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# דुe Fibonacci Quarterly <br> THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION <br> DEVOTED TO THE STUDY <br> OF INTEGERS WITH SPECIAL PROPERTIES 

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# MORE FIBONACCI FUNCTIONS 

## M. W. BUNDER

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Recently there have appeared in this Quarterly a number of generalizations of the Fibonacci number $F_{n}$ to functions $F(x)$, defined for all real $x$, and, in general, continuous everywhere.
For such a generalization two properties are particularly desirable:
(A)
$F(x)=F_{n}$ for $x=n$ a natural number
and
(B)

$$
F(x+2)=F(x)+F(x+1) .
$$

Spickerman [6] proved some general properties of functions satisfying (B).
Of the various generalizations Halsey's [1] does not generally satisfy (B) (see [7]) and even if defined for all real $x$, is not continuous at $x=1$.
Heimer's function [2] satisfies (A) and (B) but is quasilinear. Elmore's function [3] is not a generalization in the above sense, it is a function of a natural number variable and a real variable.
Parker's [4] and Scott's [5] functions which are identical are "smooth curves," satisfy both (A) and (B) but can be generalized further.
Both take

$$
F(x)=\operatorname{Re}\left(\frac{\lambda^{x}-(-1)^{x} \lambda^{-x}}{\sqrt{(5)}}\right)=\frac{\lambda^{x}-\lambda^{x} \cos \pi x}{\sqrt{(5)}}
$$

where

$$
\lambda=\frac{1+\sqrt{5}}{2} .
$$

It seems, however, that a lot is lost in taking only the real part of

$$
\frac{\lambda^{x}-(-1)^{x} \lambda^{-x}}{\sqrt{(5)}}
$$

Clearly this complex function itself (we will call it $F_{x}$ ) satisfies (A), and also (B) for any complex number $x$. Also as the real part of $F_{x}$ satisfies $(B)$ so does the imaginary part and any linear combination of these.
If we let

$$
F_{1}(x)=\operatorname{Re}\left(F_{x}\right), \quad F_{2}(x)=\|\left(F_{x}\right)=\frac{-\lambda^{-x} \sin \pi x}{\sqrt{(5)}},
$$

for $x$ real, then $F_{1}(x)+a F_{2}(x)$ satisfies (A) and (B) for each real number $a$.
Scott gives a number of identities concerning $F_{1}(x)$ and also concerning the corresponding Lucas function which we will call

$$
\left.L_{1}(x)=\operatorname{Re}\left(L_{x}\right)=\operatorname{Re} \lambda^{x}+(-1)^{x} \lambda^{-x}\right)=\lambda^{x}+\lambda^{-x} \cos \pi x .
$$

Of course $/(L x)=-F_{2}(x) \sqrt{5}$.
We now list some easily derivable properties of $F_{2}(x)$ some of which relate it to $F_{1}(x)$ :

$$
\begin{gathered}
F_{2}(x) \cdot F_{2}(-x)=\frac{-\sin ^{2} \pi x}{5}, \quad F_{2}(x+1) \cdot F_{2}(x-1)=F_{2}^{2}(x), \\
F_{2}(x+1 / 4) \cdot F_{2}(x-1 / 4)=F_{2}(2 x) \frac{\cot 2 \pi x}{2 \sqrt{(5)}}, \quad F(x+1 / 2) \cdot F_{2}(x-1 / 2)=-F_{2}^{2}(x) \cot ^{2} \pi x,
\end{gathered}
$$

$$
F_{1}(x)=\frac{-\sin \pi x}{5 F_{2}(x)}+F_{2}(x) \cot \pi x, \quad F_{2}(n x)=\frac{\sin n \pi x}{\sin ^{n} \pi x} \frac{5^{(n / 2)-1} F_{2}^{n}(x)}{(-1)^{n+1}} .
$$

Another possible generalization of $F_{n}$ for $x=n$ is $\left|F_{x}\right|$, which we will call $G_{1}(x)$.
Thus

$$
G_{1}(x)=\left|F_{x}\right|=\sqrt{F_{1}^{2}(x)+F_{2}^{2}(x)}=\frac{1}{\sqrt{(5)}} \sqrt{\lambda^{2} x-2 \cos \pi x+\lambda^{-2 x}}
$$

Another such function is

$$
G_{2}(x)=\sqrt{F_{1}^{2}(x)-F_{2}^{2}(x)}=\frac{1}{\sqrt{(5)}} \sqrt{\lambda^{2 x}-2 \cos \pi x+\lambda^{-2 x} \cos 2 \pi x}
$$

Clearly

$$
k G_{1}(x)+(1-k) G_{2}(x)=F_{n}
$$

when $x=n$ for all real $k$.
The following are some properties of these functions:

$$
\begin{gathered}
G_{1}^{2}(x+1)-G_{1}^{2}(x)=G_{1}^{2}(x+1 / 2)-2 / 5 \sin \pi x+4 / 5 \cos \pi x \\
G_{1}^{2}(2 x)=5 G_{1}^{4}(x)+4 \cos \pi x G_{1}^{2}(x) \\
G_{2}^{2}(x)=(1 / 5)\left(L_{1}(2 x)-2 \cos \pi x\right) \\
G_{1}^{2}(x)-G_{2}^{2}(x)=2 F_{2}^{2}(x) .
\end{gathered}
$$

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## PELLIAN DIOPHANTINE SEQUENCES

## A. G. SHANNON

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1. INTRODUCTION

The so-called Pellian Diophantine equation is

$$
x_{22}^{2}-2 x_{12}^{2}=1
$$

$$
\left|x_{22}^{2}-m x_{12}\right|=1
$$

or

$$
\text { abs. }\left|\begin{array}{rr}
x_{22} & m x_{12} \\
x_{12} & x_{22}
\end{array}\right|=1
$$

A generalization of this is in turn provided by
(1.1)

$$
\text { abs. }\left|\begin{array}{ccccc}
x_{r r} & m x_{1 r} & m x_{2 r} & \cdots & m x_{r, r-1} \\
x_{r-1, r} & x_{r r} & m x_{1 r} & \cdots & m x_{r, r-2} \\
& & \cdots & & \\
x_{1 r} & x_{2 r} & x_{3 r} & \cdots & x_{r r}
\end{array}\right|=1
$$

The aim of this paper is to construct a solution for this generalized Pellian Diophantine equation. The approach adopted is less general than that of Bernstein [1] but is, in a sense, more direct. For encouragement with an earlier draft of this paper thanks are due to Bernstein, whose works on pyramidal Diophantine equations [3] and the JacobiPerron algorithm [2] should be seen for further extensions. We designate the determinant in Eq. (1.1) by

$$
D\left(m ; x_{1 r}, \cdots, x_{r r}\right)
$$

## 2. SEQUENCES

We define sequences $\left\{W_{s, n}^{(r)}\right\}$ which satisfy the arbitrary order linear homogeneous recurrence relation

$$
\begin{equation*}
W_{s, n}^{(r)}=\sum_{j=1}^{r}\binom{r}{j} D^{r-j} W_{s, n-j}^{(r)}, \quad n>r \tag{2.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& D=[w], \quad w \text { an } r^{t h} \text {-degree irrational: } \\
& \begin{aligned}
w^{r} & =m \\
& =D^{r}+d, \quad m, D, d \in Z_{+}
\end{aligned}
\end{aligned}
$$

with boundary conditions determined by

$$
\begin{gather*}
W_{s, n}^{(r)}=\delta_{s, n+1} \quad\left\{\begin{array}{l}
s \leqslant n+1 \\
1 \leqslant n<r
\end{array}\right. \\
W_{s, r}^{(r)}=D^{s-1} \\
W_{s, r}^{(r)}=D W_{s-1, n}^{(r)}+W_{s-1, n-1}^{(r)} . \tag{2.2}
\end{gather*}
$$

The initial values $W_{s, 1}^{(r)}, s>2$, have not been specified because they are not used in this development. They are readily determined from Eqs. (2.1) and (2.2) if required.
[APR.

The table provides some examples of $W_{s, n}^{(2)}$ and $W_{s, n}^{(3)}$.
Each of the sequences can be expressed in terms of the fundamental sequence [6], $\left\{w_{1, n}^{(r)}\right\}$ :

$$
W_{s, n}^{(r)}=\sum_{j=0}^{s-1}\binom{s-1}{j} D^{s-j-1} W_{1, n-j}^{(r)}
$$

Proof. When $s=1,2$, we have respectively

$$
W_{1, n}^{(r)}=W_{1, n}^{(r)} \quad \text { and } \quad W_{2, n}^{(r)}=D W_{1, n}^{(r)}+W_{1, n-1}^{(r)}
$$

Suppose the result is true for $s=1,2, \cdots, t$.

$$
\begin{aligned}
W_{t+1, n}^{(r)}=D W_{t, n}^{(r)}+W_{t, n-1}^{(r)} & =\sum_{j=0}^{t-1}\binom{t-1}{j}\left\{D^{t-j} W_{1, n-j}^{(r)}+D^{t-j-1} W_{1, n-j-1}^{(r)}\right\} \\
& =\sum_{j=0}^{t}\left\{\binom{t-1}{j}+\binom{t-1}{j-1}\right\} D^{t-j} W_{1, n-j}^{(r)}=\sum_{j=0}^{t}\binom{t}{j} D^{t-j} W_{1, n-j}^{(r)},
\end{aligned}
$$

as required
We define matrices $M, N_{n}$ :
3. LEMMAS

$$
\begin{gathered}
M=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & & 1 & \cdots \\
0 & 0 \\
0 & 0 & \cdots & 0 & \cdots \\
1 & r D & \binom{r}{2} D^{2} & \cdots & r D^{r-1}
\end{array}\right], \\
N_{n}=\left[W_{\kappa, n+\rho}^{(r)}\right]
\end{gathered} \quad 1 \leqslant \kappa, \quad \rho \leqslant r .
$$

Lemma 1.

$$
N_{n+1}=M^{n} N_{1} .
$$

Proof. The result clearly follows from induction on $n$, since when $n=1$,

$$
\begin{aligned}
M N_{1} & =\left[\begin{array}{ccccc}
0 & 1 & \cdots & 0 & 0 \\
& \cdots & & \\
0 & 0 & \cdots & 1 \\
1 & r D & \cdots & r D^{r-1}
\end{array}\right]\left[\begin{array}{cccc}
0 & 1 & \cdots & 0 \\
0 & & \cdots & \\
0 & W_{2, r+1}^{(r)} & \cdots & W_{r, r+1}^{(r)}
\end{array}\right] \\
& =\left[\begin{array}{ccccc}
0 & & 0 & \cdots & W_{r, 3}^{(r)} \\
1 & \cdots & \cdots & W_{r}^{(r)} \\
1 & W_{2, r+1}^{(r)} & \cdots & W_{r}^{(r+1} \\
W_{1, r+2}^{(r)} & W_{2, r+2}^{(r)} & \cdots & W_{r, r+2}^{(r)}
\end{array}\right]=N_{2} \\
N_{3} & =M N_{2} \\
& =M^{2} N_{1}, \text { and so on. }
\end{aligned}
$$

Lemma 2.

$$
\operatorname{det} N_{n}=(-1)^{n(r-1)}
$$

Proof.

$$
\operatorname{det} M=(-1)^{r-1}=\operatorname{det} N_{1}
$$

$$
\operatorname{det} N_{n}=(-1)^{(r-1)(n-1)}(-1)^{r-1}=(-1)^{n(r-1)}
$$

Lemma 3. $\quad \sum_{k=1}^{r} \sum_{j=0}^{r-k}\binom{r-k}{j} W^{k} D^{j} W_{i, n+j+k}^{(r)}=\sum_{k=1}^{r} \sum_{j=0}^{r-k}\binom{r-k}{j} W^{k-1} D^{j} W_{i+1, n+j+k}^{(r)}$.

Proof. We consider coefficients of $w$ :

$$
\begin{aligned}
\sum_{j=0}^{r-k-1}\binom{r-k-1}{j} D^{j} W_{i+1, n+j+k+1}^{(r)} & =\sum_{j=0}^{r-k-1}\binom{r-k-1}{j} D^{j}\left(D W_{i, n+j+k+1}^{(r)}+W_{i, n+j+k}^{(r)}\right) \\
& =\sum_{j=0}^{r-k-1}\left\{\binom{r-k-1}{j} D^{j+1} W_{i, n+j+k+1}^{(r)}+\binom{r-k-1}{j} D^{j} W_{i, n+j+k}^{(r)}\right\} \\
& =\sum_{j=0}^{r-k}\left\{\binom{r-k-1}{j-1}+\binom{r-k-1}{j}\right\} D^{j} W_{i, n+j+k}^{(r)} \\
& =\sum_{j=0}^{r-k}\binom{r-k}{j} D^{j} W_{i, n+j+k}^{(r)}, \text { as required. }
\end{aligned}
$$

## 4. RESULT

Theorem. For $i, k=1,2, \cdots, r$,

$$
x_{i k}=\sum_{j=0}^{r-k}\binom{r-k}{j} D^{j} W_{i, n+j+k}^{(r)}
$$

are solutions of the Pellian Diophantine equation

$$
1=D\left(m ; x_{1 r}, \cdots, x_{r r}\right)
$$

Proof. Lemma 3 becomes

$$
\begin{equation*}
\sum_{k=1}^{r} w^{k} x_{i k}=\sum_{k=1}^{r} w^{k-1} x_{i+1, k} \tag{4.1}
\end{equation*}
$$

$(-1)^{n(r-1)}=\operatorname{det} N^{n}=\left|\begin{array}{cccc}W_{1, n+1}^{(r)} & W_{2, n+1}^{(r)} & \cdots & W_{r, n+1}^{(r)} \\ W_{1, n+2}^{(r)} & W_{2, n+2}^{(r)} & \cdots & W_{r, n+2}^{(r)} \\ & \cdots & & W_{r, n}^{(r)} \\ W_{1, n+r}^{(r)} & W_{2, n+r}^{(r)} & \cdots & W_{r, n+r}\end{array}\right|$
$=\left|\begin{array}{lll}W_{1, n+1}^{(r)}+\sum_{j=1}^{r-1}\binom{r-1}{j} D^{j} W_{1, n+j+1}^{(r)} & \cdots & W_{r, n+1}^{(r)}+\sum_{j=1}^{r}\binom{r-1}{j} D^{j} W_{r, n+j+1}^{(r)} \\ W_{1, n+2}^{(r)}+\sum_{j=1}^{r-2}\binom{r-2}{j} D^{j} W_{1, n+j+2}^{(r)} & \cdots & W_{r, n+2}^{(r)}+\sum_{j=1}^{r}\binom{r-2}{j} D^{j} W_{r, n+k+2}^{(r)} \\ & \cdots & \\ W_{1, n+r-1}^{(r)}+D W_{1, n+r}^{(r)} & \cdots & W_{r, n+r-1}^{(r)}+D W_{r, n+r}^{(r)} \\ W_{1, n+r}^{(r)} & \ldots & W_{r, n+r}^{(r)}\end{array}\right|$

$$
=\left|\begin{array}{llll}
x_{11} & x_{21} & \cdots & x_{r 1} \\
x_{12} & x_{22} & \cdots & x_{r 2} \\
x_{1 r} & x_{2 r} & \cdots & x_{r r}
\end{array}\right|=D\left(m ; x_{1 r}, \cdots, x_{r r}\right)
$$

by equating coefficients of $w^{k}$ in Eq. (4.1).

## 5. CONCLUSION

Consider, as examples: When $r=2, m=2$, we have
When $n=1$,

$$
D=[\sqrt{2}]=1, \quad \text { and } \quad x_{22}=W_{2, n+2}^{(2)}, \quad x_{12}=W_{1, n+2}^{(2)}
$$

which satisfy
when $n=0$,
which satisfy

$$
x_{22}=w_{2,3}^{(2)}=3, \quad x_{12}=w_{1,3}^{(2)}=2
$$

$$
x_{22}^{2}-m x_{12}^{2}=1
$$

$$
x_{22}=w_{2,2}^{(2)}=1, \quad x_{12}=w_{1,2}^{(2)}=1
$$

$$
x_{22}^{2}-m x_{12}^{2}=-1
$$

The relevant recurrence relation is

$$
W_{s, n}^{(2)}=2 D W_{s, n-1}^{(2)}+W_{s, n-2}^{(2)}
$$

When $r=3, m=9$, we have

$$
D=[\sqrt[3]{9}]=2, \quad \text { and } \quad x_{33}=W_{3, n+3}^{(3)}, \quad x_{23}=w_{2, n+3}^{(3)}, \quad x_{13}=W_{1, n+3}^{(3)}
$$

When $n=0$,
which satisfy

$$
x_{33}=W_{3,3}^{(3)}=4, \quad x_{23}=w_{2,3}^{(3)}=2, \quad x_{13}=W_{1,3}^{(3)}=1
$$

$$
x_{33}^{3}+m x_{23}^{3}+m^{2} x_{13}^{3}-3 m x_{13} x_{23} x_{33}=1
$$

The relevant recurrence relation is

$$
W_{s, n}^{(3)}=3 D^{2} W_{s, n-1}^{(3)}+3 D W_{s, n-2}^{(3)}+W_{s, n-3}^{(3)}, \quad n>3
$$

There is scope for further research in generalizing the properties of the second-order Pellian sequence discussed by Horadam [5]. The use of the Jacobi-Perron Algorithm in this context should be studied first [2]. The other way of generalizing the Pellian equation, namely,

$$
x^{r}-m y^{r}=1,
$$

is still an open and challenging question as Bernstein [4] remarked.

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# SOME RESULTS FOR GENERALIZED BERNOULLI, EULER, STIRLING NUMBERS 

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## SUMMARY

The present paper is a continuation of researches begun by the author in previous publications $[3,4,5]$ on three classes of generalized Bernoulli, Euler, Stirling numbers. And here, of course, will be proved some additional interesting results.

## 1. GENERALIZED BERNOULLI, EULER NUMBERS AND POLYNOMIALS

The generalized Bernoulli, Euler numbers in question, and the related polynomials, are defined by the series

$$
\begin{equation*}
f(t ; h, w)=\frac{h t}{(1+w t)^{h / w}-1}=\sum_{n=0}^{\infty} B_{n ; h, w} \frac{t^{n}}{n!} \tag{1.1}
\end{equation*}
$$

$$
\begin{equation*}
\varphi(t ; h, w)=\frac{2 h(1+w t)^{1 / w}}{(1+w t)^{2 h / w}+1}=\sum_{n=0}^{\infty} E_{n ; h w} \frac{t^{n}}{n!} \tag{1.2}
\end{equation*}
$$

$$
\begin{gather*}
F(x, t ; h, w)=\frac{h t(1+w t)^{x / w}}{(1+w t)^{h / w}-1}=\sum_{n=0}^{\infty} B_{n ; h, w}(x) \frac{t^{n}}{n!},  \tag{1.3}\\
\phi(x, t ; h, w)=\frac{2 h(1+w t)^{x / w}}{(1+w t)^{h / w}+1}=\sum_{n=0}^{\infty} E_{n ; h, w}(x) \frac{t^{n}}{n!}, \tag{1.4}
\end{gather*}
$$

where $h$ and $w$ are real parameters.
These series, for a correct treatment, will be considered in the neighborhood of the origin.
The explicit expressions of $B_{n ; h, w}(x)$ for $n=0,1, \cdots, 5$ are

$$
\begin{gathered}
B_{0 ; h, w}(x)=1, \\
B_{1 ; h, w}(x)=1 / 2(2 x-h+w), \\
B_{2 ; h, w}(x)=x(x-h)+\frac{1}{6}\left(h^{2}-w^{2}\right), \\
B_{3 ; h, w}(x)=1 / 2 x(x-h)(2 x-h-3 w)-1 / 4 w\left(h^{2}-w^{2}\right), \\
B_{4 ; h, w}(x)=x(x-h)(x-2 w)(x-h-2 w)-\frac{1}{30}\left(h^{2}-w^{2}\right)\left(h^{2}-19 w^{2}\right) \\
B_{5 ; h, w}(x)=x(x-h)\left[x^{3}-\frac{3}{2}(h+5 w) x^{2}+\frac{1}{6}\left(h^{2}+45 h w+110 w^{2}\right) x+\frac{1}{6} h^{3}-\frac{55}{6} h w^{2}\right. \\
\left.-15 w^{3}\right]+\frac{1}{4} w\left(h^{2}-w^{2}\right)\left(h^{2}-9 w^{2}\right) .
\end{gathered}
$$

$$
\sum_{r=0}^{n-1}\binom{n}{r}^{(-w)^{-r}}\left(1-\frac{h}{w}\right)_{n-r-1} B_{r ; h, w}(x)=n(-x / w)_{n-1}, \quad n>0
$$

where $(a)_{0}=1,(a)_{r}=a(a+1) \ldots(a+r-1)$.
The explicit expressions of $E_{n ; h, w}(x)$ for $n=0,1, \cdots, 5$ are

$$
\begin{gathered}
E_{0 ; h, w}(x)=h, \\
E_{1 ; h, w}(x)=1 / 2 h(2 x-h), \\
E_{2 ; h, w}(x)=h x^{2}-h(h+w) x+1 / 2 h^{2} w, \\
E_{3 ; h, w}(x)=h x^{3}-\frac{3}{2} h(h+2 w) x^{2}+h w(3 h+2 w) x+\frac{1}{4} h^{2}\left(h^{2}-4 w^{2}\right), \\
E_{4 ; h, w}(x)=h x^{4}-2 h(h+3 w) x^{3}+h w(9 h+11 w) x^{2}+h\left(h^{3}-11 h w^{2}-6 w^{3}\right) x \\
-\frac{3}{2} h^{2} w\left(h^{2}-2 w^{2}\right), \\
E_{5 ; h, w}(x)=h x^{5}-\frac{5}{2} h(h+4 w) x^{4}+5 h w(4 h+7 w) x^{3}+\frac{5}{2} h\left(h^{3}-21 h w^{2}-20 w^{3}\right) x^{2} \\
-2 h w\left(5 h^{3}-25 h w^{2}-12 w^{3}\right) x-\frac{1}{2} h^{2}\left(h^{4}-\frac{35}{2} h^{2} w^{2}+24 w^{4}\right)
\end{gathered}
$$

And these can be deduced by the recurrent relation [4]

$$
2 E_{n ; h, w}(x)+\sum_{r=0}^{n-1}\binom{n}{r}(-w)^{n-r}(-x / w)_{n-r} E_{r ; h, w}(x)=2 h(-w)^{n}(-x / w)_{n}
$$

$n>0$. For relations with generalized Bernoulli, Euler, polynomials, it is easy to see that generalized Bernoulli, Euler, numbers can be derived by the formulas

$$
B_{n ; h, w}=B_{n ; h, w}(0), \quad E_{n ; h, w}=2^{n} E_{n ; h, w / 2}(1 / 2) .
$$

The first six values of $E_{n ; h, w}$ are given by

$$
\begin{gathered}
E_{0 ; h, w}=h, \quad E_{1, h, w}=h(1-h), \\
E_{2 ; h, w}=h(1-2 h)+h(h-1) w, \\
E_{3 ; h, w}=h(h-1)\left(2 h^{2}+2 h-1\right)+3 h(2 h-1) w+2 h(1-h) w^{2}, \\
E_{4 ; h, w}=h(2 h-1)\left(4 h^{2}+2 h-1\right)+6 h(1-h)\left(2 h^{2}+2 h-1\right) w+11 h(1-2 h) w^{2}+6 h(h-1) w^{3}, \\
E_{5 ; h, w}=4 h(1-h)\left(4 h^{4}+4 h^{3}-h^{2}-h-1\right)-10\left(8 h^{4}-4 h^{2}+1\right) w+35 h(h-1)\left(2 h^{2}+2 h-1\right) w^{2} \\
\end{gathered}
$$

Moreover, it will be useful to estimate also the expression

$$
F_{n ; h, w}=2^{n} E_{n ; h, w / 2}(h / 2) .
$$

And here, of course, we introduce the particular expressions for $n=0,1, \ldots, 5$ :

$$
\begin{gathered}
F_{0 ; h, w}=h, \quad F_{1 ; h, w}=0, \quad F_{2 ; h, w}=-h^{3} \\
F_{3 ; h, w}=3 h^{3} w, \quad F_{4 ; h, w}=h^{3}\left(5 h^{2}-11 w^{2}\right), \quad F_{5 ; h, w}=-50 h^{3} w\left(h^{2}-w^{2}\right) .
\end{gathered}
$$

The theory of generalized Bernoulli, Euler, numbers and polynomials was first investigated by R. Lagrange [1], L. Tanzi Cattabianchi [2] , and later extensively in the author's paper [4].

If $h=1, w=0$, the numbers $B_{n ; h, w}, E_{n ; h, w}$, and the polynomials $B_{n ; h, w}(x), E_{n ; h, w}(x)$, reduce to the ordinary Bernoulli, Euler, numbers and polynomials, generally defined by the generating expansions

$$
\begin{equation*}
\frac{t}{e^{t}-1}=\sum_{n=0}^{\infty} B_{n} \frac{t^{n}}{n!}, \quad|t|<2 \pi \tag{1.5}
\end{equation*}
$$

(1.6)

$$
\begin{array}{ll}
\frac{2 e^{t}}{e^{2 t}+1}=\sum_{n=0}^{\infty} E_{n} \frac{t^{n}}{n!}, & |t|<\pi / 2 \\
\frac{t e^{t x}}{e^{t}-1}=\sum_{n=0}^{\infty} B_{n}(x) \frac{t^{n}}{n!}, & |t|<2 \pi \\
\frac{2 e^{t x}}{e^{t}+1}=\sum_{n=0}^{\infty} E_{n}(x) \frac{t^{n}}{n!}, & |t|<\pi
\end{array}
$$

(1.7)
(1.8)

## 2. GENERALIZED STIRLING NUMBERS

The ordinary Stirling numbers of the first and second kind $s_{n, r}, S_{n, r}$, are defined by the initial values

$$
\begin{gathered}
s_{n, 1}=(-1)^{n-1}(n-1)!, \quad s_{n, n}=1 \\
S_{n, 1}=S_{n, n}=1
\end{gathered}
$$

and the recurrences

$$
\begin{array}{cc}
s_{n, r}=s_{n-1, r-1}-(n-1) s_{n-1, r}, & 1<r<n, \\
s_{n, r}=S_{n-1, r-1}+r S_{n-1, r}, & 1<r<n,
\end{array}
$$

with

$$
\begin{array}{ll}
s_{n, 0}=0, & s_{n, 0}=0, \\
s_{n, r}=0, & s_{n, r}=0, \text { provided } r>n .
\end{array}
$$

In our paper [3], they have been generalized with the coefficients $a_{\substack{n, r}}^{(u) \text { satisfying the recurrence }}$
with

$$
a_{n, r}^{(u)}=a_{n-1, r-1}^{(u)}-[n+r(u-1)-1] a_{n-1, r}^{(u)}, \quad 1<r<n,
$$

$$
\begin{array}{ll}
a_{n, 1}^{(u)}=(-1)^{n-1}(u)_{n-1}, & a_{n, n}^{(u)}=1, \\
a_{n, 0}^{(u)}=0, & a_{n, r}^{(u)}=0 \text { provided } r>n .
\end{array}
$$

And the particular expressions of $a_{n, r}^{(u)}$ for $n=1,2, \cdots, 5, r=1,2, \cdots, 5$, are

$$
\begin{gathered}
a_{1,1}^{(u)}=1, \quad a_{2,1}^{(u)}=-u, \quad a_{2,2}^{(u)}=1, \\
a_{3,1}^{(u)}=(u)_{2}, \quad a_{3,2}^{(u)}=-3 u, \quad a_{3,3}^{(u)}=1, \\
a_{4,1}^{(u)}=-(u)_{3}, \quad a_{4,2}^{(u)}=u(7 u+4), \quad a_{4,3}^{(u)}=-6 u, \quad a_{4,4}^{(u)}=1, \\
a_{5,1}^{(u)}=(u)_{4}, \quad a_{5,2}^{(u)}=-5 u(u+1)(3 u+2), \quad a_{5,3}^{(u)}=5 u(5 u+2), \\
a_{5,4}^{(u)}=-10 u, \quad a_{5,5}^{(u)}=1 .
\end{gathered}
$$

Our paper [3] presents an extensive treatment of the coefficients $a_{n, r}^{(u)}$, and it is interesting to note here that

$$
\begin{equation*}
a_{n, r}^{(u)}=\frac{(-1)^{n-r}}{(u-1)^{r} r!} \stackrel{r}{\Delta-1}_{\Delta}^{u(x)_{n} \text { provided } x=0, ~} \tag{2.1}
\end{equation*}
$$

where $\underset{v}{\Delta x}$ is the descending difference defined by the relation ${\underset{v}{~}}_{\Delta} x f(x)=f(x+v)-f(x)$,
(2.2)

$$
a_{n, r}^{(u)}=\frac{(-1)^{n}}{(u-1)^{r} r!} \sum_{k=1}^{r}(-1)^{k}\binom{r}{k}(k u-k)_{n}
$$

(2.3)

$$
\begin{aligned}
&(-1)^{n}(x)_{n}=\sum_{r=1}^{n} a_{n, r}^{(u)}(u-1)^{r}\left(\frac{x}{1-u}\right)_{r} \\
&(-1)^{n}(x)_{n}=\sum_{r=0}^{n} a_{n+1, r+1}^{(u)}(u-1)^{r}\left(\frac{x-u}{1-u}\right)_{r}, \\
& a_{n, r}^{(u)}=\sum_{k=r}^{n} s_{n, k} s_{k, r}(1-u)^{k-r},
\end{aligned}
$$

from which
(2.6)

$$
a_{n, r}^{(1)}=s_{n, r},
$$

$$
\begin{equation*}
\lim _{u \rightarrow \infty}\left[(1-u)^{r-n} a_{n, r}^{(u)}\right]=S_{n, r} ; \tag{2.7}
\end{equation*}
$$

(2.8)

$$
a_{n, r}^{(0)}=\left\{\begin{array}{l}
0, r<n, \\
1, r=n,
\end{array}\right.
$$

$$
a_{n, r}^{(2)}=(-1)^{n-r} \frac{n!}{r!}\binom{n-1}{r-1}
$$

$$
\begin{equation*}
a_{n, r}^{(-1)}=\frac{1}{2^{n-r}} \cdot \frac{n!}{r!}\binom{r}{n-r}, \quad r \geqslant n / 2, \tag{2.10}
\end{equation*}
$$

$$
\begin{equation*}
a_{n, r}^{(1 / 2)}=\frac{(-1)^{n-r}}{2^{2 n-2 r}} \cdot \frac{(2 n-r-1)!}{(n-1)!}\binom{n-1}{r-1} . \tag{2.11}
\end{equation*}
$$

For references and applications of the coefficients $a_{n, r}^{(u)}$ to the operators satisfying the condition of permutableness of the second order, see the more recent our paper [5].
3. PARTICULAR EXPANSIONS. $n^{\text {th }}$ DERIVATIVE OF

$$
y(t)=\frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}, \quad \text { where } \quad i^{2}=-1
$$

From (1.5), placing to the left member the term $1,-t / 2$ under the summation sign of the right member, we find, as it is well known, the expansion

$$
\begin{equation*}
\tan t=\sum_{n=0}^{\infty}(-1)^{n} 2^{2 n+2}\left(2^{2 n+2}-1\right) B_{2 n+2} \frac{t^{2 n+1}}{(2 n+2)!}, \quad|t|<\pi / 2 \tag{3.1}
\end{equation*}
$$

Now, an expansion analogous to (3.1) will be derived from (1.1), proceeding similarly. First of all we have

$$
\begin{aligned}
\frac{h t}{(1+w t)^{h / w}-1}-1+1 / 2(h-w) t & =\frac{1}{2\left[(1+w t)^{h / w}-1\right]}\left(2 h t+[(h-w) t-2]\left[(1+w t)^{h / w}-1\right]\right) \\
& =\sum_{r=2}^{\infty} B_{r ; h, w} \frac{t^{r}}{r!}
\end{aligned}
$$

At once, changing at first $t$ with $4 t, w$ with $w / 4$, and after $t$ with $2 t, w$ with $w / 2$, we obtain the expansions

$$
\begin{aligned}
\frac{1}{2\left[(1+w t)^{4 h / w}-1\right]}(8 h t+[4 h t & \left.-w t-2]\left[(1+w t)^{4 h / w}-1\right]\right) \\
& =\sum_{r=2}^{\infty} 2^{2 r} B_{r ; h, w / 4} \frac{t^{r}}{r!}
\end{aligned}
$$

$$
\frac{1}{2\left[(1+w t)^{2 h / w}-1\right]}\left(4 h t+[2 h t-w t-2]\left[(1+w t)^{2 h / w}-1\right]\right)=\sum_{r=2}^{\infty} 2^{r} B_{r ; h, w / 2} \frac{t^{r}}{r!}
$$

from which it follows that

$$
\begin{align*}
& \sum_{r=2}^{\infty}\left(2^{2 r} B_{r ; h, w / 4}-2^{r} B_{r ; h, w / 2}\right) \frac{t^{r}}{r!}=\frac{1}{2\left[(1+w t)^{4 h / w}-1\right]}\left(8 h t+[4 h t-w t-2]\left[(1+w t)^{4 h / w}-1\right]\right. \\
& \left.-4 h t\left[(1+w t)^{2 h / w}+1\right]-[2 h t-w t-2]\left[(1+w t)^{4 h / w}-1\right]\right) \\
& =\frac{h t\left[(1+w t)^{2 h / w}-1\right]^{2}}{(1+w t)^{4 h / w}-1}=\frac{h t\left[(1+w t)^{2 h / w}-1\right]}{(1+w t)^{2 h / w}+1} . \\
& \frac{h\left[(1+w t)^{2 h / w}-1\right]}{(1+w t)^{2 h / w}+1}=\sum_{n=0}^{\infty} 2^{n+2}\left(2^{n+2} B_{n+2 ; h, w / 4}-B_{n+2 ; h, w / 2}\right) \frac{t^{n+1}}{(n+2)!} . \tag{3.2}
\end{align*}
$$

Moreover by (1.4), replacing $x$ by $h / 2, w$ by $w / 2, t$ by $2 t$, we obtain the other expansion

$$
\begin{equation*}
\frac{2 h(1+w t)^{h / w}}{(1+w t)^{2 h / w}+1}=\sum_{n=0}^{\infty} 2^{n} E_{n ; h, w / 2}(h / 2) \frac{t^{n}}{n!} \tag{3.3}
\end{equation*}
$$

and since

$$
\begin{aligned}
\frac{h}{i} \cdot \frac{(1+w t)^{2 h / w}-1}{(1+w t)^{2 h / w}+1}+\frac{2 h(1+w t)^{h / w}}{(1+w t)^{2 h / w}+1} & =\frac{h\left[(1+w t)^{h / w}+i\right]^{2}}{i\left((1+w t)^{2 h / w}+1\right]} \\
& =\frac{h}{i} \cdot \frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}
\end{aligned}
$$

we deduce thus the interesting expansion
(3.4) $\frac{h}{i} \cdot \frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}=\sum_{n=0}^{\infty} \frac{2^{n+2}}{i}\left(2^{n+2} B_{n+2 ; h, w / 4}-B_{n+2 ; h, w / 2}\right) \frac{t^{n+1}}{(n+2)!}+\sum_{n=0}^{\infty} 2^{n} E_{n ; h, w / 2}(h / 2) \frac{t^{n}}{n!}$.

After this expansion and for the following, it will be to estimate the $n^{\text {th }}$ derivative of

$$
y(t)=\frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}
$$

Now we consider two continuous and derivable functions $y=f(u), u=\varphi(t)$, and the formula for derivatives of a composite function

$$
\frac{d^{n} y}{d t^{n}}=\left.\sum_{r=1}^{n} \frac{(-1)^{r}}{r!} \cdot \frac{d^{r} y}{d u^{r}} \sum_{k=1}^{r}(-1)^{k}\right|_{k} ^{r} u^{r-k} \frac{d^{n} u^{k}}{d t^{n}}, \quad n>0
$$

With the assumption

$$
y=f(u)=\frac{2 i}{u-i}+1, \quad u=\varphi(t)=(1+w t)^{h / w}
$$

we arrive at

$$
\begin{aligned}
\frac{d^{n} y}{d t^{n}} & =\sum_{r=1}^{n}\left[\frac{(-1)^{r}}{r!} \cdot \frac{2 i(-1)^{r} r!}{(u-i)^{r+1}} \cdot \sum_{k=1}^{r}(-1)^{k}\binom{r}{k} u^{r-k}(-w)^{n}\left(\frac{-k h}{w}\right)_{n} u^{k-n w / h}\right] \\
& =\frac{2 i(-w)^{n}}{(1+w t)^{n}} \sum_{r=1}^{n}\left[\frac{(1+w t)^{r h / w}}{\left((1+w t)^{h / w}-i\right]^{r+1}} \cdot \sum_{k=1}^{r}(-1)^{k}\binom{r}{k}\left(\frac{-k h}{w}\right)_{n}\right]
\end{aligned}
$$

[APR.

Whence, by (2.2), we deduce, for $n>0$,
(3.5)

$$
\begin{aligned}
& \frac{d^{n}}{d t^{n}} \cdot \frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-1} \\
& \quad=\frac{2 i w^{n}}{(1+w t)^{n}}{ }_{r=1}^{n} \frac{(-1)^{r} r!h^{r}}{w^{r}} \cdot a_{n, r}^{(1-h / w)} \cdot \frac{(1+w t)^{r h / w}}{\left[(1+w t)^{h / w}-i\right]^{r+1}}
\end{aligned}
$$

Successively, putting $t=0$, we have

$$
\left[\frac{d^{n}}{d t^{n}} \cdot \frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}\right]_{t=0}=2 i w^{n} \sum_{r=1}^{n} \frac{(-1)^{r} r!h^{r}}{w^{r}} \cdot a_{n, r}^{(1-h / w)} \cdot \frac{1}{(1-i)^{r+1}} .
$$

Moreover it is

$$
\begin{gathered}
\frac{1}{1-i}=\frac{1+i}{2}=\frac{1}{\sqrt{2}}(\cos \pi / 4+i \sin \pi / 4) \\
\frac{1}{(1-i)^{r+1}}=\frac{1}{2^{(r+1) / 2}}[\cos (r+1) \pi / 4+i \sin (r+1) \pi / 4] \\
=\frac{1}{2^{(r+1) / 2}}[\cos (r-1) \pi / 4+i \sin (r-1) \pi / 4]
\end{gathered}
$$

and, however, it follows that ( $n>0$ )
(3.6) $\quad\left[\frac{d^{n}}{d t^{n}} \cdot \frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}\right]_{t=0}$

$$
=w^{n} \sum_{r=1}^{n} \frac{(-1)^{r-1} r!h^{r}}{2^{(r-1) / 2} w^{r}} \cdot a_{n, r}^{(1-h / w)}[\cos (r-1 / \pi / 4+i \sin (r-1) \pi / 4]
$$

## 4. FORMULAS FOR THE GENERALIZED BERNOULLI, EULER, STIRLING NUMBERS

We now, by (3.6), deduce the expansion of the function $y(t)$ into a series of powers of $t$,

$$
\begin{aligned}
y(t) & =\sum_{n=0}^{\infty}\left[\frac{d^{n} y(t)}{d t^{n}}\right]_{t=0} \frac{t^{n}}{n!} \\
& =i+\sum_{n=1}^{\infty} \frac{(w t)^{n}}{n!} \sum_{r=1}^{n} \frac{(-1)^{r-1} r!h^{r}}{2^{(r-1) / 2} w^{r}} a_{n, r}^{(1-h / w)}[\cos (r-1) \pi / 4+i \sin (r-1) \pi / 4]
\end{aligned}
$$

Hence, comparing with (3.4), we obtain the expansion

$$
\begin{aligned}
& \sum_{n=1}^{\infty} 2^{n+1}\left(2^{n+1} B_{n+1 ; h, w / 4}-B_{n+1 ; h, w / 2}\right) \frac{t^{n}}{(n+1)!} \\
& +i E_{0 ; h, w / 2}(h / 2)+i \cdot \sum_{n=1}^{\infty} 2^{n} E_{n ; h, w / 2}(h / 2) \frac{t^{n}}{n!}=i h \\
& +\sum_{n=1}^{\infty} \frac{(w t)^{n}}{n!} \sum_{r=1}^{n} \frac{(-1)^{r-1} r!h^{r+1}}{2^{(r-1) / 2} w^{r}} a_{n, r}^{(1-h / w)}[\cos (r-1) \pi / 4+i \sin (r-1) \pi / 4]
\end{aligned}
$$

from which, separating real and imaginary parts, and equating the coefficients of $t^{n}$ on both sides, we establish the interesting formulas

$$
\begin{align*}
& 2^{n+1}\left(2^{n+1} B_{n+1 ; h, w / 4}-B_{n+1 ; h, w / 2}\right)  \tag{4.1}\\
& \quad=(n+1) \sum_{r=1}^{n} \frac{(-1)^{r-1} r!h^{r+1} w^{n-r}}{2^{(r-1) / 2}} a_{n, r}^{(1-h / w)} \cos (r-1) \pi / 4
\end{align*}
$$

$$
\begin{align*}
& 2^{n} E_{n ; h, w / 2}(h / 2)  \tag{4.2}\\
& \quad=\sum_{r=1}^{n} \frac{(-1)^{r-1} r!h^{r+1} w^{n-r}}{2^{(r-1) / 2}} a_{n, r}^{(1-h / w)} \sin (r-1 / \pi / r
\end{align*}
$$

both for $n>0$. They realize the principal objective of the present paper.

