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CONVOLUTION ARRAYS FOR JACOBSTHAL AND FIBONACCI POLYNOMIALS

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The Jacobsthal polynomials and the Fibonacci polynomials are known to be related to Pascal's triangle and to generalized Fibonacci numbers [1]. Now, we show relationships to other convolution arrays, and in particular, we consider arrays formed from sequences arising from the Jacobsthal and Fibonacci polynomials, and convolutions of those sequences. We find infinite sequences of determinants as well as arrays of numerator polynomials for the generating functions of the columns of the arrays of Jacobsthal and Fibonacci number sequences, which are again related to the original Fibonacci numbers.

1. INTRODUCTION

The Jacobsthal polynomials $J_n(x)$,

$$J_0(x) = 0, \quad J_1(x) = 1, \quad J_{n+2}(x) = J_{n+1}(x) + xJ_n(x), \tag{1.1}$$

and the Fibonacci polynomials $F_n(x)$,

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \tag{1.2}$$

have both occurred in [1] as related to Pascal's triangle and convolution arrays for generalized Fibonacci numbers. We note that $F_n(1) = J_n(1) = F_n$, the nth Fibonacci number 1, 1, 2, 3, 5, 8, 13, ..., while $F_n(2) = P_n$, the nth Pell number 1, 2, 5, 12, 29, ... We list the first polynomials in these sequences below.

	$F_n(x)$	$J_n(x)$
$n = 1$	1	1
$n = 2$	x	1
$n = 3$	$x^2 + 1$	$1 + x$
$n = 4$	$x^3 + 2x$	$1 + 2x$
$n = 5$	$x^4 + 3x^2 + 1$	$1 + 3x + x^2$
$n = 6$	$x^5 + 4x^3 + 3x$	$1 + 4x + 3x^2$
$n = 7$	$x^6 + 5x^4 + 6x^2 + 1$	$1 + 5x + 6x^2 + x^3$
$n = 8$	$x^7 + 6x^5 + 10x^3 + 4x$	$1 + 6x + 10x^2 + 4x^3$
$n = 9$	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$	$1 + 7x + 15x^2 + 10x^3 + x^4$

Notice that the coefficients of $J_n(x)$ and $F_n(x)$ appear upon diagonals of Pascal's triangle, written as a rectangular array:

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots \\
1 & 3 & 6 & 10 & 15 & \cdots \\
1 & 4 & 10 & 20 & 35 & \cdots \\
1 & 5 & 15 & 35 & 70 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \tag{1.3}$$

The diagonals considered are formed by starting from successive elements in the left-most column and progressing two elements up and one element right throughout the array. We shall call this the 2,1-diagonal, and we shall call such a diagonal formed by moving up p units and right q units the p,q-diagonal.

The sums of the elements on the p,q-diagonals of Pascal's triangle are the numbers $u(n;\, p-1,\, q)$ of Harris & Styles [4].

We shall display sequences of convolution arrays in what follows: If

$$\left\{a_n\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{b_n\right\}_{n=0}^{\infty}$$

are two sequences of integers, then their convolved sequence

$$\left\{c_n\right\}_{n=0}^{\infty}$$

is given by

$$c_0 = a_0 b_0, \quad c_1 = a_0 b_1 + b_0 a_1, \quad c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0,$$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}. \tag{1.4}$$

Notice that this is the Cauchy product if a_n, b_n, c_n are coefficients of infinite series. The convolution array for a given sequence will contain the successive sequences formed by convolving a sequence with itself.

Pascal's triangle itself is the convolution array for powers of one. Looking back at the display (1.3), we find that the sums of elements appearing on the 1,1-diagonal are 1, 2, 4, ..., 2^n, ...; on the 2,1-diagonal are 1, 1, 2, 3, 5, ..., F_n, ...; on the 1,2-diagonal, 1, 2, 5, 13, ..., F_{2n-1}, ..., while the coefficients of $(1+x)^n$ appear on the 1,1-diagonal, and those of $F_n(x)$ and $J_n(x)$ appear on the 2,1-diagonal.

The convolution array for the powers of 2 is

(1.5)

1	1	1	1	1	...
2	4	6	8	10	...
4	12	24	40	60	...
8	32	80	160	280	...
16	80	240	560	1120	...
...	...	...	...	...	...

Notice that the sums of elements appearing on the 1,1-diagonal are powers of 3, and that the 1,1-diagonal contains coefficients of $(2+x)^n$. The 2,1-diagonals contain the coefficients of $F_{n+2}^*(x) = 2xF_{n+1}^*(x) + F_n^*(x)$, $F_1^*(x) = 1$, $F_2^*(x) = 2x$, and have the Pell numbers 1, 2, 5, 12, 29, ..., as sums, while the 1,2-diagonal sums are the sequence 1, 3, 11, 43, 171, ..., $J_{2n-1}(2)$, ... Noting that in the first array, $F_n = F_n(1)$, while in the second array the Pell numbers are given by $F_n(2)$, it would be no surprise to find that the numbers $F_n(3)$ appear as 2,1-diagonal sums in the powers of 3 convolution array. In fact,

Theorem 1.1: When the powers of k convolution array is written in rectangular form, the sums of elements appearing on the 1,1-diagonals are the powers of $(k+1)$, while the 1,1-diagonal contains the coefficients of $(k+x)^n$. The numbers given by $F_n(k)$ appear as successive sums of the elements of the 2,1-diagonals, which contain the coefficients of the polynomials $F_n^*(x)$, where

$$F_{n+2}^*(x) = kxF_{n+1}^*(x) + F_n^*(x),\ F_1^*(x) = 1,\ F_2^*(x) = kx.$$

The sums of the elements appearing on the 1,2-diagonal are given by the numbers $J_{2n-1}(x)$.

Proof: Since the powers of k are generated by $1/(1 - kx)$, the numbers $F_n(k)$ by $1/(1-kx-x^2)$, and $J_{2n-1}(x)$ by $(1-kx)/(1-(2k+1)x+k^2x^2)$, the results of Theorem 1.1 follow easily from Theorem 1.2 with proper algebra.

We need to write the generating function for the sums of elements appearing on the p,q-diagonal for any convolution array. We let $1/G(x)$ be the generating function for a sequence $\{a_n\}_{n=0}^{\infty}$. Then $[1/G(x)]^{k+1}$ is the generating function for the kth convolution of the sequence $\{a_n\}$ and thus the generating function for the kth column of the convolution array for $\{a_n\}$, where the left-most column is the 0th column.

Theorem 1.2: Let

$$1/G(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Then the sum of the elements appearing on the p,q-diagonals of the convolution array of $\{a_n\}$ has generating function given by

$$\frac{[G(x)]^{q-1}}{[G(x)]^q - x^p}.$$

Proof: We write the convolution array for $\{a_n\}$ to include the powers of x generated:

$$\begin{array}{lllll}
a_0 & b_0 & c_0 & d_0 & \cdots \\
a_1x & b_1x & c_1x & d_1x & \cdots \\
a_2x^2 & b_2x^2 & c_2x^2 & d_2x^2 & \cdots \\
a_3x^3 & b_3x^3 & c_3x^3 & d_3x^3 & \cdots \\
\cdots & \cdots & \cdots & \cdots &
\end{array}$$

We call the top-most row the 0th row and the left-most column the 0th column. In order to sum the elements appearing on the p,q-diagonal, we begin at the element a_nx^n, $n = 0, 1, 2, \ldots,$ and move p units up and q units right. We must multiply every qth column, then, successively by x , x^{2p}, x^{3p}, ..., so that the elements summed are coefficients of the same power of x. The generating functions of every qth column, then, when summed, will have the successive sums of elements found along the p,q-diagonals as coefficients of successive powers of x, so that the sum of the adjusted column generators becomes the generating function we seek. But, we notice that we have a geometric progression, so that

$$\frac{1}{G(x)} + \frac{x^p}{[G(x)]^{q+1}} + \frac{x^{2p}}{[G(x)]^{2q+1}} + \cdots = \frac{1/G(x)}{1 - x^p/[1/G(x)]^q} = \frac{[G(x)]^{q-1}}{[G(x)]^q - x^p}.$$

The sums of elements appearing on the p,q-diagonals of Pascal's triangle and generalized Pascal triangles can be found in Hoggatt & Bicknell [2], [3], and Harris & Styles [4].

2. FIBONACCI AND JACOBSTHAL CONVOLUTION ARRAYS

Returning to Pascal's triangle (1.3), since the Jacobsthal polynomials defined in (1.1) have the property that $J_n(x) = 1$ for $x = 0$ and $n = 1, 2, 3,$

..., Pascal's triangle could be considered the convolution triangle for the sequence of numbers $J_n(0)$. Recall that the 2,1-diagonal contains the coefficients of $J_n(x)$ as well as having sum $F_n = J_n(1)$. We now write the convolution array for the sequence of numbers $J_n(1)$, which, of course, is also the Fibonacci convolution array:

	1	1	1	1	1	...
	1	2	3	4	5	...
	2	5	9	14	20	...
(2.1)	3	10	22	40	65	...
	5	20	51	105	190	...
	8	38	111	256	511	...
	...	...	...	...	...	...

Observe that the sums of elements appearing along the 2,1-diagonals are 1, 1, 3, 5, 11, 21, 43, ..., $J_n(2)$, ...

If one now writes the convolution triangle for the numbers $J_n(2)$,

	1	1	1	1	1	...
	1	2	3	4	5	...
	3	7	12	18	25	...
(2.2)	5	16	34	60	95	...
	11	41	99	195	340	...
	21	94	261	...	...	...
	...	...	...	...	...	...

one finds that the sums of elements appearing on the 2,1-diagonals are 1, 1, 4, 7, 19, 40, ..., $J_n(3)$, ...

Finally, we summarize our results below.

Theorem 2.1: When the convolution array for the sequence $J_n(k)$ obtained by letting $x = k$, $k = 0, 1, 2. 3, \ldots$, in the Jacobsthal polynomials $J_n(x)$, $n = 1, 2, 3, \ldots$, is written in rectangular form, the sums of the elements appearing along successive 2,1-diagonals are the numbers $J_n(k+1)$, and the 2,1-diagonal contains the coefficients of the polynomials $J_n^*(x)$, $n = 1, 2, 3, \ldots$,

$$J_{n+2}^*(x) = J_{n+1}^*(x) + (k + x)J_n^*(x), \quad J_1^*(x) = 1, \quad J_2^*(x) = 1.$$

Proof: The Jacobsthal polynomials are generated by

$$\frac{1}{G(x)} = \frac{1}{1 - x - yx^2} = \sum_{n=0}^{\infty} J_{n+1}(y)x^n$$

From Theorem 1.2, the sums of elements on the 2,1-diagonals have generating function

$$\frac{1}{G(x) - x^2} = \frac{1}{(1 - x - kx^2) - x^2} = \frac{1}{1 - x - (k+1)x^2},$$

the generating function for the numbers $J_n(k+1)$.

If one now returns to the array given in (2.2), notice that we also have the convolution array for the Fibonacci numbers, or for the numbers $F_n(1)$. In Pascal's triangle, the 1,1-diagonal contains the coefficients of the Fibonacci polynomials, but in the Fibonacci convolution array, the 1,1-diagonals contain the coefficients of $F_n(x+1)$, where $F_n(x)$ are the Fibonacci polynomials. If one replaces x by $(x+1)$ in the display of Fibonacci polynomials given in the introduction, one obtains:

$$1,\ x+1,\ x^2+2x+2,\ x^3+3x^2+5x+3,\ x^4+4x^3+9x^2+10x+5,\ \ldots .$$

If we replace x by $(x + 2)$ in successive polynomials $F_n(x)$, we obtain:

$$1,\ x+2,\ x^2+4x+5,\ x^3+6x^2+14x+12,\ x^4+8x^3+27x^2+44x+29,\ \ldots,$$

where the constant terms are Pell numbers. We next write the convolution array for the Pell numbers, or the numbers $F_n(2)$,

$$\begin{array}{lllllll} & 1 & 1 & 1 & 1 & 1 & \ldots \\ & 2 & 4 & 6 & 8 & 10 & \ldots \\ (2.3) & 5 & 14 & 27 & 44 & 65 & \ldots \\ & 12 & 44 & 104 & 200 & 340 & \ldots \\ & 29 & \ldots & \ldots & \ldots & \ldots & \ldots \\ & \ldots & & & & & \end{array}$$

and observe that the 1,1-diagonals contain exactly those coefficients of successive polynomials $F_n(x + 2)$. Also, the sums of elements appearing in the 1,1-diagonals are 1, 3, 10, 33, 109, ..., $F_n(3)$, ..., while in the Fibonacci convolution array those sums were given by 1, 2, 5, 12, 29, ..., $F_n(2)$, ..., and in Pascal's triangle those sums were the Fibonacci numbers themselves.

We summarize as follows.

Theorem 2.2: When the convolution array for the sequence $F_n(k)$ obtained by letting $x = k$, $k = 1, 2, 3, \ldots$, in the Fibonacci polynomials $F_n(x)$, $n = 1, 2, 3, \ldots$, is written in rectangular form, the sums of the elements appearing along successive 1,1-diagonals are the numbers $F_n(k+1)$, and the 1,1-diagonals contain the coefficients of the polynomials $F_n(x+k)$.

Proof: The Fibonacci polynomials are generated by

$$\frac{1}{G(x)} = \frac{1}{1 - yx - x^2} = \sum_{n=0}^{\infty} F_{n+1}(y)x^n$$

From Theorem 1.2, the sums of elements on the 1,1-diagonals have generating function

$$\frac{1}{G(x) - x} = \frac{1}{1 - kx - x^2 - x} = \frac{1}{1 - (k + 1)x - x^2},$$

the generating function for the numbers $F_n(k + 1)$.

Rather than using the definition of convolution sequence, one can write all of these arrays by using a simple additive process. For example, each element in Pascal's rectangular array is the sum of the element in the same row, preceding column, and the element above it in the same column. In the Fibonacci convolution array, each element is the sum of the element in the same row, preceding column, and the two elements above it in the same column.

In the convolution array for $\{F_n(k)\}$, the rule of formation is to add the element in the same row, preceding solumn, to k times the element above, and the second element above, as

$$\{F_n(x)\}\ \text{convolution:}\quad \begin{array}{|c|c|} \hline & w \\ \hline & y \\ \hline x & z \\ \hline \end{array} \quad z = x + ky + w,\ k = 1, 2, \ldots .$$

The convolution array for $\{J_n(k)\}$ is formed by adding the element in the same row, preceding column, to the element above, and to k times the second element above the desired element, as

$$\{J_n(x)\} \text{ convolution: } \quad \begin{array}{|c|c|} \hline & w \\ \hline & y \\ \hline x & z \\ \hline \end{array} \qquad z = x + y + kw,\ k = 0, 1, 2, \ldots .$$

Both additive rules follow immediately from the generating function of the array. For example, for the $\{J_n(k)\}$ convolution, if $G_n(x)$ is the generating function of the nth column, then

$$G_{n+1}(x) = G_1(x)G_n(x) = [1/(1 - x - kx^2)]G_n(x),$$

$$G_{n+1}(x) = G_n(x) + xG_{n+1}(x) + kx^2G_{n+1}(x).$$

As a final example, we proceed to the Tribonacci circumstances. The Tribonacci numbers 1, 1, 2, 4, 7, 13, ..., T_n, ..., given by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad T_0 = 0, \quad T_1 = T_2 = 1, \tag{2.4}$$

appear as the sums of successive 1,1-diagonals of the trinomial triangle written in left-justified form. The trinomial triangle contains as its rows the coefficients of $(1 + x + x^2)^n$, $n = 0, 1, 2, \ldots,$

$$\begin{array}{cccccccccc} 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{2.5}$$

and the coefficients of the Tribonacci polynomials $T_n(x)$ (see [5], [6]),

$$T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x),$$

$$T_{-1}(x) = T_0(x) = 0, \quad T_1(x) = 1$$

along its 1,1-diagonals. We note that $T_n(1) = T_n$.

If we write instead three other polynomial sequences—$t_n(x)$, $t_n^*(x)$, and $t_n^{**}(x)$—which have the property that $t_n(1) = t_n^*(1) = t_n^{**}(1) = T_n$, we find a remarkable relationship to the convolution array for the Tribonacci numbers.

	$t_{n+3} = xt_{n+2} + t_{n+1} + t_n$	$t_{n+3}^* = t_{n+2}^* + xt_{n+1}^* + t_n^*$
$n = 1$	1	1
$n = 2$	x	1
$n = 3$	$x^2 + 1$	$x + 1$
$n = 4$	$x^3 + 2x + 1$	$2x + 2$
$n = 5$	$x^4 + 3x^2 + 2x + 1$	$x^2 + 3x + 3$
...	...	...

and

$$t_{n+3}^{**} = t_{n+2}^{**} + t_{n+1}^{**} + xt_n^{**}$$

$n = 1$	1
$n = 2$	1
$n = 3$	2
$n = 4$	$x + 3$
$n = 5$	$2x + 5$
...	...

We write the convolution array for the Tribonacci numbers:

	1	1	1	1	1	...
	1	2	3	4	5	...
(2.6)	2	5	9	14	20	...
	4	12	25	44	70	...
	7	26	63	...	...	...
	13	56	...			
	24	...				
	...					

If we replace x with $(x + 1)$ in $t_n(x)$, we get 1, $x + 1$, $x^2 + 2x + 2$, $x^3 + 3x^2 + 5x + 4$, ..., whose coefficients appear along the 1,1-diagonals. Putting $(x + 1)$ in place of x in $t_n^*(x)$ gives 1, 1, $x + 2$, $2x + 4$, $x^2 + 5x + 7$, ..., which coefficients are on the 2,1-diagonal, while replacing x by $(x + 1)$ in $t_n^{**}(x)$ makes 1, 1, 2, $x + 4$, $2x + 7$, $5x + 13$, $x^2 + 12x + 24$, ..., which coefficients appear on the 3,1-diagonal. The coefficients of $t_n^*(x + k)$ appear along the 1,1-diagonals of the convolution array for $t_n^*(k)$, and similarly for $t_n^*(x + k)$ and the array for $t_n^*(k)$, and for $t_n^{**}(x + k)$ and $t_n^{**}(k)$.

The Tribonacci convolution array can be generated either by the definition of convolution or by dividing out its generating functions $[1/(1 - x - x^2 - x^3)]^n$ or by the following simple additive process: each element in the array is the sum of the element in the same row but one column left and the three elements above it in the same column, or, schematically,

$$\begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & w \\ \cline{2-2} \multicolumn{1}{c|}{} & x \\ \cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline p & z \\ \hline \end{array} \qquad z = p + w + x + y.$$

Generalizations to generalized Pascal triangles are straightforward.

3. ARRAYS OF NUMERATOR POLYNOMIALS DERIVED FROM FIBONACCI AND JACOBSTHAL CONVOLUTION ARRAYS

In this section, we calculate the generating functions for the rows of the Fibonacci and Jacobsthal convolution arrays of §2. We note that, in each case, the first row is a row of constants; the second row contains elements with a constant first difference; ...; and the ith row forms an arithmetic progression of order $(i - 1)$, $i = 1, 2, \ldots$, with generating function $N_i(x)/(1 - x)^i$. We shall make use of a theorem from a thesis by Kramer [8].

Theorem 57 (Kramer): If generating function

$$A(x) = N(x)/(1 - x)^{r+1}$$

where $N(x)$ is a polynomial of maximum degree r, then $A(x)$ generates an arithmetic progression of order r, and the constant of the progression is $N(1)$.

We calculate the first few row generators for the Fibonacci convolution array (2.1) as

$$\frac{1}{1 - x}, \frac{1}{(1 - x)^2}, \frac{2 - x}{(1 - x)^3}, \frac{3 - 2x}{(1 - x)^4}, \frac{5 - 5x + x^2}{(1 - x)^5}, \frac{8 - 10x + 3x^2}{(1 - x)^6}.$$

We display the coefficients of the successive numerator polynomials:

	1			
	1			
	2	-1		
	3	-2		
(3.1)	5	-5	1	
	8	-10	3	
	13	-20	9	-1
	21	-38	22	-4
	...	...	...	...

The rising diagonal sums are 1, 1, 2, 2, 3, 3, 4, 4, ..., but if we use absolute values, they become the Tribonacci numbers 1, 1, 2, 4, 7, 13, 24, 44, ... The row sums are all 1, which means, by Theorem 57, that $N_n(1) = 1$, or that the constant of the arithmetic progression of order $(n - 1)$ found in the nth row of the Fibonacci convolution array is 1. However, the row sums, using absolute values, are 1, 1, 3, 5, 11, 21, 43, 85, ..., $J_n(2)$, ... Notice that successive columns are formed from successive columns of the Fibonacci convolution array (2.1) itself. We defer proof to the general case.

If one now turns to the convolution array (2.2) for $\{J_n(2)\}$, the first few row generators are

$$\frac{1}{1 - x}, \frac{1}{(1 - x)^2}, \frac{3 - 2x}{(1 - x)^3}, \frac{5 - 4x}{(1 - x)^4}, \frac{11 - 14x + 4x^2}{(1 - x)^5}, \ldots.$$

Displaying the coefficients of the numerator polynomials,

	1				
	1				
	3	-2			
(3.2)	5	-4			
	11	-14	4		
	21	-32	12		
	43	-82	48	-8	
	...	...	...	...	...

we find that the rising diagonal sums are 1, 1, 3, 3, 7, 7, 15, 15, ..., while, taking absolute values, they are 1, 1, 3, 7, 15, 35, 79, ..., where the kth term is formed from the sum of the preceding term and twice the sum of the two terms preceding that, a generalized Tribonacci sequence. Each row sum is again 1. However, using absolute value, the row sums become 1, 1, 5, 9, 29, 65, 181, ..., $J_n(4)$, ... Notice that successive columns are multiples of successive columns of (2.2), the second column being twice the second column of (2.2), the third column four times the original third column, and the fourth column eight times the original fourth column.

Notice that the Fibonacci numbers are also the numbers $J_n(1)$. We state and prove a theorem for the general Jacobsthal case.

