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OF INTEGERS WITH SPECIAL PROPERTIES

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GENERATING IDENTITIES FOR GENERALIZED FIBONACCI AND LUCAS TRIPLES

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BACKGROUND

In his article on generating identities for Pell triples, which involve the two Pell sequences, Serkland [5] modelled his arguments on those used by Hansen [1] for Fibonacci and Lucas sequences. Both articles suggest generalizations in a natural way.

Consider the following pairs of sequences (1) and (2), and (3) and (4):

				$n=0$	1	2	3	4	5	6	
(1)	Fibonacci	F_n	⋯	0	1	1	2	3	5	8	⋯
(2)	Lucas	L_n	⋯	2	1	3	4	7	11	18	⋯
(3)	Pell	P_n	⋯	0	1	2	5	12	29	70	⋯
(4)	Pell	R_n	⋯	2	2	6	14	34	82	198	⋯

for which the recurrence relations

(5) $$F_{n+2} = F_{n+1} + F_n$$

(6) $$L_{n+2} = L_{n+1} + L_n$$

(7) $$P_{n+2} = 2P_{n+1} + P_n$$

(8) $$R_{n+2} = 2R_{n+1} + R_n$$

and the summation relations

(9) $$F_{n+1} + F_{n-1} = L_n,$$

(10) $$P_{n+1} + P_{n-1} = R_n$$

hold.

It is natural to examine pairs of sequences $\{A_n\}$ and $\{B_n\}$ similar to (1) and (2), and (3) and (4) having the properties:

(11) $$\begin{cases} \text{(i)} & A_0 = 0,\ A_1 = 1,\ A_{n+2} = cA_{n+1} + dA_n \quad (c \neq 0,\ d \neq 0) \\ \text{(ii)} & B_0 = 2,\ B_1 = c,\ B_{n+2} = cB_{n+1} + dB_n \\ \text{(iii)} & A_{n+1} + A_{n-1} = B_n \end{cases}$$

Thus, $A_n \equiv F_n$ and $B_n \equiv L_n$ if $c = 1, d = 1$, while $A_n \equiv P_n$ and $B_n \equiv R_n$ if $c = 2,\ d = 1$.

Generally, n is any integer. From (11) (i) and (ii), we may deduce that when $d = 1$,

(12) $$A_{-n} = (-1)^{n+1} A_n$$

(13) $$B_{-n} = (-1)^n B_n$$

(14) $$A_{-n+1} + A_{-n-1} = B_{-n}\ .$$

Result (14) may be readily derived from (11) (iii), (12) and (13).

It looks as though $d = 1$ is a condition for property (11) (iii), which generalizes (9) and (10), to exist. We proceed to establish this fact.

GENERALIZATIONS

The Binet forms for A_n and B_n are

(15) $$A_n = \frac{a^n - \beta^n}{a - \beta}$$

(16) $$B_n = a^n + \beta^n\ ,$$

where a, β are the (distinct) roots of $x^2 - cx - d = 0$, so that

(17) $$a = \frac{c+D}{2},\quad \beta = \frac{c-D}{2},\quad a+\beta = c,\quad a-\beta = D,\quad D = \sqrt{c^2+4d},\quad a\beta = -d.$$

From (11) (iii), (15) and (16), we have

$$(a^{n+1} - \beta^{n+1}) + (a^{n-1} - \beta^{n-1}) = (a-\beta)(a^n + \beta^n)$$

$$(a^{n-1} - \beta^{n-1})(a\beta + 1) = 0 \quad \text{on simplification}$$

$$a\beta + 1 = 0 \quad \because \quad a^{n-1} - \beta^{n-1} \neq 0 \quad (\text{i.e., } a \neq \beta)$$

(18) $$d = 1 \quad \because \quad a\beta = -d \quad \text{by (17).}$$

Thus, the required condition is $d = 1$ with c unrestricted.

Consequently, there are infinitely many pairs of sequences $\{A_n\}$ and $\{B_n\}$ having the properties:

(11)′ $$\begin{cases} \text{(i)} & A_0 = 0,\quad A_1 = 1,\quad A_{n+2} = cA_{n+1} + A_n \quad (c \neq 0) \\ \text{(ii)} & B_0 = 2,\quad B_1 = c,\quad B_{n+2} = cB_{n+1} + B_n \\ \text{(iii)} & A_{n+1} + A_{n-1} = B_n\ . \end{cases}$$

Their Binet forms (15) and (16) now involve

(17)′ $$a = \frac{c+D}{2},\quad \beta = \frac{c-D}{2},\quad a+\beta = c,\quad a-\beta = D,\quad D = \sqrt{c^2+4},\quad a\beta = -1,$$

where a, β are now the roots of $x^2 - cx - 1 = 0$.

Some terms of these sequences are:

		⋯	$n = -3$	-2	-1	0	1	2	3	4	⋯
(19)	A_n	⋯	c^2+1	$-c$	1	0	1	c	c^2+1	c^3+2c	⋯
(20)	B_n	⋯	$-(c^3+3c)$	c^2+2	$-c$	2	c	c^2+2	c^3+3c	c^4+4c+2	⋯

Generating functions for these sequences are

(21) $$\sum_{n=1}^{\infty} A_n x^n = x(1 - cx - x^2)^{-1}$$

(22) $$\sum_{n=0}^{\infty} B_n x^n = (2 - cx)(1 - cx - x^2)^{-1}$$

The Theorems given in Serkland [5] follow directly for $\{A_n\}$ and $\{B_n\}$ by employing his methods, though in Theorems 1, 2, 3 use of the Binet forms (15) and (16) with (17)′ produces the results without difficulty.

Following Serkland's numbering [5], we have these generalized theorems:

Theorem 1. $$A_nB_m + A_{n-1}B_{m-1} = B_{m+n-1}$$

Theorem 2. $$A_nA_m + A_{n-1}A_{m-1} = A_{m+n-1}$$

Theorem 3. $$B_mB_n + B_{m-1}B_{n-1} = B_{m+n} + B_{m+n-2} = (c^2+4)A_{m+n-1}$$

Theorem 4. $$A_pA_qB_r = \sum_{k=0}^{q-1} (A_{k+1}B_{p+k+r-k} - A_{p+k+1}B_{q+r-k})$$

Theorem 5. $$A_pA_qA_r = \sum_{k=0}^{r-1} (A_{p+q+r-k}A_{k+1} - A_{p+k+1}A_{q+r-k})$$

Theorem 6. $$A_pB_qB_r = \sum_{k+0}^{p-1} ((c^2+4)A_{q+r+k+1}A_{p-k} - B_{q+k+1}B_{p+r-k})$$

Theorem 7. $$B_pB_qB_r = (c^2+4)\left[\sum_{k=0}^{p-2} (A_{q+r+k+1}B_{p-k} - A_{p+r-k}B_{q+k+1}) + cA_{p+q+r}\right] - cB_{p+q}B_{r+1}.$$

Of these theorems, we prove only the second part of Theorem 3 and all of Theorem 7 (taking the opportunity to correct some typographical errors in the original). A neater form for the expression of Theorem 3 (second part) is

$$B_{n+1} + B_{n-1} = (c^2+4)A_n$$

which should be compared with 11 (iii).

Proof of Theorem 3 (second part).

$$\begin{aligned}
B_{m+n} + B_{m+n-2} &= (A_{m+n+1} + A_{m+n-1}) + (A_{m+n-1} + A_{m+n-3}) && \text{by (11)' (iii)}\\
&= A_{m+n+1} + 2A_{m+n-1} + A_{m+n-3}\\
&= (cA_{m+n} + A_{m+n-1}) + 2A_{m+n-1} + A_{m+n-3} && \text{by (11)' (i)}\\
&= cA_{m+n} + 3A_{m+n-1} + A_{m+n-3}\\
&= c(cA_{m+n-1} + A_{m+n-2}) + 3A_{m+n-1} + A_{m+n-3} && \text{by (11)' (i)}\\
&= (c^2+3)A_{m+n-1} + (cA_{m+n-2} + A_{m+n-3})\\
&= (c^2+4)A_{m+n-1} && \text{by (11)' (i)}.
\end{aligned}$$

Proof of Theorem 7.

$$B_pB_qB_r = (A_{p+1} + A_{p-1})B_qB_r \qquad \text{by (11)' (iii)}$$

$$= A_{p+1}B_qB_r + A_{p-1}B_qB_r = \sum_{k=0}^{p} ((c^2+4)A_{q+r+k+1}A_{p-k+1} - B_{q+k+1}B_{p+r-k+1})$$

$$+ \sum_{k=0}^{p-2} ((c^2+4)A_{q+r+k+1}A_{p-k-1} - B_{q+k+1}B_{p+r-k-1}) \qquad \text{by Theorem 6}$$

$$= \sum_{k=0}^{p-2} \left[(c^2+4)A_{q+r+k+1}(A_{p-k+1} + A_{p-k-1}) - B_{q+k+1}(B_{p+r-k+1} + B_{p+r-k-1})\right]$$
$$+ (c^2+4)A_2A_{p+q+r} - B_{p+q}B_{r+2} + (c^2+4)A_1A_{p+q+r+1} - B_{p+q+1}B_{r+1}$$

$$\left.\begin{aligned}
&= \sum_{k=0}^{p-2} (c^2+4)(A_{q+r+k+1}B_{p-k} - B_{q+k+1}A_{p+r-k})\\
&\quad + (c^2+4)(cA_{p+q+r} + A_{p+q+r+1}) - (B_{p+q}B_{r+2} + B_{p+q+1}B_{r+1})
\end{aligned}\right\} \text{by (11)' (iii), (19), and Theorem 3}$$

$$= (c^2+4)\left[\sum_{k=0}^{p-2} (A_{q+r+k+1}B_{p-k} - B_{q+k+1}A_{p+r-k}) + cA_{p+q+r} + A_{p+q+r+1}\right]$$
$$- (cB_{p+q}B_{r+1} + B_{p+q}B_r + B_{p+q+1}B_{r+1}) =$$

$$= (c^2+4)\left[\sum_{k=0}^{p-2}(A_{q+r+k+1}B_{p-k} - A_{p+r-k}B_{q+k+1}) + cA_{p+q+r}\right] - cB_{p+q}B_{r+1} \qquad \text{by Theorem 3.}$$

Putting $c = 1$ in Theorems 1–7 we obtain the theorems of Hansen [1] for the Fibonacci-Lucas pair of sequences. With $c = 2$, the theorems of Serkland [5] for the two Pell sequences follow. The forms of Hansen's Theorem 5 and Serkland's Theorem 5 should be compared.

The natural extension of the special cases considered by Hansen [1] and Serkland [5] occurs when $c = 3$. Call these sequences $\{X_n\}$ and $\{Y_n\}$, some terms of which are:

		···	$n=-3$	-2	-1	0	1	2	3	4	5	6	···
(23)	X_n	···	10	−3	1	0	1	3	10	33	109	360	···
(24)	Y_n	···	−36	11	−3	2	3	11	36	119	393	1298	···

Theorems 1–7, and the associated background details, readily apply with $c = 3$ $(c^2+4 = 13)$. Interested readers may construct other pairs of related sequences from the infinitely many possibilities manifested in (19) and (20).

CONCLUDING REMARKS

Examples of familiar pairs of sequences which are excluded from our considerations (i.e., for which $d \neq 1$) are

(a) the Fermat sequences $\{2^n - 1\}$, $\{2^n + 1\}$ $(c = 3,\ d = -2)$

(b) the Chebyshev sequences

$$\{T_n = 2\cos n\theta\},\quad \left\{U_n = \frac{\sin(n+1)\theta}{\sin\theta}\right\}\quad (c = 2\cos\theta,\ d = -1).$$

(Obviously, in (a), $2^n + 1 = (2^n - 1) + 2$, i.e., the two Fermat sequences are not independent of each other.)

Comments on the excluded degenerate case which occurs when $\alpha = \beta$, i.e., $D = \sqrt{c^2+4d} = 0$, may be found in Horadam [3].

Further information on the Pell sequences, as special cases of the sequence $\{W_n\}$ for which

$$W_0 = a,\quad W_1 = b,\quad W_{n+2} = cW_{n+1} + dW_n$$

(which generalizes (11) (i) and (ii)), is given in Horadam [4]. For a partition of $\{W_n\}$ into Fibonacci-type and Lucas-type sequences the reader is referred to Hilton [2], which is generalized to r^{th}-order sequences by Shannon [6].

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ON THE EVALUATION OF CERTAIN INFINITE SERIES BY ELLIPTIC FUNCTIONS

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1. INTRODUCTION

In this paper, we will obtain closed form expressions for certain series involving hyperbolic secants and cosecants, in terms of complete elliptic integrals of the first and second kind. By specializing, we will obtain closed form expressions for series involving the reciprocals of the well known Fibonacci and Lucas sequences, thereby indicating how similar series for related sequences may be evaluated. Also, we will derive some elegant symmetrical relationships, which enable numerical evaluation of such series with a high degree of precision.

2. REVIEW

We will begin by recalling some of the basic definitions and properties of Jacobian elliptic function theory which are relevant to the topic of this paper. The notation used will be that found in [1]; the formulas quoted in this section are also taken from [1], for the most part, or in some cases from [2], with revised notation.

(1) $$u = u(\varphi, m) = \int_0^{\varphi} (1 - m \sin^2 \theta)^{-1/2} d\theta .$$

The angle φ is called *amplitude,* and we write

(2) $$\varphi = \mathrm{am}\, u .$$

In this paper, we will restrict φ to the two values 0 and $\pi/2$, and m to the open interval (0,1). Note that, in this domain of definition, u is a non-negative real number, and that $\lim_{m \to 1^-} u(\pi/2, m) = \infty$.

(3) $$K = K(m) = u(\pi/2, m); \quad K' = K'(m) = u(\pi/2, 1-m) = K(1-m).$$

(4) $$E = E(m) = \int_0^{\pi/2} (1 - m \sin^2\theta)^{1/2} d\theta; \qquad E' = E(1-m).$$

K and E are called the *complete elliptic integrals of the first and second kind,* respectively.

(5) $$\mathrm{sn}\, u = \sin\varphi ;$$

(6) $$\mathrm{cn}\, u = \cos\varphi ;$$

(7) $$\mathrm{dn}\, u = (1 - m \sin^2 \varphi)^{1/2} .$$

In (5)–(7) (as well as in the other nine Jacobian elliptic functions, which are derived from these, and not indicated here), if we wish to draw attention to the dependence of the function upon the parameter m, we write $\mathrm{sn}\,(u \mid m)$ for $\mathrm{sn}\, u$, etc.

For the values of φ with which we are concerned in this paper, we obtain the following relations:

(8) $$\mathrm{sn}\, K = 1; \qquad \mathrm{cn}\, 0 = \mathrm{dn}\, 0 = 1; \qquad \mathrm{dn}\, K = (1-m)^{1/2}.$$

We observe from the definition of $K(m)$ that it is a monotonic increasing (continuous) mapping of (0,1) onto $(\pi/2, \infty)$; it then follows that the functions x and y defined by:

(9) $$x = x(m) = \pi K'(m)/K(m), \quad \text{and} \quad y = y(m) = \pi K(m)/K'(m),$$

are one-to-one mappings of (0,1) onto $(0,\infty)$. (The notation introduced in (9) is not standard).

We also make the following definitions:

(10) $$q = \exp(-\pi K'/K) = e^{-x}; \qquad v = \pi u/2K.$$

In view of the preceding discussion, we see that $0 < q < 1$; moreover, for the two admissible values of φ which we allow, we obtain two possible triplets (u,v,φ), namely: (0,0,0) and $(K, \pi/2, \pi/2)$.

So-called q-series expansions for the functions given in (3)–(7) exist, as well as for some related functions which we will consider, and these are simply listed below:

(11) $$\mathrm{sn}\, u = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v;$$

(12) $$\mathrm{cn}\, u = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v;$$

(13) $$\mathrm{dn}\, u = \pi/2K + 2\pi/K \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2nv;$$

(14) $$(K/\pi)^2\, \mathrm{dn}^2 u - (KE)/\pi^2 = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos 2nv;$$

(15) $$\tfrac{4}{3}(2-m)(K/\pi)^2 - 4(KE)/\pi^2 + 1/3 = 8 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}};$$

(16) $$1 - 4(KE)/\pi^2 = 8 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1-q^{2n}};$$

(17) $$-1/16 \log(1-m) = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(2n-1)(1-q^{4n-2})}.$$

3. CLOSED FORMS

If, in (11)–(14), we substitute the special values of u and v indicated in the paragraph following (10), we eliminate the trigonometric terms occurring in these identities. We may also make the substitution indicated in (10), and if appropriate, extend the summation variable over all integral values. The result of these manipulations is the following list of identities:

(18) $$2 \sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{csch}(n-\tfrac{1}{2})x = \sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x = 2Km^{1/2}/\pi;$$

(19) $$\sum_{n=-\infty}^{\infty} \operatorname{sech} nx = 2K/\pi;$$

(20) $$\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} nx = 2K(1-m)^{1/2}/\pi;$$

(21) $$\sum_{n=1}^{\infty} n \operatorname{csch} nx = K(K-E)/\pi^2;$$

(22) $$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = KE/\pi^2 - (1-m)(K/\pi)^2 .$$

Since $0 < q < 1$, the following series manipulations are valid:

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nq^{(2j-1)n} = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} nq^{(2j-1)n} = \sum_{j=1}^{\infty} \frac{q^{2j-1}}{(1-q^{2j-1})^2};$$

that is,

$$\sum_{n=1}^{\infty} n \operatorname{csch} nx = \tfrac{1}{2} \sum_{n=1}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})x .$$

In a similar manner, we may prove the following identities:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = \tfrac{1}{2} \sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})x ;$$

$$\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} = \tfrac{1}{4} \sum_{n=1}^{\infty} \operatorname{csch}^2 nx ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} nq^{2n}}{1-q^{2n}} = \tfrac{1}{4} \sum_{n=1}^{\infty} \operatorname{sech}^2 nx .$$

Incorporating these results into (15), (16), (21) and (22), we obtain:

(23) $$\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 nx = 4KE/\pi^2 ;$$

(24) $$\sum_{n=1}^{\infty} \operatorname{csch}^2 nx = 1/6 + 2/3\,(2-m)(K/\pi)^2 - 2KE/\pi^2 ;$$

(25) $$\sum_{n=1}^{\infty} 2n \operatorname{csch} nx = \sum_{n=1}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})x = 2K(K-E)/\pi^2 ;$$

(26) $$\sum_{n=1}^{\infty} 2(-1)^{n-1} n \operatorname{csch} nx = \sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})x = 2KE/\pi^2 - 2(1-m)(K/\pi)^2 .$$

Finally, equation (17) may be recast as follows:

(27) $$\sum_{n=1}^{\infty} \frac{\operatorname{csch}(2n-1)x}{2n-1} = -1/8 \log(1-m).$$

The results with which we are interested are (18)–(20) and (23)–(27). These are all identities in the implicit parameter m. However, we may also view them as identities in the summand parameter x, since m, and therefore $K(m)$, $K'(m)$ and $E(m)$ are uniquely determined by (9), for any given positive x. In this sense, then, (18)–(20) and (23)–(27) represent *closed form* expressions for the indicated series, where the sums are expressed as implicit functions of x.

As a matter of interest, we include below two identities free of terms involving m, derived by inspection of (18), (19) and (23)–(26):

$$(28)\qquad \sum_{n=1}^{\infty} (\operatorname{sech}^2 nx + \operatorname{csch}^2 nx) = -1/3 + 1/3\left(\sum_{n=-\infty}^{\infty} \operatorname{sech} nx\right)^2 - 1/6\left(\sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x\right)^2, \ \forall x \neq 0.$$

$$(29)\qquad \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 (n-\tfrac{1}{2})x + \operatorname{csch}^2 (n-\tfrac{1}{2})x) = \left(\sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x\right)^2, \qquad \forall x \neq 0.$$

4. APPLICATIONS TO SERIES INVOLVING RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

Consider the sequence $\{U_n\}_0^\infty$ of non-negative integers defined by the recursion:

$$(30)\qquad U_{n+2} = aU_{n+1} + bU_n, \qquad n = 0, 1, 2, \cdots,$$

where a, b, U_0 and U_1 are given non-negative integers, with a and b not both zero, U_0 and U_1 not both zero. It is known from the theory of linear difference equations that an explicit formula for U_n exists, given by:

$$(31)\qquad U_n = U_1 G_n + bU_0 G_{n-1}, \qquad n = 1, 2, 3, \cdots,$$

where

$$(32)\qquad G_n = \frac{r^n - s^n}{r - s}, \qquad n = 0, 1, 2, \cdots,$$

and

$$(33)\qquad r = \tfrac{1}{2}(a + \sqrt{a^2 + 4b}), \qquad s = \tfrac{1}{2}(a - \sqrt{a^2 + 4b}).$$

Note that $r > 0$. If, in particular, $b = 1$, and if we let $L = \log r$, then G_n takes a form which is of interest to the topic of this paper. Specifically,

$$(34)\qquad G_{2n} = \frac{2}{\sqrt{a^2+4}} \sinh 2nL, \qquad G_{2n+1} = \frac{2}{\sqrt{a^2+4}} \cosh (2n+1)L, \quad n = 0, 1, 2, \cdots.$$

Thus, for certain special values of a, U_0 and U_1, we see that the identities of the previous section may be used to obtain closed form expressions for series involving the reciprocals of our particular sequence $\{U_n\}$.

We illustrate with a specific example, by taking $a = b = 1$. Then let

$$(35)\qquad \alpha = r = \tfrac{1}{2}(1+\sqrt{5}), \quad \beta = s = \tfrac{1}{2}(1-\sqrt{5}), \quad \lambda = L = \log \alpha.$$

The sequence $\{G_n\}$ then becomes the familiar *Fibonacci* sequence $\{F_n\}$; using (34), we see that the general term of this sequence is given by:

$$(36)\qquad F_{2n} = 2/\sqrt{5}\, \sinh 2n\lambda, \qquad F_{2n+1} = 2/\sqrt{5}\, \cosh (2n+1)\lambda, \qquad n = 0, 1, \cdots.$$

If we take $U_0 = 0$, $U_1 = 1$ as initial values, then the sequence $\{U_n\}$ coincides with $\{F_n\}$. If we take $U_0 = 2$, $U_1 = 1$, the resulting sequence is the *Lucas* sequence $\{L_n\}$, whose general term is as follows:

$$(37)\qquad L_{2n} = 2\cosh 2n\lambda, \qquad L_{2n+1} = 2\sinh (2n+1)\lambda, \qquad n = 0, 1, \cdots.$$

If, in definition (9), we let $x = 2\lambda$, this determines a unique constant μ, such that $0 < \mu < 1$, and

$$(38)\qquad \pi K'(\mu)/K(\mu) = 2\lambda.$$

Also, let $\rho = K(\mu)/\pi$, $\sigma = E(\mu)/\pi$. For this particular value of x, we may then use (18)–(20), (23)–(27) and (36)–(38) to obtain the following closed-form expressions:

$$(39)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}} = \tfrac{1}{2}\rho\sqrt{\mu};$$

$$(40)\qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \tfrac{1}{2}\rho\sqrt{5\mu};$$

(41) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \tfrac{1}{2}\rho - \tfrac{1}{4};$$

(42) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}} = \tfrac{1}{4} - \tfrac{1}{2}\rho\sqrt{1-\mu};$$

(43) $$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n}}\right)^2 = \tfrac{1}{2}\rho\sigma - 1/8;$$

(44) $$\sum_{n=1}^{\infty} \left(\frac{1}{F_{2n}}\right)^2 = \frac{5}{24} + \frac{5}{6}(2-\mu)\rho^2 - \frac{1}{2}\rho\sigma;$$

(45) $$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} = \tfrac{1}{2}\sqrt{5}\,\rho(\rho-\sigma);$$

(46) $$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n-1}}\right)^2 = \tfrac{1}{2}\rho(\rho-\sigma);$$

(47) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{F_{2n}} = \tfrac{1}{2}\sqrt{5}(\rho\sigma - (1-\mu)\rho^2);$$

(48) $$\sum_{n=1}^{\infty} \left(\frac{1}{F_{2n-1}}\right)^2 = \frac{5}{2}(\rho\sigma - (1-\mu)\rho^2);$$

(49) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{4n-2}} = -\sqrt{5}/16 \log(1-\mu).$$

Since all the series in (39)–(49) are absolutely convergent, we may obtain other formulas by combinations of the foregoing expressions. For example, if we alternately add and subtract (41) and (42), we obtain:

(50) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = \tfrac{1}{4}(1-\sqrt{1-\mu})\rho,$$

and

(51) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = \tfrac{1}{4}(1+\sqrt{1-\mu})\rho - \tfrac{1}{4}\quad.$$

A similar process on (45) and (47) yields the pair of identities:

(52) $$\sum_{n=1}^{\infty} \frac{2n-1}{F_{4n-2}} = \tfrac{1}{4}\sqrt{5}\mu\rho^2\;;$$

(53) $$\sum_{n=1}^{\infty} \frac{2n}{F_{4n}} = \tfrac{1}{4}\sqrt{5}\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\}.$$

Adding (43) and (46) yields:

(54) $$\sum_{n=1}^{\infty} \frac{1}{L_n^2} = \tfrac{1}{2}\rho^2 - 1/8.$$

Adding (44) and (48) yields:

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2} = \frac{5}{24} + \frac{5}{6}(2\mu - 1)\rho^2. \tag{55}$$

Inspection of the preceding list of closed form expressions yields a variety of interesting identities, some of which are shown below:

$$3\sum_{n=1}^{\infty} \frac{1}{F_n^2} + 5\sum_{n=1}^{\infty} \frac{1}{L_n^2} = 4\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = 80\left(\sum_{n=1}^{\infty} \frac{1}{L_{4n}}\right)\left(\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}}\right) = 5\mu\rho^2; \tag{56}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} + 5\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = 2\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = \frac{5}{2}\mu\rho^2. \tag{57}$$

The Lucas sequence may be extended to negative indices, by the following definition, which is consistent with the definition in (37):

$$L_{-n} = (-1)^n L_n, \qquad n = 0, 1, 2, \cdots. \tag{58}$$

Using (58), we obtain the following elegant identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{L_n^2} = \left(\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}}\right)^2 = \rho^2. \tag{59}$$

Note

$$\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}} = \rho.$$

One more identity is worth including, namely:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)L_{2n-1}} = 1/8 \log\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right). \tag{60}$$

This does not follow from any previous identity in this section, though similar to (49). The proof of (60) depends upon a general theorem about elliptic functions, which properly does not belong in this section; it is nevertheless instructive to include it here, illustrating how the basic identities in (18)–(20) and (23)–(27) may be made to yield other identities not previously covered.

Theorem. Suppose $2K'(m_1)/K(m_1) = K'(m_2)/K(m_2)$.

Then:

(a) $$K(m_1) = (1+\sqrt{m_2})K(m_2);$$

(b) $$K'(m_1) = \tfrac{1}{2}(1+\sqrt{m_2})K'(m_2);$$

(c) $$m_1 = 1 - \left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2 = \frac{4\sqrt{m_2}}{(1+\sqrt{m_2})^2};$$

(d) $$E'(m_1) = \frac{E'(m_2)+\sqrt{m_2}\,K'(m_2)}{1+\sqrt{m_2}};$$

(e) $$E(m_1) = \frac{2E(m_2)-(1-m_2)K(m_2)}{1+\sqrt{m_2}}.$$

Proof of (a). Let $x = \pi K'(m_1)/K(m_1)$. Observing that the series in (18) and (19) are absolutely convergent (this is actually true for all of the series in (18)–(20), (23)–(27)), provided, of course, x is real and non-zero, the following manipulation is valid:

$$\sum_{n=-\infty}^{\infty} \operatorname{sech} 2nx + \sum_{n=-\infty}^{\infty} \operatorname{sech} (2n-1)x = \sum_{n=-\infty}^{\infty} \operatorname{sech} nx .$$

Using (18), (19) and the hypothesis, this is equivalent to the following relation:

$$\frac{2}{\pi} K(m_2) + \frac{2}{\pi} m_2^{1/2} K(m_2) = \frac{2}{\pi} K(m_1).$$

This implies (a).

Proof of (b): An immediate consequence of (a) and the hypothesis.

Proof of (c): The following is Formula 17.3.29 in [1], slightly modified:

$$K(m) = \frac{2}{1+\sqrt{1-m}} K\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} .$$

Replacing m by $1-m_2$ yields:

$$K'(m_2) = \frac{2}{1+\sqrt{m_2}} K\left\{\left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2\right\} .$$

Substituting this result into (b) yields:

$$K'(m_1) = K(1-m_1) = K\left\{\left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2\right\} .$$

This result and the fact that K is a one-to-one function on (0,1) imply (c).

Proof of (d): The following is Formula 17.3.30 in [1], slightly modified:

$$E(m) = (1+\sqrt{1-m})E\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} - \frac{2\sqrt{1-m}}{1+\sqrt{1-m}} K\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} .$$

Replacing m by $1-m_2$ and incorporating the results of (b) and (c) yields:

$$E'(m_2) = (1+\sqrt{m_2})E'(m_1) - \sqrt{m_2}\,K'(m_2).$$

Rearrangement yields (d).

Proof of (e): The following is the famous relation due to Legendre:

$$EK' + E'K - KK' = \pi/2,$$

for any (implicit) parameter m. Letting $m = m_1$ and substituting the results of (a), (b) and (d) yields (e). This completes the proof of the theorem.