## 5. PARTICULAR FORMULAS

In this section we shall indicate some special cases of (4.1), (4.2).
(a) If $h=1, w=0$, the generalized numbers $B_{n ; h, w}, E_{n ; h, w}$, reduce to the ordinary Bernoulli, Euler, numbers, while

$$
w \lim _{\rightarrow \infty}\left[w^{n-r} a_{n, r}^{(1-1 / w)}\right]=S_{n, r}
$$

Moreover, it is $B_{2 n+1}=0$ for $n>0, E_{2 n+1}=0$. Consequently, by (4.1), (4.2), we deduce the formulas

$$
\begin{equation*}
\frac{2^{2 n+1}\left(2^{2 n+2}-1\right)}{n+1} B_{2 n+2}=\sum_{r=1}^{2 n+1} \frac{(-1)^{r-1} r!}{2^{(r-1) / 2}} S_{2 n+1, r} \cos (r-1) \pi / 4, \tag{5.1}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{r=1}^{2 n+1} \frac{(-1)^{r-1} r!}{2^{(r-1) / 2}} S_{2 n+1, r} \sin (r-1) \pi / 4=0 \tag{5.2}
\end{equation*}
$$

$$
\begin{equation*}
E_{2 n}=\sum_{r=2}^{2 n} \frac{(-1)^{r-1} r!}{2^{(r-1) / 2}} S_{2 n, r} \sin (r-1) \pi / 4, \quad n>0 \tag{5.4}
\end{equation*}
$$

Equations (5.1) and (5.4) are two additional formulas concerning ordinary Bernoulli, Euler, Stirling numbers.
(b) If $w=h$, we have (2.8)

$$
a_{n, r}^{(1-h / w)}=\left\{\begin{array}{l}
0, r<n, \\
1, r=n,
\end{array}\right.
$$

therefore, (4.1) reduces to

$$
\begin{equation*}
\sum_{r=1}^{2 n} \frac{(-1)^{r-1} r!}{2^{(r-1) / 2}} s_{2 n, r} \cos (r-1) \pi / 4=0, \quad n>0 \tag{5.3}
\end{equation*}
$$

$$
\begin{equation*}
2^{n+1}\left(2^{n+1} B_{n+1 ; h, h / 4}-B_{n+1 ; h, h / 2}\right)=\frac{(n+1)!(-h)^{n+1}}{2^{(n-1) / 2}} \cos (n-1) \pi / 4 \tag{5.5}
\end{equation*}
$$

Moreover, by the recurrent relation for $B_{n ; h, h / 2}$, it follows that

$$
B_{n ; h, h / 2}=-\frac{n h}{4} B_{n-1 ; h, h / 2}
$$

from which
(5.6)

$$
B_{n ; h, h / 2}=\frac{n!(-h)^{n}}{2^{2 n}}
$$

Consequently, by (5.5) we deduce

$$
\begin{equation*}
B_{n ; h, h / 4}=\frac{n!(-h)^{n}}{2^{3 n}}\left[1+2^{(n+2) / 2} \sin (n \pi / 4)\right] . \tag{5.7}
\end{equation*}
$$

This formula can be derived by the recurrent relation

$$
\begin{aligned}
B_{n ; h, h / 4} & +\frac{3 n h}{8} B_{n-1 ; h, h / 4}+\frac{n(n-1) h^{2}}{16} B_{n-2 ; h, h / 4} \\
& +\frac{n(n-1)(n-2) h^{3}}{256} B_{n-3 ; h, h / 4}=0, \quad n>0
\end{aligned}
$$

easily transformable in other forms to constant coefficients.
(c) If $w=-h$, we have (2.9)

$$
a_{n, r}^{(1-h / w)}=(-1)^{n-r} \frac{n!}{r!}\binom{n-1}{r-1},
$$

and (4.1) reduces to
(5.8)

$$
\begin{aligned}
& 2^{n+1}\left(2^{n+1} B_{n+1 ; h,-h / 4}-B_{n+1 ; h,-h / 2}\right) \\
& \quad=(n+1)!h^{n+1} \sum_{r=1}^{n} \frac{(-1)^{r-1}\binom{n-1}{r-1}}{2^{(r-1) / 2}} \cos (r-1) \pi / 4, \quad n>0 .
\end{aligned}
$$

Moreover, it is [4]

$$
B_{n ; h,-w}=(-1)^{n} B_{n ; h, w},
$$

and comparing (5.5) with (5.8) we have the identity, for $n>0$,
(5.9)

$$
\sum_{r=1}^{n}(-1)^{r-1}\binom{n-1}{r-1} 2^{(n-r) / 2} \cos (r-1) \pi / 4=\cos (n-1) \pi / 4
$$

Putting into (4.2) at first $w=h$ and after $w=-h$, and remembering that [4]

$$
E_{n ; h,-w}(h / 2)=(-1)^{n} E_{n ; h, w}(h / 2),
$$

we prove the identity
(5.10)

$$
\sum_{r=2}^{n}(-1)^{r}\binom{n-1}{r-1} 2^{(n-r) / 2} \sin (r-1) \pi / 4=\sin (n-1) \pi / 4,
$$

for $n>1$.
(d) If $w=h / 2$, we have (2.10)

$$
a_{n, r}^{(1-h / w)}=\frac{1}{2^{n-r}} \cdot \frac{n!}{r!}\binom{r}{n-r}, \quad r \geqslant n / 2,
$$

and (4.1) becomes
(5.11)

$$
\begin{aligned}
& 2^{n+1}\left(2^{n+1} B_{n+1 ; h, h / 8}-B_{n+1 ; h, h / 4}\right) \\
& =\frac{(n+1)!h^{n+1}}{2^{2 n}} \sum_{r \geqslant n / 2}^{n}(-1)^{r-1}\binom{r}{n-r} 2^{(3 r+1) / 2} \cos (r-1) \pi / 4 .
\end{aligned}
$$

Consequently, returning to (5.7), it follows
(5.12)

$$
\begin{aligned}
2^{2 n+2} B_{n+1 ; h, h / 8}= & \frac{(n+1)!h^{n+1}}{2^{2 n+2}}\left((-1)^{n+1}\left[1+2^{(n+3) / 2} \sin (n+1) \pi / 4\right]\right. \\
& \left.+\sum_{r \geqslant n / 2}^{n}(-1)^{r-1}\binom{r}{n-r} 2^{(3 r+5) / 2} \cos (r-1) \pi / 4\right)
\end{aligned}
$$

(e) If $w=2 h$, we have (2.11)

$$
a_{n, r}^{(1-h / w)}=\frac{(-1)^{n-r}}{2^{2 n-2 r}} \cdot \frac{(2 n-r-1)!}{(n-1)!} \cdot\binom{n-1}{r-1}
$$

Then (4.1), by (5.6) and the relation

$$
B_{n ; h, h}=0 \text { for } n>0
$$

reduces to the identity, for $n>0$,

$$
\begin{equation*}
\sum_{r=1}^{n} r!(2 n-r-1)!\binom{n-1}{r-1} 2^{(r+1) / 2} \cos (r-1) \pi / 4=2^{n}(n-1)!n! \tag{5.13}
\end{equation*}
$$

6. A DERIVATIVE FORMULA

Putting

$$
\begin{gathered}
P_{0 ; h, w}=i, \\
P_{n ; h, w}=\frac{2^{n+1}}{(n+1) h} \cdot\left(2^{n+1} B_{n+1 ; h, w / 4}-B_{n+1 ; h, w / 2}\right)+\frac{i 2^{n}}{h} E_{n ; h, w / 2}(h / 2), \quad n>0,
\end{gathered}
$$

the expansion (3.4) can be written in the form

$$
\begin{equation*}
F(t)=\frac{(1+w t)^{h / w}+i}{(1+w t)^{h / w}-i}=\sum_{n=0}^{\infty} P_{n ; h, w} \frac{t^{n}}{n!} . \tag{6.1}
\end{equation*}
$$

Moreover, it is not difficult to show that the function $F(t)$ satisfies the functional equation

$$
\begin{equation*}
\frac{2(1+w t)}{h} \cdot \frac{d F(t)}{d t}=1-F^{2}(t) \tag{6.2}
\end{equation*}
$$

from which the recurrent relation follows,
(6.3)

$$
2 P_{n+1 ; h, w}+2 n w P_{n ; h, w}+h \sum_{r=0}^{n}\binom{n}{r} P_{r ; h, w} P_{n-r ; h, w}=0, \quad n>0
$$

If $h=1, w=0$, we have the interesting connections with the ordinary Bernoulli, Euler numbers

$$
\begin{equation*}
P_{2 n ; 1,0}=i E_{2 n}, \quad P_{2 n+1 ; 1,0}=\frac{2^{2 n+1}\left(2^{2 n+2}-1\right)}{n+1} B_{2 n+2}, \tag{6.4}
\end{equation*}
$$

and from (6.3) we obtain, in conclusion, the special formulas
(6.5)

$$
\begin{gathered}
\sum_{r=0}^{n-1}\binom{2 n}{2 r+1} \frac{2^{2 n}\left(2^{2 r+2}-1\right)\left(2^{2 n-2 r}-1\right)}{(r+1)(n-r)} B_{2 r+2} B_{2 n-2 r}-\sum_{r=0}^{n}\binom{2 n}{2 r} E_{2 r} E_{2 n-2 r} \\
+\frac{2^{2 n+2}\left(2^{2 n+2}-1\right)}{n+1} B_{2 n+2}=0, \quad n>0
\end{gathered}
$$

$$
\begin{equation*}
E_{2 n+2}=\sum_{r=0}^{n}\binom{2 n+1}{2 r+1} \frac{2^{2 r+1}\left(1-2^{2 r+2}\right)}{r+1} B_{2 n+2} E_{2 n-2 r}, \quad n \geqslant 0 . \tag{6.6}
\end{equation*}
$$

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* 


## EXPANSION

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As every science, save one, is modified and cast aside, While mathematics only is built upon and grows, So, too, my life's whole whims and whimsies pied See their demise, while my regard for you goes
On. Like the Sieve of Eratosthenes, you sift My drifting days and sort the prime.

As determinants reflect a change, I mirror-image you And, palindromic, backward-forward go, from autumn into spring.
Approaching the limit of joy, you bring a rate of change
Which grows in my heart proportionate to you. Your range
Is my domain. By you, my worthiness a proof shows,
As solid as geometry, as crystalline as snows,
As coming-now as spring.
I am subset of you.
Happily, with you no negative numbers can deride
My existence, that foolish enterprise of sensibility;
Instead, a proper fraction of civility
Is mine. By power of example, exponent of grace,
You multiply and lace my life with life. The race
Is mine! Cantor-like you lift
Me to infinity sublime
And grant me a number prime.

# A COMBINATORIAL PROBLEM INVOLVING RECURSIVE SEQUENCES AND TRIDIAGONAL MATRICES 

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In [2], C. A. Church, Jr., shows that the total number of $k$-combinations of the first $n$ natural numbers such that no two elements $i$ and $i+2$ ippear together in the same selection is $F_{m+2}^{2}$ if $n=2 m$ and $F_{m+2} F_{m+3}$ if $n=2 m+1$. Furthermore he shows, if the $k$-combinations are arranged in a circle, so that 1 and $n$ are consecutive with no two elements $i$ and $i+2$ àpearing together in the same selection, then the number of $k$-combinations of the first $n$ natural numbers is $L_{m}^{2}$ if $n=2 m$ and $L_{m} L_{m+1}$ if $n=2 m+1$.
Letting $\left\{U_{n}\right\}_{n=0}^{\infty}$ be the sequence of $k$-combinations of the first $n$ natural numbers such that no two elements $i$ and $i+2$ appear together in the same selection, we have

$$
U_{0}=1, \quad U_{1}=2, \quad U_{2}=4, \quad U_{3}=6, \quad U_{4}=9, \quad U_{5}=15, \quad U_{6}=25, \cdots
$$

By applying standard techniques, it is easy to show that the generating function for $\left\{U_{n}\right\}_{n=0}^{\infty}$ is

$$
\begin{equation*}
\sum_{n=0}^{\infty} U_{n} x^{n}=\sum_{m=0}^{\infty}\left(F_{m+2}^{2}+F_{m+2} F_{m+3} x\right) x^{2 m}=\frac{1+2 x+2 x^{2}+2 x^{3}-x^{4}-x^{5}}{1-2 x^{2}-2 x^{4}+x^{6}} \tag{1}
\end{equation*}
$$

Although this rational function may be very interesting in its own right, it is also surprising to observe what happens if we replace $m$ by $m-1$, multiply by $x^{2}$, and then start the summation from $m=0$. Doing this we have

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left(F_{m+1}^{2}+F_{m+1} F_{m+2} x\right) x^{2 m}=\frac{1+x-x^{2}}{1-2 x^{2}-2 x^{4}+x^{6}}=\frac{1}{\left(1-x-x^{2}\right)\left(1+x^{2}\right)} \tag{2}
\end{equation*}
$$

Incidentally, it can be shown that

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left(F_{m+1}^{2} x+F_{m} F_{m+1}\right) x^{2 m}=\frac{x}{\left(1-x-x^{2}\right)\left(1+x^{2}\right)} \tag{3}
\end{equation*}
$$

The results of (2) and (3) can be generalized in a very natural way to the sequence of Fibonacci polynomials defined recursively by

$$
f_{0}(\lambda)=0, \quad f_{1}(\lambda)=1, \quad f_{n+1}(\lambda)=\lambda f_{n}(\lambda)+f_{n-1}(\lambda), \quad n \geqslant 1 .
$$

Using the well known fact that

$$
f_{n}(\lambda)=\frac{a^{n}-\beta^{n}}{a-\beta}
$$

where

$$
a=\frac{\lambda+\sqrt{\lambda^{2}+4}}{2} \quad \text { and } \quad \beta=\frac{\lambda-\sqrt{\lambda^{2}+4}}{2}
$$

together with the techniques found in part VI of [6] , we have

$$
\begin{equation*}
\sum_{m=0}^{\infty} f_{m+2}^{2}(\lambda) x^{2 m}=\frac{\lambda^{2}+\left(\lambda^{2}+1\right) x^{2}-x^{4}}{\left.\left[1-\lambda^{2}+2\right) x^{2}+x^{4}\right]\left(1+x^{2}\right)} \tag{4}
\end{equation*}
$$

and
(5)

$$
\sum_{m=0}^{\infty} f_{m+2}(\lambda) f_{m+3}(\lambda) x^{2 m+1}=\frac{\lambda^{3}+\lambda / x+\lambda^{3}+\lambda / x^{3}-\lambda x^{5}}{\left.\left[1-\lambda^{2}+2\right) x^{2}+x^{4}\right]\left(1+x^{2}\right)} .
$$

Adding (4) and (5), we obtain
(6) $\left.\sum_{m=0}^{\infty}\left[f_{m+2}^{2}(\lambda)+f_{m+2} \lambda \lambda\right) f_{m+3}(\lambda) x\right] x^{2 m}=\frac{\left.\lambda^{2}+\lambda^{3}+\lambda / x+\lambda^{2}+1\right) x^{2}+\lambda^{3}+\lambda / x^{3}-x^{4}-\lambda x^{5}}{\left[1-\left(\lambda^{2}+2\right) x^{2}+x^{4}\right]\left(1+x^{2}\right)}$

Replacing $m$ by $m-1$, multiplying by $x^{2}$, and then starting the summation from $m=0$, we have
(7) $\sum_{m=0}^{\infty}\left[f_{m+1}^{2}(\lambda)+f_{m+1}(\lambda) f_{m+2} \lambda \lambda x\right] x^{2 m}=\frac{1+\lambda x-x^{2}}{\left.\left[1-\lambda^{2}+2\right) x^{2}+x^{4}\right]\left(1+x^{2}\right)}=\frac{1}{\left(1-\lambda x-x^{2}\right)\left(1+x^{2}\right)}$.

Minor manipulations of (4) and (5) will also yield
(8)

$$
\left.\left.\sum_{m=0}^{\infty}\left[f_{m \neq 1}^{2} \lambda\right) x+f_{m}(\lambda) f_{m+1} \lambda\right)\right] x^{2 m}=\frac{x}{\left(1-\lambda x-x^{2}\right)\left(1+x^{2}\right)} .
$$

As should be the case, (7) and (8) are (2) and (3) when $\lambda=1$.
Another generalization of (2) and (3) occurs when we examine the sequence of Pellian Polynomials defined recursively by

$$
P_{0}(\lambda)=0, \quad P_{1}(\lambda)=1, \quad P_{n+1}(\lambda)=(1-\lambda) P_{n}(\lambda)-\lambda P_{n-1}(\lambda), \quad n \geqslant 2 .
$$

Since
where

$$
P_{n}(\lambda)=\frac{a^{n}-\beta^{n}}{a-\beta}
$$

$$
a=\frac{(1-\lambda)+\sqrt{\lambda^{2}-6 \lambda+1}}{2} \text { and } \beta=\frac{(1-\lambda)-\sqrt{\lambda^{2}-6 \lambda+1}}{2} \text {. }
$$

we can use the techniques found in part VI of [6] together with arguments used to develop (7) and (8) in order to show that
(9)

$$
\left.\sum_{m=0}^{\infty}\left[P_{m+1}^{2}(\lambda)+P_{m+1}(\lambda) P_{m+2} \lambda\right) x\right] x^{2 m}=\frac{1}{\left[1-(1-\lambda) x+\lambda x^{2}\left[\left(1-\lambda x^{2}\right)\right.\right.}
$$

and
(10)

$$
\left.\sum_{m=0}^{\infty}\left[P_{m+1}^{2} \lambda \lambda x+P_{m} \lambda\right) P_{m+1}(\lambda)\right] x^{2 m}=\frac{x}{\left[1-(1-\lambda) x+\lambda x^{2}\right]\left(1-\lambda x^{2}\right)}
$$

When $\lambda=-1$ we obtain the sequence of Pellian numbers.
Our final generalization of (2) and (3) is obtained by returning to subsets of a given set. Let $S_{n}=\{1,2,3, \ldots, n\}$ and $P\left(S_{n}\right)$ be the power set of $S_{n}$. Let $T_{n}$ be the number of elements of $P\left(S_{n}\right)$ with no two elements congruent modulo two. The first nine terms of $\left\{T_{n}\right\}_{n=0}^{\infty}$ with $T_{0}=1$ are

$$
1,2,4,6,9,12,16,20,25, \cdots
$$

To develop a formula for $\left\{T_{n}\right\}_{n=0}^{\infty}$ we first note that any element of $P\left(S_{n}\right)$ of order three or more is rejected. Furthermore there is one element of order zero and there are $n$ elements of order one. The number of elements of order two is

$$
\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)
$$

if $n$ is odd and $n^{2} / 4$ if $n$ is even, any even integer with any odd integer. Hence,
and

$$
T_{n}=\frac{n^{2}-1}{4}+n+1=\frac{(n+1)(n+3)}{4}, \quad n \text { odd }
$$

$$
T_{n}=\frac{n^{2}}{4}+n+1=\left(\frac{n+2}{2}\right)^{2}, \quad n \text { even } .
$$

The generating function for $\left\{T_{n}\right\}_{n=0}^{\infty}$ is

$$
\begin{equation*}
\sum_{n=0}^{\infty} T_{n} x^{n}=\sum_{m=0}^{\infty}\left[(m+1)^{2}+(m+1)(m+2) x\right] x^{2 m}=\frac{x^{2}+1}{\left(1-x^{2}\right)^{3}}+\frac{2 x}{\left(1-x^{2}\right)^{3}}=\frac{1}{(1-x)^{2}\left(1-x^{2}\right)} \tag{11}
\end{equation*}
$$

while

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left[(m+1)^{2} x+m(m+1)\right] x^{2 m}=\frac{x}{(1-x)^{2}\left(1-x^{2}\right)} \tag{12}
\end{equation*}
$$

The authors also found the generating function for the sequence of $k$-combinations of the first $n$ natural numbers arranged in a circle, so that 1 and $n$ are consecutive, with no two elements $i$ and $i+2$ appearing together in the same selection. Letting $\left\{V_{n}\right\}_{n=1}^{\infty}$ be the stated sequence, we see that

$$
v_{1}=1, \quad v_{2}=3, \quad v_{3}=9, \quad V_{4}=12, \quad v_{5}=16, \quad v_{6}=28, \quad v_{7}=49, \quad \cdots,
$$

and

$$
\begin{align*}
\sum_{n=1}^{\infty} V_{n} x^{n}=\sum_{m=1}^{\infty}\left(L_{m}^{2}+L_{m} L_{m+1} x\right) x^{2 m} & =\frac{\left(1+7 x^{2}-4 x^{4}\right) x^{2}}{1-2 x^{2}-2 x^{4}+x^{6}}+\frac{\left(3+6 x^{2}-2 x^{4}\right) x^{3}}{1-2 x^{2}-2 x^{4}+x^{6}}  \tag{13}\\
& =\frac{\left(1+3 x+7 x^{2}+6 x^{3}-4 x^{4}-2 x^{5}\right) x^{2}}{1-2 x^{2}-2 x^{4}+x^{6}}
\end{align*}
$$

Replacing $m$ by $m+1$ and summing from $m=0$, in order to obtain the same form as (2), we have

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left(L_{m+1}^{2}+L_{m+1} L_{m+2} x\right) x^{2 m}=\frac{1+3 x+7 x^{2}+6 x^{3}-4 x^{4}-2 x^{5}}{1-2 x^{2}-2 x^{4}+x^{6}} \tag{14}
\end{equation*}
$$

which does not simplify and is therefore not as appealing as the result in (2).
If we replace $m$ by $m-1$ in (13), multiply by $x^{2}$, and then sum from $m=0$ we have

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left(L_{m-1}^{2}+L_{m-1} L_{m} x\right) x^{2 m}=\frac{1-2 x+2 x^{2}+6 x^{3}-9 x^{4}+3 x^{5}}{1-2 x^{2}-2 x^{4}+x^{6}}=\frac{1-3 x+6 x^{2}-3 x^{3}}{\left(1-x-x^{2}\right)\left(1+x^{2}\right)} \tag{15}
\end{equation*}
$$

which does not simplify further and is not as appealing as equation (2).
The authors tried several other substitutions and manipulations of (13) in order to obtain a rational function whose numerator is a one or an $x$. However, they were not successful.
We now turn to the major result of this article which is the establishment of a relationship between (2), (7), (9), (11) and a sequence of determinants of tridiagonal matrices defined by the rule $P_{n}(a, b, c)=\left(a_{i j}\right)$, where

$$
a_{i j}=a \text { if } i=j, \quad a_{i j}=b \text { if } i=j-2, \quad a_{i j}=c \text { if } i=j+2, \text { and } a_{i j}=0 \text { otherwise. }
$$

The first eleven values of $P_{n}(a, b, c)$ with $P_{0}(a, b, c)$ defined to be one are

$$
\begin{gathered}
P_{0}(a, b, c)=1 \\
P_{1}(a, b, c)=a \\
P_{2}(a, b, c)=a^{2} \\
P_{3}(a, b, c)=a^{3}-a b c \\
P_{4}(a, b, c)=\left(a^{2}-b c\right)^{2} \\
P_{5}(a, b, c)=a^{5}-3 a^{3} b c+2 a b^{2} c^{2} \\
P_{6}(a, b, c)=\left(a^{3}-2 a b c\right)^{2}
\end{gathered}
$$

$$
\begin{gathered}
P_{7}(a, b, c)=a^{7}-5 a^{5} b c+7 a^{3} b^{2} c^{2}-2 a b^{3} c^{3} \\
P_{8}(a, b, c)=\left(a^{4}-3 a^{2} b c+b^{2} c^{2}\right)^{2} \\
P_{9}(a, b, c)=a^{9}-7 a^{7} b c+16 a^{5} b^{2} c^{2}-13 a^{3} b^{3} c^{3}+3 a b^{4} c^{4} \\
P_{10}(a, b, c)=\left(a^{5}-4 a^{3} b c+3 a b^{2} c^{2}\right)^{2} .
\end{gathered}
$$

It would seem at first that there is no order, except for the perfect squares, to the sequence $\left\{p_{n}(a, b, c)\right\}_{n=0}^{\infty}$. However if one were to actually evaluate the determinants he would see a nice pattern developing in the way he finds those values. In fact it can be shown by induction that

$$
\begin{equation*}
P_{n}(a, b, c)=a P_{n-1}(a, b, c)-a b c P_{n-3}(a, b, c)+b^{2} c^{2} P_{n-4} . \tag{16}
\end{equation*}
$$

The generating function for $\left\{P_{n}(a, b, c)\right\}_{n=0}^{\infty}$ is found to be

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n}(a, b, c) x^{n}=\frac{1}{\left(1-b c x^{2}\right)\left(1-a x+b c x^{2}\right)} \tag{17}
\end{equation*}
$$

When $b c=-1$ and $a=1$, (17) becomes (2). When $b c=-1$ and $a=\lambda,(17)$ becomes (7). When $b c=\lambda$ and $a=(1-\lambda)$, (17) becomes (9). When $b c=1$ and $a=2$, (17) becomes (11).

The authors were unsuccessful in trying to find a sequence of determinants whose generating function was related to (15). Similarly we had no success in trying to find such a sequence of determinants for the last two examples which we shall now discuss.
Our first example deals with a generalization of the problem of C. A. Church which can be found in [1]. Using $S_{n}$ and $P\left(S_{n}\right)$ as previously defined, we wish to determine the number of subsets of $S_{n}$ for which $3 n, 3 n+3$ or $3 n+1$, $3 n+4$ or $3 n+2,3 n+5$ are not in the same subset. Letting $U_{n}$ be the number of acceptable subsets for a given $n$, it is easy to illustrate that
$U_{0}=1, \quad U_{1}=2, \quad U_{2}=4, \quad U_{3}=8, \quad U_{4}=12, \quad U_{5}=18, \quad U_{6}=27, \quad U_{7}=45, \quad U_{8}=75, \quad U_{9}=125, \ldots$
By applying the results of [3] , it can be shown that

$$
U_{n}=\left\{\begin{array}{lll}
F_{k+2}^{3}, & \text { if } & n=3 k  \tag{18}\\
F_{k+2}^{2} F_{k+3}, & \text { if } & n=3 k+1 \\
F_{k+2} F_{k+3}^{2}, & \text { if } & n=3 k+2
\end{array}\right.
$$

where $F_{k}$ is the $k^{\text {th }}$ Fibonacci number. Hence, the generating function for $\left\{U_{n}\right\}_{n=0}^{\infty}$ is

$$
\begin{align*}
\sum_{n=0}^{\infty} U_{n} x^{n} & =\sum_{m=1}^{\infty}\left[F_{m+1}^{3}+F_{m+1}^{2} F_{m+2} x+F_{m+1} F_{m+2}^{2} x^{2}\right] x^{3 m-3}  \tag{19}\\
& =\frac{1+2 x+4 x^{2}+5 x^{3}+6 x^{4}+6 x^{5}-3 x^{6}-3 x^{7}-3 x^{8}-x^{9}-x^{10}-x^{11}}{\left(x^{6}-x^{3}-1\right)\left(x^{6}+4 x^{3}-1\right)}
\end{align*}
$$

Summing (19) from $m=0$ and multiplying both sides by $x^{3}$ we have

$$
\begin{align*}
\sum_{m=0}^{\infty}\left[F_{m+1}^{3}+F_{m+1}^{2} F_{m+2} x+F_{m+1} F_{m+2}^{2} x^{2}\right] x^{3 m} & =\frac{1+x+x^{2}-2 x^{3}-x^{4}+x^{5}-x^{6}}{\left(x^{6}-x^{3}-1\right)\left(x^{6}+4 x^{3}-1\right)}  \tag{20}\\
& =\frac{\left(1-x^{2}\right)\left(1+x+2 x^{2}-x^{3}+x^{4}\right)}{\left(x^{6}-x^{3}-1\right)\left(x^{6}+4 x^{3}-1\right)}
\end{align*}
$$

Our final example deals with counting the number of elements of $P\left(S_{n}\right)$ which have no two members of the same subset congruent modulo three. Denoting the sequence by $\left\{V_{n}\right\}_{n=0}^{\infty}$, it is easy to illustrate that
$V_{0}=1, V_{1}=2, V_{2}=4, V_{3}=8, V_{4}=12, V_{5}=18, V_{6}=27, V_{7}=36, V_{8}=48, V_{9}=64, \cdots$.
In order to determine a formula for $V_{n}$, we first note that all elements of $P\left(S_{n}\right)$ with four or more members are
rejected in the counting process. Furthermore there is always one element of $P\left(S_{n}\right)$ with no members and there are $n$ elements of $P\left(S_{n}\right)$ with one member. Let us now assume $n=3 k+1$ and arrange the numbers from 1 to $n$ as follows

| 1 | 2 | 3 |
| :---: | :---: | :---: |
| 4 | 5 | 6 |
| 7 | 8 | 9 |
| $\ldots$ | $\ldots$ | $\ldots$ |
| $3 k-2$ | $3 k-1$ | $3 k$ |
| $3 k+1$ |  |  |

An acceptable element of $P\left(S_{n}\right)$ of order two is found by taking any element of the first column with any element in columns two or three and any element of the second column with any element of the third column. Hence, the number of valid elements of $P\left(S_{n}\right)$ of order two is

$$
2 k(k+1)+k^{2}=3 k^{2}+2 k
$$

provided $n=3 k+1$. When $n=3 k+2$ there are $3 k^{2}+4 k+1$ allowable sets of order two while the number of such sets if $n=3 k$ is $3 k^{2}$.
The number of subsets of $S_{n}$ of order three is

$$
\binom{3 k+1}{3}
$$

provided $n=3 k+1$. A subset of $S_{n}$ of order three is not counted.if it contains two elements of one column and one element from either of the two remaining columns or if it contains three elements from the same column. Hence the number of valid sets of order three if $n=3 k+1$ is

$$
\binom{3 k+1}{3}-2 k\binom{k+1}{2}-(2 k+1)\binom{k}{2}-(2 k+1)\binom{k}{2}-\binom{k+1}{3}-2\binom{k}{3}=k^{3}+k^{2} .
$$

When $n=3 k+2$, the number of valid sets of order three is

$$
\binom{3 k+2}{3}-2\binom{k+1}{2}(2 k+1)-2(k+1)\binom{k}{2}-2\binom{k+1}{3}-\binom{k}{3}=k(k+1)^{2} .
$$

When $n=3 k$, the number of valid sets of order three is

$$
\binom{3 k}{3}-6 k\binom{k}{2}-3\binom{k}{3}=k^{3} .
$$

Combining the results above, we conclude that

$$
\begin{align*}
& \text { ve, we conclude that }  \tag{21}\\
& \qquad V_{n}=\left\{\begin{array}{lll}
k^{3}+3 k^{2}+3 k+1=(k+1)^{3}, & n=3 k \\
k^{3}+4 k^{2}+3 k+2=(k+1)^{2}(k+2), & & n=3 k+1 \\
k^{3}+5 k^{2}+8 k+4=(k+1)(k+2)^{2}, & & n=3 k+2 .
\end{array}\right.
\end{align*}
$$

Hence, the generating function for $\left\{V_{n}\right\}_{n=0}^{\infty}$ is

$$
\begin{align*}
\sum_{n=0}^{\infty} V_{n} x^{n} & =\sum_{m=0}^{\infty}\left[(m+1)^{3}+(m+1)^{2}(m+2) x+(m+1)(m+2)^{2} x^{2}\right] x^{3 m}  \tag{22}\\
& =\frac{x^{2}+1}{\left(x^{2}+x+1\right)^{2}(x-1)^{4}}
\end{align*}
$$

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*** **

ON THE FORMULA $\pi=2 \Sigma \operatorname{arcot} f 2 k+1$

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While questing the $n+1^{s t}$ digit of $\pi$, With series by Taylor, MacLauren, et al;
I tried the arcotan of integers high,
While old Leonardo de Pisa did call.
Old friends are a joy and, at times, a surprise
When they serependiciously drop by to chat.
They lighten our labors and open our eyes.
"Eureka!" quoth I. "Now, how about that!"
For what to my wondering eyes should appear, Intermix't with the spurious inverse contans,
Were eight Fibonacci terms standing right here,
Waiting and patiently holding their hands.
The even term's arcotangent's easily seen to equal the sum of the next pair in line. Now start back with $\pi$, and keep your eyes keen It makes 4 arcotan the unit sublime.

Note: 1 is the first and the second old friend.
So rewrite: $\pi$ equals twice this plus twice that.
"This" is the arcot of the first term of Len.
"That," which we'll split, is from the second old hat.
From 2 we get 3 , 4 ; from 4, 5 and 6 .
The evens keep splitting; the odds hang behind.
Forming convergent series: sum twice arcot $f$
Sub $2 k+1$ which is $\pi$, I remind.
We don't know the digit half-million and one.
Guiness, keep stout! There'll be other tries.
I've got half my friends in a pretty new sum.
Well worth the labor to open my eyes.

# FIBONACCI SINE SEQUENCES 

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## INTRODUCTION

The purpose of this note is to find all real numbers $x$ such that $\lim _{n \rightarrow \infty} \sin u_{n} \pi x$ exists, where $u_{n}$ is any sequence of integers satisfying the recurrence $u_{n}=u_{n-1}+u_{n-2}$ ( $u_{0}, u_{1}$ are integers, not both zero).
We will show that the sequence $\left\{\sin u_{n} \pi x\right\}$ converges only to zero and this happens precisely when $x$ is in an appropriate homothet of the set of integers in the quadratic number field $Q(\sqrt{5})$.

## MAIN RESULTS

We will use the identity $\sin a-\sin \beta=2 \cos 1 / 2(a+\beta) \sin 1 / 2(a-\beta)$ to show that if the limit

$$
\lim _{n} \sin u_{n} \pi x=\rho
$$

exists, then $\rho=0$.
Let $a=u_{n+1} \pi x, \beta=u_{n-2} \pi x$, so that $1 / 2(\alpha+\beta)=u_{n} \pi x$, and $1 / 2(\alpha-\beta)=u_{n-1} \pi x$. The identity gives

$$
\sin u_{n+1} \pi x-\sin u_{n-2} \pi x=2 \sin u_{n-1} \pi x \cos u_{n} \pi x
$$

Therefore, if $\lim \sin u_{n} \pi x=\rho \neq 0$, then

$$
\cos u_{n} \pi x=\frac{\sin u_{n+1} \pi x-\sin u_{n-2} \pi x}{2 \sin u_{n-1} \pi x}
$$

shows that $\lim \cos u_{n} \pi x=0$. However,

$$
\sin u_{n+1} \pi x=\sin \left(u_{n}+u_{n-1}\right) \pi x=\sin u_{n} \pi x \cos u_{n-1} \pi x+\cos u_{n} \pi x \sin u_{n-1} \pi x
$$

implies $\lim \sin u_{n} \pi x=0$, a contradiction.
Theorem 1. $\lim _{n} \sin u_{n} \pi x=0$ iff

$$
\lim _{n} \sin \phi^{n} \frac{\pi x}{\sqrt{5}}\left(u_{0}+u_{1} \phi\right)=0, \quad \text { where } \quad \phi=\frac{1+\sqrt{5}}{2} .
$$

Proof. Using Binet's formula for $u_{n}$, we have

$$
\begin{aligned}
\sin u_{n} \pi x & =\sin \frac{\pi x}{\sqrt{5}}\left\{\phi^{n-1}\left(u_{0}+u_{1} \phi\right)-(1-\phi)^{n-1}\left[u_{0}+u_{1}(1-\phi)\right]\right\} \\
& =\sin \frac{\pi x}{\sqrt{5}} \phi^{n-1}\left(u_{0}+u_{1} \phi\right) \cos \frac{\pi x}{\sqrt{5}}(1-\phi)^{n-1}\left[u_{0}+u_{1}(1-\phi)\right] \\
& -\sin \frac{\pi x}{\sqrt{5}}(1-\phi)^{n-1}\left[u_{0}+u_{1}(1-\phi)\right] \cos \frac{\pi x}{\sqrt{5}} \phi^{n-1}\left(u_{0}+u_{1} \phi\right) .
\end{aligned}
$$

Since $(1-\phi)^{n} \rightarrow 0$ as $n \rightarrow \infty$, the cosine in the first term tends to one, while the sine in the second term tends to zero, for any $x$. The theorem follows. ॥
Theorem 1 makes it plain that we must find the set $B$ of all real $x$ for which $\lim \sin \phi^{n} \pi x=0$. $n$

Theorem 2. $B$ is the set of all numbers of the form $a+b \phi$, where $a, b$ are integers.

Proof. We first observe that $B$ is an additive subgroup of the real numbers, for

$$
\sin \phi^{n} \pi(x-y)=\sin \phi^{n} \pi x \cos \phi^{n} \pi y-\cos \phi^{n} \pi x \sin \phi^{n} \pi y
$$

shows that $x-y$ is in $B$ if both $x$ and $y$ are in $B$. Now taking $u_{0}=-1, u_{1}=2$ in Theorem 1 and observing that $2 \phi-$ $1=\sqrt{5}$, it is apparent that 1 is in $B$ and hence the definition of $B$ shows that $\phi$ is also in $B$. It follows that $B$ contains every number of the form $a+b \phi$.
To prove that every member of $B$ has this form, we adapt an argument from Cassels [1, p. 136]. If $\lim \sin \phi^{n} \pi x=$ 0 , then $\phi^{n} x=p_{n}+r_{n}$, where $p_{n}$ is an integer and $\lim _{n} r_{n}=0$. Let $s_{n}=p_{n+2}-p_{n+1}-p_{n}$, so that $s_{n}$ is an integer. Then

$$
\begin{aligned}
s_{n} & =\left(\phi^{n+2} x-r_{n+2}\right)-\left(\phi^{n+1} x-r_{n+1}\right)-\left(\phi^{n} x-r_{n}\right) \\
& =\phi^{n} x\left(\phi^{2}-\phi-1\right)-\left(r_{n+2}-r_{n+1}-r_{n}\right)=-\left(r_{n+2}-r_{n+1}-r_{n}\right) .
\end{aligned}
$$

Since $\lim _{n} r_{n}=0$, we see that $\lim _{n} s_{n}=0$. Since $s_{n}$ is an integer, we must have $s_{n}=0$ for all $n \geqslant n_{0} \geqslant 1$. Thus $r_{n+2}=$ $r_{n+1}+r_{n}$ for $n \geqslant n_{0}$. Using Binet's formula, we have for $n \geqslant n_{0}$,

$$
r_{n}=\frac{r_{n_{0}+1}-(1-\phi) r_{n_{0}}}{\sqrt{5}} \phi^{n}-\frac{r_{n_{0}+1}-\phi r_{n_{0}}}{\sqrt{5}}(1-\phi)^{n} .
$$

Because $\phi^{n} \rightarrow \infty$ and $(1-\phi)^{n} \rightarrow 0$ as $n \rightarrow \infty$, the coefficient of $\phi^{n}$ must be zero; in other words, $r_{n_{0}+1}=(1-\phi) r_{n_{0}}$. Thus, for $n \geqslant n_{0}$,

$$
\begin{aligned}
r_{n} & =\frac{\phi r_{n_{0}}-r_{n_{0}+1}}{\sqrt{5}}(1-\phi)^{n}=\frac{\phi r_{n_{0}}-(1-\phi) r_{n_{0}}}{\sqrt{5}}(1-\phi)^{n} \\
& =\frac{r_{n_{0}}}{\sqrt{5}}(2 \phi-1)(1-\phi)^{n}=r_{n_{0}}(1-\phi)^{n} .
\end{aligned}
$$

In particular, choosing $n=n_{0}$, we find $r_{n_{0}}=r_{n_{0}}(1-\phi)^{n_{0}}$. This implies $r_{n_{0}}=0$, and therefore $\phi^{n_{0}} \boldsymbol{X}=p_{n_{0}}$, so that

$$
x=p_{n_{0}}(1 / \phi)^{n_{0}} .
$$

Using the facts that $1 / \phi=\phi-1$ and $\phi^{2}=\phi+1$, we see that $x=a+b \phi$ for suitable integers $a$ and $b$. \|

## CONCLUDING REMARKS

Combining Theorems 1 and $2, \lim _{n} \sin u_{n} \pi x$ exists iff $x$ is a member of the homothet

$$
\frac{\sqrt{5}}{u_{0}+u_{1} \phi} B=\left\{\frac{\sqrt{5}}{u_{0}+u_{1} \phi} x: x \in B\right\} .
$$

It is well known [3; p. 201] that $B$ is the set of all integers in the quadratic number field $Q(\sqrt{5})$ and this suggests comparison with other sine sequences. In [2], it is shown that $\lim _{n} \sin 2^{n} \pi x$ exists iff $2^{n_{0}} x$ is an integer for some $n_{0} \in Z$. Here we have shown that $\lim _{n} \sin \phi^{n} \pi x$ exists iff $\phi^{n_{0}} x$ is an integer for some $n_{0} \in Z$.
In closing, we suggest it would be of interest to consider the same problem for the sine sequences $\sin u_{n} \pi \dot{x}$ when the $u_{n}$ satisfies a recurrence $u_{n}=s u_{n-1}+t u_{n-2}$, where $s$ and $t$ are positive integers.