Theorem 3.1: Let $J_n^{j+1}(k)$ denote the nth element of the jth convolution of $\{J_n(k)\}$. Let $N_m(x)/(1-x)^m$ denote the generating function of the mth row, $m = 1, 2, \ldots$, in the convolution array for $\{J_n(k)\}$. Then

$$N_m(x) = \sum_{i=0}^{[(m-1)/2]} (-1)^i k^i J_{m-2i}^{i+1}(k) x^i.$$

Proof: [Note that $J_n^1(k) = J_n(k)$.] From the rule of formation of the convolution array for $\{J_n(k)\}$ derived in §2, the row generators $D_n(x)$ obey

$$D_n(x) = xD_n(x) + D_{n-1}(x) + kD_{n-2}(x) = \frac{1}{1-x}[D_{n-1}(x) + kD_{n-2}(x)] \tag{3.3}$$

$$\frac{N_n(x)}{(1-x)^n} = \frac{1}{1-x}\left[\frac{N_{n-1}(x)}{(1-x)^{n-1}} + \frac{kN_{n-2}(x)}{(1-x)^{n-2}}\right]$$

$$N_n(x) = N_{n-1}(x) + (1-x)kN_{n-2}(x) = N_{n-1}(x) + kN_{n-2}(x) - kxN_{n-2}(x). \tag{3.4}$$

Comparing (3.4) to the original recurrence for $J_n(k)$ and noting that $N_1(x) = N_2(x) = J_1(k) = J_2(k) = 1$, the constant term is given by $N_n(0) = J_n(k)$. The rule of formation of the convolution array can also be stated as

$$J_n^{i+1}(x) = J_{n-1}^{i+1}(k) + kJ_{n-2}^{i+1}(k) + J_n^i(k). \tag{3.5}$$

Let u_n be the coefficient of x in $N_n(x)$. Then

$$u_n = u_{n-1} + ku_{n-2} - kJ_{n-2}(k).$$

If $u_j = -kJ_{j-2}^2(k)$, $j = 3, 4, \ldots, n-1$, then

$$\begin{aligned} u_n &= -kJ_{n-3}^2(k) - k^2J_{n-4}^2(k) - kJ_{n-2}(k) \\ &= -k\left(J_{n-3}^2(k) + kJ_{n-4}^2(k) + J_{n-2}(k)\right) = -kJ_{n-2}^2(k) \end{aligned}$$

by (3.5). Thus, the coefficient of x has the desired form for all $n \geq 3$.

Next, let u_n be the coefficient of x^i and v_n the coefficient of x^{i-1} in $N_n(x)$. If $u_j = (-1)^i J_{j-2i}^{i+1}(k)k^i$ and $v_j = (-1)^{i-1}k^{i-1}J_{j-2i}^i(k)$ for $j = 3, 4, \ldots, n-1$, then

$$\begin{aligned} u_n &= u_{n-1} + ku_{n-2} - kv_{n-2} \\ &= (-1)^i k^i J_{n-1-2i}^{i+1}(k) + k(-1)^i k^i J_{n-2-2i}^{i+1}(k) - k(-1)^{i-1}k^{i-1}J_{n-2i}^i(k) \\ &= (-1)^i k^i\left(J_{n-1-2i}^{i+1}(k) + kJ_{n-2-2i}^{i+1}(k) + J_{n-2i}^i(k)\right) \\ &= (-1)^i k^i\left(J_{n-2i}^{i+1}(k)\right) \end{aligned}$$

by again applying (3.5), establishing Theorem 3.1, except for the number of terms summed. By Theorem 57 [8], $i \leq m$, since the degree of $N_m(x)$ is less than or equal to m. But $J_{m-2i}^{i+1}(k) = 0$ for $[(m-1)/2] < i \leq m$.

By Theorem 3.1, the generating function for the ith column of the numerator polynomial coefficient array for the generating functions of the rows of the convolution array of $\{J_n(k)\}$ is now known to be

$$\frac{k^{i-1}x^{2(i-1)}}{(1 - x - kx^2)^i}.$$

Summing the geometric series

$$\frac{1}{1 - x - kx^2} + \frac{kx^2}{(1 - x - kx^2)^2} + \frac{k^2x^4}{(1 - x - kx^2)^3} + \cdots = \frac{1}{1 - x - (2k)x^2}$$

which proves that the rows' sums, using absolute values, are given by $J_n(2k)$. However, summing for the rows as originally given, we use alternating signs in forming the geometric series, and its sum becomes $1/(1-x)$, so that $N_m(1) = 1$. That is, the ith row is an arithmetic progression of order $(i-1)$ with constant 1 in every one of the arrays for $\{J_n(k)\}$, $k = 1, 2, 3, \ldots$.

Turning to the cases of convolution arrays for the sequences $\{F_n(k)\}$, $k = 1, 2, 3, \ldots$, we look at $F_n(2)$ as in array (2.3). The first few row generators are

$$\frac{1}{1-x}, \frac{2}{(1-x)^2}, \frac{5-x}{(1-x)^3}, \frac{12-4x}{(1-x)^4}, \frac{29-14x+x^2}{(1-x)^5}, \frac{70-44x+6x^2}{(1-x)^6}.$$

The array of coefficients for the numerator polynomials is

$$\begin{array}{lllll} & 1 & & & \\ & 2 & & & \\ & 5 & -1 & & \\ (3.6) & 12 & -4 & & \\ & 29 & -14 & 1 & \\ & 70 & -44 & 6 & \\ & \cdots & \cdots & \cdots & \cdots \end{array}$$

The row sums are 1, 2, 4, 8, 16, 32, ..., 2^n, ..., and the coefficients of successive columns appear in the original array (2.3). We state the situation for the general case.

Theorem 3.2: Let $F_n^{j+1}(k)$ denote the nth element in the jth convolution of the numbers $\{F_n(k)\}$, $k = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$. Let the generating function of the mth row in the convolution array for $\{F_n(k)\}$ be $N_m^*(x)/(1-x)^m$, $m = 1, 2, \ldots$. Then

$$N_m^*(x) = \sum_{i=0}^{[(m-1)/2]} (-1)^i F_{m-2i}^{i+1}(k)x^i.$$

The proof is analogous to that of Theorem 3.1 and is omitted in the interest of brevity.

Theorem 3.2 tells us that the ith column of the numerator coefficient array form the generating functions of the rows of the convolution arrays for $F_n(k)$ is given by $(-1)^i x^{2i}/(1 - kx - x^2)^i$. Then, $N_n^*(1)$ is the sum of the rows given by the sum of the geometric series

$$\frac{1}{1 - kx - x^2} - \frac{x^2}{(1 - kx - x^2)^2} + \frac{x^4}{(1 - kx - x^2)^3} - \cdots = \frac{1}{1 - kx},$$

so that $N_n^*(1) = k^{n-1}$. By Theorem 57 [8], the $(i-1)$st order arithmetic progression formed in the ith row of the convolution array for $\{F_n(k)\}$ has constant k^{n-1}, in every one of the arrays, $k = 1, 2, 3, \ldots$.

4. ARRAYS OF SUCCESSIVE JACOBSTHAL AND FIBONACCI POLYNOMIAL SEQUENCES

In [7], Whitford considers an array whose rows are given by successive sequences derived from the Jacobsthal polynomials, such as

(4.1)

k	The sequence $\{J_n(k)\}$										
0	1	1	1	1	1	1	1	1	1	1	...
1	1	1	2	3	5	8	13	21	34	55	...
2	1	1	3	5	11	21	43	85	171	341	...
3	1	1	4	7	19	40	97	217	508	1159	...
4	1	1	5	9	29	65	181	441	1165	2929	...
...					...						

The successive elements in each column are given by $1, 1, k+1, 2k+1, k^2+3k+1, \ldots$, by the recursion relation for $\{J_n(k)\}$. The vertical sequences above are given by

$$J_n(k) = \sum_{r=0}^{n-1}\binom{n-1-r}{r}k^r = \frac{1}{2^{n-1}}\sum_{\substack{r=1\\ r\ \text{odd}}}^{n}\binom{n}{r}(4k+1)^{(r-1)/2} \tag{4.2}$$

where n is fixed, $n \geq 1$, and $k = 0, 1, 2, 3, \ldots$ (see [7]).

We now wish to obtain the generating functions for the columns of the array (4.1). Notice that the first two columns are constants, the next two columns have a constant second difference, the next two have a constant third difference, etc. This means that if $D_n(x)$ is the generating function for the nth column, $n = 1, 2, 3, \ldots$, then the denominators of $D_{2m-1}(x)$ and $D_{2m}(x)$ are each given by $(1-x)^m$. We shall again make use of Theorem 57 [8], which was quoted in §3.

One has

$$D_{2m-1}(x) = \frac{r_{2m-1}(x)}{(1-x)^m}, \quad D_{2m}(x) = \frac{r_{2m}(x)}{(1-x)^m},$$

by virtue of

$$\frac{1}{1-x-kx^2} = \sum_{n=0}^{\infty}J_{n+1}(k)x^n.$$

Now, if $J_{n+1}(k)$ has fixed $(n+1)$ and k varies, we generate the columns. If we fix k and let n vary, we generate the rows. $J_{n+1}(k)$ is a polynomial in k with coefficients lying along the 2,1-diagonal of Pascal's triangle. To get the ordinary generating function, we can note that

$$\frac{A_k(x)}{(1-x)^{k+1}} = \sum_{n=0}^{\infty}n^kx^n,$$

where the $A_k(x)$ are the Eulerian polynomials. (See Riordan [9] and Carlitz [10]). Thus, we note that the polynomials $J_{2m-1}(k)$ and $J_{2m}(k)$ are both of the same degree, and we will expect the generating functions to reflect this fact.

From careful scrutiny of the array generation, we see

$$D_{n+2}(x) = D_{n+1}(x) + xD_n'(x). \tag{4.3}$$

One then breaks this down into two cases:

$$(4.4)\quad \begin{aligned} D_{2m+2}(x) &= \frac{r_{2m+2}(x)}{(1-x)^{m+1}} = \frac{r_{2m+1}(x)}{(1-x)^{m+1}} + x\frac{d}{dx}\left(\frac{r_{2m}(x)}{(1-x)^m}\right) \\ D_{2m+3}(x) &= \frac{r_{2m+3}(x)}{(1-x)^{m+2}} = \frac{r_{2m+2}(x)}{(1-x)^{m+1}} + x\frac{d}{dx}\left(\frac{r_{2m+1}(x)}{(1-x)^{m+1}}\right) \end{aligned}$$

This leads to two simple recurrences:

$$(4.5)\quad \begin{aligned} r_{2m+2}(x) &= r_{2m+1}(x) + x(1-x)r_{2m}'(x) + mxr_{2m}(x) \\ r_{2m+3}(x) &= (1-x)r_{2m+2}(x) + x(m+1)r_{2m+1}(x) + x(1-x)r_{2m+1}'(x) \end{aligned}$$

The first fifteen polynomials $r_n(x)$ are:

n	$r_n(x)$	row sums $r_n(1)$
1	1	0!
2	1	1!
3	1	1!
4	$1 + x$	2!
5	$1 + 2x - x^2$	2!
6	$1 + 5x$	3!
7	$1 + 9x - 3x^2 - x^3$	3!
8	$1 + 17x + 7x^2 - x^3$	4!
9	$1 + 29x + 11x^2 - 17x^3$	4!
10	$1 + 50x + 76x^2 - 6x^3 - x^4$	5!
11	$1 + 83x + 164x^2 - 100x^3 - 29x^4 + x^5$	5!
12	$1 + 138x + 516x^2 + 110x^3 - 45x^4$	6!
13	$1 + 226x + 1121x^2 - 50x^3 - 571x^4 - 8x^5 + x^6$	6!
14	$1 + 370x + 2843x^2 + 2344x^3 - 421x^4 - 98x^5 + x^6$	7!
15	$1 + 602x + 6071x^2 + 4956x^3 - 5249x^4 - 1430x^5 + 89x^6$	7!

We observe that $r_n(1) = [n/2]!$, where $[x]$ is the greatest integer contained in x. This follows immediately by taking $x = 1$ in (4.5) to make a proof by mathematical induction. By Theorem 57 [8], $r_n(1)$ also is the constant of the arithmetic progression formed by the elements in the nth column of the Jacobsthal polynomial array (4.1). There is a pleasant surprise in the second column of the numerator polynomials $r_n(x)$, whose generating function is

$$1/[(1 - x - x^2)(1 - x)(1 - x^2)].$$

The sequence of coefficients is 0, 0, 0, 1, 2, 5, 9, 17, 29, 50, 83, 138, 226, 370, 602, ..., u_r, ..., $r = 1, 2, 3, \ldots$. We can prove from the recurrence relation that

$$(4.6)\quad \begin{aligned} u_{2k-1} &= F_{2k-1} - k \\ u_{2k} &= F_{2k} - k \end{aligned}$$

as r is odd or even. By returning to (4.5), we can write a recurrence for the u_r simply by looking for those terms which contain multiples of x only, so that

$$u_{2m+2} = u_{2m+1} + u_{2m} + m$$

$$u_{2m+3} = (u_{2m+2} - 1) + (m + 1) + u_{2m+1} = u_{2m+2} + u_{2m+1} + m$$

Since we know that (4.6) holds for $r = 1, 2, \ldots, 15$, we examine u_{2m+2} and u_{2m+3}, assuming that (4.6) holds for all $r < 2m + 2$. Then

$$u_{2m+2} = \left(F_{2m+1} - (m + 1)\right) + (F_{2m} - m) + m = F_{2m+2} - (m + 1)$$

$$u_{2m+3} = \left(F_{2m+2} - (m + 1)\right) + \left(F_{2m+1} - (m + 1)\right) + m = F_{2m+3} - (m + 2),$$

so that (4.6) holds for all integers r by mathematical induction.

To determine the relationship between elements appearing in the third column of the numerator polynomial array, examine (4.5) to write only those terms which contain multiples of x^2. Letting the coefficient of x^2 in $r_n(x)$ be v_n, we obtain

$$v_{2m+2} = v_{2m+1} + 2v_{2m} + (m - 1)u_{2m}$$

$$v_{2m+3} = v_{2m+2} + 2v_{2m+1} + mu_{2m+1} - u_{2m+2}$$

which, when combined with (4.6), gives us

$$\begin{aligned} v_{2m+2} &= v_{2m+1} + 2v_{2m} - u_{2m} + mF_{2m} - m^2 \\ &= v_{2m+1} + 2v_{2m} - (m - 1)(F_{2m} - m) \end{aligned} \tag{4.7}$$

$$\begin{aligned} v_{2m+3} &= v_{2m+2} + 2v_{2m+1} - u_{2m+2} + mF_{2m+1} - 2t_m \\ &= v_{2m+2} + 2v_{2m+1} + (m + 1)F_{2m+1} - F_{2m+3} - (m + 1)^2 \end{aligned}$$

where $t_m = m(m + 1)/2$, the mth triangular number.

Continuing to the fourth column, if the coefficient of x^3 in $r_n(x)$ is w_n, we can write

$$w_{2m+2} = w_{2m+1} + 3w_{2m} + (m - 2)v_{2m}$$

$$w_{2m+3} = w_{2m+2} + 3w_{2m+1} + (m - 1)v_{2m+1} - v_{2m+2},$$

and so on.

Now, if we wish to generate the columns of the numerator polynomials array, it is easy enough to write the generators for the second column if we take two cases. To write the generating function for 1, 5, 17, 50, 138, ..., u_{2n}, ..., since this is the sequence of second partial sums of the alternate Fibonacci numbers 1, 3, 8, 21, 55, ..., the generating function is

$$1/[(1 - 3x - x^2)(1 - x)^2,$$

except to use it properly, we must replace x by x^2, so that

$$\frac{1}{(1 - 3x^2 + x^4)(1 - x^2)^2} = \sum_{n=0}^{\infty} u_{2n+4}x^{2n}$$

Now, the generating function for u_{2k-1} results from combining the known generators for F_{2k-1} and for the positive integers.

Since

$$\frac{1 - x}{1 - 3x + x^2} = 1 + 2x + 5x^2 + 13x^3 + 34x^4 + \cdots$$

and

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots,$$

$$\frac{1-x}{1-3x+x^2} - \frac{1}{(1-x)^2} = \frac{2x^2 - x^3}{(1-3x+x^2)(1-x)^2} = \sum_{n=1}^{\infty}(F_{2n-1} - n)x^{n-1}.$$

To adjust the powers of x, first replace x by x^2 and then multiply each side by x, obtaining finally

$$\frac{2x^5 - x^7}{(1-3x^2+x^4)(1-x^2)^2} = \sum_{n=1}^{\infty}(F_{2n-1} - n)x^{2n-1} = \sum_{n=1}^{\infty}u_{2n-1}x^{2n-1}.$$

On the other hand, if one writes the array whose rows are given by successive sequences derived from the Fibonacci polynomials,

(4.8)

k	The sequence $\{F_n(k)\}$								
1	1	1	2	3	5	8	13	21	...
2	1	2	5	12	29	70	169	308	...
3	1	3	10	33	109	360	1189	3927	...
4	1	4	17	72	305	1292	5273	22384	...
...					...				

The successive elements in each column are given by $1,\ k,\ k^2+1,\ k^3+2k,\ k^4+3k^3+1, \ldots$, by the recursion relation for $F_n(k)$, $k = 1, 2, 3, \ldots$. The vertical sequences above are given by

$$(4.9) \qquad F_n(k) = \sum_{r=0}^{n-1}\binom{n-1-r}{r}k^{n-2r-1}$$

where n is fixed, $n \geq 1$, and $k = 1, 2, 3, \ldots$, or by

$$\frac{1}{1-kx-x^2} = \sum_{n=0}^{\infty}F_{n+1}(k)x^n,$$

which generates the rows for k fixed, $n = 1, 2, 3, \ldots$, and the columns for n fixed, $k = 1, 2, 3, \ldots$.

As before, we wish to generate the columns. We observe, since the nth column has a constant $(n-1)$st difference, that the denominators of the column generators will be $(1-x)^n$, $n = 1, 2, 3, \ldots$.

If we let $D_n^*(x)$ be the generating function for the nth column, and let

$$(4.10) \qquad D_n^*(x) = \frac{r_n^*(x)}{(1-x)^n},$$

this time we find that

$$D_{n+2}^*(x) = xD_{n+1}^{*\prime}(x) + D_n^*(x);$$

$$(4.11) \qquad r_{n+2}^*(x) = x(n+1)r_{n+1}^*(x) + x(1-x)r_{n+1}^{*\prime}(x) + (1-x)^2r_n^*(x).$$

We list the first few numerator polynomials $r_n^*(x)$:

n	$r_n^*(x)$	$r_n^*(1)$
1	1	0!
2	x	1!
3	$1 - x + 2x^2$	2!
4	$3x + 3x^3$	3!
5	$1 + 14x^2 + 4x^3 + 5x^4$	4!
6	$8x + 22x^2 + 60x^3 + 22x^4 + 8x^5$	5!
7	$1 + 6x + 99x^2 + 244x^3 + 279x^4 + 78x^5 + 13x^6$	6!
8	$21x + 240x^2 + 1251x^3 + 2016x^4 + 1251x^5 + 240x^6 + 21x^7$	7!
9	$1 + 25x + 715x^2 + 5245x^3 + 14209x^4 + 14083x^5 + 5329x^6 + 679x^7 + 34x^8$	8!
...	...	...

We find that $r_n^*(1) = (n - 1)!$, and that the coefficient of the highest power of x in $r_n^*(x)$ is F_n. It would also appear that the coefficients of x are alternate Fibonacci numbers in even-numbered rows. In fact, D. Garlick [11] observed that, if u_n is the coefficient of the linear term in $r_n^*(x)$, then

$$u_{2k} = F_{2k}$$

$$u_{2k-1} = F_{2k-1} - (2k - 1),$$

which can be proved from the recurrence relation by induction.

Let c_n be the constant term in $r_n^*(x)$. By studying (4.11) carefully to find first, constant terms only, and then just the linear terms, we can write

$$\begin{aligned} &c_{n+2} = c_n \\ (4.13)\qquad &u_{n+2} = (n + 1)c_{n+1} + u_{n+1} + u_n - 2c_n. \end{aligned}$$

Since $c_1 = 1$ and $c_2 = 0$, $c_{2k+1} = 1$ and $c_{2k} = 0$. Assume that (4.12) is true for all $n \le 2k$. Then, taking $n = 2k - 1$ in (4.13),

$$\begin{aligned} u_{2k+1} &= (2k)c_{2k} + u_{2k} + u_{2k-1} - 2c_{2k-1} \\ &= 0 + F_{2k} + F_{2k-1} - (2k - 1) - 2 = F_{2k+1} - (2k + 1). \end{aligned}$$

Similarly, from (4.13) for $n = 2k$,

$$\begin{aligned} u_{2k+2} &= (2k + 1)c_{2k+1} + u_{2k+1} + u_{2k} - 2c_{2k} \\ &= (2k + 1) + F_{2k+1} - (2k + 1) + F_{2k} - 0 = F_{2k+2}, \end{aligned}$$

so that (4.12) holds for all integers $k > 0$.

Continuing, let v_n be the coefficient of x^2 in $r_n^*(x)$. By looking only at coefficients of x^2 in (4.11), we have

$$\begin{aligned} v_{n+2} &= (n + 1)u_{n+1} + 2v_{n+1} - u_{n+1} + v_n - 2u_n + c_n \\ &= 2v_{n+1} + v_n + nu_{n+1} - 2u_n + c_n, \end{aligned}$$

which, combined with (4.12), makes

$$v_{2k+2} = 2v_{2k+1} + v_{2k} + 2k\big(F_{2k+1} - (2k + 1)\big) - 2F_{2k}$$

$$v_{2k+1} = 2v_{2k} + v_{2k-1} + (2k - 1)F_{2k} - 2\big(F_{2k-1} - (2k - 1)\big) + 1.$$

Now, to prove that the coefficient of the highest power of x is F_n, we let the coefficient of the highest power of x in $r_n^*(x)$ be h_n. As before, (4.11) gives us

$$h_{n+2} = (n + 1)h_{n+1} - nh_{n+1} + h_n = h_{n+1} + h_n.$$

Since $h_1 = 1$ and $h_2 = 1$, $h_n = F_n$.

Further, it was conjectured by Hoggatt and proved by Carlitz [12] that $r^*_{2n}(x)$ is a symmetric polynomial. Note that this also gives the linear term of $r^*_{2n}(x)$ the value F_{2n} since we have just proved that the highest power of x has F_n for a coefficient.

5. INFINITE SEQUENCES OF DETERMINANT VALUES

In [13] and [14], sequences of $m \times m$ determinants whose values are binomial coefficients were found when Pascal's triangle was imbedded in a matrix. Here, we write infinite sequences of determinant values of $m \times m$ determinants found within the rectangular arrays displayed throughout this paper. We will apply

Eves' Theorem: Consider a determinant of order n whose ith row (column) ($i = 1, 2, \ldots, n$) is composed of any n successive terms of an arithmetic progression of order ($i - 1$) with constant a_i. Then the value of the determinant is the product $a_1 a_2 \ldots a_n$.

Consider the convolution array for the powers of 2 as given in (1.5). Each row is an arithmetic progression of order ($i - 1$) and with constant 2^{i-1}, $i = 1, 2, 3, \ldots$. Thus, the determinant of any square $m \times m$ array taken to include elements from the first row of (1.5) is $2^0 2^1 2^2 \ldots 2^{m-1} = 2^{m(m-1)/2}$. Further, noticing that each element in the array is 2^{i-1} times the element of Pascal's triangle in the corresponding position in the ith row, $i = 1, 2, \ldots$, we can apply the theorems known about Pascal's triangle from [13] and [14]. However, if we form the convolution triangle for powers of k, then each element in the ith row is k^{i-1} times the corresponding element in the ith row of Pascal's triangle written in rectangular form, $i = 1, 2, \ldots$. Thus, applying the known theorems for Pascal's triangle, we could immediately evaluate determinants correspondingly placed in the powers of k convolution triangle.

Also, we notice that the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, 3, \ldots$, has its rows in arithmetic progressions of order ($i - 1$) with constant 1, $i = 1, 2, \ldots$, while the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, has its rows in arithmetic progressions of order ($i - 1$) with constant k^{i-1}, $i = 1, 2, \ldots$. From these remarks, we have the theorem given below.