If the constant μ_1 is defined by:

$$\pi K'(\mu_1)/K(\mu_1) = \lambda,$$

it follows from part (c) of the preceding theorem that μ_1 is related to μ by the following identity:

$$\mu_1 = 1 - \left(\frac{1-\sqrt{\mu}}{1+\sqrt{\mu}}\right)^2 .$$

Equation (60) then follows from this last result, by substituting $x = \lambda$ in (27) and using (37). This same substitution in the other identities of Section 3, however, results either in series which have already been treated (by decomposition into even and odd terms), or in series whose terms contain irrational numbers. Therefore, if we are interested only in obtaining closed forms for series of *rational* numbers, identity (27) is the only identity in Section 3 which yields an "interesting" result for $x = \lambda$. It would therefore appear that the theorem we have proved has very limited applicability. This is not the case, however, for if we solve for the functions of m_2 in terms of the functions of m_1, we obtain formulas for other "interesting" series not previously treated, in terms of the original parameter m_1. Theoretically, this process may be continued indefinitely, but the closed forms thereby obtained will become increasingly cumbersome at each step. To illustrate, we set $m_1 = \mu$ in the theorem of this section, and define μ'' by the relation:

$$\pi K'(\mu'')/K(\mu'') = 4\lambda\ ;$$

hence, μ'' plays the role of m_2 in the theorem. Also, let

$$\rho'' = K(\mu'')/\pi \qquad \text{and} \qquad \sigma'' = E(\mu'')/\pi\,.$$

Using the theorem, we may solve for the "double-primed" functions in terms of the unprimed functions, and obtain the following results:

(61) $$\sqrt{\mu''} = (1-\sqrt{1-\mu})/(1+\sqrt{1-\mu}\,; \qquad 1-\mu'' = \frac{4\sqrt{1-\mu}}{(1+\sqrt{1-\mu})^2}\ ;$$

(62) $$\rho'' = \tfrac{1}{2}(1+\sqrt{1-\mu})\rho\,; \qquad \sigma'' = (\sigma+\rho\sqrt{1-\mu})/(1+\sqrt{1-\mu})\ .$$

If we substitute $x = 4\lambda$ in (18) and apply (36) and (37), we obtain the formulas:

$$\frac{4}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{4n-2}} = 4\sum_{n=1}^{\infty}\frac{1}{L_{4n-2}} = 2\rho''\sqrt{\mu''}\ .$$

Now using the results of (61) and (62), we obtain the identities:

(63) $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{4n-2}} = \tfrac{1}{4}\sqrt{5}\,(1-\sqrt{1-\mu})\rho\ ;$$

(64) $$\sum_{n=1}^{\infty}\frac{1}{L_{4n-2}} = \tfrac{1}{4}(1-\sqrt{1-\mu})\rho\ .$$

Similarly, we may derive the following identities from the general ones of Section 3, by means of the same substitutions:

(65) $$\sum_{n=1}^{\infty}\frac{1}{L_{4n}} = \tfrac{1}{4}(1+\sqrt{1-\mu})\rho - \tfrac{1}{4}\ ;$$

(66) $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{4n}} = \tfrac{1}{4} - \tfrac{1}{2}(1-\mu)^{1/4}\rho\,;$$

(67) $$\sum_{n=1}^{\infty}\frac{1}{L_{4n}^2} = \tfrac{1}{4}(\rho\sigma+\rho^2\sqrt{1-\mu}) - \frac{1}{8}\ ;$$

(68) $$\sum_{n=1}^{\infty}\frac{1}{F_{4n}^2} = \frac{5}{24}\left\{1+(2-\mu)\rho^2 - 6\rho\sigma\right\}\ ;$$

(69) $$\sum_{n=1}^{\infty}\frac{4n}{F_{4n}} = \tfrac{1}{2}\sqrt{5}\,\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\}\ ;$$

(70) $$\sum_{n=1}^{\infty}\frac{1}{F_{4n-2}^2} = \frac{5}{8}\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\}\ ;$$

(71) $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{F_{4n}} = \sqrt{5}\,(\rho\sigma-\rho^2\sqrt{1-\mu})\,;$$

(72) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^2} = \tfrac{1}{4}(\rho\sigma - \rho^2\sqrt{1-\mu}) \quad ;$$

(73) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{8n-4}} = \frac{\sqrt{5}}{8} \log\left\{\frac{(1-\mu)^{1/4} + (1-\mu)^{-1/4}}{2}\right\} \quad .$$

Observe that (64) and (65) were previously derived, as indicated in (50) and (51), by a different method. Appropriate combinations of (64)–(72) yield the following identities (note that (78) and (79) were previously derived, as indicated in (43) and (44)):

(74) $$\sum_{n=1}^{\infty} \frac{1}{L_{8n}} = \frac{1}{8}\left\{1 + (1-\mu)^{1/4}\right\}^2 \rho - \tfrac{1}{4}\,;$$

(75) $$\sum_{n=1}^{\infty} \frac{1}{L_{8n-4}} = \frac{1}{8}\left\{1 - (1-\mu)^{1/4}\right\}^2 \rho\ ;$$

(76) $$\sum_{n=1}^{\infty} \frac{8n}{F_{8n}} = \tfrac{1}{4}\sqrt{5}\left\{(1+\sqrt{1-\mu})^2\rho^2 - 4\rho\sigma\right\}\ ;$$

(77) $$\sum_{n=1}^{\infty} \frac{8n-4}{F_{8n-4}} = \tfrac{1}{4}\sqrt{5}(1-\sqrt{1-\mu})^2\rho^2\ ;$$

(78) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = \tfrac{1}{2}\rho\sigma - 1/8\ ;$$

(79) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = \frac{5}{24}\left\{1 + 4(2-\mu)\rho^2 - 12\rho\sigma\right\}\ .$$

By letting $x = 8\lambda$ in (18)–(20), (23)–(27), and again using the theorem of this section, we may derive yet another set of identities, involving the reciprocals of Fibonacci and Lucas numbers of indices $8n$ or $8n-4$ (except for the identity derived from (27), which involves F_{16n-8}); the closed forms thereby derived are again functions of the three basic constants μ, ρ and σ, albeit more complicated functions. Continuing in this fashion, we may, in theory, obtain closed forms for series involving the reciprocals of Fibonacci and Lucas numbers, where their indices have one of the two forms: $2^k n$ or $2^k(2n-1)$. Note, however, that conspicuously absent from the compendium of identities in this section are formulas for the series:

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}}\ .$$

It is seen, from (36) and (37), that these, in turn, depend on an evaluation of the series

$$\sum_{n=1}^{\infty} \operatorname{csch} nx\,,$$

which is absent in Section 3. Such an evaluation does not appear to be provided by the elliptic function theory, however, and is, in fact, the subject of a separate section of this paper.

Mention should be made of recent papers by Greig and Gould ([5] and [6]), where elementary techniques are used to obtain approximations to the series

$$\sum_{n=1}^{\infty} \frac{1}{F_n},$$

and to more general series. The most significant result to the topic of this paper appears in [5] and may be expressed in the following form:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{\infty} \left(\frac{1+F_{2n}}{F_{2n+1}} + \frac{2}{F_{4n+2}} + \beta \right).$$

This formula, however, does not yield a closed form, but only a rearrangement, of the terms in

$$\sum_{n=1}^{\infty} \frac{1}{F_n},$$

albeit one which yields fairly rapid convergence.

It is clear how the formulas of this section may be extended to other sequences (U_n) of the type discussed in the beginning of this section. It is not the aim of the author to obtain an indefinite number of identities such as are listed in this section, but rather to indicate the methods by which one may proceed in so doing.

5. SYMMETRICAL RELATIONSHIPS

Although the formulas of Section 3 (and their applications in Section 4) provide closed forms for the indicated series, they are not very satisfactory from the point of view of numerical evaluation; manual computations of m (from (9), with given x), and of $K(m)$ and $E(m)$, even with the help of tables of elliptic integrals and related tables, can be quite cumbersome, and in any event cannot exceed the accuracy of the tables. There is a much more satisfactory approach, fortunately, which enables the computation of m, K and E with a high degree of precision and a minimum of effort.

Recall the definitions of x and y given in (9), and note that $xy = \pi^2$. Note also that all of the Section 3 formulas are valid if x is replaced by y, m replaced by $(1-m)$, K replaced by K', and E replaced by E' (see (3) and (4) for definitions of K' and E'). However, K, K', E and E' are not independent of each other, but rather satisfy the relations:

(80) $$K' = Kx/\pi$$

(a restatement of (9)), and

(81) $$E' = \pi/2K + x/\pi \cdot (K - E)$$

(a restatement of Legendre's relation, incorporating the result of (80); see proof of part (e) of Theorem in Section 4).

By means of (80) and (81), we may express the formulas in Section 3 as functions of y, with closed forms in terms of m, K and E. If we then equate these expressions with the original functions of x, we obtain relations between functions of x and functions of y, which display a symmetry of some sort. We illustrate this method by deriving the following symmetrical relation:

(82) $$|x|^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} nx = |y|^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} ny, \ \forall \text{ real } x,y \text{ such that } xy = \pi^2.$$

The proof of (82) follows from (19) and (80):

$$\sum_{n=-\infty}^{\infty} \operatorname{sech} ny = 2K'/\pi = 2Kx/\pi^2 = 2K/y = \pi/y \sum_{n=-\infty}^{\infty} \operatorname{sech} nx = (x/y)^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} nx,$$

provided x,y are real positive numbers such that $xy = \pi^2$. Note, however, that this result is independent of elliptic functions and is equally valid if x and y are both negative, because sech is an even function. This establishes (82).

An asymmetrical relation is obtained by applying this method to (18) and (20), which again yields a result which is independent of m, namely:

$$|x|^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x = |y|^{1/2} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny \quad (x,y \text{ real}, xy=\pi^2). \tag{83}$$

Similarly, we may derive the following formulas, where in all cases, x and y are arbitrary real numbers such that $xy=\pi^2$:

$$\sum_{n=1}^{\infty} (-1)^{n-1}(nx \operatorname{csch} nx + ny \operatorname{csch} ny) = \tfrac{1}{2}\ ; \tag{84}$$

$$\sum_{n=1}^{\infty} \left\{ |x| \operatorname{sech}^2 (n-\tfrac{1}{2})x + |y| \operatorname{sech}^2 (n-\tfrac{1}{2})y \right\} = 1\ ; \tag{85}$$

$$\sum_{n=-\infty}^{\infty} |x| \operatorname{sech}^2 nx = 2+4 \sum_{n=1}^{\infty} ny \operatorname{csch} ny = 2 + \sum_{n=1}^{\infty} 2|y| \operatorname{csch}^2 (n-\tfrac{1}{2})y\ ; \tag{86}$$

$$\sum_{n=1}^{\infty} \left\{ |x| \operatorname{csch}^2 nx + |y| \operatorname{csch}^2 ny \right\} = \frac{|x+y|}{6} - 1. \tag{87}$$

Aside from whatever elegance equations (82)–(87) possess, they are quite useful for numerical computations, for we may choose x in such a way that the series involving y converges with extreme rapidity. To see this better, we convert (82)–(87) to the forms which are more suitable for numerical computation, valid $\forall$ real $x \neq 0$:

$$\sum_{n=-\infty}^{\infty} \operatorname{sech} nx = \frac{\pi}{|x|} \sum_{n=-\infty}^{\infty} \operatorname{sech} n\pi^2/x\ ; \tag{88}$$

$$\sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x = \frac{\pi}{|x|} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} n\pi^2/x\ ; \tag{89}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = \frac{1}{2x} - \frac{\pi^2}{x^2} \sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} n\pi^2/x\ ; \tag{90}$$

$$\sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})x = \frac{1}{|x|} - \frac{\pi^2}{x^2} \sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})\pi^2/x\ ; \tag{91}$$

$$\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 nx = \frac{2}{|x|} + \frac{4\pi^2}{x^2} \sum_{n=1}^{\infty} n \operatorname{csch} n\pi^2/|x| = \frac{2}{|x|} + \frac{2\pi^2}{x^2} \sum_{n=1}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})\pi^2/x\ ; \tag{92}$$

$$\sum_{n=1}^{\infty} \operatorname{csch}^2 nx = \frac{1}{6} - \frac{1}{|x|} + \frac{\pi^2}{6x^2} - \frac{\pi^2}{x^2} \sum_{n=1}^{\infty} \operatorname{csch}^2 n\pi^2/x\ . \tag{93}$$

By choosing $0 < |x| \leq \pi$, the convergence of the series in the right members of (88)–(93) is at least as rapid as that which occurs when $|x| = |y| = \pi$, which is itself fairly rapid. If we require $|x| > \pi$, we may then reverse the roles of x and y in (88)–(93), and still obtain rapid convergence, using the series in the *left* members to evaluate the required series.

6. NUMERICAL EVALUATION OF SERIES INVOLVING RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

In this section, we will apply the results of the previous section toward numerical evaluation of the constants μ, ρ and σ defined by (38). We first need to compute $\lambda = \log\{\tfrac{1}{2}(1+\sqrt{5})\}$. The computations indicated in this section were performed manually, with the help of tables found in [1]. In all cases, the accuracy does not exceed 15 significant digits. An electronic computer would attain far greater accuracy.

(94) $$\lambda = .48121\ 18250\ 59603, \text{ approximately.}$$

Substituting $x = 2\lambda$ (or $x = \pi^2/2\lambda$, where appropriate) in (88)–(93) yields, among others, the following identities:

(95) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = -\tfrac{1}{4} + \pi/8\lambda \sum_{n=-\infty}^{\infty} \operatorname{sech} ny, \quad \text{where } y = \pi^2/2\lambda\,,$$

(96) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \frac{\pi\sqrt{5}}{8\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny\ ;$$

(97) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = \frac{5}{8\lambda} - \frac{5\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})y\ ;$$

(98) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = \frac{5}{24} - \frac{5}{8\lambda} + \frac{5\pi^2}{96\lambda^2} - \frac{5\pi^2}{16\lambda^2} \sum_{n=1}^{\infty} \operatorname{csch}^2 ny\,;$$

(99) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = -\frac{1}{8\lambda} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 ny\,;$$

(100) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = -\frac{1}{8} + \frac{1}{8\lambda} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})y\ ;$$

Adding (97) and (98) yields:

(101) $$\sum_{n=1}^{\infty} \frac{1}{F_n^2} = \frac{5}{24} - \frac{5\pi^2}{48\lambda^2} - \frac{5\pi^2}{16\lambda^2} \sum_{n=1}^{\infty} (\operatorname{sech}^2 (n-\tfrac{1}{2})y + \operatorname{csch}^2 ny).$$

Adding (99) and (100) yields:

(102) $$\sum_{n=1}^{\infty} \frac{1}{L_n^2} = -\frac{1}{8} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n-\tfrac{1}{2})y).$$

If we now compare the results of (41) and (95), we obtain:

(103) $$\rho = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} \operatorname{sech} ny\,.$$

Comparing (40) and (96) yields:

(104) $$\rho\sqrt{\mu} = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny\,,$$

from which it follows that

$$\sqrt{\mu} = \left(\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny \right) \div \left(\sum_{n=-\infty}^{\infty} \operatorname{sech} ny \right) . \tag{105}$$

The values of ρ^2 and μ can be obtained by squaring both sides of (103) and (105), respectively. An alternative approach is indicated below. If we compare (54) and (102), we obtain:

$$\rho^2 = \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n - \tfrac{1}{2})y) . \tag{106}$$

Combining (97) and (99) as indicated in (57) and comparing the results, we obtain:

$$\mu\rho^2 = \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny - \operatorname{sech}^2 (n - \tfrac{1}{2})y) . \tag{107}$$

It follows from (106) and (107) that we have:

$$\mu = \left(\sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny - \operatorname{sech}^2 (n - \tfrac{1}{2})y) \right) \div \left(\sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n - \tfrac{1}{2})y) \right) . \tag{108}$$

Again, the computation of $\rho\sqrt{1-\mu}$ may be accomplished from the values of ρ and μ obtained in (103) and (108); a somewhat more accurate result is obtained, however, if we combine the results of (37), (42) and (83), which yields:

$$\rho\sqrt{1-\mu} = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} \operatorname{sech} (n - \tfrac{1}{2})y . \tag{109}$$

In the closed form expressions occurring in Section 4, we observe that the constant σ always appears multiplied by ρ; therefore, we will indicate the numerical computation of $\rho\sigma$, rather than of σ itself. This is most easily accomplished by combining the results of (43) and (100), which yields:

$$\rho\sigma = \frac{1}{4\lambda} + \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{csch}^2 (n - \tfrac{1}{2})y . \tag{110}$$

Superficially, it would appear that the identities in (103)–(110) are very unwieldy for computational purposes. However, as mentioned previously, the infinite series in the right members of (103)–(110) converge quite rapidly; thus, at most eight terms of the series need be included to guarantee an accuracy in the result of 15 significant digits! Moreover, since the summand terms are symmetrical about the value $n = 0$, only four terms of the series, at most, need be computed for 15-digit accuracy! A summary of the computations is appended; indicated in Appendix II are the *computed* values of the series occurring in Section 4, using the constants indicated in Appendix I. As a check on the computations, the actual summations were performed by the author with the aid of a desk calculator, and all results checked with those indicated in Appendix II, to 15 significant digits! It should be emphasized that the values in Appendix II were obtained *without* performing any actual summations.

7. CONCLUSION

As mentioned previously, the series

$$\sum_{n=1}^{\infty} \operatorname{csch} nx,$$

(x real and non-zero), apparently cannot be evaluated by elliptic functions. However, the following formula in terms of Lambert functions exists:

(111) $$\sum_{n=1}^{\infty} \operatorname{csch} nx = 2\{\pounds(e^{-x}) - \pounds(e^{-2x})\}, \quad (x > 0),$$

where

(112) $$\pounds(q) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}, \quad |q| < 1,$$

is the *Lambert function.*

By decomposing (111) into even and odd-subscript terms, we may deduce the following formulas:

(113) $$\sum_{n=1}^{\infty} \operatorname{csch} 2nx = 2\{\pounds(e^{-2x}) - \pounds(e^{-4x})\};$$

and

(114) $$\sum_{n=1}^{\infty} \operatorname{csch} (2n-1)x = 2\{\pounds(e^{-x}) - 2\pounds(e^{-2x}) + \pounds(e^{-4x})\}, \text{ where } x > 0.$$

In particular, setting $x = \lambda$ in (113)–(114) and employing (36)–(37), we obtain the following formulas:

(115) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\,\{\pounds(\beta^2) - \pounds(\beta^4)\},$$

and

(116) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} = \pounds(-\beta) - 2\pounds(\beta^2) + \pounds(\beta^4), \text{ where } \beta \text{ is given in (35)}.$$

These results are not new, and were generalized by Shannon and Horadam, as well as by Brady [3]. [4]. However, their results are in terms of Lambert functions, and it is this fact which the author finds unsatisfactory, since the Lambert function is defined as an infinite series. Hence, we are using an infinite series to obtain the "closed form" sum of another infinite series; moreover, it is seen that (111) is little more than an algebraic identity, readily obtainable by manipulation of the definition in (112). It seems, therefore, that (111) is simply an artificiality, and another expression free of Lambert functions would be preferable.

It is also worth mentioning that the technique of contour integration may be used to derive identities similar to those given in Section 5. We illustrate by deriving the following identity:

(117) $$\sum_{n=1}^{\infty} \frac{\operatorname{sech} nx}{n^2 x} - \frac{\pi^2}{6x} = \frac{x}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \coth (n-\frac12)\pi^2/x}{(n-\frac12)^2}, \quad \forall x > 0.$$

Let $\underline{C}$ be the finite complex plane (z-plane), with $z = u + iv$, and consider the function $f: \underline{C} \to \underline{C}$ given by:

(118) $$f(z) = z^{-2} \operatorname{sech} xz \cot \pi z, \quad \text{where } x > \pi.$$

Let R_ξ be the residue of f at its pole ξ. Note that f is meromorphic in $\underline{C}$, with simple poles at $u_n = n$ ($n = \pm 1, \pm 2, \cdots$) and $iv_n = (n-\frac12)\pi i/x$ ($n = 0, \pm 1, \pm 2, \cdots$), and a pole of order 3 at the origin. Calculating the residues, we find:

$$R_{u_n} = \frac{\operatorname{sech} nx}{\pi n^2}; \quad R_{iv_n} = \frac{\cot(\pi i v_n)}{-v_n^2 x \sinh(xiv_n)} = \frac{(-1)^{n-1} x \coth (n-\frac12)y}{(n-\frac12)^2 \pi^2},$$

where $y = \pi^2/x$; R_0 is the coefficient of z^2 in the Taylor series expansion of

$$\frac{1}{\pi} \operatorname{sech} xz \cdot \pi z \cot \pi z = \frac{1}{\pi}\left(1 - \tfrac12 (xz)^2 + \cdots\right)\left(1 - \frac{(\pi z)^2}{3} - \cdots\right);$$

hence, $R_0 = -x^2/2\pi - \pi/3$.

Now, let $\mathcal{R}_n$ be the rectangle bounded by the lines $u = \pm(N - \tfrac12)$, $v = \pm N\pi/x$, and form the sequence $(\mathcal{R}_N)_1^\infty$. It is not difficult to show that $\sec xz \cot \pi z$ is uniformly bounded on

$$\bigcup_{N=1}^{\infty} \mathcal{R}_N :$$

from this, it follows that

$$\lim_{N\to\infty} \int_{\mathcal{R}_N} f(z)dz = 0.$$

By the Cauchy Residue Theorem:

$$\int_{\mathcal{R}_N} f(z)dz = \sum_{0<|u_n|<N-\frac12} R_{u_n} + \sum_{|v_n|<N\pi/x} R_{iv_n} + R_0 . \tag{119}$$

Allowing N to tend to ∞ in (119), we therefore obtain:

$$\sum_{n=-\infty}^{\infty}{}' \frac{\operatorname{sech} nx}{\pi n^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}\coth(n-\tfrac12)y}{(n-\tfrac12)^2 y} = x^2/2\pi + \pi/3,$$

where the first (primed) summation excludes the term for which $n = 0$. Multiplying throughout by $\pi/2x$ and simplifying, we obtain (117). The following generalization of (117) is obtained similarly, by taking

$$f(z) = z^{-2r}\operatorname{sech} xz \cot \pi z,$$

where r is a positive integer:

$$\sum_{n=1}^{\infty} \frac{\operatorname{sech} nx}{n^{2r}x^{2r}} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{r+n}\coth(n-\tfrac12)y}{(n-\tfrac12)^{2r}y^{r-1}} + \sum_{k=0}^{r} \frac{(-1)^k B_{2k}E_{2r-2k}}{(2k)!(2r-2k)!} 2^{2k-1}x^{r-k}y^k = 0 ; \tag{120}$$

here, $y = \pi^2/x$, and the B_{2k}'s and E_{2k}'s are Bernoulli and Euler numbers, respectively.

Note that if we set $r = 0$ in (120), we obtain the apparent result:

$$\sum_{n=1}^{\infty} \operatorname{sech} nx + \frac{y}{\pi}\sum_{n=1}^{\infty} (-1)^n \coth(n-\tfrac12)y + \tfrac12 = 0. \tag{121}$$

By manipulations similar to those employed after (22), we may show that, for all positive y,

$$\sum_{n=1}^{\infty} (-1)^{n-1}\coth(n-\tfrac12)y = \tfrac12 \sum_{n=-\infty}^{\infty} \operatorname{sech} ny. \tag{122}$$

Incorporating this last result into (121) and simplifying, we obtain (88), which shows that (120) is also valid for $r = 0$, though this is seemingly not justifiable by the method of contour integration. The latter method apparently provides a richer variety of identities similar to those of Section 5 than does the method of elliptic functions; on the other hand, it does not provide closed forms for the indicated series, except for special values of x and y. Thus, if we set $x = y = \pi$ in (84), (85) and (87), we obtain the results:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} n\pi = \frac{1}{4\pi} ; \tag{123}$$

$$\sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac12)\pi = \frac{1}{2\pi} ; \tag{124}$$

$$\sum_{n=1}^{\infty} \operatorname{csch}^2 n\pi = \frac{1}{6} - \frac{1}{2\pi} . \tag{125}$$

Another important observation to make is that the identities given in this paper for real values of x and y may, with certain further restrictions, be extended to the complex plane, thereby yielding results involving corresponding trigonometric expressions, instead of hyperbolic ones. This opens up a whole new area of approach, which is beyond the scope of this paper to explore. It suffices to say that there are ample avenues of research available, as suggested in this paper, as regards the series discussed. It is hoped that sufficient interest has been generated to warrant additional investigations into the indicated topics.

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APPENDIX I – TABLE OF CONSTANTS

λ = .48121 18250 59603 ;	$1/4\lambda$ = .51952 17303 08757 ;
$\pi/4\lambda$ = 1.63212 56513 1825 ;	$\pi^2/16\lambda^2$ = 2.66383 41416 9102 ;
$y = \pi^2/2\lambda$ = 10.25494 79118 337 .	
$e^{\frac{1}{2}y}$ = 168.59071 21406 95 ;	$e^{-\frac{1}{2}y}$ = .00593 15248 58649 77 ;
e^{y} = 28,422.82822 01066 ;	e^{-y} = .00003 51829 87148 8 ;
$e^{3y/2}$ = 4,791,824.85068 042;	$e^{-3y/2}$ = .00000 02086 88762 875 ;
e^{-2y} = .00000 00012 37842 58468 ;	$e^{-5y/2}$ = .00000 00000 07342 294 ;
e^{-3y} = .00000 00000 00043 55099 975 ;	
$e^{-7y/2}$ = .00000 00000 00000 25832 ;	
e^{-4y} = .00000 00000 00000 00153 ;	$e^{-9y/2}$ = .00000 00000 00000 0000
sech $y/2$ = .01186 26323 54457 871 ;	$\operatorname{sech}^2 y/2$ = .00014 07220 46377 031;
sech y = .00007 03659 74210 458 ;	$\operatorname{sech}^2 y$ = .00000 00049 51370 327 ;
sech $3y/2$ = .00000 04173 77525 749 ;	$\operatorname{sech}^2 3y/2$ = .00000 00000 00174 204 ;
sech $2y$ = .00000 00024 75685 169 ;	$\operatorname{sech}^2 2y$ = .00000 00000 00000 006 ;
sech $5y/2$ = .00000 00000 14684 588 ;	$\operatorname{sech}^2 5y/2$ = .00000 00000 00000 000 ;
sech $3y$ = .00000 00000 00087 102 ;	

APPENDIX I – (Cont'd.)

$\operatorname{sech} 7y/2 = .00000\ 00000\ 00000\ 517$;

$\operatorname{sech} 4y = .00000\ 00000\ 00000\ 003$;

$\operatorname{sech} 9y/2 = .00000\ 00000\ 00000\ 000$.

$\operatorname{csch} y/2 = .01186\ 34671\ 09510\ 403$;

$\operatorname{csch} 3y/2 = .00000\ 04173\ 77525\ 749$;

$\operatorname{csch} 5y/2 = .00000\ 00000\ 14684\ 588$;

$\operatorname{csch}^2 y/2 = .00014\ 07418\ 51858\ 435$;

$\operatorname{csch}^2 3y/2 = .00000\ 00000\ 00174\ 204$;

$\operatorname{csch}^2 5y/2 = .00000\ 00000\ 00000\ 000$.

$\sum_{n=-\infty}^{\infty} \operatorname{sech} ny = 1.00014\ 07368\ 99965$;

$\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny = .99985\ 92730\ 02775$;

$\sum_{n=-\infty}^{\infty} \operatorname{sech} (n-\tfrac{1}{2})y = .02372\ 60994\ 93337\ 4$;

$\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 ny = 1.00000\ 00099\ 02741$;

$\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})y = .00028\ 14440\ 93102\ 469$;

$\sum_{n=-\infty}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})y = .00028\ 14837\ 04065\ 278$.

$\sqrt{\mu} = .99971\ 85757\ 09592$;

$\sqrt{1-\mu} = .02372\ 27608\ 25520\ 2$;

$(1-\mu)^{-1/4} = 6.49258\ 11249\ 7349$;

$U \equiv \tfrac{1}{2}\left\{(1-\mu)^{1/4} + (1-\mu)^{-1/4}\right\} = 3.32330\ 15370\ 7076$.

$\mu = .99943\ 72306\ 18815$;

$(1-\mu)^{1/4} = .15402\ 19491\ 68033$;

$\log (1+\sqrt{\mu}) = .69300\ 64585\ 13859$;

$\sqrt{5}\log (1+\sqrt{\mu}) = 1.54960\ 95500\ 8338$;

$\log U = 1.20095\ 87276\ 7835$;

$\log (1-\sqrt{\mu}) = -8.17564\ 70971\ 5135$;

$\sqrt{5}\log (1-\sqrt{\mu}) = -18.28130\ 26692\ 792$;

$\sqrt{5}\log U = 2.68542\ 53532\ 6045$.

$\rho = 1.63235\ 53516\ 2277$;

$\rho\sqrt{5} = 3.65005\ 75296\ 6408$;

$\rho\sqrt{1-\mu} = .03872\ 39755\ 88805\ 0$;

$\rho(1-\mu)^{1/4} = .25141\ 85529\ 91809$.

$\rho\sqrt{\mu} = 1.63189\ 59671\ 7624$;

$\rho\sqrt{5\mu} = 3.64903\ 03148\ 1385$;

$\rho\sqrt{5(1-\mu)} = .08658\ 94417\ 76510\ 3$;

$\rho^2 = 2.66458\ 39939\ 7149$;

$\rho^2\sqrt{5} = 5.95819\ 09422\ 7815$;

$\rho^2\sqrt{1-\mu} = .06321\ 12887\ 88495\ 2$;

$\rho\sigma = .52027\ 15562\ 09976$;

$\rho^2\mu = 2.66308\ 44476\ 8609$;

$\rho^2\mu\sqrt{5} = 5.95483\ 78548\ 4858$;

$\rho^2\sqrt{5(1-\mu)} = .14134\ 47386\ 76446$;

$\rho\sigma\sqrt{5} = 1.16336\ 25664\ 4511$.

