a total of eight objects to be permuted while still keeping the left-hand compartment occupied.

$$\binom{N-q-1}{\frac{n_{01}-1}{2}} = \binom{N-q-1}{n-q-1\left(\frac{n_{01}-1}{2}\right)}$$

ways.

The "units" are, of course, not identical; the particles may be arranged to form "units" consisting of various numbers of particles subject to the constraint that n_{01} mixed nearest neighbor pairs must be present. To determine the number of ways the q particles can be arranged to form the $\left(\frac{n_{01}-1}{2}\right)$ "units" we consider $q-1$ lines which symbolize the separation of the q particles. (See Fig. 2.) $\left(\frac{n_{01}-1}{2}\right)$ of these lines symbolize the separation of the particles by two mixed nearest neighbor pairs and $q-1-\left(\frac{n_{01}-1}{2}\right)$ lines symbolize the adjacency of two particles. These $q-1$ lines, of which $\left(\frac{n_{01}-1}{2}\right)$ are one kind and the remainder another kind can be arranged in $\binom{q-1}{\frac{n_{01}-1}{2}}$ ways.

Thus, if we require that the compartment on the left end of the array is occupied (and the end compartment on the right is empty) then there are

$$\binom{N-q-1}{\frac{n_{01}-1}{2}}\binom{q-1}{\frac{n_{01}-1}{2}}$$

independent arrangement possible. Of course the end compartment on the right could have been occupied (and the end compartment on the left empty) so that if n_{01} is odd we obtain

$$A(n_{01}, q, N) = 2\binom{N-q-1}{\frac{n_{01}-1}{2}}\binom{q-1}{\frac{n_{01}-1}{2}} \tag{1}$$

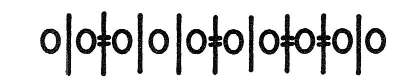

Figure 2. Figure 2 considers the particular arrangement shown in Fig. 1. There are $q-1=9$ separations between 10 particles. Of these separations $\left(\frac{n_{01}-1}{2}\right)=4$ are separations which constitute two mixed nearest neighbor pairs (two short horizontal lines) and $q-1-\left(\frac{n_{01}-1}{2}\right)=5$ represent separations between occupied nearest neighbor pairs. The q separations may be arranged in $\binom{9}{5}=126$ independent ways.

2. n_{01} even

When n_{01} is even two situations can arise:

(a) the compartments on each end of the array are empty (see Fig. 3)

(b) both end compartments are occupied (see Fig. 4).

For arrangements consistent with situation (a) there are always $\left(\frac{n_{01}}{2}\right)$ "units," each of which consists of a particle or a contiguous group of particles together with a facancy (if one is needed) to separate a "unit" from other "units," i.e., to create a mixed nearest neighbor pair. As before we initially regard these "units" as identical regardless of their composition. There are $N-q$ vacancies but not all of them are permutable; $\frac{n_{01}-2}{2}$ of

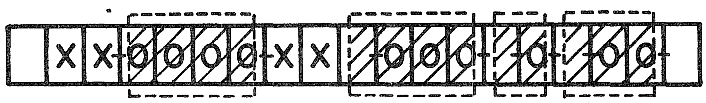

Figure 3. In this figure $N = 19$, $q = 10$, $n_{01} = 8$ and both end compartments are empty. There are $\left(\frac{n_{01}}{2}\right)$ "units" (cross-hatched) and $N - q - 1 - \left(\frac{n_{01}}{2}\right) = 4$ permutable vacancies (marked with x's) or a total of 8 objects which can be permited in $\binom{8}{4}$ ways to form independent arrangements.

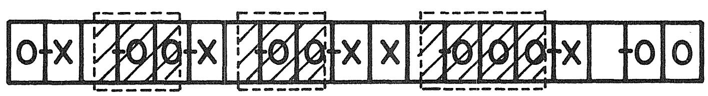

Figure 4. In this figure $N = 19$, $q = 10$, $n_{01} = 8$ and both end compartments are occupied. There are $\left(\frac{n_{01}-2}{2}\right) = 3$ permutable "units" (cross-hatched) and $N - q - 1 - \left(\frac{n_{01}-2}{2}\right) = 5$ permutable vacancies (marked with x's) or a total of 8 objects which can be permited in $\binom{8}{3}$ ways.

these vacancies are required to form mixed nearest neighbor pairs because one of the "units" (either the one to the extreme right or to the extreme left of the array) does not need a vacancy to isolate it. In addition, two vacancies, one at each end, are not permutable. Thus there are

$$N - q - \left(\frac{n_{01}}{2} - 1\right) - 2 = N - q - 1 - \frac{n_{01}}{2}$$

permutable vacancies and a total of

$$N - q - 1 - \frac{n_{01}}{2} + \frac{n_{01}}{2} = N - q - 1$$

permutable objects. These can be arranged in

$$\binom{N-q-1}{\frac{n_{01}}{2}} = \binom{N-q-1}{N-q-1-\frac{n_{01}}{2}}$$

independent ways.

The "units" are not identical as we have assumed. There are

$$\binom{q-1}{\frac{n_{01}}{2} - 1}$$

ways of arranging the q particles in the $\left(\frac{n_{01}}{2}\right)$ "units." This can be shown by the following reasoning. There are $q - 1$ lines that symbolize the separation of the q particles (see Fig. 5). Of these lines

$$\left(\frac{n_{01}-2}{2}\right)$$

represent separations of the particles by two mixed nearest neighbor pairs and

$$q - 1 - \left(\frac{n_{01}-2}{2}\right)$$

lines symbolize the separation of adjacent particles. These $q - 1$ lines can thus be arranged in

$$\binom{q-1}{\frac{n_{01}-2}{2}}$$

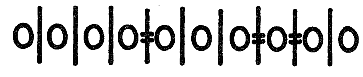

Figure 5. This figure considers the particular arrangement shown in Fig. 3. There are $q - 1 = 9$ separations between $q = 10$ particles. Of these separations $\left(\frac{n_{01}}{2}\right) - 1 = 3$ form two mixed neighbor pairs (short double horizontal lines) and

$$q - 1 - \left(\frac{n_{01}}{2} - 1\right) = 6$$

are occupied nearest neighbor pairs. The 9 separations may be arranged in $\binom{9}{6}$ independent ways. Thus the particles may be arranged within the $\frac{n_{01}}{2} = 4$ "units" in 21 ways so that there is at least one particle per unit.

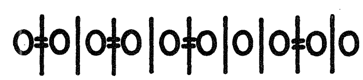

Figure 6. This figure considers the particular arrangement shown in Fig. 4. There are $q - 1 = 9$ separations between the $q = 10$ particles. Of these separations $\left(\frac{n_{01}}{2}\right) = 4$ represent separations by two mixed nearest neighbor pairs and $9 - 4 = 5$ represent separations of occupied nearest neighbor pairs. These 9 separations may be arranged in $\binom{9}{5}$ ways.

ways. This is the number of ways the q particles can be arranged to form $\left(\frac{n_{01}}{2}\right)$ "units" when the compartments on both ends are vacant.

Thus when both end compartments are empty there are

$$\binom{N - q - 1}{\frac{n_{01}}{2}}\binom{q - 1}{\frac{n_{01}}{2} - 1}$$

ways of arranging the q particles to yield exactly n_{01} nearest neighbor pairs.

For situation (b) there are $\left(\frac{n_{01} - 2}{2}\right)$ permutable "units" composed of a particle or group of contiguous particles and a vacancy to separate the "unit" for other "units." (See Fig. 4.) There are $N - q - 1 - \left(\frac{n_{01} - 2}{2}\right)$ permutable vacancies or a total of $N - q - 1$ objects which can be permuted. These objects can be arranged in

$$\binom{N - q - 1}{\frac{n_{01} - 2}{2}}$$

ways. Within the $\left(\frac{n_{01}}{2}\right)$ "units" the particles can be arranged in

$$\binom{q - 1}{\frac{n_{01}}{2}}$$

ways. This arises because there are $q - 1$ lines symbolizing the separation of the q particles (see Fig. 6); of these lines $\frac{n_{01}}{2}$ constitute separations of the "units" by two mixed nearest neighbor pairs and $q - 1 - \frac{n_{01}}{2}$ are lines which separate adjacent particles. These $q - 1$ lines can thus be arranged

$$\binom{q-1}{\frac{n_{01}}{2}}$$

ways.

Consequently, when both end compartments are occupied the q particles may be arranged in

$$\binom{N-q-1}{\frac{n_{01}}{2}-1}\binom{q-1}{\frac{n_{01}}{2}}$$

ways.

Thus the total number of arrangements possible when n_{01} is even is

$$A(n_{01}, q, N) = \binom{N-q-1}{\frac{n_{01}}{2}}\binom{q-1}{\frac{n_{01}}{2}-1} + \binom{N-q-1}{\frac{n_{01}}{2}-1}\binom{q-1}{\frac{n_{01}}{2}} \tag{2}$$

$$= 2\left(\frac{N-n_{01}}{n_{01}}\right)\binom{N-q-1}{\frac{n_{01}}{2}-1}\binom{q-1}{\frac{n_{01}}{2}-1}$$

Normalization for $A(n_{01}, q, N)$ can be shown to be

$$\sum_{n_{01}} A(n_{01}, q, N) = \binom{N}{q} \tag{3}$$

where $A(n_{01}, q, N)$ is given alternately by Eq. 1 and 2 and where the sum is over all possible values of n_{01}.

REFERENCES

1. R. B. McQuistan, *Jour. Math. Phys.,* 13, 1317 (1972).
2. E. Ising, *Z. Physik,* 31, 253 (1925).

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A FORMULA FOR FIBONACCI NUMBERS FROM A NEW APPROACH TO GENERALIZED FIBONACCI NUMBERS

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DEDICATED TO JAMES M. VAUGHN, JR.

INTRODUCTION

Ever since the establishment of the Fibonacci Association and its main publication, *The Fibonacci Quarterly,* under the devoted guidance of its founder, the Fibonacci master, Verner E. Hoggatt, Jr., [3] of San Jose State University, California, the study of the Fibonacci sequence

$$(0.1) \qquad F_1 = F_2 = 1; \qquad F_{n+2} = F_n + F_{n+1}; \qquad (n = 1, 2, \cdots)$$

has seen a new and rapid development in the last two decades. The impressive list of brilliant mathematicians who have contributed is too long to be mentioned here. But the author thinks that the time is ripe for some kind of a Dickson-survey of all the splendid results in the Fibonacci Wonderland which has fascinated mathematicians for the last 775 years, since the son of Bonacci wrote his *Liber abaci* in 1202.

Together with the study of the original Fibonacci sequence (0.1) went the generalization of these sequences. This was a result of pure mathematical curiosity and speculative creativity, without any application to the freightening population explosion of rabbits. This generalization could lead into various directions. First—the initial values of F_1, F_2 in (0.1) could be arbitrarily chosen, and this gave birth to the Lucas numbers, in addition to many other step-children. The most reckless, most general generalization, taking us to dimensions beyond the imagination or needs of Leonardo da Pisa, would be the following: let

$$(0.2) \qquad \begin{cases} F_j = a_j & (j = 1, \cdots, n) \\ F_{n+v} = \sum_{i=0}^{n-1} b_i F_{v+i}, & (v = 1, 2, \cdots) \\ a_j, b_i \in Z; & a_j, b_i \text{ fixed.} \end{cases}$$

Of course, it is possible to drive this inconsiderateness still further and choose a_j, b_i from C. But one should make a halt somewhere. In a previous paper the author [1,a], and in a joint paper Hasse and the author [1,b] have investigated the most simple case of the general generalization of the Fibonacci numbers, viz.

$$(0.3) \qquad \begin{cases} F_1 = 1, \quad F_i = 0 & (i = 2, \cdots, n), \\ F_{n+v} = \sum_{i=0}^{n-1} F_{v+i} & (v = 1, 2, \cdots). \end{cases}$$

The author succeeded to calculate F_{n+v} in a comparatively simple explicit formula. In principle, this is possible also for F_{n+v} from (0.2), by means of Euler's generating functions. The author applied, for the calculation of F_{n+v} from (0.3), the Jacobi-Perron algorithm [1,c], which led him to suggest that the sequence of the original Fibonacci numbers should actually be defined by

$$(0.4) \qquad F_{-1} = 1; \quad F_0 = 0; \quad F_{n+2} = F_n + F_{n+1} \quad (n = -1, 0, 1, \cdots).$$

While trying to generalize the original Fibonacci number to higher dimensions, one is immediately exposed to the danger of losing the royal property of the original Fibonacci numbers, viz., $F_m | F_{mk}$. This damage has not yet been repaired for Fibonacci numbers of dimension $n > 2$, and the author conjectures that this will remain a utopia.