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# GENERAL IDENTITIES FOR LINEAR FIBONACCI AND LUCAS SUMMATIONS 

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Many well known identities involving the first $n$ terms of the Fibonacci sequence $\left\{F_{j}\right\}_{j=0}^{\infty}$ and the Lucas sequence $\left\{L_{j}\right\}_{j=0}^{\infty}$ have extensions to the sequences $\left\{F_{j+r}\right\}_{j=0}^{\infty},\left\{L_{j+r}\right\}_{j=0}^{\infty},\left\{F_{j k}\right\}_{j=0}^{\infty}$, and $\left\{L_{j k}\right\}_{j=0}^{\infty}$, where $r$ and $k$ are fixed integers. Any such result may be considered as a special case of an identity related to sequences $\left\{F_{j k+r}\right\}_{j=0}^{\infty}$ and $\left\{L_{j k+r}\right\}_{j=0}^{\infty}$, and hence it is with these latter sequences that we are principally concerned. Since the subscripts are linear functions of $j$, these identities are called linear Fibonacci and Lucas summations.
A variety of techniques are used in deriving many of these summations. We begin by considering several basic results which are quickly deduced from the Binet definition of the terms of the given sequences. This approach is introduced in [1] and [2] , with extensions via a difference equation route given in [3]. We have

$$
\begin{equation*}
F_{j k+r}=\frac{a^{j k+r}-\beta^{j k+r}}{a-\beta} \quad \text { and } \quad L_{j k+r}=a^{j k+r}+\beta^{j k+r} \tag{0}
\end{equation*}
$$

where

$$
a=\frac{1+\sqrt{5}}{2} \quad \text { and } \quad \beta=\frac{1-\sqrt{5}}{2} \text {. }
$$

Note that $a$ and $\beta$ are the roots of the equation $x^{2}-x-1=0$, and hence $a+\beta=1$ and $a \beta=-1$. Using the summation formula for the first $n$ terms of a geometric progression, the following results are obtained:

$$
\begin{align*}
& \sum_{j=0}^{n} F_{j k+r}=F_{r}+F_{k+r}+F_{2 k+r}+\cdots+F_{n k+r}=\left(\frac{a^{r}-\beta^{r}}{a-\beta}\right)+\left(\frac{a^{k+r}-\beta^{k+r}}{a-\beta}\right)+\left(\frac{a^{2 k+r}-\beta^{2 k+r}}{a-\beta}\right)  \tag{1}\\
& +\cdots+\left(\frac{a^{n k+r}-\beta^{n k+r}}{a-\beta}\right) \\
& =\frac{1}{a-\beta}\left[a^{r}\left(\frac{a^{(n+1) k}-1}{a^{k}-1}\right)-\beta^{r}\left(\frac{\beta^{(n+1) k}-1}{\beta^{k}-1}\right)\right]=\frac{F_{(n+1) k+r}+(-1)^{k+1} F_{n k+r}+(-1)^{r} F_{k-r}-F_{r}}{L_{k}-1+(-1)^{k+1}} .
\end{align*}
$$

Similarly, one may find

$$
\begin{align*}
\sum_{j=0}^{n} L_{j k+r} & =\frac{L_{(n+1) k+r}+(-1)^{k+1} L_{n k+r}+(-1)^{r} L_{k-r}-L_{r}}{L_{k}-1+(-1)^{k+1}}  \tag{2}\\
\sum_{j=0}^{n}(-1)^{j} F_{j k+r} & =\frac{(-1)^{n} F_{(n+1) k+r}+(-1)^{n+k} F_{n k+r}+(-1)^{r} F_{k-r}+F_{r}}{L_{k}+1+(-1)^{k}} \\
\sum_{j=0}^{n}(-1)^{j} L_{j k+r} & =\frac{(-1)^{n} L_{(n+1) k+r}+(-1)^{n+k} L_{n k+r}+(-1)^{r} L_{k-r}+L_{r}}{L_{k}+1+(-1)^{k}} .
\end{align*}
$$

These identities are used to simplify any summation expression that may be represented as a linear combination of Fibonacci and/or Lucas numbers. One direction to take is to observe by (0) that

$$
\begin{aligned}
F_{j k+r} F_{j u+v} & =\frac{1}{5}\left[L_{j(k+u)+(r+v)}-(-1)^{j u+v} L_{j(k-u)+(r-v)}\right] \\
F_{j k+r} L_{j u+v} & =F_{j(k+u)+(r+v)}+(-1)^{j u+v} F_{j(k-u)+(r-v)} \\
L_{j k+r} L_{j u+v} & =L_{j(k+u)+(r+v)}+(-1)^{j u+v} L_{j(k-u)+(r-v)} .
\end{aligned}
$$

[APR.

Identities (1) to (4) yield an expression for the sum of the first $n+1$ terms ( $0 \leqslant j \leqslant n$ ) of each product given above. Let us explicitly consider only the second such product.
(5) $\sum_{j=0}^{n} F_{j k+r} L_{j u+v}=\sum_{j=0}^{n} F_{j(k+u)+(r+v)}+(-1)^{v} \sum_{j=0}^{n}(-1)^{j u} F_{j(k-u)+(r-v)}$

$$
=\frac{F_{(n+1)(k+u)+(r+v)}+(-1)^{k+u+1} F_{n(k+u)+(r+v)}}{L_{k+u}-1+(-1)^{k+u+1}}+\frac{(-1)^{r+v} F_{(k+u)-(r+v)}-F_{r+v}}{L_{k+u}-1+(-1)^{k+u+1}}
$$

$$
+\left\{\begin{array}{l}
\frac{(-1)^{v}\left[F_{(n+1)(k-u)+(r-v)}+(-1)^{k-u+1} F_{n(k-u)+(r-v)}+(-1)^{r-v} F_{(k-u)-(r-v)}-F_{r-v}\right]}{L_{k-u}-1+(-1)^{k-u+1}} \\
\text { if } u \text { is even, } \\
\frac{(-1)^{v}\left[(-1)^{n} F_{(n+1)(k-u)+(r-v)}+(-1)^{n+k-u} F_{n(k-u)+(r-v)}+(-1)^{r-v} F_{(k-u)-(r-v)}+F_{r-v}\right]}{L_{k-u}+1+(-1)^{k-u}} \\
\text { if } u \text { is odd. }
\end{array}\right.
$$

Specifying $k, u, r$, and $v$ as particular integers leads the reader to a countable number of interesting special cases. The known (see [3] and [4]) generating functions for sequences $\left\{F_{j k+r}\right\}_{j=0}^{\infty}$ and $\left\{L_{j k+r}\right\}_{j=0}^{\infty}$ are now used to find several additional classes of general linear Fibonacci and Lucas summation identities. We now list these generating functions for reference-with the first be derived from (0) to show a general approach to such calculations.
(6)

$$
\begin{aligned}
\sum_{j=0}^{\infty} F_{j k+r} x^{j} & =\sum_{j=0}^{\infty} \frac{a^{j k+r}-\beta^{j k+r}}{a-\beta} x^{j}=\frac{1}{a-\beta}\left[a^{r} \sum_{j=0}^{\infty} a^{j k} x^{j}-\beta^{r} \sum_{j=0}^{\infty} \beta^{j k} x^{j}\right] \\
& =\frac{1}{a-\beta}\left[\frac{a^{r}}{1-a^{k} x}-\frac{\beta^{r}}{1-\beta^{k} x}\right]=\frac{1}{a-\beta}\left[\frac{\left(a^{r}-\beta^{r}\right)+\left(-a^{r} \beta^{k}+a^{k} \beta^{r}\right) x}{1-\left(a^{k}+\beta^{k}\right) x+a^{k} \beta^{k} x^{2}}\right] \\
& =\frac{F_{r}+(-1)^{r} F_{k-r} x}{1-L_{k} x+(-1)^{k} x^{2}}=\frac{F_{r}+\left(F_{k+r}-F_{r} L_{k}\right) x}{1-L_{k} x+(-1)^{k} x^{2}} .
\end{aligned}
$$

$$
\begin{equation*}
\sum_{j=0}^{\infty}(-1)^{j} F_{j k+r} x^{j}=\frac{F_{r}+(-1)^{r+1} F_{k-r} x}{1+L_{k} x+(-1)^{k} x^{2}}=\frac{F_{r}+\left(F_{r} L_{k}-F_{k+r}\right) x}{1+L_{k} x+(-1)^{k} x^{2}} \tag{7}
\end{equation*}
$$

(8)

$$
\sum_{j=0}^{\infty} L_{j k+r} x^{j}=\frac{L_{r}+(-1)^{r+1} L_{k-r} x}{1-L_{k} x+(-1)^{k} x^{2}}=\frac{L_{r}+\left(L_{k+r}-L_{r} L_{k}\right) x}{1-L_{k} x+(-1)^{k} x^{2}}
$$

(9)

$$
\sum_{j=0}^{\infty}(-1)^{j} L_{j k+r} x^{j}=\frac{L_{r}+(-1)^{r} L_{k-r} x}{1+L_{k} x+(-1)^{k} x^{2}}=\frac{L_{r}+\left(L_{r} L_{k}-L_{k+r}\right) x}{1+L_{k} x+(-1)^{k} x^{2}} .
$$

The derivative of these generating functions leads to identities which are of interest in themselves, and these in turn yield additional summation results. We begin by differentiating both sides of (6) with respect to $x$.

$$
\frac{d}{d x}\left[\sum_{j=0}^{\infty} F_{j k+r} x^{j}\right]=\frac{d}{d x}\left[\frac{F_{r}+(-1)^{r} F_{k-r} x}{1-L_{k} x+(-1)^{k} x^{2}}\right]
$$

$$
\begin{aligned}
\sum_{j=0}^{\infty}(j+1) F_{(j+1) k+r} x^{j}= & \frac{d}{d x}\left[\frac{F_{r}}{1-L_{k} x+(-1)^{k} x^{2}}\right]+\frac{d}{d x}\left[\frac{(-1)^{r} F_{k-r} x}{1-L_{k} x+(-1)^{k} x^{2}}\right] \\
= & \frac{F_{r}}{1-L_{k} x+(-1)^{k} x^{2}} \frac{L_{k}+2(-1)^{k+1} x}{1-L_{k} x+(-1)^{k} x^{2}}+\frac{(-1)^{r} F_{k-r}}{1-L_{k} x+(-1)^{k} x^{2}} \frac{1-(-1)^{k} x^{2}}{1-L_{k} x+(-1)^{k} x^{2}} \\
= & \sum_{j=0}^{\infty} \frac{F_{r} F_{(i+1) k}}{F_{k}} x^{j} \cdot \sum_{j=0}^{\infty} L_{(j+1) k} x^{j} \\
& +\sum_{j=0}^{\infty} \frac{(-1)^{r} F_{k-r} F_{(j+1) k}}{F_{k}} \cdot x^{j}\left[-1+\frac{2-L_{k} x}{1-L_{k} x+(-1)^{k} x^{2}}\right]
\end{aligned}
$$

by special cases of (6) and (8),

$$
\begin{aligned}
=\sum_{j=0}^{\infty} \sum_{s=0}^{j} \frac{F_{r} F_{(s+1) k}}{F_{k}} L_{(j-s+1) k} x^{j} & +\sum_{j=0}^{\infty} \frac{(-1)^{r+1} F_{k-r} F_{(j+1) k}}{F_{k}} x^{j} \\
& +\sum_{j=0}^{\infty} \frac{(-1)^{r} F_{k-r} F_{(j+1) k}}{F_{k}} x^{j} \cdot \sum_{j=0}^{\infty} L_{j k} x^{j},
\end{aligned}
$$

by convolution of the series and by (8),

$$
=\sum_{j=0}^{\infty}\left[\frac{(-1)^{r+1} F_{k-r} F_{(j+1) k}}{F_{k}}+\sum_{s=0}^{j} \frac{F_{r} F_{(s+1) k} L_{(i-s+1) k}}{F_{k}}+\sum_{s=0}^{j} \frac{(-1)^{r} F_{k-r} F_{(s+1) k} L_{(j-s) k}}{F_{k}}\right] x^{j}
$$

By equating the corresponding coefficients of the above series, the identity

$$
\begin{equation*}
(j+1) F_{(j+1) k+r}=\frac{1}{F_{k}}\left\{(-1)^{r+1} F_{k-r} F_{(j+1) k}+\sum_{s=0}^{j} F_{(s+1) k}\left[F_{r} L_{(j-s+1) k}+(-1)^{r} F_{k-r} L_{(j-s) k}\right]\right\} \tag{10}
\end{equation*}
$$

is found, which in turn yields

$$
\begin{equation*}
\sum_{j=0}^{n}(j+1) F_{(j+1) k+r}=\frac{1}{F_{k}} \sum_{j=0}^{n}\left\{(-1)^{r+1} F_{k-r} F_{(j+1) k}+\sum_{s=0}^{j} F_{(s+1) k}\left[F_{r} L_{(j-s+1) k}+(-1)^{r} F_{k-r} L_{(j-s) k}\right]\right\} \tag{11}
\end{equation*}
$$

Performing the same operations as above on identities (7), (8), and (9) yields results similar to (10) and (11). These results related to (8) are as follows:

$$
\begin{equation*}
(j+1) L_{(j+1) k+r}=\frac{1}{F_{k}}\left\{(-1)^{r} L_{k-r} F_{(j+1) k}+\sum_{s=0}^{j} F_{(s+1) k}\left[L_{r} L_{(j-s+1) k}+(-1)^{r+1} L_{k-r} L_{(j-s) k}\right]\right\} \tag{12}
\end{equation*}
$$

and
(13) $\sum_{j=0}^{n}(j+1) L_{(j+1) k+r}=\frac{1}{F_{k}} \sum_{j=0}^{n}\left\{(-1)^{r} L_{k-r} F_{(j+1) k}+\sum_{s=0}^{j} F_{(s+1) k}\left[L_{r} L_{(j-s+1) k}+(-1)^{r+1} L_{k-r} L_{(j-s) k}\right]\right\}$.

Taking higher order derivatives of (6), (7), (8), and (9) leads the reader to additional summation identities that are similar in form to those listed above. Further, numerous special cases of each identity given may be quickly deduced.
The relationships between binomial coefficients and terms of sequences $\left\{F_{j k+r}\right\}_{j=0}^{\infty}$ and $\left\{L_{j k+r}\right\}_{j=0}^{\infty}$ take the form of rather simple but elegant summation identities. To begin we return to definition (0).

$$
\begin{equation*}
F_{j k+r}=\frac{a^{j k} a^{r}-\beta^{j k} \beta^{r}}{a-\beta}=\frac{\left(a^{2}-1\right)^{j k} a^{r}-\left(\beta^{2}-1\right)^{j k} \beta^{r}}{a-\beta}= \tag{14}
\end{equation*}
$$

$=\frac{\sum_{t=0}^{j k}\binom{j k}{t}(a)^{2 t+r}(-1)^{j k-t}-\sum_{t=0}^{j k}\binom{j k}{t}(\beta)^{2 t+r}(-1)^{j t}}{a-\beta}=\sum_{t=0}^{j k}\binom{j k}{t}(-1)^{j k-t}\left[\frac{a^{2 t+r}-\beta^{2 t+r}}{a-\beta}\right]$
$=\sum_{t=0}^{j k}\binom{j k}{t}(-1)^{j k-t} F_{2 t+r}=\sum_{t=0}^{j k}\binom{j k}{t}(-1)^{j k-t} F_{t+r / 2} L_{t+r / 2}$.
If $j=2 j^{\prime}$, then an even more elementary summation results.

$$
\begin{equation*}
F_{2 j^{\prime} k+r}=\frac{\left(a^{2}\right)^{j^{\prime \prime k}} \dot{a}^{r}-\left(\beta^{2}\right)^{j^{\prime} k} \beta^{r}}{a-\beta}=\frac{(a+1)^{j^{\prime} k} a^{r}-(\beta+1)^{j^{\prime} k} \beta^{r}}{a-\beta}=\sum_{t=0}^{j^{\prime \prime k}}\binom{j^{\prime} k}{t} F_{t+r} . \tag{15}
\end{equation*}
$$

For the Lucas numbers, the corresponding results are

$$
\begin{equation*}
L_{j k+r}=\sum_{t=0}^{j k}\binom{j k}{t}(-1)^{j k-t} L_{2 t+r}=\sum_{t=0}^{j k}\binom{j k}{t}(-1)^{j k-t} \sum_{s=0}^{t}\binom{t}{s} L_{s+r} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{2 j^{\prime} k+r}=\sum_{t=0}^{j^{\prime} k}\binom{j^{\prime} k}{t} L_{t+r} \tag{17}
\end{equation*}
$$

Taking the view that a summation identity is "improved" by reducing the number of addends (even if the addends become more complicated), we now consider several methods of approach in an attempt to find additional "improved" results linking binomial coefficients and Fibonacci and Lucas numbers.
The column generators of the columns in the left-justified Pascal Triangle shown below are most useful in this endeavor, as was first shown by V. E. Hoggatt, Jr., in [5]

$$
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \\
\frac{1}{1-x} & \frac{x}{(1-x)^{2}} & \frac{x^{2}}{(1-x)^{3}} & \frac{x^{3}}{(1-x)^{4}} & & \\
\text { Column Generators }
\end{array}
$$

That is, defining the binomial coefficient $\binom{n}{j}=0$ for $n<j$ we observe
$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^{n}=\sum_{n=0}^{\infty}\binom{n}{0} x^{n}, \quad \frac{x}{(1-x)^{2}}=\sum_{n=0}^{\infty} n x^{n}=\sum_{n=0}^{\infty}\binom{n}{1} x^{n}, \quad \frac{x^{j}}{(1-x)^{j+1}}=\sum_{n=0}^{\infty}\binom{n}{j} x^{n}, \quad j \geqslant 0$.
Hence,

$$
\sum_{j=0}^{\infty} F_{j k+r} \frac{x^{j}}{(1-x)^{j+1}}=\sum_{j=0}^{\infty} F_{j k+r} \sum_{n=0}^{\infty}\binom{n}{j} x^{n}=\sum_{n=0}^{\infty} \sum_{j=0}^{n}\binom{n}{j} F_{j k+r} x^{n}
$$

By identity (6),

$$
\sum_{j=0}^{\infty} F_{j k+r} x^{j}=\frac{F_{r}+(-1)^{r} F_{k-r} x}{1-L_{k} x+(-1)^{k} x^{2}}
$$

and thus, we also have
$\sum_{j=0}^{\infty} F_{j k+r} \frac{x^{j}}{(1-x)^{j+1}}=\frac{1}{1-x} \frac{F_{r}+(-1)^{r} F_{k-r}(x / 1-x)}{1-L_{k}(x / 1-x)+(-1)^{k}(x / 1-x)^{2}}=\frac{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x}{1-\left[2+L_{k}\right] x+\left[1+L_{k}+(-1)^{k}\right] x^{2}}$.
There are two cases of the above identity to consider:
(i) $k$ even. Then

$$
\begin{aligned}
\sum_{j=0}^{\infty} F_{j k+r} \frac{x^{j}}{(1-x)^{j+1}} & =\frac{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x}{1-\left(2+L_{k}\right)(1-x)_{x}}=\left\{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x\right\} \sum_{j=0}^{\infty}\left(2+L_{k}\right)^{j}(1-x)^{j} x^{j} \\
& =\sum_{j=0}^{\infty}\left\{F_{r} \sum_{s=0}^{j}\left(2+L_{k}\right)^{j}\binom{j}{s}(-1)^{s} x^{s+j}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] \sum_{s=1}^{j+1}\left(2+L_{k}\right)^{j}\binom{j}{s-1}(-1)^{s-1} x^{s+j}\right\}_{j}
\end{aligned}
$$

Now let $m=s+j$. Then we have

$$
\left.=\sum_{m=0}^{\infty} F_{r} \sum_{s=0}^{[m / 2]}\left(2+L_{k}\right)^{m-s}\binom{m-s}{s}(-1)^{s}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] \sum_{s=1}^{[(m+1) / 2]}\left(2+L_{k}\right)^{m-s}\binom{m-s}{s-1}(-1)^{s-1}\right\} x^{m} .
$$

Hence, equating like coefficients of $x$ in the above two series yields, for $k$ even,

(ii) $k$ odd. Then

$$
\begin{aligned}
& \sum_{j=0}^{\infty} F_{j k+r} \frac{x^{j}}{(1-x)^{j+1}}=\frac{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x}{1-\left[\left(2+L_{k}\right)-L_{k} x\right] x}=\left\{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x\right\} \sum_{j=0}^{\infty}\left[\left(2+L_{k}\right)-L_{k} x\right]^{j} x^{j} \\
& =\left\{F_{r}+\left[-F_{r}+(-1)^{r} F_{k-r}\right] x\right\} \sum_{j=0}^{\infty} \sum_{s=0}^{j}\binom{j}{s}\left(2+L_{k}\right)^{j-s}\left(L_{k} x\right)^{s}(-1)^{s} x^{j}=\sum_{m=0}^{\infty}\left\{F_{r} \sum_{s=0}^{[m / 2]}(-1)^{s}\binom{m-s}{s}\left(2+L_{k}\right)^{m-2 s} L_{k}^{s}\right. \\
& \left.\quad+\left[-F_{r}+(-1)^{r} F_{k-r}\right] \sum_{s=1}^{[(m+1) / 2]}(-1)^{s-1}\binom{m-s}{s-1}\left(2+L_{k}\right)^{m-2 s+1} L_{k}^{s-1}\right\} x^{m}
\end{aligned}
$$

and thus, for $k$ odd,

$$
\begin{align*}
\sum_{j=0}^{n}\binom{n}{j} F_{j k+r}= & F_{r} \sum_{s=0}^{[n / 2]}(-1)^{s}\binom{n-s}{s}\left(2+L_{k}\right)^{n-2 s} L_{k}^{s}  \tag{19}\\
& +\left[-F_{r}+(-1)^{r} F_{k-r}\right] \sum_{s=1}^{[(n+1) / 2]}(-1)^{s-1}\binom{n-s}{s-1}\left(2+L_{k}\right)^{n-2 s+1} L_{k}^{s-1}
\end{align*}
$$

Using the column generators in the left-justified Pascal Triangle with generating functions (7), (8), and (9) leads to three pairs of summation identities which are similar in form to (18) and (19).

Several special cases of (18) and (19) are given which show the inherent simplicity of these identities.
Letting $r=0$ and $k=2$ in (18) gives

$$
\sum_{j=0}^{n}\binom{n}{j} F_{j 2}=\sum_{s=0}^{[(n+1) / 2]}(-1)^{s-1} 5^{n-s}\binom{n-s}{s-1}
$$

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If $k=r=1$ in (19), then

$$
\sum_{j=0}^{n}\binom{n}{j} F_{j-1}=\sum_{s=0}^{[n / 2]}(-1)^{s}\binom{n-s}{s} 3^{n-2 s}+\sum_{s=1}^{[(n+1) / 2]}(-1)^{s}\binom{n-s}{s-1} 3^{n-2 s+1}
$$

Taking $r=0$ and $k=3$ in (19) yields

$$
\sum_{j=0}^{n}\binom{n}{j} F_{j 3}=2 \sum_{s=1}^{[(n+1) / 2]}(-1)^{s-1}\binom{n-s}{s-1} 6^{n-2 s+1} 4^{s-1}
$$

More generally, we deduce from (18) and (19) that

$$
\sum_{j=0}^{n}\binom{n}{j} F_{j k}=\left\{\begin{array}{l}
F_{k} \sum_{s=1}^{[(n+1) / 2]}(-1)^{s-1}\left(2+L_{k}\right)^{n-s}\binom{n-s}{s-1}, \text { for } k \text { even }, \\
F_{k} \sum_{s=1}^{[(n+1) / 2]}(-1)^{s-1}\left(1+L_{k}\right)^{n-2 s+1}\binom{n-s}{s-1} L_{k}^{s-1}, \text { for } k \text { odd }
\end{array}\right.
$$

One of the nicest results linking binomial coefficients and Fibonacci numbers is given in [6]. Here, using the fact that

$$
\begin{gathered}
Q^{n}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)^{n}=\left(\begin{array}{lr}
F_{n+1} & F_{n} \\
F_{n} & F_{n-1}
\end{array}\right), \text { for any integer } n, \text { and } \\
Q^{n k+r}=\sum_{j=0}^{k}\binom{k}{j} Q^{j+r} F_{n}^{j} F_{n-1}^{k-j},
\end{gathered}
$$

the identity

$$
\begin{equation*}
F_{n k+r}=\sum_{j=0}^{k}\binom{k}{j} F_{j+r} F_{n}^{j} F_{n-1}^{k-j} \tag{20}
\end{equation*}
$$

is deduced by equating upper right elements in the previous matrix equation. This identity is actually a special case of the next result.
Since for any integer $t$,

$$
a^{n}=a^{n-t} a^{t}=\left(F_{n-t} a+F_{n-t-1} I\right) a^{t}
$$

where $I=\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$, it follows that

$$
\begin{aligned}
Q^{n k+r} & \left.=Q^{r}\left[F_{n-t} Q+F_{n-t-1} I\right) Q^{t}\right]^{k}=Q^{t k+r} \sum_{j=0}^{k}\binom{k}{j} Q^{j} F_{n-t}^{j} F_{n-t-1}^{k-j}, \text { for } t \neq n \\
& =\sum_{j=0}^{k}\binom{k}{j} Q^{t k+r+j} F_{n-t}^{j} F_{n-t-1}^{k-j} .
\end{aligned}
$$

By equating the upper right elements in this matrix equation we obtain, for any integer $t \neq n$,

$$
\begin{equation*}
F_{n k+r}=\sum_{j=0}^{k}\binom{k}{j} F_{t k+r+j} F_{n-t}^{j} F_{n-t-1}^{k-j} \tag{21}
\end{equation*}
$$

The companion results for Lucas numbers are deduced by either using the identity $L_{m}=F_{m+1}+F_{m-1}$ or the matrix result

They are

$$
\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)^{m-1}\left(\begin{array}{ll}
3 & 1 \\
1 & 2
\end{array}\right)=\left(\begin{array}{cc}
L_{m+1} & L_{m} \\
L_{m} & L_{m-1}
\end{array}\right), \text { for any integer } m .
$$

(22)

$$
L_{n k+r}=\sum_{j=0}^{k}\binom{k}{j} L_{j+r} F_{n}^{j} F_{n-1}^{k-j}
$$

and

$$
L_{n k+r}=\sum_{j=0}^{k}\binom{k}{j} L_{t k+r+j} F_{n-t}^{j} F_{n-t-1}^{k-j}, \text { for } t \neq n
$$

The final approach we take to find additional linear Fibonacci and Lucas identities is via exponential generating functions. This productive technique stems from the Maclaurin series expansion for $e^{x}$ :
$e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots$ and hence $\mathrm{e}^{\alpha x}=1+\frac{(a x)}{1!}+\frac{(a x)^{2}}{2!}+\frac{(a x)^{3}}{3!}+\cdots$ and $e^{\beta x}=1+\frac{(\beta x)}{1!}+\frac{(\beta x)^{2}}{2!}+\frac{(\beta x)^{3}}{3!}+\cdots$.
It follows that the basic Fibonacci and Lucas generating functions are

$$
\sum_{n=0}^{\infty} F_{n} \frac{x^{n}}{n!}=\frac{e^{\alpha x}-e^{\beta x}}{a-\beta} \text { and } \sum_{n=0}^{\infty} L_{n} \frac{x^{n}}{n!} \dot{=} e^{\alpha x}+e^{\beta x} .
$$

The exponential generating functions of the sequences of interest in this paper are found to be

$$
\begin{equation*}
\sum_{n=0}^{\infty} F_{n k+r} \frac{x^{n}}{n!}=\frac{a^{r} e^{\alpha^{k} x}-\beta^{r} e^{\beta^{k} x}}{a-\beta} \tag{24}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=0}^{\infty}(-1)^{n} F_{n k+r} \frac{x^{n}}{n!}=\frac{a^{r} e^{-\alpha^{k} x}-\beta^{r} e^{-\beta^{k} x}}{a-\beta} \tag{26}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=0}^{\infty} L_{n k+r} \frac{x^{n}}{n!}=a^{r} e^{\alpha^{k} x}+\beta^{r} e^{\beta^{k} x} \tag{25}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=0}^{\infty}(-1)^{n} L_{n k+r} \frac{x^{n}}{n!}=a^{r} e^{-\alpha^{k} x}+\beta^{r} e^{-\beta^{k} x} \tag{27}
\end{equation*}
$$

Convoluting series (24) and (25) and equating like coefficients yields an interesting identity. We proceed as follows:
and $\quad \sum_{n=0}^{\infty} F_{n k+r} \frac{x^{n}}{n!} \sum_{n=0}^{\infty} L_{n k+r} \frac{x^{n}}{n!}=\sum_{n=0}^{\infty} \sum_{j=0}^{n}\binom{n}{j} F_{j k+r} L_{(n-j) k+r} \frac{x^{n}}{n!}$

$$
\begin{aligned}
\left(\frac{a^{r} e^{\alpha^{k} x}-\beta^{r} e^{\beta^{k} x}}{a-\beta}\right)\left(a^{r} e^{\alpha^{k} x}+\beta^{r} e^{\beta^{k} x}\right) & =\frac{a^{2 r} e^{2 \alpha^{k} x}-\beta^{2 r} e^{2 \beta^{k} x}}{a-\beta}=\frac{a^{2 r} \sum_{n=0}^{\infty}\left(2 a^{k}\right)^{n} \frac{x^{n}}{n!}-\beta^{2 r} \sum_{n=0}^{\infty}\left(2 \beta^{k}\right)^{n} \frac{x^{n}}{n!}}{a-\beta} \\
& =\sum_{n=0}^{\infty} 2^{n}\left(\frac{a^{n k+2 r}-\beta^{n k+2 r}}{a-\beta}\right) \frac{x^{n}}{n!}=\sum_{n=0}^{\infty} 2^{n} F_{n k+2 r} \frac{x^{n}}{n!}
\end{aligned}
$$

Hence,
(28)

$$
\sum_{j=0}^{n}\binom{n}{j} F_{j k+r} L_{(n-j) k+r}=2^{n} F_{n k+2 r}
$$

Many additional identities may be deduced using the generating functions (24), (25), (26), and (27). By convoluting each of (24) and (25) with itself, the following results are deduced:

$$
\begin{equation*}
\sum_{j=0}^{n}\binom{n}{j} F_{j k+r} F_{(n-j) k+r}=\frac{1}{5}\left[2^{n} L_{n k+2 r}+2(-1)^{r+1} L_{k}^{n}\right] \tag{29}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{j=0}^{n}\binom{n}{j} L_{j k+r} L_{(n-j) k+r}=2^{n} L_{n k+2 r}+2(-1)^{r} L_{k}^{n} \tag{30}
\end{equation*}
$$

We invite the reader to explore the special cases of the results given and also to use the procedures introduced to discover additional identities.

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## AN INEQUALITY FOR A CLASS OF POLYNOMIALS

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## 1. INTRODUCTION

Recently, Klamkin and Newman [1], using double induction, proved that

$$
\begin{equation*}
\sum_{k=1}^{n} A_{k}^{3} \leqslant\left(\sum_{k=1}^{n} A_{k}\right)^{2} \quad(n=1,2, \cdots) \tag{1.1}
\end{equation*}
$$

where $A_{k}$ is a non-decreasing sequence with $A_{0}=0$ and $A_{k}-A_{k-1} \leqslant 1$. For $A_{k}=k$, (1.1) gives the well known elementary identity

$$
\begin{equation*}
\sum_{k=1}^{n} k^{3}=\left(\sum_{k=1}^{n} k\right)^{2} \quad(n=1,2, \cdots) \tag{1.2}
\end{equation*}
$$

Our inequality (2.1) for polynomials in a single variable $x$ gives (1.1) for $x=1$.

## 2. A POLYNOMIAL INEQUALITY

Our first general result is given by
Theorem 1. Let $C_{k}$ be a non-decreasing sequence with $C_{0}=0$ and $C_{k}-B C_{k-1} \leqslant 1, k=1,2, \cdots$, where $B$ is a constant, $0 \leqslant B \leqslant 1$. Then, for $x \geqslant 1$, we have the inequality

$$
\begin{equation*}
\sum_{k=1}^{n} c_{k}^{3} x^{k} \leqslant\left(\sum_{k=1}^{n} c_{k} x^{k}\right)^{2} \quad(n=1,2, \cdots) \tag{2.1}
\end{equation*}
$$

Proof. We will use double induction. For $n=1$, (2.1) requires that $C_{1}^{3} x \leqslant C_{1}^{2} x^{2}$, or $C_{1}^{2} x\left(C_{1}-x\right) \leqslant 0$, which is true, since $C_{1} \leqslant 1$ and $x \geqslant 1$. Assuming (2.1) is true for $k=1,2, \cdots, n$, we must now show that

$$
\sum_{k=1}^{n+1} c_{k}^{3} x^{k}=C_{n+1}^{3} x^{n+1}+\sum_{k=1}^{n} c_{k}^{3} x^{k} \leqslant c_{n+1}^{3} x^{n+1}+\left(\sum_{k=1}^{n} c_{k} x^{k}\right)^{2} \leqslant\left(\sum_{k=1}^{n+1} c_{k} x^{k}\right)^{2}
$$

which requires the truth of
(2.2)

$$
2 \sum_{k=1}^{n} c_{k} x^{k} \geqslant c_{n+1}^{2}-C_{n+1} x^{n+1} \quad(n=1,2, \cdots)
$$

For $n=1$, (2.2) gives

$$
c_{2}^{2}-C_{2} x^{2} \leqslant 2 C_{1} x
$$

Since $x \geqslant 1, x^{2} C_{2} \geqslant C_{2}$,

$$
c_{2}^{2}-c_{2} x^{2} \leqslant c_{2}^{2}-c_{2}
$$

but $C_{2}-B C_{1} \leqslant 1$, and so

$$
C_{2}^{2}-C_{2} \leqslant C_{1} B C_{2} \leqslant C_{1} B\left(1+B C_{1}\right) \leqslant C_{1} B(1+B) \leqslant 2 C_{1} \leqslant 2 C_{1 x}
$$

which is true since $B(1+B) \leqslant 2$ for $0 \leqslant B \leqslant 1$. Assuming (2.2) is true for $k=1,2, \cdots, n$, we must show that

$$
2 \sum_{k=1}^{n+1} c_{k} x^{k} \geqslant 2 C_{n+1} x^{n+1}+\left(C_{n+1}^{2}-C_{n+1} x^{n+1}\right) \geqslant c_{n+2}^{2}-C_{n+2} x^{n+2}
$$

which requires that

$$
x^{n+1}\left(x C_{n+2}+C_{n+1}\right) \geqslant C_{n+2}^{2}-C_{n+1}^{2}, \quad n=1,2, \cdots .
$$

Since $B \leqslant 1,-B C_{n+1} \geqslant-C_{n+1}$, and so

$$
C_{n+2}-C_{n+1} \leqslant C_{n+2}-B C_{n+1} \leqslant 1 .
$$

Hence

$$
C_{n+2}^{2}-C_{n+1}^{2} \leqslant C_{n+2}+C_{n+1} \leqslant x^{n+1}\left(x C_{n+2}+C_{n+1}\right)
$$

since $x \geqslant 1$. Thus, the truth of (2.2) completes the proof of Theorem 1 .
In [1, p. 29], the following,
Lemma. If $x, y \geqslant 0, p \geqslant 2$, then $p(x-y)\left(x^{p-1}+y^{p-1}\right) \geqslant 2\left(x^{p}-y^{p}\right)$,
was used to generalize (1.1) (see [1, (18), p. 29]). Using the above lemma and double induction, we now obtain a generalization of Theorem 1, i.e.,
Theorem 2. Let $C_{k}$ be a non-decreasing sequence with $C_{0}=0$ and $C_{k}-B C_{k-1} \leqslant 1, k=1,2, \cdots$, where $B$ is a constant, $0 \leqslant B \leqslant 1$. Then, for $x \geqslant 1$ and $p=2,3, \cdots$, we have the polynomial inequality

$$
\begin{equation*}
2 \sum_{k=1}^{n} c_{k}^{2 p-1} x^{k} \leqslant p\left(\sum_{k=1}^{n} c_{k}^{p-1} x^{k}\right)^{2} \quad(n=1,2, \cdots) . \tag{2.3}
\end{equation*}
$$

Remarks. For $p=2,(2.3)$ gives (2.1). For $B=1$ and $x=1$, (2.3) gives (18) of [1, p. 29] , and (2.1) gives (1.1). The proof of Theorem 2, similar to the proof of Theorem 1 , is omitted. We note that when $C_{k}-B C_{k-1}=1$ for $k=1,2$, $\cdots$, then

$$
C_{k}=\left(1-B^{k}\right) /(1-B)
$$

$B \neq 1$ and $C_{k}=k$ for $B=1$. For $B=0$ and $C_{k}=1, k=1,2, \cdots,(2.1)$ gives

$$
1 \leqslant \sum_{k=1}^{n} x^{k}
$$

so that for $n=1,1 \leqslant x$, as required.

## [Continued on page 146.]

# A PRIMER FOR THE FIBONACCI NUMBERS XVII: GENERALIZED FIBONACCI NUMBERS SATISFYING $u_{n+1} u_{n-1}-u_{n}^{2}= \pm 1$ 

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There are many ways to generalize the Fibonacci sequence. Here, we examine some properties of integral sequences $\left\{u_{n}\right\}$ satisfying

$$
\begin{equation*}
u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}, \tag{1}
\end{equation*}
$$

where necessarily $u_{0}=0$ and $u_{1}= \pm 1$. The Fibonacci polynomials $f_{n}(x)$ given by

$$
\begin{equation*}
f_{n+1}(x)=x f_{n}(x)+f_{n-1}(x), \quad f_{0}(x)=0, \quad f_{1}(x)=1 \tag{2}
\end{equation*}
$$

evaluated at $x=b$ provide special sequences $\left\{u_{n}\right\}$. Of course, $f_{n}(1)=F_{n}$, the Fibonacci numbers $0,1,1,2,3,5, \cdots$, and $f_{n}(2)=P_{n}$, the Pell numbers $0,1,2,5,12,29, \ldots$. Divisibility properties of the Fibonacci polynomials [1] and properties of the Pell numbers and the general sequences $\left\{f_{n}(b)\right\}$ [2] have been examined in earlier Primer articles.
In the course of events, we will completely solve the Diophantine equations $y^{2}-\left(a^{2} \pm 4\right) x^{2}= \pm 4$ and show that all of our generalized Fibonacci polynomials are special cases of Chebyshev polynomials of the first and second kinds.

$$
\text { 1. SOLUTIONS TO } y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4
$$

Theorem 1. Let $\left\{u_{n}\right\}$ be a sequence of integers such that $u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}$ for all integers $n$. Then there exists an integer a such that
(3)

$$
u_{n+2}=a u_{n+1}+u_{n} .
$$

Proof. Set

$$
u_{2}=a u_{1}+b u_{0}, \quad u_{3}=a u_{2}+b u_{1}
$$

for some real numbers $a$ and $b$. By Cramer's rule,

$$
b=\left|\begin{array}{ll}
u_{1} & u_{2} \\
u_{2} & u_{3}
\end{array}\right| \div\left|\begin{array}{ll}
u_{1} & u_{0} \\
u_{2} & u_{1}
\end{array}\right|=\frac{u_{1} u_{3}-u_{2}^{2}}{u_{1}^{2}-u_{0} u_{2}}=1
$$

since $u_{1} u_{3}-u_{2}^{2}=(-1)^{2}$ and $u_{0} u_{2}-u_{1}^{2}=(-1)^{1}$ by definition of $\left\{u_{n}\right\}$. Thus, $a$ is an integer. In fact, $u_{2}=a u_{1}+u_{0}$ and $u_{3}=a u_{2}+u_{1}$ yield

$$
a=\frac{u_{3}-u_{1}}{u_{2}}=\frac{u_{2}-u_{0}}{u_{1}}
$$

Assume that $u_{n+1}=a u_{n}+u_{n-1}$. Then

$$
a=\frac{u_{n+1}-u_{n-1}}{u_{n}}
$$

and

$$
a u_{n+1}+u_{n}=\frac{u_{n+1}-u_{n-1}}{u_{n}} \cdot u_{n+1}+u_{n}=\frac{u_{n+1}^{2}-u_{n-1} u_{n+1}+u_{n}^{2}}{u_{n}}=\frac{u_{n+1}^{2}+(-1)^{n+1}}{u_{n}}
$$

But, $u_{n+2} u_{n}-u_{n+1}^{2}=(-1)^{n+1}$ by definition of the sequence, so that

$$
u_{n+2}=\left[u_{n+1}^{2}+(-1)^{n+1}\right] / u_{n}, \quad \text { and } \quad u_{n+2}=a u_{n+1}+\dot{u}_{n}
$$

for an integer $a$ by the Axiom of Mathematical Induction.
Corollary 1.1. The sequence $\left\{u_{n}\right\}$ has starting values $u_{0}=0, u_{1}= \pm 1$.