APPENDIX II – COMPUTED FORMULAS FOR SERIES IN SECTION A

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}} = .81594\ 79835\ 88122\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = .56617\ 76758\ 11385\ ;$$

$$\sum_{n=1}^{\infty} \frac{2n}{F_{2n}} = 4.79482\ 83758\ 3304\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = .13513\ 57781\ 04988\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = 1.07215\ 62188\ 8076\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = 1.82451\ 51574\ 0692\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}} = .23063\ 80122\ 05598\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2n}{F_{2n}} = 1.16000\ 94790\ 1554\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = 1.12939\ 07263\ 5581\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = 1.29693\ 00248\ 1143\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{4n-2}} = 1.04573\ 08199\ 4974\ ;$$

$$\sum_{n=1}^{\infty} \frac{4n-2}{F_{4n-2}} = 2.97741\ 89274\ 2429\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = .39840\ 78440\ 08491\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = 1.20729\ 19969\ 8575\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{4n}} = .12429\ 07235\ 04095\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n}^2} = .02087\ 07112\ 49618\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^2} = .11426\ 50668\ 55370\ ;$$

$$\sum_{n=1}^{\infty} \frac{4n}{F_{4n}} = 1.81740\ 94484\ 0875\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = .16776\ 98318\ 02894\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{n}^2} = 2.42632\ 07511\ 6724\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{4n-2}} = .89086\ 70219\ 72118\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2} = .11342\ 79589\ 57717\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2} = 1.01596\ 27673\ 9809\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4n}{F_{4n}} = 1.02201\ 78277\ 6866\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)L_{2n-1}} = 1.10858\ 16944\ 5815\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{8n}} = .02173\ 95541\ 49399\ ;$$

$$\sum_{n=1}^{\infty} \frac{8n}{F_{8n}} = .39769\ 58103\ 20044\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{8n-4}} = .33567\ 81691\ 57557\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{8n-4}} = .14603\ 02776\ 53494\ ;$$

$$\sum_{n=1}^{\infty} \frac{8n-4}{F_{8n-4}} = 1.41971\ 36380\ 8871.$$

****

SEQUENCES ASSOCIATED WITH t-ARY CODING OF FIBONACCI'S RABBITS

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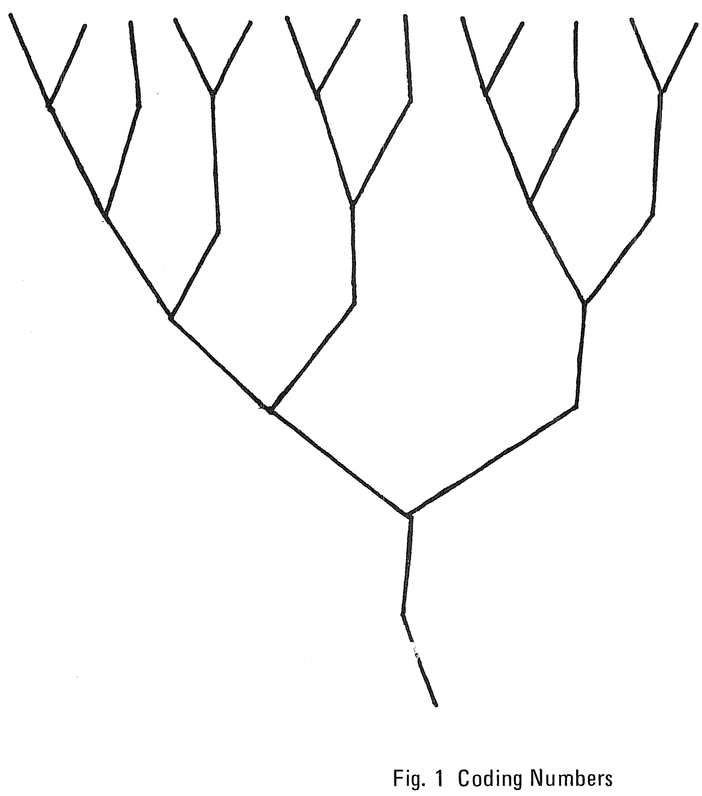
The object of this note is to point out a curious kind of sequence which arises in connection with a binary coding of the tree diagram for the production of rabbits by Fibonacci's recurrence.

At the left below is a standardized way of drawing the usual Fibonacci rabbit tree. At the right is a binary code for each level. The code is assigned by a very simple rule. On each level, a single segment | is coded by 0 and a branched segment V is coded by 1. It is clear that this establishes a unique binary coding for each level of the Fibonacci rabbit tree (or any other tree for that matter). We suspect that this is not a new idea, but do not have a reference.

TREE CODING



10110101

10110

101

10

1

0

0

Fig. 1 Coding Numbers

In the next table we give a summary of initial values of the binary coding, first in base 2 and then converted into base ten. In each case notice that the coding number for a given level can be expressed in terms of the coding for the two previous levels.

Table 1
Coding Numbers and Recurrences

n	$(C_n)_2$	$(C_n)_{10} = A_n$
1	0	0
2	0	0
3	1	1
4	10	$2 = 2^1(1) + 0$
5	101	$5 = 2^1(2) + 1$
6	10110	$22 = 2^2(5) + 2$
7	10110101	$181 = 2^3(22) + 5$
8	1011010110110	$5{,}814 = 2^5(181) + 22$
9	101101011011010110101	$1{,}488{,}565 = 2^8(5{,}814) + 181$
10		$12{,}194{,}330{,}294 = 2^{13}(1{,}488{,}565) + 5{,}814$

We put $(C_n)_2$ for the coding number in binary form, and $(C_n)_{10}$ or A_n for the coding number expressed in base ten.

It is evident from the formation of the rabbit tree that the base ten coding numbers satisfy the recurrence

$$A_{n+2} = 2^{F_{n-1}}A_{n+1} + A_n, \qquad n \geqslant 2, \tag{1}$$

where F_n is the ordinary Fibonacci sequence, $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$, $F_1 = 1$. Again, from the law of formation it is evident that $(C_n)_2$ has exactly F_{n-1} digits. Thus also

$$2^{F_{n-1}} > A_n \geqslant 2^{F_{n-1}-1}, \qquad \text{for} \quad n \geqslant 3. \tag{2}$$

Formula (1), together with initial values defines the sequence A_n uniquely. Starting with the sequence A_n we may recover the Fibonacci numbers from the formula

$$F_n = \log_2 \frac{A_{n+3} - A_{n+1}}{A_{n+2}}. \tag{3}$$

Special sums involving the sequence A_n may be found in closed form. From (1) we can get almost at once

$$A_{n+3} + A_{n+2} - 1 = \sum_{k=1}^{n} 2^{F_k}A_{k+2}, \qquad n \geqslant 1. \tag{4}$$

Multiply each side of (1) by 2^{F_n} and use the fact that $F_n + F_{n-1} = F_{n+1}$. We find then

$$2^{F_n}A_{n+2} = 2^{F_{n+1}}A_{n+1} + 2^{F_n}A_n, \qquad n \geqslant 2, \tag{5}$$

and this form of the recurrence is the clue to the proof of the next formula:

$$\sum_{k=2}^{n} (-1)^k 2^{F_k}A_{k+2} = (-1)^n 2^{F_{n+1}}A_{n+1}, \qquad n \geqslant 2. \tag{6}$$

We have not found a generating function for A_n and this is posed as a research problem for the reader.

We have also not found the sequence A_n in Sloane's book [2]. Does any reader know any previous appearance of A_n?

The process by which we have obtained A_n is not restricted to the standard Fibonacci sequence. Here is another example yielding a different sequence with the same behavior. Define a third-order recurrent sequence by the recurrence

(7) $$G_{n+1} = G_n + G_{n-2}, \quad \text{with} \quad G_1 = G_2 = G_3 = 1.$$

The reader may draw the corresponding rabbit tree and verify that the coding numbers and recurrence values in the next table are correct.

Table 2
Coding Numbers for G_n

n	G_n	$(D_n)_2$	$(D_n)_{10} = B_n$
1	1	0	0
2	1	0	0
3	1	0	0
4	2	1	1
5	3	10	2 = 2(1) + 0
6	4	100	4 = 2(2) + 0
7	6	1001	9 = 2(4) + 1
8	9	100110	38 = 4(9) + 2
9	13	100110100	308 = 8(38) + 4
10	19	1001101001001	4,937 = 16(308) + 9
11	28		158,022 = 64(4,937) + 38

Here it is evident that the law of formation is

(8) $$B_{n+3} = 2^{G_{n-1}} B_{n+2} + B_n, \qquad n \geqslant 3.$$

Again sums such as (4) and (6) can be established.

It appears that the behavior of these sequences can be predicted to follow in similar fashion for other recurrent sequences for which we can draw a suitable tree.

Recalling that the Lucas numbers are related to the Fibonacci numbers by the formula $L_n = F_{n-1} + F_{n+1}$, we see that we can devise a Lucas rabbit tree by adding together two Fibonacci trees. We can call this method allowing twins to occur once in the Fibonacci tree. It is then evident that the binary coding must correspond to

(9) $$(E_n)_2 = 2^{F_{n-3}}(C_n)_2 + (C_{n-2})_2 ,$$

and we have the associated sequence $H_n = (E_n)_{10}$ satisfying

(10) $$H_n = 2^{F_{n-3}} A_n + A_{n-2}$$

in terms of our original coding. The corresponding Lucas rabbit tree is exhibited on the following page.

Because we start the twinning at level 3, we have defined $(E_3)_2 = 1$ and $H_3 = (E_3)_{10} = 1$ which is consistent with $H_3 = 2^{F_0}A_3 + A_1 = 1 + 0 = 1$.

We make some further remarks about the coding of the original Fibonacci rabbit tree. The sequence defined by

(11) $$U_n = 2^{F_{n-1}} - 1$$

satisfies

(12) $$U_{n+2} = 2^{F_{n-1}} U_{n+1} + U_n,$$

because

$$U_{n+2} = 2^{F_{n+1}} - 1 = 2^{F_{n-1}}(2^{F_n} - 1) + 2^{F_{n-1}} - 1 = 2^{F_{n-1}} U_{n+1} + U_n,$$

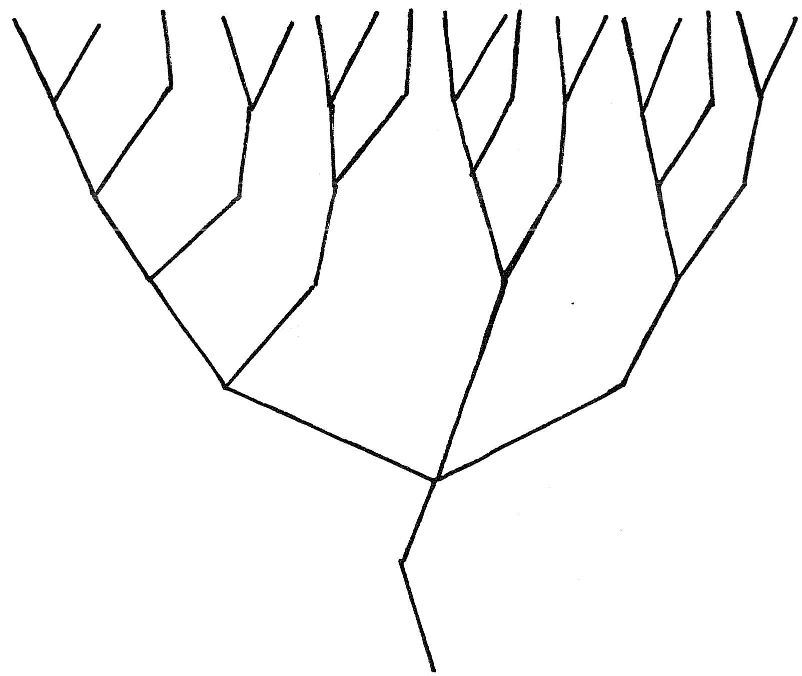
and so U_n is another solution of the equation (1).

In fact U_n and A_n can be found as numerator and denominator, respectively, of the partial convergents of the continued fraction

(13) $$1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{8+}\,\frac{1}{32+}\,\frac{1}{256+}\cdots ,$$

where the terms are defined from $2^{F_{n-1}}$. Thus the partial convergents of (13) turn out to be:

TREE



CODING

$(E_n)_2$	$H_n = (E_n)_{10}$
10110101101	1453
1011010	90
1011	11
100	4
1	1
0	0
0	0

Fig. 2 Coding of Lucas Rabbit Tree

$$(14)\qquad \frac{U_n}{A_n} = 1, \frac{3}{2}, \frac{7}{5}, \frac{31}{22}, \frac{255}{181}, \frac{8191}{5814}, \cdots$$

The sequence of values

$$\frac{3}{2} = 1.5$$

$$\frac{7}{5} = 1.4$$

$$\frac{31}{22} = 1.4090909\cdots$$

$$\frac{255}{181} = 1.408839779\cdots$$

$$\frac{8191}{5814} = 1.408840729\cdots$$

$$\frac{2097151}{1488565} = 1.408840729\cdots$$

suggests that there exists a limit of the form

$$(15)\qquad \lim_{n\to\infty} \frac{U_n}{A_n} = \lim_{n\to\infty} \frac{2^{F_{n-1}}}{A_n} = 1.40884073\cdots$$

which would be somewhat analogous to the well-known limit

$$(16)\qquad \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = \frac{2.236068+1}{2} = 1.6180339^{+}\ .$$

Formula (15) would also yield the asymptotic formula

$$(17)\qquad A_n \sim (0.709803442\cdots)2^{F_{n-1}} \quad \text{as } n\to\infty\ .$$

Davison [1] has just proved that with $a = (1+\sqrt{5})/2$ then

$$(18)\qquad T(a) = \sum_{n=1}^{\infty} \frac{1}{2^{[na]}} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{8+}\,\frac{1}{32+}\cdots$$

is transcendental. This remarkable result combines two things, the equivalence of the series and continued fraction, and the fact that the number so defined *is* transcendental. $T(a)$ is the reciprocal of the continued fraction in (13), so we have the transcendental limit

$$(19)\qquad \lim_{n\to\infty} \frac{A_n}{U_n} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{8+}\,\frac{1}{32+}\cdots = \sum_{n=1}^{\infty} \frac{1}{2^{[na]}} = 0.709803442\cdots$$

with $a = (1+\sqrt{5})/2$, and where square brackets denote the greatest integer function.

So far we have restricted our attention to binary coding. We return now to Table 1 and consider ternary coding. Actually what we do is to interpret the numbers $(C_n)_2 = (C_n)_3$ as if they were in ternary rather than binary form. Translating the ternary code to base ten, and writing $(C_n)_3 = A_n(3)$, we get the following sequence of numbers:

$$(20)\qquad A_n(3) = 0, 0, 1, 3, 10, 93, 2521, 612696, 4019900977, \cdots$$

and this sequence enjoys most of the properties belonging to the original sequence $A_n = A_n(2)$ derived from binary coding. Thus $2521 = 3^3(93) + 10$, $612696 = 3^5(2521) + 93$, etc., and in general

$$(21)\qquad A_{n+2}(3) = 3^{F_{n-1}}A_{n+1}(3) + A_n(3), \qquad n \geqslant 2.$$

As a matter of fact it is just as easy to consider the original coding with 0's and 1's as being t-ary coding, i.e., numbers in base t, where $t = 2, 3, 4, \cdots$. We write $(C_n)_t = A_n(t)$ for this form of the sequence. It is not difficult to see then that the formulas we developed for the binary case become in general:

$$A_{n+2}(t) = t^{F_{n-1}}A_{n+1}(t) + A_n(t), \qquad n \geqslant 2, \tag{22}$$

$$t^{F_{n-1}} > A_n(t) \geqslant t^{F_{n-1}-1}, \qquad n \geqslant 3, \tag{23}$$

$$F_n = \log_t \frac{A_{n+3}(t) - A_{n+1}(t)}{A_{n+2}(t)}, \tag{24}$$

$$A_{n+3}(t) + A_{n+2}(t) - 1 = \sum_{k=1}^{n} t^{F_k}A_{k+2}(t), \qquad n \geqslant 1, \tag{25}$$

$$t^{F_n}A_{n+2}(t) = t^{F_{n+1}}A_{n+1}(t) + t^{F_n}A_n(t), \qquad n \geqslant 2, \tag{26}$$

$$\sum_{k=2}^{n} (-1)^k t^{F_k} A_{k+2}(t) = (-1)^n t^{F_{n+1}} A_{n+1}(t), \qquad n \geqslant 2, \tag{27}$$

and in place of the sequence U_n we have the corresponding extension

$$U_n(t) = t^{F_{n-1}} - 1, \tag{28}$$

which satisfies the recurrence

$$X_{n+2}(t) = t^{F_{n-1}}X_{n+1}(t) + X_n(t), \tag{29}$$

as an extension of (1).

We also have an asymptotic result of the form

$$A_n(t) \sim K \cdot t^{F_{n-1}}, \qquad n \to \infty.$$

We shall find K in terms of continued fractions.

The continued fraction (13) with partial convergents (14) has a very interesting form in the general t-ary case:

$$\frac{U_n(t)}{A_n(t)} = (t-1) + \frac{t-1}{t^1+}\,\frac{1}{t^1+}\,\frac{1}{t^2+}\,\frac{1}{t^3+}\cdots\frac{1}{t^{F_{n-3}}}. \tag{30}$$

For $t = 3$ we have the case

$$2 + \frac{2}{3+}\,\frac{1}{3+}\,\frac{1}{9+}\,\frac{1}{27+}\,\frac{1}{243+}\cdots = 2.602142009\cdots \tag{31}$$

The reciprocal of this is $0.3842987802\cdots$, and it is now remarkable to note that if we extend the series of Davison (18) in the obvious way, we find that

$$\sum_{n=1}^{\infty} \frac{1}{3^{[n\alpha]}} = \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^9} + \frac{1}{3^{11}} + \cdots = 0.3842987802\cdots \tag{32}$$

and this is correct to at least as many decimals as shown since we have calculated the sum to 20 terms and the 21^{st} term adds only about 1.798865×10^{-16} to this.

It is natural to conjecture that Davison's theorem can be extended to show that this number also is transcendental and moreover that the limit of (30) as $n \to \infty$ is probably transcendental for every natural number $t \geqslant 2$.

Some of the first few partial convergents of (31) are:

$$2, \frac{8}{3}, \frac{26}{10}, \frac{242}{93}, \frac{6560}{2521}, \frac{1594322}{612696}, \cdots. \tag{33}$$

The general theorem which we claim is that for the continued fraction in (30),

$$\left(\sum_{n=1}^{\infty}\frac{1}{t^{[na]}}\right)^{-1} = \lim_{n\to\infty}\frac{U_n(t)}{A_n(t)} = (t-1)+\frac{t-1}{t+}\;\frac{1}{t+}\;\frac{1}{t^2+}\;\frac{1}{t^3+}\;, \tag{34}$$

where the exponents in the continued fraction are the successive Fibonacci numbers.

The first few partial convergents of the general continued fraction in (30) are:

$$\frac{U_4(t)}{A_4(t)} = \frac{t^2-1}{t},\quad \frac{U_5(t)}{A_5(t)} = \frac{t^3-1}{t^2+1},\quad \frac{U_6(t)}{A_6(t)} = \frac{t^5-1}{t^4+t^2+t},$$

$$\frac{U_7(t)}{A_7(t)} = \frac{t^8-1}{t^7+t^5+t^4+t^2+1},\quad \frac{U_8(t)}{A_8(t)} = \frac{t^{13}-1}{t^{12}+t^{10}+t^9+t^7+t^5+t^4+t^2+t},$$

etc., where, of course, the numerator is $t^{F_{n-1}}-1$, and the exponents of the t's in the denominator are precisely the powers of 2 appearing in the original binary coding of the rabbit tree as given in Fig. 1 or Table 1.

The first 50 values of $[na]$ for use in writing out the series (34) are: 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, 42, 43, 45, 46, 48, 50, 51, 53, 55, 56, 58, 59, 61, 63, 64, 66, 67, 69, 71, 72, 74, 76, 77, 79, 80. This agrees with sequence No. 917 in Sloane [2], where it is called a Beatty sequence because of the fact that $a_n = [na]$ and $b_n = [nb]$, where a and b are irrational with $1/a + 1/b = 1$ makes a_n and b_n disjoint subsequences of the natural numbers whose union is precisely the set of all natural numbers. Such sets are called complementary sequences.

Relations (30) and (34) may be put in more attractive form. Dividing each side of (30) by $t-1$ we get

$$\frac{U_n(t)}{(t-1)A_n(t)} = 1+\frac{1}{t+}\;\frac{1}{t+}\;\frac{1}{t^2+}\;\frac{1}{t^3+}\;\frac{1}{t^5+}\cdots\frac{1}{t^{F_{n-3}}}, \tag{35}$$

and taking reciprocals on both sides we find

$$\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\;\frac{1}{t+}\;\frac{1}{t+}\;\frac{1}{t^2+}\cdots\frac{1}{t^{F_{n-3}}}. \tag{36}$$

Then the limiting case (34) becomes more elegantly

$$(t-1)\sum_{n=1}^{\infty}\frac{1}{t^{[na]}} = \lim_{n\to\infty}\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\;\frac{1}{t+}\;\frac{1}{t+}\;\frac{1}{t^2+}\;\frac{1}{t^3+}\;\frac{1}{t^5+}\;\frac{1}{t^8+}\cdots, \tag{37}$$

apparently valid for all real $t > 1$.

Although the series diverges when $t = 1$, still the continued fraction makes sense, giving the familiar special case

$$\lim_{t\to 1}\lim_{n\to\infty}\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots = \frac{1+\sqrt{5}}{2}. \tag{38}$$

For $t = 1$, the sequence $A_n(1) = F_n$ so that we have in the general sequence an extension of the Fibonacci sequence.

Let us now make the definition

$$T(x,t) = \sum_{n=1}^{\infty}\frac{1}{t^{[nx]}} \tag{39}$$

for arbitrary real $t > 1$ and real $x > 0$.

This function has interesting properties, some of which we shall exhibit here. Take $x = a - 1 = 1/a$, a being as defined before. Then the sequence of values of $[na-n] = [na]-n$ begins: 0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 11, 11, 12, 12, 13, 14, 14, 15, 16, 16, 17, 17, 18, 19, 19, 20, ⋯. It does not seem to be tabulated in Sloane [2]. Taking $t = 2$, one finds that $T(a-1, 2) = 2.7098016\cdots$ and it seems evident that in fact $T(a-1, 2) = 2 + T(a,2)$. For $t = 3$ we find that

$$T(a-1,3) = 1.884298779\cdots = 1.5 + 0.384298779\cdots = 3/2 + T(a,3).$$

For $t = 7$, we find that

$$T(a-1,7) = 1.312864454\cdots = 7/6 + T(a,7).$$

The general result appears to be

$$T(a-1,t) = \frac{t}{t-1} + T(a,t), \qquad t > 1. \tag{40}$$

This appears to depend on the value of a being $(1+\sqrt{5})/2$. Indeed,

$$T(\pi,7) = 2.923976609\cdots \quad \text{and} \quad T(\pi-1,7) = 0.02083333\cdots$$

while $7/6 = 1.16666\cdots$ so that (40) does not hold.

Here is another numerical result that may be of some interest:

$$T(a,a) = \sum_{n=1}^{\infty} \frac{1}{a^{[na]}} = 1.1004127718\cdots. \tag{41}$$

Some of the partial convergents from the continued fraction are:

$$\frac{A_6(a)}{U_6(a)} = \frac{11.09016995}{10.09016995} = 1.099106358\cdots.$$

Note that $(11/10) = 1.1$;

$$\frac{A_7(a)}{U_7(a)} = \frac{50.59674778}{45.97871383} = 1.10043852\cdots.$$

Note that $(50.6/46) = 1.1$;

$$\frac{A_8(a)}{U_8(a)} = \frac{572.2107019}{520.0019205} = 1.10041267\cdots.$$

Note that $(572/520) = 1.1$.

It is interesting to note that $T(a,a)$ is just slightly larger than 1.1, suggesting this as a dominant term.

Here is still another numerical example of (40): Let $e = 2.7182818\cdots$.

$$T(a,e) = 0.438943611\cdots, \quad T(a-1,e) = 2.020920317\cdots, \quad e/(e-1) = 1.581976707\cdots,$$

so that

$$T(a,e) + e/(e-1) = 2.020920318 = T(a-1,e)$$

as closely as we could compute the numbers.

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FIBONACCI SEQUENCES AND ADDITIVE TRIANGLES OF HIGHER ORDER AND DEGREE

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It is often desirable for either ease of calculation or nicety of expression to represent a function in terms of positive integers only. For example, the Binet formula for the N^{th} term of the Fibonacci series quite easily reduces to the expansion:

$$F_N = \frac{\binom{N}{1}5^0 + \binom{N}{3}5^1 + \binom{N}{5}5^2 + \binom{N}{7}5^3 + \cdots}{2^{N-1}} .$$

The last term, of course, will be $5^{(N-1)/2}$ if N is odd, or $N5^{(N-2)/2}$ if N is even.

However, it is well known that the sums of the terms of the ascending diagonals of the Pascal triangle also produce the Fibonacci numbers, thus providing another simple expansion,

$$F_N = \binom{N-1}{0} + \binom{N-2}{1} + \binom{N-3}{2} + \binom{N-4}{3} + \cdots ,$$

the last term being $N/2$ if N is even and 1 if N is odd. (This comes as no surprise since a common method of constructing the triangle is by a two-step additive process.)

	$F_N = 0$	1	1	2	3	5	8	13	21	34	55	$\sum = 2^N =$
$N = 0$	1											1
1	1	1										2
2	1	2	1									4
3	1	3	3	1								8
4	1	4	6	4	1							16
5	1	5	10	10	5	1						32
6	1	6	15	20	15	6	1					64
7	1	7	21	35	35	21	7	1				128
8	1	8	28	56	70	56	28	8	1			256
9	1	9	36	84	126	126	84	36	9	1		512
10												

It is interesting to note that since the sum of the terms of the N^{th} diagonal is equal to F_N, and the sum of the terms of the N^{th} row is equal to 2^N, then the product of those two sums is equal to twice the numerator of the first expansion:

$$2^N\left[\binom{N}{1}5^0 + \binom{N}{3}5^1 + \binom{N}{5}5^2 + \binom{N}{7}5^3 + \cdots\right]$$

or

$$\frac{(1+\sqrt{5})^N - (1-\sqrt{5})^N}{\sqrt{5}} .$$

A Tribonacci triangle, constructed by a three-step additive process, has as the sum of the terms of the N^{th} row 3^N, provides the coefficients of the expansion $(x^0 + x^1 + x^2)^N$, and has the Tribonacci numbers as sums of the ascending diagonals:

$T_N =$	0	1	1	2	4	7	13	24	44							$\sum = 3^N =$
$N = 0$	1															1
1	1	1	1													3
2	1	2	3	2	1											9
3	1	3	6	7	6	3	1									27
4	1	4	10	16	19	16	10	4	1							81
5	1	5	15	30	45	51	45	30	15	5	1					243
6	1	6	21	50	90	126	141	126	90	50	21	6	1			729
7	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1	2187
8	1	8	36	...												

Fig. 2 Tribonacci Triangle

Just as the terms in each row of the Pascal triangle are the binomial coefficients, the terms of each row of the Tribonacci triangle are the trinomial coefficients; that is, if the trinomial expression $(x^0 + x^1 + x^2)$ is raised to a given power such as three,

$$(x^0 + 3x^1 + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6),$$

the coefficients of the resulting terms are the terms of the corresponding row $(N = 3)$ of the Tribonacci triangle, 1, 3, 6, 7, 6, 3, 1.

An easy method of constructing the triangle, rather than actually multiplying the trinomials or using a generating formula for each term (which is simple for Pascal's triangle, but much more complex for higher order triangles), is to simply create each term by adding the three terms immediately above and to the left in the preceding row. For example, the fourth row is derived from the third row as follows:

$N = 3$	1	3	6	7	6	3	1			$\sum = 3^3$
4	1	4	10	16	19	16	10	4	1	3^4

(0 + 0 + 1 = 1)
(0 + 1 + 3 = 4)
(1 + 3 + 6 = 10)
(3 + 6 + 7 = 16)
(6 + 7 + 6 = 19)
(7 + 6 + 3 = 16)
(6 + 3 + 1 = 10)
(3 + 1 + 0 = 4)
(1 + 0 + 0 = 1)

Any additive triangle must begin with the number 1, since any quantity with an exponent of zero is by definition 1. The second row is also composed of ones, since the coefficients of the given trinomial are 1, 1 and 1. From this point onward all terms can easily be calculated by the process described above, which is in effect an arithmetical short-cut through the lengthy process of multiplying polynomials of ascending powers of X.

The ascending diagonals of a four-step or Quadronacci triangle provide the terms of that series in a similar manner:

$Q_N =$	0	1	1	2	4	8	15	29						$\sum = 4^N =$
$N = 0$	1													1
1	1	1	1	1										4
2	1	2	3	4	3	2	1							16
3	1	3	6	10	12	12	10	6	3	1				64
4	1	4	10	20	31	40	44	40	31	20	10	4	1	256
5	1	5	...											
6	1	...												
7														

Fig. 3. Quadranacci Triangle $(x^0 + x^1 + x^2 + x^3)^N$

The terms in the Quadronacci triangle are derived just as are those in the Tribonacci triangle, except that the terms are added in groups of four instead of three.

Indeed, any order of additive triangle can be thus constructed to give similar results. If the order, K, of an additive triangle is defined to be the number of terms in its polynomial base, which is also the number of additive steps necessary to derive each term from the terms of the preceding row, then it may be noted that for an additive triangle of K^{th} order: each row will have $(K-1)$ more terms than the preceding row; the sum of the terms of each row will equal K^N; and the terms in each row will provide the coefficients of the expansion:

$$(x^0 + x^1 + x^2 + x^3 + \cdots + x^{K-1})^N.$$

The sums of the terms of the ascending diagonals of a K^{th} order triangle produce the series:

$$T_N + T_{N+1} + T_{N+2} + T_{N+3} + \cdots + T_{N+K-1} = T_{N+K}$$

$$T_1 = 1, \quad T_2 = 1; \quad (K-1) \geqslant 1.$$

Since for Pascal's triangle $K = 2$ (see Fig. 1), the series produced is $T_N + T_{N+1} = T_{N+2}$, the Fibonacci series. Similarly, in the Quadranacci triangle, since $K = 4$, the series produced is

$$T_N + T_{N+1} + T_{N+2} + T_{N+3} = T_{N+4}, \quad T_3 = 2, \quad T_4 = 4.$$

If an additive triangle is altered by arranging the ascending diagonals as rows, a corresponding alteration results in the series produced by the sums of the terms of the new ascending diagonals. The new series consists of the same number of steps, but of different terms. For example, the Pascal triangle (see Fig. 1), when altered in this manner, now becomes:

$F'_N =$	0	1	1	1	2	3	4	6	9	13	19	$\sum = F_{N+1} =$
$N = 0$	1											1
1	1											1
2	1	1										2
3	1	2										3
4	1	3	1									5
5	1	4	3									8
6	1	5	6	1								13
7	1	6	10	4								21
8	1	7	15	10	1							34
9	1	8	21	20	5							55
10	1	...										

Fig. 4 Second-Degree Pascal Triangle

The ascending diagonals provide the terms of the series

$$T_N + T_{N+2} = T_{N+3} \qquad (T_1 = T_2 = T_3 = 1).$$

A similar treatment of the Tribonacci triangle produces the series

$$T_N + T_{N+2} + T_{N+4} = T_{N+5} \qquad (T_1 = T_2 = T_3 = 1).$$

It can readily be seen that these two new additive series skip every other term.