In this context the question arises: what is the natural generalization of the Fibonacci numbers, if any? For this purpose, one should look into another direction than (0.2).

As is known, the generating polynomial for the original Fibonacci numbers is

$$(0.5)\qquad \begin{cases} P(x) = x^2 - x - 1; & P(\alpha) = P(\beta) = 0 \\ \alpha = \dfrac{1+\sqrt{5}}{2}, & \beta = \dfrac{1-\sqrt{5}}{2} \end{cases}$$

from which, by easy calculations, the two well known formulas are derived:

$$(0.6)\qquad \begin{cases} F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} = \displaystyle\sum_{i=0} \binom{n-1-i}{i}\ ; \\ n = 1, 2, \cdots;\ \binom{0}{0} \overset{\text{def}}{=} 1. \end{cases}$$

As is seen from (0.5), α and β are units in $Q(\sqrt{5})$. Generalizing (0.5), and demanding that the two roots of the new polynomial be units, one would suggest that the natural generalization of the generating polynomial would be

$$(0.7)\qquad P(x) = x^2 - ax - 1, \quad a \in N.$$

By a technique which will be developed in the next chapter, one obtains generalized Fibonacci numbers (of dimension two) $F_{a,n}$ of the form

$$(0.8)\qquad \begin{cases} F_{a,n} = \displaystyle\sum_{i=0} \binom{n-1-i}{i} a^{n-1-2i}, & (n = 1, 2, \cdots) \\ F_{a,n+2} = F_{a,n} + aF_{a,n+1}\ . \end{cases}$$

For $a = 1$, (0.8) become the original Fibonacci numbers, as should be. But, alas, we had hoped to arrive at a new formula for F_n. So the generalization (0.7) does not supply the natural generalization for the original Fibonacci numbers, and new horizons must be searched.

1. THE NEW APPROACH

In two previous papers [1,d), e)] the author has established a few new combinatorial identities by means of a new technique. These identities are of a quite complicated nature, and only a combinatorial master like Leonard Carlitz [2] could have succeeded to prove them by elementary tools. The basic ideas of this new technique, together with a few illustrations, will soon appear in a paper; an abstract [1,f)] of this paper has been published. The author doubts not that mathematicians, once they have become familiar with this technique, will come up with a treasure of new and interesting combinatorial identities which could probably not be proved with elementary means.

A word about its contents. Since the new technique is based on the knowledge of one or more independent units in an algebraic number field of any degree $n \geqslant 2$, these units must, of course, be explicitly stated. Now, there are many elaborate methods to find the basis (the maximal set of fundamental units) of the multiplicative group of units in a numeric, given algebraic number field $Q(w)$,

$$(1.1)\qquad \begin{cases} w^n + k_1 w^{n-1} + \cdots + k_{n-1} w + k_n = 0, \\ k_i \in Z, \quad k_i \text{ fixed} \quad (i = 1, \cdots, n). \end{cases}$$

The situation is entirely different, if the $k_1, \cdots, k_n$ from (1.1) are any free parameters. In this case we speak about $Q(w) = Q(w; k_1, \cdots, k_n)$ as a functional algebraic number field. In this case it seems almost impossible to state one or more independent units in $Q(w; k_1, \cdots, k_n)$ explicitly (they must not be fundamental). We do dare to think that the author, and in a few joint papers the author and Hasse [1,c] were the first pioneers to state explicitly units in functional algebraic number fields of any degree $n \geqslant 2$. Of course, if $k_n = \pm 1$ in (1.1), then w is always a unit. This led to the original generating polynomial (0.5) for the original Fibonacci numbers, and its generalization (0.7), which, as we have seen led to essentially nothing new, but an identity of the numbers $F_{a,n}$ from (0.8) and a polynomial in $\frac{1}{2}(1 \pm \sqrt{a^2+4})$. For generalized Fibonacci numbers of dimension $n \geqslant 3$, it will be definitely worthwhile investigating the case $|k_n| = 1$ in (1.1), and the author is sure that, by means of his new technique, many new combinatorial identities can be obtained, and his Ph.D. students are already working on this subject. This technique, as exposed in the *a/m* abstract, proceed as follows: let

$$\left\{\begin{array}{c} P(x) = x^n + k_1x^{n-1} + \cdots + k_{n-1}x + k_n, \quad k_i \in Z \quad (i = 1, \cdots, n) \\ P(w) = 0, \quad w \in R. \end{array}\right. \tag{1.2}$$

Let $Q(w)$ be the algebraic number field over Q, obtained by adjunction of w to Q. Let further

$$\left\{\begin{array}{l} e = a_1 + a_2w + \cdots + a_nw^{n-1}, \\ a_i \in Q, \quad (i = 1, \cdots, n) \end{array}\right. \tag{1.3}$$

where e is explicitly stated. By means of Euler's generating functions, one calculates first the positive powers of e, explicitly, viz.

$$\left\{\begin{array}{l} e^m = b_{1,m} + b_{2,m}w + \cdots + b_{n,m}w^{n-1}, \\ b_{i,m} \in Q, \quad (i = 1, \cdots, n;\ m = 0, 1, \cdots) \end{array}\right. \tag{1.4}$$

$$\left\{\begin{array}{l} e^{-m} = c_{1,m} + c_{2,m}w + \cdots + c_{n,m}w^{n-1}, \\ c_{i,m} \in Q, \quad (i = 1, \cdots, n;\ m = +1, 2, \cdots). \end{array}\right. \tag{1.5}$$

Then from

$$\left\{\begin{array}{l} 1 = e^m \cdot e^{-m} = g_{1,m} + g_{2,m}w + \cdots + g_{n,m}w^{n-1} \\ g_{i,m} \in Q, \quad (i = 1, \cdots, n;\ m = 0, 1, \cdots) \end{array}\right. \tag{1,6}$$

one obtains, by comparison of coefficients of powers of w, the necessary identities, which usually involve n's order determinant with combinatorial coefficients (or their linear combinations) as entries. Thus, in [1,d)] the author obtained the combinatorial identity

$$\left\{\begin{array}{l} \left(\sum_{k=0} \binom{m-k-1}{2k-1+2s}\right) = \left[\sum_{i=0} (-1)^i \binom{n-3-2i}{i}\right]^2 \\ \qquad - \sum_{i=0} (-1)^i \binom{n-4-2i}{i} \sum_{i=0} (-1)^i \binom{n-2-2i}{i}\ ; \\ m = \left[\frac{n}{2}\right]; \qquad 2s = n-2m; \quad n = 4, 5, \cdots . \end{array}\right. \tag{1.7}$$

2. THE GENERATING POLYNOMIAL

A polynomial over Z of the form

$$P_n(x) = (x-D_1)(x-D_2)\cdots(x-D_n) - d, \quad n \geqslant 2, \tag{2.1}$$

has been investigated by the author [1,g)] for the purpose of constructing periodic Jacobi-Perron algorithm, and by the author and Hasse [1,h)] for the purpose of obtaining $n-1$ independent units in a functional algebraic number field of degree n. In this paper, in order to obtain the natural generalization of two-dimensional Fibonacci numbers and a new formula for the original ones, we shall investigate the case $n = 2$ of (2.1). The case $n = 3$ has been investigated by my Ph.D. student Seeder [5], for the purpose of obtaining combinatorial identities. We are now taking the liberty of marking the following

Statement. The generating polynomial for the natural generalization of the original Fibonacci numbers to the dimension two, has the form

(2.2)
$$\begin{cases} F_2(x) = (x-D_1)(x-D_2)-d; \\ D_1, D_2 \in Z; \quad d \in N. \\ D_1 > D_2; \quad D_1 - D_2 \equiv 0 \pmod d; \\ D_1 - D_2 \equiv 1(2); \quad d \neq m^2; \quad m \in Z \end{cases}$$

The last two restrictions on $F_2(x)$ from (2.2) are chosen for convenience sake, as the reader will see later, and can, generally, be dropped for the definition of $F_2(x)$. From $D_1 - D_2 \equiv 1(2)$ would follow, since $d \mid D_1 - D_2$, that d is odd. The restriction $d \neq m^2$ is convenient in another context; both are not necessary conditions. Since $F_2(x) = x^2 - (D_1 + D_2)x + D_1 D_2 - d$, the two roots of $F_2(x)$ are

(2.3)
$$\begin{cases} w_1 = \dfrac{D_1 + D_2 + \sqrt{(D_1 - D_2)^2 + 4d}}{2}; \quad w_2 = \dfrac{D_1 + D_2 - \sqrt{(D_1 - D_2)^2 + 4d}}{2} \\ w_1, w_2 \in R; \quad w_1 > w_2. \end{cases}$$

We now prove

Lemma 1. $(D_1 - D_2)^2 + 4d$ is not a perfect square.

Proof. The lemma holds, as we shall see, even without the restrictions $D_1 - D_2 \equiv 1(2)$, $d \neq m^2$. The other restrictions of (2.2) must remain valid. We have

(2.4)
$$D_1 - D_2 = td, \quad t \in Z - \{0\}.$$

For $|t| = 1$, we have $(D_1 - D_2)^2 = d^2$, $(D_1 - D_2)^2 + 4d = d^2 + 4d$, and $(d+1)^2 < d^2 + 4d < (d+2)^2$.

For $|t| > 1$, we have $(D_1 - D_2)^2 = t^2 d^2$,

$$t^2d^2 < t^2d^2 + 4d < t^2d^2 + 2|t|d + 1 = (|t|d+1)^2,$$

$$(|t|d)^2 < (D_1 - D_2)^2 + 4d < (|t|d+1)^2.$$

This proves the Lemma 1 completely. From Lemma 1, we immediately derive

Theorem 1. The polynomial

$$F_2(x) = (x-D_1)(x-D_2)-d; \quad D_1 - D_2 \in Z; \quad d \in N; \quad D_1 > D_2; \quad D_1 - D_2 \equiv 0 \pmod d$$

is irreducible over Q (over Z). The roots of $F_2(x)$ are real quadratic irrationals.

Notation 1. The greater of the two roots of $F_2(x)$ will be denoted by

(2.5)
$$w = w_1 = \frac{D_1 + D_2 + \sqrt{(D_1 - D_2)^2 + 4d}}{2}.$$

For later purposes we shall need the expansion of w as a simple continued periodic fraction. We have from (2.2), (2.4), (2.5)

(2.6)
$$(w - D_1)(w - D_2) = d,$$

and make the restriction

(2.7)
$$d \neq 1.$$

Then, as the reader can easily verify,

$$w = D_1 + \frac{1}{x_1}, \quad x_1 = \frac{1}{w - D_1} = \frac{w - D_2}{(w-D_1)(w-D_2)} = \frac{w-D_2}{d} = \frac{D_1 - D_2}{d} + \frac{w - D_1}{d} = \frac{D_1 - D_2}{d} + \frac{1}{x_2};$$

$$x_2 = \frac{d}{w - D_1} = \frac{d(w-D_2)}{(w-D_1)(w-D_2)} = w - D_2 = D_1 - D_2 + w - D_1 = D_1 - D_2 + \frac{1}{x_3} = D_1 - D_2 + \frac{1}{x_1}.$$

We have thus obtained: $x_1 = x_3$

(2.8)
$$w = \left[D_1, \overline{\frac{D_1 - D_2}{d}, D_1 - D_2}\right]$$

If we now drop restriction (2.7), we obtain

(2.9) $$w = [D_1, \overline{D_1 - D_2}]\ ; \quad d = 1.$$

If we set

(2.10) $$\begin{cases} D_1 = 1,\ D_2 = 0;\ d = 1, \text{ we obtain} \\ x^2 - x - 1 = 0; \quad w^2 - w - 1 = 0; \\ w = \dfrac{1+\sqrt{5}}{2} = [1]\,. \end{cases}$$

Thus formula (2.9) is valid also for the conditions of (2.10). Formula (2.10) leads, as was mentioned, to the original Fibonacci numbers. We return to the original case. As is known, a unit in

$$Q(w) = Q(\sqrt{(D_1 - D_2)^2 + 4d}$$

is given by

$$e_1 = x_1 x_2 = \frac{d}{(w - D_1)^2} \quad ,$$

and since, from (2.6)

$$\frac{(w - D_1)^2 (w - D_2)^2}{d^2} = 1,$$

we obtain

(2.11) $$e_1 \cdot 1 = e = \frac{(w - D_2)^2}{d} \text{ is a unit in } Q(w).$$

If $d = 1$, $w - D_2 \in Q(w)$, so that

(2.11a) $$w - D_2 \text{ is a unit in } Q(w),\ d = 1.$$

We shall, for the time being, eliminate the case $d = 1$, but shall return to it later. That

$$e = \frac{(w - D_2)^2}{d}, \quad e > 1$$

is a unit in $Q(w)$ can also be proved directly; we have

(2.12) $$\begin{cases} e = \dfrac{w^2 - 2D_2 w + D_2^2}{d} = \dfrac{(D_1 + D_2)w - D_1 D_2 + d - 2D_2 w + D_2^2}{d} \\ \quad = \dfrac{-D_2(D_1 - D_2) + (D_1 - D_2)w + d}{d}\ ; \end{cases}$$

thus e is an integer, since $D_1 - D_2 \equiv 0(d)$.