Theorem 5.1: Form the $m \times m$ matrix A such that it contains m consecutive rows of the original array, with its first row the first row of the original array, and m consecutive columns of the original array with its first column the jth column of the original array. In the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, \ldots$, $\det A = 1$. In the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, or in the convolution array for the powers of k, $\det A = k^{m(m-1)/2}$.

Determinants whose values are binomial coefficients also appear within these arrays. To apply the results of [13] and [14], we must first express our convolution arrays in terms of products of infinite matrices. Let the rectangular convolution array for $\{F_n(k)\}$ be imbedded in an infinite matrix $\mathcal{F}_k$, and similarly, let $\mathcal{J}_k$ be the infinite matrix formed from the convolution array for $\{J_n(k)\}$. Let P be the infinite matrix formed by Pascal's triangle written in rectangular form. Consider the convolution array for the powers of k, written in rectangular form. Each successive 1,1-diagonal contains the coefficients of $(x + k)^n$. Form the matrix A_k such that the coefficients of $(k + n)^n$ appear in its columns on and beneath the main diagonal, and the matrix

B_k in exactly the same way, except use the coefficients of $(1 + kx)^n$. Then, $A_kP = \mathcal{F}_k$ and $B_kP = \mathcal{J}_k$. We illustrate, using 5 x 5 matrices, for $k = 2$:

$$A_2P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 8 & 0 & \cdots \\ 0 & 0 & 1 & 12 & 16 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 4 & 6 & 8 & 10 & \cdots \\ 5 & 14 & 27 & 44 & 65 & \cdots \\ 12 & 44 & 104 & 200 & 340 & \cdots \\ 29 & 121 & 366 & 810 & 1555 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \mathcal{F}_2$$

$$B_2P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 4 & 6 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 3 & 7 & 12 & 18 & 25 & \cdots \\ 5 & 16 & 34 & 60 & 95 & \cdots \\ 11 & 41 & 99 & 195 & 340 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \mathcal{J}_2$$

Using the methods of [13] and [14], since the generating function for the jth column of A_k is $[x(k+x)]^{j-1}$ while the jth column of P is $1/(1-x)^j$, the jth column of A_kP is $1/[1-x(k+x)]^j = [1-kx-x^2]^j$, where we recognize the generating functions for the columns of the convolution array for $\{F_n(k)\}$, so that $A_kP = \mathcal{F}_k$. Similarly, since the jth column of B_k is generated by $[x(1+kx)]^{j-1}$, B_kP is generated by $1/[1-x(1+kx)]^{j-1} = 1/[1-x-kx]^{j-1}$, so that $B_kP = \mathcal{J}_k$.

Each submatrix of $\mathcal{J}_k$ taken with its first row anywhere along the first row or second row of $\mathcal{J}_k$ is the product of a similarly placed submatrix of P and a matrix with unit determinant. The case for $\mathcal{F}_k$ is similar, except that an $m \times m$ submatrix of P is multiplied by an $m \times m$ matrix whose determinant is $k^{m(m-1)/2}$. Since we know how to evaluate determinants of submatrices of P [13], [14], we write

Theorem 5.2: Form an $m \times m$ matrix B from m consecutive rows and columns of the original array by starting its first row along the second row of the original array and its first column along the jth column of the original array. In the convolution array for the sequence $\{J_n(k)\}_{n=0}^{\infty}$, $k = 0, 1, 2, \ldots$, $\det B = \binom{j-1+m}{m}$. In the convolution array for the sequence $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, or in the convolution array for the powers of k, $\det B = k^{m(m-1)/2}\binom{j-1+m}{m}$.

We could extend the results of Theorem 5.1 to apply to any convolution array for a sequence with first term 1 and second term k, since Hoggatt and Bergum [15] have shown that such convolution arrays always have the ith row an arithmetic progression of order $(i-1)$ with constant k. It is conjectured that Theorem 5.2 also holds for the convolution array of any increasing sequence whose first term is 1 and second term is k.

Proceeding to the array formed from the Jacobsthal sequences themselves, as given in (4.1), the nth column is an arithmetic progression of order $[(n - 1)/2]$, where $[x]$ is the greatest integer contained in x. That makes determinants of value zero very easy to find. Any determinant formed with its first column the first, second, or third column of the original array containing any m consecutive rows of m consecutive columns, $m > 3$, is zero. Det A = det B = 0 whenever $m > j$, for matrices A and B formed as in Theorems 5.1 and 5.2. However, determinants formed from m consecutive rows taken from alternate columns have value $(0!)(1!)(2!) \ldots (m - 1)!$ or $(1!)(2!) \ldots (m!)$ depending upon whether one takes the first column and then successive odd columns or begins with the second column and then successive even columns.

Similarly, the array (4.8) formed of the sequences $\{F_n(k)\}_{n=0}^{\infty}$, $k = 1, 2, 3, \ldots$, has its ith column an arithmetic progression of order $(i - 1)$ with constant $(i - 1)!$, so that any determinant formed from any m consecutive rows of the first m columns has determinant $(0!)(1!) \ldots (m - 1)!$.

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THE FIBONACCI SEQUENCE MODULO N

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Let n be a positive integer. The Fibonacci sequence, when considered modulo n, must repeat. In this note we investigate the period of repetition and the related unsolved problem of finding the smallest Fibonacci number divisible by n. The results given here are similar to those of the simple problem of determining the period of repetition of the decimal representation of $1/p$. If p is a prime other than 2 or 5, it is an easy matter to verify that the period of repetition is the order of the element 10 in the multiplicative group $\mathbf{Z}_p^*$ of residues modulo p. Analogously, the period of repetition of the Fibonacci sequence modulo p is the order of an element ε in a group to be defined in §1. This result will allow us to estimate the period of repetition and the least Fibonacci number divisible by n. Sections 2 and 3 contain the exact statements of these theorems; in §4, related topics are discussed.

1. DEFINITIONS AND PRELIMINARY RESULTS

The Fibonacci sequence is defined recursively: $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 2$. If we define

$$\varepsilon = (1 + \sqrt{5})/2,$$

then it is easy to verify the following by induction.

Lemma 1: $\varepsilon^m = (f_{m-1} + f_{m+1})/2 + (f_m/2)$.

Letting $\mathbf{Z}_n$ be the ring of residue classes of integers modulo n, define

$$\mathbf{Z}_n[\sqrt{5}] = \left\{a + b\sqrt{5} \middle| a,\ b \in \mathbf{Z}_n\right\}.$$

This becomes a ring with respect to the usual addition and multiplication. For n relatively prime to 5 define the norm as a mapping $N:\mathbf{Z}_n[\sqrt{5}] \to \mathbf{Z}_n$ given by $N(a + b\sqrt{5}) = a^2 - 5b^2$. If $\mathbf{Z}_n^*[\sqrt{5}]$ denotes the multiplicative group of invertible elements of $\mathbf{Z}_n[\sqrt{5}]$, then the norm restricted to $\mathbf{Z}_n^*[\sqrt{5}]$ is a surjective homomorphism $N:\mathbf{Z}_n^*[\sqrt{5}] \to \mathbf{Z}_n^*$. That the mapping is onto can be verified by observing that the number of elements in the image of N is over half the order of $\mathbf{Z}_n^*$.

Now consider the Fibonacci sequence modulo n. Define $\rho(n)$ to be the least integer m such that $f_m \equiv 0 \pmod{n}$. Let $\sigma(n)$ be the period of repetition of the Fibonacci sequence modulo n, i.e., σ is the least positive integer m such that $f_{m+1} \equiv 1$ and $f_{m+2} \equiv 1$. The following fact is well known [5].

Lemma 2: $f_m = 0 \pmod{n} \iff \rho | m$.

This implies that $\rho | \sigma$, and define $D(n) = \sigma(n)/\rho(n)$.

2. THE PERIOD OF REPETITION

Let $n = p_1^{r_1} p_2^{r_2} \ldots p_m^{r_m}$ be the prime decomposition of n. The first theorem relates $\sigma(n)$ to the structure of the group $\mathbf{Z}_n^*[\sqrt{5}]$. The second reduces the problem to a study of the groups $\mathbf{Z}_{p_1^{r_1}}[\sqrt{5}]$, and the third further reduces it to properties of the groups $\mathbf{Z}_{p_1}[\sqrt{5}]$.

Theorem 1: If n is odd then $\sigma(n)$ is equal to the order of ε in the group $\mathbf{Z}_n^*[\sqrt{5}]$.

Theorem 2: $\sigma(n) = [\sigma(p_1^{r_1}), \sigma(p_2^{r_2}), \ldots, \sigma(p_m^{r_m})]$, where [,] denotes the least common multiple.

Theorem 3: Let s be the greatest integer $\leq r$ such that $\sigma(p^s) = \sigma(p)$. Then $\sigma(p^r) = p^{r-s}\sigma(p)$.

Proof of Theorem 1: By Lemma 1,

$$\varepsilon^\sigma = (f_{\sigma-1} + f_{\sigma+1})/2 + (f_\sigma/2)\sqrt{5} = (f_\sigma + 2f_{\sigma-1})/2 + (f_\sigma/2)\sqrt{5} = f_{\sigma-1} = 1.$$

Conversely, if $\varepsilon^m = 1$, then, again by Lemma 1, it follows that $f_m = 0$ and $f_{m-1} = 1$. Hence, m is a multiple of σ. □

Proof of Theorem 2: The proof is immediate since, for any integers a and b,

$$a \equiv b \pmod{n}$$

if and only if

$$a \equiv b \pmod{p_{1^{r_i}}}$$

for all i. □

For any group G let $|G|$ denote its order. The following result will be helpful in the next proof.

Lemma 3:

$$|\mathbf{Z}_{p^r}^*[\sqrt{5}]| = \begin{cases} p^{2r-2}(p-1)(p+1) & \text{if } p \equiv \pm 2 \pmod{5} \\ p^{2r-2}(p-1) & \text{if } p \equiv \pm 1 \pmod{5}. \end{cases}$$

Proof: By the law of quadratic reciprocity, if $p \equiv \pm 2 \pmod 5$, then 5 has no square root modulo p. A quick calculation then reveals that the elements $a + b\sqrt{5}$ in the ring $\mathbf{Z}_{p^r}^*[\sqrt{5}]$ without multiplicative inverse are of the form $a = up$ and $b = vp$ for any integers u and v with $0 \leq u < p^{r-1}$ and $0 \leq v < p^{r-1}$. Hence, $|\mathbf{Z}_{p^r}^*[\sqrt{5}]| = p^{2r} - p^{2(r-1)}$. On the other hand, if $p \equiv \pm 1 \pmod 5$), then 5 does have a square root mod p and hence a square root mod p^r. The criteria for $a + b\sqrt{5}$ to have no multiplicative inverse in $\mathbf{Z}_{p^r}^*[\sqrt{5}]$ is that

$$(a + b\sqrt{5})(a - b\sqrt{5}) \equiv a^2 - 5b^2 \equiv 0 \text{ modulo } p.$$

There are $p^{2(r-1)}(2p-1)$ solutions to this congruence, so that

$$|\mathbf{Z}_{p^r}^*[\sqrt{5}]| = p^2 \quad - p^2 \quad ^{-1} (2p-1).$$

Proof of Theorem 3: Let p be an odd prime and consider

$$g: \mathbf{Z}_{p^r}^*[\sqrt{5}] \to \mathbf{Z}_p^*[\sqrt{5}],$$

the homomorphism which takes an element of $\mathbf{Z}_{p^r}^*[\sqrt{5}]$ into its residue in $\mathbf{Z}_p^*[\sqrt{5}]$. Theorem 1 implies that $\sigma(p)|\sigma(p^r)$ and also that $\varepsilon^{\sigma(p)}$ lies in H, the kernel of g. A calculation using Lemma 3 indicates that $|H| = p^{2r-2}$ and hence the order of $\varepsilon^{\sigma(p)}$ in $\mathbf{Z}_{p^r}^*[\sqrt{5}]$ is a power of p. Since $\varepsilon^{\sigma(p)}$ belongs to H it may be represented as

$$\varepsilon^{\sigma(p)} = (1 + a_1p + a_2p^2 + \cdots + a_{r-1}p^{r-1}) + (b_1p + b_2p^2 + \cdots + b_{r-1}p^{r-1})\sqrt{5}$$

where $0 \leq a_i < p$ and $0 \leq b_i < p$ for all i. Let s be the smallest integer such that either $a_s \neq 0$ or $b_s \neq 0$. A simple induction then suffices to show that $r - s$ is the least integer k such that $\varepsilon^{\sigma(p)p^k} = 1$ in $\mathbf{Z}_{p^r}^*[\sqrt{5}]$. The above definition of s is equivalent to $\sigma(p^s) = \sigma(p)$, which completes the proof. We leave to the reader the slight alteration of method needed to show that $\sigma(2^r) = 3 \cdot 2^{r-1}$. □

These three theorems show that the problem of determining σ is equivalent to the determination of s and the order of ε in the group $\mathbf{Z}_p^*[\sqrt{5}]$ for odd primes p. Comments on the conjecture that s is always 1 will be made in §4. The next theorem gives bounds for σ in the case of an odd prime.

Theorem 4: Let $p \equiv \pm 2 \pmod 5$ and $p + 1 = 2^v \cdot k$, where k is odd. Then $\sigma | 2(p+1)$ and $2^{v+1} | \sigma$. If $p \equiv \pm 1 \pmod 5$, then $\sigma | p - 1$; furthermore, $\sqrt{5}$ exists in $\mathbf{Z}_p^*$ and σ equals the order of ε^2 as an element of $\mathbf{Z}_p^*$.

It is not always true that $\sigma = 2(p+1)$ or $\sigma = p-1$. For example, $\sigma(47) = 32$ and $\sigma(101) = 50$.

Proof of Theorem 4: Let $p = 2 \pmod 5$. Since $\mathbf{Z}_p[\sqrt{5}]$ is a finite field, $\mathbf{Z}_p^*[\sqrt{5}]$ is a cyclic group [2]. Consider the elements of norm 1, i.e., the kernel K of the map N. As a subgroup of $\mathbf{Z}_p^*[\sqrt{5}]$, K is also cyclic, and since N is surjective, $|K| = (p^2 - 1)/(p - 1) = p + 1$. The norm of ε is -1, which implies that ε^2 is an element of K. This shows that $\sigma | 2(p+1)$. Now let α be a generator of the group $\mathbf{Z}_p^*[\sqrt{5}]$. Any element of K must be of the form $\alpha^{(p-1)j}$ for some integer j. Since ε^2 belongs to K but ε does not, there must be an integer j such that $\varepsilon = \alpha^{(p-1)(j+1/2)}$. Therefore, $\sigma(p)$ is equal to the smallest positive integer m such that $p^2 - 1 | m(p-1)(j+1/2)$, which is equivalent to $2(p+1) | m(2j+1)$. Since $2j+1$ is odd, this concludes the proof for the case $p \equiv \pm 2 \pmod 5$.

Now let $p \equiv \pm 1 \pmod 5$. The fact that 5 has a square root modulo p gives rise to a canonical homomorphism $h: \mathbf{Z}_p^*[\sqrt{5}] \to \mathbf{Z}_p^*$, which takes any element of $\mathbf{Z}_p^*[\sqrt{5}]$ into its residue mod p. We can then define a map $f: \mathbf{Z}_p^*[\sqrt{5}] \to \mathbf{Z}_p^* \times \mathbf{Z}_p^*$ by $f(\alpha) = (N(\alpha), h(\alpha))$. Routine calculation bears out that f is one-one and onto and thus an isomorphism. Since $|\mathbf{Z}_p^*| = p-1$, the order of any member of $\mathbf{Z}_p^*[\sqrt{5}]$ divides $p-1$; in particular, $\sigma | p-1$. The last statement in the theorem becomes apparent by noting that the first coordinate of $f(\varepsilon^2)$ is 1. □

3. THE SMALLEST FIBONACCI NUMBER DIVISIBLE BY n

By Lemma 1, the value of $\rho(n)$ is the least positive integer m such that ε^m lies in the subgroup

$$J_1 = \left\{a + b\sqrt{5} \in \mathbf{Z}_n[\sqrt{5}] \,|\, b = 0\right\}.$$

In addition, $N(\varepsilon^\rho) = (N\varepsilon)^\rho = (-1)^\rho = \pm 1$ indicates that ρ is actually the least positive integer m such that ε^m lies in the subgroup

$$J = \left\{a + b\sqrt{5} \in \mathbf{Z}_n^*[\sqrt{5}] \,|\, b = 0 \text{ and } a^2 = \pm 1\right\}.$$

If we define $V_n = \mathbf{Z}_n^*[\sqrt{5}]/J$, and carry out proofs exactly as in §2, we obtain three theorems concerning the value of ρ corresponding to Theorems 1, 2, and 3 of §2.

Theorem 5: If n is odd, then $\rho(n)$ is equal to the order of n in the group V_n.

Theorem 6: $\rho(n) = [\rho(p_1^{r_1}), \rho(p_2^{r_2}), \ldots, \rho(p_m^{r_m})]$ where $n = p_1^{r_1} p_2^{r_2} \ldots p_m^{r_m}$ is the prime decomposition of n.

Theorem 7: For an odd prime p let t be the greatest integer $\leq r$ such that $\rho(p^t) = \rho(p)$. Then $\rho(p^r) = \rho^{r-t}(p)$. Also

$$\rho(2^r) = \begin{cases} 3 \cdot 2^{r-1} & \text{if } r = 1 \text{ or } 2 \\ 3 \cdot 2^{r-2} & \text{if } r \geq 3. \end{cases}$$

The final theorems describe the relationship between ρ and σ and give bounds for ρ in the case of an odd prime.

Theorem 8: If $n = 2^{r_0} p_1^{r_1} p_2^{r_2} \ldots p_m^{r_m}$ where the p_i are distinct odd primes, then $\rho = \sigma / D(n)$ with

$$D(n) = \begin{cases} 1 & \text{if } r_0 \le 2 \text{ and } D(p_i) = 1 \text{ for all } i \\ 4 & \text{if } r_0 \le 1 \text{ and } D(p_i) = 4 \text{ for all } i \\ 2 & \text{otherwise} \end{cases}$$

and for an odd prime p,

$$D(p) = \begin{cases} 1 & \text{if } p \equiv 11 \text{ or } 19 \pmod{20} \\ 2 & \text{if } p \equiv 3 \text{ or } 7 \pmod{20} \\ 4 & \text{if } p \equiv 13 \text{ or } 17 \pmod{20} \\ 1 \text{ or } 4 & \text{if } p \equiv 21 \text{ or } 20 \pmod{40} \\ 1,\ 2, \text{ or } 4 & \text{if } p \equiv 1 \text{ or } 9 \pmod{40}. \end{cases}$$

Theorem 9: Let p be an odd prime and express $p + 1 = 2^v \cdot k$, where k is odd.

If $p \equiv 3$ or $7 \pmod{20}$, then $\rho | p + 1$ and $2^v | \rho$
If $p \equiv 13$ or $17 \pmod{20}$, then $\rho | (p + 1)/2$ and $2^{v-1} | \rho$
If $p \equiv 1 \pmod{5}$, then $\rho | p - 1$.

The proofs will utilize the following lemma.

Lemma 4: For n odd,

$$\begin{aligned} D(n) = 1 &\Longleftrightarrow \rho \equiv 2 \pmod{4} \Longleftrightarrow \sigma \equiv 2 \text{ or } 6 \pmod{8} \\ D(n) = 2 &\Longleftrightarrow \rho \equiv 0 \pmod{4} \Longleftrightarrow \sigma \equiv 0 \pmod{8} \\ D(n) = 4 &\Longleftrightarrow \rho \equiv 1 \text{ or } 3 \pmod{4} \Longleftrightarrow \sigma \equiv 4 \pmod{8}. \end{aligned}$$

Proof: By Lemma 1, we have in $\mathbf{Z}_n[\sqrt{5}]$,

$$\begin{aligned} \varepsilon^{\rho} &= f_{\rho-1} \\ \varepsilon^{2\rho} &= f_{\rho-1}^2 = f_{\rho} f_{\rho-2} + (-1)^{\rho} = (-1) \\ \varepsilon^{4\rho} &= 1 \end{aligned}$$

so that $D = 1$, 2, or 4. We will prove the above equivalences in the following order.

$D = 4 \Longleftrightarrow \rho \equiv 1$ or $3 \pmod{4}$: $\rho \equiv 1 \pmod{2} \Longleftrightarrow \varepsilon^{2\rho} = -1 \Longleftrightarrow D = 4$.

$D = 1 \Longleftrightarrow \rho \equiv 2 \pmod{4}$: If $D = 1$, then $(\varepsilon^{\rho/2})^2 = \varepsilon^{\rho} = 1$. Now $\varepsilon^{\rho/2} = \pm 1$ would contradict the fact that f_{ρ} is the least Fibonacci number divisible by n. Since $+1$ and -1 are the only square roots of 1 with norm 1, $\varepsilon^{\rho/2}$ has norm -1. Then $-1 = N(\varepsilon^{\rho/2}) = (N\varepsilon)^{\rho/2} = (-1)^{\rho/2}$ implies $\rho \equiv 2 \pmod{4}$.

$D = 2 \Longleftrightarrow \rho \equiv 0 \pmod{4}$: Assume $D = 2$. Since $D \ne 4$, ρ is even and $N(\varepsilon^{\rho}) = (N\varepsilon)^{\rho} = 1$. Therefore, $\varepsilon^{2\rho} = 1$ implies $\varepsilon^{\rho} = -1$. Then $\varepsilon^{\ 2}$ is a square root of -1. A small calculation shows that the only square roots of -1 in $[\sqrt{5}]$ with norm -1 lie in J. However, $\varepsilon^{\rho/2}$ cannot lie in J by Theorem 5 and thus has norm $+1$. Now $1 = N(\varepsilon^{\rho/2}) = (N\varepsilon)^{\rho/2} = (-1)^{\ 2}$ implies $\rho \equiv 0 \pmod{4}$. The remaining implications follow logically and immediately from the above. □

Proof of Theorem 8: Let p be an odd prime. If $p \equiv 3$ or $7 \pmod{20}$, then by Theorem 4, $\sigma \equiv 0 \pmod{8}$ and by Lemma 4, $D = 2$. If $p \equiv 13$ or $17 \pmod{20}$, then $\sigma \equiv 4 \pmod{8}$ by Theorem 4 and $D = 4$ by Lemma 4. If $p = 11$ or $19 \pmod{20}$, then by Theorem 4, $\sigma | p - 1$, which implies that $\sigma \equiv 2$ or $6 \pmod{8}$. Then by Lemma 4, $D = 1$. If $p \equiv 21$ or 29 (mod 40), then $\sigma | p - 1$ implies that $\sigma \not\equiv 0$ (mod 8). By Lemma 4, $D \neq 2$. This concludes the proof of the second part of the theorem. By Theorems 2 and 6, a formula for $D(n)$ is obtained:

$$D(n) = \frac{[D(2^{r_0})\rho(2^{r_0}),\ D(p_1^{r_1})\rho(p_1^{r_1}),\ \ldots,\ D(p_1^{r_m})\rho(p_m^{r_m})]}{[\rho(2^{r_0}),\ \rho(p_1^{r_1}),\ \ldots,\ \rho(p_m^{r_m})]}.$$

For an odd prime p, we have, by Theorems 3 and 7,

$$\sigma(p^r)/\rho(p^r) = p^{r-s}\sigma(p)/p^{r-t}\rho(p) = p^{t-s}\sigma(p)/\rho(p).$$

Since this value is either 1, 2, or 4, it must be the case that $s = t$, and hence, $D(p^r) = D(p)$. The formula above reduces to

$$D(n) = \frac{[D(2^{r_0})\rho(2^{r_0}),\ D(p_1)\rho(p_1),\ \ldots,\ D(p_m)\rho(p_m)]}{[\rho(2^{r_0}),\ \rho(p_1),\ \ldots,\ \rho(p_m)]}.$$

A routine checking of all cases—using Lemma 4, the formula above, and the formulas for $\sigma(2^r)$ and $\rho(2^r)$—verifies the remainder of Theorem 8. □

Theorem 9 is now an immediate consequence of Theorems 4 and 8.

