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THE TRIBONACCI SEQUENCE

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By definition, a Fibonacci sequence consists of numbers equal to the sum of the preceding two. Symbolically, this means that any term

$$F_n = F_{n-1} + F_{n-2}.$$

This definition can be expanded to define any term as the sum of the preceding three.

It is the purpose of this paper to examine this new sequence that we will call the TRIBONACCI SEQUENCE (the name obviously resulting from "tri" meaning three (3)). Therefore, let us define this new sequence as T and consisting of terms:

$$T_1, T_2, T_3, T_4, T_5, \cdots, T_n, \cdots,$$

where we will define

$$T_1 = 1, \quad T_2 = 1, \quad T_3 = 2$$

and any following term as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}.$$

For any further study of this sequence, it will be useful to know the generating function of these numbers. To find this generating function, let the terms of the sequence be the coefficients of an infinite polynomial $T(x)$ giving

$$T(x) = T_1 + T_2x + T_3x^2 + T_4x^3 + \cdots + T_nx^{n-1} + \cdots.$$

By multiplying this infinite polynomial first by $-x$, then by $-x^2$ and finally by $-x^3$, and then collecting like terms and substituting in appropriate values of $T_1, T_2, T_3, \cdots$, we get the following:

$$\begin{aligned}
T(x) &= T_1 + T_2x + T_3x^2 + T_4x^3 + T_5x^4 + \cdots \\
-xT(x) &= \quad - T_1x - T_2x^2 - T_3x^3 - T_4x^4 - \cdots \\
-x^2T(x) &= \qquad\quad - T_1x^2 - T_2x^3 - T_3x^4 - \cdots \\
-x^3T(x) &= \qquad\qquad\quad - T_1x^3 - T_2x^3 - \cdots
\end{aligned}$$

$$\begin{aligned}
T(x) - xT(x) - x^2T(x) - x^3T(x) &= T_1 = 1 \\
T(x)(1 - x - x^2 - x^3) &= 1 \\
T(x) &= \frac{1}{1 - x - x^2 - x^3}
\end{aligned}$$

Therefore, we have found the generating function of the Tribonacci sequence as $T(x)$ and can be verified by simple long division.

This Tribonacci sequence can be further examined in a convolution array. The first column of this array will be defined as the coefficients of $T(x)$. The second and subsequent columns can be found in two (2) ways:

(1) The first method is by convolution* (thus giving the title of the array). By convolving the first column with itself, the second column will result; by convolving the first with the second, we will get the third; the first and third to get the fourth and so on. It will also be noticed that the even-numbered columns are actually

---

**Convolution*: a folding upon itself.

It will be recalled that a mathematical convolution is as follows:

Given: Sequence 1 as $S_1, S_2, S_3, S_4, S_5, S_6, \cdots$
Sequence 2 as $P_1, P_2, P_3, P_4, P_5, P_6, \cdots$.

To find the sixth term of the resulting sequence:

$$(S_1)(P_6) + (S_2)(P_5) + (S_3)(P_4) + (S_4)(P_3) + (S_5)(P_2) + (S_6)(P_1).$$

squares. That is to say, to get the second column the first is convolved with itself; to get the fourth, the second is convolved with itself; the third with itself to arrive at the sixth and so on.

(2) The second method for deriving the same array clearly shows why the convolution array can also be called a power array. Recall that the first column is the Tribonacci sequence and is generated by the function

$$\frac{1}{1-x-x^2-x^3}\,.$$

To derive the second column, then, the first column generating function is squared. The third column is $T^3(x)$, the fourth column is $T^4(x)$ and so forth. Therefore we can represent the array as:

I.

Power of $T(x)$

Powers of x	1	2	3	4	5	6	7	8	⋯
0									
1									
2									
⋮									

And our specific array as:

II.

	1	2	3	4	5	6	7	8	9	10	11	⋯
0	1	1	1	1	1	1	1	1	1	1	1	⋯
1	1	2	3	4	5	6	7	8	9	10	11	⋯
2	2	5	9	14	20	27	35	44	54	⋯		
3	4	12	25	44	70	104	147	200	264	⋯		
4	7	26	63	125	220	⋯						
5	13	56	153	336	646	⋯						
6	⋮	⋮	⋮	⋮	⋮							
⋮												

This specific array can be found and verified in either of the two ways described above.

A third more simple method of deriving this same array is by the use of a recursion pattern or template. To find this template pattern, one must recall the power array (method 2 of getting the convolution array). We then realize that:

$$T(x) = \frac{1}{1-x-x^2-x^3}$$

generates the first column

$$T^2(x) = \left(\frac{1}{1-x-x^2-x^3}\right)^2$$

generates the second column and

$$T^3(x) = \left(\frac{1}{1-x-x^2-x^3}\right)^3$$

generates the third column or, we can rewrite this as:

$$T^n(x) = \left(\frac{1}{1-x-x^2-x^3}\right)^n$$

which itself can be rewritten as

$$T^n(x) = \left(\frac{1}{1-x-x^2-x^3}\right)\left(\frac{1}{1-x-x^2-x^3}\right)^{n-1}$$

$$T^n(x) = \left(\frac{1}{1-x-x^2-x^3}\right)T^{n-1}(x)$$

By multiplying both sides of this equation by $(1-x-x^2-x^3)$ we will get:

(a) $$T^n(x) = xT^n(x) + x^2T^n(x) + x^3T^n(x) + T^{n-1}(x)$$

or by collecting all the $T_n(x)$ terms, we get:

(b) $$T^{n-1}(x) = T^n(x) - xT^n(x) - x^2T^n(x) - x^3T^n(x).$$

In words, this means that the n^{th} column is equal to x times itself plus x^2 times itself plus x^3 times itself plus the previous column. For a specific example, let us examine $T^4(x)$.

Therefore:

$$T^n(x) = T^4(x) = 1 + 4x + 14x^2 + 44x^3 + 125x^4 + \cdots$$
$$T^{n-1}(x) = T^3(x) = 1 + 3x + 9x^2 + 25x^3 + 63x^4 + \cdots.$$

By substituting this in Eq. (b) above:

$$T^4(x) - xT^4(x) - x^2T^4(x) - x^3T^4(x) = T^3(x)$$

$$\begin{array}{rl}
T^4(x) = & 1 + 4x + 14x^2 + 44x^3 + 125x^4 + \cdots \\
-xT^4(x) = & \quad - x - 4x^2 - 14x^3 - 44x^4 - \cdots \\
-x^2T^4(x) = & \quad\quad - x^2 - 4x^3 - 14x^4 - \cdots \\
-x^3T^4(x) = & \quad\quad\quad - x^3 - 4x^4 - \cdots \\
\hline
= & 1 + 3x + 9x^2 + 25x^3 + 63x^4 - \cdots
\end{array}$$

which indeed is $T^3(x)$.

What we would like to do, however, is apply this method to a specific element in any column or row, rather than to entire columns. Let us again refer to the equation

$$T^n(x) = xT^n(x) + x^2T^n(x) + x^3T^n(x) + T^{n-1}(x)$$

and a specific element in the column. To translate this equation, refer to Array 1 on the previous page, and remember what each item in the array represents. Pictorially, then, the equation means the following (we will consider each element in the equation separately):

- $T^n(x)$: the specific element in a row and column that we are interested in. We will call it X.
- $xT^n(x)$: the element in the same column but up one row. The multiplier x has the effect of shifting it down one row. We will call this U.
- $x^2T^n(x)$: the element in the same column but up two rows. The x^2 has the effect of shifting it down two rows. We will call this V.
- $x^3T^n(x)$: the element in the same column but up three rows, shifted down by the factor of x^3. Call this W.
- $T^{n-1}(x)$: the element in the same row but the previous column. Call this Z.

Therefore, by this pattern we can find any element in the array through the use of a single template. The template (from the above equation) is:

	W
	V
	U
Y	X

$$X = U + V + W + Y$$

This template, then, because it is so general, will help to see relationships between other convolution arrays and numerator polynomial arrays which will be discussed now.

As we have seen, we know of a function that when expanded, will yield an infinite polynomial whose coefficients correspond to the Tribonacci numbers. We also know that this function, namely

$$\frac{1}{1-x-x^2-x^3}$$

when squared and expanded will yield the coefficients of the second column of the convolution array. We have seen that this function can also be cubed and expanded to give the entries in the third column of the array, and so on.

Suppose we wish to find a function or series of functions that will generate the *rows* of this convolution array.

Let us, then, consider the first row (actually called the zero$^{\text{th}}$ row, since rows correspond to the powers of x in the polynomials and the "first" row is the row of constants) of the array as coefficients of the infinite polynomial $R(x)$, giving

$$R(x) = 1+x+x^2+x^3+\cdots.$$

By mutliplying $R(x)$ by $-x$ and adding to $R(x)$, the following is obtained:

$$\begin{aligned} R(x) &= 1+x+x^2+x^3+x^4+\cdots \\ -xR(x) &= -x-x^2-x^3-x^4-\cdots \\ \hline (1-x)R(x) &= 1 \\ R(x) &= \frac{1}{1-x}\quad. \end{aligned}$$

Thus, $1/(1-x)$ will generate an infinite polynomial whose coefficients correspond to the zero$^{\text{th}}$ row of the Tribonacci array. It is also true that the function $(1/(1-x))^2$ will generate the first row of the array. However, $(1/(1-x))^3$ does not generate the second row.

As a result, the row generating function must be generalized to give all the rows. Let us call, then, the numerator of this function $r_n(x)$, giving:

$$R_n(x) = \frac{r_n(x)}{(1-x)^{n+1}}\quad.$$

The numerators then for row 0 and row 1 are simply equal to 1. For row 2, we will find $r_2(x)$ by simple algebra as follows:

$$\begin{aligned} \frac{r_2(x)}{(1-x)^3} &= 2+5x+9x^2+14x^3+20x^4+\cdots \\ r_2(x) &= (2+5x+9x^2+14x^3+20x^4+\cdots)(1-x)^3 \\ r_2(x) &= (2+5x+9x^2+14x^3+20x^4+\cdots)(1-3x+3x^2-x^3) \\ r_2(x) &= 2+5x+9x^2+14x^3+20x^4+\cdots \\ & -6x-15x^2-27x^3-42x^4-\cdots \\ & 6x^2+15x^3+27x^4+\cdots \\ & -2x^3-5x^4-\cdots \\ \hline r_2(x) &= 2-x \end{aligned}$$

and

$$R_2(x) = \frac{2-x}{(1-x)^3}\quad.$$

In a similar manner, we can find $r_3(x)$, $r_4(x)$ and so on. These polynomials henceforth will be known as the *numerator polynomials.* A listing of these is as follows:

$$\begin{aligned} r_0(x) &= 1 \\ r_1(x) &= 1 \\ r_2(x) &= 2 - x \\ r_3(x) &= 4 - 4x + x^2 \\ r_4(x) &= 7 - 9x + 3x^2 \\ r_5(x) &= 13 - 22x + 12x^2 - 2x^3 \end{aligned}$$

etc. If one were to take the time and calculate this data, it would soon be realized that there is a considerable amount of arithmetic involved. The $r_{n(x)}$ numerator polynomial is obtained by expanding $(1-x)^{n+1}$ and using it to multiply an infinite polynomial. It turns out, that when this is done and like terms are collected, all but a finite number of terms result in zero. Nevertheless, it is quite a time-consuming task.

The coefficients of these polynomials can themselves be formed into an array similar to our original convolution array. Like the original convolution array, this array can also be formed in several methods. The first method we have already examined: finding $r_n(x)$. The other method is by also developing a template pattern. This template can be found as follows:

We know that if we let $R_n(x)$ (where $n = 0, 1, 2, 3, 4, \cdots$) denote the rows of the Tribonacci convolution array, then

$$R_n(x) = \frac{r_n(x)}{(1-x)^{n+1}}$$

Similarly:

$$R_{n+1}(x) = \frac{r_{n+1}(x)}{(1-x)^{n+2}}$$

$$R_{n+2}(x) = \frac{r_{n+2}(x)}{(1-x)^{n+3}}$$

$$R_{n+3}(x) = \frac{r_{n+3}(x)}{(1-x)^{n+4}}$$

Also looking at the row polynomial in terms of the pattern discussed

$$\begin{matrix} R_{n+3}(x) & = & xR_{n+3}(x) & + & R_{n+2}(x) & + & R_{n+1}(x) & + & R_n(x) \\ X & = & (Y & + & U & + & V & + & W) \end{matrix}$$

By simple substitution:

$$\frac{r_{n+3}(x)}{(1-x)^{n+4}} = \frac{xr_{n+3}(x)}{(1-x)^{n+4}} + \frac{r_{n+2}(x)}{(1-x)^{n+3}} - \frac{r_{n+1}(x)}{(1-x)^{n+2}} - \frac{r_n(x)}{(1-x)^{n+1}}.$$

By simple algebra:

$$\frac{r_{n+3}(x)}{(1-x)^{n+4}}(1-x) = \frac{r_{n+2}(x)}{(1-x)^{n+3}} + \frac{r_{n+1}(x)}{(1-x)^{n+2}} + \frac{r_n(x)}{(1-x)^{n+1}}$$

$$\frac{r_{n+3}(x)}{(1-x)^{n+3}} = \frac{r_{n+2}(x)}{(1-x)^{n+3}} + \frac{r_{n+1}(x)}{(1-x)^{n+2}} + \frac{r_n(x)}{(1-x)^{n+1}}$$

$$\begin{aligned} r_{n+3}(x) &= r_{n+2}(x) + (1-x)r_{n+1}(x) + (1-x)^2 r_n(x) \\ &= r_{n+2}(x) + r_{n+1}(x) - xr_{n+1}(x) + r_n(x) - 2xr_n(x) + x^2 r_n(x). \end{aligned}$$

From this information and remembering the procedure for converting this equation to a template pattern, the following template for the array of coefficients of the numerator polynomial is

$r_n(x)$	W	U	V
$r_{n-1}(x)$		T	Z
$r_{n-2}(x)$			Y
$r_{n-3}(x)$			X

$$X = Y + Z + V + W - T - 2U$$

We have already discussed a specific Tribonacci sequence and its related convolution and numerator polynomial arrays. Our goal in this portion is to generalize our conclusions from the specific case. We would like to examine and investigate the general case and see if any generalized conclusions can be reached.

Two (2) general Tribonacci sequences exist: $1, 1, p, 2+p, \cdots$ or $1, p, q, 1+p+q, \cdots$. Since the second is more general, we will use it for further investigation. The sequence, then, is as follows:

$$1, p, q, 1+p+q, 1+2p+2q, \cdots,$$

where each term is defined as the sum of the previous three.

As in the specific case, a generating function can also be found for the general case. Again, let the terms of the sequence be coefficients of an infinite polynomial, giving:

$$G(x) = 1 + px + qx^2 + (1+p+q)x^3 + (1+2p+2q)x^4 + \cdots.$$

By multiplying by $-x$, $-x^2$ and $-x^3$ and collecting like terms, we get:

$$\begin{aligned}
G(x) &= 1 + px + qx^2 + (1+p+q)x^3 + (1+2p+2q)x^4 + \cdots \\
-xG(x) &= \quad - x - px^2 - \quad qx^3 - (1+p+q)\,x^4 - \cdots \\
-x^2G(x) &= \qquad - x^2 - \quad px^3 - \quad qx^4 - \cdots \\
-x^3G(x) &= \qquad - \quad x^3 - \quad px^4 - \cdots \\
\hline
(1-x-x^2-x^3)G(x) &= 1 + (p-1)x + (q-p-1)x^2 \\
G(x) &= \frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3},
\end{aligned}$$

where $G(x)$ defines the generalized generating function and "p" is the second term in the sequence and "q" is the third.

Again, using the specific case as an example, we can expand the sequence into a convolution array. The first column is given and defined as the generalized sequence, with the generating function of

$$G(x) = \frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}$$

The subsequent columns can be found by convolution or by giving appropriate powers of the generating function (as discussed earlier in the specific case). By either method, the resulting array is shown in the table on the following page. The columns represent the power of the generating function and the rows are the corresponding powers of x. Therefore, we are guaranteed a way of generating this array—by either convolution or raising the generating function to a power—two rather tedious, time-consuming methods. If we could find a template pattern for this generalized convolution array, it could be used for *any* Tribonacci sequence.

To find this template pattern, recall that the generating function for the first column is

$$\frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}.$$

For any n^{th} column, the generating function is:

$$G^n(x) = \left(\frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}\right)^n$$

Powers of $G(x)$

Powers of x	1	2	3	4	5	6	
0	1	1	1	1	1	1	$\cdots$
1	p	$2p$	$3p$	$4p$	$5p$	$6p$	$\cdots$
2	q	p^2+2q	$3p^2+3q$	$6p^2+4q$	$10p^2+5q$	$15p^2+6q$	$\cdots$
3	$p+q+1$	$2p+2q+2pq+8$	$p^3+3p+3q+6pq+3$	$4p^3+4p+4q+12pq+4$	$\cdots$		
4	$2p+2q+1$	$2p^2+6p+q^2+4p+2pq+2$	$6p^2+12p+3q^2+6q+3p^2q+6pq^2$	$p^4+12p^2+20p+6q^2+8q+12p^2q+12pq+4$	$\cdots$		
5	$3p+4q+2$	$4p^2+6p+2q^2+10q+6pq+4$	$\cdots$				
6	$6p+7q+4$	$\vdots$					
$\vdots$							

or

$$G^n(x) = \left(\frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}\right)\left(\frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}\right)^{n-1}$$

which can be rewritten as:

$$G^n(x) = \frac{1+(p-1)x+(q-p-1)x^2}{1-x-x^2-x^3}\,G^{n-1}(x).$$

By multiplying both sides of the equation by $1-x-x^2-x^3$ we will get:

$$G^n(x)(1-x-x^2-x^3) = (1+(p-1)x+(q-p-1)x^2)G^{n-1}(x)$$

$$G^n(x)-xG^n(x)-x^2G^n(x)-x^3G^n(x) = G^{n-1}(x)+(p-1)xG^{n-1}(x)+(q-p-1)x^2G^{n-1}(x)$$

$$G^n(x) = xG^n(x)+x^2G^n(x)+x^3G^n(x)+G^{n-1}(x)+(p-1)xG^{n-1}(x)+$$
$$+(q-p-1)x^2G^{n-1}(x)$$

Let us represent this symbolically as:

$$X = Y+U+V+W+(p-1)Z+(q-p-1)Q.$$

Then, as we discussed earlier, this can be translated pictorally to give our template for the generalized Tribonacci sequence:

	V
$(q-p-1)Q$	U
$(p-1)Z$	Y
W	X

Naturally, in extending this discussion, we can also discuss the numerator polynomials that will generate the rows of the $1, p, q, \cdots$ array. Again, by sheer arithmetic, we can generate the numerator polynomials:

$$r_0(x) = 1$$
$$r_1(x) = p$$
$$r_2(x) = q + (p^2 - q)x$$
$$r_3(x) = (p+q+1) + (-2p - 2q - 2pq - 2)x + (p^2 + p + q - 2pq + 1)x^2$$
$$r_4(x) = (2p + 2q + 1) + (2p^2 - 4p + q^2 - 6q + 2pq - 3)x + (-4p^2 + 2p - 2q^2 + 6q + 3p^2q - 4pq + 3)x^2 + (p^4 + 2p^2 + 2pq - 3p^2q - 2q - 1)x^3$$

etc.

Using the same method utilized in discussing the specific case, we can determine a pattern for the coefficients of these numerator polynomials.

First let us translate the pattern for the columns to pattern for the rows. This gives us:

$R_{n-2}(x)$		U
$R_{n-1}(x)$	V	Z
$R_n(x)$		Y
$R_{n+1}(x)$	W	X

$$X = Y + Z + U + W + (p-2)V$$

$$R_{n-1}(x) = xR_{n+1}(x) + R_n(x) + R_{n-1}(x) + (p-2)xR_{n-1}(x) + R_{n-2}(x)$$

or

$$R_{n+1}(x)(1-x) = R_n(x) + R_{n-1}(x)(1 + (p-2)x) + R_{n-2}(x).$$

We still have the relation that

$$R_n(x) = \frac{r_n(x)}{(1-x)^{n+1}}$$

By substituting:

$$\frac{r_{n+1}(x)}{(1-x)^{n+2}}(1-x) = \frac{r_n(x)}{(1-x)^{n+1}} - \frac{r_{n-1}(x)}{(1-x)^n}(1 + (p-2)x) + \frac{r_{n-2}(x)}{(1-x)^{n-1}}$$

$$r_{n+1}(x) = r_n(x) + (1-x)(r_{n-1}(x))(1 + (p-2)x) + (1-x)^2 r_{n-2}(x)$$

$$r_{n+1}(x) = r_n(x) + r_{n-1}(x) + (p-3)xr_{n-1}(x) + (2-p)x^2 r_{n-1}(x) + r_{n-2}(x) - 2xr_{n-2}(x) + x^2 r_{n-2}(x).$$

This yields a pattern for the array of the numerator polynomials:

$r_{n-2}(x)$	N	$(-2)T$	U
$r_{n-1}(x)$	$M(2-p)$	$(p-3)V$	Z
$r_n(x)$			Y
$r_{n+1}(x)$			X

$$X = Y + Z + U + (p-3)V + (2-p)M - 2T + N.$$

There are some interesting features of these numerator polynomials. First of all, this pattern does not hold for the entire array. To use the pattern to get the $(p^2 + q)$ coefficient of the x term of the $r_2(x)$ polynomial, some "special" terms must be added to the top of the array. Rather than discuss this at length, it will suffice to say that if one were interested in generating this array one could generate the first three rows by the method of equating coefficients and then utilize the pattern derived.

It can also be noted that the sum of the coefficients of each numerator polynomial sums to a power of p, the second element of the Tribonacci sequence. Specifically, the sum of the coefficients of the r_n numerator polynomial is p^n. (Note that the sum of the coefficients for the numerator polynomials of the 1, 1, 2, ··· Tribonacci array is always 1. This is logical since the second element of the array is 1 and 1^n is always 1.)

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THE PASCAL MATRIX

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The $n \times n$ matrix P or $P(n)$ whose coefficients are the elements of Pascal's triangle has been suggested as a test datum for matrix inversion programs, on the grounds that both itself and its inverse have integer coefficients.

For example, if $n = 4$

(1) $$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 6 & 4 & 1 \\ 6 & 14 & 11 & 3 \\ 4 & 11 & 10 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix},$$

$$|P - \lambda I| = \lambda^4 - 29\lambda^3 + 72\lambda^2 - 29\lambda + 1.$$

It occurred to us to take a closer look at this entertaining object. We shall require a couple of binomial coefficient identities, both of which are easily proved by induction from the fundamental relation

$$\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j} = (i+j)!/i!j!,$$

or 0 unless $0 \leqslant j \leqslant i$.

(2) $$\sum_k \binom{s}{k+u}\binom{t}{k} = \binom{s+t}{s-u}.$$

(3) $$\sum_k \binom{s+k}{u}\binom{t}{k}(-)^k = \binom{s}{u-t}(-)^t.$$

(Here and subsequently all summations over i, j, k, etc., are implicitly over the values 0 to $n-1$. Notice that our matrix subscripts are also taken over this domain.) P is defined by

$$P_{ij} = \binom{i+j}{i}.$$

First notice that the determinant of P is unity. For subtracting from each row the row above, and similarly differencing the columns, we find

(4) $$P(n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P(n-1) & \\ 0 & & & \end{vmatrix} = |P(n-1)| = |P(0)| = 1$$

It follows that P^{-1} has integer coefficients, since they are signed minors of P divided by $|P|$. As it happens, there is a nice explicit formula for them:

(5) $$(P^{-1})_{ij} = (-)^{i+j} \sum_k \binom{k}{i}\binom{k}{j}.$$

Proof of (5). Let the RHS temporarily define a matrix Q. Then

$$(PQ)_{ij} = \sum_p \binom{i+p}{i}\left[(-)^{p+j} \sum_k \binom{k}{p}\binom{k}{j}\right]$$

$$= \sum_k \binom{k}{j}(-)^j\left[\sum_p \binom{i+p}{i}\binom{k}{p}(-)^p\right]$$

$$= \sum_k \binom{k}{j}\binom{i}{k}(-)^{i+j} \quad \text{by (3)}$$

$$= \binom{0}{i-j}(-)^{i+j} = \delta_{ij} \quad \text{by (3) again.}$$

That is, $PQ = I$ and $Q = P^{-1}$.

The decomposition of P into lower- and upper-triangular factors is simply

(6) $$P = LU, \quad \text{where } L_{ij} = \binom{i}{j}, \quad U_{ij} = \binom{j}{i};$$

since $(LU)_{ij}$ is immediately reducible to P_{ij} via (2). And from (5) it is immediate that

(7) $$(UL)_{ij} = |(P^{-1})_{ij}|,$$

or the coefficients of UL are the moduli of those of P^{-1}.

Turning to the characteristic polynomial of P, we need the following method of computing

$$|A - \lambda I| = \sum_m c_m \lambda^{n-m}$$

for any matrix A :—

(8) $$\text{Let } d_k = \text{trace}\,(A^k) = \sum_i (A^k)_{ii} \quad \text{for } k > 0.$$

$$d_0 = m \quad \text{(instead of } n\text{)},$$

$$c_0 = 1,$$

then

$$\sum_k c_{m-k} d_k = 0.$$

This relation enables us to compute the c's in terms of the d's or vice versa, e.g.,

$$c_0 = 1$$

$$c_1 = -d_1$$

$$c_2 = -\tfrac{1}{2}(c_1 d_1 + c_0 d_2) = \tfrac{1}{2}(d_1^2 - d_2)$$

$$c_3 = \tfrac{1}{3}(c_2 d_1 + c_1 d_2 + c_0 d_3) = \tfrac{1}{6}(-d_1^3 + 3d_1 d_2 - 2d_3).$$

Proof of (8). The eigenvalues of A^k are just the k^{th} powers of the eigenvalues of A and our relation is simply a special case of Newton's identity, which relates the coefficients of a polynomial to the sums of k^{th} powers of its roots, etc. (In numerical computation this formula suffers from heavy cancellation.)

Notice also that, by the definition of matrix multiplication, for $m > 0$

$$(9) \qquad d_m = \sum_i \sum_j \sum_k \cdots \sum_q \sum_r A_{ij} A_{jk} \cdots A_{ri}$$

(over m summations and factors).

Now suppose that $A = P(n)$, and denote by C_m and D_k the values of $(-)^m c_m$ and d_k; the former are tabulated for a few small n at the end. The first thing to strike the eye is their symmetry:—

$$(10) \qquad C_m = C_{n-m} .$$

To prove this, it is by (8) enough to show that $D_m = D_{n-m}$. Also since the eigenvalues of P^{-1} are the reciprocals of those of P, and the determinant of P is unity (4), the characteristic polynomial of P^{-1} is just the reverse of that of P. So it is enough to show that $D_m = d_m(P^{-1})$. But by (9) and (5)

$$\begin{aligned} d_m(P^{-1}) &= \sum_{i,j,k,\ell} \left[\sum_p \binom{p}{i}\binom{p}{j}\right]\left[\sum_q \binom{q}{j}\binom{q}{k}\right]\left[\sum_r \binom{r}{k}\binom{r}{\ell}\right] \cdots \\ &= \sum_{p,q,r} \left[\sum_j \binom{p}{j}\binom{q}{j}\right]\left[\sum_k \binom{q}{k}\binom{r}{k}\right] \cdots \\ &= \sum_{p,q,r} \binom{p+q}{p}\binom{q+r}{q} \cdots \quad \text{by (2)} \\ &= \sum_{p,q,r} P_{pq} P_{qr} \cdots = D_m \quad \text{by (6) and (9), QED.} \end{aligned}$$

Incidentally, setting $m = 2$ shows that the sums of squares of coefficients (the d_2) are the same for P and P^{-1}.

The next striking feature is

$$(11) \qquad C_m > 0.$$

If the characteristic polynomial of some A is expanded explicitly in the form $|A - \lambda I|$, it is easily seen that $(-)^m c_m$ is the sum of all principal $m \times m$ minors of A. So (11) is a consequence of the more general result

(12) Every minor of P is positive.

We denote by $M = M(i,k, \cdots, o, q; j, \ell, \cdots, p,r)$ the $m \times m$ minor of P

$$\begin{vmatrix} P_{ij} & & & & & \\ & P_{k\ell} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & P_{op} & \\ & & & & & P_{qr} \end{vmatrix}$$

and define the "type" of M to be the triple (m,q,r). One triple is said to be "less than or equal to" another if this relation holds between corresponding pairs of elements. With this ordering we prove (12) by induction on the type.

The result is clearly true for $m = 0$, since any 0×0 determinant has value 1. Suppose then that $m > 0$ and the result is true for all types less than (m,p,r). According to the fundamental relation $P_{qr} = P_{q-1,r} + P_{q,r-1}$ etc., so we can decompose the final row of M to obtain $M =$

$$M(i, \cdots, q-1; j, \cdots, r) + \begin{vmatrix} P_{ij} & & \\ & \ddots & \\ & & P_{op} \\ P_{q,j-1} & \cdots & P_{q,r-1} \end{vmatrix},$$

where the final row of the latter determinant has been shifted one place to the left. Repeating the decomposition on the new minor, we eventually reach a zero minor when the final row coincides with row o, and so

$$M = \sum_{q'=o+1}^{q} \begin{vmatrix} P_{ij} & & \\ & \ddots P_{op} & \\ P_{q',j-1} & \cdots & P_{q',r-1} \end{vmatrix}.$$

Decomposing all the other rows of the summand in turn, we finally get them lined up again to form a respectable minor, thus

$$(13) \qquad M = \sum_{i',k',\cdots,o',q'} M(i',k', \cdots, o',q'; j-1, \ell-1, \cdots, p-1, r-1),$$

where $-1 < i' \leqslant i < k' \leqslant k < \cdots < o' \leqslant o < q' \leqslant q$.

If $j > 0$, each summand is of type at most $(m,q,r-1)$. If $j = 0$, we need to introduce another row and column for P, defined by $P_{-1,k} = P_{k,-1} = \delta_{0k}$, to preserve the sense of (13): we need then only consider the case $i' = 0$, and (13) becomes

$$M = \sum_{k',\cdots,o',q'} M(k', \cdots o',q'; \ell-1, \cdots, p-1, r-1),$$

in which each summand is of type at most $(m-1, q, r-1)$. In either case M is a sum of minors of lesser type and therefore is positive, QED.

We can squeeze more than (11) out of (12): since $C_m(n+1)$ includes all the minors in $C_m(n)$, it follows that

(14) $C_m(n)$ is an increasing function of n.

A squint at the data suggests the tougher conjecture

(15) $C_m(n)$ is an increasing function of m for $m < \frac{1}{2}n$?

Concerning P in general, some further questions suggest themselves. The maximum element of P is clearly $P_{nn} \sim 4^n/\sqrt{(\frac{1}{2}\pi n)}$ by Stirling's approximation; but what about that of P^{-1}?

How are the eigenvalues of P distributed? By (10) they occur in inverse pairs, with 1 an eigenvalue for all odd n; how big is the largest? Since $P = LL'$, it is positive definite and they are all positive.

1							
1	1						
1	3	1					
1	9	9	1				
1	29	72	29	1			
1	99	626	626	99	1		
1	351	6084	13869	6084	351	1	
1	1275	64974	347020	347020	64974	1275	1

Coefficients of $|P(n) + \lambda I|$, n (descending) = 0(1)7.

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ZERO-ONE SEQUENCES AND STIRLING NUMBERS OF THE SECOND KIND

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Let $x_1, x_2, \cdots, x_n$ denote a sequence of zeros and ones of length n. Define a polynomial of degree $(n-m) \geqslant 0$ as follows

(1) $$\beta_{m+1,n+1}(d) = \sum d^{1-x_1}(d+x_1)^{1-x_2} \cdots (d+x_1+x_2+\cdots+x_{n-1})^{1-x_n}$$

with $\beta_{1,1}(d) = 1$, where the summation is over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m.$$

Summing over x_n we have the following recurrence relation

(2) $$\beta_{m+1,n+1}(d) = (m+d)\beta_{m+1,n}(d) + \beta_{m,n}(d),$$

where $\beta_{0,0}(d) = 1$.

Summing over x_1 we have the following recurrence relation

(3) $$\beta_{m+1,n+1}(d) = d \cdot \beta_{m+1,n}(d) + \beta_{m,n}(d+1),$$

where $\beta_{0,0}(d) = 1$.

Now we introduce the following theorems to establish relationships between the polynomials defined in (1) and Stirling numbers of the second kind; see Riordan [1, pp. 32–34].

Theorem 1. $\beta_{m,n}(1)$ defined in (1) is Stirling numbers of the second kind, i.e., $\beta_{m,n}(1)$ is the coefficient of $t^n/n!$ in the expansion of $(e^t - 1)^m/m!$, $m,n \geqslant 1$.

Proof. From (1) we have $\beta_{1,1}(1) = 1$ and from (2) we have

(4) $$\beta_{m+1,n+1}(1) = (m+1)\beta_{m+1,n}(1) + \beta_{m,n}(1),$$

which is the recurrence relation for Stirling numbers of the second kind; see Riordan [1, p. 33]. Thus Theorem 1 is proved.

Using (2), (3), and (4), we have

Corollary 1.