We further have

$$\begin{aligned} N(e) &= \frac{(N(w - D_2))^2}{d^2} = \frac{((w_1 - D_2)(w_2 - D_2))^2}{d^2} \\ &= d^{-2} \left(\frac{(D_1 - D_2) + \sqrt{(D_1 - D_2)^2 + 4d}}{2} \cdot \frac{(D_1 - D_2) - \sqrt{(D_1 - D_2)^2 + 4d}}{2} \right)^2 \\ &= d^{-2} \cdot d^2 = 1. \end{aligned}$$

We shall operate, in the sequel, only with the unit

$$e = \frac{(w - D_2)^2}{d}$$

regardless of whether e is fundamental or not, though this question could be easily answered. Since e is in the ring $R[w]$, we also do not need to construct a basis for $Q(w)$, and shall operate with integers of the form

(2.13) $$\beta = x + yw; \quad x, y \in Z.$$

A last question remains to be resolved, viz.: are there indeed infinitely many real quadratic fields of the form $Q(\sqrt{(D_1 - D_2)^2 + 4d})$? To prove this, let us presume

$$(2.14)\qquad \begin{cases} D_1 - D_2 \equiv 1(d); & d \neq m^2, \quad m \in \mathbb{Z}, \\ D_1 - D_2 = td; & t \in \mathbb{Z}; \quad t \text{ fixed.} \end{cases}$$

Then

$$(D_1 - D_2)^2 + 4d = t^2 d^2 + 4d.$$

Now, Erdös [4] has proved, that for infinitely many n,

$$t^2 n^2 + 4n, \qquad (tn \equiv 1(2), \; n \neq m^2, \; t, n, m \in \mathbb{Z})$$

has no square factor. This proves that there are infinitely many real quadratic fields of the form

$$Q(\sqrt{(D_1 - D_2)^2 + 4d}).$$

3. THE POWERS OF e

In this chapter we shall give formulas for the explicit calculation of e^n and e^{-n}, $(n = 0, 1, \cdots)$. This is the central result of this paper from which the new formula for the original Fibonacci numbers will be derived.

We have from (2.12)

$$e = \frac{-D_2(D_1 - D_2) + d + (D_1 - D_2)w}{d},$$

and with $D_1 - D_2 = td$, we obtain

$$(3.1)\qquad e = -D_2 t + 1 + tw.$$

From $w^2 = (D_1 + D_2)w - D_1 D_2 + d$, we obtain, with $D_1 = D_2 + td$,

$$(3.2)\qquad w^2 = -(D_2^2 + D_2 dt) + d + (2D_2 + dt)w.$$

One calculates easily from (3.1), taking into account (3.2)

$$(3.3)\qquad e^2 = -D_2 t(dt^2 + 2) + dt^2 + 1 + (dt^3 + 2t)w.$$

We now denote

$$(3.4)\qquad e^n = x_n + y_n w, \quad n = 0, 1, \cdots.$$

With (3.1), (3.3) we have

$$(3.5)\qquad \begin{cases} x_0 = 1, \; y_0 = 0; \quad x_1 = -D_2 t + 1, \; y_1 = t; \\ x_2 = -D_2(dt^3 + 2t) + dt^2 + 1; \; y_2 = dt^3 + 2t. \end{cases}$$

From (3.4), (3.1), we further obtain

$$e^{n+1} = e^n \cdot e = (x_n + y_n w)[(-D_2 t + 1) + tw].$$

An easy calculation, taking into account (3.2), yields

$$e^{n+1} = x_{n+1} + y_{n+1}w = (-D_2 t + 1)x_n + (-D_2^2 t - D_2 dt^2 + dt)y_n + [(-D_2 t + 1)y_n + (2D_2 t + dt^2)y_n + tx_n]\,w,$$

hence

$$(3.6)\qquad \begin{cases} x_{n+1} = (-D_2 t + 1)x_n + (-D_2^2 t - D_2 dt^2 + dt)y_n, \\ y_{n+1} = tx_n + (D_2 t + dt^2 + 1)y_n. \end{cases}$$

From the second formula of (3.6) we obtain

$$y_{n+2} = tx_{n+1} + (D_2 t + dt^2 + 1)y_{n+1},$$

and substituting here the value of x_{n+1} from (3.6),

$$y_{n+2} = (-D_2 t + 1)tx_n + (-D_2^2 t^2 - D_2 dt^3 + dt^2)y_n + (D_2 t + dt^2 + 1)y_{n+1}.$$

Substituting here the value of tx_n from the second formula of (3.6), we obtain

$$y_{n+2} = (-D_2 t + 1)[y_{n+1} - (D_2 t + dt^2 + 1)y_n] + (-D_2^2 t^2 - D_2 dt^3 + dt^2)y_n + (D_2 t + dt^2 + 1)y_{n+1},$$

and, after simple calculations

(3.7) $$y_{n+2} = (dt^2+2)\, y_{n+1} - y_n .$$

From (3.6) we obtain

$$x_n = \frac{1}{t}\,[y_{n+1} - (D_2 t + dt^2 + 1)y_n]\,,$$

or, raising the index by one,

$$x_{n+1} = \frac{1}{t}\,[y_{n+2} - (D_2 t + dt^2 + 1)y_{n+1}]\,,$$

and substituting here the value of y_{n+2} from (3.7),

$$x_{n+1} = \frac{1}{t}\,[(dt^2+2)y_{n+1} - y_n - (D_2 t + dt^2 + 1)y_{n+1}]$$

$$x_{n+1} = \frac{1}{t}\,[(-D_2 t + 1)y_{n+1} - y_n]$$

$$x_{n+1} = -D_2 y_{n+1} + t^{-1}(y_{n+1} - y_n),$$

(3.8) $$x_n = D_2 y_n + t^{-1}(y_n - y_{n-1}).$$

Thus

(3.9) $$e^n = [-D_2 y_n + t^{-1}(y_n - y_{n-1})] + y_n w,$$

and, to complete the calculation of e we have to calculate y_n. This is done by means of the recurrency formula (3.7). We obtain, taking into account the values of y_0 and y_1 from (3.5)

$$\sum_{n=0}^{\infty} y_n u^n = y_0 + y_1 u + \sum_{n=2}^{\infty} y_n u^n = tu + \sum_{n=0}^{\infty} y_{n+2} u^{n+2} = tu + \sum_{n=0}^{\infty} [(dt^2+2)y_{n+1} - y_n]u^{n+2}$$

$$= tu - u^2 \sum_{n=0}^{\infty} y_n u^n + (dt^2+2)u \sum_{n=0}^{\infty} y_{n+1} u^{n+1}$$

$$= tu - u^2 \sum_{n=0}^{\infty} y_n u^n + (dt^2+2)u \left[\left(\sum_{n=0}^{\infty} y_n u^n\right) - y_0 u^0\right]$$

$$= tu - u^2 \sum_{n=0}^{\infty} y_n u^n + (dt^2+2)u \sum_{n=0}^{\infty} y_n u^n .$$

We have obtained,

$$\sum_{n=0}^{\infty} y_n u^n = tu - u^2 \sum_{n=0}^{\infty} + (dt^2+2)u \sum_{n=0}^{\infty} y_n u^n$$

(3.10) $$\sum_{n=0}^{\infty} y_n u^n = \frac{tu}{1 - au + u^2}\,, \qquad a = dt^2 + 2.$$

From (3.10) we obtain, for u sufficiently small

$$\sum_{n=0}^{\infty} y_n u^n = tu \sum_{k=0}^{\infty} (au - u^2)^k$$

(3.11) $$\sum_{n=0}^{\infty} y_n u^n = t \sum_{k=0}^{\infty} u^{k+1}(a-u)^k .$$

Collecting on the right side of (3.11) powers of u^n, we obtain, by comparison of coefficients, taking $k = n-1, n-2, \cdots,$

$$y_n = t\left[a^{n-1} - \binom{n-2}{1} a^{n-3} + \binom{n-3}{2} a^{n-5} - + \cdots\right];$$

$$y_n = t \sum_{i=0} (-1)^i \binom{n-1-i}{i} a^{n-1-2i},$$

and finally

(3.12) $$y_n = t \sum_{i=0} (-1)^i \binom{n-1-i}{i} (dt^2+2)^{n-1-2i}, \quad y_0 = 0; \; y_1 = t; \; n = 2, 3, \cdots.$$

From (3.8) and (3.12) we now also obtain the value of x_n, viz.

(3.13) $$\begin{cases} x_n = -D_2 t \sum_{i=0} (-1)^i \binom{n-1-i}{i} (dt^2+2)^{n-1-2i} \\ \quad + \sum_{i=0} (-1)^i \left[\binom{n-1-i}{i} (dt^2+2)^{n-1-2i} - \binom{n-2-i}{i} (dt^2+2)^{n-2-2i}\right], \\ x_0 = 1; \; x_1 = -D_2 t + 1; \; x_2 = -D_2(dt^3+2t) + dt^2 + 1; \; n = 3, 4, \cdots. \end{cases}$$

We shall now proceed to calculate the negative powers of e and use a Kunstgriff for this purpose. We remember that

$$w = w_1; \quad w_1 + w_2 = D_1 + D_2.$$

We further have

$$e^{-n} = \frac{1}{x_n + y_n w} = \frac{x_n + y_n w_2}{(x_n + y_n w_1)(x_n + y_n w_2)}.$$

Now, the whole trick consists of

$$N(e^n) = N(x_n + y_n w) = (x_n + y_n w_1)(x_n + y_n w_2);$$

but $N(e^n) = (N(e))^n = 1^n = 1$, so that

$$e^{-n} = x_n + y_n w_2 = x_n + y_n(D_1 + D_2 - w)$$

$$e^{-n} = x_n + y_n(D_1 + D_2) - y_n w.$$

But from (3.8), $x_n = -D_2 y_n + t^{-1}(y_n - y_{n-1})$, so that finally

(3.14) $e^{-n} = D_1 y_n + t^{-1}(y_n - y_{n-1}) - y_n w; \quad y_0 = 0; \; y_1 = t; \; n = 2, 3, \cdots; \; y_n$ from (3.12).

The reader will easily verify that the norm equation of e^n yields

(3.14a) $$x_n^2 + (D_1 + D_2)x_n y_n + (D_1 D_2 - d)y_n^2 = 1.$$

4. THE "NEW" FORMULA

We return to the generating polynomial of the original Fibonacci numbers, $P(x) = x^2 - x - 1$. We have

(4.1) $P(w_1) = P(w_2) = 0; \; w_1 = \frac{1+\sqrt{5}}{2}, \; w_2 = \frac{1-\sqrt{5}}{2}, \; w^2 - w - 1 = 0; \; w^2 = w + 1; \; w = w_1$

In $Q(w)$, w is a (fundamental) unit; we shall calculate its non-negative integral powers.

(4.2) $$w^n = g_n + f_n w; \quad g_0 = 1; \quad g_1 = 0; \quad f_0 = 0; \quad f_1 = 1.$$

Multiplying in (4.2) both sides of w, we obtain

$$w^{n+1} = g_n w + f_n w^2 = g_n w + f_n(w+1), \quad w^{n+1} = f_n + (g_n + f_n)w = g_{n+1} + f_{n+1}w$$

$$g_{n+1} = f_n; \quad f_{n+1} = f_n + g_n = f_n + f_{n-1}$$

(4.3) $$w^n = f_{n-1} + f_n w$$

(4.4) $$f_{n+2} = f_n + f_{n+1}.$$

Since $w^2 = g_2 + f_2 w = 1 + w$, we have $f_2 = 1$, so that (4.2), (4.4) and $f_2 = 1$ yield,

$$f_{n+2} = f_n + f_{n+1}; \quad f_1 = f_2 = 1; \quad n = 1, 2, \cdots$$

which shows that the f_n are the original Fibonacci numbers,

(4.5) $$f_n = F_n; \quad n = 1, 2, \cdots.$$

If we set in (2.2)

$$D_1 = 1; \quad D_2 = 0; \quad d = 1; \quad t = 1,$$

we obtain, from (2.11), (3.4), (4.2), (4.5) and (3.12)

$$e^n = w^{2n} = x_n + y_n w = g_{2n} + f_{2n} w, \quad F_{2n} = y_n;$$

(4.6) $$\underline{\underline{F_{2n} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i},}}$$

since $dt^2 + 2 = 3$.