If the new ascending diagonals are converted into rows a second time for the Tribonacci triangle, the sums of the terms of the resulting diagonals will produce the series

$$T_N + T_{N+3} + T_{N+6} = T_{N+7} \qquad (T_1 = 1,\ T_2 = 1,\ T_3 = 1,\ T_4 = 1).$$

Here the series skips twice between each term.

If the degree, R, of an additive triangle is defined to be the number of times the triangle has been altered by rearranging the ascending diagonals into rows (beginning with $R = 1$ for the triangle in unaltered form), it may then be said that for an additive triangle of K^{th} order and R^{th} degree, the sums of the terms of the ascending diagonals produce the series:

$$T_N + T_{N+R} + T_{N+2R} + T_{N+3R} + \cdots + T_{N+R(K-1)} = T_{N+R(K-1)+1}$$
$$(T_1 = 1,\ T_2 = 1,\ T_3 = 1, \cdots, T_{R+1} = 1) \qquad (K-1) \geqslant 1,\ R \geqslant 1.$$

For the standard Pascal triangle, since $K = 2$ and $R = 1$, the series is the normal Fibonacci series (1, 1, 2, 3, 5, 8, ⋯), where

$$T_N + T_{N+1} = T_{N+2} \qquad (T_1 = T_2 = 1).$$

For a five-step additive triangle, the diagonals of which have been twice rearranged into rows $(K = 5,\ R = 3)$, the series produced is

1, 1, 1, 1, 2, 3, 4, 6, 8, 13, 19, 28, 41, 60, 88, 129, 188, ⋯,

where

$$T_N + T_{N+3} + T_{N+6} + T_{N+9} + T_{N+12} = T_{N+13} \qquad (T_1 = T_2 = T_3 = T_4 = 1).$$

Comparing Fig. 1 with Fig. 4, it will be observed that the terms in each column remained unaltered by a change in the degree of the triangle; each column is merely lowered with respect to the column to its left. Consequently, if the terms of the N^{th} row (and hence the terms of the columns) of a first degree K^{th} order triangle can be expressed in terms of N, then it follows that the N^{th} term of the additive series produced by the sums of the terms of the ascending diagonals of a K^{th} order R^{th} degree triangle can be expressed as a seriers in N and R. For example, the sums of the terms of the ascending diagonals of the Pascal triangle $(K = 2)$ of R^{th} degree produce the series:

$$T_N + T_{N+R} = T_{N+R+1}.$$

The N^{th} term of this series is the expansion

$$T_N = \binom{N-1}{0} + \binom{N-1-R}{1} + \binom{N-1-2R}{2} + \binom{N-1-3R}{3} + \cdots$$

It is easy to conjecture that a general expansion in terms of N, K and R is possible for the N^{th} term of the series generated by the sums of the terms of the ascending diagonals of a triangle of K^{th} order and R^{th} degree, but that requires a treatment much more advanced than is offered here.

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A PROBLEM OF FERMAT AND THE FIBONACCI SEQUENCE

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1. INTRODUCTION

Fermat observed that the numbers 1, 3, 8, 120 have the following property: The product of any two increased by one is a perfect square. Davenport showed that for 1, 3, 8, x to have the same property x must be 120 and that it is impossible to find integers x and y such that the five numbers 1, 3, 8, x, y have this unique property.

In [1] and [2], B. W. Jones extends the problem to polynomials by showing

Theorem 1.1. Let $w^2 - 2(x+1)w + 1 = 0$ have $\alpha(x)$ and $\beta(x)$ as roots. Let $f_k(x) = (\alpha^k - \beta^k)/(\alpha - \beta)$. Let $c_k(x) = 2f_k(x)f_{k+1}(x)$. Then the polynomials x, $x+2$, $c_k(x)$, $c_{k+1}(x)$ have the property that the product of any two plus one is a perfect square.

Any enthusiast of the sequence of Fibonacci numbers would quickly observe that 1, 3, and 8 are terms of that sequence whose subscripts are consecutive even integers. That is, they are respectively F_2, F_4, and F_6. Using the Binet formula it is easy to show that the property enjoyed by 1, 3, and 8 is shared with any three terms of the Fibonacci sequence whose subscripts are consecutive even integers. In fact, we have

(1.1) $$F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$$

(1.2) $$F_{2n}F_{2n+4} + 1 = F_{2n+2}^2$$

and

(1.3) $$F_{2n+2}F_{2n+4} + 1 = F_{2n+3}^2 .$$

One might now ask if there exists an integer x such that $F_{2n}x + 1$, $F_{2n+2}x + 1$ and $F_{2n+4}x + 1$ are perfect squares. In order to show that the answer is yes we proceed as follows. From (1.1) we see that

$$1 = F_{2n+1}^2 - F_{2n}F_{2n+2} = F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3}$$

so that

(1.4) $$4F_{2n}F_{2n+1}F_{2n+2}F_{2n+3} + 1 = (2F_{2n+1}F_{2n+2} - 1)^2 .$$

Replacing n by $n+1$ in (1.1), we have

$$1 = F_{2n+3}^2 + F_{2n+1}F_{2n+4} - F_{2n+3}F_{2n+4} = F_{2n+1}F_{2n+4} - F_{2n+3}F_{2n+2}$$

so that

(1.5) $$4F_{2n+1}F_{2n+2}F_{2n+3}F_{2n+4} + 1 = (2F_{2n+2}F_{2n+3} + 1)^2 .$$

Using the Binet formula show that $F_{2n+2}^2 = F_{2n+1}F_{2n+3} - 1$. Multiply both sides of this equation by $4F_{2n+2}^2$ to obtain

(1.6) $$4F_{2n+1}F_{2n+2}^2F_{2n+3} + 1 = (2F_{2n+2}^2 + 1)^2.$$

Combining (1.1) through (1.6) we have

Theorem 1.2. For $n \geqslant 1$, the four numbers F_{2n}, F_{2n+2}, F_{2n+4}, and $x = 4F_{2n+1}F_{2n+2}F_{2n+3}$ have the property that the product of any two increased by one is a perfect square.

For n respectively 1, 2, and 3 we obtain the quadruples (1, 3, 8, 120), the result of Fermat, (3, 8, 21, 2080) and (8, 21, 55, 37128). The authors conjecture that the value x of Theorem 1.2 is unique.

Although three terms of the Fibonacci sequence whose subscripts are consecutive odd numbers do not have the property of those with even subscripts we do have the following.

Theorem 1.3. Let $n \geq 1$ and $x = 4F_{2n+2}F_{2n+3}F_{2n+4}$ then the numbers F_{2n+1}, F_{2n+3}, F_{2n+5} and x are such that

$$\begin{aligned} F_{2n+1}F_{2n+3} - 1 &= F_{2n+2}^2 \\ F_{2n+1}F_{2n+5} - 1 &= F_{2n+3}^2 \\ F_{2n+3}F_{2n+5} - 1 &= F_{2n+4}^2 \\ F_{2n+1}x + 1 &= (2F_{2n+2}F_{2n+3} + 1)^2 \\ F_{2n+3}x + 1 &= (2F_{2n+3}^2 - 1)^2 \\ F_{2n+5}x + 1 &= (2F_{2n+3}F_{2n+4} - 1)^2 . \end{aligned}$$

Here again the authors conjecture that the value of x in Theorem 1.3 is unique. Letting n respectively be 1, 2, and 3 in Theorem 1.3 we obtain the quadruples (2, 5, 13, 480), (5, 13, 34, 8136), and (13, 34, 89, 157080).

We now turn our attention to several problems which arose in our investigation of the results of Theorems 1.2 and 1.3.

First we wanted to know if there exists an x such that

$$\begin{cases} F_{2n}x - 1 = P^2 \\ F_{2n+2}x - 1 = M^2 \\ F_{2n+4}x - 1 = N^2 . \end{cases}$$

If such an x exists then by eliminating that value between pairs of equations, we have

$$\begin{cases} F_{2n}M^2 - F_{2n+2}P^2 = F_{2n+1} \\ F_{2n}N^2 - F_{2n+4}P^2 = L_{2n+2} \\ F_{2n+2}N^2 - F_{2n+4}M^2 = F_{2n+3} , \end{cases}$$

where L_i is the i^{th} Lucas number. One and only one of F_{2n}, F_{2n+1}, F_{2n+2} is even. Furthermore there exists an integer k such that $n = 3k$, $n = 3k + 1$, or $n = 3k + 2$. If $n = 3k$ then P is odd and the first equation becomes $-1 \equiv -F_{6k+2} \equiv F_{6k+1} \equiv 1 \pmod 4$ which is impossible. If $n = 3k + 1$ the first equation becomes $F_{6k+2}M^2 - F_{6k+4}P^2 = F_{6k+3}$. Since F_{6k+3} is even either M and P are both even or both odd. If both are even then $0 \equiv F_{6k+3} \equiv 2 \pmod 4$ which is impossible. If both are odd then $-2 \equiv F_{6k+2} - F_{6k+4} \equiv 2 \pmod 8$ which is impossible. When $n = 3k + 2$ M is odd and the first equation becomes $3 \equiv F_{6k+4} \equiv F_{6k+5} \equiv 1 \pmod 4$ which is impossible. Hence, the first equation is never solvable. Therefore no x can be found which satisfies the original system of equations. Following an argument similar to that given above it is easy to show that

$$F_{2n}N^2 - F_{2n+4}P^2 = L_{2n+2}$$

is impossible.

Next we tried to determine if more than one solution exists for

$$\text{(A)} \qquad \begin{cases} F_{2n}x + 1 = P^2 \\ F_{2n+2}x + 1 = M^2 \\ F_{2n+4}x + 1 = N^2 \end{cases} .$$

By eliminating the x we see that a necessary condition for a solution is

$$\text{(A}'\text{)} \qquad \begin{cases} F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1} \\ F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2} \\ F_{2n+2}N^2 - F_{2n+4}M^2 = -F_{2n+3} \end{cases} .$$

Recognizing that the first and last equations of (A′) are essentially the same, we conclude that a necessary condition for (A) to be solvable is that there exist a common solution of the Diophantine equations of the form

(1.7) $$F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1}$$

and

(1.8) $$F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2}.$$

Because of the relationships that exist between Diophantine equations of the form $Ax^2 - By^2 = \pm C$, continued fractions, and linear recurrences, we were led to consider the auxiliary polynomials

(1.9) $$w^2 - 2F_{2n+1}w + 1 = 0$$

and

(1.10) $$w^2 - 2F_{2n+2}w + 1 = 0.$$

Using these auxiliary polynomials we will develop a sequence of solutions to (1.7) and (1.8). In this and future developments we need the following lemma all of whose parts can be verified by using the Binet formula or formulas found in [1], [2].

Lemma 1.1. For all $k \geq 1$

(a) $$F_kF_{k+3}^2 - F_{k+2}^3 = (-1)^{k+1}F_{k+1}.$$

(b) $$F_{k+3}F_k^2 - F_{k+1}^3 = (-1)^{k+1}F_{k+2}.$$

(c) $$F_kF_{k+3}^2 - F_{k+4}F_{k+1}^2 = (-1)^{k+1}L_{k+2}.$$

(d) $$F_kL_{k+3}^2 - F_{k+4}L_{k+1}^2 = (-1)^{k+1}L_{k+2}.$$

2. SOLUTIONS OF $F_{2n}M^2 - F_{2n+2}P^2 = -F_{2n+1}$

We first turn our attention to (1.9) whose roots throughout this section are denoted by

$$\alpha = F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} \quad \text{and} \quad \beta = F_{2n+1} - \sqrt{F_{2n+1}^2 - 1}$$

Let $H_m = (\alpha^m - \beta^m)/(\alpha - \beta)$ then $\{H_m\}_{m=0}^{\infty}$ is given by

(2.1) $$H_0 = 0,\ H_1 = 1,\ H_m = 2F_{2n+1}H_{m-1} - H_{m-2}, \qquad m \geq 2$$

and it can be verified that

(2.2) $$H_{m-1}^2 - H_mH_{m-2} = 1.$$

With $M_m = AH_m + BH_{m-1}$, $P_m = A^*H_m + B^*H_{m-1}$ and (2.1), we see that

(2.3) $$\begin{cases} M_m = -M_1H_{m-2} + M_2H_{m-1} \\ P_m = -P_1H_{m-2} + P_2H_{m-1}. \end{cases}$$

Requiring that (M_m, P_m) be a solution of (1.7), provided (M_1, P_1) and (M_2, P_2) are, we have

$$(H_{m-2}^2 + H_{m-1}^2 - 1)F_{2n+1} = 2H_{m-1}H_{m-2}(F_{2n+2}P_1P_2 - F_{2n}M_1M_2)$$

which by using (2.1) and (2.2) becomes

(2.4) $$F_{2n+1}^2 = F_{2n+2}P_1P_2 - F_{2n}M_1M_2.$$

Obviously (±1, ±1) is a solution of (1.7) and Lemma 1.1, part (a), tells us that $(\pm F_{2n+3}, \pm F_{2n+2})$ is also a solution. Checking the sixteen possible combinations respectively for (M_1, P_1) and (M_2, P_2) in (2.4) we find only four solutions which are

$$\{(1,1), (F_{2n+3}, F_{2n+2})\}, \quad \{(1,-1), (F_{2n+3}, -F_{2n+2})\}, \quad \{(-1,1), (-F_{2n+3}, F_{2n+2})\},$$

and

$$\{(-1,-1), \quad (-F_{2n+3}, -F_{2n+2})\}.$$

Each of these four solutions, when used in conjunction with (2.3), gives us a sequence $\{(M_m, P_m)\}_{m=1}^{\infty}$ of solutions to (1.7) which, except for signs, are the same. Because of the exponents in (1.7) we consider only those pairs given by

$$(2.5)\qquad \begin{cases} M_m = 2F_{2n+1}M_{m-1} - M_{m-2} \\ P_m = 2F_{2n+1}P_{m-1} - P_{m-2} \end{cases}$$

where $M_1 = P_1 = 1$, $M_2 = F_{2n+3}$, and $P_2 = F_{2n+2}$.

Noting that the auxiliary polynomial for $\{M_m\}_{m=1}^{\infty}$ and $\{P_m\}_{m=1}^{\infty}$ is $w^2 - 2F_{2n+1}w + 1 = 0$, it is easy to show by standard techniques that

$$(2.6)\qquad \begin{cases} M_m = [(F_{2n}\beta + 1)\alpha^m - (F_{2n}\alpha + 1)\beta^m]/(\alpha - \beta) \\ P_m = [(-F_{2n-1}\beta + 1)\alpha^m - (-F_{2n-1}\alpha + 1)\beta^m]/(\alpha - \beta) \end{cases}$$

Let $D_M = M_m^2 - M_{m-1}M_{m+1}$ be the characteristic of $\{M_m\}_{m=1}^{\infty}$. Using (2.6), it can be shown that

$$(2.7)\qquad D_M = F_{2n+1}F_{2n+2} \quad\text{and}\quad D_P = -F_{2n}F_{2n+1}$$

or

$$(2.8)\qquad F_{2n}D_M = -F_{2n+2}D_P\,.$$

Using

$$(2.9)\qquad \begin{cases} M_{m-2} = 2F_{2n+1}M_{m-1} - M_m \\ P_{m-2} = 2F_{2n+1}P_{m-1} - P_m \end{cases}$$

together with part (b) of Lemma 1.1 it can be verified that $\{(\overline{M}_m, \overline{P}_m)\}_{m=1}^{\infty}$ is another sequence of solutions of (1.7) where

$$(2.10)\qquad \begin{cases} \overline{M}_m = 2F_{2n+1}\overline{M}_{m-1} - \overline{M}_{m-2} \\ \overline{P}_m = 2F_{2n+1}\overline{P}_{m-1} - \overline{P}_{m-2} \end{cases}$$

with

$$\overline{M}_1 = \overline{P}_1 = 1, \quad \overline{M}_2 = -F_{2n} \quad\text{and}\quad \overline{P}_2 = F_{2n-1}\,.$$

The sequences $\{\overline{M}_m\}_{m=1}^{\infty}$ and $\{\overline{P}_m\}_{m=1}^{\infty}$ are called conjugate sequences of $\{M_m\}_{m=1}^{\infty}$ and $\{P_m\}_{m=1}^{\infty}$. Since the auxiliary polynomial for $\{\overline{M}_m\}_{m=1}^{\infty}$ and $\{\overline{P}_m\}_{m=1}^{\infty}$ is

$$w^2 - 2F_{2n+1} + 1 = 0,$$

we see by standard techniques that

$$(2.11)\qquad \begin{cases} \overline{M}_m = [(-F_{2n+3}\beta + 1)\alpha^m - (-F_{2n+3}\alpha + 1)\beta^m]/(\alpha - \beta) \\ \overline{P}_m = [(-F_{2n+2}\beta + 1)\alpha^m - (-F_{2n+2}\alpha + 1)\beta^m]/(\alpha - \beta) \end{cases}$$

$$(2.12)\qquad D_{\overline{M}} = F_{2n+1}F_{2n+2} = D_M$$

and

$$(2.13)\qquad D_{\overline{P}} = -F_{2n}F_{2n+1} = D_P\,.$$

3. SOLUTIONS OF $F_{2n}N^2 - F_{2n+4}P^2 = -L_{2n+2}$

We now turn our attention to (1.10) whose roots throughout this section are denoted by

$$\gamma = F_{2n+2} + \sqrt{F_{2n+2}^2 - 1} \quad\text{and}\quad \delta = F_{2n+2} - \sqrt{F_{2n+2}^2 - 1}\,.$$

Let

$$H_m = (\gamma^m - \delta^m)/(\gamma - \delta)$$

then $\{H_m\}_{m=0}^{\infty}$ is given by

$$(3.1)\qquad H_0 = 0,\quad H_1 = 1,\quad H_m = 2F_{2n+2}H_{m-1} - H_{m-2},\quad m \geqslant 2$$

and it can be shown that

$$(3.2)\qquad H_{m-1}^2 - H_mH_{m-2} = 1.$$

Let (N_1, P_1), (N_2, P_2) be solutions of (1.8). Let $\{(N_m, P_m)\}_{m=3}^{\infty}$ be given by

$$(3.3)\qquad \begin{cases} N_m = -N_1H_{m-2} + N_2H_{m-1} \\ P_m = -P_1H_{m-2} + P_2H_{m-1} \end{cases}.$$

Let (N_m, P_m) be a solution of (1.8). By an argument similar to that given in Section 2, we find

$$(3.4)\qquad L_{2n+2}F_{2n+2} = F_{2n+4}P_1P_2 - F_{2n}N_1N_2.$$

Lemma 1.1, part (c), yields $(\pm F_{2n+3}, \pm F_{2n+1})$ as a solution of (1.8). Obviously $(\pm 1, \pm 1)$ is a solution of (1.8). Letting these pairs be (N_1, P_1) and (N_2, P_2) we obtain sixteen possible values for (3.4). Using

$$(3.5)\qquad L_{2n+2}F_{2n+2} = F_{2n+4}F_{2n+1} + F_{2n}F_{2n+3}$$

it is easy to check that only four solutions exist which, except for signs, are the same. The solution we use gives rise to

$$(3.6)\qquad \begin{cases} N_m = 2F_{2n+2}N_{m-1} - N_{m-2} \\ P_m = 2F_{2n+2}P_{m-1} - P_{m-2} \end{cases},$$

where

$$N_1 = P_1 = 1,\quad N_2 = -F_{2n+3} \quad\text{and}\quad P_2 = F_{2n+1}.$$

Furthermore,

$$(3.7)\qquad \begin{cases} N_m = [(-L_{2n+3}\delta + 1)\gamma^m - (-L_{2n+3}\gamma + 1)\delta^m]/(\gamma - \delta) \\ P_m = [(-L_{2n+1}\delta + 1)\gamma^m - (-L_{2n+1}\gamma + 1)\delta^m]/(\gamma - \delta) \end{cases}$$

$$(3.8)\qquad D_N = F_{2n+4}L_{2n+2},\quad D_P = -F_{2n}L_{2n+2}$$

and

$$(3.9)\qquad F_{2n}D_N = -F_{2n+4}D_P.$$

The conjugate sequences $\{\bar{N}_m\}_{m=1}^{\infty}$ and $\{\bar{P}_m\}_{m=1}^{\infty}$ are given by

$$(3.10)\qquad \begin{cases} \bar{N}_m = 2F_{2n+2}\bar{N}_{m-1} - \bar{N}_{m-2} \\ \bar{P}_m = 2F_{2n+2}\bar{P}_{m-1} - \bar{P}_{m-2} \end{cases}$$

with

$$\bar{N}_1 = \bar{P}_1 = 1,\quad \bar{N}_2 = L_{2n+3},\quad\text{and}\quad \bar{P}_2 = L_{2n+1}.$$

Using Lemma 1.1, part (d), it can be shown that $\{(\bar{N}_m, \bar{P}_m)\}_{m=1}^{\infty}$ is a sequence of solutions to (1.8). Furthermore,

$$(3.11)\qquad \begin{cases} \bar{N}_m = [(F_{2n+3}\delta + 1)\gamma^m - (F_{2n+3}\gamma + 1)\delta^m]/(\gamma - \delta) \\ \bar{P}_m = [(-F_{2n+1}\delta + 1)\gamma^m - (-F_{2n+1}\gamma + 1)\delta^m]/(\gamma - \delta) \end{cases}$$

$$(3.12)\qquad D_{\bar{N}} = F_{2n+4}L_{2n+2} = D_N$$

and

$$(3.13)\qquad D_{\bar{P}} = -F_{2n}L_{2n+2} = D_P.$$

Although the results of Sections 2 and 3 do not directly give a solution to (A), we can generate an infinite sequence of solutions for each of the equations of (A') by using (2.5), (2.10), (3.6) and (3.10).

4. CONCLUDING REMARKS

By eliminating the x value between pairs of equations we see that a necessary condition for

$$(\text{B})\qquad \begin{cases} F_{2n+1}x + 1 = R^2 \\ F_{2n+3}x + 1 = S^2 \\ F_{2n+5}x + 1 = T^2 \end{cases}$$

or

(C)
$$\begin{cases} F_{2n+1}x - 1 = R^2 \\ F_{2n+3}x - 1 = S^2 \\ F_{2n+5}x - 1 = T^2 \end{cases}$$

to be solvable is

(B′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = -F_{2n+4} \end{cases}$$

or

(C)
$$\begin{cases} F_{2n+1}x - 1 = R^2 \\ F_{2n+3}x - 1 = S^2 \\ F_{2n+5}x - 1 = T^2 \end{cases}$$

to be solvable is

(B′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = -F_{2n+4} \end{cases}$$

or

(C′)
$$\begin{cases} F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2} \\ F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3} \\ F_{2n+3}T^2 - F_{2n+5}S^2 = F_{2n+4} \end{cases}$$

Recognizing the similarity of several of the equations we are led to consider only solutions of Diophantine equations of the form

(4.1)
$$F_{2n+1}S^2 - F_{2n+3}R^2 = \mp F_{2n+2}$$

and

(4.2)
$$F_{2n+1}T^2 - F_{2n+5}R^2 = \mp L_{2n+3}\,.$$

CASE I: $F_{2n+1}S^2 - F_{2n+3}R^2 = \mp F_{2n+2}$.

In this case we consider the auxiliary polynomial

$$w^2 - 2F_{2n+2}w - 1 = 0$$

whose roots are denoted by

$$\epsilon = F_{2n+2} + \sqrt{F_{2n+2}^2 + 1} \quad \text{and} \quad \sigma = F_{2n+2} - \sqrt{F_{2n+2}^2 + 1}\,.$$

Following the techniques of Section 2 it can be shown that

(4.3)
$$\begin{cases} S_m = 2F_{2n+2}S_{m-1} + S_{m-2} \\ R_m = 2F_{2n+2}R_{m-1} + R_{m-2} \end{cases},$$

with

$$S_1 = R_1 = 1, \quad S_2 = F_{2n+4}, \quad \text{and} \quad R_2 = F_{2n+3},$$

is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2}$$

when m is odd and

$$F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2}$$

when m is even. Furthermore

$$(4.4)\qquad \begin{cases} S_m = [(-F_{2n+1}\sigma + 1)\epsilon^m - (-F_{2n+1}\epsilon + 1)\sigma^m]/(\epsilon - \sigma) \\ R_m = [(F_{2n}\sigma + 1)\epsilon^m - (F_{2n}\epsilon + 1)\sigma^m]/(\epsilon - \sigma) \end{cases}$$

$$(4.5)\qquad D_S = (-1)^m F_{2n+3}F_{2n+2} \qquad D_R = (-1)^{m+1}F_{2n+1}F_{2n+2}$$

and

$$(4.6)\qquad F_{2n+3}D_R = -F_{2n+1}D_S .$$

The conjugate sequences $\{\overline{S}_m\}_{m=1}^{\infty}$ and $\{\overline{R}_m\}_{m=1}^{\infty}$ are given by

$$(4.7)\qquad \begin{cases} \overline{S}_m = -2F_{2n+2}\overline{S}_{m-1} + \overline{S}_{m-2} \\ \overline{R}_m = -2F_{2n+2}\overline{R}_{m-1} + \overline{R}_{m-2} \end{cases}$$

with

$$\overline{S}_1 = \overline{R}_1 = 1, \qquad \overline{S}_2 = F_{2n+1} \quad\text{and}\quad \overline{R}_2 = -F_{2n} .$$

When m is odd, $(\overline{S}_m, \overline{R}_m)$ is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = -F_{2n+2}$$

while it is a solution of

$$F_{2n+1}S^2 - F_{2n+3}R^2 = F_{2n+2}$$

when m is even.

Furthermore

$$(4.8)\qquad \begin{cases} \overline{S}_m = [(F_{2n+4}\epsilon + 1)(-\sigma)^m - (F_{2n+4}\sigma + 1)(-\epsilon)^m]/(\epsilon - \sigma) \\ \overline{R}_m = [(F_{2n+3}\epsilon + 1)(-\sigma)^m - (F_{2n+3}\sigma + 1)(-\epsilon)^m]/(\epsilon - \sigma) \end{cases}$$

$$(4.9)\qquad D_{\overline{S}} = D_S = (-1)^m F_{2n+3}F_{2n+2}$$

and

$$(4.10)\qquad D_{\overline{R}} = D_R = (-1)^{m+1}F_{2n+1}F_{2n+2} .$$

CASE II: $F_{2n+1}T^2 - F_{2n+5}R^2 = \mp L_{2n+3}$.

In this case we consider the auxiliary polynomial $w^2 - 2F_{2n+3}w - 1 = 0$ whose roots are

$$\psi = F_{2n+3} + \sqrt{F_{2n+3}^2 + 1} \quad\text{and}\quad \xi = F_{2n+3} - \sqrt{F_{2n+3}^2 + 1} .$$

Following the techniques of Section 2, it can be shown that

$$(4.11)\qquad \begin{cases} T_m = 2F_{2n+3}T_{m-1} + T_{m-2} \\ R_m = 2F_{2n+3}R_{m-1} + R_{m-2} \end{cases},$$

with

$$T_1 = R_1 = 1, \qquad T_2 = -F_{2n+4}, \quad\text{and}\quad R_2 = F_{2n+2},$$

is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3}$$

when m is odd and

$$F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3}$$

when m is even. Furthermore

$$(4.12)\qquad \begin{cases} T_m = [(L_{2n+4}\xi + 1)\psi^m = (L_{2n+4}\psi + 1)\xi^m]/(\psi - \xi) \\ R_m = [(L_{2n+2}\xi + 1)\psi^m = (L_{2n+2}\psi + 1)\xi^m]/(\psi - \xi) \end{cases}$$

(4.13) $$D_T = (-1)^m F_{2n+5} L_{2n+3}, \qquad D_R = (-1)^{m+1} F_{2n+1} L_{2n+3}$$

and

(4.14) $$F_{2n+1} D_T = -F_{2n+5} D_R\ .$$

The conjugate sequences $\{\bar{T}_m\}_{m=1}^{\infty}$ and $\{\bar{R}_m\}_{m=1}^{\infty}$ are given by

(4.15) $$\begin{cases} \bar{T}_m = -2F_{2n+3}\bar{T}_{m-1} + \bar{T}_{m-2} \\ \bar{R}_m = -2F_{2n+3}\bar{R}_{m-1} + \bar{R}_{m-2} \end{cases}$$

with

$$\bar{T}_1 = \bar{R}_1 = 1, \qquad \bar{T}_2 = -L_{2n+4} \quad \text{and} \quad \bar{R}_2 = -L_{2n+2}\,.$$

When m is odd $(\bar{T}_m, \bar{R}_m)$ is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = -L_{2n+3}$$

while it is a solution of

$$F_{2n+1}T^2 - F_{2n+5}R^2 = L_{2n+3}$$

when m is even. Furthermore

(4.8) $$\begin{cases} \bar{T}_m = [(-F_{2n+4}\psi + 1)(-\xi)^m - (-F_{2n+4}\xi + 1)(-\psi)^m]/(\psi - \xi) \\ \bar{R}_m = [(F_{2n+2}\psi + 1)(-\xi)^m - (F_{2n+2}\xi + 1)(-\psi)^m]/(\psi - \xi) \end{cases}$$

(4.9) $$D_{\bar{T}} = D_T = (-1)^m F_{2n+5} L_{2n+3}$$

and

(4.10) $$D_{\bar{R}} = D_R = (-1)^{m+1} F_{2n+1} L_{2n+3}\,.$$

In closing, we observe that if you choose $m = 3$ in (2.5) and (3.6) you obtain

$$M_3 = 2F_{2n+1}F_{2n+3} - 1 = 2F_{2n+2}^2 + 1, \qquad P_3 = 2F_{2n+1}F_{2n+2} - 1, \quad \text{and} \quad N_3 = -2F_{2n+2}F_{2n+3} - 1$$

which are equivalent to the values in (1.6), (1.4), and (1.5). Letting $m = 3$ in (4.3) and (4.11) you obtain

$$S_3 = 2F_{2n+2}F_{2n+4} + 1 = 2F_{2n+3}^2 - 1, \qquad R_3 = 2F_{2n+2}F_{2n+3} + 1, \quad \text{and} \quad T_3 = -2F_{2n+3}F_{2n+4} + 1$$

which are equivalent to the values in Theorem 1.3.

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MORE REDUCED AMICABLE PAIRS

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INTRODUCTION

Perfect, amicable and sociable numbers are the fixed points of the arithmetic function L and its iterates. $L(n) = \sigma(n) - n$, where σ is the sum of divisors function. Recently, there has been interest in fixed points of functions L_+, L_-, $L_\pm(n) = L(n) \pm 1$, and their iterates. Jerrard and Temperley [1] studied the fixed points of L_+ and L_-. Lal and Forbes [2] conducted a computer search for fixed points of $(L_-)^2$. For earlier references to L_-, see the bibliography in [2].