(a) $$\beta_{m+1,n+1}(0) = \beta_{m,n}(1),$$

(b) $$\beta_{m+1,n+1}(1) = \beta_{m+1,n}(1) + \beta_{m,n}(2),$$

(c) $$\beta_{m,n}(2) = m\beta_{m+1,n}(1) + \beta_{m,n}(1)\quad.$$

Theorem 2. The polynomial defined in (1) can be written

$$\beta_{m+1,n+1}(d) = \sum_{y=0}^{(n-m)} \binom{n}{y} d^y \beta_{m,n-y}(1).$$

Proof. Assume that n distinguishable balls are randomly distributed into N distinguishable cells such that the probability a ball falls in a specified cell is $1/N$. Assume that $d = \theta N \leqslant N$, $0 \leqslant \theta \leqslant 1$, of the cells are previously occupied.

Define $x_i = 1$ if i^{th} ball falls in an empty cell and $x_i = 0$ otherwise. The joint probability function of $(x_1, x_2, \cdots, x_n)$ can be written

(5)
$$\left(\frac{N-d}{N}\right)^{x_1}\left(\frac{d}{N}\right)^{1-x_1}\left(\frac{N-d-x_1}{N}\right)^{x_2}\left(\frac{d+x_1}{N}\right)^{1-x_2}\cdots$$
$$\cdots\left(\frac{N-d-x_1-x_2-\cdots-x_{n-1}}{N}\right)^{x_n}\left(\frac{d+x_1+x_2+\cdots+x_{n-1}}{N}\right)^{1-x_n}.$$

Let $E_{m,j,k}$ be the event that m additional cells will be occupied when j balls are randomly distributed into k cells such that the probability that a ball falls in a specified cell is $1/k$. Now summing (5) over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m,$$

we have

(6)
$$P[E_{m,n,N}] = \frac{1}{N^n}\frac{(N-d)!}{(N-d-m)!}\beta_{m+1,n+1}(d).$$

Let $F_{y,n}$ denote the event that y out of n balls will fall in the previously occupied cells, d out of N cells. Then

(7)
$$P[F_{y,n}] = \binom{n}{y}\left(\frac{d}{N}\right)^{y}\left(1-\frac{d}{N}\right)^{n-y}, \qquad y = 0, 1, \cdots, n.$$

But we have

$$P[E_{m,n,N}] = \sum_{y=0}^{(n-m)} P[F_{y,n}]P[E_{m,n,N}|F_{y,n}],$$

where using similar expression as (5) and (a) of Corollary 1,

(8)
$$P[E_{m,n,N}|F_{y,n}] = P[E_{m,n-y,N-d}] = \frac{1}{(N-d)^{n-y}}\frac{(N-d)!}{(N-d-m)!}\beta_{m,n-y}(1).$$

Thus using (7) and (8)

(9)
$$P[E_{m,n,N}] = \frac{1}{N^n}\frac{(N-d)!}{(N-d-m)!}\left\{\sum_{y=0}^{(n-m)}\binom{n}{y}d^{y}\beta_{m,n-y}(1)\right\}.$$

Equating (6) and (9), Theorem 2 follows.

From Theorem 2, we have the following recurrence relation for Stirling numbers of the second kind.

Corollary 2.

$$\beta_{m+1,n+1}(1) = \sum_{y=0}^{(n-m)}\binom{n}{y}\beta_{m,n-y}(1)$$

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ON POWERS OF THE GOLDEN RATIO*

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The golden ratio $\underline{G}$ is peculiar in that it is the number $\underline{X}$ such that $\underline{X}^2 = \underline{X} + 1$. This characteristic permits deduction of properties of $\underline{G}^n$ not unlike those of Fibonacci numbers $\underline{F}$. Also, interesting relations of $\underline{F}$ numbers are derivable from properties of $\underline{G}^n$. Some of these properties and relations are given below.

First, a given $\underline{n}^{th}$ power of $\underline{G}$ is the sum of G^{n-1} and G^{n-2} for

$$G^{n-1} + G^{n-2} = G^{n-2}(G+1) = G^n. \tag{1}$$

Furthermore, for $\underline{n}$ a positive integer, $G^n = F_n G + F_{n-1}$ which implies that G^n approaches an integer as $\underline{n}$ increases. For proof, determine that

$$G^1 = 1G + 0$$

$$G^2 = G + 1 = 1G + 1$$

$$G^3 = G(G+1) = 2G + 1$$

and from (1), $G^4 = (1+2)G + (1+1)$, $G^5 = (3+2)G + (2+1)$, etc.

The coefficient of $\underline{G}$ on the right for each successive power of $\underline{G}$ is the sum of the two preceding F_{n-1} and F_{n-2} coefficients, and the number added to the multiple of $\underline{G}$ is the sum of F_{n-2} and F_{n-3}. Hence,

$$G^n = F_n G + F_{n-1}.$$

As $\underline{n}$ increases, $F_n G \to F_{n+1}$, so

$$G^n \to F_{n+1} + F_{n-1}. \tag{2}$$

Hence, G^n approaches an integer as $\underline{n}$ incraeases, and thus approximates all properties of $F_{n+1} + F_{n-1}$.

No restrictions were placed on $\underline{n}$ in (1), so the equation holds for $n \leqslant 0$. For example, given $\underline{n} = 0$,

$$G^{n-1} + G^{n-2} = \frac{1}{G} + \frac{1}{G^2} = \frac{G+1}{G^2} = 1 = G^0.$$

Hence, sums of reciprocals of F numbers assume F properties as $F_{n+1}/F_n \to G$. Generally, let $F_n G$ represent F_{n+1}, and $F_n G^2$ represent F_{n+2}. Then

$$\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} \to \frac{1}{F_n G} + \frac{1}{F_n G^2} = \frac{1}{F_n}\left(\frac{G+1}{G^2}\right) = \frac{1}{F_n} \tag{3}$$

Equation (3) is a special case of a much more general interpretation of (1), for positive or negative fractional exponents may be used. To reveal the general application to $\underline{F}$ numbers, derive from the general equation for F_n,

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} = \frac{G^n - \frac{1}{(-G)^n}}{\sqrt{5}}$$

that $F_n\sqrt{5} \to G^n$ as $\underline{n}$ increases. Hence, for any positive integers $\underline{n}$ and $\underline{m}$,

*We wish to thank Mary Ellen Deese for her help in discerning patterns in computer printouts.

$$G^{\frac{n}{m}} = G^{\frac{n}{m}-1} + G^{\frac{n}{m}-2}$$

(4) $$(G^n)^{\frac{1}{m}} = (G^{n-m})^{\frac{1}{m}} + (G^{n-2m})^{\frac{1}{m}}$$

(5) $$F_n^{\frac{1}{m}} \to F_{n-m}^{\frac{1}{m}} + F_{n-2m}^{\frac{1}{m}} \ .$$

To illustrate Eq. (4), let $n = 1$ and $m = 3$.

$$G^{\frac{1}{3}} = G^{-\frac{1}{3}} + G^{-\frac{5}{3}} \ .$$

Cubing both sides gives

$$G = G^{-\frac{6}{3}} + 3G^{-\frac{9}{3}} + 3G^{-\frac{12}{3}} + G^{-\frac{15}{3}} = G^{-5}(G^6) = G \, .$$

The proximity of the relation in (5) even for $\underline{n}$ small can be illustrated by letting $\underline{n} = 10$ and $\underline{m} = 2$, or

$$F_{10}^{1/2} \to F_8^{1/2} + F_6^{1/2}$$

$$\sqrt{55} = 7.416 \to \sqrt{21} + \sqrt{8} = 7.411.$$

Equation (4) adapts readily to $-1/m$, for

$$(G^n)^{-\frac{1}{m}} = (G^{n+m})^{-\frac{1}{m}} + (G^{n+2m})^{-\frac{1}{m}}$$

and from (5),

$$F_n^{-\frac{1}{m}} \to F_{n+m}^{-\frac{1}{m}} + F_{n+2m}^{-\frac{1}{m}} \ .$$

Again, letting $n = 10$ and $m = 2$,

$$F_{10}^{-1/2} = .134839 \qquad \text{and} \qquad F_{12}^{-1/2} + F_{14}^{-1/2} = .134835.$$

An additional insight regarding $\underline{F}$ relations derives from (2) and the fact that $F_n\sqrt{5} \to G^n$, for

$$F_n\sqrt{5} \to G^n \to F_{n+1} + F_{n-1}$$

$$F_n\sqrt{5} \to F_{n+1} + F_{n-1} \ .$$

Hence, $F_n\sqrt{5}$ approaches an integer as $\underline{n}$ increases.

These relations of $\underline{F}$ and powers of $\underline{G}$, especially those involving negative exponents, permit greater perspective for $\underline{F}$ numbers. For example, Vorob'ev [1] states that the condition $U_n = U_{n-1} + U_{n-2}$ does not define all terms in the $\underline{F}$ sequence because not every term has two preceding it. Specifically, 1,1,2 ··· does not have two terms before 1,1. Such is not true of G^n where $-\infty < n < \infty$. F_n properties approach those of G^n as $n \to \pm\infty$, with maximum discrepancy at $\underline{n} = 0$. $\underline{G}$ is usually viewed as the limit of F_{n+1}/F_n as $\underline{n} \to \infty$; perhaps the more mystical concept of a guiding essence for harmonic variations of F_n is in order. G^n brings F_n to taw. The distortion in F_n relations relative to G^n is never great so long as $\underline{n}$ is a positive or negative integer. And G^n properties surmount even $\underline{n} = 0$.

A last look at G^n will be made in terms of logarithms of $\underline{F}$ numbers to the base $\underline{G}$. Because $F_n \to G^n/\sqrt{5}$,

$$\log_G F_n \to n - \tfrac{1}{2}\log_G 5 = n - 1.6722759\cdots = (n-2) + .3277240\cdots.$$

Therefore,

(8) $$F_n \to G^{n-2}G^{.3277240\cdots}.$$

Hence, $\log_G F_n - \log_G F_{n-1}$ harmonically approaches unity, and rapidly.

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UNIFORM DISTRIBUTION FOR PRESCRIBED MODULI

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In [1] the author proves the following

Theorem. Let p be an odd prime and $\{T_n\}$ be the sequence defined by

$$T_{n+1} = (p+2)T_n - (p+1)T_{n-1}$$

and the initial values $T_1 = 0$, $T_2 = 1$. Then $\{T_n\}$ is uniformly distributed (mod m) if and only if m is a power of p.

The proof of the theorem rests on a lemma which states that if p is an odd prime and k is a positive integer, $p+1$ belongs to the exponent p^k (mod p^{k+1}). The lemma is also proved in [1].

Since for each positive integer k, 3 belongs to the exponent 2^{k-1}(mod 2^{k+1}), (see [2, §90]), the lemma and the theorem cannot be extended to the case $p = 2$. It is the object of this paper to find a sequence of integers which is uniformly distributed (mod m) if and only if m is a power of 2.

We will need the following

Lemma. For each positive integer k, 5 belongs to the exponent 2^k (mod 2^{k+2}).

Proof. See [2, §90].

Theorem. The sequence $\{T_n\}$ defined by

$$T_{n+1} = 6T_n - 5T_{n-1}$$

and the initial values $T_1 = 0$ and $T_2 = 1$ is uniformly distributed (mod m) if and only if m is a power of 2.

Proof. The formula of the Binet type for the terms of $\{T_n\}$ is

$$T_n = \tfrac{1}{4}(5^{n-1} - 1) \qquad n = 1, 2, 3, \cdots.$$

To prove this, note that the zeros of the quadratic polynomial

$$x^2 - 6x + 5$$

associated with $\{T_n\}$ are 5 and 1. Solving for c_1 and c_2 in

$$c_1 \cdot 5 + c_2 = 0$$
$$c_1 \cdot 5^2 + c_2 = 1,$$

we find $c_1 = 1/20$ and $c_2 = -1/4$. Therefore

$$T_n = \frac{1}{20}5^n - \frac{1}{4} \qquad n = 1, 2, 3, \cdots,$$

which agrees with the result above. Similar derivations are discussed in [3].

PART 1. We show in this part of the proof that $\{T_n\}$ is uniformly distributed (mod 2^k) for $k = 1, 2, 3, \cdots$.

First we prove that $\{T_i : i = 1, \cdots, 2^k\}$ is a complete residue system (mod 2^k). Accordingly, suppose that

$$T_i \equiv T_j \pmod{2^k},$$

where $1 \leqslant i, j \leqslant 2^k$. Then

$$\tfrac{1}{4}(5^{i-1} - 1) \equiv \tfrac{1}{4}(5^{j-1} - 1) \pmod{2^k}$$

or

$$5^{i-1} \equiv 5^{j-1} \pmod{2^{k+2}}.$$

Assuming $i \geqslant j$, we write

$$5^{j-1} \cdot 5^e \equiv 5^{j-1} \pmod{2^{k+2}},$$

where $0 \leqslant e \leqslant 2^k - 1$. Then

$$5^e \equiv 1 \pmod{2^{k+2}}.$$

But by the lemma, 5 belongs to the exponent 2^k (mod 2^{k+2}), so $e = 0$ and $i = j$.

Next, we note that as a consequence of the lemma,

$$5^{2^k+i-1} \equiv 5^{i-1} \pmod{2^{k+2}} \quad i = 1, 2, 3, \cdots$$

or

$$T_{2^k+i} \equiv T_i \pmod{2^{k+2}} \quad i = 1, 2, 3, \cdots.$$

Thus we see that the complete residue system (mod 2^k) occurs in the first and all successive blocks of length 2^k in $\{T_n\}$, proving that $\{T_n\}$ is uniformly distributed (mod 2^k).

PART 2. We prove in this part that $\{T_n\}$ is not uniformly distributed (mod m) unless m is a power of 2.

If $\{T_n\}$ is uniformly distributed (mod m), it is uniformly distributed (mod q) for each prime divisor q of m. We show that $\{T_n\}$ is not uniformly distributed (mod q) if $q \neq 2$.

Suppose first that $q = 5$. Then

$$T_{n+1} = 6T_n - 5T_{n-1} \equiv T_n \pmod 5.$$

Hence $\{T_n\}$ (mod 5) is $\{0, 1, 1, 1, \cdots\}$.

Suppose finally that $q \neq 2,5$. We show that

(1) $$T_q \equiv 0 \pmod q$$

and

(2) $$T_{q+1} \equiv 1 \pmod q.$$

Note (1) is equivalent to

$$\tfrac{1}{4}(5^{q-1} - 1) \equiv 0 \pmod q$$

or

(3) $$5^{q-1} \equiv 1 \pmod{4q}$$

which is equivalent to the pair

$$5^{q-1} \equiv 1 \pmod 4$$

and

$$5^{q-1} \equiv 1 \pmod q$$

both of which are elementary. Eq. (2) also reduces to (3). Equations (1) and (2) imply that the period of $\{T_n\}$ (mod q) divides $q - 1$, so at least one residue will not occur in the sequence. Therefore, the distribution of $\{T_n\}$ (mod q) is not uniform.

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LIMITING RATIOS OF CONVOLVED RECURSIVE SEQUENCES

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It is a well known result that, for the Fibonacci numbers $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} .$$

See [1]. Our main result in this paper is that convolving linear recurrent sequences leaves limiting ratios unchanged. Some particular cases of our theorem prove an interesting study. It is indeed surprising that such striking limiting cases have been left unnoticed.

Definition 1. If $\{u_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers and if

$$\lambda = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} ,$$

then λ is defined to be the limiting ratio of the sequence $\{u_n\}$.

Definition 2. If $\{u_n\}$ is a linear recurrence sequence

(1) $$a_0u_{n+r} + a_1u_{n+r-1} + a_2u_{n+r-2} + \cdots + a_ru_n = 0$$

then

$$a_0x^r + a_1x^{r-1} + \cdots + a_r = P_u(x)$$

is called the auxiliary polynomial for the sequence $\{u_n\}$.

Definition 3. If $\{u_n\} = U$ and $\{v_n\} = V$ are two linear recurrence sequences with generating functions

$$\frac{P(x)}{Q(x)} \quad \text{and} \quad \frac{R(x)}{S(x)},$$

respectively, we say $\{u_n\}$ and $\{v_n\}$ are *relatively prime* if

$$(P(x), S(x)) = (R(x), Q(x)) = 1.$$

The following classic result was known to Euler:

Lemma. If the auxiliary polynomial $P_u(x)$ for the sequence $\{u_n\}$ in (1) has a single root of largest absolute value, say λ, then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda .$$

Let us call such a λ as a *dominant root* of $P_u(x)$. Moreover, let $\mathrm{Dom}(\alpha,\beta)$ represent the number with bigger absolute value.

The Lemma stated above leads to the following general theorem.

Theorem 1. Let

$$\{u_n\}_{n=0}^{\infty} \quad \text{and} \quad \{v_n\}_{n=0}^{\infty}$$

be two relatively prime linear recurrence sequences with auxiliary polynomials $P_u(x)$ and $P_v(x)$ whose dominant roots are λ_u, and λ_v. Then, if $\{w_n\}_{n=0}^{\infty}$ is the convolution sequence of $\{u_n\}$ and $\{v_n\}$,

$$w_n = \sum_{k=0}^{n} v_k u_{n-k}, \tag{2}$$

then

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = \mathrm{Dom}\,(\lambda_u, \lambda_v).$$

Proof. Consider a polynomial $P(x)$ with non-zero roots $a_1, a_2, \cdots, a_n$. Let $P^*(x)$ denote a polynomial with roots $1/a_1$, $1/a_2, \cdots, 1/a_n$. We call $P^*(x)$ the *reciprocal* of $P(x)$. Now denote the reciprocals of $P_u(x)$ and $P_v(x)$ by $P_u^*(x)$ and $P_v^*(x)$, respectively. It is known from the theory of linear recurrence that

$$\sum_{n=0}^{\infty} u_n x^n = \frac{R(x)}{P_u^*(x)} \tag{3}$$

and

$$\sum_{n=0}^{\infty} v_n x^n = \frac{S(x)}{P_v^*(x)} \tag{4}$$

for some polynomials $R(x)$ and $S(x)$.

It is quite clear from (2), (3) and (4) that

$$\sum_{n=0}^{\infty} w_n x^n = \frac{R(x)S(x)}{P_u^*(x)P_v^*(x)} = \frac{T(x)}{P_u^*(x)P_v^*(x)} \tag{5}$$

which reveals that $\{w_n\}$ is also a linear recurrence sequence. It is easy to prove that if $P_w(x)$ denotes the auxiliary polynomial of $\{w_n\}$, then its reciprocal $P_w^*(x)$ obeys

$$P_w^*(x) = P_u^*(x)P_v^*(x). \tag{6}$$

It is clear that $1/\lambda_u$ and $1/\lambda_v$ are the roots of $P_u^*(x)$ and $P_v^*(x)$ with minimum absolute value, so that min $(1/\lambda_u, 1/\lambda_v)$ is the root of $P_w^*(x)$ with minimum absolute value. But, since $P_w^*(x)$ is the reciprocal of $P_w(x)$, Dom (λ_u, λ_v) is the dominant root of $P_w(x)$. This together with the lemma proves

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = \lambda .$$

We state below some particular cases of the above theorem.

Theorem 2. Let $\{u_n\}_{n=0}^{\infty}$ be a linear recurrence sequence

$$u_{n+1} = u_n + u_{n-r}, \quad u_0 = 0, \quad u_1 = u_2 = u_3 = \cdots = u_r = 1, \quad r \in Z^+.$$

Let $g_{n,1}$ denote the first convolution sequence of $\{u_n\}_{n=0}^{\infty}$

$$u_n \sum_{n=0}^{\infty}$$

$$g_{n,1} = \sum_{k=0}^{n} u_k u_{n-k} \tag{7}$$

and $g_{n,r}$ the r^{th} convolution $(u_n = g_{n,0})$

$$g_{n,r} = \sum_{k=0}^{n} g_{k,r-1} u_{n-k}. \tag{8}$$

Then $\lim_{n\to\infty} u_{n+1}/u_n$ exists and

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{g_{n+1,r}}{g_{n,r}}$$

for every $r\in Z^+$.

Proof. The auxiliary polynomial for $\{u_n\}_{n=0}^{\infty}$ is $x^{r+1}-x^r-1$. We will first prove that the root with largest absolute value is real. Denote the auxiliary polynomial by

$$P_u(x)=x^{r+1}-x^r-1.$$

Clearly, $P_u(1)=-1<0$ and $P_u(\infty)=\infty$. Further,

$$\frac{dP_u(x)}{dx}=(r+1)x^r-rx^{r-1}>0$$

for $1<x<\infty$ so that $P_u(x)=0$ for $1\leqslant x<\infty$ at precisely one point, say λ_u. It is also clear that $P_u(x)>0$ for $x>\lambda_u$ implies

(9) $$|x^{r+1}|>|x^r+1|$$

for $x>\lambda_u$.

Let z_0 be a complex root of $P_u(x)=0$ with $|z_0|>\lambda_u$. Now, since z_0 is a root of $P_u(x)=0$,

$$|z_0^{r+1}|=|z_0^r+1|.$$

But $|z_0|>\lambda_u$, and comparing with (9) we have

$$|z_0^{r+1}|\leqslant|z_0^r|+|1|,$$

a contradiction. One may also show similarly that there is no other root z_0 with $|z_0|=\lambda_u$ proving that λ_u is a dominant root of $P_u(x)$. This proves that the limiting ratio of $\{u_n\}$ exists and that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lambda_u.$$

Further, Theorem 1 gives

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{g_{n+1,r}}{g_{n,r}}$$

by induction on r and the definition of $g_{n,r}$ in (8).

Theorem 3. If $t,s\in Z^+$ and $t<s$, then

$$\lim_{n\to\infty}\frac{g_{n,t}}{g_{n,s}}=0.$$

Proof. For the linear recurrence sequence $\{u_n\}$ satisfying

$$u_{n+1}=u_n+u_{n-r},\quad u_0=0,\quad u_1=u_2=\cdots=u_r=1,$$

define a companion sequence of polynomials

(10) $$u_{n+1}(x)=xu_n(x)+u_{n-r}(x)$$

$$u_0(x)=0,\quad u_1(x)=1,\quad u_2(x)=x,\ \cdots,u_r(x)=x^{r-1}.$$

Denote by $g_{n,0}(x)=u_n(x)$,

$$g_{n,1}(x)=\sum_{k=0}^{n}u_k(x)u_{n-k}(x),$$

and

(11) $$g_{n,t}(x)=\sum_{k=0}^{n}g_{k,r-1}(x)u_{n-k}(x).$$

One of us (K. A.) has established in [2] that

$$\frac{d^t u_n(x)}{t!\,dx^t} = g_{n,t}(x). \tag{12}$$

We know from (10) that

$$\frac{d^t u_{n+1}(x)}{dx^t} = x\,\frac{d^t u_n(x)}{dx^t} + t\cdot\frac{d^{t-1}u_n(x)}{dx^{t-1}} + \frac{d^t u_{n-r}(x)}{dx^t}. \tag{13}$$

Now, (12) makes (13) reduce to

$$g_{n+1,t}(x) = xg_{n,t}(x) + g_{n-r,t}(x) + g_{n,t-1}(x). \tag{14}$$

Note from (11) that $g_{n,t}(1) = g_{n,t}$ so that (14) can be rewritten as

$$g_{n+1,t} = g_{n,t} + g_{n-r,t} + g_{n,t-1}. \tag{15}$$

Dividing (15) throughout by $g_{n,t}$ we get

$$\frac{g_{n+1,t}}{g_{n,t}} = 1 + \frac{g_{n-r,t}}{g_{n,t}} + \frac{g_{n,t-1}}{g_{n,t}}. \tag{16}$$

We know from Theorem 2 that

$$\lim_{n\to\infty} g_{n+1,t}/g_{n,t} = \lambda_u \quad\text{and}\quad \lim_{n\to\infty} g_{n-r,t}/g_{n,t} = 1/\lambda_u^r,$$

so that (16) reduces to

$$\lambda_u = 1 + \frac{1}{\lambda_u^r} + \lim_{n\to\infty}\frac{g_{n,t-1}}{g_{n,t}}. \tag{17}$$

But, λ_u is the dominant root of $x^{r+1} - x^r - 1 = 0$ so that

$$\lim_{n\to\infty}\frac{g_{n,t-1}}{g_{n,t}} = 0.$$

This gives by induction

$$\lim_{n\to\infty}\frac{g_{n,t}}{g_{n,s}} = 0 \quad\text{for}\quad t < s,$$

proving Theorem 3.

Corollary. If $\{u_n\}$ is the Fibonacci sequence, then

$$\lim_{n\to\infty}\frac{g_{n+1,r}}{g_{n,r}} = \frac{1+\sqrt{5}}{2}$$

and

$$\lim_{n\to\infty}\frac{g_{n,t}}{g_{n,s}} = 0 \quad\text{for}\quad t < s.$$

We include the unproved theorem:

Theorem 4. If

$$g_{n+1,r}g_{n-1,r} - g_{n,r}^2 = w_n,$$

then

$$\lim_{n\to\infty}\frac{w_{n+1}}{w_n} = \lambda_u^2.$$

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AN APPLICATION OF THE CHARACTERISTIC OF THE GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In [1], Hoggatt and Bicknell discuss the numerator polynomial coefficient arrays associated with the row generating functions for the convolution arrays of the Catalan sequence and related sequences [2], [3]. In this paper, we examine the numerator polynomials and coefficient arrays associated with the row generating functions for the convolution arrays of the generalized Fibonacci sequence $\{H_n\}_{n=1}^{\infty}$ defined recursively by

$$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geq 3, \tag{1}$$

where the characteristic $D = P^2 - P - 1$ is a prime. A partial list of P for which the characteristic is a prime is given in Table 1. A zero indicates that the characteristic is composite, while $P^2 - P - 1$ is given if the characteristic is a prime.

Table 1
Characteristic $P^2 - P - 1$ is Prime, $1 \leq P \leq 179$

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	5	11	19	29	41	0	71
1	89	109	131	0	181	0	239	271	0	0
2	379	419	461	0	0	599	0	701	0	811
3	0	929	991	0	0	0	1259	0	0	1481
4	1559	0	1721	0	0	1979	2069	2161	0	2351
5	0	2549	0	0	2861	2969	3079	3191	0	0
6	3539	3659	0	0	0	4159	4289	4421	0	4691
7	0	4969	0	0	0	0	0	5851	0	0
8	0	0	0	0	6971	0	7309	7481	0	0
9	8009	0	0	0	8741	8929	0	9311	0	0
10	0	10099	10301	0	10711	0	0	0	0	0
11	0	0	0	0	0	13109	13339	0	0	0
12	0	14519	0	0	0	0	15749	16001	0	0
13	0	17029	17291	0	0	18089	0	0	0	19181
14	0	19739	20021	0	0	20879	21169	0	0	22051
15	22349	0	0	0	23561	23869	24179	0	0	25121
16	25439	25759	0	0	26731	27059	0	0	0	0
17	28729	0	29411	0	0	30449	0	31151	0	0

Examining Table 1, we see that $P^2 - P - 1$ is never prime, with the exception of $P = 3$, whenever P is an integer whose units digit is a 3 or an 8. This is so because $P^2 - P - 1 \equiv 0 \pmod 5$ if $P \equiv 3 \pmod 5$. Furthermore, we note that there are some falling diagonals which are all zeros. This occurs whenever $P \equiv -3 \pmod{11}$ or $P \equiv 4 \pmod{11}$.

If P is an integer whose units digit is not congruent to 3 modulo 5, then $P^2 - P - 1 \equiv \pm 1 \pmod 5$ and we see why no prime, in fact no integer, of the form $5k \pm 2$ would occur in Table 1.

There also exist primes of the form $5k \pm 1$ which are not of the form $P^2 - P - 1$. Such primes are 31, 61, 101, 59, 79, and 119. The last observation leads one to question the cardinality of P for which $P^2 - P - 1$ is a prime. The authors believe that there exist an infinite number of values for which the characteristic is a prime. However, the proof escapes discovery at the present time and is not essential for the completion of this paper.

2. A SPECIAL CASE

The convolution array, written in rectangular form, for the sequence $\{H_n\}_{n=1}^{\infty}$, where $P = 3$ is

Convolution Array when $P = 3$

1	1	1	1	1	1	1	1	...
3	6	9	12	15	18	21	24	...
4	17	39	70	110	159	217	284	...
7	38	120	280	545	942	1498	2240	...
11	80	315	905	2120	4311	7910	13430	...
18	158	753	2568	7043	16536	34566	66056	...
...	...	...	...	...	...	...	...	...

The generating function $C_m(x)$ for the m^{th} column of the convolution array is given by

(2) $$C_m(x) = \left[\frac{1+2x}{1-x-x^2}\right]^m$$

and it can be shown that

(3) $$(1+2x)C_{m-1}(x) + (x+x^2)C_m(x) = C_m(x).$$

Using $R_{n,m}$ as the element in the n^{th} row and m^{th} column of the convolution array, we see from (3) that the rule of formation for the convolution array is

(4) $$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1} + 2R_{n-1,m-1}.$$

Pictorially, this is given by

	a
c	b
d	x

where

(5) $$x = a+b+d+2c.$$

Letting $R_m(x)$ be the generating function for the m^{th} row of the convolution array and using (4), we have

(6) $$R_1(x) = \frac{1}{1-x}$$

(7) $$R_2(x) = \frac{3}{(1-x)^2}$$

and

(8) $$R_m(x) = \frac{(1+2x)N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \quad m \geqslant 3,$$

where $N_m(x)$ is a polynomial of degree $m - 2$.

The first few numerator polynomials are found to be

$$\begin{aligned}
N_1(x) &= 1\\
N_2(x) &= 3\\
N_3(x) &= 4+5x\\
N_4(x) &= 7+10x+10x^2\\
N_5(x) &= 11+25x+25x^2+20x^3\\
N_6(x) &= 18+50x+75x^2+60x^3+40x^4.
\end{aligned}$$

Recording our results by writing the triangle of coefficients for these polynomials, we have

Table 2

Numerator Polynomial $N_m(x)$ Coefficients when $P = 3$

1						
3						
4	5					
7	10	10				
11	25	25	20			
18	50	75	60	40		
29	100	175	205	140	80	
47	190	400	540	530	320	160

It appears as if 5 divides every coefficient of every polynomial $N_m(x)$ except for the constant coefficient.

Using (6), (7), and (8), we see that the constant coefficient of $N_m(x)$ is H_m and it can be shown by induction that

$$H_{n-1}H_{n+1} - H_n^2 = 5(-1)^{n+1}. \tag{9}$$

If 5 divides H_{n-1} then 5 divides H_n and by (1) H_{n-2}. Continuing the process, we have that 5 divides $H_1 = 1$ which is obviously false. Hence, 5 does not divide H_n for any n.

Using (8), we see that the rule of formation for the triangular array of coefficients of the numerator polynomials follows the scheme

$$\begin{array}{|c|c|} \hline d & a \\ \hline c & b \\ \hline \multicolumn{1}{c|}{} & x \\ \cline{2-2} \end{array}$$

where

$$x = a + b + 2c - d. \tag{10}$$

By mathematical induction, we see that

$$H_{n+1} = 3F_n + F_{n-1}, \tag{11}$$

where F_n is the n^{th} Fibonacci number.

From (10) and (11), we now know that the values in the second column are given by

$$x = a + b + 5F_n. \tag{12}$$

Since 5 divides the first two terms of the second column of Table 2, we conclude using (12), (10), and induction that 5 divides every element of Table 2 which is not in the first column. By induction and (10), it can be shown that the leading coefficient of $N_m(x)$ is given by

$$2^{m-3} \cdot 5, \quad m \geqslant 3. \tag{13}$$

Now in [4], we find

Theorem 1. Eisenstein's Criterion. Let

$$q(x) = \sum_{i=0}^{n} a_i x^i$$

be a polynomial with integer coefficients. If p is a prime such that $a_n \not\equiv 0 \pmod p$, $a_i \equiv 0 \pmod p$ for $i < n$, and $a_0 \not\equiv 0 \pmod{p^2}$ then $q(x)$ is irreducible over the rationals.

In [5], we have

Theorem 2. If the polynomial

$$g(x) = \sum_{i=0}^{n} a_i x^i$$

is irreducible then the polynomial

$$h(x) = \sum_{i=0}^{n} a_{n-i}x^i$$

is irreducible.

Combining all of these results, we have the nice result that $N_m(x)$ is irreducible for all $m \geqslant 3$. In fact, we shall now show that these results are true for any P such that the characteristic $P^2 - P - 1$ is a prime.

3. THE GENERAL CASE

Throughout the remainder of this paper, we shall assume that P is an integer where $P^2 - P - 1$ is a prime.

By standard techniques, it is easy to show that the generating function for the sequence $\{H_n\}_{n=1}^{\infty}$ is

$$\frac{1+(p-1)x}{1-x-x^2} \tag{14}$$

By induction, one can show that

$$(1+(p-1)x)\left(\frac{1+x(p-1)}{1-x-x^2}\right)^n + (x+x^2)\left(\frac{1+(p-1)x}{1-x-x^2}\right)^{n+1} = \left(\frac{1+(p-1)x}{1-x-x^2}\right)^{n+1}. \tag{15}$$

Hence, the rule of formation for the convolution array associated with the sequence $\{H_n\}_{n=1}^{\infty}$ is

$$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1} + (p-1)R_{n-1,m-1}\,. \tag{16}$$

Since

$$R_1(x) = \frac{1}{1-x} \tag{17}$$

and

$$R_2(x) = \frac{p}{(1-x)^2} \tag{18}$$

we have, by (16) and induction,

$$R_m(x) = \frac{(1+(p-1)x)N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \quad m \geqslant 3. \tag{19}$$

The triangular array for the coefficients of the polynomials $N_m(x)$, with $D = P^2 - P - 1$, is

Table 3

Numerator Polynomial $N_m(x)$ Coefficients when $H_2 = P$

1					
P					
$P+1$	D				
$2P+1$	$2D$	$(P-1)D$			
$3P+2$	$5D$	$(3P-4)D$	$(P-1)^2D$		
$5P+3$	$10D$	$(9P-12)D$	$(4P^2-10P+6)D$	$(P-1)^3D$	
$8P+5$	$20D$	$(22P-31)D$	$(14P^2-36P+23)D$	$(5P^3-18P^2+21P-8)D$	$(P-1)^4D$

By (19), we see that the rule of formation for the triangular array of coefficients of the numerator polynomials $N_m(x)$ follows the scheme

d	a
c	b
	x

where

$$x = a + b + (P-1)c - d. \tag{20}$$

By induction, we see that

(21) $$H_{n-1}H_{n+1} - H_n^2 = D(-1)^{n+1}$$
and
(22) $$H_{n+1} = PF_n + F_{n-1},$$
where F_n is the n^{th} Fibonacci number while using (17) through (19) we conclude that the constant term of $N_m(x)$ is H_m.