(4.6) is the new, and surprising, beautiful formula for F_{2n}; F_{2n+1} is then obtained from the relation

$$F_{2n+1} = F_{2n+2} - F_{2n} = \left(\sum_{i=0}^{\infty} (-1)^i 3^{n-2i} \binom{n-1}{i} \right) - \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i},$$

so that, by the new approach to Fibonacci numbers, we obtain the sequence (which is, of course, identical with the original one)

(4.7) $$\begin{cases} F_1 = F_2 = 1; \quad F_{2n} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i}; \quad n = 2, 3, \cdots; \\[2ex] F_{2n+1} = \left(\sum_{i=0} (-1)^i \binom{n-i}{i} 3^{n-2i} \right) - \left(\sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-i} \right); n = 1, 2, \cdots \end{cases}$$

In (4.7), for $n = 1$ in F_{2n+1}, we have to define $\binom{0}{0} \overset{\text{def}}{=} 1$.

From (4.3), we have, with (4.5)

$$w^n = F_{n-1} + F_n w.$$

Now. since $w^2 - w - 1 = 0$, we have

$$N(w) = -1,$$

so that

$$\begin{aligned} N(w^n) &= (-1)^n = N(F_{n-1} + F_n w) = (F_{n-1} + F_n w_1)(F_{n-1} + F_n w_2) \\ &= F_{n-1}^2 + (w_1 + w_2) F_{n-1} F_n + F_n^2 w_1 w_2 = F_{n-1}^2 + F_{n-1} F_n - F_n^2 \\ &= F_{n-1}^2 + F_{n-1}(F_{n+1} - F_{n-1}) - F_n^2 = F_{n-1} F_{n+1} - F_n^2, \end{aligned}$$

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1},$$

a well known formula.

The analogue for the generalized generating polynomial $F_2(x)$ from (2.2) is obtained from (3.4), with $N(e) = 1$, viz.

$$x_n^2 + (D_1 + D_2) x_n y_n + (D_1 D_2 - d) y_n^2 = 1,$$

which solves the Diophantine equation

$$x^2 + (D_1 + D_2) xy + (D_1 D_2 - d) y^2 = 1,$$

$$D_1 > D_2; \quad D_1 - D_2 \equiv 0(d); \quad d, D_1, D_2 \in Z; \quad d \geqslant 1.$$

5. ALTNEULAND*—AN EPILOGUE

The new Formula for the original Fibonacci numbers, as the author has called it with unforgivable self-styled praise, is actually an old formula which could be achieved by elementary means, as was kindly remarked to the author in a private correspondence by Professor Verner E. Hoggatt, Jr., of San Jose State University. Here is the way it can be obtained from the original Fibonacci numbers:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + F_{2n-1} = 2F_{2n} + F_{2n} - F_{2n-2};$$

$$F_{2n+2} = 3F_{2n} - F_{2n-2}\,. \tag{5.1}$$

Eq. (5.1) leads to the generating polynomial

$$x^2 - 3x + 1 = 0, \tag{5.2}$$

and from (5.2) the new formula for F_{2n+2} is easily obtained by the use of Euler's generating functions, as used in this paper. But finding a new formula for F_{2n+2} was not the idea of this paper, as was pointed out in the introduction. The aim was two-fold—first finding the most natural generalization for Fibonacci numbers, of which the original ones would be a special case; second—to demonstrate the powerful use of units to finding combinatorial identities, since, after all, what we have found—and again, this may be considered Altneuland—is the combinatorial identity

$$\sum_{i=0} \binom{2n-1-i}{i} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i}\,. \tag{5.3}$$

Besides the technique used in this paper, the author has found a new, and, as he believes, powerful different technique by using units in algebraic functional fields of any degree for finding new combinatorial identities of higher dimension which surely cannot be proved by elementary combinatorial means. These new results will appear in a book by the author which is now in preparation.

REFERENCES

1(a). L. Bernstein, "The Linear Diophantine Equation in n Variables and its Application," *The Fibonacci Quarterly,* Special Issue (June 1968), pp. 1–52.

1(b). L. Bernstein and H. Hasse, "Explicit Determination of the Matrices in Periodic Algorithms of the Jacobi-Perron Type, etc.," *The Fibonacci Quarterly,* Vol. 7, No. 4, Special Issue (Nov. 1969), pp. 293–365.

1(c). L. Bernstein, "The Jacobi-Perron Algorithm, its Theory and Application," *Lecture Notes in Mathematics,"* No. 207, Springer-Verlag (1871), I–IV, pp. 1–160.

1(d). L. Bernstein, "Zeroes of the Function $F(n) = \sum_{i=0} (-1)^i \binom{n-2i}{i}$," *Journal of Number Theory,* Vol. 6, No. 4 (August 1974), pp. 264–270.

1(e). L. Bernstein, "Zeros of Combinatorial Functions and Combinatorial Identities," *Houston Journal of Math,* Vol. 2, No. 1 (1976), pp. 9–16.

1(f). L. Bernstein, "Units in Algebraic Number Fields and Combinatorial Identities," *Notices of the A.M.S.,* July issue, 1976.

1(g). L. Bernstein, "New Infinite Classes of Periodic Jacobi-Perron Algorithms," *Pacific Journal of Math,* 16 (1965), pp. 1–31.

1(h). L. Bernstein and H. Hasse, "An Explicit Formula for the Units of an Algebraic Number Field of Degree $n > 2$," *Pacific Journal of Math.,* Vol. 30, No. 2 (1969), pp. 293–365.

2. L. Carlitz, "Some Combinatorial Identities of Bernstein," in print.

3. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton-Mifflin Co. (1969), pp. 1–91.

*Title of the famous novel by the Austrian writer, Theodor Herzl.

4. P. Erdös, "Arithmetical Properties of Polynomials," *J. London Math. Soc.,* 28 (1953), pp. 416–425.
5. S. Seeder, "Units and Their Application to Diophantine Equations and Combinatorial Identities," Ph.D. Thesis, Illinois Institute of Technology (1976) unpublished.

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LETTER TO THE EDITOR

Dear Editor:

I am teaching a survey course at the Pennsylvania State University. After two days of studying the elementary properties of the Fibonacci sequence, I asked my class to write a poem about Fibonacci. One very talented student submitted the following:

FIBONACCI'S PARTY
by Cynthia Ellis

The great mathematician Fibonacci
Went out to the market and bought a Hibachi.
He decided to give a small Bar-B-Que
For himself, his wife, and a good friend or two.

So he called his friend Joe and he asked him to come
With a small jug of wine or a bottle of rum.
"My wife (one) and I (one) make two" figured he,
"And with Joseph attending, the total is three."

But then the telephone rang in the hall:
His parents would be there, making five guests in all.
And his wife told him also her parents were coming.
With sister Loretta—now eight was his summing.

But, oh, he'd forgotten Joe's girlfriend Eileen.
With her and her family the total's thirteen.
And Loretta brings friends to wherever there's fun.
So he counted it up and he got twenty-one.

Just then he remembered the neighbors next door.
They'd certainly be there to make thirty-four.
And then his club's football teams pulled in the drive.
And he tore at his hair as he thought "Fifty-five!"

While out in the street he saw line after line
Of neighborhood moochers to make eighty-nine.
And 'round from the alley there came at a trot
His boss and co-workers, the whole bloomin' lot.

Fib went to the gameroom and sat on the floor
And figured the total as one-forty-four.
So he crawled to the bar and swalled a dose
And started to wonder how three grew to gross.

So he pulled out his list and he started to count,
Carefully writing down every amount.
And discovered the sequence that now bears his name,
Thanks to the party where everyone came.

I hope you like the poem and decide to publish it.

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DIVISIBILITY PROPERTIES OF RECURRENT SEQUENCES

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1. INTRODUCTION

The Fibonacci numbers, the Fibonacci polynomials, and the generalized Fibonacci polynomials, these latter defined by

$$u_n(x,y) = xu_{n-1}(x,y) + yu_{n-2}(x,y); \qquad u_0(x,y) = 0, \quad u_1(x,y) = 1,$$

all have the following divisibility property:

(1) $$\text{If } m \mid n, \text{ then } u_m \mid u_n .$$

In their recent paper [3], Hoggatt and Long prove (1) and more:

(2) $$\text{If } m \geqslant 1 \text{ and } n \geqslant 1, \text{ then } (u_m, u_n) = u_{(m,n)} .$$

Further, if p is a prime, then $u_p(x,y)$ is irreducible over the rational number field, a result originating with Webb and Parberry [5]. Similar results for Lucas polynomials and generalized Lucas polynomials are proved by Bergum and Hoggatt [2].

In this present paper, we consider divisibility properties of certain polynomials which include the generalized Fibonacci polynomials and a modification of the generalized Lucas polynomials as special cases.

Let x, y, z be indeterminants and let

(3) $$f_n = (x+y)f_{n-1} - xyf_{n-2}; \qquad f_0 = 0, \quad f_1 = 1.$$

Define $\ell_0 = 0$, $\ell_1(f) = f = f_1$, and

$$\ell_n(x,y,z) = \ell_n(f) = \begin{cases} f\ell_{n-1}(f) + z\ell_{n-2}(f) & \text{for even } n \\ f\ell_{n-1}(f) + z\ell_{n-2}(f) + 2z^{(n-1)/2} & \text{for odd } n, \end{cases}$$

where f^i is replaced by f_i for $i \geqslant 0$ after the multiplications involving f are carried out. Since

$$f_n(x,y) = (x^n - y^n)/(x-y)$$

for $n \geqslant 1$, it is easy to write out the first few $\ell_n(x,y,z)$ as follows:

$$\begin{aligned}
\ell_0 &= 0 = f_0 \\
\ell_1 &= (x-y)/(x-y) = f_1 \\
\ell_2 &= (x^2-y^2)(x-y) = f_2 \\
\ell_3 &= [x^3 - y^3 + 3z(x-y)]/(x-y) = f_3 + 3zf_1 \\
\ell_4 &= [x^4 - y^4 + 4z(x^2-y^2)]/(x-y) = f_4 + 4zf_2 \\
\ell_5 &= f_5 + 5zf_3 + 5z^2f_1 \\
\ell_6 &= f_6 + 6zf_4 + 9z^2f_2 \\
\ell_7 &= f_7 + 7zf_5 + 14z^2f_3 + 7z^3f_1 \\
\ell_8 &= f_8 + 8zf_6 + 20z^2f_4 + 16z^3f_2 \\
\ell_9 &= f_9 + 9zf_7 + 27z^2f_5 + 30z^3f_3 + 9z^4f_1
\end{aligned}$$

In general,

(4) $$\ell_n(x,y,z) = \sum_{i=0}^{w} \left[\binom{n+1-i}{i} - \binom{n-1-i}{i-2} \right] z^i f_{n-2i}, \quad \text{where } w = \begin{cases} (n-2)/2 & \text{for even } n \\ (n-1)/2 & \text{for odd } n \end{cases}.$$

*Supported by University of Evansville Faculty Research Grant A6D-0240.

Several special cases of the polynomials $\ell_n(x,y,z)$ are as follows:

Fibonacci numbers	$\ell_n(\alpha,\beta,0)$;	$\alpha+\beta = 1, \quad \alpha\beta = -1$
Fibonacci polynomials	$\ell_n(a,b,0)$;	$a+b = x, \quad ab = -1$
generalized Fibonacci polynomials	$\ell_n(A,B,0)$;	$A+B = x, \quad AB = -y$
modified Lucas numbers	$\ell_n(1,0,1)$	
modified Lucas polynomials	$\ell_n(x,0,1)$	
generalized modified Lucas polynomials	$\ell_n(x,0,z)$	

For comparison with (unmodified) sequences of generalized Lucas polynomials $L_n(x,z)$, Lucas polynomials $L_n(x,1)$, and Lucas numbers $L_n(1,1)$, we have, for $n = 0, 1, \cdots$,

$$L_n(1,1)\text{: } 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \cdots,$$

$$\ell_n(1,0,1)\text{: } 0, 1, 1, 4, 5, 11, 16, 29, 45, 76, 121, 199, 320, 521, \cdots;$$

$$L_n(x,z) = xL_{n-1}(x,z) + zL_{n-2}(x,z), \quad L_0(x,z) = 2, \quad L_1(x,z) = x;$$

(5)
$$x\ell_n(x,0,z) = \begin{cases} L_n(x,z) & \text{for odd } n \\ L_n(x,z) - 2z^{n/2} & \text{for even } n; \end{cases}$$

(6)
$$\ell_n(x,y,z) = \frac{L_n(x,z) - L_n(y,z)}{x-y}.$$

In Section 2 we prove that the divisibilities in (1) hold for the polynomials $\ell_n(x,y,z)$. In Section 3 we prove that consecutive terms of the sequence $\ell_n(x,y,z)$ are relatively prime. In Section 4 we prove the same for sequences of the form ℓ_{mn}/ℓ_m, where m is fixed. In Section 5 we prove that (2) holds for the sequence $\ell_n(x,y,z)$. In Section 6 we consider the irreducibility of some of the $\ell_n(x,y,z)$.

