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PROPERTIES OF GENERATING FUNCTIONS OF A CONVOLUTION ARRAY

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A sequence of sequences S_k which arise from inverses of matrices containing certain columns of Pascal's triangle provided a fruitful study reported by Hoggatt and Bicknell [1], [2], [3], [4]. The sequence $S_1 = \{1, 1, 2, 5, 14, 42, \ldots\}$ is the sequence of Catalan numbers. Convolution arrays for these sequences were computed, leading to classes of combinatorial and determinant identities and a web of inter-relationships between the sequences S_k. The inter-relationships of the generating functions of these related sequences led to the H-convolution transform of Hoggatt and Bruckman [5], which provided proof of all the earlier results taken together as well as generalizing to any convolution array. The development required computations with infinite matrices by means of the generating functions $S_k(x)$ for the columns containing the sequences S_k. In this paper, properties of the generating functions $S_k(x)$ are studied and extended.

1. INTRODUCTION

We define $S_k(x)$ as in Hoggatt and Bruckman [5]. Let $f(x)$ be the generating function for a sequence $\{f_i\}$ so that

$$f(x) = \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} a_{i,0} x^i \tag{1.1}$$

where $f(0) = f_0 = a_{00} \neq 0$ and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij} x^i, \quad j = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{1.2}$$

where $a_{i,-1} = 1$ if $i = 0$ and $a_{i,-1} = 0$ if $i \neq 0$. Form a new sequence with generating function $S_1(x)$ given by

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i = \sum_{i=0}^{\infty} s_i x^i, \tag{1.3}$$

where $\{a_{ii}\}$ was generated in the convolution array by $f(x)$ as in (1.2). Then if we let $f(x) = S_0(x)$, from [5] we have $f(xS_1(x)) = S_1(x)$,

$$f(xS_k(x)) = S_k(x) \tag{1.4}$$

and

$$f(xS_k^k(x)) = S_k(x), \tag{1.5}$$

as well as

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki + j} a_{i,\,ki+j-1} x^i, \quad k = 0, 1, 2, \ldots \tag{1.6}$$

In particular, if $f(x) = 1/(1 - x)$, we have the generating functions for the columns of Pascal's triangle and the sequences S_k are the Catalan and related sequences reported in [1], [2], [3], [4], and $a_{i,\,ki+j-1}$ is the binomial $\binom{(i+1)k + j - 1}{k}$. The sequence generated by $S_k^j(x)$ is the $(j - 1)$st convolution of the sequence S_k. The sequence S_k is formed by taking the absolute values of the elements of the first column of the matrix inverse of a matrix P_k, where P_k is formed by placing every $(k + 1)$st column of Pascal's triangle on and below the main diagonal, with zeroes elsewhere. P_0 is Pascal's triangle itself, and P_1 contains every other column of Pascal's triangle and gives the Catalan numbers 1, 1, 2, 5, 14, 42, ..., as the sequence S_1.

We now discuss properties of the generating functions $S_k(x)$.

2. THE GENERATING FUNCTIONS $S_k(x)$

We begin with

$$f(xS(x)) = S(x) \tag{2.1}$$

by assuming that $f(x)$ is analytic about $x = 0$ and $f(0) = 1$. We also note that $S(x) \neq 0$ for finite x, since $S(x) = 0$ would violate $f(0) = 1$.

Theorem 2.1: If $f(xS(x)) = S(x)$, then $S(x/f(x)) = f(x)$.

Proof: Note that $f(x) \neq 0$ for finite x. Let $y = xS(x)$ so that $f(y) = S(x)$ and $x = y/S(x) = y/f(y)$. Therefore, $f(y) = S(y/f(y))$. Changing to x we get $S(x/f(x)) = f(x)$.

Theorem 2.2: If $S(x/f(x)) = f(x)$, then $f(xS(x)) = S(x)$.

Proof: Let $y = x/f(x)$. Then $S(y) = f(x)$, $x = yf(x) = yS(y)$ which implies $f(yS(y)) = f(x) = S(y)$ so that $f(xS(x)) = S(x)$.

Theorem 2.3: The solution to $f(xS(x)) = S(x)$ is unique.

Proof: Assume $f(xS(x)) = S(x)$ and $f(xT(x)) = T(x)$. We shall show that $T(x) = S(x)$. By Theorem 2.1, $S(x/f(x)) = f(x)$. Let $x = xT(x)$ so that

$$S(xT(x))/f(xT(x)) = S(xT(x)/T(x)) = S(x).$$

But also

$$S(xT(x))/f(xT(x)) = f(xT(x)) = T(x).$$

Thus, $S(x) \equiv T(x)$.

Theorem 2.4: In $S(x/f(x)) = f(x)$, $f(x)$ is unique.

Proof: Assume $S(x/f(x)) = f(x)$ and $S(x/g(x)) = g(x)$. Apply Theorem 2.1, $S(x) = f(xS(x))$, letting $x = x/g(x)$. Then $S(x/g(x))$ becomes

$$S(x/g(x)) = f\left[(x/g(x))S(x/g(x))\right] = f\left[(x/g(x)) \cdot g(x)\right] = f(x),$$

but $S(x/g(x)) = g(x)$ so that $f(x) \equiv g(x)$.

3. THE GENERATING FUNCTIONS $S_k(x)$ WHERE $S_0(x)$ GENERATES PASCAL'S TRIANGLE

We now go on to another phase of this problem. Let

$$S_0(x) = \frac{1}{1 - x} = f(x) \tag{3.1}$$

and $S_0(xS_1(x)) = S_1(x)$ be the unique solution, and from $S_1(x/S_0(x)) = S_1(x)$, when $x = 0$ we have $S_1(0) = S_0(0) = 1$. From

$$S_k\left(xS_{k+1}(x)\right) = S_{k+1}(x) \tag{3.2}$$

one can easily prove

$$S_0\left(xS_k^k(x)\right) = S_k(x) \tag{3.3}$$

for all integral k as in Hoggatt and Bruckman [5].

Thus from $S_0(x) = 1/(1 - x)$, we have

$$S_0\left(xS_k^k(x)\right) = \frac{1}{1 - xS_k^k(x)} = S_k(x)$$

or

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0, \quad k > 0,$$

and from

$$S_0\left(x/S_{-k}^k(x)\right) = \frac{1}{1 - x/S_{-k}^k(x)} = S_{-k}(x),$$

$$xS_{-k}^{-k+1}(x) - S_{-k}(x) + 1 = 0, \quad k \geq 0.$$

Clearly, $S_0\left(xS_0^0(x)\right) = S_0(x)$. Thus, uniformly

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0 \tag{3.4}$$

for all integral k.

In particular, by (3.4),

$$xS_1^2(x) - S_1(x) + 1 = 0,$$

$$S_1(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Clearly, $S_1(x)$ is undefined for $x > 1/4$. The solution with the positive radical is unbounded at the origin, while

$$S_1(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is bounded at the origin, and $\lim_{x \to \infty} S_1(x) = 1$. $S_1(x)$ is the generating function for the Catalan numbers. Note that $S_1\left(xS_2(x)\right) = S_2(x)$ leads to

$$S_2(x) = \frac{1 - \sqrt{1 - 4xS_2(x)}}{2xS_2(x)}$$

defined for $xS_2(x) < 1/4$, where $\lim_{x \to \infty} xS_2(x) = 0$ while $S_2(x) \neq 0$ for any x.

We now proceed to the proof that

$$zS^k(z) - S(z) + 1 = 0$$

has only one continuous bounded function in the neighborhood of the origin. We first need a theorem given by Morris Marden [6, p. 3, Theorem (1.4)]:

Theorem: The zeroes of a polynomial are continuous functions of the coefficients.

Theorem 3.1: There is one and only one continuous solution to

$$zS^k(z) - S(z) + 1 = 0$$

which is bounded in the neighborhood of the origin, and this solution is such that $\lim_{x \to 0} S(x) = 1$.

Proof: Let $S_1^*(z), S_2^*(z), \ldots, S_k^*(z)$ be the continuous zeroes (solutions) to $zS^k(z) - S(z) + 1 = 0$, and rewrite this as

$$S^k(z) - S(z)/z + 1/z = 0, \quad z \neq 0.$$

$$(S - S_1^*)(S - S_2^*) \ldots (S - S_k^*) = S^k - S/z + 1/z = 0.$$

Therefore, $S_1^*S_2^*S_3^* \ldots S_k^* = (-1)^k/z$ as the last coefficient, and

$$S_1^*S_2^*S_3^* \cdots S_k^*\left(\frac{1}{S_1^*} + \frac{1}{S_2^*} + \frac{1}{S_3^*} + \cdots + \frac{1}{S_k^*}\right) = (-1)^k/z$$

from the next-to-last coefficient. Therefore,

$$\frac{1}{S_1^*} + \frac{1}{S_2^*} + \frac{1}{S_3^*} + \cdots + \frac{1}{S^*} = 1. \tag{3.5}$$

Let $S_1^*(z)$ be bounded in the neighborhood; then

$$\lim_{z \to 0}\left(zS_1^{*k}(z) - S_1^*(z) + 1\right) = \lim_{z \to 0} zS_1^{*k}(z) - \lim_{z \to 0} S_1^*(z) + 1 = 0,$$

$\lim_{z \to 0} zS_1^{*k}(z) = 0$, and $\lim_{z \to 0} S_1^*(z) = S_1^*(0) = 1$. Thus $\lim_{z \to 0} 1/S_1^*(z) = 1$.

Suppose $S_j^*(z)$ is continuous but unbounded in the neighborhood of $z = 0$. Then $\lim_{z \to 0} 1/S_j^*(z) = 0$. From (3.5), we therefore conclude that $S_1^*(z)$ is the only continuous and bounded solution to our equation as $z \to 0$. We also note that since the right side is indeed 1 for all $z \neq 0$, there is *one* bounded solution. This concludes the proof of Theorem 3.1.

Theorem 3.2: $S_{-m}(x) = \dfrac{1}{S_{m-1}(-x)}$.

Proof: $S_{-m}(x)$ satisfies

$$xS_{-m}^{-m+1}(x) - S_{-m}(x) + 1 = 0.$$

Multiply through by $S_{-m}^{-1}(x)$ to yield

$$xS_{-m}^{-m}(x) - 1 + S_{-m}^{-1}(x) = 0.$$

Replace x by $(-x)$,

$$-xS_{-m}^{-m}(-x) - 1 + S_{-m}^{-1}(-x) = 0,$$

which can be rewritten as

$$x\left(S_{-m}^{-1}(-x)\right)^m - \left(S_{-m}^{-1}(-x)\right) + 1 = 0.$$

This is precisely the polynomial equation satisfied by $S_{m-1}(x)$, which is

$$xS_{m-1}^m(x) - S_{m-1}(x) + 1 = 0.$$

Since $S_{m-1}(0) = 1$, it is the unique continuous solution which is bounded in the neighborhood of the origin. If $S_{-m}(x)$ is such that

$$\lim_{x \to 0} S_{-m}(x) = S_{-m}(0) = 1,$$

then

$$\lim_{x \to 0} S_{-m}^{-1}(x) = S_{-m}^{-1}(0) = 1.$$

Therefore, by Theorem 3.1, we conclude that

$$S_{-m}^{-1}(-x) = S_{m-1}(x) \quad \text{or} \quad S_{-m}(x) = \frac{1}{S_{m-1}(-x)}$$

which concludes the proof of Theorem 3.2.

Theorem 3.3: If $S_k(x)$ obeys

$$xS_k^{k+1}(x) - S_k(x) + 1 = 0,$$

then $S_k(x) \neq 0$ for any finite x, $k \neq -1$.

Proof: Let $S_k(x)$ be the continuous solution as a function of x. Then,

$$\lim_{x \to x_0} S_k(x) = S_k(x_0),$$

where $S_k(x_0)$ is finite. If $S_k(x_0) = 0$, then

$$\lim_{x \to x_0} \left[xS_k^{k+1}(x) - S_k(x) + 1 \right] = 1 \neq 0,$$

which contradicts the fact that $xS_k^{k+1}(x) - S_k(x) + 1 = 0$. However, if $k = -1$, then $S_{-1}(x) = 1 + x$, which is zero for $x = -1$. For all other k, $S_k(x) = 0$ for all finite x.

4. EXTENDED RESULTS FOR GENERALIZED PASCAL TRIANGLES

The results of Section 3 can be extended. Let

$$f(x) = \frac{1}{1 + cxg(x)}, \; c \neq 0, \; f(0) = 1, \tag{4.1}$$

$g(x)$ a polynomial in x. Then $f(xS(x)) = S(x)$ yields

$$\frac{1}{1 + cxS(x)g(xS(x))} = S(x) \tag{4.2}$$

or

$$1 - S(x) + cxS(x)g(xS(x)) = 0$$

which is a polynomial in $S(x)$. Because of the 1 and $-S(x)$ relationships in the equation, all of the previous results hold. For example, all of the generalized Fibonacci numbers from the generalized Pascal triangles arising from the coefficients generated in the expansions of the multinomials $(1 + x + x^2 + \cdots + x^m)^n$ will have convolution arrays governed by the results of this paper and similar to those reported for Pascal's triangle in [1] through [4].

Now, looking at (4.2), since $g(0) = 1$, the polynomial in S is of the form

$$\frac{1}{x^k} + \frac{cx - 1}{x^k}S + \cdots + S^k(x) = 0.$$

As before, inspecting the coefficients yields, for roots $S_1^*, S_2^*, \ldots, S_k^*$,

$$S_1^*S_2^*S_3^* \ldots S_k^* = (-1)^k/x^k$$

and

$$S_1^*S_2^*S_3^* \ldots S_k^* \left(\frac{1}{S_1^*} + \frac{1}{S_2^*} + \cdots + \frac{1}{S_k^*} \right) = \frac{(cx - 1)(-1)^k}{x^k}$$

so that

$$\frac{1}{S_1^*} + \frac{1}{S_2^*} + \cdots + \frac{1}{S_k^*} = 1 - cx.$$

Now,

$$\operatorname*{limit}_{x \to 0} \left(\frac{1}{S_1^*(x)} + \frac{1}{S_2^*(x)} + \cdots + \frac{1}{S_k^*(x)} \right) = 1;$$

Thus, $\operatorname*{limit}_{x \to 0} 1/S_1^*(0) = 1$ and $\operatorname*{limit}_{x \to 0} 1/S_j^*(x) = 0$, and we again have one and only one bounded and continuous solution near the origin.

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SOME MORE PATTERNS FROM PASCAL'S TRIANGLE

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1. INTRODUCTION

Over the years, much use has been made of Pascal's triangle, part of which is shown in Table 1.1. The original intention was to read the table horizontally, when its nth row gives, in order, the coefficients of x^m $\{m = 0, 1, \ldots, n\}$ for the binomial expansion of $(1 + x)^n$.

Pargeter [1] pointed out that the consecutive elements, read downwards, in the nth column gave the coefficients of x^m $\{m = 0, 1, \ldots, \infty\}$ for the infinite expansion of $(1 - x)^n$. More recently, Fletcher [2] has considered the series whose coefficients are obtained (in the representation of Table 1.1) by starting on one of the diagonal unities and making consecutive "knight's moves" of two steps down and one to the right. Again, moving down the diagonals of Table 1.1, we obtain consecutive series of the so-called "figurative numbers," for instance, see Beiler [3]; and the ingenious reader will be able to find other interesting series, which can be simply generated. As with all work on integer sequences, Sloane [4] will be found invaluable.

Table 1.1 Pascal's Triangle

1											
1	1										
1	2	1									
1	3	3	1								
1	4	6	4	1							
1	5	10	10	5	1						
1	6	15	20	15	6	1					
1	7	21	35	35	21	7	1				
1	8	28	56	70	56	28	8	1			
1	9	36	84	126	126	84	36	9	1		
1	10	45	120	210	252	210	120	45	10	1	
1	11	55	165	330	462	462	330	165	55	11	1
⋮											⋮

However, the object of this paper is to draw attention to some equally striking, but rather more subtle patterns, obtainable from Pascal's triangle. The results emerged from the study of the determinants of a class of matrices which occurred naturally in a piece of statistical research, as reported by Anderson [5].

2. THE DETERMINANTAL VALUE FOR A FAMILY OF MATRICES

Consider any two positive integers r and s, and define the rth order determinant $D_r(s)$ as having $\binom{2s}{s+i-j}$ for its general i,jth element. Then it is well-known that

$$D_r(1) \equiv r+1, \quad r \geq 1$$

and, interestingly enough, it can be shown that $D\ (s)$

$$\equiv \frac{(r+1)(r+2)^2 \cdots (r+s-1)^{s-1}(r+s)^s(r+s+1)^{s-1} \cdots (r+2s-2)^2(r+2s-1)}{1.2^2 \cdots (s-1)^{s-1}s^s(s+1)^{s-1} \cdots (2s-2)^2(2s-1)},$$

$r,\ s \geq 1$. For instance, see Anderson [6].

If we write out the family of determinants $D_r(s)$: $r,\ s \geq 1$ as a doubly infinite two-dimensional array, we get Table 2.1.

Table 2.1 Values for Family of Determinants $D_r(s)$

		s 1	2	3	4	$\cdots$
	1	2	6	20	70	$\cdots$
	2	3	20	175	1764	
r	3	4	50	980	24696	
	4	5	105	4116	232848	$\cdots$
	$\vdots$	$\vdots$			$\vdots$	$\ddots$

What is really pretty is that this table can be written down quite simply, and in several ways, from Pascal's triangle.

3. GENERATING THE DETERMINANTS FROM PASCAL'S TRIANGLE

3.1 Generating the Rows of Table 2.1

The first row is obtained just by making the "knight's moves" from the apex of the triangle—see Table 3.1.1. If we then evaluate the second-order determinants, with leading terms at these "knight's moves," we obtain the second row—see Table 3.1.2. Similarly, if we evaluate the third-order determinants, shown in Table 3.1.3, and the fourth-order determinanys, shown in Table 3.1.4, we get the third and fourth rows, respectively. In both cases, the determinants have the "knight's moves" for their leading terms. Continuing in this way, Table 2.1 can be extended to as many rows as we like.

Table 3.1.1 Generation of the First Row

$$\begin{tikzcd} 2 \arrow[dr] & & & & \\ & 6 \arrow[dr] & & & \\ & & 20 \arrow[dr] & & \\ & & & 70 & \\ & & & & \ddots \end{tikzcd}$$

Table 3.1.2 Generation of the Second Row

$$\begin{array}{ccccc}
2 & 1 & & & \\
3 & 3 & & & \\
 & 6 & 4 & & \\
 & 10 & 10 & & \\
 & & 20 & 15 & \\
 & & 35 & 35 & \\
 & & & 70 & 56 \\
 & & & 126 & 126 \\
 & & & & \ddots
\end{array}$$

Table 3.1.3 Generation of the Third Row

$$\begin{array}{ccccccc}
2 & 1 & 0 & & & & \\
3 & 3 & 1 & & & & \\
4 & 6 & 4 & 1 & & & \\
 & 10 & 10 & 5 & & & \\
 & 15 & 20 & 15 & 6 & & \\
 & & 35 & 35 & 21 & & \\
 & & 56 & 70 & 56 & 28 & \\
 & & & 126 & 126 & 84 & \\
 & & & 210 & 252 & 210 & \\
 & & & & & & \ddots
\end{array}$$

Table 3.1.4 Generation of the Fourth Row

2	1	0	0			
3	3	1	0			
4	6	4	1	0		
5	10	10	5	1		
	15	20	15	6	1	
	21	35	35	21	7	
		56	70	56	28	8
		84	126	126	84	36
			210	252	210	120
			330	462	462	330
						⋱

3.2 Generating the Columns of Table 2.1

The first column can be picked out as the natural number diagonal of Table 2.1, shown in Table 3.2.1. For the second column, the overlapping second-order determinants, shown in Table 3.2.2, are evaluated; while the third and fourth columns are obtained from the determinants of the overlapping arrays in Tables 3.2.3 and 3.2.4, respectively. And so on.

Table 3.2.1 Generation of the First Column

2				
	3			
		4		
			5	
				⋱

Table 3.2.2 Generation of the Second Column

3	3				
4	6	4			
	10	10	5		
		20	15	6	
			35	21	
					⋱

Table 3.2.3 Generation of the Third Column

4	6	4				
5	10	10	5			
6	15	20	15	6		
	21	35	35	21	7	
		56	70	56	28	
			126	126	84	
						⋱

Table 3.2.4 Generation of the Fourth Column

5	10	10	5			
6	15	20	15	6		
7	21	35	35	21	7	
8	28	56	70	56	28	8
	36	84	126	126	84	36
		120	210	252	210	120
			330	462	462	330
						$\ddots$

3.3 Generating the Diagonals of Table 2.1

Finally, the diagonals of Table 2.1 can be obtained as follows. The tth member of the main diagonal is found by evaluating the tth-order determinant with leading terms given by the tth "knight's move" from the apex of the triangle. The first super-diagonal is achieved using the same principle, but starting with the second "knight's move." Thus its tth term is the tth-order determinant whose leading term is the $(t+1)$th "knight's move." The tth term of the second super-diagonal is given by the tth-order determinant, starting with the $(t+2)$th "knight's move." And so on for all the other super-diagonals. The sub-diagonals can be obtained in a similar way; by using, instead of the "knight's move" sequence, a sequence diagonally down from it in the triangle. For the first sub-diagonal, the new sequence is one step down; for the second, two steps down, and so on.

4. GENERATING THE DETERMINANTS FROM A DIFFERENT REPRESENTATION OF PASCAL'S TRIANGLE

If we represent Pascal's triangle as in Table 4.1, where the ones have been omitted, we get a more meaningful row and column array. We then find that Table 2.1 can be generated in still further ways, as the reader can readily verify.

TABLE 4.1 Alternative Representation of Pascal's Triangle

2	3	4	5	6	$\cdots$
3	6	10	15	21	
4	10	20	35	56	
5	15	35	70	126	
6	21	56	126	252	
$\vdots$				$\vdots$	$\ddots$

5. IN CONCLUSION

All the patterns discussed can, of course, be verified by combinatorial algebra. Thus, for instance, in Section 3, the second column of Table 2.1 is claimed to have for its nth element the second-order determinant:

$$\begin{vmatrix} \binom{n+1}{n-1} & \binom{n+1}{n} \\ \binom{n+2}{n-1} & \binom{n+2}{n} \end{vmatrix}.$$

On evaluation, this gives

$$\frac{(n+1)(n+2)^2(n+3)}{12}$$

as required.

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COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL $F_{3qr}(x)$

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Let F_m be the mth cyclotomic polynomial. Bang [1] has shown that for $m = pqr$, a product of three odd primes with $p < q < r$, the coefficients of $F_m(x)$ do not exceed $p-1$ in absolute value. The smallest such m is 105 and the coefficient of x^7 in F_{105} is -2. It might be assumed that coefficients 2 and/or -2 occur in every F_{3qr}. This is not so. It is the purpose of this paper to characterize the pairs q,r in $m = 3qr$ such that no coefficient of absolute value 2 can occur in F_{3qr}.

1. PRELIMINARIES

Let $F_m(x) = \sum_{n=0}^{\varphi(m)} c_n x^n$. Then for $m = 3qr$, c_n is determined [1] by the number of partitions of n of the form:

$$n = a + 3\alpha q + 3\beta r + \gamma qr + \delta_1 q + \delta_2 r, \tag{1}$$

$0 \le a < 3$; α, β, γ, nonnegative integers; $\delta_i \in \{0, 1\}$. If n has no such partition, $c_n = 0$. Each partition of n in the form (1) contributes +1 to the value of c_n if $\delta_1 = \delta_2$, but -1 if $\delta_1 \ne \delta_2$. Because $F_m(x)$ is symmetric, we consider only $n \le \varphi(m)/2 = (q-1)(r-1)$. For $n > (q-1)(r-1)$, $c_n = c_{n'}$, with $n' = \varphi(m) - n$. We note that for $n \le (q-1)(r-1)$, γ in (1) must be zero.

A permissible partition of n is therefore one of these four:

$$\begin{aligned} P_1 &= a_1 + 3\alpha_1 q + 3\beta_1 r, & P_2 &= a_2 + 3\alpha_2 q + 3\beta_2 r + q + r, \\ P_3 &= a_3 + 3\alpha_3 q + 3\beta_3 r + q, & P_4 &= a_4 + 3\alpha_4 q + 3\beta_4 r + r. \end{aligned} \tag{2}$$

Partitions P_1 and P_2 will each contribute +1 to c_n, while P_3 and P_4 will each contribute -1. When $n \le (q-1)(r-1)$, only one partition for each P_i, $i = 1, \ldots, 4$, is possible [1].

Lemma 1: For any β_i in (2), $3\beta_i \le q - 2$ for all q.

Proof: Following Bloom [3] we have $3\beta_i r \le (q-1)(r-1) < (q-1)r$. Thus, $3\beta_i < q - 1$.

Corollary: $3\beta_i \le q - 3$ for $i = 2, 4$.

Lemma 2: Either $r + q \equiv 0 \pmod 3$ or $r - q \equiv 0 \pmod 3$, for all primes q and r with $3 < q < r$.

Proof: Let $q = 2k + 1$, $r = 2k_1 + 1$. Since 3 divides one and only one of the numbers $2t$, $2(t+1)$ when $2t+1$ is a prime, it follows that 3 divides one and only one of the numbers $r + q = 2(k + k_1 + 1)$ or $r - q = 2(k - k_1)$.

2. BOUNDS ON THE COEFFICIENTS

We set $3 < q < r$ and make repeated use of the expressions:

$$P_2 - P_1 = a_2 - a_1 + 3(\alpha_2 - \alpha_1)q + 3(\beta_2 - \beta_1)r + q + r = 0; \tag{3}$$

$$P_4 - P_3 = a_4 - a_3 + 3(\alpha_4 - \alpha_3)q + 3(\beta_4 - \beta_3)r + r - q = 0. \tag{4}$$

Theorem 1: In $F_{3qr}(x)$,

(a) if $r - q \equiv 0 \pmod 3$, then $-1 \le c_n \le 2$,

(b) if $r + q \equiv 0 \pmod 3$, then $-2 \le c_n \le 1$.