Following the argument when P was 3 and using (21), we see that D does not divide H_m for any m or that the constant term of $N_m(x)$ is never divisible by D.

By (20) and (22), the elements in the second column of Table 3 are given by

(23) $$x = a + b + F_nD.$$

Since D divides the first two terms of the second column of Table 3, we can conclude by using (23), (20), and induction that D divides every element of Table 3 which is not in the first column. Using (20) and induction, we see that the leading coefficient of $N_m(x)$ is given by

(24) $$(P-1)^{m-3}D, \quad m \geqslant 3.$$

By the preceding remarks, together with Theorems 1 and 2, we conclude that $N_m(x)$ is irreducible for all $m \geqslant 3$, provided D is a prime.

4. CONCLUDING REMARKS

If one adds the rows of Table 2 he obtains the sequence 1, 3, 9, 27, 81, 243, 729, and 2187. Adding the rows of Table 3 we obtain the sequence $1, P, P^2, P^3, P^4, P^5, P^6$, and P^7. This leads us to conjecture that the sum of the coefficients of the numerator polynomial $N_m(x)$ is P^{m-1}.

From (19), we can determine the generating function for the sequence of numerator polynomials $N_m(x)$ and it is

(25) $$\frac{1+(P-1)(1-x)\lambda}{1-(1+(P-1)x)\lambda-(1-x)\lambda^2} = \sum_{m=0}^{\infty} N_{m+1}(x)\lambda^m.$$

Letting $x = 1$, we obtain

(26) $$\frac{1}{1-P\lambda} = \sum_{m=0}^{\infty} (P\lambda)^m = \sum_{m=0}^{\infty} N_{m+1}(1)\lambda^m$$

and our conjecture is proved.

We now examine the generating functions for the columns of Table 3. The generating function for the first column is already given in (14). Using (23), we calculate the generating function for the second column to be

(27) $$C_2(x) = \frac{D}{(1-x-x^2)^2}$$

while when using (20) we see that

(28) $$C_n(x) = \frac{P-1-x}{1-x-x^2}\, C_{n-1}(x), \quad n \geqslant 3.$$

Hence, we have

(29) $$C_1(x) + x^2C_2(x)\sum_{k=0}^{\infty}\left(\frac{x(P-1)-x^2}{1-x-x^2}\right)^k = \frac{1}{1-xP}.$$

In conclusion, we observe that there are special cases when the characteristic D is not a prime and the polynomials $N_m(x)$ are still irreducible.

In [7], it is shown that

(30) $$D = 5^eP_1^{\alpha_1}P_2^{\alpha_2}\cdots P_n^{\alpha_n}, \quad e = 0 \text{ or } 1,$$

where the P_i are primes of the form $10m \pm 1$.

Assume either $e = 1$ or some $a_i = 1$. Following the argument when P was 3 and using (21), we conclude that neither 5 nor P_i divides the constant term of $N_m(x)$. We have already shown that D divides every nonconstant coefficient of every polynomial $N_m(x)$ so that either 5 or P_i divides every nonconstant coefficient of every polynomial $N_m(x)$.

By Theorems 1 and 2 together with (24), we now know that the polynomials $N_m(x)$ are irreducible whenever 5 or P_i does not divide $P - 1$. However, it is a trivial matter to show that neither 5 nor P_i can divide both $P - 1$ and $P^2 - P - 1 = D$. Hence, $N_m(x)$ is irreducible for all $m \geqslant 3$ provided $e = 1$ or $a_i = 1$ for some i.

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METRIC PAPER TO FALL SHORT OF "GOLDEN MEAN"

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If the Greeks were right that the most pleasing of rectangles were those having their sides in medial section ratio, $\sqrt{5} + 1 : 2$, the classic "Golden Mean," then the world is missing a golden opportunity in standardizing its paper sizes for the anticipated metric conversion.

Metric paper sizes have their dimensions in the ratio $1 : \sqrt{2}$, an ingenious arrangement that permits repeated halvings without altering the ratio. But the 1.414 ratio of length to width falls perceptively short of the "golden" 1.612, as have most paper sizes with which North Americans are familiar. Thus, 8½ × 11 inch typing paper has the ratio 1.294. Popular sizes for photographic paper include 5 × 7 inches (1.400), 8 × 10 inches (1.250), and 11 × 14 inches (1.283). Closest to the Golden Mean, perhaps, was "legal" size typing paper, 8½ × 14 inches (1.647).

With a number of countries, including the United Kingdom, South Africa, Canada, Australia, and New Zealand, making marked strides into "metrication," office typing paper now is being seen that is a little narrower, a little longer, and notably closer to what the Greeks might have chosen.

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GENERATING FUNCTIONS FOR POWERS OF CERTAIN SECOND-ORDER RECURRENCE SEQUENCES

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1. INTRODUCTION

Let $u(n)$ and $v(n)$ be two sequences of numbers defined by

(1) $$u(n) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}, \quad n = 0, 1, 2, \cdots$$

and

(2) $$v(n) = r_1^n + r_2^n, \quad n = 0, 1, 2, \cdots,$$

where r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$.

It is known that the generating functions of these sequences are

$$u_1(x) = \left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1} \quad \text{and} \quad v_1(x) = \left(2 + \frac{b}{a}x\right)\left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1}.$$

We put

(3) $$u_k(x) = \sum_{n=0}^{\infty} u^k(n)x^n$$

and

(4) $$v_k(x) = \sum_{n=0}^{\infty} v^k(n)x^n.$$

J. Riordan [1] found a recurrence for $u_k(x)$ in the case $b = c = -a$. L. Carlitz [2] generalized the result of Riordan giving the recurrence relations for $u_k(x)$ and $v_k(x)$. A. Horadam [3] obtained a recurrence which unifies the preceding ones. He and A. G. Shannon [4] considered third-order recurrence sequences, too.

The object of this paper is to give the new recurrence relations for $u_k(x)$ and $v_k(x)$ such as the explicit form of the same generating functions. The generating functions of $u(n)$ and $v(n)$ for the multiple argument will be given, too. We use the result of E. Lucas [5].

2. RELATIONS OF $u(n)$ AND $v(n)$

From (1) and (2) we have

$$4r_i^{m+n+2} = \Delta u(n)u(m) + v(n+1)v(m+1) + (-1)^{i-1}\sqrt{\Delta}\,(u(n)v(m+1) + u(m)v(n+1)), \quad i = 1, 2,$$

with $\Delta = (b^2 - 4ac)/a^2$.

Then it follows that

$$2u(m+n+1) = u(n)v(m+1) + u(m)v(n+1)$$

$$2v(m+n+2) = v(n+1)v(m+1) + \Delta u(n)u(m).$$

Since

$$u(-n-1) = -q^{-n}u(n-1), \quad v(-n) = -q^{-n}v(n),$$

we find the relations

(5) $$u((n+2)m-1) = u((n+1)m-1)v(m) - q^m u(nm-1),$$

(6) $$v(nm) = v((n-1)m)v(m) - q^m v((n-2)m).$$

From the identity

$$r_1^{kn} + r_2^{kn} = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r (r_1^n + r_2^n)^{k-2r}(r_1 r_2)^{rn},$$

if we put $u(n)$ and $v(n)$ we get

$$v(kn) = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r q^{rn} v^{k-2r}(n), \quad k \geqslant 1. \tag{7}$$

Similarly, from

$$2r_i^{n+1} = v(n+1) + (-1)^{i-1}\sqrt{\Delta}\, u(n), \quad i = 1, 2,$$

and taking into consideration

$$\sum_{s=0} \binom{p+s}{s}\binom{2p+m}{2p+2s} = 2^{m-1}\, \frac{2p+m}{m} \binom{m+p-1}{p},$$

we obtain

$$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^{r(n+1)} u^{k-2r}(n) = \lambda_k(n), \tag{8}$$

where

$$\lambda_k(n) = \begin{cases} u(k(n+1)-1), & k \text{ odd,} \\ v(k(n+1)), & k \text{ even.} \end{cases}$$

3. GENERATING FUNCTIONS OF $u(n)$ AND $v(n)$ FOR MULTIPLE ARGUMENT

The relations (5) and (6) give us the possibility to find the generating functions of $u(n)$ and $v(n)$ when the argument is a multiple. Indeed, we obtain from (5)

$$(1 - v(m)x + q^m x^2)u(m,x) = u(m-1), \tag{9}$$

where

$$u(m,x) = \sum_{n=0}^{\infty} u((n+1)m-1)x^n. \tag{10}$$

From (6) we have

$$(1 - v(m)x + q^m x^2)v(m,x) = v(m) - q^m v(0)x, \tag{11}$$

where

$$v(m,x) = \sum_{n=0}^{\infty} v((n+1)m)x^n. \tag{12}$$

We find also

$$(1 - v(m)x + q^m x^2)\tilde{v}(m,x) = v(0) - v(m)x, \tag{13}$$

with

$$\tilde{v}(m,x) = v(0) + v(m,x)x.$$

4. RECURRENCE RELATIONS OF $u_k(x)$ AND $v_k(x)$

Let us now return to (8) and consider the sum

$$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^r \sum_{n=0}^{\infty} u^{k-2r}(n)(q^r x)^n = \sum_{n=0}^{\infty} \lambda_k(n) x^n$$

which by (3), (10) and (12) yields the following relation

$$\Delta^{[k/2]}u_k(x) = \lambda(k,x) - \sum_{r=1}^{[k/2]} \Delta^{[k/2]-r} \frac{k}{k-r} C_{k-r}^r q^r u_{k-2r}(q^r x),$$

where

$$\lambda(k,x) = \begin{cases} u(k,x), & k \text{ odd} \\ v(k,x), & k \text{ even}. \end{cases}$$

Similarly from (7) for $v_k(x)$ follows

$$v_k(x) = \tilde{v}(k,x) + \sum_{r=1}^{[k/2]} (-1)^{r-1} \frac{k}{k-r} C_{k-r}^r v_{k-2r}(q^r x).$$

5. EXPLICIT FORM OF $u_k(x)$ AND $v_k(x)$

Next we construct the powers for $u(n)$ and $v(n)$. From (1) and (2) we obtain

$$\Delta^{[k/2]}u^k(n) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^{r(n+1)} \lambda_{k-2r}(n), \tag{14}$$

and

$$v^k(n) = \sum_{r=0}^{[k/2]} C_k^r q^{rn} \tilde{v}((k-2r)n), \tag{15}$$

where

$$\tilde{v}(t) = \begin{cases} v(t), & t \neq 0, \\ \tfrac{1}{2}v(t), & t = 0. \end{cases}$$

Hence we multiply each member of the equations (14) and (15) by x^n and sum from $n = 0$ to $n = \infty$. By (3) and (4) the following generating functions for powers of $u(n)$ and $v(n)$ are obtained:

$$\Delta^{[k/2]}u_k(x) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^r \lambda(k-2r, q^r x),$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} C_k^r v(k-2r, q^r x).$$

If we replace $u(m,x)$, $v(m,x)$ and $\tilde{v}(m,x)$ from (9), (11) and (13), we get

$$\Delta^{[k/2]}u_k(x) = \sum_{r=0}^{[k/2]} \frac{(-1)^r C_k^r q^r \mu_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\mu_{kr} = \begin{cases} u(k-2r-1), & k \text{ odd}, \\ v(k-2r) - q^r v(0)x, & k \text{ even}, k \neq 2r \\ \tilde{v}(k-2r) - q^r v(0)x, & k = 2r, \end{cases}$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} \frac{C_k^r \omega_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\omega_{kr} = \begin{matrix} v(0) - q^r v(k-2r)x, & k \neq 2r, \\ \tilde{v}(0) - q^r \tilde{v}(k-2r)x, & k = 2r. \end{matrix}$$

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A SET OF GENERALIZED FIBONACCI SEQUENCES SUCH THAT EACH NATURAL NUMBER BELONGS TO EXACTLY ONE

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1. INTRODUCTION

We shall prove there is an infinite array

1	2	3	5	8	·	·	·
4	6	10	16	26	·	·	·
7	11	18	29	47	·	·	·
9	15	24	39	63	·	·	·
·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·

in which every natural number occurs exactly once, such that past the second column every number in a given row is the sum of the two previous numbers in that row.

2. PROOF

Let α be the largest root of $z^2 - z - 1 = 0$, so $\alpha = (1 + \sqrt{5})/2$. For every positive integer x let $f(x) = [\alpha x + \frac{1}{2}]$ where $[u]$ denotes the greatest integer in u. We require two lemmas: the first asserts that $f(x)$ is one-to-one, and the second asserts that the iterates of $f(x)$ form a sequence with the Fibonacci property.

Lemma 1. If x and y are positive integers and $x > y$ then $f(x) > f(y)$.

Proof. Since $\alpha(x - y) > 1$ we have $(\alpha x + \frac{1}{2}) - (\alpha y + \frac{1}{2}) > 1$, so $f(x) > f(y)$.

Lemma 2. If x and y are integers, and $y = [\alpha x + \frac{1}{2}]$, then $x + y = [\alpha y + \frac{1}{2}]$.

Proof. Write $\alpha x + \frac{1}{2} = y + r$, where $0 < r < 1$. Then

$$(1 + \alpha)x + \frac{\alpha}{2} = \alpha y + \alpha r$$

so

$$x + y + r - \frac{1}{2} + \frac{\alpha}{2} = \alpha y + \alpha r \quad \text{and} \quad \alpha y + \frac{1}{2} = x + y + \frac{\alpha}{2} + (1 - \alpha)r.$$

Since $1 < \alpha = 1.618 \cdots < 2$ we have $0 < \alpha - 1 < \frac{\alpha}{2} < 1$ and the result follows.

We now prove the theorem. Let the first row of the array consist of the Fibonacci numbers $1, 2 = f(1)$, $3 = f(2)$, $5 = f(3)$, $8 = f(5)$, and so on. The first positive integer not in this row is 4; let the second row be $4, 6 = f(4)$, $10 = f(6)$, $16 = f(10)$, and so on. The first positive integer not in the first or second row is 7; let the third row be $7, 11 = f(7)$, $18 = f(11)$, and so on. We see by Lemma 1 that there is no repetition. By Lemma 2 each row has the Fibonacci property. Finally, this process cannot terminate after a finite number of steps since the distances between successive elements in a row increase without bound. This completes the proof.

For the array just constructed, let a_n be the n^{th} number in the first column and b_n the n^{th} number in the second column. I conjecture that for $n \geqslant 2$ the difference $b_n - a_n$ is either a_i or b_i for some $i < n$.

We comment that the fact that $F_{n+1} = [\alpha F_n + \frac{1}{2}]$, where F_n is the n^{th} Fibonacci number, is Theorem III on p. 34 of the book *Fibonacci and Lucas Numbers*, Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969.

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PERIODIC CONTINUED FRACTION REPRESENTATIONS OF FIBONACCI-TYPE IRRATIONALS

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Consider the sequence $\{a_k\}_{k=1}^{\infty}$, where $a_k \geqslant 1 \; \forall \; k$, and also consider the sequence of *convergents*

(1) $$\frac{P_k}{Q_k} = [a_1, a_2, \cdots, a_k] \equiv a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k}, \qquad k = 1, 2, \cdots.$$

It is known from continued fraction theory that $P_k = P_k(a_1, a_2, \cdots, a_k)$ and $Q_k = P_{k-1}(a_2, a_3, \cdots, a_k)$ are polynomial functions of the indicated arguments, with $Q_1 = 1$; moreover, the condition $a_k \geqslant 1 \; \forall \; k$ is sufficient to ensure that $\lim_{k \to \infty} P_k/Q_k$ exists. We call this limit the *value* of the infinite continued fraction $[a_1, a_2, a_3, \cdots]$; where no confusion is likely to arise, we will use the latter symbol to denote both the infinite continued fraction and its value. Clearly, this value is at least as great as unity, which is also true for all values of

$$P_k, \quad Q_k \quad \text{and} \quad \frac{P_k}{Q_k}, \quad k = 1, 2, \cdots.$$

The computation of the convergents of the infinite continued fraction $[a_1, a_2, a_3, \cdots]$ is facilitated by considering the matrix products

(2) $$\begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}, \qquad k = 1, 2, \cdots,$$

where $P_0 \equiv 1$, $Q_0 \equiv 0$. Relation (2) is easily proved by induction, using the recursions

(3) $$P_{k+1} = a_{k+1}P_k + P_{k-1},$$

(4) $$Q_{k+1} = a_{k+1}Q_k + Q_{k-1}, \qquad k = 1, 2, \cdots.$$

Now, given a positive integer $n \geqslant 2$, suppose that we define the sequence $\{a_k\}_{k=1}^{\infty}$ as follows:

(5) $$a_1 = z, \quad a_2 = a_3 = \cdots = a_n = x, \quad a_{n+1} = 2z, \quad a_{k+n} = a_k, \qquad k = 2, 3, \cdots,$$

where $z \geqslant 1$, $x \geqslant 1$. Also, given that $n = 1$, we may define the sequence $\{a_k\}_{k=1}^{\infty}$ as follows:

(6) $$a_1 = z, \quad a_k = 2z, \qquad k = 2, 3, \cdots, \quad \text{where} \quad z \geqslant 1.$$

Let ϕ_n denote the value of the corresponding *periodic* infinite continued fraction; that is,

(7) $$\phi_n = [z; \overline{\underbrace{x, x, \cdots, x}_{n-1}, 2z}], \qquad n = 1, 2, \cdots.$$

Also, define θ_n as follows:

(8) $$\theta_n = z + \phi_n.$$

Thus, θ_n has a *purely periodic* continued fraction representation, namely

(9) $$\theta_n = [\overline{2z, \underbrace{x, x, \cdots, x}_{n-1}}].$$

We let P_k/Q_k denote the k^{th} convergent of the continued fraction given in (9) $(k = 1, 2, \cdots)$. In view of (2), note that

$$\begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, each matrix in the right member of the last expression is symmetric. Taking transposes of both sides leads to the result that the product matrix is itself symmetric, i.e.,

(10) $$P_n = Q_{n+1}.$$

We will return to this result later. Our concern is to evaluate θ_n, and thus ϕ_n, in terms of z, x and n. Another result which will be useful later is the special case of (4) with $k = n$, namely

(11) $$Q_{n+1} = 2zQ_n + Q_{n-1}.$$

Returning to (9), note that this is equivalent to the following:

(12) $$\theta_n = [2z, \underbrace{x, x, \cdots, x}_{n-1}, \theta_n].$$

This implies the equation

(13) $$\theta_n = \frac{\theta_n P_n + P_{n-1}}{\theta_n Q_n + Q_{n-1}}.$$

Clearing fractions in (13), we obtain a quadratic in θ_n, namely

(14) $$Q_n\theta_n^2 - (P_n - Q_{n-1})\theta_n - P_{n-1} = 0.$$

Rejecting the negative root of (14), we obtain the unique solution:

(15) $$\theta_n = \frac{P_n - Q_{n-1} + \sqrt{(P_n - Q_{n-1})^2 + 4P_{n-1}Q_n}}{2Q_n}.$$

Therefore, using (8), (11) and (10) in order, we obtain an expression for ϕ_n, which we shall find convenient to express in the form

(16) $$\phi_n = \sqrt{\frac{\left(\frac{P_n - Q_{n-1}}{2}\right)^2 \Big/ Q_n + P_{n-1}}{Q_n}}.$$

We will now show that (16) may be further simplified, and that depending on our choice of z, may be expressed in terms of a Fibonacci polynomial, with argument x. We digress for a brief review of these polynomials. The Fibonacci polynomials $F_m(x)$ are defined by the recursion:

(17) $$F_{m+2}(x) = xF_{m+1}(x) + F_m(x), \qquad m = 0, \pm 1, \pm 2, \cdots,$$

with initial values

(18) $$F_0(x) = 0, \qquad F_1(x) = 1.$$

The characteristic equation

(19) $$f^2 = xf + 1$$

has the two solutions

(20) $$\alpha(x) = \tfrac{1}{2}(x + \sqrt{x^2+4}), \qquad \beta(x) = \tfrac{1}{2}(x - \sqrt{x^2+4}),$$

which satisfy the relations

(21) $$\alpha(x)\beta(x) = -1, \quad \alpha(x) + \beta(x) = x, \quad \alpha(x) - \beta(x) = \sqrt{x^2+4}.$$

Closed form expressions for the F_m's are given by:

(22) $$F_m(x) = \frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)},$$

for all integers m. The Lucas polynomials are also defined by (17), but with initial values

(23) $$L_0(x) = 2, \qquad L_1(x) = x.$$

Closed forms for the Lucas polynomials $L_m(x)$ are given by:

(24) $$L_m(x) = \alpha^m(x) + \beta^m(x),$$

for all integers m. A convenient pair of formulas for extending the F_m's and L_m's to negative indices is the following.

(25) $$F_{-m}(x) = (-1)^{m-1}F_m(x),$$

(26) $$L_{-m}(x) = (-1)^m L_m(x), \qquad m = 0, 1, 2, \cdots.$$

Note that $F_m(1) = F_m$, $L_m(1) = L_m$, the familiar Fibonacci and Lucas *numbers*, respectively. The following additional relations may be verified by the reader:

(27) $$\alpha^r(x) = F_r(x)\cdot\alpha(x) + F_{r-1}(x);$$

(28) $$F_{m+2r}(x) = F^2_{r+1}(x)F_m(x) + 2F_{r+1}(x)F_r(x)F_{m-1}(x) + F^2_r(x)F_{m-2}(x);$$

(29) $$(x^2+4)F_{m+2r}(x) = L^2_{r+1}(x)F_m(x) + 2L_{r+1}(x)L_r(x)F_{m-1}(x) + L^2_r(x)F_{m-2}(x);$$

(30) $$\lim_{m\to\infty} \sqrt{\frac{F_{m+2r}(x)}{F_m(x)}} = \alpha^r(x), \quad \text{provided} \quad x > 0.$$

From (19),

$$\alpha^2(x) = x\alpha(x) + 1, \quad \text{or} \quad \alpha(x) = x + \frac{1}{\alpha(x)}.$$

Assuming $x \geqslant 1$, by iteration of the last expression, we ultimately obtain the purely periodic continued fraction expression for $\alpha(x)$, namely:

(31) $$\alpha(x) = [\bar{x}], \quad x \geqslant 1.$$

More generally, from (27),

$$\alpha^r(x)/F_r(x) = \alpha(x) + F_{r-1}(x)/F_r(x),$$

provided $F_r(x) \neq 0$. If, in particular, r is natural and $x \geqslant 1$, then in view of (31), we have:

$$\alpha^r(x)/F_r(x) = [\bar{x}] + F_{r-1}(x)/F_r(x) = [x + F_{r-1}(x)/F_r(x); \bar{x}] = [(xF_r(x) + F_{r-1}(x))/F_r(x); \bar{x}],$$

or, using (17) with $m = r-1$,

(32) $$\alpha^r(x)/F_r(x) = [F_{r+1}(x)/F_r(x); \bar{x}], \quad r \text{ natural}, \quad x \geqslant 1.$$

Comparing (30) and (32), it therefore seems reasonable to suppose that, for r natural and $x \geqslant 1$, the continued fraction expression for

$$\frac{1}{F_r(x)}\sqrt{\frac{F_{m+2r}(x)}{F_m(x)}}$$

should approximate, in some sense, the right member of (32). The exact relationship is both startling and elegant, and is our first main result. Before proceeding to it, however, we will develop a pair of useful lemmas.

Lemma 1. For all natural numbers r, let

(33) $$A_r(x) = \begin{pmatrix} F_{r+1}(x) & F_r(x) \\ F_r(x) & F_{r-1}(x) \end{pmatrix}.$$

Then

(34) $$A_r(x) = \{A_1(x)\}^r = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^r.$$

Proof. Let S be the set of natural numbers r for which (34) holds. Clearly, $1 \in S$. Suppose $r \in S$. Then, using the inductive hypothesis and (17), we obtain

$$\{A_1(x)\}^{r+1} = \{A_1(x)\}^r\cdot A_1(x) = \begin{pmatrix} F_{r+1}(x) & F_r(x) \\ F_r(x) & F_{r-1}(x) \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} xF_{r+1}(x) + F_r(x) & F_{r+1}(x) \\ xF_r(x) + F_{r-1}(x) & F_r(x) \end{pmatrix} = \begin{pmatrix} F_{r+2}(x) & F_{r+1}(x) \\ F_{r+1}(x) & F_r(x) \end{pmatrix} = A_{r+1}(x).$$

Hence, $r \in S \Rightarrow (r+1) \in S$.

By induction, Lemma 1 is proved.

Lemma 2. Suppose $[a_1, a_2, a_3, \cdots]$ converges. Then, for all $c > 0$,

$$c[a_1, a_2, a_3, \cdots] = \left[ca_1, \frac{a_2}{c}, ca_3, \frac{a_4}{c}, \cdots\right]. \tag{35}$$

Proof. Consider the convergents

$$\frac{P_k}{Q_k} = [a_1, a_2, a_3, \cdots, a_k], \qquad k = 1, 2, 3, \cdots.$$

Then

$$cP_k/Q_k = c\left\{a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k}\right\} = ca_1 + \frac{c}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k} = ca_1 + \frac{1}{(a_2/c)+}\,\frac{1/c}{a_3+}\cdots\frac{1}{a_k} = \cdots$$

$$= \left[ca_1, \frac{a_2}{c}, ca_3, \frac{a_4}{c}, \cdots, c^{(-1)^{k-1}}a_k\right].$$

Let

$$\phi = \lim_{k\to\infty} \frac{P_k}{Q_k} = [a_1, a_2, a_3, \cdots].$$

Then

$$\left[ca_1, \frac{a_2}{c}, ca_3, \cdots\right] = \lim_{k\to\infty}\left[ca_1, \frac{a_2}{c}, ca_3, \cdots, c^{(-1)^{k-1}}a_k\right] = \lim_{k\to\infty} cP_k/Q_k$$

$$= c\lim_{k\to\infty} P_k/Q_k = c\phi = c[a_1, a_2, a_3, \cdots]. \text{ Q.E.D.}$$

Before proceeding to the main theorems, we conclude the preliminary discussion with a brief table of $F_m(x)$ and $L_m(x)$, for ready reference:

m	$F_m(x)$	$L_m(x)$
0	0	2
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x

Returning to (16), we may compute the required quantities from the matrix identity:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1}\end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0\end{pmatrix}^{n-1}.$$

However, using Lemma 1, this becomes:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1}\end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} F_n(x) & F_{n-1}(x) \\ F_{n-1}(x) & F_{n-2}(x)\end{pmatrix} = \begin{pmatrix} 2zF_n(x)+F_{n-1}(x) & 2zF_{n-1}(x)+F_{n-2}(x) \\ F_n(x) & F_{n-1}(x)\end{pmatrix}.$$

Substituting these quantities in (16), we thus obtain the result:

$$[z; \overline{\underbrace{x, x, \cdots, x}_{n-1}, 2z}] = \sqrt{\frac{z^2F_n(x)+2zF_{n-1}(x)+F_{n-2}(x)}{F_n(x)}}, \tag{36}$$

for all natural n, provided $z \geqslant 1$, $x \geqslant 1$.

The following two theorems are easy consequences of (36):

Theorem 1. For all natural n and r, $x \geqslant 1$,

$$\frac{1}{F_r(x)}\sqrt{\frac{F_{n+2r}(x)}{F_n(x)}} = \left[\frac{F_{r+1}(x)}{F_r(x)}; \overline{\underbrace{x, x, \cdots, x}_{n-1}, \frac{2F_{r+1}(x)}{F_r(x)}}\right] \tag{37}$$

Proof: Let

$$z = \frac{F_{r+1}(x)}{F_r(x)}$$

in (36) and apply (28), with $m = n$. Since

$$z = x + \frac{F_{r-1}(x)}{F_r(x)} \geqslant x,$$

the condition $z \geqslant 1$ is clearly satisfied.

Theorem 2. For all natural n and r, $x \geqslant 1$,

$$(38)\qquad \frac{1}{L_r(x)}\sqrt{\frac{(x^2+4)F_{n+2r}(x)}{F_n(x)}} = \left[\frac{L_{r+1}(x)}{L_r(x)};\overline{\underbrace{x, x, \cdots x}_{n-1}, \frac{2L_{r+1}(x)}{L_r(x)}}\right].$$

Proof. Let

$$z = \frac{L_{r+1}(x)}{L_r(x)}$$

in (36) and apply (29), with $m = n$. Since

$$z = x + \frac{L_{r-1}(x)}{L_r(x)} \geqslant x,$$

the condition $z \geqslant 1$ is clearly satisfied.

Corollary 1.

$$(39)\qquad \sqrt{\frac{F_{n+2}(x)}{F_n(x)}} = [x;\overline{\underbrace{x, x, \cdots, x}_{n-1}, 2x}]\ ,$$

for all natural n, $x \geqslant 1$.

Proof. Set $r = 1$ in Theorem 1.

Corollary 2.

$$(40)\qquad \sqrt{\frac{F_{n+4}(x)}{F_n(x)}} = \begin{cases} [x^2+1;\overline{\underbrace{1, x^2, \cdots, 1, x^2}_{(\frac{1}{2}n-1)\text{ pairs}}, 1, 2x^2+2}], & n = 2, 4, 6, \cdots; \\ \left[x^2+1;\overline{\underbrace{1, x^2, \cdots, 1, x^2}_{\frac{1}{2}(n-1)\text{ pairs}}, \frac{2x^2+2}{x^2}, \underbrace{x^2, 1, \cdots, x^2, 1}_{\frac{1}{2}(n-1)\text{ pairs}}, 2x^2+2}\right], & n = 1,3,5,\cdots, x \geqslant 1. \end{cases}$$

Proof. Set $r = 2$ in Theorem 1. Then multiply both sides by $F_2(x) = x$, applying Lemma 2. Distinguishing between the cases n even and n odd leads to (40).

Corollary 3.

$$(41)\qquad \sqrt{\frac{F_{n+2}}{F_n}} = [1;\overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 2}],\quad \text{for all natural } n,$$

Proof. Set $x = 1$ in Corollary 1.

Corollary 4.

$$(42)\qquad \sqrt{\frac{F_{n+4}}{F_n}} = [2;\overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 4}],\quad \text{for all natural } n.$$

Proof. Set $x = 1$ in Corollary 2.

Corollary 5.

$$(43)\qquad \sqrt{\frac{(x^2+4)F_{n+2}(x)}{F_n(x)}} = \begin{cases} [x^2+2;\overline{\underbrace{1, x^2, \cdots, 1, x^2}_{(\frac{1}{2}n-1)\text{ pairs}}, 1, 2x^2+4}], & n = 2, 4, 6, \cdots; \\ \left[x^2+2;\overline{\underbrace{1, x^2, \cdots 1, x^2}_{\frac{1}{2}(n-1)\text{ pairs}}, \frac{2x^2+4}{x^2}, \underbrace{x^2, 1, \cdots, x^2, 1}_{\frac{1}{2}(n-1)\text{ pairs}}, 2x^2+4}\right], & n = 1,3,5,\cdots x \geqslant 1. \end{cases}$$

Proof. Set $r = 1$ in Theorem 2. Then multiply both sides by $L_1(x) = x$, applying Lemma 2. Distinguishing between the cases n even and n odd leads to (43).

Corollary 6.

$$\sqrt{\frac{5F_{n+2}}{F_n}} = [3; \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 6}], \quad \text{for all natural } n. \tag{44}$$

Proof. Set $x = 1$ in Corollary 5.

The continued fraction representations of corresponding expressions involving the *Lucas* polynomials are somewhat more complicated, since they contain fractions with numerators other than unity. The theory of such general continued fractions is more complex, and is not considered here. The interested reader may pursue this topic further, but will probably discover that the results found thereby will not be as elegant as those given in this paper.

The primary motivation for this paper came out of the diophantine equations studied in Bergum and Hoggatt [1].

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PI-OH-MY!

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Though Π in circles may be found,
It's far from being a number round.
Not three, as thought in times Hebraic
(Indeed, this value's quite archaic!);
Not seven into twenty-two---
For engineers, this just won't do!
Three-three-three over one-oh-six
Is closer; but exactly? Nix!
The Hindus made a bigger stride
In valuing Π; if you divide
One-one-three into three-three-five.
This closer value you'll derive.
But Π's not even algebraic,
And so the previous lot are fake.
For those who deal in the abstract
Know it can never be exact
And are content to leave it go
Right next to omicron and rho.
As for the others, not as wise,
In circle-squarers' paradise,
They strain their every resource mental
To rationalize the transcendental!