Proof. By Theorem $1, u_{2}=a u_{1}+u_{0}$. Thus,

$$
u_{2}^{2}=a^{2} u_{1}^{2}+2 a u_{1} u_{0}+u_{1}^{2}=a u_{1}\left(a u_{1}+u_{0}\right)+u_{0}^{2}=a u_{1} u_{2}+u_{0}^{2} .
$$

Since also $u_{0}=u_{2}-a u_{1}$, substituting above for $u_{0}^{2}$, we have

$$
u_{2}^{2}=a u_{1} u_{2}+\left(u_{2}^{2}-2 a u_{1} u_{2}+a^{2} u_{1}^{2}\right), \quad 0=a u_{1}\left(a u_{1}-u_{2}\right)
$$

Now, either $a=0$, or $u_{1}=0$, or $u_{2}=a u_{1}$. If $a=0, u_{2}=u_{0}$, and from $u_{2} u_{0}-u_{1}^{2}=-1, u_{0}=0$ and $u_{1}= \pm 1$ give the only possible solutions. If $u_{1}=0$, then $u_{2}=u_{0}$ leads to $u_{2}^{2}=-1$, clearly impossible for integers. If $u_{2}=a u_{1}$, then $u_{2}=a u_{1}=a u_{1}+u_{0}$ forces $u_{0}=0$, and again $u_{1}= \pm 1$.
Theorem 2. Let $\left\{u_{n}\right\}$ be a sequence of integers such that $u_{n+1} u_{n+1}-u_{n}^{2}=(-1)^{n}$ for all $n$. Then $x=u_{n}$ and $y=u_{n+1}+u_{n-1}$ are solutions for the Diophantine equation

$$
\begin{equation*}
y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4 \tag{4}
\end{equation*}
$$

where also $u_{n+1}=a u_{n}+u_{n-1}$.
Proof. From Theorem 1, $u_{n+1}=a u_{n}+u_{n-1}$. If $y=u_{n+1}+u_{n-1}$ and $x=u_{n}$, then

$$
u_{n+1}=y-u_{n-1}=y-\left(u_{n+1}-a u_{n}\right)=y-u_{n+1}-a x
$$

yielding

$$
u_{n+1}=(y-a x) / 2
$$

Then

$$
u_{n-1}=y-u_{n+1}=y-(y-a x) / 2=(y+a x) / 2 .
$$

By definition of the sequence $\left\{u_{n}\right\}$,

$$
\begin{aligned}
& u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n} \\
& \frac{y+a x}{2} \cdot \frac{v-a x}{2}-x^{2}= \pm 1 \\
& \left(y^{2}-a^{2} x^{2}\right)-4 x^{2}= \pm 4 \\
& y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4
\end{aligned}
$$

Now, let the generalized Lucas and Fibonacci numbers $£_{n}$ and $\xi_{n}$ be defined in terms of Fibonacci polynomials as in Eq. (2):

$$
\begin{gather*}
\mathcal{L}_{n}=f_{n+1}(a)+f_{n-1}(a)  \tag{5}\\
F_{n}=f_{n}(a) .
\end{gather*}
$$

Since [2]

$$
\begin{gather*}
f_{n+1}(x) f_{n-1}(x)-f_{n}^{2}(x)=(-1)^{n},  \tag{6}\\
L_{n}^{2}-\left(a^{2}+4\right) \bar{F}_{n}^{2}= \pm 4 \tag{7}
\end{gather*}
$$

by Theorem 2. Thus, the generalized Lucas and Fibonacci numbers give solutions to the Diophantine equation (4).
Theorem 3. The generalized Lucas and Fibonacci numbers $£_{n}$ and $z_{n}$ are the only solutions to the Diophantine equation

$$
\begin{equation*}
y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4 \tag{4}
\end{equation*}
$$

Proof. Now, $y^{2}-\left(a^{2}+4\right) x^{2}=+4$ has solution $x=0, y=2$, as well as a solution $x=1, y=3$ if $a=1$, but no solution for $x=1$ when $a>1$. The other equation $y^{2}-\left(a^{2}+4\right) x^{2}=-4$ has solution $x=1, y=a$. The case $a=1$ was solved by Ferguson [3]. We use a method of infinite descent which is an extension of the method of Ferguson [3], and take $a>1, x>1$. Thus, $y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4$ implies that

$$
a x<y<(a+2) x
$$

since
forces

$$
y^{2}=\left(a^{2}+4\right) x^{2} \pm 4=a^{2} x^{2}+4 x^{2} \pm 4<a^{2} x^{2}+4 a x^{2}+4 x^{2}
$$

,

$$
(a x)^{2}<y^{2}<(a+2)^{2} x^{2}
$$

Since $y$ and $a x$ must have the same parity, let

$$
y=a x+2 t, \quad 1 \leqslant t<x .
$$

Assume that $x$ is the smallest non-Fibonacci solution. Replace $y$ with $a x+2 t$ in (4), yielding

$$
\begin{gathered}
(a x+2 t)^{2}-\left(a^{2}+4\right) x^{2} \pm 4=0 \\
4 x^{2}-4 a x t-4 t^{2} \pm 4=0
\end{gathered}
$$

Solve the quadratic for $2 x$, yielding

$$
2 x=a t \pm \sqrt{\left(a^{2}+4\right) t^{2} \pm 4}
$$

But, $2 x$ is an integer, and therefore

$$
\left(a^{2}+4\right) t^{2} \pm 4=s^{2}
$$

for an integer $s$ so that $t=u_{n}$ and $s=u_{n+1}+u_{n-1}$ are solutions by Theorem 2. Since $x>0$,

$$
\begin{aligned}
2 x & =a t+\sqrt{\left(a^{2}+4\right) t^{2} \pm 4} \\
& =a t+s \\
& =a u_{n}+\left(u_{n+1}+u_{n-1}\right) \\
& =\left(a u_{n}+u_{n-1}\right)+u_{n-1} \\
& =2 u_{n+1}
\end{aligned}
$$

so that $x=u_{n+1}$. But, if $x$ is the smallest non-Fibonacci solution, then $x$ cannot be the next larger Fibonacci solution after $t$. This is a contradiction, and there is no first non-Fibonacci solution. Thus, the Diophantine equation

$$
y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4
$$

has solutions in integers if and only if

$$
y= \pm £_{n}=f_{n+1}(a)+f_{n-1}(a) \quad \text { and } \quad x= \pm f_{n}=f_{n}(a) .
$$

2. SPECIAL SEQUENCES $\left\{u_{n}\right\}$ AND THE EQUATION $y^{2}-\left(a^{2}-4\right) x^{2}= \pm 4$

Now, all of these sequences $\left\{u_{n}\right\}$ have starting values $u_{0}=0$ and $u_{1}= \pm 1$. It in interesting to note some special cases. Notice that the sequence

$$
\ldots, 1,0,1,0,1,0,1,0,1,1,2,3,5, \ldots
$$

due to Bergum [4] satisfies $u_{0}=0, u_{1}=1$, and

$$
u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}
$$

where the left-hand part of the sequence has

$$
u_{n+2}=u_{n}=0 \cdot u_{n+1}+u_{n}
$$

while the right-hand part has

$$
u_{n+2}=1 \cdot u_{n+1}+u_{n} .
$$

It is interesting to note that special cases of the sequences $\left\{u_{n}\right\}$ satisfying $u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}$ occur from [2]

$$
\begin{equation*}
\tau_{n-k} £_{n+k}-\tau_{n}^{2}=(-1)^{n+k+1} v_{k}^{2} \tag{8}
\end{equation*}
$$

for the generalized Fibonacci numbers given in Eq. (5). Let

$$
f_{n k-k-1}=x_{n k+k}^{2}=(-1)^{n k+k+1_{-}}{ }_{k}^{2}
$$

be rewritten

$$
\frac{\tau(n-1) k}{\tau_{k}} \frac{\tau(n+1) k}{\tau_{k}}-\frac{\tau_{n k}^{2}}{\tau_{k}^{2}}=(-1)^{(n+1) k+1}
$$

Now, since $\tau_{n k} / \tau_{k}$ is known to be an integer [1], let $u_{n}=\tau_{n k} / \tau_{k}$, and the equation above becomes

$$
u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{(n+1) k+1}
$$

where $(-1)^{(n+1) k+1}$ is $(-1)^{n}$ if $k$ is odd but $(-1)$ if $k$ is even. In particular, if $k=2$, the sequence of Fibonacci numbers with even subscripts, $\{0,1,3,8,21, \cdots\}$, gives a solution to $u_{n+1} u_{n-1}-u_{n}^{2}=-1$. Another solution is $u_{n}=n$, since $(n+1)(n-1)-n^{2}=-1$ for all $n$.
Is there a sequence $\left\{u_{n}\right\}$ of positive terms for which $u_{n+1} u_{n-1}-u_{n}^{2}=+1$ ? Considering Fibonacci numbers with odd subscripts, $\{1,2,5,13,34, \ldots\}$, we observe that $u_{n}=F_{2 n+1}$ is a solution, and that $u_{n+1}=3 u_{n}-u_{n-1}$, Using $u_{n+1} u_{n-1}-u_{n}^{2}=1$ and solving $u_{n+1}=a u_{n}+b u_{n-1}$ as in Theorem 1 yields $u_{n+1}=a u_{n}-u_{n-1}$. If we let $y=u_{n+1}-$ $u_{n-1}$ and $x=u_{n}$, proceeding as in Theorem 2, we are led to the Diophantine equation $y^{2}-\left(a^{2}-4\right) x^{2}=-4$. We summarize as
Theorem 4. If $\left\{u_{n}\right\}$ is a sequence of integers such that

$$
u_{n+1} u_{n-1}-u_{n}^{2}=+1
$$

for all $n$, then there exists an integer $a$ such that

$$
u_{n+2}=a u_{n+1}-u_{n}
$$

and $y=u_{n+1}-u_{n-1}$ and $x=u_{n}$ are solutions of the Diophantine equation
(9)

$$
y^{2}-\left(a^{2}-4\right) x^{2}=-4
$$

Theorem 5. The odd-subscripted Fibonacci and Lucas numbers give the only solutions to the Diophantine equation
(9)

$$
y^{2}-\left(a^{2}-4\right) x^{2}=-4
$$

Proof. We show that (9) has no integral solutions if $|a| \neq 3$, proceeding in the manner of the proof of Theorem 3. Here,

$$
(a-2) x<y<a x
$$

Since $y$ and $a x$ must have the same parity, let

$$
y=a x-2 t, \quad 1 \leqslant t<x .
$$

Notice that, if $x=1, y^{2}-\left(a^{2}-4\right)=-4$ becomes $a^{2}-y^{2}=8$, which is solved only by $a=3, y=1$.
Let $x$ be the first solution greater than one: Replace $y$ with $a x-2 t$ in (9), yielding

$$
\begin{gathered}
(a x-2 t)^{2}-\left(a^{2}-4\right) x^{2}+4=0 \\
4 x^{2}-4 a x t+4 t^{2}+4=0
\end{gathered}
$$

Solving the quadratic for $2 x$ gives

$$
2 x=a t \pm \sqrt{\left(a^{2}-4\right) t^{2}-4}
$$

Since $2 x$ is integral, we must have $\left(a^{2}-4\right) t^{2}-4=s^{2}$ for some integer $s$. By Theorem $4, t=u_{n}$ is a solution where $t>1$. But, since $x$ is the first solution greater than 1 , and $x>t$, we have a contradiction, and

$$
y^{2}-\left(a^{2}-4\right) x^{2}=-4
$$

is not solvable in positive integers unless $a=3$. When $a=3$, the equation becomes $y^{2}-5 x^{2}=-4$, which is solved only by

$$
y=L_{2 n+1}, \quad x=F_{2 n+1}
$$

odd-subscripted Lucas and Fibonacci numbers [5].

Theorem 6. If $\left\{u_{n}\right\}$ is a sequence of integers such that

$$
u_{n+1} u_{n-1}-u_{n}^{2}=-1
$$

for all $n$, then there exists an integer a such that

$$
u_{n+2}=a u_{n+1}-u_{n} \quad \text { and } \quad y=u_{n+1}-u_{n-1} \quad \text { and } \quad x=u_{n}
$$

are solutions of the Diophantine equation
(10)

$$
y^{2}-\left(a^{2}-4\right) x^{2}=+4
$$

Proof. Proceed as in Theorem 4.
Theorem 7. The Fibonacci and Lucas numbers with even subscripts give solutions to the Diophantine equation

$$
y^{2}-\left(a^{2}-4\right) x^{2}=+4
$$

Proof. Set $a=3$ and refer to Lind [5].

## 3. GENERALIZED FIBONACCI POLYNOMIALS

Next, in order to write solutions for the Diophantine equation (10), we consider a type of generalized Fibonacci polynomial. Let
(11)

$$
h_{0}(x)=0, \quad h_{1}(x)=1, \quad \text { and } \quad h_{n+2}(x)=x h_{n+1}(x)-h_{n}(x)
$$

and

$$
g_{0}(x)=2, \quad g_{1}(x)=x
$$

where

$$
g_{n+2}(x)=x g_{n+1}(x)+g_{n-1}(x) .
$$

We note that $\left\{h_{n}(a)\right\}$ is a special sequence $\left\{u_{n}\right\}$ since

$$
h_{n+1}(a) h_{n-1}(a)-h_{n}^{2}(a)=-1 .
$$

Then

$$
\begin{gathered}
h_{n}(x)=\frac{a_{1}^{n}(x)-a_{2}^{n}(x)}{a_{1}(x)-a_{2}(x)}, \quad x \neq 2 ; \quad h_{n}(2)=n \\
g_{n}(x)=a_{1}^{n}(x)+a_{2}^{n}(x)=h_{n+1}(x)-h_{n-1}(x),
\end{gathered}
$$

where $a_{1}(x)$ and $a_{2}(x)$ are roots of

$$
\lambda^{2}-\lambda x+1=0 .
$$

(By way of comparison, the Fibonacci polynomials $f_{n}(x)$ have the analogous relationship to the roots of

$$
\lambda^{2}-\lambda x-1=0
$$

Also note that $h_{n}(3)=F_{2 n}$.)
It is easy to establish from $a_{1}(x) a_{2}(x)=1$ that

$$
\begin{aligned}
& 2 a_{1}^{n}=g_{n}(x)+\left[a_{1}(x)-a_{2}(x)\right] h_{n}(x) \\
& 2 a_{2}^{n}=g_{n}(x)-\left[a_{1}(x)-a_{2}(x)\right] h_{n}(x)
\end{aligned}
$$

with $a_{1}(x)-a_{2}(x)=\sqrt{x^{2}-4}$. From this it readily follows that

$$
1=a_{1}^{n}(x) a_{2}^{n}(x)=\left[g_{n}^{2}(x)-\left(x^{2}-4\right) h_{n}^{2}(x)\right] / 4
$$

or

$$
g_{n}^{2}(x)-\left(x^{2}-4\right) h_{n}^{2}(x)=+4 .
$$

Now, we are interested in the sequences of integers formed by evaluating $h_{n}(x)$ and $g_{n}(x)$ at $x=a$. Thus

$$
\begin{equation*}
g_{n}^{2}(a)-\left(a^{2}-4\right) h_{n}^{2}(a)=+4 . \tag{12}
\end{equation*}
$$

and we do have solutions to

$$
y^{2}-\left(a^{2}-4\right) x^{2}=+4
$$

Theorem 8. The generalized Fibonacci numbers $\left\{h_{n}(a)\right\}$ and generalized Lucas numbers $\left\{g_{n}(a)\right\}$ provide the only solutions to the Diophantine equation

$$
\begin{equation*}
y^{2}-\left(a^{2}-4\right) x^{2}=+4 \tag{10}
\end{equation*}
$$

Proof. Note that if $x=1$, then $y=a$, and if $x=0$, then $y=2$, Now one can proceed as follows. We can write, as before,

$$
(a-2) x<y \leqslant a x .
$$

Clearly, $y$ and $a x$ must have the same parity, so that we can let

$$
y=a x-2 t, \quad 1 \leqslant t<x
$$

where $x$ is the first positive integer which is greater than 1 , not equal to $h_{m}(a)$, and a solution. Then, as before, replace $y$ with $a x-2 t$ in (10), yielding

$$
\begin{gathered}
(a x-2 t)^{2}-\left(a^{2}-4\right) x^{2}-4=0 \\
4 x^{2}-4 a x t+4 t^{2}-4=0
\end{gathered}
$$

Solving the quadratic for $2 x$,

$$
\begin{equation*}
2 x=a t \pm \sqrt{\left(a^{2}-4\right) t^{2}+4} \tag{13}
\end{equation*}
$$

Since $2 x$ is an integer, there exists an integer $s$ such that

$$
\left(a^{2}-4\right) t^{2}+4=s^{2}
$$

with a solution given by

$$
t=h_{n}(a) \quad \text { and } \quad s=g_{n}(a)=h_{n+1}(a)-h_{n-1}(a)
$$

by Eq. (12). Then, (13) taken with the plus sign gives

$$
2 x=a h_{n}(a)+h_{n+1}(a)-h_{n-1}(a)=2 h_{n+1}(a)
$$

and $x=h_{n+1}(a)$, a contradiction, since $x$ was defined as not having the form $h_{m}(a)$.
Next, we consider the case of Eq. (13) taken with the minus sign. The cases $a=1$ or $a=0$ are not very interesting. We need a lemma:
Lemma. For $a>1$, the sequence $\left\{h_{n}(a)\right\}$ is a strictly increasing sequence.
Proof of the Lemma.

$$
h_{0}(a)=0, \quad h_{1}(a)=1, \quad h_{2}(a)=a, \quad h_{n+2}(a)=a h_{n+1}(a)-h_{n}(a) .
$$

Since

$$
h_{n+1}(a)=a h_{n}(a)-h_{n-1}(a)>(a-1) h_{n}(a)
$$

if

$$
h_{n-1}(a)<h_{n}(a)
$$

then

$$
h_{n+1}(a)>h_{n}(a)
$$

Thus, if we choose the minus sign in Eq. (13), then we have

$$
\begin{aligned}
2 x & =a h_{n}(a)-\left(h_{n+1}(a)-h_{n-1}(a)\right) \\
& =a h_{n}(a)-h_{n+1}(a)+h_{n-1}(a)=2 h_{n-1}(a)
\end{aligned}
$$

or $x=h_{n-1}(a)$ which contradicts the restriction that $t<x$. Thus, we must choose the plus sign in (13), which yielded $x=h_{n+1}(a)$. So, even if $x$ is the first integer greater than one for which we have a solution for

$$
y^{2}-\left(a^{2}-4\right) x^{2}=+4
$$

and where $x \neq h_{m}(a)$, we find $x=h_{n+1}(a)$. This shows that there is no first positive integer which solves Eq. (10) which is not of the form $x=h_{m}(a)$. This concludes the proof of Theroem 8.

We note that the case $a=2$ yields $y= \pm 2$ and $x$ any integer. The recurrence

$$
u_{n+2}=2 u_{n+1}-u_{n}
$$

is satisfied by any arithmetic progression $b, b+d, b+2 d, \cdots, B+n d, \cdots$. However, the restriction

$$
u_{n+1} u_{n-1}-u_{n}^{2}=-1
$$

limits these to the integers $n=u_{n}$.
In summary, we have set down the complete solutions to the Diophantine equations

$$
y^{2}-\left(a^{2} \pm 4\right) x^{2}= \pm 4
$$

$y^{2}-\left(a^{2}+4\right) x^{2}$ has solution $x=0, y=2$, for all $a$. For

$$
y^{2}-\left(a^{2}+4\right) x^{2}=-4
$$

we get $x=1, y=a$. Both solutions are starting pairs for the recurrence

$$
u_{n+2}=a u_{n+1}+u_{n},
$$

and $y=2, a, \cdots$ leads to $f_{n+1}(a)+f_{n-1}(a)$, and $x=0,1, \ldots$, leads to $f_{n}(a)$, where $f_{n}(x)$ are the Fibonacci polynomials. Here, $u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}$ lead to $y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4$ via $u_{n+2}=a u_{n+1}+u_{n}$. But either

$$
u_{n+1} u_{n-1}-u_{n}^{2}=-1 \quad \text { or } \quad u_{n+1} u_{n-1}-u_{n}^{2}=+1
$$

lead to the recurrence $u_{n+2}=a u_{n+1}-u_{n}$, and lead to $y^{2}-\left(a^{2}-4\right) x^{2}= \pm 4$. Now $y^{2}-\left(a^{2}-4\right) x^{2}=+4$ allows $x$ $=0, y=2$ and $x=1, y=a$ as starting solutions, where $x=0,1, \cdots$, leads to $h_{n}(a)$, and $y=2, a, \cdots$, leads to $h_{n+1}(a)-$ $h_{n-1}(a)$ for the generalized Fibonacci polynomials $h_{n}(x)$. Finally, $y^{2}-\left(a^{2}-4\right) x^{2}=-4$ has solution $x=1, y=1$ when $|a|=3$, but no solution if $|a| \neq 3$. This then becomes $y^{2}-5 x^{2}=-4$ which is satisfied only by the oddly subscripted Fibonacci and Lucas numbers, which satisfy the recurrence $u_{n+1}=3 u_{n}-u_{n-1}$, so that

$$
F_{2 n+1}=h_{n+1}(3)-h_{n}(3),
$$

and, of course, $F_{2 n+1}=f_{2 n+1}(1)$. In all cases, the only solutions arise from sequences of Fibonacci polynomials $f_{n}(x)$ evaluated at $x=a$, or generalized Fibonacci polynomials $h_{n}(x)$ evaluated at $x=a$. We can then state
Theorem 9. The Diophantine equations

$$
\begin{aligned}
& y^{2}-\left(a^{2}-4\right) x^{2}= \pm 4 \\
& y^{2}-\left(a^{2}+4\right) x^{2}= \pm 4
\end{aligned}
$$

have solutions in positive integers if and only if

$$
y^{2}-\left(a^{2}-4\right) x^{2}=-4
$$

has a solution $x=1$ or

$$
y^{2}-\left(a^{2}+4\right) x^{2}=-4
$$

has a solution $x=1$. Every solution is given by terms of a sequence of Fibonacci polynomials evaluated at $a,\left\{f_{n}(a)\right\}$, or generalized Fibonacci polynomials evaluated at $x=a,\left\{h_{n}(a)\right\}$.

## 4. CHEBYSHEV POLYNOMIALS

There are Chebyshev polynomials of two kinds:

$$
\begin{aligned}
& U_{n+2}(x)=2 x U_{n+1}(x)-U_{n}(x) \\
& T_{n+2}(x)=2 x T_{n+1}(x)-T_{n}(x)
\end{aligned}
$$

with $T_{0}(x)=1$ and $T_{1}(x)=x$, and $U_{0}(x)=1$ and $U_{1}(x)=2 x$. The $T_{n}(x)$ are the Chebyshev polynomials of the first kind, and the $U_{n}(x)$ are the Chebyshev polynomials of the second kind [8]. There are also related polynomials

$$
S_{n}(x)=U_{n}(x / 2) \quad \text { and } \quad C_{n}(x)=2 T_{n}(x / 2)
$$

which are tabulated in [8]. Our $h_{n}(x)$ and $g_{n}(x)$ are related to $S_{n}(x)$ and $C_{n}(x)$ as follows:

$$
h_{n}(x)=S_{n+1}(x) \quad \text { and } \quad g_{n}(x)=C_{n}(x)
$$

An early article by Paul F. Byrd [10] explains the close connection between Fibonacci and Lucas polynomials and the $U_{n}(x)$ and $T_{n}(x)$. See also Hoggatt [9] , and Buschman [11].

## 5. ANOTHER CONSEQUENCE OF $u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}$

Finally, we examine another consequence of

$$
u_{n+1} u_{n-1}-u_{n}^{2}=(-1)^{n}
$$

We note that

$$
\left(u_{n}, u_{n+1}\right)=1, \quad\left(u_{n}, u_{n-1}\right)=1 .
$$

Note that $1,-1,-u_{n-1}, u_{n-1}$ are incongruent modulo $u_{n}, u \geqslant 5$, and form a multiplicative subgroup of the multiplicative group of integers modulo $u_{n}$. Since the order of the multiplicative group of integers $\bmod u_{n}$ is $\varphi\left(u_{n}\right)$, where $\varphi(n)$ denotes the number of integers less than $n$ and prime to $n$, and since the order of subgroup divides the order of a group, $4 \mid \varphi\left(u_{n}\right)$. This method of proof was given by Montgomery [6] as solution to the problem of showing that $\varphi\left(F_{n}\right)$ is divisible by 4 if $n \geqslant 5$. The same problem also appeared in a slightly different form in the Fibonacci Quarterly [7]. We can generalize to

$$
2^{m+2} \mid \varphi\left(\tau_{2} m_{n}\right), \quad n \geqslant 5,
$$

for the generalized Fibonacci numbers $\tau_{n}=f_{n}(a)$ by virtue of $\varphi(s)=2 k \geqslant 2$ for positive integers $s>2$, and $\tau_{2 t}=$ $\tau_{t} \mathfrak{f}_{t}$. Since $\left(\tau_{t}, \mathcal{L}_{t}\right)=1$ or 2 , then

$$
\varphi\left(\tau_{2 t}\right)=\varphi\left(\tau_{t}\right) \varphi(a)
$$

where $a=\mathcal{L}_{t}$ or $\mathcal{L}_{t} / 2$ so that $\varphi(a)=2 k \geqslant 2$. Thus,

$$
\tau_{2} m_{n}=\tau_{n} \delta_{n} £ 2 n £_{4 n}, \cdots,
$$

where

$$
\varphi\left(\tau_{n}\right) \varphi\left(\delta_{n} \mathcal{L}_{2 n} \AA_{4 n} \cdots\right)=4.2^{m} r
$$

for some integer $r \geqslant 1$.

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# GENERALIZED EULERIAN NUMBERS AND POLYNOMIALS 

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## 1. INTRODUCTION

Put
(1.1)

$$
\sum_{k=0}^{\infty} k^{n} x^{k}=\frac{A_{n}(x)}{(1-x)^{n+1}} \quad(n \geqslant 0)
$$

It is well known (see for example [1], [2, Ch. 2] that, for $n \geqslant 1, A_{n}(x)$ is a polynomial of degree $n$ :

$$
\begin{equation*}
A_{n}(x)=\sum_{k=1}^{n} A_{n, k} x^{k} \tag{1.2}
\end{equation*}
$$

the coefficients $A_{n, k}$ are called Eulerian numbers. They are positive integers that satisfy the recurrence
(1.3)
and the symmetry relation
(1.4)

$$
A_{n+1, k}=(n-k+2) A_{n, k-1}+k A_{n, k}
$$

$$
A_{n, k}=A_{n, n-k+1} \quad(1 \leqslant k \leqslant n)
$$

There is also the explicit formula

$$
\begin{equation*}
A_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{n+1}{j}(k-j)^{n} \quad(1 \leqslant n \leqslant k) \tag{1.5}
\end{equation*}
$$

Consider next

$$
\begin{equation*}
\sum_{k=0}^{\infty}\left(\frac{k(k+1)}{2}\right)^{n} x^{k}=\frac{G_{n}(x)}{(1-x)^{2 n+1}} \quad(n \geqslant 0) \tag{1.6}
\end{equation*}
$$

We shall show that, for $n \geqslant 1, G_{n}(x)$ is a polynomial of degree $2 n-1$ :

$$
\begin{equation*}
G_{n}(x)=\sum_{k=0}^{2 n-1} G_{n, k} x^{k} \tag{1.7}
\end{equation*}
$$

The $G_{n, k}$ are positive integers that satisfy the recurrence
(1.8) $G_{n+1, k}=1 / 2 k(k+1) G_{n, k}-k(2 n-k+2) G_{n, k-1}+1 / 2(2 n-k+2)(2 n-k+3) G_{n, k-2} \quad(1 \leqslant k \leqslant 2 n+1)$
and the symmetry relation
(1.9)

$$
G_{n, k}=G_{n, 2 n-k} \quad(1 \leqslant k \leqslant 2 n-1) .
$$

There is also the explicit formula

$$
\begin{equation*}
G_{n, k}=\sum_{j=0}^{k}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^{n} \quad(1 \leqslant k \leqslant 2 n-1) \tag{1.10}
\end{equation*}
$$

The definitions (1.1) and (1.6) suggest the following generalization. Let $p \geqslant 1$ and put

$$
\begin{equation*}
\sum_{k=0}^{\infty} T_{k, p^{n}}^{n} x^{k}=\frac{G_{n}^{(p x)}(x)}{(1-x)^{p n+1}} \quad(n \geqslant 0) \tag{1.11}
\end{equation*}
$$

where
(1.12)

$$
T_{k, p}=\binom{k+p-1}{p}
$$

We shall show that $G_{n}^{(p)}(x)$ is a polynomial of degree $p n-p+1$.

$$
\begin{equation*}
G_{n}^{(p)}(x)=\sum_{k=1}^{p n-p+1} G_{n, k}^{(p)} x^{k} \quad(n \geqslant 1), \tag{1.13}
\end{equation*}
$$

where the $G_{n, k}^{(p)}$ are positive integers that satisfy the recurrence

$$
\begin{equation*}
G_{n+1, m}^{(p)}=\sum_{\substack{k=1 \\ k \geqslant m-p}}^{m}\binom{k+p-1}{m-1}\binom{p n-k+1}{m-k} G_{n, k}^{(p)} \quad(1 \leqslant m \leqslant p n+1), \tag{1.14}
\end{equation*}
$$

and the symmetry relation
(1.15)

$$
G_{n, k}^{(p)}=G_{n, p n-p-k+2}^{(p)} \quad(1 \leqslant k \leqslant p n-p-k+1) .
$$

There is also the explicit formula

$$
\begin{equation*}
G_{n, k}^{(p)}=\sum_{j=0}^{k}(-1)^{j}(\underset{j}{p n+1}) T_{k-j, p}^{n} \quad(1 \leqslant k \leqslant p n-p+1) \tag{1.16}
\end{equation*}
$$

with $T_{k, p}$ defined by (i.12).
Clearly

$$
G_{n}^{(1)}(x)=A_{n}(x), \quad G_{n}^{(2)}(x)=G_{n}(x)
$$

The Eulerian numbers have the following combinatorial interpretation. Put $Z_{n}=\{1,2, \cdots, n\}$, and let $\pi=\left(a_{1}, a_{2}\right.$, $\cdots, a_{n}$ ) denote a permutation of $Z_{n}$. A rise of $\pi$ is a pair of consecutive elements $a_{i}, a_{i+1}$ such that $a_{i}<a_{i+1}$; in addition a conventional rise to the left of $a_{1}$ is included. Then [6, Ch. 8] $A_{n, k}$ is equal to the number of permutations of $Z_{n}$ with exactly $k$ rises.
To get a combinatorial interpretation of $G_{n, k}^{(p)}$ we recall the statement of the Simon Newcomb problem. Consider sequences $\sigma=\mid\left(a_{1}, a_{2}, \cdots, a_{N}\right)_{\mid}$of length $N$ with $a_{i} \in Z_{n}$. For $1 \leqslant i \leqslant n$, let $i$ occur in $\sigma$ exactly $e_{i}$ times; the ordered set $\left(e_{1}, e_{2}, \cdots, e_{n}\right)$ is called the specification of $\sigma$. A rise is a pair of consecutive elements $a_{i}, a_{i+1}$ such that $a_{i}<a_{i+1}$; a fall is a pair $a_{i}, a_{i+1}$ such that $a_{i}>a_{i+1}$; a leve/ is a pair $a_{i}, a_{i+1}$ such that $a_{i}=a_{i+1}$. A conventional rise to the left of $a_{1}$ is counted, also a conventional fall to the right of $a_{N}$. Let $\sigma$ have $r$ rises, $s$ falls and $t$ levels, so that $r+s+t=$ $N+1$. The Simon Newcomb problem [5, IV, Ch. 4] , [6, Ch. 8] asks for the number of sequences from $Z_{n}$ of length $N$, specification $\left[e_{1}, e_{2}, \cdots, e_{n}\right.$ ] and having exactly $r$ rises. Let $A\left(e_{1}, e_{2}, \cdots, e_{n!} r\right)$ denote this number. Dillon and Roselle [4] have proved that $A\left(e_{1}, \cdots, e_{n} \mid r\right)$ is an extended Eulerian number [2] defined in the following way. Put

$$
\frac{1-\lambda}{\zeta(s)-\lambda}=\sum_{m=1}^{\infty} m^{-s}(\lambda-1)^{-N} \sum_{r=1}^{N} A^{*}(m, r) \lambda^{N-r},
$$

where $\zeta(s)$ is the Riemann zeta-function and

$$
m=p_{1}^{\varphi_{1}} p_{2}^{e_{2}} \cdots p_{n}^{e_{n}}, \quad N=e_{1}+e_{2}+\cdots+e_{n} ;
$$

then

$$
A\left(e_{1}, e_{2}, \cdots, e_{n} \mid r\right)=A^{*}(m, r),
$$

Moreover

$$
\begin{equation*}
A\left(e_{1}, e_{2}, \cdots, e_{n} \mid r\right)=\sum_{j=0}^{r}(-1)^{j}\binom{N+1}{j} \prod_{i=1}^{n}\binom{e_{i}+r-j-1}{e_{i}} \tag{1.17}
\end{equation*}
$$

A refined version of the Simon Newcomb problem asks for the number of sequences from $z_{n}$ of length $N$, specification $\left[e_{1}, e_{2}, \cdots, e_{r}\right.$ ] and with $r$ rises and $s$ falls. Let $A\left(e_{1}, \cdots, e_{n} \mid r, s\right)$ denote this enumerant. It is proved in [3] that

$$
\begin{equation*}
\sum_{e_{1}, \cdots, e_{n}=0}^{\infty} \sum_{r+s \leqslant N+1} A\left(e_{1}, \cdots, e_{n} \mid r, s\right) z_{1}^{e_{1}} \cdots z_{n}^{e_{n}} x^{r} y^{s}=x y \frac{\prod_{i=1}^{n}\left(1+(y-1) z_{i}\right)-\prod_{i=1}^{n}\left(1+(x-1) z_{i}\right)}{v \prod_{i=1}^{n}\left(1+(x-1) z_{i}\right)-x \prod_{i=1}^{n}\left(1+(y-1) z_{i}\right)} . \tag{1.18}
\end{equation*}
$$

However explicit formulas were not obtained for $A\left(e_{i}, \cdots, e_{n}(r, s)\right.$.

Returning to $G_{n, k}^{(p)}$, we shall show that

Thus (1.17) gives

$$
\begin{equation*}
G_{n, k}^{(p)}=A(\underbrace{p, \cdots, p}_{n} \mid k) . \tag{1.19}
\end{equation*}
$$

$$
\begin{equation*}
G_{n, k}^{(p)}=\sum_{j=0}^{k}(-1)^{j}\binom{p n+1}{j}\binom{p+k-j-1}{p}^{n} \tag{1.20}
\end{equation*}
$$

in agreement with (1.16).
It follows from (1.6) that

$$
G_{n}(x)=\sum_{j=0}^{2 n+1}(-1)^{j}\binom{2 n+1}{j} x^{j} \sum_{k=0}^{\infty}\left(\frac{k(k+1)}{2}\right)^{n} x^{k}=\sum_{k=0}^{\infty} x^{k} \sum_{\substack{j=0 \\ j \leqslant k}}^{2 n+1}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^{n} .
$$

$$
\begin{equation*}
G_{n, k}=\sum_{\substack{j=0 \\ j \leqslant k}}^{2 n+1}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^{n} \tag{2.1}
\end{equation*}
$$

Since the $(2 n+1)^{\text {th }}$ difference of a polynomial of degree $\leqslant 2 n$ must vanish identically, we have

$$
\text { (2.2) } \quad G_{n, k}=0 \quad(k \geqslant 2 n+1)
$$

Let $k \leqslant 2 n$. Then
(2.3) $0=\sum_{j=0}^{2 n+1}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^{n}=G_{n, k}+\sum_{j=k+1}^{2 n+1}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^{n}$

$$
=G_{n, k}-\sum_{j=0}^{2 n-k}(-1)^{j}\binom{2 n+1}{2 n-j+1}\left(\frac{(k+j-2 n-1)(k+j-2 n)}{2}\right)^{n}
$$

Therefore

$$
=G_{n k}-\sum_{j=0}^{2 n-k}(-1)^{j}\binom{2 n+1}{j}\left(\frac{(2 n-k-j)(2 n-k-j+1)}{2}\right)^{n}=G_{n, k}-G_{n, 2 k-k}
$$

(2.4)

Note also that, by (2.3),
(2.5)

Since by (2.4)

$$
\begin{gathered}
G_{n, k}=G_{n, 2 n-k} \quad(1 \leqslant k \leqslant 2 n-1) \\
G_{n, 2 n}=0
\end{gathered}
$$

it is clear that $G_{n}(x)$ is of degree $2 n-1$.

$$
G_{n, 2 n-1}=G_{n, 1}=1
$$

$$
\begin{aligned}
& \text { In the next place, by (1.7), } \\
& \qquad 2 \frac{G_{n+1}(x)}{(1-x)^{2 n+3}}=x \frac{d^{2}}{d x^{2}}\left\{\frac{x G_{n}(x)}{(1-x)^{2 n+1}}\right\}=\frac{x^{2} G_{n}^{\prime \prime}(x)+2 x G_{n}^{\prime}(x)}{(1-x)^{2 n+1}}+2(2 n+1) \frac{x^{2} G_{n}^{\prime}(x)+x G_{n}(x)}{(1-x)^{2 n+2}} \\
& \qquad \quad+(2 n+1)(2 n+2) \frac{x^{2} G_{n}(x)}{(1-x)^{2 n+3}} . \\
& \text { Hence } \\
& \text { (2.6) } 2 G_{n+1}(x)=(1-x)^{2}\left(x^{2} G_{n}^{\prime \prime}(x)+2 x G_{n}^{\prime}(x)\right)+3(3 n+1)(1-x)\left(x^{2} G_{n}^{\prime}(x)+x G_{n}(x)\right)+(2 n+1)(2 n+2) x^{2} G_{n}(x) .
\end{aligned}
$$

Comparing coefficients of $x^{k}$, we get, after simplification,
(2.7) $G_{n+1, k}=1 / 2 k(k+1) G_{n, k}-k(2 n-k+2) G_{n, k-1}+1 / 2(2 n-k+2)(2 n-k+3) G_{n, k-2} \quad(1 \leqslant k \leqslant 2 n-1)$.

For computation of the $G_{n}(x)$ it may be preferable to use (2.6) in the form
(2.8) $2 G_{n+1}(x)=(1-x)^{2} x\left(x G_{n}(x)\right)^{\prime \prime}+2(2 n+1)(1-x) x\left(x G_{n}(x)\right)^{\prime}+(2 n+1)(2 n+2) x^{2} G_{n}(x)$.

The following values were computed using (2.8):
(2.9)

$$
\left\{\begin{array}{l}
G_{0}(x)=1, \quad G_{1}(x)=x \\
G_{2}(x)=x+4 x^{2}+x^{3} \\
G_{3}(x)=x+20 x^{2}+48 x^{3}+20 x^{4}+x^{5} \\
G_{4}(x)=x+72 x^{2}+603 x^{3}+1168 x^{4}+603 x^{5}+72 x^{6}+x^{7}
\end{array}\right.
$$

Note that, by (2.1),

$$
\begin{gathered}
G_{n, 2}=3^{n}-(2 n+1), \quad G_{n, 3}=6^{n}-(2 n+1) \cdot 3^{n}+n(2 n+1) \\
G_{n, 4}=10^{n}-(2 n+1) \cdot 6^{n}+n(2 n+1) \cdot 3^{n}-\frac{1}{3} n\left(4 n^{2}-1\right)
\end{gathered}
$$

## and so on.

By means of (2.7) we can evaluate $G_{n}(1)$. Note first that (2.7) holds for $1 \leqslant k \leqslant 2 n+1$. Thus, summing over $k$, we get

$$
\begin{aligned}
G_{n+1}(1) & =\sum_{k=1}^{2 n-1} 1 / 2 k(k+1) G_{n, k}-\sum_{k=2}^{2 n} k(2 n-k+2) G_{n, k-1}+\sum_{k=3}^{2 n+1} 1 / 2(2 n-k+3)(2 n-k+3) G_{n, k-2} \\
& =\sum_{k=1}^{2 n-1}\{1 / 2 k(k+3)-(k+1)(2 n-k+1)+1 / 2(2 n-k)(2 n-k+1)\} G_{n, k}=\sum_{k=1}^{2 n-1}(n+1)(2 n+1) G_{n, k}
\end{aligned}
$$

so that
(2.10) It follows that

$$
\begin{aligned}
& G_{n+1}(1)=(n+1)(2 n+1) G_{n}(1) . \\
& G_{n}(1)=2^{-n}(2 n)!\quad(n \geqslant 0) .
\end{aligned}
$$

(2.11)

$$
G_{1}(1)=1, \quad G_{2}(1)=6, \quad G_{3}(1)=90, \quad G_{4}(1)=2520,
$$

in agreement with (2.9).