We conducted computer searches for fixed points n of iterates of L_- and L_+. Fixed points occur in sets where the number of elements in the set equals the power of L_- or L_+ in question.

In §1, we describe the results of L_-. The previous work of Lal and Forbes [2] discovered the fixed points of $(L_-)^2$ with one element of each pair $\leqslant 10^5$. We extend the results to $n \leqslant 10^6$. No other types of fixed points were discovered

The results for L_+ are described in §2. Again only pairs were found.

1. THE FUNCTION L_-

Lal and Forbes [2] discovered nine pairs of fixed points of $(L_-)^2$, where at least one element was less than, or equal to, 10^5. In fact, for all pairs, both numbers were less than 10^5.

If n is a fixed point of $(L_-)^k$: i.e. $(L_-)^k(n) = n$, for $k \geqslant 1$, then $(L_-)(n), (L_-)^2(n), \cdots, (L_-)^{k-1}(n)$ are also fixed points of $(L_-)^k$. Thus fixed points of iterates of L_- occur in sets of cardinality k. For at least one integer n in such a set, $L_-(n) > n$. Thus it suffices to search among n with $L_-(n) > n$.

A computer search was conducted using an IBM 370, Model 135. All natural numbers n, $0 < n \leqslant 10^6$, $L_-(n) > n$ were examined. The iterates $(L_-)^k(n)$, $1 \leqslant k \leqslant 50$, were calculated. Calculation of iterates stopped if

(a) $$(L_-)^m(n) = 0, \qquad 1 \leqslant m \leqslant 50;$$

or

(b) $$(L_-)^{m+k}(n) = (L_-)^m(n), \qquad 1 \leqslant k \leqslant 4.$$

The printout was to list all iterates calculated in case (b) and for the case where $(L_-)^{50}(n) > 0$. The program would discover any sets of fixed points arising from iterating L_- on integers n, $10^5 < n \leqslant 10^6$. We found six new pairs of reduced amicable numbers. There were no sets of fixed points of cardinality other than 2. Of the twelve numbers, only one exceeded 10^6. The pairs are listed in Table 1 with the prime factorization.

Table 1
L_-

(a)	186615	=	3(2)5·11·13·29
	206504	=	2(3)83·311
(b)	196664	=	2(3)13·31·61
	219975	=	3·5(2)7·419
(c)	199760	=	2(4)5·11·227
	309135	=	3·5·37·557
(d)	266000	=	2(4)5(3)7·19
	507759	=	3·7·24179
(e)	312620	=	2(2)5·7(2)11·29
	549219	=	3·11(2)17·89
(f)	587460	=	2(2)3·5·9791
	1057595	=	5·7·11·41·67

2. THE FUNCTION L_+

Jerrard and Temperley [2] ran a search for fixed points of L_+. Every power of 2 is a fixed point. But they discovered no others. They did not examine fixed points of iterates of L_+.

We call natural numbers *augmented perfect numbers, augmented amicable numbers* and *augmented sociable numbers* as they are fixed points of L_+, of $(L_+)^2$ or of $(L_+)^k$, $k > 2$. The names are suggested by the name *reduced amicable numbers* for fixed points of $(L_-)^2$ as used in [2].

A computer search for fixed points was run in the range, $0 < n \leqslant 10^6$. No augmented perfect numbers, no augmented sociable numbers were found. Eleven pairs of augmented amicable numbers were found. They are listed in Table 2. Two pairs have both elements over 10^6. They arose from iterating L_+ on 532512, 844740 and 869176.

TABLE 2

L_+

(a)	6160	=	2(4)5·7·11
	11697	=	3·7·557
(b)	12220	=	2(2)5·13·47
	16005	=	3·5·11·97
(c)	23500	=	2(2)5(3)47
	28917	=	3(5)7·17
(d)	68908	=	2(2)7·23·107
	76245	=	3·5·13·17·23
(e)	249424	=	2(4)7·17·131
	339825	=	3·5(2)23·197
(f)	425500	=	2(2)5(3)23·37
	570405	=	3·5·11·3457
(g)	434784	=	2(5)3·7·647
	871585	=	5·11·13·23·53
(h)	649990	=	2·5·11·19·311
	697851	=	3(2)7·11·19·53
(i)	660825	=	3(3)5(2)11·89
	678376	=	2(3)19·4463
(j)	1017856	=	2(11)7·71
	1340865	=	3(2)5·83·359
(k)	1077336	=	2(3)3(2)13·1151
	2067625	=	5(3)7·17·139

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FIBONACCI SEQUENCE AND EXTREMAL STOCHASTIC MATRICES

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ABSTRACT

The purpose of this note is to exhibit an interesting connection between the Fibonacci sequence and a class of three-dimensional extremal plane stochastic matrices.

1. A *three-dimensional matrix* of order n is a real valued function A defined on the set $J_{3,n}$ of points (i,j,k), where $1 \leqslant i,j, \ k \leqslant n$. It is customary to say that the value of this function at the point (i,j,k) is an entry of the matrix and to denote it by $a_{i,j,k}$. A *plane* is defined to be a subset of which results when one of i,j,k is held fixed. A plane is called a *row, column, or horizontal plane* according as to whether i, j, or k is held fixed. A matrix A is *plane stochastic* if its entries are non-negative numbers and the sum of the entries in each plane is equal to one. If A and B are plane stochastic matrices of order n and $0 \leqslant \alpha \leqslant 1$, then $\alpha A + (1-\alpha)B$ is also a plane stochastic matrix. Thus the collection of all plane stochastic matrices of order n is a convex set. The extreme points of this convex set are called *extremal plane stochastic matrices.* Jurkat and Ryser [3] have raised the question of determining all the extremal stochastic matrices. This appears to be a very difficult problem. One class of extremal plane stochastic matrices is formed by the permutation matrices (with precisely one non-zero entry in each plane). But unfortunately very little is known about other extremal matrices.

In what follows we construct a class of extremal plane stochastic matrices using Fibonacci numbers.

2. If A is a three-dimensional matrix of order n, then the *pattern* of A is the set of all points (i,j,k) for which $a_{ijk} \neq 0$. Jurkat and Ryser [3] observed that a *plane Stochastic matrix* A *is extremal if and only if there is no plane stochastic matrix other than* A *which has the same pattern as* A.

We are now ready to construct a class of extremal stochastic matrices. Let $S_n \subseteq J_{3,n}$ $(n = 1, 2, \cdots)$ be the pattern defined as follows: The points $(n, n, n-1)$ and $(1, n, n)$ belong to S_n. In addition $(i,j,k) \in S_n$ whenever one of the following holds:

(i) $\quad i = j = k \quad \text{for} \quad i = 1, \cdots, n-1;$

(ii) $\quad i = j+1 \quad \text{and} \quad k = n, \quad \text{for} \quad i = 2, \cdots, n;$

(iii) $\quad i = j-1 = k+1, \quad \text{for} \quad i = 2, \cdots, n-1.$

The matrix T_n in Figure 1 is a two-dimensional representation of this pattern. The (i,j)-entry of T_n equals k if and only if $(i,j,k) \in T_n$. Fortunately, T_n is such that $(i,j,k), (i,j,k') \in T_n$ implies $k = k'$.

The (two-dimensional) matrix B_n indicated in Figure 2 represents a three-dimensional matrix A_n of order n. If $(i,j,k) \in S_n$, then $a_{ijk} = b_{ij}$; if $(i,j,k) \notin S_n$, then $a_{ijk} = 0$. The sequence $f_1, f_2, f_3, f_4, \cdots$ is the Fibonacci sequence $1, 1, 2, 3, \cdots$.

Theorem. The matrix A_n is an extremal plane stochastic matrix of order n.

Proof. We observe that all the indicated entries of B_n are positive so that the pattern of A_n is S_n. In order to verify that A_n is plane stochastic, we compute the plane sums of A_n. First we observe that the row

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$$\begin{bmatrix} 1 & & & & & & & & n \\ n & 2 & 1 & & & & & & \\ & n & 3 & 2 & & & & & \\ & & \cdot & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & n & n-3 & n-4 & & \\ & & & & & n & n-2 & n-3 & \\ & & & & & & n & n-1 & n-2 \\ & & & & & & & n & n-1 \end{bmatrix}$$

Figure 1

$$B_n = \frac{1}{f_{n+2}-1}\begin{bmatrix} f_{n+2}-f_3 & & & & & & & f_1 \\ f_2 & f_{n+2}-f_4 & f_3-1 & & & & & \\ & f_3 & f_{n+2}-f_5 & f_4-1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & f_{n-3} & f_{n+2}-f_{n-1} & f_{n-2}-1 & \\ & & & & & f_{n-2} & f_{n+2}-f_n & f_{n-1}-1 \\ & & & & & & f_{n-1} & f_{n+2}-f_{n+1} \quad f_n-1 \\ & & & & & & & f_n \quad f_{n+1}-1 \end{bmatrix}$$

Figure 2

and column plane sums of A_n are the row and column sums of B_n. It is more convenient to verify that the row and column sums of $C_n = (f_{n+2} - 1)B_n$ are all $f_{n+2} - 1$. The first row sum of C_n equals

$$f_{n+2} - f_3 + f_1 = f_{n+2} - 1;$$

the last row sum of C_n is clearly $f_{n+2} - 1$. The i^{th} row sum of C_n, $2 \leqslant i \leqslant n-1$, equals

$$f_i + (f_{n+2} - f_{i+2}) + (f_{i+1} - 1) = f_{n+2} - 1.$$

The first, second, and last column sums of C_n equal $f_{n+2} - 1$. The j^{th} column sum of C_n, $3 \leqslant j \leqslant n-1$, equals

$$(f_j - 1) + (f_{n+2} - f_{j+2}) + f_{j+1} = f_{n+2} - 1.$$

Thus far we have verified that the row and column plane sums of A_n are one. Now we compute the horizontal plane sums of A_n. The k^{th} horizontal plane sum of A_n, $1 \leqslant k \leqslant n-1$, equals

$$\frac{(f_{n+2} - f_{k+2}) + (f_{k+2} - 1)}{f_{n+2} - 1} = 1.$$

The n^{th} horizontal plane sum equals

$$\frac{f_1 + f_2 + \cdots + f_n}{f_{n+2} - 1} = 1.$$

Thus A_n is a plane stochastic matrix.

To show that A_n is extremal, it suffices to show that there is no other plane stochastic matrix with pattern S_n.

Let E be a plane stochastic matrix with pattern S_n. Let $a > 0$ be the $(1,n,n)$-entry of E. Let $G = [g_{ij}]$ be the two-dimensional representation of E. We claim that G has the form indicated in Figure 3.

$$G_n = \begin{bmatrix} 1-(f_3-1)a & & & & & f_1 a \\ f_2 a & 1-(f_4-1)a & (f_3-1)a & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & f_{n-1}a & 1-(f_{n+1}-1)a & (f_n-1)a \\ & & & & f_n a & 1-f_n a \end{bmatrix}$$

Figure 3

We verify this by using step-by-step the fact that the plane sums of E are one. For the first row of G we have $g_{1n} = a = f_1 a$, and from $g_{11} + g_{1n} = 1$ we conclude that

$$g_{11} = 1 - a = 1 - (f_3 - 1)a.$$

For the second row of G we have $g_{21} = 1 - g_{11} = a = f_2 a$. Since the first horizontal plane sum is one,

$$g_{23} = 1 - g_{11} = a = (f_3 - 1)a.$$

Finally, $g_{22} = 1 - (f_4 - 1)a$ can be determined by considering the second row sum of G. Suppose it has been verified that the first i rows of G are as claimed for some i with $2 \leqslant i \leqslant n-1$. Considering the i^{th} column sum of G we compute that

$$g_{i+1,i} = 1 - g_{i-1,i} - g_{ii} = 1 - (f_i - 1)a - (1 - (f_{i+2} - 1)a) = f_{i+1}a.$$

Considering the i^{th} horizontal plane sum of E, we compute that

$$g_{i+1,i+2} = 1 - g_{ii} = 1 - (1 - (f_{i+2} - 1)a) = (f_{i+2} - 1)a.$$

Finally, considering the $(i+1)^{st}$ row sum of G, we compute that

$$g_{i+1,i+1} = 1 - g_{i+1,i} - g_{i+1,i+2} = 1 - f_{i+1}a - (f_{i+2} - 1)a = 1 - (f_{i+3} - 1)a.$$

Thus by induction we have verified our claim up to and including the $(n-1)^{st}$ row of G. By considering the $(n-1)^{st}$ column sum and n^{th} row sum of g in turn, we calculate that

$$g_{n,n-1} = 1 - g_{n-2,n-1} - g_{n-1,n-1} = 1 - (f_{n-1} - 1)a - (1 - (f_{n+2} - 1)a) = f_n a,$$

and

$$g_{n,n} = 1 - g_{n,n-1} = 1 - f_n a.$$

Thus our claim is verified.

Now by considering the n^{th} horizontal plane sum of E, we see that a is uniquely determined. Hence E is unique, and thus $E = A_n$. This completes the proof of the theorem.

Constructions for other extremal matrices and additional properties of planar stochastic matrices can be found in [1, 2].

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BELL'S IMPERFECT PERFECT NUMBERS

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A perfect number is one which, like 6 or 28, is the sum of its aliquot parts. Euclid proved that $2^{p-1}(2^p - 1)$ is perfect when $(2^p - 1)$ is a prime; and it has been shown that this formula includes all perfect numbers which are even.[1]

In Eric Temple Bell's fascinating book[2], the seven perfect numbers after 6 are listed as follows:

28, 496, 8128, 130816, 2096128, 33550336, 8589869056.

Checking these numbers by Euclid's formula, I found that

$$2^8(2^9 - 1) = 256 \times 511 = 130816$$

and

$$2^{10}(2^{11} - 1) = 1024 \times 2047 = 2096128.$$

However, $511 = 7 \times 73$; and $2047 = 23 \times 89$.

Inasmuch as 511 and 2047 are not primes, it follows that 130816 and 2096128 are not perfect numbers, and they should not have been included in Bell's list.

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[1] *Encyclopedia Britannica,* Eleventh Edition, Vol. 19, page 863.
[2] *The Last Problem*, Simon and Schuster, New York, 1961, page 12.

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FIXED POINTS OF CERTAIN ARITHMETIC FUNCTIONS

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INTRODUCTION

Perfect, amicable and sociable numbers are fixed points of the arithemetic function L and its iterates.

$$L(n) = \sigma(n) - n,$$

where σ is the sum of divisor's function. Recently there have been investigations into functions differing from L by 1; i.e., functions L_+, L_-, defined by $L_\pm(n) = L(n) \pm 1$. Jerrard and Temperley [1] studied the existence of fixed points of L_+ and L_-. Lal and Forbes [2] conducted a computer search for fixed points of $(L_-)^2$. For earlier references to L_-, see the bibliography in [2].

We consider the analogous situation using σ^*, the sum of unitary divisors function. Let L_+^*, L_-^*, be arithmetic functions defined by

$$L_\pm^*(n) = \sigma^*(n) - n \pm 1.$$

In § 1, we prove, using parity arguments, that L_-^* has no fixed points.

Fixed points of iterates of L_-^* arise in sets where the number of elements in the set is equal to the power of L_-^* in question. In each such set there is at least one natural number n such that $L_-^*(n) > n$. In § 2, we consider conditions n must satisfy to enjoy the inequality and how the inequality acts under multiplication. In particular if n is even, it is divisible by at least three primes; if odd, by five. If n enjoys the inequality, any multiply by a relatively prime factor does so. There is a bound on the highest power of n that satisfies the inequality. Further if n does not enjoy the inequality, there are bounds on the prime powers multiplying n which will yield the inequality.

In § 3, we describe a computer search for fixed points of iterates of L_-^*. In the range $0 < n < 110{,}000$, we found no sets of fixed points.

In § 4, we summarize theory and a computer search for L_+^*. Again, by a parity argument, we prove there are only two fixed points, 1, 2, for L_+^*. The computer search, $0 < n < 100{,}000$, found no fixed points of iterates of L_+^*.

1. THE FUNCTION L_-^*

Let $\underline{\underline{Z}}$ be the integers and $\underline{\underline{N}}$ the natural numbers. The arithmetic function $\sigma^*: \underline{\underline{N}} \to \underline{\underline{Z}}$ is the sum of unitary divisors function. For $n = \prod p^\alpha$,

$$\sigma^*(n) = \prod (1 + p^\alpha). \tag{1}$$

Define new arithmetic functions L^*, L_-^*, L_+^*, by

$$L^*(n) = \sigma^*(n) - n; \tag{2}$$

$$L_-^*(n) = L^*(n) - 1; \tag{3}$$

$$L_+^*(n) = L^*(n) + 1. \tag{4}$$

We are interested in the fixed points of L_-^*, L_+^*, and their iterates. For L_-^*, we call these fixed points *reduced unitary perfect* and *reduced unitary amicable* and *sociable numbers.* For L_+^*, *augmented unitary perfect, amicable* and *sociable numbers.* The names are suggested by [2]. We consider L_-^* in detail.

Note that $L_-^*(1) = -1$ and $L_+^*(2) = 0$.

Lemma 1. For $n \in N$, $L_-^*(n) = 0$ if, and only if, $n = p^\alpha$, p a prime, $p \geq 2$, $\alpha \geq 1$.

Proof. If $n = p^\alpha$, $L_-^*(n) = \sigma^*(n) - (1+n) = (1+n) - (1+n) = 0.$
If $n \neq p^\alpha$, $n = p^\alpha m$, p a prime, $\alpha \geqslant 1$; $(p,m) = 1$, $m > 1$. Then $L_-^*(n) = \sigma^*(n) - (1+n)$.
But
$$\sigma^*(n) = \sigma^*(p^\alpha)\sigma^*(m)$$
$$\geqslant (1+p^\alpha)(1+m) = 1 + p^\alpha + m + p^\alpha m$$
$$L_-^*(n) \geqslant p^\alpha + m > 0.$$

Lemma 2. For $n \in N$, $L_-^*(n)$ has the same parity as n if, and only if, $n = 2^\alpha$, $\alpha \geqslant 0$.

Proof. If $n = 2^0 = 1$; $L_-^*(1) = -1$.
If $n = 2^\alpha > 1$; $L_-^*(2^\alpha) = 0$ by Lemma 1.
If $n = \Pi p^\alpha$, all p odd primes:
$L_-^*(n) = \Pi(1+p^\alpha) - (1+n)$.
Both terms on the right are even, so $L_-^*(n)$ is even.
If $n = 2^\beta \Pi p^\alpha$, all p odd primes:
$L_-^*(n) = (1+2^\beta)\Pi(1+p^\alpha) - (1+n)$. The terms on the right are of opposite parity; and $L_-^*(n)$ is odd.

Theorem A. L_-^* has no fixed points.

Proof. By Lemma 2, need only consider cases where parity of n and $L_-^*(n)$ are the same. By Lemma 1, in these cases $L_-^*(n) < n$.

2. THE INEQUALITY $L_-^*(n) > n$

If $(L_-^*)^k(n) = n$, $k \geqslant 2$, then the images $L_-^*(n), (L_-^*)^2(n), \cdots, (L_-^*)^{k-1}$ are also fixed points of $(L_-^*)^k$. Thus fixed points of $(L_-^*)^k$, $k \geqslant 2$, arise in sets of k distinct points. In each set of fixed points, there is at least one integer m such that $L_-^*(m) > m$. The following propositions deal with the behavior of this inequality.

Proposition 3. If $k = nm$, $(n,m) = 1$, then $L_-^*(k) > L_-^*(n)L_-^*(m) + nL_-^*(m) + mL_-^*(n)$.

Proof.
$$L_-^*(k) = \sigma^*(k) - (1+k) = \sigma^*(n)\sigma^*(m) - (1+mn)$$
$$= [L_-^*(n) + (1+n)][L_-^*(m) + (1+m)] - (1+mn)$$
$$= L^*(n)L_-^*(m) + (1+n)L_-^*(m) + (1+m) + (1+m)$$
$$> L_-^*(n) + m + n$$
$$> L_-^*(n)L_-^*(m) + nL_-^*(m) + mL_-^*(n).$$

Corollary 4. If $k = nm$, $(n,m) = 1$, then
$$L_-^*(k) < L_-^*(m)L_-^*(n) + (1+m)L_-^*(n) + (1+n)L_-^*(m) + (1+m)(1+n).$$

For $k = 210 = 2\cdot3\cdot5\cdot7$, let $m = 6$, $n = 35$.
The inequality of the proposition is:
$$365 = L_-^*(210) > L_-^*(6)L_-^*(35) + 6L_-^*(35) + 35L_-^*(6) = 5\cdot12 + 5\cdot35 + 12\cdot6 = 307.$$
The corollary inequality, with these numbers is:
$$365 < 5\cdot12 + 5\cdot35 + 12\cdot6 = 307.$$
Relative primeness is necessary in the proposition. For $k = 90$, $m = 6$, $n = 15$, the required inequality is
$$80 > 5\cdot8 + 6\cdot8 + 15\cdot5 = 163$$
which is false.

Proposition 5. Let $m = p^\alpha n$, $(p,n) = 1$. If $L_-^*(n) > n$, then $L_-^*(m) > m$.

Proof.
$$L_-^*(n) > n \Rightarrow \sigma^*(n) - (1+n) > n \Rightarrow \sigma^*(n) - n > n$$
$$L_-^*(m) = \sigma^*(p^\alpha n) - (1+p^\alpha n) = (1+p^\alpha)\sigma^*(n) - 1 - p^\alpha n$$
$$= \sigma^*(n) - (1+n) + p^\alpha[\sigma^*(n) - n] + n = L_-^*(n) + p^\alpha[\sigma^*(n) - n] + n$$
$$> p^\alpha n + 2n > m.$$

If $m = p^\alpha n$, $(p,n) = p$, the result does not necessarily follow.

$$L_-^*(30) = 41 > 30; \qquad L_-^*(60) = 59 < 60.$$

The inequality fails.

Proposition 6. Let $n = \prod p^\alpha$. If $L_-^*(n) > n$, then

$$\prod\left(1 + \frac{1}{p^\alpha}\right) > 2.$$

Proof. The inequality $\sigma^*(n) - (1+n) > n$ is also written as $\sigma^*(n) > 2n + 1$. Then

$$\frac{\sigma^*(n)}{n} > 2 + \frac{1}{n} > 2.$$

But

$$\frac{\sigma^*(n)}{n} = \frac{\prod(1+p^\alpha)}{\prod p^\alpha} = \prod\left(1 + \frac{1}{p^\alpha}\right).$$

Corollary 7. Let $n = \prod p^\alpha$. If $L_-^*(n) > n$, then

$$\prod\left(1 + \frac{1}{p}\right) > 2.$$

The results in Proposition 6 and Corollary 7 are necessary conditions but not sufficient. The inequalities are first satisfied by an integer n with exponents α equal to 1. Among even integers, $n = 30 = 2\cdot 3\cdot 5$ is the smallest. $L_-^*(30) = 41$. Among odd integers, $n = 15015 = 3\cdot 5\cdot 7\cdot 11\cdot 13$ is the smallest. $L_-^*(15015) = 17240$.

Corollary 8. If n is even and $L_-^*(n) > n$, then n is divisible by at least three distinct primes.

Corollary 9. If n is odd and $L_-^*(n) > n$, then n is divisible by at least five distinct primes.

Proposition 10. For each natural number n, there is a natural number $t = t(n)$ such that for $k \geqslant t$,

$$L_-^*(n^k) < n^k.$$

Proof. Let $n = \prod p^\alpha$; and $\underline{\underline{Q}}$, the rationals. The function $\theta : \underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{Q}}$ defined by

$$\theta(n,s) = \prod\left[1 + \left(\frac{1}{p^\alpha}\right)^s\right]$$

is, for fixed n, a decreasing function of s bounded below by 1. Let t be the first integer such that

$$\theta(n,t) \leqslant 2.$$

Proposition 11. Let $m = p^\alpha n$, $(p,n) = 1$ with $L_-^*(n) < n$. If $L_-^*(m) > m$, then

$$\frac{2n}{2n - \sigma^*(n)} > p^\alpha.$$

Proof.

$$L_-^*(n) < n \Rightarrow \sigma^*(n) - (1+n) < n \Rightarrow \sigma^*(n) - 1 < 2n$$

$$L_-^*(m) > m \Rightarrow \sigma^*(p^\alpha n) - (1 + p^\alpha n) > p^\alpha n$$

$$\Rightarrow \sigma^*(n) - 1 > 2p^\alpha n - p^\alpha \sigma^*(n).$$

Then using the first inequality

$$2n > \sigma^*(n) - 1 > p^\alpha[2n - \sigma^*(n)] \quad \text{and} \quad \frac{2n}{2n - \sigma^*(n)} > p.$$

This proposition sets the bound on the multiples of a natural number n, $L_-^*(n) < n$, which enjoy the reverse inequality. For $n = 10$, $\sigma^*(n) = 18$,

$$\frac{2n}{2n - \sigma^*(n)} = \frac{20}{20 - 18} = \frac{20}{2} = 10.$$

The possible p^α are $3, 7, 3^2$, $L_-^*(30) = 41$. $L_-^*(70) = 73$; $L_-^*(90) = 89$.

Note that for 90,

$$\Pi\left(1+\frac{1}{p^\alpha}\right) = 2.$$

3. THE COMPUTER SEARCH FOR FIXED POINTS OF ITERATES OF L_-^*

A computer search for natural numbers n such that $L_-^*(n) > n,\ 0 < n < 110{,}000,$ was run on an IBM 370, model 135. For each such natural number n, the iterated values $(L_-^*)^k(n)$ were calculated, until 0 was reached. The program allowed fifty iterations. The values under the iterations were printed out. The process thus identifies any set of fixed points with an element less than, or equal to, 110,000.

Table 1 summarizes the results. There were no fixed points discovered. For all integers examined, iterations of L_-^* eventually reached zero. For each n, the order of n is the first integer k such that $(L_-^*)^k(n) = 0$. For each value of the order, we list the first occurrence of the order and the frequency, or count, of the natural numbers with that order. The first natural number examined was 30; the last, 109,986. Note that there are no numbers of order 3 in the internal. Further the count of odd orders is relatively small. This can be explained, in part, by the few odd numbers under 110,000 satisfying $L_-^*(n) > n$. Recall that the first such is 15015. A total of 7697 numbers were examined.

It is desirable to develop upper and lower bounds for the first integer which is fixed under $(L_-^*)^2$.

Table 1
L_-^*

Order	1st	Frequency
2	30	2203
3	—	0
4	66	1947
5	1596	10
6	294	1733
7	3290	38
8	854	1133
9	1190	46
10	4854	446
11	15890	20
12	14630	121
13	21945	8
14	38570	5
15	76670	4
16	104510	1
17	107030	1

4. THE FUNCTION L_+^*

In this section we examine L_+^*. For any natural number n, $L_+^*(n) = L_-^*(n) + 2$. So

$$L_-^*(n) > n \Rightarrow L_+^*(n) > n.$$

$$L_+^*(1) = 1; \quad L_+^*(2) = 2.$$

Thus L_+^* has at least two fixed points.

Proofs of the following results parallel those above.

Lemma 12. For $n \in \underline{N}$, $L_+^*(n) = 2$ if, and only if, $n = p^\alpha$, p a prime, $\alpha \geqslant 1$.

Lemma 13. For n in $\underline{N}$, $L_+^*(n)$ has the same parity as n if, and only if, $n = 2^\alpha$; $\alpha > 0$.

Theorem B. L_+^* has exactly two fixed points, 1 and 2.

Proposition 14. If $k = mn$, $(m,n) = 1$ then $L_+^*(k) < L_+^*(m)L_+^*(n) + mL_+^*(n) + nL_+^*(m)$.

Proposition 15. Let $m = p^\alpha n$, $(p,n) = 1$. If $L_+^*(n) > n$, then $L_+^*(m) > m$.

Proposition 16. Let $n = \Pi p^\alpha$. If

$$\Pi\left(1+\frac{1}{p^\alpha}\right) > 2,$$

then $L_+^*(n) > n$.

Corollary 17. Let $n = \Pi p^\alpha$. If

$$\Pi\left(1+\frac{1}{p}\right) > 2,$$

then $L_+^*(n) > n$.

Recall that in Proposition 6 and Corollary 7, the condition

$$\Pi\left(1+\frac{1}{p}\right) > \Pi\left(1+\frac{1}{p^\alpha}\right) > 2$$

was necessary but not sufficient. Here it is sufficient but not necessary.

Proposition 18. For each natural number n, there is a natural number $t = t(n)$ such that for $k > t$, $L_+^*(n^k) < n^k$.

Proof. Using the notation of the proof of Proposition 10, it suffices to let t be the first integer such that

$$\theta(n,t) \leqslant \frac{3}{2}.$$

Proposition 19. Let $m = p^\alpha n$, $(p,n) = 1$ with $L_+^*(n) < n$. If $L_+^*(m) > m$, then

$$\frac{2n}{2n - \sigma^*(n)} > p^n.$$

A computer search for natural numbers n such that $L_+^*(n) > n$ was run, $0 < n < 100{,}000$. The iteration values were calculated and printed up to fifty iterations. The end value for iterations is 2 rather than 0. The search would have discovered any set of fixed points of an iterate of L_+^* where one element of the set was less than, or equal to, 100,000. None were found. The results are in Table 2. The organization is as for Table 1.

Table 2

Order	L_+^* 1st	Frequency
1	1,2	2
2	6	2020
3	82005	2
4	42	1274
5	498	27
6	78	1213
7	2530	144
8	402	1154
9	10650	72
10	1518	698
11	19635	19
12	2470	289
13	15015	2
14	10158	85
15	–	0
16	57030	15
17	84315	1

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2. M. Lal and A. Forbes, "A Note on Chowla's Function," *Math. Comp.,* 25 (1971), pp. 923–925. MR 45-6737.

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FIBONACCI ASSOCIATION RESEARCH CONFERENCE

October 22, 1977

Host: MENLO COLLEGE
(El Camino Real)
Menlo Park, Calif.