Proof of (a): Assume $c_n = -2$ for some n, i.e., partitions of n of forms P_3 and P_4 exist. Taking (4), modulo 3, we obtain $a_4 - a_3 \equiv 0 \pmod 3$. But $a < 3$, so that $a_4 = a_3$. Now taking (4), modulo q, we obtain $[3(\beta_4 - \beta_3) + 1]r \equiv 0 \pmod q$. Then $3(\beta_4 - \beta_3) + 1 = \beta q$, for some integer $\beta \ne 0$. Either $3(\beta_4 - \beta_3) = \beta q - 1 \ge q - 1$, or $3(\beta_3 - \beta_4) = |\beta| q + 1 \ge q + 1$. But $3\beta_i \le q - 2$ by Lemma 1. Therefore, P_3 and P_4 cannot both exist and we have $c_n \ne -2$.

The proof of (b) follows from a similar argument by considering (3), modulo 3, and then modulo q.

Remark 1: F_{3qr} may have a coefficient of 2 or of -2 but not of both.

Remark 2: If q and r are twin primes, $c_r = -2$ with $P_3 = 2 + q$, $P_4 = r$.

3. SPECIAL CASES

Before taking up the general case, we consider $r = kq \pm 1$ and $r = kq \pm 2$. We prove a theorem about $r = kq \pm 1$.

Theorem 2: Let $r = kq \pm 1$. In $F_{3qr}(x)$, $|c_n| \le 1$ if and only if $k \equiv 0 \pmod 3$.

Proof: To show the sufficiency of the condition, let $r = 3hq + 1$, with $q \equiv 1 \pmod 3$. Then $r - q \equiv 0 \pmod 3$, and $c_n \ne -2$ by Theorem 1. We show $c_n \ne 2$, i.e., there is no n for which partitions P_1 and P_2 can both exist. Taking (3), modulo 3, we obtain $a_2 - a_1 = 1$ or -2. We note that $r \equiv 1 \pmod q$. Then (3), modulo q, leads to one of the equations:

$$3(\beta_2 - \beta_1) = \beta q - 2 \quad \text{or} \quad 3(\beta_2 - \beta_1) = \beta q + 1$$

with $\beta \equiv 2 \pmod 3$. Obviously, there is no value of β which satisfies Lemma 1. Hence there is no n, $0 \le n \le (q-1)(r-1)$, for which partitions P_1 and P_2 both exist. Similarly, with $q \equiv 2 \pmod 3$, it can be shown that there is no n for which partitions P_3 and P_4 can both exist. When $r = 3hq - 1$, $r \equiv 2$ (mod

3). If $q \equiv 2$, the proof leads to the same two equations as above with $\beta \equiv 1$. Thus both equations are inconsistent with Lemma 1. If $q \equiv 1$, the same equations appear with β_2 and β_1 replaced by β_4 and β_3, respectively, and $\beta \equiv 2$. Thus $|c_n| \leq 1$.

The necessity of the condition $k \equiv 0 \pmod 3$ is shown by the counterexamples in Table 1. Values of k are given modulo 3. For each n, other partitions are not possible. We illustrate with the first counterexample, $r = kq + 1$ with $k \equiv 1$. The only possible r and q are $r \equiv 2$ and $q \equiv 1 \pmod 3$. Note that for $n = r$, $n \equiv 2 \pmod 3$. Thus in partitions P_1 or P_2, $a_1 = a_2 = 2$. Then $P_1 = 2 + 3\alpha_1 q + 3\beta_1 r = r = P_2 = 2 + 3\alpha_2 q + 3\beta_2 r + q + r$. In neither P_1 nor P_2 is it possible to find nonnegative α and β to satisfy the equations. Hence, the coefficient of x^r in F_{3qr} is -2.

Table 1 $r = kq \pm 1$

k (mod 3)	r	Partitions of n		Examples c_n	q	r	n
1	$kq + 1$	$P_3 = 1 + (k - 1)q + q$	$P_4 = r$	-2	7	29	29
1	$kq - 1$	$P_3 = (k - 1)q + q$	$P_4 = 1 + r$	-2	5	19	20
2	$kq + 1$	$P_1 = 1 + (k + 1)q$	$P_2 = q + r$	2	5	41	46
2	$kq - 1$	$P_1 = (k + 1)q$	$P_2 = 1 + q + r$	2	7	13	21

Theorem 3: Let $r = kq \pm 2$. In $F_{3qr}(x)$, $|c_n| \leq 1$ if and only if $k \equiv 0$ and $q \equiv 1 \pmod 3$.

The proof follows the method in Theorem 2 and is omitted here. Table 2 gives counterexamples to show the necessity.

Table 2 $r = kq \pm 2$

k (mod 3)	r	Partitions of n		Examples c_n	q	r	n
$q \equiv 2 \pmod 3$ 0	$kq + 2$	$P_1 = 2 + (q + 1)r/2$	$P_2 = 1 + (q - 1)kq/2 + q + r$	2	5	17	53
$q \equiv 2 \pmod 3$ 0	$kq - 2$	$P_3 = (q + 1)r/2 + q$	$P_4 = 1 + (q - 1)kq/2 + r$	-2	5	13	44
1	$kq + 2$	$P_3 = (k - 1)q + q + 2$	$P_4 = r$	-2	5	37	37
1	$kq - 2$	$P_3 = (k - 1)q + q$	$P_4 = r + 2$	-2	7	47	49
2	$kq + 2$	$P_1 = (k + 1)q + 2$	$P_2 = q + r$	2	7	37	44
2	$kq - 2$	$P_1 = (k + 1)$	$P_2 = q + r + 2$	2	5	23	30

4. THE GENERAL CASE

More generally, for all primes q and r with $3 < q < r$, we have $r = (kq + 1)/h$, or $r = (kq - 1)/h$, $h \leq (q - 1)/2$. If $h = 1$, Theorem 2 applies. Therefore we set $1 < h$. In $r = (kq \pm 1)/h$, we may consider r, q, k, ± 1 as four independent variables with h dependent. Since r and q each have two possible values modulo 3 and k has three, there are 24 cases to be examined. We shall examine one of them. Then we shall present Table 3 showing all 24 cases and from the table we form a theorem which states conditions on q and r so that $|c_n| \leq 1$ in F_{3qr}.

First we take $r \equiv q \equiv 1$, $k \equiv 0 \pmod 3$ in $r = (kq - 1)/h$, $1 < h \leq (q - 1)/2$. Note that $h \equiv 2$. Since $r - q \equiv 0 \pmod 3$, $c \neq -2$ by Theorem 1. We show $c_n \neq 2$. Taking (3), modulo 3, we find $a_2 - a_1 = -2$ or 1. Then taking (3), modulo q, we obtain two possible congruences:

$$-2 + [3(\beta_2 - \beta_1) + 1](-1/h) \equiv 0 \quad \text{and} \quad 1 + [3(\beta_2 - \beta_1) + 1](-1/h) \equiv 0.$$

The first leads to the equation $3(\beta_2 - \beta_1) = \beta q - 2h - 1$ with $\beta \equiv 2$. No such value of β will satisfy Lemma 1. The second congruence leads to the equation $3(\beta_2 - \beta_1) = \beta q + h - 1$ with $\beta \equiv 2$. If $h = 2$, there is no value of β which satisfies Lemma 1, and $c_n \neq 2$. If $h > 2$, then $3\beta_1 = q - h + 1$ satisfies Lemma 1. Substituting this value in (3), we obtain $3\alpha_2 = r - k - 1$. Then $P_1 = (q - h + 1)$ and $P_2 = (r - k - 1)q + q + r$ with $a_1 = 0$, $a_2 = 1$. But when we set $a_3 + 3\alpha_3 q + 3\beta_3 r + q = (q - h + 1)$, we obtain $P_3 = 2 + (r - 2k - 1) + (h + 1)r + q$. Moreover, if we let $a_1 = 1$, $a_2 = 2$, partitions P_1 and P_2 exist but also P_4 exists. Thus, there is no n for which $c_n = 2$.

In Table 3 the values for r, q, k, and h are all modulo 3. From an inspection of Table 3 for the cases when $\max|c_n| = 1$, we state

Theorem 4: Let $r = (kq \pm 1)/h$, $1 < h \leq (q - 1)/2$. In $F_{3qr}(x)$, $|c_n| \leq 1$ if and only if one of these conditions holds: (a) $k \equiv 0$ and $h + q \equiv 0 \pmod 3$ or (b) $h \equiv 0$ and $k + r \equiv 0 \pmod 3$.

Table 3 $r = (kq \pm 1)/h$, $1 < h < (q - 1)/2$

(Values for q, r, h, k are modulo 3)

	k	h	± 1	Partitions of n		$\max\lvert c_n\rvert$
$r \equiv q \equiv 1$	0	1	+	$P_1 = 2 + (q - 2h + 1)r$	$P_2 = (r - 2k - 1)q + q + r$	2
	1	2	+	$P_1 = 2 + (2k + 1)q$	$P_2 = (2h - 1)r + q + r$	2
	2	0	+			1
	0	2	-			1
	1	0	-	$P_1 = 2 + (2h + 1)r$	$P_2 = (2k - 1)q + q + r$	2
	2	1	-	$P_1 = 2 + (r - 2k + 1)q$	$P_2 = (q - 2h - 1)r + q + r$	2

(continued)

Table 3—continued

	k	h	± 1	Partitions of n		max $\lvert c_n \rvert$
$r \equiv q \equiv 2$	0	2	+	$P_1 = (r - 2k + 1)q$	$P_2 = 2 + (q - 2h - 1)r + q + r$	2
	1	0	+			1
	2	1	+	$P_1 = (2h + 1)r$	$P_2 = 2 + (2k - 1)q + q + r$	2
	0	1	-			1
	1	2	-	$P_1 = (2k + 1)q$	$P_2 = 2 + (2k - 1)r + q + r$	2
	2	0	-	$P_1 = (q - 2h + 1)r$	$P_2 = 2 + (r - 2k - 1)q + q + r$	2
$r \equiv 1,\ q \equiv 2$	0	1	+			1
	1	0	+	$P_3 = 2 + (q - 2h + 1)r + q$	$P_4 = (r - 2k + 1)q + r$	2
	2	2	+	$P_3 = 2 + (2k - 1)q + q$	$P_4 = (2h - 1)r + r$	2
	0	2	-	$P_3 = 2 + (r - 2k - 1)q + q$	$P_4 = (q - 2h - 1)r + r$	2
	1	1	-	$P_3 = (k - 1)q + q$	$P_4 = 1 + (h - 1)r + r$	2
	2	0	-			1
$r \equiv 2,\ q \equiv 1$	0	2	+			1
	1	1	+	$P_3 = 1 + (k - 1)q + q$	$P_4 = (h - 1)r + r$	2
	2	0	+	$P_3 = (r - 2k - 1)q + q$	$P_4 = 2 + (q - 2h - 1)r + r$	2
	0	1	-	$P_3 = (q - 2h + 1)r + q$	$P_4 = 2 + (r - 2k + 1)q + r$	2
	1	0	-			1
	2	2	-	$P_3 = (q - 2h + 1)r + q$	$P_4 = 2 + (r - 2k + 1)q + r$	2

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SIMPLIFIED PROOF OF A GREATEST INTEGER FUNCTION THEOREM

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The purpose of this paper is to give a simple proof of a result, due originally to Anaya and Crump [1], involving the greatest integer function $[\cdot]$. In the following, $a = \frac{1+\sqrt{5}}{2} \doteq 1.618$, $b = \frac{1-\sqrt{5}}{2} \doteq -0.618$, and $F_n = \frac{a^n - b^n}{\sqrt{5}}$ defines the nth Fibonacci number for $n \geq 1$.

Definition: Let δ be defined by $\delta = \frac{1}{2} - \frac{b^2}{\sqrt{5}} > 0$.

Lemma 1: For $n \geq 2$, $\delta \leq \frac{1}{2} \pm \frac{b^n}{\sqrt{5}}$.

Proof: Equivalent to $\frac{-b^2}{\sqrt{5}} \leq \frac{\pm b^n}{\sqrt{5}}$ or $\pm b^n \leq b^2$, which is clearly true for $n \geq 2$, since $|b| < 1$.

Lemma 2: For $n \geq 2$ and any γ satisfying $|\gamma| < \delta$,

$$\left[\frac{a^n}{\sqrt{5}} + \gamma + \frac{1}{2}\right] = F_n.$$

Proof: We must show $F_n < \frac{a^n}{\sqrt{5}} + \gamma + \frac{1}{2} < F_n + 1$, or using $F_n = \frac{a^n - b^n}{\sqrt{5}}$ and $-|\gamma| \leq \gamma \leq |\gamma|$, the required inequality will be true if

$$\frac{-b^n}{\sqrt{5}} \leq -|\gamma| + \frac{1}{2} \leq |\gamma| + \frac{1}{2} < \frac{-b^n}{\sqrt{5}} + 1.$$

The extreme left and right inequalities reduce to $|\gamma| \leq \frac{1}{2} + \frac{b^n}{\sqrt{5}}$ and $|\gamma| < \frac{1}{2} - \frac{b^n}{\sqrt{5}}$, respectively, both valid for $|\gamma| < \delta$ by Lemma 1.

Theorem 1: (Cf. [1]): For $n \geq 1$ and $1 \leq k < n$, $\left[a^k F_n + \frac{1}{2}\right] = F_{n+k}$.

Proof: $\left[a^k F_n + \frac{1}{2}\right] = \left[\frac{a^k(a^n - b^n)}{\sqrt{5}} + \frac{1}{2}\right] = \left[\frac{a^{k+n}}{\sqrt{5}} - \frac{(-1)^k b^{n-k}}{\sqrt{5}} + \frac{1}{2}\right]$ (using $ab = -1$) $= F_{n+k}$ by Lemma 2 since $\left|\frac{(-1)^{k+1} b^{n-k}}{\sqrt{5}}\right| \leq \frac{|b|}{\sqrt{5}} < \delta$.

Corollary 1: ([2], pp. 34-35): $F_{n+1} = \left[aF_n + \frac{1}{2}\right]$ for $n = 2, 3, 4, \ldots$

Proof: Take $k = 1$ in the theorem and note $1 = k < n$ for $n = 2, 3, 4, \ldots$

Corollary 2: (Cf. [3], p. 22): $\left[\frac{a^n}{\sqrt{5}} + \frac{1}{2}\right] = F_n$ for all $n \geq 1$.

Proof: Clearly true for $n = 1$, since $a \doteq 1.618$. For $n \geq 2$, the result follows from Lemma 2 with $\gamma = 0$.

Note: The case $k = n$ is not treated in Theorem 1, and in fact the result of the theorem fails for $n \geq 1$, $1 \leq k \leq n$ when $n = 1$ and $k = 1$, since

$$\left[aF_1 + \frac{1}{2}\right] = \left[a + \frac{1}{2}\right] = [2.118] = 2 \neq F_2 = 1$$

(thus the statement of the theorem in [1] requires modification). However, we can easily prove the following:

Theorem 2: Let $n \geq 2$ and $k = n$. Then $\left[a^n F_n + \frac{1}{2}\right] = F_{2n}$.

Proof: $\left[a^n F_n + \frac{1}{2}\right] = \left[\frac{a^n}{\sqrt{5}}(a^n - b^n) + \frac{1}{2}\right] = \left[\frac{a^{2n}}{\sqrt{5}} - \frac{(-1)^n}{\sqrt{5}} + \frac{1}{2}\right] = \left[\frac{a^{2n}}{\sqrt{5}} \pm \frac{1}{\sqrt{5}} + \frac{1}{2}\right]$

which will be F_{2n} $\left(\text{since } \frac{a^{2n}}{\sqrt{5}} > F_{2n} \text{ and } \pm\frac{1}{\sqrt{5}} + \frac{1}{2} > 0\right)$ if $\frac{a^{2n}}{\sqrt{5}} \pm \frac{1}{\sqrt{5}} + \frac{1}{2} < F_{2n} + 1$

or $\frac{-1}{2} \pm \frac{1}{\sqrt{5}} < \frac{-b^{2n}}{\sqrt{5}}$, an inequality which is easily verified for $n \geq 2$.

With both n and k unrestricted positive integers, we can also state two simple inequalities which depend on the fact that $[\cdot]$ is a nondecreasing function of its argument.

Corollary 3:

(i) For n even, $\left[a^k F_n + \frac{1}{2}\right] \leq F_{n+k}$ $(n \geq 1,\ k \geq 1)$

(ii) For n odd, $\left[a^k F_n + \frac{1}{2}\right] \geq F_{n+k}$ $(n \geq 1,\ k \geq 1)$.

Proof:

(i) With n even, $\frac{a^n}{\sqrt{5}} > F_n$ and $\left[a^k F_n + \frac{1}{2}\right] \leq \left[\frac{a^{k+n}}{\sqrt{5}} + \frac{1}{2}\right] = F_{n+k}$ by Cor. 2.

(ii) Similarly, n odd implies $\frac{a^n}{\sqrt{5}} < F_n$ and $\left[a^k F_n + \frac{1}{2}\right] \geq \left[\frac{a^{k+n}}{\sqrt{5}} + \frac{1}{2}\right] = F_{n+k}$ again by application of Cor. 2.

We may also obtain a similar result on Lucas numbers due to Carlitz [4] by an analogous approach (recall $L_n = a^n + b^n$ for $n \geq 1$).

Lemma 2: For all $n \geq 4$ and γ satisfying $|\gamma| \leq b^2$, $\left[a^n + \gamma + \frac{1}{2}\right] = L_n$.

Proof: We must show $L_n \leq a^n + \gamma + \frac{1}{2} < L_n + 1$, or, using $L_n = a^n + b^n$,

$b^n \leq \gamma + \frac{1}{2} < b^n + 1$. Since $|\gamma| \leq b^2$, the required inequality is satisfied if

$$b^n \leq -b^2 + \frac{1}{2} < b^2 + \frac{1}{2} < b^n + 1.$$

But $b^n + b^2 < \frac{1}{2}$ and $b^n - b^2 > -\frac{1}{2}$ for $n \geq 4$, so the result follows.

Theorem 3: (Cf. [4]): For $k \geq 2$ and $n \geq k + 2$, $\left[a^k L_n + \frac{1}{2}\right] = L_{n+k}$.

Proof: $\left[a^k L_n + \frac{1}{2}\right] = \left[a^k(a^n + b^n) + \frac{1}{2}\right] = \left[a^{n+k} + (-1)^k b^{n-k} + \frac{1}{2}\right] = L_{n+k}$ by Lemma 2, since $|(-1)^k b^{n-k}| \leq b^2$ and $n + k \geq 4$.

Corollary 4: $\left[a^n + \frac{1}{2}\right] = L_n$ for $n \geq 2$.

Proof: For $n \geq 4$, result is established by Lemma 2 on taking $\gamma = 0$. For $n = 2, 3$, a direct verification suffices. [Recall $a^2 = a + 1$, so that $a^3 = (a + 1)a = 2a + 1$]. The result is also immediate from the fact that

$$|a^n - (a^n + b^n)| = |b|^n < \frac{1}{2} \text{ for } n \geq 2,$$

which shows that L_n is the closest integer to a^n for $n \geq 2$. It then follows that

$$\left[a^n + \frac{1}{2}\right] = L_n \text{ for } n \geq 2.$$

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ON SQUARE PSEUDO-FIBONACCI NUMBERS

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If the Fibonacci numbers are defined by

$$u_1 = u_2 = 1,\ u_{n+2} = u_{n+1} = u_n,$$

then J. H. E. Cohn [1] has shown that

$$u_1 = u_2 = 1 \quad \text{and} \quad u_{12} = 144$$

are the only square Fibonacci numbers.

If n is a positive integer, we shall call the numbers defined by

$$u_1 = 1,\ u_2 = 4,\ u_{n+2} = u_{n+1} + u_n \tag{1}$$

pseudo-Fibonacci numbers.

The object of this paper is to show that the only square pseudo-Fibonacci numbers are

$$u_1 = 1,\ u_2 = 4, \text{ and } u_4 = 9.$$

If we remove the restriction $n > 0$, we obtain exactly one more square,

$$u_{-8} = 81.$$

It can easily be shown that the general solution of the difference equation (1) is given by

$$u_n = \frac{7}{5 \cdot 2^n}(\alpha^n + \beta^n) - \frac{1}{5 \cdot 2^{n-1}}(\alpha^{n-1} + \beta^{n-1}), \tag{2}$$

where

$$\alpha = 1 + \sqrt{5},\ \beta = 1 - \sqrt{5},$$

and n is an integer. Let

$$\eta_r = \frac{\alpha^r + \beta^r}{2^r},\ \xi_r = \frac{\alpha^r - \beta^r}{2^r\sqrt{5}}.$$

Then we easily obtain the following relations:

$$u_n = \frac{1}{5}(7\eta_n - \eta_{n-1}), \tag{3}$$

$$\eta_r = \eta_{r-1} + \eta_{r-2},\ \eta_1 = 1,\ \eta_2 = 3, \tag{4}$$

$$\xi_r = \xi_{r-1} + \xi_{r-2},\ \xi_1 = 1,\ \xi_2 = 1, \tag{5}$$

$$\eta_r^2 - 5\xi_r^2 = (-1)^r 4, \tag{6}$$

$$\eta_{2r} = \eta_r^2 + (-1)^{r+1}2, \tag{7}$$

$$2\eta_{m+n} = 5\xi_m\xi_n + \eta_m\eta_n, \tag{8}$$

$$2\xi_{m+n} = \eta_n\xi_m + \eta_m\xi_n, \tag{9}$$

$$\xi_{2r} = \eta_r\xi_r. \tag{10}$$

The following congruences hold:

$$u_{n+2r} \equiv (-1)^{r+1}u_n (\text{mod } \eta_r 2^{-S}), \tag{11}$$

$$u_{n+2r} \equiv (-1)^r u_n (\text{mod } \xi_r 2^{-S}), \tag{12}$$

where $S = 0$ or 1.

Let $\phi_t = \eta_{2^t}$, where t is a positive integer. Then we get

$$\phi_{t+1} = \phi_t^2 - 2. \tag{13}$$

We also need the following results concerning ϕ_t:

$$\phi_t \text{ is an odd integer}, \tag{14}$$

$$\phi_t \equiv 3 \ (\text{mod } 4), \tag{15}$$

$$\phi_t \equiv 2 \ (\text{mod } 3),\ t \geq 3. \tag{16}$$

We also have the following tables of values:

n	-8	0	1	2	3	4	5	7	9	11	12	13	15
u_n	81	3	1	4	5	9	14	37	97	254	411	665	1741

t	7	14
η_t	29	$3 \cdot 281$

t	4	7	8
ξ_t	3	13	$3 \cdot 7$

Let

$$x^2 = u_n. \tag{17}$$

The proof is now accomplished in sixteen stages:

(a) (17) is impossible if $n \equiv 3 \ (\text{mod } 8)$. For, using (12) we find that

$$u_n \equiv u_3 \ (\text{mod } \xi_4) \equiv 5 \ (\text{mod } 3).$$

Since $\left(\frac{5}{3}\right) = -1$, (17) is impossible.

(b) (17) is impossible if $n \equiv 5 \pmod 8$. For, using (12) we find that

$$u_n \equiv u_5 \pmod{\xi_4} \equiv 14 \pmod 3.$$

Since $\left(\frac{14}{3}\right) = -1$, (17) is impossible.

(c) (17) is impossible if $n \equiv 0 \pmod{16}$. For, using (12) in this case

$$u_n \equiv u_0 \pmod{\xi_8} \equiv 3 \pmod 7, \text{ since } 7 \mid \xi_8.$$

Since $\left(\frac{3}{7}\right) = -1$, (17) is impossible.

(d) (17) is impossible if $n \equiv 15 \pmod{16}$. For, using (12) we find that

$$u_n \equiv u_{15} \pmod{\xi_8} \equiv 1741 \pmod 7, \text{ since } 7 \mid \xi_8.$$

Since $\left(\frac{1741}{7}\right) = -1$, (17) is impossible.

(e) (17) is impossible if $n \equiv 12 \pmod{16}$. For, using (12) in this case

$$u_n \equiv u_{12} \pmod{\xi_8} \equiv 411 \pmod 7, \text{ since } 7 \mid \xi_8.$$

Since $\left(\frac{411}{7}\right) = -1$, (17) is impossible.

(f) (17) is impossible if $n \equiv 7 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_7 \pmod{\xi_7} \equiv \pm 37 \pmod{13}.$$

Since $\left(\frac{-37}{13}\right) = \left(\frac{37}{13}\right) = -1$, (17) is impossible.

(g) (17) is impossible if $n \equiv 3 \pmod{14}$. For, using (12) in this case

$$u_n \equiv \pm u_3 \pmod{\xi_7} \equiv \pm 5 \pmod{13}.$$

Since $\left(\frac{-5}{13}\right) = \left(\frac{5}{13}\right) = -1$, (17) is impossible.

(h) (17) is impossible if $n \equiv 5 \pmod{14}$. For, using (11) we find that

$$u_n \equiv u_5 \pmod{\eta_7} \equiv 14 \pmod{29}.$$

Since $\left(\frac{14}{29}\right) = -1$, (17) is impossible.

(i) (17) is impossible if $n \equiv 13 \pmod{14}$. Sor, using (12) in this case

$$u_n \equiv \pm u_{13} \pmod{\xi_7} \equiv \pm 665 \pmod{13}.$$

Since $\left(\frac{-665}{13}\right) = \left(\frac{665}{13}\right) = -1$, (17) is impossible.

(j) (17) is impossible if $n \equiv 11 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_{11} \pmod{\xi_7} \equiv \pm 254 \pmod{13}.$$

Since $\left(\frac{-254}{13}\right) = \left(\frac{254}{13}\right) = -1$, (17) is impossible.