ZERO-ONE SEQUENCES AND STIRLING NUMBERS OF THE FIRST KIND

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This is a dual note to the paper [1]. Let $x_1, x_2, \cdots, x_n$ denote a sequence of zeros and ones of length n. Define a polynomial of degree $(n-m) \geqslant 0$ as follows

(1) $$a_{m+1,n+1}(d) = \sum (x_1 - d)^{1-x_1}(x_2-(d+1))^{1-x_2} \cdots (x_n - (d+n-1))^{1-x_n}$$

with

$$a_{1,1}(d) = 1 \quad \text{and} \quad a_{m+1,n+1}(d) = 0, \quad n < m,$$

where the summation is over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m.$$

Summing over x_n we have the following recurrence relation

(2) $$a_{m+1,n+1}(d) = -(d+n-1)a_{m+1,n}(d) + a_{m,n}(d),$$

where

$$a_{0,0}(d) = 1 \quad \text{and} \quad a_{0,n}(d) = 0, \quad n > 0.$$

Summing over x_1, we have the following recurrence relation

(3) $$a_{m+1,n+1}(d) = -d\,a_{m+1,n}(d+1) + a_{m,n}(d+1),$$

where

$$a_{0,0}(d) = 1 \quad \text{and} \quad a_{0,n}(d) = 0, \quad n > 0.$$

The following theorem establishes a relationship between the polynomials defined in (1) and Stirling numbers of the first kind; see Riordan [2, pp. 32–34].

Theorem 1. $a_{m,n}(1)$ defined in (1) are Stirling numbers of the first kind.

Proof. From (1) $a_{1,1}(d) = 1$ and from (2)

(4) $$a_{m+1,n+1}(1) = -n\,a_{m+1,n}(1) + a_{m,n}\,,$$

which is the recurrence relation for Stirling numbers of the first kind, see Riordan [2, p. 33]. Thus Theorem 1 is proved.

Using (2), (3) and (4) the following Corollary can be shown.

Corollary. (a) $$a_{m+1,n+1}(0) = a_{m,n}(1)$$

(b) $$a_{m+1,n+1}(1) = -a_{m+1,n}(2) + a_{m,n}(2)$$

(c) $$a_{m,n}(2) - a_{m+1,n}(2) = -na_{m+1,n}(1) + a_{m,n}(1).$$

Theorem 2. Let $\beta_{m+1,n+1}(d)$ be a polynomial of degree $(n-m) \geqslant 0$ given by Park [1]. Then

(5) $$\sum a_{m+1,k+1}(d)\beta_{k+1,n+1}(d) = \delta_{m+1,n+1} \quad \text{with } \delta_{m,n} \text{ the Kronecker delta.}$$

$\delta_{m,n} = 1$, $\delta_{m,n} = 0$, $m \neq n$, and summed over all values of k for which $a_{m+1,k+1}(d)$ and $\beta_{k+1,n+1}(d)$ are nonzero.

Proof. It can be verified that the polynomial defined in (1) has a generating function

$$(6)\qquad (t-d)^{(n)} = \sum_{m=0}^{n} t^m a_{m+1,n+1}(d), \quad \text{where} \quad (t-d)^{(n)} = (t-d)(t-d-1)\cdots(t-d-n+1).$$

The generating function of $\beta_{m+1,n+1}(d)$ can be written

$$(7)\qquad t^n = \sum_{m=0}^{n} (t-d)^{(m)} \beta_{m+1,n+1}(d).$$

Using (6) and (7), (5) follows. This completes the proof of Theorem 2.

EXAMPLE: For $n = 3$, let

$$A = \begin{bmatrix} a_{1,1}(d) & 0 & 0 & 0 \\ a_{1,2}(d) & a_{2,2}(d) & 0 & 0 \\ a_{1,3}(d) & a_{2,3}(d) & a_{3,3}(d) & 0 \\ a_{1,4}(d) & a_{2,4}(d) & a_{3,4}(d) & a_{4,4}(d) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 \\ d(d+1) & -(2d+1) & 1 & 0 \\ -d(d+1)(d+2) & (3d^2+6d+2) & -3(d+1) & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta_{1,1}(d) & 0 & 0 & 0 \\ \beta_{1,2}(d) & \beta_{2,2}(d) & 0 & 0 \\ \beta_{1,3}(d) & \beta_{2,3}(d) & \beta_{3,3}(d) & 0 \\ \beta_{1,4}(d) & \beta_{2,4}(d) & \beta_{3,4}(d) & \beta_{4,4}(d) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ d^2 & (2d+1) & 1 & 0 \\ d^3 & (3d^2+3d+1) & 3(d+1) & 1 \end{bmatrix}$$

Then $A \cdot B = I$.

ACKNOWLEDGEMENT

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PROBLEMS

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Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Prove that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{n^2+n+1} = \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{F_{2n+1}}.$$

Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Show that

$$\text{(a)}\qquad \sum_{n=1}^{\infty} \frac{1}{n^2 F_{n+2}} > \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} + \frac{1}{48}$$

$$\text{(b)}\qquad \sum_{n=0}^{\infty} \frac{1}{F_{n+2}} \left(\tan \frac{\pi}{2^{n+2}}\right) > \frac{4}{\pi} + 0.0166.$$

[Continued on p. 257.]

GAUSSIAN FIBONACCI NUMBERS

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The purpose of this note is to present a natural manner of extension of the Fibonacci numbers into the complex plane. The extension is analogous to the analytic continuation of solutions of differential equations. Although, in general, it does not guarantee permanence of form, in case of the Fibonacci numbers even that requirement is satisfied. The resulting complex Fibonacci numbers are, in fact, Gaussian integers. The applicability of this generalization will be demonstrated by the derivation of two interesting identities for the classical Fibonacci numbers.

The notion of monodiffricity was introduced by Rufus P. Isaacs [1, 2] in 1941; for references to the more recent literature the reader is directed to two papers by the present author [3, 4]. The domain of definition of monodiffric functions is the set of Gaussian integers; a complex-valued function f is said to be monodiffric at $z = x + yi$ if

(1) $$\frac{1}{i}\,[f(z+i) - f(z)] = f(z+1) - f(z).$$

As Isaacs already observed, if f is defined on the set of integers, then the requirement of monodiffricity determines f uniquely at the Gaussian integers of the upper half-plane. We term this extension monodiffric continuation. Kurowsky [5] showed that the functional values of f may be calculated by use of the formula

(2) $$f(x+yi) = \sum_{k=0}^{y} \binom{y}{k} i^k \Delta^k f(x),$$

where the operator Δ^k is defined by the relations

$$\Delta^0 f(x) = f(x), \quad \Delta^1 f(x) = f(x+1) - f(x) \quad \text{and} \quad \Delta^k f(x) = \Delta^{k-1}(\Delta^1 f(x)) \quad \text{for } k \geqslant 2.$$

When applied to the Fibonacci numbers Δ^k behaves especially nicely; one may easily prove that

$$\Delta^k F_n = F_{n-k}\,.$$

Therefore, via Eq. (2), one may define the Gaussian Fibonacci numbers, F_{n+mi}, for n an integer, m a nonnegative integer by

(3) $$F_{n+mi} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k}\,.$$

The first few values of F_{n+mi} are tabulated below:

$-6+8i$	$3-4i$	$-3+4i$	0	$-3+4i$	$-3+4i$	$-6+8i$	$-9+12i$	$-15+20i$
-5	$3+i$	$-2+i$	$1+2i$	$-1+3i$	$5i$	$-1+8i$	$-1+13i$	$-2+21i$
$-1-2i$	$1+2i$	0	$1+2i$	$1+2i$	$2+4i$	$3+6i$	$5+10i$	$8+16i$
$1-i$	i	1	$1+i$	$2+i$	$3+2i$	$5+3i$	$8+5i$	$13+8i$
1	0	1	1	2	3	5	8	13

y ↑ $\quad$ x →

Figure 1

On the basis of Eq. (3) it is easily shown that

(4) $$F_{n+mi} = F_{(n-1)+mi} + F_{(n-2)+mi} ,$$

that is, for each fixed m, the sequences $\{Re(F_{n+mi})\}$ and $\{Im(F_{n+mi})\}$ are generalized Fibonacci sequences in the sense of Horadam [6].

Our first aim will be to utilize Eq. (4) in order to find a closed form for the Gaussian Fibonacci numbers. The development hinges upon the observation (easily proven by induction via Eq. (1)) that for each $m = 0,1,2,\cdots$,

$$F_{m+2mi} = 0,$$

and, consequently, with the help of Eq. (4), one can prove that

(5) $$F_{n+2mi} = F_{m+1+2mi}F_{n-m}$$

for each $n = 0, \pm 1, \pm 2, \cdots, m = 0, 1, 2, \cdots$.

Although one could show directly that

(6) $$F_{m+1+2mi} = (1+2i)^m ,$$

we shall provide a more insightful derivation. It is well known that if

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{then} \quad Q^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$

for each $k = 0, \pm 1, \pm 2, \cdots$. Since a matrix must satisfy its characteristic equation, one may then write

$$Q^2 = Q + I.$$

With the help of this one finds that

$$(Q+iI)^2 = Q^2 + 2iQ - I = (1+2i)Q,$$

or, more generally, for $m = 0,1,2,\cdots$

$$(Q+iI)^{2m} = (1+2i)^m Q^m .$$

Expansion of the left member of this identity and multiplication by Q^{n-2m} yields

$$\sum_{k=0}^{2m} \binom{2m}{k} i^k Q^{n-k} = (1+2i)^m Q^{n-m} .$$

Finally, equating the first row second column entries of the two members of this matrix identity gives

(7) $$\sum_{k=0}^{2m} \binom{2m}{k} i^k F_{n-k} = (1+2i)^m F_{n-m} .$$

Since, in view of Eq. (3), the left members of Eqs. (5) and (7) are identical, Eq. (6) is proven.

The evaluation of the right member of Eq. (3) for odd m is easily accomplished now with the help of Eq. (1). The results may be summarized as follows:

(8a) $$F_{n+2mi} = (1+2i)^m F_{n-m}$$

(8b) $$F_{n+(2m+1)i} = (1+2i)^m [F_{n-m} + iF_{n-1-m}] .$$

It may be observed that for fixed odd positive integers, m, the sequences $\{F_{n+mi}\}$ are closely related to the generalized complex Fibonacci sequences studied by Horadam [7] and possess similar properties. One may also observe that Eq. (6) is a special case of Eq. (8a), arising when $n = m + 1$.

The identities,

(9a) $$\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k F_{n-2k} = a_m F_{n-m}$$

(9b) $$\sum_{k=0}^{m} \binom{2m+2}{2k+1} (-1)^k F_{n-2k} = b_{m+1} F_{n-m} ,$$

promised earlier in the paper, are obtained by equating the real and the imaginary parts of Eq. (7). The numbers a_k and b_k, defined by

$$(1+2i)^k = a_k + b_k i,$$

may also be obtained with the help of the following algorithm (which is more in the spirit of the present publication): $a_0 = 1$, $b_0 = 0$ and for $k \geqslant 1$,

$$a_k = a_{k-1} - 2b_{k-1} \quad \text{and} \quad b_k = b_{k-1} + 2a_{k-1}.$$

The table below lists the first few values of a_k and b_k obtained in this manner:

n	a_n	b_n
1	1	2
2	−3	4
3	−11	−2
4	−7	−24
5	41	−38
6	117	44
7	29	278
8	−527	336
9	−1,199	−718
10	237	−3,116

n	a_n	b_n
11	6,469	−2,642
12	11,753	10,296
13	−8,839	33,802
14	−76,443	16,124
15	−108,691	−136,762
16	164,833	−354,144
17	873,121	−24,478
18	922,077	1,721,764
19	−2,521,451	3,565,918
20	−9,653,287	−1,476,984

Figure 2

To illustrate the results, we list below the evaluation of Eqs. (9a) and (9b) for $m = 5$:

$$F_n - 45F_{n-2} + 210F_{n-4} - 210F_{n-6} + 45F_{n-8} - F_{n-10} = 41F_{n-5},$$

$$12F_n - 220F_{n-2} + 792F_{n-4} - 792F_{n-6} + 220F_{n-8} - 12F_{n-10} = 44F_{n-5},$$

which, upon simplification, may be combined into the following elegant relationship:

(11) $$F_n - 5F_{n+2} - 9F_{n+5} + 5F_{n+8} - F_{n+10} = 0.$$

Other simple identities arising as special cases include:

(12) $$F_n - 3F_{n+2} + F_{n+4} = 0,$$

(13) $$F_n + 4F_{n+3} - F_{n+6} = 0,$$

and

(14) $$F_n - 12F_{n+2} + 29F_{n+4} - 12F_{n+6} + F_{n+8} = 0.$$

In conclusion we note that the entire development can be extended to the study of generalized Fibonacci numbers. In fact, if the sequence H_n is defined by

$$H_0 = p, \quad H_1 = q, \quad H_n = H_{n-1} + H_{n-2} \quad \text{for} \quad n \geqslant 2,$$

where p and q are arbitrary integers, then Eqs. (9a) and (9b) will readily generalize to

(15a) $$\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k H_{n-2k} = a_m H_{n-m}$$

and

(15b) $$\sum_{k=0}^{m} \binom{2m+2}{2k+1} (-1)^k H_{n-2k} = b_{m+1} H_{n-m},$$

respectively.

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CONSTANTLY MEAN

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The golden mean is quite absurd;
It's not your ordinary surd.
If you invert it (this is fun!),
You'll get itself, reduced by one;
But if increased by unity,
This yields its square, take it from me.

Alone among the numbers real,
It represents the Greek ideal.
Rectangles golden which are seen,
Are shaped such that this golden mean,
As ratio of the base and height,
Gives greatest visual delight.

Expressed as a continued fraction,
It's one, one, one, ···, until distraction;
In short, the simplest of such kind
(Doesn't this really blow your mind?)
And the convergents, if you watch,
Display the series Fibonacc'
In both their bottom and their top,
That is, until you care to stop.

Since it belongs to F-root-five
Its value's tedious to derive.
These properties are quite unique
And make it something of a freak.
Yes, one-point-six-one-eight-oh-three,
You're too irrational for me.

ON MINIMAL NUMBER OF TERMS IN REPRESENTATION OF NATURAL NUMBERS AS A SUM OF FIBONACCI NUMBERS

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Let $f(k)$ denote this number for any natural number k. It is shown that $f(k) \leqslant n$ for $k < F_{2n+2} - 2$, $f(k) = n$ for $k = F_{2n+2} - 2$ and $f(k) = n + 1$ for $k = F_{2n+2} - 1$.

1. A *base* for natural numbers is any sequence S of positive integers for which numbers n and N may be found such that any positive integer $\geqslant N$ may be represented as a sum of $\leqslant n$ members of S. Any arithmetical progression

$$(1) \qquad 1,\ 1+d,\ 1+2d,\ \cdots,$$

where d is an integer and $d > 1$, is a base (it is enough to take $n = d$, $N = 1$). A geometrical progression

$$(2) \qquad 1,\ q,\ q^2,\ \cdots,$$

where q is an integer and $q > 1$, is not a base; if we take for any positive integers n and N the number

$$\sum_{i=0}^{m} q^i = \frac{q^{m+1} - 1}{q - 1},$$

where

$$m = \max\left(n, [lg_q\{1 + N(q-1)\}]\right),$$

is greater than N, but may not be represented as a sum of $\leqslant n$ numbers of progression (2). The sequence of the Fibonacci numbers is defined as $F_i = i$, where $i = 1,2$; $F_i = F_{i-1} + F_{i-2}$, where $i > 2$. This sequence may be considered additive by definition, but it increases faster than any arithmetical progression of type (I). On the other hand a specific characteristic of Fibonacci numbers

$$\lim_{i \to \infty} \frac{F_{i+1}}{F_i} = \frac{\sqrt{5}+1}{2}$$

shows that they increase asymptotically as a geometrical progression with a denominator

$$\frac{\sqrt{5}+1}{2} = q^*;$$

however, $q^* < 2$, i.e., Fibonacci numbers increase more slowly than any geometrical progression of type (2). We show that Fibonacci numbers, in the representation of the positive integers as a sum of these numbers, act as a geometrical progression of type (2). Let us call

$$k = \sum_{i=1}^{f} F_{m_i}, \qquad m_i \leqslant m_{i-1},$$

a *correct* decomposition, if $f = 1$, or if $f > 1$ we have $m_i < m_{i-1} - 1$ for all $i \in [2,f]$.

The theorem of Zeckendorf gives that for any positive integer there exists a correct decomposition; moreover any decomposition of the positive integer into a sum of Fibonacci numbers contains no fewer terms than its correct decomposition.

2. *Theorem 1.*

(1) For any positive integer n the number $F_{2n+2} - 1$ is the smallest number which is not representable as a sum of $\leqslant n$ Fibonacci numbers.

(2) Number $F_{2n+2} - 1$ may be represented as a sum of $n + 1$ Fibonacci numbers.

(3) Number $F_{2n+2} - 2$ is not representable as a sum of $< n$ Fibonacci numbers.

Indeed, if $n = 1$, theorem is evident. Let us assume that the theorem is correct for $n \leq m$. The numbers of segment $[1, F_{2m+2} - 2]$ may be represented for part (1) of the theorem, as a sum of $\leq m$ Fibonacci numbers. Number $(F_{2m+2} - 2) + 1 = F_{2m+2} - 1$ may be represented for part (2) as a sum of $m + 1$ Fibonacci numbers. Number $(F_{2m+2} - 2) + 2 = F_{2m+2}$ is a Fibonacci number. The numbers of segment

$$[F_{2m+2} + 1, \quad F_{2m+2} + (F_{2m+1} - 1)] \tag{3}$$

are sums of number F_{2m+2} and of the corresponding numbers of segment $[1, F_{2m+1} - 1]$, which for part (1) of the theorem (since $F_{2m+1} - 1 \leq F_{2m+2} - 2$) are representable as a sum of $\leq m$ Fibonacci numbers. Number $F_{2m+2} + (F_{2m+1} - 1) + 1 = F_{2m+3}$ is a Fibonacci number. The numbers of the segment

$$[F_{2m+3} + 1, \quad F_{2m+3} + (F_{2m+2} - 2)]$$

are representable as a sum of $\leq m + 1$ Fibonacci numbers for the same reason as for the numbers of segment (3); though in this case we have the number F_{2m+3} and not F_{2m+2}. Thus all numbers not greater than

$$F_{2m+3} + (F_{2m+2} - 2) = F_{2(m+1)+2} - 2$$

are representable as sums of $\leq m + 1$ Fibonacci numbers. A correct decomposition of numbers $F_{2m+2} - 2$ and $F_{2m+2} - 1$ contains respectively (on the basis of the inductive assumptions) m and $m + 1$ terms. If to these decompositions we add on the left-hand side the term F_{2m+3} we obtain the correct decomposition of numbers $F_{2m+4} - 2$ and $F_{2m+4} - 1$. These latter contain respectively $m + 1$ and $m + 2$ terms. From this and from the theorem of Zeckendorf it follows that numbers $F_{2(m+1)+2} - 2$ and $F_{2(m+1)+2} - 1$ may be represented respectively as the sums of $m + 1$ (but not less) and respectively $m + 2$ (but not less) Fibonacci numbers.

By the way, it is clear that

$$F_{2n+2} - 2 = \sum_{i=1}^{2n} F_i = \sum_{i=1}^{n} F_{2i+1} .$$

One of more detailed works on these problems is [2].

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LETTER TO THE EDITOR

April 28, 1970

In regard to the two articles, "A Shorter Proof" by Irving Adler (December, 1969 *Fibonacci Quarterly*) and "1967 as the Sum of Three Squares," by Brother Alfred Brousseau (April, 1967 *Fibonacci Quarterly*), the general result is as follows:

$$x^2 + y^2 + z^2 = n$$

is solvable if and only if n is not of the form $4^t(8k + 7)$, for $t = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots$. See [1].

Since $1967 = 8(245) + 7$, $1967 \neq x^2 + y^2 + z^2$. A lesser result known to Fermat and proven by Descartes is that no integer $8k + 7$ is the sum of three rational squares [2]. The *really* short and usual proof is:

For x, y, and z any integers, $x^2 \equiv 0, 1, \text{or } 4 \pmod 8$ so that $x^2 + y^2 + z^2 \equiv 0, 1, 2, 3, 4, 5, \text{or } 6 \pmod 8$ or $x^2 + y^2 + z^2 \not\equiv 7 \pmod 8$.

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COMPOSITIONS AND RECURRENCE RELATIONS II

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In an earlier paper by the same authors [1] properties of the compositions of an integer with 1 and 2 were discussed. This paper is a sequel to the earlier one and contains results on modes and related concepts. We stress once again as before that the word "compositions" refers only to compositions with ones and twos unless specially mentioned.

Definition 1. To every composition of a positive integer N we add an unending string of zeroes at both ends. The transition $\cdots 0+1+\cdots$ is a rise while $\cdots + 1 + 0 + \cdots$ is a fall. We also defined in [1] that a one followed by a two is rise while it is a fall if they occur in reverse order. We also define $\cdots 0 + 1 + \cdots + 1 + 2$ as *a* rise and $\cdots 2 + 1 + \cdots + 1 + 0 + \cdots$ as *a* fall.

Definition 2. A composition of a positive integer N is called "unimaximal" if there is exactly one rise and one fall. In other words it is unimaximal if there is no 1 occurring between two 2's. (All the 2's are bunched together.) Let $M^1(N)$ denote the number of unimaximal (unimax in short) compositions of N.

Definition 3. A composition of a positive integer is called "uniminimal" if there is no 2 occurring between two 1's. (All the 1's are bunched together.) Let $m^1(N)$ denote the number of uniminimal (unimin in short) compositions of N.

We shall now investigate some of the properties of $m^1(N)$ and $M^1(N)$ and make an asymptotic estimate of $m^1(N)/M^1(N)$.

Theorem 1.

(a) $$M^1(N) = M^1(N-1) + [N/2]$$

(b) $$m^1(N) = m^1(N-2) + [N/2]$$

(c) $$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2}$$

(d) $$m^1(2N) + m^1(2N-1) = m^1(2N+1) + m^1(2N-2),$$

where $[x]$ represents the largest integer $\leqslant x$.

Proof. Let $M^1(N,1)$ and $M^1(N,2)$ denote the number of unimax compositions ending with 1 and 2, respectively. Clearly $M^1(N) = M^1(N,1) + M^1(N,2)$. By Definition 2 we see that

(1) $$M^1(N,1) = M^1(N-1)$$

since the 1 at the end of the compositions counted by $M^1(N,1)$ will not affect the bunching of twos. However a 2 at the end preserves unimax if and only if it is preceded by another 2 or a complete string of ones only. Thus

(2) $$M^1(N,2) = M^1(N-2, 2) + 1$$

so that decomposing (2) further we arrive at

$$M^1(2N+1) = N$$

and

(3) $$M^1(2N) = N.$$

Putting (1) and (3) together we get

(4) $$M^1(N) = M^1(N-1) + [N/2]\,.$$

Now using similar combinatorial arguments for m^1 with similar notation for $m^1(N,1)$ and $m^1(N,2)$ we see

(5) $$m^1(N) = m^1(N,1) + m^1(N,2)$$

and

(6) $$m^1(N,2) = m^1(N-2)$$

while

$$m^1(N,1) = m^1(N-1,1) + 1 \quad \text{if} \quad N-1 \equiv 0 \pmod 2$$
$$m^1(N,1) = m^1(N-1,1) \quad \text{if} \quad N \equiv 1 \pmod 2$$

which gives

(7) $$m^1(2N) = m^1(2N-2) + N$$

(8) $$m^1(2n+1) = m^1(2N-1) + N$$

or

$$m^1(N) = m^1(N-2) + [n/2]\,.$$

From (4) we deduce

$$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2}$$

for

$$M^1(2N) = M^1(2N-1) + N$$
$$M^1(2N+1) = M^1(2N) + N.$$

Finally (7) and (8) together imply

$$m^1(2N) + m^1(2N-1) = m^2(2N+1) + m^1(2N-2)$$

proving Theorem 1.

Theorem 2.

$$\lim_{N\to\infty} \frac{m^1(N)}{M^1(N)} = \frac{1}{2}\,.$$

Proof. Let Δ_n denote the n^{th} triangular number

$$\Delta_n = \frac{n(n+1)}{2}\,.$$

In general for real x let

(9) $$\Delta_x = \frac{x(x+1)}{2}\,.$$

It is not difficult to establish using induction and Theorem 1 that

(10) $$m^1(2N+1) = \Delta_{N+1}$$

(11) $$m^1(2N) = m^1(2N-1) + 1$$

so that (10) and (11) together imply

(12) $$m^1(N) = \Delta_{N/2} + O(1)\,.$$

One can also show similarly that

(13) $$M^1(2N+1) = \Delta_{N+1} + \Delta_{N-1}$$

and

(14) $$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2} = \frac{\Delta_{n+1} + 2\Delta_{N-1} + \Delta_{N-3}}{2}$$

which give

(15) $$M^1(N) = 2\Delta_{N/2} + O(N)$$

for

$$\lim_{N\to\infty} \Delta_N/\Delta_{N+1} = 1.$$

Now (12) and (15) together imply

$$\lim_{N \to \infty} \frac{m^1(N)}{M^1(N)} = \frac{1}{2}$$

proving Theorem 2.

Definition 4. Every rise and a fall determines a maximum. Every fall and a rise determines a minimum. Let $M(N)$ and $m(N)$ denote the number of maximums and minimums in the compositions of N.

Theorem 3.

$$M(N) = M(N-1) + M(N-2) + F_{N-2} - 1$$
$$m(N) = m(N-1) + m(N-2) + F_{N-2} - 1$$

$$\lim_{N \to \infty} \frac{m(N)}{M(N)} = 1.$$

Proof. As before split $M(N)$ as

$$M(N) = M(N,1) + M(N,2).$$

It is clear that the "1" at the end of the compositions counted by $M(N,1)$ does not record a max and so

$$M(N,1) = M(N-1).$$

Clearly the "2" at the end of the compositions counted by $M(N,2)$ records an extra max if and only if the corresponding composition counted by $N-2$ ends in a 1 but not for $N-2 = 1+1+\cdots 1$ a string of ones. Thus

$$M(N,2) = M(N-2) + C_{N-2}(1) - 1$$
$$= M(N-2) + F_{n-2} - 1$$

giving

(16) $$M(N) = M(N-1) + M(N-2) + F_{N-2} - 1.$$

Proceeding similarly for $m(N)$ we have

$$m(N) = m(N,1) + m(N,2) \quad \text{and} \quad m(N,1) = m(N-1) + C_{N-1}(2) - 1 = m(N-1) + F_{N-2} - 1$$

while $m(N,2) = m(N-2)$ giving

(17) $$m(N) = m(N-1) + m(N-2) + F_{N-2} - 1.$$

It is quite clear from (16) and (17) that $m(N)$ and $M(N)$ are Fibonacci Convolutions so that [see Hoggatt and Alladi [2]].

(18) $$\lim_{N \to \infty} \frac{F_N}{m(N)} = \lim_{N \to \infty} \frac{F_N}{M(N)} = 0.$$

Now pick any composition of N say N_C. Let $M(N_C)$ and $m(N_C)$ denote the number of max and min, respectively in N_C. Since there is a fall between two rises and a rise between two falls we have

(19) $$|M(N_C) - m(N_C)| \leqslant 1.$$

Now from the definition of N_C it is obvious that

(20) $$|M(N) - m(N)| = \left|\sum_C M(N_C) - \sum_C m(N_C)\right| = \left|\sum_C (M(N_C) - m(N_C))\right| \leqslant \sum_C |M(N_C) - m(N_C)| \leqslant C_N = F_{N+1}$$

by (19). Now if we use (18) we get

$$\lim_{N \to \infty} \frac{m(N)}{M(N)} = 1.$$

In other words the number of maximums and the number of minimums are asymptotically equal.

Let us now find the asymptotic distribution of 1's and 2's in unimax compositions. Let $M_1(N)$ and $M_2(N)$ denote the number of ones and number of twos in the unimax compositions of N.

Theorem 4.

$$M_1(2N+1) = M_1(2N) + M^1(2N) + N^2, \quad M_1(2N) = M_1(2N-1) + M^1(2N-1) + N(N-1).$$

Proof. As before, let

(21) $$M_1(N) = M_1(N,1) + M_1(N,2).$$

Clearly we have

$$M_1(N,1) = M_1(N-1) + M^1(N-1)$$

while

(22) $$M_1(N,2) = M_1(N-2,2) + (N-2)$$

for the compositions $1 + 1 + 1 \cdots 1 = N-2$, and $1 + 1 + \cdots 1 + 2 = N$ are both unimax. Now if we decompose (22) further we sum alternate integers. Then (21) gives the two equations of Theorem 4.

Theorem 5. $$M_2(2N+1) = M_2(2N) + N + \frac{(N-1)N}{2}$$

$$M_2(2N) = M_2(2N-1) + N + \frac{(N-1)N}{2}.$$

Proof. By combinatorial arguments similar to Theorem 4 we get

$$M_2(N) = M_2(N,1) + M_2(N,2)$$

giving $M_2(N,1) = M_2(N-1)$ and

$$M_2(N,2) = M_2(N-2,2) + M^1(N-2,2) + 1 = N/2 + M^1(N-2,2) + M^1(N-4,2) + \cdots$$

on further decomposition. We also know from (3) that

$$M^1(2N+1,2) = M^1(2N,2) = N$$

so that

$$M_2(2N+1) = M_2(2N) + \frac{N(N+1)}{2}, \quad M_2(2N) = M_2(2N-1) + \frac{N(N+1)}{2}.$$

Theorem 6.

$$\lim_{N\to\infty} \frac{M_2(N)}{M_1(N)} = \frac{1}{2}.$$

Proof. It is easy to prove that for real x

(23) $$f(x) = \sum_{N \leqslant x} N^2 \sim \frac{x^3}{3}$$

We know from Theorem (4) that

(24) $$M_1(2N+1) = M_1(2N) + M^1(2N) + N^2$$

(25) $$M_1(2N) = M_1(2N-1) + M^1(2N-1) + N(N-1).$$

From (4) one can deduce without trouble that

(26) $$M^1(2N+1) = N^2 + N + 1$$

(27) $$M^1(2N) = N^2 + 1.$$

Now substituting (26) and (27) in (24) and (25) and continuing the decomposition using the recursion on Theorem 4 we get

(28) $$M_1(N) = \sum_{m \leqslant N/2} m^2 + \sum_{m \leqslant N/2} m^2 + O(N^2) = \frac{2}{3}\left(\frac{N}{2}\right)^3 + O(N^2) \sim \frac{2}{3}\left(\frac{N}{2}\right)^3$$

using (23). If we adopt the same decomposition procedure to the two equations in Theorem (5) we get by virtue of

(29) $$M_2(N) = \sum_{m \leqslant N/2} m^2 + O(N^2) = \frac{1}{3}\left(\frac{N}{2}\right)^3 + O(N^2).$$

Now (28) and (29) together imply

$$\lim_{N \to \infty} \frac{M_2(N)}{M_1(N)} = \frac{1}{2}$$

establishing Theorem 6,

We now state theorems analogous to (4) and (5) and (6) for the uniminimal compositions.

Theorem 7.

$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [N/2]$$

$$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]\,.$$

Proof. With the usual notation $m_1(N,1)$ and $m_1(N,2)$ we find

$$m_1(N) = m_1(N,1) + m_1(N,2)$$

$$m_1(N_12) = m_1(N-2)$$

since the "2" at the end of the compositions counted by $m(N,2)$ will not affect the counting of minimums or ones. However for $m_1(N,1)$ we find

$$\begin{aligned} m_1(N,1) &= m_1(N-1,\,1) + m^1(N-1,\,1) + 1 \\ &\qquad \text{if } N-1 \equiv 0 \pmod 2 \\ &= m_1(N-1,\,1) + m^1(N-1,\,1) \\ &\qquad \text{if } N-1 \equiv 1 \pmod 2 \end{aligned}$$

so putting these together we get

$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [N/2]\ .$$

With similar use of notation for m_2 we get

$$m_2(N) = m_2(N,1) + m_2(N,2)$$

giving

$$m_2(N_12) = m_2(N-2) + m^1(N-2)$$

while

$$\begin{aligned} m_2(N,1) &= m_2(N-1,\,1) + 1 \quad \text{if } N-1 \equiv 0 \pmod 2 \\ &= m_2(N-1) \qquad\qquad \text{if } N-1 \equiv 1 \pmod 2 \end{aligned}$$

so that these give

$$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]\,.$$

Theorem 8.

$$\lim_{N \to \infty} \frac{m_2(N)}{m_1(N)} = \frac{1}{2}\,.$$

Proof. We know from Theorem 7 that

$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [n/2]\,. \tag{29}$$

Now from Theorem 1 we deduce that

$$m^1(n,1) = [n/2]$$

so that

$$\sum_{n=1}^{N-1} m^1(n,1) = \sum_{n=1}^{N-1} [n/2] = \sum_{n=1}^{N-1} [n/2] + O(N-1) = \frac{\Delta_{N-1}}{2} + O(N-1)\,. \tag{30}$$

If we continue to decompose $m_1(N-2)$ in (29) and use (30) we will finally get

(31) $$m_1(N) = \left\{\frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \frac{\Delta_{N-5}}{2} + \cdots\right\} + O(N^2) \sim \frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \cdots .$$

We also know from Theorem 7 that

(32) $$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]$$

It is easy to establish from Theorem 1 that

$$m^1(2N+1) = \Delta_{N+1}, \qquad m^1(2N+2) = m^1(2N+1) + 1$$

giving

(33) $$m^1(N) = \Delta_{N/2} + O(N) \sim \Delta_{N/2} .$$

Now decomposing $m_2(N-2)$ in (32) further and using (33) we get

(34) $$m_2(N) = \left\{\frac{\Delta_{N-2}}{2} + \frac{\Delta_{N-4}}{2} + \cdots\right\} + O(N^2) = \frac{1}{2}\left\{\frac{\Delta_{N-2}}{2} + \frac{\Delta_{N-4}}{2} + \cdots\right\} + O(N^2)$$

$$= \frac{1}{2}\left\{\frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \cdots\right\} + O(N^2)$$

since $x \sim y$ implies $\Delta_x \sim \Delta_y$. Now if we compare (34) and (31) we get

$$\lim_{N\to\infty} \frac{m_2(N)}{m_1(N)} = \frac{1}{2}$$

proving Theorem 8.