9:30	Opening Remarks: V. E. Hoggatt, Jr.
9:40 to 10:20	ARITHMETIC DIVISORS OF HIGHER ORDER Krishna Alladi (UCLA)
10:30 to 11:00	FIBONACCI CHROMATOLOGY or HOW TO COLOR YOUR RABBIT Marjorie Johnson (Wilcox High School)
11:10 to 12:00	PARTITIONS SUMS OF GENERALIZED PASCAL'S TRIANGLES (A Second Report) Claudia Smith (SJSU) or Verner E. Hoggatt, Jr. (SJSU)
12:00	LUNCH – to each his own
1:30 to 2:10	APPLICATIONS OF CERTAIN BASIC SEQUENCE CONVOLUTIONS TO FIBONACCI NUMBERS Rodney Hansen (MSU, Bozeman, Montana)
2:20 to 3:00	GAMBLER'S RUIN AND FIBONACCI NUMBERS Fred Stern (SJSU)
3:10 to 3:30	ENUMERATION OF CHESS GAME ENDINGS George Ledin, Jr. (USF)
3:40 to 4:20	PRIMER ON STERN'S DIATOMIC SEQUENCE Bob and Tina Giuli (SJSU)

GENERALIZED QUATERNIONS OF HIGHER ORDER

I. L. IAKIN
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In a previous article [2], we conjectured that the idea of a quaternion with quaternion components could be extended to include higher order quaternions. The purpose of this article is to investigate this concept and to obtain further generalizations of the results in [2].

PROPERTIES

Firstly, to be able to denote higher order quaternions, we need to introduce an operator notation. Thus for λ a positive integer we define the *quaternions of order* λ, after λ operations, as:

(1) $$\Omega^\lambda W_n = \Omega(\Omega(\Omega\cdots(\Omega W_n)\cdots)) = \Omega^{\lambda-1}W_n + i\Omega^{\lambda-1}W_{n+1} + j\Omega^{\lambda-1}W_{n+2} + k\Omega^{\lambda-1}W_{n+3}$$

(2) $$\Delta^\lambda W_n = \Delta(\Delta(\Delta\cdots(\Delta W_n)\cdots)) = \Delta^{\lambda-1}W_n + iq\Delta^{\lambda-1}W_{n-1} + jq^2\Delta^{\lambda-1}W_{n-2} + kq^3\Delta^{\lambda-1}W_{n-3}$$

where we also define

(3) $$\Omega^0 W_n = W_n,\quad \Delta^0 W_n = W_n,\qquad \Omega^1 W_n = \Omega W_n,\quad \Delta^1 W_n = \Delta W_n$$

and the quaternion vectors i,j,k have the following properties

(4) $$i^2 = j^2 = k^2 = -1,\quad ij = -ji = k,\qquad jk = -kj = i,\quad ki = -ik = j$$

and where from Horadam [1] we have that for integers $a,b,p,q,$

(5a) $$W_n \equiv W_n(a,b;p,q)$$

$$W_n = pW_{n-1} - qW_{n-2} \quad \text{for } n \geqslant 2$$

$$W_0 = a,\quad W_1 = b$$

(5b) $$U_n \equiv W_n(1,p;p,q)$$

(5c) $$V_n \equiv W_n(2,p;p,q)$$

(5d) $$\epsilon = pab - qa^2 - b^2\ .$$

Thus we see from (1), (3), (5b) and (5c) that for $\lambda = 1$ we obtain the special cases 1(a), 1(b) and 1(c) of [2], while $\lambda = 2$ gives us 7(a) and 7(b) of [2]. Equation (11) of [2] is obtained from (2) and (3) for $\lambda = 1$.

We can now combine the operators Ω and Δ to define quaternions of the type $\Omega\Delta W_n$ and $\Delta\Omega W_n$, i.e.,

(6) $$\Omega\Delta W_n = \Omega(\Delta W_n) = \Delta W_n + i\Delta W_{n+1} + j\Delta W_{n+2} + k\Delta W_{n+3}$$

(7) $$\Delta\Omega W_n = \Delta(\Omega W_n) = \Omega W_n + iq\Omega W_{n-1} + jq^2\Omega W_{n-2} + kq^3\Omega W_{n-3}\ .$$

If we expand (6) and (7) we see that

$$\Omega\Delta W_n \neq \Delta\Omega W_n\ .$$

Since quaternion vector multiplication is non-commutative we also know that

$$i\cdot\Omega W_m\Omega W_n \neq \Omega W_m\cdot i\cdot\Omega W_n \neq \Omega W_m\Omega W_n\cdot i\ .$$

To overcome some of the problems associated with calculations involving higher order quaternions, resulting from the failure of the commutative law for quaternion multiplication, we introduce two new operators, namely Ω^* and Δ^*. We thus define

(8) $$\Omega^*\Omega W_n = \Omega^*(\Omega W_n) = \Omega W_n + \Omega W_{n+1}\cdot i + \Omega W_{n+2}\cdot j + \Omega W_{n+3}\cdot k$$

(9) $$\Delta^*\Delta W_n = \Delta^*(\Delta W_n) = \Delta W_n + q\Delta W_{n-1}\cdot i + q^2\Delta W_{n-2}\cdot j + q^3\Delta W_{n-3}\cdot k$$

Hence we see that the operators Ω^* and Δ^* are the same as the operators Ω and Δ except that they create quaternions by post-multiplication of the quaternion vectors. Obviously

$$\Omega^* W_n = \Omega W_n$$

and since, say

$$\Delta\Omega^* W_n = \Delta(\Omega^* W_n) = \Delta\Omega W_n$$

it follows that the star operators are only meaningful when applied to the L.H.S. of quaternions of order $\geqslant 1$.

If we now expand the R.H.S. of Eq. (8) we see that we have result 8(a) of [2], i.e.,

(10) $$\Omega^*\Omega W_n = \Omega^2 W_n .$$

Similarly from (9) it follows that

(11) $$\Delta^*\Delta_n = \Delta^2 W_n .$$

We leave it to the reader to show, by expanding, that

(12) $$\Omega\Delta W_n = \Delta^*\Omega W_n$$

(13) $$\Delta\Omega W_n = \Omega^*\Delta W_n$$

and to prove the associative laws for the operators, e.g.,

(14) $$(\Omega\Delta)\Omega W_n = \Omega(\Delta\Omega)W_n, \qquad (\Delta\Omega)\Delta W_n = \Delta(\Omega\Delta)W_n .$$

Now for μ a positive integer we know

$$\begin{aligned} \Omega^*\Omega^\mu W_n &= \Omega^*(\Omega\Omega^{\mu-1})W_n && \text{(by (1))} \\ &= (\Omega^*\Omega)\Omega^{\mu-1}W_n && \text{(Associative laws)} \\ &= \Omega^2\Omega^{\mu-1}W_n && \text{(by (10))} \end{aligned}$$

(15) $$\Omega^*\Omega^\mu W_n = \Omega^{\mu+1}W_n \qquad \text{(by (1))}.$$

If we replace Ω by Δ in the above proof we obtain the result

(16) $$\Delta^*\Delta^\mu W_n = \Delta^{\mu+1}W_n .$$

Next, induction on μ produces the results

(17) $$\Omega\Delta^\mu W_n = (\Delta^*)^\mu\Omega W_n$$

(18) $$\Delta\Omega^\mu W_n = (\Omega^*)^\mu\Delta W_n .$$

Using the above results and induction on λ we can prove the following

(19) $$\Lambda^\lambda\Omega W_n = \Omega^*\Delta^\lambda W_n$$

(20) $$\Omega^\lambda\Delta W_n = \Delta^*\Omega^\lambda W_n$$

(21) $$(\Omega^*)^\lambda\Omega^\mu W_n = \Omega^{\lambda+\mu}W_n$$

(22) $$(\Delta^*)^\lambda\Delta^\mu W_n = \Delta^{\lambda+\mu}W_n$$

(23) $$\Omega^\lambda\Delta^\mu W_n = (\Delta^*)^\mu\Omega^\lambda W_n$$

(24) $$\Delta^\lambda\Omega^\mu W_n = (\Omega^*)^\mu\Delta^\lambda W_n$$

EXTENDED GENERALIZED RESULTS

In this section we extend some of the identities given in Iakin [2]. We commence by proving the generalization of Eq. (10) of [2].

(25) $$\Omega^\lambda U_{-n} = -q^{-n+1}\Delta^\lambda U_{n-2} .$$

Proof. We prove this result using induction on λ. For $\lambda = 1$ we have Eq. (10) of [2]. Assume the result is true for $\lambda = h$, i.e.,

$$\Omega^h U_{-n} = -q^{-n+1}\Delta^h U_{n-2}\ .$$

Now for $\lambda = h+1$ we have from (1)

$$\Omega^{h+1}U_{-n} = \Omega^h U_{-n} + i\Omega^h U_{-n+1} + j\Omega^h U_{-n+2} + k\Omega^h U_{-n+3}$$

which becomes on using the assumption

$$\begin{aligned}\Omega^{h+1}U_{-n} &= -q^{-n+1}\Delta^h U_{n-2} - iq^{-n+2}\Delta^h U_{n-3} - jq^{-n+3}\Delta^h U_{n-4} - kq^{-n+4}\Delta^h U_{n-5}\\ &= -q^{-n+1}(\Delta^h U_{n-2} + iq\Delta^h U_{n-3} + jq^2\Delta^h U_{n-4} + kq^3\Delta^h U_{n-5}) = -q^{-n+1}\Delta^{h+1}U_{n-2}\ \text{(by (2))}\end{aligned}$$

Since the result holds for $\lambda = 1$ and is true for $\lambda = h+1$ providing it is true for $\lambda = h$, then by the principle of induction the result holds for all positive integer values of λ.

Similarly we can show by induction on λ that

(26) $$\Omega^\lambda V_{-n} = q^{-n}\Delta^\lambda V_n$$

(27) $$\Omega^\lambda W_{-n} = q^{-n}(a\Delta^\lambda U_n - b\Delta^\lambda U_{n-1})$$

(28) $$\Omega^\lambda W_{n+r} + q^r\Omega^\lambda W_{n-r} = V_r\Omega^\lambda W_n\ .$$

After a lengthy proof using induction on $\lambda + \mu$ we have

(29) $$\Omega^{\lambda+\mu}W_{m+n} = \Omega^\lambda W_m\Omega^\mu U_n - q\Omega^\lambda W_{m-1}\Omega^\mu U_{n-1}$$

for which we obtain the special cases

(30) $$2\Omega^{\lambda+\mu}U_{m+n-1} = \Omega^\lambda U_{m-1}\Omega^\mu V_n + \Omega^\lambda V_m\Omega^\mu U_{n-1}$$

(31) $$2\Omega^{\lambda+\mu}V_{m+n} = \Omega^\lambda V_m\Omega^\mu V_n + d^2\Omega^\lambda U_{m-1}\Omega^\mu U_{n-1}$$

where $d^2 = p^2 - 4q$.

If we again use induction on $\lambda + \mu$ we can arrive at

(32) $$\Omega^\lambda W_m\Omega^\mu W_n - q\Omega^\lambda W_{m-1}\Omega^\mu W_{n-1} = a\Omega^{\lambda+\mu}W_{m+n} + (b-pa)\Omega^{\lambda+\mu}W_{m+n-1}$$

(33) $$\Omega^\lambda W_{m+1}\Omega^\mu W_{n+1} - q^2\Omega^\lambda W_{m-1}\Omega^\mu W_{n-1} = b\Omega^{\lambda+\mu}W_{m+n+1} + (b-pa)q\Omega^{\lambda+\mu}W_{m+n-1}\ .$$

Now letting $m = n$ and $\lambda = \mu$ in both (32) and (33) gives us

(34) $$(\Omega^\lambda W_n)^2 - q(\Omega^\lambda W_{n-1})^2 = a\Omega^{2\lambda}W_{2n} + (b-pa)\Omega^{2\lambda}W_{2n-1}$$

(35) $$(\Omega^\lambda W_{n+1})^2 - (q\Omega^\lambda W_{n-1})^2 = b\Omega^{2\lambda}W_{2n+1} + (b-pa)q\Omega^{2\lambda}W_{2n-1}\ .$$

Note that Eqs. (28), (29), (30), (31), (32), (33), (34) and (35) give, as special cases, Eqs. 24(a) and (b), 22(a) and (b), 21(a) and (b), (23), (16) and (17), (20), (18) and (19), respectively.

We now list a set of identities whose proofs we omit due to their length and repetitiveness. We leave it to the reader to prove by induction the following results:

$$W_{n-r}\Omega^{\lambda+\mu}W_{n+r+t}$$

(36) $$= \Omega^\lambda W_n\Omega^\mu W_{n+t} + \epsilon q^{n-r}\Omega^\lambda U_{r-1}\Omega^\mu U_{r+t-1}$$

(37) $$= \Omega^\lambda W_{n+t}\Omega^\mu W_n + \epsilon q^{n-r}\Omega^\lambda U_{r+t-1}\Omega^\mu U_{r-1}$$

$$\Omega^\lambda W_{n-r}\Omega^\mu W_{n+r+t}$$

(38) $$= W_n\Omega^{\lambda+\mu}W_{n+t} + \epsilon q^{n-r}\Delta^\lambda U_{r-1}\Omega^\mu U_{r+t-1}$$

(39) $$= \Omega^\lambda W_n\Omega^\mu W_{n+t} + \epsilon q^{n-r}U_{r-1}\Delta^\lambda\Omega^\mu U_{r+t-1}$$

(40) $$= \Omega^\lambda W_{n+t}\Omega^\mu W_n + \epsilon q^{n-r}U_{r+t-1}\Delta^\lambda\Omega^\mu U_{r-1}$$

(41) $$= W_{n+t}\Omega^{\lambda+\mu}W_n + \epsilon q^{n-r}\Delta^\lambda U_{r+t-1}\Omega^\mu U_{r-1}$$

and finally $\Omega^\lambda W_{m-r+t}\Omega^\mu W_{n+r+s}$

(42) $$= W_{n-r+t}\Omega^{\lambda+\mu}W_{m+r+s} + \epsilon q^{n-r}\Delta^\lambda U_{n-m-1}\Omega^\mu U_{2r-t+s-1}$$

(43) $$= \Omega^\lambda W_{n-r+t}\Omega^\mu W_{m+r+s} + \epsilon q^{n-r}U_{n-m-1}\Delta^\lambda\Omega^\mu U_{2r-t+s-1}$$

(44) $$= \Omega^{\lambda} W_{m+r+s} \Omega^{\mu} W_{n-r+t} + \epsilon q^{n-r} U_{2r-t+s-1} \Delta^{\lambda} \Omega^{\mu} U_{n-m-1}$$

(45) $$= W_{m+r+s} \Omega^{\lambda+\mu} W_{n-r+t} + \epsilon q^{n-r} \Delta^{\lambda} U_{2r-t+s-1} \Omega^{\mu} U_{n-m-1}\ .$$

Putting $\lambda = 1$ and $\mu = 1$ in (36), (39) and (40) gives us, respectively, (13), (26) and (27) of [2], while letting $\lambda = 1$, $\mu = 2$ in (39) and (40) gives, respectively, 28(a) and (b). If, however, we let $t = 0$, $s = 0$, $\lambda = 1$ and $\mu = 1$ in (43) we have as a special case result (29) of [2].

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2. A.L. Iakin, "Generalized Quaternions with Quaternion Components," *The Fibonacci Quarterly,* 1974 preprint.

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LETTER TO THE EDITOR

16 September 1977

Dear Professor Hoggatt:

In a recent article with Claudia Smith (*The Fibonacci Quarterly,* Vol. 14, No. 4, p. 343), you referred to the question whether a prime p and its square p^2 can have the same rank of apparition in the Fibonacci sequence, and mentioned that Wall (1960) had tested primes up to 10,000 and not found any with this property.

I have recently extended this search and found that no prime up to 1,000,000 (one million) has this property.

My computations in fact test the Lucas sequence for the property

(1) $$L_p \equiv 1 \pmod{p^2} \qquad p = \text{prime}\ .$$

For $p > 5$ this is easily shown to be a necessary and sufficient condition for p and p^2 to have the same rank of apparition in the Fibonacci sequence, because of the identity

(2) $$(L_p - 1)(L_p + 1) = 5F_{p-1}F_{p+1}\ .$$

So far I have shown that the congruence (1) does not hold for any prime less than one million; I hope to extend the search further at a later date.

You may wish to publish these results in *The Fibonacci Quarterly.*

Yours sincerely,

s/ Dr. L. A. G. Dresel
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ON THE CONNECTION BETWEEN THE RANK OF APPARITION OF A PRIME p IN FIBONACCI SEQUENCE AND THE FIBONACCI PRIMITIVE ROOTS

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Let the number g be a primitive root (mod p). If $x = g$ satisfies the congruence

(1) $$x^2 \equiv x + 1 \pmod{p},$$

then the g is called *Fibonacci Primitive Root.* D. Shanks [1] and D. Shanks, L. Taylor [2] dealt with the condition of existence of the Fibonacci Primitive Roots and they proved a few theorems.

In connection with the Fibonacci sequence

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \cdots (F_n = F_{n-1} + F_{n-2}),$$

the natural number $a = a(p)$ is called by D. Jarden [3] the rank of apparition of p if F_a is divisible by p and F_i is not divisible by p in case $i < a$.

In this article, we shall deal with the connections between the rank of apparition of prime p in the Fibonacci sequence and the Fibonacci Primitive Roots. We shall prove the following theorems:

Theorem 1. The congruence $x^2 \equiv x + 1 \pmod{p}$ is solvable if and only if $p - 1$ is divisible by $a(p)$ or $p=5$.

Theorem 2. If $p = 10k \pm 1$ is a prime number and there exist two Fibonacci Primitive Roots (mod p) or no Fibonacci Primitive Root exists, then $a(p) < p - 1$.

Theorem 3. There is exactly one Fibonacci Primitive Root (mod p) if and only if $a(p) = p - 1$ or $p = 5$.

D. Shanks [1] proved that if (1) is solvable then $p = 5$ or $p = 10k \pm 1$. But D. H. Halton [4] proved that $F_{p-(5/p)}$ is divisible by the prime p $(p \neq 5)$, where $(5/p)$ is the Legendre's symbol, and it is well known that if $p = 10k \pm 1$, then $(5/p) = 1$, therefore F_{p-1} is divisible by p. So it is enough to prove the following lemma for the verification of the first part of Theorem 1:

Lemma 1. If F_n is divisible by number p, then n is divisible by the rank $a(p)$ of p and if n is divisible by $a(p)$, then F_n is divisible by p.

Let $a = a(p)$ and $n = a \cdot m + r$, where $0 < r \leqslant a$. N. N. Vorobev proved that $F_{b+c} = F_b \cdot F_{c+1} + F_{b-1} \cdot F_c$ ([5], p. 10) and $F_{b \cdot c}$ is divisible by F_b for every natural numbers b and c ([5], p. 29). For this reason p is a divisor of $F_{a \cdot m}$ and if p is a divisor of F_n, then

$$F_n = F_{am+r} = F_{am} \cdot F_{r+1} + F_{am-1} \cdot F_r \equiv F_{am-1} \cdot F_r \equiv 0 \pmod{p}.$$

But F_{am} and F_{am-1} are neighboring numbers of the Fibonacci sequence, for that very reason F_{am-1} is prime to F_{am} (see [5], p. 30). So p is not a divisor of F_{am-1} because p is a divisor of F_{am} and $F_r \equiv 0 \pmod{p}$. From this follows $a = r$ by reason of definition of $a = a(p)$. Thus n is divisible by $a = a(p)$. Should it happen that n is divisible by $a = a(p)$, then, due to the Vorobev's previous theorem, F_n is divisible by $F_{a(p)}$ and so F_n is divisible by p, too. With this we proved the Lemma 1 and from this follows the proof of the first part of Theorem 1.

If $p - 1$ is divisible by $a(p)$, then by reason of Lemma 1 F_{p-1} is divisible by p. From this follows that $(5/p) = 1$. Namely, if $(5/p) = -1$, then F_{p+1} is divisible by p, too, and so $F_p = F_{p+1} - F_{p-1}$ also is divisible by p. But F_i and F_{i+1} are relatively prime for every natural number i, therefore $(5/p) = 1$. From this follows that $p = 10k \pm 1$ and so the congruence (1) is solvable. It completes the proof of Theorem 1.

Before the proof of Theorem 2 and Theorem 3, we shall prove two Lemmas.

Lemma 2. If the congruence $x^2 \equiv x + 1 \pmod{p}$ is solvable, $p \neq 5$ and the two roots are g_1, g_2, then $g_1 - g_2 \not\equiv 0 \pmod{p}$.

Lemma 3. If x is a solution of the congruence $x^2 \equiv x+1 \pmod p$, then

$$x^k \equiv F_k \cdot x + F_{k-1} \pmod p$$

for every natural exponent k.

Let us prove the Lemma 2 first. If (1) has solutions g_1 and g_2, then $g_1 + g_2 \equiv 1 \pmod p$ and $g_2 \equiv 1 - g_1 \pmod p$, respectively (see [1]). Let us suppose that $g_1 - g_2 \equiv 0 \pmod p$, that is

$$(2) \qquad 2g_1 \equiv 1 \pmod p.$$

g_1 is a root of (1) and so $g_1^2 \equiv g_1 + 1 \pmod p$. Let us add this congruence to (2). Then we get $g_1^2 + g_1 \equiv 2 \pmod p$ and from this $4g_1^2 + 4g_1 \equiv 8 \pmod p$ and $(2g_1+1)^2 \equiv 9 \pmod p$, respectively. From the later congruence we get $2g_1 + 1 \equiv 3$ or $2g_1 + 1 \equiv -3 \pmod p$ and from these subtracting the congruence (2) we get $5 \equiv 0$ or $1 \equiv 0 \pmod p$. But these are true only if $p = 5$ according to $p > 1$, which proves the Lemma 2. In case $p = 5$ really $g_1 - g_2 \equiv 0 \pmod p$ because $g_1 = 3$ and $g_2 \equiv 1 - g_1 = -2 \equiv g_1 \pmod 5$.

We shall carry out the proof of the Lemma 3 by induction over k. In the cases $k = 1$ and $k = 2$ indeed

$$x = x + 0 = F_1 \cdot x + F_0 \quad \text{and} \quad x^2 \equiv x + 1 = F_2 \cdot x + F_1 \pmod p.$$

After this if $k > 2$ and the statement is true for exponents smaller than k, then

$$\begin{aligned} x^k &= x^2 \cdot x^{k-2} \equiv (x+1) \cdot x^{k-2} = x^{k-1} + x^{k-2} \equiv F_{k-1} \cdot x + F_{k-2} + F_{k-2} \cdot x + F_{k-3} \\ &= F_k \cdot x + F_{k-1} \pmod p \end{aligned}$$

which proves Lemma 3.

Now let us suppose that $p = 10k \pm 1$. In this case by reason of [1], (1) is solvable. If both roots g_1 and g_2 are primitive $(\text{mod } p)$, then, according to Lemma 3 (using for every primitive root $g^{(p-1)/2} \equiv -1 \pmod p$)

$$g_1^{(p-1)/2} \equiv F_{(p-1)/2} \cdot g_1 + F_{(p-1)/2-1} \equiv -1 \pmod p$$

$$g_2^{(p-1)/2} \equiv F_{(p-1)/2} \cdot g_2 + F_{(p-1)/2-1} \equiv -1 \pmod p.$$

The difference of the congruences gives: $F_{(p-1)/2}(g_1 - g_2) \equiv 0 \pmod p$ and from this follows by reason of Lemma 2 $(p \neq 5)$ that $F_{(p-1)/2} \equiv 0 \pmod p$ which by reason of Lemma 1 proves the first part of Theorem 2.

Let us suppose that neither g_1 nor g_2 is primitive root $(\text{mod } p)$ and g_1 belongs to the exponent n_1 and g_2 belongs to the n_2. Then n_1 and n_2 are divisors of $p - 1$ $(n_1, n_2 < p - 1)$ and

$$(3) \qquad g_1^{n_1} \equiv 1, \qquad g_2^{n_2} \equiv 1 \pmod p.$$

If $n_1 = n_2 = n$, then similarly to the previous cases, using the congruences (3) and the Lemma 3, we get $F_n \equiv 0 \pmod p$ and so n is divisible by $a(p)$, that is $a(p) \leqslant n < p - 1$.

If $n_1 \neq n_2$, then we can suppose that $n_1 > n_2$. But $g_1 \cdot g_2 \equiv -1 \pmod p$ (see [1]) for this reason, using the congruences (3),

$$g_1^{n_2} \equiv g_1^{n_2} \cdot g_2^{n_2} = (g_1 \cdot g_2)^{n_2} \equiv (-1)^{n_2} \pmod p.$$

g_1 belongs to the exponent $n_1 \pmod p$ and $n_1 > n_2$, so n_2 must be an odd number and $g_1^{n_2} \equiv -1 \pmod p$. In this case $g_1^{2n_2} \equiv 1 \pmod p$ and from this follows that n_1 is a divisor of $2n_2$. But $2n_2 < 2n_1$, so $n_1 = 2n_2$ and

$$(4) \qquad g_2^{n_1} = g_2^{2n_2} \equiv 1 \pmod p.$$

According to congruences (3) and (4) and Lemma 3:

$$g_1^{n_1} \equiv F_{n_1} \cdot g_1 + F_{n_1 - 1} \equiv 1 \pmod p$$

$$g_2^{n_2} \equiv F_{n_1} \cdot g_2 + F_{n_1 - 1} \equiv 1 \pmod p$$

and from this we get, as above, using Lemma 2: $F_{n_1} = 0 \pmod p$ and so by reason of Lemma 1 n_1 is divisible by $a(p)$. Thus $a(p) \leqslant n_1 < p - 1$ which proves the second part of Theorem 2.

Theorem 3 is true in the case $p = 5$ (see [1]), therefore we can suppose further on that $p \neq 5$. Let it be now $a(p) = p - 1$. In this case, by reason of Theorem 1, the congruence (1) is solvable. There is exactly one primitive root $(\text{mod } p)$ between the two roots because otherwise $a(p) < p - 1$ would follow according to Theorem 2.

And conversely, if congruence (1) is solvable, one of the roots is primitive and the other is not (mod p), that is $n_1 = p - 1$, then it follows from the foregoing that $n_2 = (p-1)/2$ and n_2 is an odd number. Let us suppose that $a(p) < p - 1$ as opposed to Theorem 3 and let q denote the least common multiple of n_2 and $a(p)$. q is divisible by n_2 and $a(p)$ therefore

$$1 \equiv g_2^q \equiv F_q \cdot g_2 + F_{q-1} \equiv F_{q-1} \pmod p$$

(because p is a divisor of F_q according to Lemma 1). Using this congruence we get

$$g_1^q \equiv F_q \cdot g_1 + F_{q-1} \equiv F_{q-1} \equiv 1 \pmod p.$$

From this follows $q = p - 1$ because n_2 and $a(p)$ are divisors of $p - 1$ and g_1 is a primitive root (mod p). But $q = p - 1$ is an even number and n_2 is odd, therefore $a(p)$ is an even number.

N. N. Vorobev proved that for every natural number n $F_{n+1}^2 = F_n \cdot F_{n+2} + (-1)^n$ ([5], p. 11). Let us use this equation for the case $n = a(p) - 1$, it derives

$$F_{a(p)-1} \cdot F_{a(p)+1} = F_{a(p)}^2 + (-1)^{a(p)}.$$

But, on the one hand, $a(p)$ is an even number, on the other hand,

$$F_{a(p)+1} = F_{a(p)} + F_{a(p)-1} \equiv F_{a(p)-1} \pmod p,$$

so $F_{a(p)-1}^2 \equiv 1 \pmod p$. From this $F_{a(p)-1} \equiv -1 \pmod p$ follows because in the case $F_{a(p)-1} \equiv 1 \pmod p$ g_1 cannot be a primitive root (mod p) by reason of

$$g_1^{a(p)} \equiv F_{a(p)} \cdot g_1 + F_{a(p)-1} \equiv F_{a(p)-1} \equiv 1 \pmod p \tag{5}$$

and the condition $a(p) < p - 1$. From the latter it follows that, similarly to (5),

$$g_1^{a(p)} \equiv -1 \pmod p.$$

But g_1 is a primitive root (mod p) and $a(p) < p - 1$ therefore $a(p) = (p-1)/2 = n_2$. However, $a(p) = n_2$ is impossible, for $a(p)$ is even and n_2 is an odd number, so the condition $a(p) < p - 1$ is impossible. Then $a(p) = p - 1$, which completes the proof of Theorem 3.

The reverse of Theorem 2 follows from Theorem 3 as well: If the congruence $x^2 \equiv x + 1 \pmod p$ is solvable and $a(p) < p - 1$, then both roots are primitive (mod p) or neither of them is primitive. The point is that in this case, by reason of Theorem 3, there cannot be exactly one primitive root.

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GENERALIZED QUATERNIONS WITH QUATERNION COMPONENTS

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The relations connecting generalized Fibonacci Quaternions obtained by Iyer [3], following earlier work by Horadam [2], together with the recent article by Swamy [4], prompted this note on further generalized quaternions, as well as an investigation of generalized quaternions whose components are quaternions.

Following the ideas of [3] we define

1. (a) $$P_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$$

(b) $$Q_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}$$

(c) $$R_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3},$$

where

(d) $$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k$$
$$jk = -kj = i, \qquad ki = -ik = j$$

and where

2. (a) $$W_n = pW_{n-1} - qW_{n-2} \qquad W_0 = a, \quad W_1 = b$$

(b) $$U_n = pU_{n-1} - qU_{n-2} \qquad U_0 = 1, \quad U_1 = p$$

(c) $$V_n = pV_{n-1} - qV_{n-2} \qquad V_0 = 2, \quad V_1 = p\,.$$

Thus from 1(a) and 2(a) we have that

3. $$P_n = pP_{n-1} - qP_{n-2}\,.$$

Analogous results to equations 2.14 and 2.15 of Horadam [1] are, respectively:

4. $$P_n = aQ_n + (b - pa)Q_{n-1}$$

5. $$R_n = 2Q_n - pQ_{n-1}\,.$$

The conjugate quaternion of P_n is given by

6. $$\overline{P}_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$$

We now define the quaternions T_n and S_n as the quaternions whose components are the quaternions P_n and Q_n, respectively, viz.

7. (a) $$T_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

(b) $$S_n = Q_n + iQ_{n+1} + jQ_{n+2} + kQ_{n+3}$$

which on expanding give

8. (a) $$T_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} + 2iW_{n+1} + 2jW_{n+2} + 2kW_{n+3}$$

and similarly for S_n.

