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FIBONACCI AND LUCAS NUMBERS AND THE COMPLEXITY OF A GRAPH

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1. TERMINOLOGY

In this note we shall use the following notation and terminology:

$$\text{the Fibonacci numbers } F_n : F_1 = F_2 = 1,$$
$$F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 1;$$
$$\text{the Lucas numbers } \quad L_n : L_1 = 1, \ L_2 = 3,$$
$$L_{n+2} = L_{n+1} + L_n, \quad n \geqslant 1;$$

α,β: zeros of the associated auxiliary polynomial;

a *composition* of a positive integer n is a vector $(a_1, a_2, \cdots, a_k)$ of which the components are positive integers which sum to n;

a *graph* G, is an ordered pair (V,E), where V is a set of vertices, and E is a binary relation on V; the ordered pairs in E are called the edges of the graph.

a *cycle* is a sequence of three or more edges that goes from a vertex back to itself;

a graph is *connected* if every pair of vertices is joined by a sequence of edges;

a *tree* is a connected graph which contains no cycles;

a *spanning tree* of a graph is a tree of the graph that contains all the vertices of the graph;

two spanning trees are *distinct* if there is at least one edge not common to them both;

the *complexity*, $k(G)$, of a graph is the number of distinct spanning trees of the graph.

For relevant examples see Hilton [2] and Rebman [4], and for details see Harary [1].

2. RESULTS

Hilton and Rebman have used combinatorial arguments to establish a relation between the complexity of a graph and the Fibonacci and Lucas numbers. Rebman showed that

$$K(W_n) = L_{2n} - 2, \tag{2.1}$$

where W_n, the n-wheel, is a graph with $n + 1$ vertices obtained from a cycle on n points by joining each of these n points to a further point.

Hilton also established this result and

$$L_{2n} - 2 = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} F_{2a_1} \cdots F_{2a_k}, \tag{2.2}$$

in which $\gamma(n)$ indicates summation over all compositions $(a_1, \cdots, a_k)$ of n, the number of components being variable. It is proposed here to prove (2.1) by a number theoretic approach.

To do so we need the following preliminary results which will be proved in turn:

$$F_{2n} = F_{2n+2} - 2F_{2n} + F_{2n-2}, \tag{2.3}$$

$$1 - 2x^2 + x^4 = \exp\left(-2 \sum_{m=1}^{\infty} x^{2m}/m\right), \tag{2.4}$$

(2.5) $$\sum_{n=0}^{\infty} F_{2n}x^{2n} = x^2 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right),$$

(2.6) $$1+\sum_{n=0}^{\infty} F_{2n}x^{2n} = (1-2x^2+x^4)\exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right),$$

(2.7) $$1+\sum_{n=1}^{\infty} F_{2n}x^{2n} = \exp\left(\sum_{m=1}^{\infty} (L_{2m}-2)x^{2m}/m\right),$$

wherein it is assumed that all power series are considered formally.

3. PROOFS

Proof of (2.3).

$$\begin{aligned} F_{2n} &= F_{2n} + F_{2n-1} - F_{2n-1} \\ &= F_{2n+1} - F_{2n} + F_{2n} - F_{2n-1} \\ &= F_{2n+1} - F_{2n} + F_{2n-2} \\ &= F_{2n+2} - 2F_n + F_{2n-2}\,. \end{aligned}$$

Proof of (2.4).

$$\begin{aligned} 1-2x^2+x^4 &= (1-x^2)^2 \\ &= \exp\ln(1-x^2)^2 \\ &= \exp(-2\ln(1-x^2)^{-1}) \\ &= \exp\left(-2\sum_{m=1}^{\infty} x^{2m}/m\right). \end{aligned}$$

Proof of (2.5).

$$\begin{aligned} \sum_{n=0}^{\infty} F_{2n}x^{2n} &= x^2/(1-3x^2+x^4) \\ &= x^2/(1-\alpha^2x^2)(1-\beta^2x^2) \end{aligned}$$

$$\begin{aligned} \ln\left(\sum_{n=0}^{\infty} F_{2n}x^{2n-2}\right) &= -\ln(1-\alpha^2x^2)(1-\beta^2x^2) \\ &= -\ln(1-\alpha^2x^2) - \ln(1-\beta^2x^2) \\ &= \sum_{m=1}^{\infty}\frac{\alpha^{2m}x^{2m}}{m} + \sum_{m=1}^{\infty}\frac{\beta^{2m}x^{2m}}{m} \\ &= \sum_{m=1}^{\infty}(\alpha^{2m}+\beta^{2m})x^{2m}/m \\ &= \sum_{m=1}^{\infty} L_{2m}x^{2m}/m\,. \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} F_{2n}x^{2n-2} = \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) \quad\text{and}\quad \sum_{n=0}^{\infty} F_{2n}x^{2n} = \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right).$$

Proof of (2.6).

$$\sum_{n=0}^{\infty} F_{2n}x^{2n-2} = \sum_{n=1}^{\infty} F_{2n}x^{2n-2}$$

$$= \sum_{n=0}^{\infty} F_{2n+2}x^{2n}$$

$$= \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right)$$

$$\sum_{n=0}^{\infty} F_{2n-2}x^{2n} = -1 + \sum_{n=0}^{\infty} F_{2n}x^{2n+2}$$

$$= -1 + x^2 \sum_{n=0}^{\infty} F_{2n}x^{2n}$$

$$= -1 + x^4 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) .$$

Now

$$\sum_{n=0}^{\infty} F_{2n}x^{2n} = \sum_{n=0}^{\infty} (F_{2n+2} - 2F_{2n} + F_{2n-2})x^{2n} .$$

So

$$1 + \sum_{n=1}^{\infty} F_{2n}x^{2n} = (1 - 2x^2 + x^4)\exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) .$$

Proof of (2.7).

$$1 + \sum_{n=1}^{\infty} F_{2n}x^{2n} = (1-x^2)^2 \exp\left(\sum_{m=1}^{\infty} L_{2m}x^{2m}/m\right) = \exp\left(\sum_{m=1}^{\infty} (L_{2m}-2)x^{2m}/m\right)$$

from (2.4).

4. MAIN RESULT

To prove the result (2.2) we let

$$W_n = \sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} F_{2a_1} \cdots F_{2a_k} .$$

Then

$$\sum_{n=1}^{\infty} W_n x^{2n} = \sum_{n=1} \left\{ \sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} F_{2a_1} \cdots F_{2a_k} \right\} x^{2n}$$

$$= \sum_{k=1}^{\infty} -\left(-\sum_{n=1}^{\infty} F_{2n}x^{2n}\right)^k \Big/ k$$

$$= \ln\left(1 + \sum_{n=1}^{\infty} F_{2n}x^{2n}\right) = \sum_{n=1}^{\infty} (L_{2n}-2)x^{2n}/n$$

from which we get that

$$W_n = (L_{2n} - 2)/n$$

or

$$L_{2n} - 2 = \sum_{\gamma(n)} \frac{(-1)^{k-1}n}{k} F_{2a_1} \cdots F_{2a_k} .$$

These properties have been generalized elsewhere for arbitrary order recurrence relations [5].

Hoggatt and Lind [3] have also developed similar results in an earlier paper.

The author would like to thank Dr. A. J. W. Hilton of the University of Reading, England, for suggesting the problem.

REFERENCES

1. F. Harary, *Graph Theory,* Addison-Wesley, Reading, Massachusetts, 1969.
2. A. J. W. Hilton, "Spanning Trees and Fibonacci and Lucas Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 3 (October 1974), pp. 259–262.
3. V. E. Hoggatt, Jr., and D. A. Lind, "Fibonacci and Binomial Properties of Weighted Compositions," *J. of Combinatorial Theory,* Vol. 4, No. 2 (March 1968), pp. 121–124.
4. Kenneth R. Rebman, "The Sequence 1 5 16 45 121 320 ⋯ in Combinatorics," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), pp. 51–55.
5. A. G. Shannon, "Ordered Partitions and Arbitrary Order Linear Recurrence Relations," *Mathematics Student,* in press.

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EMBEDDING A GROUP IN THE p^{th} POWERS

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In a finite group *G,* the set of squares, cubes, or p^{th} powers in general, does not necessarily constitute a subgroup. However, we can always embed a finite group into the set of squares, cubes, or any p^{th} powers of another group.

A subgroup H of a group G is said to be a *subgroup of* p^{th} *powers* if for every $y \in H$, there is an $x \in G$ such that $x^p = y$.

Theorem. Every finite group G is isomorphic to a subgroup of p^{th} powers of some permutation group.

Proof. Let G be a finite group, and let P be an isomorphic permutation group on n elements, say $a_{11}, a_{12}, \cdots, a_{1n}$.

Consider a permutation group Q on pn elements

$$a_{11}, a_{12}, \cdots, a_{1n};\quad a_{21}, a_{22}, \cdots, a_{2n};\quad \cdots,\quad a_{p1}, a_{p2}, \cdots, a_{pn},$$

defined in the following manner: For any permutation

$$\sigma = (a_{1i_1} a_{1i_2} \cdots a_{1i_k}) \cdots (a_{1j_1} a_{1j_2} \cdots a_{1j_m})$$

in P corresponds the permutation

$$\hat{\sigma} = (a_{1i_1} a_{1i_2} \cdots a_{1i_k})(a_{2i_1} a_{2i_2} \cdots a_{2i_n}) \cdots (a_{pi_1} a_{pi_2} \cdots a_{pi_k})$$
$$\cdots (a_{1j_1} a_{1j_2} \cdots a_{1j_m})(a_{2j_2} \cdots a_{2j_m}) \cdots (a_{pj_1} a_{pj_2} \cdots a_{pj_m})$$

in the symmetric group S_{pn}. Q is clearly isomorphic to P and each elemenr in Q is the p^{th} power of an element in S_{pn}. In fact, $\hat{\sigma} = \tau^p$, where

$$\tau = (a_{1i_1} a_{2i_1} \cdots a_{pi_1} a_{1i_2} a_{2i_2} \cdots a_{pi_2} \cdots a_{1i_k} a_{2i_k} \cdots a_{pi_k})$$
$$\cdots (a_{1j_1} a_{2j_1} \cdots a_{pj_1} a_{1j_2} a_{2j_2} \cdots a_{pj_2} \cdots a_{1j_m} a_{2j_m} \cdots a_{pj_m}).$$

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IDENTITIES RELATING THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS, II

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Definition. If $i \geqslant 0$ and $n \geqslant 1$, let $q_i^e(n)$ be the number of partitions of n into an even number of parts, where each part occurs at most i times. Let $q_i^o(n)$ be the number of partitions of n into an odd number of parts, where each part occurs at most i times. If $i \geqslant 0$, let $q_i^e(0) = 1$ and $q_i^o(0) = 0$.

Definition. If $i \geqslant 0$ and $n \geqslant 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

The purpose of this paper is to determine $\Delta_i(n)$ when i is any odd positive integer. The only cases previously known were $i = 1$, proved by Euler (see [1]), $i = 3$, proved by this writer (see [2]), and $i = 5$ and 7, proved by Alder and Muwafi (see [3]).

Definition. If s, t, u are positive integers with s odd and $1 \leqslant s < t$, and n is an integer, let $f_{s,t,u}(n)$ be the number of partitions of n in which each odd part occurs at most once and is $\not\equiv \pm s \pmod{2t}$ and in which each even part is divisible by $2t$ and occurs $< u$ times.

Theorem. If s, t, u are positive integers with s odd and $1 \leqslant s < t$, and n is an integer, then

$$\Delta_{2tu-1}(n) = (-1)^n \sum_j f_{s,t,u}(n - tj^2 - (t-s)j).$$

Proof.

$$\sum_n \Delta_{2tu-1}(n)x^n = \prod_{\substack{j \\ j\geqslant 1}} \frac{1-x^{2tuj}}{1+x^j} = \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j}} (1-x^j)\cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2t\mid j}} (1-x^j)(1+x^j+x^{2j}+\dots+x^{(u-1)j})$$

$$= \prod_{\substack{j \\ j\geqslant 0}} (1-x^{2tj+s})(1-x^{2tj+2t-s})(1-x^{2tj+2t})\cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j \\ j\not\equiv \pm s \,(\mathrm{mod}\ 2t)}} (1-x^j)\cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2t\mid j}} (1+x^j+x^{2j}+\dots+x^{(u-1)j})$$

$$= \sum_j (-1)^j x^{tj^2+(t-s)j}\cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j \\ j\not\equiv \pm s \,(\mathrm{mod}\ 2t)}} (1-x^j)\cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2t\mid j}} (1+x^j+x^{2j}+\dots+x^{(u-1)j}),$$

where the last equality follows from Jacobi's identity with $k = t$ and $\ell = t - s$. Since s is odd,

$$tj^2 + (t-s)j \equiv j \pmod 2.$$

Hence, when we substitute $-x$ for x, we obtain

$$\sum_n (-1)^n \Delta_{2tu-1}(n)x^n = \sum_j x^{tj^2+(t-s)j} \cdot \prod_{\substack{j \\ j \geq 1 \\ 2 \nmid j \\ j \not\equiv \pm s \pmod{2t}}} (1+x^j) \cdot \prod_{\substack{j \\ j \geq 1 \\ 2t \mid j}} (1+x^j+x^{2j}+\cdots+x^{(u-1)j})$$

$$= \sum_j x^{tj^2+(t-s)j} \cdot \sum_m f_{s,t,u}(m)x^m,$$

from which the theorem follows immediately.

Corollary 1. If s and t are positive integers with s odd and $1 \leq s < t$, and n is an integer, then

$$\Delta_{2t-1}(n) = (-1)^n \sum_j f_{s,t,1}(n - tj^2 - (t-s)j).$$

Note that $f_{s,t,1}(n)$ is the number of partitions of n into distinct odd parts $\not\equiv \pm s \pmod{2t}$.

Proof. Let $u = 1$ in the theorem.

Letting $s = 1$ and $t = 3$ yields Theorem 1 of [3].

Corollary 2. If $i \geq 2$ and n is an integer, then $(-1)^n\Delta_i(n) \geq 0$.

Proof. For even i, this follows from Theorem 3 of [2]; for odd i, it follows by letting $s = 1$ and $t = (i+1)/2$ in Corollary 1.

Corollary 3. If s and t are positive integers with s odd and $1 \leq s < t$, and n is an integer, then

$$\Delta_{4t-1}(n) = (-1)^n \sum_j f_{s,t,2}(n - tj^2 - (t-s)j).$$

Note that $f_{s,t,2}(n)$ is the number of partitions of n into distinct parts which are either odd but $\not\equiv \pm s \pmod{2t}$ or which are divisible by $2t$.

Proof. Let $u = 2$ in the theorem.

Corollary 4. If u is a positive integer and n is an integer, then

$$\Delta_{4u-1}(n) = (-1)^n \sum_j f_{1,2,u}(n - 2j^2 - j).$$

Note that $f_{1,2,u}(n)$ is the number of partitions of n into parts divisible by 4, where each part occurs $< u$ times.

Proof. Let $s = 1$, $t = 2$ in the theorem.

Letting $u = 1$ yields Theorem 2 of [2] and $u = 2$, Theorem 2 of [3].

REFERENCES

1. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers,* 3rd ed., pp. 221–222, Wiley, New York, 1972.
2. Dean R. Hickerson, "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," *J. Combinatorial Theory, Section A* (1973), pp. 351–353.
3. Henry L. Alder and Amin A. Muwafi, "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," *The Fibonacci Quarterly,* Vol. 13, No. 2 (1975), pp. 337–339.

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ON THE EXISTENCE OF THE RANK OF APPARITION OF m IN THE LUCAS SEQUENCE

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Let m be an arbitrary positive integer. According to the notation of Vinson [1, p. 37] let $s(m)$ denote the period of F_n modulo m and let $f(m)$ denote the rank of apparition of m in F_n.

It is easily verified that

$$(1) \qquad F_{2n+1} = (-1)^n + F_nL_{n+1} = (-1)^{n+1} + F_{n+1}L_n$$

for all integers n.

In the sequel we shall use, without explicit reference, the well known facts that

$$F_{2n} = F_nL_n ,$$

and that F_n and L_n are both odd or both even and

$$(F_n, L_n) = d \leqslant 2, \quad \text{and} \quad F_m \mid F_{mn}$$

for all integers n and $m \neq 0$.

Lemma 1. $F_{2n} \equiv 0 \pmod m$ and $F_{2n+1} \equiv (-1)^n \pmod m$ if and only if $F_n \equiv 0 \pmod m$.

Proof. Let $F_{2n} \equiv 0 \pmod m$ and $F_{2n+1} \equiv (-1)^n \pmod m$. Then by (1), $F_nL_{n+1} \equiv 0 \pmod m$. Since $F_{2n} = F_nL_n \equiv 0 \pmod m$, we have

$$F_nL_{n+2} = F_nL_{n+1} + F_nL_n \equiv 0 \equiv F_nL_{n+1} - F_nL_n = F_nL_{n-1} \pmod m .$$

So whether n is negative or non-negative we obtain after finitely many steps that $F_nL_1 = F_n \equiv 0 \pmod m$.

Conversely, let $F_n \equiv 0 \pmod m$. Then $F_{2n} = F_nL_n \equiv 0 \pmod m$ and by (1), $F_{2n+1} \equiv (-1)^n \pmod m$.

Lemma 2. $F_{2n} \equiv 0 \pmod m$ and $F_{2n+1} \equiv (-1)^{n+1} \pmod m$ if and only if $L_n \equiv 0 \pmod m$.

Proof. Analogous to the proof of Lemma 1.

The following lemma can be found in Wall [2, p. 526]. We give an alternative proof.

Lemma 3. If $m > 2$, then $s(m)$ is even.

Proof. Suppose $s(m)$ is odd. We have by definition of $s(m)$ that

$$F_{2s(m)+1} = F_{s(m)+s(m)+1} \equiv F_{s(m)+1} \equiv 1 = (-1)^{s(m)+1} \pmod m .$$

Also

$$F_{2s(m)} = F_{s(m)}L_{s(m)} \equiv 0 \pmod m .$$

Therefore by Lemma 2, $L_{s(m)} \equiv 0 \pmod m$. But

$$(F_{s(m)}, L_{s(m)}) = d \leqslant 2$$

which contradicts the fact that $m > 2$.

An equivalent form of the following theorem, but with a different proof can be found in Vinson [1, p. 42].

Theorem 1. We have

i) $m > 2$ and $f(m)$ is odd if and only if $s(m) = 4f(m)$
ii) $m = 1$ or 2 or $s(m)/2$ is odd if and only if $s(m) = f(m)$
iii) $f(m)$ is even and $s(m)/2$ is even if and only if $s(m) = 2f(m)$.

Proof. We first prove the sufficiency in each case.

Case i): Let $m > 2$ and $f(m)$ be odd. From Vinson [1, p. 37] we have $f(m)|s(m)$. Since $s(m)$ is even for $m > 2$ we know that $s(m) \neq f(m)$ and $s(m) \neq 3f(m)$. We have $F_{2f(m)} \equiv 0 \pmod m$ and by (1),

$$F_{2f(m)+1} \equiv (-1)^{f(m)} = -1 \pmod m.$$

Therefore $s(m) \neq 2f(m)$ since $m > 2$. But $F_{4f(m)} \equiv 0 \pmod m$ and by (1),

$$F_{4f(m)+1} \equiv (-1)^{2f(m)} = 1 \pmod m.$$

Therefore $s(m) = 4f(m)$.

Case ii): The conclusion is clear for $m = 1$ or 2. Let $m > 2$ and $s(m)/2$ be odd. Then by Case i), $f(m)$ is even. So $F_{2f(m)} \equiv 0 \pmod m$ and by (1),

$$F_{2f(m)+1} \equiv (-1)^{f(m)} = 1 \pmod m$$

which implies that $s(m) \leqslant 2f(m)$. $s(m) \neq 2f(m)$ since $s(m)/2$ is odd and $f(m)$ is even. Therefore since $f(m)|s(m)$, we have $s(m) = f(m)$.

Case iii): Let $f(m)$ be even and $s(m)/2$ be even. Then $m > 2$. We have $F_{2f(m)} \equiv 0 \pmod m$ and by (1),

$$F_{2f(m)+1} \equiv (-1)^{f(m)} = 1 \pmod m.$$

Therefore $s(m) \leqslant 2f(m)$. Now, $F_{s(m)} \equiv 0 \pmod m$ and $F_{s(m)+1} \equiv 1 = (-1)^{s(m)/2} \pmod m$. So by Lemma 1, $F_{s(m)/2} \equiv 0 \pmod m$. Thus $s(m) \neq f(m)$ and therefore since $f(m)|s(m)$ we have $s(m) = 2f(m)$.

The necessity in each case follows directly from the implications already proved.

The following corollary is part of a theorem by Vinson [1, p. 39].

Corollary 1. Let p be any odd prime and e any positive integer. Then we have

i). $f(p^e)$ is odd if and only if $s(p^e) = 4f(p^e)$

ii). $f(p^e)$ is even and $f(p^e)/2$ is odd if and only if $s(p^e) = f(p^e)$

iii). $f(p^e)$ is even and $f(p^e)/2$ is even if and only if $s(p^e) = 2f(p^e)$.

Proof. By Theorem 1, we need only prove that $s(p^e)/2$ is odd if and only if $f(p^e)$ is even and $f(p^e)/2$ is odd. The sufficiency is clear by Theorem 1, ii).

Conversely, let $f(p^e)$ be even and $f(p^e)/2$ be odd. Then

$$F_{f(p^e)} = F_{f(p^e)/2} L_{f(p^e)/2} \equiv 0 \pmod{p^e}.$$

Since

$$(F_{f(p^e)/2}, L_{f(p^e)/2}) = d \leqslant 2 < p$$

we have $L_{f(p^e)/2} \equiv 0 \pmod{p^e}$. Therefore by (1),

$$F_{f(p^e)+1} \equiv (-1)^{(f(p^e)/2)+1} = 1 \pmod{p^e}.$$

Thus $s(p^e) = f(p^e)$ and so $s(p^e)/2$ is odd.

Definition. If m divides some member of the Lucas sequence, let $g(m)$ denote the smallest positive integer n such that $m|L_n$.

If m divides no member of the Lucas sequence, we shall say that $g(m)$ does not exist.

From Vinson [1, p. 37] we have

(2) $$F_n \equiv 0 \pmod m \text{ if and only if } f(m)|n.$$

It is interesting to note from the following proof that if $4|f(4n)$, then $g(4n)$ does not exist.

Lemma 4. If n is an odd integer and $g(4n)$ exists, then $4|L_{f(4n)/2}$.

Proof. By observing the residues of the Lucas sequence modulo 4 we find that $4|L_{g(4n)}$ implies $g(4n) = 3 + 6k$ for some integer k. Therefore $g(4n)$ is odd. We have $4n|L_{g(4n)}|F_{2g(4n)}$. So by (2), $f(4n)|2g(4n)$. Hence $4 \nmid f(4n)$. Since $4|F_{f(4n)}$ we have by (2) that $6 = f(4)|f(4n)$. Since $f(4n)/2$ is odd and $3|f(4n)/2$ we have from Carlitz [3, p. 15] that $4 = L_3|L_{f(4n)/2}$.

Theorem 2. If $m > 2$ and $g(m)$ exists, then $2g(m) = f(m)$.

Proof. We have $m|L_{g(m)}|F_{2g(m)}$. So by (2), $f(m)|2g(m)$. Suppose $f(m)$ is odd. Then $f(m)|g(m)$ and therefore by (2), $m|F_{g(m)}$. Thus $m|(L_{g(m)}, F_{g(m)}) = d \leqslant 2$, a contradiction since $m > 2$. Hence $f(m)$ is even.

To complete the proof it suffices to show that $m|L_{f(m)/2}$ which implies $g(m) = f(m)/2$. We have

$$m|F_{f(m)} = F_{f(m)/2}L_{f(m)/2}.$$

Let $m = m_1m_2$ where $m_1|F_{f(m)/2}$ and $m_2|L_{f(m)/2}$. Since $f(m)/2|g(m)$ we have $m_1|F_{f(m)/2}|F_{g(m)}$. Therefore $m_1|(F_{g(m)}, L_{g(m)}) = d \leqslant 2$. So $m_1 = 1$ or 2. If $m_1 = 1$, then $m_2 = m|L_{f(m)/2}$, the desired conclusion. Assume $m_1 = 2$. Then m is even. Since $2|F_{f(m)/2}$ we have $2|L_{f(m)/2}$. If $m_2 = m/2$ is odd, then $2m_2 = m|L_{f(m)/2}$, the desired conclusion. Assume $m_2 = m/2$ is even. Since $g(8)$ does not exist we know that $8 \nmid m$. Therefore $m_2/2 = m/4$ is odd. Since $g(4(m_2/2)) = g(m)$ exists we have by Lemma 4 that $4|L_{f(m)/2}$. Thus $m = 4(m_2/2)|L_{f(m)/2}$. The proof is complete.

Corollary 2. For any odd prime p and any positive integer e, $g(p^e)$ exists if and only if $f(p^e)$ is even.

Proof. The sufficiency follows from Theorem 2 and the necessity follows from the facts $F_{2n} = F_nL_n$ and $(F_n, L_n) = d \leqslant 2 < p$ for all integers n.

Theorem 3. We have

i) $g(m)$ exists and is odd if and only if $s(m) = f(m)$
ii) $g(m)$ exists and is even if and only if $s(m) = 2f(m)$ and $F_{f(m)+1} \equiv -1 \pmod{m}$
iii) $g(m)$ does not exist if and only if either $s(m) = 2f(m)$ and $F_{f(m)+1} \not\equiv -1 \pmod{m}$ or $s(m) = 4f(m)$.

Proof. Case i): Let $g(m)$ exist and be odd. The case $m = 1$ or 2 is clear. Assume $m > 2$. By Theorem 2, $f(m) = 2g(m)$. Therefore by (1),

$$F_{f(m)+1} \equiv (-1)^{g(m)+1} = 1 \pmod{m}.$$

Hence $s(m) = f(m)$.

Conversely, let $s(m) = f(m)$. The case $m = 1$ or 2 is clear. Assume $m > 2$. By Theorem 1, $s(m)/2$ is odd. Therefore

$$F_{s(m)} \equiv 0 \pmod{m} \quad \text{and} \quad F_{s(m)+1} \equiv 1 = (-1)^{(s(m)/2)+1} \pmod{m}.$$

Hence by Lemma 2, $L_{s(m)/2} \equiv 0 \pmod{m}$ and thus $g(m)$ exists. By Theorem 2, $s(m) = f(m) = 2g(m)$. Therefore $g(m)$ is odd.

Case ii): Let $g(m)$ exist and be even. Then $m > 2$ and by Theorem 2, $f(m) = 2g(m)$. Thus $4|f(m)$ and so by Theorem 1, $s(m) = 2f(m)$. By (1), $F_{f(m)+1} \equiv (-1)^{g(m)+1} = -1 \pmod{m}$.

Conversely, let $s(m) = 2f(m)$ and $F_{f(m)+1} \equiv -1 \pmod{m}$. We have $F_{f(m)} \equiv 0 \pmod{m}$. By Theorem 1, $m > 2$ and $f(m)$ is even. If $f(m)/2$ is odd, then $F_{f(m)+1} \equiv (-1)^{f(m)/2} \pmod{m}$ which implies by Lemma 1 that $F_{f(m)/2} \equiv 0 \pmod{m}$, a contradiction. Hence $f(m)/2$ is even. Therefore $F_{f(m)+1} \equiv (-1)^{(f(m)/2)+1} \pmod{m}$ which implies by Lemma 2 that $L_{f(m)/2} \equiv 0 \pmod{m}$. Thus $g(m)$ exists and by Theorem 2, $f(m)/2 = g(m)$ is even.

Case iii): Follows from Cases i) and ii) and from Theorem 1.

Corollary 3. For any odd prime p and any positive integer e we have

i) $g(p^e)$ exists and is odd if and only if $s(p^e) = f(p^e)$
ii) $g(p^e)$ exists and is even if and only if $s(p^e) = 2f(p^e)$
iii) $g(p^e)$ does not exist if and only if $s(p^e) = 4f(p^e)$.

Proof. In view of Theorem 3 we need only prove that $s(p^e) = 2f(p^e)$ implies $F_{f(p^e)+1} \equiv -1 \pmod{p^e}$. By Corollary 1, if $s(p^e) = 2f(p^e)$, then $f(p^e)$ is even and $f(p^e)/2$ is even. We have

$$F_{f(p^e)} = F_{f(p^e)/2}L_{f(p^e)/2} \equiv 0 \pmod{p^e} \quad \text{and} \quad (F_{f(p^e)/2}, L_{f(p^e)/2}) = d \leqslant 2 < p.$$

Therefore $L_{f(p^e)/2} \equiv 0 \pmod{p^e}$. So by (1),

$$F_{f(p^e)+1} \equiv (-1)^{(f(p^e)/2)+1} = -1 \pmod{p^e}.$$

Theorem 4. Let p be an odd prime and e be any positive integer. Then

i) $g(p^e)$ exists and is odd if $p \equiv 11$ or $19 \pmod{20}$

ii) $g(p^e)$ exists and is even if $p \equiv 3$ or $7 \pmod{20}$

iii) $g(p^e)$ does not exist if $p \equiv 13$ or $17 \pmod{20}$

iv) $g(p^e)$ is odd or does not exist if $p \equiv 21$ or $29 \pmod{40}$.

Proof. Follows from Vinson [1, p. 43] and Corollary 3.

Wall [2, p. 525] has shown that the period of L_n modulo m exists for all positive integers m.

Let $h(m)$ denote the period of L_n modulo m.

Corollary 4. Let $g(m)$ exist. Then

i) $m = 1$ or 2 if and only if $h(m) = g(m)$

ii) $m > 2$ and $g(m)$ is odd if and only if $h(m) = 2g(m)$

iii) $g(m)$ is even if and only if $h(m) = 4g(m)$.

Proof. Since $g(m)$ exists and $g(5)$ does not exist we have $(m, 5) = 1$. So from the corollary to Theorem 8 of Wall [2, p. 529] we have $s(m) = h(m)$. We first prove the sufficiency in each case.

Case i) is clear.

Case ii): By Theorems 2 and 3, $2g(m) = f(m) = s(m) = h(m)$.

Case iii): By Theorems 2 and 3, $4g(m) = 2f(m) = s(m) = h(m)$.

The necessity in each case follows directly from the implications already proved.

REFERENCES

1. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," *The Fibonacci Quarterly*, Vol. 1, No. 2 (April 1963), pp. 37–45.
2. D. D. Wall, "Fibonacci Series Modulo m," *Amer. Math. Monthly*, 67 (1960), pp. 525–532.
3. L. Carlitz, "A Note on Fibonacci Numbers" *The Fibonacci Quarterly*, Vol. 2, No. 1 (Feb. 1964), pp. 15–28.

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RECURRENCES OF THE THIRD ORDER AND RELATED COMBINATORIAL IDENTITIES

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1. Let g be a rational integer such that $\Delta = 4g^3 + 27$ is squarefree and let w denote the real root of the equation

$$x^3 + gx - 1 = 0 \qquad (g > 1). \tag{1.1}$$

Clearly w is a unit of the cubic field $Q(w)$.

Following Bernstein [1], put

$$w^n = r_n + s_n w + t_n w^2 \qquad (n \geqslant 0) \tag{1.2}$$

and

$$w^{-n} = x_n + y_n w + z_n w^2 \qquad (n \geqslant 0). \tag{1.3}$$

Making use of the theory of units in an algebraic number field, Bernstein obtained some combinatorial identities. He showed that

$$s_n = r_{n+2}, \quad t_n = r_{n+1}, \quad y_n = x_{n-2}, \quad z_n = x_{n-1}$$

and

$$\sum_{n=0}^{\infty} r_n u^n = \frac{1+gu^2}{1+gu-u^3}, \qquad \sum_{n=0}^{\infty} x_n u^n = \frac{1}{1-gu^2-u^3} . \tag{1.4}$$

Moreover, it follows from (1.2) and (1.3) that

$$\begin{cases} r_n^2 - r_{n-1}r_{n+1} = x_{n-3} \\ x_n^2 - x_{n-1}x_{n+1} = r_{n+3} \end{cases} . \tag{1.5}$$

Explicit formulas for r_n and x_n are implied by (1.4). Substituting in (1.5) the combinatorial identities result. Since $\Delta = 4g^3 + 27$ is squarefree for infinitely many values of g, the identities are indeed polynomial identities.

The present writer [2] has proved these and related identities using only some elementary algebra. For example, if we put

$$1 + gx^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

and define

$$\sigma_n = \alpha^n + \beta^n + \gamma^n \qquad (\text{all } n)$$

and

$$\rho_n = \begin{cases} r_n & (n \geqslant 0) \\ x_{-n} & (n \geqslant 0) \end{cases},$$

then various relations are found connecting these quantities. For example

$$\sigma_m \sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n} . \tag{1.6}$$

Each relation of this kind implies a combinatorial identity.

In the present paper we consider a slightly more general situation. Let u, v denote indeterminates and put

$$1 - ux + vx^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x).$$

We define σ_n by means of

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(1.7) $$\sigma_n = \alpha^n + \beta^n + \gamma^n \qquad \text{(all } n\text{)}$$

and ρ_n by

(1.8) $$\rho_n = A\alpha^n + B\beta^n + C\gamma^n \qquad \text{(all } n\text{)},$$

where A, B, C are determined by

$$\frac{1}{1 - vx + ux^2 - x^3} = \frac{A}{1 - \beta\gamma x} + \frac{B}{1 - \gamma\alpha x} + \frac{C}{1 - \alpha\beta x}$$

Thus

(1.9) $$\sum_{n=0}^{\infty} \rho_{-n} x^n = \frac{1}{1 - vx + ux^2 - x^3}$$

and

(1.10) $$\sum_{n=0}^{\infty} \rho_n x^n = \frac{1 - ux + vx^2}{1 - ux + vx^2 - x^3} ,$$

while

(1.11) $$\sum_{n=0}^{\infty} \sigma_n x^n = \frac{3 - 2ux + vx^2}{1 - ux + vx^2 - x^3}$$

and

(1.12) $$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \frac{3 - 2vx + ux^2}{1 - vx + ux^2 - x^3}$$

Since $\alpha^3 - \alpha^2 u + \alpha v - 1 = 0$, it is clear from the definition of σ_n, ρ_n that

$$\sigma_{n+3} - u\sigma_{n+2} + v\sigma_{n+1} - \sigma_n = 0$$

and

$$\rho_{n+3} - u\rho_{n+3} + v\rho_{n+1} - \rho_n = 0$$

for arbitrary n.

If we use the fuller notation

$$\sigma_n = \sigma_n(u,v), \qquad \rho_n = \rho_n(u,v),$$

it follows from the generating functions that

(1.13) $$\sigma_{-n}(u,v) = \sigma_n(v,u), \qquad \rho_n(u,v) = \rho_{3-n}(v,u).$$

We show that

(1.14) $$\sigma_m\sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n} ,$$

for arbitrary m,n. Similarly

(1.15) $$\sigma_m\rho_n = \rho_{m+n} + \rho_{m-n}\alpha_{-n} - \rho_{m-2n} .$$

As for the product $\rho_m\rho_n$, we have first

(1.16) $$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \rho_{n-3}\sigma_{n-3} .$$

The more general result is

(1.17) $$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1}$$
$$= \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{n-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6} ,$$

again for arbitrary m,n.

Each of the functions $\sigma_n(u,v)$, $\sigma_{-n}(u,v)$, $\rho_n(u,v)$, $\rho_{-n}(u,v)$, $n \geqslant 0$, is a polynomial in u,v. Explicit formulas for these polynomials are given in (2.9), (2.10), (4.5), (4.6) below. Moreover σ_{pn} is a polynomial in σ_n, σ_{-n}; indeed we have

(1.18) $$\sigma_{pn}(u,v) = \sigma_p(\sigma_n, \sigma_{-n}) \qquad (p \geqslant 0).$$

The corresponding formula for ρ_{pn} is somewhat more elaborate; see (4.3) and (4.4) below.