We now shift our attention to compositions called "Zeckendorf compositions." A composition of N in which no two consecutive ones appear is called a Zeckendorf composition (1) and if no two consecutive twos appear it is called a Zeckendorf composition (2). We denote them in short as z_1 and z_2 compositions respectively. Note that in a z_1 composition there *should be a* 2 between ones while in a unimin there *should not* similarly z_2 is the opposite of unimax. Now denote by

$Z(N)$ = the number of Z_2 compositions of N
$z(N)$ = the number of Z_1 compositions of N.

Theorem 9. $Z(N) = Z(N-1) + Z(N-3), \qquad z(N) = z(N-2) + z(N-3).$

$$\lim_{N\to\infty} \frac{z(N)}{Z(N)} = 0.$$

Proof. As usual partition

$$z(N) = z(N,1) = z(N,2)$$

clearly $z(N,2) = z(N-2)$, while

$$z(N,1) = z(N-1,2) = z(N-3)$$

this proves

$$z(N) = z(N-2) + z(N-3).$$

Again

$$Z(N) = Z(N_1 1) + Z(N,2) \quad \text{and} \quad Z(N,1) = Z(N-1)$$

while

$$Z(N,2) = Z(N-2,1) = Z(N-3)$$

giving

$$Z(N) = Z(N-1) + Z(N-3).$$

It can be shown that

$$\lim_{N\to\infty} \frac{Z(N+1)}{Z(N)} = \alpha$$

and

$$\lim_{N\to\infty} \frac{Z(N+1)}{z(N)} = \beta,$$

where α and β are the dominant roots of the auxiliary polynomials $x^3 - x^2 - 1 = 0$ and $x^3 - x - 1 = 0$ $(\alpha > \beta)$. See Hoggatt and Alladi [2]. This implies that there exist constants $c_\alpha, c_\beta > 0$ so that

$$Z(N) > C_\alpha \alpha^N$$

and

and

$$z(N) < C_\beta \beta^N$$

giving

$$\lim_{N \to \infty} \frac{z(N)}{Z(N)} = 0.$$

Corollary. On similar lines

$$\lim_{N \to \infty} \frac{Z(N)}{C_N} = \lim_{N \to \infty} \frac{z(N)}{C_N} = 0.$$

NOTE. Given a partition of N in terms of 1 and 2, if we rearrange the summands so as to get the maximum number of max we get a Z_2 composition. If we rearrange to get the maximum number of min we get a Z_1 composition. Roughly a Zeckendorf composition is either a maximax or a maximin composition.

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A TOPOLOGICAL PROOF OF A WELL KNOWN FACT ABOUT FIBONACCI NUMBERS

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Theorem. Let p be a prime. Then there is a sequence $\{m_j\}$ of positive integers such that

$$F_{m_j} \equiv 1 - F_{m_j-1} \equiv 1 - F_{m_j+1} \equiv 0 \pmod{p^j}.$$

The proof depends on the following lemma.

Lemma. Let G be a topological group whose completion (in the natural uniformity) is compact. Let $g \in G$. Then the sequence $g, g^2, g^3, \cdots$ has a subsequence which converges to 1.

Proof. The sequence of powers of g has an accumulation point $h = \lim_{j \to \infty} g^{n_j}$ in the compact completion $\overline{G}$ of G. Let $m_j = n_{j+1} - n_j$. Then $g^{m_j} \to 1$ in $\overline{G}$ and hence in G.

To prove the theorem we shall apply the lemma to

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

in the group G of 2×2 integer matrices of determinant ± 1 topologized p-adically. That is, for every integer n write $n = p^k m$, $(p,m) = 1$ and set $\|n\|_p = p^{-k}$. Then for $A, B \in G$ let

$$d(A,B) = \max\left\{ \|A_{ij} - B_{ij}\|_p : i, j = 1, 2 \right\}$$

G equipped with the metric d satisfies the hypotheses of the lemma.

It is easy to check inductively that

$$g^m = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}.$$

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ZERO-ONE SEQUENCES AND FIBONACCI NUMBERS

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1. INTRODUCTION

It is well known that the number of zero-one sequences of length n:

(1.1) $$(a_1, a_2, \cdots, a_n) \qquad (a_i = 0 \text{ or } 1)$$

with consecutive ones forbidden is equal to the Fibonacci number F_{n+2}. Moreover the number of such sequences with $a_n = a_1 = 1$ also forbidden is equal to the Lucas number L_n. This suggests the following two problems.

1. Let $n_{00}, n_{01}, n_{10}, n_{11}$ be non-negative integers such that

$$n_{00} + n_{01} + n_{10} + n_{11} = n - 1.$$

We seek the number of sequences (1.1) with exactly n_{00} occurrences of 00, n_{01} occurrences of 01, n_{10} occurrences of 10 and n_{11} occurrences of 11.

2. Let $n_{00}, n_{01}, n_{10}, n_{11}$ be non-negative integers such that

$$n_{00} + n_{01} + n_{10} + n_{11} = n.$$

We again seek the number of sequences (1.1) with n_{ij} occurrences of ij, but now $a_n a_1$ is counted as a consecutive pair.

Let $a(n_{00}, n_{01}, n_{10}, n_{11})$ denote the number of solutions of Problem 1 and $b(n_{00}, n_{01}, n_{10}, n_{11})$ denote the number of solutions of Problem 2. Put

$$f_n = f_n(x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{n_{ij}=0}^{\infty} a(n_{00}, n_{01}, n_{10}, n_{11}) x_{00}^{n_{00}} x_{01}^{n_{01}} x_{10}^{n_{10}} x_{11}^{n_{11}},$$

$$g_n = g_n(x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{n_{ij}=0}^{\infty} b(n_{00}, n_{01}, n_{10}, n_{11}) x_{00}^{n_{00}} x_{01}^{n_{01}} x_{10}^{n_{10}} x_{11}^{n_{11}}.$$

It is convenient to take

$$f_0 = g_0 = 0, \qquad f_1 = g_1 = 2.$$

Put

$$F(u) = \sum_{n=0}^{\infty} f_n u^n, \qquad G(u) = \sum_{n=0}^{\infty} g_n u^n.$$

We show that

(1.2) $$F(u) = \frac{2u + (x_{01} + x_{10} - x_{00} - x_{11})u^2}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2}$$

and

(1.3) $$2 + G(u) = \frac{2 - (x_{00} + x_{11})u}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2}.$$

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The special case

$$x_{00} = x_{01} = x_{10} = 1, \qquad x_{11} = x \tag{1.4}$$

is of some interest. In this case (1.2) and (1.3) reduce to

$$1 + F(u) = \frac{1+(1-x)u}{1-(1+x)u-(1-x)u^2} \tag{1.5}$$

$$2 + G(u) = \frac{2-(1+x)u}{1-(1+x)u-(1-x)u^2}, \tag{1.6}$$

respectively. These generating functions evidently contain the enumeration of zero-one sequences with a given number of occurrences of 11.

For $x = 0$, (1.5) and (1.6) reduce to the generating functions for F_{n+2} and L_n, respectively. Thus it is natural to put

$$1 + F(u) = \sum_{n=0}^{\infty} f_{n+2}(x)u^n, \qquad f_n(x) = \sum_k F_{n,k}x^k,$$

$$2 + G(u) = \sum_{n=0}^{\infty} g_n(x)u^n, \qquad g_n(x) = \sum_k L_{n,k}x^k .$$

We find that $f_n(x)$, $g_n(x)$ both satisfy

$$v_{n+2} = (1+x)v_{n+1} + (1-x)v_n ,$$

which implies

$$F_{n+2,k} = F_{n+1,k} + F_{n,k} + F_{n+1,k-1} - F_{n,k-1}$$

and similarly for $L_{n,k}$. Moreover there is the striking relation

$$g_n(x) = f_{n+3}(x) - 2f_{n+2}(x) + 2f_{n+1}(x) \qquad (n \geqslant 0).$$

2. PROBLEM 1

In order to enumerate the number of sequences of Problem 1 it is convenient to define

$$a^i_{rs}(n_{00}, n_{01}, n_{10}, n_{11}) \qquad (i = 0, 1) \tag{2.1}$$

as the number of zero-one sequences with r zeros, s ones, n_{jk} occurrences of jk and ending with i, where

$$n_{00} + n_{01} + n_{10} + n_{11} = r + s - 1.$$

Put

$$f_i(r,s) = f_i(r,s|x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{r,s} a^i_{rs}(n_{00}, n_{01}, n_{10}, n_{11})x_{00}^{n_{00}}x_{01}^{n_{01}}x_{10}^{n_{10}}x_{11}^{n_{11}} . \tag{2.2}$$

It is convenient to take

$$\begin{cases} f_0(0,0) = 0, & f_0(1,0) = 1, \quad f_0(0,1) = 1 \\ f_1(0,0) = 0, & f_1(1,0) = 0, \quad f_1(0,1) = 1 . \end{cases} \tag{2.3}$$

Deleting the final element in a given sequence, we obtain the following recurrences:

$$\begin{cases} f_0(r,s) = x_{00}f_0(r-1, s) + x_{10}f_1(r-1, s) \\ f_1(r,s) = x_{01}f_0(r, s-1) + x_{11}f_1(r, s-1) \end{cases} \quad (r+s > 1). \tag{2.4}$$

Put

$$F_i = F_i(u,v) = \sum_{r,s=0}^{\infty} f_i(r,s)u^rv^s \qquad (i = 0, 1). \tag{2.5}$$

Then by the first of (2.4)

$$F_0(u,v) = uf_0(1,0) + vf_0(0,1) + x_{00}u \sum_{r+s\geq 2} u^{r-1}v^s f_0(r-1,s) + x_{10}v \sum_{r+s\geq 2} u^{r-1}v^s f_1(r-1,s),$$

so that

(2.6) $$F_0(u,v) = u + x_{00}uF_0(u,v) + x_{10}uF_1(u,v).$$

Similarly

(2.7) $$F_1(u,v) = v + x_{01}vF_0(u,v) + x_{11}vF_1(u,v).$$

This pair of formulas can be written compactly in matrix form:

(2.8) $$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + M \begin{pmatrix} F_0 \\ F_1 \end{pmatrix},$$

where

(2.9) $$M = \begin{pmatrix} x_{00}u & x_{10}u \\ x_{01}v & x_{11}v \end{pmatrix}.$$

It follows at once from (2.8) that

$$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = (I-M)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Since

$$(I-M)^{-1} = \frac{1}{D}\begin{pmatrix} 1-x_{11}v & x_{10}u \\ x_{01}v & 1-x_{00}u \end{pmatrix},$$

where

(2.10) $$D = \det M = 1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv,$$

we get

(2.11) $$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} u + (x_{10}-x_{11})uv \\ v + (x_{01}-x_{00})uv \end{pmatrix}.$$

Hence

(2.12) $$F(u,v) = F_0(u,v) + F_1(u,v) = \frac{u + v + (x_{01}+x_{10}-x_{00}-x_{11})uv}{1 - x_{00}u - x_{11}v + (x_{00}x_{11}-x_{01}x_{10})uv}.$$

This furnishes a generating function for the enumeration of sequences with a given number of zeros and a given number of ones and n_{ij} occurrences of ij.

Finally, taking $u = v$, we get the desired solution of Problem 1.

(2.13) $$F(u) = F(u,u) = \frac{2u + (x_{01}+x_{10}-x_{00}-x_{11})u^2}{1-(x_{00}+x_{11})u + (x_{00}x_{11}-x_{01}x_{10})u^2}.$$

Explicit formulas for

$$f(r,s) = f_0(r,s) + f_1(r,s)$$

can be obtained from (2.12). The extreme right member is equal to

$$\frac{u(1-x_{11}v) + v(1-x_{00}u) - (x_{01}+x_{11})uv}{(1-x_{00}u)(1-x_{11}v) - x_{01}x_{10}uv} = \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})^k u^{k+1}v^k}{(1-x_{00}u)^{k+1}(1-x_{11}v)^k}$$

$$+ \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})^k u^k v^{k+1}}{(1-x_{00}u)^k(1-x_{11}v)^{k+1}} - (x_{01}+x_{10}) \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})^k u^{k+1}v^{k+1}}{(1-x_{00}u)^{k+1}(1-x_{11}v)^{k+1}}.$$

Expanding, we get after some manipulation

(2.14) $$f(r,s) = \sum_{k>0} \binom{r-1}{k}\binom{s-1}{k-1}(x_{01}x_{10})^k x_{00}^{r-k-1}x_{11}^{s-k} + \sum_{k>0}\binom{r-1}{k-1}\binom{s-1}{k}(x_{01}x_{10})^k x_{00}^{r-k}x_{11}^{s-k-1}$$

$$- (x_{01}+x_{10})\sum_{k>0}\binom{r-1}{k-1}\binom{s-1}{k-1}(x_{01}x_{10})^k x_{00}^{r-k-1}x_{11}^{s-k-1} \quad (r>0,\ s>0,\ r+s>2).$$

3. SPECIAL CASES OF PROBLEM 1

If we take

(3.1) $$x_{00} = x_{01} = x_{10} = 1, \quad x_{11} = x,$$

(2.3) reduces to

(3.2) $$1 + F(u) = \frac{1+(1-x)u}{1-(1+x)u-(1-x)u^2}.$$

For $x = 0$ the right-hand side becomes

$$\frac{1+u}{1-u-u^2} = \sum_{n=0}^{\infty} F_{n+2}u^n$$

as anticipated. We now define $F_{n,j}$ by means of

(3.3) $$\frac{1+(1-x)u}{1-(1+x)u-(1-x)u^2} = \sum_{n=0}^{\infty} f_{n+2}(x)u^n,$$

where

(3.4) $$f_n(x) = \sum_{j \geq 0} F_{n,j}x^j.$$

It follows from (3.3) that $f_n(x)$ satisfies

(3.5) $$f_{n+2}(x) = (1+x)f_{n+1}(x) + (1-x)f_n(x) \quad (n \geq 2)$$

together with $f_2(x) = 1$, $f_3(x) = 2$; if we take $f_1(x) = 1$, then (3.5) holds for $n \geq 1$. From (3.5) we get the recurrence

(3.6) $$F_{n+2,k} = F_{n+1,k} + F_{n+1,k-1} + F_{n,k} - F_{n,k-1} \quad (n \geq 1).$$

The following table is now easily computed.

$F_{n,k}$:

n \ k	0	1	2	3	4	5	6	7
1	1							
2	1							
3	2							
4	3	1						
5	5	2	1					
6	8	5	2	1				
7	13	10	6	2	1			
8	21	20	13	7	2	1		
9	34	38	29	16	8	2	1	
10	55	71	60	39	19	9	2	1

Note that

(3.7) $$f_n(1) = \sum_{j \geq 0} F_{n,j} = 2^{n-2} \quad (n \geq 2).$$

This follows at once by taking $x = 1$ in (3.3). If we take $x = -1$ we get

$$\sum_{n=0}^{\infty} f_{n+2}(-1)u^n = \frac{1+2u}{1-2u^2},$$

which yields

(3.8) $$f_{2n}(-1) = 2^{n-1}, \quad f_{2n+1}(-1) = 2^n \quad (n \geq 1).$$

The table suggests

(3.9)
$$\begin{cases} F_{n,n-3} = 1 & (n > 3) \\ F_{n,n-4} = 2 & (n > 4) \\ F_{n,n-5} = n-1 & (n \geqslant 5) \end{cases}$$

Since

$$\frac{1+(1-x)u}{1-(1+x)u-(1-x)u^2} = \frac{1}{1-u-u^2} + \sum_{k=1}^{\infty} \frac{u^{k+1}(1-u)^{k-1}x^k}{(1-u-u^2)^{k+1}},$$

we have also

(3.10)
$$\sum_{n=k+3}^{\infty} F_{n,k}u^n = \frac{u^{k+1}(1-u)^{k-1}}{(1-u-u^2)^{k+1}} \qquad (k \geqslant 1).$$

Replacing x by x/u in (3.3) we get

(3.11)
$$\frac{1-x+u}{1-x-(1-x)u-u^2} = \sum_{n=0}^{\infty} u^n \sum_{k=0}^{\infty} F_{n+k+2,k}x^k,$$

which furnishes a generating function for diagonals, namely

(3.12)
$$D_n(x) \equiv \sum_{k=0}^{\infty} F_{n+k+2,k}x^k = \sum_{2s \leqslant n+1} \binom{n-s+1}{s}(1-x)^{-s}.$$

For example

$$D_0(x) = 1, \quad D_1(x) = 1+\frac{1}{1-x}, \quad D_2(x) = 1+\frac{2}{1-x}, \quad D_3(x) = 1+\frac{3}{1-x}+\frac{1}{(1-x)^2},$$

in agreement with (3.9). Also,

$$D_4(x) = 1+\frac{4}{1-x}+\frac{3}{(1-x)^2}, \quad D_5(x) = 1+\frac{5}{1-x}+\frac{6}{(1-x)^2}+\frac{1}{(1-x)^3}, \text{ etc.}$$

The special case

(3.10)
$$x_{00} = x_{10} = x_{11} = 1, \quad x_{01} = x$$

is considerably simpler than (3.1). Using (3.10), (2.13) reduces to

(3.11)
$$1+F(u) = \frac{1}{1-2u+(1-x)u^2}.$$

Since

$$\frac{1}{1-2u+(1-x)u^2} = \frac{1}{(1-u)^2-xu^2} = \sum_{k=0}^{\infty} \frac{x^k u^{2k}}{(1-u)^{2k+2}} = \sum_{k=0}^{\infty} x^k u^{2k} \sum_{j=0}^{\infty} \binom{2k+k+1}{j} u^j$$

$$= \sum_{n=0}^{\infty} u^n \sum_{2k \leqslant n} \binom{n+1}{2k+1} x^k,$$

so that (3.11) becomes

(3.12)
$$1+F(u) = \sum_{n=0}^{\infty} u^n \sum_{2k \leqslant n} \binom{n+1}{2k+1} x^k.$$

It follows from (3.12) that the number of sequences of length n with k occurrences of 01 is equal to the binomial coefficient $\binom{n+1}{2k+1}$. It is not difficult to give a direct combinatorial proof of this result.

4. PROBLEM 2

Let

(4.1)
$$a_{rs}^{ij}(n_{00}, n_{01}, n_{10}, n_{11}) \qquad (i,j = 0)$$

denote the number of sequences with r zeros and s ones, where $r+s = n_{00}+n_{01}+n_{10}+n_{11}+1$, with n_{hk} occurrences of hk, beginning with i and ending with j. Also put

$$f_{ij}(r,s) = f_{ij}(r,s\,|\,x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{n_{hk}=0}^{\infty} a_{rs}^{ij}(n_{00}, n_{01}, n_{10}, n_{11})x_{00}^{n_{00}}x_{01}^{n_{01}}x_{10}^{n_{10}}x_{11}^{n_{11}}, \tag{4.2}$$

$$F_{ij} = F_{ij}(u,v) = \sum_{r,s=0}^{\infty} f_{ij}(r,s)u^r v^s . \tag{4.3}$$

Exactly as in § 2, we have

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} + M \begin{pmatrix} F_{00} & F_{01} \\ F_{01} & F_{11} \end{pmatrix}, \tag{4.4}$$

where M is defined in (2.9). Thus

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = (I-M)^{-1}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} .$$

It follows that

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} u - x_{11}uv & x_{10}uv \\ x_{01}uv & v - x_{00}uv \end{pmatrix}, \tag{4.5}$$

where as before

$$D = 1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv. \tag{4.6}$$

For Problem 2 we require

$$G(u,v) = x_{00}F_{00} + x_{10}F_{01} + x_{01}F_{10} + x_{11}F_{11}. \tag{4.7}$$

Hence, by (4.5) and (4.6),

$$G(u,v) = \frac{x_{00}u + x_{11}v - 2(x_{00}x_{11} - x_{01}x_{10})uv}{1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv} .$$

It is convenient to replace this by

$$2 + G(u,v) = \frac{2 - x_{00}u - x_{11}v}{1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv} . \tag{4.8}$$

In particular, for $u = v$, (4.8) becomes

$$2 + g(u,u) = \frac{2 - (x_{00} + x_{11})u}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2} . \tag{4.9}$$

Thus (4.9) furnishes a generating function for Problem 2.

If we put

$$2 + G(u,u) = \sum_{n=0}^{\infty} g_n u^n, \qquad F(u) = \sum_{n=0}^{\infty} f_n u^n,$$

where, by (2.13)

$$F(u) = \frac{2u + (x_{01} + x_{10} - x_{00} - x_{11})u^2}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2},$$

then it is clear that

$$(2 - (x_{00} + x_{11})u) \sum_{0}^{\infty} f_n u^n = (2u + (x_{01} + x_{10} - x_{00} - x_{11})u^2) \sum_{0}^{\infty} g_n u^n .$$

Comparison of coefficients gives

$$f_n - (x_{00} + x_{11})f_{n-1} = 2g_{n-1} + (x_{01} - x_{10} - x_{00} - x_{11})g_{n-2} . \tag{4.10}$$

5. SPECIAL CASES OF PROBLEM 2

We take

(5.1) $$x_{00} = x_{01} = x_{10} = 1, \qquad x_{11} = x.$$

Then (4.9) reduces to

(5.2) $$2 + G(u,u) = \frac{2-(1+x)u}{1-(1+x)u-(1-x)u^2} .$$

For $x = 0$ the right side of (5.2) becomes

$$\frac{2-u}{1-u-u^2} = \sum_0^\infty L_n u^n$$

as was expected. We now define $L_{n,j}$ by means of

(5.3) $$\frac{2-(1+x)u}{1-(1+x)u-(1-x)u^2} = \sum_{n=0}^\infty g_n(x)u^n ,$$

where

(5.4) $$g_n(x) = \sum_{j\geqslant 0} L_{n,j}x^j .$$

It follows from (5.3) that $g_n(x)$ satisfies

(5.5) $$g_{n+2}(x) = (1+x)g_{n+1}(x) + (1-x)g_n(x) \quad (n \geqslant 0)$$

together with $g_0(x) = 2, g_1(x) = 1 + x$. It is also clear that $L_{n,k}$ satisfies the recurrence

(5.6) $$L_{n+2,k} = L_{n+1,k} + L_{n+1,k-1} + L_{n,k} - L_{n,k-1} \quad (n \geqslant 0)$$

which is of course the same as (3.6).

The following table is easily computed.

$L_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	2										
1	1	1									
2	3	0	1								
3	4	3	0	1							
4	7	4	4	0	1						
5	11	10	5	5	0	1					
6	18	18	15	6	6	0	1				
7	29	35	28	21	7	7	0	1			
8	47	64	60	40	28	8	8	0	1		
9	76	117	117	93	54	36	9	9	0	1	
10	123	210	230	190	135	70	45	10	10	0	1

It is easily proved by means of (5.3) and (5.4) that

(5.7) $$g_n(1) = \sum_{k=0}^n L_{n,k} = 2^n \qquad (n \geqslant 1),$$

(5.8) $$g_{2n}(-1) = 2^{n+1}, \qquad g_{2n+1}(-1) = 0 \qquad (n \geqslant 0).$$

The table suggests that $L_{nn} = 1,$

$$(5.9)\qquad \begin{cases} L_{n,n-1} = 0 & (n > 1), \\ L_{n,n-2} = n & (n > 2), \\ L_{n,n-3} = n & (n > 3). \end{cases}$$

These results are easily proved by induction using (5.6).

Comparison of (5.3) with (3.3) gives

$$(5.10)\qquad g_n(x) + (1-x)g_{n-1}(x) = 2f_{n+2}(x) - (1+x)f_{n+1}(x).$$

In view of (3.5), this implies

$$(5.11)\qquad g_n(x) + (1-x)g_{n-1}(x) = f_{n+2}(x) + (1-x)f_n(x) \qquad (n \geqslant 1).$$

In particular (5.11) contains the familiar relation $L_{n+1} = F_{n+2} + F_n$. It would be of interest to express $g_n(x)$ in terms of $f_k(x)$.

We find that

$$g_0(x) = f_3(x), \quad g_1(x) = f_4(x) - f_3(x), \quad g_2(x) = f_5(x) - 2f_4(x) + 2f_3(x),$$

$$g_3(x) = f_6(x) - 2f_5(x) + 2f_4(x), \quad g_4(x) = f_7(x) - 2f_6(x) + 2f_5(x), \quad g_5(x) = f_8(x) - 2f_7(x) + 2f_6(x),$$

$$g_6(x) = f_9(x) - 2f_8(x) + 2f_7(x), \quad g_7(x) = f_{10}(x) - 2f_9(x) + 2f_8(x).$$

This suggests that

$$(5.12)\qquad g_n(x) = f_{n+3}(x) - 2f_{n+2}(x) + 2f_{n+1}(x) \qquad (n = 0, 1, 2, \cdots).$$

To prove (5.12) we make use of the identity

$$u(2-(1+x)u) = (1-2u+2u^2)(1+(1-x)u) - (1-2u)(1-(1+x)u-(1-x)u^2).$$

Dividing both sides by $D = 1-(1+x)u-(1-x)u^2$, this becomes

$$u\,\frac{2-(1+x)u}{D} = (1-2u+2u^2)\,\frac{1+(1-x)u}{D} - 1 + 2u.$$

Hence, by (3.3) and (5.3),

$$u\sum_{n=0}^{\infty} g_n(x)u^n = (1-2u+2u^2)\sum_{n=0}^{\infty} f_{n+2}(x)u^n - 1 + 2u.$$

Comparing coefficients of u^n, we get

$$g_{n-1}(x) = f_{n+2}(x) - 2f_{n+1}(x) + 2f_n(x) \qquad (n \geqslant 1),$$

which is equivalent to (5.12).

From (5.12) we get

$$(5.13)\qquad L_{n,k} = F_{n+3,k} - 2F_{n+2,k} + 2F_{n+1,k} \qquad (k = 0, 1, 2, \cdots).$$

Note that, for $k = 0$, (5.13) reduces to the familiar

$$L_n = F_{n+3} - 2F_{n+2} + 2F_{n+1} = -F_{n+2} + 3F_{n+1} = 2F_{n+1} - F_n = F_{n+1} + F_{n-1}.$$

Finally, replacing x by x/u in (5.3), we get

$$\frac{2-x-u}{1-x-(1-x)u-u^2} = \sum_{n=0}^{\infty} u^n \sum_{k=0}^{\infty} L_{n+k,k}x^k.$$

This yields

$$(5.14)\qquad \sum_{k=0}^{\infty} L_{n+k,k}x^k = \frac{3-2x}{1-x}\sum_{2s\leqslant n}\frac{1}{(1-x)^s} - \sum_{2s\leqslant n+1}\binom{n-s+1}{s}\frac{1}{(1-x)^s}.$$

For example

$$\sum_{k=0}^{\infty} L_{k+1,k}x^k = \frac{3-2x}{1-x} - \left(1+\frac{1}{1-x}\right) = 1,$$

which is correct.

REFERENCE

1. L. Carlitz, "Zero-One Sequences and Fibonacci Numbers of Higher Order," *The Fibonacci Quarterly*, Vol. 12 (1974), pp. 1–10.

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THE UNIFIED NUMBER IDENTITY

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The identity illustrated below shows a relation connecting all of the most important constants and numbers in mathematics.

$$e^{i\pi}\left(2\beta + \sum_{n=0}^{\infty} (-1)^n(\sqrt{5}F_{n+1} - L_{n+1})\right) + \alpha \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} \sum\limits_{k=1}^{\infty} (1/k)^{2n}}{B_n (10)^{2n}} + 1 = 0.$$

In the usual notation the above identity has the following constants and numbers:

CONSTANTS

$$0,\ 1,\ -1,\ 2,\ \sqrt{5},\ i = \sqrt{-1},\ e,\ \pi,\ \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ 10.$$

NUMBERS

Notation		Explanation
n	$n = 0, 1, \cdots$	denotes zero and the set of positive integers.
$1/k$	$k = 1, 2, \cdots$	is the collection of fractions of the form $1/k$.
F_{n+1}	$n = 0, 1, \cdots$	denotes the $(n+1)^{th}$ Fibonacci number.
L_{n+1}	$n = 0, 1, \cdots$	" " " Lucas number.
B_n	$n = 0, 1, \cdots$	" " n^{th} Bernoulli number.
E_{2n}	$n = 0, 1, \cdots$	" " $2n^{th}$ even Euler number.

The author of this note wishes to point out that since the letter n denotes zero and the set of positive integers, then it must denote most of the conceivable numbers defined by mathematicians so far. Let us name some of these numbers. Prime, Fermat, Guy Moebius, Perfect, Pythagorean, Random, Triangular, Amicable, Automorphic, Palindromic, and the list goes on and on $\cdots$.

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POLYNOMIALS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

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BACKGROUND

Jaiswal [1] investigated certain polynomials $p_n(x)$ related to Chebyshev polynomials of the second kind $U_n(x)$ for which

(1) $$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad U_0(x) = 1, \quad U_1(x) = 2x$$

with

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} .$$

In this article, similar properties are derived for the corresponding polynomials $q_n(x)$ related to Chebyshev polynomials of the first kind $T_n(x)$ for which

(2) $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 2, \quad T_1(x) = 2x$$

with

$$T_n(\cos\theta) = 2\cos n\theta .$$

The first few Chebyshev polynomials of the first kind are

(3) $$\begin{cases} T_0(x) = 2 \\ T_1(x) = 2x \\ T_2(x) = 4x^2 - 2 \\ T_3(x) = 8x^3 - 6x \\ T_4(x) = 16x^4 - 16x^2 + 2 \\ T_5(x) = 32x^5 - 40x^3 + 10x \\ T_6(x) = 64x^6 - 96x^4 + 36x^2 - 2 \\ T_7(x) = 128x^7 - 224x^5 + 112x^3 - 14x \\ T_8(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2 \\ \text{-----------------------------} \end{cases}$$

THE ASSOCIATED POLYNOMIALS

Now take the sums along the rising diagonals on the right-hand side of (3). We obtain polynomials $q_n(x)$ which bear a close relationship to the Fibonacci numbers F_n. It is natural to define $q_0(x) = 0$.

From (3), the first few polynomials $q_n(x)$ are

(4) $$\begin{cases} q_1(x) = 2, \quad q_2(x) = 2x, \quad q_3(x) = 4x^2, \quad q_4(x) = 8x^3 - 2, \quad q_5(x) = 16x^4 - 6x, \\ q_6(x) = 32x^5 - 16x^2, \quad q_7(x) = 64x^6 - 40x^3 + 2, \quad q_8(x) = 128x^7 - 96x^4 + 10x, \\ q_9(x) = 256x^8 - 224x^5 + 36x^2, \quad q_{10}(x) = 512x^9 - 512x^6 + 112x^3 - 2 . \end{cases}$$

Observe in (4) the recurrence relation

(5) $$q_{n+3}(x) = 2xq_{n+2}(x) - q_n(x) \quad (n \geqslant 0)$$

which is (not unexpectedly) similar to Jaiswal's recurrence relation.

SOME PROPERTIES OF THE POLYNOMIALS

The $q_n(x)$ are seen to be connected with Jaiswal's $p_n(x)$ by the formula

(6) $$q_n(x) = p_n(x) - p_{n-3}(x) \quad (n \geqslant 3,\ p_0(x) = 0)$$

leading to

(7) $$\sum_{n=3}^{\infty} q_n(x)t^n = \sum_{n=3}^{\infty} p_n(x)t^n - \sum_{n=3}^{\infty} p_{n-3}(x)t^n \quad (n \geqslant 3)$$

i.e., by Jaiswal's generating function, to the generating function

(8) $$\sum_{n=3}^{\infty} q_n(x)t^n = (t - t^4)(1 - 2xt + t^3)^{-1}$$

For convenience, write the left-hand side of (8) as

(9) $$Q(x,t) = \sum_{n=3}^{\infty} q_n(x)t^n$$

from which we have (abbreviating $Q(x,t)$ as Q)

(10) $$\frac{\partial Q}{\partial t} = \frac{1 - 6t^3 - t^6 + 6xt^4}{(1 - 2xt + t^3)^2}, \qquad \frac{\partial Q}{\partial x} = \frac{t - t^4}{(1 - 2xt + t^3)^2}$$

Manipulation with (10) leads to the partial differential equation

(11) $$2t\,\frac{\partial Q}{\partial t} - (2x - 3t^2)\,\frac{\partial Q}{\partial x} - 8Q + 6G_1 = 0,$$

where, adjusting Jaiswal's notation slightly, we write

$$G_1(x,t) = \sum_{n=3}^{\infty} p_n(x)t^n = \frac{t}{1 - 2xt + t^3}\ .$$

But from (9),

(12) $$\frac{\partial Q}{\partial t} = \sum_{n=3}^{\infty} nq_n(x)t^{n-1}, \qquad \frac{\partial Q}{\partial x} = \sum_{n=3}^{\infty} q_n'(x)t^n\ .$$

Substitution in (11) yields

(13) $$2xq_{n+2}'(x) - 3q_n'(x) = 2(n-2)q_{n+2}(x) + 6p_{n+2}(x) \quad (n \geqslant 0).$$

Comparing coefficients of t^{n+1} in (8), we obtain

$$q_{n+1}(x) = (2x)^n - \binom{n-2}{1}(2x)^{n-3} + \binom{n-4}{2}(2x)^{n-6} - \cdots - \left\{(2x)^{n-3} - \binom{n-5}{1}(2x)^{n-6} + \cdots\right\}$$

that is,

(14) $$q_{n+1}(x) = \sum_{r=0}^{\left[\frac{n}{3}\right]} \binom{n-2r}{r}(-1)^r(2x)^{n-3r} \sum_{r=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2r}{r}(-1)^r(2x)^{n-3-3r}\ .$$

SPECIAL CASE $x = 1$

Putting $x = 1$ in (4) and writing $Q_n \equiv q_n(1)$, we obtain the sequence

(15)

	n=0	1	2	3	4	5	6	7	8	9	10	⋯
Q_n:	0	2	2	4	6	10	16	26	42	68	110	⋯
	≡2(0	1	1	2	3	5	8	13	21	34	55	⋯)

Clearly

(16) $$Q_n = 2F_n \quad (n \geqslant 0),$$

where F_n is the n^{th} Fibonacci number.

