ON SOME INVERSE TANGENT SUMMATIONS

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In this note, we derive the sums of a number of infinite series, some apparently new, in a rather simple manner.
It is a simple result that

(1) $$\sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2+n+x^2} = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1} \right\} = \tan^{-1} x \, .$$

More generally, we have

$$\sum_{n=0}^{\infty} \left\{ \tan^{-1} F(n) - \tan^{-1} F(n+1) \right\} = \sum_{n=0}^{\infty} \tan^{-1} \frac{F(n) - F(n+1)}{1+F(n)F(n+1)} = \tan^{-1} F(0) - \lim_{n\to\infty} \tan^{-1} F(n)$$

for arbitrary F. In particular, for $F(n) = (an+b)/(cn+d)$, we obtain

(2) $$\tan^{-1} \frac{bc-ad}{ab+cd} = \sum_{n=0}^{\infty} \tan^{-1} \frac{bc-ad}{n^2+An+B} \, ,$$

where

$$A = 2(ab+cd)+1, \qquad B = b^2+d^2+ab+cd, \qquad a^2+c^2 = 1.$$

If in (2), we let $bc - ad = x$, $ab + cd = y$, then $b^2 + d^2 = x^2 + y^2$ giving

(3) $$\tan^{-1} \frac{x}{y} = \sum_{n=0}^{\infty} \tan^{-1} \frac{x}{n^2+(2y+1)n+x^2+y^2+y} \, .$$

Then by differentiating (3) with respect to x and y, separately we obtain

(4) $$\frac{y}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{n^2+(2y+1)n+y^2+y-x^2}{[n^2+(2y+1)n+y^2+y+x^2]^2+x^2}$$

(5) $$\frac{1}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{(2n+2y+1)}{[n^2+(2y+1)n+y^2+y+x^2]^2+x^2} \, ,$$

and also the following interesting special cases:

(6) $$\tan^{-1} \frac{x}{x+1} = \sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2+(2x+1)n+2x^2+x} \, ,$$

(7) $$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{2n^2-1}{4n^4+1} \, ,$$

(8) $$\frac{1}{4} = \sum_{n=1}^{\infty} \frac{n}{4n^4+1},$$

(9) $$\frac{1}{x^2+1} = \sum_{n=1}^{\infty} \frac{n^2+n-x^2}{(n^2+n+x^2)^2+x^2},$$

(10) $$\frac{1}{x^2+1} = \sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n+x^2)^2+x^2}.$$

To obtain analogous alternating sums, we let

$$F(n) = (-1)^n \left\{ \tan^{-1} \frac{an+b}{cn+d} - \tan^{-1} \frac{a}{c} \right\}$$

which leads to

(11) $$\tan^{-1} \frac{x}{y} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1} \frac{x(2n+2y+1)}{n^2+(2y+1)n+y^2+y-x^2}$$

and then by differentiating to

(12) $$\frac{y}{x^2+y^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left\{ n^2+(2y+1)n+y^2+y+x^2 \right\}(2n+2y+1)}{\left\{ n^2+(2y+1)n+y^2+y-x^2 \right\}^2+(2n+2y+1)x^2},$$

(13) $$\frac{1}{2(x^2+y^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n \left\{ n^2+(2y+1)n+y^2+y+x^2 \right\}}{\left\{ n^2+(2y+1)n+y^2+y-x^2 \right\}^2+(2n+2y+1)x^2}.$$

These three latter formulae include the following special cases:

(14) $$\pi - \tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1} \frac{(2n+1)x}{n^2+n-x^2},$$

(15) $$\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1} \frac{4nx}{4n^2-x^2-1},$$

(16) $$\tanh^{-1} \frac{1}{y} = \sum_{n=0}^{\infty} (-1)^n \tanh^{-1} \frac{1}{2n+y+1},$$

(17) $$\frac{\pi}{2} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1} \frac{2n+3}{n^2+3n+1},$$

(18) $$\frac{1}{8} = \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{8n^4-4n^2+1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{8n^4-4n^2+1},$$

(19) $$\frac{1}{2x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n^2+n+x^2)}{(n^2+n-x^2)^2+(2n+1)^2 x^2}.$$

To obtain a class of sums complementary to (2), we use another simple general method. Consider any product (finite or infinite)

$$P = \prod_n (a_n + ib_n), \qquad (a_n, b_n - \text{real}).$$

Then,

$$\arg P = \sum_n \tan^{-1} \frac{b_n}{a_n}, \tag{20}$$

$$|P|^2 = \prod_n (a_n^2 + b_n^2). \tag{21}$$

Applying (20) and (21) to the infinite products

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \qquad \cos \pi z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right)$$

$$\frac{e^{\gamma a}\Gamma(z+1)}{\Gamma(z-a+1)} = \prod_{k=1}^{\infty} \left\{ e^{a/k} \left(1 - \frac{a}{x+k+iy}\right)\right\}, \quad J_0(xe^{3\pi i/4}) = \prod_{k=1}^{\infty} \left(1 + \frac{ix^2}{j_{0,k}^2}\right),$$

we obtain,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{k^2 - x^2 + y^2} = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{\tanh \pi y}{\tan \pi x}, \tag{22}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{(2k-1)^2 - x^2 + y^2} = \tan^{-1} \left\{ \tan \frac{\pi x}{2} \tanh \frac{\pi y}{2} \right\}, \tag{23}$$

$$\prod_{k=1}^{\infty} \left\{ 1 - \frac{2(x^2 - y^2)}{k^2} + \frac{(x^2+y^2)^2}{k^4} \right\} = \frac{\sin^2 \pi x + \sinh^2 \pi y}{\pi(x^2+y^2)} \tag{24}$$

$$\prod_{k=1}^{\infty} \left\{ 1 - \frac{8(x^2-y^2)}{(2k-1)^2} + \frac{16(x^2+y^2)^2}{(2k-1)^4} \right\} = \cos^2 \pi x + \sinh^2 \pi y \tag{25}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{ay}{(x+k)^2 - a(x+k) + y^2} = \arg \Gamma(z+1)\Gamma(\bar{z} - a + 1) \tag{26}$$

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{x^2}{j_{0,k}^2} = \tan^{-1} \left\{ \frac{bei(x)}{ber(x)} \right\}, \tag{27}$$

$$\prod_{k=1}^{\infty} \left(1 + \frac{x^4}{j_{0,k}^4}\right) = ber^2 x + bei^2 x. \tag{28}$$

The right-hand side of (26) can be explicitly evaluated if either a or $\bar{z} + z - a$ is integral. If a is a positive integer,

$$\arg \Gamma(z+a)/\Gamma(z-a+1) = \sum_{k=0}^{a-1} \tan^{-1} \frac{y}{x-k}.$$

If $\bar{z} + z - a + 2 = m$ (positive integer), then $a = 2 + 2x - m$ and

$$\arg \Gamma(z+1)\Gamma(m-z-1) = \tan^{-1} \frac{\tanh \pi y}{\tan \pi x} - \sum_{k=2}^{m} \tan^{-1} \frac{y}{m-x-k}$$

(the last sum is to be taken as zero if $m = 1$). Further sums can be obtained by continued differentiation of all the previous sums containing at least one parameter.

Some of the sums given here appeared as problems in the Mathematical Tripos. A number of these are given among the exercises in Chapter XI of T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series,* MacMillan, London, 1947.

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A NOTE ON THE GOLDEN ELLIPSE

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In [1], Huntley discusses some of the properties of the golden ellipse; that is, an ellipse whose ratio of the major axis to the minor axis is ϕ, the golden ratio. For example, Huntley shows that the eccentricity, e, of the golden ellipse is $1/\sqrt{\phi}$. This note is an examination of the golden ellipse as a conic section; see Fig. 1. It will be assumed that the plane does not pass through the vertex of the cone.

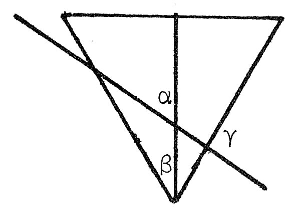

Figure 1

In [2] p. 355, it is shown that the eccentricity is determined by $\cos\alpha/\cos\beta = e$, where α and β are the angles in Fig. 1. Furthermore, for ellipses, $\beta < \alpha < 90°$.

In Fig. 1, the angle γ is formed by the intersection of the plane and the cone, in the plane through the axis of the cone and the main axis of the ellipse (easier seen than said). This angle will be referred to as the angle formed by the intersection of the plane and the cone.

Theorem. If α and β are such that $\sec\alpha = \phi$ and $\csc\beta = \phi$, then α and β are complementary, and the plane intersects the cone at a right angle, forming a golden ellipse. Conversely, if the plane intersects the cone at a right angle, forming a golden ellipse, then α and β are complementary, $\sec\alpha = \phi$, and $\csc\beta = \phi$.

Proof. Firstly, $\sin\beta = 1/\phi = \cos\alpha = \sin(\pi/2 - \alpha)$. Therefore, $\beta = \pi/2 - \alpha$. From Fig. 1 it follows that $\gamma = \pi - (\alpha+\beta)$. Hence, α and β are complementary and γ is a right angle.

Recalling that $\phi^2 - \phi - 1 = 0$,

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-1/\sqrt{\phi^2}} = \sqrt{(\phi^2-1)/\phi^2} = \sqrt{\phi/\phi^2} = 1/\sqrt{\phi}\,.$$

Since $\cos\alpha = 1/\phi$, $e = \cos\alpha/\cos\beta = 1/\sqrt{\phi}$, and so the ellipse is golden.

Suppose that γ is a right angle and the ellipse is golden. Then, $\cos\alpha/\cos\beta = 1/\sqrt{\phi}$ and since

$$\pi/2 = \gamma = \pi - (\alpha+\beta),$$

α and β are complementary. Thus, $\cos\beta = \sin\alpha$. Now, $\sqrt{\phi}\cos\alpha = \cos\beta$ implies that

$$\phi\cos^2\alpha = \cos^2\beta = \sin^2\alpha = 1-\cos^2\alpha\,.$$

Therefore, $\cos^2\alpha = 1/(\phi+1) = 1/\phi^2$ and so $\sec\alpha = 1/\cos\alpha = \phi$. Also,

$$\csc\beta = 1/\sin\beta = 1/\cos\alpha = \phi\,.$$

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A DIRECT METHOD OF OBTAINING FAREY-FIBONACCI SEQUENCES

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1. Krishnaswami Alladi [1], [2] has recently considered the problem of arranging in ascending order of magnitude the fractions $F_j / F_k,\ 2 \leqslant j < k \leqslant n$ that can be obtained from the first n Fibonacci numbers by the relations

$$F_1 = F_2 = 1; \quad F_{m+1} = F_m + F_{m-1}, \quad m \geqslant 2$$

and discussed the symmetries and properties of this arrangement. As a consequence of these properties he gives a rapid method of constructing the Farey-Fibonacci sequence.

In this note we offer a direct method of obtaining such a Farey-Fibonacci sequence of fractions for $n \geqslant 3$. In fact once we prove the order of arrangement, the array on page 1 would give various properties with which Alladi started.

2. For n even, arrange the numbers from 2 to n in the order:

$$2 \quad 4 \quad 6 \quad \cdots \quad n \quad n-1 \quad n-3 \quad n-5 \quad \cdots \quad 3;$$

and for n odd, arrange them in the order:

$$3 \quad 5 \quad 7 \quad \cdots \quad n \quad n-1 \quad n-3 \quad n-5 \quad \cdots \quad 2.$$

The method is now best described with the help of an example. Let $n = 10$, then the numbers from 2 to 10 are written in the order

$$(1) \qquad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 9 \quad 7 \quad 5 \quad 3.$$

With (1) as the base, complete the structure

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & & 3 & 2 & & & \\
 & & & 2 & 4 & 3 & & & \\
 & & & 3 & 5 & 4 & 2 & & \\
 & & 2 & 4 & 6 & 5 & 3 & & \\
 & & 3 & 5 & 7 & 6 & 4 & 2 & \\
 & 2 & 4 & 6 & 8 & 7 & 5 & 3 & \\
 & 3 & 5 & 7 & 9 & 8 & 6 & 4 & 2 \\
\hline
2 & 4 & 6 & 8 & 10 & 9 & 7 & 5 & 3
\end{array}$$

The building plan of the structure is simple and readily understood. Each figure in the configuration stands for a suffix of F. Thus, 5 stands for F_5, so to say. The base is separated from the superstructure by a line. The figures above the line provide the numerators, those on the base the denominators. For any numerator the figure vertically below it on the base provides the denominator. Thus 5 of the sixth row will give the fraction F_5/F_8. We start reading the figures from the top. The even numbered rows are read from right to left, the odd numbered rows from left to right. In other words, 2 is regarded as the first entry in each row. The configuration now gives the Farey-Fibonacci sequence straight away. In our example, it is:

$$F_2/F_{10}, \quad F_2/F_9, \quad F_3/F_{10}, \quad F_2/F_8, \quad F_4/F_{10}, \quad F_3/F_9, \quad \cdots, \quad F_3/F_4 .$$

In our scheme, there is no loss of labour in extending the structure. Thus, for $n = 11$, we obtain

				2					
				3	2				
			2	4	3				
			3	5	4	2			
		2	4	6	5	3			
		3	5	7	6	4	2		
	2	4	6	8	7	5	3		
	3	5	7	9	8	6	4	2	
2	4	6	8	10	9	7	5	3	
3	5	7	9	11	10	8	6	4	2

3. To show that our scheme does give the fractions in ascending order of magnitude, we have just to prove that

(i) $$F_2/F_3 < F_4/F_5 < \cdots < F_5/F_6 < F_3/F_4\ ;$$

the two terms at the point of change-over being

$$F_{n-1}/F_n,\quad F_{n-2}/F_{n-1} \qquad \text{or} \qquad F_{n-2}/F_{n-1},\quad F_{n-1}/F_n$$

according as n is odd or even.

(ii) If $F_j/F_{j+1} > F_k/F_{k+1}$ then $F_j/F_{j+h} > F_k/F_{k+h}$, for every $h > 2$, and

(iii) $$F_3/F_{k+2} < F_2/F_k, \qquad k \geqslant 3.$$

The proof of (iii) is straightforward and is left to the reader.

Proof of (i).

$$1/1,\quad F_2/F_3,\quad F_3/F_4,\quad \cdots,\quad F_{n-1}/F_n$$

are convergents of the simple continued fraction

$$C_n = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{1} \quad \text{(with } n-1 \text{ partial quotients)}.$$

The well known properties of even and odd convergents provide immediately the proof of (i).

Proof of (ii). We have

(2) $$F_{j+1}/F_j < F_{k+1}/F_k.$$

Adding 1 on both sides of the inequality, we get

(3) $$F_{j+2}/F_j < F_{k+2}/F_k.$$

From (2) and (3) by addition, we obtain

(4) $$F_{j+3}/F_j < F_{k+3}/F_k.$$

The process can be continued to establish (ii).

We leave it to the reader to suggest a rule for obtaining the Farey-Fibonacci sequence for $n = m + 1$ from that for $n = m$.

4. We conclude with a formula which gives the position of the fraction F_j/F_k in the Farey-Fibonacci sequence for a given n, $2 \leqslant j < k \leqslant n$.

First observe that there are in all $\tfrac12(n-1)(n-2)$ fractions in the sequence. It is now easy to see that F_j/F_{j+1} is the t^{th} term in the sequence, where

$$t = \begin{cases} \tfrac12\left\{(n-2)(n-3)+j\right\}, & \text{when } j \text{ is even,} \\ \tfrac12\left\{(n-1)(n-2)-(j-3)\right\}, & \text{when } j \text{ is odd.} \end{cases}$$

All that we need note now is that the position of F_j/F_{j+h}, $2 \leqslant j \leqslant m$, in the sequence for $n = m + h$, $h \geqslant 2$, is the same as the position of F_j/F_{j+1} in the sequence for $n = m + 1$.

These results follow at once from our scheme.

EXAMPLES: F_6/F_7 is the 31st term in the sequence for $n = 10$;

F_7/F_8 is the 43rd term in the sequence for $n = 11$;

and F_4/F_9 has the same position in the sequence for $n = 11$, as F_4/F_5 has in that for $n = 7$. This means that F_4/F_9 is the 12th term in the sequence for $n = 11$.

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ON CONSECUTIVE PRIMITIVE ROOTS

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The purpose of this note is to determine which positive integers have their primitive roots consecutive. Of course, if "consecutive primitive roots" is taken to include integers which have only one primitive root, then 2, 3, 4, and 6 would qualify with primitive roots 1, 2, 3, and 5, respectively. It will be shown that 5, with primitive roots 2 and 3, is the only positive integer which has its primitive roots (plural) consecutive. It is well known that the only positive integers m, greater than 4, which have primitive roots are of the form p^n or $2p^n$, $n \geq 1$, p an odd prime. Most of these can be eliminated by the first two theorems.

Theorem 1. If $m = 2p^n$ $(m > 6)$, $n \geq 1$, p an odd prime, then the primitive roots are not consecutive.

Proof. Primitive roots must have inverses, and, consequently, must be relatively prime to the modulus. With $m > 6$, there will be at least two primitive roots. Therefore, there are at least two odd primitive roots and no even primitive roots; they are not consecutive.

Theorem 2. If $m = p^n$, $n \geq 2$, p an odd prime, then the primitive roots are not consecutive.

Proof. For $n \geq 3$,

$$p < p^{n-2}(p-1)\phi(p-1) = \phi(\phi(p^n)).$$

This implies that multiples of p occur within a span less than $\phi(\phi(p^n))$. Now, multiples of p are not relatively prime to the modulus, and are, therefore, not primitive roots. Since there are $\phi(\phi(p^n))$ primitive roots, they cannot be consecutive. For n = 2, $\phi(\phi(p^2)) = (p-1)\phi(p-1)$. For $p > 3$, $\phi(p-1) \geq 2$, and so,

$$(p-1)\phi(p-1) \geq 2(p-1) = 2p-2 = p+p-2 > p.$$

The conclusion follows as in the case $n \geq 3$. For $m = 3^2$, the primitive roots are 2 and 5, and not consecutive.

Lemma. If p is an odd prime greater than 5 and not equal to 7, 11, 13, 19, 31, 43, 61, then $2\sqrt{p-1} \leq \phi(p-1)$.

Proof. The conclusion is equivalent to $4(p-1) \leq [\phi(p-1)]^2$. Let $p - 1 = 2^a p_1^{a_1} \cdots p_n^{a_n}$, and suppose that $4(p-1) > [\phi(p-1)]^2$. Then,

$$2^{a+2}p_1^{a_1}\cdots p_n^{a_n} > 2^{2(a-1)}p_1^{2(a_1-1)}\cdots p_n^{2(a_n-1)}(p_1-1)^2\cdots(p_n-1)^2. \tag{1}$$

If $p - 1 = 2^a$, then (1) reduces to $2^{a+2} > 2^{2(a-1)}$. This implies that $16 > 2^a$, or $a < 4$. Thus, $p = 3$ or 5.

Otherwise, (1) reduces to

$$8 > 2^{a-1}p_1^{a_1-2}(p_1-1)^2 p_2^{a_2-2}(p_2-1)^2\cdots p_n^{a_n-2}(p_n-1)^2. \tag{2}$$

[Continued on page 394.]

ON THE INFINITE MULTINOMIAL EXPANSION, II

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In a previous note (Hilliker [7]) we derived, by an iterative argument, the following version of the Multinomial Expansion: If the inequalities

$$(1) \qquad |a_j| < |a_1 + a_2 + \cdots + a_{j-1}|,$$

for $j = 2, 3, \cdots, r$ all hold, then

$$(2) \qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{r-1}+1)}{n_1!\,n_2!\cdots n_{r-1}!} a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\cdots-n_{r-1}},$$

where the summation is an iterated summation taken under all $n_i \geqslant 0$, where i first takes on the value $r-1$, then $r-2$, and so on until the last value, 1, is taken on. Here $n, a_1, a_2, \cdots, a_r$ are complex numbers with n not equal to a non-negative integer. On the other hand, one can assume a single inequality

$$(3) \qquad |a_2 + a_3 + \cdots + a_r| < |a_1|$$

and avoid the more complicated iterative argument by direct employment of the Multinomial Theorem for non-negative integral exponents. The result is that the same formal expansion (2) holds, but this time the summation is taken under all $n_i \geqslant 0$ with $n_1 + n_2 + \cdots + n_{r-1} = j$ for $j = 0, 1, 2, \cdots$. See, for example, Chrystal [2], where a similar version is established. In this note we shall view these two forms from the perspective of a single Multinomial Expansion valid under a certain divisibility condition on r.

Let p *be an integer with* $1 \leq p \leq r-1$, *and assume that the congruence*

$$(4) \qquad r \equiv 1 \pmod{p}$$

holds. If the inequalities

$$(5) \qquad |a_{r-(i+1)p+1} + a_{r-(i+1)p+2} + \cdots + a_{r-ip}| < |a_1 + a_2 + \cdots + a_{r-(i+1)p}|,$$

for $i = 0, 1, 2, \cdots q$, *all hold, where the non-negative integer* q *is given by* $r = 1 + (q+1)p$, *then the formal expansion (2) holds. Here the summation is taken under all* $n_i \geqslant 0$, $1 \leqslant i \leqslant r-1$, *with*

$$(6) \qquad n_{jp+1} + n_{jp+2} + \cdots + n_{jp+p} = t_j,$$

where $t_j = 0, 1, 2, \cdots$, *and where* j *first takes on the value* q *then* $q-1$, *and so on until the last value, 0, is taken on.*

Our argument rests upon Abel's proof of about 1825 of the Binomial Theorem:

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

for n and z complex and with $|z| < 1$. See Abel [1]. See also Markushevich [9], I, for this Maclaurin expansion. Here, as usual, we define z^n as being that branch of the function $f(z) = e^{n \log z}$ defined over the complex z-plane with the non-positive real axis excluded, and with $f(1) = 1$. That is, the logarithmic function is given by $\log z = \log |z| + i \arg z$ with $|\arg z| < \pi$. The quantities $a_1 + a_2 + \cdots + a_r$ and a_1 are not 0 by the inequalities (5) with $i = 0$ and $i = q$, respectively. We will need to assume that they are not negative real numbers. Likewise, in the course of the proof we will need to assume that the quantities $a_1 + a_2 + \cdots + a_{r-(i+1)p}$, for $0 \leqslant i \leqslant q-1$, are not negative real numbers. If n is a (negative) integer, these restrictions which guarantee single-valuedness, may, of course, be ignored.

As a first example, let $p = 1$. Then (4) automatically holds and $q = r - 2$. The inequalities (5) become identical with those of (1), and the summation conditions (6) become $n_{j+1} = t_j$ for $j = r-2, r-1, \cdots, 0$. Thus the first mentioned form is covered.

As a second example, let $p = r - 1$. Then (4) holds, and $q = 0$. The inequalities (5) reduce to the single inequality (3). The summation conditions (6) reduce to the single condition $n_1 + n_2 + \cdots + n_{r-1} = t_0$. Consequently, the second mentioned form is also covered.

We begin by writing

$$(7)\qquad \begin{aligned}(a_1 + a_2 + \cdots + a_r)^n &= [(a_1 + a_2 + \cdots + a_{r-p}) + (a_{r-p+1} + a_{r-p+2} + \cdots + a_r)]^n \\ &= \sum_{t_0=0}^{\infty} \binom{n}{t_0} \left(\sum_{k=r-p+1}^{r} a_k\right)^{t_0} \left(\sum_{\ell=1}^{r-p} a_\ell\right)^{n-t_0} .\end{aligned}$$

Here we have used the inequality (5) for the case $i = 0$.

Since $n - t_0 \neq 0$, we may apply Formula (7) to the summation under ℓ on the right side of (7). We may repeat this iterative process. After m iterations of (7), $m \geqslant 0$ and not too large, one obtains, by using (5) for $i = 0, 1, \cdots, m$,

$$(8)\qquad \begin{aligned}(a_1 + a_2 + \cdots + a_r)^n = &\sum_{t_0,t_1,\cdots,t_m=0}^{\infty} \prod_{j=0}^{m} \binom{n - t_0 - \cdots - t_{j-1}}{t_j} \left(\sum_{k=r-(j+1)p+1}^{r-jp} a_k\right)^{t_j} \\ &\times \left(\sum_{\ell=1}^{r-(m+1)p} a_\ell\right)^{n-t_0-t_1-\cdots-t_m} .\end{aligned}$$

First we apply the Multinomial Theorem for non-negative integral exponents to the summation under k on the right side of (8). Since this summation contains p terms, we can write

$$(9)\qquad \left(\sum_{k=r-(j+1)p+1}^{r-jp} a_k\right)^{t_j} = \sum \frac{t_j!}{n_{jp+1}!\,n_{jp+2}!\cdots n_{jp+p}!}\, a_{r-jp}^{n_{jp+1}} a_{r-jp-1}^{n_{jp+2}} \cdots a_{r-jp-p+1}^{n_{jp+p}} ,$$

where the summation is taken under all non-negative values of the p integers $n_{jp+1}, n_{jp+2}, \cdots, n_{jp+p}$ subject to the restruction (6).

Secondly we observe that

$$(10)\qquad \prod_{j=0}^{m} \binom{n - t_0 - t_1 - \cdots - t_{j-1}}{t_j} = \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{mp+p}+1)}{t_0!\,t_1!\cdots t_m!}$$

since, by (6), $t_0 + t_1 + \cdots + t_m = n_1 + n_2 + \cdots + n_{mp+p}$.

Finally we note that from (4) we can choose m in such a way that $r - (m+1)p = 1$, so that the summation under ℓ on the right side of (8) reduces to a single term.

Thus it follows from (8), (9) and (10) that

$$\begin{aligned}(a_1 + a_2 + \cdots + a_r)^n = &\sum_{t_0,t_1,\cdots,t_m=0}^{\infty} \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{r-1}+1)}{n_1!\,n_2!\cdots n_{r-1}!} \\ &\times a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\cdots-n_{r-1}} ,\end{aligned}$$

where the summation is first taken under t_m, then under t_{m-1}, and so on until the last summation is taken under t_0.

Our expository sequence of papers on the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [3], [4], [5], [6], [7] and the present paper) will continue (Hilliker [8]).

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[Continued from page 391.]

Let q^b denote one of the $p_i^{a_i}$ and P denote $q^{b-2}(q-1)^2$. Now,

$$q^{b-2}(q-1)^2 = q^{b-1}(q-2+1/q). \tag{3}$$

From (3), it can be seen that $P > 1$, for all q, and that $P > 8$, for all $q \geqslant 11$. Furthermore, for $q < 11$, the following table can be obtained, by checking the right side of (3) for the case $b = 1$, and the left side of (3) for the case $b \geqslant 2$.

Prime q	3	3	5	5	7	7
Exponent b	2	3	1	2	1	2
P greater than or equal to	4	8	2	8	4	8

Hence, (2) holds for $p-1$ possibly equal to $2\cdot3$, $2\cdot3^2$, $2\cdot5$, $2\cdot7$, $2\cdot3\cdot5$, $2\cdot3\cdot7$ ($a = 1$); $4\cdot3$, $4\cdot5$, $4\cdot3\cdot5$ ($a = 2$); or $8\cdot3$ ($a = 3$); and (2) fails to hold for all other choices. These combinations lead to the primes 7, 11, 13, 19, 31, 43, 61.

Theorem 3. If p is a prime greater than 5, then the primitive roots are not consecutive.

Proof. For the primes excluded in the Lemma, the primitive roots are: for 7 – 3, 5; for 11 – 2, 6, 7, 8; for 13 – 2, 6, 7, 11; for 19 – 2, 3, 10, 13, 14, 15; for 31 – 3, 11, 12, 13, 17, 21, 22, 24; for 43 – 3, 5, 12, 18, 19, 20, 26, 28, 29, 30, 33, 34; for 61 – 2, 6, 7, 10, 17, 18, 26, 30, 31, 35, 43, 44, 51, 54, 55, 59. None of these primes have consecutive primitive roots.

Now, let p denote a prime for which the Lemma applies and suppose that k is a positive integer for which $k^2 \leqslant p-1$. Then,

$$k^2-(k-1)^2 = 2\cdot k-1 < 2\cdot k \leqslant 2\sqrt{p-1} \leqslant \phi(p-1).$$

Therefore, consecutive squares appear within a span less than $\phi(p-1)$. Since squares are quadratic residues, and therefore not primitive roots, no string of consecutive primitive roots can be of length $\phi(p-1)$. Consequently, the primitive roots are not consecutive.

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CATALAN AND RELATED SEQUENCES ARISING FROM INVERSES OF PASCAL'S TRIANGLE MATRICES

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Here is recorded a fascinating sequence of sequences which arise in the first column of matrix inverses of matrices containing certain columns of Pascal's triangle. The convolution arrays of these sequences are computed, leading to determinant relationships, a general formula for any element in the convolution array for any of these sequences, and a class of combinatorial identities.

The sequence $S_1 = \{1, 1, 2, 5, 14, 42, \cdots\}$ is the sequence of Catalan numbers [1], and the sequence $S_2 = \{1, 1, 3, 12, 55, \cdots\}$ appeared in an enumeration problem given by Carlitz [2, p. 125].

1. SEQUENCES ARISING FROM INVERSES OF PASCAL'S TRIANGLE MATRICES

We form a series of $n \times n$ matrices P_i, $i = 0, 1, 2, 3, \cdots$, by placing every $(i+1)^{st}$ column of Pascal's triangle on and below the main diagonal, and zeroes elsewhere. Then, P_0 contains Pascal's triangle itself, while P_1 contains every other column of Pascal's triangle and P_2 every third column. We call the inverse of P_i the matrix P_i^{-1} and record the convolution arrays for the sequences S_i which arise as the absolute values of the elements in the first column of P_i^{-1} in the tables which follow.

Table 1.1 Non-Zero Elements of the Matrices P_i^{-1} and P_i

	P_i^{-1}						P_i					
	1						1					
	−1	1					1	1				
$i = 0$	1	−2	1				1	2	1			
	−1	3	−3	1			1	3	3	1		
	1	−4	6	−4	1		1	4	6	4	1	
	−1	5	−10	10	−5	1	1	5	10	10	5	1
	⋯						⋯					
	1						1					
	−1	1					1	1				
$i = 1$	2	−3	1				1	3	1			
	−5	9	−5	1			1	6	5	1		
	14	−28	20	−7	1		1	10	15	7	1	
	⋯						⋯					
	1						1					
	−1	1					1	1				
$i = 2$	3	−4	1				1	4	1			
	−12	18	−7	1			1	10	7	1		
	55	−88	42	−10	1		1	20	28	10	1	
	⋯						⋯					
	1						1					
	−1	1					1	1				
$i = 3$	4	−5	1				1	5	1			
	−22	30	−9	1			1	15	9	1		
	140	−200	72	−13	1		1	35	45	13	1	
	⋯						⋯					

Next, we will compute the convolution arrays for the sequences S_i which are tabulated below as well as establish the form of the n^{th} term.

Table 1.2 The Sequences S_i Arising from Matrices P_i^{-1}

i	S_i	n^{th} term
0	1, 1, 1, 1, 1, ···	$\binom{n}{n}$
1	1, 1, 2, 5, 14, ···	$\frac{1}{n+1}\binom{2n}{n}$
2	1, 1, 3, 12, 55, ···	$\frac{1}{2n+1}\binom{3n}{n}$
3	1, 1, 4, 22, 140, ···	$\frac{1}{3n+1}\binom{4n}{n}$
4	1, 1, 5, 35, 285, ···	$\frac{1}{4n+1}\binom{5n}{n}$
...	...	...
k	1, 1, $k+1$, ···	$\frac{1}{kn+1}\binom{(k+1)n}{n} = \frac{1}{n}\binom{(k+1)n}{n-1}$

It is important to note that convolutions of the sequences S_i arising from P_i^{-1} have as their i^{th} convolution that same sequence, less its first element. Let $S_i(x)$ be the generating function for the sequence S_i, and let * denote a convolution. We easily calculate:

$$i = 1: \quad (1, 1, 2, 5, 14, \cdots)*(1, 1, 2, 5, 14, \cdots) = (1, 2, 5, 14, \cdots)$$

(1.1) $$xS_1^2(x) = S_1(x) - 1$$

$$i = 2: \quad (1, 1, 3, 12, 55, \cdots)*(1, 1, 3, 12, 55, \cdots)*(1, 1, 3, 12, 55, \cdots) = (1, 3, 12, 55, \cdots)$$

(1.2) $$xS_2^3(x) = S_2(x) - 1$$

$$i = 3: \quad (1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots) = (1, 4, 22, \cdots)$$

(1.3) $$xS_3^4(x) = S_3(x) - 1.$$

In fact, it will be shown by the Lemma [3] following, that

(1.4) $$xS_i^{i+1} = S_i(x) - 1,$$

which will allow an easy construction of the convolution array for S_i.

Lemma: Two infinite matrices (denoted by giving successive column generators),

$$(f^m(x), xf^{m+k}(x), x^2f^{m+2k}(x), \cdots) \quad \text{and} \quad (A^m(x), xA^{m+k}(x), x^2A^{m+2k}(x), \cdots)$$

are inverses if

$$A(x)f(xA^k(x)) = 1.$$

Here, we take $f(x) = 1/(1-x)$, the generating function for the first column of the Pascal matrix, and let $A(x) = S_i(-x)$, where $S_i(x)$ is the generating function for the sequence S_i, and take $k = i + 1$. Then

$$1 = A(x)f(xA^k(x)) = S_i(-x)[1 - xS_i^{i+1}(-x)]^{-1},$$

or

$$1 - xS_i^{i+1}(-x) = S_i(-x)$$

which establishes (1.4) upon replacing $(-x)$ by x and rearranging terms.