Substitution of the explicit formulas for σ_n, σ_{-n}, ρ_n, ρ_{-n} in any of the relations such as (1.14), (1.15), (1.16), (1.17) gives rise to a large number of polynomial identities.

The introduction of two indeterminates u, v in σ_n, ρ_n leads to somewhat more elaborate formulas than those in [1]. However the greater symmetry implied by (1.13) is gratifying.

2. It follows from

$$1 - ux + vx^2 - x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x) \tag{2.1}$$

that

$$\begin{cases} \alpha + \beta + \gamma = u \\ \beta\nu + \gamma\alpha + \alpha\beta = v \\ \alpha\beta\gamma = 1 \end{cases} \tag{2.2}$$

Since $\alpha\beta\nu = 1$, (2.1) is equivalent to

$$1 - vx + ux^2 - x^3 = (1 - \beta\gamma x)(1 - \gamma\alpha x)(1 - \alpha\beta x). \tag{2.3}$$

We have defined

$$\sigma_n = \alpha^n + \beta^n + \gamma^n, \tag{2.4}$$

for n an arbitrary integer. Thus

$$\sum_{n=0}^{\infty} \sigma_n x^n = \sum \frac{1}{1 - \alpha x} = \frac{\Sigma(1 - \beta x)(1 - \gamma x)}{1 - ux + vx^2 - x^3},$$

which, by (2.2), reduces to

$$\sum_{n=0}^{\infty} \sigma_n x^n = \frac{3 - 2ux + vx^2}{1 - ux + vx^2 - x^3}. \tag{2.5}$$

Similarly

$$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \sum \frac{1}{1 - \beta\gamma x} = \frac{(1 - \alpha\beta x)(1 - \alpha\gamma x)}{1 - vx + ux^2 - x^3}.$$

so that

$$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \frac{3 - 2vx + ux^2}{1 - vx + ux^2 - x^3}. \tag{2.6}$$

Using the fuller notation

$$\sigma_n = \sigma_n(u,v), \quad \sigma_{-n} = \sigma_{-n}(u,v),$$

it is clear from (2.5) and (2.6) that

$$\sigma_{-n}(u,v) = \sigma_n(v,u). \tag{2.7}$$

By (2.1), α, β, ν are the roots of

$$z^3 - uz^2 + vz - 1 = 0$$

and so

$$\sigma_{n+3} - u\sigma_{n+2} + v\sigma_{n+1} - \sigma_n = 0, \tag{2.8}$$

for all n.

Next,

$$(1 - ux + vx^2 - x^3)^{-1} = \sum_{k=0}^{\infty} (ux - vx^2 + x^3)^k = \sum_{i,j,k=0}^{\infty} (-1)^j (i,j,k) u^i v^j x^{i+2j+3k}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{i+2j+3k=n} (-1)^j (i,j,k) u^i v^j,$$

where

$$(i,j,k) = \frac{(i+j+k)!}{i!\,j!\,k!} .$$

Thus, by (2.5),

$$\sigma_n = 3 \sum_{i+2j+3k=n} (-1)^j (i,j,k) u^i v^j - 2u \sum_{i+2j+3k=n-1} (-1)^j (i,j,k) u^i v^j + v \sum_{i+2j+3k=n-2} (-1)^j (i,j,k) u^i v^j$$

$$= \sum_{i+2j+3k=n} (-1)^j u^i v^j \{3(i,j,k) - 2(i-1,j,k) - (i,j-1,k)\} .$$

Hence

(2.9) $$\sigma_n = \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \qquad (n > 0).$$

By (2.7) the corresponding formula for σ_{-n} is

(2.10) $$\sigma_{-n} = \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j \qquad (n > 0).$$

It follows that, for n prime, coefficients of all terms—except the leading term—in σ_n are divisible by n.

Returning to (2.4), we have

$$\sigma_m \sigma_n = \Sigma \alpha^m \Sigma \alpha^n = \Sigma \alpha^{m+n} + \Sigma \alpha^m (\beta^n + \gamma^n) = \sigma_{m+n} + \Sigma \alpha^{m-n} (\alpha^n \beta^n + \alpha^n \gamma^n)$$
$$= \sigma_{m+n} + \Sigma \alpha^{m-n} (\alpha_{-n} - \beta^n \gamma^n),$$

which gives

(2.11) $$\sigma_m \sigma_n = \sigma_{m+n} + \sigma_{m-n} \sigma_{-n} - \sigma_{m-2n} ,$$

valid for all m, n. Replacing m by $m+2n$, (2.11) becomes

(2.12) $$\sigma_{m+3n} - \sigma_{m+2n} \sigma_n + \sigma_{m+n} \sigma_{-n} - \sigma_m = 0 .$$

For $m = n$, (2.11) reduces to

(2.13) $$\sigma_n^2 = \sigma_{2n} + 2\sigma_{-n} .$$

Hence, for $m = 2n$,

$$\sigma_n \sigma_{2n} = \sigma_{3n} + \sigma_n \sigma_{-n} - 3,$$

so that

(2.14) $$\sigma_{3n} = \sigma_n^3 - 3\sigma_n \sigma_{-n} + 3 .$$

To get the general formula we take

$$\sum_{p=0}^{\infty} \sigma_{pn} x^k = \sum \frac{1}{1-\alpha^n x} = \frac{\Sigma (1-\beta^n x)(1-\gamma^n x)}{(1-\alpha^n x)(1-\beta^n x)(1-\gamma^n x)} = \frac{3 - 2\sigma_n x + \sigma_{-n} x^2}{1 - \sigma_n x + \sigma_{-n} x^2 - x^3} .$$

Comparing with (2.5), it is evident from (2.9) that

(2.15) $$\sigma_{pn} = \sum_{i+2j+3k=p} (-1)^j \frac{p}{i+j+k} (i,j,k) \sigma_n^i \sigma_{-n}^j \qquad (p > 0).$$

Substitution from (2.9) and (2.10) in (2.11), (2.12), (2.13), (2.14), (2.15) evidently results in a number of combinatorial identities. We state only

(2.16) $$\left\{ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \right\}^2 = \sum_{i+2j+3k=2n} (-1)^j \frac{2n}{i+j+k} (i,j,k) u^i v^j + 2 \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j$$
$$(n \geqslant 0).$$

3. Put

$$\frac{1}{1 - vx + ux^2 - x^3} = \frac{A}{1-\beta\gamma x} + \frac{B}{1-\gamma a x} + \frac{C}{1-a\beta x}, \tag{3.1}$$

where A, B, C are independent of x. Then

$$(1-a^2\beta)(1-a^2\gamma)A = 1. \tag{3.2}$$

Since

$$(1-a^2\beta)(1-a^2\gamma) = 1 - a^2(\beta+\gamma) + a^4\beta\gamma = 1 - a^2(u-a) + a^3 = 1 - a^2u + 2a^3,$$

it follows from $a^3 - a^2u + av - 1 = 0$ that

$$A = \frac{1}{3 - 2av + a^2u} \tag{3.3}$$

with similar formulas for B and C.

Replacing x by $1/x$ in (3.1) and simplifying, we get

$$\frac{x^3}{1 - ux + vx^2 - x^3} = -\sum \frac{Ax}{\beta\gamma - x} = \sum \frac{Aax}{1-ax} = \sum \frac{A}{1-ax} - \sum A.$$

Since $\Sigma A = 1$, it follows that

$$\frac{1 - ux + vx^2}{1 - ux + vx^2 - x^3} = \sum \frac{A}{1-ax}. \tag{3.4}$$

We now define ρ_n, ρ_{-n} by means of

$$\frac{1 - ux + vx^2}{1 - ux + vx^2 - x^3} = \sum_{n=0}^{\infty} \rho_n x^n \tag{3.5}$$

and

$$\frac{1}{1 - vx + ux^2 - x^3} = \sum_{n=0}^{\infty} \rho_{-n} x^n. \tag{3.6}$$

It then follows from (3.1) and (3.4) that

$$\rho_n = \Sigma A a^n, \tag{3.7}$$

for all n.

By (3.6), we have, for arbitrary m and n,

$$\rho_m\rho_n = \Sigma A a^m \cdot \Sigma A a^n = \Sigma A^2 a^{m+n} + \Sigma BC(\beta^m\gamma^n + \gamma^m\beta^n).$$

Thus

$$\rho_{m+1}\rho_{n-1} = \Sigma A^2 a^{m+n} = BC(\beta^{m+1}\gamma^{n-1} + \gamma^{m+1}\beta^{n-1}),$$

so that

$$\rho_m\rho_n - \rho_{m+1}\rho_{n-1} = \Sigma BC\left\{(\beta^m\gamma^n + \gamma^m\beta^n) - (\beta^{m+1}\gamma^{n-1} + \gamma^{m+1}\beta^{n-1})\right\}. \tag{3.8}$$

The quantity in braces is equal to

$$-(\beta-\gamma)(\beta^m\gamma^{n-1} - \gamma^m\beta^{n-1}).$$

Hence

$$\begin{cases} \rho_m\rho_n - \rho_{m+1}\rho_{n-1} = -\Sigma BC(\beta-\gamma)(\beta^m\gamma^{n-1} - \gamma^m\beta^{n-1}) \\ \rho_m\rho_n - \rho_{m-1}\rho_{n+1} = -\Sigma BC(\beta-\gamma)(\beta^n\gamma^{m-1} - \gamma^n\beta^{m-1}) \end{cases}.$$

It follows that

$$\begin{aligned} &2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} \\ &= -\Sigma BC(\beta-\gamma^2(\beta^{m-1}\gamma^{n-1} + \gamma^{m-1}\beta^{n-1}). \end{aligned} \tag{3.9}$$

By (3.2),

$$BC(\beta-\gamma)^2 = -Aa^2,$$

so that (3.9) becomes

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \Sigma A(\beta^{m-3}\gamma^{n-3} + \gamma^{m-3}\beta^{n-3}). \tag{3.10}$$

In particular, if $m = n$, (3.10) reduces to

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \Sigma A\beta^{n-3}\gamma^{n-3} = \Sigma Aa^{-n+3}$$

and so

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{-n+3} \qquad (\text{all } n). \tag{3.11}$$

To get a more general result consider

$$\beta^m\nu^n + \gamma^m\beta^n = (\beta^m + \gamma^m)(\beta^n + \gamma^n) - (\beta^{m+n} + \gamma^{m+n}) = (\sigma_m - a^m)(\sigma_n - a^n) - (\sigma_{m+n} - a^{m+n})$$
$$= \sigma_m\sigma_n - \sigma_m a^n - \sigma_n a^m - \sigma_{m+n} + 2a^{m+n}.$$

Thus

$$\Sigma A(\beta^m\gamma^n + \gamma^m\beta^n) = \sigma_m\sigma_n - \sigma_{m+n} - \sigma_m\beta_n - \sigma_n\beta_m + 2\rho_{m+n}. \tag{3.12}$$

Combining (3.10) and (3.12) we get

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{n-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6}. \tag{3.13}$$

For $m = n$, (3.13) reduces to

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \sigma_{n-3}\rho_{n-3} + \sigma_{-n+3}. \tag{3.14}$$

It is not evident that (3.14) is equivalent to (3.11). This is proved immediately below.

4. We now take

$$\rho_m\sigma_n = \Sigma Aa^m\Sigma a^n = \Sigma Aa^{m+n} + \Sigma Aa^m(\beta^n + \gamma^n) = \rho_{m+n} + \Sigma Aa^{m-n}(a^n\beta^n - a^n\gamma^n)$$
$$= \rho_{m+n} + \Sigma Aa^{m-m}(\sigma_{-n} - a^{-n}),$$

which gives

$$\rho_m\sigma_n = \rho_{m+n} + \rho_{m-n}\sigma_{-n} - \rho_{m-2n}. \tag{4.1}$$

In particular, for $m = n$,

$$\rho_n\sigma_n = \rho_{2n} + \sigma_{-n} - \rho_{-n}, \tag{4.2}$$

which shows that (3.14) is indeed equivalent to (3.11).

For $m = 2n$, (4.1) gives

$$\rho_{3n} = \rho_{2n}\sigma_n - \rho_n\sigma_{-n} + 1 = \rho_n\sigma_n^2 - \sigma_n\sigma_{-n} + \rho_{-n}\sigma_n - \rho_n\sigma_{-n} + 1.$$

To get a general formula for ρ_{pn} take

$$\sum_{p=0}^{\infty} \rho_{pn}x^p = \sum_{p=0}^{\infty} x^p \sum Aa^{pn} = \sum \frac{A}{1-a^n x} = \frac{\Sigma A(1-\beta^n x)(1-\gamma^n x)}{(1-a^n x)(1-\beta^n x)(1-\gamma^n x)}$$

$$= \frac{1-(\sigma_n - \rho_n)n + \rho_{-n}x^2}{1 - \sigma_n x + \sigma_{-n}x^2 - x^3}.$$

Then, as in the proof of (2.15), we have

$$\rho_{pn} = c_{p,n} - (\sigma_n - \rho_n)c_{p-1,n} + \rho_{-n}c_{p-2,n} \qquad (p \geqslant 0), \tag{4.3}$$

where

$$c_{p,n} = \sum_{i+2j+3k=p} (-1)^j(i,j,k)\sigma_n^i\sigma_{-n}^j. \tag{4.4}$$

Since

$$\rho_1 = \Sigma A\alpha = 0, \qquad \rho_2 = \Sigma A\alpha^2 = 0,$$

we have in particular

(4.5) $$\rho_p = \sum_{i+2j+k=p-3} (-1)^j (i,j,k) u^i v^j \qquad (p \geq 3)$$

and

(4.6) $$\rho_{-p} = \sum_{i+2j+3k=p} (-1)^j (i,j,k) v^i u^j \qquad (p \geq 0).$$

With the fuller notation

$$\rho_n = \rho_n(u,v), \qquad \rho_{-n} = \rho_{-n}(u,v),$$

it is clear from (4.5) and (4.6) that

(4.7) $$\rho_n(u,v) = \rho_{3-n}(v,u).$$

Moreover (4.4) becomes

(4.8) $$c_{p,n} = \rho_p(\sigma_n, \sigma_{-n}) \qquad (p \geq 0).$$

We may now substitute from the explicit formulas (2.9), (2.10), (4.5), (4.6) in various formulas of Sections 3 and 4 to obtain a large number of polynomial identities in two indeterminants. To give only one relatively simple example, we take (4.2). Thus

(4.9) $$\left\{ \sum_{i+2j+3k=n-3} (-1)^j (i,j,k) u^i v^j \right\} \left\{ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \right\}$$

$$= \sum_{i+2j+3k=2(n-3)} (-1)^j (i,j,k) u^i v^j - \sum_{i+2j+3k=n} (-1)^j (i,j,k) v^i u^j$$

$$+ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j \qquad (n \geq 0).$$

5. For small n, σ_n and ρ_n can be computed without much labor by means of the recurrences. Moreover the results are extended by the symmetry relations

$$\sigma_{-n}(u,v) = \sigma_n(v,u), \qquad \rho_n(u,v) = \rho_{3-n}(v,u).$$

A partial check on σ_n is furnished by the result, that, for prime n,

$$\sigma_n(u,v) \equiv u^n \pmod{n}.$$

Also, by (2.5),

$$\sum_{n=0}^{\infty} \sigma_n(1,1) x^n = \frac{3 - 2x + x^2}{1 - x + x^2 - x^3} = \frac{3 + x - x^2 + x^3}{1 - x^4},$$

which implies

$$\sigma_n(1,1) = 3, \quad \sigma_{4n+1}(1,1) = \sigma_{4n+3}(1,1) = 1, \quad \sigma_{4n+2}(1,1) = -1.$$

As for $\rho_n(1,1)$, we have by (3.5)

$$\sum_{n=0}^{\infty} \rho_n(1,1) x^n = \frac{1 - x + x^2}{1 - x + x^2 - x^3} = \frac{1 + x^3}{1 - x^4},$$

so that

$$\rho_{4n}(1,1) = \rho_{4n+3}(1,1) = 1, \qquad \rho_{4n+1}(1,1) = \rho_{4n+2}(1,1) = 0.$$

Table 1

$\sigma_0 = 3, \quad \sigma_1 = u, \quad \sigma_2 = u^2 - 2v$
$\sigma_3 = u^3 - 3uv + 3$
$\sigma_4 = u^4 - 4u^2v + 2v^2 + 4u$
$\sigma_5 = u^5 - 5u^3v + 5uv^2 + 5u^2 - 5v$
$\sigma_6 = u^6 - 6u^4v + 9u^2v^2 + 6u^3 - 2v^3 + 12uv + 3$
$\sigma_7 = u^7 - 7u^5v + 14u^3v^2 + 7u^4 - 7uv^3 - 21u^2v + 7v^2 + 7u$
$\sigma_8 = u^8 - 8u^6v + 20u^4v^2 + 8u^5 - 16u^2v^3 - 32u^3v + 2v^4 + 24uv^2 + 12u^2 - 8v$
$\sigma_9 = u^9 - 9u^7v + 27u^5v^2 + 9u^6 - 30u^3v^3 - 45u^4v + 9uv^4 + 54u^2v^2 + 18u^3 - 9v^2 - 27uv + 3$
$\sigma_{10} = u^{10} - 10u^8v + 35u^6v^2 + 10u^7 - 50u^4v^3 - 60u^5v + 25u^2v^4 + 100u^3v^2 - 2v^5 + 25u^4 - 40uv^3 - 60u^2v + 15v^2 + 10u$

Table 2

$\rho_0 = 1, \quad \rho_1 = \rho_2 = 0, \quad \rho_3 = 1$
$\rho_4 = u, \quad \rho_5 = u^2 - v$
$\rho_6 = u^3 - 2uv + 1$
$\rho_7 = u^4 - 3u^2v + v^2 + 2u$
$\rho_8 = u^5 - 4u^3v + 3uv^2 + 3u^2 - 2v$
$\rho_9 = u^6 - 5u^4v + 6u^2v^2 + 4u^3 - v^3 - 6uv + 1$
$\rho_{10} = u^7 - 6u^5v + 10u^3v^2 + 5u^4 - 4uv^3 - 12u^2v + 3v^2 + 3u$

REFERENCES

1. L. Bernstein, "Zeros of Combinatorial Functions and Combinatorial Identities," *Houston Journal of Math.*, Vol. 2 (1976), pp. 9–16.
2. L. Carlitz, "Some Combinatorial Identities of Bernstein," *SIAM Journal of Mathematical Analysis*, to appear.

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SOME SEQUENCE-TO-SEQUENCE TRANSFORMATIONS WHICH PRESERVE COMPLETENESS

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1. INTRODUCTION

A sequence $\{s_i\}_1^\infty$ of positive integers is termed *complete* if *every* positive integer N can be expressed as a distinct sum of terms from the sequence; it is well known ([1], Theorem 1) that if $\{s_i\}_1^\infty$ is nondecreasing with $s_1 = 1$, then a necessary and sufficient condition for completeness is

$$s_{n+1} \leqslant 1 + \sum_1^n s_i \quad \text{for } n \geqslant 1. \tag{1}$$

Using this criterion for completeness, we will exhibit several transformations which convert a given complete sequence of positive integers into another sequence of positive integers without destroying completeness. Since the Fibonacci numbers $(F_1 = F_2 = 1,\ F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 2)$ and the sequence of primes with unity adjoined $(P_1 = 1,\ P_2 = 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ \cdots)$ are examples of complete sequences, our results will yield as special cases some new complete sequences associated with the Fibonacci numbers and the primes.

2. QUANTIZED LOGARITHMIC TRANSFORMATION

Let $[x]$ denote the greatest integer contained in x, and define the function $<\cdot>$ by

$$<x> = 1 + [x] \quad \text{for all real } x.$$

Thus $<x>$ is the least integer $>x$ in contrast to $[x]$, the greatest integer $\leqslant x$. Both $<\cdot>$ and $[\cdot]$ may be thought of as quantizing characteristics in the sense that a non-integral x is rounded off to the integer immediately following x in the case of $<\cdot>$ or to the integer immediately preceding x when $[\cdot]$ is used. If x is an integer, then $[x] = x$ and $<x> = 1 + x$. The following lemma shows that $<\cdot>$ is subadditive:

Lemma 1. $$<x+y> \leqslant <x> + <y>.$$

Proof. If $x = [x] + \eta_x$ and $y = [y] + \eta_y$ with $0 \leqslant \eta_x, \eta_y < 1$, then

$$<x+y> = <[x] + [y] + \eta_x + \eta_y> \leqslant [x] + [y] + 2 = 1 + [x] + [y] + 1 = <x> + <y>.$$

Lemma 2. Let $\ln x$ denote the natural logarithm of x. Then for $x, y \geqslant 2$,

$$\ln (x+y) \leqslant \ln x + \ln y$$

that is, the logarithm is subadditive on the domain $[2, \infty)$.

Proof. For $x,y \geqslant 2$,

$$x + y \leqslant 2 \cdot \max (x,y) \leqslant \min (x,y) \max (x,y) = xy,$$

and $\ln (x+y) \leqslant \ln (xy) = \ln x + \ln y$, from the nondecreasing property of the logarithm.

Theorem 1. Let $\{s_i\}_1^\infty$ be a strictly increasing, complete sequence of positive integers. Then the sequence $\{<\ln s_i>\}_2^\infty$ is also complete.

Proof. By the assumed completeness,

$$s_{n+1} \leqslant 1 + \sum_1^n s_i \quad \text{for } n \geqslant 1.$$

Since $s_1 = 1$, we may write

$$s_{n+1} \leqslant 2 + \sum_2^n s_i \quad \text{for } n \geqslant 1;$$

hence,

$$\ln s_{n+1} \leqslant \ln\left(2 + \sum_2^n s_i\right),$$

and, on noting $s_i \geqslant 2$ for $i \geqslant 2$, it follows from Lemma 2 (by induction) that

$$\ln s_{n+1} \leqslant \ln 2 + \sum_2^n \ln s_i .$$

Now we may use the nondecreasing and subadditive (lemma 1) properties of $<\cdot>$ to conclude

$$<\ln s_{n+1}> \leqslant \left\langle \ln 2 + \sum_2^n \ln s_i \right\rangle \leqslant <\ln 2> + \sum_2^n <\ln s_i> = 1 + \sum_2^n <\ln s_i> \quad \text{for } n \geqslant 2.$$

Hence (noting $<\ln s_2> = <\ln 2> = 1$) by the completeness criterion, the sequence $\{<\ln s_i>\}_2^\infty$ is complete, proving the theorem.

The following theorem yields a similar conclusion for a class of functions ϕ where each ϕ possesses properties similar to that of the logarithmic function.

Theorem 2. Let $\{s_i\}_1^\infty$ be a nondecreasing complete sequence of positive integers and let $\phi(\cdot)$ be a function defined on the domain $x \geqslant 1$, nondecreasing and subadditive on that domain with $0 \leqslant \phi(1) < 1$. Then $\{<\phi(s_i)>\}_1^\infty$ is complete.

Proof. From

$$s_{n+1} \leqslant 1 + \sum_1^n s_i ,$$

it follows that

$$\phi(s_{n+1}) \leqslant \phi\left(1 + \sum_1^n s_i\right) \leqslant \phi(1) + \sum_1^n \phi(s_i).$$

Then

$$<\phi(s_{n+1})> \leqslant <\phi(1)> + \sum_1^n <\phi(s_i)> = 1 + \sum_1^n <\phi(s_i)>,$$

so that, with $<\phi(1)> = 1$ and the completeness criterion, the sequence $\{\phi(s_i)\}_1^\infty$ is complete.

NOTE. Theorem 1 is not a special case of Theorem 2 since the logarithm is not subadditive on $[1, \infty)$. It is also clear that the domain of ϕ could be restricted to only those integers lying in $[1, \infty)$.

EXAMPLE. If $\phi(x) = \sqrt{x - 1/2}$ for $x \geqslant 1$, the reader may easily verify that ϕ is nondecreasing, subadditive and $0 \leqslant \phi(1) = \sqrt{1/2} < 1$. Therefore $\{<\sqrt{s_i - 1/2}>\}_1^\infty$ is complete whenever $\{s_i\}_1^\infty$ is a nondecreasing complete sequence of positive integers.

EXAMPLE. The function $\phi(x) = ax$ for $x \geqslant 1$ and some fixed $a > 0$ is nondecreasing and subadditive, and if

$0 < a < 1$, then $\phi(1) = a$ and ϕ satisfies the conditions of Theorem 2. Thus, for example, the sequence

$$\left\{\binom{s_i}{2}\right\}_1^\infty$$

is complete whenever $\{s_i\}_1^\infty$ is a nondecreasing complete sequence of positive integers.

EXAMPLE: If $P_1 = 1, P_2 = 2, 3, 5, 7, 11, \cdots$ denotes the sequence of primes (with unity adjoined); then it is well known [2] that $\{P_i\}_1^\infty$ is complete. Hence by Theorem 1, the sequence $\{\langle \ln P_i \rangle\}_2^\infty$ is also complete, and thus each positive integer N has an expansion of the form

$$N = \sum_2^\infty a_i \langle \ln P_i \rangle ,$$

where each a_i is binary (zero or one). The series is clearly finite, since $a_i = 0$ for $i \geqslant k$, where k is such that $\langle \ln P_k \rangle$ exceeds N.

It is of interest to prove the completeness of $\{\langle \ln P_i \rangle\}_2^\infty$ directly without using the completeness of $\{P_i\}_1^\infty$. In this manner, we avoid the implicit use of Bertrand's postulate which is normally invoked in showing the primes are complete.

Theorem 3. The sequence $\{\langle \ln P_i \rangle\}_2^\infty$ is complete.

Proof. Using Euler's classical argument, we observe that

$$1 + \prod_2^n P_i$$

is not divisible by $P_1, P_2, \cdots, P_n$ and therefore must have a prime divisor larger than P_n; that is

$$1 + \prod_2^n P_i \geqslant P_{n+1},$$

or

$$P_{n+1} \leqslant 1 + \prod_1^n P_i \leqslant 2 \prod_1^n P_i \quad \text{for } n \geqslant 1.$$

Since the logarithm is an increasing function,

$$\ln P_{n+1} \leqslant \ln 2 + \sum_1^n \ln P_i$$

and consequently,

$$\langle \ln P_{n+1} \rangle \leqslant \langle \ln 2 \rangle + \sum_1^n \langle \ln P_i \rangle = 1 + \sum_1^n \langle \ln P_i \rangle$$

establishing the result by the completeness criterion.

3. LUCAS TRANSFORMATION

The transformation defined in the following theorem is called a Lucas Transformation since it corresponds to the manner in which the Lucas sequence is generated from the Fibonacci sequence.

Theorem 4. Let $\{u_i\}_1^\infty$ be a nondecreasing complete sequence with $u_1 = u_2 = 1$. Define a sequence $\{v_i\}_0^\infty$ by

$$\begin{cases} v_0 = 1 \\ v_1 = 2 \\ v_n = u_{n-1} + u_{n+1} \quad \text{for } n \geqslant 2. \end{cases}$$

Then $\{v_i\}_0^\infty$ is complete.

Proof. For $n \geqslant 1$,

$$v_{n+1} = u_n + u_{n+2} \leqslant 1 + \sum_{1}^{n-1} u_i + 1 + \sum_{1}^{n+1} u_i = (u_{n+1} + u_{n-1}) + (u_n + u_{n-2}) + \cdots + (u_3 + u_1) + u_2 + u_1 + 2$$

$$= v_n + v_{n-1} + \cdots + v_2 + u_2 + u_1 + 2 = v_n + v_{n-1} + \cdots + v_2 + v_1 + v_0 + 1 = 1 + \sum_{0}^{n} v_i\,,$$

where we have used $u_2 + u_1 + 2 = 4 = v_1 + v_0 + 1$. Thus $v_0 = 1$ and

$$v_{n+1} \leqslant 1 + \sum_{0}^{n} v_i$$

for $n \geqslant 0$ which implies that $\{v_i\}_0^\infty$ is complete.

EXAMPLE: Let $u_i = F_i$, where $\{F_i\}_1^\infty$ is the Fibonacci sequence. Then the sequence defined by

$$v_0 = 1, \quad v_1 = 2, \quad v_n = F_{n-1} + F_{n+1} \quad \text{for } n \geqslant 2$$

is complete by Theorem 4. Moreover, recalling that the Lucas numbers $\{L_n\}_0^\infty$, defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geqslant 1,$$

are also expressible by

$$L_n = F_{n-1} + F_{n+2} \quad \text{for } n \geqslant 2,$$

we see that $\{v_n\}_0^\infty$ is simply the sequence $\{L_n\}_0^\infty$ put in nondecreasing order by an interchange of L_0 and L_1. Completeness is not affected by a renumbering of the sequence; however, the inequality criterion for completeness must be applied only to nondecreasing sequences.

4. SUMMARY

If S denotes the set of all nondecreasing complete sequences of positive integers, we have considered certain transformations which map S into itself. In particular, it was shown, as special cases of the general results, that the sequences $\{\langle \ln F_n \rangle\}_3^\infty$, $\{\langle \ln P_n \rangle\}_2^\infty$ and $\{\langle \alpha F_n \rangle\}_2^\infty$ are complete sequences, where $\langle\cdot\rangle$ is defined by $\langle x \rangle = 1 + [x]$, $\{F_n\} = \{1, 1, 2, 3, 5, \cdots\}$ is the Fibonacci sequence, $\{P_n\} = \{1, 2, 3, 5, 7, 11, \cdots\}$ is the sequence of primes with unity adjoined and α is a fixed constant satisfying $0 < \alpha < 1$.

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," *American Math. Monthly,* Vol. 68, No. 6, June–July, 1961, pp. 557–560.
2. V.E. Hoggatt, Jr., and Bob Chow, "Some Theorems on Completeness," *The Fibonacci Quarterly,* Vol. 10, No. 5, 1972, pp. 551–554.

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AN IDENTITY RELATING COMPOSITIONS AND PARTITIONS

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The following partition identity was proved in [1]:

Theorem. If $f(r,n)$ denotes the number of partitions of n of the form $n = b_0 + b_1 + \cdots + b_s$, where for $0 \leqslant i \leqslant s-1$, $b_i \geqslant rb_{i+1}$, and $g(r,n)$ denotes the number of partitions of n, where each part is of the form $1 + r + r^2 + \cdots + r^i$ for some $i \geqslant 0$, then $f(r,n) = g(r,n)$.

In this paper, we will give a generalization of this theorem.

In [1], the parts of the partitions were listed in non-increasing order. It will, however, be more convenient for our purposes to list them in non-decreasing order.

The main result of this paper is given in the following theorem.

Theorem 1. Let $r_1, r_2, \cdots$ be integers. Let $c_0 = 1$ and, for $i \geqslant 1$, let $c_i = r_1c_{i-1} + r_2c_{i-2} + \cdots + r_ic_0$. Suppose that, for all $i \geqslant 0$, $c_i > 0$. For $i \geqslant 0$, let $t_i = c_0 + \cdots + c_i$ and define $T = \{t_0, t_1, t_2, \cdots\}$. Then, for $n \geqslant 0$, the number, $f(n)$, of compositions $b_0 + \cdots + b_s$ of n in which $b_i \geqslant r_1b_{i-1} + r_2b_{i-2} + \cdots + r_ib_0$ for $1 \leqslant i \leqslant s$, is equal to the number, $g(n)$, of partitions of n with parts in T.

Proof. Let $n = a_0t_0 + \cdots + a_st_s$ be a partition of n counted by $g(n)$, where $a_s > 0$. Define, for $0 \leqslant i \leqslant s$,

$$b_i = \sum_{0 \leqslant j \leqslant i} a_{j+s-i}\, c_j\, .$$

Then

$$b_0 + \cdots + b_s = \sum_{0 \leqslant i \leqslant s} b_{s-i} = \sum_{0 \leqslant i \leqslant s}\ \sum_{0 \leqslant j \leqslant s-i} a_{i+j}c_j = \sum_{0 \leqslant k \leqslant s}\left(a_k \sum_{0 \leqslant j \leqslant k} c_j\right) = \sum_{0 \leqslant k \leqslant s} a_k t_k = n\, .$$

Also, for $0 \leqslant i \leqslant s$,

$$b_i = \sum_{0 \leqslant j \leqslant i-1} a_{j+s-i}c_j + a_sc_i > \sum_{0 \leqslant j \leqslant i-1} a_{j+s-i}c_j \geqslant 0\, .$$

Therefore, $b_0 + \cdots + b_s$ is a composition of n. Moreover, for $1 \leqslant i \leqslant s$,

$$b_i \geqslant \sum_{1 \leqslant j \leqslant i} a_{j+s-i}c_j = \sum_{1 \leqslant j \leqslant i}\left(a_{j+s-i} \sum_{1 \leqslant k \leqslant j} r_kc_{j-k}\right) = \sum_{1 \leqslant k \leqslant i}\left(r_k \sum_{k \leqslant j \leqslant i} a_{j+s-i}c_{j-k}\right)$$

$$= \sum_{1 \leqslant k \leqslant i}\left(r_k \sum_{0 \leqslant j \leqslant i-k} a_{j+s-(i-k)}c_j\right) = \sum_{1 \leqslant k \leqslant i} r_kb_{i-k}\, .$$

Thus, $b_0 + \cdots + b_s$ is a composition of n counted by $f(n)$.

This constitutes a mapping ϕ from the set of partitions counted by $g(n)$ into the set of compositions counted by $f(n)$. It suffices to show that ϕ is one-to-one and onto.

If ϕ is not one-to-one, then there exist distinct partitions $a_ot_o + \cdots + a_st_s$ and $a'_ot_o + \cdots + a'_{s'}t_{s'}$ of n which yield the same composition. From the definition of ϕ, it follows that $s = s'$. Let i_o be the least $i \geqslant 0$ such that $a_{s-i} \neq a'_{s-i}$. Then

$$a_{s-i_0} = a_{s-i_0}c_0 = b_{i_0} - \sum_{1\le j\le i_0} a_{s-(i_0-j)}c_j = b_{i_0} - \sum_{i\le j\le i_0} a'_{s-(i_0-j)}c_j = a'_{s-i_0}c_0 = a'_{s-i_0},$$

a contradiction. Hence ϕ is one-to-one.

We will now show that ϕ is onto. Let $b_0 + \cdots + b_s$ be a composition counted by $f(n)$. Define, for $0 \le i \le s$,

$$a_{s-i} = b_i - \sum_{1\le j\le i} r_j b_{i-j}\ .$$

We claim that $a_0t_0 + \cdots + a_st_s$ is a partition counted by $g(n)$ whose image under ϕ is the composition $b_0 + \cdots + b_s$.

Clearly, $a_s = b_0 > 0$. Also, for $1 \le i \le s$,

$$b_i \ge r_1b_{i-1} + \cdots + r_ib_0 = \sum_{1\le j\le i} r_jb_{i-j}$$

so $a_{s-i} \ge 0$. Also,

$$a_0t_0 + \cdots + a_st_s = \sum_{0\le i\le s} a_{s-i}t_{s-i} = \sum_{0\le i\le s}\left(b_i - \sum_{1\le j\le i} r_jb_{i-j}\right)t_{s-i} = \sum_{0\le i\le s} b_it_{s-i} - \sum_{0\le j<i\le s} r_{i-j}b_jt_{s-i}$$

$$= \sum_{0\le j\le s} b_jt_{s-j} - \sum_{0\le j\le s}\left(b_j \sum_{j<i\le s} r_{i-j}t_{s-i}\right) = \sum_{0\le j\le s} b_j\left(t_{s-j} - \sum_{j<i\le s} r_{i-j}t_{s-i}\right)$$

$$= \sum_{0\le j\le s} b_{s-j}\left(t_j - \sum_{s-j<i\le s} r_{i-s+j}t_{s-i}\right) = \sum_{0\le j\le s} b_{s-j}\left(t_j - \sum_{1\le i\le j} r_it_{j-i}\right)\ .$$

For $0 \le j \le s$, we have

$$t_j - \sum_{1\le i\le j} r_it_{j-i} = \sum_{0\le k\le j} c_k - \sum_{1\le i\le j}\left(r_i \sum_{i\le k\le j} c_{k-i}\right)$$

$$= \sum_{0\le k\le j}\left(c_k - \sum_{1\le i\le k} r_ic_{k-i}\right) = c_0 + \sum_{1\le k\le j}\left(c_k - \sum_{1\le i\le k} r_ic_{k-i}\right).$$

By definition,

$$c_0 = 1 \quad \text{and} \quad c_k = \sum_{1\le i\le k} r_ic_{k-i} \quad \text{for} \quad k \ge 1,$$

so

$$t_j - \sum_{1\le i\le j} r_it_{j-i} = 1 \quad \text{and} \quad a_0t_0 + \cdots + a_st_s = \sum_{0\le j\le s} b_{s-j} = n.$$

Therefore, $a_0t_0 + \cdots + a_st_s$ is a partition counted by $g(n)$.