(k) (17) is impossible if $n \equiv 9 \pmod{14}$. For, using (12) we find that

$$u_n \equiv \pm u_9 \pmod{\xi_7} \equiv \pm 97 \pmod{13}.$$

Since $\left(\frac{-97}{13}\right) = \left(\frac{97}{13}\right) = -1$, (17) is impossible.

(l) (17) is impossible if $n \equiv 15 \pmod{28}$. For, using (11) we find that

$$u_n \equiv \pm u_{14} \pmod{\eta_{14}} \equiv \pm 1741 \pmod{281}, \text{ since } 281/\eta_4.$$

Since $\left(\frac{-1741}{281}\right) = \left(\frac{1741}{281}\right) = -1$, (17) is impossible.

(m) (17) is impossible if $n \equiv 1 \pmod 4$, $n \neq 1$, that is, if $n = 1 + 2^t r$, where r is odd and t is a positive integer ≥ 2. For, using (11) in this case

$$u_n \equiv -u_1 \pmod{\eta_2 t - 1} \equiv -1 \pmod{\phi_{t-1}}.$$

Now, using (15) we have $\phi_{t-1} = 4k + 3$, where k is a nonnegative integer. Since $\left(\frac{-1}{\phi_{t-1}}\right) = \left(\frac{-1}{4k+3}\right) = -1$, (17) is impossible.

(n) (17) is impossible if $n \equiv 2 \pmod 4$, $n \neq 2$, that is, if $n = 2 + 2^t r$, where r is odd and t is a positive integer ≥ 2. For, using (11) we find that

$$u_n \equiv -u_2 \pmod{\eta_2 t - 1} \equiv -4 \pmod{\phi_{t-1}}.$$

Now, using (15) we have $\phi_{t-1} = 4k + 3$, where k is a nonnegative integer. By virtue of (14), $(2, \phi_{t-1}) = 1$. Since $\left(\frac{-4}{\phi_{t-1}}\right) = \left(\frac{-4}{4k+3}\right) = -1$, (17) is impossible.

(o) (17) is impossible if $n \equiv 4 \pmod{16}$, $n \neq 4$, that is, if $n = 4 + 2^t r$, where r is odd and t is a positive integer ≥ 4. For, using (11) we find that

$$u_n \equiv -u_4 \pmod{\eta_{2^{t-1}}} \equiv -9 \pmod{\phi_{t-1}}.$$

Now, using (16), we get $(\phi_{t-1}, 3) = 1$, and by virtue of (15), $\phi_{t-1} = 4k + 3$, where k is a positive integer ≥ 11.

Next, since $\left(\frac{-9}{\phi_{t-1}}\right) = \left(\frac{-9}{4k+3}\right) = -1$, (17) is impossible.

(p) (17) is impossible if $n \equiv -8 \pmod{16}$, $n \neq -8$, that is, if $n = -8 + 2^t r$, where r is odd and t is a positive integer ≥ 4. For, using (11) in this case

$$u_n \equiv -u_{-8} \pmod{\eta_2 t - 1} \equiv -81 \pmod{\phi_{t-1}}.$$

Now, using (16) we get $(\phi_{t-1}, 3) = 1$, and by virtue of (15), $\phi_{t-1} = 4k + 3$, where k is a positive integer ≥ 11.

Next, since $\left(\dfrac{-81}{\phi_{t-1}}\right) = \left(\dfrac{-81}{4k+3}\right) = -1$, (17) is impossible.

We have now four further cases, $n = -8$, 1, 2, and 4, to consider.

(1) When $n = -8$, $u_n = 81$ is a perfect square.

(2) When $n = 1$, $u_n = 1$ is a perfect square.

(3) When $n = 2$, $u_n = 4$ is a perfect square.

(4) When $n = 4$, $u_n = 9$ is a perfect square.

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FIBONACCI, INSECTS, AND FLOWERS

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It remains an interesting phenomenon that elements of the Fibonacci and Lucas sequences appear in numerous structural entities belonging to varied species of higher plants. McNabb [3] cites the abundance of flower species with numbers of petals (up to 89 in Michaelmas daisies) that correspond to Fibonacci numbers. Karchmar [1] obtained the commonly observed angle between adjacent leaf primordia (137°30') by applying the limiting value of the following Fibonacci ratio:

$$\frac{F_n}{F_{n+1}} \tag{1}$$

where F_n and F_{n+1} denote, respectively, the nth and $(n+1)$th elements of the Fibonacci sequence.

Although there exists a considerable body of literature pertaining to plant structure and Fibonacci sequences, the above references are singled out for their use of expression (1). As pointed out by McNabb [3], phyllotaxic descriptions are often denoted in the form of expression (1). It is to expression (1) that we give most of our concern in relation to insects which reside on flowers of field thistle (*Circium discolor*). Specifically, we are interested in the sequences of lengths among these insects. Table 1 lists the species of insect, sample size, mean length, and standard deviation.

Table 1

Length Statistics of Five Insect Species Resident on Flowers of *Circium discolor*

Insect	Sample Size	Mean Length (mm)	Standard Deviation
Diabrotica longicornis (beetle)	15	6.0	0.58
Plagiognathus (bug)	13	3.7	0.23
Olibrus semistriatus (beetle)	17	2.2	0.25
Orius insidiosus (bug)	14	2.0	0.10
Frankliniella tritici (thrip)	15	0.9	0.12

Let us assume that because flowers are of a limited volume, insects are competing for space. Another alternative is that of competition for food, but since we rarely observe flowers devastated by insects, we presently reject this alternative. We can further speculate that if competition is for space, we expect the appearance of ecological and evolutionary mechanisms aimed at the avoidance of physical encounter. Such an avoidance may be realized if each insect were to possess a "refuge" (i.e., a volumetric space) for the avoidance of larger insects. Within a complex flower, such as field thistle, smaller insects could avoid larger insects by seeking crevices which larger insects could not enter. This mechanism does not exclude other means of avoidance, although if we accept the mechanism of avoidance by spatial refuge, then there should arise constraints on the size of each insect species. We can thus imagine that, of a pair of insects, the larger will "push" the smaller (over evolutionary time) to a reduced size. We assume here that, upon encounter, the smaller insect is more likely to move away from the larger than the larger move away from the smaller. In this manner, the largest insect residing on the flower will determine, at a first approximation, the entire size sequence of the remaining insects.

From the above consideration, we make use of the Fibonacci sequence in an unusual manner. Since it is assumed that the largest insect determines the length sequence, we start our sequence backwards, setting our largest number as the first term in the sequence. We then define our sequence, on the basis of the first term (u_1), as:

$$u_n = u_1(\beta^{n-1}) \tag{2}$$

where $\beta = 1/\alpha$ and α (the Fibonacci ratio in the limit) approximates the value of 1.62. Thus, $\beta \doteq .62$.

We are now able, given the first term, to calculate elements of (2). Recalling that the length of the largest insect is 6.0 mm, we may set this value as the first term in the sequence, and then proceed to calculate the next four terms. A comparison of the empirical and predicted sequences is impressive.

Predicted Sequence	Empirical Sequence
6.0	6.0
3.7	3.7
2.3	2.2
1.4	2.0
0.9	0.9

We may imply from this comparison that the length ratio of two neighboring insects in the sequence, taking the larger to the smaller, should approximate 1.62. This ratio can then be viewed as a "limiting similarity" [2] for two species, i.e., how similar can two species be in the utilization of a resource (this resource being space in our consideration) before one excludes the other.

If we accept the above comparison of sequences as noncoincidental, we can go on to hypothesize that the refuge volumes occupied by these five species of insects may be a function of the insects' lengths. If the volume occupied is simply related to the insect's length by a constant (k), then we can denote a volume sequence (u_n') as:

$$u_n' = ku_1(\beta^{n-1})$$

which is qualitatively identical to (2). That is, these insects may possess refuge volumes which correspond, in magnitude, to a Fibonacci sequence.

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ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—I

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1. INTRODUCTION

Throughout what follows, we will let n denote an arbitrary nonnegative integer, $S(n)$ a nonnegative integer-valued function of n, and $T(n) = n + S(n)$. We also let $\mathcal{R} = \{x \mid x = T(n) \text{ for some } n\}$ and $\mathbf{C}$ = complement of $\mathcal{R} = \{n \geq 0 \mid n \notin \mathcal{R}\}$.

It is of interest to ask whether or not the set $\mathbf{C}$ is infinite. We can also pose the question: does the set $\mathcal{R}$ have asymptotic density and, if so, does $\mathcal{R}$ (or $\mathbf{C}$) have positive density? It might be suspected that if $S(n)$ is "small" there is a good chance that $\mathcal{R}$ has density. However, this suspicion is incorrect, as can be seen from the following example: for a given $n \geq 1$, let k be the unique integer satisfying $k! \leq n \leq (k+1)! - 1$ and define

$$S(n) = \begin{cases} 0 \text{ if } k \text{ is odd} \\ 1 \text{ if } n = k! + k_1,\ k \text{ and } k_1 \text{ even},\ 0 \leq k_1 \leq (k+1)! - 1 \\ 0 \text{ if } n = k! + k_1,\ k \text{ even},\ k_1 \text{ odd and as above} \end{cases}$$

Then n or $n+1$ belongs to $\mathcal{R}$ for every natural number n, so if δ and Δ denote the lower and upper density of $\mathcal{R}$, respectively, we have $\frac{1}{2} \leq \delta \leq \Delta \leq 1$. Now if $D(n) = \{x \leq n \mid x = T(y) \text{ for some } y\}$ then

$$\frac{D\big((k+1)! - 1\big)}{(k+1)! - 1} = \frac{\frac{1}{2}\big((k+1)! - k!\big) - \big(k! - 1 - (k-1)!\big) - \cdots}{(k+1)! - 1} \leq \tfrac{1}{2} + o(1)$$

if k is even, and

$$\frac{D\big((k+1)! - 1\big)}{(k+1)! - 1} = \frac{(k+1)! - k! - \frac{1}{2}\big(k! - 1 - (k-1)!\big) - \cdots}{(k+1)! - 1} \geq 1 + o(1)$$

if k is odd. Hence, $\delta = \frac{1}{2}$ and $\Delta = 1$. Therefore, even if $S(n)$ can take on only the values 0 and 1, it is possible for $\mathcal{R}$ not to have density.

Let $b \geq 2$ be arbitrary and let $n = \sum_{j=0}^{k} d_j b^j$ be the unique representation of n in base b. Define $S(n) = \sum_{j=0}^{k} f(d_j, j)$, where $f(d, j)$ is a nonnegative integer-valued function of the digit d and the place where the digit occurs, and $T(n) = n + S(n)$. The consideration of functions of this form is motivated by the problem (which was posed in [1]) of showing that $\mathbf{C}$ is infinite when $T(n) = n + \sum_{j=0}^{k} d_j$. A solution, as given in [2], was obtained by recursively constructing an infinite sequence of integers in $\mathbf{C}$ for all bases b. It was also observed in [2] that if b is odd then $T(n)$ is always even. In fact, $\mathcal{R}$ is precisely the set of all nonnegative even integers when b is odd. To see this, observe that $n \equiv S(n) \pmod{b-1}$ and, therefore, $T(n) \equiv 2S(n) \pmod{b-1}$ where $S(n) = \sum_{j=0}^{k} d_j$. Hence $T(n)$ is even if b is odd. Since $T(0) = 0$, $T(n+1) \le T(n) + 2$ for every natural number n, and $T(n) \to \infty$ as $n \to \infty$, the result is proved.

2. EXISTENCE AND COMPUTABILITY OF THE DENSITY

Again, letting $n = \sum_{j=0}^{k} d_j b^j$, $S(n) = \sum_{j=0}^{k} f(d_j, j)$, and $T(n) = n + S(n)$, we prove that the density of $\mathfrak{R}$ exists and is in fact computable when suitable hypotheses are placed on the function f. We will adhere to the following notation:

$$\begin{aligned}
\Omega(k, r) &= \{T(x) \mid k \le x \le r\} \\
\Omega(r) &= \Omega(0, r) \\
D(k, r) &= |\Omega(k, r)| \\
D(r) &= |\Omega(r)|.
\end{aligned}$$

Theorem 2.1: Let $f(d, j)$ $(d = 0, 1, \ldots, b-1)$ be a family of nonnegative integer-valued functions satisfying

(a) $f(0, j) = 0$, $j = 0, 1, 2, \ldots$
(b) $f(d, j) = o(b^j)$, $1 \le d \le b - 1$.

Then the density of $\mathfrak{R}$ exists.

Proof: First, we show that

$$D(db^k, db^k + r) = D(r), \quad 0 \le r \le b^k - 1, \quad 0 \le d \le b - 1. \tag{2.2}$$

To prove this, suppose that

$$x = db^k + \sum_{j=0}^{k-1} d_j b^j \quad \text{and} \quad y = db^k + \sum_{j=0}^{k-1} d_j' b^j.$$

Clearly $T(x) = T(y)$ if and only if

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j' b^j\right).$$

Now if $d_{k-1} = d_{k-2} = \cdots = d_{k-t} = 0$ (or if $d_{k-1}' = d_{k-2}' = \cdots = d_{k-t}' = 0$), then, by assumption (a), we see that

$$T\left(\sum_{j=0}^{k-t-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(\sum_{j=0}^{k-1} d_j' b^j\right).$$

We therefore have a one-one correspondence between the elements of $\Omega(db^k, db^k + r)$ and $\Omega(r)$, $0 \le r \le b^k - 1$, from which (2.2) follows. In particular, if $r = b^k - 1$, we have

$$D\big(db^k, (d+1)b^k - 1\big) = D(b^k - 1). \tag{2.3}$$

Our next lemma will enable us to relate $D(b^{k+1} - 1)$ to

$$\sum_{d=0}^{b-1} D\big(db^k,\ (d+1)b^k - 1\big).$$

Lemma 2.4: There exists an integer k_0 such that for all $k \geq k_0$ the sets $\Omega(0,\ b^k - 1),\ \Omega(b^k,\ 2b^k - 1),\ \ldots,\ \Omega\big((b-1)b^k,\ b^{k+1} - 1\big)$ are pairwise disjoint, except possibly for adjacent pairs.

Proof: The maximum value of any element in $\Omega\big(db^k,\ (d+1)b^k - 1\big)$ is at most $(d+1)b^k - 1 + M_k(k+1)$, where $M_k = \max\{f(d,\ j)\,|\,0 \leq j \leq k\}$ and the minimum value of any element in $\Omega\big((d+2)b^k,\ (d+3)b^k - 1\big)$ is at least $(d+2)b^k$. Because of assumption (b), there exists k_0' such that $f(d,\ j) < b^j/2$ for all $j \geq k_0'$ and there exists $k_0 \geq k_0'$ such that $f(d,\ j) < b^j/2 - M_{k_0'}(k_0' + 1)$, whenever $k_0 \geq k_0'$, where

$$M_{k_0'} = \max\{f(d,\ j)\,|\,0 \leq j \leq k_0'\}.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{k} f(d_j,\ j) &= \sum_{j=0}^{k_0'} f(d_j,\ j) + \sum_{j=k_0'+1}^{k_0} f(d_j,\ j) + \sum_{j=k_0+1}^{k} f(d_j,\ j)\\ &< M_{k_0'}(k_0' + 1) + \sum_{j=k_0'+1}^{k} b^j/2 - M_{k_0'}(k_0' + 1)(k - k_0)\\ &\leq \sum_{j=k_0'+1}^{k} b^j/2 < b^k \text{ for all } k \geq k_0,\end{aligned}$$

so, in particular, $M_k(k+1) < b^k$. Hence,

$$(d+1)b^k - 1 + M_k(k+1) < (d+2)b^k$$

whenever $k \geq k_0$, which completes the proof of the lemma.

Now $D(b^{k+1} - 1) = \sum_{d=0}^{b-1} D\big(db^k,\ (d+1)b^k - 1\big) - Q$, where Q depends on the size of the intersections of the sets

$$\Omega(0,\ b^k - 1),\ \Omega(b^k,\ 2b^k - 1),\ \ldots,\ \Omega\big((b-1)b^k,\ b^{k+1} - 1\big).$$

Define

$$\lambda_{d,k} = \big|\Omega\big(db^k,\ (d+1)b^k - 1\big) \cap \Omega(d+1)b^k,\ \big((d+2)b^k - 1\big)\big|,\ 0 \leq d \leq b-2.$$

Using Lemma 2.4 and Equation (2.3), we obtain

$$D(b^{k+1} - 1) = bD(b^k - 1) - \sum_{d=0}^{b-1} \lambda_{d,k},\ k \geq k_0. \tag{2.5}$$

Let

$$A_k = D(b^k - 1)/b^k \quad\text{and}\quad \varepsilon_k = \sum_{d=0}^{b-2} \lambda_{d,k}/b^{k+1},\ k \geq k_0.$$

Then 2.5 can be rewritten as

$$A_{k+1} - A_k = -\varepsilon_k.$$

Therefore,

$$A_{k+1} - A_k = -\varepsilon_k$$
$$A_k - A_{k-1} = -\varepsilon_{k-1}$$
$$\vdots$$
$$A_{k_0+1} - A_{k_0} = -\varepsilon_{k_0}$$

and by telescoping, we obtain

$$A_{k+1} = A_{k_0} - \sum_{j=k_0}^{k} \varepsilon_j.$$

Replacing $k + 1$ by k yields

$$A_k = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j, \; k \geq k_0. \tag{2.6}$$

Obviously, $1/b^k \leq A_k \leq 1$ and $\sum_{j=k_0}^{k-1} \varepsilon_j = A_{k_0} - A_k < A_{k_0} \leq 1$. Thus $\sum_{j=k_0}^{k} \varepsilon_j$ is a series of nonnegative terms bounded above by A_{k_0}, hence is convergent. Let

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j \tag{2.7}$$

(We have just shown that $0 \leq L \leq 1$). Then, (2.6) yields

$$A_k = L + \sum_{j=k}^{\infty} \varepsilon_j, \; k \geq k_0;$$

i.e.,

$$A_k = L + o(1). \tag{2.8}$$

Hence

$$D(b^k - 1) = Lb^k + o(b^k). \tag{2.9}$$

Using (2.3), (2.4), (2.9), and recalling the definition of the $\lambda_{d,k}$ and the ε_k, we have

$$\begin{aligned} D(db^k - 1) &= \sum_{c=0}^{d-1} D\big(cb^k, \; (c+1)b^k - 1\big) - \sum_{c=0}^{d-2} \lambda_{d,k} \\ &= \sum_{c=0}^{d-1} \big(Lb^k + o(b^k)\big) + O(b^{k+1}\varepsilon_k) = db^k L + o(b^k); \end{aligned}$$

i.e.,

$$D(db^k - 1) = db^k L + o(b^k). \tag{2.10}$$

Now let $n = \sum_{j=0}^{k} d_j b^j$ be any nonnegative integer. Then

$$\begin{aligned} D(n) &= D\left(\sum_{j=0}^{k} d_j b^j\right) \\ &= D(d_k b^k - 1) + D\left(d_k b^k, \; \sum_{j=0}^{k} d_j b^j\right) - Q, \end{aligned}$$

where Q is the number of elements that the sets

$$\Omega(d_k b^k - 1) \quad \text{and} \quad \Omega\left(d_k b^k, \sum_{j=0}^{k} d_j b^j\right)$$

have in common. Therefore, if n is sufficiently large, then by using (2.10), (2.2), and the definition of the $\lambda_{d,k}$, we have

$$D(n) = d_k b^k L + o(b^k) + D\left(\sum_{j=0}^{k-1} d_j b^j\right) + o(b^k) = d_k b^k L + D\left(\sum_{j=0}^{k-1} d_j b^j\right) + o(b^k).$$

Applying the same reasoning to the quantities $D\left(\sum_{j=0}^{t} d_j b^j\right)$, $k_0 \le t \le k - 1$, we eventually obtain

$$D(n) = L\left(\sum_{j=k_0}^{k} d_j b^j\right) + D\left(\sum_{j=0}^{k_0-1} d_j b^j\right) + \sum_{j=k_0}^{k} o(b^j);$$

i.e.,

$$D(n) = L\left(n - \sum_{j=0}^{k_0-1} d_j b^j\right) + D\left(\sum_{j=0}^{k_0-1} d_j b^j\right) + o(n).$$

Dividing both sides of this equation by n yields

$$D(n)/n = L + o(1),$$

which proves the density of $\mathfrak{R}$ is L.

Remark: It should be noted that Equation (2.2), and therefore the above proof of Theorem 2.2, breaks down if we lift the condition $f(0, j) = 0$.

A particular case of Theorem 2.1 of interest occurs when we assume that f depends only on d:

Corollary 2.11: If $f(d)$ is an arbitrary nonnegative function of d, $1 \le d \le b - 1$, and $f(0) = 0$, then the density of $\mathfrak{R}$ exists and is equal to L, where L is defined as in Equation (2.7).

We also easily obtain the following two corollaries to Theorem 2.1:

Corollary 2.12: $L < 1$ if and only if the function $T(n)$ is not one-one.

Proof: We have

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j = A_k - \sum_{j=k}^{\infty} \varepsilon_j, \text{ for all } k \ge k_0,$$

where k_0 is defined as in Lemma 2.4. If $T(x) = T(y)$, $x \ne y$, and k is such that $k \ge k_0$ and $x \le b^k - 1$, $y \le b^k - 1$, then, since $A_k = D(b^k - 1)/b^k$, it follows that $L \le A_k < 1$. If T is one-one, then it follows from the definition of the A_k and the ε_k that $A_k = 1$ and $\varepsilon_k = 0$ for all k, so $L = 1$.

Corollary 2.13: If $f(d, j) = f(d)$ depends only on d and if $f(0) = 0$ and $f(b - 1) \ne 0$, then $L < 1$.

Proof: Let $f(b - 1) = s > 0$. Then $T(b^k - 1) = T\big((b-1)b^{k-1} + (b-1)b^{k-2} + \cdots + b - 1\big) = b^k - 1 + ks$.

Now, if k is such that $ks - 1 - f(1) < b^k$ and $n = \sum_{j=0}^{r} d_j b^j$ satisfies $T(n) = ks - 1 - f(1)$, then $n < b^k$ since $T(n) \geq n$. Hence $T(b^k + n) = T(b^k) + T(n) = b^k + f(1) + ks - 1 - f(1) = b^k - 1 + ks = T(b^k - 1)$. Therefore T is not one-one, so $L < 1$ by the above corollary. If there is never any n which satisfies the equation $T(n) = ks - 1 - f(1)$, then almost all integers of the form $ks - 1 - f(1)$, $k = 1, 2, 3, \ldots$, do not belong to $\mathfrak{R}$, hence, $\mathcal{C}$ has positive density, so $L < 1$ in this case also.

Remark: The problem posed in [1] is now an immediate consequence of the above corollary.

More generally, it seems to be true that if $f(d)$ is not identically 0 and $f(0) = 0$, then we again have $L < 1$. We let this statement stand as a conjecture. Note that the hypothesis $f(0) = 0$ is essential; for example, if f is any nonzero constant, then $T(n)$ is strictly increasing and therefore $L = 1$.

There is another question which can be raised about the value of the density L: must one always have $L > 0$ under the hypotheses of Theorem 2.1? Again, the proof of this result, if true, seems to be elusive. Since

$$L = A_k - \sum_{j=k}^{\infty} \varepsilon_j \text{ for } k \geq k_0,$$

we see that $L = 0$ if and only if $A_k = o(1)$, which means that the function $T(n)$ must be very far from being one-one.

3. EXISTENCE OF THE DENSITY WHEN $f(d, j) = O(b^j/j^2 \log^2 j)$

The main drawback to Theorem 2.1 is the condition $f(0, j) = 0$. It seems to be difficult to prove that the density of $\mathfrak{R}$ exists if we assume only that $f(d, j) = o(b^j)$ for all digits d. On the other hand, it also seems to be difficult to find an example of an image set $\mathfrak{R}$ which does not have density under the latter assumption on f, so that the statement that $\mathfrak{R}$ does have density under this assumption will be left as a conjecture. However, the following weaker result does hold:

Theorem 3.1: If $f(d, j) = O(b_j/j^2 \log^2 j)$ for all d, then the density of $\mathfrak{R}$ exists.