Also notice that, in a convolution triangle, the generating function for the i^{th} column is the i^{th} power of the generating function for the first column. Putting this together with (1.4) gives us a neat way to generate the convolution triangle for any one of the sequences S_i. For example, for $i = 1$,

$$xS_1^2(x) = S_1(x) - 1$$

(1.5) $$xS_1^{k+1}(x) = S_1^k(x) - S_1^{k-1}(x)$$

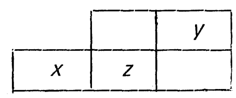
$$S_1^k(x) = S_1^{k-1}(x) + xS_1^{k+1}(x)$$

which means that we have a Pascal-like rule of formation for the elements of the convolution triangle. An element in the k^{th} column in the sum of elements in the $(k-1)^{st}$ and $(k+1)^{st}$ columns as shown in the convolution triangle for S_1 (the Catalan numbers) given below:

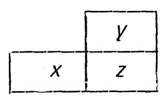
Table 1.3 Convolution Triangle for S_1 : 1, 1, 2, 5, 14, 42, ···

1	1	1	1	1	1	···
1	2	3	4	5	6	···
2	5	9	14	20	27	···
5	14	28	48	75	110	···
14	42	90	165	275	429	···
···	···	···	···	···	···	···

Scheme: $z = x + y$



Notice that, except for spacing, the rule of formation is the same as that for Pascal's triangle. For Pascal's triangle in rectangular form, the scheme would be a diagram like below, where $z = x + y$:



Similarly, for $i = 2$, we obtain

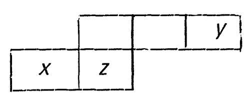
(1.6) $$S_2^k(x) = S_2^{k-1}(x) + xS_2^{k+2}(x)$$

which leads to the generation of the convolution triange for S_2 below.

Table 1.4 Convolution Triange for S_2 : 1, 1, 3, 12, 55, ···

1	1	1	1	1	1	···
1	2	3	4	5	6	···
3	7	12	18	25	33	···
12	30	55	88	130	182	···
55	143	273	455	700	1020	···
···	···	···	···	···	···	···

Scheme: $z = x + y$



For $i = 3$, we have

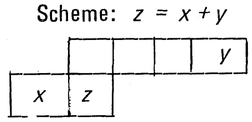
(1.7) $$S_3^k(x) = S_3^{k-1}(x) + xS_3^{k+3}(x)$$

which gives a scheme similar to those preceding, using a grid in which the column entries to be added are separated by three spaces, as computed below:

Table 1.5 Convolution Triangle for S_3 : 1, 1, 4, 22, 140, ···

1	1	1	1	1	1	···
1	2	3	4	5	6	···
4	9	15	22	30	39	···
22	52	91	140	200	272	···
140	340	612	969	1425	1995	···
···	···	···		···	···	···

Scheme: $z = x + y$



Returning for a moment to the matrices P_i^{-1} and comparing them to the convolution arrays for the sequences just given, notice that, ignoring signs, P_1^{-1} contains the alternate columns of the Catalan convolution array, and that P_i^{-1} is always composed of columns of a convolution array for the sequence in the first column. In fact, except for signs, the matrix P_i^{-1} always contains the zeroth column, the $(i+1)^{st}$ column, the $2(i+1)^{nd}$ column,

$\cdots$, and has its $(k+1)^{st}$ column given by the $k(i+1)^{st}$ column of the convolution array for the sequence S_i. (Notice that the count of the columns for matrices begins with one, but for convolution arrays begins with zero.) We have proved this already in applying the Lemma.

Now, to generalize, the formulation of the convolution triangle for S_i would require a grid in which column entries to be added were separated by i spaces, so that the generating function $S_i(x)$ for the zero th column of the convolution array for S_i satisfies

$$S_i^k(x) = S_i^{k-1}(x) + xS_i^{k+1}(x), \tag{1.8}$$

where, of course, $S_i^k(x)$ is the generating function for the $(k-1)^{st}$ column, $k = 1, 2, 3, \cdots$.

Then, notice that this means that each row in the convolution array for any of the sequences S_i is the partial sum of the previous row from some point on. Thus, each convolution array written in rectangular form has its i^{th} row an arithmetic progression of order i, $i = 0, 1, 2, 3, \cdots$, and the constant of each of these progressions is 1. By previous results [4], we have

Theorem 1.1. The determinant of any $n \times n$ array taken to include the row of 1's in the convolution array written in rectangular form for any of the sequences S_i has value one.

It will also be shown in a later paper that the determinant of any $n \times n$ array taken to include the row of integers $(1, 2, 3, 4, \cdots)$ and its first column the $(j-1)^{st}$ column of the convolution array has value

$$\binom{n+j-1}{n}, \qquad j = 1, 2, 3, \cdots.$$

2. GENERATION OF CONVOLUTION TRIANGLES FOR SEQUENCES S_i FROM PASCAL'S TRIANGLE

The convolution triangles for these sequences S_i are also available from Pascal's triangle in a reasonable way. If one looks at Pascal's triangle as given in Table 2.1,

Table 2.1 Pascal's Triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

and takes diagonals parallel to the central diagonal

$$1, 2, 6, 20, 20, 70, 252, \cdots, \binom{2n}{n}, \cdots,$$

one sees that

$$1/1, 2/2, 6/3, 20/4, 70/5, 252/6, \cdots = 1, 1, 2, 5, 14, 42, \cdots$$

$$2(1/2, 3/3, 10/4, 35/5, 126/6, \cdots) = 1, 2, 5, 14, 42, \cdots$$

$$3(1/3, 4/4, 15/5, 56/6, 210/7, \cdots) = 1, 3, 9, 28, 90, \cdots$$

$$4(1/4, 5/5, 21/6, 84/7, 330/8, \cdots) = 1, 4, 14, 48, 165, \cdots,$$

where successive parallel diagonals of Pascal's triangle produce successive columns of the Catalan convolution triangle.

To write the convolution triangle for the sequence S_2, one uses the diagonal

$$1, 3, 15, 84, 495, \cdots, \binom{3n}{n}, \cdots,$$

and diagonals parallel to it:

$$1/1, 3/3, 15/5, 84/7, 495/9, \cdots = 1, 1, 3, 12, 55, \cdots$$

$$2(1/2, 4/4, 21/6, 120/8, \cdots) = 1, 2, 7, 30, \cdots$$

$$3(1/3,\ 5/5,\ 28/7,\ 165/9,\ \cdots) = 1,\ 3,\ 12,\ 55,\ \cdots$$
$$4(1/4,\ 6/6,\ 36/8,\ 220/10,\ \cdots) = 1,\ 4,\ 18,\ 88,\ \cdots$$
$$5(1/5,\ 7/7,\ 45/9,\ 286/11,\ \cdots) = 1,\ 5,\ 25,\ 130,\ \cdots.$$

Notice that we again produce successive columns of the convolution triangle from successive diagonals of Pascal's triangle.

As a final example, we write the convolution triangle for S_3 from the diagonal

$$1,\ 4,\ 28,\ 220,\ 1820,\ \cdots,\ \binom{4n}{n},\ \cdots$$

and diagonals parallel to it:

$$1/1,\ 4/4,\ 28/7,\ 220/10,\ 1820/13,\ \cdots = 1,\ 1,\ 4,\ 22,\ 140,\ \cdots$$
$$2(1/2,\ 5/5,\ 36/8,\ 286/11,\ 2380/14,\ \cdots) = 1,\ 2,\ 9,\ 52,\ 340,\ \cdots$$
$$3(1/3,\ 6/6,\ 45/9,\ 364/12,\ \cdots) = 1,\ 3,\ 15,\ 91,\ \cdots$$
$$4(1/4,\ 7/7,\ 55/10,\ 455/13,\ \cdots) = 1,\ 4,\ 22,\ 140,\ \cdots$$
$$5(1/5,\ 8/8,\ 66/11,\ 560/14,\ \cdots) = 1,\ 5,\ 25,\ 200,\ \cdots.$$

Before we continue to the general case, observe the arithmetic progressions appearing in the denominators. For the Catalan numbers, the sequence S_1, the common difference is one; for S_2, two; and for S_3, three. For S_3, for example, we find the parallel diagonals from Pascal's rectangular array by beginning in the leftmost column and counting to the right one and down 4 throughout the array. To get the sequence S_3 itself, we multiply the Pascal diagonal 1, 4, 28, 220, ··· by 1 and divide by 1, 4, 7, 10, 13, ··· ; to get the first convolution or S_3^2, we multiply the first diagonal parallel to 1, 4, 28, 220, ··· by 2 and divide by 2, 5, 8, 11, ··· ; for the second convolution or S_3^3, we take the next parallel diagonal, multiply by 3, and divide by 3, 6, 9, 12, ··· ; and for S_3^k, we multiply the k^{th} diagonal by k and divide by $k,\ k+3,\ k+6,\ k+9,\ \cdots$.

To find the diagonals easily, write Pascal's triangle in rectangular form:

Table 2.2 Pascal's Triangle in Rectangular Form

1	1	1	1	1	1	1	1	...
1	2	3	4	5	6	7	8	...
1	3	6	10	15	21	28	36	...
1	4	10	20	35	56	84	120	...
1	5	15	35	70	126	210	330	...
1	6	21	56	126	252	462	792	...
1	7	28	84	210	462	924	1716	...
...	...	...	...	...	...	...	...	...

Then the sequence S_i is given by

$$\frac{1}{ni+1}\binom{(i+1)n}{n},$$

which diagonal is found by beginning in the leftmost column and counting to the right one and down $(i+1)$ throughout the rectangular Pascal array. The diagonals which lead to the convolution array for S_i are parallel and below this first diagonal. To find the $(k-1)^{st}$ convolution S_i^k, we multiply the k^{th} diagonal by k and divide by $k,\ k+i,\ k+2i,\ k+3i,\ \cdots$. The diagonals used to find the convolution triangle for S_2 are marked in the array above.

Now, we can find all the positive integral powers of the Catalan sequence in the convolution sense. However, let us not neglect the zero or negative powers. Here, we must adopt a convention, and call 0/0 = 1 and −0/0 = −1. We find S_1^0, S_1^{-1}, and S_1^{-2} by following the same process as given above but using an extended Pascal's triangle which includes coefficients for the binomial expansion of $(1+x)^{-k}$.

To see this in perspective, let us write down the parallel diagonals and the numbers to multiply and divide by.

$\cdots$

$S_1^3:\quad 3(1/3, 4/4, 15/5, 56/6, 210/7, \cdots) = 1, 3, 9, 28, 90, \cdots$

$S_1^2:\quad 2(1/2, 3/3, 10/4, 35/5, 126/6, \cdots) = 1, 2, 5, 14, 42, \cdots$

$S_1:\quad 1(1/1, 2/2, 6/3, 20/4, 70/5, \cdots) = 1, 1, 2, 5, 14, \cdots$

$S_1^0:\quad 0(1/0, 1/1, 3/2, 10/3, 35/4, \cdots) = 1, 0, 0, 0, 0, \cdots$

$S_1^{-1}:\quad -1(1/-1, 0/0, 1/1, 4/2, 15/3, 56/4, \cdots) = 1, -1, -1, -2, -5, -14, \cdots$

$S_1^{-2}:\quad -2(1/-2, -1/-1, 0/0, 1/1, 5/2, \cdots) = 1, -2, -1, -2, -5, \cdots$

$\cdots$

Thus, you see that if we write down the extra terms from "Pascal's attic," the process works in reverse to obtain all columns of the Catalan convolution triangle. This process will provide the zero and negative powers for any of the sequences S_i. One can also complete the Catalan convolution array to the left to provide negative integral powers by using its rule of formation in reverse, which is the following scheme:

		y
x	z	

$$x = z - y$$

The rule of formation can be rewritten to work to the left for the convolution array for any of the sequences S_i.

Now, write the complete Pascal array down in rectangular form as

⋯	1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	⋯
⋯	21	15	10	6	3	1	0	0	1	3	6	10	15	21	⋯
⋯	−35	−20	−10	−4	−1	0	0	0	1	4	10	20	35	56	⋯
⋯	35	15	5	1	0	0	0	0	1	5	15	35	70	126	⋯
⋯	−21	−6	−1	0	0	0	0	0	1	6	21	56	126	252	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

This is the regular arrangement. Now, if we move the i^{th} row i places to the left, $i = 0, 1, 2, \cdots$, we form

⋯	1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	⋯
⋯	10	6	3	1	0	0	1	3	6	10	15	21	28	36	⋯
⋯	−4	−1	0	0	0	1	4	10	20	35	56	84	120	165	⋯
⋯	0	0	0	0	1	5	15	35	70	126	210	330	495	715	⋯
⋯	0	0	0	1	6	21	56	126	252	462	792	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Now, all diagonals which are parallel to 1, 2, 6, 20, 70, $\cdots$ are all vertical. By proper processing, as just described, we can obtain all columns of the Catalan convolution triangle. To obtain the column which gives us S_1^k, we multiply the column above which starts with $1, k+1, \cdots$, by k and divide successive terms by $k, k+1, k+2, k+3, k+4, \cdots$, for $k = 0, \pm 1, \pm 2, \pm 3, \cdots$, where we adopt the convention that $0/0 = 1$ and $-0/0 = -1$. If we begin again with the regular arrangement, but this time move the i^{th} row $2i$ spaces to the right, we obtain an arrangement which has column $1, -2, 6, -20, 70, \cdots$, as

⋯	1	1	1	1	1	1	1	1	1	1	1
⋯	−5	−4	−3	−2	−1	0	1	2	3	4	5
⋯	21	15	10	6	3	1	0	0	1	3	6
⋯	−84	−56	−35	−20	−10	−4	−1	0	0	0	1
⋯	330	210	126	70	35	15	5	1	0	0	0
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

With the same processing as above, we obtain the Catalan convolution array with alternating signs. This shows that Pascal's triangle itself contains all that the inverses of the Pascal matrices gives from properly processed columns in the Pascal convolution array. By similar movement of the rows of Pascal's triangle and proper processing, we can obtain S_i^k, $i = 0, 1, 2, 3, \cdots; k = 0, \pm 1, \pm 2, \cdots$.

As we already know, the Catalan sequence S_1 and its convolution triangle are obtained by processing properly the diagonal 1, 2, 6, 20, 70, $\cdots$, and those diagonals parallel to it. Since Pascal's triangle has symmetry, we can use the parallel diagonals either above or below the central diagonal, when Pascal's triangle is written in rectangular form as in Table 4.4. Then, S_1^k is obtained by multiplying the parallel diagonal which begins with $1, k+1, \cdots$ by k and dividing successive entries by $k, k+1, k+2, \cdots$. Now, suppose that we try the same process for the Catalan convolution array, using diagonals parallel to $1, 2, 9, 48, 275, \cdots$, the central diagonal of the array, as given in Table 1.3.

$$1/1, 2/2, 9/3, 48/4, 275/5, \cdots = 1, 1, 3, 12, 55, \cdots = S_2$$

$$2(1/2, 3/3, 14/4, 75/5, 429/6, \cdots) = 1, 2, 7, 30, 143, \cdots = S_2^2$$

$$3(1/3, 4/4, 20/5, 110/6, 637/7, \cdots) = 1, 3, 12, 55, 273, \cdots = S_2^3$$

$$4(1/4, 5/5, 27/6, 154/7, \cdots) = 1, 4, 18, 88, \cdots = S_2^4$$

Surely you recognize the convolution array for the next of our sequences, S_2! If this same process is used on the convolution array for S_i, one obtains the convolution array for S_{i+1}. See [8], [9], [10].

3. A SECOND GENERATION OF THE SEQUENCES S_i FROM PASCAL'S TRIANGLE

These arrays can be obtained in yet another way from the diagonals of Pascal's triangle written in rectangular form. To obtain the convolution array for $S_2 = \{1, 1, 3, 12, 55, 273, \cdots\}$, we multiply successive diagonals and divide by successive members of an arithmetic progression with constant difference 3 as follows:

$$1(1/1, 4/4, 21/7, 120/10, \cdots) = 1, 1, 3, 12, \cdots = S_2$$

$$2(1/2, 5/5, 28/8, 165/11, \cdots) = 1, 2, 7, 30, \cdots = S_2^2$$

$$3(1/3, 6/6, 36/9, 220/12, \cdots) = 1, 3, 12, 55, \cdots = S_2^3$$

$$4(1/4, 7/7, 45/10, 286/13, \cdots) = 1, 4, 18, 88, \cdots = S_2^4$$

The diagonals are obtained by beginning in the row of ones in the Pascal rectangular array and counting down one and right two, or by beginning in the column of ones and counting to the right one and down two. The multiplier is the same as the exponent of S_2^k, and the arithmetic progression used is $k, k+3, k+6, \cdots, k+3n, n = 0, 1, 2, \cdots$.

To obtain the Catalan sequence, and its convolution triangle, we can use the diagonals obtained by counting down one and right one beginning in the column of ones (or in the row of ones) so that the beginning diagonal is 1, 3, 10, 35, $\cdots$, and dividing by successive terms of arithmetic progressions with constant difference two as follows:

$$1(1/1, 3/3, 10/5, 35/7, 126/9, \cdots) = 1, 1, 2, 5, 14, \cdots = S_1$$

$$2(1/2, 4/4, 15/6, 56/8, 210/10, \cdots) = 1, 2, 5, 14, 42, \cdots = S_1^2$$

$$3(1/3, 5/5, 21/7, 84/9, 330/11, \cdots) = 1, 3, 9, 28, 90, \cdots = S_1^3$$

Again the multiplier is the same as the exponent for S_1^k, and the arithmetic progression used for the divisors is $k+2n, n = 0, 1, 2, \cdots$.

Then, we have a dual system working here for extracting the convolution array of the sequence S_i from Pascal's triangle written in rectangular form. To obtain the convolution array for S_i, we find successive diagonals from Pascal's array by beginning in the column of ones and counting right one and down i, taking the first diagonal as $1, i+2, \cdots$. (Or, we can work to the right, taking the diagonals successively that are parallel to the diagonal beginning with $1, i+2, \cdots$, obtained by counting down one and right i throughout the array.)

To write S_i^k, we take the k^{th} diagonal which begins $1, k+i+1, \cdots$, multiply by k, and divide successively by the successive terms of the arithmetic progression $k+in$, $n = 0, 1, 2, \cdots$. Explicitly, we write the m^{th} element of S_i^k as

$$\frac{k}{k+im}\binom{(i+1)m+k-1}{m}$$

for $i = 0, 1, 2, \cdots;\ k = 1, 2, 3, \cdots;\ m = 0, 1, 2, \cdots$.

Many cases were shown which verify that the m^{th} term of the $(k-1)^{st}$ convolution of the sequence S_i, denoted by $s_i(m,k)$, is given by

$$s_i(m,k) = \frac{k}{k+im}\binom{(i+1)m+k-1}{m}, \tag{3.1}$$

$m = 0, 1, 2, \cdots;\ k = 1, 2, 3, \cdots;\ i = 0, 1, 2, \cdots$. Applying (1.8) leads to a rule of formation for the convolution array for any sequence S_i,

$$s_i(m,k) = s_i(m,k-1) + s_i(m-1,k+i). \tag{3.2}$$

Assume that (3.1) holds for all convolutions for the first $(m-1)$ terms, and holds for the first $(k-2)$ convolutions for the first m terms. Then $s_i(m,k)$ again will have the desired form of (3.1) as shown by

$$s_i(m,k) = s_i(m,k-1) + s_i(m-1,k+i) = \frac{k-1}{k-1+im}\binom{(i+1)m+k-2}{m} + \frac{k+i}{k+im}\binom{(i+1)m+k-2}{m-1}$$

$$= \binom{(i+1)m+k-1}{m}\left[\frac{k-1}{k-1+im}\cdot\frac{im+k-1}{im+m+k-1} + \frac{k}{k+im}\cdot\frac{m}{im+m+k-1}\right]$$

$$= \binom{(i+1)m+k-1}{m}\cdot\frac{k(im+m+k-1)}{(k+im)(im+m+k-1)}\,.$$

4. THE SEQUENCE OF SEQUENCES S_i TAKEN AS A RECTANGULAR ARRAY

Next, suppose one simply considers the sequence of sequences S_i as the rows of a rectangular array, and considers the progressions appearing in the columns. We omit the first term for each sequence S_i

Table 4.1 The Sequences S_i

S_0:	1	1	1	1	1	1	1	1	⋯
S_1:	1	2	5	14	42	132	429	1,430	⋯
S_2:	1	3	12	55	273	1,428	7,752	43,263	⋯
S_3:	1	4	22	140	969	7,084	53,820	420,732	⋯
S_4:	1	5	35	285	2,530	23,751	231,880	2,330,445	⋯
S_5:	1	6	51	506	5,481	62,832	749,398	9,203,634	⋯
S_6:	1	7	70	819	10,472	141,776	1,997,688	28,989,675	⋯
S_7:	1	8	92	1,240	18,278	285,384	4,638,348	77,652,024	⋯
	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
Order of AP:	0	1	2	3	4	5	6	7	
Constant:	1	1	3	16	125	1296	16807	262,144	⋯
Form:	1^{-1}	2^0	3^1	4^2	5^3	6^4	7^5	8^6	n^{n-2}

Notice that the k^{th} column is an arithmetic progression of order $(k-1)$, with common difference k^{k-2}. This means, using Eves' Theorem [4], [5],

Eves' Theorem: Consider a determinant of order n whose i^{th} column $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an arithmetic progression of order $(i-1)$ with constant a_i. The value of the determinant is the product $a_1a_2 \cdots a_n$.

that we can write Theorems 4.1 and 4.2.

Theorem 4.1: The determinant of any $n \times n$ array taken to include the column of 1's in the sequence of sequences S_i rectangular array has value

$$\prod_{j=1}^{n} j^{j-2}.$$

Theorem 4.2: Take a determinant of order n with its first column in the column of integers, and its first row along the row of ones of the rectangular sequence of sequences S_i array. The value of the determinant is

$$\prod_{j=1}^{n+1} j^{j-2}.$$

Proof: Subtract the $(i-1)^{st}$ row from the i^{th} row, $i = n, n-1, \cdots, 2$, to obtain a determinant whose k^{th} column is an arithmetic progression of order $k-1$ with constant $(k+1)^{(k+1)-2}$ and apply Eve's Theorem.

Further, the following result seems to be true.

Conjecture: Take an $n \times n$ determinant such that its first column is the column of integers in the sequence of sequences S_i rectangular array and its first row is the k^{th} row, $k = 1, 2, 3, \cdots$. Then its determinant is given by

$$\left(\prod_{j=1}^{n+1} j^{j-2}\right) \cdot \binom{n+k-1}{n} .$$

To prove that the constants of the arithmetic progressions have the form given, we quote Hsu [6, p. 480]:

$$\sum_{r=0}^{n'} (-1)^r \binom{n'}{r} \binom{sr+t}{m} = \begin{cases} 0, & m < n' \\ (-s)^{n'}, & m = n' \end{cases}$$

and substitute $n' = n-1,\ t = n^2,\ s = -n,\ m = n-1$, to obtain

$$\frac{1}{n}\sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r}\binom{n^2-rn}{n-1} = \frac{1}{n}\left(n^{n-1}\right) = n^{n-2}, \tag{4.1}$$

where we also make use of the known general form for the m^{th} term of S_i.

5. A CLASS OF COMBINATORIAL IDENTITIES

Returning to the first section, in Table 1.1 we computed matrices P_i^{-1}. Now, since $P_iP_i^{-1} = I$, we can write an entire class of combinatorial identities. Notice that, since we are dealing with infinite matrices such that all non-zero elements appear on and below the main diagonals, $P_iP_i^{-1} = I$ for any $n \times n$ matrices P_i, P_i^{-1}, and I formed from the $n \times n$ blocks in the upper left of the original infinite matrices. Since P_i contains elements taken from Pascal's triangle, it is a simple matter to write the element in its $(n+1)^{st}$ row and $(j+1)^{st}$ column as

$$p_i(n,j) = \binom{n+ij}{j+ij}, \qquad n = 0, 1, 2, \cdots;\ j = 0, 1, 2, \cdots. \tag{5.1}$$

Now, the elements in P_i^{-1} are the same as those in the convolution array for S_i, except for sign. When $i = 1$, we have the Catalan convolution array, and the element of P_1^{-1} in its $(r+1)^{st}$ row and $(p+1)^{st}$ column is given by the $(r-p)^{th}$ element of the $(2p)^{th}$ convolution of S_1, or the $(r-p)^{th}$ element in the sequence S_1^{2p+1}, which is, by (3.1),

$$p_1^*(r,p) = (-1)^{r-p}s_1(r-p, 2p+1) = \frac{(-1)^{r-p}(2p+1)}{p+1+r}\binom{2r}{r-p}$$

while

$$p_1(n,j) = \binom{n+j}{2j} .$$

Since $P_1P_1^{-1} = I$, the element in the $(n+1)^{st}$ row and $(p+1)^{st}$ column of I is given by

$$0 = \sum_{j=0}^{n} p_1(n,j)p_1^*(j,p), \qquad n \neq p.$$

Now, when $p = 0$, we have the first column of P_1^{-1}, of the sequence S_1 of Catalan numbers, and

$$0 = \sum_{j=0}^{n} \frac{(-1)^j}{j+1}\binom{2j}{j}\binom{n+j}{2j} \tag{5.2}$$

which was given as (3.100) by Gould [7].

Since $n \geqslant p+1$ gives non-diagonal elements of I, we also have the more general

$$0 = \sum_{j=p}^{n} \frac{(-1)^{j-p}(2p+1)}{p+1+j}\binom{2j}{j-p}\binom{n+j}{2j} \quad . \tag{5.3}$$

We can further generalize by not restricting i. Let the element in the $(r+1)^{st}$ row and $(p+1)^{st}$ column of P_i^{-1} be

$$p_i^*(r,p) = (-1)^{r-p}s_i(r-p, ip+1) = \frac{(-1)^{r-p}[(i+1)p+1]}{p+1+ir}\binom{r+ir}{r-p} \quad . \tag{5.4}$$

Since $P_iP_i^{-1} = I$, for $n \geqslant p+1$ we obtain a non-diagonal element, giving the very general identity

$$0 = \sum_{j=p}^{n} \frac{(-1)^{j-p}[(1+i)p+1]}{p+1+ij}\binom{j+ij}{j-p}\binom{n+ij}{j+ij} \ , \tag{5.5}$$

for $i = 0, 1, 2, 3, \cdots; p = 0, 1, 2, \cdots$; and $n \geqslant p+1$.

Notice that, for $i = 0$, we have Pascal's triangle in both P_i and P_i^{-1}, leading to

$$0 = \sum_{j=p}^{n} (-1)^{j-p}\binom{j}{j-p}\binom{n}{j} \ , \tag{5.6}$$

and, when $i = 0$ and $p = 0$, to the familiar identity,

$$0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ . \tag{5.7}$$

For $p = 0$ in (5.5), we are in the first column, and

$$0 = \sum_{j=0}^{n} \frac{(-1)^j}{1+ij}\binom{j+ij}{j}\binom{n+ij}{j+ij} = \sum_{j=0}^{n} \frac{(-1)^j}{1+ij}\binom{n+ij}{ij}\binom{n}{j} \tag{5.8}$$

gives a recursion relation for the terms of S_i, as

$$0 = \sum_{j=0}^{n} (-1)^j s_i(j,1)\binom{n+ij}{j+ij} \ , \tag{5.9}$$

where $s_i(j, 1)$ is the j^{th} term of the sequence S_i^1.

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EXPONENTIALS AND BESSEL FUNCTIONS

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A Bessell function of order n may be defined as follows:

$$J_n(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{\Gamma(\lambda+1)\Gamma(\lambda+n+1)} \left(\frac{x}{2}\right)^{n+2\lambda} \tag{1}$$

It may be easily shown that for integral $n, J_n(x)$ is the coefficient of U^n in the expansion of

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)\right]$$

i.e.,

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)\right] = \sum_{n=-\infty}^{\infty} U^n J_n(x) \tag{2}$$

Now let

$$u - \frac{1}{u} = L_{2k+1}, \tag{3}$$

where L_{2k+1} is a Lucas number defined by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2}, \tag{4}$$

where n is any integer.

Equation (3) becomes $u^2 - uL_{2k+1} - 1 = 0$ with roots

$$\left(\frac{1+\sqrt{5}}{2}\right)^{2k+1} = \alpha^{2k+1} \qquad \text{and} \qquad \left(\frac{1-\sqrt{5}}{2}\right)^{2k+1} = \beta^{2k+1},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the well known quadratic

$$\phi^2 = \phi + 1. \tag{5}$$

[Continued on page 418.]

THE GOLDEN SECTION AND THE ARTIST

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The readers of *The Fibonacci Quarterly,* interested for the most part in ramifications of their fascinating subject as expressed in mathematical terms, may also be interested in seeing what happens when the geometric harmonies inherent in the series are made visible to the eyes.

The ratio of the Fibonacci series, 1.618 or ϕ, reciprocal 0.618, when drawn out rectangular form, produces the golden section rectangle (Fig. 1). The rectangle can be constructed geometrically by drawing a square, marking the center of the base and drawing a diagonal from this center to an opposite corner; then with this diagonal as a radius and the center base as center, drawing an arc that cuts a line extended from the base of the square. This will mark the end of a rectangle whose side will be in 1.618 ratio to the end. The end will be in 0.618 ratio to the side. The excess will itself be a ϕ rectangle.

A line parallel to the side through the point where the diagonal intersects the side of the square will mark off another ϕ rectangle with a square on its side in the excess, and a ϕ rectangle in the square; the remainder of the square will contain a ϕ rectangle with a square on its end.

Many instances of the presence of the golden section relation can be found in fine works of art preserved for their merits through the centuries. Some works of art can be found that have dimensions whose quotients are close to the ratio 1.618. In the cases studied, when these areas were subdivided geometrically as in Fig. 2, all main lines of the pictorial designs, and all minor directions and details were found to fall along lines of the diagram and diagonals to further subdivisions.

The subdivision of the ϕ rectangle can be accomplished geometrically by drawing lines parallel to the sides through the intersections of diagonals with the side of the square, and lines parallel to side and end through intersections of these lines with diagonals of square and excess, and through any other intersections that may occur (Fig. 2).

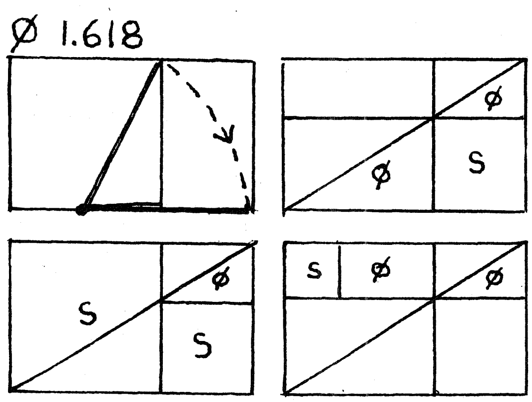

Figure 1

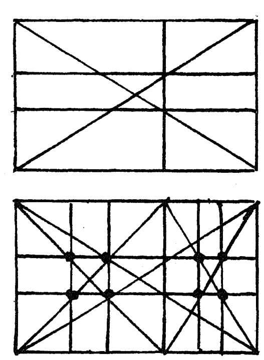

Figure 2

Or, it can be done perhaps more precisely by using the Fibonacci series.

The measurements of Giotto's *Ognissanti Madonna,* cl310, in the Uffizi, Fig. 3, fall just a little short of the 1.618 ratio. They are given as,

$$10'8'' \times 6'8'' = 128'' \times 80'' = 1.6$$

photos measure,

$$25.9 \times 16.1 \text{ cm} = 1.618 - .0093$$
$$13.4 \times 8.3 \text{ cm} = 1.618 - .0036$$

subdivision in the Fibonacci series:

		for practical purposes
$.618 \times 8.3 = 5.1294$	8.3	8.3
	5.1294	5.113
	3.1706	3.17
	1.9588	1.96
	1.2118	1.21
	.747	.75
	.4648	.46

When the golden section rectangle is applied to the photo of the painting, and the main divisions drawn, and the Fibonacci subdivisions are marked off on the edges, it will be found that the area occupied by the Madonna and Child lies precisely within a main ϕ division of the excess at the top, and a main ϕ division of the square at the bottom, and $\phi/2$ divisions at the sides. Architectural details, the angles of the steepled frame, vertical supports, centers of arcs, divisions of the platform, fall along subdivisions or along obliques from one subdivision to another. The lines of the top of the painting extend to center of golden section excess. The hands of the Madonna and Child, all lines of the angels, the tilt of their faces, their arms wings, the folds of their garments, fall along directions from one ϕ subdivision to another.

In making a study of the apparent incidence of certain geometric patterns in fine art, over 400 paintings of accepted excellence were analyzed. All but a few yielded to analysis. The majority clearly showed the presence of the ϕ relationship, or of its related shape, the $\sqrt{5}$ rectangle (Fig. 5). However, the overall shape of only a small number was in the simple 1.618 proportion. All followed the diagram lines in their designs. Among them we can mention: (Measurements starred are from photos of pictures shown within frames or borders, and are in centimeters. All others are dimensions given in catalogues or art histories, and are in inches.)

Duccio	*Madonna Enthroned* (Rucellai) 1285, Florence	$14.32 \times 8.85^* = 1.618$
Duccio	*Madonna and Child,* Academy, Siena	$5.82 \times 3.6^* = 1.618$
Martini	*Road to Calvary*, c1340 Louvre	$9\text{-}7/8 \times 6\text{-}1/8 = 1.618 - .0058$
da Vinci	*Virgin of the Rocks,* 1483, Louvre	$78 \times 48 = 1.618 - .007$
Turner	*Bay of Baise,* Tate Gal.	$57\frac{1}{2} \times 93\frac{1}{2} = 1.618 - .0008$
Cole	*Florence from San Marco,* Cleveland Museum of Art, 1837	$39 \times 63\text{-}1/8 = 1.618 - .0001$
Romney	*Anne, Lady de la Pole,* 1786, MFA Boston	$95\frac{1}{2} \times 59 = 1.618 + .0006$

The photo of an Egyptian stele c. 2150 B.C., in the Metropolitan Museum of Art shows dimensions that have the 1.618 ratio. A seated figure fits exactly within the excess, heiroglyphic details fit in subdivisions of the square.

There is a bas-relief of an Assyrian winged demi-god of the 9th Century B.C. in the Metropolitan Museum of Art that fits perfectly into a 1.618 rectangle, and the strong lines of the wings, legs, beak follow divisions and diagonals of the ϕ diagram.

The Babylonian *Dying Lioness,* Ninevah, c. 600 B.C., in the British Museum, London, can also be contained exactly in a 1.618 rectangle. All lines of the figure, the directions of the arrows, fit on the lines of the diagram.

In a slab from the frieze of the Parthenon, c. 440 B.C., in the British Museum, showing two youths on prancing horses, the design also can be contained exactly in a ϕ rectangle and all lines conform to the pattern of the diagram.