We have

$$\sum_{0\le j\le i} a_{j+s-i}c_j = \sum_{0\le k\le i} a_{s-k}c_{i-k} = \sum_{0\le k\le i} c_{i-k}\left(b_k - \sum_{1\le j\le k} r_jb_{k-j}\right) = \sum_{0\le m\le i} c_{i-m}b_m$$

$$- \sum_{\substack{0\le k\le i\\0\le m\le k}} c_{i-k}r_{k-m}b_m = b_i + \sum_{0\le m<i} b_m\left(c_{i-m} - \sum_{m<k\le i} c_{i-k}r_{k-m}\right) = b_i + \sum_{0\le m<i} b_m\left(c_{i-m} - \sum_{1\le j\le i-m} r_jc_{(i-m)-j}\right) = b_i$$

Therefore, the image under ϕ of the partition $a_0t_0 + \cdots + a_st_s$ is the composition $b_0 + \cdots + b_s$, so the proof is complete.

We will now determine when Theorem 1 is a partition identity. This occurs if and only if, for every $n \geqslant 0$, all compositions counted by $f(n)$ are partitions. Since $c_0 + c_1 + \cdots + c_i$ is a composition counted by $f(t_i)$, a necessary condition is that $c_0 \leqslant c_1 \leqslant c_2 \leqslant \cdots$. We now show that this condition is also sufficient.

Theorem 2. Suppose the hypotheses of Theorem 1 are satisfied, and, in addition, $c_0 \leqslant c_1 \leqslant c_2 \leqslant \cdots$. Then, for $n \geqslant 0$, the number of partitions $b_0 + \cdots + b_s$ of n in which $b_i \geqslant r_1b_{i-1} + \cdots + r_ib_0$, for $1 \leqslant i \leqslant s$, is equal to the number of partitions of n with parts in T.

Proof. It suffices to show that all compositions counted by $f(n)$ are partitions. Suppose $b_0 + \cdots + b_s$ is such a composition. Let $1 \leqslant k \leqslant s$. We will show, by induction on i, that, for $1 \leqslant i \leqslant k$,

$$b_k - b_{k-1} \geqslant (c_i - c_{i-1})b_{k-i} + \sum_{0\leqslant j<k-i} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<i} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) .$$

Applying this with $i = k$ gives

$$b_k - b_{k-1} \geqslant (c_k - c_{k-1})b_0 \geqslant 0,$$

which will complete the proof.

We have

$$b_k - b_{k-1} \geqslant \sum_{0\leqslant j<k} b_jr_{k-j} - b_{k-1} = (c_1 - c_0)b_{k-1} + \sum_{0\leqslant j<k-1} b_jr_{k-j} ,$$

so the inequality holds for $i = 1$. Suppose it holds for $i = m - 1$, where $2 \leqslant m \leqslant k$. Then

$$\begin{aligned} b_k - b_{k-1} &\geqslant (c_{m-1} - c_{m-2})b_{k-m+1} + \sum_{0\leqslant j<k-m+1} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<m-1} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) \\ &\geqslant (c_{m-1} - c_{m-2})\left(\sum_{0\leqslant j<k-m+1} b_jr_{k-j-m+1}\right) + \sum_{0\leqslant j<k-m+1} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<m-1} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) \\ &= \sum_{0\leqslant j\leqslant k-m} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) = b_{k-m}\left(r_m + \sum_{1\leqslant \ell<m} (c_\ell - c_{\ell-1})r_{m-\ell}\right) \\ &\quad + \sum_{0\leqslant j<k-m} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right). \end{aligned}$$

But

$$r_m + \sum_{1\leqslant \ell<m} (c_\ell - c_{\ell-1})r_{m-\ell} = \sum_{0\leqslant \ell<m} c_\ell r_{m-\ell} - \sum_{1\leqslant \ell<m} c_{\ell-1}r_{m-\ell} = c_m - c_{m-1} ,$$

so

$$b_k - b_{k-1} \geqslant (c_m - c_{m-1})b_{k-m} + \sum_{0\leqslant j<k-m} b_j\left(r_{k-j} + \sum_{1\leqslant \ell<m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right),$$

and the inequality holds for $i = m$. This completes the induction and the proof.

The following is an important corollary of Theorem 2.

Corollary. Suppose $r_1, r_2, \cdots$ are non-negative integers with $r_1 \geqslant 1$. Define T as above. Then, for $n \geqslant 0$, the number of partitions $b_0 + \cdots + b_s$ of n in which $b_i \geqslant r_1b_{i-1} + \cdots + r_ib_0$, for $1 \leqslant i \leqslant s$, is equal to the number of partitions of n with parts in T.

Proof. For $i \geqslant 1$, $c_i = r_1c_{i-1} + r_2c_{i-2} + \cdots + r_ic_0 \geqslant c_{i-1}$, and Theorem 2 applies.

We will now illustrate Theorems 1 and 2 and the corollary to Theorem 2 by some examples.

EXAMPLE 1. In the corollary, let $r_1 = r \geq 1$ and $r_2 = r_3 = \cdots = 0$. Then, for $i \geq 0$, $c_i = r^i$ and $t_i = 1 + r + \cdots + r^i$. Hence, for $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of n in which $b_i \geq rb_{i-1}$ for $1 \leq i \leq s$ is equal to the number of partitions of n with parts of the form $1 + r + \cdots + r^i$ for $i \geq 0$. This is the result of [1].

EXAMPLE 2. In the corollary, let $r_1 = r_2 = 1$ and $r_3 = r_4 = \cdots = 0$. Then, for $i \geq 0$, $c_i = F_{i+1}$ and $t_i = F_{i+3} - 1$. Thus,

$$T = \{F_3 - 1, F_4 - 1, \cdots\} = \{1, 2, 4, 7, 12, \cdots\}.$$

For $n \geq 0$, the number of partitions of n in which each part is greater than or equal to the sum of the two preceding parts is equal to the number of partitions of n in which each part is 1 less than a Fibonacci number.

EXAMPLE 3. In the Corollary, let $r_1 = r_2 = \cdots = 1$. Then $c_0 = 1$ and, for $i \geq 1$, $c_i = 2^{i-1}$. Hence $t_i = 2^i$, for $i \geq 0$, and $T = \{1, 2, 4, 8, \cdots\}$. For $n \geq 0$, the number of partitions of n in which each part is greater than or equal to the sum of all preceding parts is equal to the number of partitions of n into powers of 2.

EXAMPLE 4. In Theorem 2, let $r_1 = -2$, $r_2 = -1$, $r_3 = r_4 = \cdots = 0$. Then, for $i \geq 0$, $c_i = i + 1$ and

$$t_i = \frac{(i+1)(i+2)}{2},$$

so $T = \{1, 3, 6, 10, 15, \cdots\}$. For $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of n in which $b_1 \geq 2b_0$ and, for $2 \leq i \leq s$, $b_i \geq 2b_{i-1} - b_{i-2}$ is equal to the number of partitions of n into triangular numbers.

EXAMPLE 5. In Theorem 1, let $r_1 = (-1)^{i+1}F_{i+2}$, for $i \geq 1$. Then $c_0 = 1$, $c_1 = 2$, $c_2 = c_3 = \cdots = 1$, so $t_0 = 1$ and $t_i = i+2$ for $i \geq 1$. Hence, $T = \{1, 3, 4, 5, 6, \cdots\}$. For $n \geq 0$, the number of compositions $b_0 + \cdots + b_s$ of n in which

$$b_i \geq 2b_{i-1} - 3b_{i-2} + 5b_{i-3} + \cdots + (-1)^{i+1}F_{i+2}b_0,$$

for $1 \leq i \leq s$, is equal to the number of partitions of n with no part equal to 2.

REFERENCE

1. Dean R. Hickerson, "A Partition Identity of the Euler Type," *Amer. Math. Monthly*, 81 (1974), pp. 627–629.

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ON THE MULTIPLICATION OF RECURSIVE SEQUENCES

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1. INTRODUCTION

The object of this note is to generalize the results of Catlin [1] and Wyler [3] for the multiplication of recurrences. They studied second-order recurrences whereas the aim here is to set up definitions for their arbitrary order analogues.

The work is also related to that of Peterson and Hoggatt [2]. They considered a type of multiplication of series in their exposition of the characteristic numbers of Fibonacci-type sequences. In the last section of this paper we see how a definition of a characteristic arises from the earlier definition of multiplication.

We define an arbitrary order recursive sequence $\{W_n\}$ by the recurrence relation

(1.1) $$W_n = \sum_{j=1}^{r} (-1)^{j+1} P_j W_{n-j}, \qquad n > r,$$

in which the P_j are arbitrary integers, and there are suitable initial values, $W_1, W_2, \cdots, W_r$. (Suppose $W_n = 0$ for $n \leqslant 0$.)

We shall need to consider some particular cases of these as well as some results associated with the product sums of the roots, a_t, of the associated auxiliary equation

(1.2) $$a_t^r = \sum_{j=1}^{r} (-1)^{j+1} P_j a_t^{r-j} .$$

2. PRODUCT SUMS

We define the product sum

$$S_{tm} = \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m}$$

with $S_{to} = 1$. For example, when $r = 3$,

$$S_{31} = a_1 + a_2 \quad \text{and} \quad S_{32} = a_1 a_2 .$$

Some results we shall use now follow.

(2.1) $$S_{tm} = P_m - a_t S_{t,m-1} .$$

Proof.

$$P_m - a_t S_{t,m-1} = \sum a_{j_1} a_{j_2} \cdots a_{j_m} - a_t \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m} = \sum_{j \neq t} a_{j_1} a_{j_2} \cdots a_{j_m} .$$

For example, when $r = 3$,

$$P_2 - a_1 S_{11} = a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 (a_2 + a_3) = a_2 a_3 = S_{12} .$$

(2.2) $$S_{tr} = 0$$

Proof.

$$P_j = S_{tj} + a_t S_{t,j-1}$$

$$\sum_{j=1}^{r} (-1)^{j+1} P_j a_t^{r-j} = \sum_{j=1}^{r} (-1)^{j+1} S_{tj} a_t^{r-j} - \sum_{j=1}^{r} (-1)^{j+1} S_{t,j-1} a_t^{r-j+1} \;;$$

that is

$$a_t^r = S_{tr} + S_{to} a_t^r \,,$$

which yields the result.

We note out of interest that:

$$S_{tm} = \sum_{j=0}^{m} (-1)^{m-j} P_j a_t^{m-j}, \qquad P_0 = 1 \tag{2.3}$$

Proof. We use induction on m.

$$S_{t0} = 1, \quad S_{t1} = P_1 - a_t\,, \quad \cdots,$$

$$S_{tm} = P_m - a_t S_{t,m-1} = P_m - a_t P_{m-1} + a_t^2 S_{t,m-2} = \sum_{j=0}^{m} (-1)^{m-j} P_j a_t^{m-j}\,.$$

$$\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j} = a_t^n \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{r-j}\,, \qquad n \geqslant 0. \tag{2.4}$$

Proof. We use induction on n. When n is zero, the result is obvious. Suppose the result is true for $n = 1, 2, \cdots, k-1$. Then

$$\begin{aligned}
\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{k+r-j} &= A_{k+r} + \sum_{j=1}^{r-1} (-1)^j S_{tj} A_{k+r-j} \\
&= \sum_{j=1}^{r} (-1)^{j+1} P_j A_{k+r-j} + \sum_{j=1}^{r-1} (-1)^j S_{tj} A_{k+r-j} \\
&= (-1)^{r+1} P_r A_k + \sum_{j=1}^{r-1} (-1)^j (S_{tj} - P_j) A_{k+r-j} \\
&= (-1)^{r+1} a_t S_{t,r-1} A_k + \sum_{j=1}^{r-1} (-1)^{j-1} a_t S_{t,j-1} A_{k+r-j} \\
&= (-1)^{r-1} a_t S_{t,r-1} A_k + \sum_{j=0}^{r-2} (-1)^j a_t S_{tj} A_{k+r-j-1} \\
&= a_t \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{k+r-j-1} \\
&= a_t^{k-r} \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{r-j} \quad \text{(by the inductive hypothesis)},
\end{aligned}$$

and so the result follows. In particular, it follows that

$$\sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j} = a_t \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j-1}\,. \tag{2.5}$$

Result (2.4) is a generalization of Wyler's:

$$A_{n+1} - a_1 A_n = a_2^n (A_1 - a_1 A_0).$$

For ease of notation we shall write

$$\sum (t, A_n) = \sum_{j=0}^{r-1} (-1)^j S_{tj} A_{n+r-j}.$$

3. MATRIX RESULTS

We define matrices with rows i and columns j, $1 \leq i,\ j \leq r$:

(3.1) $$W^{(n)} = [W_{n+r-i+j}],$$

(3.2) $$M = [(-1)^{i+j} P_{j-i}], \quad \text{with } P_n = \begin{cases} 0 \text{ for } n < 0 \\ 1 \text{ for } n = 0 \end{cases},$$

(3.3) $$S^{(t)} = [(-1)^{i+j} S_{t,j-i}], \quad \text{with } S_{tn} = 0 \text{ for } n < 0,$$

(3.4) $$E = [\delta_{i,j-1}] \quad \text{(Kronecker delta)},$$

(3.5) $$Q = [q_{ij}], \quad \text{with } q_{ij} = \begin{cases} (-1)^{j+1} P_j & \text{for } i = 1 \\ \delta_{i-1,j} & \text{for } i > 1 \end{cases}.$$

It follows from definitions (3.2), (3.3) and result (2.1) that

$$M = [(-1)^{i+j} P_{j-i}] = [(-1)^{i+j} S_{t,j-1}] - a_t[(-1)^{i+j} S_{t,j-i-1}] = S^{(t)} - a_t E S^{(t)} = (I - a_t E) S^{(t)}.$$

It can be readily proved by induction on n that

(3.6) $$W^{(n)} = Q^n W^{(0)}.$$

Furthermore,

$$S^{(t)} A^{(0)} = [\Sigma(t, A_{j-i})],$$

and so by using property (2.5), we find

$$S^{(t)} A (I - a_t E) = [S_{1j} \Sigma(t, A_{1-i})].$$

4. MULTIPLICATION

We can define a product $\{A_n\}\{B_n\}$ of two of these sequences to be the sequence $\{C_n\}$:

(4.1) $$C^{(0)} = A^{(0)} M B^{(0)}.$$

It follows from result (2.4) that

(4.2) $$C^{(m+n)} = Q^m C^{(0)} {Q^n}^T = A^{(m)} M B^{(n)}.$$

We can see how these generalize Catlin and Wyler. When $r = 2$:

$$W^{(0)} = \begin{bmatrix} W_2 & W_3 \\ W_1 & W_2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} P_1 & -P_2 \\ 1 & 0 \end{bmatrix}.$$

Result (4.2) becomes

$$\begin{bmatrix} C_{m+n+2} & C_{m+n+3} \\ C_{m+n+1} & C_{m+n+2} \end{bmatrix} = \begin{bmatrix} A_{m+2} & A_{m+3} \\ A_{m+1} & A_{m+2} \end{bmatrix} \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B_{n+2} & B_{n+3} \\ B_{n+1} & B_{n+2} \end{bmatrix}$$
$$= \begin{bmatrix} A_{m+2} & A_{m+3} - P_1 A_{m+2} \\ A_{m+1} & A_{m+2} - P_1 A_{m+1} \end{bmatrix} \begin{bmatrix} B_{n+2} & B_{n+3} \\ B_{n+1} & B_{n+2} \end{bmatrix}.$$

from which we get, after equating corresponding matrix entries:

$$C_{m+n+2} = A_{m+2} B_{n+2} - P_2 A_{m+1} B_{n+1},$$
$$C_{m+n+1} = A_{m+1} B_{n+2} + A_{m+2} B_{n+1} - P_1 A_{m+1} B_{n+1},$$

in which we have used the recurrence relation

$$A_{m+3} = P_1A_{m+2} - P_2A_{m+1} .$$

These results agree with Catlin and Wyler.

For $r = 3$, we have

$$W^{(0)} = \begin{bmatrix} W_3 & W_4 & W_5 \\ W_2 & W_3 & W_4 \\ W_1 & W_2 & W_3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -P_1 & P_2 \\ 0 & 1 & -P_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} P_1 & -P_2 & P_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Result (4.2) now becomes

$$\begin{bmatrix} C_{m+n+3} & C_{m+n+4} & C_{m+n+5} \\ C_{m+n+2} & C_{m+n+3} & C_{m+n+4} \\ C_{m+n+1} & C_{m+n+2} & C_{m+n+3} \end{bmatrix}$$

$$= \begin{bmatrix} A_{m+3} & A_{m+4} - P_1A_{m+3} & A_{m+5} - P_1A_{m+4} + P_2A_{m+3} \\ A_{m+2} & A_{m+3} - P_1A_{m+2} & A_{m+4} - P_1A_{m+3} + P_2A_{m+2} \\ A_{m+1} & A_{m+2} - P_1A_{m+1} & A_{m+3} - P_2A_{m+2} + P_2A_{m+1} \end{bmatrix} \cdot \begin{bmatrix} B_{n+3} & B_{n+4} & B_{n+5} \\ B_{n+2} & B_{n+3} & B_{n+4} \\ B_{n+1} & B_{n+2} & B_{n+3} \end{bmatrix}$$

from which we obtain, for example,

$$C_{m+n+3} = A_{m+3}B_{n+3} + A_{m+4}B_{n+2} - P_1A_{m+3}B_{n+2} + P_3A_{m+2}B_{n+1} .$$

We further obtain

$$\text{(4.3)} \qquad \sum_{j=0}^{r-1} (-1)^j S_{tj}C_{r-j} = \sum_{i=0}^{r-1} (-1)^i S_{ti}A_{r-i} \sum_{j=0}^{r-1} (-1)^j S_{tj}B_{r-j} .$$

Proof. We premultiply each side of definition (4.1) by $S^{(t)}$:

$$S^{(t)}C^{(0)} = S^{(t)}A^{(0)}MB^{(0)} = S^{(t)}A^0(I - a_tE)S^{(t)}B^{(0)} = S_{ij}\Sigma(t, A_{1-i})S^{(t)}B^{(0)},$$

or

$$\begin{bmatrix} \Sigma(t, C_0) & \Sigma(t, C_1) & \cdots & \Sigma(t, C_{r-1}) \\ \Sigma(t, C_{-1}) & \Sigma(t, C_{-2}) & \cdots & \Sigma(t, C_{r-2}) \\ & \cdots & \cdots & \\ \Sigma(t, C_{1-r}) & \Sigma(t, C_{-r}) & \cdots & \Sigma(t, C_0) \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma(t, A_0) & 0 & \cdots & 0 \\ \Sigma(t, A_{-1}) & 0 & \cdots & 0 \\ & & \cdots & \\ \Sigma(t, A_{1-r}) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \Sigma(t, B_0) & \Sigma(t, B_1) & \cdots & \Sigma(t, B_{r-1}) \\ \Sigma(t, B_{-1}) & \Sigma(t, B_{-2}) & \cdots & \Sigma(t, B_{r-2}) \\ & & \cdots & \\ \Sigma(t, B_{1-r}) & \Sigma(t, B_{-r}) & \cdots & \Sigma(t, B_0) \end{bmatrix}$$

and so,

$$\Sigma(t, C_0) = \Sigma(t, A_0)\,\Sigma(t, B_0),$$

as required. When $r = 2$, $t = 1$, result (4.3) becomes

$$(C_2 - a_2C_1) = (A_2 - a_2A_1)(B_2 - a_2B_1)$$

as in Wyler and Catlin. When $r = 3$, $t = 1$:

$$(C_3 - (a_2 + a_3)C_2 + a_2a_3C_1) = (A_3 - (a_2 + a_3)A_2 + a_2a_3A_1)(B_3 - (a_2 + a_3)B_2 + a_2a_3B_1).$$

Using property (2.4), we get

$$\sum_{i=0}^{r-1} (-1)^i S_{ti}A_{m+r-i} \sum_{j=0}^{r-1} (-1)^j S_{tj}B_{n+r-j} = a_t^{m+n} \sum_{i=0}^{r-1} (-1)^i S_{ti}A_{r-i} \sum_{j=0}^{r-1} (-1)^j S_{tj}B_{r-j}$$

$$= a_t^{m+n} \sum_{j=0}^{r-1} (-1)^j S_{tj}C_{r-j} = \sum_{j=0}^{r-1} (-1)^j S_{tj}C_{m+n+r-j} ,$$

as a generalization of Wyler's:

$$C_{m+n+2} - a_1 C_{m+n+1} = a_2^{m+n}(A_2 - a_1 A_1)(B_2 - a_1 B_1) = (A_{m+2} - a_1 A_{m+1})(B_{n+2} - a_1 B_{n+1}).$$

5. NORMS AND DUALS

As in Catlin, we can define norms and duals. We define the norm or characteristic of $\{W_n\}$ as

$$N\{W_n\} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j s_{tj} W_{r-j} . \tag{5.1}$$

For example, for the "basic" sequences $\{U_{s,n}\}$ which satisfy the recurrence relation (1.1) but have initial conditions

$$U_{s,n} = S_{s,n}, \qquad n = 1, 2, \cdots, r,$$

we have

$$N\{U_{s,n}\} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j s_{tj} U_{s,r-j} = \prod_{t=1}^{r} (-1)^{r-s} s_{t,r-s} ;$$

in particular, $N\{U_{r,n}\} = 1$. (The "basic" properties are seen in

$$W_n = \sum_{s=1}^{r} U_{s,n} W_s ,$$

for instance.)

$$N\{A_n\} N\{B_n\} = N\{A_n\}\{B_n\} . \tag{5.2}$$

Proof.

$$N\{A_n\}N\{B_n\} = \prod_{t=1}^{r} \sum_{i=0}^{r-1} (-1)^i s_{ti} A_{r-i} \sum_{j=0}^{r-1} (-1)^j s_{tj} B_{r-j} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j s_{tj} C_{r-j} = N\{C_n\} = N\{A_n\}\{B_n\}.$$

As

$$\Sigma(t, C_0) = \Sigma(t, A_0)\Sigma(t, B_0)$$

is related to $C^{(0)} = A^{(0)} M B^{(0)}$, so is

$$N\{C_n\} = N\{A_n\} N\{B_n\}$$

related to $|C^{(0)}| = |A^{(0)}|\,|B^{(0)}|$.

When $r = 2$, we have in fact that

$$N\{W_n\} = \begin{vmatrix} W_2 & W_3 \\ W_1 & W_2 \end{vmatrix} = W_2^2 - W_1 W_3 = (W_2 - a_1 W_1)(W_2 - a_2 W_1).$$

Furthermore, from definition (5.1) we have that

$$P_r^n N\{W_n\} = P_r^n \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j s_{tj} W_{r-j} = \prod_{t=1}^{r} a_t^n \sum_{j=0}^{r-1} (-1)^j s_{tj} W_{r-j} = \prod_{t=1}^{r} \sum_{j=0}^{r-1} (-1)^j s_{tj} W_{n+r-j}$$

as a generalization of Wyler's:

$$W_{n+2}^2 - W_{n+1} W_{n+3} = P_2^n N\{W_n\}.$$

We can compare this with

$$|W^{(n)}| = |Q^n|\,|W^{(0)}| \quad \text{in Eq. (3.6)}$$

$$= P_r^n |W^0|.$$

Similarly, we can form a dual as in Catlin. Given the recursive sequence $\{W_n\}$, we form its dual $\{W_n^*\}$ from the initial values

$$Wn, \;\; n = 1, 2, \cdots, r:$$

(5.3)
$$\underset{\sim}{w}^* = \left(I - \sum_{k=1}^{r-1} (E^T)^k\right)\underset{\sim}{w}$$

where
$$w = [W_1, W_2, \cdots, W_r]^T,$$

and E is the nilpotent matrix of order r defined in (3.4). For example, when $r = 2$,

$$\begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

and
$$W_1^* = W_1, \quad W_2^* = W_2 - W_1,$$

as in Catlin. When $r = 3$,

$$\begin{bmatrix} W_1^* \\ W_2^* \\ W_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix},$$

and so on. Essentially, what has been done here is to illustrate how the work for the second-order recurrences can be extended to any order. It may interest others to develop the algebra further by considering the canonical forms of elements in various extension fields and rings.

Another line of approach is to consider the treatment here as a generalization of Simson's (second-order) relation:

$$A_{n+1}^2 - A_n A_{n+2} = P_2^n N\{A_n\},$$

or, since $N\{F_n\} = 1$,

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n$$

for the Fibonacci numbers.

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REFERENCES

1. Paul A. Catlin, "On the Multiplication of Recurrences," *The Fibonacci Quarterly*, Vol. 12 (1974), pp. 365–368.
2. Brian Peterson and V. E. Hoggatt, Jr., "On the Solutions to the Diophantine Equation $x^2 + xy - y^2 = \pm D$, or the Number of Fibonacci-Type Sequences with a Given Characteristic," *The Fibonacci Quarterly*, Vol. 13 (1975), pp. 243–251.
3. Oswald Wyler, "On Second-order Recurrences," *Amer. Math. Monthly*, Vol. 72 (1965), pp. 500–506.

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DIAGONAL FUNCTIONS

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INTRODUCTION

The object of this article is to combine and generalize some of the ideas in [1] and [2] which dealt with extensions to the results of Jaiswal, and of Hansen and Serkland. [See [1] and [2] for the references.]

We commence with the pair of sequences $\{A_n\}$ and $\{B_n\}$ for which

(1) $$A_{n+2} = xA_{n+1} + A_n, \qquad A_0 = 0, \quad A_1 = 1 \quad (x \neq 0)$$

(2) $$B_{n+2} = xB_{n+1} + B_n, \qquad B_0 = 2, \quad B_1 = x$$

with the special properties

(3) $$A_{n+1} + A_{n-1} = B_n$$

(4) $$B_{n+1} + B_{n-1} = (x^2+4)A_n\ .$$

[See [2], where c has been replaced by x.]

The first few terms of these sequences $\{A_n\}$ and $\{B_n\}$ are

(5) $$\begin{array}{l} 0 \\ 1 \\ x \\ x^2+1 \\ x^3+2x \\ x^4+3x^2+1 \\ x^5+4x^3+3x \\ x^6+5x^4+6x^2+1 \\ x^7+6x^5+10x^3+4x \\ x^8+7x^6+15x^4+10x^2+1 \\ \text{--------------------} \end{array}$$

(6) $$\begin{array}{l} 2 \\ x \\ x^2+2 \\ x^3+3x \\ x^4+4x^2+2 \\ x^5+5x^3+5x \\ x^6+6x^4+9x^2+2 \\ x^7+7x^5+14x^2+7x \\ x^8+8x^6+20x^4+16x^2+2 \\ x^9+9x^7+27x^5+30x^3+9x \\ \text{--------------------} \end{array}$$

RISING DIAGONAL FUNCTIONS

Consider the *rising diagonal functions* of x, $R_i(x)$, $r_i(x)$ for (5) and (6), respectively (indicated by unbroken lines):

(7) $$\begin{cases} R_1(x) = 1 & R_2(x) = x \qquad R_3(x) = x^2 \qquad R_4(x) = x^3+1 \\ R_5(x) = x^4+2x & R_6(x) = x^5+3x^2 \qquad R_7(x) = x^6+4x^3+1 \qquad R_8(x) = x^7+5x^4+3x \\ R_9(x) = x^8+6x^5+6x^2 & R_{10}(x) = x^9+7x^6+10x^3+1, \cdots \end{cases}$$

(8) $$\begin{cases} r_1(x) = 2 & r_2(x) = x \qquad r_3(x) = x^2 \qquad r_4(x) = x^3+2 \\ r_5(x) = x^4+3x & r_6(x) = x^5+4x^2 \qquad r_7(x) = x^6+5x^3+2 \qquad r_8(x) = x^7+6x^4+5x \\ r_9(x) = x^8+7x^5+9x^2 & r_{10}(x) = x^9+8x^6+14x^3+2, \cdots \end{cases}$$

Define

(9) $$R_0(x) = r_0(x) = 0\,.$$

Observe that, in (7), (8) and (9), for $n \geq 3$,

(10)
$$\begin{cases} r_n(x) = R_n(x) + R_{n-3}(x) \\ R_n(x) = xR_{n-1}(x) + R_{n-3}(x) \\ r_n(x) = xr_{n-1}(x) + r_{n-3}(x) \quad . \end{cases}$$

Generating functions for the rising diagonal polynomials are

(11)
$$A \equiv A(x,t) \equiv (1 - xt - t^3)^{-1} = \sum_{n=1}^{\infty} R_n(x)t^{n-1}$$

and

(12)
$$B \equiv B(x,t) \equiv (1 + t^3)(1 - xt - t^3)^{-1} = \sum_{n=2}^{\infty} r_n(x)t^{n-1} \; .$$

Calculations with (11) and (12) yield the partial differential equations

(13)
$$t\,\frac{\partial A}{\partial t} - (x + 3t^2)\,\frac{\partial A}{\partial x} = 0$$

and

(14)
$$t\,\frac{\partial B}{\partial t} - (x + 3t^2)\,\frac{\partial B}{\partial x} - 3B + 3A = 0 .$$

leading to

(15)
$$xR'_{n+2}(x) + 3R'_n(x) - (n+1)R_{n+2}(x) = 0$$

(16)
$$xr'_{n+2}(x) + 3r'_n(x) - (n-2)r_{n+2}(x) - 3R_{n+2}(x) = 0 \quad (n \geqslant 2),$$

where the prime denotes differentiation with respect to x.

Comparing coefficients of t^n in (11) we deduce that

(17)
$$R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} \quad (n \geqslant 3)$$

where $[n/3]$ is the integral part of $n/3$.

Similarly, from (12) we derive

(18)
$$r_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i} x^{n-3i} \quad (n \geqslant 3)$$

as may also be readily seen from the first statement in (10).

Simple examples of rising diagonal sequences are:

(a) for the Fibonacci and Lucas sequences $(x = 1)$:

(19) 0 1 1 1 2 3 4 6 9 13 19 ...

(20) 2 1 1 3 4 5 8 12 17 25 ...

and

(b) for the Pell sequences $(x = 2)$:

(21) 0 1 2 4 9 20 44 97 214 ...

(22) 2 2 4 10 22 48 106 234 ...

DESCENDING DIAGONAL FUNCTIONS

From (5) and (6), the descending diagonal functions of $x, D_i(x), d_i(x)$ (indicated by broken lines) are:

(23)
$$\begin{cases} D_1(x) = 1 & D_2(x) = x+1 & D_3(x) = (x+1)^2 & D_4(x) = (x+1)^3 \\ D_5(x) = (x+1)^4 & D_6(x) = (x+1)^5 & D_7(x) = (x+1)^6 & D_8(x) = (x+1)^7, \quad \cdots \end{cases}$$

$$(24)\begin{cases} d_1(x) = 2 \qquad d_2(x) = (x+1)+(x+1)^0 = (x+2)(x+1)^0 = x+2 \\ d_3(x) = (x+1)^2+(x+1) = (x+2)(x+1) \qquad d_4(x) = (x+1)^3+(x+1)^2 = (x+2)(x+1)^2 \\ d_5(x) = (x+1)^4+(x+1)^3 = (x+2)(x+1)^3 \qquad d_6(x) = (x+1)^5+(x+1)^4 = (x+2)(x+1)^4, \end{cases}$$

Define

$$D_0(x) = d_0(x) = 0. \tag{25}$$

Obviously $(n \geqslant 2)$

$$(26)\quad \begin{cases} D_n = (x+1)D_{n-1} = (x+1)^{n-1} \\ d_n = D_n + D_{n-1} = (x+2)D_{n-1} = (x+2)(x+1)^{n-2} \\ d_n = (x+1)d_{n-1} \quad (n > 2) \\ \dfrac{D_n}{D_{n-1}} = \dfrac{d_n}{d_{n-1}} \; (= x+1) \quad (n > 2) \\ \dfrac{D_n}{d_n} = \dfrac{x+1}{x+2}, \end{cases}$$

where, for visual ease, we have temporarily written $D_n \equiv D_n(x)$ and $d_n \equiv d_n(x)$.

Generating functions for the descending diagonal polynomials are

$$A \equiv A(x,t) = [1-(x+1)t]^{-1} = \sum_{n=1}^{\infty} D_n(x)t^{n-1} \tag{27}$$

and

$$B \equiv B(x,t) = (x+2)[1-(x+1)t]^{-1} = \sum_{n=1}^{\infty} d_{n+1}(x)t^{n-1} \tag{28}$$

from which are obtained the partial differential equations

$$t\frac{\partial A}{\partial t} - (x+1)\frac{\partial A}{\partial x} = 0 \tag{29}$$

$$t\frac{\partial B}{\partial t} - (x+1)\frac{\partial B}{\partial x} + (x+1)A = 0, \tag{30}$$

leading to

$$(x+1)D'_n(x) = (n-1)D_n(x) \tag{31}$$

$$(x+1)d'_{n+2}(x) - (n+1)d_{n+2}(x) + (x+1)D_n(x) = 0. \tag{32}$$

Descending diagonal sequences for some well known sequences are:

(a) for the Fibonacci and Lucas sequences $(x = 1)$:

(33)	1	2	4	8	16	32	64	128	···	2^n	···
(34)	2	3	6	12	24	48	96	192	···	$3 \cdot 2^{n-1}$	···

and

(b) for the Pell sequences $(x = 2)$:

(35)	1	3	9	27	81	243	729	2187	···	3^n	···
(36)	2	4	12	36	108	324	972	2916	···	$4 \cdot 3^{n-1}$	···

CONCLUDING COMMENTS

1. The above results proceed only as far as corresponding work in [1] and [2]. Undoubtedly, more work remains to be done on functions R_i, r_i, D_i, d_i.
2. Excluded from our consideration in this article are the pair of Fermat sequences and the pair of Chebyshev sequences for both of which the criteria (1) and (2) do not hold. [See [2].]

3. Jaiswal, and the author [1], deal only with the rising diagonal functions of Chebyshev polynomials of the first and second kinds.
4. Our special criteria (3) and (4) prevent the use of the more general sequences $\{U_n\}$, $\{V_n\}$ for which

$$U_{n+2} = xU_{n+1} + yU_n \qquad U_0 = 0, \quad U_1 = 1 \quad (x \neq 0,\ y \neq 0)$$
$$V_{n+2} = xV_{n+1} + yV_n \qquad V_0 = 2, \quad V_1 = x\ .$$

See [2] and Lucas [3] pp. 312–313.

5. Finally, in passing, we note that the Pell sequence obtained from (1) with $x = 2$, namely, the sequence 1,2,5,12,29,70, ···, arises from rising diagonals in the "arithmetical square" of Delannoy [Lucas [3] p. 174]

Can any reader inform me, along with a suitable reference, whether Delannoy's "arithmetical square" has been generalized?

REFERENCES

1. A. F. Horadam, "Polynomials Associated with Chebyshev Polynomials of the First Kind," *The Fibonacci Quarterly,*
2. A. F. Horadam, "Generating Identities for Generalized Fibonacci and Lucas Triples," *The Fibonacci Quarterly,*
3. E. Lucas, *Théorie des Nombres,* Gauthier-Villars et Fils, Paris, 1891.