Proof: Letting $n = \sum_{j=0}^{k} d_j b^j$, we have

$$S(n) = \sum_{j=0}^{k} O(b^j/j^2 \log^2 j) = O(b^k/k^2 \log^2 k). \tag{3.2}$$

Now if $r \leq s \leq t$ $(r < t)$ and $s < b^{k+1}$, then, letting D and Ω be the same as in the proof of Theorem 2.1, we see that

$$D(r, t) = D(r, s) + D(s + 1, t) - |\Omega(r, s) \cap \Omega(s + 1, t)|.$$

Hence, by (3.2),

$$D(r, t) = D(r, s) + D(s + 1, t) + O(b^k/k^2 \log^2 k). \tag{3.3}$$

In particulr, if $r = 0$, $s = b^{k-1} - 1$, and $t = b^k - 1$, then

$$D(b^k - 1) = D(0,\ b^{k-1} - 1) + D(b^{k-1},\ b^k - 1) + 0\big(b^{k-1}/(k - 1)^2 \log^2(k - 1)\big).$$

Similarly, we see that

$$D(b^q - 1) = D(0,\ b^{q-1} - 1) + D(b^{q-1},\ b^q - 1) + 0\big(b^{q-1}/(q - 1)^2 \log^2(q - 1)\big),\quad 1 \le q \le k - 1.$$

Using the two latter equations and (3.2), we obtain

$$D(b^k - 1) = D(0) + D(1,\ b - 1) + \cdots + D(b^{q-1},\ b^q - 1) + \cdots + D(b^{k-1},\ b^k - 1) + 0(b^k/k^2 \log^2 k). \tag{3.4}$$

Let us now consider the quantity $D\big(db^k,\ (d + 1)b^k - 1\big)$. From (3.3), we have

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1) + D\big(db^k + b,\ (d + 1)b^k - 1\big) + 0(b^k/k^2 \log^2 k).$$

A second application of (3.3) yields

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1) + D(db^k + b,\ db^k + b^2 - 1) + D(db^k + b^2,\ (d + 1)b^k - 1) + 0(b^k/k^2 \log^2 k),$$

and by repeatedly applying (3.3), we eventually obtain

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(db^k,\ db^k) + D(db^k + 1,\ db^k + b - 1) + \cdots + D(db^k + b^q,\ db^k + b^{q+1} - 1) + \cdots + D(db^k + b^{k-1},\ db^k + b^k - 1) + 0(b^k/k \log^2 k). \tag{3.5}$$

Since all integers x satisfying

$$db^k + b^q \le x \le db^k + b^{q+1} - 1 \quad (0 \le q \le k - 1)$$

have the same number of leading zeros, there is a one-one correspondence between the elements of $\Omega(db^k + b^q,\ db^k + b^{q+1} - 1)$ and $\Omega(b^q,\ b^{q+1} - 1)$, i.e.,

$$D(db^k + b^q,\ db^k + b^{q+1} - 1) = D(b^q,\ b^{q+1} - 1).$$

Using this fact, (3.5) becomes

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(0) + D(1,\ b - 1) + \cdots + D(b^{k-1},\ b^k - 1) + 0(b^k/k \log^2 k), \tag{3.6}$$

and (3.4) and (3.6) imply that

$$D\big(db^k,\ (d + 1)b^k - 1\big) = D(b^k - 1) + 0(b^k/k \log^2 k). \tag{3.7}$$

Now, from (3.7),

$$\begin{aligned} D(b^{k+1} - 1) &= D(b^k - 1) + D(b^k,\ b^{k+1} - 1) + 0(b^k/k^2 \log^2 k) \\ &= D(b^k - 1) + D(b^k,\ 2b^k - 1) + D(2b^k,\ b^{k+1} - 1) + 0(b^k/k^2 \log^2 k) \\ &= 2D(b^k - 1) + D(2b^k,\ b^{k+1} - 1) + 0(b^k/k \log^2 k). \end{aligned}$$

By repeated application of (3.7), we have

$$D(b^{k+1} - 1) = bD(b^k - 1) + 0(b^k/k \log^2 k). \tag{3.8}$$

Letting $A_k = D(b^k - 1)/b^k$, (3.8) becomes

$$b^{k+1}A_{k+1} - b^{k+1}A_k = 0(b^k/k \log^2 k)$$

and therefore

$$A_{k+1} - A_k = 0(1/k \log^2 k).$$

Since $\sum_{j=0}^{k} 0(1/j \log^2 j) = 0(1/\log k)$, there exists a constant L such that

$$A_k = L + 0(1/\log k). \tag{3.9}$$

Let $n = d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots$ be any integer, each $d_{k_j} \neq 0$. Then

$$D(n) = D(d_{k_1}b^{k_1} - 1) + D(d_{k_1}b^{k_1}, n) + 0(b^{k_1}/k_1^2 \log^2 k_1).$$

By the same reasoning used to obtain (3.8), we see that

$$D(d_{k_1}b^{k_1} - 1) = d_{k_1}D(b^{k_1} - 1) + 0(b^{k_1}/k_1 \log^2 k_1).$$

Therefore, by (3.9), we have

$$D(n) = d_{k_1}b^{k_1}(L + 0(1/\log k_1)) + 0(b^{k_1}/k_1 \log^2 k_1)$$
$$+ D(d_{k_1}b^{k_1}, d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots).$$

Since $d_{k_j} \neq 0$ for any j, we know that

$$D(d_{k_1}b^{k_1}, d_{k_1}b^{k_1} + d_{k_2}b^{k_2} + \cdots) = D(d_{k_2}b^{k_2} + \cdots)$$

[c.f. the reasoning applied between equations (3.5) and (3.6)]. Hence,

$$D(n) = d_{k_1}b^{k_1}(L + 0(1/\log k_1)) + 0(b^{k_1}/k_1 \log^2 k_1) + D(d_{k_2}b^{k_2} + \cdots).$$

Continuing in this manner, we have

$$D(n) = nL + 0(b^{k_1}/k_1 \log^2 k_1) + \sum_{j=0}^{k_1} 0(b^j/\log j) = nL + 0(b^{k_1}/\log k_1).$$

This last equation shows that the density of $\mathfrak{R}$ is L, q.e.d.

Remark I: This theorem, in contrast to Theorem 2.1, has the drawback that no formula for the density of $\mathfrak{R}$ has been derived.

Remark II: It is interesting to note that there exist sets $\mathfrak{R}$ which do not have density under the assumption that $f(d, j) = 0(b^j)$. For example, let $f(d, j) = 0$ if j is even and $f(d, j) = b^j$ if j is odd. Evidently,

$$T\left(b^k + \sum_{j=0}^{k-1} d_j b^j\right) = b^k + \sum_{j=0}^{k-1} d_j b^j + b^k + b^{k-2} + \cdots + b \geq 2b^k$$

if k is odd, and

$$T\left(b^k + \sum_{j=0}^{k-1} d_j b^j\right) = b^k + \sum_{j=0}^{k-1} d_j b^j + b^{k-1} + b^{k-3} + \cdots + b$$

if k is even.

Therefore, the number of integers between b^k and $2b^k$ in $\mathfrak{R}$ if k is odd is at most $1 + b^{k-2} + b^{k-4} + \cdots + b$, and the number of integers between b^k and $2b^k$ in $\mathfrak{R}$ if k is even is at least $b^k - b^{k-1} - b^{k-3} - \cdots - b$. Hence, if we let δ and Δ denote the lower and upper density of $\mathfrak{R}$, respectively, we see that

$$\delta \leq 1/b^2 + 1/b^4 + 1/b^6 + \cdots = 1/(b^2 - 1)$$

and

$$\Delta \geq 1 - 1/b - 1/b^3 - 1/b^5 - \cdots = 1 - b/(b^2 - 1).$$

Since $1 - b/(b^2 - 1) > 1/(b^2 - 1)$ when $b > 2$, it follows that $\mathfrak{R}$ does not have density if $b \neq 2$.

It is also interesting that we can obtain examples in which the set $\mathfrak{R}$ is of density 0 if $f(d, j) = 0(b^j)$. For example, if $b = 10$ and $f(d, j) = 0$ if $d \neq 1$ and $f(d, j) = 8 \cdot 10^j$ if $d = 1$, then no member of $\mathfrak{R}$ has a 1 anywhere in its decimal representation, and the set

$$n = \left\{\sum_{j=0}^{k} d_j 10^j,\ d_j \neq 1,\ 0 \leq j \leq k\right\}$$

is a set which is well known to have density 0.

Corollary 3.10: If $f(d)$ is an arbitrary nonnegative function of the digit d, then the density of $\mathfrak{R}$ exists.

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MATRICES AND CONVOLUTIONS OF ARITHMETIC FUNCTIONS

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1. INTRODUCTION

The purpose of this paper is to relate certain matrices with integer entries to convolutions of arithmetic functions.

Let n be a positive integer, let α, β, and γ be arithmetic functions (complex-valued functions with domain the set of positive integers), and let $\alpha_{[n]}$ denote the $1 \times n$ matrix $[\alpha(1)\ \alpha(2)\ \ldots\ \alpha(n)]$.

We define the $n \times n$ divisor matrix $D_n = (d_{ij})$ by $d_{ij} = 1$ if $i|j$, $d_{ij} = 0$ otherwise. Both D_n and its inverse, D_n^{-1}, are upper triangular matrices. The arithmetic functions ν_k, σ, and ε are defined by $\nu_k(n) = n^k$ for $k = 0, 1, 2$, $\sigma(n) = \sum_{d|n} d$, and $\varepsilon(n) = 1$ if $n = 1$, $\varepsilon(n) = 0$ if $n > 1$. We also consider the divisor function τ, the Moebius function μ, and Euler's ϕ-function. We observe that

$$\nu_{0[n]}D = \tau_{[n]}, \tag{1}$$

$$\nu_{1[n]}D = \sigma_{[n]}, \tag{2}$$

$$\varepsilon_{[n]}D_n^{-1} = \mu_{[n]}, \tag{3}$$

$$\nu_{1[n]}D_n^{-1} = \phi_{[n]}. \tag{4}$$

These matrix formulas, which can be used to evaluate arithmetic functions as in [2], are consequences of the following equations which involve the Dirichlet convolution, $*_D$.

$$\nu_0 *_D \nu_0 = \tau, \tag{1'}$$

$$\nu_1 *_D \nu_0 = \sigma, \tag{2'}$$

$$\varepsilon *_D \mu = \mu, \quad \varepsilon = \mu *_D \nu_0, \tag{3'}$$

$$\nu_1 *_D \mu = \phi, \quad \phi *_D \nu_0 = \nu_1. \tag{4'}$$

As an illustration, consider matrices D_6 and D_6^{-1} which appear below.

$$D_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}, \qquad D_6^{-1} = \begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 1 \\ & 1 & 0 & -1 & 0 & -1 \\ & & 1 & 0 & 0 & -1 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}.$$

Any omitted entry is assumed to be zero. By (2),

$$[1\ 2\ 3\ 4\ 5\ 6]D_6 = [\sigma(1)\ \sigma(2)\ \sigma(3)\ \sigma(4)\ \sigma(5)\ \sigma(6)],$$

so that $\sigma(6) = \sum_{d|6} d = \sum_{d|6} \nu_1(d) = (\nu_1 *_D \nu_0)(6)$. And by (4),

$$[1\ 2\ 3\ 4\ 5\ 6]D_6^{-1} = [\phi(1)\ \phi(2)\ \phi(3)\ \phi(4)\ \phi(5)\ \phi(6)],$$

so that $\phi(6) = 1 - 2 - 3 + 6 = (\nu_1 *_D \mu)(6)$.

These observations lead us to define and illustrate matrix-generated convolutions.

2. MATRIX-GENERATED CONVOLUTIONS

Suppose that $G = (g_{ij})$ is an infinite dimentional (0, 1)-matrix with $g_{ij} = 1$ if $i = j$ and $g_{ij} = 0$ if $i > j$, and that the 1's in column n of G appear in rows $n_1, n_2, \ldots, n_k$ ($n_1 < n_2 < \ldots < n_k = n$). We say that G generates the convolution $*_G$ defined by

$$(\alpha *_G \beta)(n) = \sum_{v=1}^{k} d(n_v)\beta(n_{k+1-v}),\ n = 1, 2, 3, \ldots.$$

Clearly, $*_G$ is a commutative operation on the set of arithmetic functions. We denote by G_n the $n \times n$ submatrix of $G = (g_{ij})$ with $1 \leqq i \leqq n$, $1 \leqq j \leqq n$.

The convolutions in Examples 1-4 below are defined and referenced in [3].

Example 1: The matrix $D = (d_{ij})$, with $d_{ij} = 1$ if $i|j$, $d_{ij} = 0$ otherwise, generates the Dirichlet convolution $*_D$. D_n is the $n \times n$ divisor matrix, and the set $\{n_1, n_2, \ldots, n_k\}$ is the set of positive divisors of n.

Example 2: The unitary convolution is generated by the matrix $U = (u_{ij})$ with $u_{ij} = 1$ if $i \leqq j$ and $i|j$ and i and j/i are relatively prime, $u_{ij} = 0$ otherwise.

Example 3: The matrix $C = (c_{ij})$ defined by $c_{ij} = 1$ if $i \leqq j$, $c_{ij} = 0$ otherwise, generates a convolution $*_C$ related to the Cauchy product. Since $\{n_1, n_2, \ldots, n_k\} = \{1, 2, \ldots, n\}$, we have

$$(\alpha *_C \beta)(n) = \alpha(1)\beta(n) + \alpha(2)\beta(n - 1) + \cdots + \alpha(n)\beta(1).$$

Example 4: For a fixed prime p, let the matrix $L = (l_{ij})$ be defined by $l_{ij} = 1$ if $i \leqq j$ and $p \nmid \binom{j-1}{i-1}$, $l_{ij} = 0$ otherwise. The convolution $*_L$ generated by L is related to the Lucas product. The entries shown in the matrix L_{14} for $p = 3$ are easily determined by the use of a basis representation criterion given in [1].

$$L_{14} = \atop (p = 3) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ & & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & & & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & & & & 1 & 1 & 0 & 1 \\ & & & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & & & & 1 & 1 \\ & & & & & & & & & & & & & 1 \end{bmatrix}$$

3. A GENERAL MOEBIUS FUNCTION

In view of (3′), we next define a general Moebius function μ_G by $\nu_0 *_G \mu_G = \varepsilon$. It is immediate from $G_n^{-1}G_n = I_n$ (the $n \times n$ identity matrix) that

$$\text{if } G_n^{-1} = (\overline{g}_{ij}) \text{ then } \overline{g}_{ij} = \mu(j) \text{ for } j = 1, 2, \ldots, n \text{ and } n = 1, 2, 3, \ldots \tag{5}$$

For example, the elements in row one of D_6^{-1} are $\mu_D(1) = \mu(1), \mu(2), \ldots, \mu(6)$ (in that order). The values of the unitary, Cauchy, and Lucas Moebius functions given in [3] agree with corresponding entries in row one of U_n, C_n, and L_n, respectively. Property (5) implies $\varepsilon_{[n]}G_n^{-1} = \mu_{G[n]}$, which is a generalization of (3).

The following three properties are related to the Moebius function and are stated for future reference.

$$\alpha *_G \varepsilon = \alpha \text{ for all arithmetic functions } \alpha. \tag{6}$$

$$*_G \text{ is an associative operation on the set of arithmetic functions.} \tag{7}$$

$$\text{If } g_{ij} = 0 \text{ then } \overline{g}_{ij} = 0, \text{ where } G_n^{-1} = (\overline{g}_{ij}),\ n = 1, 2, 3, \ldots. \tag{8}$$

Property (6) is equivalent to

$$g_{1j} = 1 \text{ for } j = 1, 2, 3, \ldots. \tag{6'}$$

For (6′) clearly implies (6); and if $g_{1n} = 0$ for some n, and α is such that $\alpha(n) \neq 0$, then $(\alpha *_G \varepsilon)(n) = 0 \neq \alpha(n)$.

Example 5: Let the matrix $P = (p_{ij})$ be defined by $p_{ij} = 1$ if $i \leq j$ and i and j are of the same parity, $p_{ij} = 0$ otherwise. Evidently, (6′) and (6) do not hold here. For example, $(\nu_0 *_P \varepsilon)(2) = \nu_0(2)\varepsilon(2) = 0 \neq \nu_0(2)$. Although ε', defined by $\varepsilon'(1) = \varepsilon'(2) = 1$, $\varepsilon'(n) = 0$ if $n > 2$, satisfies $\alpha *_P \varepsilon' = \alpha$ for all arithmetic functions α, ε' is not related to matrix multiplication in $G_n^{-1}G_n = I_n$ in the desirable way that ε is.

We note that if (6) and (7) hold then we can apply Moebius inversion in the form $\alpha = \nu_0 *_G \beta$ iff $\beta = \mu_G *_G \alpha$ [as illustrated in (4′)]. It is clear that

(6) holds and well known that (7) holds for the convolutions in Examples 1-4; so (8) holds as well, as can be verified by direct computation or by application of the following theorem.

Theorem 1: Property (7) implies property (8).

Proof: Assume that (8) is false. Let j be the smallest positive integer such that for some i we have $g_{ij} = 0$ and $\overline{g}_{ij} \neq 0$; let this $j = n$. Consider the largest value of i such that $g_{in} = 0$ and $\overline{g}_{in} \neq 0$; let this $i = t$. It follows by the assumptions and $G_n G_n^{-1} = I_n$ that $g_{tt} = 1$, $g_{tn} = 0$, $\overline{g}_{tn} \neq 0$, there is an integer r such that $t < r < n$ and $g_{tr} = 1$, and $g_{rn} = 1$. Since $r \in \{n_1, \ldots, n_k\}$ and $g_{tr} = 1$, then $\alpha(t)$ is a factor in some term of

$$((\alpha *_G \beta) *_G \gamma)(n).$$

But no term of $(\alpha *_G (\beta *_G \gamma))(n)$ has a factor $\alpha(t)$ because $t \notin \{n_1, \ldots, n_k\}$. Therefore, (7) is false and the proof is complete. ■

4. THE MAIN THEOREM

We now define some special functions and matrices leading to the main result in this paper. Assume that the matrix G generates the convolution $*_G$ and define the arithmetic functions A and B by

$$A(n) = \sum_{i=1}^{n} g_{in}\alpha(i) \quad \text{and} \quad B(n) = \sum_{i=1}^{n} \overline{g}_{in}\beta(i).$$

Then for $n = 1, 2, 3, \ldots$, we have

$$\alpha_{[n]} G_n = A_{[n]} \tag{9}$$

and

$$\beta_{[n]} G_n^{-1} = B_{[n]}. \tag{10}$$

Define $G_n^S = (s_{ij})$ to be the $n \times n$ matrix with $s_{ij} = 1$ if $i = n_v$ and $j = n_{k+1-v}$, $v = 1, 2, \ldots, k$, $s_{ij} = 0$ otherwise. Note that G_n^S is a symmetric (0, 1)-matrix with at most one nonzero entry in any row or column. If M^t denotes the transpose of a matrix M, then

$$(\alpha *_G \beta)(n) = \alpha_{[n]} G_n^S \left(\beta_{[n]}\right)^t \tag{11}$$

and

$$(A *_G B)(n) = A_{[n]} G_n^S \left(B_{[n]}\right)^t. \tag{12}$$

The matrix $G_n G_n^S$ is of special interest and can be characterized as follows.

Column n_v of $G_n G_n^S$ equals column n_{k+1-v} of G_n, for $v = 1, 2, \ldots, k$;

the other columns (if any) of $G_n G_n^S$ are zero columns. (13)

Although $G_n G_n^S$ is symmetric (for all positive integers n) for the matrices defined in Examples 1-5, $G_n G_n^S$ is not symmetric for $G_n = E_3$ given below.

$$E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 E_3^S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Theorem 2: The matrix $G_n G_n^S$ is symmetric for $n = 1, 2, 3, \ldots$ if and only if $(\alpha *_G \beta)(n) = (A *_G B)(n)$ for all arithmetic functions α and β, and for all positive integers n.

Proof:

1. Assume that $G_n G_n^S$ is symmetric for $n = 1, 2, 3, \ldots$. This and the symmetry of G_n^S imply that $(G_n G_n^S)^t = G_n (G_n^S)^t$. In view of (9), (10), (11), and (12), we have

$$\begin{aligned}(A *_G B)(n) &= A_{[n]} G_n^S \left(B_{[n]}\right)^t \\ &= \alpha_{[n]} G_n G_n^S \left(\beta_{[n]} G_n^{-1}\right)^t \\ &= \alpha_{[n]} G_n^S (G_n)^t (G_n^{-1})^t \left(\beta_{[n]}\right)^t \\ &= \alpha_{[n]} G_n^S \left(\beta_{[n]}\right)^t \\ &= (\alpha *_G \beta)(n), \; n = 1, 2, 3, \ldots .\end{aligned}$$

2. Assume that there is a positive integer n such that $G_n G_n^S$ is not symmetric. Then $G_n G_n^S \neq (G_n G_n^S)^t$ implies that $G_n G_n^S (G_n^{-1})^t \neq G_n^S$ and that $(A *_G B)(n) = \alpha_{[n]} G_n G_n^S (G_n^{-1})^t \left(\beta_{[n]}\right)^t$ and $(\alpha *_G \beta)(n)$ are not identically equal. Therefore, there exist arithmetic functions α and β such that

$$(A *_G B)(n) \neq (\alpha *_G \beta)(n).$$

This completes the proof of the theorem. ∎

Next, we give an application of this theorem.

Example 6: Since $P_n P_n^S$ is symmetric for $n = 1, 2, 3, \ldots$ for P in Example 5, we can apply Theorem 2 with $n = 2t - 1$ (for t a positive integer), $\alpha = \nu_1$, $\beta(2k - 1) = k$ for $k = 1, 2, \ldots, t$, to obtain the identity

$$\sum_{k=1}^{t} \nu_2(k) = \sum_{k=1}^{t} (2k-1)(t-k+1),$$

which can be expressed in the form

$$t^3 = \sum_{k=1}^{t} \nu_2(k) + \sum_{k=1}^{t-1} k(2k+1).$$

5. A GENERAL EULER FUNCTION

Assume that the matrix G generates the convolution $*_G$. In §3, we defined a general Moebius function μ_G and obtained a generalization of (3). In this section, we define a general Euler function ϕ_G for G such that $*_G$ satisfies (6) and (7), and derive a generalization of (4).

First, we consider the property

$$G_n G_n^S \text{ is symmetric for } n = 1, 2, 3, \ldots \tag{14}$$

and some preliminary theorems.

Theorem 3: Property (7) implies Property (14).

Proof: Assume that $G_n G_n^S = (h_{ij})$ is not symmetric.

Case 1: Suppose that column w of $G_n G_n^S$ is a zero column and that $h_{wq} = 1$ for some $q \in \{1, 2, \ldots, n\}$. By (13), $g_{wn} = 0$ and $q \in \{n_1, \ldots, n_k\}$; say $q = n_{k+1-t}$. Then $g_{wn_t} = 1 = g_{n_t n} = g_{n_t n_t}$ and $((\alpha *_G \beta) *_G \gamma)(n)$ has a term with factor $\alpha(w)$; but $(\alpha *_G (\beta *_G \gamma))(n)$ has no term with factor $\alpha(w)$ and (7) is false.

Case 2: Suppose that $h_{n_s n_r} = 0$ and $h_{n_r n_s} = 1$, where n_s and n_r belong to $\{n_1, \ldots, n_k\}$. Then $g_{n_s n_{k+1-r}} = 0$, $g_{n_r n_{k+1-s}} = 1$, and $g_{n_s n} = 1 = g_{n_r n}$. Therefore, $(\alpha *_G \beta)(n_{k+1-s})\gamma(n_s)$ has a term with factors $\alpha(n_r)$ and $\gamma(n_s)$, but $\alpha(n_r)(\beta *_G \gamma)(n_{k+1-r})$ has no term with a $\gamma(n_s)$ factor. Again, (7) is false. ■

Theorem 4: Property (14) implies Property (8).

Proof: Assume that (8) is false and let t and r be defined as in the proof of Theorem 1. Column t of $G_n G_n^S$ is a zero column (since $g_{tn} = 0$); but a 1 entry appears in row t of $G_n G_n^S$ (because $g_{tr} = 1 = g_{rn}$), so that $G_n G_n^S$ is not symmetric. ■

We note that (7) implies (8) and (14), and that (14) implies (8); there are no other implications among the properties (6), (7), (8), and (14) (as will be shown in §5).

It follows from (9) that $A = \nu_0 *_G \alpha$. If G and $*_G$ satisfy (6) and (7), then (by Theorems 3 and 2) we have $(\alpha *_G \beta)(n) = (\alpha *_G \nu_0 *_G B)(n)$ for all arithmetic functions α and β and for $n = 1, 2, 3, \ldots$. Therefore, we have

$$\beta(n) = (\nu_0 *_G B)(n);$$

and

$$B(n) = (\beta *_G \mu_G)(n) \tag{15}$$

for all arithmetic functions β and for $n = 1, 2, 3, \ldots$ follows by Moebius inversion.

Theorem 5: If properties (6) and (7) hold for G and $*_G$, then

$$\overline{g}_{n_v n} = \mu_G(n_{k+1-v}), \; v = 1, 2, \ldots, k.$$

Proof: Define the arithmetic functions β_v, $v = 1, 2, \ldots, k$, by $\beta_v(n) = 1$ if $n = n_v$, $\beta_v(n) = 0$ otherwise. Property (15) implies that

$$\sum_{i=1}^{n} \beta(i)\overline{g}_{in} = \sum_{v=1}^{k} \beta(n_v)\mu_G(n_{k+1-v}) \tag{16}$$

for all arithmetic functions β and for $n = 1, 2, 3, \ldots$. Let $G = *_G$ in (16) to obtain $\overline{g}_{n_v n} = \mu_G(n_{k+1-v})$; this is valid for $v = 1, 2, \ldots, k$. ■

For G and $*_G$ which satisfy (6) and (7) we define the general Euler function ϕ_G by $\phi_G = \nu_1 *_G \mu_G$. We can now generalize (4).

Theorem 6: If G and $*_G$ satisfy (6) and (7), then $\nu_{1[n]} G_n^{-1} = \phi_{G[n]}$.

Proof: This is a direct consequence of Theorem 5 and Property (8) (which follow from (6), (7), and Theorems 3 and 4). ■

Other general functions such as τ_G and σ_G can be defined analogously.

6. REMARKS

First, we show that there are no implications among properties (6), (7), (8), and (14) except (7) implies (8) and (14), and (14) implies (8). If R_5 is as shown and $R = (r_{ij})$ is defined for $i > 5$ and $j > 5$ by $r_{ij} = 1$ if $i = j$ or $i = 1$, $r_{ij} = 0$ otherwise, then R satisfies (6) but not (7), (8), and (14). The matrix P defined in

$$R_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Example 5 satisfies (7), (8), and (14) but not (6). A matrix $M = (m_{ij})$ which satisfies (8) but not (7) and (14) can be defined for $i > 5$ and $j > 5$ by $m_{ij} = 1$ if $i = j$, $m_{ij} = 0$ otherwise, with M_5 as shown. If K_{10} is as shown and $K = (k_{ij})$ is defined for $i > 10$ and $j > 10$ by $k_{ij} = 1$ if $i = j$, $k_{ij} = 0$ otherwise, then (14) holds, but (7) is false since, for example,

$$((\nu_1 *_K \nu_1) *_K \nu_0)(10) \neq (\nu_1 *_K (\nu_1 *_K \nu_0))(10).$$

$$K_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 & 1 & 1 \\ & & & & & 1 & 0 & 0 & 0 & 1 \\ & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & 1 & 0 & 0 \\ & & & & & & & & 1 & 1 \\ & & & & & & & & & 1 \end{bmatrix}.$$

Properties (6), (7), (8), and (14) all hold for the matrices (and generated convolutions) in Examples 1-4 as well as for those defined in our concluding example.