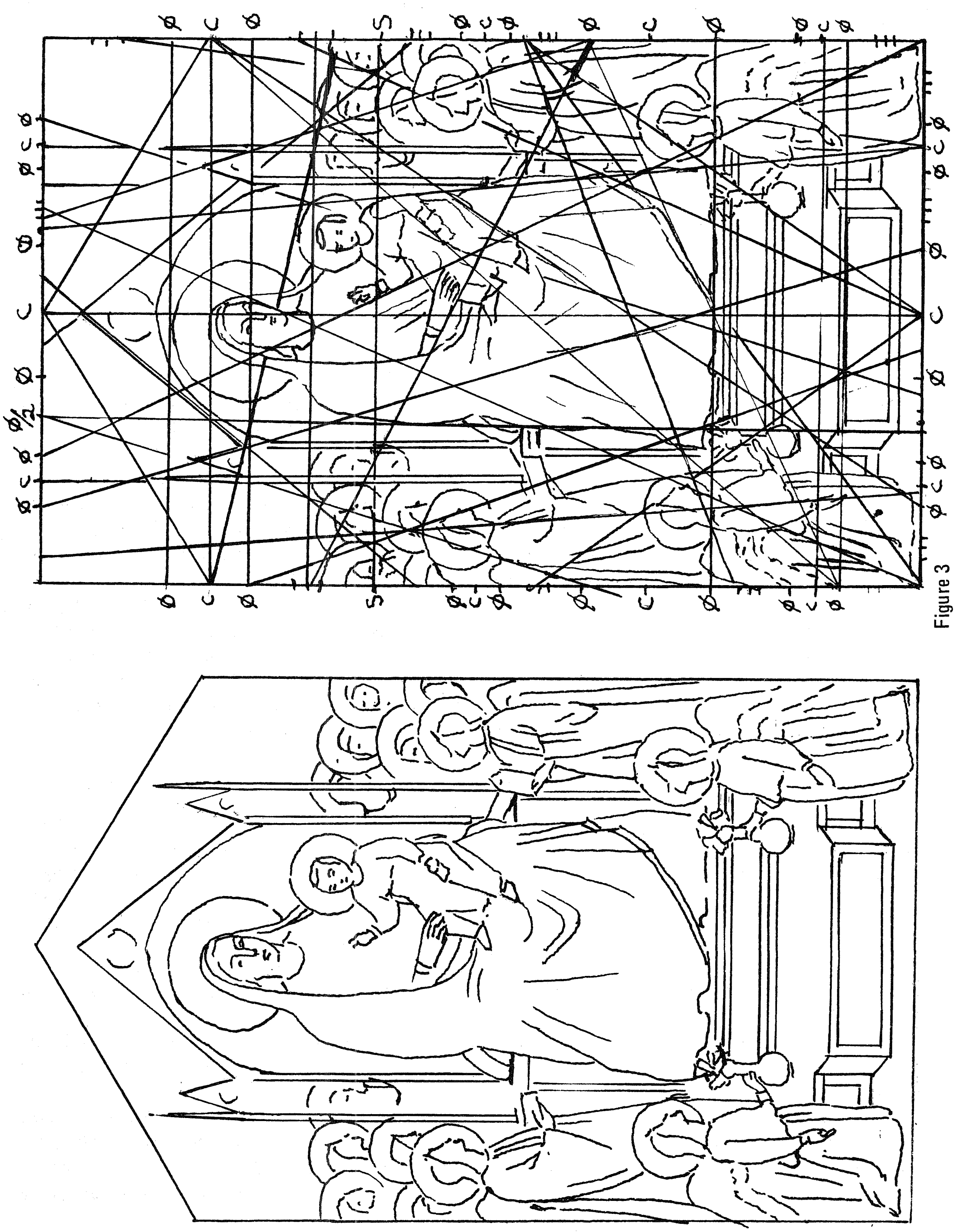

Figure 3

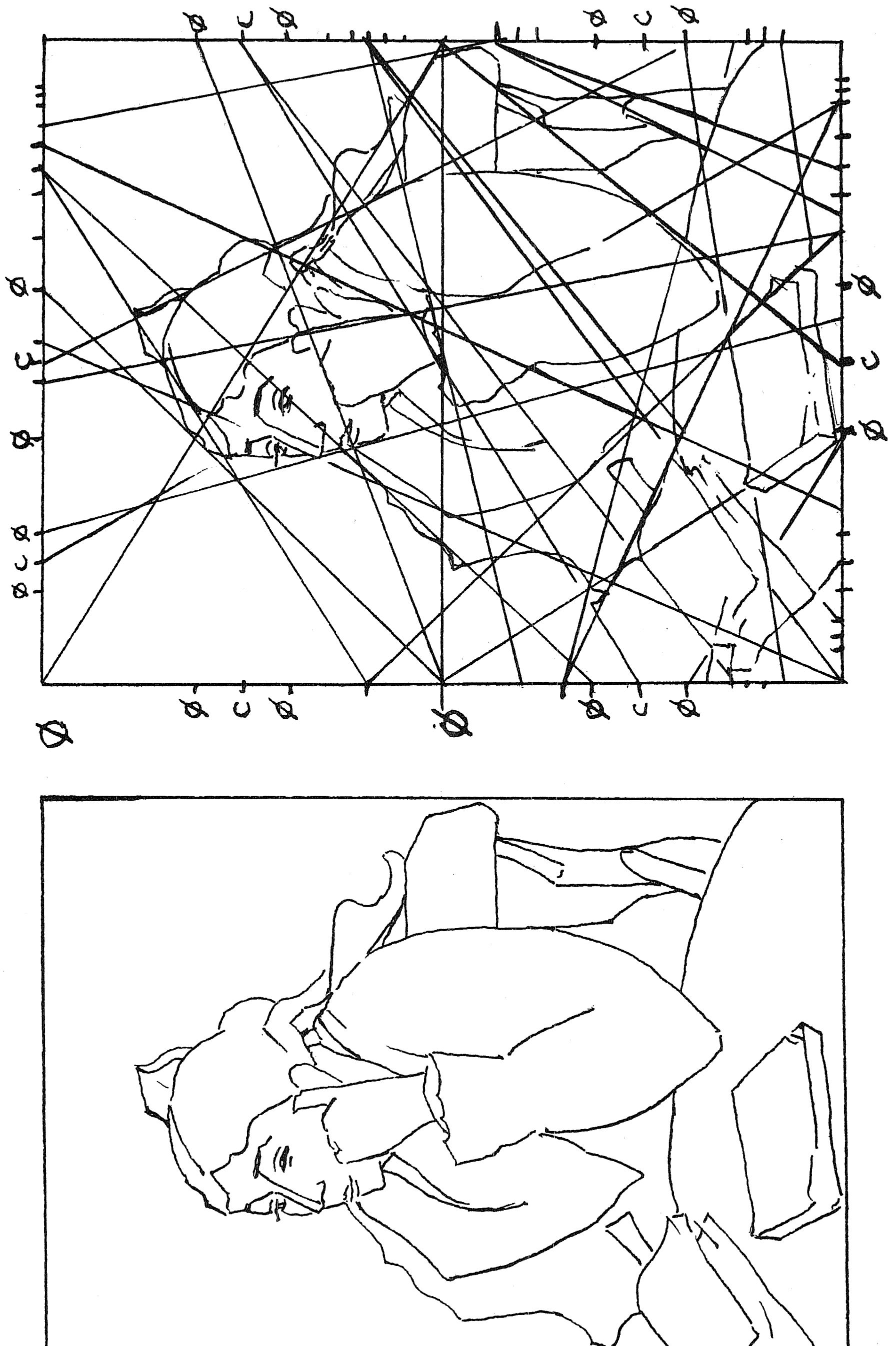

Figure 4

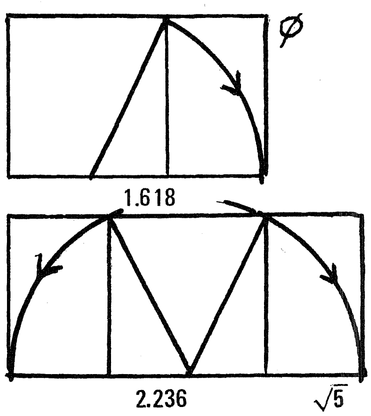

Figure 5

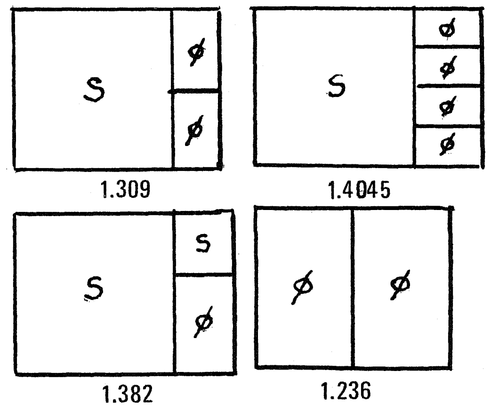

Figure 6

The measurements given for a marble balustrade relief in the Cathedral Baptistry, Civitale, Italy, c 725–750 A.D., are "about 3′ × 5′." In the photo, the border measures 8.2 × 13.25* = 1.618 – .001. All directions and details fit into the ϕ subdivisions.

The dimensions of many of the paintings studied yielded quotients close to the ratios of figures that consisted of sections of the ϕ rectangle, often combined with squares (Fig. 6):

$$1.309 = 1 + \frac{.618}{2}$$

$$1.4045 = 1 + \frac{1.618}{4}$$

$$1.302 = 1 + (1 - .618)$$

$$.809 = \frac{1.618}{2} \quad \text{(reciprocal, } 1.236 = 2 \times .618\text{)}.$$

We can see an example of one of these combined areas in *Yellow Accent,* 1947, private collection, by Jacques Villon (Fig. 7). The measurements of the photo of the picture shown in its frame are:

$$9.3 \times 11.5^{*} = 1.236 + .0004.$$

This couldn't be much closer to 1.236. To get subdivisions in the proportions of the Fibonacci series:

$$.618 \times 9.3 = 5.7474 \qquad \begin{array}{r} 9.3 \\ \underline{-5.7474} \\ 3.5526 \\ 2.1948 \\ 1.3578 \\ .837 \\ .5208 \\ .316 \\ .2046 \end{array}$$

When the edges of the painting are subdivided in these proportions, lines of the painting will be found to extend from one point of division to another precisely.

The same 1.236 framework can be found in *L'Arlesienne,* painted by Van Gogy in 1888. Its measurements are given as

$$36 \times 29 = 1.236 + .0053$$

$$\text{Photo } 10.5 \times 8.5^{*} = 1.236 - .0007$$

All lines outlining areas and giving directions to details go from one ϕ division on the edge to another.

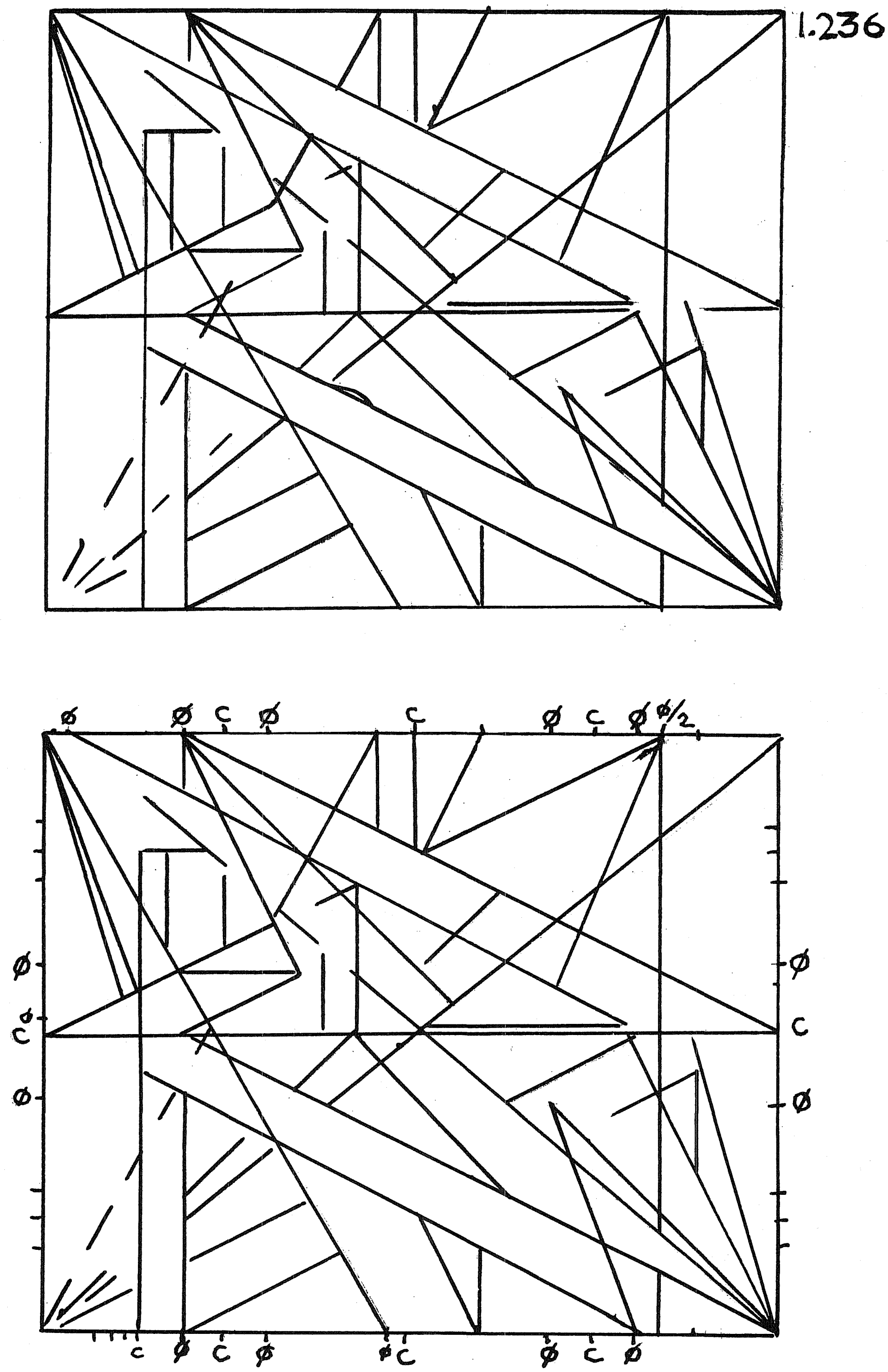

Figure 7

Among paintings that have ratios close to 1.236 and can be analyzed on that there are

Gos. Bk. of Ebbo	*St. Luke,* a. 823, Epernay	5-3/8 × 6-7/8 = 1.236 − .0001
Cloisters Apocalypse	*Opening of Book,* c 1320, Cloisters, N. Y.	13.4 × 16.6* = 1.236 + .0028
Cezanne	*Still Life,* c 1890, N G A Wash.	25½ × 31½ = 1.236 − .0008
Seurat	*Fishing Fleet,* c 1885, M Mod. A N Y	8.85 × 10.9* = 1.236 − .0044
Picasso	*Lady With Fan,* 1905, Harriman Col.	39-3/4 × 32 = 1.236 − .0045
Gris	*Painter's Window,* 1925, Baltimore M A	39-1/4 × 31-3/4 = 1.236 − .0063

Many more complicated combinations were found. A figure made of a square plus an excess containing two $\sqrt{5}$ rectangles with a square on their side has the ratio 1.528 (Fig. 8).

An .809 shape with a ϕ rectangle across its side has the ratio 1.427.

Two $\sqrt{5}$ rectangles side-by-side has the ratio 1.118 (2.236/2).

All but a few paintings with dimensions that give quotients close to these ratios yielded to rigorous analysis.

The mathematical system on which this study was based was worked out in the early 1900's by Jay Hambidge, a minor American artist, who was interested in investigating several phases of art, particularly that of the classic Greek, in search of a possible mathematical basis for its apparent perfection. He measured hundreds of Greek vases in the Boston Museum of Art and the Metropolitan Museum in New York, and defined a series of figures basic to the combinations whose ratios kept recurring in the measurements of the vases. They were rectangles in the proportions of 1 to $\sqrt{2}$ (1.4142), $\sqrt{3}$ (1.732), $\sqrt{5}$ (2.236), and the golden rectangle, 1.618 or ϕ.

To identify the various combinations that he found, and to properly subdivide them, he calculated their ratios and obtained their reciprocals. This mathematical material was not new, but his application of it to Greek art and his suggestion that artists should use it in their own work were new, and his clarification of the series of root rectangles, and their properties and interrelations evidently took even mathematicians by surprise.

He presented his discoveries in *Dynamic Symmetry: The Greek Vase* and *The Parthenon,* Yale University Press, 1920 and 1922. The general substance originally published in his review, *The Diagonal,* 1919–1920, and in *Elements of Dynamic Symmetry* is available now in a Dover publication, 1967.

In this study I have applied Hambidge's method of finding the specific geometric figure present in a work of art by identifying the quotient of its dimensions with the ratio of known geometric figures. As far as I know, this is an approach to the subject that has not been made before to works of art other than that of the Egyptians and Classic Greeks.

Hambidge thought that the system of planning works of art, vases, statues, murals, buildings, by the use of geometric frameworks disappeared with the classic Greeks, and that the Romans and others used what he called "static" symmetry, or a squared-off frame, which gave proportion in *line,* rather than in *area* (Fig 9).

However, it seems that evidences of the Greek knowledge of this process of geometric design can be found in later periods in many areas within the Greek sphere of influence. The first statues of Buddha were made in Gandhara in northwest India, which was settled by officers and soldiers from the remnants of Alexander's army and remained to some extent in contact with the western world.

There is a seated *Buddha,* c. 3rd Century A.D., in the Seattle Art Museum (Fig. 10), that shows the Greek influence in the treatment of hair and drapery. A ϕ rectangle can be applied to a front view photo of it, and all parts will be found to conform to the ϕ framework. This tradition seems to have persisted, as correlation with figures consisting of more complicated combinations of ϕ rectangles and squares can be found in a *Teaching Buddha* in Benares of the 5th Century A.D., and in an icon from South India, *Shiva* as *King of Dancers,* of the 12th Century A.D.

Other examples of works of art done in areas under Greek influence in which the ϕ rectangle or its combinations are apparent can be cited:

floor tiles	*Diana the Huntress*	square ÷ ϕ
	Still Life	square ÷ ϕ
wall panels,	*Fish*	6.4 × 8.6* = 1.3455 − .0018 (Fig. 11)
	Man and Lions	7.9 × 5.9* = 1.3455 − .0015

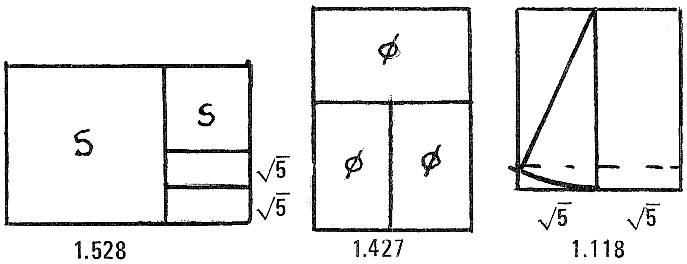

Figure 8

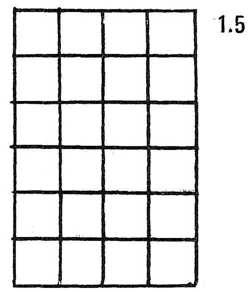

Figure 9

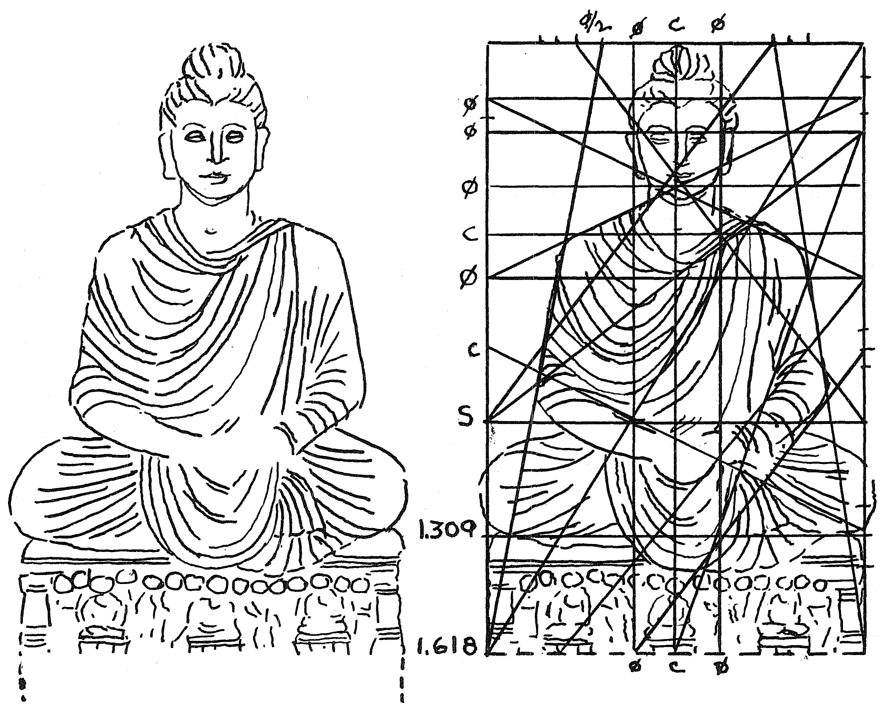

Figure 10

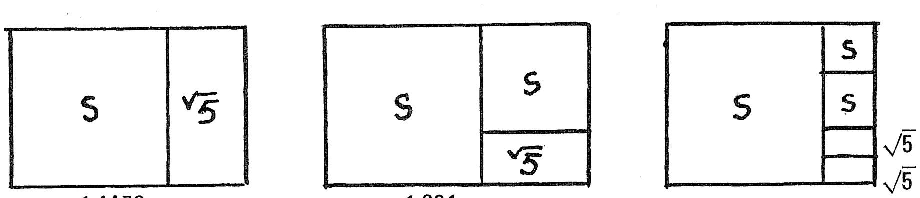

Figure 11

wall painting, *Hercules and Telephus* 9.9 × 8* = 1.236 + .0017
mms. *Georgics,* Bk. 111, 5th Century A.D., Vatican Library
Shepherds Tending Flocks 19 × 19.5* = 1.0225 + .0038 (.618 + .4045)

As the Graeco-Roman merges into the Early Christian culture, manuscript paintings, mosaics and frescoes still give evidence of the presence of geometric pattern on various ϕ arrangements, and now more frequently, on the $\sqrt{2}$ and $\sqrt{3}$ themes:

Mosaics, 5th Century A.D., Santa Maria Maggiore, Rome	
Abraham and Angels	19.8 × 17.2* = 1.1545 + .0046
Melchizedek and Abraham	19.3 × 14.8* = 1.309 + .0018
Manuscripts	
Echternach Gospels, Ireland (?) c. 690	
Symbol of St. Mark	19 × 14.6* = 1.309 − .0008
Book of Durrow, Irish, 7th Century, Trinity College, Dublin	
Symbol of St. Matthew	6 × 13.9* = 2.309 + .0076
Irish Gospel Book, St. Gall, 8th Century	
St. Mark and Four Evangelists	19.7 × 14.65* = 1.3455 − .0008
Registrum Gregoril, Trier, c. 985, Musee Conde, Chantilly	
Emperor Otto II or III	20.8 × 15.4* = 1.3455 + .0051
	given 10-5/8 × 7-7/8″ = 1.3455 + .0037
Fresco, Catacomb of Commodilla, Rome, 7th Century	
St. Luke	19 × 18 = 2 squares ÷ $\sqrt{2}$

All conform in their design to the geometric patterns indicated by the quotients of their dimensions.

The ϕ presence continues through the centuries unfolding into the Renaissance with the works of Duccio and Cimabue. Most of the paintings analyzed in this study fell within the Renaissance and Baroque periods, c. 1300 – c. 1660. Most of the artists were born in, or spent time in special areas, Venice, Florence, Milan, Umbria, Rome. One or another of these were also the dwelling places from time to time of the mathematicians Luca Paciola, Alberti, Bramanti, and the artist-mathematicians da Vinci, della Francesca, and Durer. The ratios found in this period included many combinations of the ϕ and $\sqrt{5}$ rectangles, of varying degrees of intricacy.

One of the combinations found is the 1.691 shape (Fig. 11). This consists of a square and an excess that contains a $\sqrt{5}$ rectangle with a square on its side. (Hambidge found this to be part of the floor plan of the Parthenon.) The ratio of the $\sqrt{5}$ rectangle is 2.236, its reciprocal is .4472. The ratio of the excess of the 1.691 figure will be 1.4472, reciprocal .691. Among works whose dimensions give a quotient close to 1.691, and yield to analysis are:

Rembrandt	*Goldweigher's Field* (etching)	6.75 × 18.15* = 2.691 − .0022
Sassetta	*Wolf of Gubbio*	25.3 × 15* = 1.691 − .0044
Sassetta	*St. Francis and the Bishop*	26.09 × 15.4* = 1.691

If the excess of the 1.691 shape is divided in half longitudingly, the ratio of the square and this section will be

$$1 + \frac{.691}{2} = 1.3455$$

The excess will contain two squares and two $\sqrt{5}$ rectangles.

Among works whose dimensions yield quotients close to this figure and that analyze precisely are:

	Avignon Pieta, c. 1460, Louvre	64 × 86 = 1.3455 − .0018
Pollaiuolo (?)	*Portrait of Man,* Nat. Ga. Wash.	20½ × 15¼ = 1.3455 − .0013
Clouet	*Francis I,* Louvre c. 1525	28-1/8 × 32-3/4 = 1.3455 − .0033
David	*Sabines,* 1799, Louvre	152 × 204-3/4 = 1.3455 − .0011
Beardsley	*Flosshilde*	10.1 × 7.5* = 1.3455 + .0011

The *Isenheim Altarpiece,* 1511–1515, by Mather Grunewald, consists of a center panel, two side panels, and a base. Dimensions given are for the paintings within the frames, and are meaningless as geometric ratios. However, if the frames are included and the work is considered as a single plan, as sometimes happened in Medieval and Early Renaissance art, the overall dimensions measured on a photo of the complete work (Fig. 12), are:

$$26.55 \times 35.72^* = 1.3455.$$

The center panel plus the sides are contained in an area cut off by a ϕ division in the lower part of the square. Such are the interrelations of areas in the dynamic shapes that this area has the proportions

$$20.28 \times 35.72^* = 1.764\ (-.0022)(.764 = \text{r } 1.309).$$

The center panel, *The Crucifixion,* including the frame, is

$$20.28 \times 22.72^* = 1.118\ (-.0028).$$

The painting itself has strong lines of action, all of which coincide with divisions of the 1.118 shape or diagonals to prominent intersections.

The side panels, *St. Sebastian* and *St. Anthony* measure

$$17.5 \times 6.5^* = 2.691\ (+.0013).$$

The area remaining in the overall 1.3455 shape after the three panels are cut off consists of 2 ϕ rectangles, 2 squares, and a .4677 shape, reciprocal 2.1382 (the shape that Hambidge found to be the floor plan of the Parthenon). *The Entombment,* pictured on the stand, has areas and line directions that conform to subdivisions of the ϕ rectangles and squares in which they occur.

As far as I know, there is no concrete proof to show that the geometric relations found in the works of art were the result of deliberate planning on the part of the artists. The evidence is circumstantial.

There is a time pattern found in those examined. Pictorial designs on the $\sqrt{2}$ theme occurred c. 1200 – c. 1450, then seldom appeared again until the late 1800's. The ϕ theme was found throughout, peaking c. 1550, the $\sqrt{5}$ was most prevalent in the 1600's, the $\sqrt{\phi}$ in the 1700's, reappearing in the late 1800's.

There is the phenomenon of the irregularity of dimensions of paintings. Of the 400 studied, only about 1/8 had regular proportions, as 1-1/2, 1-1/3, etc. All the rest had odd measurements, as 70-1/2 × 53-1/2, 33 × 26, 18-1/2 × 16. The ratios of all could be closely related to ratios of geometric figures which were combinations of squares and $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ or ϕ rectangles. When the figures appropriate to the dimensions were applied to the paintings and properly subdivided, all lines of direction and demarcation of areas to smallest detail, fell into place on the parts of the diagram. The experience of finding this correlation tends to be very convincing to one who sees it happening over and over again.

Only a few clear clues were found. Fragments of dotted lines, vertical, horizontal, oblique, that fitted into a 1.472 shape, in background and design of a drawing by Poussin; an engraving by Durer in a 1.427 rectangle, a close copy by Raimondi in a 1.382 shape; construction lines of ϕ rectangles showing in the background of a 16th Century Japanese screen, whose panels had the ratios of 3.236 and 2.809.

Matila Ghyka, in his *Geometry in Art and Life,* has a chapter in which he presents evidence that a secret geometry based on the circle and pentagram was passed on from early Medieval times by secret ceremonies in the masons' guilds. He infers that a similar practice could have passed the knowledge down through the artists guilds. Ghyka shows instances of the ϕ rectangles in Renaissance art and architecture. He thought that knowledge of the system disappeared in the late 17th Century after van Dyke, and was rediscovered from time to time by individual artists, like Seurat, or by small cults.

However, instances of the presence of the ϕ rectangle, and of the special figure of the $\sqrt{\phi}$ (1.273) (Fig. 14) can be discerned in some 18th Century paintings, as,

Pater	*Bathera*	c1735	Grenoble	25-1/2 × 32-1/2 = 1.273 + .0015
Boucher	*Bath of Diana*	1742	Louvre	22-1/2 × 28-3/4 = 1.273 + .0047
David	*Death of Marat*	1793	Brussells	64 × 49 = 1.309 - .0029
Watteau	*Gilles*	c1720	Louvre	58-3/8 × 72-1/4 = 1.235 - .0062
Chardin	*Dessert*	1741	Louvre	18-1/2 × 22 = 1.191 - .0018

All elements of the compositions relate closely to appropriate subdivisions. (.382)/2

Paintings by the early 19th Century artists working in the academic tradition also show ϕ relationships:

Ingres	*M. Bertin*	1832	Louvre	37-1/2 × 46 = 1.236 - .009
Delacroix	*Massacre at Scio*	1824	Louvre	166 × 138-1/2 = 1.191 - .004
Goya	*May 3, 1808*	1814	Prado	104-3/4 × 135-7/8 = 1.309 - .0119
Gericault	*Raft of the Medusa*	1819	Louvre	193 × 282 = 1.4635 - .0024 (1.618 – .618/4)

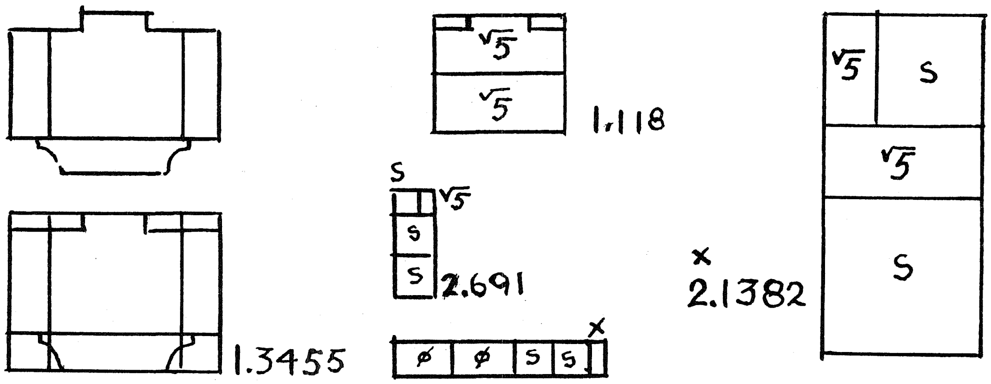

Figure 12

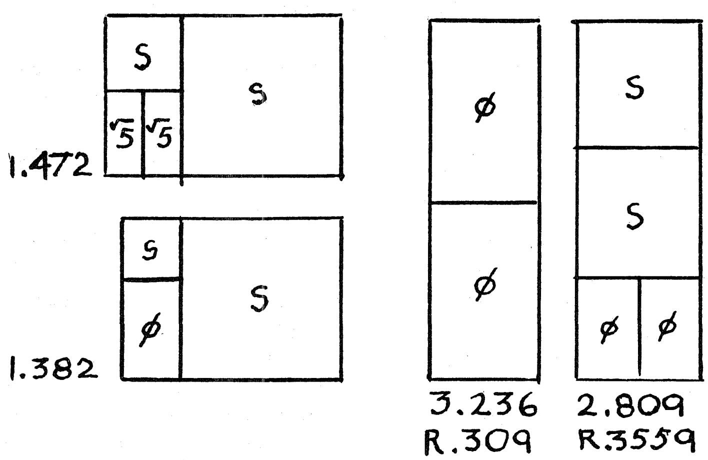

Figure 13

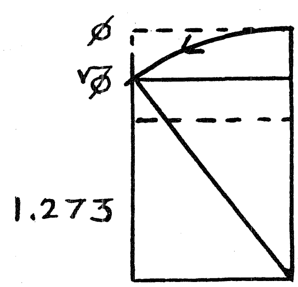

Figure 14

The upheaval in the art world, the split away from the academics that occurred in the middle of the 19th Century, is usually interpreted in terms of subject matter and technique. From my study, I am inclined to think that it was also partly related to the "liberation" of the knowledge of geometric design from the confines of the tight academic circle. Who was responsible for the disclosure? One of the Barbizon painters? Courbet? Someone made it known to the outsiders. From internal evidence, Manet had it, and Renoir, Degas, Toulouse-Lautrec, Cezanne, Seurat, Van Gogh.

In the late 80's, there was a group of artists led by Serusier, devoted to the study and application of the golden section. Bonnard and Vuillard were members of the group. They centered at Pont Avon, where Gauguin was in contact with them. His first well known painting, *Jacob Wrestling with the Angel,* was made there in 1888. It measures 28-3/4 $\times$ 36.5 = 1.273 $-$ .0034 ($\sqrt{\phi}$), and analyzes perfectly on this pattern divided in ϕ ratio.

All of these artists were greatly interested in the newly revealed arts of Japan. We wonder to what extent they discerned the presence of geometric relationships in Japanese prints. These can be found clearly and definitely in the few examples of Japanese art examined. In four from a series *The Manga,* in the Metropolitan Museum, by Hokusai, 1817, the borders measure:

			10 $\times$ 14.45* = 1.4472 $-$ .0022 (1 + $\sqrt{5}$)
sketches in style of Hokusai		1818	10.5 $\times$ 14.5* = 1.382 $-$ .0022
Anecdotes by Hokusai		1850	10.2 $\times$ 14.4* = 1.4142 $-$ .0025 ($\sqrt{2}$)
Red and White Peppers, Freer Gallery		18th Century	47 $\times$ 19 = 1.472 $-$ .0017 1 + (1.118 + 1)
Horses, Baltimore Museum of Art	screen	17th Century	4.5 $\times$ 12.2* = 2.7071 + .004 (.7071 = r 1.4142)
Landscapes of the 4 Seasons	screen		6.73 $\times$ 12.3* = 1.8284 $-$ .0008 (.4142 + .4142)
Han-Shan and Shih-te	screen	16th Century	2.38 $\times$ 7.7* = 3.236 (r..309)
Dai-itoku M F A Boston, painting		11th Century	6'3½" $\times$ 46½ = 1.618 + .005

All analyze precisely.