It might be remarked that when $x = 1$, Eq. (5) becomes

$$Q_{n+3} = 2Q_{n+2} - Q_n \quad (n \geqslant 0)$$

which is a characteristic feature of the Fibonacci sequence of numbers.

[Setting $x = 1$ in $\{U_n\}$ and $\{T_n\}$ gives, on using (1) and (2) (or (3)), the sequences 1, 2, 3, 4, 5, 6, ⋯ and 2, 2, 2, 2, 2, 2, ⋯ , respectively.]

Further, one may notice that

$$P_n = Q_n + F_{n-1} - 1, \tag{17}$$

where P_n are the numbers obtained from Jaiswal's polynomials $p_n(x)$ by putting $x = 1$, i.e., $P_n \equiv p_n(1)$.

$$(P_{n+1} = P_{n+1} + P_n - 1, \qquad P_0 = 1, \qquad P_1 = 1.)$$

Finally, $x = 1$ in (14) yields, with (16),

$$F_{n+1} = \tfrac{1}{2}\left\{ \sum_{r=0}^{[n/3]} \binom{n-2r}{r}(-1)^r 2^{n-3r} - \sum_{r=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2r}{r}(-1)^r 2^{n-3-3r} \right\}. \tag{18}$$

Our results should be compared with the corresponding results produced by Jaiswal. The generating function (8), and the properties which flow from it such as (11) and (13), are slightly less simple than we might have wished. However, the Fibonacci property (16) could hardly be simpler. What we lose on the swings we gain on the roundabouts!

REFERENCE

1. D. V. Jaiswal, "On Polynomials Related to Tchebichef Polynomials of the Second Kind," *The Fibonacci Quarterly*, Vol. 12, No. 3 (Oct. 1974), pp. 263–265.

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[Continued from p. 232.]

Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Show that

$$\text{(a)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \tan^{-1} \frac{2F_{2n+1}}{F_{2n}F_{2n+2}}$$

$$\text{(b)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \cos^{-1} \frac{F_{2n}F_{2n+2}}{F_{2n}F_{2n+2}+2}$$

$$\text{(c)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \sin^{-1} \frac{2F_{2n+1}}{F_{2n}F_{2n+2}+2} .$$

Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Find a function A_k in terms of k alone for the following expression.

$$F_n = \sum_{k=1}^{F_n} p_k - \sum_{k=1}^{F_n} A_k ,$$

where p_k denotes the k^{th} prime and F_n denotes the n^{th} Fibonacci number.

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SEMI-ASSOCIATES IN $Z[\sqrt{2}]$ AND PRIMITIVE PYTHAGOREAN TRIPLES*

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1. INTRODUCTION

Waclaw Sierpinski [2, p. 6], [3, p. 94] raised the following question:

SIERPINSKI'S PROBLEM: Are there an infinite number of primitive pythagorean triples with both the hypotenuse and the odd leg equal to a prime?

This question is equivalent to asking for an infinite number of solutions, in primes, to the Diophantine equation $q^2 = 2p - 1$. Other than this simple transformation it seems that no progress has been made toward a solution to Sierpinski's problem.

As a result of his work on Sierpinski's Problem, I. A. Barnett raised the following questions:

QUESTION A: Are there an infinite number of primitive pythagorean triples for which the sum of the legs is a prime?

QUESTION B: Are there an infinite number of primitive pythagorean triples for which the absolute value of the difference of the legs is a prime?

QUESTION C: Are there an infinite number of primitive pythagorean triples for which both the sum of the legs and the absolute value of the difference of the legs are prime?

For a complete discussion and characterization of primitive pythagorean triangles with either the sum or the difference of legs equal to a prime consult [4]. The more interesting aspects of [4] are summarized in the following.

Every prime divisor of either the sum or the difference of the legs of a primitive pythagorean triangle is congruent to ± 1 modulo 8. Conversely, if p $\equiv \pm 1 \pmod 8$ *is prime, there is a unique primitive pythagorean triangle with the sum of the legs equal to* p. *However, there are two disjoint infinite sequences of primitive pythagorean triangles, with the difference of the legs equal to* p, *for every triangle in these sequences. Moreover, every triangle with the difference of the legs equal to* p, *is in one of these sequences.*

In Section 2 of this paper, we define "α is a semi-associate of β" for $\alpha, \beta \in Z[\sqrt{2}]$ and present some elementary properties of this concept. These properties are used in Section 3 to show the equivalence of Question C to four questions about primes in $Z[\sqrt{2}]$.

In this paper we use the integral domain $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$, where Z denotes the usual set of integers. A detailed discussion of this integral domain is available [1, pp. 231–244], but some of the basic facts and some notations are presented in this section.

I will follow the usual custom of referring to elements of $Z[\sqrt{2}]$ as integers and elements of Z as rational integers.

If $\epsilon = 1 + \sqrt{2}$, then the set of units of $Z[\sqrt{2}]$ is precisely the set $\{\pm\epsilon^n \mid n \in Z\}$.

The primes in $Z[\sqrt{2}]$ are all associates of:

(1) $\sqrt{2}$

(2) All rational primes of the form $8k \pm 3$.
These are called primes of the second degree.

(3) All conjugate factors of rational primes of the form $8k \pm 1$.
These are called primes of the first degree.

The following notation and terminology will be used. If $\alpha = a + b\sqrt{2}$, then

*This research is a portion of the author's doctoral dissertation written at Ohio University, Athens, Ohio.

$\bar{\alpha} = a - b\sqrt{2}$ is called the conjugate of α.
$N(\alpha) = \alpha\bar{\alpha}$ is called the norm of α.
$R(\alpha) = a$ is called the rational part of α.
$I(\alpha) = b$ is called the irrational part of α.
$\epsilon = 1 + \sqrt{2}$ is called the fundamental unit in $Z[\sqrt{2}]$.
$\epsilon^{-1} = -1 + \sqrt{2}$ is called the inverse of ϵ.

Each of the properties listed in Lemma 1 is an elementary consequence of the definitions of the symbols involved but are useful in later sections. Proofs can easily be supplied by the reader.

Lemma 1. If α and β are integers, then

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$$

$$N(\alpha\beta) = N(\alpha)N(\beta), \qquad \alpha + \bar{\alpha} = 2R(\alpha), \qquad \alpha - \bar{\alpha} = 2\sqrt{2}I(\alpha)$$

$$R(\alpha\beta) = R(\alpha)R(\beta) + 2I(\alpha)I(\beta), \qquad I(\alpha\beta) = I(\alpha)R(\beta) + R(\alpha)I(\beta)$$

$$R(\alpha\bar{\beta}) = R(\alpha)R(\beta) - 2I(\alpha)I(\beta), \qquad I(\alpha\bar{\beta}) = R(\beta)I(\alpha) - R(\alpha)I(\beta)$$

$$R(\alpha^2) = R^2(\alpha) + 2I^2(\alpha), \qquad I(\alpha^2) = 2R(\alpha)I(\alpha)$$

$$R(\alpha\epsilon) = R(\alpha) + 2I(\alpha), \qquad I(\alpha\epsilon) = R(\alpha) + I(\alpha)$$

$$R(\alpha\epsilon^{-1}) = 2I(\alpha) - R(\alpha), \qquad I(\alpha\epsilon^{-1}) = R(\alpha) - I(\alpha)$$

$$N(\alpha) = 2I(\alpha)I(\alpha\epsilon^{-1}) - R(\alpha)R(\alpha\epsilon^{-1}), \qquad N(\alpha) = R(\alpha)I(\alpha\epsilon) - R(\alpha\epsilon)I(\alpha).$$

The following lemma summarizes all of the information needed about Pell-type equations.

Lemma 2. If p is a rational prime of the form $8k \pm 1$, the equation $x^2 - 2y^2 = p$ has exactly one solution $x = a$, $y = b$ such that the following two equivalent statements are true:

(i) $$\sqrt{p} < a < \sqrt{2p}$$

(ii) $$0 < b < \sqrt{p/2}.$$

The equation $x^2 - 2y^2 = p$ has infinitely many solutions, all of which are obtained from $(a + b\sqrt{2})\epsilon^{2t}$, where t is any rational integer and $x = a$, $y = b$ is any solution of $x^2 - 2y^2 = p$.

The unique solution which satisfies (i) and (ii) will be called the fundamental solution.

2. SEMI-ASSOCIATES IN $Z[\sqrt{2}]$

Theorem 1. If α and β are integers in $Z[\sqrt{2}]$, then the following are equivalent.

(1) Some associate, call it γ, of β has the same irrational part as α and $\gamma\epsilon$ has the same rational part as α.

(2) There is a rational integer n such that either:

(a) $$I(\beta\epsilon^n) = I(\alpha) \quad \text{and} \quad R(\beta\epsilon^{n+1}) = R(\alpha)$$

or

(b) $$I(-\beta\epsilon^n) = I(\alpha) \quad \text{and} \quad R(-\beta\epsilon^{n+1}) = R(\alpha).$$

(3) β is an associate of $[R(\alpha) - 2I(\alpha)] + I(\alpha)\sqrt{2}$.

(4) β is an associate of $\alpha - 2I(\alpha)$.

(5) $\pm N(\beta) = N(\alpha) + 4I(\alpha)[I(\alpha) - R(\alpha)]$.

(6) $\pm N(\beta) = N(\alpha) - 4I(\alpha)I(\alpha\epsilon^{-1})$.

(7) $N(\alpha) \pm N(\beta) = 4I(\alpha)[R(\alpha) - I(\alpha)]$.

(8) $N(\alpha) \pm N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})$.

Proof. It is clear from the characterization of associates in $Z[\sqrt{2}]$ and from Lemma 1 that: (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8). To complete the proof we show (1) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5).

To see that (3) ⇒ (1), let $\gamma = [R(\alpha) - 2I(\alpha)] + I(\alpha)\sqrt{2}$ and observe that $I(\gamma) = I(\alpha)$ and $R(\gamma\epsilon) = R(\alpha)$.

To see that (1) ⇒ (3), assume β is an associate of γ with $I(\gamma) = I(\alpha)$, and $R(\gamma\epsilon) = R(\alpha)$. Then γ must be of the form $\gamma = r + I(\alpha)\sqrt{2}$ and hence

$$\gamma\epsilon = [r + 2I(\alpha)] + [I(\alpha) + r]\sqrt{2}.$$

Now $R(\gamma\epsilon) = R(\alpha)$ implies $r = R(\alpha) - 2I(\alpha)$. Hence

$$\gamma = [R(\alpha) - 2I(\alpha)] + I(\alpha)\sqrt{2}.$$

To prove (4) ⇔ (5), note β is an associate of $\alpha - 2I(\alpha)$ if and only if

$$\pm N(\beta) = N[\alpha - 2I(\alpha)] = [R(\alpha) - 2I(\alpha)]^2 - 2I^2(\alpha) = R^2(\alpha) - 4I(\alpha)R(\alpha) + 4I^2(\alpha) - 2I^2(\alpha)$$
$$= N(\alpha) + 4I(\alpha)[I(\alpha) - R(\alpha)].$$

Definition 1. If α and β are integers in $Z[\sqrt{2}]$ which satisfy any one, and hence all, of the conditions of Theorem 1, then α is called a semi-associate of β.

It is clear that the relation "is a semi-associate of" is not an equivalence relation. The next sequence of theorems characterizes those elements for which the relation is either reflexive, symmetric, or transitive.

Theorem 2. Let α be an integer in $Z[\sqrt{2}]$. α is a semi-associate of itself if and only if

$$R(\alpha)I(\alpha)R(\alpha\epsilon^{-1})I(\alpha\epsilon^{-1}) = 0.$$

Proof. The theorem follows easily from the fact that α is a semi-associate of itself if and only if

$$4I(\alpha)I(\alpha\epsilon^{-1}) = N(\alpha) + N(\alpha) = 4I(\alpha)I(\alpha\epsilon^{-1}) - 2R(\alpha)R(\alpha\epsilon^{-1})$$

or

$$4I(\alpha)I(\alpha\epsilon^{-1}) = N(\alpha) - N(\alpha) = 0.$$

Corollary. The primes in $Z[\sqrt{2}]$ which are semi-associates of themselves are $\pm\sqrt{2}$, $\pm\epsilon\sqrt{2}$, $\pm p$, $\pm\epsilon p$, where $p\epsilon\{p \mid p$ is a rational prime of the form $8k \pm 3\}$.

Proof. That each of the primes listed is a semi-associate of itself follows directly from the theorem. To see that these are the only possibilities, consider the four cases:

(i) $$R(\alpha) = 0$$

(ii) $$I(\alpha) = 0$$

(iii) $$R(\alpha) - I(\alpha) = I(\alpha\epsilon^{-1}) = 0.$$

(iv) $$2I(\alpha) - R(\alpha) = R(\alpha\epsilon^{-1}) = 0.$$

Theorem 3. Two integers α and β are semi-associates of each other if and only if one of the following four pairs of conditions is true:

(i) $I(\alpha)I(\alpha\epsilon^{-1}) = I(\beta)I(\beta\epsilon^{-1})$, $\quad R(\alpha)R(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1})$

(ii) $I(\alpha)I(\alpha\epsilon^{-1}) = -I(\beta)I(\beta\epsilon^{-1})$, $\quad R(\alpha)R(\alpha\epsilon^{-1}) = R(\beta)R(\beta\epsilon^{-1})$

(iii) $2I(\alpha)I(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1})$, $\quad R(\alpha)R(\alpha\epsilon^{-1}) = 2I(\beta)I(\beta\epsilon^{-1})$

(iv) $2I(\alpha)I(\alpha\epsilon^{-1}) = R(\beta)R(\beta\epsilon^{-1})$, $\quad R(\alpha)R(\alpha\epsilon^{-1}) = -2I(\beta)I(\beta\epsilon^{-1})$.

Proof. If α is a semi-associate of β and simultaneously β is a semi-associate of α, then by Theorem 1, part 8,

$$N(\alpha) \pm N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1}) \quad \text{and} \quad N(\beta) \pm N(\alpha) = 4I(\beta)I(\beta\epsilon^{-1}).$$

This leads to the following four cases:

Case 1. $N(\alpha) + N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})$, $\quad N(\alpha) + N(\beta) = 4I(\beta)I(\beta\epsilon^{-1})$.

Case 2. $N(\alpha) - N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})$, $\quad N(\beta) - N(\alpha) = 4I(\beta)I(\beta\epsilon^{-1})$.

Case 3. $N(\alpha) + N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})$, $\quad N(\beta) - N(\alpha) = 4I(\beta)I(\beta\epsilon^{-1})$.

Case 4. $N(\alpha) - N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})$, $\quad N(\beta) + N(\alpha) = 4I(\beta)I(\beta\epsilon^{-1})$.

In Case 1 it is clear that

$$I(\alpha)I(\alpha\epsilon^{-1}) = I(\beta)I(\beta\epsilon^{-1})$$

and then by Lemma 1,

$$4I(\alpha)I(\alpha\epsilon^{-1}) = N(\alpha)+N(\beta) = -R(\alpha)R(\alpha\epsilon^{-1})+2I(\alpha)I(\alpha\epsilon^{-1}) - R(\beta)R(\beta\epsilon^{-1})+2I(\beta)I(\beta\epsilon^{-1})$$
$$= -R(\alpha)R(\alpha\epsilon^{-1}) - R(\beta)R(\beta\epsilon^{-1})+4I(\alpha)I(\alpha\epsilon^{-1}).$$

It now follows that

$$R(\alpha)R(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1}).$$

Conversely if

$$I(\alpha)I(\alpha\epsilon^{-1}) = I(\beta)I(\beta\epsilon^{-1}) \quad \text{and} \quad R(\alpha)R(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1})$$

then by Lemma 1,

$$N(\alpha)+N(\beta) = -R(\alpha)R(\alpha\epsilon^{-1})+2I(\alpha)I(\alpha\epsilon^{-1}) - R(\beta)R(\beta\epsilon^{-1})+2I(\beta)I(\beta\epsilon^{-1}) = 4I(\alpha)I(\alpha\epsilon^{-1})$$
$$= 4I(\beta)I(\beta\epsilon^{-1}).$$

Thus by Theorem 1, α and β are semi-associates of each other. In Case 2, it is clear that

$$I(\alpha)I(\alpha\epsilon^{-1}) = -I(\beta)I(\beta\epsilon^{-1})$$

and as in Case 1, Lemma 1 implies that

$$R(\alpha)R(\alpha\epsilon^{-1}) = R(\beta)R(\beta\epsilon^{-1}).$$

The converse again follows from Lemma 1. In Case 3, addition of the two equalities yields

$$N(\beta) = 2I(\alpha)I(\alpha\epsilon^{-1})+2I(\beta)I(\beta\epsilon^{-1})$$

and then by Lemma 1,

$$-R(\beta)R(\beta\epsilon^{-1})+2I(\beta)I(\beta\epsilon^{-1}) = N(\beta) = 2I(\alpha)I(\alpha\epsilon^{-1})+2I(\beta)I(\beta\epsilon^{-1}).$$

Thus

$$2I(\alpha)I(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1}).$$

On the other hand if the second equality is subtracted from the first and Lemma 1 is used we get

$$-R(\alpha)R(\alpha\epsilon^{-1})+2I(\alpha)I(\alpha\epsilon^{-1}) = N(\alpha) = 2I(\alpha)I(\alpha\epsilon^{-1}) - 2I(\beta)I(\beta\epsilon^{-1}) .$$

Thus

$$R(\alpha)R(\alpha\epsilon^{-1}) = 2I(\beta)I(\beta\epsilon^{-1}),$$

Conversely if both conditions in (iii) are true, then direct computation, using Lemma 1, shows

$$N(\alpha)+N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1}) \quad \text{and} \quad N(\beta)-N(\alpha) = 4I(\beta)I(\beta\epsilon^{-1})$$

and hence α and β are semi-associates of each other. In Case 4, addition of the two equalities and Lemma 1 yields

$$R(\alpha)R(\alpha\epsilon^{-1}) = -2I(\beta)I(\beta\epsilon^{-1}).$$

Subtraction of the first equality from the second and Lemma 1 yields

$$2I(\alpha)I(\alpha\epsilon^{-1}) = R(\beta)R(\beta\epsilon^{-1}).$$

The converse is proved by direct computation as indicated in Case 3. This completes the proof.

Integers α and β which are semi-associates of each other may also be characterized in terms of norms and rational parts of integers.

Theorem 4. Two integers α and β are semi-associates of each other if and only if one of the following four pairs of conditions is true:

(i) $\quad N(\alpha) = R(\beta^2\epsilon^{-1}), \quad N(\beta) = R(\alpha^2\epsilon^{-1})$

(ii) $\quad N(\alpha) = -R(\beta^2\epsilon^{-1}), \quad N(\beta) = -R(\alpha^2\epsilon^{-1})$

(iii) $\quad N(\alpha) = -R(\beta^2\epsilon^{-1}), \quad N(\beta) = R(\alpha^2\epsilon^{-1})$

(iv) $\quad N(\alpha) = R(\beta^2\epsilon^{-1}), \quad N(\beta) = -R(\alpha^2\epsilon^{-1}).$

Proof. If the conditions in part (i) of Theorem 3 are true, then from Lemma 1,

$$R(\alpha^2\epsilon^{-1}) = R(\alpha)R(\alpha\epsilon^{-1}) + 2I(\alpha)I(\alpha\epsilon^{-1}),$$

and hence,

$$N(\beta) = 2I(\beta)I(\beta\epsilon^{-1}) - R(\beta)R(\beta\epsilon^{-1}) = R(\alpha)R(\alpha\epsilon^{-1}) + 2I(\alpha)I(\alpha\epsilon^{-1}) = R(\alpha^2\epsilon^{-1}).$$

Similarly

$$N(\alpha) = R(\beta^2\epsilon^{-1}).$$

Conversely, if

$$N(\beta) = R(\alpha^2\epsilon^{-1}) \quad \text{and} \quad N(\alpha) = R(\beta^2\epsilon^{-1}),$$

then

$$2I(\beta)I(\beta\epsilon^{-1}) - R(\beta)R(\beta\epsilon^{-1}) = N(\beta) = R(\alpha^2\epsilon^{-1}) = 2I(\alpha)I(\alpha\epsilon^{-1}) + R(\alpha)R(\alpha\epsilon^{-1}),$$

and

$$2I(\alpha)I(\alpha\epsilon^{-1}) - R(\alpha)R(\alpha\epsilon^{-1}) = N(\alpha) = R(\beta^2\epsilon^{-1}) = 2I(\beta)I(\beta\epsilon^{-1}) + R(\beta)R(\beta\epsilon^{-1}).$$

Addition of these two equalities yields

$$I(\alpha)I(\alpha\epsilon^{-1}) = I(\beta)I(\beta\epsilon^{-1})$$

and subtraction yields

$$R(\alpha)R(\alpha\epsilon^{-1}) = -R(\beta)R(\beta\epsilon^{-1}).$$

Thus condition (i) of Theorem 3 is true and α and β are semi-associates of each other. Similar arguments show that conditions (ii), (iii), and (iv) of this theorem are equivalent to conditions (ii), (iii), and (iv) of Theorem 3 and the proof is complete.

The property of transitivity for the relation "is a semi-associate of" is closely related to reflexivity. This relation is expressed in Theorem 5.

Theorem 5. If α, β, and γ are integers in $Z[\sqrt{2}]$ such that α is a semi-associate of β and β is a semi-associate of γ, then α is a semi-associate of γ if and only if β is a semi-associate of itself.

Proof. If α is a semi-associate of β and β is a semi-associate of γ and itself, then β and γ are associates and hence α is a semi-associate of γ. Conversely if β is a semi-associate of γ and α is a semi-associate of both β and γ, then β and γ are associates and thus β is an associate of $\beta - 2I(\beta)$ because γ is. Hence β is a semi-associate of itself.

The following results will be particularly useful in the next section.

Lemma 3.
$$R(\beta^2\epsilon^{2k+1}) = N[\beta\epsilon^k + 2I(\beta\epsilon^k)].$$

Proof.
$$\begin{aligned} R(\beta^2\epsilon^{2k+1}) &= R^2(\beta\epsilon^k) + 4R(\beta\epsilon^k)I(\beta\epsilon^k) + 2I(\beta\epsilon^k) \\ &= [R(\beta\epsilon^k) + 2I(\beta\epsilon^k)]^2 - 2I^2(\beta\epsilon^k) = N[\beta\epsilon^k + 2I(\beta\epsilon^k)]. \end{aligned}$$

Theorem 6. If α is a semi-associate of β, then

$$N(\alpha) = R(\beta^2\epsilon^{2k+1})$$

for some rational integer k.

Proof. Since α is a semi-associate of β, it follows from Theorem 1, that there is a rational integer k such that exactly one of the following cases is true:

Case 1. $\quad I(\beta\epsilon^k) = I(\alpha) \quad$ and $\quad R(\beta\epsilon^k) = R(\alpha) - 2I(\alpha).$

Case 2. $\quad I(-\beta\epsilon^k) = I(\alpha) \quad$ and $\quad R(-\beta\epsilon^k) = R(\alpha) - 2I(\alpha).$

In Case 1 we have

$$\begin{aligned} R(\beta^2\epsilon^{2k+1}) &= R(\beta^2\epsilon^{2k}) + 2I(\beta^2\epsilon^{2k}) = R^2(\beta\epsilon^k) + 2I^2(\beta\epsilon^k) + 4R(\beta\epsilon^k)I(\beta\epsilon^k) \\ &= [R(\alpha) - 2I(\alpha)]^2 + 2I^2(\alpha) + 4I(\alpha)[R(\alpha) - 2I(\alpha)] \\ &= R^2(\alpha) - 2I^2(\alpha) = N(\alpha). \end{aligned}$$

In Case 2 note that

$$I(-\eta) = -I(\eta) \quad \text{and} \quad -R(\eta) = R(-\eta)$$

for any $\eta \in Z[\sqrt{2}]$ and then

$$\begin{aligned} R(\beta^2\epsilon^{2k+1}) &= R(\beta^2\epsilon^{2k}) + 2I(\beta^2\epsilon^{2k}) = R^2(\beta\epsilon^k) + 2I^2(\beta\epsilon^k) + 4R(\beta\epsilon^k)I(\beta\epsilon^k) \\ &= R^2(-\beta\epsilon^k) + 2I^2(-\beta\epsilon^k) + 4R(-\beta\epsilon^k)I(-\beta\epsilon^k) \\ &= [R(a) - 2I(a)]^2 + 2I^2(a) + 4I(a)[R(a) - 2I(a)] \\ &= R^2(a) - 2I^2(a) = N(a)\,. \end{aligned}$$

Theorem 6 gives a necessary condition for one integer to be a semi-associate of another integer. This condition does not seem to be sufficient, but a partial result in this direction is given in Theorem 7.

Theorem 7. If α is a prime and

$$N(\alpha) = R(\beta^2\epsilon^{2k+1})$$

for some rational integer k, then some associate of α or some associate of $\bar{\alpha}$ is a semi-associate of β.

Proof. If

$$N(\alpha) = R(\beta^2\epsilon^{2k+1}),$$

then by Lemma 3

$$N(\alpha) = N[\beta\epsilon^k + 2I(\beta\epsilon^k)]$$

so that either α or $\bar{\alpha}$ is an associate of

$$\beta\epsilon^k + 2I(\beta\epsilon^k).$$

Consider the case where α is an associate of

$$\beta\epsilon^k + 2I(\beta\epsilon^k).$$

Then there is a rational integer t such that

$$\pm\alpha\epsilon^t = \beta\epsilon^k + 2I(\beta\epsilon^k).$$

Hence

$$\beta\epsilon^k = \beta\epsilon^k + 2I(\beta\epsilon^k) - 2I(\beta\epsilon^k) = \pm\alpha\epsilon^t - 2I(\beta\epsilon^k) = \pm\alpha\epsilon^t - 2I(\pm\alpha\epsilon^t).$$

Thus β is an associate of

$$\pm\alpha\epsilon^t - 2I(\pm\alpha\epsilon^t)$$

and hence $\pm\alpha\epsilon^t$ is a semi-associate of β. The remaining case follows in a similar fashion.

3. EQUIVALENT FORMS OF QUESTION C

The term "generators" of a primitive pythagorean triple will mean the quantities m and n in the familiar formulae:

$$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2,$$

where m and n are of opposite parity, $(m,n) = 1$, and $m > n$.

Theorem 8. Let p and q be rational primes of the form $8k \pm 1$ (not necessarily of the same form). Let $u = a$, $v = b$ be the fundamental solution of $u^2 - 2v^2 = p$, and let $\alpha = a + b\sqrt{2}$. Let $u = c$, $v = d$ be the fundamental solution of $u^2 - 2v^2 = q$ and let $\beta = c + d\sqrt{2}$. If (x,y,z) is a primitive pythagorean triangle such that $x + y = p$ and $|x - y| = q$, then α is a semi-associate of β or $\bar{\beta}$.

Proof. Let m and n be the generators of (x,y,z). Since

$$p = x + y = (m+n)^2 - 2n^2 > (2n)^2 - 2n^2 = 2n^2$$

it follows that $u = m + n$ and $v = n$ is the fundamental solution of $u^2 - 2v^2 = p$. Hence $a = m + n$ and $b = n$. Now note that

$$N(\beta) = q = |y - x| = |(m-n)^2 - 2n^2| = |N[(m-n) + n\sqrt{2}]| = |N[\alpha - 2I(\alpha)]|\,.$$

Since β is prime it follows that β or $\bar{\beta}$ is an associate of $\alpha - 2I(\alpha)$ and hence α is a semi-associate of β or $\bar{\beta}$.

Theorem 9. Let α and β be primes of the first degree in $Z[\sqrt{2}]$. Let p and q be the rational primes such that $N(\alpha) = p$ and $N(\beta) = q$. If α is a semi-associate of β, then there is a primitive pythagorean triangle (x,y,z) such that $x + y = p$ and $|x - y| = q$.

Proof. Let $\alpha = a + b\sqrt{2}$ and $\beta = c + d\sqrt{2}$. Let $m = a - b$ and $n = b$. Then m and n generate a primitive pythagorean triangle (x,y,z) such that

$$x + y = (m+n)^2 - 2n^2 = a^2 - 2b^2 = N(\alpha) = p.$$

Since

$$\alpha = a + b\sqrt{2} = (m+n) + n\sqrt{2}$$

is a semi-associate of $\beta = c + d\sqrt{2}$, there is a rational integer n_0 such that the conditions in one of the following cases is true:

Case 1. $$\beta\epsilon^{n_0} = r + n\sqrt{2} \quad \text{and} \quad \beta\epsilon^{n_0+1} = (m+n) + s\sqrt{2},$$

where r and s are rational integers.

Case 2. $$\beta(-\epsilon^{n_0}) = r + n\sqrt{2} \quad \text{and} \quad \beta(-\epsilon^{n_0+1}) = (m+n) + s\sqrt{2},$$

where r and s are rational integers.

In Case 1 we have

$$(m+n) + s\sqrt{2} = \beta\epsilon^{n_0+1} = (r + n\sqrt{2})\epsilon = (r+2n) + (r+n)\sqrt{2}.$$

Comparing rational parts yields $r = m - n$. Thus

$$\beta\epsilon^{n_0} = (m-n) + n\sqrt{2}.$$

Now we have

$$q = N(\beta) = \pm N(\beta\epsilon^{n_0}) = \pm N[(m-n) + n\sqrt{2}] = \pm[(m-n)^2 - 2n^2] = \pm[(m+n)^2 - 2m^2] = \pm(y - x).$$

Hence, in this case, $|x - y| = q$. In Case 2 we have

$$(m+n) + s\sqrt{2} = (r + n\sqrt{2})\epsilon = (r+2n) + (r+n)\sqrt{2},$$

and as before we conclude $q = |x - y|$.

Combining the results of Theorems 1, 8, and 9 yields the following theorem.

Theorem 10. The following questions are each equivalent to Question C.

QUESTION D: Are there infinitely many pairs of primes of the first degree in $Z[\sqrt{2}]$ such that one member of the pair is a semi-associate of the other member of the pair?

QUESTION E: Are there infinitely many pairs α and $\alpha - 2I(\alpha)$, of primes of the first degree in $Z[\sqrt{2}]$?

QUESTION F: Are there infinitely many pairs (α,β) of primes of the first degree in $Z[\sqrt{2}]$ such that either

$$N(\alpha) + N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1}) \quad \text{or} \quad N(\alpha) - N(\beta) = 4I(\alpha)I(\alpha\epsilon^{-1})\,?$$

Combining the results of Theorems 6, 7, and 10 yields the final theorem.

Theorem 11. Questions C, D, E, and F are all equivalent to:

QUESTION G: Are there infinitely many pairs (α,β) of primes of the first degree in $Z[\sqrt{2}]$ such that

$$N(\alpha) = R(\beta^2\epsilon^{2k+1})$$

for some rational integer k, depending on α and β?

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UNIFORM DISTRIBUTION (MOD m) OF RECURRENT SEQUENCES

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In this paper it is shown that, for any odd prime p, a sequence of integers can be found which is uniformly distributed (mod m) if and only if m is a power of p.

Suppose m is an integer greater than 1. We say that an infinite sequence of integers $\{T_n\}$ is *uniformly distributed* (mod m) if for $j = 0, 1, \cdots, m-1$

$$\lim_{n\to\infty} \frac{1}{n} A(n,j,m) = \frac{1}{m},$$

where $A(n,j,m)$ denotes the number of terms among $T_1, \cdots, T_n$ which satisfy the congruence

$$T_i \equiv j \pmod{m}.$$

The combined results of Kuipers and Shiue [1] and Niederreiter [2] establish the fact that the Fibonacci sequence $\{F_n\}$ is uniformly distributed (mod m) if and only if m is a power of 5. In this paper we show that, for any odd prime p, a sequence of integers can be defined by a linear recurrence of the second order which is uniformly distributed (mod m) if and only if m is a power of p.

We first prove

Lemma. Suppose p is an odd prime and that k is a positive integer. Then $p+1$ belongs to the exponent p^k (mod p^{k+1}).

Proof. We use induction.

For the case $k = 1$, note that

$$(p+1)^p = p^p + \cdots + \binom{p}{2}p^2 + p^2 + 1 \equiv 1 \pmod{p^2}.$$

Now if $p+1$ belongs to e (mod p^2), it follows that $e \mid p$, hence $e = p$.