Example 7: Let $\hat{F} = \{1, 2, 3, 5, 8, \ldots\}$ be the set of positive Fibonacci numbers. Define $\hat{F} = (f_{ij})$ by $f_{ij} = 1$ if $i = j$ or if $i < j$ and $i \varepsilon \hat{F}$, $f_{ij} = 0$ otherwise. $\hat{F}$ can be replaced by any finite or infinite set of positive integers which includes 1, and properties (6), (7), (8), and (14) will be satisfied. If $\hat{F}$ is replaced by the set of all positive integers, we obtain the matrix C in Example 3.

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A NEW SERIES

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There is a series very similar to Fibonacci's that also displays some interesting properties. An article by Marjorie Bicknell [1] in *The Fibonacci Quarterly* (February 1971) casually mentions the series as a result of more investigation of Pascal's Triangle. The series is 0, 1, 1, 1, 2, 3, 4, 6, 9, 13, ... Each term is found from the relationship

$$F^*_{n+1} = F^*_n + F^*_{n-2}.$$

The series resulted from my research in the Great Pyramid of Gizeh, where the base-to-height ratio is $\pi/2$ and the slant height of a side to the height approximates $\sqrt{\phi}$. ϕ represents the series limit of the Fibonacci Series (see Figure 1).

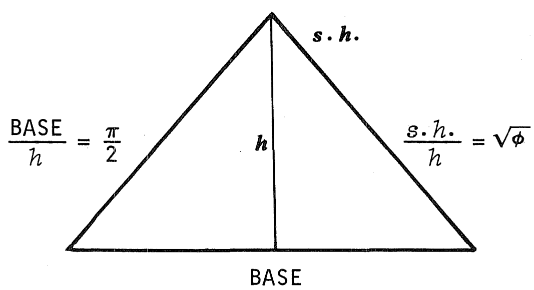


Fig. 1

One of the properties of the new series presented in this paper is that it better fits into the design features of the pyramid than does the accepted fact that Fibonacci's Series limit is intended to be decoded.

The series limit of the new series is represented by the symbol ψ and represents the number 1.46557123..., which will be used as

$$\psi = 1.465571232$$

in this paper.

Referring to Figure 1 again, the ratio of slant height to height is much better represented by the following relationship,

$$\frac{s.h.}{h} = \sqrt{\frac{1 - \ln\psi}{\ln\psi}}$$

This relationship yields a slant height which is only 1.67 inches from the measured values. Fibonacci's Series limit being employed yields a slant height that is 2.7 inches greater than the measured value; the new series limit yielding 1.67 inches less. This is not to dispute the existence of the Fibonacci Series limit as being intended, but to confirm that both expressions are intended by the Designer of the Great Pyramid.

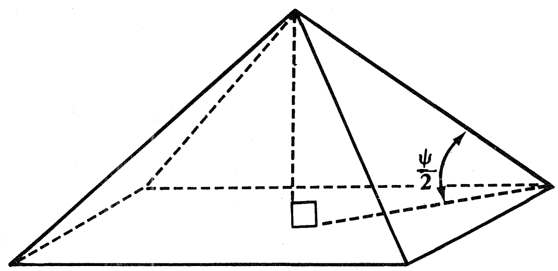


Fig. 2

Figure 2 shows one other place where the new series limit is found in the Great Pyramid. From the corner angles the outside edges of the Pyramid that follow the diagonals form the series limit divided by two as shown and expressed in radian measure.

To find any number in the new series, the recursion formula

$$F^*_{n+1} = F^*_n + F^*_{n-2} \tag{1}$$

can be used, where

$$F^*_0 = 0, \quad F^*_1 = F^*_2 = F^*_3 = 1.$$

The ratio, $\frac{F^*_{n+1}}{F^*_n}$, reaches a definite limit as one uses latter numbers of the series. This ratio is

$$\psi = \lim_{n\to\infty} \frac{F^*_{n+1}}{F^*_n} = 1.46557123\ldots\ . \tag{2}$$

Further investigation reveals that

$$\psi^3 - \psi^2 - 1 = 0 \tag{3}$$

and

$$\psi^{n+1} = \psi^2 F^*_n + \psi F^*_{n-2} + F^*_{n-1}. \tag{4}$$

Equation (3) reveals that ψ is a root of the equation

$$X^3 - X^2 - 1 = 0 \tag{5}$$

The roots of (5) are of considerable interest. These are easily verified to be the true roots of (5). Let the roots be α, β, and γ.

$$\alpha = \psi \tag{6}$$

$$\beta = -\frac{1}{2\psi^2}\left(1 + i\sqrt{4\psi^3 - 1}\right) \tag{7}$$

$$\gamma = -\frac{1}{2\psi^2}\left(1 - i\sqrt{4\psi^3 - 1}\right) \tag{8}$$

Roots (6), (7), and (8) will be used in (4) to develop a formula for F_m^*.

$$\alpha^{n+1} = \alpha^2 F_n^* + \alpha F_{n-2}^* + F_{n-1}^* \tag{9}$$

$$\beta^{n+1} = \beta^2 F_n^* + \beta F_{n-2}^* + F_{n-1}^* \tag{10}$$

$$\gamma^{n+1} = \gamma^2 F_n^* + \gamma F_{n-2}^* + F_{n-1}^* \tag{11}$$

Solving,

$$F_n^* = \frac{\begin{vmatrix} \alpha^{n+1} & \alpha & 1 \\ \beta^{n+1} & \beta & 1 \\ \gamma^{n+1} & \gamma & 1 \end{vmatrix}}{\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}} \tag{12}$$

yielding,

$$F_n^* = \frac{\alpha^{n+1}(\beta - \gamma) + \beta^{n+1}(\gamma - \alpha) + \gamma^{n+1}(\alpha - \beta)}{\alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta)} \tag{13}$$

which reduces to

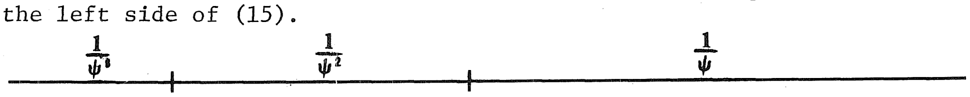
$$F_n^* = \frac{\alpha^{n+1}(\beta - \gamma) + \beta^{n+1}(\gamma - \alpha) + \gamma^{n+1}(\alpha - \beta)}{-i\sqrt{31}} \tag{14}$$

Equation (14) successfully computes F_n^*. The algebra gets fairly involved for higher numbers of the series, but the results agree with the established series.

Geometrical considerations are next. If one considers the relationship

$$\frac{1}{\psi^3} + \frac{1}{\psi^2} + \frac{1}{\psi} = \psi, \tag{15}$$

a line of length ψ can be thought of as divided into three parts as indicated on the left side of (15).



Line of Length ψ

Fig. 3

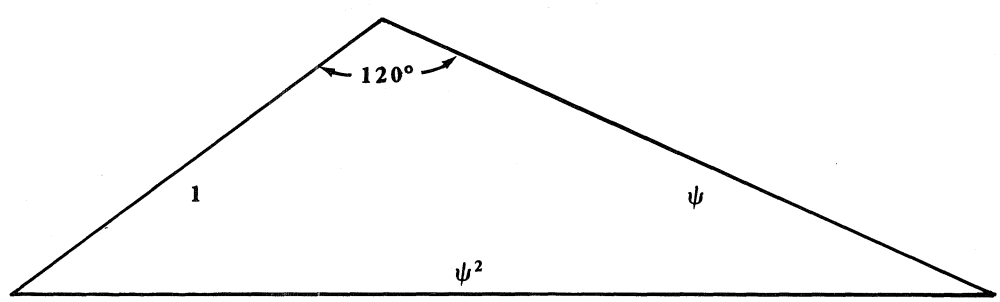
The result of (15) also leads to another interesting fact. The parts of the line in Figure 3 can be used to establish a proportion

$$\frac{1}{\psi^3}:\frac{1}{\psi^2}:\frac{1}{\psi} \tag{16}$$

which is better represented by the proportion

$$\psi^2:\psi:1 \tag{17}$$

The proportion in (17) established the sides of a special triangle which will be named the $\psi^2:\psi:1$ triangle.



$\psi^2:\psi:1$ Triangle

Fig. 4

The $\psi^2:\psi:1$ triangle incorporates an angle of 120° as its largest angle. This fact suggests that it can be placed into the vertices of the regular hexagon.

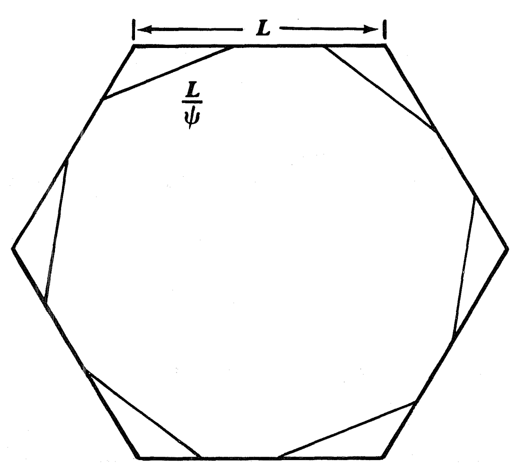


Fig. 5

Figure 5 represents a unit hexagon with six of the $\psi^2:\psi:1$ triangles at the vertices such that $\frac{L}{\psi}:\frac{L}{\psi^2}:\frac{L}{\psi^3}$ proportions are maintained in each.

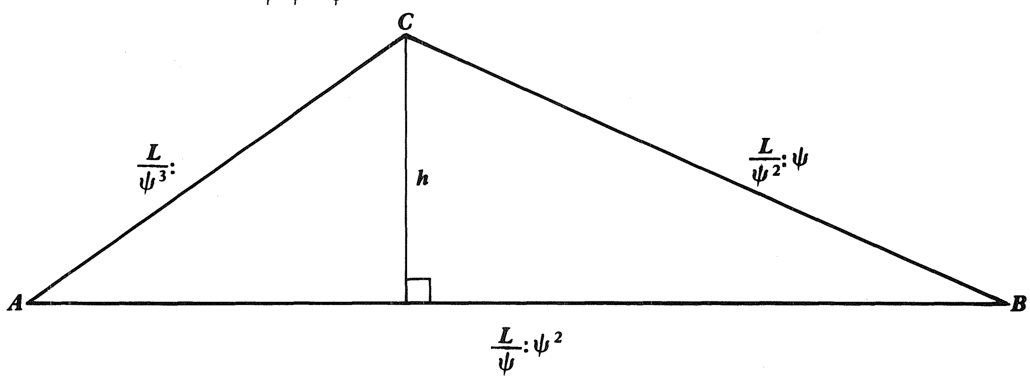


Fig. 6

Using Figure 6, we see that

$$\sin A = \frac{\sin C}{\psi} = \frac{\sqrt{3}}{2\psi}. \tag{18}$$

and

$$h = \frac{\sqrt{3}L}{2\psi^4} \tag{19}$$

The area of each small triangle in Figure 5 then is

$$\text{area} = \frac{1}{2}\,\frac{L}{\psi}\,\frac{\sqrt{3}L}{2\psi^4} = \frac{\sqrt{3}L^2}{4\psi^5}, \tag{20}$$

and for the total area represented by the six triangles

$$\text{area total} = \frac{3\sqrt{3}L^2}{2\psi^5}; \tag{21}$$

the area of the hexagon

$$\text{area hexagon} = \frac{3\sqrt{3}L^2}{2}. \tag{22}$$

A comparison of (22) to (21) yields

$$\frac{\text{area hexagon}}{\text{area six triangles}} = \psi^5 \tag{23}$$

This further indicates that the area of each small $\psi^2:\psi:1$ triangle is given by

$$\text{area of small } \psi^2:\psi:1 \text{ triangle} = \frac{\text{area hexagon}}{6\psi^5} \tag{24}$$

Rearranging equation (3),

$$\psi^3 = \psi^2 + 1 \tag{3}$$

gives the suggestion of volume as indicated in Figure 7.

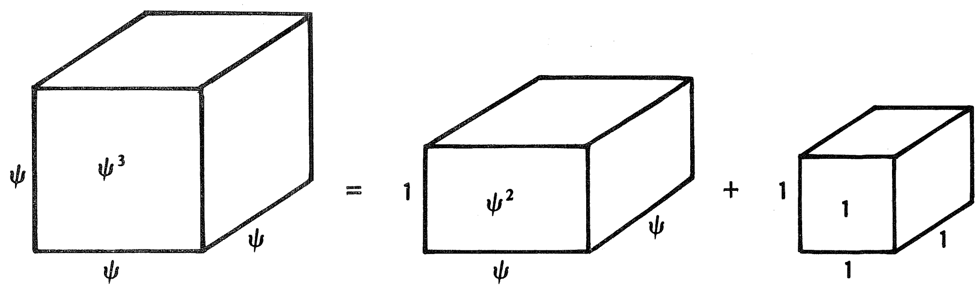


Fig. 7

The next property of the series limit is stated as a theorem.

Theorem: Given any triangle, choose any one of its sides and divide the length of that side by the factor ψ, the resulting length by ψ; and the final resulting length by ψ, so as to have three new lengths from the original side of the triangle. The three resulting lengths, when placed inside the triangle parallel to the side chosen will create equal perpendicular distances between the longest resulting length and side chosen as well as the shortest length of the vertex opposite the chosen side.

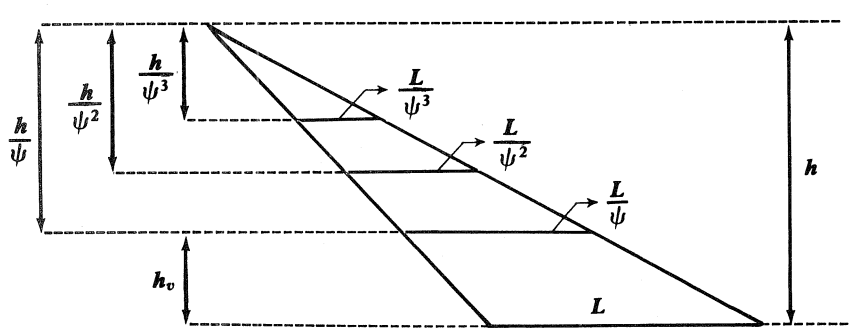


Fig. 8

Figure 8 represents one orientation of a given triangle with a side L. It is to be shown that

$$h_v = \frac{h}{\psi^3} \tag{25}$$

By similar triangles the height of the topmost triangle is $\frac{h}{\psi^3}$; the second is $\frac{h}{\psi^2}$; the third is $\frac{h}{\psi}$. It is easily seen that

$$h - \frac{h}{\psi} = h_v$$

or

$$h\left(1 - \frac{1}{\psi}\right) = h_v. \tag{26}$$

From the identity in (3),

$$\psi^3 - \psi^2 - 1 = 0, \tag{3}$$

then

$$1 = \psi^3 - \psi^2$$

and

$$\frac{1}{\psi^3} = 1 - \frac{1}{\psi}. \tag{27}$$

Substituting (27) into (26) yields

$$h_v = \frac{h}{\psi^3}. \tag{25}$$

A similar analysis can be used to prove the other orientations of the triangle.

Clarles Funk-Hellet [3], a French mathematician, constructed an additive series similar to Fibonacci's by replacing the second one in the series by a five and adding as in the original series. The series was developed into 36 rows, each row containing 18 entries. Table 1 illustrates Funk-Hellet's results in part.

Row 2 of the table contains the $1/\phi$, 1, ϕ, and ϕ^2 values, while the 14th entry of the 24th row yields a very precise value for π. The 7th column represents the one-eighth divisions of a circle. Other results were found by Funk-Hellet concerning other matters.

We construct a similar series using the series concerned in this paper by replacing the third one by six and adding as in the original series. Table 2 shows some of the results. This table was constructed in the same manner as Funk-Hellet's.

The 10th, 11th, and 12th entries of row 1 are values for $1/5\psi^3$, $1/5\psi^2$, and $1/5\psi$, respectively; the 14th through 18th places represent 2ψ, $2\psi^2$, $2\psi^3$, $2\psi^4$, and $2\psi^5$. The last entries of the 5th row give values for $1/\psi^5$, $1/\psi^4$, ..., $1/\psi$, 1, ψ, ..., ψ^4, ψ^5. The 9th entry of the 26th row represents $\phi - 1/2$; the 9th entry of the 29th row represents one-half the value of twice the height of the Great Pyramid less its base. The 6th entry of the 31st row yields the value for the log e.

One might wonder why Funk-Hellet chose to add the number five in his table and six was chosen in the newest case. It could be because the pentagon relates the Fibonacci limit and the hexagon relates the ψ-number limit. For whatever reason, the chosen numbers in conjunction with the related series to each yield some unexpected results.

As a final note:

$$\alpha_1 = \psi = \frac{1}{6}\left[4(29 + 3\sqrt{3}\sqrt{31})\right]^{1/3} + \frac{2}{3}\left[4(29 + 3\sqrt{3}\sqrt{31})\right]^{-1/3} + \frac{1}{3} \tag{6}$$

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Table 1

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII
1	5	6	11	17	28	45	73	118	191	309	500	809	1309	2118	3427	5545	8972
2	10	12	22	34	56	90	146	236	382	618	1000	1618	2618	4236	6854	11090	17944
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
24	120	144	264	408	672	1080	1752	2832	4584	7416	12000	19416	31416	50832	82248	133080	215328
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 2

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII
1	1	6	7	8	14	21	29	43	64	93	136	200	293	429	629	922	1351
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	5	30	35	40	70	105	145	215	320	465	680	1000	1465	2145	3145	4610	6755
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
26	26	156	182	208	364	546	754	1118	1664	2418	3536	5200	7618	11154	16354	23972	35126
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
29	29	174	203	232	406	609	881	1247	1856	2697	3944	5800	8497	12441	18241	26738	39179
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
31	31	186	217	248	434	651	899	1333	1984	2883	4216	6200	9083	13299	19499	28582	41881
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

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5. A special thanks to the Hewlett-Packard Company for the development of their HP-35 calculator. This paper would not have been possible without its use.

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SOME PROPERTIES OF A GENERALIZED FIBONACCI SEQUENCE MODULO m

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The Fibonacci sequence reduced by the modulus m has been examined by Wall [1], Dynkin and Uspenskii [2], and others. In this paper we investigate the generalized Fibonacci sequence $\{K_n\}$, where $K_0 = 0$, $K_1 = K_2 = 1$, and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \; n > 2. \tag{1}$$

We reduce $\{K_n\}$ modulo m, taking least nonnegative residues.

Definition: Let $h = h(m)$, where $h(m)$ denotes the number of terms in one period of the sequence $\{K_n\}$ modulo m before the terms start to repeat, be called the *length of the period* of $\{K_n\}$ (mod m).

Example: The values of $\{K_n\}$ (mod 7) are

0, 1, 1, 2, 4, 0, 6, 3, 2, 4, 2, 1, 0, 3, 4, 0, 0, 4, 4, 1, 2, 0, 3, 5,
1, 2, 1, 4, 0, 5, 2, 0, 0, 2, 2, 4, 1, 0, 5, 6, 4, 1, 4, 2, 0, 6, 1, 0,

and then repeat. Consequently, we conclude that $h(7) = 48$. Note that $K_{46} \equiv 1$, $K_{47} \equiv K_{48} \equiv 0$, $K_{49} \equiv 1 \pmod 7$. Hence the sequence has started to repeat when we reach the triple 1, 0, 0. Note also that $K_{15} \equiv K_{16} \equiv 0$, $K_{31} \equiv K_{32} \equiv 0 \pmod 7$, so that the 48 terms in one period are divided by adjacent double zeros into three sets of 16 terms each. This example illustrates a general principle contained in

Theorem 1: The sequence $\{K_n\}$ (mod m) forms a simply periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values 0, 1, 1.

Proof: If we consider any three consecutive terms in the sequence reduced modulo m, there are only m^3 possible such distinct triples. Hence at some point in the sequence, we have a repeated triple. A repeated triple results in the recurrence of K_0, K_1, K_2, for from the defining relation (1),

$$K_{n-2} = K_{n+1} - K_n - K_{n-1}.$$

Therefore, if

$$K_{t+1} \equiv K_{s+1}, \; K_t \equiv K_s, \text{ and } K_{t-1} \equiv K_{s-1} \pmod m,$$

then

$$K_{t-2} = K_{t+1} - K_t - K_{t-1} \equiv K_{s+1} - K_s - K_{s-1} \equiv K_{s-2} \pmod m$$

and, similarly (assuming that $t > s$),

$$
\begin{aligned}
K_{t-3} &\equiv K_{s-3} \pmod{m} \\
K_{t-4} &\equiv K_{s-4} \pmod{m}, \\
&\ldots\ldots\ldots\ldots \\
K_{t-s+2} &\equiv K_2 \pmod{m} = 1 \\
K_{t-s+1} &\equiv K_1 \pmod{m} = 1 \\
K_{t-s} &\equiv K_0 \pmod{m} = 0.
\end{aligned}
$$

Hence, any repeated triple implies a repeat of 0, 1, 1 and a return to the starting point of the sequence.

If h is the length of the period of $\{K_n\}$ (mod m), then clearly $K_h \equiv 0$, $K_{h+1} \equiv K_{h+2} \equiv 1$ (mod m). From the defining relation (1), it also follows that $K_{h-1} \equiv 0$, $K_{h-2} \equiv 1$, $K_{h-3} \equiv m-1$, and $K_{h-4} \equiv 0$ (mod m). We now list some identities for the sequence $\{K_n\}$ which will be useful in the sequel. These identities and their proofs may be found in [3].

$$K_{n+p} = K_n K_{p+1} + K_{n-1}(K_p + K_{p-1}) + K_{n-2}K_p,\ n \geq 2,\ p \geq 1; \tag{2}$$

$$K_{n+p} = K_{n-r}K_{p+r+1} + K_{n-r-1}(K_{p+r} + K_{p+r-1}) + K_{n-r-2}K_{p+r}, \quad n \geq 2,\ p \geq 1,\ -p+2 \leq r \leq n-1; \tag{3}$$

$$L_n = K_{n-1} + K_{n-2}; \tag{4}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} K_{n+1} & L_{n+1} & K_n \\ K_n & L_n & K_{n-1} \\ K_{n-1} & L_{n-1} & K_{n-2} \end{bmatrix},\ n \geq 2; \tag{5}$$

$$K_n^2 K_{n-3} + K_{n-1}^3 + K_{n-2}^2 K_{n+1} - K_{n+1}K_{n-1}K_{n-3} - 2K_nK_{n-1}K_{n-2} = 1,\ n \geq 3. \tag{6}$$

The following theorem gives an unusual property about the terms which immediately precede and follow adjacent double zeros in the sequence $\{K_n\}$ (mod m).

Theorem 2: If $K_n \equiv K_{n-1} \equiv 0$ (mod m), then $K_{n-2}^3 \equiv K_{n+1}^3 \equiv 1$ (mod m).

Proof: The fact that $K_{n-2}^3 \equiv K_{n+1}^3$ (mod m) follows from the defining relation (1) and the fact that $K_n \equiv K_{n-1} \equiv 0$ (mod m). To prove the other part, we observe by (6) that

$$K_{n-1}^2 K_{n-4} + K_{n-2}^3 + K_{n-3}^2 K_n - K_n K_{n-2} K_{n-4} - 2K_{n-1}K_{n-2}K_{n-3} = 1.$$

All terms on the left side of this equation, except K_{n-2}^3, are congruent to 0 modulo m. Hence we have

$$K_{n-2}^3 \equiv K_{n+1}^3 \equiv 1 \pmod{m}.$$

Theorem 3: If j is the least positive integer such that $K_{j-1} \equiv K_j \equiv 0 \pmod{m}$, then

(a) $K_{nj-1} \equiv K_{nj} \equiv 0 \pmod{m}$, for all positive integers n and

(b) if $K_{t-1} \equiv K_t \equiv 0 \pmod{m}$, then $t = nj$ for some positive integer n.

Proof of (a): The proof is by induction on n. For $n = 1$, the conclusion is immediate from the hypothesis. If we assume as induction hypothesis that $K_{nj-1} \equiv K_{nj} \equiv 0 \pmod{m}$, then by (2)

$$K_{(n+1)j} = K_{nj+j} = K_{nj}K_{j+1} + K_{nj-1}(K_j + K_{j-1}) + K_{nj-2}K_j \equiv 0 \pmod{m}.$$

A similar argument shows that $K_{nj-1} \equiv 0 \pmod{m}$.

Proof of (b): Let t be such that $K_t \equiv K_{t-1} \equiv 0 \pmod{m}$. We have $t > j$ since j was least such that $K_j \equiv K_{j-1} \equiv 0 \pmod{m}$. If j does not divide t, then by the division algorithm,

$$t = jq + r,\ 0 < r < j.$$

We have by (2),

$$K_t = K_{jq+r} = K_{jq}K_{r+1} + K_{jq-1}(K_{r-1} + K_r) + K_{jq-2}K_r \equiv 0 \pmod{m}.$$

But since $K_{jq} \equiv K_{jq-1} \equiv 0 \pmod{m}$, this equation implies that

$$K_{jq-2}K \equiv 0 \pmod{m}.$$

By Theorem 2,

$$K_{jq-2}^3 \equiv 1 \pmod{m},$$

which implies that no divisors of m divide K_{jq-2}. Thus,

$$K_r \equiv 0 \pmod{m}.$$

Similarly, we can show that

$$K_{r-1} \equiv 0 \pmod{m}.$$

But $r < j$, and so these last two congruences contradict the choice of j as least such that

$$K_j \equiv K_{j-1} \equiv 0 \pmod{m}.$$

The following theorem shows that in considering properties about the length of the period of $\{K_n\} \pmod{m}$ we can, without loss of generality, restrict the choice of m to p^t, where p is a prime and t a positive integer.