In Paris, about 1910 there was a group that called itself "Section d'Or," that investigated the use of this proportion. The group included Duchamp, Villon and Picabia. Matisse and Picasso were in contact with them:

Duchamp	*Nude Descending Staircase*	1912	58-3/8 $\times$ 35-3/8 = 1.644 + .006 (.809 + squares)
Villon	*Dinner Table*	1912	27-3/4 $\times$ 32 = 1.236 + .0067
Matisse	*Variation on de Heem*	1915	71 $\times$ 87-3/4 = 1.236 $-$ .0008
Picasso	*Lady with a Fan*	1905	39-3/4 $\times$ 32 = 1.236 $-$ .0045

One wonders also how the revelation of the geometric system by the publication of Hambidge's investigations of Greek art, the probable original source, affected those who were in possession of the secret, who were still an "elect" group. At about the time of the revelation of Hambidge's discoveries, some artists in Paris, and Duchamp and Picabia in New York, started in a new direction, leading to Dada and Surrealism, the antithesis of the ideal of the order of Cubism and Dynamic Symmetry. This movement succeeded in the predominance of Surrealism in the 30's, which to some extent dampened interest in the order of geometric design.

The theory behind this study is that down through the ages from Classic Greek times, the knowledge of the process of geometric design was the possession of carefully chosen groups sworn to secrecy. That, of all art produced at any one time, their works are the ones that have mostly survived, partly because those chosen would naturally be the better artists, partly because of the superior effect the ordered proportions gave to their works.

What effect will the placing of this knowledge at the disposal of all artists have? Hambidge seemed to expect that artists would eagerly sieze upon his findings and use them in their work, and thus raise the quality of art on all levels. This did happen to a certain extent in illustration, advertising design and layout, industrial design, architecture and interior design, paralleling similar developments stemming from the Cubist movement in Europe. Hambidge was obviously unaware of the experiments with the golden section of Seurat, or of the Serusier group, or of the Section d'Or. Among outstanding American painters of the time who adopted the system we can mention Leon Kroll, George Bellows, Robert Henri and Jonas Lie.

Many others in the art field closed their eyes to the whole idea. If they should find it to be true, they would have to rethink all their concepts about art. Artists, art critics and historians often are not inclined to mathematics, and tend to shy away from it as something they don't know much about, and would have to make an effort to understand.

It takes some mental effort to understand and use the geometrical diagrams. Some can't do it; some, who must "paint as the bird sings" find it confusing to the point of interrupting their intuitive inspiration. Many artists resented the proposition that proportion and line direction, that they had worked so hard to master, could be achieved easily and perhaps more effectively by the use of a diagram. Many, not versed in mathematics, cannot appreciate the beauty of order in mathematics, and interpret it as "mechanical."

Will the situation resolve itself as before—the survival of the fittest—only now with the means of survival open to all those equal to grasping it? Or will the secret handed down through the ages as a "precious jewel" to those carefully selected for ability and responsibility, be diffused and list in indifference and sloth?

SOURCES

Jay Hambidge, *Dynamic Symmetri The Greek Vase,* Yale University Press, 1920.
Jay Hambidge, *The Diagonal,* monthly review, 1920.
Jay Hambidge, *The Elements of Dynamic Symmetry,* Dover Publications, Inc., New York, 1967.
Matila Ghyka, *The Geometry of Art and Life,* Sheed and Ward, New York, 1946.

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[Continued from page 405.]

From (5) it can be shown by induction that

$$\alpha^n = \alpha F_n + F_{n-1} \quad \text{and} \quad \beta^n = \beta F_n + F_{n-1}, \tag{6}$$

where F_n and F_{n-1} are Fibonacci numbers defined for integral n by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}. \tag{7}$$

From (2) and (3) we may write

$$\exp \frac{x}{2} L_{2k+1} = \sum_{n=-\infty}^{\infty} U^n J_n(x) \tag{8}$$

From (6), we specialize

$$U^n = \alpha^{(2k+1)n} = \alpha F_{(2k+1)n} + F_{(2k+1)n-1}$$
$$U^n = \beta^{(2k+1)n} = \beta F_{(2k+1)n} + F_{(2k+1)n-1}.$$

Therefore (8) becomes

$$\exp\left(\frac{x}{2} L_{2k+1}\right) = \alpha \sum_{n=-\infty}^{\infty} F_{(2k+1)n} J_n(x) + \sum_{n=-\infty}^{\infty} F_{(2k+1)n-1} J_n(x) \tag{9a}$$

and

$$\exp\left(\frac{x}{2} L_{2k+1}\right) = \beta \sum_{n=-\infty}^{\infty} F_{(2k+1)n} J_n(x) + \sum_{n=-\infty}^{\infty} F_{(2k+1)n-1} J_n(x) \tag{9b}$$

[Continued on page 426.]

GOLDEN SEQUENCES OF MATRICES WITH APPLICATIONS TO FIBONACCI ALGEBRA

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1. INTRODUCTION

As is well known, the problem of finding a sequence of real numbers, $\{a_n\}$, $n = 0, 1, 2, \cdots$, which is both geometric $(a_{n+1} = ka_n,\ n = 0, 1, 2, \cdots)$ and "Fibonacci" $(a_{n+1} = a_n + a_{n-1},\ n = 1, 2, \cdots$, with $a_0 = 1)$ admits a solution—in fact, a unique solution. (Cf. [1]; for some extensions and geometric interpretations, see [2].) This "golden sequence" [1] is:

$$1,\ \phi,\ \phi^2, \cdots, \phi^n, \cdots,$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the "golden mean," and satisfies the Fibonacci equation

$$x^2 - x - 1 = 0.$$

In this paper, we pose an equivalent problem for a sequence of real, non-singular 2×2 matrices. Curiously, we will show that this problem admits an infinitude of solutions (i.e., that there exist infinitely many such "golden sequences"); that each such sequence is naturally related to each of the others (the relation given in familiar, algebraic terms of the generators of the sequences); and that these sequences are essentially the only such "golden sequences" of matrices (this, a simple consequence of a classical theorem of linear algebra). Finally, by applying two basic tools from the theory of matrices to the generators of these golden sequences, we deduce simply and naturally, some of the more familiar Fibonacci/Lucas identities [3] (including several which appear to be new); and the celebrated Binet formulas for the general terms of the Fibonacci and Lucas sequences.

2. THE DEFINING EQUATIONS

Let

$$A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where x, y, u, v are to be determined subject to the constraint that $xv - yu \neq 0$. Clearly, a necessary and sufficient condition for the geometric sequence

$$I, A, A^2, A^3, \cdots, A^n, \cdots$$

to be "Fibonacci" is that

(0) $$A^2 = A + I;$$

that is, that

(1) $$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \cdot \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} x & y \\ u & v \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(The necessity of (0) is clear; further, (0) implies that

$$A^{n+1} = A^n + A^{n-1}, \qquad n = 1, 2, 3, \cdots,$$

so long as A is not nilpotent. This will be the case since we're restricting A to be nonsingular.) A simple calculation shows that the matrix equation in (1) is equivalent to the following system of scalar equations:

(2) $$x^2 + yu = x + 1, \qquad xy + yv = y, \qquad xu + uv = u, \qquad yu + v^2 = v + 1,$$

which we write in the following more convenient form:

(3.1) $$x^2 - x - 1 + yu = 0$$

(3.2) $$(x + v - 1)y = 0$$

$$(3.3)\qquad (x+v-1)u = 0$$

$$(3.4)\qquad v^2 - v - 1 + yu = 0.$$

We now investigate possible solution sets.

Case 1. $y = 0$. Equations (3.1), (3.4) reduce to the Fibonacci equation, implying $x = \{\phi, \phi'\}$, $v = \{\phi, \phi'\}$, where $\phi = \tfrac{1}{2}(1+\sqrt{5})$, $\phi' = \tfrac{1}{2}(1-\sqrt{5})$.

(a) If $u = 0$, solution matrices of (1) are

$$\Phi_0 = \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} \phi' & 0 \\ 0 & \phi \end{pmatrix}, \qquad \Phi_3 = \begin{pmatrix} \phi' & 0 \\ 0 & \phi' \end{pmatrix}.$$

(The reader not familiar with the elementary identities involving ϕ and ϕ' is referred to either [1, 3]. The easily proved identities we will need in the sequel are

$$\phi + \phi' = 1, \quad \phi - \phi' = \sqrt{5}, \quad 2\phi - 1 = \sqrt{5}, \quad \phi \cdot \phi' = -1, \quad \phi^2 = \phi + 1,$$
$$\phi'^2 = \phi' + 1, \quad \phi^{n+1} = \phi^n + \phi^{n-1}, \quad \phi'^{n+1} = \phi'^n + \phi'^{n-1}, \quad n = 0, \pm 1, \pm 2, \cdots.)$$

Application of the appropriate identities shows that each of the sequences

$$\{\Phi_0^n\}, \quad \{\Phi_1^n\}, \quad \{\Phi_2^n\}, \quad \{\Phi_3^n\}$$

is golden (the second and fourth of these sequences are said to be trivial).

(b) If $u \neq 0$, equation (3.3) implies $x + v = 1$, and hence, that

$$\Phi_{0u} = \begin{pmatrix} \phi & 0 \\ u & \phi' \end{pmatrix}, \qquad \Phi_{2u} = \begin{pmatrix} \phi' & 0 \\ u & \phi \end{pmatrix}$$

are solution matrices of (1). The general term of the golden sequence generated by Φ_{0u} is easily shown to be

$$\Phi_{0u}^n = \begin{pmatrix} \phi^n & 0 \\ F_n u & \phi'^n \end{pmatrix},$$

where

$$F_1 = 1, \qquad F_2 = 1, \qquad F_3 = 2, \qquad F_4 = 3, \cdots, \qquad F_n = F_{n-1} + F_{n-2}, \cdots,$$

the Fibonacci sequence. (For elementary properties of the latter, cf. [1, 4].)

Case 2. $y \neq 0$.

(a) If $u = 0$, Eqs. (3.1) and (3.4) reduce to the Fibonacci equation, and Eq. (3.3) implies $x + v = 1$. The situation is similar to the one in Case 1(b). We will return, however, to the matrix Φ_{0y} in Section 4.

(b) Suppose $u \neq 0$. Equation (3.3) implies $x = 1 - v$ (consistent with Eq. (3.2)). Substitution for x in Eq. (3.1) results in

$$(1-v)^2 - (1-v) - 1 + yu = 0,$$

which after simplification reduces to $v^2 - v - 1 + yu = 0$, consistent with Eq. (3.4). Thus, the assumptions $y \neq 0$, $u \neq 0$ reduce the system (3.1) to (3.4) to the following equivalent system:

$$(4.1)\qquad v = \tfrac{1}{2}(1 \pm \sqrt{5 - 4yu})$$

$$(4.2)\qquad x = 1 - v,$$

where $y \neq 0$, $u \neq 0$, are otherwise arbitrary. It is in this form of the equations that we will systematically investigate various sets of solutions of (1) in the next section.

3. EXAMPLES OF GOLDEN SEQUENCES

Example 1: If we limit y, u to positive integer values in (4.1), then there is a unique pair which keeps the radicand positive: $y = u = 1$. In this case, we have two sets of solutions:

$$x = 0, \quad v = 1, \quad y = 1, \quad u = 1; \quad \text{and} \quad x = 1, \quad v = 0, \quad y = 1, \quad u = 1.$$

The latter set results in the so-called "Q-matrix" [3, 4]:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and the corresponding golden sequence

$$I, Q, Q^2, \cdots, Q^n, \cdots,$$

where [3, 4]

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Example 2: As we observed in the previous section, and as we may now corroborate from Eq. (4.1), the pair $y = u = 0$, results in the matrix Φ_0, and the corresponding golden sequence

$$I, \Phi_0, \Phi_0^2, \cdots, \Phi_0^n, \cdots,$$

where

$$\Phi_0^n = \begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

A natural question is whether or not Q and Φ_0 are related. A calculation shows that the characteristic equation for Q is

(5) $$\lambda^2 - \lambda - 1 = 0,$$

(the Fibonacci equation), the roots of which are ϕ and ϕ', the diagonal entries of Φ_0.

Thus, Eq. (5) is the characteristic equation for both Q and Φ_0, and by the Cayley-Hamilton theorem, each of these matrices satisfies this equation. A comparison of Eqs. (5) and (0) shows that we have in fact a characterization for all matrices which give rise to golden sequences:

Theorem 1. A necessary and sufficient condition for a matrix A to be a generator of a golden sequence is that its characteristic equation is the Fibonacci equation.

Since our hypotheses on the matrix A imply that the characteristic equation is, in fact, the minimal equation for A, we have

Corollary 1. Any two matrix generators of non-trivial golden sequences of matrices are similar.

Corollary 2. Q is similar to Φ_0; i.e., there exists a non-singular matrix T such that

$$Q = T\Phi_0 T^{-1},$$

where the columns of T are eigenvectors of Q corresponding respectively to the eigenvalues ϕ and ϕ'.

In what follows (see Section 4) we will require the matrix T. A straight-forward computation shows that

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix};$$

this is easily checked by observing that, in fact, $QT = T\Phi_0$.

From Corollary 1, we infer that

$$Q^n = T\Phi_0^n T^{-1},$$

and hence, that Q^n is similar to Φ_0^n. Hence,

$$\det(Q^n) = \det(\Phi_0^n), \qquad \text{trace}(Q^n) = \text{trace}(\Phi_0^n),$$

and we have our first pair of Fibonacci identities:

Corollary 3. (i) $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

(ii) $$F_{n+1} + F_{n-1} = \phi^n + \phi'^n,$$

$n = 1, 2, 3, \cdots$.

Remark 1: Since $L_n = F_{n+1} + F_{n-1}$ [3], where L_n is the general term of the Lucas sequence [3]

(6) $$1, 3, 4, 7, 11, \cdots,$$

it would appear that line (ii) in Corollary 3 establishes a proof for the Binet formula [3] : $L_n = \phi^n + \phi'^n$. However, the formula, $L_n = F_{n+1} + F_n$ is generally established from the principal Binet formula [3] :

$$F_n = (\phi^n + \phi'^n)/(\phi - \phi').$$

Although we have enough machinery at this point to establish the latter, the proof is not an immediate consequence of the similarity invariants, "trace" and "determinant" (which we would like to limit ourselves to in this section); thus, we defer this proof until Section 4. We do, however, establish the formula: $L_n = F_{n+1} + F_n$ in the next example, within our own framework.

Remark 2: Motivated from the general term, Q^n, in Example 1, where

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

it is natural to inquire as to whether the sequence with general term

$$P^n = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix},$$

is golden. However, since for $n = 1$, and setting $P = P^1$,

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix},$$

we see that P does not satisfy the Fibonacci equation; thus we conclude by Theorem 1 that P^n is not a golden sequence. Nevertheless, we will show in the next example that the Lucas numbers, (6), do, in fact, enter the picture in a natural way.

Example 3: Referring again to Eqs. (4.1), (4.2), we take $y = 1$, $u = 5/4$; then $v = ½$, $x = ½$, and we obtain the sequence generator

$$H = \begin{pmatrix} 1/2 & 1 \\ 5/4 & 1/2 \end{pmatrix},$$

and the corresponding golden sequence

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{5}{4} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 3 & 1 \\ \frac{5}{4} & \frac{1}{2}\cdot 3 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 4 & 2 \\ \frac{5}{4}\cdot 2 & \frac{1}{2}\cdot 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 7 & 3 \\ \frac{5}{4}\cdot 3 & \frac{1}{2}\cdot 7 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 11 & 5 \\ \frac{5}{4}\cdot 5 & \frac{1}{2}\cdot 11 \end{pmatrix}, \cdots,$$

where the general term is easily shown to be

$$H^n = \begin{pmatrix} \frac{1}{2}L_n & F_n \\ \frac{5}{4}F_n & \frac{1}{2}L_n \end{pmatrix}.$$

Similarity of Q^n (see Example 1) with H^n implies, by the invariance of trace, that

(7) $$L_n = F_{n+1} + F_{n-1},$$

and by determinant invariance, that

$$\frac{1}{4}L_n^2 - \frac{5}{4}F_n^2 = F_{n+1}F_{n-1} - F_n^2,$$

which after simplification becomes

(8) $$L_n^2 = 4F_{n+1}F_{n-1} + F_n^2.$$

Whereas, similarity of Φ^n with H^n implies

(9) $$L_n = \phi^n + \phi'^n \quad \text{(Binet)},$$

(10) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

Example 4: In (4.1), take $y = 1$, $u = -1$; then one set of solutions is $v = 2$, $x = -1$, and we obtain the matrix

$$F = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}.$$

The general term of the corresponding golden sequence is easily seen to be

$$\boldsymbol{F}^n = \begin{pmatrix} -F_{n-2} & F_n \\ -F_n & F_{n+2} \end{pmatrix}.$$

Similarity with Φ^n gives

(11) $$F_{n+2} - F_{n-2} = \phi^n + \phi'^n$$

and

(12) $$F_n^2 - F_{n+2}F_{n-2} = (-1)^n .$$

NOTE: In what follows, we shall only use those similarity results which produce identities not already established.

Similarity with Q^n gives

(13) $$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2},$$

and

(14) $$F_{n+1}F_{n-1} + F_{n+2}F_{n-2} = 2F_n^2 .$$

Similarity with H^n gives

(15) $$L_n = F_{n+2} - F_{n-2}$$

and

$$\frac{1}{4}L_n^2 - \frac{5}{4}F_n^2 = F_n^2 - F_{n+2}F_{n-2},$$

which after simplification becomes

(16) $$L_n^2 = 9F_n^2 - 4F_{n+2}F_{n-2}.$$

Example 5: By taking $y = 1$, $u = -5$ in (4.1), we obtain $v = -2$, $x = 3$, and the generator

$$\boldsymbol{L} = \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix} .$$

The corresponding golden sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ -5\cdot 2 & -3 \end{pmatrix}, \begin{pmatrix} 11 & 3 \\ -5\cdot 3 & -4 \end{pmatrix}, \cdots,$$

with general term

$$\boldsymbol{L}^n = \begin{pmatrix} L_{n+1} & F_n \\ -5F_n & -L_{n-1} \end{pmatrix} .$$

Similarity with Q^n implies

(17) $$L_{n+1}L_{n-1} + F_{n+1}F_{n-1} = 6F_n^2 .$$

Similarity with Φ^n implies

(18) $$5F_n^2 - L_{n+1}L_{n-1} = (-1)^n .$$

Similarity with H^n gives

(19) $$L_n^2 + 4L_{n+1}L_{n-1} = 25F_n^2 .$$

and similarity with $\boldsymbol{F}^n$ gives

(20) $$L_{n+1} - L_{n-1} = F_{n+2} - F_{n-2},$$

and

(21) $$L_{n+1}L_{n-1} - F_{n+2}F_{n-2} = 4F_n^2 .$$

Remark 3: Although there appear to be infinitely many more golden sequences we could investigate, subject only to the constraining equations (4.1) and (4.2), and thus, a limitless supply of Fibonacci identities to discover (or, rediscover) via the similarity invariants, "trace" and "determinant," we switch our direction at this point.

In Section 4, we offer two final examples of generators of golden sequences, and compute their eigenvectors. With this new tool we will then establish Binet's formula for F_n in terms of ϕ, ϕ' and their powers, and the formulas [3] for ϕ and ϕ' in terms of F_n and L_n.

4. PROOFS OF SOME CLASSICAL FORMULAS

In (4.1) take $u = 0$, $y \neq 0$, but for the time being arbitrary. Then $v = \phi$, $x = \phi'$, and we have the matrix (cf. Section 1)

$$\Phi_{oy} = \begin{pmatrix} \phi & y \\ 0 & \phi' \end{pmatrix}.$$

Setting $\Phi_y = \Phi_{oy}$, one easily checks that we generate the golden sequence

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \phi & y \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} \phi^2 & y \\ 0 & \phi'^2 \end{pmatrix}, \begin{pmatrix} \phi^3 & 2y \\ 0 & \phi'^3 \end{pmatrix}, \cdots,$$

where the general term is easily seen to be

$$\Phi_y^n = \begin{pmatrix} \phi^n & F_n y \\ 0 & \phi'^n \end{pmatrix},$$

The eigenvectors, corresponding to the eigenvalue ϕ, are computed to be $\begin{pmatrix} a \\ 0 \end{pmatrix}$, $a \neq 0$; we single out the eigenvector corresponding to $a = 1$; while the eigenvectors corresponding to ϕ' are of the form

$$\begin{pmatrix} a \\ \frac{1}{y}(\phi' - \phi)a \end{pmatrix}, \quad a \neq 0.$$

Since $\phi' - \phi = -\sqrt{5}$ (see Section 2, or [3]), we take $y = \sqrt{5}$ (so that $\Phi_y = \Phi_{\sqrt{5}}$), and $a = 1$. Thus we have the two eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

corresponding to the eigenvalues ϕ and ϕ', respectively. Set

$$S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then by Corollary 1,

$$\Phi_{\sqrt{5}} = S\Phi_0 S^{-1},$$

which implies that

$$\Phi^n_{\sqrt{5}} = S\Phi^n_0 S^{-1},$$

and hence, that

(22) $$S\Phi^n_{\sqrt{5}} = S\Phi^n_0.$$

We write out Eq. (22):

$$\begin{pmatrix} \phi^n & F_n\sqrt{5} \\ 0 & \phi'^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

Multiplying out gives

$$\begin{pmatrix} \phi^n & \phi^n - F_n\sqrt{5} \\ 0 & -\phi'^n \end{pmatrix} = \begin{pmatrix} \phi^n & \phi'^n \\ 0 & -\phi'^n \end{pmatrix},$$

which implies that

$$\phi^n - F_n\sqrt{5} = \phi'^n;$$

or

(*) $$F_n = \frac{\phi^n - \phi'^n}{\phi - \phi'}, \quad \text{(Binet)}$$

For our final example, we will permit our generator matrix to be complex. In (4.1), take $y = ½$, $u = 3$; then take $v = (1 + i)/2$, so that $x = (1 - i)/2$, and we obtain the matrix

$$C = \begin{pmatrix} (1-i)/2 & 1/2 \\ 3 & (1+i)/2 \end{pmatrix}.$$

The corresponding golden sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \tfrac12(1-i) & \tfrac12 \\ 3 & \tfrac12(1+i) \end{pmatrix}, \begin{pmatrix} \tfrac12(3-i) & \tfrac12 \\ 3 & \tfrac12(3+i) \end{pmatrix}, \begin{pmatrix} \tfrac12(4-2i) & \tfrac12\cdot 2 \\ 3\cdot 2 & \tfrac12(4+2i) \end{pmatrix}, \cdots,$$

with general term

$$C^n = \begin{pmatrix} \tfrac12(L_n - F_n i) & \tfrac12 F_n \\ 3F_n & \tfrac12(L_n + F_n i) \end{pmatrix}.$$

Proceeding as in the previous example, we take as eigenvector corresponding to the eigenvalue ϕ, the vector

$$\begin{pmatrix} \tfrac13[\phi - \tfrac12(1+i)] \\ 1 \end{pmatrix};$$

and corresponding to ϕ', we take the eigenvector

$$\begin{pmatrix} \tfrac13[\phi' - \tfrac12(1+i)] \\ 1 \end{pmatrix}.$$

Setting

$$B = \begin{pmatrix} \tfrac13[\phi - \tfrac12(1+i)] & \tfrac13[\phi' - \tfrac12(1+i)] \\ 1 & 1 \end{pmatrix},$$

we have by Corollary 1 that

$$C^n B = B\Phi_0^n. \tag{23}$$

Performing the indicated multiplication in (23) results in the matrix equation

$$\begin{pmatrix} \tfrac16[L_n - F_n i][\phi - \tfrac12(1+i)] + \tfrac12 F_n & \tfrac16[L_n - F_n i][\phi' - \tfrac12(1+i)] + \tfrac12 F_n \\ F_n[\phi - \tfrac12(1+i)] + \tfrac12[L_n + F_n i] & F_n[\phi' - \tfrac12(1+i)] + \tfrac12[L_n + F_n i] \end{pmatrix}$$
$$= \begin{pmatrix} \phi^n/3[\phi - \tfrac12(1+i)] & \phi'^n/3[\phi' - \tfrac12(1+i)] \\ \phi^n & \phi'^n \end{pmatrix}.$$

(a) Equating the corresponding entries in the second row, first column, and simplifying gives

$$L_n = 2\phi^n + (1-2\phi)F_n.$$

Solving for ϕ^n gives

$$\phi^n = \tfrac12(L_n + \sqrt5 F_n). \tag{24}$$

(b) Equating the corresponding terms in the first row, second column, and noting that these are identical to those obtained in (a) except that ϕ is replaced by ϕ', we have

$$L_n = 2\phi'^n + (1-2\phi')F_n;$$

or, solving for ϕ'^n, that

$$\phi'^n = \tfrac12(L_n - \sqrt5 F_n). \tag{25}$$

Remark 4: Equating the two remaining pairs of corresponding entries in the above matrix equation results in lines (24) and (25).

Remark 5: We chose the matrix

$$\Phi_{\sqrt5} = \begin{pmatrix} \phi & 5 \\ 0 & \phi' \end{pmatrix}$$

to establish the principal Binet formula (line (*)) because of the simplicity of the proof. It should be noted, however, that a proof within the framework of the Q-matrix [4] is also possible. Since the machinery has already been set up in Example 2, and because of the historical importance of this matrix, we give the proof. We have already established (similarity) that

$$Q^n T = T\Phi_0^n;$$

i.e., that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

Multiplying out gives

$$\begin{pmatrix} \phi F_{n+1} + F_n & F_{n+1} - \phi F_n \\ \phi F_n + F_{n-1} & F_n - \phi F_{n-1} \end{pmatrix} = \begin{pmatrix} \phi^{n+1} & \phi'^n \\ \phi^n & \phi'^{(n-1)} \end{pmatrix}.$$

Equating corresponding terms results in the following equivalent system of equations:

$$\phi F_{n+1} + F_n = \phi^{n+1}$$

$$F_{n+1} - \phi F_n = \phi'^n$$

$$\phi F_n + F_{n-1} = \phi^n$$

$$F_n - \phi F_{n-1} = \phi'^{(n-1)}.$$

Solving the second equation for F_{n+1} and substituting this into the first equation, gives

$$\phi(\phi'^n + \phi F_n) + F_n = \phi^{n+1}.$$

Multiplying through by $-\phi'$ gives

$$\phi'^n + \phi F_n - \phi' F_n = \phi^n.$$

Finally, solving for F_n gives the desired result:

$$F_n = \frac{\phi^n - \phi'^n}{\phi - \phi'}.$$

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[Continued from page 418.]

From (9a) and (9b), we obtain

(10a) $$\sum_{n=-\infty}^{\infty} F_{(2k+1)n} J_n(x) = 0$$

and

(10b) $$\sum_{n=-\infty}^{\infty} F_{(2k+1)n-1} J_n(x) = \exp\left(\frac{x}{2} L_{2k+1}\right).$$

Equations (10a) and (10b) can be combined in the following equation, as may be shown by induction

(11) $$\sum_{n=-\infty}^{\infty} F_{(2k+1)n+m} J_n(x) = F_m \exp\left(\frac{x}{2} L_{2k+1}\right).$$

With $k = 0$ and $m = 1$, (11) becomes

$$\sum_{n=-\infty}^{\infty} F_{n+1} J_n(x) = \exp \frac{x}{2}.$$

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SOME SUMS OF MULTINOMIAL COEFFICIENTS

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1. Recent interest in some lacunary sums of binomial coefficients (see for example [2], [3]) suggests that it may be of interest to consider some simple sums of multinomial coefficients.

Put

$$(i, j, k) = \frac{(i+j+k)!}{i!\,j!\,k!} ,$$

so that

$$(x+y+z)^n = \sum_{i+j+k=n} (i, j, k) x^i y^j z^k . \tag{1.1}$$

Let $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and define

$$S_{000} = S_{000}(n) = \sum_{i,j,k \text{ even}} (i, j, k),$$

$$S_{100} = S_{100}(n) = \sum_{\substack{i \text{ odd} \\ j,k \text{ even}}} (i, j, k), \text{ etc.,}$$

where in each case the summation is over all non-negative i, j, k such that $i+j+k = n$. Since

$$S_{100} = S_{010} = S_{001}, S_{011} = S_{101} = S_{110} ,$$

it is evident from (1.1) that

$$S_{000} + S_{100}(\epsilon_1 + \epsilon_2 + \epsilon_3) + S_{011}(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) + S_{111}\epsilon_1\epsilon_2\epsilon^3 = (\epsilon_1 + \epsilon_2 + \epsilon_3)^n . \tag{1.2}$$

Specializing the ϵ_i we get

$$\begin{aligned}
(1, 1, 1) &: S_{000} + 3S_{100} + 3S_{110} + S_{111} = 3^n \\
(-1, 1, 1) &= S_{000} + S_{100} - S_{110} - S_{111} = 1 \\
(1, -1, -1) &: S_{000} - S_{100} - S_{110} + S_{111} = (-1)^n \\
(-1, -1, -1) &: S_{000} - 3S_{100} + 3S_{110} - S_{111} = (-3)^3 .
\end{aligned}$$

Solving for the S_{ijk}, we get

$$\left\{\begin{aligned}
8S_{000} &= 3^n + 3 + 3(-1)^n + (-3)^n \\
8S_{100} &= 3^n + 1 - (-1)^n - (-3)^n \\
8S_{110} &= 3^n - 1 - (-1)^n + (-3)^n \\
8S_{111} &= 3^n - 3 + 3(-1)^n - (-3)^n .
\end{aligned}\right. \tag{1.3}$$

Tabulating even and odd values of n separately, (1.3) reduces to

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$$\text{(1.4)}\qquad \begin{cases} S_{000}(2n) = \tfrac14(3^{2n}+3) \\ S_{100}(2n) = 0 \\ S_{110}(2n) = \tfrac14(3^{2n}-1) \\ S_{111}(2n) = 0 \end{cases}$$

$$\text{(1.5)}\qquad \begin{cases} S_{000}(2n+1) = 0 \\ S_{100}(2n+1) = \tfrac14(3^{2n+1}+1) \\ S_{110}(2n+1) = 0 \\ S_{111}(2n+1) = \tfrac14(3^{2n+1}-3). \end{cases}$$

It follows from (1.4) and (1.5) that

$$\text{(1.6)}\qquad S_{000}(2n) = S_{110}(2n)+1, \qquad S_{100}(2n+1) = S_{111}(2n+1)+1.$$

We also have the generating functions

$$\text{(1.7)}\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} S_{000}(n)x^n = \tfrac14\left(\frac{1}{1-9x^2}+\frac{3}{1-x^2}\right) = \frac{1-7x^2}{(1-x^2)(1-9x^2)} \\ \displaystyle\sum_{n=0}^{\infty} S_{100}(n)x^n = \tfrac14\left(\frac{3x}{1-9x^2}+\frac{x}{1-x^2}\right) = \frac{1-3x^2}{(1-x^2)(1-9x^2)} \\ \displaystyle\sum_{n=0}^{\infty} S_{110}(n)x^n = \tfrac14\left(\frac{1}{1-9x^2}-\frac{1}{1-x^2}\right) = \frac{2x^2}{(1-x^2)(1-9x^2)} \\ \displaystyle\sum_{n=0}^{\infty} S_{111}(n)x^n = \tfrac14\left(\frac{3x}{1-9x^2}-\frac{3x}{1-x^2}\right) = \frac{6x^3}{(1-x^2)(1-9x^2)} \end{cases} .$$

2. Let $m > 1$ and define

$$\text{(2.1)}\qquad S_{ijk} = S_{ijk}^{(m)}(n) = \sum_{r+s+t=n} (r,s,t),$$

where the summation is restricted to non-negative r, s, t such that

$$r \equiv i, \quad s \equiv j, \quad t \equiv k \pmod m.$$

We may also assume that

$$0 \leqslant i < m, \quad 0 \leqslant j < m, \quad 0 \leqslant k < m.$$

Clearly S_{ijk} is symmetric in the three indices i, j, k. Also it is evident from the definition that

$$\text{(2.2)}\qquad S_{ijk}^{(m)}(n) = 0 \quad (n \not\equiv i+j+k \pmod m).$$

Hence in what follows it will suffice to assume that $n \equiv i+j+k \pmod m$.

Let ζ denote a fixed primitive m^{th} root of 1. Then it is clear from (1.1) and (2.1), that for arbitrary integers, $a, b, c,$

$$\text{(2.3)}\qquad (\zeta^a+\zeta^b+\zeta^c) = \sum_{r+s+t=n} \zeta^{ra+sb+tc}(r,s,t) = \sum_{i,j,k=0}^{m-1} \zeta^{ia+jb+kc} S_{ijk}^{(m)}(n).$$

Since

$$\sum_{a=0}^{m-1} \zeta^{ra} = \begin{cases} m & (r=0) \\ 0 & (0<r<m), \end{cases}$$

it follows from (2.3) that

$$m^3 S_{ijk}^{(m)}(n) = \sum_{a,b,c=0}^{m-1} (\zeta^a + \zeta^b + \zeta^c)^n \zeta^{-ai-bj-ck}. \tag{2.4}$$

While this theoretically evaluates $S_{ijk}^{(m)}(n)$, it is not really satisfactory. For $m = 3$ more explicit results are obtainable without a great deal of computation.