## 3. THE GENERAL CASE

It follows from

$$
\begin{equation*}
\frac{G_{n}^{(p)}(x)}{(1-x)^{p n+1}}=\sum_{k=0}^{\infty} T_{k, p}^{n} x^{k} \quad(p \geqslant 1, n \geqslant 0) \tag{3.1}
\end{equation*}
$$

that

$$
G_{n}^{(p)}(x)=\sum_{j=0}^{p n+1}(-1)^{j}\binom{p n+1}{j} x^{j} \sum_{k=0}^{\infty} x^{k} \sum_{\substack{j=0 \\ j \leqslant k}}^{p n+1}(-1)^{j}\left(p_{j}^{n+1}\right) T_{k-j, p}^{n}
$$

Since

$$
T_{k, p}=\binom{k+p-1}{p}
$$

is a polynomial of degree $p$ in $k$ and the ( $p n+1$ )th difference of a polynomial of degree $\leqslant p n$ vanishes identically, we have

$$
\begin{equation*}
\sum_{j=0}^{p n+1}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n}=0 \tag{3.3}
\end{equation*}
$$

Thus, for $p n-p+1<k \leqslant p n$,

$$
\begin{equation*}
\sum_{j=0}^{k}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n}=-\sum_{j=k+1}^{p n+1}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n} \tag{3.4}
\end{equation*}
$$

Since, for $p n-p+1<k \leqslant p n, k<p \leqslant p+1$, we have $-p<k-j<0$, so that $T_{k-j, p}=0(k+1 \leqslant j \leqslant p n+1)$. That is, every term in the right member of (3.4) is equal to zero. Hence (3.3) gives

$$
\begin{equation*}
\sum_{j=0}^{k}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n}=0 \quad(p n-p+1<k \leqslant p n) \tag{3.5}
\end{equation*}
$$

It follows that $G_{n}^{(p)}(x)$ is of degree $\leqslant p n-p+1$ :

$$
\begin{equation*}
G_{n}^{(p)}(x)=\sum_{k=0}^{p n-p+1} G_{n, k}^{(p)} x^{k} \quad(n \geqslant 1) \tag{3.6}
\end{equation*}
$$

where

By (3.3) and (3.7),

$$
\begin{equation*}
G_{n, k}^{(p)}=\sum_{j=0}^{k}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n} \quad(1 \leqslant k \leqslant p n-p+1) \tag{3.7}
\end{equation*}
$$

$$
\begin{equation*}
G_{n, k}^{(p)}=-\sum_{j=k+1}^{p n+1}(-1)^{j}\binom{p n+1}{j} T_{k-j, p}^{n}=(-1)^{p n} \sum_{j=0}^{p n-k}(-1)^{j}\binom{p n+1}{j} T_{k+j-p n-1, p}^{n} \tag{3.8}
\end{equation*}
$$

For $m \geqslant 0$, we have

$$
T_{-m, p}=\frac{(-m)(-m+1) \cdots(-m+p-1)}{p!}=(-1)^{p}\binom{m}{p}=(-1)^{p} T_{m-p+1, p} .
$$

Substituting in (3.8), we get

$$
G_{n, k}^{(p)}=(-1)^{p n} \sum_{j=0}^{p n-k}(-1)^{j}\binom{p n+1}{j} \cdot(-1)^{p n} T_{p n-k-j-p+2, p}^{n}=\sum_{j=0}^{p n-k}(-1)^{j}\binom{p n+1}{j} T_{(p n-k-p+2)-j, p}^{n}
$$

This evidently proves the symmetry relation
(3.9)

$$
G_{n, k}^{(p)}=G_{n, p n-k-p+2}^{(p)} \quad(1 \leqslant k \leqslant p n-p+1)
$$

For $p=1$, (3.9) reduces to (1.4); for $p=2$, it reduces to (1.9).
In the next place, it follows from (3.1) and (3.2) that

$$
\begin{aligned}
& p!\frac{G_{n+1}^{(p)}(x)}{(1-x)^{p(n+1)+1}}=x \frac{d^{p}}{d x^{p}} \quad x^{p-1}\left\{\frac{G_{n}^{(p)}(x)}{(1-x)^{p n+1}}\right\}=x \sum_{j=0}^{p}\binom{p}{j} \frac{d^{p-j}}{d x^{p-1}}\left(x^{p-1} G_{n}^{(p)}(x)\right) \cdot \frac{d^{j}}{d x^{p}}\left((1-x)^{-p n-1}\right) \\
& =x \sum_{j=0}^{p}\binom{p}{j}(p n+1)_{j}(1-x)^{-p n-j-1} \frac{d^{p-j}}{d x^{p-j}}\left(x^{p-1} G_{n}^{p)}(x)\right),
\end{aligned} \quad \begin{aligned}
& \text { where }
\end{aligned}
$$

$$
(p n+1)_{j}=(p n+1)(p n+2) \cdots(p n+j)
$$

We have therefore
(3.10)

$$
p!G_{n+1}^{(p)}(x)=x \sum_{j=0}^{p}\binom{p}{j}(p n+1)_{j}(1-x)^{p-j} \frac{d^{p-j}}{d x^{p-j}}\left(x^{p-1} G_{n}^{(p)}(x)\right)
$$

Substituting from (3.6) in (3.10), we get
$\rho!\sum_{m=1}^{p n+1} G_{n+1, m}^{(p)} x^{m}=x \sum_{j=0}^{p}\binom{p}{j}(p n+1)_{j}(1-x)^{p-j} \cdot \frac{d^{p-j}}{d x^{p-j}} \sum_{k=0}^{p n-p+1} G_{n, k}^{(p)} x^{k+p-1}=x \sum_{j=0}^{p}\binom{p}{j}(p n+1) \sum_{s=0}^{p-j}(-1)^{s}\binom{p-j}{s} x^{s}$
(3.11)

$$
\begin{aligned}
& \cdot \sum_{k=1}^{p n-p+1} G_{n, k}^{(p)}(k+j)_{p-j} x^{k+j-1}=\sum x^{m} \sum_{k+j+s=m}(-1)^{s}\binom{p}{j}\binom{p-j}{s}(p n+1)_{j}(k+j)_{p-j} G_{n, k}^{(p)} \\
= & \sum_{m=1}^{p n+1} x^{m} \sum_{\substack{k=1 \\
k \geqslant m-p}}^{m} G_{n, k}^{(p)} \sum_{j+s=m-k}(-1)^{s}\binom{p}{j}\binom{p-j}{s}(p n+1)_{j}(k+j)_{p-j} .
\end{aligned}
$$

The sum on the extreme right is equal to $\}$

$$
\begin{align*}
& \sum_{j+s=m-k}(-1)^{s} \frac{p!(p n+1)_{j}(k+j)_{p-j}}{j!s!(p-j-s)!}=\sum_{j=0}^{m-k}(-1)^{m-k-j} \frac{p!(p n+1)_{j}(k+p-1)!}{j!(m-k-j)!(k+p-m)!(k+j-1)!}  \tag{3.12}\\
&=(-1)^{m-k} \frac{p!(k+p-1)!}{(k-1)!(m-k)!(k+p-m)!} \sum_{j=0}^{m-k} \frac{(-m+k)_{j}(p n+1)_{j}}{j!(k)_{j}} .
\end{align*}
$$

By Vandermonde's theorem, the sum on the right is equal to

$$
\frac{(k-p n-1)_{m-k}}{(k)_{m-k}}=(-1)^{m-k} \frac{(p n-k+1)!(k-1)!}{(p n-m+1)!(m-1)!} .
$$

Hence, by (3.11) and (3.12),

$$
\begin{equation*}
G_{n+1, m}^{(p)}=\sum_{\substack{k=1 \\ k \geqslant m-p}}\binom{k+p-1}{m-1}\binom{p n-k+1}{m-k} G_{n, k}^{(p)} \quad(1 \leqslant m \leqslant p n+1) \tag{3.13}
\end{equation*}
$$

Summing over $m$, we get

$$
G_{n+1}^{(p)}(1)=\sum_{k=1}^{p n-p+1} G_{n, k}^{(p)} \sum_{m=k}^{k+p}\binom{k+p-1}{k+p-m}\binom{p n-k+1}{m-k}
$$

By Vandermonde's theorem, the inner sum is equal to

$$
\binom{p n+p}{p}
$$

so that
(3.14)

$$
G_{n+1}^{(p)}(1)=\binom{p n+p}{p} G_{n}^{(p)}(1)
$$

Since $G_{1}^{(p)}(x)=x$, it follows at once from (3.14) that (3.15)

By (3.10) we have

$$
G_{n}^{(p)}(1)=(p!)^{-n}(p n)!
$$

$$
p!G_{2}^{(p)}(x)=x \sum_{j=0}^{p}\binom{p}{j}(p+1)_{j}(1-x)^{p-j} \cdot \frac{p!}{j!} x^{j}
$$

so that

$$
\begin{equation*}
G_{2}^{(p)}(x)=x \sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} x^{j}(1-x)^{p-j} \tag{3.16}
\end{equation*}
$$

The sum on the right is equal to

$$
\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} x^{j} \sum_{s=0}^{p-j}(-1)^{s}\binom{p-j}{s} x^{s}=\sum_{k=0}^{p}\binom{p}{k} x^{k} \sum_{j=0}^{p-j}(-1)^{k-j}\binom{k}{j}\binom{p+j}{j}
$$

The inner sum, by Vandermonde's theorem or by finite differences, is equal to $\binom{p}{k}$. Therefore

$$
\begin{equation*}
G_{2}^{(p)}(x)=x \sum_{k=0}^{p}\binom{p}{k}^{2} x^{k} \tag{3.17}
\end{equation*}
$$

An explicit formula for $G_{3}^{(p)}(x)$ can be obtained but is a good deal more complicated than (3.17). We have, by (3.10) and (3.17),

$$
\begin{aligned}
p!G_{3}^{(p)}(x)=x \sum_{j=0}^{p}\binom{p}{j}(1-x)^{p-j} \cdot & \cdot \frac{d^{p-j}}{d x^{p-j}}\left\{\sum_{k=0}^{p}\binom{p}{k}^{2} x^{k+p}\right\}
\end{aligned}=x \sum_{j=0}^{p}(2 p+1) \cdot\binom{p}{j} \sum_{s=0}^{p-j}(-1)^{s}\binom{p-j}{s} x^{s} .
$$

The inner sum is equal to

$$
\begin{gathered}
\sum_{k+j+s=m}(-1)^{s} \frac{p!}{j!s!(p-s-j)!}\binom{p}{k}^{2} \frac{(k+p)!}{(k+j)!}(2 p+1)_{j}=\sum_{k+t=m}\binom{p}{k}^{2}\binom{p}{t} \frac{(k+p)!}{k!} \sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j} \frac{(2 p+1)_{j}}{(k+1)_{j}} \\
=\sum_{k+t=m}(-1)^{t}\binom{p}{k}^{2}\binom{p}{t} \frac{(k+p)!}{k!} \frac{(k-2 p)_{t}}{(k+1)_{t}}=\sum_{k+t=m}\binom{p}{k}^{2}\binom{p}{t} \frac{(k+p)!}{m!} \frac{(2 p-k)!}{(2 p-m)!} .
\end{gathered}
$$

Therefore

$$
\begin{equation*}
G_{3}^{(p)}(x)=x \sum_{m=0}^{2 p} x^{m} \sum_{k=0}^{m}\binom{p}{k}^{2}\binom{p}{m-k} \frac{(k+p)!(2 p-k)!}{p!m!(2 p-m)!} . \tag{3.18}
\end{equation*}
$$

## 4. COMBINATORIAL INTERPRETATION

As in the Introduction, put $Z_{n}=\{1,2, \cdots, n\}$ and consider sequences $\sigma=\left(a_{1}, a_{2}, \cdots, a_{N}\right)$, where the $a_{i} \in Z_{n}$ and the element $j$ occurs $e_{j}$ times in $\sigma, 1 \leqslant j \leqslant n$. A rise in $\sigma$ is a pair $a_{i}, a_{i+1}$ such that $a_{i}<a_{i+1}$, also a conventional rise to the left of $a_{1}$ is counted. The ordered set of nonnegative integers $\left[e_{1}, e_{2}, \cdots, e_{n}\right]$ is called the signature of $\sigma$. Clearly $N=e_{1}+e_{2}+\cdots+e_{n}$.
Let

$$
A\left(e_{1}, e_{2}, \cdots, e_{n} \mid r\right)
$$

denote the number of sequences $\sigma$ of specification $\left[e_{1}, e_{2}, \cdots, e_{n} \mid r\right]$ and having $r$ rises. In particular, for $e_{1}=e_{2}=$ $\ldots=e_{n}=p$, we put
(4.1)

The following lemma will be used.

$$
A(n, p, r)=A(p, \underbrace{p, \cdots, p}_{n}(r) .
$$

Lemma. For $n \geqslant 1$, we have

$$
\begin{equation*}
A(n+1, p, r)=\sum_{\substack{j=1 \\ j \geqslant r-p}}^{r}\binom{p n-j+1}{r-j}\binom{p+j-1}{r-1} A(n, p, j) \quad(1 \leqslant r \leqslant p n+1) . \tag{4.2}
\end{equation*}
$$

It is easy to see that the number of rises in sequences enumerated by $A(n+1, p, r)$ is indeed not greater than $p n+1$.
To prove (4.2), let $\sigma$ denote a typical sequence from $z_{n}$ of specification $[p, p, \cdots, p]$ with $j$ rises. The additional $p$ elements $n+1$ are partitioned into $k$ nonvacuous subsets of cardinality $f_{1}, f_{2}, \cdots, f_{k} \geqslant 0$ so that

$$
\begin{equation*}
f_{1}+f_{2}+\cdots+f_{k}=p, \quad f_{i}>0 . \tag{4.3}
\end{equation*}
$$

Now when $f$ elements $n+1$ are inserted in a rise of $\sigma$ it is evident that the total number of rises is unchanged, that is, $j \rightarrow j$. On the other hand, if they are inserted in a nonrise (that is, a fall or level) then the number of rises is increased by one: $j \rightarrow j+1$. Assume that the additional $p$ elements have been inserted in a rises and $b$ nonrises. Thus we have $j+b=r, a+b=k$, so that

$$
a=k+j-r, \quad b=r-j
$$

The number of solutions $f_{1}, f_{2}, \cdots, f_{k}$ of $(4.3)$, for fixed $k$, is equal to $\binom{p-1}{k-1}$. The a rises of $\sigma$ are chosen in

$$
\binom{j}{a}=\binom{j}{k+j-r}=\binom{j}{r-k}
$$

ways; the $b$ nonrises are chosen in

$$
\binom{p n-j+1}{b}=\binom{p n-j+1}{r-j}
$$

ways.
It follows that

The inner sum is equal to

$$
A(n+1, p, r)=\sum_{j} A(n, p, j) \cdot \sum_{k=1}^{p}\binom{p-1}{k-1}\binom{j}{r-k}\binom{p n-j+1}{r-j} .
$$

$$
\binom{p n-j+1}{r-j} \sum_{k=0}^{p-1}\binom{p-1}{k}\binom{j}{r-k-1}=\binom{p n-j+1}{r-j}\binom{p+j-1}{r-1}
$$

by Vandermonde's theorem. Therefore

$$
A(n+1, p, r)=\sum_{j=1}^{r}\binom{p n-j+1}{r-j}\binom{p+j-1}{r-1} A(n, p, j)
$$

This completes the proof of (4.2). The proof may be compared with the proof of the more general recurrence (2.9) for $A\left(e_{1}, \cdots, e_{n} \mid r, s\right)$ in [3].
It remains to compare (4.2) with (3.13). We rewrite (3.13) in slightly different notation to facilitate the comparison:

Since

$$
\begin{equation*}
G_{n+1, r}^{(p)}=\sum_{j=1}^{r}\binom{p n-j+1}{r-j}\binom{p+j-1}{r-1} G_{n, j}^{(p)} . \tag{4.4}
\end{equation*}
$$

$$
A_{n, 1}^{(p)}=G_{n, 1}^{(p)}=1 \quad(n=1,2,3, \cdots)
$$

it follows from (4.2) and (4.4) that (4.5)

$$
G_{n, r}^{(p)}=A(n, p, r)
$$

To sum up, we state the following
Theorem. The coefficient $G_{n, k}^{(p)}$ defined by

$$
G_{n}^{(p)}(x)=\sum_{k=1}^{p n-p+1} G_{n, k}^{(p)} x^{k}
$$

is equal to $A(n, p, k)$, the number of sequences $\sigma=\left(a_{1}, a_{2}, \cdots, a_{p n}\right)$ from $Z_{n}$, of specification $[p, p, \cdots, p]$ and having exactly $k$ rises.
As an immediate corollary we have

$$
\begin{equation*}
G_{n}^{(p)}(1)=\sum_{k=1}^{p n-p+1} G_{n, k}^{(p)}=(p!)^{-n}(p n)! \tag{4.6}
\end{equation*}
$$

Clearly $G_{n}^{(p)}(1)$ is equal to the total number of sequences of length $p n$ and specification $[p, p, \cdots, p]$, which, by a familiar combinatorial result, is equal to ( $p!$ ) $-n(p n)$ ! The previous proof (4.6) given in $\S 3$ is of an entirely different nature.

## 5. RELATION OF $G_{n}^{p}(x)$ TO $A_{n}(x)$

The polynomial $G_{n}^{(p)}$ can be expressed in terms of the $A_{n}(x)$. For simplicity we take $p=2$ and, as in $\S 2$, write $G_{n}(x)$ in place of $G(2)(x)$.
By (1.6) and (1.1) ${ }^{n}$ we have
$2^{n} \frac{G_{n}(x)}{(1-x)^{2 n+1}}=\sum_{k=0}^{\infty}(k(k+1))^{n} x^{k}=\sum_{k=0}^{\infty} x^{k} \sum_{j=0}^{n}\binom{n}{j} k^{n+j}=\sum_{j=0}^{n}\binom{n}{j} \sum_{k=0}^{\infty} k^{n+j_{x} k}=\sum_{j=0}^{n}\binom{n}{j} \frac{A_{n+j}(x)}{(1-x)^{n+j+1}}$,
so that
(5.1)

$$
2^{n} G_{n}(x)=\sum_{j=0}^{n}\binom{n}{j}(1-x)^{n-j} A_{n+j}(x)
$$

The right-hand side of (5.1) is equal to

$$
\sum_{j=0}^{n}\binom{n}{j} \sum_{s=0}^{n-j}(-1)^{s}\binom{n-j}{s} x^{s} \sum_{k=1}^{n+j} A_{n+j, k} x^{k}=\sum_{m=1}^{2 n} x^{m} \sum_{j=0}^{n} \sum_{\substack{k=1 \\ k \leqslant m}}^{n+j}(-1)^{m-k}\binom{n}{j}\binom{n-j}{n-k} A_{n+j, k}
$$

Since the left-hand side of (5.1) is equal to

$$
2^{n} \sum_{m=1}^{2 n-1} G_{n, m} x^{m}
$$

it follows that

$$
\begin{equation*}
2^{n} G_{n, m}=\sum_{k=1}^{m}(-1)^{m-k} \sum_{j=0}^{n-m+k}\binom{n}{j}\binom{n-j}{m-k} A_{n+j, k} \quad(1 \leqslant m \leqslant 2 n-1) \tag{5.2}
\end{equation*}
$$

and
(5.3)

$$
0=\sum_{k=n}^{2 n}(-1)^{k} \sum_{j=0}^{k-n}\binom{n}{j}\binom{n-j}{2 n-k} A_{n+j, k}
$$

In view of the combinatorial interpretation of $A_{n, k}$ and $G_{n, m}$, (5.2) implies a combinatorial result; however the result in question is too complicated to be of much interest.
For $p=3$, consider
$6^{n} x \frac{G_{n}^{(3)}(x)}{(1-x)^{3 n+1}}=\sum_{k=0}^{\infty} k^{n}\left(k^{2}-1\right)^{n} x^{k}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} \sum_{k=0}^{\infty} k^{n+2 j} x^{k}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} \frac{A_{n+2 j}(x)}{(1-x)^{n+2 j+1}}$.
Thus we have

$$
\begin{equation*}
6^{n} x G_{n}^{(3)}(x)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1-x)^{2 n-2 j} A_{n+2 j}(x) \tag{5.4}
\end{equation*}
$$

The right-hand side of (5.4) is equal to
$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} \sum_{s=0}^{2 n-2 j}(-1)^{s}\binom{2 n-2 j}{s} x^{s} \sum_{k=1}^{n+2 j} A_{n+2 j, k} x^{k}=\sum_{m=1}^{3 n} x^{m} \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} \sum_{k=1}^{n+2 j}(-1)^{m-k}\binom{2 n-2 j}{m-k} A_{n+2 j, k}$.
It follows that
(5.5)

$$
6^{n} G_{n, m-1}^{(3)}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} \sum_{k=1}^{n+2 j}(-1)^{m-k}\binom{2 n-2 k}{m-k} A_{n+2 j, k}
$$

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## [Continued from page 129.]

Recalling [2, p. 137] that

$$
(j+1) \sum_{k=1}^{n} k^{j}=B_{j+1}(n+1)-B_{j+1}
$$

where $B_{j}(x)$ are Bernoulli polynomials with $B_{j}(0)=B_{j}$, the Bernoulli numbers, we obtain from (2.3) with $x=1, B=$ 1, and $C_{k}=k$ the inequality

$$
\text { (2.4) } \quad B_{2 p}(n+1)-B_{2 p} \leqslant\left(B_{p}(n+1)-B_{p}\right)^{2} \quad(n=1,2, \cdots)
$$

For $p=2 k+1, k=1,2, \cdots, B_{2 k+1}=0$, and so (2.4) gives the inequality

$$
\begin{equation*}
B_{4 k+2}(n+1)-B_{4 k+2} \leqslant B_{2 k+1}^{2}(n+1) \quad(n, k=1,2, \cdots) \tag{2.5}
\end{equation*}
$$

3. AN INEQUALITY FOR INTEGER SEQUENCES

Noting that $U_{k}=k$ satisfies the difference equation

## [Continued on page 151.]

$$
U_{k+2}=2 U_{k+1}-U_{k}
$$

# WYTHOFF PAIRS 

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## 1. INTRODUCTION

The author had been working on the safe combinations (Wythoff pairs) in Wythoff's game [11] when the researches of Silber $[9,10]$ came to his attention. As the two approaches differ somewhat, it is probably worthwhile to indicate briefly the author's alternative treatment, which may throw a little light on the general problem.
Both Silber and the author use the fundamental idea of the canonical Fibonacci representation of an integer. While much work has been done recently on Fibonacci representation theory and on Nim-related games, we will attempt to minimize our reference list.
Wythoff pairs have been analyzed in detail by Carlitz, Scoville and Hoggatt, e.g., in [3, 4] , though without specific reference to Wythoff's game. For a better understanding of the principles used in our reasoning which follows, it is desirable to present a description of the nature and strategy of Wythoff's game.

## 2. WYTHOFF'S GAME

Wythoff's game was first investigated by W. A. Wythoff [11] in 1907. It is similar to Nim (see Bouton [2]) and may be described thus (Ball [1]):
Unspecified numbers of counters occur in each of two heaps. In each draw, a player may freely choose counters from either (i) one heap, or (ii) two heaps, provided that in this case he must take the same number from each.
For example, heaps of 1 and 2 can be reduced to 0 and 2 , or 1 and 1 , or 1 and 0 . The player who takes the last counter wins the game.
As Coxeter [5] remarks: "An experienced player, playing against a novice, can nearly always win by remembering which pairs of numbers are "safe combinations": safe for him to leave on the table with the knowledge that, if he does not make any mistake later on, he is sure to win. (If both players know the safe combinations, the outcome depends on whether the initial heaps form a safe or unsafe combination.)"
The safe combinations for Wythoff's game are known to be the pairs:
(1)

| $n=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,2)$, | $(3,5)$, | $(4,7)$, | $(6,10)$, | $(8,13)$, | $(9,15)$, | $(11,18)$, | $(12,20)$, | $\cdots$ |

A safe pair may also be called a Wythoff pair.
There are several interesting things about the integers occurring in these safe combinations. They are:
(I) Members of the first pair of integers differ by 1 , of the second pair by 2 , of the third by $3, \ldots$, of the $n^{\text {th }}$ pair by $n$.
(II) The $n^{\text {th }}$ pair is $\left(a_{n}, b_{n}\right) \equiv\left([n a],\left[n a^{2}\right]\right)$, where the symbol $[x]$ denotes the greatest integer which is less than, or equal to, $x$, and $a=(1+\sqrt{5}) / 2 \doteq 1.618$ (so that $\left.a^{2}=(3+\sqrt{5}) / 2 \doteq 2.618\right)$. We recognize $a$ as the "golden section" number which is a root of $x^{2}-x-1=0$ (i.e., $a^{2}=a+1$ ). Note that $b_{n}=a_{n}+n$, i.e., $\left[n a^{2}\right]=[n a]+n$.
(III) In the list of integers occurring in the ordered pairs for safe combinations, each integer appears exactly once (i.e., every interval between two consecutive positive integers contains just one multiple of either $a$ or $a^{2}$, as Ball [1] observes).
(IV) In every pair of a safe combination, the smaller number is the smallest integer not already used and the larger number is chosen so that the difference in the $n$th pair is $n$.

It might reasonably be asked: How does the "golden section" number a come into the solution of Wythoff's game? The answer is detailed in Coxeter [5] where the solution given by Hyslop (Glasgow) and Ostrowski (Göttingen) in 1927, in response to a problem proposed by Beatty (Toronto) a year earlier, is reproduced. (Coxeter notes that Wythoff himself obtained his solution "out of a hat" without any mathematical justification). Basically, the answer to our query, as given by Hyslop and Ostrowski quoted in [5] , depends on the occurrence of the equations ( $1 / x$ ) + $(1 / y)=1, y=x+1$ which, when $y$ is eliminated, yield our quadratic equation $x^{2}-x-1=0$.
That the Wythoff pairs are ultimately connected with Fibonacci numbers should not surprise us since, by Hoggatt [6], the $n^{\text {th }}$ Fibonacci number

$$
F_{n}=\left[\frac{a^{n}}{\sqrt{5}}+\frac{1}{2}\right]
$$

for $n=1,2,3, \cdots$, i.e., both Wythoff pairs and Fibonacci numbers involve $[x]$. (See (II) above.)
The first forty Wythoff pairs are listed in Silber [9]. From the rules of construction (I)-(IV) it is only a matter of patience for the interested reader to form as long a list of Wythoff pairs as is desired.

## 3. WYTHOFF PAIRS AS MEMBERS OF $\left\{H_{m}(p, q)\right\}$

Consider the generalized Fibonacci sequence $\left\{H_{m}(p, q)\right\}$ of integers (Horadam [7]):

$$
\begin{array}{cccccccc}
\mathrm{H}_{0} & \mathrm{H}_{1} & \mathrm{H}_{2} & \mathrm{H}_{3} & \mathrm{H}_{4} & \mathrm{H}_{5} & \mathrm{H}_{6} & \mathrm{H}_{7}  \tag{2}\\
q & p & p+q & 2 p+q & 3 p+2 q & 5 p+3 q & 8 p+5 q & 13 p+8 q
\end{array} \cdots
$$

where

$$
\begin{equation*}
H_{m}=H_{m-1}+H_{m-2} \quad(m \geqslant 2) \tag{3}
\end{equation*}
$$

in which we omit $p, q$ when there is no possible ambiguity. The restriction $m \geqslant 2$ in (3) may be removed, if desired, to allow for negative subscripts.
The ordinary Fibonacci sequence $\left\{F_{m}\right\}$ with $F_{0}=0, F_{1}=F_{2}=1$ occurs when $p=1, q=0$, i.e.,

$$
\begin{equation*}
F_{m}=H_{m}(1,0) . \tag{4}
\end{equation*}
$$

It is known [7] that
(5)

$$
H_{m}=p F_{m}+q F_{m-1} .
$$

Every positive integer $N$
(a) produces a $H_{1}(=[\mathrm{Na}]=p)$ which is the first member of a Wythoff pair, i.e., sequences $\left\{H_{m}([\mathrm{Na}], N)\right\}$ yield all the Wythoff pairs; and
(b) is, by (IV), a member of a Wythoff pair and a member of some $H$-sequence (in fact, of infinitely many $H$ sequences of which the given H -sequence forms a part),
e.g., $52=H_{1}(52,32)=H_{2}(32,20)=H_{3}(20,12)=H_{4}(12,8)=H_{5}(8,4)=H_{6}(4,4)=H_{7}(4,0)=\ldots$
with infinite extension through negative values of $m$ if the restriction $m \geqslant 2$ in (3) is removed.
$\ulcorner$ Every positive integer $N$ is obviously also a member of infinitely many non-Wythoff pairs belonging to infinitely many different $t$-sequences, e.g.,

$$
N=2000=20 F_{11}+4 F_{10}\left(=H_{11}(20,4)\right)=20 \times 89+4 \times 55
$$

is a member of all the $H$-sequences resulting from the solution, by Euclid's algorithm or by congruence methods, of the Diophantine equation $89 x+55 y=2000$. Some instances of this are

$$
2000=H_{11}(75,-85)=H_{11}(-35,93)
$$

yielding the non-Wythoff pairs $(1235,2000)$ and $(1237,2000)$ whereas $(1236,2000)$ is a Wythoff pair $\left(\equiv\left(a_{764}, b_{764}\right)\right.$ ).」
Now make the identification with the notation in [9]:

$$
\begin{equation*}
a_{n}=H_{m}, \quad b_{n}=H_{m+1}, \quad a_{b_{n}}=H_{m+2} . \text { for some } p, q \tag{6}
\end{equation*}
$$

For example, $n=6$ yields the Wythoff pair $\left(a_{6}, b_{6}\right) \equiv(9,15) \equiv\left(H_{3}, H_{4}\right)$ for $p=3, q=3$.
To save space, we will assume the results in [9] expressed in our notation:

Theorem. All pairs in an $H$-sequence after a Wythoff pair are Wythoff pairs, i.e., each Wythoff pair generates a sequence of Wythoff pairs.
Theorem. A Wythoff pair $\left(H_{m}, H_{m+1}\right)$ is primitive if and only if (for $\left.H_{m}=a_{n}\right) n=a_{k}$ for some positive integer $k$, and for some positive integers $p, q$.
A primitive Wythoff pair is a Wythoff pair which is not generated by any other Wythoff pairs. Thus, (1,2), (4.7), $(6,10),(9,15),(88,143)$ are primitive Wythoff pairs.

## 4. ZECKENDORF'S CANONICAL REPRESENTATION

Zeckendorf's Theorem, quoted in Lekkerkerker [8], states: (Zeckendorf's Theorem) Every positive integer $N$ can be represented as the sum of distinct Fibonacci numbers, using no two consecutive numbers, and such a representation is unique.
Symbolically, this canonical (Zeckendorf) representation of $N$ is
(7)

$$
N=F_{k_{1}}+F_{k_{2}}+\ldots+F_{k_{r}},
$$

where
(8)

$$
k_{1}>k_{2}>\cdots>k_{r} \geqslant 2(r \text { depending on } N)
$$

and
(9)

$$
k_{j}-k_{j+1} \geqslant 2 \quad(j=1,2, \cdots, r-1)
$$

From [9] , the criteria for a Wythoff pair ( $H_{m}, H_{m+1}$ ) are that, in the canonical representation (7), with (8) and (9),
A.
(i)

$$
\left\{\begin{array}{c}
H_{m}=F_{k_{1}}+F_{k_{2}}+\ldots+F_{k_{r}} \\
H_{m+1}=F_{k_{1}+1}+F_{k_{2}+1}+\ldots+F_{k_{r}+1}
\end{array}\right.
$$

(ii) $k_{r}$ is even.
For a primitive Wythoff pair, we have further that
B.
(i) $\quad k_{r}=2$
(ii)

$$
n=F_{k_{1-z}}+F_{k_{2}-z}+\ldots+F_{k_{r} z}
$$

where
(10)

$$
z=k_{r}-1 \quad\left(k_{r}-z=1\right)
$$

so the last term in $n(\mathrm{~B}(\mathrm{ii}))$ is $F_{1}=1$.
Examples. (1) Non-Wythoff pair $(62,100)$
$62=F_{10}+F_{5}+F_{3}$ so $k_{r}=3$ which is not even, and A (ii) is therefore invalid (though $A(i)$ holds).
(2) Non-Wythoff pair $(62,101)$

$$
101=F_{11}+F_{6}+F_{4}+F_{1} \text { so } A(i) \text { is invalid (and so is } A(i i) \text { ). }
$$

(3) Non-primitive Wythoff pair $(1236,2000) \equiv\left(a_{764}, b_{764}\right)$ $1236=F_{16}+F_{13}+F_{7}+F_{4}$ so $B(i)$ is invalid (and so is $B(i i)$ ), though $A(i), A(i i)$ are valid.
(4) Primitive Wythoff pair $(108,175) \equiv\left(a_{67}, b_{67}\right)$

$$
108=F_{11}+F_{7}+F_{5}+F_{2}, \quad 175=F_{12}+F_{8}+F_{6}+F_{3}
$$

so $A(i), A(i i), B(i)$ are valid and $\left(175-108=67=F_{10}+F_{6}+F_{4}+F_{1}\right.$
so $B$ (ii) holds.
5. WYTHOFF PAIRS, ZECKENDORF'S REPRESENTATION AND $\left\{H_{m}(p, q)\right\}$

From (5) and (7) we have, for $k_{1} \geqslant m>m-1 \geqslant k_{r}$,

$$
N=H_{m}(p, q)=p F_{m}+q F_{m-1}=F_{k_{1}}+F_{k_{2}}+\ldots+F_{k_{r}}
$$

A little thought reveals that
where
(13)

$$
\begin{equation*}
N=H_{m}(p, q)=H_{m}\left(p^{\prime}, q^{\prime}\right), \tag{12}
\end{equation*}
$$

$$
\begin{gathered}
m^{\prime}=z \\
p^{\prime}=H_{m-z+1}(p, q) \\
q^{\prime}=H_{m-z}(p, q)
\end{gathered}
$$

(15)
in which $\left(p^{\prime}, p^{\prime}+q^{\prime}\right)$ is a primitive Wythoff pair. That is, the sequence $\left\{H_{m}^{\prime}\left(p^{\prime}, q^{\prime}\right)\right\}$ is generated by a primitive Wythoff pair.
The explanation of (12)-(15) is as follows. If, in (12), $p^{\prime}$ is the first member of a primitive Wythoff pair, then by $\mathrm{B}(\mathrm{i})$ its canonical representation must end with $F_{2}$. Thus, by (11), $p^{\prime}$ precedes $N$ by $k_{r}-2=z-1$ places by (10), i.e., $p^{\prime}$ is located in term position $m-(z-1)=m-z+1$ in the $H$-sequence. Hence, we have (14) and consequently (15). Clearly, $m^{\prime}=(z-1)+1=z$ giving (13).

It is now possible to determine, for any positive integer $N$, exactly which Wythoff pair generates the $H$-sequence in which that $N$ appears, as well as the location of $N$ in that sequence (as is done in [9]).

## Examples.

(1)
(so $z=4$ by (10))

$$
\begin{aligned}
N & =52=F_{9}+F_{7}+F_{5} \\
& =4 F_{7}+0 . F_{6}
\end{aligned}
$$

by repeated use of $F_{m}=F_{m-1}+F_{m-2}$

$$
=H_{7}(4,0)
$$

by (11) ( so $m=7, p=4, q=0$ )

$$
=H_{4}(12,8)
$$

by (12)-(15) since

$$
m^{\prime}=z=4, \quad p^{\prime}=H_{4}(4,0)=12, \quad q^{\prime}=H_{3}(4,0)=8 .
$$

That is, $N=52$ is the $4 t h$ term in the sequence $\{H(12,8)\}$ generated by the primitive Wythoff pair $(12,20) \equiv\left(a_{8}, b_{8}\right)$ :

$$
N=1000=F_{16}+F_{7}=H_{11}(10,2)=H_{6}(90,56)
$$

(2)
i.e., 1000 is the $6^{\text {th }}$ term in the sequence generated by the primitive Wythoff pair $(90,146) \equiv\left(a_{56}, b_{56}\right)$ :


Example (1) is given by Silber [9]. Comparing his zero-unit notation with our $H$-notation in relation to canonical representations, we see that our $z$ is a suggestive symbol as it is also the number of zeros at the right-hand end of the zero-unit notation for a canonical representation. Checking that $(12,20)$ and $(90,146)$ in the examples above are indeed primitive Wythoff pairs is straightforward.

## ACKNOWLEDGEMENT

Thanks are due to Professor Robert Silber for sending the author preprints of some of his forthcoming articles on Fibonacci representations, thus making available further information, including references.

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[Continued from page 146.]
and that $V_{n}=2$ satisfies
we can rewrite (1.2) as

$$
V_{n+2}=2 V_{n+1}-V_{n},
$$

$$
2 \sum_{k=1}^{n} u_{k}^{3}=v_{n}\left(\sum_{k=1}^{n} u_{k}\right)^{2}
$$

This suggests the following result for integer sequences.
Conjecture. Let $U_{k}$, with $U_{0}=0, U_{1}=1$, and $V_{k}$, with $V_{0}=2, V_{1}=P$, be two solutions of

$$
W_{k+2}=P W_{k+1}+Q W_{k}, \quad k=0,1, \cdots
$$

where $P$ and $Q$ are integers with $P \geqslant 2$ and $P+Q \geqslant 1$. We then claim that

$$
\begin{equation*}
2 \sum_{k=1}^{n} U_{k}^{3} \leqslant V_{n}\left(\sum_{k=1}^{n} U_{k}\right)^{2} \quad(n=1,2, \cdots) . \tag{3.1}
\end{equation*}
$$

Remarks. For $P=2$ and $Q=-1,(3.1)$ gives (1.2). Using double induction, one can prove the conjecture for $P+Q$ $\geqslant 3$, which leaves the two cases $P+Q=2$ and $P+Q=1$ open.

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## *****

# GEOMETRIC SEQUENCES AND THE INITIAL DIGIT PROBLEM 

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Recently, R. L. Duncan discussed the initial digit problem for the sequence of positive integers, J [1]. The subsequence of positive integers with initial digit $a \in\{1,2, \ldots, 9\}$ is denoted by $A=\{a \gamma\}$. Although the asymptotic density of $A$ in $J^{\prime}$ does not exist, the logarithmic density of $A$ in $J$ is $\log (1+1 / a)$, where $\log x$ is the common $\log$ arithm of $x$.
The purpose of this note is to show that the relative asymptotic density of $A$ in certain geometric sequences is also $\log (1+1 / a)$.
Let $c$ denote a positive integer which is not a power of ten. We adopt the following definitions.
Definition 1.
Definition 2.
Definition 3.
Definition 4.