Theorem 4: If m has prime factorization

$$m = p_1^{t_1} p_2^{t_2} \ \ldots \ p_s^{t_s},$$

and if h_i denotes the length of the period of $\{K_n\}$ (mod $p_i^{t_i}$), then the length of the period of $\{K_n\}$ (mod m) is equal to l.c.m. $[h_i]$, the least common multiple of the h_i.

Proof: For all i, if h_i denotes the length of the period of $\{K_n\}$ (mod $p_i^{t_i}$),

$$K_{h_i - 1} \equiv K_{h_i} \equiv 0 \pmod{p_i^{t_i}}$$

$$K_{h_i - 2} \equiv K_{h_i + 1} \equiv 1 \pmod{p_i^{t_i}},$$

and also

$$K_{rh_i - 1} \equiv K_{rh_i} \equiv 0 \pmod{p_i^{t_i}},$$

$$K_{rh_i - 2} \equiv K_{rh_i + 1} \equiv 1 \pmod{p_i^{t_i}}$$

for all positive integers r. If j = l.c.m. $[h_i]$, it follows then that

$$K_j \equiv K_{j-1} \equiv 0 \pmod{m},$$

$$K_{j-2} \equiv K_{j+1} \equiv 1 \pmod{m}.$$

Conversely, if h is the length of the period of $\{K_n\}$ (mod m), then

$$K_h \equiv K_{h-1} \equiv 0 \pmod{m},$$

$$K_{h-2} \equiv K_{h+1} \equiv 1 \pmod{m},$$

which implies that for all i,

$$K_h \equiv K_{h-1} \equiv 0 \pmod{p_i^{t_i}},$$

$$K_{h-2} \equiv K_{h+1} \equiv 1 \pmod{p_i^{t_i}}.$$

By Theorem 3, $h = h_i r_i$ for all h_i and an appropriate r_i. That is, h is a common multiple of the h_i. By definition of h then, $h = j$, the l.c.m. $[h_i]$.

Theorem 5: If $K_t \equiv K_{t-1} \equiv 0 \pmod{m}$, then $K_{t-4} \equiv 0 \pmod{m}$.

Proof: By the defining relation (1) and the hypothesis, we have

$$K_t = K_{t-1} + K_{t-2} + K_{t-3} \equiv 0 \pmod{m}, \tag{7}$$

$$K_{t-1} = K_{t-2} + K_{t-3} + K_{t-4} \equiv 0 \pmod{m}. \tag{8}$$

Now subtracting (8) from (7), we have

$$K_{t-1} - K_{t-4} \equiv 0 \pmod{m},$$

or

$$K_{t-1} \equiv K_{t-4} \pmod{m}.$$

The next theorem gives an interesting transformation of a certain factor from the subscript to a power in moving from the modulus m to m^2 when the subscript is a specified function of the length of the period of $\{K_n\}$ (mod m). This theorem is useful in establishing the length of the period of $\{K_n\}$ (mod p^r) relative to the length of the period of $\{K_n\}$ (mod p), p a prime.

Theorem 6: If h is the length of the period of $\{K_n\}$ (mod m), then the following identities hold in terms of the modulus m^2.

$$K_{sh+1} \equiv K_{h+1}^{s} \pmod{m^2}, \tag{9}$$

$$K_{sh-1} \equiv sK_{h-2}^{s-1}K_{h-1} \pmod{m^2}, \tag{10}$$

$$K_{sh-2} \equiv K_{h-2}^{s} \pmod{m^2}, \tag{11}$$

$$K_{sh} \equiv (K_{h+1}^{s} - sK_{h-2}^{s-1}K_{h-1} - K_{h-2}^{s}) \pmod{m^2}. \tag{12}$$

Proof of (9): The proof is by induction on s. For $s = 1$, the conclusion is immediate. If we assume that

$$K_{sh+1} \equiv K_{h+1}^{s} \pmod{m^2},$$

then, by (2),

$$K_{(s+1)h+1} = K_{(sh+1)+h} = K_{sh+1}K_{h+1} + K_{sh}(K_h + K_{h-1}) + K_{sh-1}K_h.$$

Since h is the length of the period of $\{K_n\}$ (mod m), and also using Theorem 3, we have

$$K_h \equiv K_{h-1} \equiv K_{sh} \equiv K_{sh-1} \equiv 0 \pmod{m}.$$

But these congruences imply that

$$K_{sh}(K_h + K_{h-1}) \equiv K_{sh-1}K_h \equiv 0 \pmod{m^2},$$

which together with the induction hypothesis implies that

$$K_{(s+1)h+1} \equiv K_{sh+1}K_{h+1} \equiv K_{h+1}^{s}K_{h+1} \equiv K_{h+1}^{s+1} \pmod{m^2},$$

and the result is proved.

The proofs of (10) and (11) follow in a similar manner.

Proof of (12): Using the defining relation (1) and (9), (10), and (11), we have

$$K_{sh} = K_{sh+1} - K_{sh-1} - K_{sh-2} \equiv (K_{h+1}^s - sK_{h-2}^{s-1}K_{h-1} - K_{h-2}^s) \pmod{m^2}.$$

We return now to the question of the relation between the length of the period of $\{K_n\}$ (mod p) and the length of the period of $\{K_n\}$ (mod p^r) where r is an arbitrary positive integer. First, a preliminary theorem:

Theorem 7: If p is a prime and $h = h(p)$ is the length of the period of $\{K_n\}$ (mod p), then $K_{h+1}^p \equiv 1 \pmod{p^2}$.

Proof: If $K_{h+1} \equiv 1 \pmod{p^2}$, then $K_{h+1}^p \equiv 1 \pmod{p^2}$ trivially. If $K_{h+1} \not\equiv 1 \pmod{p^2}$, then

$$K_{h+1}^p - 1 = (K_{h+1} - 1)(K_{h+1}^{p-1} + K_{h+1}^{p-2} + \cdots + K_{h+1}^p + 1).$$

Now

$$K_{h+1} - 1 \equiv 0 \pmod{p} \tag{13}$$

and

$$K_{h+1}^s \equiv 1 \pmod{p}$$

for any s. Therefore,

$$K_{h+1}^{p-1} + K_{h+1}^{p-2} + \cdots + K_{h+1} + 1 \equiv 1 + 1 + \cdots + 1 \equiv 0 \pmod{p}. \tag{14}$$

Using (13) and (14), we see that

$$K_{h+1}^p - 1 \equiv 0 \pmod{p^2}.$$

We now state the main theorem.

Theorem 8: If p is a prime and $h(p^2) \neq h(p)$, then $h(p^r) = p^{r-1}h(p)$ for any positive integer $r > 1$.

Proof: We prove the case when $r = 2$. The general case follows in a similar manner by means of induction. Since $h = h(p)$ is the length of the period of $\{K_h\}$ (mod p), then using (5), we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^h = \begin{bmatrix} K_{\ +1} & L_{h+1} & K_h \\ K_h & L_{h+1} & K_{h-1} \\ K_{h-1} & L_{h-1} & K_{h-2} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \pmod{p},$$

where h is the smallest sech power for which this property holds. [The sequence $\{L_n\}$ is defined by (4).] Now also by (5),

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{hp} = \begin{bmatrix} K_{hp+1} & L_{hp+1} & K_{hp} \\ K_{hp} & L_{hp} & K_{hp-1} \\ K_{hp-1} & L_{hp} & K_{hp-2} \end{bmatrix}.$$

If $h(p) \neq h(p^2)$, then using (9), (10), and (12), the first column of the matrix on the right has values as follows:

$$K_{hp+1} \equiv K_{h+1}^p \pmod{p^2}, \tag{15}$$

$$K_{hp} \equiv (K_{h+1}^p - pK_{h-2}^{p-1}K_{h-1} - K_{h-2}^p) \pmod{p^2}, \tag{16}$$

$$K_{hp-1} \equiv pK_{h-2}^{p-1}K_{h-1} \pmod{p^2}. \tag{17}$$

By Theorem 7 and (15), it follows that

$$K_{hp+1} \equiv 1 \pmod{p^2}. \tag{18}$$

Using Theorem 7, (16), and (17), it follows that

$$K_{hp} \equiv K_{hp-1} \equiv 0 \pmod{p^2}, \tag{19}$$

From (18) and (19) we conclude that the length of the period of $\{K_n\}$ (mod p^2) is hp if

$$K_{t+1} \equiv 1 \pmod{p^2}, \tag{20}$$

$$K_t \equiv K_{t-1} \equiv 0 \pmod{p^2}, \tag{21}$$

for no $t < hp$. To see that this is indeed the case, we observe that since 20) and (21) also imply that

$$K_{t+1} \equiv 1 \pmod{p}, \tag{22}$$

$$K_t \equiv K_{t-1} \equiv 0 \pmod{p}, \tag{23}$$

then by Theorem 3, $t = hq$ for some q. Now assuming that (22) and (23) hold,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{t} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{hq} = \begin{bmatrix} K_{h+1} & L_{h+1} & K_{h} \\ K_{h} & L_{h} & K_{h-1} \\ K_{h-1} & L_{h-1} & K_{h-2} \end{bmatrix}^{q} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \pmod{p^2}$$

Since $h(p) \neq h(p^2)$,

$$A = \begin{vmatrix} K_{h+1} & L_{h+1} & K_{h} \\ K_{h} & L_{h} & K_{h-1} \\ K_{h-1} & L_{h-1} & K_{h-2} \end{vmatrix} \not\equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \pmod{p^2},$$

but by (18) and (19),

$$A^p \equiv A^q \equiv I \pmod{p^2}.$$

Since $A \not\equiv I \pmod{p^2}$ and p is prime, this implies that p divides q or $p \leq q$ and $t = hq \geq hp$. Thus hp is the length of the period of $\{K_n\} \pmod{p^2}$.

Whether the hypothesis $h(p) \neq h(p^2)$ is necessary or whether $h(p^2)$ can never equal $h(p)$ is an open question. No example of $h(p) = h(p^2)$ was found, yet a proof that none exists was not found either. In the event that k is largest such that $h(p^k) = h(p)$, it can be shown that $h(p^r) = p^{r-k}h(p)$.

Theorem 9: If $K_t \equiv K_{t-1} \equiv 0 \pmod{m}$, then $K_{st+1} \equiv K_{t+1}^s \pmod{m}$ for all positive integers s.

Proof: The proof is similar to the proof of (9).

The example illustrates that at the end of a period, the triple 1, 0, 0 occurs in the sequence $\{K_n\} \pmod{m}$. In the example, we also saw that adjacent double zeros (not necessarily preceded by 1) occur at equally spaced intervals throughout the period and that adjacent double zeros occur three times within the period. For one period of $\{K_n\} \pmod{3}$, we have

$$0,\ 1,\ 1,\ 2,\ 1,\ 1,\ 1,\ 0,\ 2,\ 0,\ 2,\ 1,\ 0$$

in which the adjacent double zeros occur once (at the beginning or end) of a period cycle. A general principle is given by

Theorem 10: If t is the least positive integer such that $K_t \equiv K_{t-1} \equiv 0 \pmod{m}$, then either $K_{t+1} \equiv 1 \pmod{m}$ or $K_{3t+1} \equiv 1 \pmod{m}$ and the length of the period is t or $3t$.

Proof: Suppose $K_{t+1} \equiv 1 \pmod{m}$. Then using (6) and $K_t \equiv K_{t-1} \equiv 0 \pmod{m}$, we have

$$\begin{aligned} 1 &= K_t^2K_{t-3} + K_{t-1}^3 + K_{t-2}^2K_{t+1} - K_{t+1}K_{t-1}K_{t-3} - 2K_tK_{t-1}K_{t-3} \\ &\equiv K_{t-2}^2K_{t+1} \equiv K_{t+1}^3 \pmod{m}, \end{aligned}$$

since $K_{t-2} \equiv K_{t+1} \pmod{m}$. By Theorem 9,

$$K_{3t+1} \equiv K_{t+1}^3 \pmod{m}$$

and so we have

$$K_{3t+1} \equiv 1 \pmod{m}$$

as required.

To show that $K_{2t+1} \not\equiv 1 \pmod{m}$ if $K_{t+1} \not\equiv 1 \pmod{m}$ we assume the contrary and observe that

$$K_{3t+1} \equiv K_{t+1}^3 \equiv 1 \equiv K_{2t+1} \equiv K_{t+1}^2 \pmod{m}.$$

Hence,

$$K_{t+1}^3 \equiv K_{t+1}^2 \pmod{m},$$

which implies that $K_{t+1} \equiv 1 \pmod{m}$ since the g.c.d. $(K_{t+1}^2, m) = 1$. This is a contradiction of our assumption that $K_{t+1} \not\equiv 1 \pmod{m}$.

Remark: If $K_{t-1} \equiv K_t \equiv 0 \pmod{m}$ and $K_{t+1} \not\equiv 1 \pmod{m}$, then we showed in the proof of Theorem 10 that

$$K_{t+1}^3 \equiv 1 \pmod{m}.$$

It also follows that

$$K_{2t+1}^3 \equiv 1 \pmod{m},$$

since

$$K_{2t+1}^3 \equiv (K_{t+1}^2)^3 \equiv (K_{t+1}^3)^2 \equiv 1^2 \equiv 1 \pmod{m}.$$

Theorem 10 and the Remark would imply that only integers n which can occur in the sequence $\{K_n\}$ (mod m) immediately preceding and following adjacent double zeros are such that

$$n \equiv 1 \pmod{m},$$

or

$$n^3 \equiv 1 \pmod{m}.$$

The Remark would also imply that if $n \not\equiv 1 \pmod{m}$, then there exist at least two distinct values n_1, n_2 such that

$$n_1^3 \equiv n_2^3 \equiv 1 \pmod{m},$$

where n_1, n_2 are the immediate predecessors and successors of adjacent double zeros in the sequence $\{K_n\}$ (mod m).

Theorem 11: If p is prime, $h = h(p)$, and $K_t \equiv K_{t-1} \equiv 0 \pmod{p}$ where $t < h$, then $h = 3t$ and

$$K_r + K_{r+t} + K_{r+2t} \equiv 0 \pmod{p}.$$

Proof: That $h = 3t$ is an immediate consequence of Theorem 9 since $t < h$. To prove the second statement, we have by (2),

$$K_{r+t} = K_{r+1}K_t + (K_r + K_{r-1})K_{t-1} + K_rK_{t-2} \equiv K_rK_{t-2} \pmod{p} \tag{24}$$

since $K_t \equiv K_{t-1} \equiv 0 \pmod{p}$.

$$K_{r+2t} = K_{r+t+1}K_t + (K_{r+t} + K_{r+t-1})K_{t-1} + K_{r+t}K_{t-2} \equiv K_{r+t}K_{t-2} \pmod{p} \tag{25}$$

$$\begin{aligned} K_{r+3t} &= K_{r+2t+1}K_t + (K_{r+2t} + K_{r+2t-1})K_{t-1} + K_{r+2t}K_{t-2} \\ &\equiv K_{r+2t}K_{t-2} \pmod{p}. \end{aligned} \tag{26}$$

Now adding the left and right sides of (24), (25), and (26) and using the fact that $K_{r+3t} \equiv K_r \pmod{p}$, we get

$$K_{r+t} + K_{r+2t} + K_r \equiv (K_r + K_{r+t} + K_{r+2t})K_{t-2} \pmod{p}. \tag{27}$$

Since $K_{t-2} \not\equiv 1 \pmod{p}$ and p is prime, (20) implies that

$$K_r + K_{r+t} + K_{r+2t} \equiv 0 \pmod{p}$$

as required.

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AN INVARIANT FOR COMBINATORIAL IDENTITIES

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Dedicated to Dr. Thomas L. Martin, Jr.

0. INTRODUCTION

By an invariant of a mathematical structure—a matrix, an equation, a field—we usually understand a relation, or a formula emerging from that structure—which remains unaltered if certain operations are performed on this structure. An invariant is, so to speak, the calling card of some mathematical pattern, it is a fixed focus around which the infinite elements of this pattern revolves. Matrices, the general quadratic, and many other mathematical configurations have their invariants. So do groups, if they are not simple. A prima donna invariant is the class number of algebraic number fields. She is far from having been unveiled. Some serenades have been sung to her from the quadratic, and to a much lesser extent, the cubic fields. Higher fields are absolutely taboo for their class number, and will probably remain so for many decades to come. With certain restrictions, also the set of fundamental units of an algebraic number field is an invariant.

This paper states a new invariant for all cubic fields. In a further paper a similar invariant will be stated for all algebraic number fields of any degree. Here the cubic case is singled out, and completely solved, since the technique, used in this paper, will carry over, step by step, to the general case. We shall outline the idea of this new invariant, as obtained here in the cubic case. Let e be any unit (not necessarily a fundamental one) of a cubic number field. Since e and e^{-1} are of third degree, both can be used as bases for the field. This must not be a minimal basis, so that we can put

$$e^v = x_v + y_v e + z_v e^2,\ x_v, y_v, z_v \in \mathbb{Z},\ v = 0, 1, \ldots,\ e^{-v} = r_v + s_v e^{-1} + t_v e^{-2}.$$

x_v and r_v are then calculated explicitly as arithmetic functions of v. From $e^v \cdot e^{-v} = 1$, we obtain the combinatorial identity

$$x_v = r_v^2 - r_{v-1} r_{v+1},$$

and this is an invariant, regardless of how the cubic field and one of its units is chosen. We also obtain a second invariant, viz.,

$$r_v = x_v^2 - x_{v-1} x_{v+1}.$$

Few invariants can please better the heart of a mathematician.

1. POWERS OF UNITS

Let

$$F(x) = x^3 + c_1x^2 + c_2x + c_3; \; c_1, c_2, c_3 \in \mathbb{Z} \tag{1.1}$$

be an irreducible polynomial in x over $\mathbb{Z}$ of negative discriminant, having one real root w, and one pair of conjugate roots. By Dirichlet's theorem, $Q(w)$ has exactly one fundamental unit e, viz.,

$$e = r_1 + r_2w + r_3w^2; \; r_1, r_2, r_3 \in Q.$$

Of course, e is a third-degree algebraic irrational. Since

$$0 = w^3 + c_1w^2 + c_2w + c_3,$$

we find the field equation of e by the known method

$$\begin{aligned} e &= r_1 + r_2w + r_3w^2, \\ we &= r_1' + r_2'w + r_3'w^2, \; (r_1', r_2', r_3' \in Q) \\ w^2e &= r_1'' + r_2''w + r_3''w^2, \; (r_1'', r_2'', r_3'' \in Q) \end{aligned}$$

and obtain

$$\begin{cases} e^3 - a_1e^2 - a_2e - a_3 = 0, \\ a_1, a_2, a_3 \in \mathbb{Z}, \; a_3 = \pm 1. \end{cases} \tag{1.2}$$

Here we investigate, w.e.g., the case $a_3 = 1$, hence

$$\begin{aligned} &e^3 - a_1e^2 - a_2e - 1 = 0 \\ &e^3 = 1 + a_2e + a_1e^2; \; a_1, a_2 \neq 0, \text{ by presumption.} \end{aligned} \tag{1.2a}$$

Our further aim is to obtain explicit expressions for the positive and negative powers of e. To achieve this, we take refuge to a very convenient trick which makes the calculations uncomparably easier. We use as a basis for $Q(w)$ the triples 1, e, e^2 and 1, e^{-1}, e^{-2}; the question whether these are minimal bases is not relevant here. We put

$$e^v = x_v + y_ve + z_ve^2; \;\; x_v, y_v, z_v \in \mathbb{Z}; \; v = 0, 1, \ldots, \tag{1.3}$$

$$x_0 = 1, \; x_1 = x_2 = 0. \tag{1.3a}$$

We obtain from (1.3), multiplying by e, and with (1.2a)

$$\begin{aligned} e^{v+1} &= x_ve + y_ve^2 + z_v(1 + a_2e + a_1e^2) \\ &= z_v + (x_v + a_2z_v)e + (y_v + a_1z_v)e^2 \\ &= x_{v+1} + y_{v+1}e + z_{v+1}e^2, \end{aligned}$$

Hence, by comparison of coefficients,

$$\begin{aligned} x_{v+1} &= z_v, \\ y_{v+1} &= x_v + a_2 z_v, \\ z_{v+1} &= y_v + a_1 z_v. \end{aligned}$$

Thus, we obtain

$$\begin{cases} y_v = x_{v-1} + a_2 x_v; \quad (v = 1, 2, \ldots) \\ z_v = x_{v+1} \\ e^v = x_v + (x_{v-1} + a_2 x_v)e + x_{v+1}e^2 \end{cases} \tag{1.4}$$

$$x_{v+2} = x_{v-1} + a_2 x_v + a_1 x_{v+1};$$

$$x_{v+3} = x_v + a_2 x_{v+1} + a_1 x_{v+2}; \quad (v = 0, 1, \ldots). \tag{1.4a}$$

Formula (1.4a) is the recurrence relation which will enable us to calculate explicitly x_v, and with it e^v. We set

$$\sum_{v=0}^{\infty} x_v u^v = x_0 + x_1 u + x_2 u^2 + \sum_{v=3}^{\infty} x_v u^v,$$

and, with the initial values from (1.3a),

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \sum_{v=3}^{\infty} x_v u^v = 1 + \sum_{v=0}^{\infty} x_{v+3} u^{v+3}. \tag{1.4b}$$

Substituting on the right side the value of x from (1.4a) [and taking into account (1.3a)], we obtain

$$\begin{aligned} \sum_{v=0}^{\infty} x_v u^v &= 1 + \sum_{v=0}^{\infty} (x_v + a_2 x_{v+1} + a_1 x_{v+2}) u^{v+3} \\ &= 1 + u^3 \sum_{v=0}^{\infty} x_v u^v + a_2 u^2 \sum_{v=0}^{\infty} x_{v+1} u^{v+1} + a_1 u \sum x_{v+2} u^{v+2} \\ &= 1 + u^3 \sum_{v=0}^{\infty} x_v u^v + a_2 u^2 \left[\left(\sum_{v=0}^{\infty} x_v u^v\right) - x_0\right] + a_1 u \left[\left(\sum_{v=0}^{\infty} x_v u^v\right) - x_0 - x_1 u\right] \\ &= 1 + (u^3 + a_2 u^2 + a_1 u) \sum_{v=0}^{\infty} x_v u^v - a_2 u^2 - a_1 u. \end{aligned}$$

We have thus obtained

$$(1 - a_1 u - a_2 u^2 - u^3) \sum_{v=0}^{\infty} x_v u^v = 1 - a_1 u - a_2 u^2. \tag{1.4c}$$

Since u is an indeterminate, and can assume any value, we choose

$$1 - a_1 u - a_2 u^2 - u^3 \neq 0, \tag{1.4d}$$

and obtain from (1.4c) and (1.4d)

$$\sum_{v=0}^{\infty} x_v u^v - \frac{1 - a_1 u - a_2 u^2}{1 - a_1 u - a_2 u^2 - u^3},$$

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

and from (1.4b)

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \sum_{v=0}^{\infty} x_{v+3} u^{v+3} = 1 + \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

$$\sum_{v=0}^{\infty} x_{v+3} u^{v+3} = \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

and since $u \neq 0$,

$$\sum_{v=0}^{\infty} x_{v+3} u^v = \frac{1}{1 - u(a_1 + a_2 u + u^2)}. \tag{1.4e}$$

Choosing, additionally to (1.4d),

$$0 < |u(a_1 + a_2 u + u^2)| < 1,$$

we obtain, from (1.4d)

$$\sum_{v=0}^{\infty} x_{v+3} u^v = \sum_{j=0}^{\infty} u^j (a_1 + a_2 u + u^2)^j. \tag{1.5}$$

To calculate x_v explicitly, we shall compare the coefficients of u^m ($m = 0, 1, \ldots$) on each side of (1.5). On the left, this equals to x_{m+3}. On the right side we investigate

$$\left\{\begin{aligned} &\sum_{i=0} u^{m-i}(a_1 + a_2 u + u^2)^{m-i} = \\ &\sum_{i=0}^{m-i} u^{m-i} \sum_{y_1 + y_2 + y_3 = m-i} \binom{m-i}{y_1, y_2, y_3} a_1^{y_1} (a_2 u)^{y_2} u^{2y_3}. \end{aligned}\right. \tag{1.5a}$$

Since we demand that the element u have the exponent m, we obtain

$$m - i + y_2 + 2y_3 = m.$$

$$y_2 + 2y_3 = i, \tag{1.5b}$$

$$y_2 = i - 2y_3, \tag{1.5c}$$

and

$$y_1 + y_2 + y_3 = m - i \tag{1.5d}$$

yield

$$y_1 = m - i - i + 2y_3 - y_3,$$

$$y_1 = m - 2i + y_3. \tag{1.5e}$$

We further have

$$\binom{m-i}{y_1, y_2, y_3} = \frac{(m-i)!}{y_1!y_2!y_3!} = \frac{(m-i)!}{(m-2i+y_3)!(i-2y_3)!y_3!}$$

$$= \frac{(m-i)!(i-y_3)!}{(i-y_3)!(m-2i+y_3)!(y-2y_3)!y_3!}$$

$$= \binom{m-i}{i-y_3}\binom{i-y_3}{y_3},$$

$$\binom{m-i}{y_1, y_2, y_3} = \binom{m-i}{i-y_3}\binom{i-y_3}{y_3}. \tag{1.5f}$$

Writing j for y_3, we thus obtain

$$x_{m+3} = \sum_{i=0}\sum_{j=0}\binom{m-i}{i-j}\binom{i-j}{j}a_1^{m-2i+j}a_2^{i-2j}. \tag{1.5g}$$

We shall determine the upper bounds of i and j. From the binomial coefficient $\binom{i-j}{j}$, we obtain

$$j \le i - j, \quad 2j \le i, \quad j \le \frac{i}{2}; \tag{1.5h}$$

From the binomial coefficient $\binom{m-i}{i-j}$, we obtain

$$m - i \ge i - j, \quad m - 2i \ge -j,$$

and from (1.5h), $-j \ge -\frac{i}{2}$, so that

$$m - 2i \ge -\frac{i}{2}, \quad m \ge \frac{3}{2}i, \quad i \le \frac{2}{3}m. \tag{1.5i}$$

From (1.5h) and (1.5i), we have thus obtained

$$i \le \left[\frac{2m}{3}\right]; \quad j \le \left[\frac{i}{2}\right],$$

hence,

$$x_{m+3} = \sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{m-2i+j} a_2^{i-2j}; \quad (m = 0, 1, \ldots) \tag{1.6}$$

We shall verify formula (1.6) which does not lack harmony in its simple structure. From (1.3a) and (1.4a), we obtain, for $v = 0, 1, \ldots,$

$$\begin{aligned} x_3 &= 1, \\ x_4 &= a_1, \\ x_5 &= a_2 + a_1^2, \\ x_6 &= 1 + 2a_1a_2 + a_1^3. \end{aligned}$$

From (1.6), we obtain, for $m = 0, 1, 2, 3$,

$m = 0,\ x_3 = 1,$ since $i = j = 0,\ \binom{0}{0} \overset{\text{def}}{=} 1.$

$m = 1;\ i = j = 0,\ x_4 = a_1;$

$m = 2;\ i = 0,\ j = 0;\ i = 1,\ j = 0,\ x_5 = a_1^2 + a_2;$

$m = 3;\ i = 0,\ j = 0;\ i = 1,\ j = 0;\ i = 2,\ j = 1,\ x_6 = a_1^2 + 2a_1a_2 + 1.$

We shall proceed to calculate the negative powers of e, and put

$$e^{-v} = r_v + s_v e^{-1} + t_v e^{-2}. \tag{1.7}$$

For the initial values, we obtain again

$$v = 0, 1, 2;\quad r_0 = 1;\quad r_1 = r_2 = 0. \tag{1.8}$$

For the field equation of e^{-1}, we obtain, from (1.2a),

$$e^{-3} = 1 - a_1e^{-1} - a_2e^{-2};\ a_1,\ a_2 \neq 0. \tag{1.9}$$

If we compare (1.9) with (1.2a), we see that the recursion formula for e^{-v}, with the same initial values for $v = 0, 1, 2$, is the same as that for e^v, substituting only $-a_1$ for a_2 and $-a_2$ for a_1; hence we obtain, in complete analogy with (1.4), (1.4a), and (1.6),

$$\left\{\begin{aligned} s_v &= r_{v-1} - a_1r_v, \\ t_v &= r_{v+1} \\ e^{-v} &= r_v + (r_{v-1} - a_1r_v)e^{-1} + r_{v+1}e^{-2} \\ r_{v+3} &= r_v - a_1r_{v+1} - a_2r_{v+2} \end{aligned}\right. \tag{1.9a}$$

$$r_{m+3} = \sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j}(-a_2)^{m-2i+j}(-a_1)^{i-2j}$$

$$r_{m+3} = \sum_{i=0}^{[\frac{2m}{3}]} \sum_{j=0}^{[\frac{i}{2}]} (-1)^{m-i-j} \binom{m-i}{i-j} \binom{i-j}{j} a^{i-2j} a^{m-2i+j}; \quad (m = 0, 1, \ldots). \tag{1.9b}$$

Formulas (1.6) and (1.9b) are our main tools in establishing new identities of combinatorial structures. Both x_m and r_m are arithmetic functions, and we shall show that there exist simple relations between them.