By (2.4) we have

$$27S_{000}^{(3)}(n) = \sum_{a,b,c=0}^{2} (\omega^a + \omega^b + \omega^c)^3,$$

where $\omega^2 + \omega + 1 = 0$. This reduces to

$$\begin{aligned} 27S_{000}^{(3)}(n) = {} & 3^n + 3(\omega + 2)^n + 3(\omega^2 + 2)^n + 3(2\omega + 1)^n + 3(2\omega^2 + 1)^n \\ & + 3(2\omega^2 + \omega)^n + 3(2\omega + \omega^2)^n + 6(\omega^2 + \omega + 1)^n + (3\omega)^n + (3\omega^2)^n. \end{aligned}$$

By (2.2),

$$S_{000}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod 3).$$

For n a multiple of 3 we get, replacing n by $3n$,

$$27S_{000}^{(3)}(3n) = 3^{3n+1} + 9(2\omega + 1)^{3n} + 9(2\omega^2 + 1)^{3n} \quad (n > 0).$$

This reduces to

$$\begin{cases} S_{000}^{(3)}(6n) = 3^{6n-2} + 2(-1)^n 3^{3n-1} & (n > 0) \\ S_{000}^{(3)}(6n+3) = 3^{6n+1} & (n \geq 0). \end{cases} \tag{2.5}$$

Check.

$$S_{000}^{(3)}(6) = \frac{3\cdot 6!}{6!\,0!\,0!} + \frac{3\cdot 6!}{3!\,3!\,0!} = 3 + 3\cdot 60 = 63 = 3^4 - 2\cdot 3^2, \quad S_{000}^{(3)}(3) = \frac{3\cdot 3!}{3!\,0!\,0!} = 3,$$

$$S_{000}^{(3)}(9) = \frac{9!}{3!\,3!\,3!} + \frac{6\cdot 9!}{6!\,3!\,0!} + \frac{3\cdot 9!}{9!\,0!\,0!} = 5\cdot 6\cdot 7\cdot 8 + 7\cdot 8\cdot 9 + 3 = 3\cdot 729 = 3^7.$$

Similarly we have

$$\begin{aligned} 27S_{111}^{(3)}(n) = {} & \sum_{a,b,c=0}^{2} (\omega^a + \omega^b + \omega^c)^n \omega^{-a-b-c} = 3^n + 3(\omega + 2)^n \omega^{-1} + 3(\omega^2 + 2)^n \omega^{-2} \\ & + 3(2\omega + 1)^n \omega^{-2} + 3(2\omega^2 + 1)^n \omega^{-1} + 3(2\omega^2 + \omega)^n \omega^{-2} + 3(2\omega + \omega^2)\omega^{-1} \\ & + 6(\omega^2 + \omega + 1)^n + (3\omega)^n + (3\omega^2)^n. \end{aligned}$$

As in the previous case,

$$S_{111}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod 3),$$

while

$$\begin{aligned} 27S_{111}^{(3)}(3n) = {} & 3\cdot 3^{3n} + 3(2\omega^2 + 1)^{3n}\omega^{-1} + 3(2\omega + 1)^{3n}\omega^{-2} + 3(2\omega + 1)^{3n}\omega^{-2} + 3(2\omega^3 + 1)^{3n}\omega^{-1} \\ & + 3(2\omega + 1)^{3n}\omega^{-2} + 3(2\omega^2 + 1)^{3n}\omega^{-1} + 6(\omega^2 + \omega + 1)^{3n} \\ = {} & 3^{3n+1} + 9(\sqrt{-3})^{3n}\omega^{-2} + 9(-\sqrt{-3})^{3n}\omega^{-1} + 6(\omega^2 + \omega + 1)^{3n}. \end{aligned}$$

It follows that

$$\begin{cases} S_{111}^{(3)}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n > 0) \\ S_{111}^{(3)}(6n+3) = 3^{6n+1} + (-1)^n 3^{3n+1} & (n \geq 0). \end{cases} \tag{2.6}$$

Check.

$$S_{111}^{(3)}(6) = \frac{3\cdot 6!}{4!\,1!\,1!} = 3\cdot 6\cdot 5 = 90 = 3^4 + 3^2, \qquad S_{111}^{(3)}(3) = \frac{3!}{1!\,1!\,1!} = 6 = 3+3,$$

$$S_{111}^{(3)}(9) = \frac{3\cdot 9!}{7!\,1!\,1!} + \frac{3\cdot 9!}{4!\,4!\,1!} = 3\cdot 8\cdot 9 + 3\cdot 9\cdot 70 = 3^4\cdot 26 = 3^7 - 3^4.$$

We find also that

$$\text{(2.7)} \qquad \begin{cases} S_{222}^{(3)}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n > 0) \\ S_{222}^{(3)}(6n+3) = 3^{6n+1} - (-1)^n 3^{3n+1} & (n \geq 0). \end{cases}$$

Check.

$$S_{222}^{(3)}(6) = \frac{6!}{2!\,2!\,2!} = 90 = 3^4 + 3^2, \qquad S_{222}^{(3)}(9) = \frac{3\cdot 9!}{5!\,2!\,2!} = 3^4\cdot 28 = 3^7 + 3^4.$$

Note that it follows from (2.6) and (2.7) that

$$\text{(2.8)} \qquad S_{111}^{(3)}(6n) = S_{222}^{(3)}(6n)$$

and from (2.5), (2.6), (2.7),

$$\text{(2.9)} \qquad S_{111}^{(3)}(6n+3) + S_{222}^{(3)}(6n+3) = 2S_{000}^{(3)}(6n+3).$$

3. Since

$$(r, s, t) = (r-1, s, t) + (r, s-1, t) + (r, s, t-1),$$

it follows from (2.1) that

$$\text{(3.1)} \qquad S_{i,j,k}^{(m)}(n) = S_{i-1,j,k}^{(m)}(n-1) + S_{i,j-1,k}^{(m)}(n-1) + S_{i,j,k-1}^{(m)}(n-1),$$

where

$$S_{i,j,k}^{(m)}(n) = S_{i',j',k'}^{(m)}(n)$$

when

$$i \equiv i', \quad j \equiv j', \quad k \equiv k' \pmod m.$$

In particular

$$\text{(3.2)} \qquad S_{i,i,i}^{(m)}(n) = 3S_{i,i,i-1}^{(m)}(n-1).$$

For example, when $m = 2$, we have

$$\begin{cases} S_{000}^{(2)}(2n) = 3S_{100}^{(2)}(2n-1) \\ S_{111}^{(2)}(2n+1) = 3S_{110}^{(2)}(2n) \\ S_{100}^{(2)}(2n+1) = S_{000}^{(2)}(2n) + 2S_{110}^{(2)}(2n) \\ S_{110}^{(2)}(2n) = S_{111}^{(2)}(2n-1) + 2S_{100}^{(2)}(2n-1) \end{cases}$$

in agreement with previous results.

The case $m = 3$ is more interesting as well as more involved. We have, to begin with,

$$\begin{cases} S_{000}^{(3)}(3n) = 3S_{200}^{(3)}(3n-1) \\ S_{111}^{(3)}(3n) = 3S_{110}^{(3)}(3n-1) \\ S_{222}^{(3)}(3n) = 3S_{221}^{(3)}(3n-1). \end{cases}$$

It therefore follows from (2.5), (2.6), and (2.7) that

$$\text{(3.3)} \qquad \begin{cases} S_{200}^{(3)}(6n+5) = 3^{6n+3} - 2(-1)^n 3^{3n+1} \\ S_{200}^{(3)}(6n+2) = 3^{6n} \end{cases} \qquad (n \geq 0)$$

(3.4)
$$\begin{cases} S^{(3)}_{110}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S^{(3)}_{110}(6n+2) = 3^{6n} + (-1)^n 3^{3n} \end{cases} \qquad (n \geqslant 0)$$

(3.5)
$$\begin{cases} S^{(3)}_{221}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S^{(3)}_{221}(6n+2) = 3^{6n} - (-1)^n 3^{3n} \end{cases} \qquad (n \geqslant 0).$$

Check.

$$S^{(3)}_{200}(5) = \frac{5!}{5!\,0!\,0!} + \frac{2\cdot 5!}{2!\,3!\,0!} = 1 + 20 = 21 = 3^3 - 2\cdot 3,$$

$$S^{(3)}_{200}(8) = \frac{8!}{8!\,0!\,0!} + \frac{2\cdot 8!}{5!\,3!\,0!} + \frac{2\cdot 8!}{2!\,6!\,0!} + \frac{8!}{2!\,3!\,3!}$$
$$= 1 + 112 + 56 + 560 = 729 = 3^6;$$

$$S^{(3)}_{110}(5) = \frac{5!}{1!\,1!\,3!} + \frac{2\cdot 5!}{4!\,1!\,0!} = 20 + 10 = 30 = 3^3 + 3,$$

$$S^{(3)}_{110}(8) = \frac{2\cdot 8!}{7!\,1!\,0!} + \frac{8!}{4!\,4!\,0!} + \frac{2\cdot 8!}{1!\,4!\,3!} + \frac{8!}{1!\,1!\,6!} = 16 + 70 + 560 + 56 = 702 = 3^6 - 3^3;$$

$$S^{(3)}_{221}(5) = \frac{5!}{2!\,2!\,1!} = 30 = 3^3 + 3,$$

$$S^{(3)}_{221}(8) = \frac{2\cdot 8!}{5!\,2!\,1!} + \frac{8!}{2!\,2!\,4!} = 27\cdot 28 = 3^6 + 3^3.$$

In the next place, it follows from

$$S^{(3)}_{210}(3n) = S^{(3)}_{110}(3n-1) + S^{(3)}_{200}(3n-1) + S^{(3)}_{221}(3n-1)$$

that

$$S^{(3)}_{210}(6n) = S^{(3)}_{110}(6n-1) + S^{(3)}_{200}(6n-1) + S^{(3)}_{221}(6n-1)$$
$$= (3^{6n-3} - (-1)^n 3^{3n-2}) + (3^{6n-3} + 2(-1)^n 3^{3n-2}) + (3^{6n-3} - (-1)^n 3^{3n-2}) = 3^{6n-2} \quad (n > 0).$$

$$S^{(3)}_{210}(6n+3) = S^{(3)}_{110}(6n+2) + S^{(3)}_{200}(6n+2) + S^{(3)}_{221}(6n+2)$$
$$= (3^{6n} + (-1)^n 3^{3n}) + 3^{6n} + (3^{6n} - (-1)^n 3^{3n}) = 3^{6n+1} \qquad (n \geqslant 0)$$

that is,

(3.6)
$$\begin{cases} S^{(3)}_{210}(6n) = 3^{6n-2} & (n > 0) \\ S^{(3)}_{210}(6n+3) = 3^{6n+1} & (n \geqslant 0). \end{cases}$$

Check.

$$S^{(3)}_{210}(6) = \frac{6!}{2!\,1!\,3!} + \frac{6!}{5!\,1!\,0!} + \frac{6!}{2!\,4!\,0!} = 60 + 6 + 15 = 3^4,$$

$$S^{(3)}_{210}(3) = \frac{3!}{2!\,1!\,0!} = 3,$$

$$S^{(3)}_{210}(9) = \frac{9!}{2!\,1!\,6!} + \frac{9!}{5!\,1!\,3!} + \frac{9!}{2!\,4!\,3!} + \frac{9!}{8!\,1!\,0!} + \frac{9!}{5!\,4!\,0!} + \frac{9!}{2!\,7!\,0!}$$
$$= 9\cdot 4\cdot 7 + 9\cdot 8\cdot 7 + 9\cdot 7\cdot 20 + 9 + 9\cdot 7\cdot 2 + 9\cdot 4 = 9\cdot 243 = 3^7.$$

Next it follows in like manner from

$$\begin{cases} S^{(3)}_{211}(3n+1) = S^{(3)}_{111}(3n) + 2S^{(3)}_{210}(3n) \\ S^{(3)}_{220}(3n+1) = S^{(3)}_{222}(3n) + 2S^{(3)}_{210}(3n) \\ S^{(3)}_{100}(3n+1) = S^{(3)}_{000}(3n) + 2S^{(3)}_{210}(3n) \end{cases}$$

that

(3.7) $$\begin{cases} S^{(3)}_{211}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S^{(3)}_{211}(6n+4) = 3^{6n+2} + (-1)^n 3^{n+1} \end{cases} \quad (n \geqslant 0)$$

(3.8) $$\begin{cases} S^{(3)}_{220}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S^{(3)}_{220}(6n+4) = 3^{6n+2} - (-1)^n 3^{3n+1} \end{cases} \quad (n \geqslant 0)$$

(3.9) $$\begin{cases} S^{(3)}_{100}(6n+1) = 3^{6n-1} + 2(-1)^n 3^{3n-1} \\ S^{(3)}_{100}(6n+4) = 3^{6n+2} \end{cases} \quad (n \geqslant 0).$$

Check.

$$S^{(3)}_{211}(7) = \frac{7!}{5!\,1!\,1!} + \frac{2!\,7!}{2!\,4!\,1!} = 42 + 210 = 252 = 3^5 + 3^2, \quad S^3_{211}(4) = \frac{4!}{2!\,1!\,1!} = 12 = 3^2 + 3,$$

$$S^{(3)}_{211}(10) = \frac{10!}{8!\,1!\,1!} + \frac{2\cdot 10!}{5!\,4!\,1!} + \frac{10!}{2!\,4!\,4!} + \frac{2\cdot 10!}{2!\,7!\,1!}$$
$$= 9\cdot 10 + 9\cdot 280 + 9\cdot 350 + 9\cdot 80 = 3^4\cdot 80 = 3^8 - 3^4;$$

$$S^{(3)}_{220}(7) = \frac{7!}{2!\,2!\,3!} + \frac{2\cdot 7!}{5!\,2!\,0!} = 3^2\cdot 28 = 3^5 + 3^2,$$

$$S^{(3)}_{220}(4) = \frac{4!}{2!\,2!\,0!} = 6 = 3^2 - 3,$$

$$S^{(3)}_{220}(10) = \frac{10!}{2!\,2!\,6!} + \frac{2\cdot 10!}{5!\,2!\,3!} + \frac{10!}{5!\,5!\,0!} + \frac{2\cdot 10!}{8!\,2!\,0!} = 9^2\cdot 82 = 3^8 + 3^4.$$

$$S^{(3)}_{100}(7) = \frac{7!}{7!\,0!\,0!} + \frac{2\cdot 7!}{4!\,3!\,0!} + \frac{7!}{1!\,3!\,3!} + \frac{2\cdot 7!}{1!\,6!\,0!} = 1 + 70 + 140 + 14 = 3^2\cdot 25 = 3^5 - 2\cdot 3^2,$$

$$S^{(3)}_{100}(4) = \frac{4!}{4!\,0!\,0!} + \frac{2\cdot 4!}{1!\,3!\,0!} = 1 + 8 = 3^2,$$

$$S^{(3)}_{100}(10) = \frac{10!}{10!\,0!\,0!} + \frac{2\cdot 10!}{7!\,3!\,0!} + \frac{10!}{4!\,3!\,3!} + \frac{2\cdot 10!}{4!\,6!\,0!} + \frac{2\cdot 10!}{1!\,9!\,0!} + \frac{2\cdot 10!}{1!\,6!\,3!}$$
$$= 1 + 240 + 4200 + 420 + 20 + 1680 = 3^8.$$

This completes the evaluation of the ten functions $S^{(3)}_{ijk}(n)$.

4. The five functions $S^{(2)}_{ijk\ell}(n)$ can be evaluated without much computation. To begin with, we have

$$2^4 S^{(2)}_{0000}(2n) = (1+1+1+1)^{2n} + 4(1+1+1-1)^{2n} + 6(1+1-1-1)^{2n}$$
$$+ 4(1-1-1-1)^{2n} + (-1-1-1-1)^{2n},$$

which reduces to

(4.1) $$S^{(2)}_{000}(2n) = 2^{4n-3} + 2^{2n-1} \qquad (n > 0).$$

Since

$$S^{(2)}_{0000}(2n) = 4S^{(2)}_{1000}(2n-1),$$

we get

(4.2) $$S^{(2)}_{1000}(2n+1) = 2^{4n-1} + 2^{2n-1} \qquad (n \geqslant 0).$$

Next, since

$$S^{(2)}_{1000}(2n+1) = S^{(2)}_{0000}(2n) + 3S^{(2)}_{1100}(2n),$$

it follows that

(4.3) $$S^{(2)}_{1100}(2n) = 2^{4n-3} \qquad (n > 0).$$

Similarly, from

$$S^{(2)}_{1100}(2n) = 2S^{(2)}_{1000}(2n-1) + 2S^{(2)}_{1110}(2n-1)$$

we get

(4.4) $$S^{(2)}_{1110}(2n+1) = 2^{4n-1} - 2^{2n-1} \qquad (n \geqslant 0).$$

Finally, it follows from

$$S^{(2)}_{1110}(2n+1) = S^{(2)}_{1111}(2n) + 3S^{(2)}_{1100}(2n)$$

that

(4.5) $$S^{(2)}_{1111}(2n) = 2^{4n-3} - 2^{2n-1} \qquad (n \geq 1).$$

For example

$$S^{(2)}_{1111}(6) = \frac{4\cdot 6!}{3!\,1!\,1!\,1!} = 480 = 2^9 - 2^5.$$

Note that it follows from the above results that

(4.6) $$S^{(2)}_{0000}(2n) + S^{(2)}_{1111}(2n) = 2S^{(2)}_{1100}(2n)$$

and

(4.7) $$S^{(2)}_{1000}(2n+1) + S^{(2)}_{1110}(2n+1) = 8S^{(2)}_{1100}(2n).$$

5. The results of §4 suggest that it would be of interest to evaluate

(5.1) $$f_{j,k}(n) = S^{(2)}_{\underbrace{1\cdots 1}_{j}\underbrace{0\cdots 0}_{k}}(n),$$

where j, k are arbitrary non-negative integers and the right-hand side of (5.1) has the obvious meaning. Clearly

(5.2) $$f_{j,k}(n) = 0 \qquad (n \not\equiv j \pmod 2).$$

To begin with, we have

$$\begin{aligned} 2^k f_{0,k}(n) &= (1+1+\cdots+1)^n + \binom{k}{1}(1+\cdots+1-1)^n \\ &\quad + \binom{k}{2}(1+\cdots+1-1-1)^n + \cdots + \binom{k}{k}(-1-1-\cdots-1)^n \\ &= k^n + \binom{k}{1}(k-2)^n + \binom{k}{2}(k-4)^n + \cdots + \binom{k}{k}(-k)^n. \end{aligned}$$

Thus

(5.3) $$f_{0,k}(n) = 2^{-k}\sum_{j=0}^{k}\binom{k}{j}(k-2j)^n.$$

Since

(5.4) $$f_{0,k}(n) = kf_{1,k-1}(n-1),$$

it follows at once that $S_{1,k-1}$ can be evaluated explicitly by means of (5.3). Next, since

$$f_{1,k-1}(n-1) = f_{0,k}(n-2) + (k-1)f_{2,k-2}(n-2),$$

we get

(5.5) $$k(k-1)f_{2,k-2}(n-2) = f_{0,k}(n) = kf_{0,k}(n-2).$$

Similarly it follows from

$$f_{2,k-2}(n-2) = 2f_{1,k-1}(n-3) + (k-2)f_{3,k-3}(n-3)$$

that

(5.6) $$k(k-1)(k-2)f_{3,k-3}(n-3) = f_{0,k}(n) - (3k-2)f_{0,k}(n-2).$$

We also find that

(5.7) $$k(k-1)(k-2)(k-3)f_{4,k-4}(n-4) = f_{0,k}(n) - 2(3k-4)f_{0,k}(n-2) + 3k(k-2)f_{0,k}(n-4),$$

(5.8) $$\begin{aligned} k(k-1)(k-2)(k-4)f_{5,k-5}(n-5) &= f_{0,k}(n) - 2(5k-10)f_{0,k}(n-2) \\ &\quad + (15k^2 - 50k + 24)f_{0,k}(n-4). \end{aligned}$$

These results suggest the following general formula:

(5.9) $$\frac{k!}{(k-j)!}f_{j,k-j}(n-j) = \sum_{2s\leq j}(-1)^s P_{j,s}(k) f_{0,k}(n-2s) \qquad (0 \leq j \leq k),$$

where $P_{j,s}(k)$ denotes a polynomial in k of degree s. Since

$$f_{j,k-j}(n-j) = jf_{j-1,k-j+1}(n-j-1) + (k-j)f_{j+1,k-j-1}(n-j-1),$$

it follows that

$$\frac{(k-j)!}{k!}\sum_{2s\leqslant j}(-1)^s P_{j,s}(k)f_{0,k}(n-2s) = j\,\frac{(k-j+1)!}{k!}\sum_{2s<j}(-1)^s P_{j-1,s}(k)f_{0,k}(n-2s-2)$$

$$+ (k-j)\,\frac{(k-j-1)!}{k!}\sum_{2s\leqslant j+1}(-1)^s P_{j+1,s}(k)f_{0,k}(n-2s).$$

Hence we take

(5.10) $$P_{j+1,s}(k) = P_{j,s}(k) + j(k-j+1)P_{j-1,s-1}(k).$$

$P_{j,s}(k)$				
$j \backslash s$	0	1	2	3
0	1			
1	1			
2	1	k		
3	1	$3k-2$		
4	1	$6k-8$	$3k(k-2)$	
5	1	$10k-20$	$15k^2-50k+24$	
6	1	$15k-40$	$45k^2-210k+184$	$15k(k-2)(k-4)$
7	1	$21k-70$	$105k^2-630k+784$	$105k^3-840k^2+1764k-720$

It is evident that

(5.11) $$P_{j,0}(k) = 1 \qquad (j \geqslant 0).$$

Also it follows easily from (5.10) that

(5.12) $$P_{2j,j}(k) = 1\cdot 3\cdot 5\cdots(2j-1)k(k-2)(k-4)\cdots(k-2j+2) \qquad (j \geqslant 0).$$

Since

$$P_{j+1,1}(k) = P_{j,1}(k) + j(k-j+1) \qquad (j \geqslant 1),$$

we get

$$P_{j+1,1}(k) = \sum_{t=1}^{j} t(k-t+1).$$

This gives

(5.13) $$P_{j,1}(k) = \frac{1}{2}j(j-1)k - \frac{1}{3}j(j-1)(j-2) \qquad (j \geqslant 0).$$

Similarly, since

$$P_{j+1,2}(k) = P_{j,2}(k) + j(k-j+1)P_{j-1,1}(k) = P_{j,2}(k) + \tfrac{1}{2}j(j-1)(j-2)k^2 - \left\{\frac{5}{6}j(j-1)(j-2)(j-3) + j(j-1)(j-2)\right\}k + \frac{1}{3}j(j-1)(j-2)(j-3)(j-4) + j(j-1)(j-2)(j-3),$$

we find that

$$(5.14)\qquad P_{j,2}(k) = 3\binom{j}{4}k^2 - \left[20\binom{j}{5}+6\binom{j}{4}\right]k + \left[40\binom{j}{6}+24\binom{j}{5}\right]$$
$$= \frac{1}{8}j(j-1)(j-2)(j-3)k^2 - \frac{1}{12}j(j-1)(j-2)(j-3)(2j-5)k^2$$
$$+\frac{1}{90}j(j-1)(j-2)(j-3)(j-4)(5j-7).$$

For example

$$P_{6,2}(k) = 3\cdot 15k^2 - (20\cdot 6+6\cdot 15)k + (40+24\cdot 6) = 45k^2 - 210k + 184.$$

We also find that

$$(5.15)\qquad P_{j,3}(k) = 15\binom{j}{6}k^3 - \left[210\binom{j}{7}+90\binom{j}{6}\right]k^2$$
$$+\left[1120\binom{j}{8}+924\binom{j}{7}+120\binom{j}{6}\right]k - \left[2240\binom{j}{9}+2688\binom{j}{8}+720\binom{j}{7}\right]$$

For example

$$P_{7,3}(k) = 15\cdot 7k^3 - (210+90\cdot 7)k^2 + (924+120\cdot 7) - 720 = 105k^3 - 840k^2 + 1764k - 720.$$

We have noted above that $P_{j,s}(k)$ is a polynomial in k of degree s. In addition we can assert that $P_{j,s}(k)$ i polynomial in j of degree $3s$. More precisely, if we put

$$(5.16)\qquad P_{j,s}(k) = \sum_{t=0}^{s}(-1)^s c_{s,t}(j)k^{s-t},$$

then $c_{s,t}(j)$ is a polynomial in j of degree $2s+t$. If we substitute from (4.7) in (4.1) we get

$$\sum_{t=0}^{s}(-1)^t[c_{s,t}(j+1)-c_{s,t}(j)]k^{s-t} = j(k-j+1)\sum_{t=0}^{s-1}(-1)^t c_{s-1,t}(j-1)k^{s-t-1}.$$

This gives

$$(5.17)\qquad c_{s,t}(j+1)-c_{s,t}(j) = jc_{s-1,t}(j-1)+j(j-1)c_{s-1,t-1}(j-1).$$

The table of values of $P_{j,s}(k)$ suggests that

$$(5.18)\qquad \begin{cases} \displaystyle\sum_{s=0}^{j}(-1)^{j-s}P_{2j,s}(k) = 1\cdot 3\cdot 5\cdots(2j-1)(k-1)(k-3)\cdots(k-2j+1) \\ \displaystyle\sum_{s=0}^{j}(-1)^{j-s}P_{2j+1,s}(k) = 1\cdot 3\cdot 5\cdots(2j-1)(2j+1)\cdot(k-1)(k-3)\cdots(k-2j+1) \end{cases}$$

These formulas are easily proved by means of (5.10).

The explicit results (5.13), (5.14), (5.15) also suggest that

$$(5.19)\qquad P_{j,s}(k) = 0 \qquad (j = 0, 1, \cdots, 2s-1).$$

This can be proved inductively using (5.10) in the form

$$(5.20)\qquad P_{j,s}(k) = P_{j+1,s}(k) - j(k-j+1)P_{j-1,s-1}(k).$$

Thus, to begin with,

$$P_{2s-1,s}(k) = P_{2s,s}(k) - (2s-1)(k-2s+2)P_{2s-2,s-1}(k) = 0,$$

by (5.12). In the next place, taking $j = 2s-2$, we get

$$P_{2s-2,s}(k) = P_{2s-1,s}(k) - (2s-2)(k-2s+3)P_{2s-s,s-1}(k) = 0.$$

Continuing in this way, we get

$$P_{j,s}(k) = 0 \qquad (1 \leqslant j \leqslant 2s-1).$$

Finally, taking $j = 1$ and replacing s by $s + 1$ in (5.10), we have

$$P_{2,s+1}(k) = P_{1,s+1}(k) + kP_{0,s}(k),$$

which gives $P_{0,s}(k) = 0$.

6. We now put

$$P_j(k,x) = \sum_{2s \leqslant j} (-1)^s P_{j,s}(k) x^{j-2s}, \qquad P_0 = 1, \qquad P_1 = x, \tag{6.1}$$

and

$$F(z) = F(k, x, z) = \sum_{j=0}^{\infty} P_j(k,x) \frac{z^j}{j!} . \tag{6.2}$$

By (5.10),

$$P_{j+1}(k,x) = \sum_{2s \leqslant j+1} (-1)^s P_{j,s}(k) x^{j-2s+1} + j(k-j+1) \sum_{2s \leqslant j+1} (-1)^s P_{j-1,s-1}(k) x^{j-2s+1},$$

so that

$$P_{j+1}(k,x) = xP_j(k,x) - j(k-j+1)P_{j-1}(k,x). \tag{6.3}$$

It follows from (6.2) and (6.3) that

$$\begin{aligned} F'(z) &= \sum_{j=0}^{\infty} P_{j+1}(k,x) \frac{z^j}{j!} = x \sum_{j=0}^{\infty} P_j(k,x) \frac{z^j}{j!} - z \sum_{j=0}^{\infty} (k-j) P_j(k,x) \frac{z^j}{j!} \\ &= xF(z) - kzF(z) + z^2 F'(z). \end{aligned}$$

Hence

$$\frac{F'(z)}{F(z)} = \frac{x - kz}{1 - z^2} ,$$

which gives

$$F(k, x, z) = (1+z)^{\frac{1}{2}(x+k)} (1-z)^{-\frac{1}{2}(x-k)} . \tag{6.4}$$

It follows from the recurrence (6.3) that the polynomials

$$P_n(k,x) \qquad (n = 0, 1, 2, \cdots) \tag{6.5}$$

constitute a set of orthogonal polynomials in x. The polynomials have been discussed in [1, § 9]; in that paper the relationship with Euler numbers of higher order is stressed. If we put

$$(1+z)^{\frac{1}{2}x} (1-z)^{-\frac{1}{2}x} = \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!} ,$$

so that $A_n(x) = P_n(0,x)$, then, by (6.4),

$$P_n(k,x) = \sum_{2s \leqslant n} \frac{(-1)^s}{(n-2s)!} \binom{\frac{1}{2}k}{s} A_{n-2s}(x). \tag{6.6}$$

Returning to (5.9) and using (5.3), we have

$$\begin{aligned} j! \binom{k}{j} f_{j,k-j}(n) &= \sum_{2s \leqslant j} (-1)^s P_{j,s}(k) \cdot 2^{-k} \sum_{t=0}^{k} \binom{k}{t} (k-2t)^{n-2s} \\ &= 2^{-k} \sum_{t=0}^{k} \binom{k}{t} (k-2t)^n \sum_{2s \leqslant j} (-1)^s P_{j,s}(k)(k-2t)^{j-2s} , \end{aligned}$$

so that

$$(6.7)\qquad j!\binom{k}{j}f_{j,k-j}(n) = 2^{-k}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^nP_j(k,k-2t)\qquad (0\le j\le k).$$

We shall show that (6.7) holds for all j, that is, the right-hand side vanishes indentically for $j > k$. To prove this, consider the sum

$$2^{-k}\sum_{j=0}^{\infty}\frac{z^j}{j!}\sum_{n=0}^{\infty}\frac{y^n}{n!}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^nP_j(k,k-2t) = 2^{-k}\sum_{t=0}^{k}\binom{k}{t}\sum_{n=0}^{\infty}\frac{y^n}{n!}(k-2t)^n\sum_{j=0}^{\infty}\frac{z^j}{j!}P_j(k,k-2t)$$

$$= 2^{-k}\sum_{t=0}^{k}\binom{k}{t}e^{(k-2t)y}(1+z)^{k-t}(1-z)^t = 2^{-k}e^{-ky}((1+z)e^{2y}+1-z)^k$$

$$= 2^{-k}((1+z)e^{y}+(1-z)e^{-y})^k = (\cosh y + z\sinh y)^k.$$

Since this is a polynomial of degree k in z, it follows that the right-hand side of (6.7) does indeed vanish for $j > k$ and all n. For example, for $k = 1$, we get

$$P_j(1,1) + (-1)^nP_j(1,-1) = 0\qquad (j > 1).$$

Since this holds for all n, we have

$$(6.8)\qquad P_j(1,1) = P_j(1,-1) = 0\qquad (j > 1).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty}P_j(1,1)\frac{z^j}{j!} = 1+z,\qquad \sum_{j=0}^{\infty}P_j(1,-1)\frac{z^j}{j!} = 1-z,$$

in agreement with (6.8).

For $k = 2$ we get

$$2^nP_j(2,2) + 4\delta_{n,0}P_j(2,0) + (-2)^nP_j(2,-2) = 0\qquad (j > 2).$$

This implies

$$P_j(2,2) = P_j(2,0) = P_j(2,-2) = 0\qquad (j > 2).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty}P_j(2,2)\frac{z^j}{j!} = (1+z)^2,\qquad \sum_{j=0}^{\infty}P_j(2,-2)\frac{z^j}{j!} = (1-z)^2,\qquad \sum_{j=0}^{\infty}P_j(2,0)\frac{z^j}{j!} = 1-z^2.$$

Since the determinant

$$\left|\binom{k}{t}(k-2t)^n\right| \ne 0\qquad (t,n = 0,1,\cdots,k),$$

the identical vanishing of the right-hand side of (6.7) implies

$$(6.9)\qquad P_j(k,k-2t) = 0\qquad (j > k;\quad 0\le t\le k).$$

This is indeed implied by (6.4), since

$$F(k,k-2t,z) = (1+z)^{k-t}(1-z)^t.$$

It follows from (6.9) and (6.1) that

$$(6.10)\qquad \sum_{2s\le j}(-1)^sP_{j,s}(k)(k-2t)^{j-2s} = 0\qquad (j > k;\ 0\le t\le k);$$

it evidently suffices to take $t \le k/2$. In particular, for $j = k+1$, (6.10) becomes

(6.11) $$\sum_{2s \leqslant j} (-1)^s P_{j,s}(j-1)(j-2r-1)^{j-2s} = 0 \qquad (2t < j)$$

For j even we consider

$$\sum_{s=0}^{j} (-1)^s P_{2j,s}(2j-1)(2j-2t-1)^{2j-2s} = 0 \qquad (0 \leqslant t < j).$$

Since $P_{j,0}(k) = 1$ this may be written in the form

(6.12) $$\sum_{s=1}^{j} (-1)^{s-1} P_{2j,s}(2j-1)(2r-1)^{2j-2s} = (2r-1)^{2j} \qquad (1 \leqslant r \leqslant j).$$

By Cramer's rule the system (6.12) has the solution

(6.13) $$P_{2j,s}(2j-1) = \frac{N_s}{D} \qquad (1 \leqslant s \leqslant j),$$

where

$$D = \det\left((2r-1)^{2s-2}\right) \qquad (r, s = 1, 2, \cdots, j)$$

and N_s is obtained from D by replacing the s^{th} column by $(2r-1)^{2t}$. Making use of a familiar theorem on the quotient of two alternants [4, Ch. 11], we get

(6.14) $$P_{2j,s}(2j-1) = c_s(1^2, 3^2, 5^2, \cdots, (2j-1)^2) \qquad (1 \leqslant s \leqslant j),$$

where $c_s(x_1, x_2, \cdots, x_j)$ denotes the s^{th} elementary symmetric function of the x_j.

For odd j in (6.11) we consider

$$\sum_{s=0}^{j} (-1)^s P_{2j+1,s}(2j)(2j-2t)^{2j-2s+1} = 0 \qquad (0 \leqslant t < j).$$

This may be written in the form

(6.15) $$\sum_{s=1}^{j} (-1)^{s-1} P_{2j+1,s}(2j)(2r)^{2j-2s+1} = (2r)^{2j+1} \qquad (1 \leqslant r \leqslant j).$$

Exactly as in the case of (6.12), the solution of the system (6.16) is given by

(6.16) $$P_{2j+1,s}(2j) = c_s(2^2, 4^2, 6^2, \cdots, (2j)^2) \qquad (1 \leqslant s \leqslant j).$$

where again c_s denotes the s^{th} symmetric function of the indicated arguments.