4. RELATED TOPICS

Several questions remain open. We would like to know, for example, whether a formula for $D(p)$ is possible when $p \equiv 1$ or 9 (mod 20).

One may also ask whether $\sigma(p^2) \neq \sigma(p)$ for all odd primes p. If so, our formulas of Theorems 3 and 7 would be simplified so that $s = t = 1$. This question has been asked earlier by D. D. Wall [6]. Penny & Pomerance claim to have verified it for $p \leq 177{,}409$ [4]. Using Theorem 1, the conjecture is equivalent to $\varepsilon^{p^2-1} \neq 1$ in $\mathbf{Z}_{p^2}^*[\sqrt{5}]$. A similar equality $2^{p-1} = 1$ in $\mathbf{Z}_{p^2}^*$ has been extensively studied, and the first counterexample is $p = 1093$. The analogy between the two makes the existence of a large counterexample to $\sigma(p^2) \neq \sigma(p)$ seem likely.

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CONGRUENT PRIMES OF FORM $(8r + 1)$

J. A. H. HUNTER

An integer e is congruent if there are known integral solutions for the system $X^2 - eY^2 = Z^2$, and $X^2 + eY^2 = Z^2$. At present, we can be sure that a particular number is congruent only if corresponding X, Y values have been determined.

However, it has been stated and accepted that integers of certain forms cannot be congruent. Proofs exist for most of those excluding conditions, but not for all—no counterexamples having been discovered as regards the latter. For example, a prime of form ($8r + 3$), or the product of two such primes, cannot be congruent.

L. Bastien and others have stated that a prime of form ($8r + 1$), representable as ($k^2 + t^2$) cannot be congruent if ($k + t$) is not a quadratic residue of that prime. But no proof of this has been known to exist in the literature.

The necessary proof will be developed in this paper.

We first show that the situations regarding primes of form ($8r + 1$), and those of form ($8r + 5$), are not the same. For this we use the Collins analysis method.

It is well known that every congruent number must be of form $uv(u^2 - v^2)/g^2$. Then, if e be a prime of form ($8r + 5$) or ($8r + 1$), for congruent e we must have solutions to $uv(u^2 - v^2) = eg^2$: from which it follows that one of u, v, $(u - v)$, $(u + v)$ must be ea^2, say, and the other three must all be squares.

Consider each of the four possibilities.

(1) $u + v = ea^2$, $u - v = b^2$, $u = c^2$, $v = d^2$.

Then, $b^2 - 2c^2 = -ea^2$:
possible with $e = 8r + 1$; impossible with $e = 8r + 5$.

Similarly, $b^2 + 2d^2 = ea^2$:
possible with $e = 8r + 1$; impossible with $e = 8r + 5$.

Also, $c^2 + d^2 = ea^2$, and $c^2 - d^2 = b^2$:
both possible for $e = 8r + 1$ and for $e = 8r + 5$.

Hence, this case (1) applies to $e = 8r + 1$, but not to $e = 8r + 5$.

(2) $u - v = ea^2$, $u + v = b^2$, $u = c^2$, $v = d^2$.

Then, $b^2 - 2c^2 = -ea^2$:
possible with $e = 8r + 1$; impossible with $e = 8r + 5$.

Similarly, $b^2 - 2d^2 = ea^2$:
possible with $e = 8r + 1$; impossible with $e = 8r + 5$.

Also, $c^2 - d^2 = ea^2$, and $c^2 + d^2 = b^2$:
both possible for $e = 8r + 1$ and for $e = 8r + 5$.

Hence, this case (2) applies to $e = 8r + 1$, but not to $e = 8r + 5$.

(3) $u = ea^2$, $u + v = b^2$, $u - v = c^2$, $v = d^2$.

Then, $b^2 + c^2 = 2ea^2$, $b^2 - c^2 = 2d^2$, $b^2 - d^2 = ea^2$, and $c^2 + d^2 = ea^2$:
All possible for both $e = 8r + 1$ and $e = 8r + 5$.

Hence, this case (3) applies to both.

(4) $v = ea^2$, $u + v = b^2$, $u - v = c^2$, $u = d^2$.

Then, $b^2 - c^2 = 2ea^2$, $b^2 + c^2 = 2d^2$, $b^2 - d^2 = ea^2$, and $d^2 - c^2 = ea^2$:
All possible for both $e = 8r + 1$ and $e = 8r + 5$.

Hence, this case (4) applies to both.

So, for $e = 8r + 5$, we have possible:

Case (3) $\left.\begin{array}{l} x^2 + y^2 = 2ez^2 \\ x^2 - y^2 = 2w^2 \end{array}\right\}$ Case (4) $\left.\begin{array}{l} x^2 + y^2 = 2z^2 \\ x^2 - y^2 = 2ew^2 \end{array}\right\}$

But, for $e = 8r + 1$, we have possible:

Case (1) $\left.\begin{array}{l} x^2 + y^2 = ez^2 \\ x^2 - y^2 = w^2 \end{array}\right\}$ Case (2) $\left.\begin{array}{l} x^2 + y^2 = z^2 \\ x^2 - y^2 = ew^2 \end{array}\right\}$

Case (3) $\left.\begin{array}{l} x^2 + y^2 = 2ez^2 \\ x^2 - y^2 = 2w^2 \end{array}\right\}$ Case (4) $\left.\begin{array}{l} x^2 + y^2 = 2z^2 \\ x^2 - y^2 = 2ew^2 \end{array}\right\}$

We now show that each of the subsidiary-equation systems (1), (2), and (3) will provide a solution for the system (4) for any congruent number prime $(8r + 1)$.

From (1) to (4):

Say $x^2 + y^2 = ez^2$, $x^2 - y^2 = w^2$, and $A^2 + B^2 = 2C^2$, $A^2 - B^2 = 2eD^2$.

Setting $A = x^4 + 2x^2y^2 - y^4$, $B = x^4 - 2x^2y^2 - y^4$, we have

$$A^2 + B^2 = 2(x^4 + y^4)^2,\ A^2 - B^2 = 2e \cdot (2xyzw)^2.$$

As an example,

$$\left.\begin{array}{l} 5^2 + 4^2 = 41 \cdot 1^2 \\ 5^2 - 4^2 = 3^2 \end{array}\right\} \qquad \left.\begin{array}{l} 1169^2 + 431^2 = 2 \cdot 881^2 \\ 1169^2 - 431^2 = 2 \cdot 41 \cdot 120^2 \end{array}\right\}$$

From (2) to (4):

Say $x^2 + y^2 = z^2$, $x^2 - y^2 = ew^2$, and $A^2 + B^2 = 2C^2$, $A^2 - B^2 = 2eD^2$.

Setting $A = x^4 + 2x^2y^2 - y^4$, $B = x^4 - 2x^2y^2 - y^4$; we have

$$A^2 + B^2 = 2(x^4 + y^4)^2,\ A^2 - B^2 = 2e \cdot (2xyzw)^2.$$

As an example,

$$\left.\begin{array}{l} 21^2 + 20^2 = 29^2 \\ 21^2 - 20^2 = 41 \cdot 1^2 \end{array}\right\} \qquad \left.\begin{array}{l} 387281^2 + 318319^2 = 2 \cdot 354481^2 \\ 387281^2 - 318319^2 = 2 \cdot 41 \cdot 24360^2 \end{array}\right\}$$

From (3) to (4):

Say $x^2 + y^2 = 2ez^2$, $x^2 - y^2 = w^2$, and $A^2 + B^2 = 2C^2$, $A^2 - B^2 = 2eD^2$.

Setting $A = (ez^2)^2 + 2ez^2w^2 - w^4$, $B = (ez^2)^2 - 2ez^2w^2 - w^4$, we have

$$A^2 + B^2 = 2[(ez^2)^2 + w^4]^2,\ A^2 - B^2 = 2e \cdot (2xyzw)^2.$$

As an example,

$$\left.\begin{array}{l} 33^2 + 31^2 = 82 \cdot 5^2 \\ 33^2 - 31^2 = 2 \cdot 8^2 \end{array}\right\} \qquad \left.\begin{array}{l} 1177729^2 + 915329^2 = 2 \cdot 1054721^2 \\ 1177729^2 - 915329^2 = 2 \cdot 41 \cdot 81840^2 \end{array}\right\}$$

We may also consider the system (4) itself:

Say $x^2 + y^2 = 2z^2$, $x^2 - y^2 = 2ew^2$.

From the first of the two equations we require

$$x = u^2 + 2uv - v^2,\ y = u^2 - 2uv - v^2,\ z = u^2 + v^2.$$

Then $x^2 - y^2 = (2u^2 - 2v^2)4uv$,
whence, $4uv(u^2 - v^2) = ew^2$, which we know has solutions if e is a congruent number.

Now, having shown that each of the four possible systems of subsidiary equasions, for prime e of form $8r + 1$, must have solutions if e is to be congruent—and that system (4) is linked to each of the other three systems—a proof that any one of the four systems will not have solutions for any particular value of e must be proof that no other of the four systems can have solutions. Accordingly, we now show that e cannot be congruent if $e = k^2 + t^2$, and $(k + t)$ is not a quadratic residue of e. For this we investigate the subsidiary-equation system (1).

Say e is a prime of form $(8r + 1)$, represented uniquely as $k^2 + t^2$.

We have the system: $x^2 + y^2 = ez^2$, $x^2 - y^2 = w^2$. Thence,

$$(kz)^2 = x^2 + y^2 - (tz)^2,$$

with solution

$$\left.\begin{aligned} kz &= a^2 + b^2 - c^2 \\ tz &= 2ac \end{aligned}\right\} \qquad \left.\begin{aligned} x &= a^2 - b^2 + c^2 \\ y &= 2ab \end{aligned}\right\} \; \ldots \; (M)$$

hence,

$$2kac = ta^2 + tb^2 - tc^2,$$

making

$$t^2c^2 + 2ktac - t^2a^2 = t^2b^2$$

whence,

$$(tc + ka)^2 - (ka)^2 - (ta)^2 = (tb)^2$$

so

$$(tc + ka)^2 - ea^2 = (tb)^2,$$

with solution

$$\left.\begin{aligned} tc + ka &= m^2 + en^2 \\ ka &= 2kmn \end{aligned}\right\} \qquad \left.\begin{aligned} tc &= m^2 - 2kmn + en^2 \\ tb &= m^2 - en^2 \end{aligned}\right\}$$

Without loss of generality, that becomes

$$a = 2tmn,\ b = m^2 - en^2,\ c = m^2 - 2kmn + en^2.$$

Substituting in (M), and omitting the common term $4mn$, we get

$$x = km^2 - 2emn + ken^2,\ y = t(m^2 - en^2),$$

whence

$$x + y = (k + t)m^2 - 2emn + (k - t)^2$$

and

$$x - y = (k - t)m^2 - 2emn + (k + t)^2.$$

Now, since we have $x^2 + y^2 = ez^2$, with e an odd prime, x and y cannot be of same parity. Hence, each of $(x + y)$ and $(x - y)$ must be a square.

So, say, $x + y = p^2$. Then,

$$[(k + t)m - en]^2 - 2e(tn)^2 = (k + t)p^2,$$

which is possible only if $(k + t)$ is a quadratic residue of e.

That completes the proof that a prime of form $(8r + 1)$, uniquely represented as $(k^2 + t^2)$, cannot be congruent if $(k + t)$ is a quadratic nonresidue of e.

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SOME CLASSES OF FIBONACCI SUMS

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1. INTRODUCTION

Layman [3] recalled the formulas [2]

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k, \tag{1.1}$$

$$2^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{3k}, \tag{1.2}$$

$$3^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{4k}, \tag{1.3}$$

where, as usual, the F_n are the Fibonacci numbers defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

As Layman remarks, the three identities suggest the possibility of a general formula of which these are special instances. Several new sums are given in [2]. Many additional sums occur in [1].

Layman does not obtain a satisfactory generalization; however, he does obtain a sequence of sums that include (1.1), (1.2), and (1.3). In particular, the following elegant formulas are proved:

$$5^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k}, \tag{1.4}$$

$$8^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{6k}, \tag{1.5}$$

$$F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{2k}, \tag{1.6}$$

$$5^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{5k}. \tag{1.7}$$

He notes also that each of the sums he obtains remains valid when F_n is replaced by L_n, where the L_n are the Lucas numbers defined by

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1} \qquad (n \geq 1).$$

In the present paper, we consider the following question. Let p, q be fixed positive integers. We seek all pairs λ, μ such that

$$\lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \qquad (n = 0, 1, 2, \ldots). \tag{1.8}$$

It is easily seen that $p \neq q$. We shall show that (1.8) holds if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{1.9}$$

Since (1.8) is equivalent to

$$(-\mu)^n F_{qn} = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^k F_{pk} \qquad (n = 0, 1, 2, \ldots), \tag{1.10}$$

we may assume that $p < q$. However, this is not necessary since we may take $F_{-n} = (-1)^{n-1} F_n$. Also, the final result is in fact for all p, q, $p \neq q$.

For the Lucas numbers, we consider

$$\lambda^n L_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk} \qquad (n = 0, 1, 2, \ldots). \tag{1.11}$$

We show that (1.11) holds if and only if λ, μ satisfy (1.9) or

$$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}. \tag{1.9$'$}$$

In the next place, if w denotes a root of $x^2 = x + 1$, we show that

$$\lambda^n w^{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k w^{qk} \qquad (n = 0, 1, 2, \ldots), \tag{1.12}$$

if and only if λ, μ satisfy (1.9).

The stated results concerning (1.8) and (1.11) can be carried over to the more general

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots) \tag{1.13}$$

and

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk+r} \qquad (n = 0, 1, 2, \ldots), \tag{1.14}$$

where r is an arbitrary integer. We show that (1.13) holds if and only if λ, μ satisfy (1.9); thus, the result for (1.13) includes that for (1.8). However, (1.14), with $r \neq 0$, holds if and only if λ, μ satisfy (1.9); thus, the result for (1.13) includes that for (1.8). But (1.14), with $r \neq 0$, holds if and only if λ, μ satisfy (1.9); thus, the values (1.9)$'$ for λ, μ apply only in the case $r = 0$.

As for

$$\lambda^n w^{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k w^{qk+r} \qquad (n = 0, 1, 2, \ldots),$$

it is obvious that this is equivalent to (1.12) for all r.

The formulas (1.8), (1.11), (1.12), (1.13), (1.14) with λ, μ satisfying (1.9) can all be written in such a way that they hold for all p, q. For example, (1.8) becomes

$$F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{p(n-k)} \binom{n}{k} F_p F_{q-p}^{n-k} F_{qk}. \tag{1.15}$$

For $p = q$, this reduces to a mere tautology. However, for (1.11) with λ, μ defined by (1.9), we have

$$F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p F_{p+q}^{n-k} L_{qk}. \tag{1.16}$$

For $q = p$, this reduces to

$$L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} L_{pk}. \tag{1.17}$$

Note that (1.15) and (1.16) had been obtained in [1].

For some remarks concerning (1.17) see §7 below. In particular, the following pair of formulas is obtained:

$$(-1)^r L_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} L_{pk+r}, \tag{1.18}$$

$$(-1)^{r-1} F_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} F_{pk+r}, \tag{1.19}$$

where r is an arbitrary integer.

Formulas (1.18) and (1.19) differ from (1.13) and (1.14) in a rather essential way. The former pair suggest the problem of determining λ, μ, C_r such that

$$C_r \lambda^n L_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu^k L_{pk+r},$$

and similarly for

$$C_r \lambda^n F_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu^k F_{pk+r},$$

where C_r depends only on r. This is left for another paper.

SECTION 2

Let a, b denote the roots of $x^2 = x + 1$. We recall that

$$F_n = \frac{a^n - b^n}{a - b}, \quad L_n = a^n + b^n. \tag{2.1}$$

Thus, the equation

$$\lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \tag{2.2}$$

becomes

(2.3) $$\lambda^n(a^{pn} - b^{pn}) = \sum_{k=0}^{n} \binom{n}{k}\mu^k(a^{qk} - b^{qk}).$$

Multiplying both sides of (2.3) by x and summing over n we get

$$\begin{aligned}\frac{1}{1 - \lambda a^p x} - \frac{1}{1 - \lambda b^p n} &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k}\mu^k(a^{qk} - b^{qk}) \\ &= \sum_{k=0}^{\infty}\mu^k(a^{qk} - b^{qk})x^k \sum_{n=0}^{\infty}\binom{n+k}{k}x^n \\ &= \sum_{k=0}^{\infty}\mu^k(a^{qk} - b^{qk})\frac{x^k}{(1 - x)^{k+1}} \\ &= \frac{1}{1 - x}\left\{\frac{1}{1 - \frac{\mu a^q x}{1 - x}} - \frac{1}{1 - \frac{\mu b^q x}{1 - x}}\right\}.\end{aligned}$$

Since

$$\frac{1}{a - b}\left(\frac{1}{1 - a^p z} - \frac{1}{1 - b^p z}\right) = \frac{1}{1 - L_p z + (-1)^p z^2},$$

it follows that

(2.4) $$\frac{\lambda F_p}{1 - \lambda L_p x + (-1)^p\lambda^2 x^2} = \frac{\mu F_q}{(1 - x)^2 - \mu L_q x(1 - x) + (-1)^q\mu^2 x^2}.$$

For $x = 0$, this reduces to

(2.5) $$\lambda F_p = \mu F_q.$$

Thus,

(2.6) $$1 - \lambda L_p x + (-1)^p\lambda^2 x^2 = (1 - x)^2 - \mu L_q x(1 - x) + (-1)^q\mu^2 x^2.$$

Equating coefficients of x and x^2, we get

(2.7) $$\lambda L_p = 2 + \mu L_q$$

and

(2.8) $$(-1)^p\lambda^2 = 1 + \mu L_q + (-1)^q\mu^2,$$

respectively.

Now by (2.5) and (2.7), we have

$$\lambda L_p F_q = 2F_q + \mu L_p L_q = 2F_q + \lambda F_p L_q,$$

so that

(2.9) $$\lambda(L_p F_q - F_p L_q) = 2F_q.$$

It is easily verified that

(2.10) $$L_p F_q - F_p L_q = 2(-1)^{q-1}F_{p-q}.$$

Hence, (2.9) yields

(2.11) $$\lambda = (-1)^p\frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p\frac{F_p}{F_{q-p}};$$

the second equality is of course a consequence of (2.5).

It remains to consider the condition (2.8). We shall show that (2.8) is implied by (2.11), or, what is the same, by (2.5) and (2.7). To do this with a minimum of computation, note that (2.5), (2.7), (2.8) can be replaced by

$$(2.5)' \qquad \lambda(a^p - b^p) = (1 + \mu a^q) - (1 + \mu b^q),$$

$$(2.7)' \qquad \lambda(a^p + b^p) = (1 + \mu a^q) + (1 + \mu b^q),$$

$$(2.8)' \qquad \lambda^2(ab)^p = (1 + \mu a^q)(1 + \mu b^q),$$

respectively. Subtracting the square of (2.5)′ from the square of (2.7)′, we get (2.8)′.

We have therefore proved that (2.5) and (2.7) imply both (2.8) and (2.11). Conversely, (2.11) implies (2.5) and (2.7). The first implication, (2.11) (2.5) is immediate. As for (2.11) $\rightarrow$ (2.7), we have

$$\lambda L_p - \mu L_q = (-1)^p \cdot \frac{L_pF_q - F_pL_q}{F_{q-p}} = (-1)^q \cdot \frac{2(-1)^pF_{q-p}}{F_{q-p}},$$

by (2.10). Hence, $\lambda L_p - \mu L_q = 2$.

This completes the proof of the following:

Theorem 1: Let p, q be fixed positive integers, $p \neq q$. Then,

$$(2.12) \qquad \lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \qquad (n = 0, 1, 2, \ldots),$$

if and only if

$$(2.13) \qquad \lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

Thus, we have the explicit identities

$$(2.14) \qquad (-1)^{pn} F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{pk} \binom{n}{k} F_p F_{q-p}^{n-k} F_{qk} \qquad (n = 0, 1, 2, \ldots).$$

If we use the fuller notation $\lambda(p, q)$, $\mu(p, q)$ for λ, μ in (2.13), then,

$$\mu(q, p) = (-1)^{-p-1} \frac{F_p}{F_{q-p}},$$

so that

$$(2.15) \qquad \mu(q, p) = -\lambda(p, q).$$

In proving Theorem 1, we have not made any use of the positivity of p and q. All that is required is that p and q are distinct nonzero integers. This observation gives rise to additional identities. Replacing p by $-p$ in (2.13) we get

$$(2.16) \qquad \lambda(-p, q) = (-1)^p \frac{F_q}{F_{p+q}}, \quad \mu(-p, q) = -\frac{F_p}{F_{p+q}},$$

and (2.14) becomes

$$(2.17) \qquad F_q F_{pn} = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p^k F_{p+q}^{n-k} F_{qk} \qquad (n = 0, 1, 2, \ldots).$$

Comparison of (2.17) with (2.14) yields

$$(2.18) \qquad (-1)^{pn-1}\sum_{k=0}^{n}(-1)^{pk}\binom{n}{k}F_p F_{p+q}^{n-k}F_{qk} = \sum_{k=0}^{n}(-1)^k F_p^k F_{p+q}^{n-k}F_{qk}$$

$$(n = 0, 1, 2, \ldots; p^2 \neq q^2).$$

Similarly,

$$(2.19) \qquad \begin{aligned} \lambda(p, -q) &= \frac{F_q}{F_{p+q}} = (-1)^p \lambda(-p, q), \\ \mu(p, -q) &= (-1)^{q-1}\frac{F_p}{F_{p+q}} = (-1)^q \mu(-p, q) \end{aligned}$$

and we again get (2.17).

Finally, the formulas

$$(2.20) \qquad \begin{aligned} \lambda(-p, -q) &= \frac{F_q}{F_{p+q}} = (-1)^p \lambda(p, q), \\ \mu(-p, -q) &= (-1)^q \frac{F_p}{F_{p+q}} = (-1)^{p+q}\mu(p, q) \end{aligned}$$

again lead to (2.14).