The conjugate for T_n is

9. (a) $$\overline{T}_n = P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}$$

which becomes on expansion

(b) $$\overline{T}_n = W_n + W_{n+2} + W_{n+4} + W_{n+6}$$

so that the conjugate quaternion can be expressed solely in terms of W_n's and is independent of the vectors i, j, k.

Now consider

$$Q_{-n} = U_{-n} + iU_{-n+1} + jU_{-n+2} + kU_{-n+3}\ .$$

Using equation 2.17 of [1] and noting that the result should be

$$U_{-n} = -q^{-n+1}U_{n-2}$$

we obtain

$$Q_{-n} = -q^{-n+1}[U_{n-2} + iqU_{n-3} + jq^2U_{n-4} + kq^3U_{n-5}]$$

10. $$Q_{-n} = -q^{-n+1}Q^*_{n-2}\ ,$$

where we define

11. $$Q^*_n = U_n + iqU_{n-1} + jq^2U_{n-2} + kq^3U_{n-3}\ .$$

Similarly we have that

12. $$Q^*_{-n} = -q^{-n+1}Q_{n-2}\ .$$

Using the above we shall now establish some relations between these quaternions. The first of these is

13. $$P_nP_{n+t} + eq^{n-r}Q_{r-1}Q_{r+t-1} = W_{n-r}T_{n+r+t}\ .$$

The proof for this is lengthy and is left to the reader. A direct proof uses 1(a), 1(b), 7(a) and equation 4.18 of Horadam [1].

Now letting $t = 0$ in equation (13) above we have

14. $$P_n^2 + eq^{n-r}Q^2_{r-1} = W_{n-r}T_{n+r}\ .$$

If we let $r = 1$ in equation (14) we obtain

15. $$eq^{n-1}\sum_{j=0}^{3} U_j^2 = P_n^2 + 2eq^{n-1}Q_0 - W_{n-1}T_{n+1}\ .$$

Another identity is

16. $$aP_{m+n} + (b-pa)P_{m+n-1} = W_mP_n - qW_{m-1}P_{n-1}\ .$$

The proof uses 1(a) and equation 4.1 of Horadam [1].

Further results are

17. $$P_mP_n - qP_{m-1}P_{n-1} = aT_{m+n} + (b-pa)T_{m+n-1} = W_mT_n - qW_{m-1}T_{n-1}\ .$$

For $m = n$ in (17)

18. $$P_n^2 - qP_{n-1}^2 = aT_{2n} + (b-pa)T_{2n-1} = W_nT_n - qW_{n-1}T_{n-1}$$

19. $$P_{n+1}^2 - q^2P_{n-1} = bT_{2n+1} + (b-pa)qT_{2n-1}$$

20. $$bP_{2n+1} + (b-pa)qP_{2n-1} = W_{n+1}P_{n+1} - q^2W_{n-1}P_{n-1}\ .$$

Now from 7(b)

21. (a) $$2S_{m+n-1} = R_nQ_{m-1} + Q_{n-1}R_m$$

(b) $$2Q_{m+n-1} = U_{m-1}R_n + Q_{n-1}V_m = Q_{m-1}V_n + U_{n-1}R_m$$

22. (a) $$P_{n+r} = U_nP_r - qU_{n-1}P_{r-1} = W_nQ_r - qW_{n-1}Q_{r-1}$$

(b) $$T_{n+r} = P_nQ_r - qP_{n-1}Q_{r-1} = U_nT_r - qU_{n-1}T_{r-1} = W_nS_r - qW_{n-1}S_{r-1}$$

23. $$2R_{m+n} = V_mR_n + d^2U_{m-1}Q_{n-1}\ ,$$

where $d^2 = p^2 - 4q$.

24. (a) $$P_{n+r} + q^rP_{n-r} = P_nV_r$$

(b) $$T_{n+r} + q^rT_{n-r} = T_nV_r$$

Now recalling the notation we established in equation (11) we let

25. $$P^*_n = W_n + iqW_{n-1} + jq^2W_{n-2} + kq^3W_{n-3}\ .$$

We are thus able to establish the interesting relations

26. $$P_{n-r}P_{n+r+t} - P_nP_{n+t} = eq^{n-r}U_{r-1}S^*_{r+t-1}$$

27. $$P_{n-r}P_{n+r+t} - P_{n+t}P_n = eq^{n-r}U_{r+t-1}S^*_{r-1}\,.$$

Thus we note the change in the R.H.S. expressions for equations (26) and (27) when the only difference in the L.H.S. is that the elements in the subtracted product term have been commuted. This is to be expected as quaternion multiplication is non-commutative.

Similarly we obtain

28. (a) $$P_{n-r}T_{n+r+t} - P_nT_{n+t} = eq^{n-r}U_{r-1}(S_{r+t-1} + iqS_{r+t-2} + jq^2S_{r+t-3} + kq^3S_{r+t-4})$$

(b) $$P_{n-r}T_{n+r+t} - P_{n+t}T_n = eq^{n-r}U_{r+t-1}(S_{r-1} + iqS_{r-2} + jq^2S_{r-3} + kq^3S_{r-4})$$

29. $$P_{m-r}P_{n+r} - P_{n-r}P_{m+r} = eq^{m-r}U_{n-m-1}S^*_{2r-1}$$

and where $e = pab - qa^2 - b^2$ from equation (2).

At this point it is interesting to note the correlation of the above equations (13), (14), (16), (17), (18), (19), (20), (21), (22), (23), (24) and {(26), (27), (28)} with equations 4.18, 4.5, 4.1, 4.1, 4.2, 4.17, 4.17, 4.8, 3.14, 4.9, 3.16, 4.18 of Horadam [1], respectively. The equations listed from Horadam were in fact used to obtain the corresponding results for the generalized quaternions.

From 9(b) we have for the conjugate quaternion $\overline{T}_{2n}$

$$\overline{T}_{2n} = W_{2n} + W_{2n+2} + W_{2n+4} + W_{2n+6}$$

and thus

$$a\overline{T}_{2n} = aW_{2n} + aW_{2n+2} + aW_{2n+4} + aW_{2n+6}\,.$$

Using equation 4.5 of Horadam [1] we have

$$a\overline{T}_{2n} = W_n^2 + W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 + e(U_{n-1}^2 + U_n^2 + U_{n+1}^2 + U_{n+2}^2)$$

but

$$P_n^2 = W_n^2 - W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2 + 2iW_nW_{n+1} + 2jW_nW_{n+2} + 2kW_nW_{n+3}$$

and similarly for Q_n^2.

Therefore

30. $$a\overline{T}_{2n} + P_n^2 + eQ_{n-1}^2 = 2(W_nP_n + eU_{n-1}Q_{n-1})\,.$$

Many more results can be obtained for the above-defined quaternions. By use of a functional notation the ideas expressed in this article can be easily extended.

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FIBONACCI PRIMITIVE ROOTS AND THE PERIOD OF THE FIBONACCI NUMBERS MODULO P

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One says g is a *Fibonacci primitive root* modulo p, where p is a prime, iff g is a primitive root modulo p and $g^2 \equiv g+1 \pmod p$. In [1], [2], and [3] some interesting properties of Fibonacci primitive roots were developed. In this paper, we shall show that a necessary and sufficient condition for a prime $p \neq 5$ to have a Fibonacci primitive root is $p \equiv 1$ or $9 \pmod{10}$ and $A(p) = p-1$, where $A(p)$ is the period of the Fibonacci numbers modulo p (Theorem 1); for $p \equiv 11$ or $19 \pmod{20}$, we shall explicitly determine the Fibonacci primitive root if it exists (Proposition 1). In the sequel, F_n will denote the n^{th} Fibonacci number and p will denote a prime greater than five.

Theorem 1. There exists a Fibonacci primitive root modulo p Iff $p \equiv 1$ or $9 \pmod{10}$ and $A(p) = p-1$.

Before proving six lemmas needed to prove Theorem 1, we shall remark (see [2] for a proof) that the congruence equation $x^2 \equiv x+1 \pmod p$ has no solutions for $p \equiv 3$ or $7 \pmod{10}$, one solution modulo 5, and two solutions modulo p for $p \equiv 1$ or $9 \pmod{10}$.

Lemma 1. If $g^2 \equiv g+1 \pmod p$ then $g^n \equiv F_n g + F_{n-1} \pmod p$.

The proof of Lemma 1 follows easily by induction.

Lemma 2. If $g^2 \equiv g+1 \pmod p$ and if g has order n then $n = A(p)$ or $n = \dfrac{A(p)}{2}$.

Proof. Since

$$g^{A(p)} \equiv F_{A(p)}g + F_{A(p)-1} \equiv 1 \pmod p,$$

$n \mid A(p)$. Thus $n \leqslant A(p)$.

If $F_n \equiv 0 \pmod p$, then

$$1 \equiv g^n \equiv F_n g + F_{n-1} \equiv F_{n-1} \pmod p.$$

Thus $A(p) \leqslant n$ and hence in this case $n = A(p)$.

If $F_n \not\equiv 0 \pmod p$ then

$$g \equiv \frac{1-F_{n-1}}{F_n} \pmod p.$$

Thus

$$\begin{aligned} 0 &\equiv 0\cdot F_n^2 \equiv (g^2-g-1)F_n^2 \\ &\equiv -(F_n^2 - F_nF_{n-1} - F_{n-1}^2) - (F_n + F_{n-1}) - F_{n-1} + 1 \\ &\equiv (-1)^n - L_n + 1 \pmod p. \end{aligned}$$

For n even, we have $L_n \equiv 2 \pmod p$ and this implies, since $L_n^2 - 5F_n^2 = 4(-1)^n$, that $F_n \equiv 0 \pmod p$. Thus we must have n odd and hence $L_n \equiv 0 \pmod p$. Since

$$0 \equiv L_n \equiv 3F_{n-1} + F_{n-2} \pmod p,$$

we see that

$$1 = -(F_{n-1}^2 - F_{n-1}F_{n-2} - F_{n-2}^2) \equiv 5F_{n-1}^2 \pmod p.$$

Also we see that

$$-5 = L_n^2 - 5F_n^2 - 1 \equiv -5F_n^2 - 1 \equiv -5(F_n^2 + F_{n-1}^2) = -5F_{2n-1} \pmod p.$$

Thus $F_{2n-1} \equiv 1 \pmod p$. Also $F_{2n} = F_n L_n \equiv 0 \pmod p$. Hence $A(p) \leqslant 2n$. Thus, since $n \mid A(p)$, $A(p) = n$ or $A(p) = 2n$. In fact, since $F_n \not\equiv 0 \pmod p$, $A(p) = 2n$.

Lemma 3. If $g^2 \equiv g + 1 \pmod p$, $g^n \equiv 1 \pmod p$, and $n < A(p)$, then g is uniquely determined modulo p.

Proof. By Lemma 1,

$$1 \equiv g^n \equiv F_n g + F_{n-1} \pmod p.$$

Thus, if $F_n \equiv 0 \pmod p$ then $F_{n-1} \equiv 1 \pmod p$. Whence $A(p) \leqslant n$. Thus $F_n \not\equiv 0 \pmod p$. This implies that

$$g \equiv \frac{1 - F_{n-1}}{F_n} \pmod p$$

and therefore g is uniquely determined modulo p.

Lemma 4. Assume $p \equiv 1$ or $9 \pmod{10}$ and assume g_1 and g_2 are two distinct solutions modulo p to the congruence equation $x^2 \equiv x + 1 \pmod p$. If $A(p) \equiv 2 \pmod 4$ then one of g_1, g_2 has order $A(p)$ modulo p and the other has order $A(p)/2$ modulo p. If $A(p) \not\equiv 2 \pmod 4$ then g_1 and g_2 both have order $A(p)$ modulo p.

Proof. By Lemmas 2 and 3, g_1 and g_2 both have order $A(p)$, or one has order $A(p)$ and the other has order $A(p)/2$. Thus, we may say that at least one of g_1, g_2 has order $A(p)$ and, without loss of generality, let us assume g_1 has order $A(p)$.

If $A(p) \equiv 2 \pmod 4$ then

$$-1 \equiv (-1)^{A(p)/2} \equiv (g_1 g_2)^{A(p)/2} = g_1^{A(p)/2} g_2^{A(p)/2} \equiv -g_2^{A(p)/2} \pmod p.$$

Thus the order of g_2 is not $A(p)$ so it must be $A(p)/2$.

If $A(p) \equiv 0 \pmod 4$ then

$$1 \equiv (-1)^{A(p)/2} \equiv (g_1 g_2)^{A(p)/2} = g_1^{A(p)/2} g_2^{A(p)/2} \equiv -g_2^{A(p)/2} \pmod p.$$

Thus g_2 does not have order $A(p)/2$ so g_2 has order $A(p)$.

If $A(p)$ is odd then neither g_1 nor g_2 has order $A(p)/2$ so both g_1 and g_2 have order $A(p)$.

Lemma 5. If there exists a Fibonacci primitive root modulo p then $p \equiv 1$ or $9 \pmod{10}$ and $A(p) = p - 1$.

Proof. Assume g is a Fibonacci primitive root modulo p. By the remark after Theorem 1, $p \equiv 1$ or $9 \pmod{10}$. Since g has order $p - 1$, by Lemma 4, $p - 1 = A(p)$, or $p - 1 = A(p)/2$ and $A(p) \equiv 2 \pmod 4$. This second possibility must be excluded since $p - 1$ is even.

Lemma 6. If $p \equiv 1$ or $9 \pmod{10}$ and $A(p) = p - 1$, then there exists a Fibonacci primitive root modulo p.

Proof. Since $p \equiv 1$ or $9 \pmod{10}$, there exists two solutions to $x^2 \equiv x + 1 \pmod p$. By Lemma 4, at least one of these two solutions has order $A(p) = p - 1$.

As a final result we prove

Proposition 1. If $p \equiv 11$ or $19 \pmod{20}$ and if g is a Fibonacci primitive root modulo p then

$$g \equiv -\frac{1 + F_{n-1}}{F_n} \pmod p,$$

where $n = (p-1)/2$.

Proof. Let g_2 be the solution other than g to $x^2 \equiv x + 1 \pmod p$ and let $n = (p-1)/2$. By Lemma 5, $A(p) = p - 1 \equiv 2 \pmod 4$. Thus, by Lemma 4, g_2 has order $A(p)/2 = n$. If $F_n \equiv 0 \pmod p$ then

$$-1 \equiv g^n \equiv F_n g + F_{n-1} \equiv F_{n-1} \equiv F_n g_2 + F_{n-1} \equiv g_2^n \equiv 1 \pmod p.$$

Hence $F_n \not\equiv 0 \pmod p$ and the result follows.

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ON SUMS OF FIBONACCI-TYPE RECIPROCALS

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Letting j be an integer, consider sequences of the form

(1) $$P_{n+1} = jP_n + P_{n-1} ,$$

where $P_0 = 0$. Without loss of generality take $P_1 = 1$. As an example think of 0, 1, 3, 10, 33, 109, $\cdots$. We can define the Lucas complement of (1) to be

(2) $$P_n^* = P_{n+1} + P_{n-1} .$$

The solution of these via the characteristic equation for the roots of [1] is well known. Let the roots of

(3) $$(q^2 - jq - 1) = 0$$

be a,b. The theory of equations tells us that $ab = -1$ and that $(a+b) = j$. This gives $(a-b) = 2a - j$. Using the initial conditions it can easily be shown following the method of Vorob'ev [1] that

(4) $$P_n = (a^n - b^n)/(a-b) \qquad \text{and} \qquad P_n^* = (a^n + b^n).$$

A few manipulations suffice to show that

(5) $$P_{j,2n} = P_{j,n}P_{j,n}^*$$

and using $(ab)^n = (-1)^n$ we can prove

(6) $$P_{j,2n-1} = P_{j,n-1}P_{j,n}^* - \cos(\pi n).$$

Although well known for the Fibonacci and Lucas sequences when $j = 1$, their validity when $j \neq 1$ has not been appreciated. Similarly we can derive

(7) $$P_{j,4n+1} = P_{j,2n+1}P_{2n}^* - 1 .$$

Good [2] has derived the harmonic sum

$$\sum_{m=0}^{n} (1/F_b) = 3 - F_{B-1}/F_B ,$$

where $b = 2^m$ and $B = 2^n$ have the virtue of conciseness. A double generalization follows introducing j as above and k a natural number arbitrary multiplier.

Theorem.

(8) $$\sum_{m=0}^{n} (1/P_{j,kb}) = C_{j,k} - P_{j,kB-1}/P_{j,kB} \quad \text{for} \quad n \geqslant 1.$$

Proof. Let j have any value, then as the basis for induction the proposition is certainly true for $n = 1$ since that merely defines the parameter $C_{j,k}$. Now assume that it is true for some $B = 2^n$ and add the next term $(1/P_{j,k2B})$ to each side. Hence the added term will equal the new minus the old right-hand side.

$$1/P_{j,k2B} = (P_{j,kB-1}/P_{j,kB}) - (P_{j,k2B-1}/P_{j,k2B}).$$

Cross-multiplying we have

$$P_{j,kB} = P_{j,2kB}P_{j,kB-1} - P_{j,2kB-1}P_{j,kB}$$

which is easy to prove using a Binet type of formula (4) as only the cross-product terms are non-zero. But it would be more aesthetically appealing to keep the proof in the realm of integers. This is easily done by substitution of first (5) and then (6) into the above equation. This completes the inductive transition.

Recall that $C_{j,k}$ is found from (8) when $n = 1$. The numerators of $C_{j,k}$ are thus

$$P_{j,2k}C_{j,k} = (1 + P^*_{j,k} + P_{j,2k-1}). \tag{9}$$

Successive application of (1) shows that

$$\{P_{j,k}\} = 0,\ 1,\ j,\ (j^2+1),\ (j^3+2j),\ (j^4+3j^2+1),\ (j^5+4j^3+3j),\ \cdots. \tag{10}$$

And using the definition (2) for the Lucas complement one finds

$$\{P^*_{j,k}\} = 2,\ j,\ (j^2+2),\ (j^3+3j),\ (j^4+4j+2),\ (j^5+5j^3+5j),\ \cdots. \tag{11}$$

And using (9) the numerators of $C_{j,k}$ are

$$\{P_{j,2k}C_{j,k}\} = 4, (j+2),\ (2j^2+4),\ (j^4+j^3+3j^2+3j+2),\ (j^6+6j^4+10j^2+4),$$
$$(j^8+7j^6+j^5+15j^4+5j^3+10j^2+5j+2),\ \cdots. \tag{12}$$

Table of $C_{j,k}$ Values

(written in the form with denominator $P_{j,2k}$ as in Eq. (9))

j/k	1	2	3	4	5	6
1	3/1	6/3	10/8	21/21	46/55	108/144
2	4/2	12/12	44/70	204/408	1068/2378	
3	5/3	22/33	146/360	1309/3927	13364/42837	
4	6/4	36/72	382/1292	5796/23188	99574/416020	
5	7/5	54/135	843/3640	19629/98145	513402/2646275	
j		$2/j$		$1/j$		

There are some simplifications. When $k \equiv 0 \pmod 4$ then using (5) gives $C_{j,k} = P_{j,kh-1}/P_{j,kh}$, where $h = ½$ and for $k = 4, 8, \cdots C_{j,k} = (1/j), (1/j - 1/P_{j,4}), \cdots$. When $k \equiv 2 \pmod 4$ then using (7) one finds

$$C_{j,k} = P^*_{j,kh-1}/P^*_{j,kh},$$

where $h = ½$ and for $k = 2, 6, \cdots C_{j,k} = 2/j, (1/j - 1/P^*_{j,3}), \cdots$. A short table of $C_{j,k}$ values is given and the interested reader can extend it with some patience.

Returning to the point of this paper, if we sum both sides of (8) over all odd k then the left-hand side is intuitively obviously a sum over all the natural numbers. The right-hand side of (8) is merely a sum over all odd k and so the sum of the reciprocals of numbers satisfying (1) (which I call coprime sequences) has been reduced to half the number of terms. The special case of Fibonacci numbers, $j = 1$, was derived by the author in October 1975 and is [3]. Gould [4, Eq. 2] expresses the rearrangement array as a sum and goes on to generalize it into partition arrays, his equation (9). So from (8) I write

$$\sum_{n=1}^{\infty} (1/P_{j,n}) = \sum_{k=1}^{\infty} (C_{j,k} - 1/a) \quad \text{for } k \text{ odd} \tag{13}$$

$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} [1/P_{j,k} + 1/P_{j,2k} + (a-b)/(a^{4k}-1)] \tag{14}$$

as but two of several expressions that can be derived using Binet's expressions (4), where $a + b = j$ and $ab = -1$. All of the equations in the author's earlier paper [3] are valid here by merely replacing $\sqrt{5}$ by the more general $(a - b)$ and I do not see any point in taking up space to repeat them. I refer to sequences satisfying (1) as coprime sequences because they fulfill a generalization of a theorem in Vorob'ev [1] showing that only in this case are adjacent terms always coprime. The author used the generalization of this theorem in an earlier work [5]

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ROW AND RISING DIAGONAL SUMS FOR A TYPE OF PASCAL TRIANGLE

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As has been noted by Hoggatt [1], Pascal's Triangle can be thought of as having been generated by column generators. This provides insight into the row sums and rising diagonal sums of this triangle. Let $\{a_{i,0}\}_{i=0}^{\infty}$ denote a real number sequence and consider the following array:

$$\begin{array}{cccccccc} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,m} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & & a_{n,m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{array}$$

which has the Pascal-like property [3]

$$a_{ij} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if } i \geqslant j \geqslant 1, \\ 0 & \text{if } j > i \geqslant 0. \end{cases}$$

Under these conditions, it follows readily that

$$a_{ij} = \sum_{k=0}^{i-1} a_{k,j-1}$$

for all i and j such that $i \geqslant j \geqslant 1$. For the following assume that $f(x)$ is the generating function for the sequence $\{a_{i,0}\}_{i=1}^{\infty}$.

Theorem 1. The generating function for the k^{th} column $(k = 0,1,2,\cdots)$ in the above array is

$$g_k(x) = f(x)[x/(1-x)]^k.$$

Proof. Let

$$f(x) = a_{0,0} + a_{1,0}x + a_{2,0}x^2 + \cdots$$

denote the generating function for the zeroth column $\{a_{i,0}\}_{i=1}^{\infty}$. Suppose that

$$f(x)[x/(1-x)]^m = \sum_{i=0}^{\infty} (a_{i,m})x^i$$

for some positive integer m. Then by the comment preceding Theorem 1

$$f(x)[x/(1-x)]^{m+1} = f(x)[x/(1-x)]^m[x/(1-x)] = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{i-1} a_{k,m}\right)x^i = \sum_{i=0}^{\infty} (a_{i,m+1})x^i$$

and the proof is complete by induction on m.

Theorem 2. The generating function for the row sums of the above array is

$$[f(x)(1-x)]/(1-2x).$$

Proof. Since $g_k(x) = f(x)[x/(1-x)]^k$, the generating function for the row sums is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k = f(x)\left[\frac{1}{1-\frac{x}{1-x}}\right] = f(x)\,\frac{1-x}{1-2x}\,.$$

Theorem 3. The generating function for the rising diagonal sums of the above array is

$$[f(x)(1-x)]/(1-x-x^2).$$

Proof. Consider the new array:

$$\begin{array}{cccccc}
a_{0,0} & 0 & 0 & 0 & 0 & \cdots \\
a_{1,0} & a_{0,1} & 0 & 0 & 0 & \cdots \\
a_{2,0} & a_{1,1} & a_{0,2} & 0 & 0 & \cdots \\
a_{3,0} & a_{2,1} & a_{1,2} & a_{0,3} & 0 & \cdots \\
a_{4,0} & a_{3,1} & a_{2,2} & a_{1,3} & a_{0,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
f(x) & xg_1(x) & x^2g_2(x) & x^3g_3(x) & x^4g_4(x) & \cdots
\end{array}$$

Note that the column generator for the k^{th} column $(k = 0, 1, 2, \cdots)$ is $x^kg_k(x)$. Furthermore the row sums of this array are the rising diagonal sums of the original array. Thus the generating function for the rising diagonal sums in the original array is

$$D(x) = \sum_{k=0}^{\infty} x^kg_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x}\right)^k = f(x)\,\frac{1-x}{1-x-x^2}$$

Now if $f(x) = 1/(1-x)$, one has the usual Pascal Triangle and some of results in [1]. Moreover, if

$$f(x) = x[(1-x)/(1-3x+x^2)] + 1 = (1-2x)/(1-3x+x^2),$$

then the theorems above can be used to answer Problem H-183 [4] in this *Quarterly.* Indeed, since

$$H(x) = \frac{a+(b-ap)x}{1-px+qx^2}$$

is the generating function for the generalized Fibonacci sequence $w_n = w_n(a,b;p,q)$ [2], and

$$O(x) = \frac{w_1 + [w_3 - (p^2-2q)w_1]x}{1-(p^2-2q)x+q^2x^2}$$

and

$$E(x) = \frac{w_0 + [w_2 - (p^2-2q)w_0]x}{1-(p^2-2q)x+q^2x^2}$$

are the generating functions for the sequences $\{w_{2k+1}\}_{k=1}^{\infty}$ and $\{w_{2k}\}_{k=1}^{\infty}$ respectively, then questions similar to H-183 can be answered readily by considering the generating functions $xH(x)+1$, $xO(x)+1$, and $xE(x)+1$. In particular, if one considers the sequence $\{a_{i,0}\}_{i=0}^{\infty}$ where $a_{0,0} = 1$ and $a_{i,0} = L_{2i-1}$ (for $i = 1, 2, 3, \cdots$), then the array is

$$\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
4 & 2 & 1 & & \\
11 & 6 & 3 & 1 & \\
29 & 17 & 9 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

the generating function for the zeroth column is

$$(1-2x+2x^2)/(1-3x+x^2),$$

the generating function for the row sums is

$$(1-3x+4x^2-2x^3)/(1-5x+7x^2-2x^3),$$

and the generating function for the rising diagonal sums is

$$(1-3x+4x^2)/(1-4x+3x^2+2x^3-x^4).$$

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The term TRIBONACCI number was coined by Mark Feinberg in [1]above.

CERTAIN GENERAL BINOMIAL-FIBONACCI SUMS

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Numerous writers appear to have been fascinated by the many interesting summation identitites involving the Fibonacci and related Lucas numbers. Various types of formulas are discussed and various methods are used. Some involve binomial coefficients [2], [4]. Generating function methods are used in [2] and [5] and higher powers appear in [6]. Combinations of these or other approaches appear in [1], [3] and [7].

One of the most tantalizing displays of such formulas was the following group of binomial-Fibonacci identities given by Hoggatt [5]. He gives:

$$1^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k , \tag{1}$$

$$2^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{3k} , \tag{2}$$

$$3^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{4k} . \tag{3}$$

In these formulas and throughout this paper F_n denotes the n^{th} Fibonacci number defined by the recurrence:

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \qquad F_1 = 1. \tag{4}$$

Hoggatt attributes formula (2) to D. A. Lind, (3) to a special case of Problem 3-88 in the Fibonacci Quarterly and states that (1) is well known.

The three identities given above suggest, rather strongly, the possibility of a general formula of which those given are special instances. Hoggatt does obtain many new sums but does not appear to have succeeded in obtaining a satisfactory generalization of formulas (1)—(3).

In the present paper, we give elementary, yet rather powerful, methods which yield many general binomial-Fibonacci summation identities. In particular, we obtain a sequence of sums the three simplest members of which are precisely the formulas (1)—(3) given above. In addition, similar families of sums are obtained with the closed forms $a_{1,m}^n F_n$ and $a_{3,m}^n F_{3n}$ for $m = 1, 2, 3, \cdots$, as well as the general two-parameter family of sums with the closed form $(a_{r,m})^n F_{rn}$.

Our principal tools for obtaining sums will be the binomial expansion formula

$$\sum_{k=0}^{m} \binom{m}{k} (y-1)^k = y^m, \tag{5}$$

and the fact that the Fibonacci number F_n is a linear combination of a^n and b^n, where

$$a = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{5}}{2}$$

are roots of the polynomial equation

$$x^2 = x + 1. \tag{6}$$

The Fibonacci numbers are then

$$F_n = (1/\sqrt{5})(a^n - b^n). \tag{7}$$

We are already in a position to obtain a summation formula. Let w stand for a root a or b of (6). Then we have

(8) $$w^2 = w + 1.$$

Clearly, then, by (5) and (8),

$$(w^2)^n = \sum_{u=0}^{n} \binom{n}{k} (w^2 - 1)^k = \sum_{n=0}^{n} \binom{n}{k} w^k,$$

and therefore

$$(a^2)^n - (b^2)^n = \sum_{n=0}^{n} \binom{n}{k} (a^k - b^k).$$

But from (7), this is seen to be equivalent to

$$F_{2n} = \sum_{n=0}^{n} \binom{n}{k} F_k,$$

which is formula (1).