2. A MULTISECTION THEOREM

Lemma 1. The sequence $\ell_n(x,y,z)$, for $n = 1, 2, \cdots$, is generated by the function

$$G(x,y,z,t) = \frac{1+zt^2}{(1-xt-zt^2)(1-yt-zt^2)}.$$

Proof. Let

$$x(t) = 1-xt-zt^2 = (1-t_1t)(1-t_2t) \quad \text{and} \quad y(t) = 1-yt-zt^2 = (1-t_3t)(1-t_4t).$$

It is easy to check that

$$(x-y)G(x,y,z,t) = \frac{-x'(t)}{x(t)} - \frac{-y'(t)}{y(t)},$$

and it is well known [1] that $-x'(t)/x(t)$ generates a sequence of sums of powers of roots of $x(t)$. Explicitly,

(7)
$$(x-y)G(x,y,z,t) = \left\{ s_1(x) + s_2(x)t + \cdots - [s_1(y) + s_2(y)t + \cdots] \right\},$$

where $s_n(x) = t_1^n + t_2^n$ is the n^{th} (unmodified) generalized Lucas polynomial $L_n(x,z)$. Thus, the sequence $s_n(x) - s_n(y)$ generated in (7) is $L_n(x,z) - L_n(y,z)$. By (6), the proof is finished.

Theorem 1. If $m \mid n$, where $m \geqslant 1$ and $n \geqslant 1$, then $\ell_m(x,y,z) \mid \ell_n(x,y,z)$.

Proof. (The multisection procedure used here is explained in Chapter 4 of Riordan [4].) The $m-1$, m section of the series in (7) is

$$s_m(x)t^{m-1} + s_{2m}(x)t^{2m-1} + \cdots - [s_m(y)t^{m-1} + s_{2m}(y)t^{2m-1} + \cdots],$$

which we write as

$$(x-y)(\ell_m t^{m-1} + \ell_{2m} t^{2m-1} + \cdots).$$

Again as in [1], we know that $s_m(x) + s_{2m}(x)t + \cdots = -X'(t)/X(t)$, where

$$X(t) = (1-t_1^m t)(1-t_2^m t) = 1 - s_m(x)t + (-z)^m t^2,$$

and similarly,

$$s_m(y) + s_{2m}(y)t + \cdots \quad -Y'(t)/Y(t),$$

where

$$Y(t) = (1 - t_3^m t)(1 - t_4^m t) = 1 - s_m(y)t + (-z)^m t^2.$$

Thus,

$$(x - y)(\ell_m + \ell_{2m}t + \cdots) = \frac{-X'(t)}{X(t)} - \frac{-Y'(t)}{Y(t)} = \frac{XY' - X'Y}{XY}.$$

Write $1/XY$ as $H_0 + H_1 t + H_2 t^2 + \cdots$. Then

$$\begin{aligned} \ell_m + \ell_{2m}t + \cdots &= \frac{1}{x-y}(XY' - X'Y)(H_0 + H_1 t + H_2 t^2 + \cdots) \\ &= \ell_m[1 - (-z)^m t^2](H_0 + H_1 t + H_2 t^2 + \cdots). \end{aligned}$$

Therefore, if $n = km$, then $\ell_n = \ell_m[H_{k-1} - (-z)^m H_{k-3}]$.

3. CONSECUTIVE RELATIVELY PRIME POLYNOMIALS

The $m - 1$, m section of $xG(x,0,z,t) = L_1 + L_2 t + L_3 t^2 + \cdots$ readily provides a well known (e.g., [4]) recurrence relation

$$L_{nm} = L_m L_{(n-1)m} - (-z)^m L_{(n-2)m}$$

for subsequences of (unmodified) generalized Lucas polynomials. Substituting for the L's according to (5), we readily obtain the following lemma for modified generalized Lucas polynomials.

Lemma 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then for $n \geq 2$,

$$\ell_{nm} = \begin{cases} \ell_m(x\ell_{(n-1)m} + 2z^{(n-1)m/2}) + z^m \ell_{(n-2)m} & \text{for odd } m \text{ and odd } n \\ x\ell_m \ell_{(n-1)m} + z^m \ell_{(n-2)m} & \text{for odd } m \text{ and even } n; \end{cases}$$

$$\ell_{nm} = (x\ell_m + 2z^{m/2})\ell_{(n-1)m} - z^m \ell_{(n-2)m} + 2z^{(n-1)m/2}\ell_m \quad \text{for even } m.$$

Theorem 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$,

$$xz^{n-1} = \begin{cases} -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})\ell_{n+1} & \text{for odd } n \\ (x\ell_n + 4z^{n-2})\ell_n - x\ell_{n-1}\ell_{n+1} & \text{for even } n. \end{cases}$$

Proof. The proposition is obviously valid for $n = 1$. Suppose its validity for arbitrary odd n. Then for any even n,

$$\begin{aligned} xz^{n-2} &= (x\ell_{n-2} + 4z^{(n-2)/2})\ell_n - x(\ell_{n-1} + 2z^{(n-2)/2})\ell_{n-1} \\ xz^{n-1} &= (xz\ell_{n-2} + 4z^{n/2})\ell_n - x(z\ell_{n-1} + 2z^{n-2})\ell_{n-1} \\ &= (x\ell_n - x^2\ell_{n-1} + 4z^{n/2})\ell_n - x\ell_{n-1}(z\ell_{n-1} + 2z^{n/2}) \\ &= (x\ell_n + 4z^{n/2})\ell_n - x\ell_{n-1}(x\ell_n + z\ell_{n-1} + 2z^{n/2}) \\ &= (x\ell_n + 4z^{n/2})\ell_n - x\ell_{n-1}\ell_{n+1}. \end{aligned}$$

Now suppose the proposition valid for arbitrary even n. Then for any odd n,

$$\begin{aligned} xz^{n-2} &= -x\ell_{n-2}\ell_n + (x\ell_{n-1} + 4z^{(n-2)/2})\ell_{n-1} \\ xz^{n-1} &= -xz\ell_{n-2}\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} \\ &= -x(\ell_n - x\ell_{n-1} - 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} \\ &= -x(\ell_n - 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} + x^2\ell_n\ell_{n-1} \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})z\ell_{n-1} + x^2\ell_n\ell_{n-1} + 4xz^{(n-1)/2}\ell_n \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})(z\ell_{n-1} + x\ell_n) \\ &= -x(\ell_n + 2z^{(n-1)/2})\ell_n + (x\ell_{n-1} + 4z^{(n-1)/2})\ell_{n+1}. \end{aligned}$$

Corollary 2. Let $\ell_n = \ell_n(x,0,z)$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Theorem 2 shows that the only possible divisors of both ℓ_n and ℓ_{n+1} are of the form $x^i z^j$ where $0 \leq i \leq 1$ and $0 \leq j \leq n - 1$. Equation (4) shows that $j = 0$. Reading x for f in (4), we see that x divides $\ell_n(x,0,z)$ only when n is even. Since $x^i z^0$ divides consecutive ℓ's, we have $i = 0$.

Lemma 3. Let $\ell_n = \ell_n(x,y,0) = f_n$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Clearly $(f_1,f_2) = (f_2,f_3) = 1$. Suppose for arbitrary $n \geq 2$ that $(f_{n-2}, f_{n-1}) = 1$. If $d(x,y)$ divides both f_{n-1} and f_n then by (3), $d(x,y)$ divides xyf_{n-2}. Thus $d(x,y)$ divides f_{n-2}, since $(xy,f_{n-2}) = 1$. But the only divisor of both f_{n-2} and f_{n-1} is 1. Therefore $d(x,y) = 1$.

Theorem 3. Let $\ell_n = \ell_n(x,y,z)$ for $n \geq 0$. Then $(\ell_n, \ell_{n+1}) = 1$ for $n \geq 1$.

Proof. Suppose $d(x,y,z)$ divides both $\ell_n(x,y,z)$ and $\ell_{n+1}(x,y,z)$. Then $d(x,y,0)$ divides both $\ell_n(x,y,0)$ and $\ell_{n+1}(x,y,0)$. By Lemma 3, $d(x,y,z) = 1 + ze(x,y,z)$ for some $e(x,y,z)$. Since $1 + ze(x,y,z)$ divides $\ell_n(x,y,z)$, we have

$$\ell_n(x,y,z) = q(x,y,z) + ze(x,y,z)q(x,y,z)$$

for some $q(x,y,z)$. Now z does not divide $q(x,y,z)$, since z does not divide $\ell_n(x,y,z)$. Therefore the term x^{n-1} in $\ell_n(x,y,z)$ occurs in $q(x,y,z)$, Consequently, unless $e(x,y,z)$ is the zero polynomial, some nonzero multiple of zx^{n-1} occurs in the polynomial $ze(x,y,z)q(x,y,z)$. But $\ell_n(x,y,z)$ has no such term. Therefore $e(x,y,z)$ is the zero polynomial, so that $d(x,y,z) = 1$.

4. SUBSEQUENCES OF ℓ_n

In this section we consider subsequences of the form $\ell_m, \ell_{2m}, \ell_{3m}, \cdots$, where $m \geq 1$. Since each term is divisible by the first term, let us divide all terms by the first, and let $\lambda_{m,n} = \ell_{nm}/\ell_m$ for $n \geq 0$. Then by Theorem 1, for $m \geq 1$ and $k \geq 1$, $\lambda_{m,k} | \lambda_{m,n}$ whenever $k|n$. Do the λ sequences also inherit from the ℓ sequence the property that consecutive terms are relatively prime?

Lemma 4a. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$,

$$x\ell_{(n-1)m} = \begin{cases} \lambda_2\lambda_{n-1} & \text{for odd } m \\ (\lambda_2 - 4z^{m/2})\lambda_{n-1} & \text{for even } m. \end{cases}$$

Proof. By Lemma 2,

$$\ell_{2m} = \begin{cases} x\ell_m^2 & \text{for odd } m \\ x\ell_m^2 + 4z^{m/2}\ell_m & \text{for even } m. \end{cases}$$

Thus, for odd m,

$$x\ell_{(n-1)m} = \frac{\ell_{2m}}{\ell_m^2}\,\ell_{(n-1)m} = \lambda_2\lambda_{n-1}.$$

For even m,

$$x\ell_{(n-1)m} = \frac{(\ell_{2m} - 4z^{m/2}\ell_m)\ell_{(n-1)m}}{\ell_m^2} = (\lambda_2 - 4z^{m/2})\lambda_{n-1}.$$

Lemma 4b. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for odd m and $n \geq 2$,

$$\lambda_n = \begin{cases} \lambda_2\lambda_{n-1} + z^m\lambda_{n-2} + 2z^{(n-1)m/2} & \text{for odd } n \\ \lambda_2\lambda_{n-1} + z^m\lambda_{n-2} & \text{for even } n; \end{cases}$$

and for even m,

$$\lambda_n = (\lambda_2 - 2z^{m/2})\lambda_{n-1} - z^m\lambda_{n-2} + 2z^{(n-1)m/2},$$

with $\lambda_0 = 0$ and $\lambda_1 = 1$.

Proof. In Lemma 2, divide both sides of the three recurrence relations by ℓ_m, recalling that $\lambda_k = \ell_{km}/\ell_m$ for $k = n, n-1, n-2$. Now replace $x\ell_{(n-1)m}$ by $\lambda_2\lambda_{n-1}$ for odd m, and replace $x\ell_m$ by $\lambda_2 - 4z^{m/2}$ for even m.

Theorem 4a. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for odd m and $n \geq 1$,

$$\lambda_2 z^{(n-1)m} = \begin{cases} -\lambda_2(\lambda_n + 2z^{(n-1)m/2})\lambda_n + (\lambda_2\lambda_{n-1} + 4z^{(n-1)m/2})\lambda_{n+1} & \text{for odd } n \\ (\lambda_2\lambda_n + 4z^{nm/2})\lambda_n - \lambda_2\lambda_{n-1}\lambda_{n+1} & \text{for even } n. \end{cases}$$

Proof. Referring to the proof of Theorem 1, we know that for odd m,

$$\ell_m + \ell_{2m} + \cdots = \ell_m \frac{1+z^m t^2}{X(t)Y(t)},$$

so that

$$\lambda_1 + \lambda_2 t + \cdots = G[s_m(x), s_m(y), z^m, t].$$

Since $s_m(y) = L_n(y,z)$, as in the proof of Lemma 1, Eqs. (5) and (6) show that $s_m(y) = 0$ for all odd m. Further, $s_m(x) = x\ell_m$, which by Lemma 4a equals ℓ_{2m}/ℓ_m, which is λ_2. Therefore,

$$\lambda_1 + \lambda_2 t + \cdots = G(\lambda_2, 0, z^m, t) = \ell_1(\lambda_2, 0, z^m) + \ell_2(\lambda_2, 0, z^m)t + \cdots,$$

by Lemma 1. Thus Theorem 2 applies with x and z replaced by λ_2 and z^m, respectively. The result is exactly as stated above.