We remark that for $q = p + 1$ and $p + 2$, (2.14) reduces to

$$(2.21) \qquad F_{p+1}^n F_{pn} = \sum_{k=0}^{n}(-1)^{p(n-k)}\binom{n}{k}F_p^k F_{(p+1)k}$$

and

$$(2.22) \qquad F_{p+2}^n F_{pn} = \sum_{k=0}^{n}(-1)^{p(n-k)}\binom{n}{k}F_p^k F_{(p+2)k},$$

respectively.

SECTION 3

We now consider

$$(3.1) \qquad \lambda^n L_{pn} = \sum_{k=0}^{n}\binom{n}{k}\mu^k L_{qn} \qquad (n = 0, 1, 2, \ldots),$$

where p, q are distinct nonzero integers. Since $L_n = a^n + b^n$, we have

$$\lambda^n (a^{pn} + b^{pn}) = \sum_{k=0}^{n}\binom{n}{k}\mu^k (a^{qn} + b^{qn}).$$

Hence,

$$\begin{aligned} \frac{1}{1 - \lambda a^p x} + \frac{1}{1 - \lambda b^p x} &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}\binom{n}{k}\mu^k (a^{qk} + b^{qk}) \\ &= \frac{1}{1 - x}\left\{\frac{1}{1 - \frac{\mu a^q x}{1 - x}} + \frac{1}{1 - \frac{\mu b^q x}{1 - x}}\right\}, \end{aligned}$$

so that

$$\text{(3.2)} \qquad \frac{2 - \lambda L_p x}{1 - \lambda L_p x + (-1)^p \lambda^2 x^2} = \frac{2 - (2 + \mu L_q)x}{1 - (2 + \mu L_q)x + \left(1 + \mu L_q + (-1)^q \mu^2\right)x^2}.$$

Equating coefficients and simplifying, we get

$$\text{(3.3)} \qquad \begin{cases} \lambda L_p = 2 + \mu L_q \\ (-1)^p \lambda^2 = 1 + \mu L_q + (-1)^q \mu^2. \end{cases}$$

Coefficients of x^2 and of x^3 both lead to the second of (3.3).

We can rewrite (3.3) in the form

$$\text{(3.4)} \qquad \begin{cases} \lambda(a^p + b^p) = (1 + \mu a^q) + (1 + \mu b^q) \\ \lambda^2 (ab)^p = (1 + \mu a^q)(1 + \mu b^q). \end{cases}$$

Squaring the first of (3.4) and subtracting four times the second, we get

$$\lambda^2 (a^p - b^p)^2 = \mu^2 (a^q - b^q)^2,$$

and therefore,

$$\lambda F_p = \pm \mu F_q.$$

If we take $\lambda F_p = \mu L_q$, then, by the first of (3.3),

$$\lambda L_p F_q = 2F_q + \mu L_q F_q = 2F_q + \lambda L_q F_p,$$

that is,

$$\text{(3.5)} \qquad \lambda(L_p F_q - L_q F_p) = 2F_q.$$

Since, by (2.10),

$$L_p F_q - L_q F_p = 2(-1)^p F_{q-p},$$

we get

$$\text{(3.6)} \qquad \lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

On the other hand, if $\lambda L_p = -\mu L_q$, then

$$\lambda(L_p F_q + L_q F_p) = 2F_q,$$

which reduces to

$$\text{(3.7)} \qquad \lambda = \frac{F_q}{F_{p+q}} = \lambda(p, -q), \quad \mu = -\frac{F_p}{F_{p+q}} = \mu(-p, q).$$

This completes the proof of

Theorem 2: Let p, q be fixed nonzero integers, $p \neq q$. Then,

$$\text{(3.8)} \qquad \lambda^n L_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{pk} \qquad (n = 0, 1, 2, \ldots),$$

if and only if λ and μ satisfy either (3.6) or (3.7).

Thus we have the explicit identities

$$\text{(3.9)} \qquad (-1)^{pn} F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^{pk} \binom{n}{k} F_p^k F_{q-p}^{n-k} L_{qk} \qquad (n = 0, 1, 2, \ldots)$$

and

$$\text{(3.10)} \qquad F^n L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p^k F_{p+q}^{n-k} L_{qk} \qquad (n = 0, 1, 2, \ldots).$$

Note that (3.9) becomes (3.10) if p is replaced by $-p$ or q is replaced by $-q$.

SECTION 4

Let w be a root of $x^2 = x + 1$ and consider

$$\lambda^n w^{pn} = \sum_{k=0}^{n} \mu^k w^{qk} \qquad (n = 0, 1, 2, \ldots), \tag{4.1}$$

where p, q are fixed nonzero integers, $p \neq q$, and λ and μ are assumed to be rational. Since (4.1) is simply

$$\lambda^n w^{pn} = (1 + \mu w^q)^n \qquad (n = 0, 1, 2, \ldots),$$

it suffices to take $n = 1$:

$$\lambda w^p = 1 + \mu w^q. \tag{4.2}$$

Recall that

$$w^n = F_n w_n + F_{n-1} \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{4.3}$$

so that (4.2) becomes

$$\lambda(F_p w + F_{p-1}) = 1 + \mu(F_q w + F_{q-1}).$$

Since λ and μ are assumed to be rational, we have

$$\begin{cases} \lambda F_p = \mu F_q \\ \lambda F_{p-1} = 1 + \mu F_{q-1}. \end{cases} \tag{4.4}$$

Eliminating μ, we get

$$\lambda(F_{p-1}F_q - F_p F_{q-1}) = F_q.$$

It is easily verified that

$$F_{p-1}F_q - F_p F_{q-1} = (-1)^p \frac{F_p}{F_{q-p}},$$

and therefore,

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{4.5}$$

We state

Theorem 3: Let w denote a root of $x^2 = x + 1$ and let p, q be fixed nonzero integers, $p \neq q$. Then,

$$\lambda w^p = 1 + \mu w^q, \tag{4.6}$$

where p and q are rational, if and only if (4.5) is satisfied. Hence, (4.6) becomes

$$F_q w^p = (-1)^p F_{q-p} + F_p w^q. \tag{4.7}$$

It follows from (4.7) that

$$F_q^n w^{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p w^{qk},$$

and therefore we get both

$$F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p^k F_{qk} \tag{4.8}$$

and

$$F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p L_{qk}, \tag{4.9}$$

in agreement with (2.14) and (3.9). However, this does not prove Theorems 1 and 2.

SECTION 5

We now discuss

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots), \tag{5.1}$$

where $p \neq q$ but p, q, r are otherwise unrestricted. One would expect that the parameters λ, μ depend on r as well as p and q. However, as will be seen below, λ and μ are in fact independent of r.

It follows from (5.1) that

$$\frac{a^r}{1 - \lambda a^p x} - \frac{b^r}{1 - \mu b^p x} = \frac{1}{1 - x} \left\{ \frac{a^r}{1 - \frac{\mu a^q x}{1 - x}} - \frac{b^r}{1 - \frac{\mu b^q x}{1 - x}} \right\}$$

$$= \frac{a^r}{1 - (1 + \mu a^q)x} - \frac{b^r}{1 - (1 + \mu b^q)x},$$

so that

$$a^r \sum_{k=0}^{\infty} \lambda^k a^{pk} x - b^r \sum_{k=0}^{\infty} \lambda^k b^{pk} x^k = a^r \sum_{k=0}^{\infty} (1 + \mu a^q)^k x^k - b^r \sum_{k=0}^{\infty} (1 + \mu b^q)^k x^k.$$

Equating coefficients of x , we get

$$a^r(\lambda^k a^{pk} - (1 + \mu a^q)^k) = b^r(\lambda^k b^{pk} - (1 + \mu b^q)^k) \tag{5.2}$$

$$(k = 0, 1, 2, \ldots).$$

For $k = 1$, (5.2) implies

$$\lambda F_{p+r} = F_r + \mu F_{q+r}. \tag{5.3}$$

We now consider separately two possibilities:

(i) $\lambda a^p = 1 + \mu a^q$;
(ii) $\lambda a^p \neq 1 + \mu a^q$.

It is clear from

$$a^r(\lambda a^p - (1 + \mu a^q)) = b^r(\lambda b^p - (1 + \mu b^q))$$

that (i) implies

$$\lambda b^p = 1 + \mu b^q. \tag{5.4}$$

Subtracting (5.4) from (i), we get

$$\lambda F_p = \mu F_q. \tag{5.5}$$

Hence, again using (i),

$$(a^p F_q - a^q F_p)\lambda = F_q.$$

Since

$$a^pF_q - a^qF_p = \frac{1}{a-b}(a^p(a^q - b^q) - a^q(a^p - b^p))$$
$$= \frac{a^qb^p - a^pb^q}{a-b} = (-1)^pF_{q-p},$$

it follows that

(5.6) $$\lambda = (-1)^p\frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p\frac{F_p}{F_{q-p}}.$$

We now assume (ii). Take $k = 1, 2, 3$ in (5.2):

$$a^r(\lambda a^p - (1 + \mu a^q)) = b^r(\lambda b^p - (1 + \mu b^q))$$
$$a^r(\lambda^2 a^{2p} - (1 + \mu a^q)^2) = b^r(\lambda^2 b^{2p} - (1 + \mu b^q)^2)$$
$$a^r(\lambda^3 a^{3p} - (1 + \mu a^q)^3) = b^r(\lambda^3 b^{3p} - (1 + \mu b^q)^3).$$

Dividing the second and third by the first, we get

(5.7)
$$\lambda a^p + (1 + \mu a^q) = \lambda b^p + (1 + \mu b^q)$$
$$\lambda^2 a^{2p} + a^p(1 + \mu a^q) + (1 + \mu a^q) = \lambda^2 b^{2p} + \lambda b\ (1 + \mu b^q) + (1 + \mu b^q)^2.$$

The first of (5.7) yields

(5.8) $$\lambda F_p + \mu F_q = 0$$

while the second gives

(5.9) $$\lambda^2 F_{2p} + \lambda F_p + \lambda\mu F_{p+q} + 2\mu F_q + \mu^2 F_{2q} = 0.$$

Multiplying (5.9) by F_q and eliminating μ by means of (5.8), we get

$$\lambda^2 F_{2p}F_q + \lambda F_pF_q - \lambda^2 F_pF_{p+q} - 2\lambda F_pF_q + \lambda^2 F_p^2 L_q = 0,$$

that is,

$$\lambda(L_pF_q - F_{p+q} + F_pL_q) = F_q.$$

Since

$$L_pF_q - F_{p+q} + F_pL_q = F_{p+q},$$

we have, finally,

(5.10) $$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}.$$

On the other hand, it follows from (5.3) and (5.8) that

$$\lambda(F_{p+r}F_q + F_{q+r}F_p) = F_rF_q.$$

This gives

$$\lambda = \frac{F_rF_p}{F_{p+r}F_q + F_{q+r}F_p} \neq \frac{F_q}{F_{p+q}}.$$

Hence, possibility (ii) is untenable and only the value of λ and μ furnished by (5.6) need be considered.

Conversely, since (5.6) implies $\lambda a^p - (1 + \mu a^q) = 0 = \lambda b^p - (1 + \mu b^q)$, and this in turn implies (5.2), it is clear that (5.1) holds only if (5.6) is satisfied.

This completes the proof of the following

Theorem 4: Let p, q be fixed nonzero integers, $p \neq q$, and let r be an arbitrary integer. Then,

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots), \tag{5.11}$$

if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{5.12}$$

Thus, we have the explicit identity

$$F_q^n F_{pn+r} = \sum_{k=0}^{n} (-1)^{p(n-k)} \binom{n}{k} F_p^k F_{q-p}^{n-k} F_{qk+r} \qquad (n = 0, 1, 2, \ldots). \tag{5.13}$$

We note that, as stated, (5.13) holds for arbitrary integers p, q, r. In particular, for $q = -p$, (5.13) becomes

$$F_p^n F_{pn+r} = \sum_{k=0}^{n} (-1)^{(p-1)(n-k)} \binom{n}{k} F_p^k F_{2p}^{n-k} F_{-pk+r} \qquad (n = 0, 1, 2, \ldots). \tag{5.14}$$

SECTION 6

We turn finally to

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk+r} \qquad (n = 0, 1, 2, \ldots). \tag{6.1}$$

It follows from (6.1) that

$$\frac{a^r}{1 - \lambda a^p x} + \frac{b^r}{1 - \lambda b^p x} = \frac{a^r}{1 - (1 + \mu a^q)x} + \frac{b^r}{1 - (1 + \mu b^q)x}.$$

Hence,

$$a^r(\lambda^k a^{pk} - (1 + \mu a^q)^k) = -b^r(\lambda^k b^{pk} - (1 + \mu b^q)^k) \qquad (k = 0, 1, 2, \ldots). \tag{6.2}$$

For $k = 1$, (6.2) implies

$$\lambda L_{p+r} = L_r + \mu L_{q+r}. \tag{6.3}$$

As in §5, we again consider the two possibilities:

(i) $\lambda a^p = 1 + \mu a^q$;
(ii) $\lambda a^p \neq 1 + \mu a^q$.

It is clear from

$$a^r(\lambda a^p - (1 + \mu a^q)) + b^r(\lambda b^p - (1 + \mu b^q)) = 0$$

and (i) that

$$\lambda b^p = 1 + \mu b^q. \tag{6.4}$$

Adding together (i) and (6.4), we get

$$\lambda L_p = 2 + \mu L_q. \tag{6.5}$$

Again using (i),

$$\lambda(a^q L_p - a^p L_q) = a^q - b^q,$$

which gives

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}, \tag{6.6}$$

Assuming (ii), we have

$$a^r(\lambda a^p - (1 + \mu a^q)) = -b^r(\lambda b^p - (1 + \mu b^q))$$
$$a^r(\lambda^2 a^{2p} - (1 + \mu a^q)^2) = -b^r(\lambda^2 b^{2p} - (1 + \mu b^q)^2)$$
$$a^r(\lambda^3 a^{3p} - (1 + \mu a^q)^3) = -b^r(\lambda^3 b^{3p} - (1 + \mu b^q)^3).$$

This gives

$$\begin{cases} \lambda a^p + (1 + \mu a^q) = \lambda b^p + (1 + \mu b^q) \\ \lambda^2 a^{2p} + \lambda a^p(1 + \mu a^q) + (1 + \mu a^q)^2 = \lambda^2 b^{2p} + \lambda b^p(1 + \mu b^q) + (1 + \mu b^q)^2. \end{cases} \tag{6.7}$$

Then, exactly as in the previous section, we get

$$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}. \tag{6.8}$$

On the other hand, by (6.3) and the first of (6.7), that is,

$$\lambda F_p = \mu F_q = 0,$$

we get

$$\lambda(F_q L_{p+r} + F_p L_{q+r}) = L_q L_r.$$

This gives

$$\lambda = \frac{L_q L_r}{F_q L_{p+r} + F_p L_{q+r}} \neq \frac{F_q}{F_{p+q}} \qquad (r \neq 0).$$

Hence, (ii) leads to a contradiction and only (i) need be considered. Since (6.6) implies (i), it is clear that (6.1) holds only if (6.6) is satisfied.

We may state

Theorem 5: Let p, q, r be fixed nonzero integers, $p \neq q$, $r \neq 0$. Then we have

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qn+r} \qquad (n = 0, 1, 2, \ldots) \tag{6.9}$$

if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{6.10}$$

Thus, we have

$$F_q^n L_{pn+r} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_p^k F_{q-p}^{n-k} L_{qn+r} \qquad (n = 0, 1, 2, \ldots) \tag{6.11}$$

for all p, q, r.

Remark: Theorem 5 does not include Theorem 2 since, for $r = 0$, λ, μ may also take on the values (3.7).

SECTION 7

The identity

$$L_{pn} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L_p^{n-k}L_{pk} \qquad (n = 0, 1, 2, \ldots) \tag{7.1}$$

has been noted in the Introduction. This suggests the problem of finding sequences $U = \{u_0, u_1, u_2, \ldots\}$ such that

$$u_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}u_k \qquad (n = 0, 1, 2, \ldots). \tag{7.2}$$

The sequence U is not uniquely determined by (7.2). We shall assume that $u_1 \neq 0$. For $n = 1$, we have $u_1 = u_0u_1 - u_1$, so that $u_0 = 2$. For $n = 2$, we get $u_2 = u_0u_1^2 - 2u_1^2 + u_2$. For $n = 2m$, $m > 0$, (7.2) reduces to

$$\sum_{k=0}^{2m-1}(-1)^k\binom{2m}{k}u_1^{2m-k}u_k = 0 \qquad (m = 1, 2, 3, \ldots). \tag{7.3}$$

For $n = 2m - 1$, (7.2) yields

$$2u_{2m-1} = \sum_{k=0}^{2m-2}(-1)^k\binom{2m-1}{k}u_1^{2m-k-1}u_k \qquad (m = 1, 2, 3, \ldots). \tag{7.4}$$

Put

$$S_n \equiv \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}u_k.$$

Then

$$u_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}S_k,$$

so that

$$u_n - S_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}(S_k - u_k)$$

and so

$$-2(S_{2m} - u_{2m}) = \sum_{k=0}^{2m-1}(-1)^k\binom{2m}{k}u_1^{2m-k}(S_k - u_k). \tag{7.5}$$

Hence (7.4) is a consequence of the earlier relations

$$S_k = u_k \qquad (k = 1, 2, 3, \ldots, 2m - 2).$$

In the next place, if we put

$$G(x) = \sum_{n=0}^{\infty}u_n\frac{x^n}{n!},$$

it follows from (7.2) that

$$G(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}u_k = \sum_{n=0}^{\infty}(-1)^k u_k\frac{x^n}{k!}\sum_{k=0}^{n}\frac{(u_1x)^n}{n!}.$$

Thus,

$$G(x) = e^{u_1x}G(-x).$$

In particular, the sequence $\{L_0, L_p, L_{2p}, \ldots\}$, with $u_1 = L_p$, satisfies (7.2); incidentally, a direct proof of (7.1) is easy. Hence, if we put

$$G_L(x) = \sum_{n=0}^{\infty} L_{pn}\frac{x^n}{n!},$$

we have

(7.6) $$G_L(x) = e^{u_1 x} G_L(-x) \qquad (u_1 = L_p).$$

It then follows from (7.6) that

$$F(x) = G(x)/G_L(x) = F(-x).$$

Thus,

$$F(x) = \sum_{k=0}^{\infty} c_2 \frac{x^{2k}}{(2k)!} \qquad (c_0 = 1),$$

where the coefficients c_2, c_4, c_6, ... are arbitrary. We have therefore,

(7.7) $$u_n = \sum_{2k \le n} \binom{n}{2k} c_{2k} L_{p(n-2)}$$

for any sequence satisfying (7.2) with $\alpha_1 = L_p$.

This result also suggests a method for handling (7.2) when u_n is arbitrary. Put

(7.8) $$u_1 = \alpha + \beta,$$

where α, β are unrestricted otherwise. Then we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (\alpha + \beta)^{n-k} (\alpha^k + \beta^k)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k \sum_{j=0}^{n-k} \binom{n-k}{j} \alpha^{n-k-j}\beta^j + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \beta^k \sum_{j=0}^{n-k} \binom{n-k}{j} \alpha^{n-k-j}\beta^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j}\beta^j \sum_{k=0}^{n-k} (-1)^k \binom{n-j}{k} + \sum_{s=0}^{n} \binom{n}{s} \alpha^{n-s}\beta^s \sum_{k=0}^{s} (-1) \binom{s}{k} = \alpha^n + \beta^n.$$

Hence, if we define

(7.9) $$u_n = \alpha^n + \beta^n \qquad (n = 0, 1, 2, \ldots),$$

it is clear that

(7.10) $$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} u_k \qquad (n = 0, 1, 2, \ldots).$$

Thus (7.2) is satisfied with u_n defined by (7.9).

We can now complete the proof of the following theorem exactly as for the special case $u_1 = L_p$.

Theorem 6: The sequence $\{u_0 = 2, u_1, u_2, \ldots\}$ satisfies (7.10) if and only if

(7.11) $$u_n = \sum_{2k \le n} \binom{n}{2k} c_{2k} u_{n-2k} \qquad (n = 0, 1, 2, \ldots),$$

where $c_0 = 1$ and c_2, c_4, c_6, ... are arbitrary. An equivalent criterion is

(7.12) $$u_n = \alpha^n + \beta^n \qquad (n = 0, 1, 2, \ldots)$$

for some fixed α, β.

We remark that

$$\alpha^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k \qquad (n = 1, 2, 3, \ldots)$$

is not correct. For example,

$$(\alpha+\beta) - \alpha = \beta$$
$$(\alpha+\beta)^2 - 2(\alpha+\beta)\alpha + \alpha^2 = \beta^2.$$

We shall prove

$$(7.13)\qquad \begin{cases} \beta^n = \displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k \\ \alpha^n = \displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\beta^k \end{cases} \qquad (n = 0, 1, 2, \ldots).$$

It suffices to prove the first of (7.13). We have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha^k\sum_{j=0}^{n-k}\binom{n-k}{j}\alpha^{n-k-j}\beta^j$$
$$= \sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}\beta^j\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k} = \beta^n.$$

This completes the proof. Note that this result had occurred implicitly in the discussion preceding (7.10).

It follows from (7.13) after multiplication by α^r (or β^r) that

$$(7.14)\qquad (\alpha\beta)^r u_{n-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}u_{k+r} \qquad (n = 0, 1, 2, \ldots),$$

where now $u_n = \alpha^n + \beta^n$ for all integral n. Similarly, we have

$$(7.15)\qquad -(\alpha\beta)^r v_{n-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}v_{k+r} \qquad (n = 0, 1, 2, \ldots),$$

where

$$(7.16)\qquad v_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In both (7.14) and (7.16), r is an arbitrary integer.

In the case of the Lucas and Fibonacci numbers, we can improve slightly on (7.14) and (7.16) by first taking $\alpha = a^p$, $\beta = b^p$ in (7.13) and then multiplying by a^r (or b^r). Thus, we get

$$(7.17)\qquad (-1)^r L_{pn-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L^{n-k}L_{pk+r} \qquad (n = 0, 1, 2, \ldots)$$

and

$$(7.18)\qquad (-1)^{r-1}F_{pn-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L^{n-k}F_{pk+r} \quad (n = 0, 1, 2, \ldots),$$

where r is an arbitrary integer.

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FIBONACCI CHROMOTOLOGY OR HOW TO PAINT YOUR RABBIT

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Readers of this journal are aware that Fibonacci numbers have been used to generate musical compositions [1], [2], and that the Golden Section ratio has appeared repeatedly in art and architecture. However, that Fibonacci numbers can be used to select colors in planning a painting is less well-known and certainly an exciting application.

One proceeds as follows, using a color wheel based upon the color theory of Johann Wolfgang von Goethe (1749-1832) and developed and extended by Fritz Faiss [3]. Construct a 24-color wheel by dividing a circle into 24 equal parts as in Figure 1. Let 1, 7, 13, and 19 be yellow, red, blue, and green, respectively. (In this system, green is both a primary color and a secondary color.) Halfway between yellow and red, place orange at 4, violet at 10, blue-green at 16, and yellow-green at 22. The other colors must proceed by even graduations of hue. For example, 2 and 3 are both a yellow-orange, but 2 is a yellow-yellow-orange, while 3 is a more orange shade of yellow-orange. The closest colors to use are: (You must also use your eye.)