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FIBONACCI TILING AND HYPERBOLAS

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ABSTRACT

A sequence of rectangles R_n is generated by adding squares cyclically to the East, N, W, S side of the previous rectangle. The centers of R_n fall on a certain hyperbola, in a manner reminiscent of multiplication in a real quadratic number field.

INTRODUCTION

We take a special case for simplicity. Suppose R_1 is the square $-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1$. R_2 is the rectangle $-1 \leqslant x \leqslant 3, -1 \leqslant y \leqslant 1$. R_3 is the rectangle $-1 \leqslant x \leqslant 3, -1 \leqslant y \leqslant 5$. Let F_n denote the n^{th} Fibonacci number. Then R_n has sides $2F_n$ and $2F_{n-1}$ for all n.

We ask for information about the center (x_n, y_n) of R_n. This search leads us to the ring $R \otimes R$ in which $R \otimes R$ is given pointwise addition and multiplication. We close with an examination of "rotations" and linear fractional mappings of $R \otimes R$. Certain classes of hyperbolas remain invariant under such mappings.

1. DEFINITIONS AND STATEMENT OF RESULTS

Let $a,b > 0$. Suppose a sequence of rectangles is generated in the following manner. The initial rectangle has center (0,0) and positive dimensions X_1, Y_1. If the n^{th} rectangle R_n has dimensions X_n, Y_n then R_{n+1} is the union of R_n with an incremental rectangle on the East, N. W, S side of R_n according as $n \equiv 1, 2, 3, 0 \bmod 4$. The dimensions of the incremental rectangle are $aY_n + b$, Y_n if $n \equiv 1 \bmod 2$, and X_n, $aX_n + b$ if $n \equiv 0 \bmod 2$.

Theorem. Let (x_n, y_n) be the center of R_n. Let $D = \tfrac{1}{2}(aY_1 + b)$, $E = \tfrac{1}{2}(aX_1 + b)$.

Then for all $n \geqslant 1$, (x_n, y_n) lies on the right hyperbola

$$H = \left\{(x,y) : x^2 + axy - y^2 - Dx + Ey = 0\right\}.$$

Further, if h is the center of H, then the area enclosed by H and the rays

$$\overline{h, (x_n, y_n)} \quad \text{and} \quad \overline{h, (x_{n+4}, y_{n+4})}$$

is independent of n.

REMARK. The proof that the (x_n, y_n) lie on H is a rather ordinary induction. To prove that the areas enclosed by H and rays from adjacent rectangle centers to h are all equal, we introduce the ring $R \otimes R$.

Definition. $R \otimes R$ is the ring $R \otimes R$ with addition $(x,y) + (x', y') = (x + x', y + y')$ and multiplication $(x,y) \cdot (x', y') = (x \cdot x', y \cdot y')$.

Definition. If $(x,y) \in R \times R$, $N(x,y) = xy$; and $\operatorname{Arg}(x,y) = \log |(y/x)|$ if $xy \neq 0$.

Definition. If $N(x,y) \neq 0$,

$$\frac{(x', y')}{(x,y)} = \left(\frac{x'}{x}, \frac{y'}{y}\right).$$

REMARK. $N(x,y) = 1$ is the hyperbola $xy = 1$. Arg (x,y) is the area enclosed by $N(x,y) = 1$ and the rays

$$\overline{(0,0), (|x|, |y|)} \quad \text{and} \quad \overline{(0,0), (|y|, |x|)}.$$

It is for this area property, so similar to the one stated in Theorem 1, that we introduce $R \otimes R$.

Theorem 2. Let k be real, $a, b, c, d, z_0 \in R \otimes R$. Assume not both $a,b = (0,0)$ and not both $c_1, d_1 = 0$ and not both $c_2, d_2 = 0$. Let $k \neq 0$. (Here $(c_1, c_2) = c$ and $(d_1, d_2) = d$.)

Let

$$f(z) = \frac{az+b}{cz+d}$$

for all z such that $N(cz+d) \neq 0$. Then the image under f of $\{z : N(z-z_0=k\}$ is of the form

$$\{w : N(w-w_0) = k'\}$$

where no more than 4 points are missing.

REMARK. Thus except for technicalities, a linear fractional maps hyperbolas of the form $N(z-z_0) = k$ to hyperbolas of the same form. The analogy with the complex numbers, where linear fractionals map circles to circles, suggests many more similar results which space does not permit us to list.

2. PROOFS. THEOREM 1, PART 1

The reader may verify by direct calculation that the first couple of (x_n, y_n) lie on H. We now claim that

$$2x_n + ay_n + \tfrac{1}{2}(aY_n+b) = D \quad \text{if} \quad n \equiv 1 \mod 4.$$

$$-ax_n + 2y_n + \tfrac{1}{2}(aX_n+b) = E \quad \text{if} \quad n \equiv 2 \mod 4.$$

$$2x_n + ay_n - \tfrac{1}{2}(aY_n+b) = D \quad \text{if} \quad n \equiv 3 \mod 4,$$

and

$$-ax_n + 2y_n - \tfrac{1}{2}(aX_n+b) = E \quad \text{if} \quad n \equiv 0 \mod 4.$$

Observe that if $(x_n, y_n) \in H$ and the claim is true for n, then $(x_{n+1}, y_{n+1}) \in H$. Thus we need only prove the claim to show that all (x_n, y_n) are on H.

Proof of claim, $n \equiv 1 \mod 2$.

If the claim is true for some $n \equiv 1 \mod 4$, then

$$2x_n + ay_n + \tfrac{1}{2}(aY_n+b) = D.$$

We show that the claim follows for $n+2$.

For,

$$x_{n+2} = x_n + \tfrac{1}{2}(aY_n+b), \qquad y_{n+2} = y_n + \tfrac{1}{2}(b + aX_n + a^2Y_n + ab),$$

and

$$Y_{n+2} = Y_n + b + aX_n + a^2Y_n + ab.$$

Thus

$$\begin{aligned} 2x_{n+2} + ay_{n+2} - \tfrac{1}{2}(aY_{n+2}+b) &= 2(x_n + \tfrac{1}{2}(aY_n+b)) + a(y_n + \tfrac{1}{2}(b + aX_n + a^2Y_n + ab)) \\ &\quad - \tfrac{1}{2}(b + a(Y_n + b + aX_n + a^2Y_n + ab)) = \\ \text{(by claim)} \quad &= D + (aY_n+b) + (\tfrac{1}{2}ab + \tfrac{1}{2}a^2X_n + \tfrac{1}{2}a^3Y_n + \tfrac{1}{2}a^2b) - \tfrac{1}{2}b \\ &\quad - \tfrac{1}{2}aY_n - \tfrac{1}{2}ab - \tfrac{1}{2}a^2X_n - \tfrac{1}{2}a^3Y_n - \tfrac{1}{2}a^2b \\ &\quad - \tfrac{1}{2}(aY_n+b) = D. \end{aligned}$$

Similarly, if the claim is true for some $n \equiv 2 \mod 4$ it is true for $n+2$, if true for some $n \equiv 3 \mod 4$ it is true for $n+2$, and if true for $n \equiv 0 \mod 4$ it is true for $n+2$. Thus it is only necessary to check that the claim is true for $n = 1$ and $n = 2$. If $n = 1$, x_n and $y_n = 0$ and $\tfrac{1}{2}(aY_1+b) = D$ by definition. $x_2 = \tfrac{1}{2}(aY_1+b)$, and $y_2 = 0$. $X_2 = X_1 + aY_1 + b$, and $Y_2 = Y_1$. Thus

$$-ax_2 + 2y_2 + \tfrac{1}{2}(aX_2+b) = -\tfrac{1}{2}a(aY_1+b) + \tfrac{1}{2}(aX_1 + a^2Y_1 + a^2Y_1 + ab + b) = \tfrac{1}{2}(aX_1+b) = E$$

by definition. This proves the claim, and hence the centers of R_n lie on H.

For the second part of Theorem 1, we note that H is a hyperbola whose asymptotes are perpendicular. It is therefore similar, in the geometric sense, to the hyperbola $xy = 1$. Let

$$\varphi: R \otimes R \to R \otimes R$$

be a similarity mapping which takes H onto $xy = 1$.

For each n, the line $(x_{n-1}, y_{n-1}), (x_n, y_n)$ is perpendicular to $(x_n, y_n), (x_{n+1}, y_{n+1})$. This property is preserved under the similarity mapping of H onto $xy = 1$.

Let $z_n = (x'_n, y'_n) = \varphi(x_n, y_n)$. Let c be the slope of the line from (x'_1, y'_1) to (x'_2, y'_2). Let $C = (c, 1/c)$. $(c \neq 0)$. Then (with the help of a little algebra)

$$\begin{aligned} z_n &= -C^{-n+1}z_1^{-1} && \text{if} \quad n \equiv 2 \bmod 4, \\ &\ -C^{n-1}z_1 && \text{if} \quad n \equiv 3 \bmod 4 \\ &\ +C^{-n+1}z_1^{-1} && \text{if} \quad n \equiv \quad \bmod 4 \end{aligned}$$

and

$$z_n = +C^{n-1}z_1 \quad \text{if} \quad n \equiv 1 \bmod 4.$$

Now the region enclosed by the lines from (0,0) to z_n and to z_{n+4}, and by $xy = 1$, has area

$$|\tfrac{1}{2}(\operatorname{Arg}(z_{n+4}) - \operatorname{Arg}(z_n))| = |\tfrac{1}{2}\operatorname{Arg}(z_{n+4}/z_n)| = |\tfrac{1}{2}\operatorname{Arg}(C^4)| \quad \text{or} \quad |\tfrac{1}{2}\operatorname{Arg}(C^{-4})|$$

depending on whether n is odd or even. Either way, since Arg (C) = Arg (C^{-1}), all such regions have equal areas.

Thus the corresponding regions bounded by lines from the center of H to the (x_n, y_n) also have areas equal to each other's, since φ multiplies areas by a constant.

The mapping $r : z \to C^4 z$ of $R \otimes R$ onto $R \otimes R$ may be viewed as a "rotation" of $R \otimes R$, since it changes Arg (z) but not $N(z)$. Clearly r sends hyperbolas of the form $N(z) = k$ into themselves. This is reminiscent of linear fractional transformations of the complex plane. Although there is no direct further bearing on Fibonacci tiling, we are inclined to note some similarities.

Proof of Theorem 2. Fix $a,b,c,d \in R \otimes R$. Let $(c_1, c_2) = c$ and $(d_1, d_2) = d$. Suppose not both a and b = (0,0), and $(c_1, d_1) \neq (0,0)$ $(c_2, d_2) \neq (0,0)$. Fix $x_o, y_o, k \neq 0 \in R$.

Lemma 1. Under the above conditions, there exist $x_1, y_1, x_2, y_2, K \in R$ such that

$$K \neq 0,\ x_1 \neq x_0,\ x_1 \neq -d_1/c_1,\ y_1 \neq y_0,\ y_1 \neq -d_2/c_2,\ x_2 \neq x_1,$$
$$x_2 \neq -d_1/c_1,\ y_2 \neq y_1, y_2 \neq -d_2/c_2,$$

and such that $(x - x_0)(y - y_0) = k$ if and only if $(x - x_1)(y - y_1)/(x - x_2)(y - y_2) = K$ or $(x,y) = (x_1, y_2)$ or (x_2, y_1).

Proof. Select some $k \neq 0, 1$ such that

$$(K-1)(k - x_0y_0) + K^{-2}(K-1)^2x_0y_0 \neq 0.$$

Fix K. Let

$$x_2 = K^{-1}((K-1)x_0 + x_1), \quad y_2 = K^{-1}((K-1)y_0 + y_1).$$

Then the equation

$$k - x_0y_0 = (K-1)^{-1}(x_1y_1 - K^{-2}((K-1)x_0 + x_1)((K-1)y_0 + y_1)$$

has a range of solutions x_1, y_1 in which y_1 is a non-constant continuous function of x_1.

When the above conditions are satisfied, and $x_1 \neq x_0, y_1 \neq y_0$,

$$(x - x_0)(y - y_0) = k \Leftrightarrow (x - x_1)(y - y_1) = K(x - x_2)(y - y_2).$$

Thus Lemma 1.

We may restate this as saying that except for a special class of degenerate hyperbolas, every hyperbola $N(z - z_0) = k$ can be put in the form

$$\frac{N(z - z_1)}{N(z - z_2)} = K.$$

Now let $\lambda \in R \otimes R$,

$$\lambda = \left(\frac{c_1x_2 + d_1}{c_1x_1 + d_1}, \frac{c_2y_2 + d_2}{c_2y_1 + d_2}\right).$$

Let $w_1 = f(z_1)$, $w_2 = f(z_2)$. Then

$$\frac{w - w_1}{w - w_2} = \lambda \frac{z - z_1}{z - z_2} \Leftrightarrow w = f(z) \quad \text{or} \quad w = w_2,\ z = z_2\ .$$

Thus

$$\frac{N(z-z_1)}{N(z-z_2)} = K$$

has image

$$\frac{N(w-w_1)}{N(w-w_2)} = KN(\lambda).$$

By our previous results this is also a hyperbola of the same sort.

REMARK. Thus except for isolated points for which necessary divisions are impossible in $R \otimes R$, $R \otimes R$ behaves just like $\mathbb{C}$ with respect to linear fractional mappings.

One could show without great difficulty that the maps f of Theorem 2, are "conformal," in the $R \otimes R$ sense. Self mappings of the "unit circle" $N(z) \leqslant 1$ have properties analogous to their counterparts over $\mathbb{C}$. But the prospects along this line are quite limited. $R \otimes R$ is only a curiosity, and cannot (in my opinion) support a deep and rich theory.

For those familiar with the number theory of $Q(\sqrt{5})$, we remark that for the example of the introduction, by embedding $Q(\sqrt{5})$ in $R \otimes R$ one may show that the (x_n, y_n) consist of all the integer points on

$$x^2 + xy - y^2 - x + y = 0,$$

except for (0,1).

REFERENCES

1. James Hafner, "Generalized Fibonacci Tiling Made Easy," to appear.
2. Herbert Holden, "Fibonacci Tiles," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), p. 45.

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A PRIMER FOR THE FIBONACCI NUMBERS, PART XVI
THE CENTRAL COLUMN SEQUENCE

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1. INTRODUCTION

The rows of Pascal's triangle with *even* subscripts have a middle term

$$A_n = \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

since

$$\binom{n}{k} = \binom{n}{n-k},$$

for $0 \leqslant k \leqslant n$. We shall now derive the generating function

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n .$$

From

$$A_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

one easily gets

$$(n+1)A_{n+1} = 2(2n+1)A_n .$$

2. GENERATING FUNCTION

From

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = A_0 + \sum_{n=0}^{\infty} A_{n+1} x^{n+1}$$

so that by differention

$$xA'(x) = x\sum_{n=0}^{\infty} (n+1)A_{n+1}x^n = \sum_{n=0}^{\infty} nA_n x^n .$$

From tne relation

$$(n+1)A_{n+1} = 2(2n+1)A_n$$

then

$$A'(x) = \sum_{n=0}^{\infty} (n+1)A_{n+1}x^n = \sum_{n=0}^{\infty} 2(2n+1)A_n x^n = 2\left(\sum_{n=0}^{\infty} 2(nA_n)x^n + \sum_{n=0}^{\infty} A_n x^n\right)$$

so that

$$A'(x) = 2(2xA'(x) + A(x)).$$

Solving for $A'(x)$, one gets, upon dividing by $A(x)$,

$$\frac{A'(x)}{A(x)} = \frac{2}{(1-4x)}$$

from which it follows that

$$\ln A(x) = -\tfrac{1}{2}\ln(1-4x) + \ln C.$$

Thus

$$A(x) = \frac{C}{\sqrt{1-4x}},$$

but $A_0 = A(0) = 1$ implies $C = 1$, so that

$$A(x) = \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} A_n x^n .$$

3. CATALAN NUMBERS

Suppose you know that the Catalan numbers have the form

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad C_0 = 0,$$

and wish to derive the generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n .$$

Recall that

$$A(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} .$$

Then

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} A_n x^n .$$

Thus, if we integrate the series for $A(x)$, term-by-term,

$$\int \frac{dx}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{1}{n+1} A_n x^{n+1} + C^* .$$

But

$$\int \frac{dx}{\sqrt{1-4x}} = -\tfrac{1}{2}\sqrt{1-4x} = xC(x) + C^*,$$

which implies $C^* = -\tfrac{1}{2}$. This can be solved for

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} .$$

We now show how to derive the central sequence for the trinomial triangle.

4. THE TRINOMIAL TRIANGLE – CENTRAL TERM

Consider the triangular array

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & 1 & & & & \\
1 & 2 & 3 & 2 & 1 & & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & x & y & z & & & \\
 & & & w & & &
\end{array}$$

where $w = x + y + z$ shows the relation between the elements of the array. It is induced by the expansion of

$$(1+x+x^2)^n, \qquad n = 0, 1, 2, 3, \cdots .$$

Let

$$(1+x+x^2)^n = \sum_{m=0}^{2n} \beta_m x^m = \sum_{k=0}^{n} \binom{n}{k} x^{2k}(1+x)^{n-k}$$
$$= \binom{n}{0}(1+x)^n + \binom{n}{1}(1+x)^{n-1}x^2 + \cdots + \binom{n}{k}(1+x)^{n-k}x^{2k} + \cdots .$$

The coefficient β_n is the central term and is given by

$$\beta_n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n-1}{n-2} + \cdots + \binom{n}{a}\binom{n-a}{n-2a} ,$$

where $a = [n/2]$. The β_n may be written in several forms.

$$\beta_n = \sum_{k=0}^{[n/2]} \binom{n}{k}\binom{n-k}{n-2k} = \sum_{k=0}^{[n/2]} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} ,$$

since

$$\binom{n}{k}\binom{n-k}{k} = \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{k!(n-2k)!} = \frac{n!\,(2k)!}{(2k)!(n-2k)!k!k!} = \binom{n}{2k}\binom{2k}{k} .$$

We now derive the central term generating function,

$$B(x) = \frac{1}{\sqrt{1-2x-3x^2}} = \sum_{n=0}^{\infty} \beta_n x^n .$$

Thus

$$B(x) = \sum_{m=0}^{\infty} \beta_m x^m = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{[m/2]} \binom{m}{2k}\binom{2k}{k} \right) x^m = \sum_{k=0}^{\infty} \left(\sum_{m=2k}^{\infty} \binom{m}{2k} x^m \right) \binom{2k}{k},$$

since

$$\binom{m}{2k} = 0 \quad \text{if} \quad 0 \leqslant m < 2k.$$

Thus

$$= \sum_{k=0}^{\infty} \binom{2k}{k} \sum_{m=0}^{\infty} \binom{m}{2k} x^m = \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{x^{2k}}{(1-x)^{2k+1}} \right) ,$$

since

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n .$$

But

$$A(x) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

so that

$$B(x) = \frac{1}{1-x}\, A\left(\frac{x^2}{(1-x)^2} \right) = \frac{1}{1-x}\, \frac{1}{\sqrt{1-4\left(\frac{x}{1-x}\right)^2}} = \frac{1}{\sqrt{1-2x-3x^2}} .$$

This completes the derivation.

Thus, $A(x)$, the generating function for central term of the evenly subscripted rows of Pascal's triangle, is related by a transformation to the central term generating function for the trinomial triangle

$$B(x) = \frac{1}{1-x}\,A\left(\frac{x^2}{(1-x)^2}\right)\,.$$

Much more can be done with this but that is another paper and is covered in Rondeau [2] and Anaya [1]. It should be noted that the generating function $B(x)$ could also have been derived by Lagrange's Theorem as in [6].

Sequence 456, in [4], is the Catalan sequence for the Trinomial Triangle $1, 1, 2, 4, 9, 21, \cdots, C_n^*, \cdots$. This sequence C_n^* can be obtained from the regular Catalan $1, 1, 2, 5, 14, 42, \cdots$ if we truncate the first term, by repeated differencing. (See [1].)

Trinomial Catalan Nos.

$$\begin{array}{lccccccl}
 & 1 & 2 & 5 & 14 & 42 & \cdots & C_n \text{ Catalan Numbers} \\
\Delta C_n & & 1 & 3 & 9 & 28 & & \\
\Delta^2 C_n & & & 2 & 6 & 19 & & \\
\Delta^3 C_n & & & & 4 & 13 & & \\
\Delta^4 C_n & & & & & 9 & &
\end{array}$$

The Catalan generating function $C(x)$ is

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$$C^2(x) = \frac{C(x)-1}{x} = \sum_{n=0}^{\infty} C_{n+1}x^n = \frac{1-2x-\sqrt{1-4x}}{2x^2}$$

Let $C_n^*(x)$ be the generating function for Catalan numbers for the Trinomial Triangle. This is

$$C^*(x) = \frac{1}{1+x}\,C^2\left(\frac{x}{1+x}\right) = \frac{1}{1+x}\left[\frac{1-\dfrac{2x}{1+x}-\sqrt{1-\dfrac{4x}{1+x}}}{2\,\dfrac{x^2}{(1+x)^2}}\right]$$

$$= \frac{1-x-\sqrt{(1+x)(1-3x)}}{2x^2} = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}\,.$$

We can also get $C^*(x)$ from regular Catalan number generator by another transformation related to summation Webs

Trinomial Catalan Numbers ↘

Catalan Numbers →

$$\begin{array}{ccccccccccc}
1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & & \\
 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 14 & \cdots & \\
 & & 2 & 2 & 3 & 4 & 7 & 10 & & & \\
 & & & 4 & 5 & 7 & 11 & 17 & & & \\
 & & & & 9 & 12 & 18 & & & & \\
 & & & & & 21 & 30 & & & & \\
 & & & & & & 51 & & & &
\end{array}$$

Here the Catalan number generator is

$$C(x^2) = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

$$C^*(x) = \frac{1}{1-x}\,C\left(\left(\frac{x}{1-x}\right)^2\right) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

This is the same transformation we saw earlier.

5. $B(x)$ FROM THE DIFFERENCE EQUATION

In Riordan [3, p. 74], they give the recurrence for the numbers β_n, the central terms in the rows of a trinomial triangle. This is

$$n\beta_n = (2n-1)\beta_{n-1} + 3(n-1)\beta_{n-2}.$$

We shall now derive this.

We will start with the well known generating function for the Legendre Polynomials

$$\frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n.$$

We introduce a phantom parameter t in the generating function for $B(x)$.

$$B(x,t) = \frac{1}{\sqrt{1-2xt-3x^2}} = \sum_{n=0}^{\infty} M_n(t)x^n,$$

where clearly $B(x,1) = B(x)$ and $M_n(1) = \beta_n$.

Let

$$x_1 = -i\sqrt{3}x \quad \text{and} \quad t_1 = \frac{it}{\sqrt{3}},$$

then

$$\sum_{n=0}^{\infty} M_n(t)x^n = \frac{1}{\sqrt{1-2xt-3x^2}} = \frac{1}{\sqrt{1-2x_1t_1+x_1^2}} = \sum_{n=0}^{\infty} P_n(t_1)x_1^n.$$

$$= \sum_{n=0}^{\infty} P_n\left(\frac{it}{\sqrt{3}}\right)(-i\sqrt{3}x)^n.$$

We note $M_n(1) = \beta_n$, then

$$\beta_n = (-i\sqrt{3})^n P_n(i/\sqrt{3}).$$

The Legendre Polynomials obey the recurrence relation

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

for $n \geqslant 0$, with $P_0(x) = 1$ and $P_1(x) = x$. From $P_0(x) = 1$, then

$$\beta_0 = (-i\sqrt{3})^0 P_0(i/\sqrt{3}) = 1$$

and from $P_1(x) = x$, then $\beta_1 = (-i\sqrt{3})(i/\sqrt{3}) = 1$. Thus directly substituting $P_n(x)$, with $x = i/\sqrt{3}$ the recurrence relation becomes

$$nP_n(i/\sqrt{3}) = (2n-1)\frac{1}{\sqrt{3}}P_{n-1}(i/\sqrt{3}) - (n-1)P_{n-2}(i/\sqrt{3})$$

and

$$n(-\sqrt{3}i)^n P_n(i/\sqrt{3}) = (2n-1)(-\sqrt{3}i)^n \frac{i}{\sqrt{3}} P_{n-1}(i/\sqrt{3}) - (n-1)(-i\sqrt{3})^n P_{n-2}(i/\sqrt{3}).$$

Since

$$\beta_n = (-i\sqrt{3})^n P_n(i/\sqrt{3}),$$

this yields

$$n\beta_n = (2n-1)\beta_{n-1} + 3(n-1)\beta_n,$$

with $\beta_0 = 1$, $\beta_1 = 1$ as was to be shown.

We note in passing that

$$\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 3.$$

6. FROM THE RECURRENCE TO THE GENERATING FUNCTION

We now go from the recurrence relation

$$(n+2)\beta_{n+2} = (2n+3)\beta_{n+1} + 3(n+1)\beta_n\,,$$

with $\beta_0 = \beta_1 = 1$, back to the generating function.

Let

$$B(x) = \sum_{n=0}^{\infty} \beta_n x^n\,,$$

then

$$xB'(x) = \sum_{n=0}^{\infty} n\beta_n x^n\,,$$

$$3xB'(x) + 3B(x) = \sum_{n=0}^{\infty} 3(n+1)\beta_n x^n\,.$$

Further

$$xB'(x) - 0\cdot\beta_0 - x\beta_1 = \sum_{n=2}^{\infty} n\beta_n x^n$$

or

$$(B'(x) - 1)/x = \sum_{n=0}^{\infty} (n+2)\beta_{n+2} x^n\,.$$

Next,

$$B(x) = 1 + x\sum_{n=0}^{\infty} \beta_{n+1} x^n, \quad B'(x) = \sum_{n=0}^{\infty} \beta_{n+1} x^n + \sum_{n=0}^{\infty} n\beta_{n+1} x^n\,,$$

$$\frac{B(x)-1}{x} = \sum_{n=0}^{\infty} \beta_{n+1} x^n, \quad 2B'(x) + \frac{B(x)-1}{x} = \sum_{n=0}^{\infty} (2n+3)\beta_{n+1} x^n\,.$$

Thus, from the recurrence relation, we may write

$$\frac{B'(x)-1}{x} = 2B'(x) + \frac{B(x)-1}{x} + 3xB'(x) + 3B(x)$$

or

$$B'(x)(1-2x-3x^2) = (3x+1)B(x), \quad \frac{B'(x)}{B(x)} = \frac{3x+1}{1-2x-3x^2}\,.$$

Integrating, $\ln B(x) = -\frac{1}{2}\ln(1-2x-3x^2) + \ln C$. Thus

$$B(x) = \frac{C}{\sqrt{1-2x-3x^2}}\,,$$

and since $B(0) = \beta_0 = 1$, it follows that $C = 1$. This concludes the discussion.

REFERENCES

1. Janet E. Anaya, "Web Sequences and Multisecting the Column Generating Functions of Pascal's Triangle," Unpublished Master's Thesis, San Jose State University, May, 1977.
2. Michel Rondeau, "The Generating Functions for the Vertical Columns of the $(N+1)$-Nomial Triangles," Unpublished Master's Thesis, San Jose State University, December, 1973.
3. John Riordan, *Combinatorial Identities,* John Wiley and Sons, New York, N.Y., 1968, p. 74.
4. N. J. A. Sloane, *Handbook of Integer Sequences,* The Academic Press, New York, N.Y., 1973.
5. Earl D. Rainville, *Special Functions,* McMillan Co., New York, N. Y., 1960.
6. V. E. Hoggatt, Jr., and Paul S. Bruckman, "H-Convolution Transform," *The Fibonacci Quarterly,* Vol. 13, No. 4, Dec. 1975, pp. 357–368.

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ENTRY POINTS OF THE FIBONACCI SEQUENCE AND THE EULER ϕ FUNCTION

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There is an interesting analogy between primitive roots of a prime and the maximal entry points of Fibonacci numbers modulo a prime.

Expressed in terms of the periods of reciprocals of primes in various base representations, the period of the b-mal expansion of $1/p$ is of length d_i in $\phi(d_i)$ incongruent bases modulo p where $d_i | p-1$ and ϕ is Euler's totient function. A similar statement can be made about certain classes of linear recursive sequences modulo p.

1.0 Let $\Gamma^n c,q$ be the n^{th} term of a linear recursive sequence,

$$\Gamma^n c,q = \begin{cases} \dfrac{(c+\sqrt{q})^n - (c-\sqrt{q})^n}{2\sqrt{q}} & \text{for } q \not\equiv c^2 \pmod 4 \\[2ex] \dfrac{\left(\dfrac{c+\sqrt{q}}{2}\right)^n - \left(\dfrac{c-\sqrt{q}}{2}\right)^n}{\sqrt{q}} & \text{for } q \equiv c^2 \pmod 4 \end{cases}$$

yielding the sequences defined by

$$\Gamma^n = \begin{cases} 2c\Gamma^{n-1} + (q-c^2)\Gamma^{n-2} \\ c\Gamma^{n-1} + \dfrac{q-c^2}{4}\Gamma^{n-2} \end{cases}$$

with initial values $1, 2c$ or $1, c$.

For $c = 1$, $q = 5$ we have the Fibonacci sequence.

We are interested in the entry points of these sequences, modulo p, a prime.

Borrowing the analogy, we will say that $\Gamma c,q$ belongs to the exponent x modulo p, if

$$p | \Gamma^x c,q, \quad p \nmid \Gamma^y c,q \quad \text{for } y < x.$$

The main results are:

1.1 For q a quadratic non-residue of p,c ranging from 1 to p, there are $\phi(d_i)$ values c such that $\Gamma c,q$ belongs to the exponent d_i modulo p, where $d_i | p+1$, $d_i \neq 1$.

1.2 For q a quadratic residue of p,c ranging from 1 to p, there are $\phi(d_i)$ values c such that $\Gamma c,q$ belongs to d_i modulo p, $d_i | p-1$, $d_i \neq 1$, and two values for which the sequence is not divisible by p at all.

1.3 For c fixed, $c \not\equiv 0 \pmod p$, q ranging from 1 to p, for each divisor of $p-1$ and $p+1$, except 1 and 2, there are $\phi(d_i)/2$ values of q such that $\Gamma c,q$ belongs to d_i modulo p. In addition there is one value such that $\Gamma c,q$ belongs to p (for $q = p$) and one for which the sequence is not divisible by p at all (for $q \equiv c^2 \bmod p$).

1.4 Applying these results to the Fibonacci sequence, probabilistic arguments suggest that for primes of the form $10n \pm 1$ the entry point of the Fibonacci sequence should be maximal, $(p-1)$, on an average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\phi(p_i - 1)}{p_i - 3}$$

over primes of that form; and the entry point should be maximal, $(p+1)$, on an average

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{\phi(p_i+1)}{p_i-1}$$

over primes of the form $10n \pm 3$. Investigations of entry points of primes less than 3000 [1,2] show a remarkably close correspondence with these theoretical values.

Number of Maximal Entry Points for $p < 3000$

	Predicted	Observed
$\Sigma\phi(p-1)/p-3 =$	74.25	76
$\Sigma\phi(p+1)/p-1 =$	87.78	88

2.0 Consider the sequences $\{\Gamma^n c,q\}$ modulo p, where c and q range over the reduced residue classes modulo p. Let d be the exponent to which $\Gamma c,q$ belongs modulo p.

The following can easily be established:

2.1.1 If $p|\Gamma^n c,q$, then $p|\Gamma^n c,q+p$ and $p|\Gamma^n c+p,q$.

2.1.2 For $c \equiv 0 \pmod p$, $d = 2$.

2.1.3 For $q \equiv 0$, $c \not\equiv 0 \pmod p$, $d = p$.

2.1.4 For $c_i + c_j \equiv 0 \pmod p$, $d_i = d_j$.

2.1.5 For $q \equiv c^2 \pmod p$, $d = \infty$.

2.2 Let $\alpha = c + \sqrt{q}$, $\bar\alpha = c - \sqrt{q}$. If $\Gamma c,q$ belongs to the exponent $k \pmod p$, we say α has Γ-order k. That is

$$\alpha^k - \bar\alpha^k \equiv 0 \pmod p, \quad \alpha^m - \bar\alpha^m \not\equiv 0 \pmod p \text{ for } m < k,\ m \neq 0.$$

We wish to determine the smallest d such that

$$\alpha^d \equiv \bar\alpha^d \pmod p.$$

We consider two cases, q a quadratic non-residue of p, and q a residue.

3.0 Case 1, q a quadratic non-residue of p. Construct $GF(p^2)$ with typical element $c + k\sqrt{q}$ (note: $k^2q \equiv \hat{q} \pmod p$, a non-residue). For some c', q', $\alpha = c' + \sqrt{q'}$ is of order $p^2 - 1$ since the multiplicative group of $GF(p^2)$ is cyclic.

3.1 We show that $\bar\alpha = \alpha^p$.

The conjugate of α can be defined as that element $\bar\alpha$ such that $\alpha\bar\alpha$ and $\alpha + \bar\alpha$ are both rational, i.e., elements of $GF(p)$. We know that in $GF(p)$ there are $\phi(d_i)$ elements of order d_i, $d_i|p-1$, and that $\Sigma\phi(d_i) = p-1$, accounting for all the non-zero elements of $GF(p)$. Thus the elements of $GF(p^2)$ which are in $GF(p)$ are characterized by orders which divide $p-1$, i.e.,

$$\alpha^{k(p+1)}, \quad k = 1, 2, \cdots, p-1.$$

3.1.1 Since α is of order $p^2 - 1$, $\alpha\cdot\alpha^p$ is of order $p-1$, thus is rational.

3.1.2 To show: $\alpha + \alpha^p$ is of order dividing $p-1$.

Expanding $(\alpha+\alpha^p)^{p-1}$, and noticing that $\binom{p-1}{k} \equiv (-1)^k \bmod p$, we obtain

$$(\alpha+\alpha^p)^{p-1} \equiv \alpha^{p-1} + \binom{p-1}{1}\alpha^{2p-2} + \cdots + \alpha^{p(p-1)} \equiv \alpha^{p-1} - \alpha^{2p-2} + \cdots + \alpha^{p(p-1)}$$

$$\equiv \alpha^{p-1}(1 - \alpha^{p-1} + (\alpha^{p-1})^2 - \cdots + (\alpha^{p-1})^{p-1} - (\alpha^{p-1})^p + (\alpha^{p-1})^p)$$

$$\equiv \alpha^{p-1}\left[\frac{(1-(\alpha^{p-1})^{p+1})}{1+\alpha^{p-1}} + (\alpha^{p-1})^p\right] \equiv \alpha^{p-1}\left[\frac{1-\alpha^{p^2-1}}{1+\alpha^{p-1}} + \alpha^{p^2-p}\right]$$

$$\equiv \alpha^{p-1}\alpha^{p^2-p} \equiv \alpha^{p^2-1} \equiv 1 \bmod p.$$

Thus $\alpha + \alpha^p$ is of order dividing $p-1$ and is rational. It follows that $\bar\alpha = \alpha^p$.

3.1.3 It can similarly be shown that $\bar{\alpha}^a = \alpha^{ap}$, unless a is a multiple of $p+1$. In that case α^a is rational and self conjugate, cf. § 4.0.

Let $\bar{\alpha}^a = \alpha^{ap}$. Then $(\alpha^a)^k \equiv (\alpha^{ap})^k$ for $\alpha^{apk} - \alpha^{ak} \equiv 0$, $\alpha^{ak(p-1)} \equiv 1 \pmod p$, and $ak \equiv 0 \pmod{p+1}$, since α is of order p^2-1. k is a divisor of $p+1$, say, d_i. Let $nd_i = p+1$, so that n is the smallest non-zero solution to $xd_i \equiv 0 \bmod (p+1)$ (i.e., α^n has Γ-order d_i).

If $(tn)d_i \equiv 0 \pmod{p+1}$, where $(t, d_i) = m$, $t = t'm$, $d_i = d_j m$ and $d_j | p+1$ with $d_j < d_i$, then

$$(tn)d_j \equiv 0 \pmod{p+1}$$

and (tn) is a solution to $xd_j \equiv 0 \pmod{p+1}$ with $d_j < d_i$.