Theorem 4b. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for even m and $n \geq 1$,

$$z^{(n-1)m} = -(\lambda_n - 2z^{(n-1)m/2})\lambda_n + \lambda_{n-1}\lambda_{n+1}.$$

Proof. The proposition is clearly valid for $n = 1$. For arbitrary $n > 1$, suppose that

$$z^{(n-2)m} = -(\lambda_{n-1} - 2z^{(n-2)m/2})\lambda_{n-1} + \lambda_{n-2}\lambda_n.$$

Then

$$z^{(n-1)m} = -[-\lambda_{n+1} + (\lambda_2 - 2z^{m/2})\lambda_n]\lambda_{n-1} - [\lambda_n - (\lambda_2 - 2z^{m/2})\lambda_{n-1} - 2z^{(n-1)m/2}]\lambda_n$$

by Lemma 4b

$$= -(\lambda_n - 2z^{(n-1)m/2})\lambda_n + \lambda_{n-1}\lambda_{n+1}.$$

Corollary 4. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for all positive integers m and n.

Proof. For odd m, Theorem 4a shows that the only possible divisors of both λ_n and λ_{n+1} are of the form $\lambda_2^i z^j$, where $0 \leq i \leq 1$ and $0 \leq j \leq (n-1)m$. As in the proof of Theorem 4a, $\lambda_n = \ell_n(\lambda_2, 0, z^m)$, so that (4) gives (for *odd m* only),

$$\lambda_n = \sum_{i=0}^{w} \left[\binom{n+1-i}{i} - \binom{n-1-i}{i-2}\right] z^{mi} f^*_{n-2i}, \quad \text{where } w = \begin{cases} (n-2)/2 & \text{for even } n \\ (n-1)/2 & \text{for odd } n, \end{cases}$$

and $f^*_k(x,0) = f_k(\lambda_2,0) = \lambda_2^{k-1}$ for $k \geq 1$. Thus λ_2 divides λ_n only when n is even. Since λ_2^i divides consecutive λ's, we have $i = 0$.

Now for *any m*, Theorem 4b and the argument just given show that the only possible divisors of both λ_n and λ_{n+1} are of the form z^j with $0 \leq j \leq (n-1)m$. If z^j divides λ_n then z^j divides $\ell_{mn} = \ell_m\ell_n$. Thus $j = 0$, by (4).

Lemma 5. Let $\lambda_n = \lambda_{m,n}(x,y,0)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. Since $z = 0$, we have $\lambda_n = f_{nm}/f_m = f_n(x^m, y^m)$. Now (3) is used to complete the proof, just as in the proof of Lemma 3.

Theorem 5. Let $\lambda_n = \lambda_{m,n}(x,y,z)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. The method of proof is exactly as for Theorem 3. Here the exponent of x to be considered is $^{m(n-1)}$ rather than $^{n-1}$.

Theorem 6. Let m and n be odd. Let $\lambda_n = \lambda_{m,n}(x,0,z)$ for $n \geq 0$. Then for $n \geq 1$, $\lambda_n \equiv nz^{(n-1)m/2} \bmod \lambda_2$.

Proof. By Lemma 4b,

$$\lambda_n \equiv (z^m\lambda_{n-2} + 2z^{(n-1)m/2}) \bmod \lambda_2 \text{ for odd } n.$$

Repeated application of this congruence gives

$$\lambda_n \equiv 2z^{(n-1)m/2} + z^m\lambda_{n-2} \equiv 2z^{(n-1)m/2} + z^m(z^m\lambda_{n-4} + 2z^{(n-3)m/2}) \equiv 4z^{(n-1)m/2} + z^{2m}\lambda_{n-4} \equiv \cdots$$

$$\equiv 2kz^{(n-1)m/2} + z^{km}\lambda_{n-2k} \text{ for } k = 1, 2, \cdots, \frac{n-1}{2}.$$

In particular, for $k = \frac{n-1}{2}$,

$$\lambda_n \equiv (n-1)z^{(n-1)m/2} + z^{(n-1)m/2}\lambda_1 ,$$

as desired.

Corollary 6a. Let m be an odd positive integer.

Let $\lambda_n = \lambda_{m,n}(1,0,1)$ for $n \geq 0$. Then $\lambda_2 = \ell_m$. If n is odd and $n \equiv 0 \bmod \lambda_m$, then $(\lambda_n, \lambda_{n+1}) = \ell_m$. If ℓ_m is a prime, then $(\lambda_n, \lambda_{n+1}) = 1$ for each positive integer n satisfying $n \not\equiv 0 \bmod \ell_m$. For $m = 1$ and $m = 3$,

$$(\lambda_n, \lambda_{n+1}) = 1 \quad \text{for} \quad n \geq 1.$$

Proof. By Lemma 4a, $\ell_{2m} = \ell_m^2$, so that $\lambda_2 = \ell_m$. If n is odd, then $\lambda_n \equiv n \bmod \lambda_2$, by Theorem 6. Thus $\lambda_n \equiv n \bmod \ell_m$, and if $n \equiv 0 \bmod \ell_m$, then $\lambda_n \equiv 0 \bmod \ell_m$. Since $n+1$ is even, λ_2 divides λ_{n+1}, by Theorem 1. Therefore both λ_n and λ_{n+1} are divisible by λ_2. By Theorem 4a, the only divisors of both λ_n and λ_{n+1} are divisors of λ_2. Therefore $\lambda_2 = (\lambda_n, \lambda_{n+1})$.

Now for n either odd or even, Theorem 4a still shows that the only divisors of both λ_n and λ_{n+1} are divisors of λ_2. Thus if $\lambda_2 = \ell_m$ is a prime, then the conditions $\lambda_n \equiv n \bmod \ell_m$ and $n \not\equiv 0 \bmod \ell_m$ show that ℓ_m does not divide λ_n. Thus $(\lambda_n, \lambda_{n+1}) = 1$.

For $m = 1$, put $x = z = 1$ in Theorem 2.

For $m = 3$, we have $\lambda_2 = 4$, so that after dividing by 4 in Theorem 4a, we find

$$1 = \begin{cases} -(\lambda_n + 2)\lambda_n + (\lambda_{n-1} + 1)\lambda_{n+1} & \text{for odd } n \\ (\lambda_n + 1)\lambda_n - \lambda_{n-1}\lambda_{n+1} & \text{for even } n. \end{cases}$$

Corollary 6b. Let m be an even positive integer. Let $\lambda_n = \lambda_{m,n}(1,0,1)$ for $n \geq 0$. Then $(\lambda_n, \lambda_{n+1}) = 1$ for $n \geq 1$.

Proof. This is an immediate consequence of Theorem 4b.

Example. To illustrate Corollary 6a, let $m = 5$. Recalling the abbreviation

$$\lambda_5 = \lambda_{5,n}(1,0,1) = \ell_{5n}(1,0,1)/\ell_5(1,0,1)$$

for $n \geq 0$ and Lemma 4b, we have for $n \geq 2$:

$$\lambda_n = \begin{cases} 11\lambda_{n-1} + \lambda_{n-2} + 2 & \text{for odd } n \\ 11\lambda_{n-1} + \lambda_{n-2} & \text{for even } n. \end{cases}$$

We write out the first 12 λ's and factor them

$$\begin{aligned}
\lambda_0 &= 0 = 0 \\
\lambda_1 &= 1 = 1 \\
\lambda_2 &= 11 = 11 \\
\lambda_3 &= 124 = 2^2 \cdot 31 \\
\lambda_4 &= 1375 = \lambda_2(\lambda_2^2 + 4) \\
\lambda_5 &= 15251 = 101 \cdot 151 \\
\lambda_6 &= 169136 = \lambda_2 \lambda_3^2 \\
\lambda_7 &= 1875749 = 29 \cdot 71 \cdot 911 \\
\lambda_8 &= 20802375 = \lambda_4(\lambda_2\lambda_4 + 4) \\
\lambda_9 &= 230701876 = \lambda_3(\lambda_2\lambda_6 + 3) \\
\lambda_{10} &= 2558523011 = \lambda_2\lambda_5^2 \\
\lambda_{11} &= 28374454999 = \lambda_2 \cdot 199 \cdot 331 \cdot 39161.
\end{aligned}$$

In agreement with Corollary 6a, we have $(\lambda_n, \lambda_{n+1}) = 1$ for $1 \leq n \leq 9$, but $(\lambda_{10}, \lambda_{11}) = 11$.

5. THE EQUATION $(\ell_m, \ell_n) = \ell_{(m,n)}$

Lemma 7. Let $\lambda_j = \lambda_{m,j}(x,y,z)$ for $m \geq 1$ and $j \geq 0$. If k is a positive integer satisfying $(j,k) = 1$, then $(\lambda_j, \lambda_k) = 1$.

Proof. Write $1 = sj - tk$ (or $sk - tj$) where s and t are nonnegative. Let $d = (\lambda_j, \lambda_k)$. Then $d \mid \lambda_{sj}$ and $d \mid \lambda_{tk}$ by Theorem 1. Thus $d \mid (\lambda_{sj}, \lambda_{tk})$, which is to say $d \mid (\lambda_{tk}, \lambda_{tk+1})$. By Theorem 5, we have $d = 1$.

Theorem 7. Let $\ell_m = \ell_m(x,y,z)$ for $m \geq 0$. Then for $m \geq 1$ and $n \geq 1$, $(\ell_m, \ell_n) = \ell_{(m,n)}$.

Proof. Let $d = (m,n)$. Let $j = m/d$ and $k = k/d$. Then $(\lambda_j, \lambda_k) = 1$, by Lemma 7. Now $\ell_m = \lambda_j \ell_d$ and $\ell_n = \lambda_k \ell_d$. Therefore $(\ell_m, \ell_n) = \ell_d$.

Lemma 8. The following items hold true when restated in terms of $(x,0,1)$ and $(1,0,1)$, instead of $(x,0,z)$: Lemmas 4a and 4b, Theorems 4a and 4b, Theorem 6, and Lemma 7.

Proof. The resulting restatements are special cases, to which the proofs already given apply.

Theorem 8. The equations $(\ell_m, \ell_n) = \ell_{(m,n)}$ and $(\lambda_j, \lambda_n) = \lambda_{(j,k)}$ as in Theorem 7 and Corollary 7 hold if ℓ and λ are applied to $(x,0,z)$ and $(x,0,1)$. They also hold for $(1,0,1)$ if m is even or equal to 1 or 3.

Proof. The first statement follows from Lemma 8 exactly as Theorem 7 and Corollary 7 follow from Lemma 7.

For the second statement, we obtain $d \mid (\lambda_{tk}, \lambda_{tk+1})$ as in the proof of Lemma 7 and have $d = 1$ by Corollaries 6a and 6b. Then the methods of proof of Theorem 7 and Corollary 7 apply.

6. IRREDUCIBILITY OF ℓ POLYNOMIALS

Lemma 9. The polynomial $\ell_n(x,y,0)$ is irreducible over the rational number field if and only if n is a prime.

Proof. It is known [3] that the generalized Fibonacci polynomial $\ell_n(A,B,0)$, where $A + B = x$ and $AB = -y$, is irreducible if and only if n is a prime. The present lemma is an immediate consequence.

Theorem 9. The polynomial $\ell_n(x,y,z)$ is irreducible over the rational number field if and only if n is a prime.

Proof. If n is not a prime, we have Theorem 1. Suppose n is a prime and that $\ell_n(x,y,z) = d(x,y,z)q(x,y,z)$. Then one of the polynomials $d(x,y,0)$ and $q(x,y,0)$ must be the constant 1 polynomial, by Lemma 9. Supposing this one to be $d(x,y,0)$, we have $d(x,y,z) = 1 + ze(x,y,z)$ for some $e(x,y,z)$. The remainder of the proof is identical to that of Theorem 3.

Lemma 10. The polynomial $\ell_n(x,0,z) + 2z^{n/2}$, where n is even, is irreducible over the rational number field if and only if $n = 2^k$ for some $k \geq 1$. Further, the polynomial $\ell_n(x,0,z)$ for odd n is irreducible if and only if n is a prime.

Proof. These two results are proved in [2].

Theorem 10. If $n = 2^k$ for some $k \geq 1$, then the polynomial $\ell_n(x,y,z) + 2z^{n/2}$ is irreducible over the rational number field.