2. TRUNCATED FIELD EQUATIONS OF UNITS

We shall now drop the restriction (1.2a), viz., $a_1, a_2 \neq 0$, and investigate the cases when either a_1 or a_2 equal zero. We shall start with

$$a_2 = 0, \ e^3 = 1 + a_1 e^2; \quad a_1 \neq 0. \tag{2.1}$$

$$(e \text{ a cubic unit; } a_1 \in \mathbb{Z})$$

Formulas (1.9) take the form, setting

$$e^v = x_v + y_v e + z_v e^2 \ (v = 0, 1, \ldots; \ x_v, y_v, z_v \in \mathbb{Z}) \tag{2.2}$$

$$\begin{cases} y_v = x_{v-1}, \\ z_v = x_{v+1}, \\ e^v = x_v + x_{v-1} e + x_{v+1} e^2, \ (v = 1, 2, \ldots) \end{cases} \tag{2.2a}$$

and (1.4a) becomes

$$x_{v+2} = x_v + a_1 x_{v+3}, \ (x_0 = 1, \ x_1 = x_2 = 0; \ v = 0, 1, \ldots). \tag{2.3}$$

To calculate x_v explicitly from (2.3), there is no need to go through the whole process of using Euler's generating functions. Instead, we can proceed straight to formula (1.5a). Here we shall then keep in mind though, that the condition $a_2 = 0$ results in $y_2 = 0$, and we obtain

$$\sum_{i=0} u^{m-i}(a_1 + u\ )^{m-i} = \sum_{i=0} u^{m-i} \sum_{j=0} \binom{m-i}{j} a_1^{m-i-j} u^{2j} \ . \tag{2.4}$$

Since we are looking for powers u^m, we obtain

$$m - i + 2j = m,$$

$$i = 2j; \tag{2.4a}$$

since, from the binomial coefficient on the right side of (2.4),

$$m - i \geq j, \quad m - 2j \geq j, \quad j \leq \frac{m}{3} \tag{2.4b}$$

and formula (1.6) takes here the final form,

$$x_{m+3} = \sum_{j=0}^{[\frac{m}{3}]} \binom{m-2j}{j} a_1^{m-3j}, \quad m = 0, 1, \ldots . \tag{2.5}$$

Instead of proceeding to calculate the negative powers of e for the case $a_2 = 0$ from $e^3 = 1 + a_1e^2$, we shall first calculate the positive powers of e for the case $a_1 = 0$. The reasons for this will become clear in the sequel. We set again

$$\begin{cases} e^v = x_v + y_ve + z_ve^2; \; x_0 = 1; \; x_1 = x_2 = 0 \\ e^3 = 1 + a_2e; \; y_v = x_{v-1} + a_2x_v; \; z_v = x_{v+1}; \; x_{3+v} = x_v + a_2x_{v+1}; \; a_2 \neq 0. \end{cases} \tag{2.6}$$

(1.5) now takes the form

$$\sum_{v=0}^{\infty} x_{v+3}u^v = \sum_{j=0}^{\infty} u^{2j}(a_2 + u)^j. \tag{2.6a}$$

It is convenient to calculate x_{2m+3} and x_{2m+4} separately because of the factor u^{2j} under the second sigma sign. Because of the factor u^{2j}, we shall calculate separately the coefficients of u^{2m} ($v = 2m$) and u^{2m+1} ($v = 2m+1$). We obtain, after easy calculations,

$$\begin{cases} x_{2m+3} = \displaystyle\sum_{i=0}^{[\frac{m}{3}]} \binom{m-i}{2i} a_2^{m-3i}, \qquad (m = 0, 1, \ldots) \\ x_{2m+4} = \displaystyle\sum_{i=0}^{[\frac{m-i}{3}]} \binom{m-i}{2i+1} a_2^{m-3i-1}, \quad (m = 1, 2, \ldots) \\ (x_4 = 0). \end{cases} \tag{2.6b}$$

We can now easily calculate the negative powers for e^v in the cases $a_1 = 0$, and $a_2 = 0$. In the case $a_1 = 0$, we obtain, from (2.6)

$$\begin{cases} e^{-v} = r_v + s_ve^{-1} + t_ve^{-2}, \\ e^{-3} = 1 - a_2e^{-2}, \end{cases} \tag{2.7}$$

and from (2.2a),

$$e^{-v} = r_v + r_{v-1}e^{-1} + r_{v+1}e^{-2}, \tag{2.7a}$$

and from (2.5),

$$r_{v+3} = \sum_{j=0}^{[\frac{m}{3}]} \binom{m-2j}{j} (-a_2)^{m-3j}$$

$$r_{v+3} = \sum_{j=0}^{\left[\frac{m}{3}\right]} \binom{m-2j}{j}(-1)^{m-j}a_2^{m-3j}, \quad m = 0, 1, \ldots . \tag{2.7b}$$

In the case of $a_2 = 0$, we obtain, from (2.1),

$$\begin{cases} e^{-v} = r_v + s_v e^{-1} + t_v e^{-2} \\ e^{-3} = 1 - a_1 e^{-1}, \end{cases} \tag{2.7c}$$

and from (2.6),

$$e^{-v} = r_v + (r_{v-1} - a_1 r_v)e^{-1} + r_{v+1}e^{-2},$$

and from (2.6b)

$$r_{2m+3} = \sum_{i=0}^{\left[\frac{m}{3}\right]} \binom{m-i}{2i}(-a_1)^{m-3i}, \qquad (m = 0, 1, \ldots)$$

$$\underline{r_{2m+3} = \sum_{i=0}^{\left[\frac{m}{3}\right]} (-1)^{m-i}\binom{m-i}{2i}a_1^{m-3i}}, \qquad (m = 0, 1, \ldots) \tag{2.7d}$$

$$\underline{r_{2m+4} = \sum_{i=0}^{\left[\frac{m-i}{3}\right]} (-1)^{m-i-1}\binom{m-i}{2i+1}a_1^{m-3i-1}}, \quad (m = 1, 2, \ldots). \tag{2.7e}$$

3. COMBINATORIAL IDENTITIES

In this section, we shall establish the new combinatorial identities, by means of the powers of the units which we have stated explicitly in §1. We shall enumerate the main results we have obtained there in order to save the reader unnecessary backpaging.

$$\begin{cases} e^3 = 1 + a_2 e + a_1 e^2;\ a_1,\ a_2 \neq 0 \text{ (by presumption)}; \\ e^v = x_v + y_v e + z_v e^2;\ x_v, y_v, z_v, a_1, a_2 \in \mathbb{Z}; \\ y_v = x_{v-1} + a_2 x_v;\ z_v = x_{v+1}; \\ x_{v+3} = x_v + a_2 x_v \quad + a_1 x_v \quad ;\ x_0 = 1;\ x_1 = x_2 = 0; \\ e^{-v} = r_v + s_v e^{-1} + t_v e^{-2};\ r_v, s_v, t_v \in \mathbb{Z}; \\ s_v = r_{v-1} - a_1 r_v;\ t_v = r_{v+1}; \\ r_{v+3} = r_v - a_1 r_{v+1} - a_2 r_{v+2};\ r_0 = 1,\ r_1 = r_2 = 0; \\ e^{-3} = 1 - a_1 e^{-1} - a_2 e^{-2}. \end{cases} \tag{3.1}$$

From the last equation of (3.1), multiplying both sides first by e^2, then by e^{-1}, we obtain

$$\begin{aligned}
e^{-1} &= -a_2 - a_1 e + e^2, \\
e^{-2} &= -a_2(-a_2 - a_1 e + e^2) - a_1 + e, \\
e^{-1} &= -a_2 - a_1 e + e^2;\ e \quad = a_2^2 - a_1(a_1 a_2 + 1)e - a_2 e^2.
\end{aligned} \tag{3.2}$$

We now obtain, from (3.1) and (3.2),

$$\left\{\begin{aligned}
1 = e^v e^{-v} &= (x_v + y_v e + z_v e^2)(r_v + s_v e^{-1} + t_v e^{-2}) \\
&= x_v r_v + y_v s_v + z_v t_v + (y_v r_v + z_v s_v)e + z_v r_v e^2 \\
&\qquad + (x_v s_v + y_v t_v)e^{-1} + x_v t_1 e^{-2} \\
&= x_v r_v + y_v s_v + z_v t_v + (y_v r_v + z_v s_v)e + z_v r_v e^2 \\
&\qquad + (x_v s_v + y_v t_v)(-a_2 - a_1 e + e^2) \\
&\qquad + [a_2^2 - a_1 + (a_1 a_2 + 1)e - a_2 e^2]x_v t_v.
\end{aligned}\right. \tag{3.2a}$$

$$\left\{\begin{aligned}
1 = x_v r_v &+ (y_v - a_2 x_v)s_v + [z_v + (a_2^2 - a_1)x_v - a_2 y_v]t_v \\
&+ \big(y_v r_v + (z_v - a_1 x_v)s_v + [(a_1 a_2 + 1)x_v - a_1 y_v]t_v\big)e \\
&+ \big((z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v\big)e^2.
\end{aligned}\right. \tag{3.2b}$$

Comparing in (3.2b) coefficients of equal powers of e on both sides, and reminding that e is a cubic irrational, we obtain the system of three linear equations in the three indeterminates r_v, s_v, and t_v,

$$\left\{\begin{aligned}
&x_v r_v + (y_v - a_2 x_v)s_v + [z_v + (a_2^2 - a_1)x_v - a_2 y_v]t_v = 1, \\
&y_v r_v + (z_v - a_1 x_v)s_v + [(a_1 a_2 + 1)x_v - a_1 y_v]t_v = 0, \\
&z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v = 0.
\end{aligned}\right. \tag{3.3}$$

Adding to the first equation of (3.3) the a_2 multiple of the third one, we obtain, adding also to the second the a_1 multiple of the third,

$$\left\{\begin{aligned}
&(x_v + a_2 z_v)r_v + y_v s_v + (z_v - a_1 x_v)t_v = 1, \\
&(y_v + a_1 z_v)r_v + z_v s_v + x_v t_v = 0, \\
&z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v = 0.
\end{aligned}\right. \tag{3.3a}$$

Since the indeterminates r_v, s_v, t_v are to be expressed by x_v, y_v, z_v, we calculate the determinant Δ_{v+2} of the system (3.3a), viz.,

$$\begin{vmatrix}
x_v + a_2 z_v & y_v & z_v - a_1 x_v \\
y_v + a_1 z_v & z_v & x_v \\
z_v & x_v & y_v - a_2 x_v
\end{vmatrix} = \Delta_{v+2} \tag{3.3b}$$

Why this determinant has the index $v + 2$, and not v, as would seem proper, will be understood, and justified, soon.

We have, for the first row of the determinant (3.3b) from (3.1), and similarly for the second and third

$$\begin{cases} x_v + a_2 z_v = x_v + a_2 x_{v+1}, \\ y_v = x_{v-1} + a_2 x_v, \\ z_v - a_1 x = x_{v+1} - a_1 x_v = x_{v-2} + a_2 x_{v-1}. \end{cases} \tag{3.3c}$$

With (3.3c), (3.3b) becomes

$$\Delta_{v+2} = \begin{vmatrix} x_v + a_2 x_{v+1} & x_{v-1} + a_2 x_v & x_{v-2} + a_2 x_{v-2} \\ x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \end{vmatrix}. \tag{3.3d}$$

The third row of (3.3d) is obtained from (3.1) as follows:

$$z_v = x_{v+1};\; y_v - a_2 x_v = x_{v-1} + a_2 x_v - a_2 x_v = x_{v-1};$$

the first entry of the second row is obtained as follows:

$$y_v + a_1 x_v = x_{v-1} + a_2 x_v + a_1 x_{v+1} = x_{v+2}.$$

Subtracting in the determinant of (3.3d) from the first row the a_2-multiple of the third row, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_v & x_{v-1} & x_{v-2} \\ x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \end{vmatrix}. \tag{3.3e}$$

Interchanging in (3.3e) the first row with the second, and then the second with the third, we finally obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3f}$$

Substituting for the entries of the first row of (3.3f) the value from (3.1), viz.,

$$x_{k+3} = x_k + a_2 x_{k+1} + a_1 x_{k+2},\; (k = k+2,\; v+1,\; v)$$

$$\Delta_{v+2} = \begin{vmatrix} x_{v-1} + a_2 x_v + a_1 x_{v+1} & x_{v-2} + a_2 x_{v-1} + a_1 x_v & x_{v-3} + a_2 x_{v-2} + a_1 x_{v-1} \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3g}$$

Subtracting in (3.3g) from the first row the a_1-multiple of the second, and the a_2-multiple of the third, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v-1} & x_{v-2} & x_{v-3} \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3h}$$

Interchanging in (3.3h) the first row with the second, and then the second with the third, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \\ x_{v-1} & x_{v-2} & x_{v-3} \end{vmatrix}. \tag{3.3i}$$

From (3.3f) and (3.3i) we obtain the important result

$$\Delta_{v+2} = \Delta_{v+1} = \Delta_k; \; (k = 5, 6, \ldots). \tag{3.4}$$

Taking in (3.4) $k = 5$, and reminding, from (3.1), that $x_3 = 1$, $x_1 = x_2 = 0$, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_5 & x_4 & 1 \\ x_4 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1,$$

$$\Delta_{v+2} = -1. \tag{3.4a}$$

With (3.4a) we have finally calculated the determinant of the system of equations (3.3a). By Cramer's rule we now obtain from (3.3a) and (3.4a),

$$r_v = -\begin{vmatrix} z_v & x_v \\ x_v & y_v - a_2 x_v \end{vmatrix} \tag{3.5}$$

Substituting in (3.5),

$$z_v = x_{v+1}, \; y_v = x_{v-1} + a_2 x_v,$$

we obtain

$$\underline{\underline{r_v = x_v^2 - x_{v-1}x_{v+1}}}; \; (v = 1, 2, \ldots). \tag{3.6}$$

(3.6) is the desired combinatorial identity. Its full beauty will be appreciated when we substitute the values for r_v and x_v. Its simple structure in the form (3.6) is really astonishing. We must explain its remarkable origin. The reason for this harmoniousness is the fact that we have chosen to manipulate with the powers of a unit e in $Q(w)$ and a basis of the powers of e as the basis of $Q(w)$. For only this leads to the determinant Δ_{v+2} of the system of equations (3.3a) equal to ± 1. Had we chosen any other cubic irrational α in $Q(w)$, then the identity $\alpha^v \cdot \alpha^{-v} = 1$ would have led to a system of equations whose determinant is generally different from ±1. Formula (3.6) is therefore, in a certain sense, *an invariant of any cubic field* Q(w), *read of all the cubic fields*. The surprising explanation for this is the relationship

$$\Delta_{v+2} = N(e^v) = \left(N(e)\right)^v = \pm 1; \ (v = 0, 1, \ldots). \tag{3.7}$$

We shall prove (3.7). We obtain, denoting

$$\begin{aligned}
\alpha &= e^v = x_v + y_v e + z_v e^2; \ e^3 = 1 + a_2 e + a_1 e^2; \\
\alpha e &= x_v e + y_v e^2 + z_v(1 + a_2 e + a_1 e^2), \\
\alpha e &= z_v + (x_v + a_2 z_v)e + (y_v + a_1 z_v)e^2, \\
\alpha e^2 &= y_v + a_1 z_v + [a_2 y_v + (a_1 a_2 + 1)z_v]e + [x_v + a_1 y_v + (a_1^2 + a_2)z_v]e^2.
\end{aligned}$$

We thus obtain

$$\left\{\begin{array}{l} N(\alpha) = \\ (-1)^3 \begin{vmatrix} x_v & y_v & z_v \\ z_v & x_v + a_2 x_v & y_v + a_1 z_v \\ y_v + a_1 z_v & a_2 y_v + (a_1 a_2 + 1)z_v & x_v + a_1 y_v + (a_1^2 + a_2)z_v \end{vmatrix}. \end{array}\right. \tag{3.7a}$$

Subtracting in the determinant (3.7a) from the third row the a_1-multiple of the second row, we obtain

$$N(e^v) = -\begin{vmatrix} x_v & y_v & z_v \\ z_v & x_v + a_2 x_v & y_v + a_1 z_v \\ y_v & a_2 y_v - a_1 x_v + z_v & x_v + a_2 z_v \end{vmatrix},$$

and subtracting from the second column the a_2-multiple of the first column,

$$N(e^v) = -\begin{vmatrix} x_v & y_v - a_2 x_v & z_v \\ z_v & x_v & y_v + a_1 z_v \\ y_v & z_v - a_1 x_v & x_v + a_2 z_v \end{vmatrix}. \tag{3.7b}$$

Comparing (3.3b) with (3.7b), we obtain

$$N(e^v) = \Delta_{v+2} = 1. \tag{3.7c}$$

Had we chosen any $\alpha \in Q(w)$, formula (3.6) would take the form

$$N(\alpha)r_v = x_v^2 - x_{v-1}x_{v+1}, \tag{3.7d}$$

where r_v and x_v have similar meanings as before, our invariant (3.6) would be dependent on α. The corresponding combinatorial identity would be deprived of its beautiful structure. But, of course, in such a way we can obtain infinitely many combinatorial identities (3.7d) for any cubic irrational in $Q(w)$. Of course, every time $1, \alpha, \alpha^2$ and $1, \alpha^{-1}, \alpha^{-2}$ are to be taken as bases for $Q(w)$.

Now, returning to the powers e^v and e^{-v} in the general cubic cases, the reader will understand that, in principle, there is no structural difference if, in the system of linear equations (3.3a), we take x_v, y_v, z_v as indeterminates and r_v, s_v, t_v as coefficients. Carrying out the same calculations, we would then arrive at a formula, completely analogous to (3.6), viz.,

$$\underline{\underline{x_v = r_v^2 - r_{v-1}r_{v+1}}}; \quad (v = 1, 2, \ldots). \tag{3.8}$$

We shall verify (3.8) for a few values of v. We calculate from (3.1), viz.,

$$\begin{aligned}
x_{v+3} &= x_v + a_2 x_{v+1} + a_1 x_{v+2};\ x_0 = 1,\ x_1 = x_2 = 0;\\
r_{v+3} &= r_v - a_1 r_{v+1} - a_2 r_{v+2};\ r_0 = 1,\ r_1 = r_2 = 0.\\
x_4 &= a_1;\ x_5 = a_2 + a_1;\ x_6 = 1 + 2a_1a_2 + a_1^3.\\
r_4 &= -a_2;\ r_5 = -a_1^2 + a_1;\ r_6 = 1 + 2a_1a_2 - a_2^3.
\end{aligned}$$

$$\begin{cases} x_4 = r_4^2 - r_3 r_5, \\ a_1 = a_2^2 - 1(-a_1 + a_2^2) = a_1 \end{cases}$$

$$x_5 = r_5^2 - r_4 r_6,$$

$$\begin{cases} a_2 + a_1^2 = (-a_1 + a_2^2)^2 - (-a_2)(1 + 2a_1a_2 - a_2^3) \\ \qquad = a_1^2 - 2a_1a_2^2 + a_2^4 + a_2 + 2a_1a_2^2 - a_2^4 = a_2 + a_1^2. \end{cases}$$

It exposes the complicated structure of formulas (3.6) and (3.8), if we write out in full these combinatorial identities and substitute the corresponding values for x_v and r_v. We obtain from (1.6) and (1.9b)

$$r_{m+3} = x_{m+3}^2 - x_{m+2}x_{m+4}, \quad (m = 0, 1, \ldots)$$

$$\begin{cases}
\displaystyle\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{i-2j} a_2^{m-2i+j} \\
\displaystyle = \left[\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{m-2i+j} a_2^{i-2j}\right]^2 \\
\displaystyle - \left[\sum_{i=0}^{\left[\frac{2(m-1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-1-i}{i-j}\binom{i-j}{j} a_1^{m-1-2i+j} a_2^{i-2j}\right] \\
\displaystyle \times \left[\sum_{i=0}^{\left[\frac{2(m+1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m+1-i}{i-j}\binom{i-j}{j} a_1^{m+1-2i+j} a_2^{i-2j}\right];\ a_1a_2 \neq 0.
\end{cases} \tag{3.9}$$

(3.9) illustrates the complicity of these combinatorial identities, and it would be a challenging problem to prove it by "elementary" means. In the same way, we obtain

$$x_{m+3} = r_{m+3}^2 - r_{m+2}r_{m+4}, \quad (m = 0, 1, \ldots)$$

$$\left\{\begin{aligned}
&\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_2^{m-2i+j} a_1^{i-2j} \\
&= \left[\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j}\binom{m-i}{i-j}\binom{i-j}{j} a_1^{i-2j} a_2^{m-2i+j}\right]^2 \\
&- \left[\sum_{i=0}^{\left[\frac{2(m-1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j-1}\binom{m-i-1}{i-j}\binom{i-j}{j} a_2^{i-2j} a_1^{m-1-2i+j}\right] \\
&\times \left[\sum_{i=0}^{\left[\frac{2(m+1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j+1}\binom{m-i+1}{i-j}\binom{i-j}{j} a_2^{i-2j} a_1^{m+1-2i+j}\right]; \quad a_1a_2 \neq 0.
\end{aligned}\right. \tag{3.10}$$

Now let

$$\left\{\begin{aligned}
&e^3 = 1 + a_2e, \quad a_2 \neq 0, \\
&x_{2m+3} = \sum^{\left[\frac{m}{3}\right]} \binom{m-i}{2i} a_2^{m-3i}, \\
&x_{2m+4} = \sum^{\left[\frac{m-1}{3}\right]} \binom{m-i}{2i+1} a_2^{m-3i-1}, \\
&r_{m+3} = \sum^{\left[\frac{m}{3}\right]} \binom{m-2j}{j} (-1)^{m-j} a_2^{m-3j}, \quad m = 0, 1, \ldots.
\end{aligned}\right. \tag{3.11}$$

We have

$$r_{2v+3} = x_{2v+3}^2 - x_{2v+2}x_{2v+4}, \tag{3.12}$$

and substituting in (3.12) the values of (3.11), we obtain

$$\left\{\begin{aligned}&\sum_{j=0}^{\left[\frac{2m}{3}\right]}\binom{2m-2j}{j}(-1)^j a_2^{2m-3j}\\&=\left[\sum_{i=0}^{\left[\frac{m}{3}\right]}\binom{m-i}{2i}a_2^{m-3i}\right]^2\\&-\sum_{i=0}^{\left[\frac{m-2}{3}\right]}\binom{m-1-i}{2i+1}a_2^{m-2-3i}\sum_{i=0}^{\left[\frac{m-1}{3}\right]}\binom{m-i}{2i+1}a_2^{m-3i-1},\quad (m=2,3,\ldots).\end{aligned}\right. \tag{3.13}$$

Special cases of (3.11) were investigated by the author in two previous papers [1] and [2], and by L. Carlitz [3] and [4].