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RESTRICTED COMBINATIONS AND COMPOSITIONS

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INTRODUCTION

The number of k-combinations of $\{1, 2, \cdots, n\}$ with no two consecutive integers in a combination is

$$\binom{n-k+1}{k}$$

while the number of such restricted "circular" k-combinations, that is when 1 and n are also considered as consecutive integers, is

$$\frac{n}{n-k}\binom{n-k}{k} \quad .$$

These are two well known examples of restricted combinations given by Kaplansky [1943] as preliminary problems in his elegant solution of the "problème des ménages." Some other examples are given by Abramson [1971], Church [1966, 1968, 1971] and Moser and Abramson [1969a, b].

In this paper, generating functions and recurrence relations are given for a large class of restricted combinations. This method seems to be a more unified approach than using combinatorial arguments such as those of Moser and Abramson [1969a] whose main result is obtained here in Section 7 as a special case of a more general result.

We take a k-composition of an integer n to be an ordered sequence of non-negative integers $a_1, a_2, \cdots, a_k$, whose sum is n. A one-to-one correspondence between the k-compositions of n with each summand $a_i > 0$ and the $(k-1)$-combinations of $\{1, 2, \cdots, n-1\}$ is obtained by representing the combinations and compositions by binary sequences, see also Abramson and Moser [1976]. Hence there is a correspondence between restricted combinations and restricted compositions. Also, there is a correspondence between "circular" combinations and "circular" compositions.

A k-composition of n may be interpreted of course as an occupancy problem of distributing n like objects in k distinct cells, with a_i objects in cell i. Further a k-composition, $a_1, a_2, \cdots, a_k$ of n corresponds to an n-combintion, with repetitions allowed, from $\{1, 2, \cdots, k\}$ with the integer i appearing a_i times. Also since every binary sequence corresponds to a lattice path we have a 1:1 correspondence between lattice paths in a rectangular array and combinations. For example expression (2.3) of Church [1970] is case (L) of Section 3 here. Some results on combinations which have been obtained by Church and Gould [1967] by counting lattice paths have been generalized by Moser and Abramson [1969 b] and can also be derived using our approach here.

Sections 1 to 5 deal with linear compositions and combinations and Sections 6 and 7 with circular compositions and combinations. Throughout we take, as usual,

$$\binom{n}{k} = \begin{cases} n!/(n-k)!k!, & 0 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

1. RESTRICTED COMPOSITIONS

A k-composition of n

$$(1.1) \qquad a_1 + a_2 + \cdots + a_k = n, \qquad a_i \geqslant 1,$$

is an ordered sequence of k positive integers a_i, called the summands or parts satisfying (1.1) for fixed n and k. It is well known and easy to show the number of compositions (1.1) is $\binom{n-1}{k-1}$. Let

$$(1.2) \qquad A = (A_1, A_2, \cdots, A_k), \qquad A_i = \{a_{i1} < a_{i2} < a_{i3} < \cdots\}$$

denote a given collection of k, not necessarily distinct, subsets A_i, of $\{1, 2, 3, \cdots\}$. Denote by $F(n, k; A)$ the number of compositions (1.1) satisfying the restrictions $a_i \in A_i$, $i = 1, 2, \cdots, k$. That is

$$(1.3) \qquad F(n, k; A) = \sum_{\substack{a_1 + \cdots + a_k = n \\ a_i \in A_i}} 1 .$$

The enumerator generating function as is well known, see Riordan [1958] provides a general method of finding $F(n, k; A)$. This is

$$(1.4) \qquad \sum_n F(n, k; A)x^n = (x^{a_{11}} + x^{a_{12}} + \cdots)(x^{a_{21}} + x^{a_{22}} + \cdots) \cdots (x^{a_{k1}} + x^{a_{k2}} + \cdots).$$

For example, in the case $A_i = \{1, 2, 3, \cdots\}$ for all i

$$\sum_{n=1} F(n, k; A)x^n = (x + x^2 + x^3 + \cdots)^k = \sum_{i=0} \binom{k+i-1}{i} x^{i+k} = \sum_{n=1} \binom{n-1}{k-1} x^n .$$

To each of the compositions (1.1) there corresponds a unique sequence of $n - k$ 0's and $k - 1$ 1's:

$$(1.5) \qquad \underbrace{000\cdots01}_{a_1 - 1} \quad \underbrace{000\cdots01}_{a_2 - 1} \quad \cdots \quad \underbrace{000\cdots01}_{a_{k-1} - 1} \quad \underbrace{000\cdots0}_{a_k - 1}$$

Note that since $a_i \geqslant 1$ in each part of (1.5) the 1 always appears except for the last part where we have a "missing" 1. Replacing the 1's by 0's and 0's by 1's in (1.5) we have a dual representation,

$$(1.6) \qquad \underbrace{111\cdots10}_{a_1 - 1} \quad \underbrace{111\cdots10}_{a_2 - 1} \quad \cdots \quad \underbrace{111\cdots10}_{a_{k-1} - 1} \quad \underbrace{111\cdots1}_{a_k - 1}$$

corresponding to a unique sequence of $n - k$ 1's and $k - 1$ 0's.

2. RESTRICTED COMBINATIONS

We call r integers

$$(2.1) \qquad x_1 < x_2 < \cdots < x_r ,$$

chosen from $\{1, 2, \cdots, m\}$ an r-combination (choice, selection) of n. A *part* of (2.1) is a sequence of consecutive integers not contained in a longer sequence of consecutive integers. In a combination (2.1) a *succession* is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$. It is easy to see that if a combination has q parts then it has $r - q$ successions. For example

$$(2.2) \qquad 1, 3, 4, 5, 8, 9$$

is a 6-combination of 10, with parts (1), (3, 4, 5), (8, 9) of lengths 1, 3, 2, respectively. To each combination (2.1) corresponds a unique sequence of r 1's and $m - r$ 0's

$$(2.3) \qquad e_1, e_2, e_3, \cdots, e_m ,$$

where $e_i = \begin{cases} 1 \text{ if } i \text{ belongs to the } r\text{-combination} \\ 0 \text{ if } i \text{ does not belong to the } r\text{-combination.} \end{cases}$

For the combination (2.2) the corresponding sequence is

$$(2.4) \qquad 1011100110 .$$

To a given restricted composition (1.1) corresponds by the use of (1.5) a unique $(k-1)$-combination

$$(2.5) \qquad x_1 < x_2 < \cdots < x_{k-1}$$

of $n - 1$ such that

$$(2.6) \qquad x_1 = a_1, \quad n - x_{k-1} = a_k, \quad x_{i+1} - x_i = a_{i+1}, \quad i = 1, 2, \cdots, k-2.$$

Hence $F(n, k; A)$ is the number of combinations (2.5) satisfying the restrictions

(2.7) $$x_1 \in A_1, \quad n - x_{k-1} \in A_k, \quad x_{i+1} - x_i \in A_i, \quad i = 1, 2, \cdots, k-2.$$

For convenience, letting $n - 1 = m$, $k - 1 = r$, $F(m+1, r+1; A)$ is the number of combinations (2.1) satisfying

(2.8) $$x_1 \in A_1, \quad n - x_r \in A_{r+1}, \quad x_{i+1} - x_i \in A_i, \quad i = 1, 2, \cdots, r-1,$$

where

(2.9) $$A = (A_1, A_2, \cdots, A_{r+1}), \quad A_i = \{a_{i1} < a_{i2} < \cdots\}, \quad i = 1, \cdots, r+1$$

are the given restrictions.

3. EXAMPLES OF RESTRICTED COMPOSITIONS AND COMBINATIONS

Denote by $F(n, k; h_1, p_1; h_2, p_2; \cdots; h_k, p_k)$ the number of k-compositions of n satisfying the restrictions

(3.1) $$1 \leqslant h_i \leqslant a_i \leqslant p_i, \quad \text{for fixed} \quad h_i, p_i, \quad i = 1, \cdots, k.$$

Using the **sieve** formula or the enumerator generating function (1.4) with $A_i = \{h_i, h_i + 1, h_i + 2, \cdots, p_i\}$, $i = 1, \cdots, k$,

(3.2) $$F(n,k; h_1, p_1; \cdots; h_k, p_k) = \binom{n-h+k-1}{k-1}$$
$$+ \sum_{j=1}^{k} (-1)^j \sum{}^{*} \binom{n-h+k-j-1-(p_{i_1}-h_{i_1})-(p_{i_2}-h_{i_2})-\cdots-(p_{i_j}-h_{i_j})}{k-1}$$

with $h = h_1 + \cdots + h_k$ and the summation Σ^* taken over all j-combinations $i_1 < i_2 < \cdots < i_j$ of $\{1, 2, \cdots, k\}$. We consider now some special cases.

(A) The number of compositions (1.1) satisfying $1 \leqslant h_i \leqslant a_i$, $i = 1, \cdots, k$, is the case $p_i = n$, $i = 1, \cdots, k$ of (3.2),

$$F(n, k; h_1, n; \cdots; h_k, n) = \binom{n+k-1-h_1-h_2-\cdots-h_k}{k-1}.$$

(B) The number of compositions (1.1) satisfying $1 \leqslant a_i \leqslant p_i$, is the case $h_i = 1$ for all i, of (3.2) which is

$$F(n, k; 1, p_1; \cdots; 1, p_k) = \binom{n-1}{k-1} + \sum_{j=1}^{k} (-1)^j \sum{}^{*} \binom{n-1-p_{i_1}-p_{i_2}-\cdots-p_{i_j}}{k-1}$$

the summation Σ^* taken over all j-combinations $i_1 < i_2 < \cdots < i_j$ of k.

(C) The number satisfying $1 \leqslant t \leqslant a_i \leqslant w$ for all i is the case $h_i = t$, $p_i = w$ for all i,

$$F(n, k; t, w; \cdots; t, w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-k(t-1)+j(t-w-1)-1}{k-1}$$

while

(D) the number satisfying $1 \leqslant t \leqslant a_i$ is (C) with $w = n$ or (A) wirh $h_i = t$,

$$F(n, k; t, n; \cdots; t, n) = \binom{n-k(t-1)-1}{k-1}.$$

(E) The number satisfying $1 \leqslant a_i \leqslant w$ is (C) with $t = 1$ or (B) with $p_i = w$,

$$F(n, k; 1, w; \cdots; 1, w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-jw-1}{k-1}.$$

In the case $w = 2$ it is easy to obtain another expression for this number, $\binom{k}{n-k}$, so

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-2j-1}{k-1} = \binom{k}{n-k} .$$

Corresponding restricted combinations. We now give the corresponding restricted combinations to the above examples using the correspondence described in Section 2. The number of r-combinations (2.1) of m satisfying for fixed $1 \leqslant h_i \leqslant p_i \leqslant m$, $i = 1, 2, \cdots, r+1$ the conditions

(3.3) $$h_1 \leqslant x_1 \leqslant p_1, \quad m - (p_{r+1} - 1) \leqslant x_r \leqslant m - (h_{r+1} - 1)$$

and

(3.4) $$h_{i+1} \leqslant x_{i+1} - x_i \leqslant p_{i+1}, \quad i = 1, 2, \cdots, r-1$$

is equal to $F(m+1, r+1; h_1, p_1; \cdots; h_{r+1}, p_{r+1})$. We consider now some special cases.

(F) The number of r-combinations satisfying conditions (3.4) only is obtained by putting $h_1 = h_{r+1} = 1$, $p_1 = p_{r+1} = m$.

(G) The number of combinations satisfying

$$h_1 \leqslant x_1, \quad 1 \leqslant x_r \leqslant m - (h_{r+1} - 1) \quad \text{and} \quad h_{i+1} \leqslant x_{i+1} - x_i, \quad i = 1, \cdots, r-1$$

is by using (A) equal to

$$\binom{m + r + 1 - h_1 - h_2 - \cdots - h_{r+1}}{r} .$$

(H) The number satisfying $h_{i+1} \leqslant x_{i+1} - x_i$, $i = 1, \cdots, r-1$ is (G) with $h_1 = h_{r+1} = 1$,

$$\binom{m + r - 1 - h_2 - h_3 - \cdots - h_r}{r} .$$

(I) The number satisfying

$$x_1 \leqslant p_1, \quad x_r \geqslant m - (p_{r+1} - 1) \quad \text{and} \quad x_{i+1} - x_i \leqslant p_{i+1}, \quad i = 1, \cdots, r-1$$

is equal to $F(m+1, r+1; 1, p_1; \cdots; 1, p_{r+1})$ while the number of combinations satisfying $x_{i+1} - x_i \leqslant p_{i+1}$ $i = 1, \cdots, r-1$, is given by the expression in (B) with $n - 1 = m$, $k - 1 = r$, and $p_1 = p_{r+1} = m$.

(J) The number satisfying

(3.5) $$t \leqslant x_1 \leqslant w, \quad m - (w-1) \leqslant x_r \leqslant m - (w-1)$$

and

$$t \leqslant x_{i+1} - x_i \leqslant w, \quad i = 1, \cdots, r-1$$

is given in (C) with $n - 1 = m$, $k - 1 = r$,

$$\sum_{j=0}^{r+1} (-1)^j \binom{r+1}{j} \binom{m - (r+1)(t-1) + j(t-w-1)}{r} .$$

(K) The number satisfying (3.5) only is equal to (3.2) with

$$n - 1 = m, \quad k - 1 = r, \quad h_1 = h_{r+1} = 1 \quad \text{and} \quad p_1 = p_{r+1} = m,$$

$$\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{m - (r-1)(t-1) - j(1+w-t)}{r} .$$

(L) The number satisfying $t \leqslant x_{i+1} - x_i$, is (K) with $w = m$, or (H) with $h_2 = h_3 = \cdots = h_r = t$, is

$$\binom{m - (r-1)(t-1)}{r}$$

while in the case $t = 2$, no two consecutive elements in a combination, the above reduces to the familiar number

$$\binom{m - r + 1}{r} .$$

(M) The number satisfying $x_{i+1} - x_i \leqslant w$ is (K) with $t = 1$,

$$\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{m-jw}{r} \quad .$$

4. COMBINATIONS BY NUMBER AND LENGTH OF PARTS

Using correspondence (1.6) the number of $(n-k)$-combinations of $n-1$ with the length of each part less than or equal to $w-1$ is given by the expression in case (E) of Section 3. Putting $n = m+1$, $k = m-r+1$, the number of r-combinations of m with each part not greater than $w-1$ is equal to

$$\sum_{i=0}^{m-r+1} (-1)^i \binom{m-r+1}{i} \binom{m-iw}{m-r} . \tag{4.0}$$

More generally we consider the following' Given a set of q restrictions

$$A = (A_1, \cdots, A_q), \qquad A_j = \{ 2 \leqslant a_{j1} < a_{j2} < \cdots \} , \tag{4.1}$$

denote by $F_q(n, k; A)$ the number of k-compositions of n such that,

(4.2 a) $a_{i_j} \in A_j$, $j = 1, 2, \cdots, q$, for some q-combination $i_1 < i_2 < \cdots < i_q$ of $\{1, 2, \cdots k\}$.

(4.2 b) $a_i = 1$, for the remaining $k-q$ indices i.

Then

$$F_q(n, k; A) = \binom{k}{q} F(n-k+q, q; A) \tag{4.3}$$

or

$$F_q(n, k; A) = \binom{k}{q} F(n-k, q; B), \quad \text{where} \quad B = (B_1, \cdots, B_q), \tag{4.4}$$

$$B_j = \{ 1 \leqslant a_{j1} - 1 < a_{j2} - 1 < \cdots \}, \quad j = 1, \cdots, q .$$

Let a k-composition of n be given and suppose exactly q of the a_i,

$$a_{i_1}, a_{i_2}, \cdots, a_{i_q}, \quad i_1 < i_2 < \cdots < i_q ,$$

are each $\geqslant 2$. Using (1.6), to this k-composition of n corresponds a unique $(n-k)$-combination of $n-1$ with exactly q parts, the length of the j^{th} part (reading from left to right) being $a_{i_j} - 1$, $j = 1, 2, \cdots, q$. Hence $F_q(n, k; A)$ is the number of $(n-k)$-combinations of $(n-1)$ with exactly q parts, the length of the j^{th} equal to $a_j \in A_j$, $j = 1, \cdots, q$.

For convenience putting $k = m-r+1$, $n = m+1$, the number of r-combinations of m with the length of the j^{th} part equal to $a_j \in A_j$ is by substituting in (4.3) and (4.4), equal to

$$F_q(m+1, m-r+1; A) = \binom{m-r+1}{q} F(r+q, q; A) \tag{4.5}$$

or

$$F_q(m+1, m-r+1; A) = \binom{m-r+1}{q} F(r, q; B), \quad B \text{ given in (4.4).} \tag{4.6}$$

For fixed $1 \leqslant h_i \leqslant p_i \leqslant m$ and reading the parts from left to right it follows that the number of r-combinations of m having exactly q parts (or $r-q$ successions) and satisfying the restrictions,

$$h_i \leqslant \text{length of the } i^{th} \text{ part} \leqslant p_i, \quad i = 1, \cdots, q, \tag{4.7}$$

is equal to

$$\binom{m-r+1}{q} F(r, q; h_1, p_1; \cdots; h_q, p_q). \tag{4.8}$$

We consider now some special cases of (4.7). The number of combinations with exactly q parts such that the length of each part is greater or equal to t and less than or equal to w is the number (4.8) with $h_i = t$, $p_i = w$ for all i,

$$\binom{m-r+1}{q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \binom{r-q(t-1)+j(t-w-1)-1}{q-1} , \tag{4.9}$$

while the number with each part $\geqslant t$ is equal to

$$(4.10)\qquad \binom{m-r+1}{q}\binom{r-q(t-1)-1}{q-1},$$

and the number with each part $\leqslant w$ is

$$(4.11)\qquad \binom{m-r+1}{q}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\binom{r-jw-1}{q-1}.$$

Summing (4.11) over all $q \geqslant 1$ and using Vandermonde's Theorem, the number of combinations with each part $\leqslant w$ (and no restriction on the number of parts) is equal to

$$(4.12)\qquad \sum_{j=0}^{m-r+1}(-1)^j\binom{m-r+1}{j}\binom{m-j(w+1)}{m-r}$$

in agreement with (4.0) where each part is $\leqslant w-1$.

Thus we may enumerate a large class of restricted combinations using the above method. One further example is that each part is of even (odd) length while another is that the length is a multiple of a fixed number.

5. RECURRENCE RELATIONS

Denote k restrictions $A_1, \cdots, A_k$ by

$$(5.1)\qquad A^k = (A_1, \cdots, A_k), \qquad A_i = \left\{0 < a_{i1} < a_{i2} < \cdots\right\}, \quad i = 1, \cdots, k,$$

Then

$$(5.2)\quad F(n, k; A^k) = \sum_{\substack{a_1+\cdots+a_k=n\\ a_i\in A_i}} 1 = \sum_{\substack{a_k\in A_k\\ a_k\leqslant n}}\ \sum_{a_1+\cdots+a_{k-1}=n-a_k} = \sum_{\substack{a_k\in A_k\\ a_k\leqslant n}} F(n-a_k, k-1; A^{k-1}).$$

For the particular restrictions $1 \leqslant h_i \leqslant a_i \leqslant p_i$, i.e.,

$$(5.3)\qquad A_i = \left\{h_i, h_i+1, \cdots, p_i\right\}, \qquad i = 1, \cdots, k,$$

we have

$$(5.4)\qquad F(n, k; A^k) = \sum_{h_k\leqslant a_k\leqslant p_k} F(n-a_k, k-1; A^{k-1})$$

$$= F(n-h_k, k-1; A^{k-1}) + \sum_{h_k\leqslant j\leqslant p_k-1} F(n-1-j, k-1; A^{k-1})$$

$$= F(n-h_k, k-1; A^{k-1}) + F(n-1, k; A^k) - F(n-1-p_k, k-1; A^{k-1}),$$

$$(F(n, k; A^k) = 0,\ n \leqslant 0)$$

with $F(n, k; A^k)$ the same as $F(n, k; p_1, h_1; \cdots; p_k, h_k)$ of (3.2). In the case $h_i = t$ and $p_i = n$, the number of compositions with each part of length not less than t, denoted by $F(n,k; \geqslant t)$ is

$$(5.5)\qquad F(n, k; \geqslant t) = \sum_{j=t}^{n-(k-1)t} F(n-j, k-1; \geqslant t) = F(n-t, k-1; \geqslant t) + F(n-1, k; \geqslant t).$$

Denoting by $F(n, k; \leqslant w)$ the number when $1 \leqslant a_i \leqslant w$, and using (5.4) with $h_i = 1$ and $p_i = w$ for all i,

$$(5.6)\quad F(n, k; \leqslant w) = \sum_{j=1}^{w} F(n-j, k-1; \leqslant w) = F(n-1, k-1; \leqslant w) + F(n-1, k; \leqslant w) - F(n-1-w, k-1; \leqslant w).$$

If we wish to consider compositions of n with given restrictions but with the number of parts not specified, then of course we simply sum over k. That is

$$G(n;A) = \sum_{k=1}^{n} F(n,k;A^k). \tag{5.7}$$

The generating function is

$$\sum_n G(n;A)x^n = \sum_k (x^{a11}+x^{a12}+\cdots)(x^{a21}+x^{a22}+\cdots)\cdots(x^{ak1}+x^{ak2}+\cdots).$$

For example, the number of compositions of n with each part not less than t, is by summing the expression in (D) of Section 3 over all k,

$$G(n;\geqslant t) = \sum_{k=1}^{\left[\frac{n}{t}\right]} \binom{n-k(t-1)-1}{k-1} \tag{5.8}$$

and satisfies the relation

$$G(n;\geqslant t) = G(n-t;\geqslant t) + G(n-1;\geqslant t). \tag{5.9}$$

In the case $t = 2$, $G(n;\geqslant 2)$ is the $(n-1)^{th}$ Fibonacci number, since $G(n;\geqslant 2) = 1$ or each of $n = 2,3$. The number with each part of length not greater than $w < n$ is by summing the expression of (E) in Section 3 over all k,

$$G(n;\leqslant w) = \sum_{k=1}^{n-w} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\binom{n-jw-1}{k-1} \tag{5.10}$$

and satisfies the relation

$$G(n;\leqslant w) = \sum_{i=1}^{w} G(n-i;\leqslant w) = 2G(n-1;\leqslant w) - G(n-1-w;\leqslant w). \tag{5.11}$$

In the case $w = 2$,

$$F(n;\leqslant 2) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} \quad ,$$

and the above relation reduces to $G(n;\leqslant 2) = G(n-1;\leqslant 2) + G(n-2;\leqslant 2)$, $G(n;\leqslant 2)$ being the $(n+1)^{th}$ Fibonacci number since $G(n;\leqslant 2) = 1, 2$ for $n = 1, 2$, respectively.

We may obtain relations for the number counting restricted combinations by considering the number $F(n,k;A^k)$ which counts the corresponding restricted compositions.

6. CIRCULAR COMPOSITIONS AND COMBINATIONS

A (linear) composition (1.1) can be seen as a display of the integers $1, 2, \cdots, n$ in a line, with $k-1$ "dividers," no two dividers adjacent, which yield the k parts:

$$1, 2, \cdots, a_1/a_1+1, a_1+2, \cdots, a_1+a_2/\cdots/a_1+\cdots+a_{k-1}+1, \cdots, n. \tag{6.1}$$

The length of the i^{th} part (from left to right) is equal to a_i. For example the 4-composition of 9

$$2+3+1+3 = 9 \tag{6.2}$$

is seen as

$$12/345/6/789. \tag{6.3}$$

Analogously, a *circular* k*-composition of* n is a display of $1, 2, \cdots, n$ in a circle with k "dividers," no two dividers adjacent, yielding k parts each of length greater or equal to 1. We may illustrate a circular k-composition of n as

$$(6.4)\qquad \overbrace{b,\, b+1,\, \cdots,\, n-1,\, n,\, 1,\, 2,\, \cdots,\, c}^{a_1} \,/\, \overbrace{c+1,\, c+2,\, \cdots,\, c+a_2}^{a_2} \,/\, \cdots \,/\, \overbrace{b-a_k,\, \cdots,\, b-2,\, b-1}^{a_k}\,/$$
$$\text{1st part} \qquad\qquad \text{2nd part} \qquad \cdots \qquad k^{th} \text{ part}$$

placed on a circle in a clockwise direction with the integer 1 always belonging to the first part, i.e.,

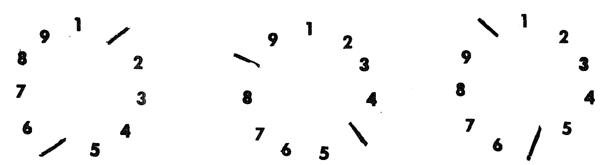
$$c \geq 1, \quad c+n-(b-1) = a_1, \quad a_i \geq 1.$$

Clearly the number of circular k-compositions (6.4) is equal to

$$\sum_{a_1+\cdots+a_k=n} a_1 = \binom{n}{k} .$$

For example,

(6.5)



or written as

(6.6) 67891/2345/, 91234/5678/, 12345/6789/,

recpectively, are three of the $\binom{9}{2}$ circular 2-compositions of 9.

To each circular composition (6.4) there corresponds a unique sequence placed on a circle in a clockwise direction,

$$(6.7)\qquad 00\overset{*}{0} \cdots 01/000 \cdots 01/ \cdots /000 \cdots 01/$$

of $n-k$ 0's and k 1's with the 0 or 1 in the first part corresponding to the integer 1 of the composition marked by "*." Replacing the 1's by 0's and 0's by 1's in (6.7) we have a dual representation of the composition,

$$(6.8)\qquad 11\overset{*}{1} \cdots 10/111 \cdots 10/ \cdots /111 \cdots 10/$$

of $n-k$ 1's and k 0's. We will call (6.7) and (6.8) "circular" sequences. For example, the circular sequences corresponding to each of (6.6), respectively, by use of (6.7) are

$$0000\overset{*}{1}/0001/, \qquad \overset{*}{0}0001/0001/, \qquad \overset{*}{0}0001/0001/,$$

and by use of (6.8) are, respectively,

$$1111\overset{*}{0}/1110/, \qquad 1\overset{*}{1}110/1110/, \qquad \overset{*}{1}1110/1110/.$$

As earlier, consider the restrictions

$$A = (A_1, \cdots, A_k), \quad A_i = \left\{1 \leq a_{i1} < a_{i2} < \cdots\right\}, \quad i = 1, \cdots, k,$$

where each A_i is some given subset of $\left\{1, 2, 3, \cdots\right\}$. Denote by $C(n, k; A)$ the number of circular compositions (6.4) with $a_i \in A_i$, $i = 1, \cdots, k$. That is

$$C(n, k; A) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \in A_i}} a_i .$$

Then the generating function is,

$$\sum_n C(n, k; A)x^n = (a_{11}x^{a_{11}} + a_{12}x^{a_{12}} + \cdots)(x^{a_{21}} + x^{a_{22}} + \cdots)(x^{a_{31}} + x^{a_{32}} + \cdots) \cdots (x^{a_{k1}} + x^{a_{k2}} + \cdots).$$

Checking for the case $A_i = \left\{1, 2, 3, \cdots\right\}$ for all i,

$$\sum_{n=k} C(n,k;A)x^n = (x+2x^2+3x^3+\cdots)(x+x^2+x^3+\cdots)^{k-1} = x(1-x)^{-2}x^{k-1}(1-x)^{-(k-1)}$$

$$= \sum_{n=k} \binom{n}{k} x^n .$$

An example of the use of the above generating function is obtained by taking $A_i = \left\{ h_i, h_i+1, h_i+2, \cdots \right\}$, $i = 1, \cdots, k$ and letting $h = h_1 + \cdots + h_k$,

$$\sum_{n=h} C(n,k;A) = (h_1 x^{h_1} + (h_1+1)x^{h_1+1} + \cdots) \prod_{i=2}^{k} (x^{h_i} + x^{h_i+1} + \cdots)$$

$$= (h_1 - h_1 x + x)x^{h_1}(1-x)^{-2}x^{h-h_1}(1-x)^{-(k-1)}$$

$$= (h_1 - h_1 x + x)x^h \sum_{i=0} \binom{k+i}{k} x^i$$

$$= h_1 x^h + \sum_{i=0} \left[h_1 \binom{k+i+1}{k} + (1-h_1)\binom{k+i}{k} \right] x^{h+i+1}$$

$$= h_1 x^h + \sum_{i=0} \frac{h_1 k + i + 1}{k+i+1} \binom{k+i+1}{k} x^{h+i+1}$$

$$= \sum_{n=h} \frac{h_1 k + n - h}{k+n-h} \binom{k+n-h}{k} x^n ,$$

and hence the number of compositions (6.4) with $1 \leqslant h_i \leqslant a_i$, $i = 1, \cdots, k$ is

$$\frac{h_1 k + n - h}{k+n-h} \binom{k+n-h}{k}, \qquad h = h_1 + \cdots + h_k . \tag{6.9}$$

We now consider a more general example which includes as a special case (6.9). Given $1 \leqslant h_i \leqslant p_i \leqslant m$, the number of circular compositions (6.4) satisfying $h_i \leqslant a_i \leqslant p_i$, $i = 1, 2, \cdots, k$ is

$$C(n,k;h_1,p_1;\cdots;h_k,p_k) = \sum_{\substack{a_1+\cdots+a_k=n \\ h_i \leqslant a_i \leqslant p_i}} a_1 = \sum_{a_1=h_1}^{p_1} a_1 \sum_{\substack{a_2+\cdots+a_k=n-a_1 \\ h_i \leqslant a_i \leqslant p_i}} 1 \tag{6.10}$$

$$= \sum_{a_1=h_1}^{p_1} a_1 F(n-a_1, k-1; h_2, p_2; \cdots; h_k, p_k),$$

where $F(n,k;h_2,p_2;\cdots;h_k,p_k)$ is given by (3.2). Using the identity

$$\sum_{i=m}^{n} i \binom{x+k-2-i}{k-2} = \binom{x+k-m}{k} \frac{x+km-m}{x+k-m} - \binom{x+k-n-1}{k} \frac{x+(n+1)(k-1)}{x+k-n-1} \tag{6.11}$$

$$= \binom{x+k-m}{k} + \binom{x+k-m-1}{k-1}(m-1) - \binom{x+k-n-1}{k} - \binom{x+k-n-2}{k-1} n$$

and (3.2), (6.10) reduces to

$$
\begin{aligned}
&C(n, k; h_1, p_1; \cdots, h_k p_k) \\
&= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{x+k-h_1}{k} \frac{x+kh_1-h_1}{x+k-h_1} - \binom{x+k-p_1-1}{k} \frac{x+(k-1)(1+p_1)}{x+k-p_1-1} \right] \\
&= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{x+k-h_1}{k} + \binom{x+k-h_1-1}{k-1}(h_1-1) - \binom{x+k-p_1-1}{k} - \binom{x+k-p_1-2}{k-1} p_1 \right],
\end{aligned} \tag{6.12}
$$

where

$$h = h_1 + \cdots + h_k, \quad x = n - h + h_1 - j - (p_{i_1} - h_{i_1}) - \cdots - (p_{i_j} - h_{i_j}) \quad \text{for } j > 0,$$

$x = n - h + h_1$ for $j = 0$ and the summation Σ^* is taken over all j combinations

$$i_1 < i_2 < \cdots < i_j \quad \text{of } \{2, 3, \cdots, k\}.$$

We consider now some of the many special cases of (6.12). The number of circular compositions satisfying:

(A) $h_i \leqslant a_i,\ i = 1, 2, \cdots, k$ is (6.12) with $p_i = n$ for all i, is the first term of second last expression for $j = 0$,

$$\frac{n-h+kh_1}{n-h+k} \binom{n-h+k}{k},$$

in agreement with (6.9),

(B) $a_i \leqslant p_i,\ i = 1, \cdots, k$ is (6.12) with $h_i = 1$ for all i,

$$
\begin{aligned}
&\sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{y}{k} - \binom{y-p_1}{k} - \binom{y-p_1-1}{k-1} p_1 \right] \\
&= \sum_{j=0}^{k-1} (-1)^j \Sigma^* \left[\binom{y}{k} - \binom{y-p_1}{k} \frac{y+p_1(k-1)}{y-p_1} \right],
\end{aligned}
$$

where $y = n - (p_{i_1} + \cdots + p_{i_j})$, the summation Σ^* taken over all j-combinations $i_1 < \cdots < i_j$ of $\{2, \cdots, k\}$ for $j \geqslant 1$ and $y = n$ when $j = 0$.

(C) $h_1 \leqslant a_1 \leqslant p_1$ and $t \leqslant a_i \leqslant w$ for $i = 2, 3, \cdots, k$ is (6.12) with $h_i = t,\ p_i = w,\ i = 2, \cdots, k$,

$$
\begin{aligned}
&\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-(k-1)t+k-h_1-j(1+w-t)}{k} \frac{n-(k-1)(t-h_1)-j(1+w-t)}{n-(k-1)t+k-h_1-j(1+w-t)} \right. \\
&\qquad \left. - \binom{n-(k-1)(t-1)-p_1-j(1+w-t)}{k} \frac{n-(k-1)(t-1-p_1)-j(1+w-t)}{n-(k-1)(t-1)-p_1-j(1+w-t)} \right] \\
&= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-(k-1)(t-1)-j(1+w-t)-h_1+1}{k} \right. \\
&\qquad + \binom{n-(k-1)(t-1)-j(1+w-t)-h_1}{k-1} (h_1-1) \\
&\qquad \left. - \binom{n-(k-1)(t-1)-j(1+w-t)-p_1}{k} + \binom{n-(k-1)(t-1)-j(1+w-t)-p_1-1}{k-1} p_1 \right].
\end{aligned}
$$

(D) $t \leqslant a_i \leqslant w$ is case (C) with $h_1 = t,\ p_1 = w$,

(E) $t \leqslant a_i$ for all i, is case (D) with $w = n$ or case (A) with $h_i = t$ for all i,

$$\frac{n}{n-k(t-1)} \binom{n-k(t-1)}{k}.$$

(F) $a_i \leqslant w$ for all i is case (D) with $t = 1$ or case (B) with $p_i = w$.

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[\binom{n-jw}{k} - \binom{n-w(j+1)}{k} \frac{n-w(j+1)+wk}{n-w(j+1)} \right]$$

$$= \binom{n}{k} + \sum_{i=1}^{k} (-1)^i \left[\binom{k-1}{i} \binom{n-iw}{k} + \binom{k-1}{i-1} \binom{n-iw}{k} \frac{n-iw+wk}{n-iw} \right]$$

$$= \frac{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-iw-1}{k-1},$$

and in the case $w = 2$ another expression is

$$\frac{n}{k} \binom{k}{n-k},$$

see case (E) of Section 3.