Proof. Suppose $\ell_n(x,y,z) + 2z^{n/2} = d(x,y,z)q(x,y,z)$. Then one of the polynomials $d(x,0,z)$ and $q(x,0,z)$ must be the constant 1 polynomial. Supposing this one to be $d(x,0,z)$, we have $d(x,y,z) = 1 + ye(x,y,z)$ for some $e(x,y,z)$, by Lemma 10. Consequently,

$$\ell_n(x,y,z) + 2z^{n/2} = q(x,y,z) + ye(x,y,z)q(x,y,z).$$

Once again, the remainder of the proof is identical to that of Theorem 3, except that here we have yx^{n-1} instead of zx^{n-1}.

REFERENCES

1. S. L. Basin, "A Note on Waring's Formula for Sums of Like Powers of Roots," *The Fibonacci Quarterly*, Vol. 2, No. 2 (April 1964), pp. 119–122.

2. G. E. Bergum and V. E. Hoggatt, Jr., "Irreducibility of Lucas and Generalized Lucas Polynomials," *The Fibonacci Quarterly,* Vol. 12, No. 1 (Feb. 1974), pp. 95–100.
3. V. E. Hoggatt, Jr., and C. T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," *The Fibonacci Quarterly,* Vol. 12, No. 2 (April 1974), pp. 113–120.
4. J. Riordan, *Combinatorial Identities,* Wiley & Sons, New York, 1968.
5. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," *The Fibonacci Quarterly,* Vol. 7, No. 5 (Dec. 1969), pp. 457–463.

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THE FIBONACCI SEQUENCE ENCOUNTERED IN NERVE PHYSIOLOGY

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The pulses travelling along the nerve fibres originate in local currents of Sodium- and Potassium-ions, across the membranes which surround the fibres. The Sodium-current is switched on and off by small amounts of Calcium-ions. In order to model the operation of Ca^{2+}, assume that the Na^{+}-current flows through identical trans-membrane pores, each made up of a string of n Na^{+}-binding sites. Also, Ca^{2+} can enter the pores, occupying two sites per ion, or, one site when entering the pore (cf. Fig. 2). Thus, a pore may momentarily look like Fig. 1.

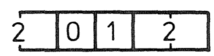

Fig. 1 A pore in one of its possible states (0: empty site; 1 : Na^{+}; 2 : Ca^{2+})

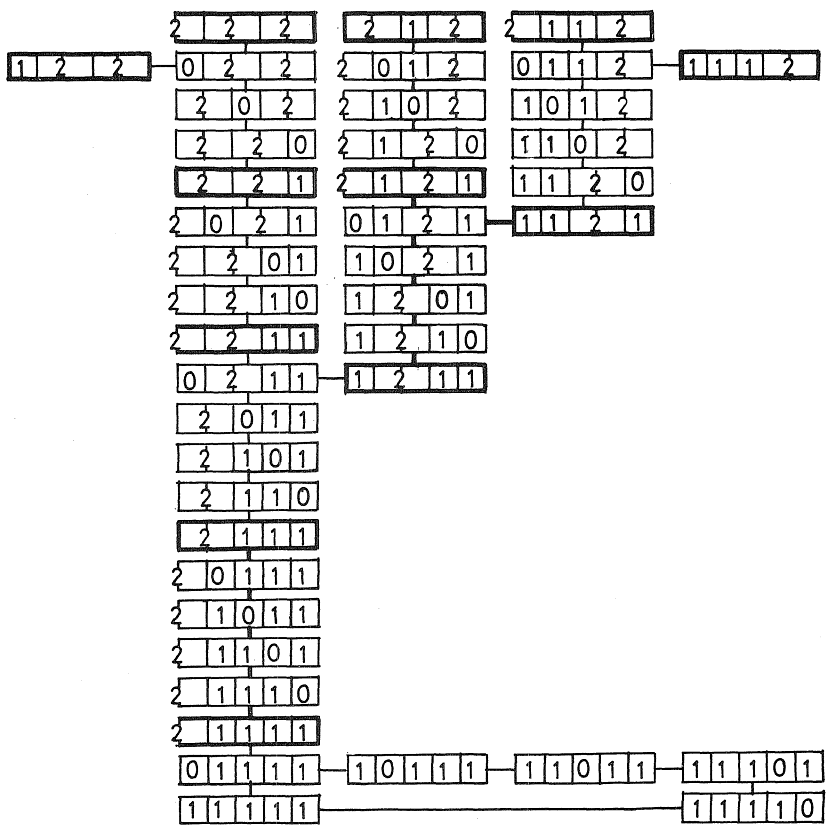

Fig. 2

Fig. 2 Graph for 5-site pore-process; only states with at most one vacant site.

Assume, further, that the particles can jump, in a stochastic manner, into a neighbouring empty site, without being able to overtake each other. During this process, Sodium may enter and leave the pores on either side, whereas Calcium may enter and leave at the left-hand side only. Thus, Calcium ions within a pore block the Sodium-current through this pore.

This model reproduces the relevant outcome of experiments (publication in preparation).

Where and how do the Fibonacci numbers come in?

Let the stochastic process described above be Markoffean. Then, the process is conveniently pictured by a graph, with its points representing the finite number of possible states of a pore, given by its occupation by 0, 1, 2 (cf. Fig. 1), and its edges representing the allowed transitions between states. Alongside one has a set of homogeneous linear differential equations of the first order, describing the time development of the states' probabilities. These are, in essence, the forward-aquations of the Markoff-process.

For the time-stationary case, these equations are conveniently solved by graph-theoretical methods (T. L. Hill, *J. Theor. Biol.* (1966), 10, 442–459). Therefore, the graph needs careful investigation.

First, consider only pores with at most one site vacant. Under feasible physical conditions, these states can be shown to be the only relevant ones: only their probabilities differ appreciably from zero. Then, the graph boils down to a single cycle (along which Na^+ is transported), and a large tree growing out of the cycle (cf. Fig. 2).

Note that the tree is made up of two types of subgraphs: each type ends in full pores, between which a vacancy travels from right to left (or vice versa). One specimen for each case is indicated by heavy edges, in Fig. 2.

It can be shown, that, in order to calculate the probabilities of the full-pore-states of the tree, one can throw out the one-vacancy-pores, too, ending up, in the case of Fig. 2, with the tree of Fig. 3.

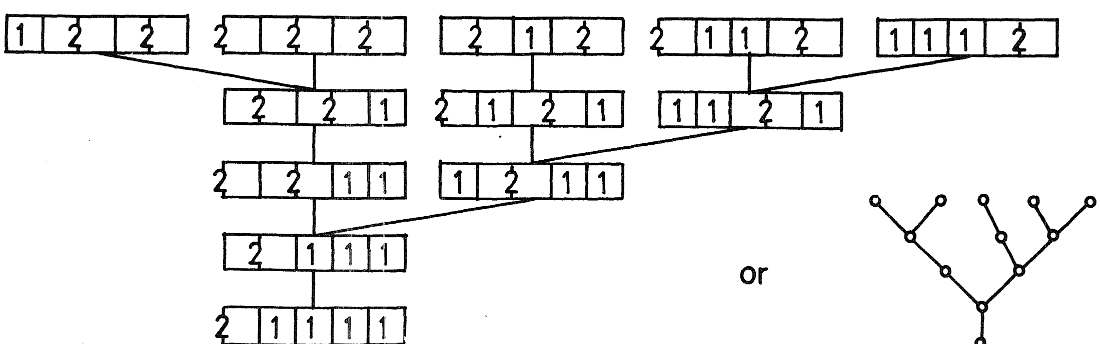

Fig . 3

Fig 3 Tree of full pores, corresponding to the tree in Fig. 2

From the general structure of the graph of Fig. 2, one infers that the tree of full-pore-states within the tree of the graph corresponding to n-site-pores $(n = 2, 3, 4, \cdots)$, has the form of Fig. 4.

Counting the number of points, $N(n)$, at level n, one finds

$$N(n) = F_n \quad (n = 2, 3, 4, \cdots),$$

the Fibonacci sequence with $F_1 = F_2 = 1$. The tree of Fig. 4 is, indeed, the graph of Fibonacci's original rabbit family. This fact is based on the Ca^{2+} entering the pore, from the left, in two steps (cf. Fig. 2), as the ion has two legs, i.e., elementary charges.

Lanthanum^{3+} is known, in its effect on the Na^+-current in nerve, as a super Calcium. If an analogous model is made, so that La^{3+}, instead of Ca^{2+}, switches the Na^+-current; and if La^{3+} enters the pores in three steps, then the graph corresponding to Fig. 4 is seen to have the structure of Fig. 5. Now the number $N^{3+}(n)$ of points on level n $(n = 2, 3, 4, \cdots)$ is given by

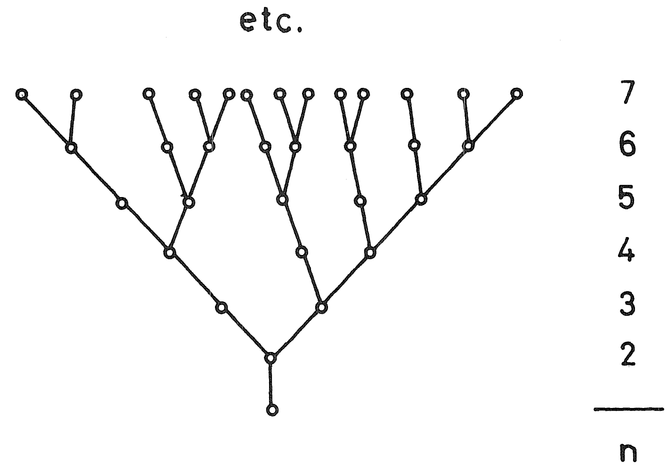

Fig. 4 Structure of full-pore-tree for *n*-sites-pores

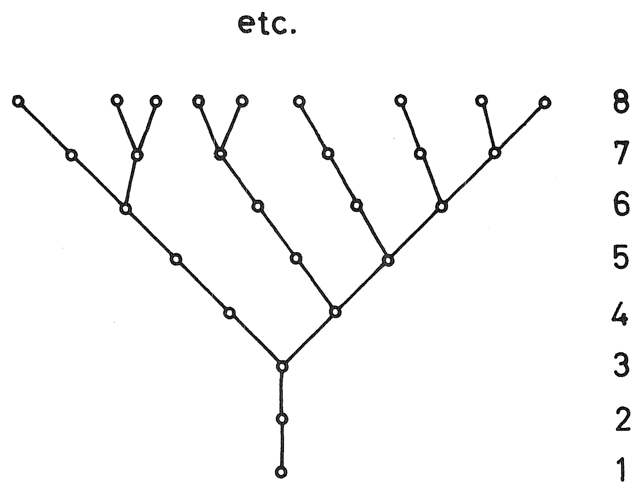

Fig. 5 Tree analogous to Fig. 4, with La^{3+} instead of Ca^{2+}

$$N^{3+}(n) = N^{3+}(n-1) + N^{3+}(n-3).$$

This can obviously be generalized.

For the membrane model maker, these findings have already been useful.

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A FIBONACCI PROPERTY OF WYTHOFF PAIRS

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In this paper we point out another of those fascinating "coincidences" which are so characteristically associated with the Fibonacci numbers. It occurs in relation to the so-called safe pairs (a_n, b_n) for Wythoff's Nim [1, 2, 3]. These pairs have been extensively analyzed by Carlitz, Scoville and Hoggatt in their researches on Fibonacci representations [4, 5, 6, 7], a context unrelated to the game of nim. The latter have carefully established the basic properties of the a_n and b_n, so that even though that which we are about to report is not described in their investigations, it is a ready consequence of them. For convenience and for reasons of precedence, we refer to the pairs (a_n, b_n) as *Wythoff pairs.*

The first forty Wythoff pairs are listed in Table 1 for reference. We recall that the pairs are defined inductively as follows: $(a_1, b_1) = (1, 2)$, and, having defined $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$, a_{n+1} is defined as the smallest positive integer not among $a_1, b_1, a_2, b_2, \cdots, a_n, b_n$ and then b_{n+1} is defined as $a_{n+1} + (n + 1)$. Each positive integer occurs exactly once as a member of some Wythoff pair, and the sequences $\{a_n\}$ and $\{b_n\}$ are (strictly) increasing.

Wythoff [1] showed that $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$, α being the golden ratio $(1 + \sqrt{5})/2$; a more elegant proof of this appears in [3]. This connection of the Wythoff pairs with the golden ratio suggests to any "Fibonaccist" that the Fibonacci numbers are not very far out of the picture. The work of Carlitz *et. al.* that we have mentioned shows that the a_n and b_n play a fundamental part in the analysis of the Fibonacci number system. We now look at another connection with Fibonacci numbers.