Suppose now that $p+1$ belongs to p^k (mod p^{k+1}). Then

$$(p+1)^{p^k} = tp^{k+1} + 1$$

and

$$(p+1)^{p^{k+1}} = (tp^{k+1}+1)^p = (tp^{k+1})^p + \cdots + \binom{p}{2}(tp^{k+1})^2 + tp^{k+2} + 1.$$

Thus

(1) $$(p+1)^{p^{k+1}} \equiv 1 \pmod{p^{k+2}}.$$

So if $p+1$ belongs to e (mod p^{k+2}), then $e \mid p^{k+1}$. But from (1) it follows that

$$(p+1)^e \equiv 1 \pmod{p^{k+1}};$$

and by the inductive supposition, $p^k \mid e$. Therefore, $e = p^k$ or $e = p^{k+1}$.

Now

(2) $$(p+1)^{p^k} \equiv \binom{p^k}{k}p^k + \cdots + \binom{p^k}{2}p^2 + p^{k+1} + 1 \pmod{p^{k+2}}.$$

We next show that

(3) $$\binom{p^k}{j}$$

is divisible by p^{k-j+2} for $j = 2, 3, \cdots, k$. It will be useful to recall

(4) $$\binom{p^k}{j} = \frac{p^k(p^k-1)\cdots(p^k-j+1)}{j!}.$$

Let $p(n)$, $p(d)$, and $p(q)$ denote, respectively, the highest power of p dividing the numerator, the denominator, and the quotient in (4). When $j = 2$, $p(n) \geq k$, $p(d) = 0$, so $p(q) \geq k$. When $j = 3$, $p(n) \geq k$, $p(q) \leq 1$, so $p(q) \geq k - 1$. In general, $p(n) \geq k$, and by the customary formula

$$p(d) = \sum_{e=1}^{\infty} \left[\frac{j}{p^e} \right] \leq j \sum_{e=1}^{\infty} \frac{j}{p^e} = \frac{j}{p-1} \quad .$$

Since $p \geq 3$, we see that

$$p(d) \leq \frac{j}{2} ;$$

and since

$$\frac{j}{2} \leq j - 2 \quad (j = 4, \cdots, k),$$

it follows that

$$p(q) \geq k - j + 2 \quad (j = 2, 3, \cdots, k).$$

Returning to (2), we see that

$$\binom{p^k}{j} p^j \quad (j = 2, \cdots, k)$$

is divisible by p^{k+2}. Hence

$$(p+1)^{p^k} \equiv p^{k+1} + 1 \not\equiv 1 \quad (\mathrm{mod}\ p^{k+2}),$$

and it follows finally that $e = p^{k+1}$, which completes the proof of the lemma.

We turn now to our major result.

Theorem. Let p be an odd prime and $\{T_n\}$ be the sequence defined by

$$T_{n+1} = (p+2)T_n - (p+1)T_{n-1}$$

and the initial values $T_1 = 0$, $T_2 = 1$. Then $\{T_n\}$ is uniformly distributed (mod m) if and only if m is a power of p.

Proof. We associate with $\{T_n\}$ the quadratic polynomial

$$x^2 - (p+2)x + p + 1$$

whose zeros over C are $p + 1$ and 1. It can be shown [3] that T_n is expressible in terms of those zeros as

$$T_n = \frac{1}{p}\{(p+1)^{n-1} - 1\} .$$

PART 1. In this part of the proof we show that $\{T_n\}$ is uniformly distributed (mod p^k), $k = 1, 2, 3, \cdots$.

As the first step we prove that $\{T_1, T_2, \cdots, T_{p^k}\}$ forms a complete residue system (mod p^k). Accordingly, suppose that $T_i \equiv T_j$ (mod p^k), or equivalently,

$$\frac{1}{p}\{(p+1)^{i-1} - 1\} \equiv \frac{1}{p}\{(p+1)^{j-1} - 1\} \quad (\mathrm{mod}\ p^k),$$

where $1 \leq i, j \leq p^k$. Then

$$(p+1)^{i-1} \equiv (p+1)^{j-1} \quad (\mathrm{mod}\ p^{k+1}).$$

Supposing $i \geq j$, we write

$$(p+1)^{j-1}(p+1)^e \equiv (p+1)^{j-1} \quad (\mathrm{mod}\ p^{k+1}),$$

where $0 \leq e \leq p^k - 1$, and it follows that

$$(p+1)^e \equiv 1 \quad (\mathrm{mod}\ p^{k+1}).$$

But by the Lemma, $p + 1$ belongs to the exponent p^k (mod p^{k+1}), so that $e = 0$ and $i = j$.

In this section of Part 1, we prove that $\{T_n\}$ (mod p^k) has period p^k. Specifically, we prove that

$$T_{p^k+1} \equiv T_1 \qquad \text{and} \qquad T_{p^k+2} \equiv T_2$$

(mod p^k). It will follow that

$$T_i \equiv T_{i+p^k} \quad (\mathrm{mod}\ p^k)$$

for $i = 1, 2, 3, \cdots$. Note first that the congruence

$$T_{p^k+1} = \frac{1}{p}\left\{(p+1)^{p^k} - 1\right\} \equiv 0 \pmod{p^k}$$

is equivalent to

(5) $$(p+1)^{p^k} \equiv 1 \pmod{p^{k+1}}$$

which follows from the Lemma. Note next that the congruence

$$T_{p^k+2} = \frac{1}{p}\left\{(p+1)^{p^k+1} - 1\right\} \equiv 1 \pmod{p^k}$$

is equivalent to

$$(p+1)^{p^k+1} \equiv p+1 \pmod{p^{k+1}}$$

which reduces to (5).

Combining the results of Part 1, we see that the complete residue system (mod p^k) occurs in the first and all successive blocks of p^k terms of $\{T_n\}$, proving that $\{T_n\}$ is uniformly distributed (mod p^k).

PART 2. In this part of the proof we show that $\{T_n\}$ is not uniformly distributed (mod m) if m is not a power of p.

If $\{T_n\}$ is uniformly distributed (mod m), then it is uniformly distributed (mod q) for every prime divisor q of m. We show here that $\{T_n\}$ is not uniformly distributed (mod q) for any prime $q \neq p$. There are two cases to consider according to whether $(p+1, q) = 1$ or q.

If $(p+1, q) = 1$, we can prove

(6) $$T_q \equiv 0 \pmod{q}$$

and

(7) $$T_{q+1} \equiv 1 \pmod{q}.$$

Equation (6) is equivalent to

$$T_q = \frac{1}{p}\{(p+1)^{q-1} - 1\} \equiv 0 \pmod{q}$$

or

(8) $$(p+1)^{q-1} \equiv 1 \pmod{pq}$$

which is equivalent to the *pair* of congruences

(9) $$(p+1)^{q-1} \equiv 1 \pmod{p}$$

and

(10) $$(p+1)^{q-1} \equiv 1 \pmod{q}.$$

Equation (9) is trivial, and (10) is proved by Fermat's theorem. Equation (7) is equivalent to

$$\frac{1}{p}\{(p+1)^q - 1\} \equiv 1 \pmod{q}$$

or

$$(p+1)^q \equiv p+1 \pmod{pq}$$

which reduces to (8). Now (6) and (7) evidently imply that the period of $\{T_n\}$ (mod q) is a divisor of $q-1$, consequently at least one residue will not occur in the sequence.

If on the other hand $(p+1, q) = q$, then

$$T_{n+1} = (p+2)T_n - (p+1)T_{n-1} \equiv T_n \pmod{q};$$

thus $\{T_n\}$ (mod q) becomes $\{0, 1, 1, \cdots\}$ which plainly is not uniformly distributed (mod q). This completes the proof of the theorem.

R. T. Bumby has found conditions for a sequence defined by a second-order linear recurrence to be uniformly distributed to all powers of a prime p.

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TRIBONACCI NUMBERS AND PASCAL'S PYRAMID

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In this paper an expression for the Tribonacci numbers discussed by the late Mark Feinberg [1] is obtained. They are expressed as sums of numbers along diagonal planes of what might be called Pascal's pyramid.

Feinberg [2] used the coefficients of a trinomial expansion as the model of a three-dimensional pyramid. He projected this pyramid onto a plane and then added the diagonal lines to get $\{T_n\}$, the Tribonacci sequence, $\{1, 1, 2, 4, 7, 13, 24, 44, \cdots\}$.

Lemma.

$$\binom{n-m-2r}{m+r}\binom{m+r}{r} = \binom{n-m-2r-1}{m+r}\binom{m+r}{r} + \binom{n-m-2r-1}{m+r-1}\binom{m+r-1}{r} + \binom{n-m-2r-1}{m+r-1}\binom{m+r-1}{r-1}.$$

Proof. The last two terms on the right-hand side

$$= \binom{n-m-2r-1}{m+r-1}\left\{\binom{m+r-1}{r} + \binom{m+r-1}{r-1}\right\} = \binom{n-m-2r-1}{m+r-1}\binom{m+r}{r}.$$

$$\binom{n-m-2r-1}{m+r-1}\binom{m+r}{r} + \binom{n-m-2r-1}{m+r}\binom{m+r}{r} = \binom{n-m-2r}{m+r}\binom{m+r}{r},$$

as required.

Theorem.

$$T_n = \sum_{m=0}^{[n/2]} \sum_{r=0}^{[n/3]} \binom{n-m-2r}{m+r}\binom{m+r}{r}.$$

Proof. We use induction.

$$T_0 = T_1 = 1.$$

$$T_2 = \sum_{m=0}^{1} \binom{2-m}{m} = \binom{2}{0} + \binom{1}{1} = 2.$$

$$T_3 = \sum_{m=0}^{1} \sum_{r=0}^{1} \binom{3-m-2r}{m+r}\binom{m+r}{r} = \binom{3}{0} + \binom{2}{1} + \binom{1}{1}\binom{1}{1} + \binom{0}{2}\binom{2}{1} = 4.$$

Assume true for $n = 4, 5, 6, \cdots, i-1$.

$$T_{i-1} = \sum_{m=0}^{\left[\frac{i-1}{2}\right]} \sum_{r=0}^{\left[\frac{i-1}{3}\right]} \binom{i-m-2r-1}{m+r}\binom{m+r}{r}.$$

$$T_{i-2} = \sum_{m=0}^{\left[\frac{i-2}{2}\right]} \sum_{r=0}^{\left[\frac{i-2}{2}\right]} \binom{i-m-2r-2}{m+r}\binom{m+r}{r} = \sum_{m=1}^{\left[\frac{i}{2}\right]} \sum_{r=0}^{\left[\frac{i-2}{2}\right]} \binom{i-m-2r-1}{m+r-1}\binom{m+r-1}{r}.$$

[Continued on p. 275.]

ON GENERATING FUNCTIONS WITH COMPOSITE COEFFICIENTS

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In this paper, we shall investigate a general problem of an interesting nature, and indicate a systematic method for obtaining at least a partial solution of it. The problem we mean is this: given the three generating functions (assuming appropriate convergence limitations are satisfied):

$$f(u) = \sum_{n=0}^{\infty} a_n u^n; \qquad g(u) = \sum_{n=0}^{\infty} b_n u^n; \qquad h(u) = \sum_{n=0}^{\infty} a_n b_n u^n . \tag{1}$$

What is the relationship, if any, which exists between $f(u)$, $g(u)$ and $h(u)$? By "relationship," we shall here mean that $h(u)$ may be expressed explicitly and in closed form as a function of u.

Many such relationships are well known, a few of which are indicated below, in tabular form:

a_n	$f(u)$	b_n	$g(u)$	a_nb_n	$h(u)$
p^n	$(1-pu)^{-1}$	q^n	$(1-qu)^{-1}$	p^nq^n	$(1-pqu)^{-1}$
$(-1)^n\binom{x}{n}$	$(1-u)^x$	q^n	$(1-qu)^{-1}$	$(-q)^n\binom{x}{n}$	$(1-qu)^x$
$\frac{1}{n!}$	e^u	$\frac{(-1)^n}{n!}$	e^{-u}	$\frac{(-1)^n}{n!n!}$	$J_0(2\sqrt{u})$
$\frac{(-1)^n}{n+1}$	$\frac{1}{u}\ln(1+u)$	$\frac{(-1)^n}{n+1}$	$\frac{1}{u}\ln(1+u)$	$\frac{1}{(n+1)^2}$	$\frac{-1}{u}\int_0^u \frac{\ln(1-t)}{t}\,dt$

As the last example illustrates, our general problem encompasses that of determining $h(u)$ when $f(u) = g(u)$, i.e., when

$$h(u) = \sum_{n=0}^{\infty} a_n^2 u^n ;$$

this latter, more specific case, is discussed briefly by Gould [2]. Our approach to the problem will depend on finite difference methods.

We recall the unit difference operators E and Δ, satisfying the following formal relationships (assuming arbitrary operand θ_o):

$$E^n\theta_0 = (1+\Delta)^n\theta_0 = \theta_n \tag{2}$$

$$\Delta^n\theta_o = (E-1)^n\theta_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E^k\theta_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\theta_k . \tag{3}$$

Using the above relationships, we may develop $h(u)$ as follows:

$$h(u) = \sum_{n=0}^{\infty} a_n u^n E^n b_0 = \sum_{n=0}^{\infty} a_n (uE)^n b_0 = f(uEb_0),$$

or

$$h(u) = f(u + u\Delta_b) = g(u + u\Delta_a) \quad \text{(the latter by symmetry).} \tag{4}$$

Naturally, we are taking great liberties in treating the formal operators E and Δ as if they were algebraic quantities, not to mention the fact that we are also ignoring convergence restrictions, if any. However, these objections may be circumvented if we treat the functions in (1) as formal generating functions, and focus our attention on their coefficients. We shall demonstrate that if due care is exercised in the manipulation of the operators and their operands, relation (4) may be made to yield results which are consistent with known relationships. In the process, we will also obtain some interesting and sometimes useful identities as by-products. Without further ado, we will illustrate the applicability of (4), first in obtaining the results already tabulated, then in developing other, more general relationships.

EXAMPLE 1. We begin by applying (3) to

$$a_o : \Delta^n a_0 = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}p^k = (p-1)^n .$$

Then, using our result in (4),

$$h(u) = g(u+u\Delta_a) = (1-qu-qu\Delta_a)^{-1} = (1-qu)^{-1}\left(1-\frac{qu\Delta_a}{1-qu}\right)^{-1} = (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n \Delta^n a_0$$

$$= (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n (p-1)^n = (1-qu)^{-1}\left(1-\frac{(p-1)qu}{1-qu}\right)^{-1} = (1-qu-(p-1)qu)^{-1}$$

$$= (1-pqu)^{-1},$$

as was previously stated. We could just as easily have used the relation in (4) obtained by reversing the roles of $f(u)$ and $g(u)$, and of a_n and b_n. The end result would have been identical.

EXAMPLE 2. By formula (3),

$$\Delta^n a_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(-1)^k\binom{x}{k} = (-1)^n\sum_{k=0}^{n}\binom{x}{n}\binom{n}{n-k} = (-1)^n\binom{x+n}{n} = \binom{-x-1}{n}$$

(using, e.g., formula (3.1) in [1]). As in Example 1,

$$h(u) = g(u+u\Delta_a) = (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n\Delta^n a_0 = (1-qu)^{-1}\sum_{n=0}^{\infty}\binom{-x-1}{n}\left(\frac{qu}{1-qu}\right)^n$$

$$= (1-qu)^{-1}\left(1+\frac{qu}{1-qu}\right)^{-x-1} = (1-qu)^x ,$$

as stated. It is instructive to reverse the roles of $f(u)$ and $g(u)$ in this example:

$$h(u) = f(u+u\Delta_b) = (1-u-u\Delta_b)^x = (1-u)^x\left(1-\frac{u\Delta_b}{1-u}\right)^x = (1-u)^x\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{-u}{1-u}\right)^n\Delta^n b_0$$

$$= (1-u)^x\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{-u}{1-u}\right)^n(q-1)^n, \quad \text{(Using Example 1)}$$

$$= (1-u)^x\left(1-\frac{(q-1)u}{1-u}\right)^x = (1-u-(q-1)u)^x = (1-qu)^x, \quad \text{as before.}$$

EXAMPLE 3.

$$\Delta^n a_0 = \sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k!},$$

using (3). This may be expressed in terms of an ordinary Laguerre polynomial as $(-1)^n L_n(1)$ (see fomula 1.115 in [1]); however, we will leave it in the summation form, to demonstrate that it is not essential for the n^{th} difference of the coefficients to be represented in closed form.

Then,

$$h(u) = e^{-u-u\Delta_a} = e^{-u}\sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\Delta^n a_0 = e^{-u}\sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k!}$$

$$= e^{-u}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{n=k}^{\infty}\binom{n}{k}\frac{u^n}{n!} = e^{-u}\sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}\sum_{n=0}^{\infty}\frac{u^n}{n!} = e^{-u}\sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}e^u = \sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}$$

$$= J_0(2\sqrt{u}),$$

by definition of the Bessel function.

EXAMPLE 4.

$$\Delta^n a_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(-1)^k}{k+1} = (-1)^n\sum_{k=0}^{n}\frac{\binom{n}{k}}{k+1}\quad .$$

We may use formula (1.37) in [1] to evaluate this expression (for the case $x = 1$), and find that

$$\Delta^n a_0 = \frac{(-1)^n}{n+1}(2^{n+1}-1).$$

Now

$$h(u) = g(u(1+\Delta_a)) = \frac{\ln(1+(1+\Delta_a)u)}{u(1+\Delta_a)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}u^n(1+\Delta_a)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}u^n\sum_{k=0}^{n}\binom{n}{k}\Delta^k a_0$$

$$= \sum_{k=0}^{\infty}\Delta^k a_0\sum_{n=k}^{\infty}\frac{(-1)^n}{n+1}\binom{n}{k}u^n = \sum_{k=0}^{\infty}\Delta^k a_0(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+k}{n}}{n+k+1}u^{n+k}\quad .$$

Denote the latter expression by y; if we differentiate uy with respect to u, we obtain:

$$(uy)' = \sum_{k=0}^{\infty}(-1)^k\Delta^k a_0\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{n}u^{n+k} = \sum_{k=0}^{\infty}(-1)^k\frac{(-1)^k}{k+1}(2^{k+1}-1)u^k(1+u)^{-k-1}$$

$$= \frac{2}{u+1}\sum_{k=0}^{\infty}\frac{\left(\frac{2u}{1+u}\right)^k}{k+1} - \frac{1}{1+u}\sum_{k=0}^{\infty}\frac{\left(\frac{u}{1+u}\right)^k}{k+1} = \frac{-2(1+u)}{(1+u)(2u)}\ln\left(1-\frac{2u}{1+u}\right) + \frac{1+u}{u(1+u)}\ln\left(1-\frac{u}{1+u}\right)$$

$$= -\frac{1}{u}\ln\left(\frac{1-u}{1+u}\right) - \frac{1}{u}\ln(1+u) = -\frac{1}{u}\ln(1-u).$$

We may now integrate with respect to u, noting that $uy = 0$ when $u = 0$, and we arrive at the desired expression:

$$uy = uh(u) = \int_0^u \frac{-1}{t}\ln(1-t)dt,\quad\text{or}\quad h(u) = \frac{-1}{u}\int_0^u\frac{\ln(1-t)}{t}dt\ .$$

We should observe that, in the foregoing examples, we could have arrived at the desired closed form of $h(u)$ by more direct methods, and thus we have not really saved ourselves any effort. On the other hand, these particular examples were chosen precisely for their simplicity, so as to enable us to check on the consistency of our results. In what follows, the value of our method will become apparent. Also, in the examples, both $f(u)$ and $g(u)$ were given in closed form (in fact, in terms of elementary functions). It is required that only one of these two functions be given in closed form; then, the n^{th} difference will be taken on the coefficients of the other function. If both $f(u)$ and $g(u)$ are given in closed form, however, as we have seen, we may develop $h(u)$ in either of two ways, which should yield the same result.

We shall begin by "proving" a well known linear transformation for the Gaussian hypergeometric function, defined as follows:

(5)
$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n}\binom{-b}{n}}{\binom{-c}{n}} (-z)^n .$$

Let

$$f(u) = \sum_{n=0}^{n} (-1)^n \binom{x}{n} u^n = (1-u)^x, \quad \text{and} \quad g(u) = \sum_{n=0}^{\infty} (-1)^n \binom{y}{n} u^n = (1-u)^y .$$

From (5), we see that

$$h(u) = \sum_{n=0}^{\infty} \binom{x}{n}\binom{y}{n} u^n = F(-x, -y, 1; u).$$

But using (4), $h(u)$ may be expressed in the alternative form

$$(1-u-u\Delta_a)^y = (1-u)^y \left(1 - \frac{u\Delta_a}{1-u}\right)^y = (1-u)^y \sum_{n=0}^{\infty} \binom{y}{n} \Delta_a^n \left(\frac{-u}{1-u}\right)^n$$

$$= (1-u)^y \sum_{n=0}^{\infty} \binom{y}{n}\binom{-x-1}{n}\left(\frac{-u}{1-u}\right)^n \quad \text{(using Example 1)},$$

$$= (1-u)^y F\left(-y, x+1, 1; \frac{-u}{1-u}\right) = (1-u)^x F\left(-x, y+1, 1; \frac{-u}{1-u}\right) \quad \text{(by symmetry)}.$$

These last relations may be found in [3], as formulas 15.3.4 and 15.3.5, setting $a = -x$, $b = -y$, $c = 1$ and $z = u$. As an interesting special case, if we set $y = -x - 1$, and $-u/(1-u) = w$, we obtain the following:

(6)
$$(1-w)^{x+1} F(x+1, x+1, 1; w) = (1-w)^{-x} F(-x, -x, 1; w)$$

This is equivalent to formula (3.141) in [1].

Other important special cases of the hypergeometric linear transformation given above occur whenever either x or y are positive integers, causing the series to terminate after a finite number of terms. For example, if $x = 3$ and $y = 2$, our relation yields:

$$F(-3, -2, 1; u) = (1-u)^2 F\left(-2, 4, 1; \frac{-u}{1-u}\right) = (1-u)^3 F\left(-3, 3, 1; \frac{-u}{1-u}\right) ,$$

each expression reducing to $1 + 6u + 3u^2$.

Another set of interesting special cases is obtained by setting $x = y = -m - 1$, where m is a non-negative integer. This yields the identity:

$$\sum_{n=0}^{\infty} \binom{m+n}{n}^2 u^n = (1-u)^{-m-1} \sum_{n=0}^{m} \binom{m}{n}\binom{m+n}{n}\left(\frac{u}{1-u}\right)^n .$$

If we obtain the convolute of (6), we obtain the identity:

(7)
$$\sum_{k=0}^{n} \binom{-x-1}{k}^2 \binom{x+1}{n-k} (-1)^k = \sum_{k=0}^{n} \binom{x}{k}^2 \binom{-x}{n-k} (-1)^k .$$

A more general identity is obtained by expanding each side of the general linear transformation formula, after making the substitution $-u/(1-u) = w$:

(8)
$$\sum_{k=0}^{n} (-1)^k \binom{-x-1}{k}\binom{y}{k}\binom{-y}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{-y-1}{k}\binom{-x}{n-k} .$$

For our second application of (4), we shall use $f(u) = e^u$, and

$$b_n = \frac{\binom{x}{n}}{\binom{y}{n}} \quad ;$$

then $a_n = 1/n!$, and $g(u) = F(-x, 1, -y; u)$. Using identity (3),

$$\Delta^n b_0 = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} \frac{\binom{x}{k}}{\binom{y}{k}} = (-1)^n \frac{\binom{y-x}{n}}{\binom{y}{n}}$$

(equivalent to (7.1) in [1]). Now

$$h(u) = e^{u+u\Delta_b} = e^u \sum_{n=0}^{\infty} \frac{u^n}{n!} \Delta^n b_0 = e^u \sum_{n=0}^{\infty} \frac{(-u)^n \binom{y-x}{n}}{\binom{y}{n} n!} = e^u M(x-y, -y, -u),$$

where $M(a,b,z)$ is the confluent hypergeometric function, or Kummer function, defined as follows:

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n}}{\binom{-b}{n}} \frac{z^n}{n!} \tag{9}$$

(see, e.g., pp. 504–505 of [3]). Since $h(u)$ is also equal to

$$\sum_{n=0}^{\infty} \frac{\binom{x}{n}}{\binom{y}{n}} \frac{u^n}{n!} = M(-x, -y, u),$$

we have derived the basic transformation formula for the Kummer function:

$$M(x, y, u) = e^u M(y-x, y, -u),$$

substituting $-x$ and $-y$ for x and y, respectively.

As another application, we will prove the following identity:

$$f(r,u) = \sum_{n=0}^{\infty} \left\{ \frac{p^n - q^n}{p-q} \right\}^r u^n = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k} q^k u} \tag{10}$$

The proof is by induction. We denote the assertion that (10) is true for a non-negative integer r as $P(r)$. We observe that $P(0)$ implies that $f(0,u) = (1-u)^{-1}$, while $P(1)$ implies that

$$f(1,u) = \frac{(1-pu)^{-1} - (1-qu)^{-1}}{p-q} \quad ,$$

each assertion readily verifiable as being true. We assume the validity of $P(r)$. Also, we define $g(u) = f(1,u)$ and $h(u) = f(r+1, u)$, consequently. By application of the result found in Example 1, since

$$b_n = \frac{p^n - q^n}{p-q}, \qquad \Delta^n b_0 = \frac{(p-1)^n - (q-1)^n}{p-q} \ .$$

Also,

$$h(u) = f(r, u + u\Delta_b) = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u - p^{r-k}q^k u\Delta_b} = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{(1-p^{r-k}q^k u)\left(1 - \dfrac{p^{r-k}q^k u\Delta_b}{1-p^{r-k}q^k u}\right)}$$

$$= (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u} \sum_{i=0}^{\infty} \left\{ \frac{p^{r-k}q^k u}{1 - p^{r-k}q^k u} \right\}^i \Delta^i b_0$$

$$= (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u} \sum_{i=0}^{\infty} \left\{ \frac{p^{r-k}q^k u}{1 - p^{r-k}q^k u} \right\}^i \left\{ \frac{(p-1)^i - (q-1)^i}{p-q} \right\} =$$

$$= (p-q)^{-r-1}\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-p^{r-k}q^ku}\left\{\left(1-\frac{(p-1)p^{r-k}q^ku}{1-p^{r-k}q^ku}\right)^{-1}-\left(1-\frac{(q-1)p^{r-k}q^ku}{1-p^{r-k}q^ku}\right)^{-1}\right\}$$

$$= (p-q)^{-r-1}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\left\{(1-p^{r+1-k}q^ku)^{-1}-(1-p^{r-k}q^{k+1}u)^{-1}\right\}$$

$$= (p-q)^{-r-1}\left\{\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-p^{r+1-k}q^ku}-\sum_{k=1}^{r+1}\frac{(-1)^{k-1}\binom{r}{k-1}}{1-p^{r+1-k}q^ku}\right\}$$

$$= (p-q)^{-r-1}\left\{\frac{1}{1-p^{r+1}u}+\sum_{k=1}^{r}\frac{(-1)^k\left\{\binom{r}{k}+\binom{r}{k-1}\right\}}{1-p^{r+1-k}q^ku}-\frac{(-1)^r}{1-q^{r+1}u}\right\}$$

$$= (p-q)^{-r-1}\left\{\frac{1}{1-p^{r+1}u}+\sum_{k=1}^{r}\frac{(-1)^k\binom{r+1}{k}}{1-p^{r+1-k}q^ku}+\frac{(-1)^{r+1}}{1-q^{r+1}u}\right\}$$

or

$$f(r+1,u) = (p-q)^{-r-1}\sum_{k=0}^{r+1}\frac{(-1)^k\binom{r+1}{k}}{1-p^{r+1-k}q^ku},$$

which equals $P(r+1)$. Therefore, $P(r)\to P(r+1)$, completing the proof.

If we set $p=\alpha$ and $q=\beta$ in (10) (the familiar Fibonacci constants), we obtain the generating function for the r^{th} power of the Fibonacci numbers, in the form of a partial fraction series:

(10a)
$$\sum_{n=0}^{\infty}F_n^ru^n = 5^{-\frac{1}{2}r}\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-\alpha^{r-k}\beta^ku}.$$

By a very similar development, we can prove the following identity:

(11)
$$\sum_{n=0}^{\infty}\left\{\frac{p^n+q^n}{p+q}\right\}^ru^n = (p+q)^{-r}\sum_{k=0}^{r}\frac{\binom{r}{k}}{1-p^{r-k}q^ku}.$$

Again, with $p=\alpha$ and $q=\beta$ in (11), we obtain the generating function for the r^{th} power of the Lucas numbers:

(11a)
$$\sum_{n=0}^{\infty}L_n^ru^n = \sum_{k=0}^{r}\frac{\binom{r}{k}}{1-\alpha^{r-k}\beta^ku}.$$

We may combine the partial fractions in (10a) and (11a), using known Fibonacci and Lucas identities, to eliminate all irrational expressions and condense the result in one closed form. For example, if $A(r,u)$ and $B(r,u)$ denote the expressions in (10a) and (11a), respectively, we may obtain the following results:

$$A(1,u) = u/(1-u-u^2);\quad A(2,u) = \frac{u-u^2}{1-2u-2u^2+u^3};$$

$$A(3,u) = (u-2u^2-u^3)/(1-3u-6u^2+3u^3+u^4);\quad B(1,u) = (2-u)/(1-u-u^2);$$

$$B(2,u) = (4-7u-u^2)/(1-2u-2u^2+u^3);\quad B(4,u) = \frac{16-79u-164u^2+76u^3+u^4}{1-5u-15u^2+15u^3+5u^4-u^5},$$

etc.

The possibilities for applying our method are virtually unlimited, provided we are careful not to separate the Δ operator in its manipulations. In this respect, Δ does not behave like an ordinary algebraic quantity, since "multiplication" is really successive application of the Δ symbol. Except for very special cases, moreover, which must be treated separately, a closed form for $h(u)$ free of symbolic operators is generally not available. The readers are invited to find other examples where the indicated method can yield useful results.

In a forthcoming paper on the topic (viz. [4]), an alternative (and more rigorous) approach is presented for the general solution of the problem proposed in this paper, under appropriate restrictions of analyticity for functions f and g.

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[Continued from p. 268.]

$$T_{i-3} = \sum_{m=0}^{\left[\frac{i-3}{2}\right]} \sum_{r=0}^{\left[\frac{i-3}{2}\right]} \binom{i-m-2r-3}{m+r}\binom{m+r}{r} = \sum_{m=2}^{\left[\frac{i+1}{2}\right]} \sum_{r=1}^{\left[\frac{i-1}{3}\right]} \binom{i-m-2r-1}{m+r-1}\binom{m+r-1}{r-1} .$$

Now,

$$T_i = T_{i-1} + T_{i-2} + T_{i-3} = \sum_{m=0}^{[i/2]} \sum_{r=0}^{[i/3]} \binom{i-m-2r}{m+r}\binom{m+r}{r}$$

(from lemma) which is what we required.

Fairly clearly when we are in the plane $r = 0,$ we have the ordinary Fibonacci numbers. Further investigations suggest themselves along the lines of Hoggatt [3] and Horner [4].

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FIBONACCI NOTES
6. A GENERATING FUNCTION FOR HALSEY'S FIBONACCI FUNCTION

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1. Halsey [2] defined a Fibonacci function by means of

$$F_u = \sum_{k=0}^{m} \left\{ (u-x) \int_0^1 x^{u-2k-1}(1-x)^k dx \right\}, \tag{1.1}$$

where m is the unique integer satisfying

$$\tfrac{1}{2}u - 1 \leqslant m < \tfrac{1}{2}u, \tag{1.2}$$

that is

$$m = \begin{cases} [\tfrac{1}{2}u] & (\tfrac{1}{2}u \neq \text{integer}) \\ \tfrac{1}{2}u - 1 & (\tfrac{1}{2}u = \text{integer}) \end{cases}. \tag{1.3}$$

The definition (1.1) is equivalent to

$$F_u = \sum_{k=0}^{m} \binom{u-k-1}{k}, \tag{1.4}$$

where again m is defined by (1.3).

In a recent note [1], Bunder has proved that F_u as defined by (1.4) satisfies the recurrence

$$F_{u+1} - F_u - F_{u-1} = \begin{cases} 0 & (2m < u \leqslant 2m+1) \\ \binom{u-m-2}{m+1} & (2m+1 < u \leqslant 2m+2) \end{cases} \tag{1.5}$$

In the present note we construct a generating function for the sequence

$$\{F_{u+n}\} \qquad (j = 0, 1, 2, \cdots; 0 < u \leqslant 1).$$

We show that

$$\sum_{n=0}^{\infty} F_{u+n}x^n = \frac{(1-x^2)^{1-u}}{1-x-x^2} \qquad (0 < u \leqslant 1), \tag{1.6}$$

This result contains (1.5). It also follows from (1.6) that

$$F_{u+n} = \sum_{0 \leqslant 2j \leqslant n} \binom{u+j-2}{j} F_{n-2j+1} \qquad (0 < u \leqslant 1), \tag{1.7}$$

where the F_{n-2j+1} on the right are ordinary Fibonacci numbers. Thus F_{u+n} is a polynomial in u of degree $\leqslant n/2$. Indeed the coefficients of the polynomials

$$P_n(u) = n!\, F_{u+2n}, \qquad Q_n(u) = n!\, F_{u+2n+1} \tag{1.8}$$

are positive integers. For some properties of these coefficients see § 4 below.