## Definition 5.

## Definition 6.

Clearly $c^{m} \in A^{\prime}$ iff
(1)

$$
a 10^{t} \leqslant c^{m}<(a+1) 10^{t} \quad(t \geqslant 0) .
$$

But (1) is equivalent to
(2)

$$
\begin{aligned}
& B(m)=\left\{y \mid y=c^{n}, n \geqslant 1, y \leqslant c^{m}, m \in J\right\} . \\
& B^{\prime}=\underbrace{}_{m \in J} B(m) . \\
& A(m)=A \cap B(m) . \\
& A^{\prime}=A \cap B^{\prime} . \\
& a(m)=\sum_{y \in A(m)} 1 .
\end{aligned}
$$

$$
b(m)=\sum_{y \in B(m)} 1=m .
$$

$$
\left[\frac{t+\log a}{\log c} \leqslant m<\frac{t+\log (a+1)}{\log c}\right) .
$$

Let

$$
t_{t+1}=\left[\frac{t+\log a}{\log c}, \frac{t+\log (a+1)}{\log c}\right) \quad(t \geqslant 0)
$$

and $\left|t_{t+1}\right|$ denote the length of $t_{t+1}$.
Obviously
(3)

$$
\left|I_{t+1}\right|=\frac{\log (1+1 / a)}{\log c} \leqslant \frac{\log 2}{\log c} \leqslant 1 .
$$

In fact, $\left|I_{t+1}\right|=1$ iff $a=1$ and $c=2$. Let $z_{t+1}$ denote the midpoint of $I_{t+1}$.

$$
\begin{equation*}
z_{t+1}=\frac{2 t+\log a(a+1)}{\log c} \quad(t \geqslant 0) . \tag{4}
\end{equation*}
$$

Lemma 1. $\left\{z_{t}\right\}_{t=1}^{\infty}$ is uniformly distributed $\bmod 1$.
Proof. $\quad \lim _{t \rightarrow \infty}\left(z_{t+1}-z_{t}\right)=\lim _{t \rightarrow \infty} \frac{2}{\log c}=\frac{2}{\log c}$ and $\frac{2}{\log c}$ is irrational [2].

$$
152
$$

Hence, $\left\{z_{t}\right\}_{t=1}^{\infty}$ is uniformly distributed $\bmod 1$ [3].
Lemina 2. $\quad a\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)=\left|I_{1}\right| n+o(n)$,
where $[x]$ denotes the greatest integer in $x$.
Proof. Obviously

$$
a\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)
$$

is the number of intervals, $I_{1}, I_{2}, \cdots, I_{n}$ which contain an integer and this is $n$ less the number of intervals which contain no integer. Since $\left|I_{t+1}\right| \leqslant 1$, it is clear that each interval contains at most one integer. If $\left|I_{t+1}\right|=1(c=2, a=1)$, then

$$
a\left(\left[\frac{n+\log 2}{\log 2}\right]\right)=n=\left|I_{1}\right| n+o(n)
$$

If $\left|I_{t+1}\right|<1, I_{t+1}$ will not contain an integer if, and only if

$$
z_{t+1} \in\left(j+\frac{\left|1_{1}\right|}{2}, j+1-\frac{\left|1_{1}\right|}{2}\right)
$$

where $z_{t+1} \in(j, j+1)$ for some integer, $j$. Using Lemma 1 and the definition of uniform distribution mod 1 [4], we have

$$
n-a\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)=\left(1-\left|I_{1}\right|\right) n+o(n)
$$

and the result follows.
Let $d(a)$ denote the relative asymptotic density of $A^{\prime}$ in $B^{\prime}$ defined as follows:

$$
\begin{equation*}
d(a)=\lim _{x \rightarrow \infty} \sum_{\substack{a \gamma \leqslant x \\ a \gamma \in A^{\prime}}} 1 / \sum_{\substack{n \leqslant x \\ n \in B^{\prime}}} 1 \tag{5}
\end{equation*}
$$

The upper and lower relative asymptotic densities of $A^{\prime}$ in $B^{\prime}$ are obtained by replacing "limit" in (6) by "limit superior" and "limit inferior," respectively, and are denoted by $\overline{d(a)}$ and $\underline{d(a)}$, respectively [5]. We conclude the discussion with our main result.

## Theorem.

$$
d(a)=\log (1+1 / a)
$$

Proof. It is clear that
(6)

$$
\underline{d(a)}=\lim _{n \rightarrow \infty} \frac{a\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)}{b\left(\left[\frac{n+1+\log (a+1)}{\log c}\right]\right)-1}
$$

(7)

$$
\overline{d(a)}=\lim _{n \rightarrow \infty} \frac{a\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)}{b\left(\left[\frac{n+\log (a+1)}{\log c}\right]\right)}
$$

Since

$$
\left[\frac{n+\log (a+1)}{\log c}\right] \sim \frac{n+\log (a+1)}{\log c},
$$

the application of Lemma 2 transforms (6) and (7) into

$$
\begin{equation*}
\underline{d(a)}=\lim _{n \rightarrow \infty} \frac{\left|I_{1}\right| n+o(n)}{\frac{n+1+\log (a+1)}{\log c}+o(n)}=\left|\iota_{1}\right| \log c=\log (1+1 / a) \tag{8}
\end{equation*}
$$

and
(9)

$$
\overline{d(a)}=\lim _{n \rightarrow \infty} \frac{\left|I_{1}\right| n+o(n)}{\frac{n+\log (a+1)}{\log c}+o(n)}=\log (1+1 / a)
$$

and the desired conclusion follows.

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****

## ADDENDA TO ADVANCED PROBLEMS AND SOLUTIONS

These problem solutions were inadvertently skipped over for a few years. Our apologies.

## FORM TO THE RIGHT

H-211 Proposed by S. Krishman, Orissa, India. (corrected)
A, Show that $\binom{2 n}{n}$ is of the form $2 n^{3} k+2$ when $n$ is prime and $n>3$.
B. Show that $\binom{2 n-2}{n-1}$ is of the form $n^{3} k-2 n^{2}-n$, when $n$ is prime.

$$
\binom{m}{j} \text { represents the binomial coefficient, } \frac{m!}{j!(m-j)!} .
$$

Solution by P. Tracy, Liverpool, New York.
A. The Vandermonde convolution identity is $\left.\binom{n}{m}=\Sigma^{\prime n} \begin{array}{c}n-L\end{array}\right)\binom{L}{m-k}$. Appling this to $\binom{2 p}{p}$ (using $\left.L=p\right)$, we get

$$
\binom{2 p}{p}=\sum_{k=0}^{p}\binom{p}{k}^{2}=2+\sum_{k=1}^{p-1}\binom{p}{k}^{2} .
$$

Since $p$ is a prime, $p \left\lvert\,\binom{ p}{k}\right.$ for $k=1,2, \cdots, p-1$. Now

$$
\binom{p}{k}^{2} \equiv p^{2} \quad \frac{(p-1)(p-2) \ldots(p-k+1)^{2}}{k!}\left(\bmod p^{3}\right)
$$

Also $(p-i) / i \equiv-1(\bmod p)$ and so

$$
\frac{1}{p^{2}} \sum_{k=1}^{p-1}\left(\frac{p}{k}\right)^{2} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{2}} \equiv 2 \quad \text { quad. res. }(\bmod p)
$$

(since every quadratic residue $\bmod p$ has exactly two roots, $\pm a$ ). Let $g$ be a primitive root, $\bmod p$, then the quadratic residues are

$$
1, g^{2}, g^{4}, \cdots, g^{\frac{p-3}{2}}
$$

To find the sum of the quadratic residues, we use the geometric sum formula to obtain (g $\left.g^{p-1}-1\right) /\left(g^{2}-1\right)$. Note that $p>3$ implies $g^{2}-1 \not \equiv 0(\bmod p)$. Hence $\Sigma$ quad. res. $\equiv 0(\bmod p)$. Therefore
[Continued on page 165.]

$$
2 p^{3} \left\lvert\, \sum_{k=1}^{p-1}\binom{p}{k}^{2} \quad\right. \text { and } \quad\binom{2 p}{p} \equiv 2\left(\bmod 2 p^{3}\right)
$$

# A SECOND VARIATION ON A PROBLEM OF DIOPHANTUS AND DAVENPORT* 

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## 1. INTRODUCTION

In a paper appearing in the Quarterly Journal of Mathematics [V.ol. 20 (1969), pp. 129-137], Harold Davenport and Alan Baker dealt with the set of numbers: 1,3,8,120. It has the property, noted by Fermat, that the product of any two increased by one is a square. We call such a set a $P$-set. Davenport and Baker proved, using the "effective" results of the latter, that if $1,3,8, c$ is a $P$-set, then $c$ must be 120 .

Long before, Diophantus noticed that the set $x, x+2,4 x+4,9 x+6$ is a $P$-set for $x=1 / 16$. Indeed, the first three have the same property considered as polynomials in $x$. In a previous paper [Quar. Jour. Math., Vol. 27 (1976), pp. 349-353] the author proved that the only $P$-sets containing $x$ and $x+2$ in $Z[x]$ are

$$
x, \quad x+2, \quad c_{r}(x), \quad c_{r+1}(x)
$$

where $r$ is a positive integer and the $c_{i}$ are certain polynomials defined recursively.
Here we consider a similar problem in a more general setting. Let $a=a(x)$ and $b=b(x)$ be two non-zero polynomials in $Z[x]$ such that $a b+1=w^{2}$, where $w$ is in $Z[x]$. [We omit the argument $x$ when there is no ambiguity.] Without loss, we may assume that $a, b$, and $w$ are in $Z^{+}[x]$, that is, have positive leading coefficients. We want to allow $a$ and $b$ to be in $Z$; in this case $Z^{+}[x]$ becomes the set of positive integers.

First we seek all solutions $c_{k}=c_{k}(x)$ in $Z^{+}[x]$ of

$$
\begin{equation*}
a c_{k}+1=y_{k}^{2}, \quad b c_{k}+1=z_{k}^{2}, \quad y_{k} \text { and } z_{k} \text { in } Z^{+}[x] . \tag{1.1}
\end{equation*}
$$

An equivalent pair of equations is

$$
\begin{equation*}
(b-a) c_{k}=z_{k}^{2}-y_{k}^{2}, \quad b-a=b y_{k}^{2}-a z_{k}^{2} . \tag{1.2}
\end{equation*}
$$

In the previous paper we considered the case when $b=a+2$. Then there was just one sequence of $c_{k}$. If $b \neq a+2$, there ate at least two such sequences. We prove that if $a, b$, and $c$ form a $P$-set and all are of the same positive degree, then there is no fourth of the-same degree which, with $a, b$, and $c$, forms a $P$-set. We prove that if $a$ and $b$ are both linear or quadratic there are exactly two sequences. If $a$ and $b$ are in $Z$ and $a<b<4 a$ we prove that there are exactly two sequences of $c_{k}$ (unless $b=a+2$ ); we also show that if $a<b<c<d$ form a $P$-set, then $d>a b+1$. Our most significant result is that when $a$ and $b$ are linear over $Z^{+}[X], c=a+b+2 w$, and $a, b, c, d$ form a $P$-set of four elements, then there is exactly one possible $d$, namely

$$
c_{2}(a, b)=\left(4 w^{2}-2\right) c+2(a+b),
$$

where $a b+1=w^{2}$. The proof of this result is an adaptation of one of B. J. Birch given in the previous paper. We show that if $a$ and $b$ are two successive even-indexed Fibonacci numbers, $c_{2}(a, b)$ reduces to $4 b\left(b^{2}+1\right)$ and is not a Fibonacci number. A final section describes some results which seem true but for which we have no proofs.
Since much of the theory is the same for integers and polynomials it is convenient to define an extension of the idea of inequality from integers to polynomials in $Z[x]$.
Definition. When we write "a is in $Z[x]$ " we mean that it is either a polynomial of positive degree or an integer. In the latter case, we call it its own "leading coefficient." The symbol $a>0$ means that the leading coefficient of $a$ is positive. Similarly if $a$ and $b$ are in $Z[x], a>b$ means that $a-b>0$. The usual fundamental properties of inequality hold for this extension.
We assume throughout that

[^0]$$
0<a<b
$$

If $n_{a}=\operatorname{deg} a$ denotes the degree of $a$ in $x$, and similarly for $n_{b}$, then (1.3) implies $n_{a} \leqslant n_{b}$. Note that $n_{a}$ and $n_{b}$ must be of the same parity and $2 n=n_{a}+n_{b}$, where $n=n_{w}$. Define $c_{0}$ to be 0 and have as a consequence that $y_{0}=z_{0}=1$.

## 2. FORMULAS FOR $c_{k}$ IN (1.1) AND (1.2)

In order to find a formula for $c_{k}$ we first seek a recursion formula for $y_{k}$ and $z_{k}$. To this end, write

$$
\begin{equation*}
\left(\sqrt{b} y_{k}+\sqrt{a} z_{k}=(w+\sqrt{a b})\left(\sqrt{b} y_{k-1}+\sqrt{a} z_{k-1}\right)\right. \tag{2.1}
\end{equation*}
$$

that is

$$
\begin{align*}
& y_{k}=w y_{k-1}+a z_{k-1} \\
& z_{k}=b y_{k-1}+w z_{k-1} . \tag{2.2}
\end{align*}
$$

To see that (2.2) defines a sequence of solutions of (1.1) suppose that $y_{k-1}, z_{k-1}$ is a solution of the second equation of (1.2). In (2.1) replace $\sqrt{a}$ by $-\sqrt{a}$ and multiply corresponding sides of the two equations to get:

$$
b y_{k}^{2}-a z_{k}^{2}=\left(w^{2}-a b\right)\left(b y_{k-1}^{2}-a z_{k-1}^{2}\right) .
$$

Another way to show this is to use Eqs. (2.2) directly in the second equation of (1.2). We show below that the first equation of (1.2) defines $c_{k}$.
Now $w y_{k}-a z_{k}=y_{k-1}$ which implies

$$
y_{k}=w y_{k-1}+w y_{k-1}-y_{k-2}=2 w y_{k-1}-y_{k-2} .
$$

Also $w z_{k}-b y_{k}=z_{k-1}$ implies $z_{k}=2 w z_{k-1}-z_{k-2}$. So

$$
\begin{equation*}
y_{k}-2 w y_{k-1}+y_{k-2}=0 \quad \text { and } \quad z_{k}-2 w z_{k-1}+z_{k-2}=0 . \tag{2.3}
\end{equation*}
$$

Note that $y_{1}=w+a$ and $z_{1}=b+w$ with (1.2) imply that $c_{1}=2 w+a+b$. By induction, deg $y_{k}=k n$. Hence, from (1.1) $\operatorname{deg} c_{k}=2 k n-n_{a}$, if $k>0$, and $\operatorname{deg} z_{k}=(k+1) n-n_{a}$.

Let $a$ and $a^{-1}$ be the zeroes of $e^{2}-2 w e+1$. Thus

$$
a=w+\sqrt{a b} \quad \text { and } \quad a^{-1}=w-\sqrt{a b} .
$$

Note that $a b \neq 0$ implies that $w \neq 1$. We seek $y_{k}, z_{k}$, and $c_{k}$ in terms of $a$ and $a^{-1}$. Thus we want to determine $r$ and $s$ so that

$$
y_{k}=\frac{r a^{k}-s a^{-k}}{a-a^{-1}}
$$

Now $r-s=a-a^{-1}$ and $r a-s a^{-1}=(w+a)\left(a-a^{-1}\right)$. This shows that

$$
r=w+a-a^{-1} \quad \text { and } \quad s=w+a-a .
$$

Hence

$$
y_{k}=(w+a) f_{k}-f_{k-1} \quad \text { and, similarly, } \quad z_{k}=(w+b) f_{k}-f_{k-1},
$$

where

$$
f_{k}=\frac{a^{k}-a^{-k}}{a-a^{-1}}
$$

Thus we have

$$
\left(z_{k}-y_{k}\right)\left(z_{k}+y_{k}\right)=(b-a) f_{k}\left[(2 w+a+b) f_{k}-2 f_{k-1}\right] .
$$

Recalling that $c_{1}=a+b+2 w$, we have, from (1.2),

$$
\begin{equation*}
c_{k}=f_{k}\left(c_{1} f_{k}-2 f_{k-1}\right) \tag{2.4}
\end{equation*}
$$

It is interesting and useful to find a recursion formula for $c_{k}$. To this end note that $e^{2}-2 w e+1=0$ implies

$$
e^{4}-\left(4 w^{2}-2\right) e^{2}+1=0
$$

Thus

$$
\begin{equation*}
\left(a^{ \pm 2}\right)^{k}-\left(4 v v^{2}-2\right)\left(a^{ \pm 2}\right)^{k-1}+\left(a^{ \pm 2}\right)^{k-2}=0, \quad \text { for } k \geqslant 2 . \tag{2.5}
\end{equation*}
$$

Then $f_{k}^{2}=a^{2 k}+a^{-2 k}+N$, where $N$ is independent of $k$, and (2.5) implies

$$
\begin{equation*}
f_{k}^{2}=\left(4 w^{2}-2\right) f_{k-1}^{2}-f_{k-2}^{2}+N^{\prime} \tag{2.6}
\end{equation*}
$$

where $N^{\prime}$ is independent of $k$. Furthermore

$$
\left(f_{k}+f_{k-1}\right)-2 w\left(f_{k-1}+f_{k-2}\right)+\left(f_{k-2}+f_{k-3}\right)=0
$$

implies that $\left(f_{k}+f_{k-1}\right)^{2}$ satisfies the same recursion formula as $f_{k}^{2}$ except for a change in $N^{\prime}$. Thus $2 f_{k} f_{k-1}$ and $f_{k} f_{k-1}$ satisfy the same recursion formula except for the term independent of $k$. Thus

$$
c_{k}=\left(4 w^{2}-2\right) c_{k-1}-c_{k-2}+L,
$$

where $L$ is in $Z[x]$ and is independent of $k$. Taking $k=2$, we have

$$
c_{2}=\left(4 w^{2}-2\right) c_{1}+L .
$$

On the other hand, (2.4), $f_{2}=2 w$, and $f_{1}=1$ imply

$$
\begin{equation*}
c_{2}=4 w^{2} c_{1}-4 w \tag{2.7}
\end{equation*}
$$

This shows that $L=2 c_{1}-4 w=2(a+b)$. Hence we have

$$
\begin{equation*}
c_{k}=\left(4 w^{2}-2\right) c_{k-1}-c_{k-2}+2(a+b) . \tag{2.8}
\end{equation*}
$$

This is the recursion formula we sought.

## 3. UNIQUENESS OF SOLUTIONS

We could hope that the $c_{k}$ as developed above would be the only solutions of the Eqs. (1.1) and (1.2), but this is not so in general. However the $c_{k}$ are the only solutions if $b-a=2$ and, with one exception, when $a$ and $b$ are both linear polynomials. To show this we develop a useful algorithm.
Let $a, b, c$ be three polynomials in $Z^{+}[x]$ such that $a<b$ and

$$
\begin{equation*}
a b+1=w^{2}, \quad a c+1=y^{2}, \quad b c+1=z^{2}, \quad \text { with } \quad x, y, z \text { in } Z^{+}[x] . \tag{3.1}
\end{equation*}
$$

Replacing $y_{k}, z_{k}, y_{k-1}, z_{k-1}$ in (2.2) by $y, z, y^{\prime}, z^{\prime}$, respectively, we have the transformation:

$$
\begin{equation*}
y=w y^{\prime}+a z^{\prime}, \quad z=b y^{\prime}+w z^{\prime} \tag{3.2}
\end{equation*}
$$

and its inverse,
(3.3)

$$
y^{\prime}=w y-a z, \quad z^{\prime}=-b y+w z .
$$

This transformation is an automorph of $b y^{2}-a z^{2}$, that is, $b y^{\prime 2}-a z^{\prime 2}=b y^{2}-a z^{2}$. We now show that if $b \leqslant a+c$ and if $c$ satisfies (3.1), then (3.3) yields a $c^{\prime}<c$. This is the basis of our algorithm.
First we show that $y^{\prime}$ is in $Z^{+}[x]$ without further condition on $a, b$, and $c$ except those in (3.1). Also $z^{\prime}$ is in $Z^{+}[x]$ if and only if $b \leqslant a+c$. From the second equation of (1.2) with subscripts suppressed, we have

$$
a(b-a)=\left(w^{2}-1\right) y^{2}-a^{2} z^{2},
$$

that is,

$$
a(b-a)+y^{2}=(w y-a z)(w y+a z)
$$

Since $b>a$, the left side is positive and since $y$ and $z$ are positive, $w y+a z$ is positive. Hence

$$
w y-a z=y^{\prime}>0 .
$$

Similarly,

$$
b(b-a)=b^{2} y^{2}-\left(w^{2}-1\right) z^{2}
$$

which shows that

$$
(w z-b y)(w z+b y)=z^{2}-b(b-a)=1+b(c+a-b)>0
$$

if and only if $b \leqslant a+c$. Thus

$$
w z-b y=z^{\prime}>0 \quad \text { if and only if } \quad b \leqslant a+c .
$$

Second, we show that $y^{\prime}$ and $z^{\prime}$ define a $c^{\prime}$ in $Z[x]$ such that
(3.4)

$$
a c^{\prime}+1=y^{\prime 2} \quad \text { and } \quad b c^{\prime}+1=z^{2}
$$

To this end we compute

$$
\begin{aligned}
z^{\prime 2}-y^{\prime 2} & =[(w-b) y+(w-a) z][(-w-b) y+(w+a) z]=(b-a)\left(b y^{2}+a z^{2}\right)+z^{2}-y^{2}-2 w(b-a) y z \\
& =(b-a) c^{\prime}, \quad \text { where } \quad c^{\prime}=b y^{2}+a z^{2}+c-2 w y z
\end{aligned}
$$

Since $b-a=b y^{\prime 2}-a z^{\prime 2}$, we have from the equivalence of equations (1.1) and (1.2) that Eqs. (3.4) hold.
Third, assume that $b$ is of positive degree and $b \leqslant a+c$. Then $w$ is of positive degree. As in the first part of our argument with $y$ and $y^{\prime}, z$ and $z^{\prime}$ interchanged, we have $w y^{\prime}-a z^{\prime}>0$. Hence (3.2) shows

$$
\begin{equation*}
n_{y^{\prime}}=n_{y}-n \tag{3.5}
\end{equation*}
$$

If $c^{\prime}=0$, then $b \leqslant a+c$ implies $y^{\prime}=z^{\prime}=1$ and hence $n_{y}=n$ and $n_{z}=n_{b}$ from (3.2). If $c^{\prime} \neq 0$, then, from (3.4)

$$
n_{a}+n_{c^{\prime}}=2 n_{y}^{\prime}=2 n_{y}-2 n=n_{a}+n_{c}-2 n=2 n_{z}-2 n .
$$

Hence the following holds

$$
\begin{equation*}
\text { If } \quad c^{\prime} \neq 0 \text {, then } \quad n_{c}^{\prime}=n_{c}-2 n \quad \text { and } \quad n_{z}^{\prime}=n_{z}-n \tag{3.6}
\end{equation*}
$$

Finally, suppose $b$ is in $Z$ and $b \leqslant a+c$. This implies that $a$ and $w$ are in $Z$. It also implies that $c$ is in $Z$ for if $c$ were of positive degree with leading coefficient $d$, then (3.1) would imply that ad and bd would be squares; this is impossible if $a b+1$ is a square. So if $b$ is in $Z$, all the letters in (3.1) represent positive integers. As in the previous paragraph, $w y^{\prime}-a z^{\prime}>0$ which implies
(3.7)

$$
y^{\prime}<y / w
$$

From (3.4) we have, using (3.7),

$$
a c^{\prime}=y^{2}-1<y^{2} / w^{2}-1=(a c+1) / w^{2}-1<a c / w^{2},
$$

since $w>1$. Hence

$$
\begin{equation*}
0 \leqslant c^{\prime}<c / w^{2} \tag{3.8}
\end{equation*}
$$

We collect all these results in the following theorem.
Theorem 1. Let $a, b, c$ be a $P-$-set over $Z^{+}[x]$ with $a<b$, let $y$ and $z$ in $Z^{+}[x]$ be defined by (1.1) with subscripts suppressed, and $y^{\prime}$ and $z^{\prime}$ defined by (3.3). Then $c^{\prime}=b y^{2}+a z^{2}-2 w y z+c$ defines a $c^{\prime}$ such that $a, b, c^{\prime}$ is a $P$-set and (3.4) holds. Also $y^{\prime}>0$ without further condition, and $z^{\prime}>0$ if and only if $b \leqslant a+c$. If $b$ is of positive degree and $b \leqslant a+c$, then conditions (3.5) and (3.6) hold. If $b$ is in $Z$ and $b \leqslant a+c$, then (3.7) and (3.8) hold. [Inequality (3.8) is sharpened in Lemma 4 of Section 6.]
The results of Theorem 1 provide the mechanism to prove two useful theorems.
Theorem 2. If $a<b<c$ are polynomials of the same degree over $Z^{+}[x]$ which satisfy Eqs. (3.1) and, when $a, b, c$ are in $Z$, the additional condition $c \leqslant w^{2}=a b+1$ holds, then $c=a+b+2 w=c_{1}(a, b)$.
Proof. The conditions of the theorem imply that $n_{a}=n=n_{c}$ and $b<a+c$. If $n>0, n_{c}=n$ and (3.6) imply $c^{\prime}=0$. If $n=0$, (3.8) implies $c^{\prime}=0$. In both cases $y^{\prime}=z^{\prime}=1$ and (3.2) shows that $y=w+a, a c+1=y^{2}$ and hence $c=a+b+2 w$. This completes the proof.
Corollary 1. If $a, b, c, d$ are four distinct polynomials of equal positive degree over $Z^{*}[x]$ they do not form a $P$-set.
The corollary follows since if they form a $P$-set we may take $a<b<c<d$ and see from Theorem 2 that $c=d$, which is a contradiction.
The corresponding result for $a$ and $b$ in $Z$ is the following.
Corollary 2. If $a$ and $b$ are in $Z$ with $a<b$ and if $a<b<c<d$ form a $P$-set, then $d>a b+1$. [In view of Lemma 4 in Section 6, $d>a b+1$ could be replaced by $d>4 a b$.]
A closely allied result is the following.
Theorem 3. If $4 a>b>a, a b+1=w^{2}$, and $a<c<b$, then $a, b, c$ do not form a $P$-set in $Z^{+}[x]$.

Proof. Note that the conditions $4 a>b>a$ and $a<c<b$ imply that $a, b, c$ are polynomials of the same degree. If $c>4, b<4 a$ implies $b<a c+1$ and hence from Theorem 2 with $b$ and $c$ interchanged,

$$
b=a+c+2 w^{\prime} \text {, where } w^{\prime 2}=1+a c .
$$

Then

$$
a b+1=a^{2}+a c+2 a w^{\prime}+1=\left(a+w^{\prime}\right)^{2}=w^{2}
$$

implies $w^{\prime}=w-a$ and $c=a+b-2 w$. . But $c>a$ implies $b(b-4 a)>4$ which denies $b<4 a$. If $c \leqslant 3$ it is easy to complete the
proof.
Theorem 3 affirms that if $a$ and $b$ are "close enough together," whether of positive degree or in $Z$, then no $c$ can be inserted between $a$ and $b$ to form a $P$-set of three elements.
Now we assume that $a$ and $b$ are of the same positive degree and seek all $c$ satisfying (3.1). [In Section 6 we consider the same problem for $a$ and $b$ in Z.] We can get explicit results if $n_{c}=k n$, where $n=n_{a}=n_{c}$. Since each time we apply transformation (3.3), Theorem 1 shows that we decrease the degree of $c$ by $2 n$, we eventually arrive at a $\tilde{c}$ of degree $n$ or in $Z$ according as $k$ is odd or even. Then if $b<\tilde{c}$, Theorem 2 shows that $\tilde{c}=c_{1}(a, b)=a+b+2 w$ and hence $c=c_{k}(a, b)$ for some $k$. If, on the other hand, $\tilde{c}<b$ we consider two cases separately.
First if $\tilde{c}<b$ and $\tilde{c}$ is of positive degree $n$, Theorem 2 with $b$ and $\widetilde{c}$ interchanged shows that $b=a+\tilde{c}+2 \tilde{y}$ where $\tilde{y}^{2}=a \tilde{c}+1$. As in the proof of Theorem 3, this implies $\tilde{c}=a+b-2 w$. This leads to a whole new sequence which we designate by $\bar{c}_{j}$. We can compute the members of this sequence by going back to Section 2 and starting with $y_{0}=$ $1=-z_{0}$ in place of $y_{0}=1=z_{0}$. Then $y_{k}$ and $z_{k}$ will satisfy the same recursion formula but will be expressed differently in terms of the $f_{k}$. Using an argument similar to that of Section 2 it can be found that

$$
\begin{equation*}
\bar{c}_{k}(a, b)=f_{k}\left(\bar{c}_{1} f_{k}+2 f_{k-1}\right), \quad \text { where } \quad \bar{c}_{1}=a+b-2 w . \tag{3.9}
\end{equation*}
$$

It can also be verified that the $\bar{c}_{j}$ satisfy the same recursion formula as $c_{k}$, given in (2.8).
Second, if $\tilde{c}<b$ and $\tilde{c}$ is in $Z$, then $\tilde{c}<a<b$ and $n$ is even. If $\tilde{c}=0$, then $\tilde{y}=1=\tilde{z}$, the $c$ before $\tilde{c}$ is $c_{1}(a, b)$ and $c=c_{k}(a, b)$ for some $k$. Then it remains to consider $0<\tilde{c}<a<b$. Now, since $a<b+\widetilde{c}$ we may use Theorem 1 with $\widetilde{c}, a, b$ in place of $a, b, c$. Since $\tilde{c} a+1=\tilde{y}^{2}, \tilde{c} b+1=\tilde{z}^{2}$, and $a b+1=w^{2}$ we define $z^{\prime}$ and $w^{\prime}$ by what corresponds to (3.3), namely

$$
\begin{aligned}
& z^{\prime}=\tilde{y} \tilde{z}-\tilde{c} w \\
& w^{\prime}=\tilde{a z}-\tilde{y} w .
\end{aligned}
$$

By Theorem 1, $\tilde{c} b^{\prime}+1=z^{\prime 2}$ defines $b^{\prime}$ which, by (3.6), must be in $Z$. Now since $\tilde{c} b^{\prime}+1, \tilde{c} a+1, b^{\prime} a+1$ are all squares with only $a$ not in $Z$, the last paragraph in the proof of Theorem 1 implies that $b^{\prime}=0$. Hence $a<b+\tilde{c}$ implies $z^{\prime}=$ $w^{\prime}=1$ and $b=a+2 \tilde{y}+\tilde{c}$. But $w=a+\tilde{y}$. Hence $\tilde{c}=a+b-2 w$ and $c=\bar{c}_{j}(a, b)$ for some $j$. We collect these results in the next theorem.
Theorem 4. If $a$ and $b$ are of the same positive degree, $c$ satisfies (3.1), and the degree of $c$ is a multiple of $n$, then $c=c_{k}(a, b)$ for some $k$ or $c=\bar{c}_{j}(a, b)$ for some $j$. The second sequence is omitted if $b=a+2$.
Corollary. If $a$ and $b$ are both linear or both quadratic in $x$ and if $c$ satisfies (3.1), then $c=c_{k}(a, b)$ for some $k$ or $c=\bar{c}_{j}(a, b)$ for some $j$. The second sequence is omitted if $b=a+2$.
The corollary follows since if $n=2$, the degree of $c$ satisfying (3.1) must be even. When $n>2$, we have in general more than two sequences. But by (3.3) we can for each $c$ find a $\widetilde{c}$ of degree not greater than $n$. From these $\widetilde{c}$, stem all the $c$ satisfying (3.1).

## 4. WHEN IS $c_{k} c_{r}+1$ A SQUARE?

To answer this question we first find a formula for $c_{k} c_{r}+1$ for $k>r$. Since we need a similar result for $\overline{c_{i}}$ we adopt a temporary notation which enables us to derive both results simultaneously. First, by use of $f_{k+1}=2 w f_{k}-f_{k-1}$, we can write (2.4) as

$$
\begin{equation*}
c_{k}=f_{k}\left(c_{1} f_{k}+2 f_{k+1}\right), \quad \text { where } \quad \bar{c}_{1}=a+b-2 w . \tag{4.1}
\end{equation*}
$$

Similarly, (3.9) can be written

$$
\begin{equation*}
\bar{c}_{k}=f_{k}\left(c_{1} f_{k}-2 f_{k+1}\right) \tag{4.2}
\end{equation*}
$$

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To treat these together, we write
(4.3)

$$
d_{k}=f_{k}\left(d_{1} f_{k} \pm 2 f_{k+1}\right)
$$

where $d_{k}=c_{k}$ or $\bar{c}_{k}$ according as $\pm$ is + or - and $\overline{\bar{d}}_{1}=d_{1}$.
Then
(4.4)

$$
d_{k} d_{r}+1=\left(d_{1} f_{k} f_{r} \pm f_{k+1} f_{r} \pm f_{r+1} f_{k}\right)^{2}-\left(f_{r+1} f_{k}-f_{k+1} f_{r}\right)^{2}+1
$$

Now

$$
\left|\begin{array}{ll}
f_{r+1} & f_{r} \\
f_{k+1} & f_{k}
\end{array}\right|=\left|\begin{array}{ll}
2 w f_{r}-f_{r-1} & f_{r} \\
2 w f_{k}-f_{k-1} & f_{k}
\end{array}\right|=\left|\begin{array}{ll}
f_{r} & f_{r-1} \\
f_{k} & f_{k-1}
\end{array}\right|=\left|\begin{array}{ll}
f_{1} & f_{0} \\
f_{k-r+1} & f_{k-r}
\end{array}\right|=f_{k-r}
$$

This shows that

$$
f_{r+1} f_{k}-f_{k+1} f_{r}=f_{k-r}
$$

Thus
(4.5)

$$
d_{k} d_{r}+1=\left(\bar{d}_{1} f_{k} f_{r} \pm 2 f_{r} f_{k+1} \pm f_{k-r}\right)^{2}-f_{k-r}^{2}+1
$$

Now if $k=r+1$, it follows that $f_{k-r}=1$ and we have

$$
\begin{equation*}
d_{r+1}^{\prime} d_{r}+1=\left(\bar{d}_{1} f_{r+1} f_{r} \pm 2 f_{r} f_{r+2} \pm 1\right)^{2} \tag{4.6}
\end{equation*}
$$

So we have the following theorem.
Theorem 5. The polynomials $c_{r+1} c_{r}+1$ and $\bar{c}_{r+1} \bar{c}_{r}+1$ are squares in $Z[x]$.

## 5. P-SETS WHEN $a$ AND $b$ ARE LINEAR

From Theorem $5, c_{k} c_{r}+1$ is a square when $k$ and $r$ are successive integers. If $a$ and $b$ are linear we can show as in the previous paper that $c_{k} c_{r}+1$ is a square in $Z[x]$ only if $k$ and $r$ are consecutive integers. The idea of the argument is the same but the needed modifications cause a little trouble. We need the same result for $\bar{c}_{k} \bar{c}_{r}+1$ but since the proof is almost the same, we omit it. We will need the following three lemmas which, as in the previous paper, we state without proof since the proofs are easy.
Lemma 1. Let $\tilde{\varphi}_{1}(a), \varphi_{2}(a)$, and $\lambda(a)$ be three polynomials in $Z\left[a, a^{-1}\right]$ such that the first $t$ coefficients of $\varphi_{1}(a)$ and $\varphi_{2}(a)$ are the same. Then the first $t$ coefficients of $\varphi_{1}(a) \lambda(a)$ and $\varphi_{2}(a) \lambda(a)$ are the same.
Lemma 2. Let the first $t$ coefficients of $\varphi_{i}(a)$ and $\psi_{i}(a)$ be the same for $i=1$ and 2 . Then the first $t$ coefficients of $\varphi_{1}(a) \varphi_{2}(a)$ and $\psi_{1}(a) \psi_{2}(a)$ are also the same.
Lemma 3. Let $\varphi_{i}(a), i=1,2$, be two polynomials in $Z\left[a, a^{-1}\right]$ whose leading coefficients are positive and such that the first $t$ coefficients of their squares are the same. Then the first $t$ coefficients of the two polynomials are the same.
Now we prove the basic theorem.
Theorem 6. If $a$ and $b$ are linear in $Z^{+}[x]$, with $a b+1=w^{2}$ and $w$ in $Z^{+}[x]$, then $a, b, c_{r}, c_{k}$ is a $P$-set if and only if $r$ and $k$ are consecutive integers. The same is true for $a, b, \bar{c}_{r}, \bar{c}_{k}$.
Proof. The "if" part is established by Theorem 5 and/or Eq. (4.6). To prove the "only if" part, first note that $e=a+b-2 w \geqslant 0$ is equivalent to $(b-a)^{2} \geqslant 4$ with equality if and only if $b=a+2$. So the case $e=0$ is covered by the previous paper. Or the reader may prefer to note the modifications needed in the following proof where we assume that $e \neq 0$.
Now $f_{r}$ can be thought of as a polynomial in $Z\left[a, a^{-1}\right]$ of degree $r-1$. It has $2 r-1$ terms with 1 and 0 alternating as coefficients. Thus if $k>r$, the sequence of $2 r-1$ coefficients of $f_{r}$ is the same as the sequence of the first $2 r-1$ coefficients for $f_{k}$. Henceforth in this proof we assume that $k>r+1$, that $c_{k} c_{r}+1$ is a square in $Z[x]$ and seek a contradiction. From what we have just noted, the first $2 r+1$ coefficients of $e f_{k}+2 f_{k+1}$ and of $e f_{r+1}+2 f_{r+2}$ are the same, where the $f_{i}$ are viewed as polynomials in $Z\left[a, a^{-1}\right]$. Note that $e=a+b-2 w$, being different from zero, is not in $Z$, for suppose this is true and write

$$
a=a_{1} x+a_{0}, \quad b=b_{1} x+b_{0}, \quad \text { and } \quad w=w_{1} x+w_{0}
$$

Then if $e$ is in $Z, a_{1} b_{1}=w_{1}$ and $a_{1}+b_{1}-2 w_{1}=0$ imply $a_{1}=b_{1}=w_{1}$. From this it follows that $b_{0}=a_{0}+2$ and hence $e=0$, contrary to hypothesis. Furthermore $e$ is not in $Z\left[a, a^{-1}\right]$ since $a$ depends only on the product $a b$ and not on the sum $a+b$. Let $Z^{\prime}=Z[e]$ and see that $c_{k}$ and $c_{r}$ are in $Z^{\prime}\left[a^{-1}, a\right]$.
Using Lemma 2 and (4.1) with $\bar{c}_{1}$ replaced by $e$, we then see that the first $2 r+1$ coefficients of $c_{k}$ and $c_{r+1}$ are the same. Then by Lemma $1, c_{k} c_{r}$ and $c_{r+1} c_{r}$ have the same first $2 r+1$ coefficients. Hence the same can be said for

$$
g_{k, r}=c_{k} c_{r}+1 \quad \text { and } \quad g_{r+1, r}=c_{r+1} c_{r}+1
$$

Suppose $g_{k, r}=\varphi^{2}(x)$, that is, $g_{k, r}$ is a square in $Z[x]$. We next show that $g_{k, r}$ is also a square in $Z^{\prime}\left[a, a^{-1}\right]$, in fact $\varphi(x)=\bar{e} \varphi_{1}+\varphi_{2}$, where $\varphi_{1}$ and $\varphi_{2}$ are in $Z\left[a, a^{-1}\right]$ and $\bar{e} t^{\prime}=e$ for some $t^{\prime}$ in $Z$. Note that $w_{1} \neq 0$ since $a$ and $b$ are linear. Since $x=\left(w-w_{0}\right) / w_{1}$,

$$
\varphi(x)=w_{1}^{-t} \sigma(w)=w_{1}^{-t}\left[w \sigma_{0}(w)+u\right],
$$

where $u$ is in $Z, t \geqslant 0$, and $\sigma(w)$ with $\sigma_{0}(w)$ are in $Z[w]$. Write $e=e_{1} x+e_{0}$ where, as we showed above, $e_{1} \neq 0$. Note that $2 w=a+a^{-1}$, and have

$$
\begin{aligned}
\varphi(x) & =e \sigma_{1}(a) / e_{1} w_{1}^{t-1} 2^{s}+\sigma_{2}(a) / e_{1} w_{1}^{t} 2^{s} \\
& =\bar{e} \sigma_{3}(a) / v_{1}+\sigma_{4}(a) / v_{2}
\end{aligned}
$$

where $s$ is a non-negative integer, $v_{1}$ and $v_{2}$ are positive factors of $e_{1} w_{1}^{t} 2^{s}$, no factor of $v_{1}$ greater than 1 divides all coefficients of $\bar{e} \sigma_{3}(a)$ and no factor of $v_{2}$ greater than 1 divides all coefficients of $\sigma_{4}(a)$. Let $v_{1}=h v_{3}, v_{2}=h v_{4}$, and $\left(v_{3}, v_{4}\right)=1$. Then
(5.1)

$$
h^{2} v_{3}^{2} v_{4}^{2} g_{k, r}=\bar{e}^{-2} v_{4}^{2} \sigma_{3}^{2}+2 \bar{e} v_{3} v_{4} \sigma_{3} \sigma_{4}+v_{3}^{2} \sigma_{4}^{2}
$$

This implies that $v_{4} \mid v_{3}^{2}$ and $v_{3} \mid v_{4}^{2}$ and hence $v_{4}=v_{3}=1$. Thus

$$
h^{2} g_{k, r}=\bar{e}^{-2} \sigma_{3}^{2}+2 \bar{e} \sigma_{3} \sigma_{4}+\sigma_{4}^{2}
$$

Hence $h^{2}=1$ and $\varphi(x)=\bar{e} \sigma_{3}(a)+\sigma_{4}(a)$, which is the result we announced at the beginning of this paragraph.
Now compare

$$
\sqrt{g_{r+1, r}}=e f_{r+1} f_{r}+2 f_{r+2} f_{r}+1
$$

from (4.6), and

$$
\sqrt{g_{k, r}}=\bar{e} \sigma_{3}+\sigma_{4}
$$

The degree of $\sqrt{g_{r+1, r}}$ in $a$ is $2 r$ and hence each of the first $2 r$ coefficients of $\sqrt{g_{r+1, r}}$ is divisible by 2 or $\bar{e}$ (or both), and the first $2 r+1$-st coefficient is the term free of $a$. Now $f_{r+1} f_{r}$ is a sum of odd powers of $a$ and hence there is no term free of $a$ in $f_{r+1} f_{r}$. This, with (4.6) shows that the $2 r+1$-st coefficient of $g_{r+1, r}$ is an odd integer. We showed above that the first $2 r+1$ coefficients of $g_{k, r}$ and $g_{r+1, r}$ are the same. Hence, by Lemma 3 , the $2 r+1$-st coefficient in $\sqrt{g_{k, r}}$ is an odd integer.
On the other hand, (4.3) with $d_{i}=c_{i}, \bar{d}_{1}=\bar{c}_{1}=e$ implies

$$
\varphi^{2}(x)=g_{k, r}=e^{2} f_{k}^{2} f_{r}^{2}+2 e f_{k} f_{r}\left(f_{r} f_{k+1}+f_{k} f_{r+1}\right)+4 f_{k} f_{r} f_{k+1} f_{r+1}+1
$$