To obtain recurrence relations we proceed as follows. Let $A^k = (A_1, \cdots, A_k)$. Then for $k \geqslant 2$,

$$(6.13)\quad C(n,k;A^k) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \in A_i}} a_1 = \sum_{\substack{a_k \in A_k \\ a_k \leqslant n}} \sum_{a_1+\cdots+a_{k-1}=n-a_k} a_1 = \sum_{\substack{a_k \in A_k \\ a_k \leqslant n}} C(n-a_k, k-1; A^{k-1}).$$

This is the same as that for the linear case (5.2) with different initial values. For the particular restrictions $1 \leqslant h_i \leqslant a_i \leqslant p_i$, i.e.,

$$A_i = \{h_i, h_i+1, \cdots, p_i\}, \qquad i = 1, \cdots, k,$$

we have

$$(6.14)\quad C(n,k;A^k) = \sum_{h_k \leqslant a_k \leqslant p_k} C(n-a_k, k-1; A^{k-1})$$

$$= C(n-h_k, k-1; A^{k-1}) + \sum_{h_k \leqslant j \leqslant p_k - 1} C(n-1-j, k-1; A^{k-1})$$

$$= C(n-h_k, k-1; A^{k-1}) + C(n-1, k; A^k) - C(n-1-p_k, k-1; A^{k-1}),$$

$$(C(n,k;A^k) = 0, \quad n \leqslant 0).$$

The number of circular compositions with each $a_i \geqslant t$, denoted by $C(n,k;\geqslant t)$ and given by the expression in case (E) above satisfies the relation

$$(6.15)\qquad C(n,k;\geqslant t) = C(n-t, k-1;\geqslant t) + C(n-1, k;\geqslant t).$$

Denoting by $C(n,k;\geqslant w)$ the number when $1 \leqslant a_i \leqslant w$ then the expression is given in case (F) above and satisfies the relation

$$(6.16)\qquad C(n,k;\leqslant w) = \sum_{j=1}^{w} C(n-j, k-1;\leqslant w)$$

$$= C(n-1, k-1;\leqslant w) + C(n-1, k;\leqslant w) - C(n-1-w, k-1;\leqslant w).$$

Summing (6.15) over all k the number of circular compositions with each part not less than t is

$$(6.17)\qquad D(n;\geqslant t) = \sum_{k=0}^{[h/t]} \frac{n}{n-k(t-1)} \binom{n-k(t-1)}{k}$$

and

(6.18) $$D(n;\geqslant t) = D(n-t;\geqslant t) + D(n-1;\geqslant t).$$

In the case $t = 2$, the above relation reduces to

$$D(n;\geqslant 2) = D(n-2;\geqslant 2) + D(n-1;\geqslant 2)$$

and $D(n;\geqslant 2)$ is the Lucas number having values 1, 3 for $n = 1, 2$, respectively. Summing (6.16) over all k the number $D(n;\leqslant w)$ of circular compositions with each part not greater than w is

(6.19) $$D(n;\leqslant w) = \sum_{k=1} \frac{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-iw-1}{k-1}$$

and satisfies the relation

(6.20) $$D(n;\leqslant w) = \sum_{j=1}^{w} D(n-j;\leqslant w).$$

In the case $w = 2$, $D(n;\geqslant 2)$ is also the Lucas number with $D(n;\geqslant 2)$ having values 1, 3 for $n = 1, 2$, respectively.

Given a set of q restrictions

$$A = (A_1, \cdots, A_q), \qquad A_j = \left\{ 2 \leqslant a_{j1} \leqslant a_{j2} \leqslant \cdots \right\},$$

denote by $C_q(n, k; A)$ the number of circular compositions (6.4) such that

(a) $a_{i_j} \in A_j,\ j = 1, 2, \cdots, q$ for some q-combination

$$i_1 < i_2 < \cdots < i_q \text{ of } \left\{1, 2, \cdots, k\right\},$$

(b) $a_1 = 1$ for the remaining $k - q$ indices i.

Then by partitioning the compositions into those with $a_1 = 1$ and $a_1 > 1$

(6.21) $$\begin{aligned} C_q(n, k; A) &= \binom{k-1}{q} F(n-k+q, q; A) + \binom{k-1}{q-1} C(n-k+q, q; A) \\ &= \binom{k-1}{q} F(n-k, q; B) + \binom{k-1}{q-1} C(n-k, q; B) + \binom{k-1}{q-1} F(n-k, q; B) \\ &= \binom{k}{q} F(n-k, q; B) + \binom{k-1}{q-1} C(n-k, q; B), \end{aligned}$$

where

$$B = (B_1, \cdots, B_q), \qquad B_j = \left\{1 \leqslant a_{j1} - 1 \leqslant a_{j2} - 1 \leqslant \cdots\right\},\ j = 1, \cdots, q$$

and $F(n, k; A)$ is the number of restricted (linear) compositions discussed earlier.

7. CIRCULAR COMBINATIONS

A *circular* k*-combination of* n is a set of k integers

(7.1) $$x_1 < x_2 < \cdots < x_k$$

chosen from the integers $1, 2, \cdots, n$ displayed in a circle. That is we consider 1 and n to be consecutive. For example the circular 6-combination 1, 3, 4, 5, 8, 9 of 9 has parts (891) and (345) each of length 3 while the same (linear) 6-combination has parts (1), (345), (89). Of course, the number $\binom{n}{k}$ of (linear) k-combinations of n is equal to the number of circular k-combinations of n. A succession here is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$ with $n, 1$ also considered a succession. As before if a combination (7.1) has q parts it has $k - q$ successions. As before to each circular combination (7.1) corresponds a unique sequence of k 1's and $n - k$ 0's.

(7.2) $$\overset{*}{e}_1, e_2, \cdots, e_n$$

with

$$e_i = \begin{cases} 1 & \text{if } i \text{ is in the combination,} \\ 0 & \text{if } i \text{ is not in the combination.} \end{cases}$$

We shall think of the sequence (7.2) placed on a circle in a clockwise direction. Hence the "circular" sequence (7.2) corresponds to the circular sequence (6.7) by agreeing to let e_1 correspond to the element of (6.7) marked by a *. To a circular composition (6.4) corresponds a unique circular combination (7.2) with

$$n-(x_k-x_1) = a_1$$

$$x_{i+1}-x_i = a_i \quad \text{for} \quad i = 1, 2, \cdots, k-1.$$

Thus the number of combinations (7.1) satisfying the restrictions

$$n-(x_k-x_1) \in A_1 \quad \text{and} \quad x_{i+1}-x_i \in A_i \quad \text{for} \quad i = 1, 2, \cdots, k-1,$$

where the A_i are given by (6.7), is simply the number $C(n, k; A)$ of Section 6. For example the number of combinations satisfying

$$h_1 \leqslant n-(x_k-x_1) \leqslant p_1 \quad \text{and} \quad t \leqslant x_{i+1}-x_i \leqslant w \quad \text{for} \quad i = 1, 2, \cdots, k-1$$

is the expression of case (C) of Section 6 and is in agreement with Moser and Abramson [1969 a, expression (14) for $C_{n,k}(t, w; h_1, p_1)$].

Using the dual representation (6.8) and (7.2) we have a one-one correspondence between the circular compositions (6.4) and circular $(n-k)$-combinations of n. For example the number of circular $(n-k)$-combinations of n with each part of length not greater than $w-1$ is the number of circular compositions with $a_i \leqslant w$ given in case (F) of Section 6. Putting $n = m$ and $k = m-r$ the number of circular r-combinations of m is

$$\frac{m}{m-r}\sum_{i=0}^{m-r}(-1)^i\binom{m-r}{i}\binom{m-iw-1}{m-r-1} \tag{7.3}$$

in agreement with Moser and Abramson [1969 a, expression (29)].

More generally the number of circular r-combinations of m having exactly q parts, or $r-q$ successions, the length of the j^{th} part (reading in a clockwise direction with the first part that part containing the smallest integer greater than or equal to 1) equal to $a_j - 1$, $a_j \in A_j$, $j = 1, 2, \cdots, q$ is $C_q(m, m-r; A)$ given by (6.21).

For example letting $A_i = \{t+1, t+2, \cdots\}$ for all i the number of circular r-combinations of m with exactly q parts and with each part of length not less than t is by using (6.21), (D) of Section 3 and (E) of Section 6,

$$\begin{aligned} C_q(m, m-r; A) &= \binom{m-r}{q}F(r, q; B) + \binom{m-r-1}{q-1}C(r, q; B) \\ &= \binom{m-r}{q}\binom{r-q(t-1)-1}{q-1} + \binom{m-r-1}{q-1}\binom{r-q(t-1)}{q}\frac{r}{r-q(t-1)} \\ &= \binom{m-r}{q}\binom{r-q(t-1)-1}{q-1}\frac{m}{m-r} \quad . \end{aligned} \tag{7.4}$$

The number with exactly q parts each of length not greater than w is obtained by taking $B_i = \{1, 2, \cdots, w\}$ for all i and using (E) of Section 3 and (F) of Section 6,

$$\begin{aligned} C_q(m, m-r; A) &= \binom{m-r}{q}F(r, q; B) + \binom{m-r-1}{q-1}C(r, q; B) \\ &= \frac{m}{m-r}\binom{m-r}{q}\sum_{i=0}(-1)^i\binom{q}{i}\binom{r-iw-1}{q-1} \\ &= \frac{m}{m-r}\sum_{i=0}(-1)^i\binom{m-r}{i}\binom{m-r-i}{q-i}\binom{r-iw-1}{q-1} . \end{aligned} \tag{7.5}$$

Summing (7.5) over all q we obtain the number of circular combinations of m with each part of length not greater than w.

$$\frac{m}{m-r}\sum_{i=0}^{m-r}(-1)^i\binom{m-r}{i}\binom{m-i(w+1)-1}{m-r-1}$$

in agreement with (7.3) where a part is of length not greater than $w - 1$.

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ODE TO PASCAL'S TRIANGLE

Pascal. . . Pascal, you genius, you,
Constructed a triangle of powers of two.
Coefficients, and powers of eleven, by base ten,
A more useful aid, there's never been.
Head, tail, tail, head,
Answers from your rows are read.
Combinations and expectations, to my delight,
Can also be proved wrong or right.
With a little less effort and a little more ease,
I might have gotten thru this course in a breeze.
So, Pascal . . . Pascal, you rascal you.
Why did you limit it to powers of two?

. . . Bob Jones
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[See p. 455 for "Response."]

A RECIPROCAL SERIES OF FIBONACCI NUMBERS WITH SUBSCRIPTS $2^n k$

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A reciprocal series of Fibonacci numbers with subscripts 2^n was summed by I. J. Good [1] and was proposed as a problem by D. A. Millin [2], and there are many proofs in [4] of

$$\sum_{n=0}^{\infty} 1/F_{2^n} = (7-\sqrt{5})/2 .$$

Here, we derive a closely related sum,

$$\sum_{n=0}^{\infty} 1/F_{2^n k} .$$

To sum $1/F_{2^n k}$ we get a good start with early examples, making use of the identity $F_{2k} = F_k L_k$.

$$\frac{1}{F_k} = \frac{1}{F_k}, \qquad \frac{1}{F_k} + \frac{1}{F_{2k}} = \frac{L_k+1}{F_{2k}} = \frac{F_{2k}/F_k+1}{F_{2k}},$$

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} = \frac{L_{2k}(L_k+1)+1}{F_{4k}} = \frac{F_{4k}/F_k + L_{2k}+1}{F_{4k}},$$

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{1}{F_{8k}} = \frac{F_{8k}/F_k + L_{4k}(L_{2k}+1)+1}{F_{8k}} .$$

From

$$L_{m+p} + L_{m-p} = L_m L_p, \quad p \text{ even},$$

and we can rewrite this as

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{1}{F_{8k}} = \frac{F_{8k}/F_k + (L_{6k}+L_{4k}+L_{2k}+1)}{F_{8k}} .$$

Now, the hinge is the Lucas identity

$$L_{2^n k}\left(L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k} + 1\right) = L_{(2^{n+1}-2)k} + L_{(2^{n+1}-4)k} + \cdots + L_{2k} .$$

Thus,

$$\sum_{i=0}^{n} \frac{1}{F_{2^i k}} = \frac{F_{2^n k}/F_k + \left(L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k} + 1\right)}{F_{2^n k}} . \tag{1}$$

But,

$$L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k}$$

can be summed and converted to a form using powers of

$$\alpha = (1+\sqrt{5})/2 \qquad \text{and} \qquad \beta = (1-\sqrt{5})/2,$$

making it possible to find the limit as $n \to \infty$.

Using a result of K. Siler [3],

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{1 - L_a + (-1)^a}$$

whence, with $a = 2k$, $k = j$ and $b = +1$;

$$\sum_{j=1}^{n} F_{2kj-1} = \frac{F_{2kn-1} - F_{2k(n+1)-1} + (-1)^{2k-1} F_1 + F_{2k-1}}{2 - L_{2k}} .$$

Now let $a = 2k$, $k = j$, and $b = -1$;

$$\sum_{j=1}^{n} F_{2kj+1} = \frac{F_{2kn+1} - F_{2k(n+1)+1} + (-1)^{2k+1} F_{-1} + F_{2k-1}}{2 - L_{2k}} .$$

Summing the preceding two series termwise,

$$\sum_{j=1}^{n} L_{2kj} = \frac{L_{2kn} - L_{2k(n+1)} - L_0 + L_{2k}}{2 - L_{2k}} = \frac{L_{2k(n+1)} - L_{2kn} - L_{2k} + 2}{L_{2k} - 2} .$$

Now, let $n = 2^{N-1} - 1$, $n + 1 = 2^{N-1}$ and return to (1):

$$\sum_{n=0}^{N} 1/F_{2^n k} = \frac{F_{2^N k}/F_k + \left(\sum_{j=1}^{2^{N-1}-1} L_{2kj} \right) + 1}{F_{2^N k}}$$

$$= \frac{1}{F_k} + \frac{L_{k(2^N)} - L_{(2^N-2)k}}{F_{2^N k}(L_{2k} - 2)} = A$$

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \lim_{N \to \infty} \frac{L_{k(2^N)} - L_{k(2^N-2)}}{F_{2^N k}(L_{2k} - 2)} .$$

Trying this for $k = 1$,

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \lim_{N \to \infty} \left(\frac{L_{2^N k}}{F_{2^N k}} - \frac{L_{(2^N-2)k}}{L_{(2^N-1)k}} \cdot \frac{L_{(2^N-1)k}}{L_{2^N k}} \cdot \frac{L_{2^N k}}{F_{2^N k}} \right) \left(\frac{1}{L_{2k} - 2} \right) \Bigg|_{k=1}$$

$$= 1 + \sqrt{5} - \sqrt{5}\beta^2 = 1 + \sqrt{5}(1 - \beta^2) = 1 + \sqrt{5}(-\beta)$$

$$= 1 + \sqrt{5}(\sqrt{5} - 1)/2 = (7 - \sqrt{5})/2 ,$$

which is the result of Millin and of Good.

Generally, we get

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \frac{1}{L_{2k} - 2}(\sqrt{5} - \sqrt{5}\beta^{2k}) = \frac{1}{F_k} + \sqrt{5} \left(\frac{1 - (L_{2k} - \sqrt{5} F_{2k})/2}{L_{2k} - 2} \right) = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2(L_{2k} - 2)} .$$

We need the identity

$$L_k^2 = L_{2k} + 2(-1)^k \tag{2}$$

which for odd k gives us

$$L_k^2 = L_{2k} - 2 .$$

For odd k, then, we can continue

$$\lim_{N \to \infty} A = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2L_k^2} = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_k}{2L_k}, \quad k \text{ odd.}$$

However, if we let k be even, then (2) gives us

$$L_k^2 = L_{2k} + 2, \qquad L_k^2 - 4 = L_{2k} - 2 = 5F_k^2,$$

so that our limit becomes

$$\lim_{N \to \infty} A = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2(5F_k^2)} = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{L_k}{2F_k}, \quad k \text{ even}$$

Finally,

$$\sum_{n=0}^{\infty} 1/F_{2^n k} = \begin{cases} \dfrac{2L_k - F_{2k}\sqrt{5} + 5F_k^2}{2F_{2k}}, & k \text{ odd;} \\[2ex] \dfrac{2 - F_k\sqrt{5} + L_k}{2F_k}, & k \text{ even.} \end{cases}$$

It would seem that the odd and even cases are closely related. First, let k be odd, or, $k = 2s + 1$. Then

$$\sum_{n=0}^{\infty} 1/F_{(2s+1)2^n} = \frac{1}{F_{2s+1}} + \frac{5F_{2(2s+1)}}{2L_{2s+1}^2} - \frac{\sqrt{5}}{2} = B.$$

Now, let k be even. Let $k = 2(2s + 1)$, making

$$\sum_{n=0}^{\infty} 1/F_{2(2s+1)2^n} = \frac{1}{F_{2(2s+1)}} + \frac{L_{2(2s+1)}}{2F_{2(2s+1)}} - \frac{\sqrt{5}}{2} = C.$$

Then, notice that $B = C + 1/F_{2s+1}$.

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[Cont. from p. 452.]

RESPONSE

We push Pascal to the left, up tight,
To see what else can be brought to light.
In flowers and trees the world around,
The Fibonacci numbers do abound.
Look up to the right while taking sums.
What you find there will strike you dumb.

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PELL'S EQUATION AND PELL NUMBER TRIPLES

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The Pell numbers are defined by

$$P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+2} = 2P_{n+1} + P_n \quad \text{for} \quad n \geqslant 0.$$

In [1] it was noted that if

$$p > q > 0 \quad \text{and} \quad p^2 - q^2 - 2pq = \pm N,$$

where N is a square or twice a square, then there exist non-negative integers a, b, and n with $a \geqslant b$ such that

$$p = aP_{n+2} - bP_{n+1} \quad \text{and} \quad q = aP_{n+1} - bP_n,$$

or

$$p = bP_{n+2} + aP_{n+1} \quad \text{and} \quad q = bP_{n+1} + aP_n.$$

We shall prove this result for $p \geqslant q \geqslant 0$ and $N > 1$ and, in addition, show that $(a+b)^2 - 2b^2 = N$ (Theorem 6). We shall also prove the converse of this result (Theorem 8). In order to prove Theorem 6 we shall need Theorem 2, which gives an interesting property of the fundamental solution(s) to Pell's Equation

(1) $$u^2 - Dv^2 = C,$$

where D is a positive integer which is not a perfect square and $C \neq 0$. The converse of Theorem 2 is also true but it is neither stated nor proved since it is not needed to prove Theorem 6.

Before proving these results we need to establish some definitions and theorems concerning (1). For this we can do no better than follow Nagel [2, 195–212] with but one exception.

If u and v are integers which satisfy (1), then we say $u + v\sqrt{D}$ is a solution to (1). If $u + v\sqrt{D}$ and $u^* + v^*\sqrt{D}$ are both solutions to (1) then they are called *associate solutions* iff there exists a solution $x + y\sqrt{D}$ to $x^2 - Dy^2 = 1$ such that

$$(u + v\sqrt{D}) = (u^* + v^*\sqrt{D})(x + y\sqrt{D}).$$

The set of all solutions associated with each other forms a class of solutions of (7). Every class contains an infinite number of solutions [2, 204].

It is possible to decide whether the two given solutions $u + v\sqrt{D}$ and $u^* + v^*\sqrt{D}$ belong to the same class or not. In fact, it is easy to see that the necessary and sufficient condition for these two solutions to be associated with each other is that the two numbers

$$\frac{uu^* - vv^*D}{C} \quad \text{and} \quad \frac{vu^* - uv^*}{C}$$

be integers.

If K is the class consisting of the solutions

$$u_i + v_i\sqrt{D}, \qquad i = 1, 2, 3, \cdots,$$

it is evident that the solutions

$$u_i - v_i\sqrt{D}, \qquad i = 1, 2, 3, \cdots,$$

also constitute a class, which may be denoted by $\bar{K}$. The classes K and $\bar{K}$ are said to be *conjugates* of each other. Conjugate classes are in general distinct, but may sometimes coincide; in the latter case we speak of *ambiguous* classes.

If the diophantine equation $u^2 - Dv^2 = C$ is solvable then from among all solutions $u + v\sqrt{D}$ in a given class K of solutions to $u^2 - Dv^2 = C$, we shall now choose a solution $u_0 + v_0\sqrt{D}$, which we shall call the *fundamental solution* of the class K. The manner of selecting this solution will depend on the value of C.

(i) For the case $C > 1$, let u_0 be the least positive value of u which occurs in K. If K is not ambiguous then the number v_0 is uniquely determined. If K is ambiguous we get a uniquely determined v_0 by prescribing that $v_0 \geqslant 0$.

(ii) For the case $C \leqslant -1$ or $C = 1$ let v_0 be the least positive value of v which occurs in K. If K is not ambiguous then the number u_0 is uniquely determined. If K is ambiguous we get a uniquely determined u_0 by prescribing that $u_0 \geqslant 0$.

In the sequel we shall always denote the fundamental solution of $u^2 - Dv^2 = 1$ by $x_1 + y_1\sqrt{D}$ instead of by $u_0 + v_0\sqrt{D}$. Since there is only one class of solutions to $u^2 - Dv^2 = 1$, we have that $x_1 > 0$ and $y_1 > 0$.

EXAMPLES. The fundamental solution to $u^2 - 2v^2 = 1$ is $3 + 2\sqrt{2}$. The fundamental solution to $u^2 - 2v^2 = -1$ is $1 + \sqrt{2}$. The two different classes of solutions to $u^2 - 2v^2 = 7$ have as their fundamental solutions $3 + \sqrt{2}$ and $3 - \sqrt{2}$. The four different classes of solutions to $u^2 - 2v^2 = 119$ have as their fundamental solution $11 + \sqrt{2}$, $11 - \sqrt{2}$, $13 + 5\sqrt{2}$, $13 - 5\sqrt{2}$.

REMARK A. It follows from the definition of fundamental solution that if $u_0 + v_0\sqrt{D}$ is a fundamental solution to a class K of solutions to $u^2 - Dv^2 = C$, where $C \neq 0$, then

(i)
$$u_0 + v_0\sqrt{D} > 0,$$

(ii) for $C \neq 1$, if $u + v\sqrt{D}$ is in K then
$$|u| \geqslant |u_0| \qquad \text{and} \qquad |v| \geqslant |v_0|, \text{ and}$$

(iii) If $C \geqslant 1$ then $u_0 > 0$ and if $C \leqslant 1$ then $v_0 > 0$.

In (ii) we must exclude $C = 1$ since for $C = 1$, $u = 1$ and $v = 0$ is a solution to $u^2 - Dv^2 = 1$ but it is not the fundamental solution.

Our definition of fundamental solution differs from Nagel's only when $v_0 < 0$. In this case, while our fundamental solution is $u_0 + v_0\sqrt{D}$ his is $-(u_0 + v_0\sqrt{D})$. Instead of satisfying $u_0 + v_0\sqrt{D} > 0$ as our fundamental solutions do Nagel's satisfy $v_0 \geqslant 0$.

If $u_0 + v_0\sqrt{D}$ is a fundamental solution to a class K of solutions to $u^2 - Dv^2 = C$, we shall sometimes simply say that $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = C$.

Lemma 1. [2, 205–207]. Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to $x^2 - Dy^2 = 1$. If $u_0 + v_0\sqrt{D}$ is a fundamental solution to the equation $u^2 - Dv^2 = -N$, where $N > 0$, then

$$0 < |v_0| \leqslant \frac{y_1\sqrt{N}}{\sqrt{2(x_1 - 1)}} \qquad \text{and} \qquad 0 \leqslant |u_0| \leqslant \sqrt{\tfrac{1}{2}(x_1 - 1)N}\,.$$

If $u_0 + v_0\sqrt{D}$ is a fundamental solution to the equation $u^2 - Dv^2 = N$, where $N > 1$, then

$$0 \leqslant |v_0| \leqslant \frac{y_1\sqrt{N}}{\sqrt{2(x_1 + 1)}} \qquad \text{and} \qquad 0 < |u_0| \leqslant \sqrt{\tfrac{1}{2}(x_1 + 1)N}\,.$$

Theorem 2. Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to $x^2 - Dy^2 = 1$. If

$$k = \frac{y_1}{x_1 - 1}$$

and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = -N$, where $N > 0$, then $v_0 = |v_0| \geqslant k|u_0|$. If

$$k = \frac{Dy_1}{x_1 - 1}$$

and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = N$, where $N > 1$, then $u_0 = |u_0| \geqslant k|v_0|$.

Proof. Assume $u_0 + v_0\sqrt{D}$ is a fundamental solution to $x^2 - Dy^2 = -N$ and assume $|v_0| < k|u_0|$. Thus

$$-N = u_0^2 - Dv_0^2 > u_0^2 - Dk^2u_0^2 = u_0^2(1 - Dk^2).$$

Hence, by Lemma 1,

$$\frac{2u_0^2}{x_1 - 1} \leqslant N < u_0^2(Dk^2 - 1).$$

Therefore we have the contradiction

$$\frac{2}{x_1 - 1} < Dk^2 - 1 = \frac{Dy_1^2}{(x_1 - 1)^2} - 1 = \frac{x_1 + 1}{x_1 - 1} - 1 = \frac{2}{x_1 - 1}.$$

Now assume $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = N$ and assume $|u_0| < k|v_0|$. Thus

$$N = u_0^2 - Dv_0^2 < k^2v_0^2 - Dv_0^2 = (k^2 - D)v_0^2 .$$

Hence, by Lemma 1,

$$\frac{2(x_1 + 1)v_0^2}{y_1^2} \leqslant N < (k^2 - D)v_0^2 .$$

Therefore we have the contradiction

$$\frac{2(x_1 + 1)}{y_1^2} < k^2 - D = \frac{D[Dy_1^2 - (x_1 - 1)^2]}{(x_1 - 1)^2} = \frac{2D}{x_1 - 1} = \frac{2(x_1 + 1)}{y_1^2} .$$

Lemma 3. Let $u_0 + v_0\sqrt{D}$ be a fundamental solution to a class of solutions to $u^2 - Dv^2 = C$, where $C \neq 1$, and let $x + y\sqrt{D}$ be a solution to the equation $x^2 - Dy^2 = 1$. In addition, let

$$u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x + y\sqrt{D}).$$

If $u \geqslant 0$ and $v \geqslant 0$ then $x > 0$ and $y \geqslant 0$ (if $C = 1$, one requires $v > 0$ instead of $v \geqslant 0$).

Proof. Since $u_0 + v_0\sqrt{D} > 0$ and $u + v\sqrt{D} > 0$, $x + y\sqrt{D} > 0$. This implies $x > 0$. If $x = 1$ then $y = 0$ and the lemma is true. Thus assume $x > 1$. We need only show $y \geqslant 0$. Since $(x + y\sqrt{D})(x - y\sqrt{D}) = 1$, $y < 0$ implies $x + y\sqrt{D} < 1$. Whence

$$u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x + y\sqrt{D}) < u_0 + v_0\sqrt{D} .$$

This is impossible since, by Remark A, $u \geqslant u_0$ and $v \geqslant v_0$.

Lemma 4. [2, 197–198]. If $x + y\sqrt{D}$ is a solution, with $x > 0$ and $y \geqslant 0$, to the diophantine equation $x^2 - Dy^2 = 1$ then

$$(x + y\sqrt{D}) = (x_1 + y_1\sqrt{D})^m,$$

where $x_1 + y_1\sqrt{D}$ is the fundamental solution to $x^2 - Dy^2 = 1$ and m is a non-negative integer.

If $u + v\sqrt{D}$ is a solution to the diophantine equation $u^2 - Dv^2 = C$ then, by the definition of a fundamental solution,

$$u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x + y\sqrt{D}),$$

where $u_0 + v_0\sqrt{D}$ is the fundamental solution to the class of solutions to $u^2 - Dv^2 = C$ to which $u + v\sqrt{D}$ belongs and $x^2 - Dy^2 = 1$. By Lemma 3, $u \geqslant 0$ and $v \geqslant 0$ imply $x > 0$ and $y \geqslant 0$. Hence by Lemma 4, we have

Theorem 5. If $u + v\sqrt{D}$ is a solution in non-negative integers to the diophantine equation $u^2 - Dv^2 = C$, where $C \neq 1$, then there exists a non-negative integer m such that

$$u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^m ,$$

where $u_0 + v_0\sqrt{D}$ is the fundamental solution to the class of solutions of $u^2 - Dv^2 = C$ to which $u + v\sqrt{D}$ belongs and $x_1 + y_1\sqrt{D}$ is the fundamental solution to $x^2 - Dy^2 = 1$.

Theorem 6. Let N be an integer greater than one. If $p \geqslant q \geqslant 0$ and $p^2 - q^2 - 2pq = \epsilon N$, where $\epsilon = 1$ or -1, then there exist non-negative integers a, b, n with $a \geqslant b$ such that either

(2) $$p = aP_{n+2} - bP_{n+1} \quad \text{and} \quad q = aP_{n+1} - bP_n ,$$

or

(3) $$p = bP_{n+2} + aP_{n+1} \quad \text{and} \quad q = bP_{n+1} + aP_n .$$

Also we have that $(a+b)^2 - 2b^2 = N$.

We shall now indicate how one can explicitly determine which of (2) or (3) is satisfied and also a, b, and n. Since $(p-q)^2 - 2q^2 = p^2 - q^2 - 2pq = \epsilon N$, by Theorem 5,

$$(p-q) + q\sqrt{2} = (u_0 + v_0\sqrt{2})(3+2\sqrt{2})^m = u_m + v_m\sqrt{2}\ , \tag{4}$$

where $u_0 + v_0\sqrt{2}$ is the fundamental solution to the class of solutions of $u^2 - 2v^2 = \epsilon N$ to which $(p-q) + q\sqrt{2}$ belongs and m is a non-negative integer.

If the product $\epsilon u_0 v_0$ is negative then p and q satisfy (2), where for $\epsilon = -1$ we have $a = v_0$, $b = v_0 - u_0$, $n = 2m$, and $a > b \geqslant 0$ whereas for $\epsilon = 1$ we have $a = u_0 + v_0$, $b = -v_0$, $n = 2m - 1$, $m \geqslant 1$, and $a \geqslant b > 0$.

If the product $\epsilon u_0 v_0$ is positive then p and q satisfy (3), where for $\epsilon = -1$ we have $a = v_0$, $b = u_0 + v_0$, $n = 2m - 1$, $m \geqslant 1$, and $a > b \geqslant 0$ whereas for $\epsilon = 1$ we have $a = u_0 - v_0$, $b = v_0$, $n = 2m$, and $a \geqslant b > 0$.

If $u_0 = 0$ then p and q satisfy (2) for $a = v_0 = b$ and $n = 2m$. Furthermore, if $m \geqslant 1$ then p and q also satisfy (3) for $a = v_0 = b$ and $n = 2m - 1$.

If $v_0 = 0$ then p and q satisfy (3) for $a = u_0$, $b = 0$, and $n = 2m$. Furthermore, if $m \geqslant 1$ then p and q also satisfy (2) with $a = u_0$, $b = 0$, and $n = 2m - 1$.

In order to prove Theorem 6, we shall need

Lemma 7. Let $u_0 + v_0\sqrt{D}$ be a fundamental solution to $u^2 - 2v^2 = C$. For $m \geqslant 0$, let

$$u_m + v_m\sqrt{2} = (u_0 + v_0\sqrt{2})(3+2\sqrt{2})^m .$$

We have that

$$u_m + v_m = v_0 P_{2m+2} + (u_0 - v_0)P_{2m+1} = (u_0 + v_0)P_{2m+1} + v_0 P_{2m} \tag{5}$$

and

$$v_m = v_0 P_{2m+1} + (u_0 - v_0)P_{2m} = (u_0 + v_0)P_{2m} + v_0 P_{2m-1} . \tag{6}$$

Proof. The second equality in both (5) and (6) follows directly from $P_{n+2} = 2P_{n+1} + P_n$. We shall prove the first equality in both (5) and (6) by induction on m. Clearly (5) and (6) are true for $m = 0$. Thus assume (5) and (6) are true for $m = k$. Now

$$u_{k+1} + v_{k+1}\sqrt{2} = (u_k + v_k\sqrt{2})(3+2\sqrt{2}) = (3u_k + 4v_k) + (2u_k + 3v_k)\sqrt{2} .$$

Hence

$$\begin{aligned} u_{k+1} + v_{k+1} = 5u_k + 7v_k &= 5(u_k + v_k) + 2v_k = 5v_0P_{2k+2} + 5(u_0 - v_0)P_{2k+1} + 2v_0P_{2k+1} + 2(u_0 - v_0)P_{2k} \\ &= 5v_0P_{2k+2} + [5(u_0 - v_0) + 2v_0]P_{2k+1} + 2(u_0 - v_0)\ (P_{2k+2} - 2P_{2k+1}) \\ &= (u_0 + v_0)(2P_{2k+2} + P_{2k+1}) + v_0P_{2k+2} \\ &= (u_0 + v_0)P_{2k+3} + v_0P_{2k+2} = v_0P_{2k+4} + (u_0 - v_0)P_{2k+3} . \end{aligned}$$

Also

$$\begin{aligned} v_{k+1} &= 2v_k + 3v_k = 2(u_k + v_k) + v_k = 2[v_0P_{2k+2} + (u_0 - v_0)P_{2k+1}] + v_0P_{2k+1} + (u_0 - v_0)P_{2k} \\ &= 2v_0P_{2k+2} + 2u_0P_{2k+1} - v_0P_{2k+1} + (u_0 - v_0)(P_{2k+2} - 2P_{2k+1}) \\ &= (u_0 + v_0)P_{2k+2} + v_0P_{2k+1} = v_0P_{2k+3} + (u_0 - v_0)P_{2k+2} . \end{aligned}$$

Now we are ready for the

Proof of Theorem 6. Assume $p \geqslant q \geqslant 0$ and $p^2 - q^2 - 2pq = \epsilon N$. By (1) – (6), we have

$$p = v_0P_{2m+2} + (u_0 - v_0)P_{2m+1} = (u_0 + v_0)P_{2m+1} + v_0P_{2m} \tag{7}$$

and

$$q = v_0P_{2m+1} + (u_0 - v_0)P_{2m} = (u_0 + v_0)P_{2m} + v_0P_{2m-1} , \tag{8}$$

where $u_0 + v_0\sqrt{2}$ is a fundamental solution to $u^2 - 2v^2 = \epsilon N$ and $m \geqslant 0$.