1	Cadmium Yellow Light
2	Cadmium Yellow Medium
3	Cadmium Yellow Deep
4	Cadmium Orange or Vermilion Orange
5	Cadmium Red Light or Vermilion
6	Cadmium Red Medium
7	Cadmium Red Deep or Acra Red
8	Alizarin Crimson Golden or Acra Crimson
9	Rose Madder or Alizarin Crimson
10	Thalo Violet or Acra Violet
11	Cobalt Violet
12	Ultramarine Violet or Permanent Mauve or Dioxine Purple
13	Ultramarine Blue
14	French Ultramarine or Cobalt Blue
15	Prussian Blue
16	Thalo Blue or Phthalocyanine Blue or Cerulean Blue or Manganese Blue
17	Thalo Blue + Thalo Green
18	Thalo Green + Thalo Blue
19	Thalo Green or Phthalocyanine Green
20	Viridian

21 Emerald Green
22 Permanent Green
23 Permanent Green Light
24 Permanent Green Light + Cadmium Yellow Light

(Note: Expect problems in mixing a true tertiary color if using acrylic paints.)

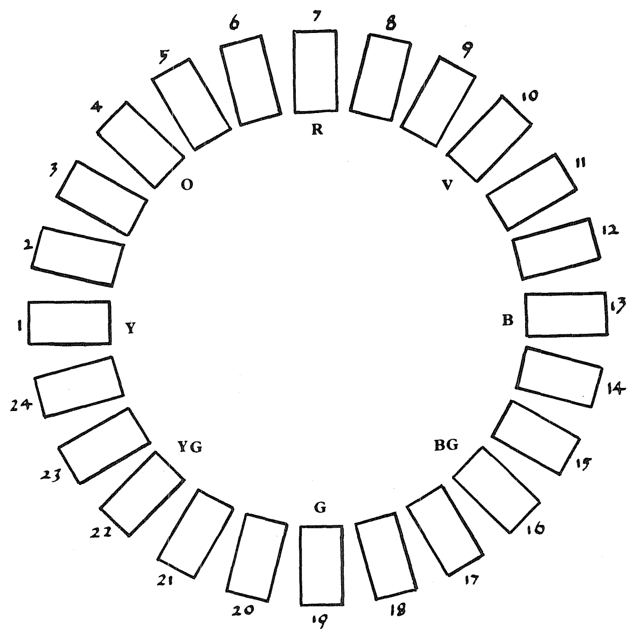

Fig. 1. 24-Color Wheel

To select colors to plan your painting, construct a second 24-color wheel but rather than coloring the spaces, cut out the spaces marked 1, 2, 3, 5, 8, 13, and 21. Place 1 at any position (primary or secondary color preferred) and use the colors thus exposed. The color under 1 should dominate, and 21 would be an accent color. This scheme solves the problem of color selection which occurs if one wishes to paint using bright, clear color; if one is accustomed to painting with "muddy" colors, he may feel that he has no problems with harmony.

Fritz Faiss has many other color schemes based upon the 24-color wheel. The color sequences based upon the Fibonacci sequence are particularly pleasing, and Fritz Faiss has done many paintings using these color sequences. Unfortunately, to fully appreciate the beauty of the color combinations that arise, one needs to actually see a properly constructed color wheel and some examples of its application.

All the color schemes generated as just described are quite lovely, and the Lucas sequence also seems to select pleasant schemes or, at least, nondiscordant ones. But, to see what a color battlefield can be constructed, make a 24-color wheel using the more familiar yellow, red, and blue as primary colors placed at 1, 9, and 17 with the in-between colors again placed in order by hue (so that, for example, 21 is green and 19 is blue-green, 20 is a green blue-green, and 18 is halfway between blue and blue-green). Then, the Fibonacci sequence does not select pleasing combinations, and one comes to appreciate the problem involved in selecting bright, true colors which harmonize.

This short article certainly will pose more questions than it answers, since mathematicians are not usually accustomed to thinking about color theory as used in painting; Fritz Faiss has devoted fifty years to the study of color theory in art. Fibonacci numbers seem to form a link from art to music; perhaps some creative person will compose a Fibonacci ballet, or harmonize Fibonacci color schemes with Fibonacci music.

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ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—II

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1. INTRODUCTION

Throughout this article, we will be using the following notation: $n \geq 0$ is an arbitrary nonnegative integer and $n = \sum_{j=0}^{k} d_j b^j$ its representation as an integer in base b, $b \geq 2$ arbitrary. Define

$$(1.1) \qquad T(n) = n + \sum_{j=0}^{k} d_j \qquad [T(0) = 0]$$

$$\mathcal{R} = \{n \mid n = T(x) \text{ for some } x\} \text{ and}$$
$$\mathcal{C} = \{n \mid n \neq T(x) \text{ for any } x\}.$$

It has been shown ([1]) that the set $\mathcal{C}$ is infinite for any base b. More generally, it is true that $\mathcal{C}$ has asymptotic density and that $\mathcal{C}$ is a set of positive density; these results are derived from the following more general theorem and its corollary (proofs of which may be found in [2]).

Theorem: Let

$$n = \sum_{j=0}^{k} d_j b^j, \; b \geq 2 \text{ arbitrary},$$

and define

$$T(n) = n + \sum_{j=0}^{k} f(d_j, j) \quad \text{and} \quad \mathcal{R} = \{n \mid n = T(x) \text{ for some } x\},$$

where $f(d_j, j)$ satisfies:

a. $f(0, j) = 0$ for all integers $j \geq 0$;
b. $f(d, j) = o(b^j)$ for all j and all digits d such that $1 \leq d \leq b - 1$.

Then the density of $\mathcal{R}$ exists and is equal to L, where L is computable, as follows: let

$$\begin{aligned}\lambda_{d,k} &= \Big|\{T(x)\,|\,db^k \leq x \leq (d+1)b^k - 1\}\\ &\quad \cap \{T(x)\,|\,(d+1)b^k \leq x \leq (d+2)b^k - 1\}\Big|,\quad 0 \leq d \leq b-2\\ \varepsilon_k &= \sum_{d=0}^{b-2} \lambda_{d,k}\,/\,b^{k+1}\\ D(b^k - 1) &= \big|\{T(x)\,|\,0 \leq x \leq b^k - 1\}\big|\\ A_k &= D(b^k - 1)/b^k.\end{aligned}$$

Then

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j = A_k - \sum_{j=k}^{\infty} \varepsilon_j$$

for all $k \geq k_0$, where k_0 is an integer having the property that for all $k \geq k_0$, the sets $\{T(x)\,|\,0 \leq x \leq b^k - 1\}$, $\{T(x)\,|\,b^k \leq x \leq 2b^k - 1\}$, ..., $\{T(x)\,|\,(b-1)b^k \leq x \leq b^{k+1} - 1\}$ are pairwise disjoint, except possibly for adjacent pairs.

Corollary: If $f(d, j) = f(d)$ depends only on the digit d and if $f(0) = 0$ and $f(b - 1) \neq 0$ then $L < 1$.

Now it is easy to see that when $T(n)$ is the function defined by formula (1.1), we have $k_0 = 0$ and that the value of the $\lambda_{d,k}$ does not depend on the digit d. Hence, if we let $\lambda_k = \lambda_{d,k}$ for each digit d, our equation for L becomes

$$L = A_0 - \sum_{j=0}^{\infty} \varepsilon_j = 1 - \sum_{j=1}^{\infty} (b-1)\lambda_j\,/\,b^{j+1}. \tag{1.2}$$

2. COMPUTATION OF THE DENSITY WHEN B IS ODD

Henceforth, let $T(n)$ be the function n + the sum of its digits, the function defined by formula (1.1). It is not difficult to prove that when b is odd, $\mathcal{R}$ is the set of all nonnegative even integers, so that $L = 1/2$ whenever b is odd ([1], [2]). We now give another proof of this fact, independent of the proof in [2], using formula (1.2).

Our principal objective is the proof of the following

Theorem 2.1: $\lambda_k = k(b - 1)/2$ for all odd bases b.

Using this result and equation (1.2), we see that

$$\begin{aligned}L &= 1 - (b-1)^2/2b\sum_{j=1}^{\infty} j/b^j = 1 - \big((b-1)^2/2b\big)\big(b/(b-1)^2\big) = 1/2\\ &= 1/2 \text{ whenever } b \text{ is odd.}\end{aligned}$$

The proof of Theorem 2.1 depends on the following two lemmas:

Lemma 2.2: If b is odd then there exists an $x < b^k$ such that $T(x) = T(b^k)$ for all natural numbers k.

Proof: The proof is by induction on k. If $k = 1$ then we have

$$T((b+1)/2) = b + 1 = T(b).$$

Assume that

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T(b^k)$$

and assume that the following claim is true.

Claim: d_0 can be chosen so that $d_0 \geq (b-1)/2$. Then

$$\begin{aligned}&T((b-1)b^k + d_{k-1}b^{k-1} + \cdots + d_1 b + d_0 - (b-1)/2)\\ &= b^{k+1} - b^k + b - 1 + T\left(\sum_{j=0}^{k-1} d_j b^j\right) - (b-1)\\ &= b^{k+1} - b^k + T(b^k) = b^{k+1} - b^k + b^k + 1 = T(b^{k+1}).\end{aligned}$$

So that all remains to be done is to prove the above claim. Observe that

$$\begin{aligned}&T(d_{k-1}b^{k-1} + \cdots + d_2 b^2 + (d_1 + 1)b + d_0' - (b+1)/2)\\ &= T(d_{k-1}b^{k-1} + \cdots + d_2 b^2 + d_1 b + d_0'),\ (b+1)/2 \leq d_0' \leq b - 1.\end{aligned}$$

Therefore the claim is proved if $d_1 \neq 0$. If $d_m = d_{m-1} = \cdots = d_1 = 0$ and if $d_{m+1} \neq 0$, we sill show that there exists

$$y = \sum_{j=0}^{m} d_j' b^j,\ d_0' \geq (b-1)/2$$

such that

$$T(d_{k-1}b^{k-1} + d_{k-2}b^{k-2} + \cdots + (d_{m+1} - 1)b^{m+1} + y) = T\left(\sum_{j=0}^{k-1} d_j b^j\right),$$

i.e.,

$$T(y) = T(b^{m+1} + d_0),\ d_0 \leq (b-3)/2$$

and this will finish the proof of the claim.

Now if $d_0 = 0$ then the existence of such a y is guaranteed by the induction hypothesis. If $d_0' = (b-1)/2$ then we have

$$T\left(\sum_{j=1}^{m} d_j' b^j + (b-1)/2\right) = T(b^{m+1}).$$

Hence

$$T\left(\sum_{j=1}^{m} d_j' b^j + (b+1)/2\right) = T(b^{m+1} + 1)$$

$$\vdots$$

$$T\left(\sum_{j=1}^{m} d_j' b^j + b - 2\right) = T(b^{m+1} + (b-3)/2)$$

and we are done if $d_0' = (b-1)/2$. Suppose now that $d_0' \geq (b+1)/2$, so that

$$T\left(\sum_{j=1}^{m} d_j' b^j + d_0'\right) = T(b^{m+1})$$

$$T\left(\sum_{j=1}^{m} d_j'b^j + d_0' + 1\right) = T(b^{m+1} + 1)$$
$$\vdots$$
$$T\left(\sum_{j=1}^{m} d_j'b^j + b - 1\right) = T(b^{m+1} + b - 1 - d_0').$$

We then obtain the following equations:

$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + (b - 1)/2\right) = T(b^{m+1} + b - d_0')$$
$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + (b + 1)/2\right) = T(b^{m+1} + b - d_0' + 1)$$
$$\vdots$$
$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + d_0' - 1\right) = T\big(b^{m+1} + (b - 3)/2\big).$$

Note that by induction we may assume that $d_1' \neq b - 1$. The claim has now been completely proved, so that the proof of the lemma is complete as well.

Remark: Lemma 2.2 is not valid in general if b is even. For example, if $k = 1$, there is no $x < b$ satisfying $T(x) = T(b) = b + 1$, since $x < b$ implies that $T(x) = 2x$ and $b + 1$ is odd.

Lemma 2.3: If b is odd and

$$x = \sum_{j=1}^{m} (b - 1)b^m + r_0,\ r_0 \geq (b + 1)/2,$$

then there exists a $y = b^{m+1} + \sum_{j=0}^{m} d_j b^j$ with $d_0 \leq (b - 1)/2$ such that $T(x) = T(y)$.

Proof: Again, the proof is inductive. If $m = 1$ then $x = (b - 1)b + r_0$, $r_0 \geq (b + 1)/2$. Now

$$T\big((b - 1)b + (b + 1)/2\big) = b^2 - b + b - 1 + b + 1 = b^2 + b$$
$$= T\big(b^2 + (b - 1)/2\big).$$

Hence,

$$T\big((b - 1)b + (b + 3)/2\big) = T\big(b^2 + (b + 1)/2\big) = T(b^2 + b)$$
$$T\big((b - 1)b + (b + 5)/2\big) = T(b^2 + b + 1)$$
$$\vdots$$
$$T\big((b - 1)b + b - 1\big) = T\big(b^2 + b + (b - 5)/2\big)$$

and therefore the statement is true for $m = 1$. Assuming that the statement is true for all natural numbers $\leq m$, consider

$$x = \sum_{j=1}^{m+1} (b - 1)b^j + r_0,\ r_0 \geq (b + 1)/2.$$

We have

$$\begin{aligned}T(x) &= T\left((b-1)b^{m+1}\right) + T\left(\sum_{j=1}^{m}(b-1)b^m + r_0\right)\\ &= b^{m+2} - b^{m+1} + b - 1 + T(y)\\ &= b^{m+2} + b - T(b^{m+1}) + T(y) = b^{m+2} + b + T\left(\sum_{j=0}^{m} d_j b^j\right)\\ &= T\left(b^{m+2} + \sum_{j=0}^{m} d_j b^j + (b-1)/2\right).\end{aligned}$$

If $d_0 = 0$, we are obviously done. If $d_0 \neq 0$, assume by induction that $d_1 \neq b - 1$ (cf. the case $m = 1$). Since

$$\begin{aligned}&T\left(b^{m+2} + \sum_{j=1}^{m} d_j b^j + d_0'\right)\\ &= T\left(b^{m+2} + \sum_{j=2}^{m} d_j b^j + (d_1+1)b + d_0' - (b+1)/2\right),\ d_0' \geq (b+1)/2,\end{aligned}$$

the result is proved.

Proof of Theorem 2.1: If m and n are integers with $m \leq n$, define

$$\Omega(m, n) = \{T(x) \mid m \leq x \leq n\}.$$

Then

$$\lambda_k = |\Omega(0, b^k - 1) \cap \Omega(b^k, 2b^k - 1)|.$$

It is easy to see that the theorem is true when $k = 1$, so it suffices to prove that $\lambda_{k+1} - \lambda_k = (b-1)/2$ for all natural numbers k. Observe that if

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right)$$

and if $d_1' \neq b - 1$, then we can choose $d_0' \leq (b-1)/2$ since

$$T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right) = T\left(b^k + \sum_{j=2}^{k-1} d_j' b^j + (d_1'+1)b + d_0' - (b+1)/2\right)$$

for all $d_0' \geq (b+1)/2$. Also, suppose that we have

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=2}^{k-1} d_j' b^j + (b-1)b + d_0'\right).$$

Since $T\left(\sum_{j=0}^{k-1} d_j b^j\right) \leq b^k - 1 + k(b-1)$, it is evident that $d_{k-1}' < b - 1$, so Lemma 2.3 says that we can choose $d_0' \leq (b-1)/2$ in this case as well.

Clearly

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right),\ d_0' \leq (b-1)/2$$

if and only if

$$T\left((b-1)b^k + \sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^{k+1} + \sum_{j=1}^{k-1} d_j' b^j + d_0' + (b-1)/2\right).$$

By the same type of reasoning used to prove Lemma 2.2, we can see that the only values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$ which we need to consider, besides the values $T(b^{k+1} + d_0')$, $0 \le d_0' \le (b - 3)/2$, are values of the form

$$T\left(b^{k+1} + \sum_{j=0}^{k} d_j' b^j\right),$$

where $d_0' \ge (b - 1)/2$.

We therefore obtain the following correspondence (corresponding values on the left-hand side belonging to $\Omega(0, b^k - 1)$ if and only if corresponding values on the right-hand side belong to $\Omega(0, b^{k+1} - 1)$:

$$T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right) \longleftrightarrow T\left(b^{k+1} + \sum_{j=0}^{k-1} d_j' b^j + (b - 1)/2\right),$$

where $d_0' \le (b - 1)/2$.

The only other values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$ which are left to consider are the values of the form $T(b^{k+1} + d_0')$, $0 \le d_0' \le (b - 3)/2$. By Lemma 2.2, there exists an integer

$$\sum_{j=0}^{k-1} d_j b^j, \; d_0 \ge (b - 1)/2,$$

such that

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T(b^k).$$

Hence

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - (b - 1)/2\right) = T(b^{k+1})$$

and therefore

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - (b - 1)/2 + 1\right) = T(b^{k+1} + 1)$$

$$\vdots$$

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - 1\right) = T(b^{k+1} + (b - 3)/2)$$

i.e., the values $T(b^{k+1} + d_0')$, $0 \le d_0' \le (b - 3)/2$ all belong to $\Omega(0, b^{k+1} - 1)$. Since each of these values are different from each other and from all the other values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$, we conclude that $\lambda_{k+1} - \lambda_k = (b - 1)/2$, Q.E.D.

3. AN ESTIMATE OF THE DENSITY WHEN B = 10

In contrast to the above result, the λ_k behave somewhat irregularly when b is even, as the following table, constructed for the case $b = 10$, shows.

The values in the table were computed essentially by finding the first integer in $\Omega(0, b^k - 1)$ which also belongs to $\Omega(b^k, 2b^k - 1)$; this appears to be difficult to do in general if b is even.

By using the table below, we obtain the following estimate of the density for base 10.

Theorem 3.1: When $b = 10$, the density of $\mathcal{R}$ is approximately 0.9022222; the error made by using this figure is less than 10^{-7}.

The Values of λ_k and $\lambda_{k+1} - \lambda_k$ for the Case $b = 10$, $1 \leq k \leq 50$

k	λ_k	$\lambda_{k+1} - \lambda_k$	k	λ_k	$\lambda_{k+1} - \lambda_k$
1	0		26	181	6
2	9	9	27	188	7
3	16	7	28	195	7
4	23	7	29	202	7
5	30	7	30	209	7
6	37	7	31	210	1
7	44	7	32	246	36
8	51	7	33	252	6
9	58	7	34	250	-2
10	65	7	35	249	-1
11	72	7	36	255	6
12	90	18	37	260	5
13	90	0	38	267	7
14	95	5	39	274	7
15	102	7	40	281	7
16	109	7	41	240	-41
17	116	7	42	321	81
18	123	7	43	327	6
19	130	7	44	313	-14
20	137	7	45	320	7
21	142	5	46	329	9
22	169	27	47	335	6
23	188	19	48	339	4
24	169	-19	49	346	7
25	175	6	50	353	7

Proof: Since $\max\{x \mid x \in \Omega(0, b^k - 1)\} = b^k - 1 + k(b - 1)$, it is clear that $\lambda_k < k(b - 1)$ for all k. Formula (1.2) says that

$$L = 1 - \sum_{k=1}^{7}(b \dot{-} 1)\lambda_k / b^{k+1} - \sum_{k=8}^{\infty}(b - 1)\lambda_k / b^{k+1}.$$

Now

$$\sum_{k=j}^{\infty}(b - 1)\lambda_k / b^{k+1} < ((b - 1)^2/b) \cdot \sum_{k=j}^{\infty} k/b^k$$

and

$$\sum_{k=8}^{\infty} k/b^k = \frac{(1 - 1/b)8(1/b)^8 + (1/b)^9}{(1 - 1/b)^2}.$$

Using the table and the above equations, our result is readily verified.

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THE FIBONACCI PSEUDOGROUP, CHARACTERISTIC POLYNOMIALS AND EIGENVALUES OF TRIDIAGONAL MATRICES, PERIODIC LINEAR RECURRENCE SYSTEMS AND APPLICATION TO QUANTUM MECHANICS

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INTRODUCTION

There are numerous applications of linear operators and matrices that give rise to tridiagonal matrices. Such applications occur naturally in mathematics, physics, and chemistry, e.g., eigenvalue problems, quantum optics, magnetohydrodynamics and quantum mechanics. It is convenient to have theoretical as well as computational access to the characteristic polynomials of tridiagonal matrices and, if at all possible, to their roots or eigenvalues. This paper produces explicitly the characteristic polynomials of general (finite) tridiagonal matrices: these polynomials are given in terms of the Fibonacci pseudogroup F_n (of order f_n, the nth Fibonacci number), a subset of the full symmetric group $\mathfrak{S}_n$. We then turn to some interesting special cases of tridiagonal matrices, those which have periodic properties: this leads directly to periodic linear recurrence systems which generalize the two-term Fibonacci type recurrence to collections of two-term recurrences defining a sequence. After some useful lemmas concerning generating functions for these systems, we return to explicitly calculate eigenvalues of periodic tridiagonal matrices. As an example of the power of the techniques, we have a theorem which gives the eigenvalues of a six-variable periodic tridiagonal matrix of odd degree explicitly as algebraic functions of these six variables, generalizing a result of Jacobi. We end with a brief discussion of how to explicitly calculate the characteristic polynomials of certain finite dimensional representations of a Hamiltonian operator of quantum mechanics.

SECTION A. THE FIBONACCI PSEUDOGROUP

We give a few essential definitions and observations about finite sets and permutations acting upon them which will be necessary in the sequel. We may think of this section as a theory of exterior powers of sets.

Let A be a finite set and let $|A|$ denote the number of distinct elements in A. Let 2^A denote the class of all subsets of A and define $\Lambda^k A$ to be the subclass of 2^A consisting of all subsets of A with exactly k distinct elements of A. Thus for $B \in 2^A$, $B \in \Lambda^k A$ iff $|B| = k$. Clearly,

$$|\Lambda^k A| = \binom{|A|}{k} \text{ (binomial coefficient) and } |2^A| = 2^{|A|}.$$

We have

$$2^A = \bigcup_{0 \le k \le |A|} \Lambda^k A \text{ (disjoint class union)}$$

which implies the usual relation

$$2^n = \sum_{0 \le k \le n} \binom{n}{k}.$$

Note that $\Lambda^0 A = \{\emptyset\}$ (empty class) and that $\Lambda^{|A|} A = A$.