The degree of $g_{k, r}$ in $a$ is $2 r+2 k-2$. Thus each of the first $2 r+2 k-2$ coefficients is divisible by $\bar{e}$ or 2 . But

$$
r+k-1 \geqslant r+r+2-1=2 r+1
$$

This is the contradiction that proves the theorem for $c_{k}$ and $c_{r}$. The proof for $\bar{c}_{k}$ and $\bar{c}_{r}$ is almost the same.
Now we prove our principal theorem for $a$ and $b$ linear.
Theorem 7. Let $a$ and $b$ be linear in $Z^{+}[x]$ and $a b+1=w^{2}, w$ in $Z^{+}[x]$. If

$$
\begin{equation*}
a, b, a+b+2 w, c \tag{5.2}
\end{equation*}
$$

is a $P$-set of four elements, then

$$
\begin{equation*}
c=c_{2}(a, b)=\bar{c}_{2}(b, a+b+2 w) . \tag{5.3}
\end{equation*}
$$

Proof. Since $a, b, c$ is a $P$-set, the corollary of Theorem 4 shows that $c=c_{k}(a, b)$ for some $k$ or $c=\bar{c}_{j}(a, b)$ for some $j$. Now $a+b+2 w=c_{1}(a, b)$ and if $c=c_{k}(a, b)$, Theorem 6 implies

$$
c=c_{2}(a, b) \quad \text { or } \quad c=\bar{c}_{j}(a, b) \quad \text { for some } j .
$$

Now use the same argument with $a$ replaced by $b$ and $b$ by $a+b+2 w$. The corollary of Theorem 4 shows that $c=c_{k}(b, a+b+2 w)$ for some $k$ or $c=\bar{c}_{j}(b, a+b+2 w)$ for some $j$. But

$$
\begin{equation*}
a=\bar{c}_{1}(b, a+b+2 w) \tag{5.5}
\end{equation*}
$$

and Theorem 6 shows that

$$
\begin{equation*}
c=\bar{c}_{2}(b, a+b+2 w) \quad \text { or } \quad c=c_{k}(b, a+b+2 w) \tag{5.6}
\end{equation*}
$$

for some $k$.
Next we prove that $\bar{c}_{2}(b, a+b+2 w)=c_{2}(a, b)$. Now $b(a+b+2 w)+1=(b+w)^{2}$. So, using the recursion formula (2.8) for $\bar{c}$ in place of $c$, we have

$$
\begin{aligned}
\bar{c}_{2}(b, a+b+2 w) & =\left[4(b+w)^{2}-2\right] a+2(a+2 b+2 w) \\
& =\left[4\left(a b+b^{2}+2 b w+1\right)-2\right] a+2(a+b+2 w)+2 b \\
& =(4 a b+2)(a+b+2 w)+2 a+2 b \\
& =\left(4 w^{2}-2\right) c_{1}(a, b)+2 a+2 b=c_{2}(a, b) .
\end{aligned}
$$

Then if $c \neq c_{2}(a, b)$ we know from (5.3), (5.4), and (5.6) that $\bar{c}_{j}(a, b)=c_{k}(b, a+b+2 w)$ for some $j$ and $k$ greater than 2. But since $c_{k}$ is of degree $2 k-1$ and $c_{j}$ of degree $2 j-1$, the equality implies $j=k$. We now reach a contradiction by showing that

$$
\begin{equation*}
c_{k}(b, a+b+2 w)>\bar{c}_{k}(a, b), \quad \text { if } \quad k>2 \tag{5.7}
\end{equation*}
$$

We showed above that $b(a+b+2 w)+1=(b+w)^{2}$, that is, $b+w$ is the " $w$ " for the pair $b, a+b+2 w$. Corresponding to $a$ for this pair is

$$
\beta=b+w+\sqrt{(b+w)^{2}-1}>a=w+\sqrt{w^{2}-1} .
$$

Let $h_{k}=\left(\beta^{k}-\beta^{-k}\right) /\left(\beta-\beta^{-1}\right)$ to see that $h_{k}$ corresponds to $f_{k}$. Thus, from (4.1) and (5.5)

$$
c_{k}(b, a+b+2 w)=h_{k}\left(a h_{k}+2 h_{k+1}\right) .
$$

Using (3.9), the inequality (5.7) may be written

$$
\begin{equation*}
a h_{k}^{2}+2 h_{k} h_{k+1}>(a+b-2 w) f_{k}^{2}+2 f_{k} f_{k-1} \tag{5.8}
\end{equation*}
$$

To show that (5.8) holds, it is sufficient to show that $a h_{k}^{2}>(a+b-2 w) f_{k}^{2}$ for $k \geqslant 2$. To this end we first show that $h_{k} / f_{k}$ increases with $k$. To do this use the recursion formulas for $h_{k}$ and $f_{k}$ to get

$$
\begin{aligned}
h_{k} f_{k-1}-f_{k} h_{k-1} & >\left(2 w h_{k-1}-h_{k-2}\right) f_{k-1}\left(2 w f_{k-1}-f_{k-2}\right) h_{k-1} \\
& =h_{k-1} f_{k-2}-f_{k-1} h_{k-2}>h_{2} f_{1}-h_{1} f_{2}=h_{2}-f_{2}>0 .
\end{aligned}
$$

Hence $h_{k} / f_{k}$ increases with $k$ and (5.8) holds if

$$
a h_{2}^{2}>(a+b-2 w) f_{2}^{2}, \quad \text { that is, } a(b+w)^{2}>(a+b-2 w) w^{2} .
$$

The last inequality is easy to verify. Hence the inequality (5.7) follows and the theorem is proved. The following corollary follows immediately from Theorem 2.
Corollary. Let $a, b, d$ be a $P$-set of three linear elements of $Z^{+}[x]$ with $a<b<d$. Then the only $P$-set containing $a, b$, and $d$ is

$$
a, b, d, c_{2}(a, b)
$$

REMARK. Notice that the part of the above where we showed $c_{2}(a, b)=\bar{c}_{2}(b, a+b+2 w)$ did not depend on $a$ and $b$ being linear. In the course of proving this result we only assumed $a b+1=w^{2}$ and (2.8) for $c_{k}$ and $\bar{c}_{j}$.

## 6. P-SETS OVER $Z$

In this section we assume that $a$ and $b$ are positive integers, $a<b$, and $a b+1=w^{2}$, where $w$ is a positive integer. Also, as in Theorem 3, we assume that $a$ and $b$ are "not too far apart," specifically, that $b<4 a$. We find all integers
$c$ such that $a, b, c$ is a $P$-set. Toward this end we first need to sharpen inequality (3.8) of Theorem 1.
Lemma 4. Let $a, b, c$ satisfy Eqs. (3.1) and let (3.3) define $y^{\prime}$, and $a c^{\prime}+1=y^{\prime 2}$ define $c^{\prime}$. Then, if $b \leqslant a+c$, it follows that
(6.1)

$$
c^{\prime}<c / 4 a b .
$$

Proof. As in the proof of Theorem 1, the condition $b \leqslant a+c$ implies that $y^{\prime}$ and $z^{\prime}$ are positive. Since $b c^{\prime}+1=$ $z^{2}$ we have

$$
a c+1=y^{2}=\left(w y^{\prime}+a z^{\prime}\right)^{2}=w^{2} a c^{\prime}+a b+1+a^{2}\left(1+b c^{\prime}\right)+2 w a y^{\prime} z^{\prime} .
$$

Hence

$$
c=\left(w^{2}+a b\right) c^{\prime}+b+a+2 w y^{\prime} z^{\prime}>2 a b c^{\prime}+a+b+2 \sqrt{a b} \sqrt{a b} c^{\prime}
$$

since $w=\sqrt{a b+1}$. Thus $c>4 a b c^{\prime}$ and the proof is complete.
The first part of the proof of the next theorem is like that of Theorem 4. After this, further details must be dealt with.
Theorem 8. If $a<b<4 a, a$ and $b$ are in $Z^{+}$. and Eqs. (3.1) hold, then $c=c_{k}(a, b)$ for some $k$ or $c=\bar{c}_{j}(a, b)$ for some $j$. The set $\bar{c}_{j}$ is omitted if $b=a+2$.
Proof. If $c \geqslant w^{2}$, then $c>b-a$ and, by Theorem 1, a sequence of transformations (3.3) yields a $c^{\prime}<w^{2}$. [We assume that the $c$ before $c^{\prime}$ in the sequence is not less than $w^{2}$. If $c<w^{2}$ the argument is what follows.] If $c^{\prime}>b$, Theorem 2 shows that $c^{\prime}=a+b+2 w=c_{1}(a, b)$ and hence $c=c_{k}(a, b)$ for some $k$. If, on the other hand, $c^{\prime}<$ $b$, Theorem 3 shows that $c^{\prime}<a<b$. Then if $b \leqslant a c^{\prime}+1$, Theorem 2 implies $b=a+c^{\prime}+2 w^{\prime}$, where $w^{\prime 2}=a c^{\prime}+1$. Then, as in the proof of Theorem 3, $c^{\prime}=a+b-2 w$ and hence $c=\bar{c}_{j}(a, b)$ for some $j$, where this sequence is omitted if $c^{\prime}=0$, that is, if $b=a+2$.
It remains to consider $0<c^{\prime}<a<b$ and $b>a c^{\prime}+1$. Then $4 a>b$ implies $c^{\prime} \leqslant 3$. Write $a c^{\prime}+1=y^{s 2}$ and $b c^{\prime}+1=$ $z^{* 2}$. Now we use (3.3) once for $c^{\prime}, a, b$ in place of $a, b, c$. By Lemma 4 the trausformation takes $b$ into $b^{\prime}$ satisfying the inequality

$$
b^{\prime}<b / 4 a c^{\prime}<1 / c^{\prime},
$$

since $a<1+b$. Hence $b^{\prime}=0$ and, as in Theorem 2, this implies

$$
\begin{equation*}
b=a+c^{\prime}+2 y^{\prime}=c_{1}\left(c^{\prime}, a\right) . \tag{6.2}
\end{equation*}
$$

First if $c^{\prime}=2$ or $3, b>a c^{\prime}+1$ implies $a+c^{\prime}+2 y^{\prime}>a c^{\prime}+1$. Then

$$
2 y^{\prime}>d a-d, \text { where } d=c^{\prime}-1 .
$$

Then

$$
\begin{gather*}
4\left(a c^{\prime}+1\right)>d^{2} a^{2}-2 d^{2} a+d^{2} \\
0>d^{2} a^{2}-2 a\left(d^{2}+2 d+2\right)+d^{2}-4 \tag{6.3}
\end{gather*}
$$

If $d=2$, ( 6.3 ) becomes $0>4 a^{2}-20 a$, that is, $a<5 / 2$ which is impossible. If $d=1$, (6.3) becomes $0>a^{2}-10 a-$ 3 which holds if and only if $a \leqslant 10$. Then, under the conditions imposed, the only possibility is $a=4, b=12, w=7$. Then $a+b-2 w=2=c^{\prime}=\bar{c}_{1}(a, b)$ and $c=\bar{c}_{j}(a, b)$ for some $j$.
Second, if $c^{\prime}=1$, (6.2) becomes $b=a+1+2 y^{\prime}$ and $1+a b=w^{2}$ implies $w=y^{\prime}+a$. Hence $a+b-2 w=c^{\prime}=\bar{c}_{1}(a, b)$. Then, as in the case when $d=1, c=\bar{c}_{j}(a, b)$ for some $j$. This completes the proof.
Theorem 8 implies the following theorem with only two little details to be filled in.
Theorem 9. If $a, b, e=a+b+2 w, d$ is a $P$-set of four distinct elements of $Z^{+}$subject to the conditions $a<b$ $<4 a$ and $a b+1=w^{2}$, then $d$ must be in each of the two following sets:

$$
\left.\begin{array}{l}
\delta_{1}=\left\{\begin{array}{lll}
c_{k}(a, b) & \cup \bar{c}_{j}(a, b)
\end{array}\right\} \\
z_{2}=\left\{c_{k}(b, e)\right. \\
\cup \\
\bar{c}_{j}(b, e)
\end{array}\right\} .
$$

One possibility is $d=c_{2}(a, b)=\bar{c}_{2}(b, e)$. If $b=a+2$, then $\mathcal{Z}_{1}=\left\{c_{k}(a, b)\right\}$.
Proof. To apply Theorem 8 to this theorem we must notice that $e<4 b$ is equivalent to $4<(9 b-a)(b-a)$ which holds since $b>a>0$. For the rest, one notes the Remark after the Corollary of Theroem 7.

## 7. P-SETS OF FIBONACCI NUMBERS

Let $F_{i}$ denote the $i^{\text {th }}$ Fibonacci number. The following well known facts can easily be verified for $a=F_{2 r-2}$, $b=F_{2 r}, r>1$ :
i) $w^{2}=a b+1=(b-a)^{2}$, that is, $a^{2}-3 a b+b^{2}=1$.
ii) If $e=F_{2 r+2}$, then $e=c_{1}(a, b)=3 b-a, a e+1=(a+w)^{2}=b^{2}, b e+1=(b+w)^{2}=(2 b-a)^{2}$, where

$$
w=b-a=F_{2 r-1}
$$

These two properties show that $a, b, e$ form a $P$-set. From i),

$$
\begin{equation*}
b=a t, \quad \text { where } \quad 2 t=3+\sqrt{5+4 / a^{2}} \tag{7.1}
\end{equation*}
$$

This shows that $b \leqslant 3 a$ with equality only if $a=1$. Hence the hypotheses of Theorem 8 hold and all the numbers $d$ such that $a, b, e, d$ form a $P$-set can be expressed as $c_{k}(a, b)$ or $\bar{c}_{j}(a, b)$. V. E. Hoggatt, Jr., and C. E. Bergum showed [1] that

$$
\begin{equation*}
F_{2 r-2}, F_{2 r}, F_{2 r+2}, c=4 F_{2 r-1} F_{2 r} F_{2 r+1} \tag{7.2}
\end{equation*}
$$

is a $P$-set. It is not hard to show that $c$ in (7.2) is, in our notation, $c_{2}(a, b)$ for $a=F_{2 r-2}$ and $b=F_{2 r}$. To this end, notice that, since $F_{2 r-1} F_{2 r+1}=F_{2 r}^{2}+1, c$ in (7.2) can also be written
(7.3)

$$
c=4 b\left(b^{2}+1\right), \quad \text { where } \quad b=F_{2 r}
$$

This can be shown to be $c_{2}(a, b)$ by using (2.8) with $w=b-a, k=2$.
Our Theorem 3 shows that there is no $c$ between $F_{2 r-2}$ and $F_{2 r}$ such that $c, F_{2 r-2}, F_{2 r}$ is a $P$-set. Theorem 2 shows that if these same three numbers form a $P$-set with $F_{2 r}<c<F_{2 r-1}^{2}$, then $c=F_{2 r+2}$. The following The orem shows that $c$ is not a Fibonacci number.
Theorem 10. If $a=F_{2 r-2}, b=F_{2 r}$, and $r>1$, then

$$
\begin{equation*}
F_{6 r-1}<c_{2}(a, b)<F_{6 r} \tag{7.4}
\end{equation*}
$$

Proof. From (7.3), $c_{2}(a, b)=4 F_{2 r}^{3}+4 F_{2 r}$. Now

$$
F_{k}=\frac{\beta^{k}-\bar{\beta}^{k}}{\beta-\bar{\beta}}, \quad \text { where } \quad \beta=(1+\sqrt{5}) / 2, \bar{\beta}=(1-\sqrt{5}) / 2
$$

Hence

$$
\begin{equation*}
F_{k}^{3}=\frac{F_{3 k}-3(-1)^{k} F_{k}}{(\beta-\bar{\beta})^{2}}=(1 / 5)\left[F_{3 k}-3(-1)^{k} F_{k}\right] \tag{7.5}
\end{equation*}
$$

Thus the two inequalities in (7.4) will follow if we can show

$$
\begin{gather*}
F_{6 r} / F_{2 r}>8  \tag{7.6}\\
F_{6 r} / F_{6 r-1}>5 / 4
\end{gather*}
$$

To show (7.6) use (7.5) to get $F_{6 r} / F_{2 r}=5 F_{2 r}^{2}+3$, which shows that $F_{6 r} / F_{2 r}$ is an increasing function of $r$. Then (7.6) follows from

$$
F_{6 r} / F_{2 r} \geqslant F_{12} / F_{4}=48>8
$$

Also (7.7) holds since $F_{2 r} / F_{2 r-1}$ is an increasing function of $r$ and

$$
F_{6 r} / F_{6 r-1} \geqslant F_{12} / F_{11}=144 / 89>5 / 4
$$

Thus the proof is complete.

## 8. UNFINISHED BUSINESS

For $b$ of degree greater than 2, there does not seem to be much of interest since in most cases there will be more than two sequences of numbers which with $a$ and $b$ form a $P$-set. For $a$ and $b$ linear it would be interesting to show that
(8.1)

$$
a, b, c_{r}(a, b), \bar{c}_{s}(a, b)
$$

is not a $P$-set for any $r$ and $s$. The difficulty in proving this is that, if one is to use the method of Birch, one first needs a pair $r, s$ for which $c_{r} \bar{c}_{s}+1$ is a square. One might at least prove that there isat most one pair $r$ and $s$ such that (8.1) is a $P$-set.
For $a$ and $b$ quadratic functions of $x$, the basic difficulty is that $g_{k, r}$ could be a square in $Z[x]$ without being a square over $Z[a, b]$. Even if that were surmounted, adapting Theorem 6 to quadratics would present some difficulties.
For $a$ and $b$ integers, this paper does not add much to present knowledge except to place the problem in a larger setting. The Davenport-Baker result shows that in Theroem 9 when $a=1, b=3$, the intersection of $\delta_{1}$ and $\delta_{2}$ is $c_{2}(1,3)=120$. A really significant result would be a proof that this is true for $a$ and $b$ any two successive Fibonacci numbers of even index. To show this independently from their result would present all the difficulties they encountered for their special case. At one time I hoped that one might by using the sequence of transformations (3.3) and a proof of "infinite descent" reduce the general result to that of the pair $a=1, b=3$, but it does not seem to work.
A somewhat weaker result would be the conjecture that if $a, b, c$ are three successive even-indexed Fibonacci numbers and if $a, b, c, d$ is a $P$-set of four numbers, then $d$ cannot be a Fibonacci number. From Theorem $10, c_{2}(a, b)$ is not a Fibonacci number. Unfortunately, for $c_{k}(a, b)$ with $k>2$ there does not seem to be such a definite inequality as (7.4). One possible approach could be to consider the set of Fibonacci numbers as dividing the line of positive reals into intervals. Perhaps one could, using Theorem 9, assume, for example, that $c_{r}(a, b)$ and $c_{s}(b, e)$ were in the same interval and thus get a relationship between $r$ and $s$ which might be fruitful. But this seems like a long hard row to hoe. Also it would be interesting to show that $a, b, c$ as defined above are not in a $P$-set of five elements. All of these results seem very plausible.

## REFERENCE

1. V. E. Hoggatt, Jr., and G. E. Bergum, "A Problem of Fermat and the Fibonacci Sequence," The Fibonacci Quarterly, Vol. 15, No. 4 (Dec. 1977), pp. 323-330.

## [Continued from page 154.]

B. We can easily obtain

$$
\binom{2 p}{p} p=2(2 p-1)\binom{2 p-2}{p-1} \quad \text { and from Part } A, \quad\binom{2 p}{p} \equiv 2\left(\bmod p^{3}\right)
$$

Thus $2 p \equiv 2(2 p-1)\binom{2 p-2}{p-1}\left(\bmod p^{3}\right)$. Since $\left(2, p^{3}\right)=\left(2 p-1, p^{3}\right)=1,2$, and $2 p-1$ we have the multiplicative inverses $\left(\bmod p^{3}\right)$ and we get $p /(2 p-1) \equiv\binom{2 p-2}{p-1}\left(\bmod p^{3}\right)$. Now $(2 p-1)-1 \equiv-1-2 p-4 p^{2}\left(\bmod p^{3}\right)$. Hence

$$
p /(2 p-1) \equiv p\left(-1-2 p-4 p^{2}\right)\left(\bmod p^{3}\right) \equiv-p-2 p^{2}\left(\bmod p^{3}\right)
$$

The result then follows.

## AN ADJUSTED PASCAL

H-213 Propased by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.
A. Let $A_{n}$ be the left adjusted Pascal triangle, with $n$ rows and columns and 0 's above the main daigonal. Thus

$$
A_{n}=\left(\begin{array}{ccccc}
1 & 0 & & \ldots & 0 \\
1 & 1 & 0 & \cdots & \ldots \\
1 & 2 & 1 & 0 & 0 \\
1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ~
\end{array}\right)_{n \times n}
$$

Find $A_{n} \cdot A_{n}^{T}$ where $A_{n}^{T}$ represents the transpose of matrix, $A_{n}$.
$B$. Let

$$
C_{n}=\left(\begin{array}{ccccccc}
1 & 0 & 0 & & & \cdots & 0 \\
0 & 1 & 0 & & & \cdots & 0 \\
0 & 1 & 1 & 0 & & \cdots & 0 \\
0 & 0 & 2 & 1 & 0 & \cdots & 0
\end{array}\right)
$$

# ON GENERALIZED $G_{j, k}$ NUMBERS 

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Most of this paper was finished prior to the author's involvement in other work [9, 10]. It is the purpose of this exegesis to find a self-contained definition of $\left\{G_{j}\right\}$ which is not dependent on other sequences. Such are (10), (12) and (16). I have defined these numbers in $[2,(3)]$ and $[3,(9)] . G$ numbers of the $j^{t h}$ order are:

$$
\begin{equation*}
G_{j, k}=1+P_{j, k}^{*}+P_{j, 2 k-1} \tag{1}
\end{equation*}
$$

where the Lucas complement is by definition
(2)

$$
P_{j, k}^{*}=P_{j, k+1}+P_{j, k-1}
$$

and where coprime sequences are by definition, $j$ an integer,

$$
\begin{equation*}
P_{j, k+1}=j P_{j, k}+P_{j, k-1} \tag{3}
\end{equation*}
$$

and where the initial conditions (IC) are by choice

$$
\begin{equation*}
P_{j, 0}=0 \quad \text { and } \quad P_{j, 1}=1 \quad \text { for all } j \tag{3a}
\end{equation*}
$$

To begin we need the following easily proven identities. The Lucas complement of the Lucas complement is

$$
\begin{equation*}
P_{j, k+1}^{*}+P_{j, k-1}^{*}=P_{j, k+2}+2 P_{j, k}+P_{j, k-2}=\left(4+j^{2}\right) P_{j, k} \tag{4}
\end{equation*}
$$

Secondly given any two point recurrence $P_{n+1}=a P_{n}+b P_{n-1}$ the recurrence among its bisection is known to be

$$
\begin{equation*}
P_{n+2}=\left(a^{2}+2 b\right) P_{n}-b^{2} P_{n-2} \tag{5}
\end{equation*}
$$

Thirdly we need the central difference operator
(6)

$$
\delta^{2} P_{n}=(\Delta-\nabla) P_{n}=P_{n+1}-2 P_{n}+P_{n-1}
$$

and fourthly I define a new operator small psi

$$
\begin{equation*}
\psi_{j}\left(P_{n}\right)=\left[\delta^{2}-j^{2}\right] P_{n} \tag{7}
\end{equation*}
$$

where $j^{2}$ is really $j^{2}$ times the identity operator. Note that if $B_{j, n}$ is any generalized bisected coprime sequence with any $B_{j, 0}$ and $B_{j, 1}$ whatsoever that $\psi_{j}$ then acts as a null operator, to wit

$$
\begin{equation*}
\psi_{j}\left(B_{j, n}\right)=0 \quad \text { for all } j \tag{8}
\end{equation*}
$$

Now when $j=1$ then (7) reduces to $\psi\left(F_{n}\right)=\left[\delta^{2}-I\right] F_{n}$. Consider

$$
\begin{equation*}
\psi_{j}\left(G_{j, k}\right)=\psi_{j}\left(P_{j, k}^{*}\right)-j^{2} \tag{9}
\end{equation*}
$$

which is obvious from (1) and (8) and the fact that $\psi_{j}(1)=-j^{2}$. In (9) elimination of $\delta^{2}$ via (6) gives
(9a)

$$
\psi_{j}\left(G_{j, k}\right)=\left(4+j^{2}\right) P_{j, k}-\left(2+j^{2}\right) P_{j, k}^{*}-j^{2}
$$

Theorem. The recurrence for $\psi_{j}\left(G_{j, k}\right)$ is Fibonacci but for the additive constant $j^{3}$.
Proof. Rewrite (9a) as $\psi_{j} G_{j, k+1}$ and substitute (3) giving
(10)

$$
\begin{aligned}
\psi_{j}\left(G_{j, k+1}\right) & =\left\{j^{2}+4\right]\left[j P_{j, k}+P_{j, k-1}\right]-\left[j^{2}+2\right]\left[j P_{j, k}^{*}+P_{j, k-1}^{*}\right]-j^{2} \\
& =j \psi_{j}\left(G_{j, k}\right)+\psi_{j}\left(G_{j, k-1}\right)+j^{3}
\end{aligned}
$$

Eliminating $j^{3}$ by calculating $\psi G_{j, k+1}-\psi G_{j, k}$ obtains
Corollary 1.

$$
\psi G_{j, k+1}=(j+1) \psi G_{j, k}-(j-1) \psi G_{j, k-1}-\psi G_{j, k-2}
$$

Inserting (7), the definition of psi, one finds the general recurrence

$$
\begin{align*}
G_{j, k+1}=\left(j^{2}+j+3\right) G_{j, k}-\left(j^{3}+j^{2}+3 j+2\right) G_{j, k-1} & +\left(j^{3}-j^{2}+3 j-2\right) G_{j, k-2}  \tag{11}\\
& +\left(j^{2}-j+3\right) G_{j, k-3}-G_{j, k-4}
\end{align*}
$$

This recurrence is not messy but instead factors into the crowning equation of this paper

$$
\begin{equation*}
\left(E^{2}-\left(j^{2}+2\right) E+1\right)\left(E^{2}-j E-1\right)(E-1) G_{j, k}=0 \tag{12}
\end{equation*}
$$

where $E$ is the forward shift operator. Note that the first, second and third parentheses of (12) are, in fact, the recurrences for bisected coprime, coprime and constant sequences respectively! A more useful expression in terms of forward and backward difference operators is

$$
\begin{equation*}
\left(\delta^{2}-I\right)(\Delta+\nabla-I) \Delta G_{j, k}=0=\left(\Delta^{3}-2 \Delta^{2}+\Delta-\nabla \delta^{2}\right) G_{j, k} \tag{13}
\end{equation*}
$$

only if $j=1$. Now (12) is more general than (1) and (13) is more general than $\left\{G_{1}\right\}=\ldots 79,42,10,9,2,4,3,6,10$, $21,46,108, \ldots$. An example of (13) is the sequence

$$
\begin{gather*}
0,0,0,0,1,5,18,56,162,450,1221,3267,8668,22880, \cdots,  \tag{13a}\\
60204,158108,414729
\end{gather*}
$$

whose falling diagonal, $\Delta^{t}$, from the first zero is

$$
\begin{equation*}
0,0,0,0,1,0,3,0,8,0,21,0, \cdots \tag{13b}
\end{equation*}
$$

Hence to obtain $j^{\text {th }}$ order $G$ numbers some IC. must be introduced. First some simplifications. When $j=1$, then Eqs. ( 9 a ), (10) and (11) become
(10a)

$$
\begin{equation*}
\left(\delta^{2}-I\right) G_{k}=5 F_{k}-3 L_{k}-1=-\left(1+2 L_{k-2}\right) \tag{9b}
\end{equation*}
$$

(11a)

$$
G_{k+1}=5 G_{k}-7 G_{k-1}+G_{k-2}+3 G_{k-3}-G_{k-4},
$$

respectively. Note that (13a) was calculated by (13) and checked by (11a). Also note that (11), (12), (13), (11a) are fifth-degree recurrences. Gould [5] found (11a) independently. Directly from (10) one can find the modified recurrence

$$
\begin{equation*}
G_{j, k+1}=\left(j^{2}+j+2\right) G_{j, k}-\left(j^{3}+2 j\right) G_{j, k-1}-\left(j^{2}-j+2\right) G_{j, k-2}+G_{j, k-3}+j^{3} \tag{14}
\end{equation*}
$$

which, when $j=1$, becomes

$$
\begin{equation*}
G_{k+1}=4 G_{k}-3 G_{k-1}-2 G_{k-2}+G_{k-3}+1 \tag{14a}
\end{equation*}
$$

and from this latter it is easy to derive the exquisite

$$
\begin{equation*}
\delta^{4} G_{k+2}=3 \delta^{2} G_{k+1}-G_{k}+1 . \tag{14b}
\end{equation*}
$$

At this point the reader should study Tables 1 and 2 . Now a curious fact results from Corollary 1 which 1 rewrite as
Corollary 1.

$$
\psi\left(G_{j, k+1}+G_{j, k-2}\right)=(1+j) \psi G_{j, k}+(1-j) \psi G_{j, k-1}
$$

This says that making both $j$ and $k$ negative reproduces the same recurrence. To be specific replace $j$ by $-j$ and let $n=(1-k)$ and the Corollary regenerates itself. Thus $4,3,6,10,21,46, \ldots$ has the same recurrence as $4,4,9,18,42$, 101, … See Table 1.
Lemma. The zeroth term of all $\left\{G_{j}\right\}$ equals the constant 4.
The proof is direct from Eqs. (1) through (3a). Omitting the subscript $j$ for simplicity and recalling that $P_{j, 1}=1$ for all $j$ we have:

$$
\begin{align*}
G_{0}=1+P_{0}^{*}+P_{-1}= & 1+P_{1}+P_{-1}+P_{-1}=1+3 P_{1}=4 \\
& G_{j, 0}=4 . \tag{15}
\end{align*}
$$

From (12) of paper [3] one may easily find

$$
\begin{equation*}
G_{j, 1}=(j+2) \quad \text { and } \quad \delta^{2} G_{j, 0}=G_{j, 1} \Delta G_{j, 0} \tag{16a,b}
\end{equation*}
$$

Table 1
Array of $G_{j, k}$ Numbers

| $j / k$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 6 | 2027452 | 53120 | 1444 | 32 | 4 | 8 | 76 | 1640 | 54796 | 2034896 |  |
| 5 | 510354 | 18761 | 729 | 22 | 4 | 7 | 54 | 843 | 19629 | 513402 |  |
| 4 | 98532 | 5392 | 324 | 14 | 4 | 6 | 36 | 382 | 5796 | 99574 |  |
| 3 | 13090 | 1154 | 121 | 8 | 4 | 5 | 22 | 146 | 1309 | 13364 |  |
| 2 | 1020 | 156 | 36 | 4 | 4 | 4 | 12 | 44 | 204 | 1068 |  |
| 1 | 42 | 10 | 9 | 2 | 4 | 3 | 6 | 10 | 21 | 46 | 108 |
| 0 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |
| -1 | 42 | 18 | 9 | 4 | 4 | 1 | 6 | 2 | 21 | 24 | 108 |
| -2 | 1020 | 184 | 36 | 8 | 4 | 0 | 12 | 16 | 204 | 804 |  |
| -3 | 13090 | 1226 | 121 | 14 | 4 | -1 | 22 | 74 | 1309 | 12578 |  |

Table 2
The Table of Differences of $G_{k}$
leaving a fourth initial condition to be chosen in order to define $G_{j, k}$. We may now take this to be

$$
\begin{equation*}
\delta^{2} G_{j, 1}=2 G_{j,-1} \tag{16c}
\end{equation*}
$$

One can also show from (1) or from (12) of paper [3] that

$$
\begin{equation*}
G_{j,-2}=\left(j^{2}+2\right)^{2} \quad \text { and } \quad G_{j,-1}=j(j-1)+2=G_{-j+1,-1} \tag{17}
\end{equation*}
$$

for all integer $j$. At this point it will help the reader to go through an example such as the $j=3$ case beginning with $P_{3, k}=\cdots 0,1,3,10,33,109,360,1189,3927, \cdots$. In fact relations stronger than Corollary 1 exist as is evident from Table 1 where we see that

$$
\begin{equation*}
G_{j, k}+G_{j,-k}=G_{-j, k}+G_{-j,-k} \tag{18}
\end{equation*}
$$

for all integer $j$ and $k$ and indeed a special case follows if $e$ is even

$$
\begin{equation*}
G_{j, e}=G_{-j, e} \tag{19}
\end{equation*}
$$

Now (18) and (19) are easily proven from (1) and the odd/even properties of $F$ and $L$ sequences.

## DIVISIBILITY PROPERTIES

For the study of divisibility properties we are able to rewrite (1) by substituting (6) of [3],
into it giving
(20)

$$
\begin{gathered}
P_{2 n-1}=P_{n}^{*} P_{n-1}-\cos (\pi n), \\
G_{j, k}=P_{j, k}^{*}\left(1+P_{j, k-1}\right)+1+(-1)^{k+1} \\
G_{k}=L_{k}\left(1+F_{k-1}\right)+1+(-1)^{k+1} .
\end{gathered}
$$

(20)

Hence the divisibility properties of the even $G_{k}$ are known since Jarden [4, p. 97] has tabulated the divisors of ( $1+$ $\left.F_{n}\right)$. The divisibility of the odd $G_{k}$ is involved. Three divides $G_{k}$ at intervals of eight starting with

$$
k=\cdots-7,1,9,17,25,33, \cdots
$$

and five divides $G_{k}$ at intervals of twenty starting with $k=\ldots-3,17,37, \ldots$ and proceeding in both directions. Divisibility properties are left for a later paper.
Conjecture 1. If $G_{k}$ is prime then $|k|$ is prime.
Conjecture 2. The number of primes in $\left\{G_{1}\right\}$ is infinite.
The known primes are $G_{-5}=79, G_{-1}=2, G_{1}=3, G_{7}=263$. $G_{31}$ may be prime.
The sequence of $G_{-k}$ is interesting. The first thirteen $G_{-k}$ numbers are placed immediately below their corresponding $G_{k}$ numbers beginning with $k=1$ in both cases.

$$
\begin{align*}
& 3,6,10,21,46,108,263,658,1674,4305,11146,28980, \quad 75547, \cdots  \tag{21}\\
& 2,9,10,42,79,252,582,1645,4106,11070,28459,75348,195898, \cdots .
\end{align*}
$$

A glance at these $G$ numbers provide another symmetry property,

$$
\begin{equation*}
G_{-2 n}-G_{2 n}=F_{4 n} \text { and } G_{d}+G_{-d}=L_{2 d}+2 \text { for } d \text { odd. } \tag{22}
\end{equation*}
$$

And more generally. it is rather easy to show via (20) that

$$
\begin{gather*}
G_{j,-2 n}-G_{j, 2 n}=P_{j, 2 n}^{*}\left(P_{j, 2 n+1}-P_{j, 2 n-1}\right)=j P_{j, 4 n}  \tag{23}\\
G_{j, d}+G_{j,-d}=P_{j, 2 d}^{*}+2 \text { for } d \text { odd } \tag{24}
\end{gather*}
$$

DIFFERENCES OF $G_{k}$
We need the following:

$$
\begin{gather*}
\nabla^{k} H_{n}=H_{n-2 k} \quad \text { and so } \quad \nabla^{k} H_{k}=H_{-k}  \tag{25}\\
\nabla^{2 k} B_{n}=B_{n-k} \quad \text { and } \quad \nabla^{2 k+1} B_{n}=\nabla B_{n-k} \\
\nabla^{k} A_{n}=\operatorname{signum}\left(A_{n}\right)\left|A_{n+k}\right| \tag{27}
\end{gather*}
$$

where $B_{n}$ is any bisection of $H_{n}$, and where (25) and (26) are easily derivable from

$$
\begin{equation*}
H_{n+1}=H_{n}+H_{n-1}, \quad \text { any } H_{0} \text { and } H_{1}, \tag{28}
\end{equation*}
$$

and where $A_{n}$ is a two-point sequence with alternating signs satisfying

$$
\begin{equation*}
A_{n+1}=-A_{n}+A_{n-1} \tag{29}
\end{equation*}
$$

corresponding to $j=-1$ in (3), and signum is the sign function.
Then application of (25) and (26) to (1) immediately gives

$$
\begin{equation*}
\nabla^{k} G_{k}=F_{k-1}+(-1)^{k} L_{k} \tag{30}
\end{equation*}
$$

which becomes $-F_{k+1}$ in the odd $k$ case. Note that these numbers lie along a falling diagonal from $G_{0}=4$ in Table 2. Equation (30) introduces a significant simplicity into the $G_{k}$ numbers. Note that (30) is reminiscent of the definition of the Bell numbers, to wit:

$$
\begin{equation*}
\nabla^{n-1} \text { Bell }_{n}=\text { Bell }_{n-1}, \quad n \geqslant 2 \tag{31}
\end{equation*}
$$

Likewise one may also show that

$$
\begin{equation*}
\nabla^{k-1} G_{k}=F_{k-4} \quad \text { for odd } k \geqslant 3 \tag{32}
\end{equation*}
$$

and these numbers $1,1,2,5, \cdots$ are a bisection of the falling diagonal from $G_{1}=3$. Note that all falling diagonals are two bisected sequences, $B_{n}$, and satisfy for all $k$ and all $n \geqslant 1$,

$$
\begin{equation*}
\Delta^{n+4} G_{k}=3 \Delta^{n+2} G_{k}-\Delta^{n} G_{k} \tag{33}
\end{equation*}
$$

I did not expect to find upon glancing at the central differences of $G_{0}$ that they would be: $-3,19,-75, \ldots$ almost Lucas numbers. We may write

$$
\begin{equation*}
\delta^{2 n} G_{0}=\nabla^{2 n} G_{n}=1+(-1)^{n} L_{3 n} \tag{34}
\end{equation*}
$$

This may be easily derived from (1) with $j=1$ by applying (25). The critical step is

$$
\begin{gather*}
\nabla^{2 k} L_{k}=L_{k-4 k}=L_{-3 k} \\
\nabla^{2 k-1} G_{k}=L_{-3 k+2}, \quad k \geqslant 1 \\
\nabla^{2 k} G_{k}=L_{-3 k}+F_{-1}, \quad k \geqslant 1  \tag{35b}\\
\nabla^{2 k+1} G_{k}=L_{-3 k-2}+F_{-2}, \quad k \geqslant 0, \tag{35c}
\end{gather*}
$$

according to (25). We obtain (35a)
where, of course, $F_{-2}=-1$ and $F_{-1}=1$. Equations (35) prove what is obvious by looking at Table 2, namely if we make a zig-zag below the 4 entry we obtain the sequence: $-1,2,-3,7,-12,19,-29,46,-75,123, \cdots$ which is almost the Lucas sequence. This makes the whole sequence easy to generate by hand. Finally the choice of letter for these sequences was Gould's [1] who suggested my name for them after seeing my paper [6].
The author appreciates some comments by Zeitlin [8] concerning (14) and (23). Zeitlin [7] has also pointed out that the subscript of the subscript of the last term of Eq. (12) of [6] should be $(k-1)$ and not ( $k-2$ ). This mis: print is obvious from the expansion in (13) of [6]
Having found that the messy looking $G_{j, k}$ sequence actually satisfies the near Fibonacci relationships (10) and (12) and further that the Lucas numbers have made their presence known, I am impelled to write down an old haiku of mine in which even the numbers of syllables in each line, namely $3,2,5,7$ are themselves a Fibonacci sequence.

PHI
Multiply
Or add
We always reach phi
Symmetries we perpetrate.

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## [Continued from page 165.]