$x = tn$, $t = 1, 2, \cdots$, are solutions to $xd_i \equiv 0 \pmod{p+1}$, and are primitive solutions for $(t, d_i) = 1$. There are exactly $\phi(d_i)$ of these less than d_i. For each of the $\phi(d_i)$ of these tn values, $tn < p+1$, α^{tn} has Γ-order d_i.

Consequently, for every divisor $d_i \neq 1$ of $p+1$, there are $\phi(d_i)$ values $a < p+1$, such that α^a has Γ-order d_i.

3.3 We wish to relate the elements in the tables below:

Table 1

$c \backslash q$	1	2	$\cdots$	q_i	$\cdots$	p
1						
2						
$\cdots$						
c_i				$c+\sqrt{q_i}$		
$\cdots$						
p						

Table 2

α	$\alpha^{1+(p+1)}$		$\alpha^{1+k(p+1)}$			
α^2						
$\cdots$		$\cdots$		$\cdots$		
α^a			$\alpha^{a+k(p+1)}$			
$\cdots$						
α^{p+1}	$\alpha^{2(p+1)}$				α^{p^2-1}	

NOTE: The elements of the last row of table two are rational. The elements of columns two through $p-1$ are rational multiples of the elements of the first column, in which for the exponent less than $(p+1)$, there are $\phi(d_i)$ elements of Γ-order d_i. Thus the Γ-orders of the elements in the first p rows are equal by rows and divide $p+1$. Since α is of order p^2-1, all $a+b\sqrt{q}$ are represented by some power of α. For $c_i+\sqrt{q_i}$, q_i a non-residue, there is some $\alpha^k = c_i + b\sqrt{q} \equiv c_i + \sqrt{q_i} \pmod p$.

3.3.1 If $\alpha^k \equiv c_i + \sqrt{q_i}$ and $\alpha^m \equiv c_j + \sqrt{q_i}$, then α^k and α^m are not in the same row in table two, for if

$$\alpha^k = \alpha^{x+y_1(p+1)} \qquad \alpha^m = \alpha^{x+y_2(p+1)} \qquad x < p+1$$

then

$$c_i + \sqrt{q_i} = \alpha^{x+y_1(p+1)}, \quad c_j + \sqrt{q_i} = \alpha^{x+y_2(p+1)}$$

subtracting,

$$c_i - c_j = \alpha^x(\alpha^{y_1(p+1)} - \alpha^{y_2(p+1)})$$

and α^x is rational, i.e., $x = p+1$, contrary to hypothesis.

3.2.2 We thus have a one-to-one mapping between elements of distinct rows of table two and elements of the q_i column of table one, indicating that for q_i a non-residue, c_i ranging from 1 to p there are $\phi(d_i)$, $d_i | p+1$ elements,

$c_i + \sqrt{q_i}$, of Γ-order d_i (Result 1.1).

4.0 Case 2, q a quadratic residue of p. Consider the elements of $GF(p)$. Let $\beta_i = a_i + b$, where $b \equiv \sqrt{q}$ (mod p), and call $\bar{\beta}_i = a_i - b$. Let $\gamma_i = \beta_i \bar{\beta}_i^{-1} = (a_i + b)/(a_i - b)$. If $(a_i + b)/(a_i - b) \equiv (a_j + b)/(a_j - b)$, then $a_i \equiv a_j$, and if a ranges through the values 0 to $p - 1$ the γ_i values generated are distinct. Provided $a \not\equiv \pm b$ (mod p), these are the elements 2 through $p - 1$ of $GF(p)$.

From $((a_i + b)/(a_i - b))^k = \gamma_i^k$ it is clear that the Γ-orders of β correspond with the orders of γ. There are $\phi(d_i)$ elements, γ_i, of order d_i for each divisor of $p - 1$ ($d_i \neq 1$), thus $\phi(d_i)$ elements β_i with Γ-orders d_i for each divisor of $p - 1$ except 1. In addition, for $a \equiv \pm b$ (mod p), i.e., $q \equiv c^2$ (mod p), the equation $(a_i + b)^k \equiv (a_i - b)^k$ has no solutions and we say the Γ-order of β is ∞. (2.1.5). (Result 1.2.)

5.0 To establish Result 1.3, relating to the rows of table one, consider $c + \sqrt{q_i}$ as q_i ranges from 1 to $p - 1$.

$c + \sqrt{q_i}$ has the same Γ-order as $ck + \sqrt{k^2 q}$ and as $(ck)' + \sqrt{k^2 q}$, where $ck + (ck)' \equiv 0$ (mod p) (2.1.4). Choose q_j a non-residue, $c_i < (p - 1)/2$, and k such that $kc_i \equiv c$. Then $k^2 q_j$ is a non-residue and $k(c_i + \sqrt{q_j}) \equiv c + \sqrt{q_i}$ and has the same Γ-order. Similarly for q_j a residue. Thus the entries in table one with $c_i < (p - 1)/2$ of a residue column and a non-residue column correspond with the entires of a row and we have Result 1.3: there are $\phi(d_i)/2$ values q such that $\Gamma c,q$ belongs to d_i (mod p) for $d_i | p - 1$, $d_i | p + 1$, with Γ-order ∞ for $q \equiv c^2$ (mod p), and Γ-order p for $q \equiv 0$ (mod p).

6.0 Results applied to the Fibonacci sequence. Let $c = 1$, $q = 5$. Since 5 is a non-residue for p of the form $10n \pm 3$ and a residue for $p = 10n \pm 1$, the maximal entry point for the former is $p + 1$ and for the latter $p - 1$. Since $c \neq p$ and $q \neq p$ for $p > 5$, the probability that the entry point is maximal for $p = 10n \pm 3$ is

$$\phi(p + 1)/(p - 1),$$

and for p of the form $10n \pm 1$,

$$\phi(p - 1)/(p - 3).$$

For $p < 3000$, over primes of the form $10n \pm 3$,

$$\sum \frac{\phi(p + 1)}{p - 1} = 87.78,$$

as compared with 88 primes of that form with maximal entry points.

Over primes of the form $10n \pm 1$,

$$\sum \frac{\phi(p - 1)}{p - 3} = 74.25,$$

as compared to 76 with maximal entry points.

Entry Points of $p = 13$ for $\{\Gamma^n c_i q_i\}$

$c \backslash q$	1	2	3	4	5	6	7	8	9	10	11	12
1	∞	7	12	6	7	14	14	14	12	3	7	4
2	3	14	6	∞	7	14	7	7	4	12	14	12
3	12	14	12	4	7	7	14	7	∞	6	14	3
4	12	7	∞	3	14	7	7	14	12	4	14	6
5	4	7	3	12	14	14	14	7	6	12	7	∞
6	6	14	4	12	14	7	7	14	3	∞	7	12

(see properties 2.1.1 – 2.1.5)

REFERENCES

1. Brother U. Alfred, "Additional Factors of the Fibonacci and Lucas Series," *The Fibonacci Quarterly*, Vol. 1, No. 1, Feb. 1963, pp. 34–42.
2. D. D. Wall, "Fibonacci Series Modulo *m*," *The American Math. Monthly*, Vol. 67, No. 6, June-July, 1960, pp. 525–432.

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MORE ON BENFORD'S LAW

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In a recent note, J. Wlodarski [1] observed that the Fibonacci and Lucas numbers tend to obey Benford's law which states: the probability that a random decimal begins with the digit p is

$$\log_{10}(p+1) - \log_{10} p.$$

(By begins, one means has extreme left digit.)

Wlodarski based his observations on the first 100 Fibonacci and Lucas numbers.

This is a report of a further investigation of the Benford phenomena. In this effort, the first 2000 representatives of both the Fibonacci and Lucas numbers were calculated and examined. The occurrences of the first digits were noted and tabulated. Further this was done for each base $b = 3$ to $b = 10$. The results of these calculations suggest an extended Benford law:

The probability that a random decimal written base b begins with p is

$$\log_{10}\frac{p+1}{p} \cdot \frac{1}{\log_{10} b} = \log_b \frac{p+1}{p}. \tag{1}$$

This result is anticipated by Flehinger [2] and is verified here.

In order to provide the statistical data concerning the Fibonacci and Lucas numbers of large magnitude and to various bases, a computer program was developed. It was written in FORTRAN-IV and has been run on an IBM 360-40. The program can develop the numbers up to $n = 5000$ base 10 using the 1000 digits provided. However, more digits would be needed for a lesser base. As a compromise $n = 2000$ was selected. The proportions of first digits to the various bases is recorded in Tables 1 and 2. Table 3 gives the corresponding results from (1) for comparison.

Table 1
Proportion of First Digits of Lucas Numbers

	Digits								
Base	1	2	3	4	5	6	7	8	9
10	.30100	.17600	.12550	.09650	.07950	.06650	.05850	.05100	.04500
9	.31800	.18150	.13300	.10250	.08300	.07000	.05900	.05300	
8	.33350	.19450	.13950	.10600	.08850	.07400	.06400		
7	.35450	.20850	.15000	.11300	.09350	.08050			
6	.37800	.22400	.16150	.12500	.10250				
5	.43050	.25100	.17950	.13900					
4	.50100	.29150	.20750						
3	.63650	.36350							

Table 2
Proportion of First Digits of Fibonacci Numbers

	Digits								
Base	1	2	3	4	5	6	7	8	9
10	.30050	.17650	.12500	.09650	.07950	.06650	.05750	.05200	.04600
9	.31400	.18650	.13200	.09900	.08300	.06950	.06200	.05400	
8	.33400	.19500	.13900	.10600	.08800	.07350	.06450		
7	.35750	.20900	.14600	.11550	.09200	.08000			
6	.38600	.22800	.16050	.12400	.10150				
5	.43100	.25250	.17800	.13850					
4	.49950	.29200	.20850						
3	.62800	.37200							

Table 3
Values of $\log_b (n+1)/n$

	n								
Base	1	2	3	4	5	6	7	8	9
10	.30103	.17609	.12494	.09691	.07918	.06695	.06099	.04815	.04576
9	.31547	.18453	.13093	.10156	.08298	.07016	.06391	.05046	
8	.33223	.19434	.13789	.10695	.08739	.07389	.06731		
7	.35621	.20837	.14784	.11467	.09369	.07922			
6	.38685	.22629	.16056	.12454	.10175				
5	.43068	.25193	.17875	.13865					
4	.50000	.29248	.20752						
3	.63093	.36907							

REFERENCES

1. J. Wlodarski, "Fibonacci and Lucas Numbers Tend to Obey Benford's Law," *The Fibonacci Quarterly,* February 1971, Vol. 9, No. 1, pp. 87–88.
2. B. J. Flehinger, "On the Probability that a Random Integer has Initial Digit A," *Amer. Math. Monthly,* Vol. 73 (1966), pp. 1056–1061.

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FORMULA DEVELOPMENT THROUGH FINITE DIFFERENCES

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FINITE DIFFERENCE CONCEPT

Given a function $f(n)$ the first difference of the function is defined

$$\Delta f(n) = f(n+1) - f(n).$$

(NOTE: There is a more generalized finite difference involving a step of size h but this can be reduced to the above by a linear transformation.)

EXAMPLES

$$f(n) = 5n + 3, \qquad \Delta f(n) = 5(n+1) + 3 - (5n+3) = 5$$

$$f(n) = 3n^2 + 7n + 2 \mid \Delta f(n) = 3(n+1)^2 + 7(n+1) + 2 - (3n^2 + 7n + 2) = 6n + 10.$$

Finding the first difference of a polynomial function of higher degree involves a considerable amount of arithmetic. This can be reduced by introducing a special type of function known as a generalized factorial.

GENERALIZED FACTORIAL

A generalized factorial

$$(x)^{(n)} = x(x-1)(x-2)\cdots(x-n+1),$$

where there are n factors each one less than the preceding. To tie this in with the ordinary factorial note that

$$n^{(n)} = n!$$

EXAMPLE

$$x^{(4)} = x(x-1)(x-2)(x-3).$$

The first difference of $x^{(n)}$ is found as follows:

$$\begin{aligned}\Delta x^{(n)} &= (x+1)x(x-1)\cdots(x-n+3)(x-n+2) - x(x-1)(x-2)\cdots(x-n+2)(x-n+1)\\ &= x(x-1)(x-2)\cdots(x-n+3)(x-n+2)[x+1-(x-n+1)] = nx^{(n-1)}.\end{aligned}$$

Note the nice parallel with taking the derivative of x^n in calculus.

To use the factorial effectively, in working with polynomials we introduce Stirling numbers of the first and second kind. Stirling numbers of the first kind are the coefficients when we express factorials in terms of powers of x. Thus

$$x^{(1)} = x, \quad x^{(2)} = x(x-1) = x^2 - x, \quad x^{(3)} = x(x-1)(x-2)(x-3) = x^3 - 3x^2 + 2x$$

$$x^{(4)} = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^6 - 6x.$$

Stirling numbers of the first kind merely record these coefficients in a table.

Stirling numbers of the second kind are coefficients when we express the powers of x in terms of factorials.

$$x = x^{(1)}$$

$$x^2 = x^2 - x + x = x^{(2)} + x^{(1)}$$

$$x^3 = x^3 - 3x^2 + 2x + (3x^2 - 3x) + x = x^{(3)} + 3x^{(2)} + x^{(1)}$$

As one example of the use of these numbers let us find the difference of the polynomial function

$$4x^5 - 7x^4 + 9x^3 - 5x^2 + 3x - 1.$$

Using the Stirling numbers of the second kind we first translate into factorials,

TABLE OF STIRLING NUMBERS OF THE FIRST KIND

power of x

n	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	2	−3	1							
4	−6	11	−6	1						
5	24	−50	35	−10	1					
6	−120	274	−225	85	−15	1				
7	720	−1764	1624	−735	175	−21	1			
8	−5040	13068	−13132	6769	−1960	322	−28	1		
9	40320	−109584	118124	−67284	22449	−4536	546	−36	1	
10	−362880	1026576	−1172700	723680	−269325	63273	−9450	870	−45	1

TABLE OF STIRLING NUMBERS OF THE SECOND KIND

Coefficients of $x^{(k)}$

k

n	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

TABLE OF FACTORIALS

	$x^{(5)}$	$x^{(4)}$	$x^{(3)}$	$x^{(2)}$	$x^{(1)}$	c
$4x^5$	4	40	100	60	4	
$-7x^4$		−7	−42	−49	−7	
$9x^3$			9	27	9	
$-5x^2$				−5	−5	
$3x - 1$					3	−1

Giving

$$4x^{(5)} + 33x^{(4)} + 67x^{(3)} + 33x^{(2)} + 4x^{(1)} - 1\,.$$

Using the formula for finding the difference of a factorial the first difference is given by

$$20x^{(4)} + 132x^{(3)} + 201x^{(2)} + 66x^{(1)} + 4\,.$$

Now we translate back to a polynomial function by using Stirling numbers of the first kind.

	x^4	x^3	x^2	x	c
$20x^{(4)}$	20	−120	220	−120	
$132x^{(3)}$		132	−396	264	
$201x^{(2)}$			201	−201	
$66x^{(1)} + 4$				66	4

The resulting polynomial function is

$$20x^4 + 12x^3 + 25x^2 + 9x + 4$$

A POLYNOMIAL FUNCTION FROM TABULAR VALUES

From the above it is evident that the first difference of a polynomial of degree n is a polynomial of degree $n - 1$; the second difference is a polynomial of degree $n - 2$; etc., so that the n^{th} difference is a constant. The $(n+1)^{st}$ difference is zero. As a matter of fact since at each step we multiply the coefficient of the first term by the power of x, the n^{th} difference of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-3} + \cdots + a_{n-1}x + a_n$$

is $a_0n!$

Conversely if we have a table of values and find that the r^{th} difference is a constant we may conclude that these values fit a polynomial function of degree r. For example for

$$f(x) = 5x^3 - 7x^2 + 3x - 8$$

we have a table of values and finite differences as follows.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	−8			
		1		
1	−7		16	
		17		30
2	10		46	
		63		30
3	73		76	
		139		30
4	212		106	
		245		30
5	457		136	
		381		30
6	838		166	
		547		
7	1385			

The problem is how to arrive at the original formula from this table.

Suppose that the polynomial is expressed in terms of factorials with undetermined coefficients $b_0, b_1, b_2, \cdots$. The problem will be solved if we find these coefficients.

$$f(x) = b_0 + b_1x^{(1)} + b_2x^{(2)} + b_3x^{(3)} + b_4x^{(4)} + b_5x^{(5)} + \cdots$$
$$\Delta f(x) = b_1 + 2b_2x^{(1)} + 3b_3x^{(2)} + 4b_4x^{(3)} + 5b_5x^{(4)} + \cdots$$
$$\Delta^2 f(x) = 2!b_2 + 3*2b_3x^{(1)} + 4*3b_4x^{(2)} + 5*4b_5x^{(3)} + \cdots$$
$$\Delta^3 f(x) = 3!b_3 + 4*3*2b_4x^{(1)} + 5*4*3b_5x^{(2)} + \cdots$$
$$\Delta^4 f(x) = 4!b_4 + 5*4*3*2b_5x^{(1)} + \cdots.$$

Set $x = 0$. Since any factorial is zero for $x = 0$ we have from the above:

$$f(0) = b_0 \quad \text{or} \quad b_0 = f(0)$$
$$\Delta f(0) = b_1 \quad \text{or} \quad b_1 = \Delta f(0)$$
$$\Delta^2 f(0) = 2!b_2 \quad \text{or} \quad b_2 = \Delta^2 f(0)/2!$$
$$\Delta^3 f(0) = 3!b_3 \quad \text{or} \quad b_3 = \Delta^3 f(0)/3!$$
$$\Delta^4 f(0) = 4!b_4 \quad \text{or} \quad b_4 = \Delta^4 f(0)/4!.$$

Hence

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \Delta^2 \frac{f(0)}{2!}x^{(2)} + \Delta^3 \frac{f(0)}{3!}x^{(3)} + \Delta^4 \frac{f(0)}{4!}x^{(4)} +$$

This is known as Newton's forward difference formula. We can find the quantities $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, $\Delta^4 f(0)$, ⋯ from the top edge of our numerical table of values provided the first value in our table is 0.

$$f(x) = -8 + x + 16x^{(2)}/2! + 30x^{(3)}/3! = -8 + x + 8x^2 - 8x + 5x^3 - 15x^2 + 10x = 5x^3 - 7x^2 + 3x - 8.$$

Stirling numbers of the first kind can be used in this evaluation.

SUMMATIONS INVOLVING POLYNOMIAL FUNCTIONS

Since a polynomial function can be expressed in terms of factorials it is sufficient to find a formula for summing any factorial. More simply by dividing the k^{th} factorial by $k!$ we have a binomial coefficient and the summation of these coefficients leads to a beautifully simple sequence of relations.

To evaluate

$$\sum_{k=1}^{n} k, \quad \text{let} \quad \sum_{k=1}^{n} k = \varphi(n)$$

meaning that the value is a function of n. Then

$$\Delta\varphi(n) = \sum_{k=1}^{n+1} k - \sum_{k=1}^{n} k = n + 1.$$

Now $\Delta n = 1$ and $\Delta n^{(2)}/2 = n$. Hence

$$\varphi(n) = \sum_{k=1}^{n} k = n^{(2)}/2 + n + C = n(n+1)/2 + C,$$

where the C is necessary in taking the anti-difference since the difference of a constant is zero. This corresponds to the constant of integration in the indefinite integral. To find the value of C let $n = 1$. Then

$$1 = 1*2/2 + C \quad \text{so that} \quad C = 0.$$

Hence

$$\sum_{k=1}^{n} k = n(n+1)/2 = \binom{n+1}{2}$$

a well-known formula. Next, let

$$\sum_{k=1}^{n} \binom{k+1}{2} = \varphi(n), \quad \Delta\varphi(n) = \sum_{k=1}^{n+1} \binom{k+1}{2} - \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{2}$$

The difference

$$\Delta\binom{n+2}{3} = \binom{n+2}{2}.$$

Hence

$$\varphi(n) = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3} + C.$$

$n = 1$ shows that $C = 0$. The sequence of formulas can be continued:

$$\sum_{k=1}^{n} \binom{k+2}{3} = \binom{n+3}{4}$$

and in general

$$\sum_{k=1}^{n} \binom{k+r}{r+1} = \binom{n+r+1}{r+2}.$$

One could derive the formula for the summation of a factorial from the above but proceeding directly:

$$\sum_{k=1}^{n} k^{(r)} = \varphi(n), \qquad \Delta\varphi(n) = (n+1)^{(r)}.$$

Hence,

$$\varphi(n) = \sum_{k=1}^{n} k^{(r)} = \frac{(n+1)^{(r+1)}}{r+1} + C.$$

Taking $n = r$,

$$r! = (r+1)^{(r+1)}/(r+1) + C$$

so that $C = 0$.

$$\sum_{k=1}^{n} k^{(r)} = \frac{(n+1)^{(r+1)}}{r+1} .$$

Again there is a noteworthy parallel with the integral calculus in this formula.

For examples we take some formulas from L. B. W. Jolley, *Summation of Series,*

EXAMPLE 1. (45) p. 8, $\displaystyle\sum_{k=1}^{n} (3k-1)(3k+2) = 2*5 + 5*8 + 8*11 + \cdots .$

This equals

$$\sum_{k=1}^{n} (9k^2 + 3k - 2) = \sum_{k=1}^{n} (9k^{(2)} + 12k^{(1)} - 2) = 9\,\frac{(n+1)^{(3)}}{3} + 12\,\frac{(n+1)^{(2)}}{2} - 2(n+1) + C.$$

Taking $n = 1$, $2*5 = 6*2 - 2*2 + c$ so that $C = 2$

$$\sum_{k=1}^{n} (3k-1)(3k+2) = 3n^3 - 3n + 6n^2 + 6n - 2n - 2 + 2 = n(3n^2 + 6n + 1).$$

EXAMPLE 2. (50) p. 10

$$\sum_{k=1}^{n} k(k+3)(k+6) = 1*4*7 + 42*5*k + 3*6*9 + \cdots = \sum_{k=1}^{n} (k^3 + 9k^2 + 18k) = \sum_{k=1}^{n} (k^{(3)} + 12k^{(2)} + 28k^{(1)})$$

$$= \frac{(n+1)^{(4)}}{4} + 12\,\frac{(n+1)^{(3)}}{3} + 28\,\frac{(n+1)^{(2)}}{2} + C = \frac{(n+1)n}{4}\,[(n-1)(n-2) + 16(n-1) + 56] + C$$

$$= n(n+1)(n+6)(n+7)/4 + C.$$

Setting $n = 1$, $1*4*7 = 1*2*7*8/4 + c$ so that $C = 0$

$$\sum_{k=1}^{n} k(k+3)(k+6) = n(n+1)(n+6)(n+7)/4.$$

EXAMPLE 3. (49) p. 10

$$\sum_{k=1}^{n} (3k-2)(3k+1)(3k+4) = 1*4*7 + 4*7*10*13 + \cdots .$$

This can be changed directly into a factorial:

$$27\sum_{k=1}^{n} (k - 2/3)(k + 1/3)(k + 4/3) = 27\sum_{k=1}^{n} (k + 4/3)^{(3)}$$

giving

$$27(n + 7/3)^{(4)}/4 + C = (3n+7)(3n+4)(3n+1)(3n-2)/12 + C.$$

Setting $n = 1$, $28 = (10*7*4*1)/12 + C$ so that $C = 56/12$

$$(3k-2)(3k+1)(3k+4) = (3n+7)(3n+4)(3n+1)(3n-2)/12 + 56/12 .$$

SUMMATIONS THROUGH NEGATIVE FACTORIALS

Starting with the relation

$$x^{(m)} * (x-m)^{(n)} = x^{(m+n)}$$

set $m = -n$.

$$x^{(-n)} * (x+n)^{(n)} = x^{(0)} = 1.$$

Therefore $x^{(-n)} = 1/(x+n)^{(n)}$.

Possibly this bit of mathematical formalism seems unconvincing. Suppose then we define the negative factorial in this fashion.

$$\begin{aligned}\Delta x^{(-n)} &= 1/[(x+n+1)(x+n)(x+n-1)\cdots(x+2)] - 1/[(x+n)(x+n-1)(x+n-2)\cdots(x+2)(x+1)]\\ &= 1/[(x+n)(x+n-1)\cdots(x+2)][1/(x+n+1) - 1/(x+1)]\\ &= -n/[(x+n+1)(x+n)(x+n-1)\cdots(x+1)] = -nx^{(-n-1)}\end{aligned}$$

showing that the difference relation that applies to positive factorials holds as well for negative factorials defined in this fashion. Consequently the anti-difference which is used in finding the value of summations can be employed with negative factorials apart from the case of -1.

EXAMPLE 1.

$$\sum_{k=1}^{n} 1/[k(k+1)(k+2)] = \sum_{k=1}^{n} (k-1)^{(-3)} = n^{(-2)}/(-2) + C = -1/[2(n+2)(n+1)] + C .$$

Setting $n = 1,\ 1/6 = -1/(2*3*2) + C$, so that $C = 1/4$

$$\sum_{k=1}^{n} 1/[k(k+1)(k+2)] = 1/4 - 1/[2(n+2)(n+1)] .$$

EXAMPLE 2. Jolley, No. 210, p. 40

$$\begin{aligned}\sum_{k=1}^{n} 1/[(3k-2)(3k+1)(3k+4)] &= (1/27)\sum_{k=1}^{n} 1/[k-2/3)(k+1/3)(k+4/3)] = (1/27)\sum_{k=1}^{n} (k-5/3)^{(-3)}\\ &= (1/27)(n-2/3)^{(-2)}/(-2) + C = -1/[6(3n+4)(3n+1)] + C .\end{aligned}$$

Setting $n = 1,\ 1/(1*4*7) = -1/(6*7*4) + C;\ C = 1/24$

$$\sum_{k=1}^{n} 1/[(3k-2)(3k+1)(3k+4)] = 1/24 - 1/[6(3n+4)(3n+1)]$$

EXAMPLE 3. Jolley, No. 213, p. 40

$$\sum_{k=1}^{n} (2k-1)/[k(k+1)(k+2)] = 2\sum_{k=1}^{n} 1/[(k+1)(k+2)] - \sum_{k=1}^{n} 1/[k(k+1)(k+2)] .$$

The second summation was evaluated in Example 1. The first gives

$$2\sum_{k=1}^{n} k^{(-2)} = 2(n+1)^{(-1)}/(-1) + C .$$

Altogether, the result is

$$-2/(n+2) - 1/4 + 1/[2(n+2)(n+1)] + C .$$

Setting $n = 1,\ 1/6 = -2/3 - 1/4 + 1/12 + C$ so that $C = 1$

$$\sum_{k=1}^{n} (2k-1)/[k(k+1)(k+2)] = 3/4 - 2/(n+2) + 1/[2(n+2)(n+1)] .$$

DIFFERENCE RELATION FOR A PRODUCT

Let there be two functions $f(n)$ and $g(n)$. Then

$$\Delta f(n)g(n) = f(n+1)g(n+1) - f(n)g(n) = f(n+1)g(n+1) - f(n+1)g(n) + f(n+1)g(n) - f(n)g(n)$$
$$= f(n+1)\Delta g(n) + g(n)\Delta f(n).$$

This will be found useful in a variety of instances.

SUMMATIONS INVOLVING GEOMETRIC PROGRESSIONS

A geometric progression with terms ar^{k-1} can be summed as follows:

$$\sum_{k=1}^{n} ar^{k-1} = \varphi(n), \qquad \Delta\varphi(n) = ar^n$$

But $\Delta r^n = r^{n+1} - r^n = r^n(r-1)$. Hence

$$\varphi(n) = \sum_{k=1}^{n} ar^{k-1} = \Delta^{-1}(ar^n) = ar^n/(r-1) + C.$$

Setting $n = 1$, $a = ar/(r-1) + C$ so that $C = -a/(r-1)$. Hence,

$$\sum_{k=1}^{n} ar^{k-1} = a(r^n - 1)/(r-1).$$

The summation

$$\sum_{k=1}^{n} kr^k = \varphi(n), \qquad \Delta\varphi(n) = (n+1)r^{n+1}, \qquad \Delta(nr^{n+1}) = (n+1)r^{n+1}(r-1) + r^{n+1}$$

using the product formula on page 8 with the first function as n and the second as r^{n+1}.

$$(n+1)r^{n+1} = \Delta[nr^{n+1}/(r-1)] - r^{n+1}/(r-1).$$

Hence

$$\Delta^{-1}(n+1)r^{n+1} = nr^{n+1}/(r-1) - r^{n+1}/(r-1)^2 + C.$$

Setting $n = 1$, $r = r^2/(r-1) - r^2/(r-1)^2 + C$; $C = r/(r-1)^2$. Accordingly

$$\sum_{k=1}^{n} kr^k = nr^{n+1}/(r-1) - r^{n+1}/(r-1)^2 + r/(r-1)^2.$$

EXAMPLE. $$\sum_{k=1}^{5} k*3^k = 1*3 + 2*9 + 3*27 + 4*81 + 5*243 = 1641.$$

By formula $$5*3^6/2 - 3^6/4 + 3/4 = 1641.$$

FIBONACCI SUMMATIONS

A Fibonacci sequence is defined by two initial terms T_1 and T_2 accompanied by the recursion relation

$$T_{n+1} = T_n + T_{n-1}.$$

SUM OF THE TERMS OF THE SEQUENCE

$$\sum_{k=1}^{n} T_k = \varphi(n), \qquad \Delta\varphi(n) = T_{n+1}, \qquad \Delta T_n = T_{n+1} - T_n = T_{n-1}.$$

Accordingly

$$\sum_{k=1}^{n} T_k = T_{n+2} + C.$$

Setting $n = 1$, $T_1 = T_3 + C$ or $C = T_1 - T_3 = -T_2$

$$\sum_{k=1}^{n} T_k = T_{n+2} - T_2$$

SUM OF THE SQUARES OF THE TERMS

$$\sum_{k=\alpha}^{n} T_k^2 = \varphi(n), \quad \Delta\varphi(n) = T_{n+1}^2 .$$

The anti-difference bears a strong resemblance to integration in the differential calculus. Just as we know integrals on the basis of differnetiation so likewise we find anti-differences on the basis of differences. Thus we try various expressions to see whether we can find one whose difference is the square of T_{n+1}.

$$\Delta T_n T_{n+1} = T_{n+1}T_{n+2} - T_n T_{n+1} = T_{n+1}(T_{n+2} - T_n) = T_{n+1}^2 .$$

Hence

$$\sum_{k=\alpha}^{n} T_k^2 = T_n T_{n+1} + C .$$

Setting $n = a$, $T_a^2 = T_a T_{a+1} + C$

$$C = T_a(T_a - T_{a+1}) = -T_a T_{a-1}, \quad \sum_{k=\alpha}^{n} T_k^2 = T_n T_{n+1} - T_a T_{a-1} .$$

SUMMATION OF ALTERNATE TERMS

$$\sum_{k=m}^{n} T_{2k+a} = \varphi(n), \quad \Delta\varphi(n) = T_{2(n+1)+a}, \quad \Delta T_{2n+a} = T_{2n+2+a} - T_{2n+a} = T_{2n+1+a} .$$

Hence

$$\Delta^{-1} T_{2(n+1)+a} = T_{2n+1+a} + C, \quad \sum_{k=m}^{n} T_{2k+a} = T_{2n+1+a} + C .$$

Setting $k = m$,

$$T_{2m+a} = T_{2m+1+a} + C, \qquad \sum_{k=m}^{n} T_{2k+a} = T_{2n+1+a} - T_{2m-1+a}$$

SUM OF EVERY FOURTH TERM

$$\sum_{k=1}^{n} T_{4k+a} = \varphi(n), \qquad \Delta\varphi(n) = T_{4n+4+a}$$

$$\Delta T_{4n+a} = T_{4n+4+a} - T_{4n+a} = T_{4n+3+a} + T_{4n+2+a} - T_{4n+2+a} + T_{4n+1+a} = T_{4n+3+a} + T_{4n+1+a}$$

To meet this situation we introduce a quantity

$$V_n = T_{n-1} + T_{n+1} .$$

Now

$$V_{n-1} + V_{n+1} = T_{n-2} + T_n + T_n + T_{n+2} = -T_{n-1} + T_n + 2T_n + T_n + T_{n+1} = 5T_n .$$

To obtain a difference which gives T we start with V. By a process similar to that for T

$$\Delta V_{4n+a} = V_{4n+3+a} + V_{4n+1+a} = 5T_{4n+2+a} .$$

Consequently,

$$\Delta^{-1} T_{4n+4+a} = (V_{4n+2+a})/5 + C = \sum_{k=1}^{n} T_{4k+a} .$$

Setting $n = 1$,

$$C = T_{4+a} - V_{6+a}/5, \quad \sum_{k=1}^{n} T_{4k+a} = (T_{4n+1+a} + T_{4n+3+a})/5 - (T_{5+a} + T_{7+a})/5 + T_{4+a} .$$

EXAMPLE. We use the terms of the sequence beginning 1,4.

1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, 1741, 2817, 4558, 7375, 11933, 19308, 31241, 50549, 81790, 132339, 214129, 346468, 560597, 907065, 1467662, 2374727.