Let S_n denote the full symmetric group of all permutations on n elements. Assume S_n acts by permuting the set of ciphers $N = \{1, 2, \ldots, n\}$. We will write the permutation as disjoint cycles; empty products will be the identity permutation. Consider the following subset $F_n \subseteq S_n$, defined by

$$F_n = \{(i_1, i_1 + 1) \ldots (i_k, i_k + 1) | 1 < i_1 + 1 < i_2, i_2 + 1 < i_3, \ldots, i_{k-1} + 1 < i_k < n\}.$$

F_n is a certain subset of disjoint two-cycle products in S_n. Observe that $(1) \varepsilon F_n$, (1) = identity of S_n. For $\sigma \varepsilon F_n$, $\sigma^2 = (1)$, thus every element of F_n is of order two and is its own inverse. Thus, if $\sigma \varepsilon F_n$, then $\sigma^{-1} \varepsilon F_n$. Suppose $\sigma, \rho \varepsilon F_n$. Then $\sigma\rho \varepsilon F_n$ iff σ and ρ are disjoint; all the two-cycle products of F_n are not disjoint. A pseudogroup is a subset of a group which contains the group identity, closed under taking inverses, but does not always have closure. In the present case $F_n = S_n$ iff $n = 0, 1, 2$. If $n < 2$, F_n is not a group, but F_n is a pseudogroup. We call F_n the Fibonacci pseudogroup because of the following lemma.

Lemma A1: Let f_n denote the nth Fibonacci number. Then

$$|F_n| = f_n, \; n \geq 0.$$

Proof: We may write

$$F_n = \bigcup_{0 \leq k \leq [n/2]} F_{k,n} \quad \text{(disjoint union)}$$

where $F_{k,n}$ consists of k disjoint two-cycles of F_n. But observe that

$$|F_{k,n}| = \binom{n-k}{k}$$

and the lemma follows. Note that $(-1)^k$ is the sign of the permutations in $F_{k,n}$. Then there are $\displaystyle\sum_{k \text{ odd}} \binom{n-k}{k}$ with negative sign and $\displaystyle\sum_{k \text{ even}} \binom{n-k}{k}$ with even sign: this gives an alternative proof with $|F_n| = |F_{n-1}| + |F_{n-2}|$, by observing that $|F_0| = 1$, $|F_1| = 1$.

Returning now to the finite set $N = \{1, 2, \ldots, n\}$ and the action of S_n on N, consider the convenient map

$$\text{Fix}: S_n \to 2^N$$

given for $\sigma \varepsilon S_n$ by $\text{Fix}\,\sigma = \{i \varepsilon N: \sigma(i) = i\}$, i.e., the set of elements of N fixed by σ. Thus, $\text{Fix}\,(1) = N$. We also define $\text{CoFix}\,\sigma = \{i \varepsilon N: \sigma(i) \neq i\}$ and note that $N = \text{Fix}\,\sigma \cup \text{CoFix}\,\sigma$ (disjoint union) for every $\sigma \varepsilon S_n$. If $n > 3$, then Fix can be onto.

Restricting Fix to F_n, the Fibonacci pseudogroup definition yields the handy facts that if $\sigma \varepsilon F_{k,n}$, then $|\text{Fix}\,\sigma| = n - 2k$ and $|\text{CoFix}\,\sigma| = 2k$.

It will be convenient to work with just half of the set CoFix σ; therefore, we define the subset of $\text{CoFix}\,\sigma$, (small c) $\text{coFix}\,\sigma = \{i \varepsilon N: \sigma(i) = i + 1\}$. Then $|\text{coFix}\,\sigma| = k$. Also, the number of elements of $\text{Fix}\,\sigma$, $\sigma \varepsilon F_{k,n}$ with $|\text{Fix}\,\sigma| = n - 2k$ is exactly $\binom{n-k}{n-2k} = \binom{n-k}{k}$. Again combining definitions, if $\sigma \varepsilon F_{k,n}$, then $|\Lambda^{\ell}\text{Fix}\,\sigma| = \binom{n-2k}{\ell}$.

SECTION B. APPLICATIONS OF THE FIBONACCI PSEUDOGROUP TO DETERMINANTS AND CHARACTERISTIC POLYNOMIALS OF TRIDIAGONAL MATRICES

We consider tridiagonal $n \times n$ matrices of the following form.

$$(1) \qquad A_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \dots & 0 & 0 & 0 \\ 0 & 0 & c_3 & a_4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & b_{n-2} & \\ 0 & 0 & 0 & 0 & \dots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \dots & & c_{n-1} & a_n \end{bmatrix}$$

We define vectors

$$a = (a_1, \dots, a_n),\ b = (b_1, \dots, b_{n-1}),\ c = (c_1, \dots, c_{n-1})$$

and regard A_n as a function of these three vectors, $A_n = A_n(a, b, c)$ or as a function of $3n - 2$ variables. Let $\det A$ denote the determinant of A. We record some simple facts about the determinant and characteristic polynomial of A_n.

Lemma B1: Let A_n be the tridiagonal matrix defined above. Then,

a. $\det A_n = a_n \det A_{n-1} - b_{n-1}c_{n-1} \det A_{n-2}$.

b. $$\begin{aligned} \det(A_n(a, b, c) - \lambda I) &= (-1)^n \det(\lambda I - A_n(a, b, c)) \\ &= \det(A_n(a, -b, -c) - \lambda I) \\ &= (-1)^n \det(\lambda I - A\ (a, -b, -c)) \\ &= (-1)^n \det(\lambda + A\ (-a, b, c)). \end{aligned}$$

Our object is to give explicit information about $\det(A_n - \lambda I)$. We summarize this information using the notation of Section A in the result.

Theorem B1: The characteristic polynomial of a tridiagonal matrix can be written as the sum of a polynomial of codegree zero and a polynomial of codegree two as follows:

$$(2) \qquad \det(A_n(a, b, c) - \lambda I) = \prod_{1 \le k \le n} (a_k - \lambda) + P_n(\lambda; a, b, c)$$

where

$$\deg P_n(\lambda; a, b, c) = n - 2$$

and

$$P_n(\lambda; a, b, c)$$

$$(3) \qquad = (-1)^n \sum_{0 \le \mu \le n-2} \lambda^\mu \sum_{1 \le k \le [n/2]} (-1)^{n-\mu-k} \left(\sum_{\sigma \in F_{k,n}} \left(\prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \left(\sum_{A \in \Lambda^{n-4-2k}\mathrm{Fix}\,\sigma} \prod_{i \in A} a_i \right)\right)\right).$$

In particular,

$$(4) \qquad \det A_n = \sum_{\sigma \in F_n} \mathrm{sgn}(\sigma) \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j.$$

This theorem gives complete closed form information about the polynomial $P_n(\lambda)$. $P_n(\lambda)$ explicitly describes the perturbation of the characteristic polynomial of A from the characteristic polynomial of the diagonal of A. Further, consider the family of hyperbolas $x_k y_k = d_k$, $1 \le k \le n - 1$ in $\mathbb{R}^{2n-2}$ space, $d_1, \ldots, d_{n-1}$ fixed constants. Then for fixed $a \in \mathbb{R}^n$, points on these hyperbolas parameterize a family of tridiagonal matrices $A_n(a, x, y)$ which all have exactly the same latent roots with the same multiplicities. The coefficients of the powers of λ in $P_n(\lambda)$ are elegantly expressed polynomials in the components of a, b, c and can be easily generated for computational purposes: the set F_n can be generated from $\{1, 2, \ldots, n\}$ in order $0 \le k \le [n/2]$, $F_{k,n}$; coFix is had immediately therefrom, and Λ^mFix can be generated from a combination subroutine.

To prove the theorem, we begin with

$$\det A_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a^1_{\sigma(1)} \cdots a^n_{\sigma(n)},$$

where $a^i_j = a_i, b_i, c_i, 0$ for $i = j$, $i + 1 = j$, $i - 1 = j$, otherwise, respectively, $1 \le i, j \le n$. However, $\det A_n$ is really a sum over $F_n \subseteq S_n$, has, in general, f_n terms, and $b_i c_i$ occurs whenever b_i occurs (Lemma B1). From the partition of F_n into k two-cycles, $0 \le k \le [n/2]$, we have

$$\det A_n = \sum_{0 \le k \le [n/2]} (-1)^k \sum_{\sigma \in F_{k,n}} a^1_{\sigma(1)} \cdots a^n_{\sigma(n)} \tag{6}$$

$$= \sum_{0 \le k \le [n/2]} (-1)^k \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j$$

because there are three cases, $j = \sigma(j)$, $j > \sigma(j)$, and $j < \sigma(j)$. If $a^j_{\sigma(j)} \ne 0$ then $|j - \sigma(j)| \le 1$. In case of equality, $a^j_{\sigma(j)}\, a^{\sigma(j)}_j = b_j c_j$ occurs in the product. For $\sigma \in F_{k,n}$, σ moves $2k$ elements and fixes $n - 2k$ elements and is characterized by its fixed elements. The most σ can fix for $k > 0$ is $n - 2$, so that (replacing each a_k by $a_k - \lambda$) we have $\deg P_n(\lambda, a, b, c) = n - 2$. Setting $P_n(\lambda) = P_n(\lambda, a, b, c)$, we have

$$P_n(\lambda) = \sum_{1 \le k \le [n/2]} (-1)^k P_{k,n}(\lambda) \tag{7}$$

where $\deg P_{k,n}(\lambda) = n - 2k$ and

$$P_{k,n}(\lambda) = \sum_{\sigma \in F_{k,n}} \prod_{i \in \mathrm{Fix}\,\sigma} (a_i - \lambda) \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \tag{8}$$

Let $M \subseteq N$, then

$$\prod_{i \in M} (a_i - \lambda) = \sum_{0 \le \ell \le |M|} (-1)^{|M| - \ell} \left(\sum_{A \in \Lambda^\ell M} \prod_{i \in A} a_i \right) \lambda^{|M| - \ell} \tag{9}$$

is simply the symmetric polynomials identity rewritten in the notation of exterior powers of sets. From this fact (9) and rearranging (8) for $M = \mathrm{Fix}\,\sigma$ we have

$$P_{k,n}(\lambda) = \sum_{\sigma \in F_{k,n}} \sum_{0 \le \ell \le n-2k} (-\lambda)^{n-2k-\ell} \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \sum_{A \in \Lambda^\ell \mathrm{Fix}\,\sigma} \prod_{i \in A} a_i. \tag{10}$$

For comparison, we note that combining equations (9) and (2) gives a direct evaluation of the traces of exterior powers of A_n (in this context, exterior powers of A_n are the compound matrices of A_n). This is so from the identity

$$\text{(11)} \qquad \det(A_n - \lambda I) = \sum_{0 \le k \le n-1} (-1)^{n-k}(\operatorname{tr}\Lambda^{n-k}A_n)\lambda^k + (-1)^n\lambda^n,$$

where A_n can be an arbitrary $n \times n$ matrix, tr is the trace of a matrix, $\Lambda^k A_n$ is the kth exterior power of A_n $\left(\text{an } \binom{n}{k} \times \binom{n}{k} \text{ matrix}\right)$. Thus, it is possible to also give $\operatorname{tr}\Lambda^k A_n(a, b, c)$ as an explicit polynomial in the components of a, b, c for $1 \le k \le n$.

We conclude this section with two examples. The first arose in a problem of positive definiteness of certain quadratic forms of interest in a plasma physics energy principle analysis.

a. Let $1 \le m \le n$ and choose $a_m = a/m$, $b_m c_m = b$. Then

$$\text{(12)} \qquad n!\,\det A_n = \sum_{0 \le k \le [n/2]} (-1)^k B_{k,n}\, a^{n-2k} b^k$$

where the $B_{k,n}$ are certain integers

$$\text{(13)} \qquad B_{k,n} = \sum_{\sigma \in F_{k,n}} \prod_{m \in \operatorname{CoFix}\sigma} m,$$

(note the upper case C on CoFix here, $|\operatorname{CoFix}\sigma| = 2k$). See Table 1 for a few of these integers.

b. Let $1 \le m \le n$ and choose $a_m = a$, $b_m c_m = b$. Then

$$\text{(14)} \qquad \det A_n = \sum_{0 \le k \le [n/2]} (-1)^k C_{k,n}\, a^{n-2k} b^k$$

where the $C_{k,n}$ are certain integers

$$\text{(15)} \qquad C_{k,n} = \sum_{\sigma \in F_{k,n}} \prod_{m \in \operatorname{Fix}\sigma} m.$$

Table 1 also contains a few of these integers.

Table 1. The First Few CoFix; Fix Integers $B_{k,n}$; $C_{k,n}$ Defined by Equations (13); (15), Respectively; $0 \le k \le [n/2]$

$n \backslash k$	0	1	2	3	4
1	1; 1				
2	1; 2	2; 1			
3	1; 6	8; 4			
4	1; 24	20; 18	24; 1		
5	1; 120	40; 96	184; 9		
6	1; 720	70; 600	784; 72	720; 1	
7	1; 5040	112; 4320	2464; 600	8448; 16	
8	1; 40320	168; 36480	6384; 5400	42272; 196	40320; 1

SECTION C. PERIODIC LINEAR RECURRENCE SYSTEMS

It is now possible to use the results and notation of Sections A and B to draw conclusions about periodic linear recurrence systems. Of course, these generalize the usual linear recurrences; however, it is surprising that the Fibonacci pseudogroup is the key idea in their description. We first state a natural corollary to Theorem B1 without restriction of periodicity.

Theorem C1: Given a pair of arbitrary sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$, then the one-parameter class of linear recurrences

(16) $$f_n(t) = a_n f_{n-1} + t b_{n-1} f_{n-2}$$

with $f_0 = 1$, $f_i = a_i$, has the general solution $n > 1$

(17) $$f_n(t) = \sum_{0 \le k \le [n/2]} t^k \sum_{\sigma \in F_{k,n}} \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j.$$

For example, taking $t = 1$, $a_k = a$, $b_k = b$, $k \ge 1$, and recalling that for $\sigma \in F_{k,n}$, $|\mathrm{Fix}\,\sigma| = n - 2k$, $|\mathrm{coFix}\,\sigma| = k$, and $|F_{k,n}| = \binom{n-k}{k}$ yields

(18) $$f_n = \sum_{0 \le k \le [n/2]} \binom{n-k}{k} a^{n-2k} b^k$$

the general solution of $f_n = a_{n-1} + bf_{n-2}, f_0 = 1, f_1 = a$. Taking $a = b = 1$ yields the well-known sum over binomial coefficients expression for the Fibonacci sequence. On the other hand, writing the generating function

(19) $$G(t) = \sum_{n \ge 0} f_n t^n$$

and recognizing that $G(t)$ is a rational function of at most two poles, indeed $G(t) = 1/(1 - at - bt^2)$, yields the alternative solution

(20) $$f_n = \frac{1}{\sqrt{a^2 + 4b}} \left\{ \left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^{n+1} \right\}.$$

Of course, from (18) we may regard $f_n = f_n(a, b)$ as a polynomial in a and b. In particular $f_n(a - \lambda, b)$ as a polynomial in λ can be written

(21) $$f_n(a - \lambda, b) = \sum_{0 \le m \le n} (-1)^m \left(\sum_{0 \le k \le [n/2]} \binom{n-k}{k} \binom{n-2k}{m} a^{n-m-2k} b^k \right) \lambda^m.$$

We see now that the zeros, λ , $1 \le k \le n$, of polynomial (21) are precisely

(22) $$\lambda_k = a + 2\sqrt{-b} \cos(\pi k/(n + 1)),\ 1 \le k \le n.$$

This follows from equation (20), for $f_n = 0$ implies that

$$a + \sqrt{a^2 + 4b} = \left(a - \sqrt{a^2 + 4b}\right) e^{2\pi 2k/n+1}$$

so that

$$\sqrt{a^2 + 4b} = -\sqrt{-1}\, a^2 \tan \pi k/n + 1.$$

Squaring gives $a^2 \sec^2 (\pi k/n + 1) = -4b$. Replacing a by $a - \lambda$ gives equation (22). We have basically done the case of a period of length one.

We now take up the case of period two.

Lemma C1: Let $\{f_n\}$, $n \geq 0$, be a sequence defined by

$$f_n = a_n f_{n-1} + b_{n-1} f_{n-2},\ f_0 = 1,\ f_1 = a_1$$

and the sequences $\{a_n\}$, $\{b_n\}$, have period two, i.e.,

$$a_{2n} = a_2,\ a_{2n-1} = a_1,\ b_{2n-1} = b_1,\ b_{2n} = b_2,\ n \geq 1.$$

Then the generating function is rational with at most four poles:

$$G(t) = \sum_{n \geq 0} f_n t^n \tag{23}$$

$$= \frac{1 + a_1 t - b_2 t^2}{1 - (b_1 + b_2 + a_1 a_2)t^2 + b_1 b_2 t^4} \tag{24}$$

$$= \frac{A(\alpha,\ \beta)}{1 - \alpha t} + \frac{A(-\alpha,\ \beta)}{1 + \alpha t} + \frac{A(\beta,\ \alpha)}{1 - \beta t} + \frac{A(-\beta,\ \alpha)}{1 + \beta t} \tag{25}$$

where for $D = b_1 + b_2 + a_1 a_2$,

$$2\alpha^2 = D + \sqrt{D^2 - 4b_1 b_2},\quad 2\beta^2 = D - \sqrt{D^2 - 4b_1 b_2} \tag{26}$$

and

$$A(\alpha,\ \beta) = (\alpha^2 + a_1\alpha - b_2)/2(\alpha^2 - \beta^2). \tag{27}$$

Proof: Write $G(t)$ in terms of its even and odd parts (two functions). Then substitute the period two relations in to get the rationality of $G(t)$ from the pair of relations

$$\left(1 - \frac{a_2 - a_1}{2} t - \frac{b_1 + b_2}{2} t^2\right) G(t) + \left(\frac{a_2 - a_1}{2} t + \frac{b_2 - b_1}{2} t^2\right) G(-t) = 1 \tag{28}$$

$$\left(-\frac{a_2 - a_1}{2} t + \frac{b_2 - b_1}{2} t^2\right) G(t) + \left(1 + \frac{a_2 + a_1}{2} t - \frac{b_2 + b_1}{2} t^2\right) G(-t) = 1 \tag{29}$$

where the determinant of this system is the denominator of the right-hand side of equation (24).

Of course, comparing coefficients will give an expression for f_n as a linear combination of powers of poles of $G(t)$ analogous to equation (20). On the other hand, there are polynomial expressions in the four variables a_1, a_2, b_1, b_2 of the type (18) which follow directly from Theorem B.

We give only one example of the former.

Let $f_{2n} = f_{2n-1} + f_{2n-2}$, $f_{2n+1} = f_{2n} + 2f_{2n-1}$, $f_0 = 1$, $f_1 = 1$, so that f_n is the sequence 1, 1, 2, 4, 6, 14, 20, 48, 68, 166, 234, ... Then, we have

$$f_{2n} = \frac{1}{2}\left((2 + \sqrt{2})^n + (2 - \sqrt{2})^n\right), \tag{30}$$

$$f_{2n+1} = \frac{1}{2\sqrt{2}}\left((2 + \sqrt{2})^{n+1} - (2 - \sqrt{2})^{n+1}\right). \tag{31}$$

Alternatively (30) and (31) can be shown by induction to satisfy the linear recurrence of period two.

We now consider the general case of rationality of generating functions of arbitrary periodic systems of linear recurrences.

Lemma C2: Let $f_n = a_n f_{n-1} + b_{n-1} f_{n-2}$ be given with $f_0 = 1$, $f_1 = a_1$. Suppose that $a_n = a_\ell$ and $b_n = b_\ell$ if $n \equiv \ell \pmod{k}$ and that

$$a_\ell,\ 1 \le \ell \le k, \quad b_\ell,\ 0 \le \ell \le k-1$$

are given as the first elements of the sequences $\{a_n\}$ and $\{b_n\}$ which are not in two k-periods. Call the system a period k system. Set

$$G(t) = \sum_{n \ge 0} f_n t^n \tag{32}$$

then $G(t)$ is a rational function of t where

$$G(t) = P(t)/Q(t) \tag{33}$$

and $P(t)$, $Q(t)$ are polynomials in t, $\deg P(t) \le 2k - 1$, $\deg Q(t) \le 2k$.

Proof: First write

$$G(t) = \sum_{1 \le \ell \le k} G_\ell(t) \tag{34}$$

where

$$G_\ell(t) = \sum_{n \equiv \ell \pmod{k}} f_n t^n \tag{35}$$

and where the sum is over integers $n \ge 0$, n congruent to ℓ modulo k. From the relations

$$f_n = a_\ell f_{n-1} + b_{\ell-1} f_{n-2} \quad \text{if} \quad n \equiv \ell \pmod{k}, \tag{36}$$

we have that

$$G_\ell(t) = a_\ell t G_{\ell-1}(t) + b_{\ell-1} t^2 G_{\ell-2}(t). \tag{37}$$

Using the modulo k relations we can write the following equations

$$G_1(t) = a_1 t G_0(t) + b_0 t^2 G_{-1}(t) = a_1 t + a_1 t G_k(t) + b_0 t\ G_{k-1}(t), \tag{38}$$

$$G_2(t) = a_2 t G_1(t) + b_1 t^2 G^0(t) = a_2 t G_1(t) + b_1 t^2 + b_1 t^2 G_k(t), \tag{39}$$

$$G_3(t) = a_3 t G_2(t) + b_2 t^2 G_1(t) \tag{40}$$

$$\vdots$$

$$G_k(t) = a_k t G_{k-1}(t) + b_{k-1} t^2 G_{k-2}(t) \tag{41}$$

This gives the system of equations in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & -b_0 t^2 & -a_1 t \\ -a_2 t & 1 & 0 & 0 & 0 & \dots & 0 & -b_1 t^2 \\ -b_2 t^2 & -a_3 t & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -b_3 t^2 & -a_4 t & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -b_4 t^2 & -a_5 t & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -b_{k-2} t^2 & -a_{k-1} t & & 1 & 0 \\ 0 & 0 & \dots & 0 & -b_{k-1} t^2 & & -a_k t & 1 \end{bmatrix} \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \\ G_5(t) \\ \vdots \\ G_{k-1}(t) \\ G_k(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{42}$$

We rewrite equation (42) as

$$HG = J, \tag{42}$$

with the obvious interpretation. Now H is invertible (in the indeterminant t) and we can solve for $G_1(t), \ldots, G_k(t)$ separately as rational functions, their sum is $G(t)$. But, clearly, deg det $H(t) = 2k$, so that the denominator of $G(t)$ must divide this, i.e., $\deg Q(t) \leq 2k$. Also, the adjoint of H is given by polynomials of degree $\leq 2k - 1$, thus, $\deg P(t) \leq 2k - 1$.

This rationality result is the starting point to produce further facts of which Lemma B1 and equation (20) are examples. The central difficulty lies in analyzing the denominator of the rational function to display sums of powers of its roots. We will apply the technique to tridiagonal matrices of periodic type in the next section.

SECTION D. APPLICATIONS OF PERIODIC RECURRENCES TO TRIDIAGONAL MATRICES

We return to tridiagonal matrices to apply the results of Section C first to recover a result of Jacobi and second to give a generalization of Jacobi's theorem.

Theorem D1 (Jacobi): The latent roots of the tridiagonal $n \times n$ matrix

$$\begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & a & b & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & a & b & 0 & \cdots & 0 & 0 \\ 0 & 0 & c & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & a \end{bmatrix} \tag{43}$$

are given for $1 \leq k \leq n$ by

$$\lambda_k = a - 2\sqrt{bc} \cos \frac{\pi k}{n + 1}.$$

Proof: This follows directly from Lemma B1 and equation (22), by recognizing that the matrix (43) defines a (period one) linear recurrence system.