## $\star \star \star \star$

where the $i^{\text {th }}$ column of $C_{n}$ is the $i^{\text {th }}$ row of Pascal's triangle adjusted to the main diagonal and the other entries are 0 's. Find $C_{n} \cdot A_{n}^{T}$.
Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.
A. Let $B_{n}=A_{n} \cdot A_{n}^{T}$. Let $a_{i j}$ and $b_{i j}$ be the entries in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of $A_{n}$ and $B_{n}$, respectively. Similarly, let $a_{i j}^{T}$ be the $j^{n}$ th entry of $A_{n}^{T}$. Then

$$
\begin{aligned}
a_{i j} & =\binom{i-1}{j-1} \quad \text { if } i \geqslant j ; \\
& =0 \quad \text { elsewhere; }
\end{aligned}
$$

therefore,

$$
\begin{aligned}
a_{i j}^{T} & =\binom{j-1}{i-1} \quad \text { if } i \leqslant j \\
& =0 \quad \cdot \text { elsewhere. }
\end{aligned}
$$

# THE FLUID MECHANICS OF BUBBLING BEDS 

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The basic criterion in establishing a pilot plant fluidized bed reactor, leading to minimal uncertainties in later scaleup, is to ensure that it simulate exactly a portion of the freely bubbling commercial bed. Though this may frequently require a larger-than-economically-justifiable design, a knowledge of the considerations surrounding its development allows some appreciation of the consequences of any compromises. In the scaling of fluidized bed reactors and the development of bubble models to describe the gas-solids contact, it is necessary that the bubble be definable from its source to the bed surface. Though an average size may be suitable in certain instances, its determination is in any event dependent on the very parameters needed to describe the bubble's entire history; hence a truer-than-average physical interpretation is to be preferred.
The idealized instantaneous picturization of a freely bubbling gas fluidized bed of solids, as illustrated in Fig. 1, gives rise to a number of questions forming the bases of realistic scaling criteria and bubble-related kinetics.

1. How do bubbles form?
2. By what mechanism do they rise?
3. What lends them stability?
4. How fast do they rise?
5. What makes them grow in size?
6. What size can they attain?
7. How do bubble and interstitial gas interact?

## BUBBLE FORMATION

Bubbles form at the ports where fluidizing gas enters the bed. They form simply because the velocity at the interface of the bed just above the hole represents a gas input rate in excess of what can pass through the interstices with a frictional resistance less than the bed weight and hence the layers of solids above the holes are pushed aside until they represent a void through whose porous surface the gas can enter at the incipient fluidization velocity. If the void attempts to grow larger the interface velocity becomes insufficient to hold back the walls of the void and hence they cave in from the sides cutting off the void and presenting a new interface to the incoming gas. This sequence is illustrated in Fig. 2. The depth of penetration of the grid gas jets has been correlated empirically and the size of the initial bubble resulting from a detached void shown, within experimental error, to be about half the penetration depth [1].

## MECHANISM OF RISE

Bubbles or "gas voids" rise in a fluidized bed by being displaced with an inflow of solids from their perimeter. Since free flowing and/or incipiently fluidized bulk solids have shallow angles of repose their walls cannot stand at $90^{\circ}$ and hence the solids slide down the bubble's walls into its bottom where all the peripheral streams collide to form a socalled "wake" as illustrated in Fig. 3. Observations [2] of this downflow of solids in a "shell" around the bubble [3] have shown it to occupy an annular thickness of $1 / 4$ of the bubble diameter so that the overall diameter within which a bubble can rise "freely" as it would in a bed of infinite diameter can be defined as $1.5 \mathrm{D}_{\mathrm{B}}$.

## SURFACE STABILITY

Since the peripheral surface of the bubble is simply a layer of particles, it is at first difficult to understand why the particles do not fall from its roof and annihilate the bubble. Danckwerts [4] simple bed support experiments, illustrated in Fig. 4, provide the physical demonstration and Rowe and Henwood's [5] experiments the classical approach.
In Fig. 4(a) the air rate is raised to the point of incipient fluidization and in Fig. 4(b) through 4(f) this same gas rate is passed through the bed in the opposite direction. Note that in position (d) the solids do not slide to their angle


Fig. 1 Idealized Freely-Bubbling Gas Fluidized Bed


Fig. 2 Bubble Formation from Bed-Penetrating Gas Jets at the Grid Ports


Fig. 3 Bubble Rise via Displacement by Inflow of a Surrounding Down-Flowing Shell of Bed Solids


Fig. 4 Bed Support Experiments of P. V. Danckwerts
of repose but instead are held at $90^{\circ}$ and that on reaching (f) the bed is held up without solids falling from what is now its lower side or conversely the upper surface of a bubble in a fluidized bed. When the surface of a bed is traversed by an incipiently fluidizing flow the particles cannot separate from each other. This not only explains the bubble's surface stability but also the integrity of the walls of a bed penetrating jet as in Fig. 2.
Rowe and Henwood carried out classical drag measurements which revealed that the drag on a downstream particle is reduced due to the presence of an adjacent upstream particle. This simply means that a particle cannot fall from the roof of a bubble because if it did then it would immediately be followed by the particle above it, and that by the particle still further above, etc., so that the entire mass or bed above the bubble would have to collapse as a unit. For this to occur, the excess gas could not be passed through the bed unless the bed were physically held down or restrained at its upper surface.

## VELOCITY OF RISE

The velocity at which bubbles rise in a gas fluidized bed has been measured photographically by several investigators. The results are in excellent agreement with what would be predicted for gas bubbles in liquids from the drag coefficient versus Reynolds number correlations of such investigators as Van Krevelen and Hoftijzer [6] illustrated in Fig. 5. Over the range of Reynolds numbers corresponding to reasonable size bubbles the drag coefficient is essentially a constant so that simple substitution shows that if gas density is small relative to the bed density:

$$
\begin{aligned}
C_{D} & =\frac{4 g D_{B}\left(\rho_{B}-\rho_{G}\right)}{3 \rho_{B} V_{B}^{2}} \\
\therefore V_{B} & =\overline{\frac{4 g}{3 k} \frac{D_{B}\left(\rho_{B}-\rho_{G}\right)}{\rho_{B}}}
\end{aligned}
$$

or

$$
V_{B}=4.01 \sqrt{D_{B}} .
$$

This has been corroborated in experiments with freely bubbling beds.
Tarmy and Matsen [7] have shown that in slugging beds the full width of the downflowing solids shell (Fig. 3) is restricted and the velocity of bubble rise then approximately $1 / 2$ that in a freely bubbling bed.

## BUBBLE GROWTH

That bubbles must grow by merger as they rise through the bed is obvious from the large and less frequent surface eruptions relative to a much higher frequency of small voids initiated from a usual multitude of grid ports. Growth by simple gas expansion resulting from the pressure reduction between bottom and top of a fluidized bed is generally relatively insignificant.
From the solids inflow model [3] of Fig. 3 it is obvious that a bed must be exceptionally homogeneous to expect the shell of downflowing solids around a bubble to be flowing at equal rate in every plane. Any bed non-uniformity can cause a shift in the bubble shape or position. Merely the prior passage of another bubble could alter local densities or distributions so as to make bed solids in one local area more readily flowable in a given direction than the bed solids in an adjacent area. The solids inflow model therefore obviates a simple mechanism of bubble merger. If two bubbles get close enough that their shells of downflowing solids begin to interact, the touching shells will represent a local downflowing stream of solids faced with more than one path to the nearest void. The stream could be squeezed to the point of being insufficient to satisfy both bubbles and thereby drain off leaving no wall between the voids and hence the appearance of a single bubble.
It is therefore readily acceptable that the idealized bubbling of Fig. 6 (a) will lead to a situation as in $6(\mathrm{~b})$ where two bubbles of unit initial volume can merge into bubbles of twice this volume. Since larger bubbles rise more rapidly these double volume bubbles will catch up and merge with other unit volume bubbles to yield bubbles of thrice the initial bubble volume. These newer bubbles will rise even more rapidly and can catch up with bubbles of 1 or 2 times the volume of the initial bubble resulting in bubbles of at most 5 times the volume of the initial bubble. The bubble of five-fold volume can now catch up with bubbles of 1,2 or 3 times the volume of the initial bubble resulting in bubbles of at most 8 times the volume of the initial bubble as illustrated in Figs. 6(c) and (d). Carrying on this process of overtaking bubbles results in a sequence of maximum multiples of the initial bubble volume in which each

Fig. 5 Rate of Rise of Gas Bubbles in Liquids


FIGURE 6. THE "CATCH-UP" MECHANISM OF bubble GROWTH
multiple is the sum of the two previous multiples. This sequence, illustrated in Fig. 7, is the well known "Fibonacci" [8] series.
Since the levels at which the maxima exist represent the summation of the diameters of their forebearers and since their diameters are proportional to the cube root of their volumes, it follows that the ratio of merged bubble diameter to initial bubble diameter is equal to the cube root of the number of initial bubbles consumed in the merger, and also that the level at which the merged bubbles exists relative to the height (or diameter) of the initial bubble is equal to the summation of the cube root of the number of initial bubbles consumed in the merged bubble. For the case of the maximum size of merged bubble this is illustrated analytically in Fig. 7 and shown graphically in Fig. 8.
That the mechanism of Figs. 7 and 8 appears in good agreement with experimental observations is illustrated in Fig. 9 where the empirical bubble growth relationships proposed by Chavarie and Grace [9], Werther [10] , and Rowe [11] are superimposed on the curve representing the Fibonacci series. In using Fig. 9 to determine the maximum bubble diameter, $D_{B}$, at any bed level, $L_{B}$, above the grid it is necessary to determine the initial bubble diameter, $D_{B i}$, which could exist at the grid level as a result of individual or merged jets. Figure 9 must also not be extrapolated beyond the maximum attainable stable bubble size.

## MAXIMUM STABLE DIAMETER

Danckwert's bed support experiments (Fig. 4) and those of Rowe and Henwood based on particle drag force measurements, demonstrated that a bed interface (and hence a bubble) should be fundamentally stable against collapse as long as it is traversed by a superficial velocity equal to its incipient fluidization rate. Since the inflowing solids shell volume usually far exceeds the incipient fluidization rate, there would appear to be no limit to the attainable bubble size, or dome, apt to collapse. Presumably, if the dome cannot collapse amid free flowing bed solids then as the bubble grows it could only be limited by particles leaving the shell and being entrained into the bubble void. Such entrainment, or particle pickup, would be most likely to occur from the bubble walls as the result of the relative velocity between gas and surface particles at the interface. Since against the downward velocity of bulk solids the bubble fluid (whether gas or liquid) rises at approximately an equal velocity, the relative flow of fluid past the particles at the bubble wall is twice the shell or bubble velocity. Equating twice the bubble velocity to the particle pickup velocity allows calculation of the minimum bubble size necessary to stir up the solids interface and thus thwart bubble appearance or growth. Since pickup velocity is approximately twice saltation [12] , this is equivalent to equating bubble velocity to saltation velocity. This procedure has given results in reasonable agreement with a broad range of observations reported to date. For example 80 micron particles of sand fluidized with air could sustain a maximum bubble diameter of the order of 24 inches whereas when fluidized with water the maximum bubble size would be indiscernable. Sand particles 600 microns in diameter when fluidized with water would permit a maximum stable bubble size of only $1 / 4$ inch and 3,000 micron lead particles a water bubble of 7 inches.

## BUBBLE GAS INTERACTION

The outside diameter of the shell of downflowing bed solids surrounding the rising bubble is the minimum reactor diameter necessary to simulate free bubbling. In addition to simulating free bubbling hydrodynamically it may be argued that gas permeation from bubble into surrounding bed should also be equalled. This only becomes significant or controlling with coarse and easily permeated beds having a high incipient fluidization velocity. The gas permeation or "cloud" diameter is calculable from the depth of gas flow at incipient fluidization velocity over the time interval required for the bubble to rise a distance of one bubble diameter. Since the bubble rises at a velocity equal to 4 times the square root of its diameter it follows that:

$$
\frac{\text { Thickness of gas penetrated "cloud" }}{\text { Thickness of downflowing solids "shell" }}=\frac{V_{m_{f}}}{\sqrt{D_{B}}}
$$

or since

$$
\begin{aligned}
& \text { "shell" 0.D. }=1.5 D_{B} \\
& \text { "cloud" 0.D. }=D_{B}+0.5 \sqrt{D_{B}} V_{m f}
\end{aligned}
$$

In applying free shell or cloud criteria in scaleup or scaledown the relationship between bubble diameter and bed depth is obtainable from Fig. 9 with the limitation of the system's maximum stable bubble size. Thus from grid design, operating superficial velocity, and fluid and particle properties, it is possible to calculate the initial bubble size
[APR.


Fig. 7 Maximum Bubble Growth by the "Catch-Up" Mechanism Resulting in a Fibonacci Sequence

Fig. 8 Bubble Growth by Merger Represented by the Fibonacci Sequence

at the grid, the maximum stable size, and the bed depth over which the bubbles may grow from their initial to their stable diameter. Once having reached their maximum stable diameter any further unlikely mergers would also lead to collapse, so that bubble diameter may be considered constant once having reached the stable size.
An unquestionably conservative approach to a minimal risk pilot plant reactor free of scaleup considerations would suggest it equal the larger of either "cloud" or "shell" diameter surrounding the system's maximum stable bubble.

## nomenclature

$C_{D}=$ Drag coefficient, dimensionless
$D_{B}=$ Bubble diameter, feet
$D_{B i}=$ Bubble diameter at grid level
$g=$ Gravitational acceleration, $32.2 \mathrm{ft} . / \mathrm{sec} .{ }^{2}$
$L_{B}=$ Bed depth, feet
$P=$ Grid jet penetration
$R e=$ Reynolds number, dimensionless
$V_{B}=$ Bubble rise velocity, $\mathrm{ft} . / \mathrm{sec}$.
$V_{m f}=$ Incipient fluidization velocity, $\mathrm{ft} . / \mathrm{sec}$.
$\rho_{B}=$ Bed density, lbs./cu. ft.
$\rho_{G}=$ Gas density, lbs./cu. ft.

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## [Continued from page 170.]

* 

Thus,

$$
b_{i j}=\sum_{k=1}^{n} a_{i k} a_{k j}^{T}=\sum_{k=1}^{n}\binom{i-1}{k-1}\binom{j-1}{j-k} .
$$

Actually, the effective upper limit of this last summation is $\min .(i, j)=m+1$. Therefore,

$$
b_{i j}=\sum_{k=0}^{m}\binom{i-1}{k}\binom{j-1}{j-1-k}=\sum_{k=0}^{m}\binom{j-1}{k}\binom{i-1}{i-1-k},
$$

which shows that $b_{i j}$ is symmetric in $i$ and $j$.
Actually, the last summation may readily be evaluated by the Vandermonde convolution theorem, so that:

$$
\begin{equation*}
b_{i j}=\binom{i+j-2}{i-1}, \quad \text { for all } i, j \leqslant n \tag{1}
\end{equation*}
$$

B. As before, let $D_{n}=C_{n} \cdot A_{n}^{T}$; let $c_{i j}$ and $d_{i j}$ be the entries in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of $C_{n}$ and $D_{n}$, respectiveiy. Then

$$
c_{i j}=\binom{j-1}{i-j}, \quad j \leqslant i \leqslant 2 j-1
$$

[Continued on p. 187]

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>A. P. HILLMAN<br>University of New Mexico, Albuquerque, New Mexico 87131

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S.E.; Albuquerque, New Mexice 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

## DEFINITIONS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
F_{n+2}=F_{n+1}+F_{n}, \quad F_{0}=0, \quad F_{1}=1 \quad \text { and } \quad L_{n+2}=L_{n+1}+L_{n}, \quad L_{0}=2, \quad L_{1}=1
$$

Also $a$ and $b$ designate the roots $(1+\sqrt{5}) / 2$ and $(1-\sqrt{5}) / 2$, respectively, of $x^{2}-x-1=0$.

## PROBLEMS PROPOSED IN THIS ISSUE

B-376 Proposed by Frank Kocher and Gary L. Mullen, Pennsy/vania State University, University Park and Sharon, Pennsy/vania.
Find all integers $n>3$ such that $n-p$ is an odd prime for all odd primes $p$ less than $n$.

## B-377 Proposed by Paul S. Bruckman, Concord, California.

For all real numbers $a \geqslant 1$ and $b \geqslant 1$, prove that

$$
\sum_{k=1}^{[a]}\left[b \sqrt{1-(k / a)^{2}}\right]=\sum_{k=1}^{[b]}\left[a \sqrt{1-(k / b)^{2}}\right],
$$

where $[x]$ is the greatest integer in $x$.
B-378 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.
Prove that $F_{3 n+1}+4^{n} F_{n+3} \equiv 0(\bmod 3)$ for $n=0,1,2, \cdots$.
B-379 Proposed by Herta T. Freitag, Roanoke, Virginià.
Prove that $F_{2 n} \equiv n(-1)^{n+1}(\bmod 5)$ for all non-negative integers $n$.
B-380 Proposed by Dan Zwillinger, Cambridge, MA.
Let $a, b$, and $c$ be non-negative integers. Prove that

$$
\sum_{k=1}^{n}\binom{k+a-1}{a}\binom{n-k+b-c}{b}=\binom{n+a+b-c}{a+b+1}
$$

Here $\binom{m}{r}=0$ if $m<r$.
B-381 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California. Let $a_{2 n}=F_{n+1}^{2}$ and $a_{2 n+1}=F_{n+1} F_{n+2}$. Find the rational function that has

$$
a_{0}+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+\cdots
$$

as its Maclaurin series.

## SOLUTIONS

## C IS EASY TO SEE

B-352 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.
Let $S_{n}$ be defined by $S_{0}=1, S_{1}=2$, and

$$
S_{n+2}=2 S_{n+1}+c S_{n} .
$$

For what value of $c$ is $S_{n}=2^{n} F_{n+1}$ for all non-negative integers $n$ ?

## Solution by Paul S. Bruckman, Concord, California.

Substituting the definition of $S_{n}$ into the given recursion yields:

$$
2^{n+2} F_{n+3}=2^{n+2} F_{n+2}+c \cdot 2^{n} F_{n+1} \text {, or } F_{n+3}=F_{n+2}+\frac{1}{4} c \cdot \cdot F_{n+1} \text {. }
$$

Since

$$
F_{n+3}=F_{n+2}+F_{n+1},
$$

it follows that $c=4$.
Also solved by George Berzsenyi, Wray G. Brady, Herta T. Freitag, Ralph Garfield, Dinh Thé' Hung, John Ivie, Graham Lord, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

## RECURSIVE SUMS

## B-353 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

For $k$ and $n$ ontegers with $0 \leqslant k \leqslant n$, let $A(k, n)$ be defined by $A(0, n)=1=A(n, n), A(1.2)=c+2$ and

$$
A(k+1, n+2)=c A(k, n)+A(k, n+1)+A(k+1, n+1) .
$$

Also let $S_{n}=A(0, n)+A(1, n)+\cdots+A(n, n)$. Show that

$$
S_{n+2}=2 S_{n+1}+c S_{n}
$$

Solution by A. G. Shannon, New South Wales, I. of T., Australia.

$$
\begin{aligned}
S_{n+2} & =\sum_{i=0}^{n+2} A(i, n+2)=2+\sum_{i=1}^{n+1} A(i, n+2)=2+c \sum_{i=1}^{n+1} A(i-1, n)+\sum_{i=1}^{n+1} A(i-1, n+1)+\sum_{i=1}^{n+1} A(i, n+1) \\
& =2+c \sum_{i=0}^{n} A(i, n)+\sum_{i=0}^{n} A(i, n+1)+\sum_{i=1}^{n+1} A(i, n+1)=2 \sum_{i=0}^{n+1} A(i, n+1)+c \sum_{i=0}^{n} A(i, n) \\
& =2 S_{n+1}+c S_{n},
\end{aligned}
$$

as required.
Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Dinh Thè' Hung, John Ivie, Graham Lord, John W. Milsom, C. B. A. Peck, Bob Prielipp, David Zeitlin and the Proposer.

## A VANISHING FACTOR

B-354 Proposed by Phil Mana, Albuquerque, New Mexico.
Show that

$$
F_{n+k}^{3}-L_{k}^{3} F_{n}^{3}+(-1)^{k} F_{n-k}\left[F_{n-k}^{2}+3 F_{n+k} F_{n} L_{k}\right]=0 .
$$

Solution by Graham Lord, Universite' Laval, Quėbec, Canada.
This follows from a special case of the algebraic identity

$$
a^{3}+b^{3}+c^{3}-3 a b c=(a+b+c)\left(a^{2}+b^{2}+c^{2}-a b-b c-c a\right),
$$

where

$$
a=F_{n+k}, \quad b=-L_{k} F_{n} \quad \text { and } \quad c=(-1)^{k} F_{n-k} .
$$

Note that

$$
F_{n+k}-L_{k} F_{n}+(-1)^{k} F_{n-k}=0
$$

Also solved by Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thè' Hung, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the Proposer.

## CUBIC IDENTITY

B-355 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsy/vania.
Show that

$$
F_{n+k}^{3}-L_{3 k} F_{n}^{3}+(-1)^{k} F_{n-k}^{3}=3(-1)^{n} F_{n} F_{k} F_{2 k}
$$

Solution by Graham Lord, Universite' Laval, Québec, Canada.
The replacement of $L_{3 k}$ by $L_{k}^{3}-3(-1)^{k} L_{k}$ and the utilization of the identity of problem B-354 changes the lefthand side above into
which is the same as

$$
3(-1)^{k} L_{k} F_{n}\left[F_{n}^{2}-F_{n+k} F_{n-k}\right],
$$

that is $3(-1)^{n} F_{n} F_{k} F_{2 k}$.
Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, and the Proposer.

## SOME SOLUTIONS

## B-356 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $S_{n}=F_{2}+2 F_{4}+3 F_{6}+\cdots+n F_{2 n}$. Find $m$ as a function of $n$ so that $F_{m+1}$ is an integral divisor of $F_{m}+S_{n}$. Solution by Paul $S$. Bruckman, Concord, California.
We first find a closed expression for $S_{n}$. Note that

$$
\begin{aligned}
S_{n} & =\sum_{k=1}^{n} k F_{2 k}=\sum_{k=1}^{n}\left\{k F_{2 k+1}-(k-1) F_{2 k-1}-F_{2 k}+F_{2 k-2}\right\}=\left.\left\{(k-1) F_{2 k-1}-F_{2 k-2}\right\}\right|_{k=1} ^{n+1} \\
& =n F_{2 n+1}-F_{2 n} .
\end{aligned}
$$

Clearly,

$$
F_{2 n}+S_{n}=n F_{2 n+1}
$$

and so $m=2 n$ is a solution of the problem. Since $F_{1}=F_{2}=1$, it is clear that $m=0$ and $m=1$ are also (trivial) solutions. The statement of the problem seems to require finding all solutions $m$, and this appears to be a difficult task, perhaps not intended by the Proposer.
Also solved by George Berzsenyi, Wray G. Brady,. Graham Lord, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin and the Proposer.

## GOLDEN RATION INEQUALITY COUNT

B-357 Proposed by Frank Higgins, Naperville, Illinois.
Let $m$ be a fixed positive integer and let $k$ be a real number such that

$$
2 m \leqslant \frac{\log (\sqrt{5} k)}{\log a}<2 m+1
$$

where $a=(1+\sqrt{5}) / 2$. For how many positive integers $n$ is $F_{n} \leqslant k$ ?

Solution by Paul S. Bruckman, Concord, California.
Since $2 m \leqslant \log (k \sqrt{5}) / \log a<2 m+1$, it follows that $a^{2 m} \leqslant k \sqrt{5}<a^{2 m+1}$; hence,

$$
a^{2 m}-b^{2 m}=a^{2 m}-\left(-a^{-1}\right)^{2 m}<k \sqrt{5}<a^{2 m+1}-\left(-a^{-1}\right)^{2 m+1}=a^{2 m+1}-b^{2 m+1}
$$

i.e.,

$$
F_{2 m}<k<F_{2 m+1}
$$

Since $\left\{F_{n}\right\}_{1}^{\infty}$ is a non-decreasing sequence of positive integers, it follows that $F_{n} \leqslant k$ for $n=1,2, \cdots, 2 m$, i.e., for $2 m$ (distinct) values of $n$.
Also solved by A. G. Shannon and the Proposer.

$$
\text { Cont. frem P. } 183
$$

## *

elsewhere.
Therefore,

$$
d_{i j}=\sum_{k=1}^{n} c_{i k} a_{k j}^{T}=\sum_{k=1}^{n}\binom{k-1}{i-k}\binom{j-1}{k-1}
$$

The effective limits of this summation are from $k=1+[1 / 2 i]$ to min . ( $i, j$ ). It will be convenient, however, to consider the upper limit to be equal to $i$; if $i>j$, the extra terms included vanish in any event. Therefore,

$$
d_{i j}=\sum_{k=[1 / 2 i]}^{i-1}\binom{k}{i-1-k}\binom{j-1}{k}=\sum_{k=0}^{[1 / 2(i-1)]}\binom{i-1-k}{k}\binom{j-1}{i-1-k}
$$

For convenience, let $i-1=r, j-1=s$.
Therefore,

$$
d_{i j}=\theta_{r s}=\sum_{k=0}^{[1 / 2 r]}\binom{r-k}{k}\binom{s}{r-k} ;
$$

let

$$
y=\sum_{r=0}^{\infty} \theta_{r s} x^{r}
$$

Then

$$
y=\sum_{r=0}^{\infty} x^{r} \sum_{k=0}^{[1 / 2 r]}\binom{r-k}{k}\binom{s}{r-k}=\sum_{k=0}^{\infty} \sum_{r=2 k}^{\infty} x^{r}\binom{r-k}{k}\binom{s}{r-k}=\sum_{k=0}^{\infty} x^{2 k} \sum_{r=0}^{\infty} x^{r}\binom{r+k}{k}\binom{s}{r+k} .
$$

Thus,

$$
y=\sum_{k=0}^{\infty}\binom{s}{k} x^{2 k} \sum_{r=0}^{\infty}\binom{s-k}{r} x^{r}
$$

by rearranging the combinatorial terms. Then,

$$
y=\sum_{k=0}^{\infty}\binom{s}{k} x^{2 k}(1+x)^{s-k}=(1+x)^{s} \sum_{k=0}^{\infty}\binom{s}{k}\left(\frac{x^{2}}{1+x}\right)^{k}=(1+x)^{s}\left(1+\frac{x^{2}}{1+x}\right)^{s}
$$

or:
(2)

$$
y=\left(1+x+x^{2}\right)^{s}
$$

Therefore, $d_{i j}$ is the coefficient of $x^{i-1}$ in $\left(1+x+x^{2}\right)^{j-1}$. From this, we may deduce that the $d_{i j}$ 's satisfy the following recursion:
(3) $\quad d_{i+2: j+1}=d_{i j}+d_{i+1: j}+d_{i+2: j}(i, j \geqslant 1) ; d_{1: j}=1, d_{2: j}=j-1(j \geqslant 1) ; d_{i: 1}=0 \quad(i>1)$.

We may readily construct a matrix (of unspecified dimensions), whose $j^{\text {th }}$ column is composed of the coefficients of $\left(1+x+x^{2}\right)^{j-1}$, written in correspondence to the ascending powers of $x$, beginning with $x^{0}$. For any given $j, d_{i j}=$ 0 for all $i \geqslant 2 j$ (since $\left(1+x+x^{2}\right)^{j-1}$ contains ( $2 j-1$ ) non-zero terms).

Also solved by the Proposer.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited By<br>RAYMOND E. WHITNEY<br>Lock Haven State College, Lock Haven, Pennsylvania 17745

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit sotutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

## H-281 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Consider matrix equation
(a)

$$
\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{array}\right)^{n}=\left(\begin{array}{ccc}
A_{n} & B_{n} & C_{n} \\
D_{n} & E_{n} & G_{n} \\
H_{n} & I_{n} & J_{n}
\end{array}\right) \quad(n \geqslant 1) .
$$

Identify $A_{n}, B_{n}, C_{n}, \cdots, J_{n}$.
Consider matrix equation
(b)

$$
\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)^{n}=\left(\begin{array}{ccc}
A_{n}^{\prime} & B_{n}^{\prime} & C_{n^{\prime}} \\
D_{n}^{\prime} & E_{n}^{\prime} & G_{n}^{\prime} \\
H_{n}^{\prime} & I_{n}^{\prime} & J_{n^{\prime}}^{\prime}
\end{array}\right) \quad(n \geqslant 1) .
$$

Identify $A_{n}, B_{n}, C_{n}^{\prime}, \cdots, J_{n}$.

## H-282 Proposed by H. W. Gould and W. E. Greig, West Virginia University.

Prove

$$
\sum_{n=1}^{\infty} \frac{a^{2 n}}{a^{4 n}-1}=\sum_{\substack{k=1 \\ k \text { odd }}}^{\infty} \frac{1}{a^{2 k}-1}
$$

where $a=(1+\sqrt{5}) / 2$, and determine which series converges the faster.

## H-283 Proposed by D. Beverage, San Diego Evening College, San Diego, California.

Define $f(n)$ as follows:

$$
f(n)=\sum_{k=0}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^{n+k} \quad(n \geqslant 0) .
$$

Express $f(n)$ in closed form.
H-284 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsy/vania.
(A generalization of R. G. Buschman's H-18)
Show that
(a)

$$
\sum_{k=0}^{n}\binom{n}{k} F_{r k} L_{r n-r k}=2^{n} F_{r n} \quad \text { or } \quad\left(\begin{array}{l}
\left(F^{r}+L^{r}\right)^{n} \underline{(U m b r a l} \text { notation) } \\
\text { (Um) }
\end{array}\right.
$$

(b)

$$
\begin{gathered}
\left.\sum_{k=0}^{n}\binom{n}{k} L_{r k} L_{r n-r k}=2^{n} L_{r n}+2 L_{r}^{n} \quad \text { or } \quad\left(L^{r}+L^{r}\right)^{n} \stackrel{=}{=} 2 L^{r}\right)^{n}+2 L_{r}^{n} \\
\sum_{k=0}^{n}\binom{n}{k} F_{r k} F_{r n-r k}=\frac{\left(2^{n} L_{r n}-2 L_{r}^{n}\right)}{D}
\end{gathered}
$$

(c)

Note. The generalization is valid for all Type I quadratic real fields, i.e, for $D=5,13,29,53,61, \cdots$.

## Remark on Problem H-123 by Henry Gould, West Virginia University.

The proposer's solution, Fibonacci Quart. 7 (1969), No. 2, 177-178, uses Stirling number expansions of factorials and powers. Since, however, it is true that

$$
\sum_{m=k}^{n} g_{n}^{(m)} S_{m}^{(k)}=\delta_{n}^{k}= \begin{cases}0, & k \neq n  \tag{1}\\ 1, & k=n\end{cases}
$$

then, for perfectly arbitrary $F_{k}$, and Fibonacci numbers in particular,

$$
\sum_{m=0}^{n} \sum_{k=0}^{m} \mathscr{S}_{n}^{(m)} S_{m}^{(k)} F_{k}=\sum_{k=0}^{n} F_{k} \sum_{m=k}^{n} S_{n}^{(m)} S_{m}^{(k)}=\sum_{k=0}^{n} F_{k} \delta_{n}^{k}=F_{n}
$$

as desired. It is also true that

$$
\begin{equation*}
\sum_{m=k}^{n} S_{n}^{(m)} \mathbb{S}_{m}^{(k)}=\delta_{n}^{k} \tag{2}
\end{equation*}
$$

so by the same argument we have the dual formula to the original problem:

$$
\begin{equation*}
\sum_{m=0}^{n} \sum_{k=0}^{m} S_{n}^{(m)} \boldsymbol{X}_{m}^{(k)} F_{k}=F_{n} \tag{3}
\end{equation*}
$$

and, what is more interesting, this and the original formula hold for any sequence $\left\{F_{n}, n \geqslant 0\right\}$, the Fibonacci numbers really having nothing whatever to do with the truth of the formulas.
Relations (1) and (2) are the standard orthogonality relations for the two kinds of Stirling numbers, and are implied by the two expansions

$$
\begin{equation*}
(x)_{n}={ }_{k=0}^{n} S(n, k) x^{k} \tag{4}
\end{equation*}
$$

and
(5)

$$
x^{n}=\prod_{k=0}^{n} \oiint(n, k)(x)_{k},
$$

where

$$
(x)_{n}=x(x-1)(x-2) \ldots 3 \cdot 2 \cdot 1, \text { with }(x)_{0}=1
$$

Expansions (4)-(5) of course are the ones used by the proposer in his solution of his problem. Formulas (1) and (2) are both in Jordan's "Calculus of Finite Differences," page 184, the same source quoted by Lind for formulas (4)(5). The essential point I am making is the generality of formulas (1)-(2) as opposed to the original solution.

EDITORIAL ACKNOWLEDGEMENT. Gregory Wulczyn, Bucknell University, submitted a solution for $\mathrm{H}-265$ as well as an extensive partial solution for $\mathrm{H}-266$.

## SOLUTIONS

## SUM SOLUTION

H-267 (Corrected) Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.
Show that

$$
S(x)=\sum_{n=0}^{\infty}(k n+1)^{n-1} \frac{x^{n}}{n!},
$$

where $k$ is any integer and $0^{0} \equiv 1$, satisfies

$$
S(x)=e^{x S^{k}(x)}
$$

Solution by P. Bruckman, Concord, California.
We identify the given series as

$$
\begin{equation*}
S(x)=\sum_{n=0}^{\infty}(k n+1)^{n-1} \frac{x^{n}}{n!} \tag{1}
\end{equation*}
$$

In "The H-Convolution Transform," V. E. Hoggatt, Jr., and Paul S. Bruckman, Fibonacci Quarterly, Vol. 13, No. 4, Dec. 1975, pp. 357-68, the following result is proved (where, to avoid confusion, we change the notation): Let

$$
\begin{equation*}
f(x)=\sum_{i=0}^{\infty} a_{i: 0} x^{i} \tag{2}
\end{equation*}
$$

(3)

$$
(f(x))^{j+1}=\sum_{i=0}^{\infty} a_{i: j} x^{i}
$$

where $f(0) \neq 0, f$ is analytic about $x=0$. Also, let

$$
\begin{equation*}
G_{s, k}(x) \equiv G(x)=\sum_{i=0}^{\infty} \frac{s}{k i+s} a_{i: k i+s-1} x^{i} \tag{4}
\end{equation*}
$$

Then
(5)

$$
G(x)=f\left\{x(G(x))^{k}\right\} .
$$

In particular, let
(6)

$$
f(x)=e^{x}, \quad s=1
$$

Then

$$
(f(x))^{j+1}=e^{(j+1) x}=\sum_{i=0}^{\infty} \frac{(j+1)^{i} x^{i}}{i!}=\sum_{i=0}^{\infty} a_{i: j} x^{i},
$$

which implies
(7)

$$
a_{i: j}=\frac{(j+1)^{i}}{i!} .
$$

Hence,

$$
\frac{s}{k i+s} a_{i: k i+s-1}=\frac{1}{k i+1} \frac{(k i+1)^{i}}{i!}=\frac{(k i+1)^{i-1}}{i!},
$$

and also $G(x)=S(x)$, as given by (1). From (5), it now follows that

$$
\begin{equation*}
\exp \left(x S^{k}(x)\right)=S(x) \tag{8}
\end{equation*}
$$

Also solved by V. E. Hoggatt, Jr.

## USE YOUR UMBRAL-AH

H-268 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Put

$$
S_{n}(x)=\sum_{k=0}^{n} S(n, k) x^{k}
$$

where $S(n, k)$ denotes the Stirling number of the second kind defined by

$$
x^{n}=\sum_{k=0}^{n} S(n, k) x(x-1) \cdots(x-k+1)
$$

Show that

$$
\left\{\begin{array}{c}
x S_{n}(x)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} S_{j+1}(x) \\
S_{n+1}(x)=x \sum_{j=0}^{n}\binom{n}{j} S_{j}(x)
\end{array}\right.
$$

More generally evaluate the coefficients $C(n, k, j)$ in the expansion

$$
x^{k} S_{n}(x)=\sum_{j=0}^{n+k} C(n, k, j) S_{j}(x) \quad(k, n \geqslant 0)
$$

## Solution by P. Bruckman, Concord, California.

For the sake of typographical convenience, we make a slight change in notation. Let $S_{1}(n, k)$ and $S_{2}(n, k)$ denote the Stirling numbers of the first and second kinds, respectively, given by:

$$
\begin{equation*}
x^{(n)} \equiv x(x-1)(x-2) \cdots(x-n+1)=\sum_{k=0}^{n} s_{1}(n, k) x^{k} \tag{1}
\end{equation*}
$$

(2)

$$
x^{n}=\sum_{k=0}^{n} S_{2}(n, k) x^{(k)}
$$

Also, we define $x^{(0)} \equiv 1$. The following orthogonality relation is satisfied by the Stirling numbers:

$$
\begin{equation*}
\delta_{m: n}=\sum_{j=m}^{n} S_{1}(n, j) S_{2}(j, m) \tag{3}
\end{equation*}
$$

Using (1)-(3), we may derive an explicit expression for the $c(n, k, j)$ 's as follows:

$$
\begin{aligned}
x^{k} S_{n}(x) & =\sum_{r=0}^{n} S_{2}(n, r) x^{r+k}=\sum_{r=0}^{n} S_{2}(n, r) \sum_{m=0}^{r+k} x^{m} \delta_{r+k: m}=\sum_{r=0}^{n} S_{2}(n, r) \sum_{m=0}^{r+k} x^{m} \sum_{j=m}^{r+k} S_{1}(r+k, j) S_{2}(j, m) \\
& =\sum_{r=0}^{n} S_{2}(n, r) \sum_{j=0}^{r+k} S_{1}(r+k, j) \sum_{m=0}^{j} S_{2}(j, m) x^{m}=\sum_{r=0}^{n} S_{2}(n, r) \sum_{j=0}^{r+k} S_{1}(r+k, j) S_{j}(x) \\
& =\sum_{j=0}^{n+k} S_{j}(x) \sum_{r=M}^{n} S_{2}(n, r) S_{1}(r+k, j)
\end{aligned}
$$

where $M=\max (j-k, 0)$. Hence,

$$
\begin{equation*}
c(n, k, j)=\sum_{r=M}^{n} S_{2}(n, r) S_{1}(r+k, j) \tag{4}
\end{equation*}
$$

A more elegant algorithm for computing $c(n, k, j)$ may be derived by employing the umbral calculus, whereby $S_{j}(x)$ is replaced by $S^{j}$, and $S$ is treated as an algebraic quantity. Returning to one of the relations preceding (4), and replacing true equality by "umbral equality," denoted by the symbol "으," we then have:

$$
\begin{aligned}
x^{k} S_{n}(x)=\sum_{r=0}^{n} S_{2}(n, r) \sum_{j=0}^{r+k} S_{1}(r+k, j) S_{j}(x) & \cong \sum_{r=0}^{n} S_{2}(n, r) \sum_{j=0}^{r+k} S_{1}(r+k, j) S^{j}=\sum_{r=0}^{n} S_{2}(n, r) S^{(r+k)} \\
& =S^{(k)} \sum_{r=0}^{n} S_{2}(n, r)(S-k)^{(r)}=S^{(k)}(S-k)^{n}
\end{aligned}
$$

More precisely, we have the generating function:
(5)

$$
\sum_{j=0}^{n+k} c(n, k, j) u^{j}=u^{(k)}(u-k)^{n}
$$

An alternative expression, derived by expanding $u^{(k)}$ in terms of $S_{1}(k, j)$ 's, is the following:

$$
\begin{equation*}
c(n, k, j)=\sum_{r=M}^{N}\binom{n}{r}(-k)^{r} S_{1}(k, j-r), \tag{6}
\end{equation*}
$$

where $M$ has been previously defined and $N=\min (j, n)$. Using the fact
(7)

$$
S_{1}(1, n)=\delta_{n: 1},
$$

we find in particular, from (6):

$$
c(n, 1, j)=\sum_{r=0}^{n}\binom{n}{r}(-1)^{r} S_{1}(1, j-r)
$$

where the summation possibly includes undefined terms, which we define to be vanishing terms. Thus,

$$
c(n, 1,0)=\binom{n}{0}(-1)^{0} S_{1}(1,0)=S_{1}(1,0)=0 ; \quad c(n, 1, n+1)=\binom{n}{n}(-1)^{n} S_{1}(1,1)=(-1)^{n} S_{1}(1,1)=(-1)^{n} ;
$$

if $1 \leqslant j \leqslant n$,

$$
c(n, 1, j)=\sum_{r=j-1}^{j}\binom{n}{r}(-1)^{r} S_{1}(1, j-r)=\binom{n}{j-1}(-1)^{j-1} S_{1}(1,1)+\binom{n}{j}(-1)^{j} S_{1}(1,0)=(-1)^{j-1}\binom{n}{j-1} .
$$

Therefore, in all cases (i.e., for $j=0,1, \cdots, n+1$ ),

$$
\begin{equation*}
c(n, 1, j)=(-1)^{j-1}\binom{n}{j-1}, \tag{8}
\end{equation*}
$$

where the binomial coefficients $\binom{r}{s}$ are defined to vanish when $s<0$ or $s>r$. Hence,

$$
\begin{equation*}
x S_{n}(x)=\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n}{j-1} S_{j}(x)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{j} S_{j+1}(x) \tag{9}
\end{equation*}
$$

By the well known technique of binomial inversion,

Also solved by F. Howard and the Proposer.

$$
\begin{equation*}
S_{n+1}(x)=x \sum_{j=0}^{n}\binom{n}{j} S_{j}(x) \tag{10}
\end{equation*}
$$

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[^0]:    *The author promises there will not be a third; he has no intention of composing a sonata.