Let $a = 2$.

$$\sum_{k=1}^{5} T_{4k+2} = T_6 + T_{10} + T_{14} + T_{18} + T_{22} = 23 + 157 + 1076 + 7375 + 50549 = 59180.$$

By formula we have

$$(T_{23} + T_{25})/5 - (T_7 + T_9)/5 + T_6 = (81790 + 214129)/5 - (37 + 97)/5 + 23 = 59180.$$

SEQUENCE WITH ALTERNATING SIGNS

$$\sum_{k=m}^{n} (-1)^k T_{2k+a} = \varphi(n), \quad \Delta\varphi(n) = (-1)^{n+1} T_{2n+2+a}, \quad V_{2n+a} = T_{2n+1+a} + T_{2n-1} + a$$

$$\Delta(-1)^n V_{2n+a} = (-1)^{n+1} V_{2n+2+a} - (-1)^n V_{2n+a} = (-1)^{n+1}[V_{2n+2+a} + V_{2n+a}] = (-1)^{n+1} 5T_{2n+1+a}.$$

Hence

$$\sum_{k=m}^{n} (-1)^n T_{2k+a} = (-1)^n (V_{2n+1+a})/5 + C = (-1)^n [T_{2n+a} + T_{2n+a+2}]/5 + C.$$

Let $n = m$.

$$(-1)^m T_{2m+a} = (-1)^m [T_{2m+a} + T_{2m+2+a}]/5 + C$$

$$\sum_{k=m}^{n} (-1)^k T_{2k+a} = (-1)^n [T_{2n+a} + T_{2n+a+2}]/5 + (-1)^{m+1}[T_{2m+a} + T_{2m+a+2}]/5 + (-1)^m T_{2m+a}.$$

Using the 1,4 sequence once more

$$\sum_{k=3}^{7} (-1)^k T_{2k+3} = -T_9 + T_{11} - T_{13} + T_{15} - T_{17} = -97 + 254 - 665 + 1741 - 4558 = -3325.$$

By formula we have

$$-(T_{17} + T_{19})/5 + (T_9 + T_{11})/5 - T_9 = -(4558 + 11933)/5 + (97 + 254)/5 - 97 = -3325.$$

GEOMETRIC-FIBONACCI SUMS

POWER of 2.

$$\sum_{k=1}^{n} 2^k T_k = \varphi(n); \qquad \Delta\varphi(n) = 2^{n+1} T_{n+1}$$

$$\Delta 2^n T_n = 2^{n+1} T_n + 2^n T_n = 2^n (2T_{n-1} + T_n) = 2^n V_n,$$

where we have used the product relation on page 8 and introduced the sequence defined by

$$V_n = T_{n-1} + T_{n+1}.$$

Since $\Delta 2^n V_n = 5 * 2^n T_n$ (following the same steps as for T_n)

$$\varphi(n) = \Delta^{-1}(2^{n+1} T_{n+1}) = 2^{n+1} V_{n+1}/5 + C.$$

Setting $n = 1$, $2T_1 = 4V_2/5 + C$. Hence

$$\sum_{k=1}^{n} 2^k T_k = 2^{n+1}(T_n + T_{n+2})/5 + (6T_1 - 4T_3)/5.$$

EXAMPLE.

$$\sum_{k=1}^{5} 2^k T_k = 2*1 + 4*4 + 8*5 + 16*9 + 32*14 = 650 \text{ (1,4 sequence)}.$$

By formula $[2^6(14+37)+6-4*5]/5 = 650.$

THE SUMMATION

$$\sum_{k=1}^{n} r^k T_k .$$

The direct approach leads to an apparent impasse. We wish to find the inverse difference of $r^{n+1}T_{n+1}$. Assume that it is of the form

$$A[r^kT_{n+1}+r^jT_n].$$

This approach parallels what is done in the solution of differential equations. $k, j,$ and A are undetermined constants. Taking the difference and setting it equal to $r^{n+1}T_{n+1}$ we have

$$A[r^{k+1}T_n+r^{j+1}T_{n-1}+r^k(r-1)T_{n+1}+r^j(r-1)T_n] = r^{n+1}T_{n+1}.$$

Replacing T_{n-1} on the left-hand side by $T_{n+1}-T_n$ and equating coefficients of T_{n+1} and T_n gives:

$$A[r^k(r-1)+r^{j+1}] = r^{n+1}, \qquad r^{k+1}+r^j(r-1)-r^{j+1} = 0.$$

From the second $j = k+1$. Then the first gives

$$A[r^{k+1}-r^k+r^{k+2}] = r^{n+1}.$$

Letting $k = n+1$ and $A = 1/(r^2+r-1)$ establishes equality. Hence

$$\sum_{k=1}^{n} r^kT_k = (r^{n+1}T_{n+1}+r^{n+2}T_n)/(r^2+r-1)+C, \quad C = (-r^2T_0-rT_1)/(r^2+r-1)$$

$$\sum_{k=1}^{n} r^kT_k = [r^{n+1}T_{n+1}+r^{n+2}T_n-r^2T_0-rT_1]/(r^2+r-1).$$

EXAMPLE (1,4 sequence)

$$\sum_{k=1}^{5} 3^kT_k = 3*1+3^2*4+3^3*5+3^4*9+3^5*14 = 4305.$$

By formula,

$$(3^6*23+3^7*14-27-3)/11 = 4305.$$

FIBONACCI-FACTORIAL SUMMATIONS

THE SUMMATION

$$\sum_{k=1}^{n} kT_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)T_{n+1}$$

$$\Delta nT_n = (n+1)T_{n-1}+T_n$$

$$\Delta nT_{n+2} = (n+1)T_{n+1}+T_{n+2}$$

$$\Delta^{-1}(n+1)T_{n+1} = nT_{n+2}-T_{n+3}+T_3+C = \sum_{k=1}^{n} kT_k$$

in which we have used the formula

$$\Delta^{-1}T_{n+2} = T_{n+3}-T_3$$

$n = 1$ gives

$$T_1 = T_3-T_4+T_3+C; \quad C = 0$$

so that

$$\sum_{k=1}^{n} kT_k = nT_{n+2} - T_{n+3} + T_3 .$$

Note that this is also $\Delta^{-1}(n+1)T_{n+1}$, a fact that is used in the next derivation.

EXAMPLE (1,4 sequence)

$$\sum_{k=1}^{5} kT_k = 1*1 + 2*4 + 3*5 + 4*9 + 5*14 = 130 .$$

By formula $5*36 - 60 + 5 = 130.$

THE SUMMATION

$$\sum_{k=1}^{n} k^{(2)}T_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)^{(2)}T_{n+1}$$

$$\Delta n^{(2)}T_{n+2} = (n+1)^{(2)}T_{n+1} + 2nT_{n+2}$$

$$\sum_{k=1}^{n} k^{(2)}T_k = n^{(2)}T_{n+2} - 2(n-1)T_{n+3} + 2T_{n+4} - 2T_4 + C$$

in which the formula for the previous case was used.

For $n = 2,$

$$2T_2 = 2T_4 - 2T_5 + 2T_6 - 2T_4 + C; \qquad C = -2T_3$$

$$\sum_{k=1}^{n} k^{(2)}T_k = n^{(2)}T_{n+2} - 2(n-1)T_{n+3} + 2T_{n+4} - 2T_4 - 2T_3$$

VERIFICATION (1,4 sequence)

$$\sum_{k=1}^{5} k^{(2)}T_k = 1*0*1 + 2*1*4 + 3*2*5 + 4*3*9 + 5*4*14 = 426 .$$

By formula

$$5*4*37 - 2*4*60 + 2*97 - 2*9 - 2*5 = 426 .$$

THE SUMMATION

$$\sum_{k=1}^{n} k^{(3)}T_k = \varphi(n)$$

$$\Delta\varphi(n) = (n+1)^{(3)}T_{n+1}$$

$$\Delta n^{(3)}T_{n+2} = (n+1)^{(3)}T_{n+1} + 3n^{(2)}T_{n+2}$$

$$\sum_{k=1}^{n} k^{(3)}T_k = n^{(3)}T_{n+2} - 3(n-1)^{(2)}T_{n+3} + 6(n-2)T_{n+4} - 6T_{n+5} + 6T_6 + C .$$

For $n = 3,$

$$6T_3 = 6T_5 - 6T_6 + 6T_7 - 6T_8 + 7T_6 + C; \qquad C = 6T_5$$

$$\sum_{k=1}^{n} k^{(3)}T_k = n^{(3)}T_{n+2} - 3(n-1)^{(2)}T_{n+3} + 6(n-2)T_{n+4} - 6T_{n+5} + 6T_7 .$$

VERIFICATION (1,4 sequence)

$$\sum_{k=1}^{6} k^{(3)}T_k = 6*5+24*9+60*14+120*23 = 3846.$$

By formula for $n = 6$,

$$120*60 - 60*97 + 24*157 - 6*254 + 6*37 = 3846.$$

The formulas for the next two cases are written down and the pattern that is emerging is noted.

$$\sum_{k=1}^{n} k^{(4)}T_k = n^{(4)}T_{n+2} - 4(n-1)^{(3)}T_{n+3} + 12(n-2)^{(2)}T_{n+4} - 24(n-3)T_{n+5} + 24T_{n+6} - 24T_9 \;.$$

$$\sum_{k=1}^{n} k^{(5)}T_k = n^{(5)}T_{n+2} - 5(n-1)^{(4)}T_{n+3} + 20(n-2)^{(3)}T_{n+4} - 60(n-3)^{(2)}T_{n+5}$$
$$+ 120(n-4)T_{n+6} - 120T_{n+7} + 120T_{11}\;.$$

The pattern may be described as follows:

For the r^{th} difference:

1. The first term is $n^{(r)}T_{n+2}$.
2. For the n portion, both n and r go down by 1 at each step.
3. For the T portion the subscript goes up by 1 at each step for $r+1$ steps.
4. The signs alternate.
5. The coefficients are the product, respectively, of the binomial coefficients for r by 0!, 1!, 2!, $\cdots$, $r!$, respectively.
6. The last term is $r!T_{2r+1}$ with sign determined by the alternation mentioned in 4.

With the aid of these factorial formulas it is now possible to find polynomial formulas. For example.

$$\sum_{k=1}^{n} k^4T_k = \sum_{k=1}^{n} [k^{(4)} + 6k^{(3)} + 7k^{(2)} + k^{(1)}]T_k\;.$$

The first few formulas for the powers are given herewith.

$$\sum_{k=1}^{n} k^2T_k = (n^2+2)T_{n+2} - (2n-3)T_{n+3} - T_6$$

$$\sum_{k=1}^{n} k^3T_k = (n^3+6n-12)T_{n+2} - (3n^2-9n+19)T_{n+3} + 6T_6 + T_3$$

$$\sum_{k=1}^{n} k^4T_k = (n^4+12n^2-48n+98)T_{n+2} - (4n^3-18n^2+76n-159)T_{n+3} - 13T_8 - 11T_7$$

$$\sum_{k=1}^{n} k^5T_k = (n^5+20n^3-120n^2+490n-1020)T_{n+2} - (5n^4-30n^3+190n^2-795n+1651)T_{n+3}$$
$$+ 120T_9 + 30T_6 + T_3\;.$$

In these formulas considerable algebra has been done to reduce the number of terms down to two main terms by using Fibonacci shift formulas.

GENERAL SECOND-ORDER RECURSION SEQUENCES

Given a second-order recursion sequence governed by the recursion relation

$$T_{n+1} = P_1T_n + P_2T_{n-1}$$

to find

$$\sum_{k=1}^{n} T_k = \varphi(n)$$

$$\Delta\varphi(n) = T_{n+1}$$

$$\Delta[T_n + P_2T_{n-1}] = T_{n+1} + P_2T_n - T_n - P_2T_{n-1} = (P_1 + P_2 - 1)T_n.$$

Provided $P_1 + P_2 - 1$ is not zero,

$$\sum_{k=1}^{n} T_k = (T_{n+1} + P_2T_n)/(P_2 + P_1 - 1) + C.$$

For $n = 1$,

$$T_1 = (T_2 + P_2T_1)/(P_2 + P_1 - 1) + C$$

$$C = [(P_1 - 1)T_1 - T_2]/(P_2 + P_1 - 1)$$

$$\sum_{k=1}^{n} T_k = [T_{n+1} + P_2T_n + (P_1 - 1)T_1 - T_2]/(P_2 + P_1 - 1).$$

EXAMPLE: $T_{n+1} = 5T_n - 3T_{n-1}$

3, 7, 26, 109, 467, 2008;

$$\sum_{k=1}^{5} T_k = 3 + 7 + 26 + 109 + 467 = 612.$$

By formula (2008 − 3*467 + 4*3 − 7)/(5 − 3 − 1) = 612.

SUM OF TERMS OF A THIRD-ORDER SEQUENCE

Such a sequence is bound by a recursion relation of the form

$$T_{n+1} = P_1T_n + P_2T_{n-1} + P_3T_{n-2}.$$

If

$$\sum_{k=1}^{n} T_k = \varphi(n), \qquad \Delta\phi(n) = T_{n+1}$$

$$\Delta(T_n + (P_3 + P_2)T_{n-1} + P_3T_{n-2}) = T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1} - T_n - (P_3 + P_2)T_{n-1} - P_3T_{n-2}$$

$$= T_{n+1} + (P_3 + P_2 - 1)T_n - P_2T_{n-1} - P_3T_{n-2} = (P_1 + P_2 + P_3 - 1)T_n.$$

Hence if $P_1 + P_2 + P_3 - 1$ is not zero,

$$\sum_{k=1}^{n} T_k = [T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1}]/(P_1 + P_2 + P_3 - 1) + C$$

$$T_1 + T_2 = [T_3 + (P_3 + P_2)T_2 + P_3T_1]/(P_1 + P_2 + P_3 - 1) + C$$

$$C = [(P_1 + P_2 - 1)T_1 + (P_1 - 1)T_2 - T_3]/(P_1 + P_2 + P_3 - 1)$$

$$\sum_{k=1}^{n} T_k = [T_{n+1} + (P_3 + P_2)T_n + P_3T_{n-1} + (P_1 + P_2 - 1)T_1 + (P_1 - 1)T_2 - T_3]/(P_1 + P_2 + P_3 - 1)$$

EXAMPLE. $T_{n+1} = 3T_n + 2T_{n-1} - T_{n-2}$

1 + 2 + 4 + 15 + 179 = 252. Next term is 624.

By formula (624 + 179 − 51 + 4*1 + 2*2 − 4)/3 = 252.

FOURTH-ORDER SEQUENCES

The recursion relation is

$$T_{n+1} = P_1 T_n + P_2 T_{n-1} + P_3 T_{n-2} + P_4 T_{n-3}.$$

An entirely similar analysis as was made for third-order sequences leads to the formula

$$T_k = [T_{n+1} + (P_2 + P_3 + P_4)T_n + (P_3 + P_4)T_{n-1} + P_4 T_{n-2}]/(P_1 + P_2 + P_3 + P_4 - 1) + C,$$

where

$$C = [(P_1 + P_2 + P_3 - 1)T_1 + (P_1 + P_2 - 1)T_2 + (P_1 - 1)T_3 - T_4]/(\Sigma P_i - 1).$$

EXAMPLE. $$T_{n+1} = 3T_n + 2T_{n-1} - 4T_{n-2} + 3T_{n-3}$$

1 + 3 + 4 + 6 + 17 + 56 + 190 + 632 = 909. Next term is 2103. By formula (2103 + 632 – 190 + 3*56 + 4*3 + 2*4 – 6)/3 = 909.

FIBONACCI-COMBINATORIAL FORMULAS

These are closely related to the Fibonacci-factorial formulas discussed on pp. 13–15. However the added simplicity of these formulas merits a listing of the first few to show the pattern.

$$\sum_{k=1}^{n} \binom{k}{1} T_k = \binom{n}{1} T_{n+2} - T_{n+3} + T_3, \qquad \sum_{k=1}^{n} \binom{k}{2} T_k = \binom{n}{2} T_{n+2} - \binom{n-1}{1} T_{n+3} + T_{n+4} - T_5$$

$$\sum_{k=1}^{n} \binom{k}{3} T_k = \binom{n}{3} T_{n+2} - \binom{n-1}{2} T_{n+3} + \binom{n-2}{1} T_{n+4} - T_{n+5} + T_7$$

$$\sum_{k=1}^{n} \binom{k}{4} T_k = \binom{n}{4} T_{n+2} - \binom{n-1}{3} T_{n+3} + \binom{n-2}{2} T_{n+4} - \binom{n-3}{1} T_{n+5} + T_{n+6} - T_9$$

$$\sum_{k=1}^{n} \binom{k}{5} T_k = \binom{n}{5} T_{n+2} - \binom{n-1}{4} T_{n+3} + \binom{n-2}{3} T_{n+4} - \binom{n-3}{2} T_{n+5} + \binom{n-4}{1} T_{n+6} - T_{n+7} + T_{11}$$

FIBONACCI EXTENSION: SUMMING MORE TERMS

Sequences governed by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

where three rather than two preceding terms are added at each step have a summation formula

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 2T_n + T_{n-1} + T_1 - T_3)/2.$$

For sequences governed by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3},$$

where the four previous terms are added

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} + 2T_1 + T_2 - T_4)/3.$$

Where five previous terms are added at each step:

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 4T_n + 3T_{n-1} + 2T_{n-2} + T_{n-3} + 3T_1 + 2T_2 + T_3 - T_5)/4.$$

Where six previous terms have been added at each step:

$$\sum_{k=1}^{n} T_k = (T_{n+1} + 5T_n + 4T_{n-1} + 3T_{n-2} + 2T_{n-3} + T_{n-4} + 4T_1 + 3T_2 + 2T_3 + T_4 - T_6)/5.$$

EXAMPLE.

$$1+2+4+5+7+8+27+53+104+204 = 415.$$

By formula

$$(403+5*204+4*104+3*53+2*27+8+4+6+8+5-8)/5 = 415.$$

CONCLUSION

Finite differences have wide application in formula development. There are, of course, many situations in which the use of this method leads to difficulties which other procedures can obviate. But where applicable the results are often obtained with such facility that other procedures seem laborious by comparison.

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A GOLDEN DOUBLE CROSTIC

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Use the definitions in the clue story which follows to write the words to which they refer; then enter the appropriate letters in the diagram to complete a quotation from a mathematician whose name appears in the last line of the diagram. The name of the book in which this quotation appeared and the author's last name appear as the first letters of the clue words. The end of each word is indicated by a shaded square following it.

CLUE STORY

The mystic Golden Section Ratio, $(1+\sqrt{5})/2$, called __(A-1, A-2)__ (the latter most commonly), occurs in several propositions in __(A-3, A-4)__ on line segments and __(A-5)__. This Golden Cut fascinated the ancient Greeks, particularly the __(D-1)__, who found this value in the ratio of lengths of segments in the __(D-2)__ and __(D-3)__ and who also made studies in __(D-4)__. The Greeks found the proportions of the Golden Rectangle most pleasing to the eye as evidenced by the ubiquitous occurrence of this form in art and architecture, such as __(C-1)__ or in sculpture as in the proportions of the famous __(C-2)__; however, they may have been copying __(C-3)__, for the Golden Proportion occurs frequently in the forms of living things and is closely related to the growth patterns of plants, as __(C-4, C-5, C-6)__, in which occur ratios of Fibonacci numbers. The Golden Section is the limiting value of the ratio of two successive Fibonacci numbers (named for __(G-1)__), being closely approximated by the __(G-2, G-3)__.

By some mathematicians, the beauty of the __(N)__ relating to the Golden Section is compared to the theorem of the __(D-1)__ and to such results from projective geometry as those seen in Pascal's "Mystic __(B)__" or even in the applications of mathematics in the *Principia Mathematica* of __(I)__ while the constant $(1+\sqrt{5})/2$ itself is rivalled by __(E-1)__ and __(E-2)__.

Unfortunately, not all persons find mathematics beautiful. __(H-1)__ was one of the four branches of arithmetic given by the Mock Turtle in *Alice in Wonderland,* and the card player's description of the sequence 2, 1, 3, 4, 7, ______, 18, 29, 47, ⋯ would be __(H-2)__, while some have to have all mathematics of practical use, such as in reading an __(M)__.

[The solution appears on page 83 of the *Quarterly*.)

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A-1: 6 10 25 40 89 127 177 176 35 153

B: 9 27 87 116 51 100 116 128

A-3: 66 25 146 177 109 140 167

D-2: 149 14 149 22 118 77 57 171 149 7 59 30

C-1: 164 148 84 38 5 160 23 133 60 65 81 148 75 83

C-2: 69 124 148 166 111 67 164 43 164 172 138

E-1: 32

C-3: 148 93 102 166 16 58

A-4: 107 177 17 115 143 155 88 167

D-1: 46 117 19 49 77 1 14 7 169 77 30 33

G-2: 110 159 141 82 147 104 136

M: 135 61 157 96 36 12 2 21

D-3: 46 175 30 105 77 1 7 77 91

A-2: 40 89 176 35 68

A-5: 63 31 146 20 122 80 15 177 129 55

N: 125 142 158 119 53 90 152 144

C-4: 164 148 111 166 148 108 172 44 13 92 123 111 160 72 41 98 172 71

C-5: 138 133 164 148 67 23 164 28 164 178 111 138 133 164 148

I: 94 39 150 34 29 94 24 76 54 120 62

D-4: 49 77 179 91 14 30 154

H-1: 132 137 139 70 45 103 163 18 112 79 85 74

H-2: 74 165 173 50 163 174

G-3: 56 170 162 99 48 8 52 99 114 141 126 37 106 82 26 64 145 134 99

G-1: 78 151 3 99 101 161 95 147 121 82 131 156 99 147

E-2: 86

C-6: 47 97 172 172 73 113 168 130 11 4 164 148 42

■	■	■	■	■	1 D	2 M	3 G	4 C	5 C	6 A	7 D	8 G	■	■	9 B	10A	11 C	■	■	■
12M	13 C	14 D	■	15 A	16 C	17A	18H	19D	■	20A	21 M	22 D	23C	24 I	25 A	26G	27B	28C	■	■
■	■	■	29 I	30 D	31 A	■	32 E	33D	■	34 I	35A	36 M	■	37 G	38C	39I	40 A	41 C	42C	43C
■	■	44C	45H	■	46D	47C	48G	49D	50H	51B	52G	53N	54I	55 A	■	■	56G	57D	58C	■
■	59 D	60 C	61M	62 I	63A	■	64G	65C	66 A	■	■	67C	68A	69C	70H	71C	72C	73C	74 H	■
■	75 C	76 I	■	77 D	■	78 G	79H	80A	81 C	■	82G	83 C	84 C	85H	■	86E	87 B	88A	89 A	90N
91D	92C	■	93C	94I	95G	■	■	96M	97C	98C	99G	■	■	■	100B	101G	102C	103H	104G	■
■	105D	106G	107A	■	108C	109A	110G	111C	112H	■	■	113C	114G	■	■	115A	116B	117D	■	■
■	118D	119N	120I	121G	122A	123C	124C	■	125N	126G	■	127A	■	128B	129A	130C	131G	132H	133C	134G
■	■	135M	136G	■	137H	138C	139H	140A	■	141G	142N	143A	■	■	144N	145G	146A	147G	148C	149D
■	■	■	150I	151G	■	152N	153A	154D	■	■	155A	156G	157M	158N	■	■	159G	■	■	■
■	■	■	■	■	160C	161G	162G	163H	164C	165H	166C	167A	■	■	■	168C	169D	170G	171D	172C
■	■	■	■	■	■	■	■	173H	■	174H	175D	176A	177A	178C	179D	■	■	■	■	■

STOLARSKY'S DISTRIBUTION OF THE POSITIVE INTEGERS

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Let F_n be the n^{th} Fibonacci number, where $F_1 = 1$, $F_2 = 2$ and $F_{n+2} = F_{n+1} + F_n$, $\forall n \in N$. It is well known that

$$\lim_{n \to \infty} F_{n+1}/F_n = \alpha = \tfrac{1}{2}(1+\sqrt{5}),$$

the larger root of the polynomial equation $x^2 = x + 1$. Using the mapping $g : N \to N$,

$$g(r) = [r\alpha + \tfrac{1}{2}],$$

i.e., $g(r)$ is the closest integer to $r\alpha$, we can give an alternate formulation of F_n. It is easy to show that,

$$g(F_n) = F_{n+1}, \forall n \in N,$$

so as $F_1 = 1$,

$$F_n = g^{n-1}(1), \forall n \in N,$$

where we set

$$g^o(r) = r, \quad \text{and} \quad g^n(r) = g(g^{n-1}(r)), \forall n \in N.$$

Hence the Fibonacci sequence is

$$(F_n) = (g^{n-1}(1)).$$

For each $r \in N$, we will show that the sequence $(g^{n-1}(r))$ has the Fibonacci recursive property

$$g^{n+1}(r) = g^n(r) + g^{n-1}(r), \forall n \in N.$$

K. Stolarsky constructed a table of these sequences to cover the positive integers in the following way. $\forall\, m,n \in N$, we define:

(a) $S(m,1)$ = least positive integer not in $T(m) = \{S(i,j) : j \in N, \ i = 1, \cdots, m-1\}$;

(b) $$S(m,n+1) = g(S(m,n)).$$

Effectively what is being constructed is a table of sequences $g^{n-1}(r)$, where r is least integer not in an earlier sequence and, $r = 1$ is the starting value for the first sequence, the Fibonacci sequence. Obviously, by construction S will cover N.

In Table 1, we list the 100 values of $S(m,n)$ for $m,n \leqslant 10$. It is easily shown (Theorem 1), that each positive integer r occurs exactly once as a value $S(m,n)$, and that $S(m,n+2) - S(m,n+1) = S(m,n)$, (Lemma 1).

Table 1

$n =$	1	2	3	4	5	6	7	8	9	10
$m =$ 1	1	2	3	5	8	13	21	34	55	89
2	4	6	10	16	26	42	68	110	178	288
3	7	11	18	29	47	76	123	199	322	521
4	9	15	24	39	63	102	165	267	432	699
5	12	19	31	50	81	131	212	343	555	898
6	14	23	37	60	97	157	254	411	665	1076
7	17	28	45	73	118	190	308	499	808	1307
8	20	32	52	84	136	220	356	576	932	1508
9	22	36	58	94	152	246	398	644	1042	1686
10	25	40	65	105	170	275	445	720	1165	1885

Stolarsky observed in his table, as far as he had calculated, that the differences between the values in columns 2 and 1 of a given row, $S(m,2) - S(m,1)$, were always integers that had previously occurred in one of these two columns. He conjectured that this was always the case. J. Butcher conjectured further, on the basis of computation, that this correspondence was one-to-one.

In this paper we prove both these conjectures, as well as constructing other interesting properties of $S(m,n)$. To facilitate our construction, we define the following functions:

$$d : N \to N,\ d(m) = S(m,2) - S(m,1);$$

$$h : N \to (-\tfrac{1}{2},\tfrac{1}{2}),\ h(r) = r\alpha - g(r);$$

and

$$k : N \to N,\ k(r) = [1 - \log_\alpha |2h(r)|].$$

Hence $d(m)$ is the difference between columns 2 and 1 in row m, and $h(r)$ is the "closeness" of $r\alpha$ to the nearest integer.

We will show firstly that S is a one-to-one and onto map $N \times N$ to N:

Theorem 1.

$$\forall r \in N,\ \exists 1\, m,n \in N : r = S(m,n).$$

We will use this result to establish Stolarsky's conjecture:

Theorem 2.

$$\forall m \in N, \exists n \in N : d(m) = S(n,1) \text{ or } d(m) = S(n,2).$$

We will then improve Theorem 1 by finding explicit invertible formulae relating m,n to $S(m,n)$:

Theorem 3.

$$S(m,1) = [m\alpha^2 - \tfrac{1}{2}\alpha],\ S(m,n) = g^{n-1}(S(m,1)),\ \forall m,n \in N,$$

$$n = k(S(m,n)),\ m = [S(m,n)\alpha^{-n-1} + \tfrac{1}{2}\alpha]$$

Further we note that the sequence m, $d(m)$, $S(m,1)$ can be approximated by m, $m\alpha$ and $m\alpha^2$, or more explicitly:

Theorem 4

For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}\alpha^{-2})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) - 1;$$

for $h(m) \in (-\tfrac{1}{2}\alpha^{-2}, \tfrac{1}{2}\alpha^{-1})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) + 1;$$

and for $h(m) \in (\tfrac{1}{2}\alpha^{-1}, \tfrac{1}{2})$,

$$d(m) = g(m), \quad S(m,1) = g(d(m)) - 1.$$

This theorem leads to explicit invertible formulae relating $d(m)$ to $S(n,1)$ and $S(n,2)$:

Theorem 5.

For $h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}\alpha^{-3})$,

$$d(m) = S([m\alpha^{-1} + \tfrac{1}{2}], 1);$$

for $h(m) \in (\tfrac{1}{2}\alpha^{-3}, \tfrac{1}{2})$,

$$d(m) = S([m\alpha^{-2} + \tfrac{1}{2}], 2);$$

while

$$S(m,1) = d([m\alpha + \tfrac{1}{2}\alpha^{-2}]), \quad \text{and} \quad S(m,2) = d([m\alpha^2 - \tfrac{1}{2}\alpha^{-1}]).$$

This leads finally to establishing Butcher's conjecture:

Theorem 6.

$$\{d(m) : m \in N\} = \{S(m,1) : m \in N\} \cup \{S(m,2) : m \in N\}.$$

We will now prove these theorems via the following lemmas. We will frequently use identities based on $a^2 = a + 1$, of the form

$$a^{n+1} = F_n a + F_{n-1}, \quad \forall n \in N,$$
$$a^{-n} = (-1)^n(F_n - F_{n-1}a), \quad \forall n \in N.$$

Lemma 1. $\forall r \in N, \quad g^2(r) = g(r) + r.$

Proof.
$$ar - \tfrac{1}{2} < g(r) < ar + \tfrac{1}{2},$$
$$\Rightarrow ag(r) - \tfrac{1}{2} < g^2(r) < ag(r) + \tfrac{1}{2},$$
$$\Rightarrow (a-1)g(r) - \tfrac{1}{2} < g^2(r) - g(r) < (a-1)g(r) + \tfrac{1}{2}.$$

But
$$a(a-1)r - \tfrac{1}{2}(a-1) < (a-1)g(r) < a(a-1)r + \tfrac{1}{2}(a-1),$$
and
$$a(a-1) = a^2 - a = 1,$$
so
$$r - \tfrac{1}{2}(a-1) - \tfrac{1}{2} < g^2(r) - g(r) < r + \tfrac{1}{2}(a-1) + \tfrac{1}{2},$$
$$\Rightarrow r - 1 < r - \tfrac{1}{2}a < g^2(r) - g(r) < r + \tfrac{1}{2}a < r + 1.$$

Hence as $g^2(r) - g(r)$ is integral, $g^2(r) - g(r) = r$, and the result of the lemma follows.

Corollary. $S(1,n) = F_n.$

Proof. $T(1) = \varphi \Rightarrow S(1,1) = 1 = F_1, \quad S(1,2) = g(1) = 2 = F_2.$

By Lemma 1,
$$S(1, n+2) = g^2(S(1,n)) = g(S(1,n)) + S(1,n) = S(1, n+1) + S(1,n), \quad \forall n \in N$$
so by induction,
$$S(1, n+2) = F_{n+1} + F_n = F_{n+2}, \quad \forall n \in N.$$

As we move from left to right across the table we find that each value $g(n)a$ gives a better approximation to an integer $(g^2(n))$ than did na, $(g(n))$. Explicitly we have the following recursive result.

Lemma 2. $\forall n \in N, \quad h(g(n)) = -a^{-1}h(n).$

Proof.
$$\begin{aligned} h(g(n)) &= ag(n) - g^2(n), \\ &\equiv ag(n) \pmod 1, \\ &\equiv a^2 n - ah(n) \pmod 1, \\ &\equiv an - ah(n) \pmod 1, \quad (\text{as } a^2 = a + 1), \\ &\equiv h(n) - ah(n) \pmod 1, \\ &\equiv (1-a)h(n) \pmod 1. \end{aligned}$$
$$1 - a = -a^{-1}, \quad |hg(n)| < \tfrac{1}{2}, \quad |-a^{-1}h(n)| < \tfrac{1}{2},$$
so $h(g(n)) = -a^{-1}h(n)$.

Lemma 2 enables us to prove the following relation between $S(m,n)$ and n, namely that r occurs in the $k(r)^{th}$ column of the table.

Lemma 3. $k(S(m,n)) = n, \quad \forall m, n \in N.$

Proof. Let $r = [S(m,1)a^{-1}]$, and set $\epsilon = ra - S(m,1)$. $0 < \epsilon < 1$.

For $m > 1$, $S(m,1) - 2 < g(r) < S(m,1) + 1$, so
$$g(r) = S(m,1) \quad \text{or} \quad S(m,1) - 1.$$
But $g(r) \in T(m)$ as $r < S(m,1)$, and $S(m,1) \notin T(m)$ so
$$g(r) = S(m,1) - 1, \quad \forall m > 1.$$

Also $g(0) = [\tfrac{1}{2}] = 0$, $S(1,1) - 1 = 0$, so

$$g(r) = S(m,1) - 1, \quad \forall\, m \in N.$$

Hence

$$S(m,1) = [ar + \tfrac{1}{2}] + 1,$$
$$= [S(m,1) + \tfrac{1}{2} - \epsilon] + 1$$
$$\Rightarrow \epsilon > \tfrac{1}{2}.$$

Further,

$$h(S(m,1)) \equiv aS(m,1) \pmod 1,$$
$$\equiv a^{-1}S(m,1) \pmod 1, \quad (\text{as } a = 1 + a^{-1}),$$
$$\equiv -\epsilon a^{-1} \pmod 1.$$

Hence, for $\epsilon < \tfrac{1}{2}a$,

$$h(S(m,1)) = -\epsilon a^{-1} < -\tfrac{1}{2}a^{-1},$$

and for $\epsilon > \tfrac{1}{2}a$,

$$h(S(m,1)) = 1 - \epsilon a^{-1} > 1 - a^{-1} > \tfrac{1}{2}a^{-1},$$

Thus in both cases

$$|h(S(m,1))| \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2}) \Rightarrow k(S(m,1)) = 1.$$

Now using Lemma 2, $k(S(m,n+1)) = k(S(m,n)) + 1$, so by induction, $k(S(m,n)) = n$.

This means an integer r cannot appear in two different columns. In the next lemma, we show that no integer can appear more than once in any given column.

Lemma 4. $$S(m+1, n) > S(m,n), \quad \forall\, m,n \in N.$$

Proof. By definition $S(m,1)$ is not the least integer in $T(m)$, and $S(m+1, 1)$ the least integer not in $T(m+1) \supseteq T(m) \cup \{S(m,1)\}$, so $S(m+1, 1) \geq S(m,1) + 1$. Also

$$S(m+1, 2) = g(S(m+1, 1)),$$
$$\geq aS(m+1, 1) - \tfrac{1}{2},$$
$$> a(S(m,1) + 1) - \tfrac{1}{2},$$
$$> aS(m,1) + \tfrac{1}{2},$$
$$\geq g(S(m,1)),$$
$$= S(m,2),$$

i.e., $S(m+1, 2) > S(m,2)$. Now by induction, using Lemma 1,

$$S(m+1, n+2) = S(m+1, n+1) + S(m+1, n) > S(m, n+1) + S(m,n)$$
$$= S(m, n+2). \qquad \forall m, n \in N.$$

Combining this final result with the two initial results we prove the lemma.

Lemmas 3 and 4 now enable us to prove Theorem 1. By the sieve type definition $S: N \times N \to N$ must be onto. If $S(m_1, n_1) = S(m_2, n_2) = r$ say, then by Lemma 3, $n_1 = n_2 = k(r)$ and then by Lemma 4 $m_1 = m_2$. Hence S is one-to-one. We have proved:

Theorem 1. $$\forall\, r \in N, \ \exists\, 1 m, n \in N : r = S(m,n).$$

Stolarsky's conjecture can now be established by proving one more Lemma.

Lemma 5. $$k(d(m)) \leq 2, \quad \forall\, m \in N.$$

Proof. $$h(d(m)) \equiv ad(m) \pmod 1,$$
$$\equiv aS(m,2) - aS(m,1) \pmod 1,$$
$$\equiv h(S(m,2)) - h(S(m,1)) \pmod 1,$$

Now by Lemma 2, $h(S(m,2)) = -a^{-1}h(S(m,1))$, so

$$h(d(m)) \equiv -(1+a^{-1})h(S(m,1)) \pmod 1,$$
$$\equiv -ah(S(m,1)) \pmod 1.$$

Further $h(S(m,1)) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-1}) \cup (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$ by Lemma **3**, so

$$h(d(m)) = 1 - ah(S(m,1)) \quad \text{if} \quad h(S(m,1)) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-1})$$

and

$$h(d(m)) = -1 - ah(S(m,1)) \text{ if } h(S(m,1)) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2}),$$

so in either case

$$|h(d(m))| = 1 - a|h(S(m,1))|,$$
$$> 1 - \tfrac{1}{2}a,$$
$$= \tfrac{1}{2}a^{-2}.$$

Hence $k(d(m)) \leqslant 2$.

As by Theorem 1, the value $r = d(m)$ can occur in only one position, and as $k(d(m)) \leqslant 2$, by Lemma 3, $d(m)$ appears in Column 1 or Column 2. Hence we have established our second theorem.