In order to obtain more general results, we proceed as follows. From (8) we see that

$$w^2 = w + 1 = F_2 w + F_1.$$
$$w^3 = w^2 + w = F_3 w + F_2,$$

and, in general, by an easy induction,

(9) $$w^m = w^{m-1} + w^{m-2} = F_m w + F_{m-1}.$$

Rewriting, we have

$$1 - \frac{w^m}{F_{m-1}} = -\frac{F_m}{F_{m-1}} w, \quad m \neq 1,$$

or, equivalently,

(10) $$-\left(\frac{F_m}{F_{m-1}}\right)^n w^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{F_{m-1}}\right)^k w^{mk},$$

where, again, w may be either a or b. Again using the fact that F_n is a linear combination of a^n and b^n, we obtain

(11) $$\left(\frac{F_m}{F_{m-1}}\right)^n F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} \left(\frac{1}{F_{m-1}}\right)^k F_{mk}, \quad m \neq 1.$$

Equation (11) takes on especially simple forms for certain values of m. For example, when $m = 2$ and 3, respectively, we have

(12) $$F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{2k}$$

and

(13) $$2^n F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+n} F_{3k}.$$

Other values of m result in non-integral ratios in (11), e.g., $m = 4$ and 5 give

(14) $$\left(\frac{3}{2}\right)^n F_n = (-1)^n \sum_{k=0}^{n} \binom{h}{k} (-1/2)^k F_{4k}$$

and

$$\left(\frac{5}{3}\right)^n F_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1/3)^k F_{5k}, \tag{15}$$

Each of the sums (12)–(15) and the general sum in (11) yield closed forms of the type

$$(a_{1,m})^n F_n .$$

In order to obtain sums with closed forms of the type

$$(a_{2,m})^n F_{2n}$$

we return to (9). If we let $m = 2$ and solve for w, $w = w^2 - 1$, we may substitute this expression into (9) to obtain

$$w^m = F_m(w^2 - 1) + F_{m-1} = F_m w^2 - (F_m - F_{m-1}) = F_m w^2 - F_{m-2}. \tag{16}$$

This is equivalent to:

$$\frac{F_m}{F_{m-2}} w^2 = \frac{1}{F_{m-2}} w^m + 1, \quad m \neq 2. \tag{17}$$

Now proceeding in the same manner as led to (11) results in the general formula

$$(F_m)^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} (F_{m-2})^{n-k} F_{mk}, \quad m \neq 2. \tag{18}$$

The special cases $m = 1, 3,$ and 4 of this general equation are found to give exactly the three sums involving F_{2n} which were listed by Hoggatt and given above in (1)–(3). All other cases can easily be seen to lead to formulas containing a power of a Fibonacci number in the summand and in this sense previous investigators can be said to have found all "easy" sums of this type. The first two cases giving new sums are thus, for $m = 5$ and 6,

$$5^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k} \tag{19}$$

and

$$8^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{6k} . \tag{20}$$

Steps similar to those leading to (16) can be followed to express w^m in terms of w^3. We find, after simplifying,

$$2w^m = F_m w^3 + F_{m-3}$$

which, following our general procedure, yields

$$(F_m)^n F_{3n} = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} 2^k (F_{m-3})^{n-k} F_{mk}, \quad m \neq 3. \tag{21}$$

For $m = 2, 4, 5, 6$ we have, respectively,

$$F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{2k}, \tag{22}$$

$$3^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{4k}, \tag{23}$$

$$5^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{5k} , \tag{24}$$

and

$$(25)\qquad 4^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{6k}.$$

Rather than continuing with these special families of sums, we now proceed to the general two-parameter family yielding closed forms of the type

$$(a_{r,m})^n F_{rn}.$$

Let $0 < r < m$. From (9) we have

$$w^r = F_r w + F_{r-1}, \qquad w^m = F_m w + F_{m-1}$$

which give, after considerable simplification,

$$(26)\qquad w^m = \frac{F_m}{F_r} w^r + (-1)^{r-1} \frac{F_{m-r}}{F_r}, \qquad 0 < r < m.$$

The result just obtained is equivalent to

$$(27)\qquad (-1)^r \left(\frac{F_m}{F_{m-r}}\right) w^r = (-1)^r \left(\frac{F_r}{F_{m-r}}\right) w^m + 1, \qquad 0 < r < m,$$

which yields the summation

$$(28)\qquad (F_m)^n F_m = \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} (F_{m-r})^{n-k} (F_r)^k F_{mk},$$

valid for all integral $m, n,$ and r satisfying $0 < r < m$.

A number of special cases of the above general formula have been given previously in this paper for $r = 1, 2,$ and 3. Another interesting case results when $m = 2r$. Using the well known fact that F_{2r}/F_r is the Lucas number L_r defined by the recurrence

$$(29)\qquad L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \qquad L_1 = 1,$$

we have, in this case,

$$(30)\qquad (L_r)^n F_{rn} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} F_{2rk}.$$

The special case $r = 2p$ has been obtained by Hoggatt in [5]. Some instances of (30) which have not been given among our previous formulas are

$$(31)\qquad 7^n F_{4n} = \sum_{k=0}^{n} \binom{n}{k} F_{8k}$$

and

$$(32)\qquad 11^n F_{5n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{10k}$$

which obtain when $r = 4$ and 5, respectively.

Of course, if we recall that the Lucas numbers L_n are linear combinations of a^n and b^n, defined in (7), specifically

$$(33)\qquad L_n = a^n + b^n,$$

then we see that each sum obtained above remains valid when L_n is substituted for F_n at the appropriate occurrences of F_n in each formula. We state some of these. From (18) we have

(34) $$(F_m)^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} (F_{m-2})^{n-k} L_{mk}, \qquad m \neq 2,$$

several specific instances being

$$1^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_k,$$

$$2^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_{3k}$$

and

$$3^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_{4k}.$$

The interested reader may obtain other Lucas number analogs of formulas given above.

Preliminary results indicate that modifications of the methods used in this paper will lead to many other quite general results on binomial Fibonacci sums. Perhaps we might be forgiven for paraphrasing Professor Moriarty (see [4]) in saying "many beautiful results have been obtained, many yet remain."

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4. H. W. Gould, "The Case of the Strange Binomial Identities of Professor Moriarty," *The Fibonacci Quarterly,* Vol. 10, No. 4 (Dec. 1972), pp. 381–391.
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7. D. Zeitlin, "On Identities for Fibonacci Numbers," *Amer. Math. Monthly,* 70 (1963), pp. 987–991.

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A NOTE ON THE SUMMATION OF SQUARES

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Consider

$$P_{n+2} = pP_{n+1} + qP_n, \quad P_0 = 0, \quad P_1 = 1.$$

We wish to find

(A) $$\sum_{j=1}^{n} P_j^2 = P_nP_{n+1} \quad \text{if } p = q = 1;$$

(B) $$\sum_{j=1}^{n} P_j^2 = \frac{P_nP_{n+1}}{p} \quad \text{if } q = 1;$$

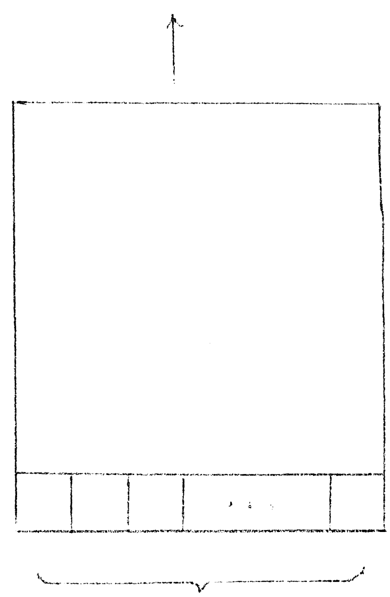
(C) $$\sum_{j=1}^{n} P_j^2 = \frac{2q^2P_{n+1}P_n + \frac{(1-q)}{p}[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2} .$$

The usual way to establish (A) is by induction after (A) has been guessed from tabular data, or by the geometric method of Brother Alfred [1]. We now establish (B) by the method of [1].

Form p unit squares horizontally. Above these add p copies of $p \times p$ squares. This yields

$$p \cdot (p^2 + 1) = P_2P_3 .$$

Add to the left p copies of the square P_2 on the edge to get a rectangle P_3P_4.



p unit squares

Since every square P_1, P_2, P_3 is used p times so far

$$P_1^2 + P_2^2 + P_3^2 = P_3P_4/p\,.$$

This obviously may be continued as far as one wishes so that

$$\sum_{j=1}^{n} P_j^2 = P_nP_{n+1}/p, \quad p \neq 0, \quad q = 1\,.$$

Second Method: $(q = 1)$

Start with

$$P_{n+2} = pP_{n+1} + P_n$$

and multiply through by P_{n+1} to get

$$P_{n+1}P_{n+2} = pP_{n+1}^2 + P_nP_{n+1}$$

$$\sum_{j=0}^{n} P_{j+2}P_{j+1} = \sum_{j=0}^{n} pP_{j+1}^2 + \sum_{j=0}^{n} P_jP_{j+1}\,.$$

Thus,

$$P_{n+2}P_{n+1} = p\sum_{j=0}^{n} P_{j+1}^2 = p\sum_{j=1}^{n+1} P_j^2 \qquad \text{and} \qquad \sum_{j=1}^{n} P_j^2 = P_nP_{n+1}/p\,.$$

Before doing the general case, let us consider the result $p = 1$ and $q \neq 0$.

$$P_{n+2} = P_{n+1} + qP_n$$

$$P_{n+2}P_{n+1} = P_{n+1}^2 + qP_{n+1}P_n$$

$$qP_{n+1}P_n = qP_n^2 + q^2P_nP_{n-1}$$

$$q^2P_nP_{n-1} = q^2P_{n-1}^2 + q^3P_{n-1}P_{n-2}$$

$$\cdots \qquad \cdots$$

$$q^{n-1}P_2P_1 = q^{n-1}P_1^2 + q^nP_1P_0\,.$$

Thus,

$$\sum_{j=0}^{n} q^jP_{n+1-j}^2 = P_{n+1}P_{n+2}\,.$$

We now proceed to the general case. From

$$P_{n+2}P_{n+1} = pP_{n+1}^2 + qP_nP_{n+1}$$

one may at once write

$$\text{(D)} \qquad \sum_{j=1}^{n+1} pP_j^2 = P_{n+2}P_{n+1} + (1-q)\sum_{j=1}^{n} P_jP_{j+1}\,,$$

while from

$$P_{j+2}^2 = p^2P_{j+1}^2 + q^2P_j^2 + 2pqP_jP_{j+1}$$

one can immediately write

$$\text{(E)} \qquad P_{n+2}^2 + P_{n+1}^2 - P_2^2 - P_1^2 = p^2(P_{n+1}^2 - P_1^2) + (p^2 + q^2 - 1)\sum_{j=1}^{n} P_j^2 + 2pq\sum_{j=1}^{n} P_jP_{j+1}\,.$$

One can now use (D) and (E) to solve directly for

$$\sum_{j=1}^{n+1} pP_j^2 = P_{n+2}P_{n+1} + (1-q)\sum_{j=1}^{n} P_jP_{j+1} = P_{n+2}P_{n+1} + \frac{(1-q)}{2pq}\Bigg\{P_{n+2}^2 + P_{n+1}^2 - p^2 - 1 - p^2P_{n+1}^2 + p^2$$

$$= (p^2 + q^2 - 1)\sum_{j=1}^{n} P_j^2\Bigg\}$$

$$pP_{n+1}^2 + \left(\sum_{j=1}^{n} P_j^2\right)\left(p - \frac{(1-q)(p^2+q^2-1)}{2pq}\right) = P_{n+2}P_{n+1} + \frac{1-q}{2pq}\,[P_{n+2}^2 + P_{n+1}^2 - 1 - p^2P_{n+1}^2]$$

$$\sum_{j=1}^{n} pP_j^2 = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2 + \dfrac{(1-q)}{2pq}\,[P_{n+2}^2 + P_{n+1}^2(1-p^2) - 1]}{(2pq - p^2 - q^2 + 1 + qp^2 + q^3 - 1)/2pq}$$

Testing $p = 1,\ q = 1,$

$$\sum_{i=1}^{n} F_i^2 = \frac{2F_{n+2}F_{n+1} - 2F_{n+1}^2}{2} = F_{n+1}F_n\ .$$

For $q = 1$ only,

$$\sum_{i=1}^{n} pP_i^2 = \frac{2pP_{n+2}P_{n+1} - 2p^2P_{n+1}^2}{p^2 + 1 - (p-1)^2} = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2}{2p} = P_{n+1}P_n$$

so that

$$\sum_{i=1}^{n} P_i^2 = P_{n+1}P_n/p\,.$$

Thus,

$$\sum_{j=1}^{n} P_j^2 = \frac{2qP_{n+2}P_{n+1} - 2pqP_{n+1}^2 + \dfrac{(1-q)}{p}\ [P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2}$$

$$= \frac{2q^2(P_{n+1}P_n) + \dfrac{(1-q)}{p}\ [P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2}\ .$$

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FIBONACCI NUMBERS IN DIATOMS?

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

In the October 1968 issue of *Pacific Discovery* there appeared an article entitled, "Nature's Opaline Gems," by G. Dallas Hanna of the California Academy of Sciences. At the beginning of the article an allusion was made to Fibonacci in connection with patterns in nature. This naturally aroused a curiosity about the possibility of such numbers having been found in diatoms (nature's opaline gems). Mr. Hanna was good enough to send over an electron microscope reproduction of a diatom that looked something like a sunflower (see Fig. 1). However, the count did not seem to work out and there were some disturbing features such as rays that started in from the edge but did not go all the way to the center.

A meeting was arranged with Mr. Hanna at the Academy of Sciences and there in the Geology Department the author encountered the world of diatoms. Mr. Hanna has been working on these algae of ancient times with their silicified cell walls since 1916. Long rows of books dealing with them as well as ponderous tomes containing drawings made of them in the past century show that this field has attracted the attention of many nature explorers.

After viewing some of the magnificent pictures that are now being produced by a special electron microscope (see Fig. 2 for another example), work was begun on going through the books, examining the pictures, counting rays and other features. After some time, the author asked Mr. Hanna whether the numbers on these specimens remained constant for a given species. He said that they did not; in fact that they varied widely without any particular pattern.

Thus the question whether there are Fibonacci patterns in diatoms seems to have a negative answer. The result is being reported here as part of the total picture of Fibonacci numbers in nature as well as to suggest that those who are interested in the world of microscopic creatures might want to examine them from this point of view.

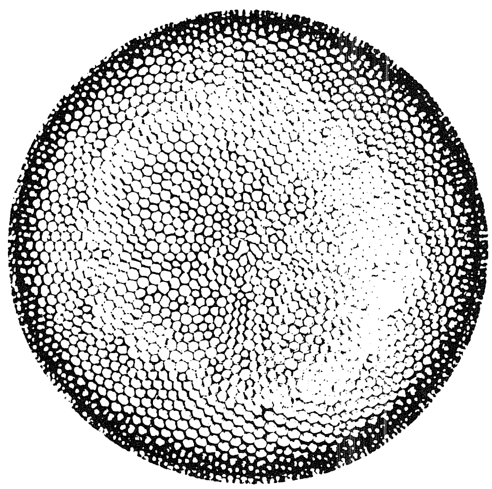

Figure 1

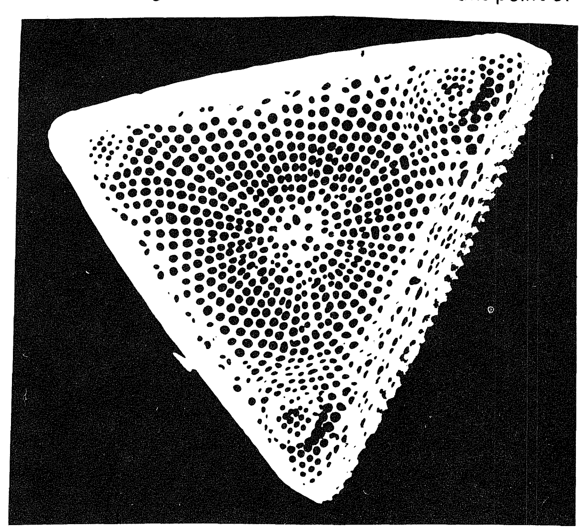

Figure 2

This note was written quite some years ago. In the meantime, the gracious and generous G. Dallas Hanna has passed away.

Shortly after we had virtually written off a connection between diatoms and Fibonacci numbers an article was received from Edward A. Parberry entitled "A Recursion Relation for Populations of Diatoms," published in *The Fibonacci Quarterly* of December, 1969, pp. 449–456.

ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-276 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that the sequence of Bell Numbers, $\{B_i\}_{i=0}^{\infty}$, is invariant under repeated differencing.

$$B_0 = 1, \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \qquad (n \geqslant 0).$$

H-277 Proposed by L. Taylor, Brentwood, New York.

If $p \equiv \pm 1 \pmod{10}$ is prime and $x \equiv \sqrt{5}$ is of even order $(\mathrm{mod}\ p)$, prove that $x-3, x-2, x-1, x, x+1$ and $x+2$ are quadratic nonresidues of p if and only if $p \equiv 39 \pmod{40}$.

SOLUTIONS

A PLAYER REP

H-261 Proposed by A. J. W. Hilton, University of Reading, Reading, England.

It is known that, given k a positive integer, each positive integer n has a unique representation in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where $t = t(n,k)$, $a_i = a_i(n,k)$, $(i = t, \cdots, k)$, $t \geqslant 1$ and, if $k > t$, $a_k > a_{k-1} > \cdots > a_t$. Call such a representation the *k-binomial representation* of n.

Show that, if $k \geqslant 2$, $n = r + s$, where $r \geqslant 1$, $s \geqslant 1$ and if the k-binomial representations of r and s are

$$r = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_u}{u}, \qquad s = \binom{c_k}{k} + \binom{c_{k-1}}{k-1} + \cdots + \binom{c_v}{v}$$

then

$$\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1} \leqslant \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{b-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}.$$

Solution by the Proposer.

Define a total order $<_s$ on the collection of all k-sized sets of positive integers as follows: If A, B are two distinct sets of k positive integers write $A <_s B$ if

$$\max\{x : x \in A/B\} < \max\{x : x \in B/A\}.$$

Let $S_k(r)$ denote the collection of the first r sets under $<_s$, and let $S'_k(s)$ denote the collection of the first s sets under $<_s$ which do not contain any of $\{1, \cdots, r\}$. If $\mathcal{A}$ is any collection of n k-sized sets of positive integers let

$$\Delta\mathcal{A} = \{B : |B| = k-1 \quad \text{and} \quad B \subset A \ \text{for some}\ A \in \mathcal{A}\}.$$

The Kruskal-Katona theorem states that $|\Delta A| \geqslant |\Delta S_k(n)|$. Thus

$$|\Delta S_k(n)| \leqslant |\Delta(S_k(r) \cup S'_k(s))| .$$

But

$$|\Delta S_k(n)| = \binom{a_k}{k-1} + \cdots + \binom{a_t}{t-1}$$

and

$$|\Delta(S_k(r) \cup S'_k(s))|$$

$$= |\Delta S_k(r)| + |\Delta S_k(s)| = \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{u-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}.$$

and the required inequality now follows.

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J. B. Kruskal, "The Number of Simplices in a Complex," *Mathematical Optimization Techniques,* University of California Press, Berkeley and Los Angeles (1963), pp. 251–278.

MODERN MOD

H-262 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that $L_{p^2} \equiv 1 \pmod{p^2}$ if and only if $L_p \equiv 1 \pmod{p^2}$.

Solution by the Proposer.

Put

$$L_n = \alpha^n + \beta^n, \qquad \alpha + \beta = 1, \qquad \alpha\beta = -1.$$

Then

$$1 = (\alpha+\beta)^n = L_n + \sum_{k=1}^{n-1} \binom{n}{k} \alpha^k \beta^{n-k}.$$

In particular

$$L_p = 1 - \sum_{k=1}^{p-1} \binom{p}{k} \alpha^k \beta^{p-k}$$

and

$$L_{p^2} = 1 - \sum_{k=1}^{p^2-1} \binom{p^2}{k} \alpha^k \beta^{p^2-k}.$$

Since

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \quad \text{and} \quad \binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p},$$

it follows that

$$L_p \equiv 1 \pmod{p^2}$$

if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \alpha^k \beta^{p-k} \equiv 0 \pmod{p}.$$

In the next place

$$\binom{p^2}{k} \equiv 0 \pmod{p^2} \qquad (p \nmid k)$$

and

$$\binom{p^2}{pk} = \frac{p}{k}\binom{p^2-1}{pk-1} \equiv (-1)^{k-1}\,\frac{p}{k} \pmod{p^2}.$$

Thus $L_{p^2} \equiv 1 \pmod{p^2}$ if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\,\alpha^{pk}\beta^{p^2-pk} \equiv 0 \pmod{p}.$$

Since

$$\left(\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\,\alpha^{k}\beta^{p-k}\right)^p \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\,\alpha^{pk}\beta^{p^2-pk} \pmod{p},$$

it follows that $L_{p^2} \equiv 1 \pmod{p}$ if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\,\alpha^{k}\beta^{p-k} \equiv 0 \pmod{p}.$$

Therefore

$$L_{p^2} \equiv 1 \pmod{p^2} \Leftrightarrow L_p \equiv 1 \pmod{p^2}.$$

REMARK: More generally, if $k \geqslant 2$, we have

$$L_{p^k} \equiv 1 \pmod{p^2} \Leftrightarrow L_p \equiv 1 \pmod{p^2}.$$

LUCAS THE SQUARE IS NOW MOD!

H-263 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $L_{2mn}^2 \equiv 4 \pmod{L_m^2}$ for every $n,m = 1, 2, 3, \cdots$.

Solution by the Proposer.

Clearly,

$$L_{(2k+1)m} = (\alpha^m)^{2k+1} + (\beta^m)^{2k+1}$$

is divisible by $L_m = \alpha^m + \beta^m$, where

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Consequently, if n is odd, say $2k+1$, then upon using formulae (I_{15}) and (I_{18}) of Hoggatt's *Fibonacci and Lucas Numbers,*

$$L_{2nm} = L_{nm}^2 - 2(-1)^{nm} = L_{(2k+1)m}^2 - 2(-1)^m$$

and, depending upon the parity of m, either $L_{2nm}+2$ or $L_{2nm}-2$ is equal to $L_{(2k+1)m}^2$. Hence the product $(L_{2nm}+2)(L_{2nm}-2) = L_{2nm}^2 - 4$ is divisible by L_m^2.

If n is even, say $n = 2k$, we proceed by induction on k. For $k = 1$,

$$L_{2nm} = L_{4m} = L_{2m}^2 - 2 = (L_m^2 - 2(-1)^m)^2 - 2 = L_m^4 - 4(-1)^m L_m^2 + 2,$$

hence, $L_{2nm} - 2$ and, therefore, $L_{2nm}^2 - 4$ is divisible by L_m^2. Assume now that the desired result holds for all even integers less than $n = 2k$. Then

$$L_{2nm} = L_{4km} = L_{2km}^2 - 2,$$

and hence

$$L_{2nm} - 2 = L_{2km}^2 - 4.$$

This latter expression is divisible by L_m^2 either by the induction hypothesis or by the proof for odd n, thus $(L_{2nm}+2)(L_{2nm}-2)$ must also be divisible by L_m^2. This completes the inductive step.

Also solved by G. Lord, D. Beverage, F. Higgins, and G. Wulczyn.

AN OLDIE!

H-256 Proposed by E. Karst, Tucson, Arizona.

Find all solutions of

(i) $$x+y+z = 2^{2n+1}-1,$$

and

(ii) $$x^3+y^3+z^3 = 2^{6n+1}-1,$$

simultaneously for $n < 5$, given that

(a) x, y, z are positive rationals

(b) $2^{2n+1}-1,\ 2^{6n+1}-1$ are integers

(c) $n = \log_2 \sqrt{t}$, where t is a positive integer.

Solution by the Proposer.

From this journal (Dec., 1972, p. 634; April, 1973, p. 188) we have the following

$$n, \qquad x+y+z = 2^{2n+1}-1, \qquad x^3+y^3+z^3 = 2^{6n+1}-1:$$

1. $1+1+5 = 7 = 2^3-1$ $\qquad 1^3+1^3+5^3 = 127 = 2^7-1$
2. $1+11+19 = 31 = 2^5-1$ $\qquad 1^3+11^3+19^3 = 8191 = 2^{13}-1$
3. $1+55+71 = 127 = 2^7-1$ $\qquad 1^3+55^3+71^3 = 524287 = 2^{19}-1$
4. $19+29+79 = 127 = 2^7-1$ $\qquad 19^3+29^3+79^3 = 524287 = 2^{19}-1$
5. $1+239+271 = 511 = 2^9-1$ $\qquad 1^3+239^3+271^3 = 33554431 = 2^{25}-1$.

Through the courtesy of Hans Riesel, Stockholm, we have also:

$$n = \log_2\sqrt{34}, \qquad x,y,z = 13/2,\ 19,\ 83/2$$
$$n = \log_2\sqrt{46}, \qquad x,y,z = 11,\ 47/2,\ 113/2$$
$$n = \log_2\sqrt{76}, \qquad x,y,z = 26,\ 31,\ 94$$
$$n = \log_2\sqrt{79}, \qquad x,y,z = 29,\ 121/4,\ 391/4.$$

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-364 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find and prove a formula for the number $R(n)$ of positive integers less than 2^n whose base 2 representations contain no consecutive 0's. (Here n is a positive integer.)

B-365 Proposed by Phil Mana, Albuquerque, New Mexico.

Show that there is a unique integer $m > 1$ for which integers a and r exist with $L_n \equiv ar^n \pmod{m}$ for all integers $n \geqslant 0$. Also show that no such m exists for the Fibonacci numbers.

B-366 Proposed by Wray G. Brady, University of Tennessee, Knoxville, Tennessee, and Slippery Rock State College, Slippery Rock, Pennsylvania

Prove that $L_i L_j \equiv L_h L_k \pmod{5}$ when $i + j = h + k$.

B-367 Proposed by Gerald E. Bergum, So. Dakota State University, Brookings, So. Dakota.

Let $[x]$ be the greatest integer in x, $a = (1+\sqrt{5})/2$, and $n \geqslant 1$. Prove that

(a) $$F_{2n} = [aF_{2n-1}],$$
and
(b) $$F_{2n+1} = [a^2 F_{2n-1}].$$

B-368 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain functions $g(n)$ and $h(n)$ such that

$$\sum_{i=1}^{n} iF_i L_{n-i} = g(n)F_n + h(n)L_n$$

and use the results to obtain congruences modulo 5 and 10.

B-369 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

For all integers $n \geqslant 0$, prove that the set

$$S_n = \{L_{2n+1}, L_{2n+3}, L_{2n+5}\}$$

has the property that if $x, y \in S_n$ and $x \neq y$ then $xy + 5$ is a perfect square. For $n = 0$ verify that there is no integer z that is not in S_n and for which $\{z, L_{2n+1}, L_{2n+3}, L_{2n+5}\}$ has this property. (For $n > 0$ the problem is unsolved.)

SOLUTIONS
BICENTENNIAL SEQUENCE

B-340 Proposed by Phil Mana, Albuquerque, New Mexico.

Characterize a sequence whose first 28 terms are:

1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875, 1880, 1886, 1897, 1909, 1915, 1920, 1926, 1937, 1943, 1948, 1954, 1965, 1971, 1976.

I. Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

It can easily be verified that the sequence consists of those years when the United States has celebrated Independence Day (July 4) on a Sunday.

II. Solution by Jeffrey Shallit, Wynnewood, Pennsylvania.

According to the World Almanac, the sequence is characterized by the years in which Christmas falls on a Saturday.

Also solved by the Proposer.

CLOSE FACTORING

B-341 Proposed by Peter Lindstrom, Genesee Community College, Batavia, New York.

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

It is well known (see, for example I_{19} in Hoggatt's *Fibonacci and Lucas Numbers*) that

$$F_{n-k}F_{n+k} - F_n^2 = (-1)^{n+k+1}F_k^2.$$

Therefore, replacing n by $2n+2$ and letting $k = 2$, one obtains

$$F_{2n}F_{2n+4} = F_{2n+2}^2 - 1 = (F_{2n+2} - 1)(F_{2n+2} + 1).$$

Thus

$$F_{2n}F_{2n+2}F_{2n+4} = (F_{2n+2} - 1)(F_{2n+2})(F_{2n+2} + 1).$$

Also solved by Gerald Bergum, Richard Blazej, Wray Brady, Michael Brozinsky, Paul S. Bruckman, Herta T. Freitag, Dinh Thế Hung, H. Turner Laquer, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

PERFECT CUBES

B-342 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Prove that

$$2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1}$$

is a perfect cube for $n = 1, 2, \cdots$.

Solution by Graham Lord, Université Laval, Québec, Canada.

$$2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1} = 2L_{n-1}^3 + (L_{n+1} - L_{n-1})^3 + 6L_{n+1}^2L_{n-1} = (L_{n+1} + L_{n-1})^3 = (5F_n)^3.$$

Also solved by Gerald Bergum, George Berzsenyi, Wray Brady, Paul S. Bruckman, Herta T. Freitag, Dinh Thế Hung, H. Turner Laquer, John W. Milsom, Carl F. Moore, C. B. A. Peck, James F. Pope, Bob Prielipp, Jeffrey Shallit, Sahib Singh, David Zeitlin, and the Proposer.

CLOSED FORM

B-343 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple expression for

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)}] .$$

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$F_{2k-1}F_{2n+1-2k} - F_{2k}F_{2n+2-2k} = \frac{1}{5}[L_{2n} + L_{4k-2n-2} - L_{2n+2} + L_{4k-2n-2}] = \frac{1}{5}[2L_{4k-2n-2} - L_{2n+1}],$$

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)})] = \frac{2}{5}\sum_{1}^{n} L_{4k-2n-2} - \frac{n}{5}L_{2n+1}$$

$$= \frac{2}{5}[F_{2(2k-1)-2r-2}]_1^{n+1} - \frac{n}{5}L_{2n+1} = \frac{2}{5}[F_{2n} - F_{-2n}] - \frac{n}{5}L_{2n+1} = \frac{1}{5}[4F_{2n} - nL_{2n+1}] .$$

Also solved by Gerald Bergum, Paul S. Bruckman, Herta T. Freitag, H. Turner Laquer, C. B. A. Peck, Bob Prielipp, Sahib Singh, and the Proposer.

AVERAGING GIVES G. P.'S

B-344 Proposed by Frank Higgins, Naperville, Illinois.

Let c and d be real numbers. Find $\lim_{n\to\infty} x_n$, where x_n is defined by

$$x_1 = c, \quad x_2 = d, \quad \text{and} \quad x_{n+2} = (x_{n+1} + x_n)/2 \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

Solution by Sahib Singh, Clarion State College, Clarion, Pennsylvania.

It is easy to see that $x_{2n+1} - x_1$ and $x_{2n} - x_2$ are both geometric progressions with ¼ as common ratio. Thus $\lim_{n\to\infty} x_n = (c + 2d)/3.$

Also solved by Gerald Bergum, George Berzsenyi, Wray Brady, Michael Brozinsky, Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Dinh Thế Hung, H. Turner Laquer, jointly by Robert McGee and Gerald Satlow and Patricia Cianfero, Carl F. Moore, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

ANOTHER LIMIT

B-345 Proposed by Frank Higgins, Naperville, Illinois.

Let $r > s > 0$. Find $\lim_{n\to\infty} P_n$, where P_n is defined by

$$P_1 = r + s \quad \text{and} \quad P_{n+1} = r + s - (rs/P_n) \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

Solution by Wray Brady, Knoxville, Tennessee.

One can establish by an induction that

$$P_n = (r^{n+1} - s^{n+1})/(r^n - s^n)$$

from which it follows that $P_n \to r$ as $n \to \infty$.

Also solved by Gerald Bergum, George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, H. Turner Laquer, jointly by Robert McGee and Gerald Satlow, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin and the Proposer.

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