Table 1
The First Forty Wythoff Pairs

n	a_n	b_n	n	a_n	b_n	n	a_n	b_n	n	a_n	b_n
1	1	2	11	17	28	21	33	54	31	50	81
2	3	5	12	19	31	22	35	57	32	51	83
3	4	7	13	21	34	23	37	60	33	53	86
4	6	10	14	22	36	24	38	62	34	55	89
5	8	13	15	24	39	25	40	65	35	56	91
6	9	15	16	25	41	26	42	68	36	58	94
7	11	18	17	27	44	27	43	70	37	59	96
8	12	20	18	29	47	28	45	73	38	61	99
9	14	23	19	30	49	29	46	75	39	63	102
10	16	26	20	32	52	30	48	78	40	64	104

In examining Table 1, it is interesting to observe that the first few Fibonacci numbers occur paired with other Fibonacci numbers:

$(a_1, b_1) = (1,2)$, $(a_2, b_2) = (3,5)$, $(a_5, b_5) = (8, 13)$, $(a_{13}, b_{13}) = (21, 34)$, $(a_{34}, b_{34}) = (55, 89)$.

It is not difficult to establish that this pattern continues throughout the sequence of Wythoff pairs, using the fact that $\lim (b_n/a_n) = \alpha$ and also $\lim (F_{n+1}/F_n) = \alpha$. However, an almost immediate proof can be had, based on Eq. (3.5) of [4], which states that

$$a_n + b_n = a_{b_n} \tag{1}$$

for each positive integer n. We defer the proof momentarily, it being our intention to state and prove a generalization of the foregoing.

Clearly there are many Wythoff pairs whose members are not Fibonacci numbers; the first such is $(a_3, b_3) = (4,7)$. The pair (4,7) can be used to generate a Fibonacci sequence in the same way that $(a_1, b_1) = (1,2)$ can be considered to determine the usual Fibonacci numbers; we take $G_1 = 4$, $G_2 = 7$, $G_{n+2} = G_{n+1} + G_n$. The first few terms of the resulting Fibonacci sequence are

$$4, 7, 11, 18, 29, 47, \cdots.$$

It is (perhaps) a bit startling to observe that

$$(a_3, b_3) = (4, 7), \qquad (a_7, b_7) = (11, 18), \qquad (a_{18}, b_{18}) = (29, 47).$$

The pair $(a_4, b_4) = (6, 10)$ similarly generates a Fibonacci sequence

$$6, 10, 16, 26, 42, 68, \cdots$$

and sure enough

$$(a_4, b_4) = (6, 10), \qquad (a_{10}, b_{10}) = (16, 26), \qquad (a_{26}, b_{26}) = (42, 68).$$

It is time for our first theorem.

Theorem. Let $G_1, G_2, G_3, \cdots$ be the Fibonacci sequence generated by a Wythoff pair (a_n, b_n). Then every pair $(G_1, G_2), (G_3, G_4), (G_5, G_6), \cdots$ is again a Wythoff pair.

Proof. By construction, every Wythoff pair satisfies

$$a_k + k = b_k \,. \tag{2}$$

Consider the first four terms of the generated Fibonacci sequence:

$$a_n, b_n, a_n + b_n, a_n + 2b_n \,.$$

According to Eq. (1)

$$a_n + b_n = a_{b_n},$$

so that

$$a_n + 2b_n = a_{b_n} + b_n \,.$$

Equation (2) with $k = b_n$ gives

$$a_{b_n} + b_n = b_{b_n} \,,$$

so that the four terms under consideration are in fact

$$a_n, b_n, a_{b_n}, b_{b_n} \,.$$

Thus we have proven that in general (G_3, G_4) is a Wythoff pair when (G_1, G_2) is. But (G_5, G_6) can be considered as consisting of the third and fourth terms of the Fibonacci sequence generated by (G_3, G_4), and the latter is already known to be a Wythoff pair. In this way, the theorem follows by induction.

Thus we see that each Wythoff pair generates a sequence of Wythoff pairs; the pairs following the first pair of the sequence will be said to be generated by the first pair. We define a Wythoff pair to be *primitive* if no other Wythoff pair generates it. It is clear that if (a_m, b_m) generates (a_n, b_n), then $m < n$. For this reason, one can determine the first few primitive pairs by the following algorithm, analogous to Eratosthenes' sieve. The first pair (1,2) is clearly primitive. All those generated by (1,2) are eliminated (up to some specified point in the table). The first pair remaining must again be primitive, and all pairs generated by that primitive are eliminated. The process is repeated.

The first few primitive pairs so determined are pair numbers

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, \cdots$$

which we recognize at once to be the sequence

$$a_1, a_2, a_3, \cdots.$$

This occasions our next theorem.

Theorem. A Wythoff pair (a_n, b_n) is primitive if and only if $n = a_k$ for some positive integer k.

Proof. We have seen that the terms of the Fibonacci sequence generated by any Wythoff pair (a_n, b_n) are of the form

$$a_n, b_n, a_{b_n}, b_{b_n}, a_{b_{b_n}}, b_{b_{b_n}}, \cdots .$$

From this it is obvious that any non-primitive pair (a_n, b_n) must have $n = b_k$ for some positive integer k, which makes every pair (a_n, b_n) with $n = a_k$ a primitive pair.

On the other hand, the sequence

$$a_n, b_n, a_{b_n}, b_{b_n}, \cdots$$

generated by (a_n, b_n) shows clearly that each pair (a_{b_k}, b_{b_k}) is generated by (a_k, b_k); thus every primitive pair (a_n, b_n) must have $n = a_k$.

This theorem shows that the number of primitive pairs is infinite, and has the following corollary.

Corollary. There exists a sequence of Fibonacci sequences which simply covers the set of positive integers.

An interesting property of the primitive pairs turns up when we calculate successively the determinants

$$\begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix} ,$$

restricting our attention to those (a_n, b_n) which are primitive. We find that

$$\begin{vmatrix} 4 & 7 \\ 1 & 2 \end{vmatrix} = 1, \qquad \begin{vmatrix} 6 & 10 \\ 1 & 2 \end{vmatrix} = 2,$$

$$\begin{vmatrix} 9 & 15 \\ 1 & 2 \end{vmatrix} = 3, \qquad \begin{vmatrix} 12 & 20 \\ 1 & 2 \end{vmatrix} = 4,$$

and so on. This suggests that the value of the determinant applied to the k^{th} primitive is $k - 1$. By the foregoing theorem, we know that the k^{th} primitive is in fact (a_{a_k}, b_{a_k}), so the suggested identity becomes

$$2a_{a_k} - b_{a_k} = k - 1,$$

which follows readily from Eq. (3.2) of [4].

We conclude by interpreting our results in terms of the findings in [4] and [6]. According to the latter, the Wythoff pairs (a_n, b_n) are those pairs of positive integers with the following two properties: first, the canonical Fibonacci representation of b_n is exactly the left shift of the canonical Fibonacci representation of a_n, and second, the right-most 1 appearing in the representation of a_n occurs in an even numbered position. (In base 2 this would be analogous to saying that $b_n = 2a_n$ and that the largest power of 2 which divides a_n is odd). No two 1's appear in succession in the representations of a_n and b_n. If we add a_n and b_n, each 1 in the representation of b_n will combine with its shift in the representation of a_n to yield a 1 in the position immediately to the left of the added pair, since $F_n + F_{n-1} = F_{n+1}$. This means that $b_n + a_n$ has a representation which is exactly the left shift of b_n. By exactly the same reasoning, $(b_n + a_n) + b_n$ has a representation which is exactly the left shift of $b_n + a_n$, and so forth. *Hence, the Fibonacci sequence generated by any Wythoff pair, when expressed canonically in the Fibonacci number system, consists of consecutive left shifts of the first term of the sequence.* In the simplest case, the pair (1,2) generates the usual Fibonacci sequence, which in the Fibonacci number system would be expressed

10, 100, 1000, 10000, 100000, ⋯

and the generated pairs would be

(10, 100), (1000, 10000), ⋯

which have the requisite properties that each b_n is the left shift of a_n and that each a_n has its right-most 1 in an even-numbered position. The next case corresponds to the sequence generated by (4, 7); 4, 7, 11, 18, ⋯. In Fibonacci, this appears as

1010, 10100, 101000, 1010000, ⋯.

This procedure can be traced back an additional step to the index n of the pair (a_n, b_n). Doing so provides in addition a simple interpretation of the primitive pairs in terms of Fibonacci representations. There is a prescription in [6] for generating a Wythoff pair (a_n, b_n) from its index n, but it necessitates the so-called second canonical Fibonacci representation. For present purposes it suffices to remark that the second canonical representation of any n can be obtained by adding 1 to the usual canonical representation of $n - 1$. (For example, the canonical representation of 7 is 10100, so the second canonical representation of 8 is 10101). *The numbers* a_n *and* b_n *are then obtained from successive left shifts of the second canonical representation of* n. Thus, in the example of (4,7), we obtain the second canonical representation of 4 as 1000 + 1 = 1001 and generate

$$\begin{array}{ccccc} n & a_n & b_n & a_n + b_n = a_{b_n}, \text{ etc.} \\ 10001: & 10010 & 100100 & 1001000 & \cdots . \end{array}$$

We have seen that the primitive pairs correspond to the case $n = a_k$. It is readily established on the basis of the results in [4] that the numbers a_k are precisely those numbers whose second canonical Fibonacci representations contain a 1 in the first position (as follows: first, a number is a b_k if and only if its canonical representation contains its right-most 1 in an odd position—which is never the first—and, second, a second canonical representation fails to be canonical if and only if it contains a 1 in the first position). It follows that *the primitive pairs* (a_n, b_n) *are precisely those for which, the second canonical representation of* n *having a 1 in the first position, the canonical representation of* a_n *ends in 10 and that of* b_n *ends in 100.* Other terms of the generated Fibonacci sequence have additional zeroes, the location of any number in the sequence being exactly dependent on the number of terminal zeroes in its canonical representation.

This enables one to determine for any positive integer n exactly which primitive Wythoff pair generates the Fibonacci sequence in which that n appears, as well as the location of n in that sequence. First determine the canonical Fibonacci representation of n. The portion of the representation between the first and last 1's, inclusive, is the second canonical representation of the number k of the primitive Wythoff pair which generates the Fibonacci sequence containing n. One left shift produces a_k; another produces b_k. Counting a_k as the first term, b_k as the second, and so forth, the i^{th} term will equal n, where i is the number of zeroes prior to the first 1 in the Fibonacci representation of n. For example, let $n = 52$. In Fibonacci, 52 is represented 101010000. Since 10101 represents 8 and the representation terminates in four zeroes, 52 must be the fourth term of the Fibonacci sequence generated by the primitive pair (a_8, b_8). As confirmation we note that this particular sequence is

12, 20, 32, 52, 84, ⋯.

A FOOTNOTE

Because of the connection of the Wythoff pairs with Wythoff's Nim, the preceding prescription for generating Wythoff pairs is clearly also a prescription for playing Wythoff's Nim using the Fibonacci number system. This gives the Fibonacci number system a role in this game quite analogous to the role of the binary number system in Bouton's Nim [8]. The analysis of Wythoff's Nim using Fibonacci representations can be made self-contained and elementary, certainly not requiring the level of mathematical sophistication required to follow the investigations in [4, 5, 6, 7]. For the benefit of those interested in mathematical recreations, we provide this analysis in a companion paper [9]. It is interesting to note that the role of the Fibonacci number system in nim games was already anticipated by Whinihan [10] in 1963.

REFERENCES

1. W. A. Wythoff, "A Modification of the Game of Nim," *Nieuw Archief voor Wiskunde,* 2nd Series, Vol. 7, 1907, pp. 199–202.
2. A. P. Domoryad, *Mathematical Games and Pastimes,* N. Y., Macmillan Co., 1964, pp. 62–65.
3. H. S. M. Coxeter, "The Golden Section, Phyllotaxis, and Wythoff's Game," *Scripta Mathematica,* Vol. 19, 1953, pp. 135–143.
4. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 1–28.
5. L. Carlitz, et. al., "Lucas Representations," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 29–42.
6. L. Carlitz, et. al., "Addendum to the Paper 'Fibonacci Representations'," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 527–530.
7. L. Carlitz, R. Scoville and T. Vaughan, "Some Arithmetic Functions Related to Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 11, 1973, pp. 337–386.
8. C. L. Bouton, "Nim, A Game with a Complete Mathematical Theory," *Annals of Mathematics,* 1901–1902, pp. 35–39.
9. R. Silber, "Wythoff's Nim and Fibonacci Representations," to appear, *The Fibonacci Quarterly.*
10. M. J. Whinihan, "Fibonacci Nim," *The Fibonacci Quarterly,* Vol. 1, No. 4, 1963, pp. 9–13.

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