If $\epsilon u_0 v_0 < 0$ and $\epsilon = -1$ then let $a = v_0$, $b = v_0 - u_0$, and $n = 2m$. For this choice of a, b, and n, by (7) and (8), we have that (2) is satisfied. We also have that $a > b$, $b \geqslant 0$ (by Theorem 2 with $D = 2$) and $n \geqslant 0$.

If $\epsilon u_0 v_0 < 0$ and $\epsilon = 1$ then let $a = u_0 + v_0$, $b = -v_0$, and $n = 2m - 1$. For this choice of a, b, and n, we have that (2) is satisfied. We also have that $a \geqslant b$ (by Theorem 2), and $b > 0$. Finally $m \neq 0$ since $m = 0$ implies, by (4), the contradiction $q = v_0 < 0$. Thus $m \geqslant 1$.

The proof for $\epsilon u_0 v_0 \geqslant 0$ and the verification that $(a+b)^2 - 2b^2 = N$ are left to the reader.

Theorem 8. If p and q are integers which satisfy (2) or (3) with $n \geqslant 0$, $a \geqslant b \geqslant 0$, and $(a+b)^2 - 2b^2 = N$, then $p \geqslant q \geqslant 0$ and $p^2 - q^2 - 2pq = \epsilon N$, where $\epsilon = 1$ or -1. We have $\epsilon = -1$ for either p and q satisfying (2) and n even or p and q satisfy (3) with n odd. Otherwise $\epsilon = 1$.

Proof. First suppose p and q satisfy (2). Thus $p = aP_{n+2} - bP_{n+1}$ and

$$q = aP_{n+1} - bP_n = -bP_{n+2} + (a+2b)P_{n+1}.$$

Hence,

$$p^2 - q^2 - 2pq = (a^2 + 2ab - b^2)(P_{n+2}^2 - 2P_{n+2}P_{n+1} - P_{n+1}^2) = N(-1)^{n+1} = \epsilon N,$$

where $\epsilon = -1$ for n even and $\epsilon = 1$ for n odd. Now we shall show that $p \geqslant q \geqslant 0$. Since $n \geqslant 0$,

$$P_{n+2} - P_{n+1} = P_{n+1} + P_n \geqslant P_{n+1} - P_n.$$

Therefore, since $a \geqslant b$,

$$aP_{n+2} - aP_{n+1} \geqslant bP_{n+1} - bP_n.$$

This implies $p \geqslant q$. Since $a \geqslant b$ and, for $n \geqslant 0$, $P_{n+1} \geqslant P_n$, we see that $aP_{n+1} \geqslant bP_n$ and this implies $q \geqslant 0$.

If p and q satisfy (3) then

$$p^2 - q^2 - 2pq = N(-1)^{n+2} = \epsilon N,$$

where $\epsilon = -1$ for n odd and $\epsilon = 1$ for n even. Since $n \geqslant 0$, $P_{n+2} \geqslant P_{n+1}$ and $P_{n+1} \geqslant P_n$. Hence

$$p = bP_{n+2} + aP_{n+1} \geqslant bP_{n+1} + aP_n = q.$$

Since $a \geqslant 0$, $b \geqslant 0$, $P_{n+1} > 0$, and $P_n \geqslant 0$, $q = bP_{n+1} + aP_n > 0$.

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ON POLYNOMIALS GENERATED BY TRIANGULAR ARRAYS

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In this paper we study a class of functions which we call Pascal functions, generated by the diagonals of triangular arrays, and discuss some of their properties. The Fibonacci polynomials become particular cases of Pascal functions, and so our results are of a fairly general nature.

1. DEFINITIONS AND GENERAL PROPERTIES

Consider a polynomial function in two variables, $p(x,y)$. It is defined to be a Pascal function of $(k-1)^{st}$ order if

$$(1) \qquad p(x,y) = \sum_{m=0}^{[n/k]} a_m x^{n-km} y^m ,$$

where the a_m are non-zero constants, and $[x]$ represents, for real x, the largest integer not exceeding x. Let us denote the set of all Pascal functions (polynomials) of k^{th} order by Π_k. (Note: k is a positive integer.)

One generalization of the famous Fibonacci polynomials is

$$F_0(x,y) = 0, \quad F_1(x,y) = 1, \quad F_{n+2}(x,y) = xF_{n+1}(x,y) + yF_n(x,y), \quad n = 0, 1, 2, \cdots .$$

We find that

$$F_n(x,y) \in \Pi_1, \qquad n = 0, 1, 2, 3, \cdots .$$

See Hoggatt and Long [1]. It is interesting to note that the following properties hold:

Lemma 1. If $p(x,y)$ and $p^*(x,y)$ are in Π_k, then $q(x,y)$ is in Π_k, where

$$q(x,y) = p(x,y)p^*(x,y).$$

This is the same as saying that Π_k is closed under multiplication.

If $p(x,y) \in \Pi_{k-1}$, and has an expansion as given in (1), then let $D(p) = n$. We then have

Lemma 2. If $p(x,y)$ and $p^*(x,y)$ are in Π_k, then

$$q(x,y) = p(x,y) + p^*(x,y)$$

is in Π_k if and only if $D(p) = D(p^*)$.

Lemma 3. If $p(x,y)$ is in Π_k, then

$$\frac{\partial p(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial p(x,y)}{\partial y}$$

are in Π_k.

The three lemmas given above can be proved easily.

We define a sequence of functions

$$\left\{ p_n(x,y) \right\}_0^\infty \in \Pi_k$$

to be proper if

$$(2) \qquad D(p_{n+1}) = D(p_n) + 1 \qquad \text{with} \qquad D(p_0) = D(p_1) = 0.$$

By a Pascal array we mean a triangular array of numbers represented in Fig. 1 below:

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$$\begin{array}{ccccc}
c_{0,0} & & & & \\
c_{1,0} & c_{1,1} & & & \\
c_{2,0} & c_{2,1} & c_{2,2} & & \\
c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Figure 1

If now we replace every $c_{i,j}$ by $c_{i,j}x^iy^j$, and take the rising diagonal sums, where the rising diagonals have a slope k, we get a proper sequence in Π_k. Conversely, to every proper sequence in Π_k, we can associate a triangular array as in Fig. 1. Note that we can get infinitely many proper sequences from Fig. 1 as k varies, and all of these sequences for different values of k, we call "associated sequences." The triangular array which generates these sequences, is called their "associated array."

We now discuss some special properties of $p(x,y) \in \Pi_k$.

2. SOME SPECIAL PROPERTIES OF PASCAL FUNCTIONS

Theorem 1. Consider the proper sequence of Pascal functions $\left\{p_n(x,y)\right\}_{n=0}^{\infty} \in \Pi_k$ satisfying

$$p_{n+1}(x,y) = axp_n(x,y) + ayp_{n-k}(x,y), \qquad n \geqslant k, \tag{3}$$

with

$$p_0(x,y) = 0, \quad p_1(x,y) = a, \quad p_2(x,y) = a^2x, \quad \cdots, \quad p_k(x,y) = a^kx^{k-1}.$$

Then

$$\frac{\partial p_n(x,y)}{\partial x} = \frac{\partial p_{n+k}(x,y)}{\partial y} = \sum_{k=0}^{n} p_k(x,y)p_{n-k}(x,y). \tag{4}$$

Proof. One can establish the first part of (4) by induction. It is clear from (3) that

$$\frac{\partial p_{n+1}(x,y)}{\partial x} = ax\frac{\partial p_n(x,y)}{\partial x} + ap_n(x,y) + ay\frac{\partial p_{n-k}(x,y)}{\partial x} \tag{5}$$

and

$$\frac{\partial p_{n+k+1}(x,y)}{\partial y} = ax\frac{\partial p_{n+k}(x,y)}{\partial y} + ap_n(x,y) + ay\frac{\partial p_n(x,y)}{\partial y}. \tag{6}$$

The form of (5) and (6) together with the fact that the first part of (4) holds for $n = 1, 2, 3, \cdots, k$, proves it by induction. We now want to show

$$\frac{\partial p_n(x,y)}{\partial x} = \sum_{k=0}^{n} p_k(x,y)p_{n-k}(x,y). \tag{7}$$

Consider the generating function

$$G(t) = \sum_{n=0}^{\infty} p_n(x,y)t^n = \frac{at}{1 - axt - ayt^{k+1}}.$$

We have

$$\sum_{n=0}^{\infty} \frac{\partial p_n(x,y)}{\partial x}t^n = \frac{\partial G(t)}{\partial x} = \frac{a^2t^2}{(1 - axt - ayt^{k+1})^2} = [G(t)]^2.$$

This proves (7) and so we have established Theorem 1.

Corollary. For the Fibonacci polynomials defined before,

$$\frac{\partial F_n(x,y)}{\partial x} = \frac{\partial F_{n+1}(x,y)}{\partial y} = \sum_{k=0}^{n} F_k(x,y)F_{n-k}(x,y).$$

Proof. The corollary follows by taking $k = 1$ in Theorem 1.

Theorem 2. If

$$\frac{\partial p_n(x,y)}{\partial x} = p_{n,1}(x,y),$$

then define

$$p_{n,r}(x,y) = \sum_{k=0}^{n} p_{k,r-1}(x,y)P_{n-k}(x,y). \tag{8}$$

Now

$$p_{n,r}(x,y) = \frac{1}{r!}\frac{\partial^r p_n(x,y)}{\partial x^r}.$$

Proof. Differentiate the generating function $G(t)$ in the proof of Theorem 1, r times. Theorem 2 follows.

Theorem 3. If a proper sequence of Pascal functions

$$\left\{p_n(x,y)\right\}_{n=0}^{\infty} \in \Pi_k$$

satisfy (4), then they satisfy (3). (Converse of Theorem 1.)

Proof. Consider the first $(k+1)$ members of the sequence

$$a_0,\ a_1,\ a_2x,\ a_3x^2,\ \cdots,\ a_kx^{k-1}.$$

Because of (4) we have

$$\frac{d}{dx}(a_0) = 2a_0a_1,$$

and $a_1 \neq 0$, which gives $a_0 = 0$.

Further,

$$\frac{d}{dx}(a_2x) = a_2 = a_1^2.$$

Similarly, one may show

$$a_r = a_1^r = a^r, \qquad r = 1, 2, \cdots, k.$$

Now assume that (3) holds for $n = 0, 1, 2, 3, \cdots, m$. Let now

$$p_{m+1}^*(x,y) = \sum_{k=1}^{m} p_k(x,y)p_{m-k+1}(x,y).$$

Clearly, by Lemmas 1 and 2, we have $p_{m+1}^*(x,y) \in \Pi_k$.

Now, denote

$$p_{m+1}^{**}(x,y) = axp_m(x,y) + ayp_{m-k}(x,y).$$

We have because of Theorem 1

$$\frac{\partial p_{m+1}^{**}(x,y)}{\partial x} = p_{m+1}^*(x,y).$$

But we know, because $p_0(x,y) = 0$,

$$\frac{\partial p_{m+1}(x,y)}{\partial x} = p_{m+1}^*(x,y)$$

and this gives

$$p_{m+1}(x,y) = p_{m+1}^{**}(x,y)$$

by (1) and by Lemma 3. This proves that (3) holds, by mathematical induction. Hence we get Theorem 3.

3. PASCAL FUNCTIONS WHICH CAN BE PASCALISED

We now shift our attention to Pascal functions which can be "pascalised." Given a proper sequence of Pascal polynomials

$$\left\{ p_n(x,y) \right\}_{n=0}^{\infty} \in \Pi_k ,$$

form the associated array $\left\{ a_{ij} \right\} = A$. Now take

$$q_n(x,y) = \frac{\partial p_n(x,y)}{\partial x}$$

to get a new proper sequence in Π_k. Let $\left\{ b_{ij} \right\} = B$ be the associated Pascal array to this sequence. If we have the relation

$$b_{ij} = a_{ij} \binom{i+1}{1} \tag{9}$$

we say $\left\{ p_n(x,y) \right\}$ can be "pascalised" to the first order. If

$$q_n(x,y) = \frac{\partial^r p_n(x,y)}{\partial x^r}$$

and

$$b_{ij} = a_{ij} \binom{i+r}{r} r! \tag{10}$$

we say that the sequence $\left\{ p_n(x,y) \right\}$ can be pascalised to the r^{th} order.

Theorem 4. A necessary and sufficient condition that a proper sequence of $(k-1)^{st}$ order Pascal functions $\left\{ p_n(x,y) \right\}_{n=0}^{\infty}$ can be pascalised to the first order is that

$$p_n(x,y) = \sum_{j=0}^{[n/k]} a_j \binom{n-(k-1)j-1}{j} x^{n-kj-1} y^j \tag{11}$$

for some sequence of constants a_j.

Proof. We will first prove the theorem for the case $k = 2$. Consider the sequence $\left\{ p_n(x,y) \right\}_{n=0}^{\infty}$, and assume that the identity holds for $n = 0, 1, 2, \cdots, m$. We have then

$$p_m(x,y) = \sum_{j=0}^{[m/2]} a_j \binom{m-j-1}{j} x^{m-2j-1} y^j . \tag{12}$$

Now let

$$p_{m+1}(x,y) = \sum_{j=0}^{[(m+1)/2]} a_{j,m} \binom{m-j}{j} x^{m-2j} y^j$$

which gives

$$\frac{\partial p_{m+1}(x,y)}{\partial x} = \sum_{j=0}^{[m/2]} a_{j,m} \binom{m-j}{j} x^{m-2j-1} y^j (m-2j) . \tag{13}$$

Now comparing coefficients in (12) and (13) and using (9) we get

$$\binom{m-j}{j} (m-2j) a_{j,m} = \binom{m-j-1}{j} a_j$$

which gives

$$a_{j,m} = a_j$$

establishing part of the theorem for $k = 2$. The converse can be proved by retracing the steps.

Now, once the theorem is proved for the first order ($k = 2$), it holds for any $k \geqslant 1$, for given a proper sequence of Pascal functions of $(k-1)^{st}$ order, we can find its associated sequence of first order. The Pascal arrays for the derivatives of these two sequences is the same since the operator $\partial/\partial x$ will operate independently in the expansion of $p_n(x,y)$ with respect to coefficients in the associated Pascal array. This completes the proof of the theorem.

Theorem 5. If a proper sequence of k^{th} order Pascal functions can be pascalised to the first order, then all their associated sequences can be pascalised to first order.

Proof. Given in the last paragraph of the proof of Theorem 4.

Theorem 6. If a proper sequence of k^{th} order Pascal functions can be pascalised to first order, they can be pascalised to any order.

Proof. By arguments similar to the above, it is enough if we prove it for $k = 1$. Furthermore, it is enough to prove the theorem for the special case $a_j = 1$ for differential operators are unaffected by constant multiples.

We know from Theorem 4 that the first-order proper sequence of Pascal functions which can be pascalised to first order can be put in the form

$$p_n(x,y) = \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} x^{n-2j-1} y^j a_j .$$

Now, as mentioned, $a_j = 1$, so that $p_n(x,y) = F_n(x,y)$, the Fibonacci polynomials. We then have

$$\frac{1}{r!}\,\frac{\partial p_{n+r+1}(x,y)}{\partial x^r} = \frac{1}{r!}\sum_{j=0}^{[(n+r)/2]} \frac{\partial}{\partial x^r}\;\frac{\binom{n+r-j}{j} x^{n+r-2j} y^j}{r!}$$

$$= \sum_{n+r-2j \geqslant 0} \binom{n+r-2j}{r}\binom{n+r-j}{j} x^{n-2j} y^j$$

which resembles (9) proving our theorem for Fibonacci polynomials, and so for Pascal functions. We demonstrate our result with the following:

Pascal Array for $F_n(x,y)$

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
...	...	...	...	...	...

Pascal Array for $\dfrac{F_n(x,y)}{x}$

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
...	...	...	...	...	...

Note 1: (2,2) = 2(1,1); (3,6,3) = 3(1,2,1); (4,12,12,4) = 4(1,3,3,1); $\cdots$. Each row has a common factor.

Note 2: Theorem 4 also says that each column has a common factor a_j. In the above all the $a_j = 1$.

Note 3: The Pascal array for $[\partial F_n(x,y)]/\partial x$ is also the Pascal array for

$$\sum_{k=0}^{n} F_k(x,y)F_{n-k}(x,y)$$

for both are equal by Theorem 1.

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1. V. E. Hoggatt, Jr., and Calvin T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," *The Fibonacci Quarterly,* Vol. 12, No. 3 (April 1974), pp. 113–120.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-267 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{kn+1} \frac{(knx)^n}{n!}$$

satisfies $S(x) = e^{xS^k(x)}$.

H-268 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind defined by

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

Show that

$$\left\{ \begin{aligned} xS_n(x) &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} S_{j+1}(x) \\ S_{n+1}(x) &= x \sum_{j=0}^{n} \binom{n}{j} S_j(x). \end{aligned} \right.$$

More generally evaluate the coefficients $c(n,k,j)$ in the expansion

$$x^k S_n(x) = \sum_{j=0}^{n+k} c(n,k,j) S_j(x) \qquad (k, n \geqslant 0).$$

SOLUTIONS
SYSTEMATIC WORK

H-244 Proposed by L. Carlitz and T. Vaughan, Durham, North Carolina and Greensboro, North Carolina.

Solve the system of equations

$$(*)\qquad x_j = a_j + \mu a_j \sum_{t=1}^{j-1} x_t + a_j x_j + \lambda a_j \sum_{t=j+1}^{n} x_t \quad (j = 1, 2, \cdots, n)$$

for $x = x_1 + x_2 + \cdots + x_n$, where $a_k \neq 0$ $(k = 1, 2, \cdots, n)$ and $\lambda \neq \mu$.

Solution by the Proposer.

For $j = 1$, (*) reduces to

$$x_1 = a_1 + a_1 x_1 + \lambda a_1 (x - x_1),$$

so that

$$\big(1 - (1-\lambda)a_1\big)x_1 = a_1(\lambda x + 1).$$

For $j = 2$, (*) becomes

$$x_2 = a_2 + \mu a_2 x_1 + a_2 x_2 + \lambda a_2 (x - x_1 - x_2),$$

so that

$$\big(1 - (1-\lambda)a_2\big)x_2 = a_2(\lambda x + 1) - a_2(\lambda - \mu)x_1 .$$

Hence

$$\big(1 - (1-\lambda)a_1\big)\big(1 - (1-\lambda)a_2\big)x_2 = a_2\big(1 - (1-\mu)a_1\big)(\lambda x + 1).$$

Similarly, for $j = 3$,

$$x_3 = a_3 + \mu a_3 (x_1 + x_2) + a_3 x_3 + \lambda a_3 (x - x_1 - x_2 - x_3).$$

After a little manipulation we get

$$\big(1 - (1-\lambda)a_1\big)\big(1 - (1-\lambda)a_2\big)\big(1 - (1-\lambda)a_3\big)x_3 = a_3\big(1 - (1-\mu)a_1\big)\big(1 - (1-\mu)a_2\big)(\lambda x + 1).$$

The general formula

$$(1)\qquad f_1(\lambda)f_2(\lambda)\cdots f_k(\lambda)x_k = a_k f_1(\mu)\cdots f_{k-1}(\mu)(\lambda x + 1) \qquad (1 \leqslant k \leqslant n),$$

where

$$f_k(\lambda) = 1 - (1-\lambda)a_k ,$$

is now easily proved by induction on k.

Returning to (*), we take $j = n$. Thus

$$x_n = a_n + \mu a_n (x - x_n) + a_n x_n ,$$

so that

$$\big(1 - (1-\mu)a_n\big)x_n(\mu x + 1).$$

For $j = n - 1$ we get

$$x_{n-1} = a_{n-1} + \mu a_{n-1}(x - x_n - x_{n-1}) + a_{n-1}x_{n-1} + \lambda a_{n-1}x_n .$$

This gives

$$\big(1 - (1-\mu)a_n\big)\big(1 - (1-\mu)a_{n-1}\big)x_{n-1} = a_{n-1}\big(1 - (1-\lambda)a_n\big)(\mu x + 1).$$

Similarly, for $j = n - 2$,

$$\big(1 - (1-\mu)a_n\big)\big(1 - (1-\mu)a_{n-1}\big)\big(1 - (1-\mu)a_{n-2}\big)x_{n-2} = a_{n-2}\big(1 - (1-\lambda)a_n\big)\big(1 - (1-\lambda)a_{n-1}\big)(\mu x + 1).$$

The general formula

$$(2)\qquad f_n(\mu)f_{n-1}(\mu)\cdots f_{n-k+1}(\mu)x_{n-k+1} = a_{n-k+1}f_n(\lambda)\cdots f_{n-k+2}(\lambda)(\mu x + 1) \quad (1 \leqslant k \leqslant n)$$

is easily proved by induction.

In (2) replace k by $n - k + 1$:

$$(3)\qquad f_n(\mu)f_{n-1}(\mu)\cdots f_k(\mu)x_k = a_k f_n(\lambda)\cdots f_{k+1}(\lambda)(\mu x + 1) \qquad (1 \leqslant k \leqslant n).$$

Comparing (3) with (1) we get

$$a_k \frac{f_1(\mu)\cdots f_{k-1}(\mu)}{f_1(\lambda)f_2(\lambda)\cdots f_k(\lambda)}(\lambda x + 1) = a_k \frac{f_n(\lambda)\cdots f_{k+1}(\lambda)}{f_n(\mu)f_{n-1}(\mu)\cdots f_k(\mu)}(\mu x + 1).$$

Since $a_k \neq 0$, it follows that

$$F_n(\mu)(\lambda x + 1) = F_n(\lambda)(\mu x + 1),$$

where

(5)
$$F_n(\lambda) = \prod_{k=1}^{n} f_k(\lambda) = \prod_{k=1}^{n} \big(1 - (1-\lambda)a_k\big).$$

Solving (4) for x, we get

(6)
$$x = \frac{F_n(\lambda) - F_n(\mu)}{\lambda F_n(\mu) - \mu F_n(\lambda)} .$$

Since, by (6),

$$x + 1 = \frac{(\lambda - \mu)F_n(\lambda)}{\lambda F_n(\mu) - \mu F_n(\lambda)} .$$

Hence (1) gives

(7)
$$x_k = a_k \frac{F_{k-1}(\mu)}{F_k(\lambda)} \frac{(\lambda-\mu)F_n(\lambda)}{\lambda F_n(\mu) - \mu F_n(\lambda)} \qquad (1 \leqslant k \leqslant n),$$

where

$$F_k(\lambda) = \prod_{j=1}^{k} f_k(\lambda) \qquad (1 \leqslant k \leqslant n).$$

PRODUCTIVE IDENTITY

H-245 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Prove the identity

(1)
$$\sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} = \frac{2\prod_{r=1}^{n-1}(1+x^r)}{(x)_n}, \qquad n = 1, 2, \cdots,$$

where

$$(x)_n = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n), \qquad n = 1, 2, \cdots; \ (x)_0 = 1.$$

Solution by the Proposer.

Lemma 1. If

$$A(w,x) = \prod_{r=1}^{\infty} (1 + x^r w), \qquad \text{then} \qquad A(w,x) = \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}n(n+1)}}{(x)_n} w^n .$$

Proof. In a previously submitted proposed problem for this section [H-236], the author established the following identity:

(2)
$$f(z,y) = \prod_{r=1}^{\infty} (1 + y^{2r-1}z) = \sum_{n=0}^{\infty} \frac{y^{n^2}}{(y^2)_n} z^n .$$

Letting $y = x^{\frac{1}{2}}$, $z = wx^{\frac{1}{2}}$ in this identity, we find that the lemma is established, with $A(w,x) = f(wx^{\frac{1}{2}}, x^{\frac{1}{2}})$.

Lemma 2. If

$$B(w,x) = \prod_{r=1}^{\infty} (1 - x^r w)^{-1}, \qquad \text{then} \qquad B(w,x) = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} w^n .$$

Proof. This is equivalent to identity (7) in the above-mentioned problem. Now, consider the product

$$F(w,x) = A(w,x)B(w,x),$$

which is also equal to

$$\prod_{r=1}^{\infty} \frac{1 + x^r w}{1 - x^r w} .$$

We observe that

$$F(wx,x) = \prod_{r=1}^{\infty} \frac{1+x^{r+1}w}{1-x^{r+1}w} = \prod_{r=2}^{\infty} \frac{1+x^r w}{1-x^r w} = \frac{1-xw}{1+xw} F(w,x).$$

Now suppose

$$F(w,x) = \sum_{n=0}^{\infty} \theta_n(x) w^n .$$

We then have

$$(1-xw)\sum_{n=0}^{\infty} \theta_n(x)w^n = (1+xw)\sum_{n=0}^{\infty} \theta_n(x)(xw)^n ,$$

which yields the recursion

$$\theta_n(x) = \frac{x(1+x^{n-1})}{1-x^n}\,\theta_{n-1}(x), \qquad n = 1, 2, \cdots .$$

Since $F(0,x) = 1 = \theta_0(x)$, we readily obtain, by induction, that

$$\theta_n(x) = \frac{2x^n(1+x)(1+x^2)\cdots(1+x^{n-1})}{(x)_n}, \quad n = 1, 2, \cdots, \text{ with } \theta_0(x) = 1.$$

Hence,

$$F(w,x) = \prod_{r=1}^{\infty} \frac{1+x^r w}{1-x^r w} = 1 + 2\sum_{n=1}^{\infty} \frac{x^n(1+x)\cdots(1+x^{n-1})}{(x)_n} w^n . \tag{3}$$

However, since

$$F(w,x) = A(w,x)B(w,x) = \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}n(n+1)}}{(x)_n} w^n \cdot \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} w^n ,$$

we also obtain the formula

$$F(w,x) = \sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k+1)}}{(x)_k} \frac{x^{n-k}}{(x)_{n-k}} . \tag{4}$$

Comparing coefficients of w in (3) and (4), we obtain for $n = 1, 2, \cdots$,

$$\frac{2x^n(1+x)\cdots(1+x^{n-1})}{(x)_n} = \sum_{k=0}^{n} \frac{x^{n+\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} .$$

Upon dividing each side by x^n, we find that (1) is established.

Also solved by P. Tracy and A. Shannon.

FIB, LUC, ET AL

H-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$F(m,n) = \sum_{i=0}^{m} \sum_{j=0}^{n} F_{i+j}F_{m-i+j}F_{i+n-j}F_{m-i+n-j}$$

$$L(m,n) = \sum_{i=0}^{m} \sum_{j=0}^{n} L_{i+j}L_{m-i+j}L_{i+n-j}L_{m-i+n-j} .$$

[Continued on page 473.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-340 Proposed by Phil Mana, Albuquerque, New Mexico.

Characterize a sequence whose first 28 terms are:

1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875, 1880, 1886, 1897, 1909, 1915, 1920, 1926, 1937, 1943, 1948, 1954, 1965, 1971, 1976.

B-341 Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, New York.

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

B-342 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Prove that $2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1}$ is a perfect cube for $n = 1, 2, \cdots$.

B-343 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple expression for

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)}].$$

B-344 Proposed by Frank Higgins, Naperville, Illinois.

Let c and d be real numbers. Find $\lim_{n \to \infty} x_n$, where x_n is defined by $x_1 = c$, $x_2 = d$, and

$$x_{n+2} = (x_{n+1} + x_n)/2 \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

B-345 Proposed by Frank Higgins, Naperville, Illinois.

Let $r > s > 0$. Find $\lim_{n \to \infty} P_n$, where P_n is defined by $P_1 = r + s$ and $P_{n+1} = r + s - (rs/P_n)$ for $n = 1, 2, 3, \cdots$.

SOLUTIONS

A FIBONACCI ALPHAMETIC

B-316 Proposed by J. A. H. Hunter, Fun with Figures, *Toronto, Ontario, Canada.*

Solve the alphametic

$$\begin{array}{ccccc} & & T & W & O \\ T & H & R & E & E \\ \underline{T} & \underline{H} & \underline{R} & \underline{E} & \underline{E} \\ E & I & G & H & T \end{array}$$

Believe it or not, there must be no 8 in this!

Solution by Charles W. Trigg, San Diego, California.

$T < 5$, and no letter represents 8. There are five cases to consider.

(1) If $2T + 1 = E$, and $T = 1$, then $E = 3$, and $O = 5$.
If $H = 6$, then $W = 9$, $I = 2$, and $2 + 2R = G$, impossible.
If $H = 7$, then $W = 0$, $I = 4$, and $I + 2R = G$, impossible.
If $H = 9$, then $I = 8$, which is prohibited.

(2) If $2T + 1 = E$ and $T = 3$, then $E = 7$ and $O = 9$.
If $H = 4$, then $W = 8$, prohibited.
If $H = 5$, then $W = 9 = O$.
If $H = 6$, then $W = 0$, and $4 + 2R = G$, impossible.

(3) If $2T + 1 = E$ and $T = 4$, then $E = 9$, $O = 6$, and $H = W$.

(4) If $2T = E$ and $T = 1$, then $E = 2$ and $O = 7$.
If $H = 3$, then $W = 8$, which is prohibited.
If $H = 4$, then $W = 9$ and $I = 8$ or 9.

(5) If $2T = E$ and $T = 3$, then $E = 6$ and $O = 1$.
If $H = 4$, then $W = 1 = O$.
If $H = 2$, then $W = 9$ and $5 + 2R = G$ or $G + 10$.

Whereupon, $R = 0$, $G = 5$, and $I = 4$. Thus the unique reconstructed addition is

$$391 + 32066 + 32066 = 65423\,.$$

Also solved by Nancy Barta, Richard Blazej, Paul S. Bruckman, John W. Milsom, C. B. A. Peck, James F. Pope, and the Proposer.

LUCAS DIVISOR

B-317 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that L_{2n-1} is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, \cdots$.

Solution by Gerald Bergum, Brookings, South Dakota.

Using the Binet formula together with $\alpha\beta = -1$ and $\alpha + \beta = 1$ we have

$$L_{2n}L_{2n-1} = (\alpha^{2n} + \beta^{2n})(\alpha^{2n-1} + \beta^{2n-1}) = \alpha^{4n-1} + \beta^{4n-1} + (\alpha\beta)^{2n-1}(\alpha+\beta) = L_{4n-1} - 1.$$

Also solved by M. D. Agrawal, George Berzsenyi, Richard Blazej, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI SQUARE

B-318 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, \cdots$.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Using well known identities (see, for example, I_{21} and I_7 in Hoggatt's *Fibonacci and Lucas Numbers)* one finds that

$$F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n}) = F_{4n}^2 + 8F_{2n}(F_{4n}L_{2n}) = F_{4n}^2 + 8F_{4n}(F_{2n}L_{2n}) = F_{4n}^2 + 8F_{4n}^2$$
$$= 9F_{4n}^2 = (3F_{4n})^2\,.$$

Also solved by M. D. Agrawal, Gerald Bergum, Richard Blazej, Wray G. Brady, Ralph Garfield, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

RERUN

B-319 Prove or disprove:

$$\frac{1}{L_2} + \frac{1}{L_6} + \frac{1}{L_{10}} + \cdots = \frac{1}{\sqrt{5}} \left(\frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots \right).$$

Solution (independently) by Carl F. Moore, Tacoma, Washington, and C. B. A. Peck, State College, Pennsylvania.

This problem is a restatement of the problem B-111, proposed and solved by L. Carlitz, *The Fibonacci Quarterly*, Vol. 5, No. 4 (Dec. 1967), p. 470.

Also solved by Paul S. Bruckman, Mike Hoffman, and the Proposer.

A SUM

B-320 Proposed by George Berzsenyi, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m}.$$

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

$$\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.$$

If n is even, we have,

$$\sum_{k=0}^{n} F_k F_{k+2m} = F_n F_{n+2m+1}.$$

If n is odd, we have

$$\sum_{k=0}^{n} F_k F_{k+2m} = F_{n-1} F_{n+2m} + F_n F_{n+2m} = F_{n+1} F_{n+2m}.$$

Also solved by M. D. Agrawal, Paul S. Bruckman, Herta T. Freitag, Frank Higgins, Graham Lord, Carl F. Moore, C. B. A. Peck, James F. Pope, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

A RELATED SUM

B-321 Proposed by George Berzsenyi, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m+1}.$$

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

$$\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.$$

If n is even, we have

$$\sum_{k=0}^{n} F_k F_{k+2m+1} = F_n F_{n+2m+2}.$$

If n is odd, we have

$$\sum_{k=0}^{n} F_k F_{k+2m+1} = F_{n-1} F_{n+2m+1} + F_n F_{n+2m+1} = F_{n+1} F_{n+2m+1}.$$

Also solved by the sovlers of B-320.

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[Continued from page 469.]

ADVANCED PROBLEMS AND SOLUTIONS

Show that

$$L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}.$$

Solution by the Proposer.

It follows from the Binet formulas

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \qquad L_m = \alpha^m + \beta^m$$

that

$$5F_m F_n = L_{m+n} - (\alpha^m \beta^n + \alpha^n \beta^m),$$

so that

$$5F_{i+j} F_{m-i+n-j} = L_{m+n} - (\alpha^{i+j} \beta^{m-i+n-j} + \alpha^{m-i+n-j})$$
$$5F_{i+n-j} F_{m-i+j} = L_{m+n} - (\alpha^{i+n-j} \beta^{m-i+j} + \alpha^{m-i+j} \beta^{i+n-j}).$$

Hence

$$\begin{aligned} 25F_{i+j} F_{m-i+j} F_{i+n-j} F_{m-i+n-j} = L^2_{m+n} - L_{m+n}(&\alpha^{i+j}\beta^{m-i+n-j} + \alpha^{m-i+n-j}\beta^{i+j} \\ &+ \alpha^{i+n-j}\beta^{m-i+j} + \alpha^{m-i+j}\beta^{i+n-j}) \end{aligned}$$
$$+ (\alpha^{2i+n}\beta^{2m-2i+n} + \alpha^{2m-2i+n}\beta^{2i+n} + \alpha^{m+2j}\beta^{m+2n-2j} + \alpha^{m+2n-2j}\beta^{m+2j}).$$

It follows that

$$25F(m,n) = (m+1)(n+1)L^2_{m+n} - 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^n (n+1) F_{2m+2} + 2(-1)^m (m+1) F_{2n+2}.$$

Similarly,

$$L(m,n) = (m+1)(n+1)L^2_{m+n} + 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^n (n+1) F_{2m+2} + 2(-1)^m (m+1) F_{2n+2}.$$

Therefore,

$$L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}.$$

Also solved by P. Bruckman.

EDITORIAL REQUEST! Send in your problem proposals!

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VOLUME INDEX

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