Theorem 2. $\forall m \in N, \exists n \in N : d(m) = S(n,1) \text{ or } d(m) = S(n,2).$

We now return to improve the result of Theorem 1 by finding an explicit relationship between m,n and $S(m,n)$. We note first

Lemma 6. $k([na^2 - \tfrac{1}{2}a]) = 1.$

Proof. Let $r = [na^2 - \tfrac{1}{2}a]$. Now

$$na^2 - \tfrac{1}{2}a \equiv na - \tfrac{1}{2}a \pmod 1,$$
$$\equiv h(n) - \tfrac{1}{2}a \pmod 1.$$

Also $-2 < -\tfrac{1}{2} - \tfrac{1}{2}a < h(n) - \tfrac{1}{2}a < \tfrac{1}{2} - \tfrac{1}{2}a < 0$, so

$$r = na^2 - h(n) - t,$$

where

$$t = 2 \quad \text{for} \quad -\tfrac{1}{2} < h(n) < \tfrac{1}{2}a - 1 = -\tfrac{1}{2}a^{-2},$$

and

$$t = 1 \quad \text{for} \quad -\tfrac{1}{2}a^{-2} < h(n) < \tfrac{1}{2}.$$

Further

$$h(r) \equiv ra \pmod 1,$$
$$\equiv na^3 - h(n)a - ta \pmod 1,$$
$$\equiv 2na - h(n)a - ta \pmod 1,$$
$$\equiv h(n)(2-a) - ta \pmod 1,$$
$$\equiv h(n)a^{-2} - ta \pmod 1.$$

For $-\tfrac{1}{2} < h(n) < -\tfrac{1}{2}a^{-2}$,

$$t = 2 \Rightarrow -ta \equiv -2a \equiv -a^3 \pmod 1,$$
$$\Rightarrow -\tfrac{1}{2} < -\tfrac{1}{2}a^{-2} - a^{-3} < h(n)a^{-2} - a^{-3} < -\tfrac{1}{2}a^{-4} - a^{-3} = -\tfrac{1}{2}a^{-1},$$
$$\Rightarrow h(r) = h(n)a^{-2} - a^{-3} \quad \text{and} \quad k(r) = 1.$$

For $-\tfrac{1}{2}a^{-2} < h(n) < \tfrac{1}{2}a^{-1}$,

$$t = 1 \Rightarrow -ta \equiv -a \equiv a^{-2} \pmod 1,$$
$$\Rightarrow \tfrac{1}{2}a^{-1} < a^{-2}(h(n)+1) < \tfrac{1}{2},$$
$$\Rightarrow h(r) = a^{-2}(h(n)+1) \quad \text{and} \quad k(r) = 1.$$

For $\tfrac{1}{2}a^{-1} < h(n) < \tfrac{1}{2}$,

$$\begin{aligned} t = 1 &\Rightarrow -ta \equiv -a \equiv -a^{-1} \pmod 1, \\ &\Rightarrow -\tfrac12 = \tfrac12 a^{-3} - a^{-1} \le h(n) < \tfrac12 a^{-2} - a^{-1} < -\tfrac12 a^{-1}, \\ &\Rightarrow h(r) = a^{-2}h(n) - a^{-1} \quad \text{and} \quad k(r) = 1. \end{aligned}$$

We can now show that the numbers $[na^2 - \frac12 a]$ are the only integers occurring in Column 1.

Lemma 7. $$S(n,1) = [na^2 - \tfrac12 a].$$

Proof. Let $r = S(n,1)$, then

$$\begin{aligned} h(r+1) &\equiv a(r+1) \pmod 1, \\ &\equiv h(r) + a \pmod 1. \end{aligned}$$

Noting $a \equiv a^{-1} \equiv -a^{-2} \pmod 1$ we find: for $-\frac12 < h(r) < -\frac12 a^{-1}$,

$$\tfrac12 a^{-3} < h(r) + a^{-1} < \tfrac12 a^{-1} \Rightarrow k(r+1) > 1;$$

and for $-\frac12 a^{-1} < h(r) < \frac12$,

$$-\tfrac12 a^{-4} < h(r) - a^{-2} < \tfrac12 a^{-3} \Rightarrow k(r+1) > 1.$$

Hence $r+1$ cannot be in Column 1, so Column 1 cannot contain two consecutive integers.

Let $t(n) = [na^2 - \frac12 a]$, then

$$na^2 - \tfrac12 a - 1 < t(n) < na^2 - \tfrac12 a,$$

$$na^2 + a^2 - \tfrac12 a - 1 < t(n+1) < na^2 + a^2 - \tfrac12 a,$$

so

$$na^2 + a^2 - \tfrac12 a - 1 - (na^2 - \tfrac12 a) < t(n+1) - t(n) < na^2 + a^2 - \tfrac12 a - (na^2 - \tfrac12 a - 1),$$

and as

$$na^2 + a^2 - \tfrac12 a - 1 - (na^2 - \tfrac12 a) = a^2 - 1 = a > 1,$$

and

$$na^2 + a^2 - \tfrac12 a - (na^2 - \tfrac12 a - 1) = a^2 + 1 = a + 2 < 4,$$

we have

$$1 < t(n+1) - t(n) < 4.$$

Hence $t(n)$ and $t(n+1)$ are distinct integers whose difference is 2 or 3. They both occur in Column 1 (Lemmas 6 and 3), so no other integer can occur in Column 1, as that would imply consecutive integers in Column 1.

We can now specify $S(m,n)$ with the following two lemmas.

Lemma 8. $S(m,n) = S(m,1)a^{n-1} + F_{n-2}h(S(m,1)),\ \forall\, n \in N$. (Putting $F_0 = 1,\ F_{-1} = 0$.)

Proof. Trivial for $n = 1$.

Assume $S(m,n) = S(m,1)a^{n-1} + F_{n-2}h(S(m,1))$, for some $n > 1$, then

$$\begin{aligned} S(m, n+1) &= g(S(m,n)), \\ &= aS(m,n) + h(S(m,n)), \\ &= S(m,1)a^n + F_{n-2}h(S(m,1))a + h(S(m,n)). \end{aligned}$$

But, by Lemma 2,

$$\begin{aligned} h(S(m,n)) &= -a^{-1}h(S(m,n-1)), \\ &= (-a^{-1})^{n-1}h(S(m,1)), \end{aligned}$$

and as $a^{-(n-1)} = (-1)^{n-1}(F_{n-1} - F_{n-2}a)$,

$$F_{n-2}a + (-a)^{-(n-1)} = F_{n-1}.$$

Hence

$$S(m, n+1) = S(m,1)a^n + F_{n-1}h(S(m,1)).$$

Thus, by induction, this result is true $\forall\, n \in N$.

From this result follows

Lemma 9. $$m = [S(m,n)a^{-n-1} + \tfrac12 a].$$

Proof. By Lemma 8,

$$|S(m,n) - S(m,1)a^{n-1}| = F_{n-2}|h(S(m,1))| < \tfrac{1}{2}F_{n-2}.$$

Also, $F_{n-2} < a^{n-2}$, so

$$|S(m,n) - S(m,1)a^{n-1}| < \tfrac{1}{2}a^{-3}.$$

From Lemma 7

$$ma^2 - \tfrac{1}{2}a - 1 < S(m,1) < ma^2 - \tfrac{1}{2}a,$$
$$\Rightarrow -\tfrac{1}{2}a^{-1} - a^{-2} < S(m,1)a^{-2} - m < -\tfrac{1}{2}a^{-1} .$$

But, from above,

$$-\tfrac{1}{2}a^{-3} < S(m,n)a^{-n-1} - S(m,1)a^{-2} < \tfrac{1}{2}a^{-3},$$

so adding

$$-\tfrac{1}{2}a = -\tfrac{1}{2}a^{-1} - a^{-2} - \tfrac{1}{2}a^{-3} < S(m,n)a^{-n-1} - m < \tfrac{1}{2}a^{-3} - \tfrac{1}{2}a^{-1}$$
$$\Rightarrow 0 < S(m,n)a^{-n-1} - m + \tfrac{1}{2}a < \tfrac{1}{2}a + \tfrac{1}{2}a^{-3} - \tfrac{1}{2}a^{-1} = a^{-1} < 1,$$
$$\Rightarrow m = [S(m,n)a^{-n-1} + \tfrac{1}{2}a] .$$

This lemma concludes the results for Theorem 3, so combining the results of Lemmas 3, 7 and 9 we have:

Theorem 3. $S(m,1) = [ma^2 - \tfrac{1}{2}a], \quad S(m,n) = g^{n-1}(S(m,1)), \quad \forall\, m,n \in N,$

$$n = k(S(m,n)), \quad m = [S(m,n)a^{-n-1} + \tfrac{1}{2}a] .$$

We now examine formulae for $d(m)$.

Lemma 10. $$d(m) = [ma - \tfrac{1}{2}a^{-1}].$$

Proof. Let

$$c(m) = [ma - \tfrac{1}{2}a^{-1}],$$

and set

$$\gamma = ma - \tfrac{1}{2}a^{-1} - c(m), \quad 0 < \gamma < 1.$$

As $S(m,1) = [ma^2 - \tfrac{1}{2}a]$, let

$$\epsilon = ma^2 - \tfrac{1}{2}a - S(m,1), \quad 0 < \epsilon < 1.$$

Now

$$\begin{aligned} \epsilon - \gamma &= m(a^2 - a) + \tfrac{1}{2}(a^{-1} - a) + c(m) - S(m,1), \\ &= m - \tfrac{1}{2} + c(m) - S(m,1), \\ &\equiv \tfrac{1}{2} \pmod 1. \end{aligned}$$

Thus for $\epsilon < \tfrac{1}{2}$, $\gamma = \epsilon + \tfrac{1}{2}$,

$$S(m,1) = c(m) + m,$$

and for $\epsilon > \tfrac{1}{2}$, $\gamma = \epsilon - \tfrac{1}{2}$,

$$S(m,1) = c(m) + m - 1 .$$

Further

$$\begin{aligned} c(m) + S(m,1) &= m(a^2 + a) - \tfrac{1}{2}(a + a^{-1}) - (\epsilon + \gamma), \\ &= ma^3 - \tfrac{1}{2}(a^3 - 2) - (\epsilon + \gamma), \\ &= (m - \tfrac{1}{2})a^3 + (1 - \epsilon - \gamma), \end{aligned}$$

and

$$\begin{aligned} S(m,2) &= g(S(m,1)), \\ &= aS(m,1) - h(S(m,1)), \\ &= ma^3 - \tfrac{1}{2}a^2 - \epsilon a - h(S(m,1)) . \end{aligned}$$

Combining these two results we find

$$\begin{aligned} c(m) + S(m,1) - S(m,2) &= \tfrac{1}{2}(a^2 - a^3) + (\epsilon a - \epsilon - \gamma) - h(S(m,1)) + 1, \\ &= 1 - \tfrac{1}{2}a + (\epsilon a - \epsilon - \gamma) - h(S(m,1)) . \end{aligned}$$

For $0 < \epsilon < \tfrac{1}{2}$, $\gamma = \epsilon + \tfrac{1}{2}$,

$$c(m) + S(m,1) - S(m,2) = 1 - \tfrac{1}{2}a + \epsilon(a - 2) - \tfrac{1}{2} - h(S(m,1)) \in (-1, 1 - \tfrac{1}{2}a),$$

and is integral, so

$$c(m) = S(m,2) - S(m,1) = d(m).$$

For $\frac{1}{2} < \epsilon < 1$,

$$\gamma = \epsilon - \tfrac{1}{2},$$

$$c(m) + S(m,1) - S(m,2) = 1 - \tfrac{1}{2}a + \epsilon(a-2) + \tfrac{1}{2} - h(S(m,1)) \in (\tfrac{1}{2}a - 1,\ 1),$$

and is integral, so

$$c(m) = S(m,2) - S(m,1) = d(m).$$

We can now formulate the relationship between m and $d(m)$.

Lemma 11. For $h(m) \in (-\frac{1}{2}, \frac{1}{2}a^{-1})$,

$$d(m) = g(m) - 1,$$

for $h(m) \in (\frac{1}{2}a^{-1}, \frac{1}{2})$

$$d(m) = g(m).$$

Proof. For $h(m) \in (-\frac{1}{2}, \frac{1}{2}a^{-1})$,

$$g(m) = ma - h(m) \in (ma - \tfrac{1}{2}a^{-1},\ ma + \tfrac{1}{2}).$$

Now this interval has length $\frac{1}{2}a^{-1} + \frac{1}{2} = \frac{1}{2}a < 1$, and $g(m)$ is integral, so

$$g(m) = [ma - \tfrac{1}{2}a^{-1}] + 1 = d(m) + 1,$$

by Lemma 10.

For $h(m) \in (\frac{1}{2}a^{-1}, \frac{1}{2})$,

$$g(m) = ma - h(m) \in (ma - \tfrac{1}{2},\ ma - \tfrac{1}{2}a^{-1}).$$

This interval has length $\frac{1}{2} - \frac{1}{2}a^{-1} = 1 - \frac{1}{2}a < 1$, so

$$g(m) = [ma - \tfrac{1}{2}a^{-1}] = d(m).$$

Lemma 12. For $h(m) \in (-\frac{1}{2}, -\frac{1}{2}a^{-2})$,

$$h(d(m)) = -a^{-1}(h(m)+1), \quad k(d(m)) = 1,$$

for $h(m) \in (-\frac{1}{2}a^{-2}, \frac{1}{2}a^{-3})$,

$$h(d(m)) = 1 - a^{-1}(h(m)+1), \quad k(d(m)) = 1,$$

for $h(m) \in (\frac{1}{2}a^{-3}, \frac{1}{2}a^{-1})$,

$$h(d(m)) = 1 - a^{-1}(h(m)+1), \quad k(d(m)) = 2,$$

for $h(m) \in (\frac{1}{2}a^{-1}, \frac{1}{2})$,

$$h(d(m)) = -a^{-1}(h(m)), \quad k(d(m)) = 2.$$

Proof. From Lemma 11

$$h(d(m)) = h(g(m) - \ell),$$

where $\ell = 0$ if $h(m) > \frac{1}{2}a^{-1}$, $\ell = 1$ otherwise. Hence

$$\begin{aligned} h(d(m)) &\equiv ag(m) - a\ell \pmod 1, \\ &\equiv ma^2 - ah(m) - a\ell \pmod 1, \\ &\equiv ma - ah(m) - a\ell \pmod 1, \\ &\equiv h(m)(1-a) - a\ell \pmod 1, \\ &\equiv -a^{-1}(h(m) + \ell) \pmod 1. \end{aligned}$$

For $h(m) \in (-\frac{1}{2}, -\frac{1}{2}a^{-2})$, $\ell = 1$,

$$\Rightarrow -\tfrac{1}{2} = -a^{-1}(1 - \tfrac{1}{2}a^{-2}) < -a^{-1}(h(m)+1) < -\tfrac{1}{2}a^{-1},$$

$$\Rightarrow h(d(m)) = -a^{-1}(h(m)+1), \qquad k(d(m)) = 1.$$

For $h(m) \in (-\frac{1}{2}a^{-2}, \frac{1}{2}a^{-1})$, $\ell = 1$,

$$\Rightarrow -\tfrac{1}{2}a = -a^{-1}(1 + \tfrac{1}{2}a^{-1}) < -a^{-1}(h(m)+1) < -\tfrac{1}{2},$$

$$\Rightarrow h(d(m)) = 1 - a^{-1}(h(m)+1).$$

In particular, if $h(m) \in (-\frac{1}{2}a^{-2}, \frac{1}{2}a^{-3})$,

$$\tfrac{1}{2}a^{-1} = 1 - a^{-1}(\tfrac{1}{2}a^{-3} + 1) < h(d(m)) < 1 - a^{-1}(-\tfrac{1}{2}a^{-2} + 1)) = \tfrac{1}{2} \Rightarrow k(d(m)) = 1,$$

and if $h(m) \in (\tfrac{1}{2}a^{-3}, \tfrac{1}{2}a^{-1})$,

$$\tfrac{1}{2}a^{-1} = 1 - a^{-1}(\tfrac{1}{2}a^{-1} + 1) < h(d(m)) < 1 - a^{-1}(\tfrac{1}{2}a^{-3} + 1) = \tfrac{1}{2}a^{-1} \Rightarrow k(d(m)) = 2.$$

For $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$, $\ell = 0$,

$$-\tfrac{1}{2}a^{-1} < -a^{-1}h(m) < -\tfrac{1}{2}a^{-2},$$
$$\Rightarrow h(d(m)) = -a^{-1}h(m), \quad k(d(m)) = 2.$$

Now we can establish the relationship between $d(m)$ and $S(m,1)$.

Lemma 13. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2}) \cup (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$S(m,1) = g(d(m)) - 1.$$

For $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$

$$S(m,1) = g(d(m)) + 1.$$

Proof. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$,

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ag(m) - a + a^{-1}(h(m) + 1), \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - ah(m) - a + a^{-1}(h(m) + 1),\\ &= ma^2 + (a^{-1} - a)(h(m) + 1),\\ &= ma^2 - (h(m) + 1),\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a = ma^2 - (1 - \tfrac{1}{2}a^{-2}) < g(d(m)) < ma^2 - \tfrac{1}{2},$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] + 1 = S(m,1) + 1.$$

For $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$,

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ag(m) - a + a^{-1}(h(m) + 1) - 1, \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - h(m) - 2,\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a^{-1} - 2 < g(d(m)) < ma^2 + \tfrac{1}{2}a^{-2} - 2 = ma^2 - \tfrac{1}{2}a - 1,$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] - 1 = S(m,1) - 1.$$

For $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$

$$\begin{aligned} g(d(m)) &= ad(m) - h(d(m)),\\ &= ma^2 - ah(m) + a^{-1}h(m), \quad \text{(Lemmas 11 and 12)},\\ &= ma^2 - h(m),\end{aligned}$$
$$\Rightarrow ma^2 - \tfrac{1}{2}a < ma^2 - \tfrac{1}{2} < g(d(m)) < ma^2 - \tfrac{1}{2}a^{-1} < ma^2 - \tfrac{1}{2}a + 1,$$
$$\Rightarrow g(d(m)) = [ma^2 - \tfrac{1}{2}a] + 1 = S(m,1) + 1.$$

We can now combine the results of Lemmas 11, 12 and 13 to give the result:

Theorem 4. For $h(m) \in (-\tfrac{1}{2}, -\tfrac{1}{2}a^{-2})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) - 1;$$

for $h(m) \in (-\tfrac{1}{2}a^{-2}, \tfrac{1}{2}a^{-1})$,

$$d(m) = g(m) - 1, \quad S(m,1) = g(d(m)) + 1;$$

and for $h(m) \in (\tfrac{1}{2}a^{-1}, \tfrac{1}{2})$,

$$d(m) = g(m) \qquad S(m,1) = g(d(m)) - 1.$$

We now turn to the problem of finding the values of i,j, so that $d(m) = S(i,j)$, for a given $m \in N$.

Lemma 14. If $d(m) = S(r,1)$, then $r = [ma^{-1} + \tfrac{1}{2}]$.

Proof. By Lemma 12,

$$\begin{aligned} k(d(m)) = 1 &\Rightarrow h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}a^{-3}),\\ &\Rightarrow d(m) = g(m) - 1, \quad \text{(Theorem 4)},\\ &= [ma + \tfrac{1}{2}] - 1,\\ &= ma - \tfrac{1}{2} - \epsilon, \quad 0 < \epsilon < 1.\end{aligned}$$

Also $S(r,1) = [ra^2 - \tfrac12 a]$, so $d(m) = S(r,1)$,

$$\Rightarrow ra^2 - \tfrac12 a - 1 < ma - \tfrac12 - \epsilon < ra^2 - \tfrac12 a,$$
$$\Rightarrow r < ma^{-1} + \tfrac12 a^{-1} + \tfrac12 a^{-2} - \epsilon a^{-2} < r + a^{-2},$$
$$\Rightarrow r < r + \epsilon a^{-2} < ma^{-1} + \tfrac12 < r + (1+\epsilon)a^{-2} < r + 2a^{-2} < r+1,$$
$$\Rightarrow r = [ma^{-1} + \tfrac12].$$

Lemma 15. If $d(m) = S(r,2)$, then $r = [ma^{-2} + \tfrac12]$.

Proof.
$$d(m) = S(r,2) \Rightarrow k(d(m)) = 2,$$
$$\Rightarrow h(m) \in (\tfrac12 a^{-3}, \tfrac12), \quad \text{(Lemma 12)}.$$

Let $r = [ma^{-2} + \tfrac12] = ma^{-2} + \tfrac12 - \epsilon, \quad 0 < \epsilon < 1.$ Now

$$\epsilon \equiv ma^{-2} + \tfrac12 \pmod 1,$$
$$\equiv -ma + \tfrac12 \pmod 1,$$
$$\equiv \tfrac12 - h(m) \pmod 1.$$

But $\tfrac12 a^{-3} < h(m) < \tfrac12$, so $\epsilon = \tfrac12 - h(m)$, and $r = ma^{-2} + h(m)$,

$$S(r,1) = [ra^2 - \tfrac12 a],$$
$$= [m + h(m)a^2 - \tfrac12 a].$$

For $h(m) \in (\tfrac12 a^{-3}, \tfrac12 a^{-1})$, $-\tfrac12 < h(m)a^2 - \tfrac12 a < 0$,

$$\Rightarrow S(r,1) = m-1$$
$$\Rightarrow S(r,2) = g(S(r,1)),$$
$$= g(m-1),$$
$$= [ma - a + \tfrac12],$$
$$= [g(m) + h(m) - a + \tfrac12].$$

Now $g(m) - 1 < g(m) + h(m) - a + \tfrac12 < g(m) - \tfrac12 a$,

$$\Rightarrow S(r,2) = g(m) - 1,$$
$$= d(m) \text{ by Theorem 4.}$$

For $h(m) \in (\tfrac12 a^{-1}, \tfrac12)$,

$$S(r,2) = g(S(r,1)),$$
$$= g(m),$$
$$= d(m) \text{ by Theorem 4.}$$

Lemma 16. $S(m,1) = d([ma + \tfrac12 a^{-2}]), \ \forall m \in N.$

Proof. Let $n = [ma + \tfrac12 a^{-2}] = ma + \tfrac12 a^{-2} - \epsilon, \quad 0 < \epsilon < 1,$

$$\Rightarrow ma + \tfrac12 a^{-2} - 1 < n < ma + \tfrac12 a^{-2},$$
$$\Rightarrow m = m + \tfrac12 a^{-3} - a^{-1} + \tfrac12 < na^{-1} + \tfrac12 < m + \tfrac12 a^{-3} + \tfrac12 = m + a^{-1},$$
$$\Rightarrow m = [na^{-1} + \tfrac12].$$

Also

$$\epsilon \equiv ma + \tfrac12 a^{-2} \pmod 1,$$
$$\equiv h(m) + \tfrac12 a^{-2} \pmod 1.$$

Hence

$$\epsilon = h(m) + \tfrac12 a^{-2} + 1 \quad \text{for} \quad h(m) \in (-\tfrac12, -\tfrac12 a^{-2}),$$
$$\epsilon = h(m) + \tfrac12 a^{-2} \quad \text{for} \quad h(m) \in (-\tfrac12 a\text{-}2, \tfrac12).$$

Further,

$$h(n) \equiv na \pmod 1,$$
$$\equiv ma^2 + \tfrac12 a^{-1} - \epsilon a \pmod 1,$$
$$\equiv h(m) + \tfrac12 a^{-1} - \epsilon a \pmod 1,$$

For

$$h(m) \in (-\tfrac12, -\tfrac12 a^{-2}), \quad \epsilon = h(m) + \tfrac12 a^{-2} + 1,$$

$$\begin{aligned}
&\Rightarrow h(n) \equiv -\alpha^{-1}h(m) - \alpha \pmod 1,\\
&\Rightarrow h(n) = -\alpha^{-1}h(m) - \alpha + 1,\\
&\qquad\quad = -\alpha^{-1}(h(m) - 1),\\
&\Rightarrow h(n) \in (-\tfrac{1}{2}, -\tfrac{1}{2}\alpha^{-1}).
\end{aligned}$$

For $h(m) \in (-\frac{1}{2}\alpha^{-2}, \frac{1}{2})$, $\epsilon = h(m) + \frac{1}{2}\alpha^{-2}$,

$$\begin{aligned}
&\Rightarrow h(n) \equiv -\alpha^{-1}h(m) \pmod 1,\\
&\Rightarrow h(n) = -\alpha^{-1}h(m),\\
&\Rightarrow h(n) \in (-\tfrac{1}{2}\alpha^{-1}, \tfrac{1}{2}\alpha^{-3}).
\end{aligned}$$

Hence in either case $h(n) \in (-\frac{1}{2}, \frac{1}{2}\alpha^{-3})$, so applying Lemma 14,

$$S(m,1) = S([n\alpha^{-1} + \tfrac{1}{2}], 1) = d(n) = d([m\alpha + \tfrac{1}{2}\alpha^{-2}]).$$

Lemma 17. $\quad S(m,2) = d([m\alpha^2 - \frac{1}{2}\alpha^{-1}]), \quad \forall m \in N.$

Proof. Let $n = [m\alpha^2 - \frac{1}{2}\alpha^{-1}] = m\alpha^2 - \frac{1}{2}\alpha^{-1} - \epsilon, \quad 0 < \epsilon < 1,$

$$\begin{aligned}
&\Rightarrow m\alpha^2 - \tfrac{1}{2}\alpha^{-1} - 1 < n < m\alpha^2 - \tfrac{1}{2}\alpha^{-1},\\
&\Rightarrow m < n\alpha^2 + \tfrac{1}{2}\alpha^{-3} + \alpha^{-2} = n\alpha^{-2} + \tfrac{1}{2} < m + \alpha^{-2},\\
&\Rightarrow m = [n\alpha^{-2} + \tfrac{1}{2}].
\end{aligned}$$

Also

$$\begin{aligned}
\epsilon &\equiv m\alpha^2 - \tfrac{1}{2}\alpha^{-1} \pmod 1,\\
&\equiv h(m) - \tfrac{1}{2}\alpha^{-1} \pmod 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\epsilon &= h(m) - \tfrac{1}{2}\alpha^{-1} + 1 \quad \text{for} \quad h(m) \in (-\tfrac{1}{2}, \tfrac{1}{2}\alpha^{-1}),\\
\epsilon &= h(m) - \tfrac{1}{2}\alpha^{-1} \qquad\ \ \text{for} \quad h(m) \in (\tfrac{1}{2}\alpha^{-1}, \tfrac{1}{2}).
\end{aligned}$$

Further

$$\begin{aligned}
h(n) &\equiv n\alpha \pmod 1,\\
&\equiv m\alpha^3 - \tfrac{1}{2} - \epsilon\alpha \pmod 1,\\
&\equiv 2m\alpha - \tfrac{1}{2} - \epsilon\alpha \pmod 1,\\
&\equiv 2h(m) - \tfrac{1}{2} - \epsilon\alpha \pmod 1,
\end{aligned}$$

For $h(m) \in (-\frac{1}{2}, \frac{1}{2}\alpha^{-1})$, $\epsilon = h(m) - \frac{1}{2}\alpha^{-1} + 1$,

$$\begin{aligned}
&\Rightarrow h(n) \equiv \alpha^{-2}h(m) - \alpha \pmod 1,\\
&\Rightarrow h(n) = \alpha^{-2}h(m) - \alpha + 2,\\
&\qquad\quad = \alpha^{-2}(1 + h(m)),\\
&\Rightarrow h(n) \in (\tfrac{1}{2}\alpha^{-2}, \tfrac{1}{2}).
\end{aligned}$$

For $h(m) \in (\frac{1}{2}\alpha^{-1}, \frac{1}{2})$, $\epsilon = h(m) - \frac{1}{2}\alpha^{-1}$,

$$\begin{aligned}
&\Rightarrow h(n) \equiv h(m)(2 - \alpha) \pmod 1,\\
&\qquad\quad = \alpha^{-2}h(m) \qquad \pmod 1,\\
&\Rightarrow h(n) = \alpha^{-2}h(m),\\
&\Rightarrow h(n) \in (\tfrac{1}{2}\alpha^{-3}, \tfrac{1}{2}\alpha^{-2}).
\end{aligned}$$

Hence in either case $h(n) \in (\frac{1}{2}\alpha^{-3}, \frac{1}{2})$, so applying Lemma 15, $S(m,2) = S([n\alpha^{-2} + \frac{1}{2}], 2) = d(n) = d([m\alpha^2 - \frac{1}{2}\alpha^{-1}])$
These four Lemmas together with Lemma 12, give us Theorem 5.

Theorem 5.

$$\begin{aligned}
d(m) &= S([m\alpha^{-1} + \tfrac{1}{2}], 1) \quad \text{if} \quad -\tfrac{1}{2} < h(m) < \tfrac{1}{2}\alpha^{-3},\\
&= S([m\alpha^{-2} + \tfrac{1}{2}], 2) \quad \text{if} \quad -\tfrac{1}{2}\alpha^{-3} < h(m) < \tfrac{1}{2},\\
S(m,1) &= d([m\alpha + \tfrac{1}{2}\alpha^{-2}]),\\
S(m,2) &= d([m\alpha^2 - \tfrac{1}{2}\alpha^{-1}]), \quad \forall m \in N.
\end{aligned}$$

We can note now from Lemma 10 that as $d(m) < m\alpha - \frac{1}{2}\alpha^{-1} < m(\alpha + 1) - \frac{1}{2}\alpha^{-1} - 1 < d(m+1)$, the sequence $d(m)$ is strictly monotonic increasing and hence by Theorem 5 we establish Butcher's conjecture.

Theorem 6.

$$\{S(m,1) : m \in N\} \cup \{S(m,2) : m \in N\} = \{d(m) : m \in N\}.$$

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ON A CONJECTURE CONCERNING A SET OF SEQUENCES SATISFYING THE FIBONACCI DIFFERENCE EQUATION

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Let $a = (1+\sqrt{5})/2$ and consider the set of sequences

$$\begin{aligned} S = \{&(1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \cdots),\\ &(2,\ 4,\ 6, 10, 16, 26,\ 42, \cdots),\\ &(4,\ 7, 11, 18, 29, 47,\ 76, \cdots),\\ &(6,\ 9, 15, 24, 39, 63, 102, \cdots),\\ &(7, 12, 19, 31, 50, 81, 131, \cdots), \cdots\}, \end{aligned}$$

where a sequence $u = (u_0, u_1, u_2, \cdots)$ is in S iff it satisfies the conditions

(1) $$u_0, u_1, u_2, \cdots \text{ are positive integers}$$

(2) $$u \text{ satisfies the Fibonacci difference equation } u_n = u_{n-1} + u_{n-2} \quad (n = 2, 3, 4, \cdots)$$

(3) $$\text{there does not exist an integer } x \text{ such that } |ax - u_1| < \tfrac{1}{2}$$

(4) $$|au_1 - u_2| < \tfrac{1}{2}.$$

Note that, for given u_1, there must exist an integer u_2 satisfying (4), because of the irrationality of a.

For $n = 0, 1, 2, \cdots$ let $S_n = \{u_n : u \in S\}$. It has been conjectured by Kenneth B. Stolarsky that for any $u \in S$, the value of $u_2 - u_1$ equals the value of either v_1 or v_2 for some $v \in S$. Since $u_2 = u_0 + u_1$, this is equivalent to saying that $S_0 \subset S_1 \cup S_2$. In this paper we prove the stronger result, that $S_0 = S_1 \cup S_2$.

Lemma 1. If $u \in S$ then for all $n = 1, 2, \cdots$

(5) $$a^{1-n}/2 < |au_{n-1} - u_n| < a^{2-n}/2$$

and

(6) $$a^{1-n}/2 < |a^2 u_{n-1} - u_{n+1}| < a^{2-n}/2.$$

Proof. We first show that, for any u, there is a constant C such that for all $n = 1, 2, \cdots$

(7) $$C = a^n |au_{n-1} - u_n|.$$

If C_n denotes the value of C given by (7) we have

$$\begin{aligned} C_{n+1} &= a^{n+1}|au_n - u_{n+1}| = a^n|a^2u_n - au_{n+1}|\\ &= a^n|(a+1)u_n - au_{n+1}| = a^n|a(u_{n+1} - u_n) - u_n|\\ &= a^n|au_{n-1} - u_n| = C_n. \end{aligned}$$

From (4) we see that $C = a^2|au_1 - u_2| < a^2/2$; also we see that $C = a|au_0 - u_1| < a/2$ since $|au_0 - u_1|$ cannot equal ½ because it is irrational, and cannot be less than ½ by (3). Combining these inequalities we obtain (5). To prove (6) we simply note that

$$|a^2u_{n-1} - u_{n+1}| = |(a+1)u_{n-1} - u_n - u_{n-1}| = |au_{n-1} - u_n|.$$

Lemma 2.

$$\bigcup_{n=1}^{\infty} S_n$$

is the set of positive integers.

Proof. If there were positive integers not in this union, let y be the lowest of these. Since y is not a member of S_1, there exists an integer x such that $|ax - y| < ½$. Since x is a positive integer less than y, it must lie in $\cup_{n=1}^{\infty} S_n$ and therefore $x = u_n$ for some $u \in S$ and n a positive integer. Since $|au_n - y| < ½$, $|au_n - u_{n+1}| < ½$ it follows that $y = u_{n+1} \in S_{n+1}$.

Lemma 3. $$S_0 \subset S_1 \cup S_2 .$$

Proof. If this result did not hold, because of Lemma 2, there would exist $u,v \in S$ and $n > 2$ such that $u_n = v_0$. By (2) we then find

$$v_2 - u_{n+2} = (v_1 + v_0) - (u_{n+1} + u_n) = v_1 - u_{n+1}$$

so that

$$|(a-1)(v_1 - u_{n+1})| = |(av_1 - v_2) - (au_{n+1} - u_{n+2})| < ½ + ½ a^{-n} \leqslant ½(1 + a^{-3}),$$

where we have used Lemma 1 to bound $|av_1 - v_2|$ and $|au_{n+1} - u_n|$ and made use of the fact that $n \geqslant 3$. Since $a^{-3} = 2a - 3$ we find

$$|v_1 - u_{n+1}| < \frac{1}{a-1} \cdot \frac{1}{2}(1 + 2a - 3) = 1$$

so that $v_1 = u_{n+1}$. Using Lemma 1 again we find that

$$½ < |av_0 - v_1| = |au_n - u_{n+1}| < a^{1-n}/2 < ½,$$

a contradiction.

Lemma 4. $$S_1 \subset S_0 .$$

Proof. Let $s = +1$ if $au_1 - u_2 > 0$ and -1 otherwise.

By Lemma 1, we have

$$\frac{a^{-1}}{2} < s(au_1 - u_2) < ½ .$$

Let $y = u_2 + s$ so that

$$\frac{1}{2} < -s(au_1 - y) < 1 - \frac{a^{-1}}{2}$$

which implies that

$$|au_1 - y| < 1 - \frac{a^{-1}}{2} = 1 - \frac{a^{-1}}{2} = 1 - \frac{a-1}{2} = \frac{a}{2} - \frac{2a-3}{2} < \frac{a}{2} .$$

If there were an x such that $|ax - y| < ½$, it would follow that

$$|au_1 - ax| < \frac{a+1}{2} = \frac{a^2}{2}$$

which implies

$$|u_1 - x| < \frac{a}{2} < 1$$

so that $u_1 = x$ and $u_2 = y$ which is impossible since $|u_2 - y| = 1$. Hence, no such x exists and therefore $y = v_1$ for some $v \in S$. Thus $|au_1 - v_1| < (a/2)$. We now find

$$|u_1 - v_0| = |u_1 - v_2 + v_1| \leqslant |u_1 - a^{-1}v_1| + |a^{-1}v_1 - v_2 + v_1| = (a-1)|au_1 - v_1| + |av_1 - v_2|$$
$$< \frac{(a-1)a}{2} + \frac{1}{2} = 1$$

so that $u_1 = v_0 \in S$.

Lemma 5. $$S_2 \subset S_0 .$$

Proof. Let $s = +1$ if $a^2u_2 - u_4 > 0$ and -1 otherwise. By Lemma 1, we have

$$\frac{a^{-2}}{2} < s(a^2u_2 - u_4) < \frac{a^{-1}}{2}$$

so that if $y = u_4 + s$ then

$$1 - \frac{a^{-1}}{2} < -s(a^2u_2 - y) < 1 - \frac{a^{-2}}{2} \quad .$$

Since

$$1 - \frac{a^{-1}}{2} > 0 \quad \text{and} \quad 1 - \frac{a^{-2}}{2} = \frac{a}{2}$$

it follows that

$$|a^2u_2 - y| < \frac{a}{2}.$$

If there were an integer w such that $|a^2w - y| < ½$ it would follow that

$$a^2|u_2 - w| < \frac{1+a}{2} = \frac{a^2}{2}$$

implying that $w = u_2$ and that $y = u_4$, contradicting the fact that $|y - u_4| = 1$. On the other hand, there is an integer $x = y - u_2$ such that $|ax - y| < ½$ since

$$|ax - y| = |(a-1)y - au_2| = (a-1)|y - a^2u_2| < \frac{a(a-1)}{2} = \frac{1}{2} \; .$$

The existence of x (and the non-existence of w) satisfying these conditions, implies that $y = v_2$ for some $v \in S$. Thus,

$$|a^2u_2 - v_2| < \frac{a}{2} \; .$$

We now find

$$\begin{aligned} |u_2 - v_0| &= |u_2 + v_1 - v_2| \leqslant |v_2a^{-2} - u_2| + |v_2(1 - a^{-2}) - v_1| \\ &= a^{-2}(|v_2 - a^2u_2| + |v_2a - a^2v_1|) < \frac{a^{-1}}{2} + \frac{a^{-1}}{2} = a^{-1} < 1 \end{aligned}$$

so that $u_2 = v_0 \in S_0$.