2. Since m as defined by (1.2), is a function of u, we put $m = m(u)$. Then clearly

$$m(u+2j) = m(u) + j \qquad (j = 0, 1, 2, \cdots). \tag{2.1}$$

Assume that

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(2.2) $$0 < u \leqslant 2.$$

Then by (1.2), $m(u) = 0$ and $m(u+2j) = j$. Thus

$$F_{u+2j} = \sum_{k=0}^{j} \binom{u+2j-k-1}{k} = \sum_{k=0}^{j} \binom{u+j+k-1}{j-k} .$$

Hence

$$\sum_{j=0}^{\infty} F_{u+2j}x^{2j} = \sum_{j=0}^{\infty} x^{2j} \sum_{k=0}^{j} \binom{u+j+k-1}{j-k} = \sum_{k=0}^{\infty} x^{2k} \sum_{j=0}^{\infty} \binom{u+j+2k-1}{j} x^{2j}$$

$$= \sum_{k=0}^{\infty} x^{2k}(1-x^2)^{-u-2k} = \frac{(1-x^2)^{-u}}{1-x^2(1-x^2)^{-2}} ,$$

so that

(2.3) $$\sum_{j=0}^{\infty} F_{u+2j}x^{2j} = \frac{(1-x^2)^{2-u}}{(1-x^2)^2 - x^2} \qquad (0 < u \leqslant 2).$$

Assume next that $0 < u \leqslant 1$, so that $m(u+1) = 0$ and $m(u+2j+1) = j$. Then as above

$$\sum_{j=0}^{\infty} F_{u+2j+1}x^{2j+1} = \sum_{j=0}^{\infty} x^{2j+1} \sum_{k=0}^{j} \binom{u+j+k}{j-k} = \sum_{k=0}^{\infty} x^{2k+1} \sum_{j=0}^{\infty} \binom{u+j+2k}{j} x^{2j}$$

$$= \sum_{k=0}^{\infty} x^{2k+1}(1-x^2)^{-u-2k-1} = \frac{x(1-x^2)^{-u-1}}{1-x^2(1-x^2)^{-2}} .$$

This gives

(2.4) $$\sum_{j=0}^{\infty} F_{u+2j+1}x^{2j+1} = \frac{x(1-x^2)^{1-u}}{(1-x^2)^2 - x^2} \qquad (0 < u \leqslant 1).$$

Combining (2.3) and (2.4), we get

(2.5) $$\sum_{j=0}^{\infty} F_{u+j}x^j = \frac{(1-x^2)^{1-u}}{1-x-x^2} \qquad (0 < u \leqslant 1).$$

3. It is clear from (2.5), to begin with, that

(3.1) $$\lim_{u=1-0} F_{u+n} = F_{n+1} \qquad (n = 0, 1, 2, \cdots),$$

where F_{n+1} denotes an ordinary Fibonacci number. In the next place, writing (2.5) in the form

$$(1-x-x^2) \sum_{j=0}^{\infty} F_{u+j}x^j = \sum_{n=0}^{\infty} \binom{u+n-2}{n} x^{2n}$$

and equating coefficients, we get

(3.2) $$\begin{cases} F_{u+2n+1} - F_{u+2n} - F_{u+2n-1} = 0 \\ F_{u+2n+2} - F_{u+2n+1} - F_{u+2n} = \binom{u+2n}{2n+2} \end{cases} \qquad (0 < u < 1).$$

Since, by (1.2),

$$m(u+2n) = m(u+2n+1) = n \qquad (0 < u \leqslant 1),$$

and

$$2n < u+2n \leqslant 2n+1, \qquad 2n+1 < u+2n+1 \leqslant 2n+2,$$

it follows that (3.2) is equivalent to (1.5).

Since the right-hand side of (2.5) is equal to

$$\sum_{n=0}^{\infty} F_{n+1}x^n \sum_{j=0}^{\infty} \binom{u+j-2}{j} x^{2j},$$

we get

$$F_{u+n} = \sum_{0 \leqslant 2j \leqslant n} \binom{u+j-2}{j} F_{n-2j+1} \qquad (0 < u \leqslant 1). \tag{3.3}$$

Alternatively, since

$$\frac{1-x^2}{1-x-x^2} = (1-x^2) \sum_{n=0}^{\infty} F_{n+1}x^n = 1+x+\sum_{n=2}^{\infty} (F_{n+1}-F_{n-1})x^n = 1+\sum_{n=1}^{\infty} F_n x^n,$$

it follows from (2.5) that

$$F_{u+n} = \sum_{0 \leqslant 2j \leqslant n}{}' \binom{u+j-1}{j} F_{n-2j} \qquad (0 < u \leqslant 1), \tag{3.4}$$

where the dash indicates that, if F_0 occurs, it is to be taken equal to 1.

From (3.3) or (3.4) we infer that, for $0 < u \leqslant 1$, F_{u+n} is a polynomial of degree $\leqslant n/2$. Since

$$\binom{u+j-1}{j} = \frac{u(u+1)\cdots(u+j-1)}{j!},$$

the coefficients of the polynomial are positive. For example

$$F_u = 1, \quad F_{u+1} = 1, \quad F_{u+2} = 1+u, \quad F_{u+3} = 2+u, \quad F_{u+4} = \tfrac{1}{2}(6+3u+u^2),$$
$$F_{u+5} = \tfrac{1}{2}(10+5u+u^2), \quad F_{u+6} = \tfrac{1}{6}(48+23u+6u^2+u^3).$$

Another corollary of (3.3) may be noted. We have

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} = \sum_{0 \leqslant 2j \leqslant n+1} \binom{u+j-2}{j} F_{n-2j+2}F_k$$

$$- \sum_{0 \leqslant 2j \leqslant n} \binom{u+j-2}{j} F_{n-2j+1}F_{k+1} = \sum_{0 \leqslant 2j \leqslant n+1} \binom{u+j-2}{j} (F_{n-2j+2}F_k - F_{n-2j+1}F_{k+1}).$$

Since

$$F_{m+1}F_n - F_mF_{n+1} = (-1)^{n+1}F_{m-n},$$

we get

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} = (-1)^{k+1} \sum_{0 \leqslant 2j \leqslant n+1} \binom{u+j-2}{j} F_{n-k-2j+1}. \tag{3.5}$$

In particular, for $0 \leqslant k \leqslant n+1$, again making use of (3.3),

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} \tag{3.6}$$

$$= (-1)^{k+1}F_{u+n+k} + (-1)^{k+1} \sum_{n-k+1 \leqslant 2j \leqslant n+1} \binom{u+j-2}{j} F_{2j-n+k-1} \qquad (0 \leqslant k \leqslant n).$$

For $k = n$ this reduces to

$$F_{u+n+1}F_n - F_{u+n}F_{n+1} = (-1)^{n+1}F_u + (-1)^{n+1} \sum_{0 < 2j \leqslant n+1} \binom{u+j-2}{j} F_{2j-1}. \tag{3.7}$$

Similar results are implied by (3.4).

4. We have noted above that, for $0 < u \leqslant 1$, F_{u+n} is a polynomial of degree $\leqslant n/2$, indeed of degree $[n/2]$. Put

(4.1) $$n!\,F_{u+2n} = P_n(u), \qquad n!\,F_{u+2n+1} = Q_n(u),$$

so that $P_n(u)$ and $Q_n(u)$ are of degree n. However we now think of them as defined for all u by means of (3.4) and (4.1). It follows from (1.5) that

(4.2) $$\begin{cases} P_n(u) = nP_{n-1}(u) + nQ_{n-1}(u) + (u+n-2)\cdots u(n-1) \\ Q_n(u) = P_n(u) + nQ_{n-1}(u) \ . \end{cases}$$

Now put

$$P_n(u) = \sum_{k=0}^{n} p(n,k)u^k, \qquad Q_n(u) = \sum_{k=0}^{n} q(n,k)u^k \ .$$

We have also

$$(u+n-1)\cdots(u+1)u = \sum_{k=0}^{n} S(n,k)u^k,$$

where $S(n,k)$ denotes a Stirling number of the second kind. Thus

$$(u+n-1)\cdots u(u-1) = \sum_{k=0}^{n+1} (S(n,k-1) - S(n,k)u^k \ .$$

Hence (4.2) gives

(4.3) $$q(n,k) = p(n,k) + nq(n-1),k)$$

and

(4.4) $$p(n,k) = np(n-1,k) + nq(n-1,k) + S(n-1,k-1) - S(n-1,k) \ .$$

Using either (4.2) or (4.3) and (4.4), the following tables are easily computed.

$p(n,k)$:

$n \backslash k$	0	1	2	3	1
0	1				
1	1	1			
2	6	3	1		
3	48	23	6	1	
4	504	242	59	10	1

$q(n,k)$:

$n \backslash k$	0	1	2	3	4
0	1				
1	2	1			
2	10	5	1		
3	78	38	9	1	
4	816	394	95	14	1

It is evident from the recurrences (4.3) and (4.4) that the $p(n,k)$ and $q(n,k)$ are integers. Moreover, by (3.4), they are positive integers.

By (3.3) and (4.1),

(4.5) $$P_n(1) = n!\,F_{2n+1}, \qquad Q_n(1) = n!\,F_{2n+2} \ .$$

This furnishes a partial check on the computed values. For example, using the table for $p(n,k)$, we get

$$\sum_{k=0}^{4} p(4,k) = 816 = 24.34 = 24F_9 .$$

Similarly

$$\sum_{k=0}^{4} q(4,k) = 1320 = 24.35 = 24F_{10} .$$

It is clear that

(4.6) $$p(n,n) = q(n,n) = 1 \qquad (n = 0, 1, 2, \cdots).$$

Taking $k = n - 1$ in (4.3) and n in (4.4), we get

(4.7) $$q(n, n-1) = p(n, n-1) + n \qquad (n \geqslant 1)$$

and

(4.8) $$p(n+1, n) = 2(n+1) + S(n, n-1) - 1,$$

respectively. Since

$$S(n, n-1) = \tfrac{1}{2}n(n-1),$$

it follows that

(4.9) $$\begin{cases} p(n, n-1) = \tfrac{1}{2}n(n+1) \\ q(n, n-1) = \tfrac{1}{2}n(n+3) = p(n+1, n) - 1 . \end{cases}$$

As for $k = 0$, it is evident from (3.4) that

$$\lim_{u=0} F_{u+n} = F_n ,$$

so that

(4.10) $$p(n,0) = n!\,F_{2n}, \qquad q(n,0) = n!\,F_{2n+1} .$$

It would be of interest to find combinatorial interpretations of $p(n,k)$ and $q(n,k)$.

REFERENCES

1. M. W. Bunder, "On Halsey's Fibonacci Function," *The Fibonacci Quarterly,* Vol. 13, No. 2 (April 1975), pp. 209–210.
2. E. Halsey, "The Fibonacci Number F_u where u is not an Integer," *The Fibonacci Quarterly,* Vol. 3, No. 2 (April 1965), pp. 147–152.

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[Continued from p. 245.]

(As a corollary, note that we have proved

$$F_{m+1}F_{m-1} - F_m^2 = \det(g^m) = (-1)^m .)$$

Then the lemma implies there is a sequence $\{m_j\}$ for which

$$g^{m_j} \to 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the p-adic topology. Thus we can choose $\{m_j\}$ so that $d(1, g^{m_j}) < p^{-j}$. Then p^j divides F_{m_j} and $1 - F_{m_j+1}$, which proves the theorem.

It is clear that one can vary G and g in the argument above to prove a class of theorems related to the well known one quoted.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-274 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

It has been shown (*The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 261–266) that if

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{then} \quad Q^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1} - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} .$$

Generalize the matrix Q to solutions of the difference equation

$$U_n = rU_{n-1} + sU_{n-2} ,$$

where r and s are arbitrary real numbers, $U_0 = 0$ and $U_1 = 1$.

H-275 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let P_n denote the Pell Sequence defined as follows: $P_1 = 1, P_2 = 2, P_{n+2} = 2P_{n+1} + P_n$ $(n \geqslant 1)$. Consider the array below.

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 5 & & 12 & & 29 & & 70\cdots & & (P_n) \\
& 1 & & 3 & & 7 & & 17 & & 41 & & \cdots & \\
& & 2 & & 4 & & 10 & & 24 & & \cdots & & \\
& & & 2 & & 6 & & 14 & & \cdots & & & \\
& & & & 4 & & 8 & & \cdots & & & & \\
& & & & & 4 & & \cdots & & & & &
\end{array}$$

Each row is obtained by taking differences in the row above.

Let D_n denote the left diagonal sequence in this array; i.e.,

$$D_1 = D_2 = 1, \quad D_3 = D_4 = 2, \quad D_5 = D_6 = 4, \quad D_7 = D_8 = 8, \cdots .$$

(i) Show $$D_{2n-1} = D_{2n} = 2^{n-1} \quad (n \geqslant 1).$$

(ii) Show that if $F(x)$ represents the generating function for $\{P_n\}_{n=1}^{\infty}$ and $D(x)$ represents the generating function for $\{D_n\}_{n=1}^{\infty}$, then

$$D(x) = \frac{1}{1+x} F\left(\frac{x}{1+x}\right) .$$

SOLUTIONS

DOUBLE YOUR FUN

H-255 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n}{j+k}\binom{2m+2n}{2m-j+k} = (-1)^{m+n}\frac{(3m+3n)!\,(2m)!\,(2n)!}{m!\,n!\,(m+n)!\,(2m+n)!(m+2n)!}\ .$$

Solution by the Proposer.

We shall use the following Saalschützian theorem for double series:

$$(1.1)\qquad \sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(a)_{r+s}(b)_r(c)_s}{r!\,s!\,(c)_{r+s}(d)_r(d')_s} = (-1)^{m+n}\frac{(c-a)_{m+n}(c-a-b')_m(c-a-b)_n}{(c)_{m+n}(c-a-b)_m(c-a-b')_n}\ ,$$

where

$$a+1 = d+d',\quad c+d = a+b-m+1,\quad c+d' = a+b'-n+1.$$

(For proof of (1) see *Journal London Math. Soc.*, 38 (1968), pp. 415–418.)

In (1) replace $b,\ b'$ by $b+m,\ b'+n,$ respectively; also replace m,n by $j,k.$ Then (1) becomes

$$\sum_{r=0}^{j}\sum_{s=0}^{k}\frac{(-j)_r(-k)_s(a)_{r+s}(b+j)_r(b'+k)_s}{r!\,s!\,(c)_{r+s}(d)_r(d')_s} = \frac{(c-a)_{j+k}(-d'-k+1)_j(-d-j+1)_k}{(c)_{j+k}(d)_j(d')_k}\ ,$$

where now

$$(2)\qquad a+1 = d+d',\quad c = b+d' = b'+d.$$

Then

$$\sum_{j,k=0}^{\infty}\frac{(b)_j(b')_k}{j!\,k!}\,\frac{(c-a)_{j+k}(-d'-k+1)_j(-d-j+1)_k}{(c)_{j+k}(d)_j(d')_k}\,x^jy^k$$

$$= \sum_{j,k=0}^{\infty}\frac{(b)_j(b')_k}{j!\,k!}\,x^jy^k\sum_{r=0}^{j}\sum_{s=0}^{k}\frac{(-j)_r(-k)_s(a)_{r+s}(b+j)_r(b'+k)_s}{r!\,s!\,(c)_{r+s}(d)_j(d')_k}$$

$$= \sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{(a)_{r+s}(b)_{2r}(b')_{2s}}{r!\,s!\,(c)_{r+s}(d)_r(d')_s}\,x^ry^s\sum_{j,k=0}^{\infty}\frac{(b+2r)_j(b+2s)_k}{j!\,k!}\,x^jy^k$$

$$= \sum_{r,s=0}^{\infty}(-1)^{r+s}\frac{(a)_{r+s}(b)_{2r}(b')_{2s}}{r!\,s!\,(c)_{r+s}(d)_r(d')_s}\,x^ry^r(1-x)^{-b-2r}(1-y)^{-b'-2s}\ ,$$

where $a,\ b,\ b',\ c,\ d,\ d'$ satisfy (2).

Now take $b=-2m,\ c=-2n.$ Then

$$d = c+2n,\qquad d' = c+2m,\qquad a+1 = 2c+2m+2n\,.$$

The above identity becomes

$$(3)\qquad \sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\frac{(-c-2m-2n+1)_{j+k}(-c-2m-k+1)_j(-c-2n-j+1)_k}{(c)_{j+k}(c+2n)_j(c+2m)_k}\,x^jy^k$$

$$= \sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\frac{(2c+2m+2n-1)_{r+s}(-2m)_{2r}(-2n)_{2s}}{r!\,s!\,(c)_{r+s}(c+2n)_r(c+2m)_s}\,x^ry^s(1-x)^{2m-2r}(1-y)^{2m-2s}\ .$$

We now take $x=y=1,\ c=p+1,$ where p is a non-negative integer. Then (3) reduces to

$$(4)\qquad \sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n+2p}{j+k+p}\binom{2m+2n+2p}{2m+p-j+k}$$

$$= (-1)^{m+n}\frac{(2m)!\,(2n)!\,(3m+3n+2p)!\,(2m+2n+2p)!}{m!\,n!\,(m+n+p)!\,(2m+2n+p)!\,(2m+n+p)!\,(m+2n+p)!}\ .$$

For $p = 0$, (4) gives the stated result.

STAGGERING SUM

H-257 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Consider the array, D, indicated below in which F_{2n+1} $(n = 0, 1, 2, \cdots)$ is written in staggered columns.

$$D: \begin{array}{cccccc} 1 \\ 2 & 1 \\ 5 & 2 & 1 \\ 13 & 5 & 2 & 1 \\ 34 & 13 & 5 & 2 & 1 \\ 89 & 34 & 13 & 5 & 2 & 1 \end{array}$$

(i) Show that the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

(ii) Show that the rising diagonal sums are $F_{n+1}F_{n+2}$ $(n = 0, 1, 2, \cdots)$.

(iii) Show that if the columns are multiplied by 1, 2, 3, ⋯ sequentially to the right, then the row sums are $F_{2n+3} - 1$ $(n = 0, 1, 2, \cdots)$.

Solution by George Brezsenyi, Lamar University, Beaumont, Texas.

(i) The sum of the entries of the n^{th} row is easily seen to be

$$\sum_{k=0}^{n} F_{2k+1},$$

which is well known to be F_{2n+2}.

(ii) The sums seemingly depend upon the parity of n. If n is odd, say $n = 2m + 1$, then the rising diagonal sum is

$$\sum_{k=0}^{m} F_{4k+3},$$

which may be shown to equal $F_{2m+2}F_{2m+3}$, or $F_{n+1}F_{n+2}$, by mathematical induction. Similarly, if n is even, say $n = 2m$, then the desired sum

$$\sum_{k=0}^{m} F_{4k+1}$$

yields upon evaluation $F_{2m+1}F_{2m+2}$, which is also equal to $F_{n+1}F_{n+2}$.

(iii) To resolve this part of the problem we show that

$$\sum_{k=0}^{n} (n+1-k)F_{2k+1} = F_{2n+3} - 1.$$

In $n = 0$, the result is trivial. Assume it for $n = m$. Then for $n = m + 1$ we have

$$\sum_{k=0}^{m+1} ((m+1)+1-k)F_{2k+1} = \sum_{k=0}^{m} (m+2-k)F_{2k+1} + F_{2m+3}$$

$$= \sum_{k=0}^{m} (m+1-k)F_{2k+1} + \sum_{k=0}^{m} F_{2k+1} + F_{2m+3}$$

$$= F_{2m+3} - 1 + F_{2m+2} + F_{2m+3} = F_{2(m+1)+3} - 1.$$

Thus the result holds for $n = m + 1$. This completes the induction.

Also solved by W. Brady, A. Shannon, G. Lord, P. Bruckman, F. Higgins and the Proposer.

THE SIGMA STRAIN

H-258 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum x^a y^b z^c t^d,$$

where the summation is over all non-negative $a, b, c, d,$ such that

$$\begin{cases} 2a \leqslant b+c+d \\ 2b \leqslant a+c+d \\ 2c \leqslant a+b+d \\ 2d \leqslant a+b+c \end{cases}$$

Solution by the Proposer.

Let

$$\begin{cases} a' = -2a+b+c+d \\ b' = a-2b+c+d \\ c' = a+b-2c+d \\ d' = a+b+c-2d. \end{cases}$$

Then a', b', c', d' are non-negative and

$$\begin{cases} 3a = b'+c'+d' \\ 3b = a'+c'+d' \\ 3c = a'+b'+d' \\ 3d = a'+b'+c' \end{cases}.$$

Thus

$$\begin{cases} b'+c'+d' \equiv 0 \\ a'+c'+d' \equiv 0 \\ a'+b'+d' \equiv 0 \\ a'+b'+c' \equiv 0 \end{cases} \pmod 3.$$

This implies

$$a' \equiv b' \equiv c' \equiv d' \pmod 3$$

and conversely.

Hence

$$S = S_0 + S_1 + S_2,$$

where

$$S_i = \sum_{\substack{a',b',c',d'=0 \\ a'\equiv b'\equiv c'\equiv d'\equiv i \pmod 3}}^{\infty} x^{\frac{1}{3}(b'+c'+d')} y^{\frac{1}{3}(a'+c'+d')} z^{\frac{1}{3}(a'+b'+d')} t^{\frac{1}{3}(a',b',c')} \qquad (i = 0, 1, 2).$$

Put $a' = 3a+i$, etc. Then

$$S_i = (xyzt)^i \sum_{a,b,c,d=0}^{\infty} x^{b+c+d} y^{a+c+d} z^{a+b+d} t^{a+b+c} = \frac{(xyzt)^i}{(1-yzt)(1-xzt)(1-xyt)(1-xyz)} \qquad (i = 0,1,2).$$

so that

$$S = \frac{1+xyzt+(xyzt)^2}{(1-yzt)(1-xzt)(1-xyt)(1-xyz)}.$$

POSITIVELY!

H-259 Proposed by R. Finkelstein, Tempe, Arizona.

Let p be an odd prime and m an odd integer such that $m \not\equiv 0 \pmod p$. Let $F_{mp} = F_p \cdot Q$. Can $(F_p, Q) > 1$?
[Continued on page 288.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-358 Proposed by Phil Mana, Albuquerque, New Mexico.

Prove that the integer u_n such that $u_n \leqslant n^2/3 < u_n + 1$ is a prime for only a finite number of positive integers n. (Note that $u_n = [n^2/3]$, where $[x]$ is the greatest integer in x and that $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $u_4 = 5$, and $u_5 = 8$.)

B-359 Proposed by R. S. Field, Santa Monica, California.

Find the first three terms T_1, T_2, and T_3 of a Tribonacci sequence of positive integers $\{T_n\}$ for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} (T_n/10^n) = 1/T_4 .$$

B-360 Proposed by T. O'Callahan, Aerojet Manufacturing Co., Fullerton, California.

Show that for all integers a, b, c, d, e, f, g, h there exist integers w, x, y, z such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (w^2 + 2x^2 + 3y^2 + 6z^2).$$

B-361 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)}$$

is a rational function of x, y, u, and v when these four variables are less than 1 in absolute value.

B-362 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let m be an integer greater than one and let R_n be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by m. Show that the sequence $R_0, R_1, R_2, \cdots$ repeats in a block $R_0, R_1, \cdots, R_t$ which reads the same from right to left as it does from left to right. (For example, if $m = 7$ the smallest repeating block is 0, 1, 3, 6, 3, 1, 0.)

B-363 Proposed by Herta T. Freitag, Roanoke, Virginia.

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n-1)/2$ also have the symmetry property stated in B-362 for their residues modulo m?

SOLUTIONS

THE PRIMES PETER OUT

B-334 Proposed by Phil Mana, Albuquerque, New Mexico.

Are all the terms prime in the sequence 11, 17, 29, 53, ⋯ defined by $u_0 = 11$, $u_{n+1} = 2u_n - 5$ for $n > 0$?

Composite of solutions by David G. Beverage, San Diego Evening College, La Mesa, California and Heiko Harborth, Technische Universität Braunschweig, West Germany.

One easily sees that $u_8 = 1541 = 23 \cdot 67$ is composite. More interestingly, one can show by induction that $u_n = 5 + 6 \cdot 2^n$. Then $u_n = 17 + 6(2^{n-1} - 1)$ and $2^4 \equiv -1 \pmod{17}$ and so $17 | u_{8k+1}$ for $k = 1, 2, \cdots$. Also, the Fermat Theorem tells us that $2^{p-1} \equiv 1 \pmod{p}$ for odd primes p and this can be used to show divisibility properties such as $11 | u_{10k}$ and $19 | u_{18k+11}$.

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Dinh Thế Hung, Sidney Kravitz, H. Turner Laquer, D. P. Laurie, Graham Lord, John W. Milsom, T. Ponnudurai, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Paul Smith, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI-LUCAS SUM

B-335 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain a closed form for

$$\sum_{i=0}^{n-k} (F_{i+k} L_i + F_i L_{i+k}).$$

Solution by Graham Lord, Université Laval, Québec, Canada.

The sum multiplied by $\sqrt{5}$ equals

$$\sum_{i=0}^{n-k} [(a^{i+k} - b^{i+k})(a^i + b^i) + (a^i - b^i)(a^{i+k} + b^{i+k})] = 2 \sum_{i=0}^{n-k} (a^{2i+k} - b^{2i+k})$$

$$= 2[a^{2n-k+1} - a^{k-1} - (b^{2n-k+1} - b^{k-1})] .$$

Hence the closed form is $2(F_{2n-k+1} - F_{k-1})$.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thế Hung, H. Turner Laquer, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

PELL SQUARES

B-336 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $Q_0 = 1 = Q_1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$. Show that $2(Q_{2n}^2 - 1)$ is a perfect square for $n = 1, 2, 3, \cdots$.

Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

By induction $2(Q_{2n}^2 - 1) = (Q_{2n} + Q_{2n-1})^2$ for $n = 1, 2, \cdots$ giving $2(Q_{2n}^2 - 1)$ as a perfect square.

Also solved by George Berzsenyi, David G. Beverage, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thế Hung, Sidney Kravitz, Graham Lord, T. Ponnudurai, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

RATIONAL POINTS ON AN ELLIPSE

B-337 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Show that there are infinitely many points with both x and y rational on the ellipse $25x^2 + 16y^2 = 82$.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

We shall establish the stronger result that if a rational number $r \neq 0$ is the sum of the squares of two rational numbers, then it has infinitely many representations as the sum of the squares of two positive rational numbers.

First, let $r = a^2 + b^2$, where a and b are rational numbers both different from zero. Without loss of generality, we may assume that a and b are both positive and that $a \geq b$. For every positive integer k,

$$(*) \qquad r = \left(\frac{(k^2-1)a - 2kb}{k^2+1}\right)^2 + \left(\frac{(k^2-1)b + 2ka}{k^2+1}\right)^2 .$$

If $k \geq 3$, $3k^2 - 8k = 3k(k-3) + k \geq 3$ and hence $3(k^2-1) \geq 8k$. Thus

$$\frac{k^2-1}{2k} \geq \frac{4}{3} > 1 \geq \frac{b}{a}$$

so $(k^2-1)a > 2kb$, from which it follows immediately that

$$a_k = \frac{(k^2-1)a - 2kb}{k^2+1} > 0.$$

If $j > k$, where j and k are positive integers then

$$(j^2 - k^2)a + b(kj-1)(j-k) > 0.$$

But this is equivalent to

$$\frac{(j^2-1)a - 2jb}{j^2+1} > \frac{(k^2-1)a - 2kb}{k^2+1} .$$

Therefore the numbers

$$a_k = \frac{(k^2-1)a - 2kb}{k^2+1}$$

increase with k so the a_k's are all different. Hence when $k \geq 3$ (*) gives different representations of r as the sum of the squares of two positive rational numbers.

Also solved by David G. Beverage, Paul S. Bruckman, H. Turner Laquer, Bob Prielipp, Sahib Singh, Paul Smith, Gregory Wulczyn, and the Proposer.

DIFFERENCE OF BINOMIAL EXPANSIONS

B-338 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let k and n be positive integers. Let $p = 4k + 1$ and let h be the largest integer with $2h + 1 \leq n$. Show that

$$\sum_{j=0}^{h} p^j \binom{n}{2j+1}$$

is an integral multiple of 2^{n-1}.

Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

Let

$$M(n,k) = \sum_{j=0}^{h} p^j \binom{n}{2j+1} .$$

As

$$(1+x)^n = \sum_{j=0}^{n} x^j \binom{n}{j} \quad \text{and} \quad (1-x)^n = \sum_{j=0}^{n} (-1)^j x^j \binom{n}{j}$$

one has

$$M(n,k) = ((1+\sqrt{p})^n - (1-\sqrt{p})^n)/(2\sqrt{p}).$$

Using this and the fact that $(1 \pm \sqrt{p})^2 = 2 \pm 2\sqrt{p} + 4k$, one obtains

$$M(n,k)/2^{n-1} = M(n-1,k)/2^{n-2} + kM(n-2,k)/2^{n-3}.$$

As $M(1,k) = 1$ and $M(2,k) = 2$ one can use induction to prove that $M(n,k)$ is divisible by 2^{n-1}.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, David Zeitlin, and the Proposer.

OPERATIONAL IDENTITY

B-339 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the validity of E. Cesàro's symbolic Fibonacci-Lucas identity $(2u+1)^n = u^{3n}$; after the binomial expansion has been performed, the powers of u are used as either Fibonacci or Lucas subscripts. (For example, when $n = 2$ one has both $4F_2 + 4F_1 + F_0 = F_6$ and $4L_2 + 4L_1 + L_0 = L_6$.)

Solution by Graham Lord, Université Laval, Québec, Canada.

For a fixed K, since both

$$F_K a + F_{K-1} = a^k \quad \text{and} \quad F_K b + F_{K-1} = b^K,$$

the n^{th} power of each when added (algebraically) will give the result

$$(F_K u + F_{K-1})^n = u^{Kn}.$$

The desired equation is the special case when $K = 3$.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, H. Turner Laquer, A. G. Shannon, David Zeitlin, and the Proposer.

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[Continued from page 284.]

Solution by David Beverage, San Diego Community College, San Diego, California.

By using the polynomials $P_{2n+1}(x)^*$ expressed explicitly as

$$(1) \qquad P_{2n+1}(x) = \sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!\,(2n+1-2r)!} x^{2n+1-2r} \; {}^{**}$$

and selecting $m = 2n+1$, obtain

$$(2) \qquad Q = \frac{F_{mp}}{F_p} = F_p \cdot H \pm m,$$

where H is a polynomial in F_p.

Clearly,

$$(F_p, m) \mid (F_p, Q).$$

Select $m > 1$ with integral coefficients and $m \nmid F_p$ $(m \not\equiv 0\ (p))$ in order that $(F_p, Q) > 1 \cdots$. The above conditions are satisfied for many numbers m and p. One example: $p = 7$ and $m = 13$ produces

$$\frac{F_{91}}{F_7} = 358465123875040793 = Q \qquad \text{and} \qquad (F_7, Q) = 13 > 1.$$

Many other interesting divisor relationships may be obtained from the polynomials $P_{2n+1}(x)$.

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*David G. Beverage, "A Polynomial Representation of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 9 No. 5 (Dec. 1971)

**David G. Beverage, "Polynomials $P_{2n+1}(x)$ Satisfying $P_{2n+1}(F_k) = F_{(2n+1)k}$," *The Fibonacci Quarterly*, Vol. 14, No. 3 (Oct. 1976), pp. 197–200.

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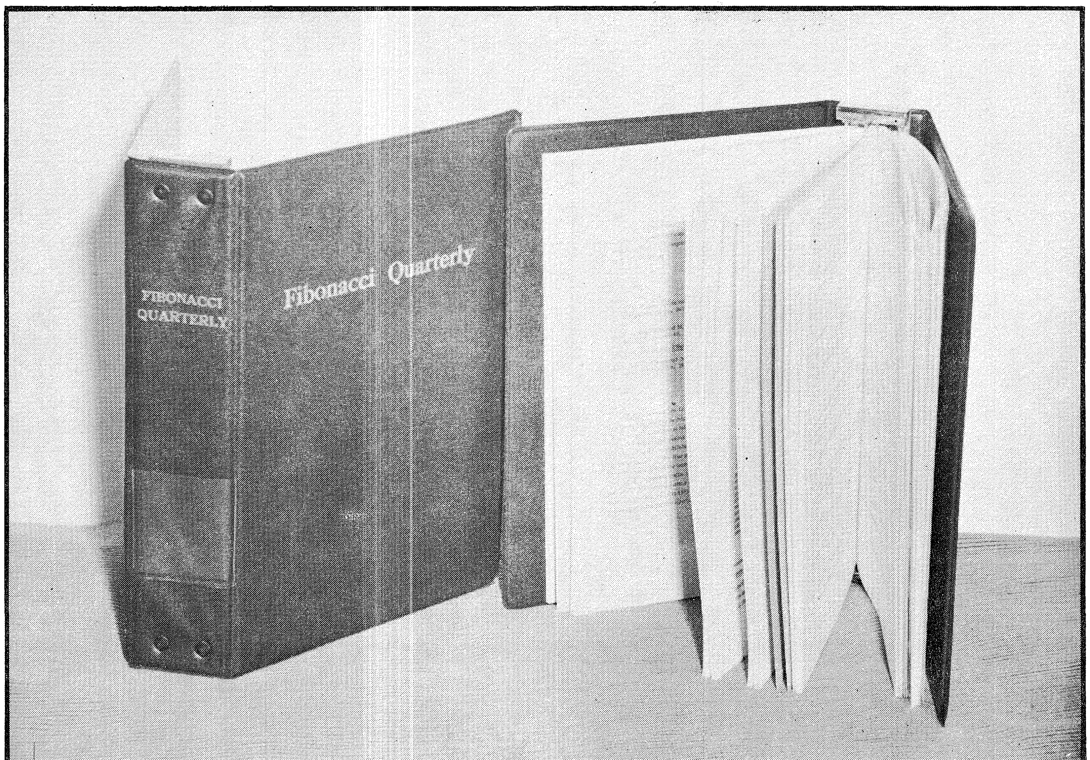
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