The case $a_1 \neq 0$, $a_2 = 0$ is treated analogously.

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SOLVED, SEMI-SOLVED, AND UNSOLVED PROBLEMS IN GENERALIZED INTEGERS: A SURVEY

E. M. HORADAM

1. DEFINITION AND DESCRIPTION OF GENERALIZED INTEGERS

The original definition of generalized integers and the name of "generalized primes" were given by Arne Beurling in 1937 (*Acta. Math.*, Vol. 68, pp. 255-291).* Translated from the French, the notation changed, and the word "finite" added, Beurling's definition was "With every sequence, finite or infinite, of real numbers $\{p\}$:

$$1 < p_1 < p_2 < \dots < p_n < \dots \tag{1}$$

we can associate a new sequence $\{g\}$:

$$1 = g_1 \leq g_2 \leq g_3 \leq \dots \leq g_n \leq \dots \tag{2}$$

formed by the set of products

$$g = p_{n_1} p_{n_2} \dots p_{n_r}, \; n_1 \leq n_2 \leq \dots \leq n_r, \; r \geq 1 \tag{3}$$

with the convention that $g_1 = 1$ and every other number g appears in (2) as many times as it has distinct representations (3). We call the p_n the generalized primes (g.p.) of the sequence $\{g\}$ and designate by $\pi(x)$ the number of $p_n \leq x$ and by $N(x)$ the number of $g_n \leq x$." It was Bertil Nyman (1949) who first used the term "generalized integer" (g.i.) to denote the numbers g_n and first referred to Beurling's paper, although V. Ramaswami (1943) seems to have independently invented generalized integers.

Thus the generalized primes need not be natural primes, nor even integers. Also, factorization of generalized integers need not be unique. From the definition, the basic properties of the g.i. are that they can be multiplied and ordered, that is, counted, but not added. The following three sequences all fit the definition of a sequence of generalized primes $\{p_n\}$ together with the corresponding sequence of generalized integers $\{g_n\}$.

$$\begin{aligned} \{p_n\} &= (2, 5, 11, \dots) \\ \{g_n\} &= (1, 2, 4, 5, 8, 10, 11, \dots) \end{aligned} \tag{4}$$

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*A bibliography of the work on generalized integers is given at the end of this paper.

$$\begin{aligned}\{p_n\} &= \left(\frac{13}{11}, \frac{7}{5}, \frac{3}{2}\right)\\ \{g_n\} &= \left(1, \frac{13}{11}, \frac{13^2}{11^2}, \frac{7}{5}, \frac{3}{2}, \frac{13^3}{11^3}, \frac{13}{11}\ \ \frac{7}{5}, \ldots\right)\end{aligned} \tag{5}$$

$$\begin{aligned}\{p_n\} &= (2, 3, 4, 5, 7, 11, 13, \ldots)\\ \{g_n\} &= (1, 2, 3, 2.2, 4, 5, 6, 7, 2.2.2, 2.4, 9, 10, 11, 2.2.3, 4.3, 13, \ldots).\end{aligned} \tag{6}$$

If unique factorization is also assumed, then analogues of the well-known multiplicative arithmetical functions, for example the Moebius function, can be defined and theorems, such as the Moebius inversion formula for g.i., can be proved. Much of this work has been carried out by the author.

However, for his work, Beurling needed an assumption on the size of $N(x)$. He and later writers on this topic were mainly concerned with the way in which $N(x)$ affected $\pi(x)$ and vice versa.

It can be seen that taking the g.p. to be a subset of the natural primes also fits the definition, but this covers a very large block of the total work done in Number Theory. Thus, this work is only included when it has been used in the context of generalized integers, even when the numbers being studied are also ordinary integers.

2. HISTORY OF THE SOLVED PROBLEMS

As succinctly stated by Beurling, his original question was "in what manner should $\varepsilon(x)$ converge to zero when x tends to infinity, so that the hypothesis $N(x) = x(A + \varepsilon(x))$, A a positive constant, infers the asymptotic law $\pi(x) \sim x/\log x$?" In fact he showed that the hypothesis $N(x) = Ax + 0(x/\log^{\gamma} x)$, $p_n \to \infty$, implies $\pi(x) \sim x/\log x$ if $\gamma > 3/2$, but it can fail to hold if $\gamma \leq 3/2$. This was proved using a complex variable and a zeta function for the g.i. Thus, Beurling was concerned with having a prime number theorem (p.n.t.) for generalized integers. Much of the later work revolved about this topic mainly in the direction of refinements in the hypothesis on $N(x)$. Amitsur (1961) gave an elementary proof of the p.n.t. when $N(x) = Ax + 0(x/\log^{\gamma} x)$ holds with $\gamma > 2$.

B. M. Bredihin (1958-1967) used the following algebraic definition: "Let G be a free commutative semi-group with a countable system P of generators. Let N be a homomorphism of G onto a multiplicative semi-group of numbers such that, for a given number x, only finitely many elements α in G have norm $N(\alpha)$ satisfying $N(\alpha) \leq x$." It can be seen that Bredihin's definition includes Beurling's definition of the g.i. except that factorisation of any α is unique but more than one element can have the same norm. Bredihin used the hypothesis

$$N(x) = Ax^{\theta} + 0(x^{\theta_1}), \ \theta > 0, \text{ and } \theta_1 < \theta;$$

and proved that

$$\lim_{x \to \infty} \pi(x) \log x/x^{\theta} = 1/\theta.$$

He was the first (1958) to publish an elementary proof of a prime number theorem for g.i.

During the 1950's, A. E. Ingham gave lectures on Beurling's work in Cambridge, England, and thus extended interest in the g.i.

The motivation for Beurling's work was to find how the number of generalized integers affected the number of generalized primes, and later on the converse problem was studied. A history, bibliography up to 1966, and demonstration using complex variable of the work done on this problem up to 1969, excluding the work of Bredihin and Rémond (1966), is given in Bateman and Diamond's article in Volume 6, MAA *Studies in Number Theory* (1969). Again, "the additive structure of the positive integers is not particularly relevant to the distribution of primes. As the g.i. have no additive structure, they are particularly useful to examine the stability of the prime number theorem." Work on the error term for $\pi(x)$ and the converse problem has been carried out by Nyman (1949), Malliavin (1961), Diamond (1969, 1970), and later writers.

A more complete history of generalized integers may be obtained by reading the reviews which W. J. Le Veque has classified under N80 in his "Reviews in Number Theory" (1974).

Recently, the work on generalized integers has been given a completely different twist by J. Knopfmacher in a series of papers (1972) to (1975) and later papers and a book *Abstract Analytic Number Theory*, Volume 12 of the North-Holland Mathematical Library and published by North-Holland, Amsterdam (1975). This book contains a complete bibliography up to 1974. In essence, Dr. Knopfmacher has used the techniques associated with generalized integers to prove an abstract prime number theorem for an "arithmetical semigroup" and has applied it to contexts not previously considered by other writers.

Segal's 1974 paper has the descriptive title "Prime Number Theorem Analogues without Primes." He states that the underlying multiplicative structure for g.i. is not all-important—a growth function is all that is needed.

There are also papers combining Beurling's generalized integers with an analogue of primes in arithmetic progressions. This means that the ideas of multiplicative semi-groups without addition had to be combined with the idea of primes in arithmetic progressions. The possibility of such an effort was worked on by Fogels (1964-1966) and Rémond (1966). Knopfmacher has used an arithmetical semi-group to attack a similar problem.

3. UNSOLVED PROBLEMS

Most unsolved problems are associated with the prime number theorem for generalized integers. One such set by Bateman and Diamond is listed on pages 198-200 in Volume 6 of *Studies in Number Theory*, mentioned previously. Of these, one was solved by Diamond and R. S. Hall in 1973. Another states that Beurling's theorem can be established by elementary methods with $\gamma < 2$, but that no one has yet succeeded in establishing it by elementary methods for $3/2 < \gamma < 2$. Dr. Diamond has also published four further problems from the Séminaire de Théorie des Nombres, 1973-74. He conjectures that

$$\int_1^\infty |N(x) - x| x^{-2}\, dx < \infty \Rightarrow \psi(x) \ll x.$$

Segal's paper of 1974 concludes with some open questions (pages 21 and 22).

A list of very general open questions is posed by Dr. Knopfmacher on pages 287-292 of his book (1975).

4. SOLVED PROBLEMS—ARITHMETICAL FUNCTIONS

In my own work on generalized integers (1961 to 1968), I have assumed the g.i. to be not necessarily integers but with unique factorization. Some of my papers on g.i. have concentrated on their arithmetical properties, that is, without a hypothesis on $N(x)$, and it is those I am concerned with here. Those needing a hypothesis on $N(x)$, I assume included in §2. However, the fact that the g.i. can be ordered and so a counting function $N(x)$ exists, is important.

Since there will now be a slight change in notation, I will repeat the definition I shall use for generalized primes and integers.

Suppose, given a finite or infinite sequence of real numbers (generalized primes) such that

$$1 < p_1 < p_2 < p_3 < \dots .$$

Form the set $\{g\}$ of all possible -products, i.e., products $p_1^{v_1}p_2^{v_2}\dots$, where $v_1, v_2, \dots$ are integers ≥ 0 of which all but a finite number are 0. Call these numbers "generalized integers" and suppose that no two generalized integers are equal if their v's are different. Then arrange $\{g\}$ as an increasing sequence:

$$1 = g_1 < g_2 < g_3 < \dots .$$

Notice that the g.i. cannot be added to give another g.i. For example, in (4), $1 + 2 = 3$ but $3 \notin \{g_n\}$.

However, division of one g.i. by another is easily defined as follows: We say $d|g_n$ if $\exists\, D$ so that $dD = g_n$ and both d and D belong to $\{g\}$. From these definitions, it follows that greatest common divisor, multiplicative functions, Moebius function, Euler ϕ-function, unitary divisors, etc., for the g.i. can be defined. These lead to further arithmetical properties mainly published by the author.

H. Gutmann (1959) and H. Wegmann (1966) also published results on properties of arithmetical functions. Gutmann worked with g.i. which were subsets of the natural numbers and he assumed that both

$$\sum_1^\infty p_n^{-1} \text{ and } \sum_1^\infty p_n^{-1} \log p_n$$

were convergent.

5. A SEMI-SOLVED PROBLEM

Consider the sequence of natural numbers

$$1, 2, 3, \dots, d, \dots, n, \dots,$$

and let $[x]$ denote the number of integers $\leq x$. Then, since every dth number is divisible by d, it follows that $[x/d]$ will give the number of integers $\leq x$ which are also divisible by d, for they are the numbers

$$1 \cdot d, 2 \cdot d, 3 \cdot d, \dots, \left[\frac{x}{d}\right] \cdot d.$$

Hence, if $n = a + b$ so that $\frac{n}{d} = \frac{a}{d} + \frac{b}{d}$, it follows that $\left[\frac{n}{d}\right] \geq \left[\frac{a}{d}\right] + \left[\frac{b}{d}\right]$, and, in particular, if d is a prime p, then

$$\left[\frac{n}{p}\right] \geq \left[\frac{a}{p}\right] + \left[\frac{b}{p}\right].$$

We now extend these ideas to generalized integers with unique factorization. In order to show the similarity between the integral value function $[x]$ and the counting function $N(x)$ for generalized integers, we change the notation and define

$$[x] = \text{number of g.i.} \leq x = N(x). \quad \text{Then } [g_n] = n.$$

Again, if in the sequence $g_1, g_2, \ldots, g_n$, $d|g_n$, there are in this sequence $\left[\frac{g_n}{d}\right]$ multiples of d, namely the numbers

$$1 \cdot d,\ g_2 \cdot d,\ \ldots,\ g \ \cdot d,\ \ldots,\ \left[\frac{g_n}{d}\right] \cdot d.$$

So $\left[\frac{x}{d}\right]$ will again give the number of g.i. $\leq x$ which are also divisible by d. Suppose now that $n = a + b$, i.e., $[g_n] = [g_a] + [g_b]$. Is it still true that $\left[\frac{g_n}{d}\right] \geq \left[\frac{g_a}{d}\right] + \left[\frac{g_b}{d}\right]$, and in particular $\left[\frac{g_n}{p}\right] \geq \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right]$, where p is a generalized prime?

Consider the following example:

$\{p\}$		5 <	7 <	8 <	11 <	13 <	29 <	...	
$\{g\}$	1 <	5 <	7 <	8 <	11 <	13 <	25 <	29 <	...
$[x]$	1	2	3	4	5	6	7	8	...

Then $[8] = 4 = 2 + 2 = [5] + [5]$.

Now take $p = 5$ and we have $\left[\frac{8}{5}\right] = 1$ and $\left[\frac{5}{5}\right] = 1$. So in this case

$$\left[\frac{g_n}{p}\right] = \left[\frac{8}{5}\right] = 1 \quad \text{and} \quad \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] = \left[\frac{5}{5}\right] + \left[\frac{5}{5}\right] = 2$$

so that

$$\left[\frac{g_n}{p}\right] < \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right].$$

So far as I know, the way in which g is affected by always (or sometimes) having the generalized primes constructed so that

$$\left[\frac{g_n}{p}\right] < \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] \quad \text{when} \quad [g_n] = [g_a] + [g_b]$$

has not been investigated.

We now consider the way in which g is affected by always having the generalized primes constructed so that

$$\left[\frac{g_n}{p}\right] \geq \left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right] \quad \text{when} \quad [g_n] = [g_a] + [g_b]. \tag{7}$$

Given the sequence $p_1 < p_2 < p_3 < \ldots$, the sequence of g.i. must begin

$$1 < p_1 < p_2 < \cdots \tag{8}$$

or

$$1 < p_1 < p_1^2 < p_2 < \cdots \tag{9}$$

or

$$1 < p_1 < p_1^2 < p_1^3 < p_2 < \cdots, \text{ etc.} \tag{10}$$

Suppose we assume (9) to be the case; then

$$\begin{array}{lccccc} \{g\} & 1 < & p_1 < & p_1^2 < & p_2 < & \cdots \\ [x] & 1 & 2 & 3 & 4 & \cdots \end{array}$$

and g_5 has to be found.

First notice that if $\left[\dfrac{g_a}{p}\right] \geq [g_c]$, then $\dfrac{g_a}{p} \geq g_c$. Now

$$[g_5] = 5 = \begin{matrix} 1 + 4 \\ 2 + 3 \end{matrix} = \begin{matrix} [1] + [p_2] \\ [p_1] + [p_1^2] \end{matrix}.$$

So

$$\left[\frac{g_5}{p_1}\right] \geq \begin{matrix} \left[\dfrac{1}{p_1}\right] + \left[\dfrac{p_2}{p_1}\right] \\ \left[\dfrac{p_1}{p_1}\right] + \left[\dfrac{p_1^2}{p_1}\right] \end{matrix} = \begin{matrix} 0 + 2 \\ 1 + 2 \end{matrix} = 3 = [p_1^2].$$

Hence, $g_5 \geq p_1^3$, and since p_1^3 has not yet occurred, it follows that $g_5 = p_1^3$.

Repeating the process, we have

$$[g_6] = 6 = \begin{matrix} 1 + 5 \\ 2 + 4 \\ 3 + 3 \end{matrix} = \begin{matrix} [1] + [p_1^3] \\ [p_1] + [p_2] \\ [p_1^2] + [p_1^2] \end{matrix}.$$

Hence,

$$\left[\frac{g_6}{p_1}\right] \geq \begin{matrix} 0 + 3 \\ 1 + 2 \\ 2 + 2 \end{matrix} = 4 = [p_2].$$

Hence, $g_6 = p_1 p_2$.

$$[g_7] = 7 = \begin{matrix} 1 + 6 \\ 2 + 5 \\ 3 + 4 \end{matrix} = \begin{matrix} [1] + [p_1 p_2] \\ [p_1] + [p_1^3] \\ [p_1^2] + [p_2] \end{matrix}.$$

So

$$\left[\frac{g_7}{p_1}\right] \geq \begin{matrix} 0+4 \\ 1+3 \\ 2+2 \end{matrix} = 4 = [p_2]$$

Hence $g_7 \geq p_1p_2$, which we knew already. It is certainly true that

$$\left[\frac{g_7}{p_2}\right] \geq \left[\frac{p_1p_2}{p_2}\right] = 2$$

is not less than any combination of $\left[\frac{g}{p_2}\right] + \left[\frac{g}{p_2}\right]$. The value for g_7 is therefore not precisely determined. If it is not to be p_3, since the sequence is now

$$\begin{matrix} 1 & < & p_1 & < & p_1^2 & < & p_2 & < & p_1^3 & < & p_1p_2 \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{matrix},$$

then g_7 must be p_1^4. However, since there is room for another prime, and the assumption will still be satisfied, I take $g_7 = p_3$. Although we will not go through the routine, the next number must be p_1^4.

$$[g_8] = \begin{matrix} 1+7 & [1] + [p_3] \\ 2+6 & [p_1] + [p_1p_2] \\ 3+5 & [p_1^2] + [p_1^3] \\ 4+4 & [p_2] + [p_2] \end{matrix}$$

So

$$\left[\frac{g_8}{p_1}\right] \begin{matrix} 0+4 \\ 1+4 \\ 2+3 \\ 2+2 \end{matrix} = 5 = [p_1^3]$$

Hence $g_8 = p_1^4$, as stated previously. Notice that the maximum value of $\left[\frac{g_a}{p}\right] + \left[\frac{g_b}{p}\right]$ can only increase by 1 at each step in the routine.

If this process is continued with primes being inserted wherever there is room for them, what sequence is developed and is the inequality sufficient to determine it?

(a) If the sequence starts as (8), then the total sequence becomes the natural numbers when $p_1 = 2$ and $p_2 = 3$. This was proved by induction by the author and D. E. Daykin. Dr. Daykin also proved that a sequence generated by only two generalized primes always satisfies the inequality. He conjectured that a sequence generated by three generalized primes only cannot satisfy the inequality.

(b) In his Master's thesis for the University of Melbourne (1968), R. B. Eggleton solved a variant of the problem in an algebraic context. He also proved that if the sequence starts as (8) then the total sequence is isomorphic to the natural numbers under multiplication.

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THE NUMBER OF PRIMES IS INFINITE

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For the theorem used as the title of this paper, many proofs exist, some simple, some erudite. For earlier proofs, we refer to [1]. We present here three interesting proofs of the above theorem, and believe that they are new in some sense.

Theorem 1: Let $A_0 = \alpha + m$, where α and m are positive integers with $(\alpha, m) = 1$. Let A_n be defined recursively by

$$A_{n+1} = A_n^2 - mA_n + m.$$

Then each A_i is prime to every A_j, $j \neq i$.

Proof: By definition,

$$A_1 = A_0^2 - mA_0 + m = \alpha A_0 + m.$$

Again,

$$A_2 = A_1^2 - mA_1 + m = A_1(A_1 - m) + m = \alpha A_0 A_1 + m.$$

Assume that

$$A_k = \alpha A_0 A_1 \ldots A_{k-1} + m.$$

By hypothesis, we have

$$A_{k+1} = A_k^2 - mA_k + m = A_k(A_k - m) + m.$$

Now, substituting $\alpha A_0 A_1 \ldots A_{k-1}$ for $A_k - m$ in the preceding line, we obtain

$$A_{k+1} = \alpha A_0 A_1 \ldots A_k + m.$$

So, by induction hypothesis, we get

$$A_n = \alpha A_0 A_1 \ldots A_{n-1} + m \text{ for all } n.$$

Again, from $A_0 \equiv \alpha \pmod{m}$, it follows that $A_1 \equiv \alpha^2 \pmod{m}$. Suppose that

$$A_k \equiv \alpha^{2^k} \pmod{m}.$$

Since we have $A_{k+1} = A_k^2 - mA_k + m$, we have

$$A_{k+1} \equiv (\alpha^{2^k})^2 \pmod{m},$$

that is,

$$A_{k+1} = \alpha^{2^{k+1}} \pmod{m}.$$

Hence, by induction,

$$A_i \equiv \alpha^{2^i} \pmod{m}.$$

Next, let $d = (A_i, A_j)$, $j > i$. Since

$$A_j = \alpha A_0 A_1 \ldots A_{j-1} + m,$$

we have $d|m$. But d divides $A_i \equiv \alpha^{2^i} \pmod{m}$. Now $d|A_i$ and $d|m$ together imply $d = 1$ for $(\alpha, m) = 1$. Hence,

$$(A_i, A_j) = 1, \; j > i,$$

and the theorem is proved.

Corollary 1: The number of primes is infinite.

Proof: It is easy to see that $A_1, A_2, \ldots$ are all odd. Since each A_i is prime to every A_j by Theorem 1, each of the numbers $A_1, A_2, \ldots$ is divisible by an odd prime which does not divide any of the others, and hence there are at least n distinct primes $\leq A_n$. This proves the corollary.

We note that Pólya's proof of the theorem using Fermat numbers [2] is a particular case of the above theorem. Taking $\alpha = 1$, $m = 2$ in the above theorem, we have $A_0 = 3$, $A_1 = 5$, $A_2 = 17$, etc. These are Fermat numbers defined by $F_n = 2^{2^n} + 1$, satisfying $F_{n+1} = F_n^2 - 2F_n + 2$ with $F_0 = 3$. Again, the theorem in [3] is obtained when we put $\alpha = 1$ and $m = 1$.

Theorem 2: Every prime divisor of $\frac{1}{3}(2^p + 1)$, where p is a prime > 3, is greater than p.

Proof: First, we show that $\frac{1}{3}(2^p + 1)$ where p is a prime >3 is not divisible by 3. Now,

$$\frac{1}{3}(2^p + 1) = \frac{2^p + 1}{2 + 1} = 2^{p-1} - 2^{p-2} + \cdots + 1$$

is an integer. Again

$$\frac{1}{3}(2^p + 1) = (2^{p-1} + 2^{p-3} + \cdots + 1) - (2^{p-2} + 2^{p-4} + \cdots + 2)$$

$$\equiv \frac{p+1}{2} - 2 \cdot \frac{p-1}{2} \pmod 3 \equiv \frac{-p+3}{2} \pmod 3.$$

Since p is a prime >3 we have $p = 6k + 1$ or $6k + 5$. Then,

$$\frac{1}{3}(2^p + 1) \equiv \frac{-6k - 1 + 3}{2} \equiv 1 \pmod 3$$

or

$$\frac{1}{3}(2^p + 1) \quad \frac{-6k - 5 + 3}{2} \quad -1 \pmod 3.$$

Next, suppose that $\frac{1}{3}(2^p + 1) \equiv 0 \pmod q$, where q is a prime $\leq p$. Clearly, q is odd and $q \neq 3$ when $p > 3$. Now, by Fermat's little theorem

$$2^{q-1} \equiv 1 \pmod q.$$

If $q = p > 3$, we have $2^{p-1} \equiv 1 \pmod q$, whence $2^p \equiv 2 \pmod q$. But

$$\frac{1}{3}(2^p + 1) \equiv 0 \pmod q$$

by assumption. Hence, we obtain $3 \equiv 0 \pmod q$, a contradiction. Therefore, $q < p$. Now, $(q - 1, p) = 1$ implies that there exist integers a and b such that $ap + b(q - 1) = 1$. Then

a $$2 = 2^{ap+b(q-1)} = (2^p)^a \cdot (2^{q-1})^b \equiv (-1)^a(1)^b \pmod q \equiv -1 \pmod q$$

for a odd. Hence, $2 \equiv -1 \pmod q$ or $3 \equiv 0 \pmod q$. Since q is a prime and $q \neq 3$, we have again a contraction; hence, $q > p$. Therefore, every prime divisor of $\frac{1}{3}(2^p + 1)$ is greater than p. Now it is a corollary that the number of primes is infinite.

We note that "Every prime divisor of $2^p - 1$ where p is a prime is greater than p" was a problem in the *American Mathematical Monthly*.

Theorem 3: Let p be an odd prime > 5. Then every prime divisor of U_p is greater than p where U_p is the pth Fibonacci number.

The Fibonacci numbers are definable by $U_1 = U_2 = 1$ and $U_{n+1} = U_n + U_{n-1}$. We use the following facts to prove the theorem:

(1) $U_{n+m} = U_{n-1}U_m + U_n U_{m+1}$.

(2) If $m|n$ then $U_m|U_n$ and conversely.

(3) Neighboring Fibonacci numbers are relatively prime to each other.

(4) For any m, n we have $(U_m, U_n) = U_{(m,n)}$, where (a, b) means g.c.d. of a and b.

(5) If p is an odd prime, then $p|U_p$, $p|U_{p-1}$, or $p|U_{p+1}$, according as $p = 5$, $p = 10m \pm 1$, or $p = 10m \pm 3$.

Proof: For $p = 2$, $2|F_{p+1} = 2$. Let p be an odd prime > 5. Then U_p is odd since only U_{3t}'s are even. So, every divisor of U_p is odd. Let $q|U_p$ where q is a prime. If $q = p$, then $p|U_p$. This is impossible for $p > 5$. Suppose $q < p$. Now, $(U_p, U_q) = U_{(p,q)} = U_1 = 1$, $(U_{q-1}, U_p) = U_{(q-1,p)} = U_1 = 1$ and $(U_{q+1}, U_p) = U_{(q+1,p)} = U_1 = 1$. Hence, $q \nmid U_q$, $q \nmid U_{q-1}$, and $q \nmid U_{q+1}$. This contradicts (5). Therefore, $q > p$ and the theorem is proved.

Thus, it is a corollary that the number of primes is infinite.

A Request: The author is trying to collect all the proofs on infinitude of primes. Any information in this regard will be very much appreciated.

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