Combining the results of Lemmas 3, 4, 5 we have

Theorem. $\qquad S_0 = S_1 \cup S_2$.

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A GOLDEN DOUBLE CROSTIC: SOLUTION

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"Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel." J. Kepler. Quotation given in *The Divine Proportion* by Huntley (Dover, New York, 1970, p. 23).

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BINARY SEQUENCES WITHOUT ISOLATED ONES

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Liu [2] asks for the number of sequences of zeros and ones of length five, such that every digit 1 has at least one neighboring 1. The solution [1] uses the principle of inclusion-exclusion, although it is easier in this particular case to enumerate the twelve sequences:

00000, 11000, 01100, 00110, 00011, 11100, 01110, 00111, 11110, 01111, 11011, 11111.

In order to obtain a general result it seemed to us easier to find a recurrence relation.

Call a sequence *good* if each one in it has a neighboring one, and let a_n be the number of good sequences of length n. For example,

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 4, \quad a_4 = 7 \quad \text{and} \quad a_5 = 12.$$

Good sequences of length n are obtained from other good sequences of length $n-1$ by appending 0 or 1 to them, except that

(a) some not good sequences are also produced, namely those which end in 01, but are otherwise good, and
(b) there are good sequences which are not produced in this way; those obtained by appending 011 to a good sequence of length $n-3$.

So

(1) $$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}.$$

Alternatively, all good sequences are obtained from shorter good sequences by appending 0, 11 or 0111, so that

(2) $$a_n = a_{n-1} + a_{n-2} + a_{n-4}.$$

The characteristic equation for (2) is the same as that for (1), namely

(3) $$x^3 - 2x^2 + x - 1 = 0,$$

except for the additional root -1. The equation (3) has one real root, $\gamma \approx 1.754877666247$ and two complex roots, $\alpha \pm i\beta$, the square of whose modulus, $1/\gamma$, is less than 1.

$$a_n = c\gamma^n + (a+ib)(\alpha+i\beta)^n + (a-ib)(\alpha-i\beta)^n,$$

where

$$\alpha = 1 - \tfrac{1}{2}\gamma \approx 0.122561166876, \quad \beta = \sqrt{3\gamma^2 - 4\gamma}/2 \approx 0.744861766619,$$

$$a = (\gamma^2 - 2\gamma + 2)/2(2\gamma^2 - 2\gamma + 3) \approx 0.138937790848, \quad b = (2\gamma+1)(\gamma-1)/2\beta \approx 0.202250124098,$$

$$c = (\gamma^2 + 1)/(2\gamma^2 - 2\gamma + 3) \approx 0.722124418303,$$

and a_n is the nearest integer to $c\gamma^n$.

The sequence $\{a_n\}$ does not appear in Neil Sloane's book [3]; nor do the corresponding sequences $\{a_n^{(k)}\}$ of numbers of binary sequences of length n in which the ones occur only in blocks of length at least k. The problem so far considered is $k = 2$. The more general analogs of (1), (2), (3) are

(1′) $$a_n = 2a_{n-1} - a_{n-2} + a_{n-k-1},$$

(2′) $$a_n = a_{n-1} + a_{n-k} + a_{n-k-2} + a_{n-k-3} + \cdots + a_{n-2k},$$

(3′) $$x^{k+1} - 2x^k + x^{k-1} - 1 = 0.$$

Then

$$a_{-1}^{(k)} = a_0^{(k)} = a_1^{(k)} = \cdots = a_{k-1}^{(k)} = 1; \quad a_{k+r}^{(k)} = 1 + \tfrac{1}{2}(r+1)(r+2)$$

for $0 \leq r \leq k$; and for larger values of n, $a_n^{(k)}$ is the nearest integer to $c_k \gamma_k^n$, where γ_k is the real root of (3') which lies between 1 and 2, and c_k is an appropriate constant. Approximate values of γ_k and c_k for $k = 1(1)9$ are shown in Table 1.

Table 1

k	1	2	3	4	5	6	7	8	9
γ_k	2	1.7549	1.6180	1.5289	1.4656	1.4178	1.3803	1.3499	1.3247
c_k	1	0.7221	0.5854	0.5033	0.4481	0.4082	0.3778	0.3539	0.3344

The sequence $\{a_n^{(3)}\}$ is similar to the Lucas sequence associated with the Fibonacci numbers, since $\gamma_3 = (1+\sqrt{5})/2$, the golden number.

The characteristic polynomial for (2') is the product of that for (1') with the cyclotomic polynomial $x^{k-1} + x^{k-2} + \cdots + x + 1$. When k is odd, (3') is of even degree and is reducible and has a second real root between 0 and -1. Table 2 gives the values of $a_n^{(k)}$ for $n = 0(1)26$, $k = 2(1)9$. Of course, $a_n^{(1)} = 2^n$, the number of unrestricted binary sequences of length n.

Table 2

n	$a_n^{(2)}$	$a_n^{(3)}$	$a_n^{(4)}$	$a_n^{(5)}$	$a_n^{(6)}$	$a_n^{(7)}$	$a_n^{(8)}$	$a_n^{(9)}$
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1
3	4	2	1	1	1	1	1	1
4	7	4	2	1	1	1	1	1
5	12	7	4	2	1	1	1	1
6	21	11	7	4	2	1	1	1
7	37	17	11	7	4	2	1	1
8	65	27	16	11	7	4	2	1
9	114	44	23	16	11	7	4	2
10	200	72	34	22	16	11	7	4
11	351	117	52	30	22	16	11	7
12	616	189	81	42	29	22	16	11
13	1081	305	126	61	38	29	22	16
14	1897	493	194	91	51	37	29	22
15	3329	798	296	137	71	47	37	29
16	5842	1292	450	205	102	61	46	37
17	10252	2091	685	303	149	82	57	46
18	17991	3383	1046	443	218	114	72	56
19	31572	5473	1601	644	316	162	94	68
20	55405	8855	2452	936	452	232	127	84
21	97229	14328	3753	1365	639	331	176	107
22	170625	23184	5739	1999	897	467	247	141
23	299426	37513	8771	2936	1257	650	347	191
24	525456	60697	13404	4316	1766	894	484	263
25	922111	98209	20489	6340	2493	1220	667	364
26	1618192	158905	31327	9300	3536	1660	907	502

Since these are recurring sequences, they have many divisibility properties. Examples are $5 \mid a_n^{(2)}$ just if $n \equiv -4$ or -2, mod 12; $8 \mid a_n^{(2)}$ just if $n \equiv -4$ or -2, mod 14 and $2 \mid a_n^{(k)}$ according to the residue class to which n belongs, mod $2(2^{(k+1)/2} - 1)$, k odd, or mod $2^{k+1} - 1$, k even.

REFERENCES

1. Murray Edelberg, Solutions to Problems in 2, McGraw-Hill, 1968, p. 74.
2. C.L.Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, 1968, Problem 4–4, p. 119.
3. N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973, p. 59.

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ON THE EQUALITY OF PERIODS OF DIFFERENT MODULI IN THE FIBONACCI SEQUENCE

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Let m be an arbitrary positive integer. According to the notation of Vinson [1, p. 37] let $s(m)$ denote the period of F_n modulo m and let $f(m)$ denote the rank of apparition of m in the Fibonacci sequence.

Let p be an arbitrary prime. Wall [2, p. 528] makes the following remark: "The most perplexing problem we have met in this study concerns the hypothesis $s(p^2) \neq s(p)$. We have run a test on a digital computer which shows that $s(p^2) \neq s(p)$ for all p up to 10,000; however, we cannot yet prove that $s(p^2) = s(p)$ is impossible. The question is closely related to another one, "can a number x have the same order mod p and mod p^2?," for which rare cases give an affirmative answer (e.g., $x = 3,\ p = 11;\ x = 2,\ p = 1093$); hence, one might conjecture that equality may hold for some exceptional p."

Based on Ward's Last Theorem [3, p. 205] we shall give necessary and sufficient conditions for $s(p^2) = s(p)$.

From Robinson [4, p. 30] we have for $m,n > 0$

(1) $$F_{n+r} \equiv F_r \pmod m \text{ for all integers } r \text{ if and only if } s(m)\,|\,n.$$

If $m,n > 0$ and $m\,|\,n$, then $F_{s(n)+r} \equiv F_r \pmod m$ for all r. Therefore by (1), $s(m)\,|\,s(n)$. So we have for $m,n > 0$

(2) $$m\,|\,n \text{ implies } s(m)\,|\,s(n).$$

It is easily verified that for all integers n

(3) $$F_{2n+1} = (-1)^{n+1} + F_{n+1}L_n.$$

From Theorem 1 of [1, p. 39] we have that $s(m)$ is even if $m > 2$.

An equivalent form of the following theorem can be found in Vinson [1, p. 42].

Theorem 1. We have

i) $s(m) = 4f(m)$ if and only if $m > 2$ and $f(m)$ is odd.
ii) $s(m) = f(m)$ if and only if $m = 1$ or 2 and $s(m)/2$ is odd.
iii) $s(m) = 2f(m)$ if and only if $f(m)$ is even and $s(m)/2$ is even.

To prove the above theorem it is sufficient, in view of Theorem 3 by Vinson [1, p. 42], to prove the following:

Lemma. $m = 1$ or 2 or $s(m)/2$ is odd if and only if $8 \nmid m$ and $2\,|\,f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m.

Proof. Let $m = 1$ or 2 or $s(m)/2$ be odd. If $m = 1$ or 2, then the conclusion is clear. So we may assume that $m > 2$ and $s(m)/2$ is odd. Suppose $8\,|\,m$. Then by (2), $12 = s(8)\,|\,s(m)$. Therefore $s(m)/2$ is even, a contradiction. Hence $8 \nmid m$.

Let p be any odd prime which divides m. From [1, p. 37] and (2), $f(p)\,|\,s(p)\,|\,s(m)$. Therefore $4 \nmid f(p)$. Suppose $2 \nmid f(p)$. Then by Theorem 1 of [1, p. 39] and (2), we have $4f(p) = s(p)\,|\,s(m)$, a contradiction. Thus $2\,|\,f(p)$.

Conversely, let $8 \nmid m$ and $2\,|\,f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m. Let p be any odd prime which divides m and let e be any positive integer. From [1, p. 40] we have that $f(p)$ and $f(p^e)$ are divisible by the same power of 2. Therefore $2\,|\,f(p^e)$ and $4 \nmid f(p^e)$. Then since

$$p^e \mid F_{f(p^e)} = F_{f(p^e)/2} L_{f(p^e)/2}$$

and $p^e \nmid F_{f(p^e)/2}$ and $(F_n, L_n) = d \leq 2 < p$ for all integers n, we have $p^e \mid L_{f(p^e)/2}$. So by (3),

$$F_{f(p^e)+1} \equiv (-1)^{(f(p^e)/2)+1} = 1 \pmod{p^e}.$$

Therefore by definition, $f(p^e) = s(p^e)$.

Now, suppose that $m > 2$ and $s(m)/2$ is even. Let m have the prime factorization $m = 2^a p_1^{a_1} \cdots p_r^{a_r}$ with $a \geq 0$. Then by [1, p. 41]

$$s(m) = \underset{1 \leq i \leq r}{\text{l.c.m.}} \{s(2^a), s(p_i^{a_i})\} .$$

Then $4 \mid s(m)$ implies $4 \mid s(2^a)$ or $4 \mid s(p_j^{a_j})$ for some j such that $1 \leq j \leq r$. If $4 \mid s(2^a)$, then $a \geq 3$. Thus $8 \mid m$, a contradiction. If $4 \mid s(p_j^{a_j}) = f(p_j^{a_j})$, then we have another contradiction. Therefore $s(m)/2$ is odd or $m = 1$ or 2.

Various relationships of equality between integral multiples of $s(m)$, $f(m)$, $s(t)$ and $f(t)$ for arbitrary positive integers m and t can be obtained as corollaries to Theorem 1. We mention only the following:

Corollary 1. If $m > 2$ and $t > 2$ and

i) $s(m)/2$ and $s(t)/2$ are both odd, or
ii) $f(m)$ and $f(t)$ are both odd, or
iii) $s(m)/2$, $s(t)/2$, $f(m)$ and $f(t)$ are all even,

then $s(m) = s(t)$ if and only if $f(m) = f(t)$.

Theorem 2. Let m and t be positive integers such that $m \mid L_{f(m)/2}$ if $f(m)$ is even and $t \mid L_{f(t)/2}$ if $f(t)$ is even. Then $s(m) = s(t)$ if and only if $f(m) = f(t)$.

Proof. Let $s(m) = s(t)$. We have $m = 1$ iff $t = 1$ and $m = 2$ iff $t = 2$, so we may assume that $m > 2$ and $t > 2$. By Corollary 1, we need only consider the case; $s(m)/2 = s(t)/2$ is even and $f(m)$ and $f(t)$ have different parity, say $f(m)$ is odd and $f(t)$ is even. Then by Theorem 1, $4f(m) = s(m) = s(t) = 2f(t)$. Therefore $f(t)/2 = f(m)$ is odd. Since $f(t)$ is even we have by hypothesis that $t \mid L_{f(t)/2}$. Thus by (3),

$$F_{f(t)+1} \equiv (-1)^{(f(t)/2)+1} \equiv 1 \pmod{t}.$$

But $t \mid F_{f(t)}$ and $f(t) < s(t)$. This contradicts the definition of $s(t)$. Therefore the case under consideration cannot occur.

Conversely, let $f(m) = f(t)$. As before we may assume that $m > 2$ and $t > 2$. By Corollary 1, we need only consider the case; $f(m) = f(t)$ is even and $s(m)/2$ and $s(t)/2$ have different parity, say $s(m)/2$ is odd and $s(t)/2$ is even. Then by Theorem 1,

$$2s(m) = 2f(m) = 2f(t) = s(t).$$

Therefore $f(t)/2$ is odd. Since $f(t)$ is even we have $t \mid L_{f(t)/2}$. Thus by (3), $F_{f(t)+1} \equiv 1 \pmod{t}$. But $t \mid F_{f(t)}$ and $f(t) < s(t)$. This is a contradiction and therefore the case under consideration cannot occur.

Corollary 2. Let p and q be arbitrary odd primes and e and a be arbitrary positive integers. Then $s(p^e) = s(q^a)$ if and only if $f(p^e) = f(q^a)$.

Proof. By Theorem 2 we need only show that if $f(p^e)$ is even then $p^e \mid L_{f(p^e)/2}$. We have

$$F_{f(p^e)} = F_{f(p^e)/2} L_{f(p^e)/2} \quad \text{and} \quad p^e \nmid F_{f(p^e)/2} \quad \text{and} \quad (F_{f(p^e)/2}, L_{f(p^e)/2}) = d \leq 2 < p.$$

Thus $p^e \mid L_{f(p^e)/2}$.

Corollary 3. Let $\phi_n(x) = x + x^2/2 + \cdots + x^n/n$, and let $k(x) = k_p(x) = (x^{p-1} - 1)/p$, where p is an odd prime greater than 5. Then $s(p^2) = s(p)$ if and only if $\phi_{(p-1)/2}(5/9) \equiv 2k(3/2) \pmod{p}$.

[Continued on page 96.]

ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-370 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Solve the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = F_n.$$

B-371 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let

$$S_n = \sum_{k=1}^{F_n} \sum_{j=1}^{k} T_j,$$

where T_j is the triangular number $j(j+1)/2$. Does each of $n \equiv 5 \pmod{15}$ and $n \equiv 10 \pmod{15}$ imply that $S_n \equiv 0 \pmod{10}$? Explain.

B-372 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let S_n be as in B-371. Does $S_n \equiv 0 \pmod{10}$ imply that n is congruent to either 5 or 10 modulo 15? Explain.

B-373 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California, and P. L. Mana, Albuquerque, New Mexico.

The sequence of Chebyshev polynomials is defined by

$$C_0(x) = 1, \quad C_1(x) = x, \quad \text{and} \quad C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x)$$

for $n = 2, 3, \cdots$. Show that $\cos[\pi/(2n+1)]$ is a root of

$$[C_{n+1}(x) + C_n(x)]/(x+1) = 0$$

and use a particular case to show that $2\cos(\pi/5)$ is a root of

$$x^2 - x - 1 = 0.$$

B-374 Proposed by Frederick Stern, San Jose State University, San Jose, California.

Show both of the following:

$$F_n = \frac{2^{n+2}}{5}\,[(\cos(\pi/5))^n \cdot \sin(\pi/5) \cdot \sin(3\pi/5) + (\cos(3\pi/5))^n \cdot \sin(3\pi/5) \cdot \sin(9\pi/5)]\,,$$

$$F_n = \frac{(-2)^{n+2}}{5}\,[(\cos(2\pi/5))^n \cdot \sin(2\pi/5) \cdot \sin(6\pi/5) + (\cos(4\pi/5))^n \cdot \sin(4\pi/5) \cdot \sin(12\pi/5)]\,.$$

B-375 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Express

$$\frac{2^{n+1}}{5}\sum_{k=1}^{4}\,[(\cos(k\pi/5))^n \cdot \sin(k\pi/5) \cdot \sin(3k\pi/5)]$$

in terms of Fibonacci number, F_n.

SOLUTIONS

TRIANGULAR CONVOLUTION

B-346 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a closed form for

$$\sum_{k=1}^{n} F_{2k}T_{n-k} + T_n + 1\,,$$

where T_k is the triangular number

$$\binom{k+2}{2} = (k+2)(k+1)/2\,.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Using well-known generating functions one finds that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{2k}T_{n-k} + T_n + 1\right)x^n &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{2k}T_{n-k}\right)x^n + \sum_{n=0}^{\infty} T_nx^n + \sum_{n=0}^{\infty} x^n\\ &= \left(\sum_{n=0}^{\infty} F_{2n}x^n\right)\left(\sum_{n=0}^{\infty} T_nx^n\right) + \sum_{n=0}^{\infty} T_nx^n + \sum_{n=0}^{\infty} x^n\\ &= \frac{x}{1-3x+x^2}\cdot\frac{1}{(1-x)^3} + \frac{1}{(1-x)^3} + \frac{1}{1-x}\\ &= \frac{2-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+3}x^n\,.\end{aligned}$$

Since for $k = 0$, $F_{2k}T_{n-k} = 0$, this implies that

$$\sum_{k=1}^{n} F_{2k}T_{n-k} + T_n + 1 = F_{2n+3}\,.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the proposer.

A THIRD-ORDER ANALOGUE OF THE *F*'s

B-347 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let a, b, and c be the roots of $x^3 - x^2 - x - 1 = 0$. Show that

$$\frac{a^n-b^n}{a-b}+\frac{b^n-c^n}{b-c}+\frac{c^n-a^n}{c-a}$$

is an integer for $n = 0, 1, 2, \cdots$.

Solution by Graham Lord, Université Laval, Québec, Canada.

For $n = 0, 1, 2$ and 3 the expression, $E(n)$, above has the values 0, 3, 2 and 5, for all integers and demonstrating the recursion relation when

$$n = 0: E(n+3) = E(n+2)+E(n+1)+E(n).$$

This latter equation is readily proven since $a^3 = a^2 + a + 1$, etc. That $E(n)$ is an integer follows immediately, by induction, from this recursion relation.

Also solved by George Berzsenyi, Wray Brady, Clyde A. Bridger, Paul S. Bruckman, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the proposer.

PENTAGON RATIO

B-348 Proposed by Sidney Kravitz, Dover, New Jersey.

Let $P_1, \cdots, P_5$ be the vertices of a regular pentagon and let Q_1 be the intersection of segments $P_{i+1}P_{i+3}$ and $P_{i+2}P_{i+4}$ (subscripts taken modulo 5). Find the ratio of lengths Q_1Q_2/P_1P_2.

Solution by Charles W. Trigg, San Diego, California.

Extend P_4P_3 and P_4P_5 to meet P_1P_2 extended in A and B, respectively. Draw P_2P_5.

All diagonals of a regular pentagon of side e are equal, say, to d. Each diagonal is parallel to the side of the pentagon with which it has no common point. So, $AP_3P_5P_2$ is a rhombus. It follows that $AP_3 = AP_2 = d = BP_1 = BP_5$.

From similar triangles,

$$e/d = P_4P_3/P_3P_5 = P_4A/AB = (e+d)/(e+2d),$$

so, $d^2 - ed - e^2 = 0$ and $d = (\sqrt{5}+1)e/2$.

Then,

$$Q_1Q_2/P_1P_2 = P_4Q_1/P_4P_2 = P_4P_3/P_4A = e/(e+d) = 2/(3+\sqrt{5}) = (3-\sqrt{5})/2 = 0.382 = \beta^2.$$

Furthermore,

$$Q_1Q_2/P_3P_5 = (Q_1Q_2/P_1P_2)(P_1P_2/P_3P_5) = (3-\sqrt{5})/(\sqrt{5}+1) = \sqrt{5}-2 = 0.236 = -\frac{L_3-F_3\sqrt{5}}{2} = -\beta^3.$$

Also solved by George Berzsenyi, Wray Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Dinh Thê' Hûng, C. B. A. Peck, and the Proposer.

GENERATING TWINS

B-349 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots$ be the sequence 1, 1, 2, 2, 3, 3, $\cdots$, i.e., let a_n be the greatest integer in $1 + (n/2)$. Give a recursion formula for a_n and express the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

as a quotient of polynomials.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Since the sequence of integers satisfies the relation $x_n = 2x_{n-1} - x_{n-2}$, the given sequence obviously satisfies the recursion formula $a_n = 2a_{n-2} - a_{n-4}$. The corresponding generating function is

$$\frac{x+1}{x^4-2x^2+1},$$

which may be proven by multiplying

$$\sum_{n=0}^{\infty} a_n x^n$$

by $x^4 - 2x^2 + 1$ and utilizing the above recurrence relation.

Also solved by Wray Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, David Zeitlin, and the Proposer.

CUBES AND TRIPLE SUMS OF SQUARES

B-350 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let a_n be as in B-349. Find a closed form for

$$\sum_{k=0}^{n} a_{n-k}(a_k + k)$$

in the case (a) in which n is even and the case (b) in which n is odd.

Solution by Graham Lord, Université Laval, Québec, Canada.

A closed form for the sum in case (a) is $(n+2)^3/8$, and in case (b) $(n+1)(n^2+5n+6)/8$. The proofs of these two are similar, only that of case (a) is given. With $n = 2m$,

$$\sum_{k=0}^{n} a_{n-k}(a_k + k) = \sum_{\ell=0}^{m} [1+m-\ell]\{[1+\ell]+2\ell\} + \sum_{\ell=0}^{m-1} [1+m-\ell-\tfrac{1}{2}]\{[1+\ell+\tfrac{1}{2}]+2\ell+1\}$$

$$= \sum_{0}^{m} (1+m-\ell)(1+3\ell) + \sum_{0}^{m-1} (m-\ell)(2+3\ell)$$

$$= (3m+1)(m+1) + 6m \sum_{0}^{m} \ell - 6 \sum_{0}^{m} \ell^2 = (m+1)^3.$$

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, and the Proposer.

NON-FIBONACCI PRIMES

B-351 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_4 = 3$ is the only Fibonacci number that is a prime congruent to 3 modulo 4.

Solution by Graham Lord, Université Laval, Québec, Canada.

As $F_n \equiv 3 \pmod 4$ IFF $n = 6m + 4 = 2k$, then such an F_n factors $F_k L_k$, and so F_n is a prime IFF $F_k = 1$, that is IFF $n = 4$.

Also solved by Paul S. Bruckman, Michael Bruzinsky, Herta T. Freitag, Dinh Thê' Hũng, Bob Prielipp, Gordon Sinnamon, Lawrence Somer, and the Proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-278 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show

$$\sqrt{\frac{5F_{n+2}}{F_n}} = \langle 3, \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 6} \rangle$$

(Continued fraction notation, cyclic part under bar).

H-279 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the F-L identities:

(a) $$F^4_{n+6r} - (L_{4r}+1)(F^4_{n+4r} - F^4_{n+2r}) - F^4_n = F_{2r}F_{4r}F_{6r}F_{4n+12r}$$

(b) $$F^4_{n+6r+3} + (L_{4r+2}-1)(F^4_{n+4r+2} - F^4_{n+2r+1}) - F^4_n = F_{2r+1}F_{4r+2}F_{6r+3}F_{4n+12r+6}\ .$$

H-280 Proposed by S. Bruckman, Concord, California.

Prove the congruences

(1) $$F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{n+3}};$$

(2) $$L_{3\cdot 2^n} \equiv 2 + 2^{2n+2} \pmod{2^{2n+4}},\ n = 1, 2, 3, \cdots$$

SOLUTIONS

SUM-ARY CONCLUSION

H-264 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-s-i+1}{n-s} = \sum_{i=0}^{n-s}\binom{r+i}{i}\binom{m+n-r-i+1}{m-r}\ .$$

Solution by P. Bruckman, Concord, Calif.

Let

(1) $$\theta(m,n,r,s) = \sum_{i=0}^{m-r}\binom{s+i}{i}\binom{m+n-s-i+1}{n-s}\ ,$$

and

(2) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(m,n,r,s) w^m x^n y^r z^s .$$

Then

$$F(w,x,y,z) = \sum_{m,n,r,s=0}^{\infty} \theta(m+r, n+s, r, s) w^{m+r} x^{n+s} y^r z^s$$

$$= \sum_{m,n,r,s=0}^{\infty} \sum_{i=0}^{m} \binom{s+i}{i} \binom{m+r+n-i+1}{n} w^m x^n (wy)^r (xz)^s$$

$$= \sum_{m,n,r,s,i=0}^{\infty} \binom{s+i}{i} \binom{m+r+n+1}{n} w^{m+i} x^n (wy)^r (xz)^s$$

$$= \sum_{m,n,r,s,i=0}^{\infty} \binom{-s-1}{i} \binom{-m-r-2}{n} w^m (-x)^n (wy)^r (xz)^s (-w)^i$$

$$= \sum_{m,r,s=0}^{\infty} (1-w)^{-s-1} (1-x)^{-m-r-2} w^m (wy)^r (xz)^s$$

$$= (1-w)^{-1}(1-x)^{-2} \sum_{m,r,s=0}^{\infty} \left(\frac{w}{1-x}\right)^m \left(\frac{wx}{1-x}\right)^r \left(\frac{xz}{1-w}\right)^s$$

$$= (1-w)^{-1}(1-x)^{-2} \left(1-\frac{w}{1-x}\right)^{-1} \left(1-\frac{wy}{1-x}\right)^{-1} \left(1-\frac{xz}{1-w}\right)^{-1}$$

or

(3) $$F(w,x,y,z) = (1-w-x)^{-1}(1-x-wy)^{-1}(1-w-xz)^{-1} .$$

From (3), the following symmetry relation is evident:

(4) $$F(w,x,y,z) = F(x,w,z,y) .$$

Hence,

(5) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(m,n,r,s) x^m w^n z^r y^s .$$

In the last expression, we may make the following substitutions:

(6) $$m \to N, \quad n \to M, \quad r \to S, \quad s \to R .$$

Then

$$F(w,x,y,z) = \sum_{N,M=0}^{\infty} \sum_{S=0}^{N} \sum_{R=0}^{M} \theta(N,M,S,R) x^N w^M z^S y^R .$$

Now reversing the orders of summation and converting capital letters to small letters again, we obtain:

(7) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(n,m,s,r) w^m x^n y^r z^s .$$

Now comparing coefficients of (2) and (7), treating F as a function of each of its variables, in order, we conclude

(8) $$\theta(m,n,r,s) = \theta(n,m,s,r). \quad \text{Q.E.D.}$$

Also solved by D. Beverage and the Proposer.

ANOTHER CONGRUENCE

H-265 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that

$$F_{2^3 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}, \quad \text{where} \quad k \geqslant 1 .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let p be an odd prime, $p \neq 5$ and let m be a positive integer such that $p \mid F_m$. We shall prove that

(*) $$F_{mp^{k-1}} \equiv 0 \pmod{p^k} \quad (k = 1, 2, 3, \cdots).$$

Proof of (*). We have (Binet representation)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad F_{pn} = \frac{\alpha^{pn} - \beta^{pn}}{\alpha - \beta},$$

so that

$$\frac{F_{pn}}{F_n} = \frac{\alpha^{pn} - \beta^{-n}}{\alpha^n - \beta^n} \quad (\alpha^n - \beta^n)^{p-1} \pmod{p}.$$

Thus

(**) $$\frac{F_{pn}}{F_n} \equiv (\alpha - \beta)^{p-1} F_n^{p-1} \pmod{p}.$$

Now assume that (*) holds up to and including the value k. By (**),

$$\frac{F_{mp^k}}{F_{mp^{k-1}}} (\alpha - \beta)^{p-1} F^{p-1}_{mp^{k-1}} \equiv 0 \pmod{p},$$

$$F_{mp^k} \equiv 0 \pmod{p\, F_{mp^{k-1}}}.$$

Hence, by the inductive hypothesis,

$$F_{mp^k} \equiv 0 \pmod{p^{k+1}}.$$

This evidently completes the proof.

It is known that the smallest positive m such that $p \mid F_m$ is a divisor of $\tfrac{1}{2}(p^2 - 1)$. It follows that

$$F_M \equiv 0 \pmod{p^k}, \quad (M = \tfrac{1}{2}(p^2 - 1)p^{k-1}, \ k > 1).$$

Indeed, if $p \equiv \pm 1 \pmod 5$, then

$$F_M \equiv 0 \pmod{p^k} \quad (M = (p-1)p^{k-1}, \ k > 1).$$

In particular we have

$$F_{4 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}.$$

Also solved by P. Bruckman and D. Beverage.

IDENTIFY!

H-266 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Find all identities of the form

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = s^n F_{tn}$$

with positive integral r, s and t.

Solution by P. Bruckman, Concord, California

(1) $$\sum_{k=0}^{n}\binom{n}{k}F_{rk} = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\alpha^{rk}-\beta^{rk}) = \frac{(1+\alpha^r)^n-(1+\beta^r)^n}{\sqrt{5}}\quad .$$

If this expression is to equal s^nF_{nt}, for some natural triplet (r,s,t), it must hold for *all* non-negative n. The case $n = 0$ yields no information, merely confirming the trivial identity $0 = 0$. The cases $n = 1$, 2 and 3 yield, respectively:

(2) $$F_r = sF_t\ ;$$

(3) $$2F_r + F_{2r} = s^2F_{2t}\ ;$$

(4) $$3F_r + 3F_{2r} + F_{3r} = s^3F_{3t}.$$

Using (2) and (3), we obtain:

$$2F_r + F_rL_r = s^2F_tL_t = sF_rL_t,$$

or, since $r > 0$,

(5) $$L_r + 2 = sL_t\ .$$

Finally, using (2), (4) and the identity:

$$F_{3m} = F_m(L_m^2 - (-1)^m),$$

we have:

$$3F_r + 3F_rL_r + F_r(L_r^2-(-1)^r) = s^3F_t(L_t^2-(-1)^t) = s^2F_r(L_t^2-(-1)^t);$$

dividing throughout by F_r and using the result of (5), we obtain:

$$3 + 3L_r + L_r^2 - (-1)^r = (L_r+2)^2 - s^2(-1)^t,$$

or upon simplification:

(6) $$1 + (-1)^r + L_r = s^2(-1)^t.$$

We consider two mutually exclusive and exhaustive cases:

<u>CASE I : r is even</u>

From (5) and (6),

$$L_r + 2 = s^2(-1)^t = sL_t\ ;$$

hence, since $s > 0$,

$$s = (-1)^tL_t.$$

Since also t and $L_t > 0$, thus t is even, and $s = L_t$. Then by (2),

$$F_r = L_tF_t = F_{2t},$$

which implies $r = 2t$. We have shown that the triplet $(4m, L_{2m}, 2m)$ is a solution of the desired identity for $n = 0, 1, 2, 3$. It remains to verify this as a solution for all n. Substituting $r = 4m$ in the right member of (1), that expression becomes:

$$\frac{1}{\sqrt{5}}\{(1+\alpha^{4m})^n-(1+\beta^{4m})^n\} = \frac{1}{\sqrt{5}}\{\alpha^{2mn}(\alpha^{2m}+\beta^{2m})^n-(\alpha^{2m}+\beta^{2m})^n\beta^{2mn}\} = L_{2m}^nF_{2mn},$$

which is of the desired form, with $s = L_{2m}$, $t = 2m$. Hence,

(7) $$(r,s,t) = (4m, L_{2m}, 2m),\qquad m = 1, 2, 3, \cdots$$

is a sequence of solutions, the only ones yielded by this case.

<u>CASE II : r is odd</u>

From (6),

$$L_r = s^2(-1)^t.$$

Hence, t must be even and $L_r = s^2$. Substituting this result in (5), we obtain: $sL_t - s^2 = 2$, which implies $s|2$, and so s = 1 or 2.

<u>SUBCASE A : $s = 1$</u>

Thus, $L_r = 1^2 = 1$, and $r = 1$. Thus, by (2), $F_1 = 1 = F_t$. Since t must be even, thus $t = 2$. Hence, (1,1,2) is another possible solution. Since

$$\frac{1}{\sqrt{5}}\left\{(1+\alpha)^n - (1+\beta)^n\right\} = \frac{1}{\sqrt{5}}\left\{\alpha^{2n} - \beta^{2n}\right\} = F_{2n} = 1^n F_{2n},$$

thus (1,1,2) is a valid solution, the only one yielded by this subcase.

<u>SUBCASE B : $s = 2$</u>

Then $L_r = 2^2 = 4$, so $r = 3$. Thus, by (2), $F_3 = 2 = 2F_t$. As in Subcase A above, $t = 2$. This yields the possible solution (3,2,2). Now

$$(1+\alpha^3) = 2\alpha + 2 = 2\alpha^2;$$

similarly, $(1+\beta^3) = 2\beta^2$. Hence,

$$\frac{1}{\sqrt{5}}\left\{(1+\alpha^3)^n - (1+\beta^3)^n\right\} = \frac{2n}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = 2^n F_{2n},$$

which shows that (3,2,2) is indeed a valid solution, the only one yielded by this subcase.

Therefore, <u>all</u> solutions (r,s,t) of the desired identity are given by (7), and also by (1,1,2) and (3,2,2).

Also solved by the Proposer.

Late Acknowledgements:

P. Bruckman solved H-258, H-259, H-262, H-263.

S. Singh solved H-263.

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[Continued from page 87.]

Proof. From Corollary 2 and [4, p. 205] we have $s(p^2) = s(p)$ if and only if $f(p^2) = f(p)$ if and only if

$$\phi_{(p-1)/2}(5/9) \equiv 2k(3/2) \pmod{p}.$$

From Wall's remark we note that $\phi_{(p-1)/2}(5/9) \not\equiv 2k(3/2) \pmod{p}$ for all primes p such that $5 < p < 10{,}000$.

REFERENCES

1. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," *The Fibonacci Quarterly,* Vol. 1, No. 2 (1963), pp. 37–45.
2. D. D. Wall, "Fibonacci Series Modulo m," *Amer. Math. Monthly,* 67 (1960), pp. 525–532.
3. H-24, *The Fibonacci Quarterly,* Vol. 2, No. 3 (1964), pp. 205–207.
4. D. W. Robinson, "The Fibonacci Matrix Modulo m," *The Fibonacci Quarterly,* Vol. 1, No. 2 (1963), pp. 29-36.

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