Theorem D2: The latent roots of the $(2n+1) \times (2n+1)$ tridiagonal matrix

$$\begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & c & e & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & f & a & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & e & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & f & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & c & e \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & f & a \end{bmatrix} \tag{44}$$

lie among the values ($1 \leq k \leq n + 1$ with the plus sign, $1 \leq k \leq n$ with the minus sign):

$$\lambda = \frac{a + c}{2} \pm \sqrt{\left(\frac{a - c}{2}\right)^2 + bd + ef + 2\sqrt{bdef}\,\cos\frac{\pi k}{n + 1}}\,. \tag{45}$$

Proof: Note that when $a = c$, $b = e$, and $d = f$ this reduces to the case of the period one theorem. By Lemma B1, we recognize (44) as defining a period *two* linear recurrence system. Take therefore the odd case in Lemma C1, thus $(-1)^{2n-1} = -1$ and

$$\frac{A(\alpha,\ \beta)\ -\ A(-\alpha,\ \beta)}{A(\beta,\ \alpha)\ -\ A(-\beta,\ \alpha)} = \frac{\alpha}{\beta}\,. \tag{46}$$

Then f_n is zero iff $(\alpha/\beta)^{2n+2} = e^{2\pi ik}$, $0 \le k \le n + 1$. Reasoning as with equation (22) yields

$$bd + ef + ac = 2\sqrt{bdef}\,\cos\frac{\pi k}{n + 1}\,. \tag{47}$$

Replacing ac by $(a - \lambda)(c - \lambda)$ and solving for λ gives (45). Thus we have all latent roots of a five-parameter family of matrices.

Again, to apply similar techniques to families of matrices with more parameters involves analyzing the denominator in Lemma C2. We point out that for large periodic matrices of special type (particular sparse matrices) the root analysis is relatively easy to do numerically, say, for periods small relative to the size of the matrix.

SECTION E. THE APPLICATION TO A HAMILTONIAN OPERATOR OF QUANTUM MECHANICS

The differential equation of the quantum mechanical asymmetric rotor may be written as $(D - E\ )\Psi\ = 0$. (Schroedinger equation) where the matrix corresponding to the inertia tensor is

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{48}$$

Define the variables α, β, δ by the equation

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \delta \end{bmatrix} \tag{49}$$

so that $\beta = C -\ \ (A + B)$, and the differential equation becomes (single variable representation)

$$P(x)\frac{d^2}{dz^2} + A(z)\frac{d}{dz} + R(z)\ = 0 \tag{50}$$

where

$$\begin{aligned} P(z) &= \alpha z^6 + \beta z^4 + \alpha z^2, \\ Q(z) &= 2\alpha(j + 2)z^5 + \beta z^3 = 2(j + 1)z, \\ R(z) &= \ (j + 1)(j + 2)z^4 - E\ z^2 + \alpha(j + 1)(j + 2). \end{aligned} \tag{51}$$

After choosing a convenient z-basis of eigenfunctions, getting the corresponding difference equation with respect to that basis we have a tridiagonal matrix appear. This tridiagonal matrix, however, is tridiagonal with the main diagonal and second upper and lower diagonals, but it is possible to reduce it to direct sums of the usual tridiagonals that we have already treated in Section B. We are not concerned here with giving the representation theory, and so we will sketch briefly the facts we need.

The difference equation alluded to above becomes

(52) $$P_{j,m}A_{m+2} + (Q_{j,m} - E)A_m - R_{j,m}A_{m-2} = 0,$$

where

$$\begin{aligned} P_{j,m} &= (j - m)(j - m - 1), \\ Q_{j,m} &= \beta m^2, \\ R_{j,m} &= (j + m)(j + m - 1). \end{aligned}$$

We have here for convenience replaced $\frac{\beta}{\alpha}$ by β, $\frac{E}{\alpha}$ by E; note that $P_{j,m} = R_{j,m}$, where m varies through $-j \le m \le j$, j may be a half integer. We choose the variable $n = 2j + 1$, so that $j = \frac{n-1}{2}$ and the matrix of interest is the $n \times n$ matrix $A = (a_{ij})$, where

(53) $$a_{ij} = \begin{cases} \beta \dfrac{n - 2i + 1}{2} & i = j, \\ (n - i)(n - i - 1) & j = i + 2, \\ (i - 1)(i - 2) & i = j + 2, \\ 0 & \text{otherwise} \end{cases}$$

This is a nonstandard tridiagonal matrix with off diagonal integer entries. Generalizing this situation slightly, we define

(54) $$A = \begin{vmatrix} a_1 & 0 & b_{n-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_{n-3} & \ddots & & & \\ b_1 & 0 & a_3 & 0 & \ddots & \ddots & & \\ 0 & b_2 & 0 & a_4 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & a_4 & 0 & b_2 & 0 \\ 0 & & \ddots & \ddots & 0 & a_3 & 0 & b_1 \\ 0 & & & \ddots & b_{n-3} & 0 & a_2 & 0 \\ 0 & & & & 0 & b_{n-2} & 0 & a_1 \end{vmatrix}$$

We see immediately that the directed graph of this matrix has two components each of which is the directed graph of a standard tridiagonal matrix. This observation will give the first direct sum splitting: we shall see that each of these splits for sufficiently large n.

Lemma E1: The $n \times n$ matrix A is similar to a direct sum of four tridiagonal matrices if n is not trivially small. Alternatively, the characteristic polynomial of the $n \times n$ matrix A factors into four polynomials whose degrees differ by no more than one.

Proof: It is sufficient to exhibit the similarity transformations that convert the generalized supertridiagonal matrix A into similar standard tridiagonal matrices. For the first stage define the permutation σ,

(55) $$\sigma(k) = \begin{cases} 2k - 1 & \text{if} \quad k \le \dfrac{n+1}{2} \\ 2k - \left[\dfrac{n}{2}\right] & \text{if} \quad k > \dfrac{n+1}{2} \end{cases}$$

where $1 \leq k \leq n$ and $[x]$ denotes the greatest integer in x function. Associated with σ is an $n \times n$ permutation matrix S_σ. Then, $S_\sigma A S_\sigma^{-1}$ will be a standard tridiagonal matrix, i.e., zero entries everywhere except the main diagonal, first above and first below diagonals. Further, setting $B = S_\sigma A S_\sigma^{-1}$, B will be, in general, $(n \geq 3)$, a direct sum of two tridiagonals:

$$k \times k \quad \text{and} \quad (n - k) \times (n - k) \quad \text{where} \quad m = \left[\frac{n+1}{2}\right].$$

But these tridiagonals are of a special kind, in fact, of the form

$$(56) \qquad B' = \begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+1} & 0 & 0 \\ & b_{m-1} & a_m & b_m & 0 \\ & 0 & b_m & a_m & b_{m-1} \\ & 0 & 0 & b_{m+1} & a_{m-1} \\ & & & & \cdots \end{bmatrix}$$

for the even case and

$$(57) \qquad B'' = \begin{bmatrix} \cdots & & & \\ & a_{m-1} & b_{m+2} & 0 \\ & b_{m-1} & a_m & b_{m+1} \\ & 0 & b_m & a_{m+1} \\ & & & \cdots \end{bmatrix}$$

for the odd case. Because of the special up and down features, we can split these matrices by means of the similarity matrices:

$$(58) \qquad P' = \left[\begin{array}{c|c} I & J \\ \hline -J & I \end{array}\right] \text{ for } n \text{ even; } P'' = \left[\begin{array}{c|c|c} I & 0 & J \\ \hline 0 & 1 & 0 \\ \hline -J & 0 & I \end{array}\right] \text{ for } n \text{ odd;}$$

where I is the identity matrix of appropriate size and J is zero everywhere except for ones on the main cross diagonal. Thus, PBP^{-1} (with appropriate primes on the P and B) is a direct sum of two matrices and of the form

$$(59) \qquad \begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+1} & & \\ & b_{m-1} & a_m - b_m & & \\ & & & a_m + b_m & b_{m-1} \\ & & & b_{m+1} & a_{m-1} \\ & & & & \cdots \end{bmatrix} \text{ for } n \text{ even, and}$$

$$(60) \qquad \begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+2} & & \\ & b_{m-1} & a_m & & \\ & & & a_{m+1} & 2b_m \\ & & & b_{m+1} & a_m \\ & & & & \cdots \end{bmatrix} \text{ when } n \text{ is odd.}$$

We can now apply the lemmas of Section B to write down explicitly the characteristic polynomials of these quantum mechanical Hamiltonian operators; from such explicit forms one expects to elicit information about energy levels and spectra, viz., the eigenvalues are roots of these polynomials.

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VECTORS WHOSE ELEMENTS BELONG TO A GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In a recent paper, D. V. Jaiswal [1] considered some geometrical properties associated with Generalized Fibonacci Sequences. In this paper, we shall extend some of his concepts to n dimensions and generalize his Theorems 2 and 3. We do this by considering column vectors with components that are elements of a G(eneralized) F(ibonacci) S(equence) whose indices differ by fixed integers. We prove two theorems: first, the "area" of the "parallelogram" determined by any two such column vectors is a function of the differences of the indices of successive components; second, any column vectors of the same type form a matrix of rank 2.

2. PRELIMINARY RESULTS

We shall be considering submatrices of an $N \times N$ matrix $T = [T_{i+j-1}]$ where T_s is an element of a GFS with $T_1 = a$ and $T_2 = b$. For the special case $a = b = 1$, we denote the sequence as F_s. We shall indicate the kth column vector of the matrix T as $T_{0k} = [T_{i+k-1}]$. In particular, the first two column vectors of T are $T_{01} = [T_i]$ and $T_{02} = [T_{i+1}]$. We shall now prove a basic property of the matrix T.

Lemma 2.1: The matrix $T = [T_{i+j-1}]$ is of rank 2.

From the fundamental identity for GFS,

$$T_{r+s} = F_{r+1}T_s + F_r T_{s-1},$$

it follows that

$$T_{0k} = F_{k-1}T_{02} + F_{k-2}T_{01}.$$

Hence, the first two column vectors of T span the column space of T. Further, these two vectors are linearly independent, for if $T_{02} = cT_{01}$, it would follow that $cT_i = T_{i+1}$ for all indices i. This implies that

$$c^2 T_i = c(cT_i) = cT_{i+1} = T_{i+2}.$$

However,

$$T_{i+2} = T_{i+1} + T_i = cT_i + T_i = (c+1)T_i,$$

so that

$$(c^2 - c - 1)T_i = 0.$$

The solutions for c are irrational, so the components of T_{0k} would also be irrational. Thus, there is no c and the vectors are linearly independent.

In the next lemma, we evaluate the determinant of some 2 x 2 matrices.

Lemma 2.2: For any k,

$$\begin{vmatrix} T_k & T_{k+1} \\ T_{k+1} & T_{k+2} \end{vmatrix} = (-1)^k(b^2 - a^2 - ab) \neq 0.$$

To prove this, we first show in two steps that the subscripts can be reduced by 2 without changing the value of the determinant. For this, we replace one column by the other column plus a column with subscripts decreased by 2. This gives the determinant of a matrix with two equal columns plus another determinant. The first determinant is zero and is omitted.

$$\begin{vmatrix} T_k & T_{k+1} \\ T_{k+1} & T_{k+2} \end{vmatrix} = \begin{vmatrix} T_k & T_k + T_{k-1} \\ T_{k+1} & T_{k+1} + T_k \end{vmatrix} = \begin{vmatrix} T_k & T_{k-1} \\ T_{k+1} & T_k \end{vmatrix}$$

$$= \begin{vmatrix} T_{k-1} + T_{k-2} & T_{k-1} \\ T_k + T_{k-1} & T_k \end{vmatrix} = \begin{vmatrix} T_{k-2} & T_{k-1} \\ T_{k-1} & T_k \end{vmatrix}$$

In the case where k is even, repeated application of the process yields the determinant

$$\begin{vmatrix} T_2 & T_3 \\ T_3 & T_4 \end{vmatrix} = \begin{vmatrix} T_2 & T_2 + T_1 \\ T_3 & T_3 + T_2 \end{vmatrix} = \begin{vmatrix} T_2 & T_1 \\ T_3 & T_2 \end{vmatrix}$$

Recalling that $T_1 = a$, $T_2 = b$, so $T_3 = a + b$, the determinant is equal to $b^2 - a^2 - ab$.

In the case where k is odd, repeated application of the process yields the determinant

$$\begin{vmatrix} T_1 & T_2 \\ T_2 & T_3 \end{vmatrix} = (-1)\begin{vmatrix} T_2 & T_1 \\ T_3 & T_2 \end{vmatrix} = (-1)(b^2 - a^2 - ab)$$

This proves the first part of the lemma. For the last condition, it is easy to verify that if the determinant were zero, then a and/or b would be irrational.

The final lemma is concerned with the "area" of the "parallelogram" formed by any two column vectors. The proof is based on the property that the determinant of the inner product matrix for the two column vectors is the square of the area of the parallelogram they determine.

Lemma 2.3: The square of the "area" of the "parallelogram" formed by the two $n \times 1$ vectors $\alpha = [a_i]$ and $\beta = [b_i]$ is

$$\sum_{j>1} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2$$

The inner product matrix for α and β is

$$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n a_k^2 & \sum_{k=1}^n a_k b_k \\ \sum_{k=1}^n b_k a_k & \sum_{k=1}^n b_k^2 \end{bmatrix}$$

The determinant of this matrix is

$$\begin{aligned} &\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k\right)^2 \\ &= \sum_{k=1}^n a_k^2 b_k^2 + \sum_{j>i}\left(a_i^2 b_j^2 + a_j^2 b_i^2\right) - \sum_{k=1}^n a_k^2 b_k^2 - 2\sum_{j>i} a_i b_i a_j b_j \\ &= \sum_{j>i} a_i^2 b_j^2 - 2a_i b_i a_j b_j + a_j^2 b_i^2 = \sum_{j>i}\left(a_i b_j - a_j b_i\right)^2 = \sum_{j>i} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2. \end{aligned}$$

3. MAJOR RESULTS

We shall be concerned in this section with two $n \times 1$ submatrices of T of the form $[T_{d_i + c_1 - 1}]$ and $[T_{d_i + c_2 - 1}]$. Because these are submatrices of T, the d_i will form a monotonic increasing sequence. They are in fact the indices of the rows of T appearing in the submatrix. The c_1 and c_2 are the column indices for the submatrix. For convenience, we shall assume that $c_2 > c_1$.

Theorem 3.1: The area of the parallelogram formed by $\alpha = [T_{d_i + c_1 - 1}]$ and $\beta = [T_{d_i + c_2 - 1}]$ is

$$|b^2 - a^2 - ab| F_{c_2 - c_1} \sqrt{\sum_{j>i}\left(F_{d_j - d_i}\right)^2} \neq 0.$$

By Lemma 2.3, the square of the area is given by

$$\sum_{j>i} \begin{vmatrix} T_{d_i + c_1 - 1} & T_{d_i + c_2 - 1} \\ T_{d_j + c_1 - 1} & T_{d_j + c_2 - 1} \end{vmatrix}^2$$

Using the fundamental identity for GFS,

$$T_{d_k + c_2 - 1} = F_{c_2 - c_1 + 1} T_{d_k + c_1 - 1} + F_{c_2 - c_1} T_{d_k + c_1 - 2},\ k = i,\ j.$$

We can replace the second column vector in our determinant by a sum of two vectors. The first gives a zero determinant, while the second gives

$$\sum_{j>i} F^2_{c_2-c_1} \begin{vmatrix} T_{d_i+c_1-1} & T_{d_i+c_1-2} \\ T_{d_j+c_1-1} & T_{d_{j_i}+c_1-2} \end{vmatrix}^2$$

for the square of the area.

In a similar manner, we can express the second row vector as a linear combination using the identity,

$$T_{d_j+c_1-k} = F_{d_j-d_i+1} T_{d_i+c_1-k} + F_{d_j-d_i} T_{d_i+c_1-k-1}, \quad k = 1, 2.$$

This reduces our expression to

$$\sum_{j>i} F^2_{c_2-c_1} \; F^2_{d_j-d_i} \begin{vmatrix} T_{d_i+c_1-1} & T_{d_i+c_1-2} \\ T_{d_i+c_1-2} & T_{d_i+c_1-3} \end{vmatrix}^2$$

By Lemma 2.2, this determinant has the constant value $(b^2 - a^2 - ab)^2$. Thus, the area of the parallelogram is

$$|b^2 - a^2 - ab| F_{c_2-c_1} \sqrt{\sum_{j>i} \left(F_{d_j-d_i}\right)^2}.$$

This area is nonzero, since none of the factors can be zero.

The next theorem follows from the theorem just proved.

Theorem 3.2: Any $r \times s$ submatrix of $T = [T_{i+j-1}]$, where $r, s > 1$, is of rank 2.

By Theorem 3.2, any two column vectors of the submatrix form a parallelogram of nonzero area. Hence, they must be linearly independent, so the rank must be at least 2. But by Lemma 2.1, the matrix T has rank 2 and hence the rank of any submatrix cannot exceed 2. Therefore, the rank is exactly 2.

The result given in Theorem 3.2 would seem to indicate that the geometry associated with GFS is necessarily of dimension 2. A check of the results of the Jaiswal paper confirms this observation.

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ON NTH POWERS IN THE LUCAS AND FIBONACCI SERIES

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A. INTRODUCTION

Let F_n be the nth term in the Fibonacci series defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,$$

and let L_n be the nth term in the Lucas series defined by

$$L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$$

In a previous paper [3], H. London and the present author considered the problem of finding all the Nth powers in the Lucas and Fibonacci series. It was shown that the problem reduces to solving certain Diophantine equations, and all the cubes in both series were found. However, the problem of finding all the cubes in the Fibonacci sequence depended upon the solutions of the equations $y^2 \pm 100 = x^3$, and the finding of all these solutions is quite a difficult matter.

In the present paper we first present a more elementary proof of this fact which does not depend on the solution of $y^2 \pm 100 = x^3$. We then show that if p is a prime and $p \geq 5$, then L_{3k} and L_{2k} are never pth powers. Further, we show that if $F_{2^t k}$ is a pth power then $t \leq 1$, and we find all the 5th powers in the sequence F_{2m}. Finally, we close with some discussion of Lucas numbers of the form $y^p + 1$.

In our work we shall require the following theorems, which we state without proof:

Theorem 1: The Lucas and Fibonacci numbers satisfy the relations

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad \text{and} \quad L_m F_m = F_{2m}.$$

Theorem 2 (Nagell [6]): The equation $Ax^3 + By^3 = C$, where $3 \nmid AB$ if $C = 3$ has at most one solution in nonzero integers (u, v). There is a unique exception for the equation $x^3 + 2y^3 = 3$, which has exactly the two solutions $(u, v) = (1, 1)$ and $(-5, 4)$.

Theorem 3 (Nagell [7, p. 28]): If n is an odd integer ≥ 3, A is a square-free integer ≥ 1, and the class number of the field $Q(\sqrt{-A})$ is not divisible by n then the equation $Ax^2 + 1 = y^n$ has no solutions in integers x and y for y odd and ≥ 1 apart from $x = \pm 11$, $y = 3$ for $A = 2$ and $n = 5$.

Theorem 4 (Nagell [7, p. 29]): Let n be an odd integer ≥ 3 and let A be a square-free integer ≥ 3. If the class number of the field $Q(\sqrt{-A})$ is not divisible by n, the equation $Ax^2 + 4 = y^n$ has no solutions in odd integers A, x, and y.

Theorem 5 (Af Ekenstam [1], p. 5]): Let ε be the fundamental unit of the ring $R(\sqrt{m})$. If $N(\varepsilon) = -1$, the equation $x^{2n} - My^{2n} = 1$ has no integer solutions with $y \neq 0$.

Theorem 6 [5, p. 301]: Let p be an odd prime. Then the equation $y^2 + 1 = x^p$ is impossible for $x > 1$.

B. CUBES IN THE LUCAS SEQUENCE

By Theorem 1, we have

(1) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

If $L_n = y^3$, $F_n = x$, we get

$$y^6 - 5x^2 = 4(-1)^n$$

with $u > 0$ and $x > 0$. Suppose first n is even, then we get

$$y^6 - 4 = 5x^2.$$

If y is even, this equation is impossible mod 32. Thus, y is odd, and

$$(y^3 + 2)(y^3 - 2) = x^2$$

with

$$(y^3 + 2,\ y^3 - 2) = 1;$$

this implies that either

$$\begin{cases} y^3 + 2 = u^2 \\ y^3 - 2 = 5v^2 \end{cases}$$

or

$$\begin{cases} y^3 + 2 = 5u^2 \\ y^3 - 2 = v^2. \end{cases}$$

But it is well known (see, e.g., [9], pp. 399-400) that the only solution of the equation $y^3 + 2 = u^2$ is $y = -1$, $u = 1$. This, however, does not yield any value for v. Further, the equation $y^3 - 2 = v^2$ has only the solution $v = \pm 5$, $y = 3$. But this does not yield a value for u. Therefore, there are no cubes in the sequence L_{2m}.

Note: This result also follows immediately from Theorem 5 since the class number of $Q(\sqrt{5})$ is 1.

Next, suppose n is odd, then we get

(2) $$5x^2 - 4 = y^6.$$

If y is even, this equation is impossible mod 32. Thus x and y are odd, and (2) reduces to

(3) $$5x^2 - 4 = u^3,$$

with u a square. Equation (3) may be written

$$(2 + \sqrt{5}x)(2 - \sqrt{5x}) = u^3,$$

and since x is odd,

$$(2 + \sqrt{5}x,\ 2 - \sqrt{5x}) = 1.$$

Thus we conclude

(4) $$2 + \sqrt{5}x = \left(\frac{a + b\sqrt{5}}{2}\right)^3$$

or

(5) $$2 + \sqrt{5}x = \left(\frac{a + b\sqrt{5}}{2}\right)^3 \left(\frac{1 + \sqrt{5}}{2}\right).$$

Equation (4) yields

$$3a^2b + 5b^3 = 8,$$

which, in turn, yields

$$a = b = 1,\ v = 1.$$

To solve (5), we note that

$$N\left(\frac{a + b\sqrt{5}}{2}\right) = u, \text{ i.e., } a^2 - 5b^2 = 4u.$$

Since u is odd, a and b are odd, thus $\left(\frac{a + b\sqrt{5}}{2}\right)^3$ is of the form $S + T\sqrt{5}$, with S even and T odd. And

$$\left(\frac{1 + \sqrt{5}}{2}\right)(S + T\sqrt{5})$$

can never be of the form $A + B\sqrt{5}$. Thus, (5) is impossible. We have proved

Theorem 7: The only cube in the Lucas sequence is $L_1 = 1$.

C. CUBES IN THE FIBONACCI SEQUENCE

First, suppose m is even. Then $F_{2n} = x^3$ implies $F_n L_n = x^3$. If $n \not\equiv 0 \pmod 3$, $(F_n, L_n) = 1$. Thus, $L_n = t^3$ and $n = 1$.

If $n \equiv 0 \pmod 3$, $(F_n, L_n) = 2$ and either:

a. L_n is a cube, which is impossible;
b. $L_n = 2z^3$, $F = 4y^3$; or
c. $L_n = 4z^3$, $F = 2y^3$.

We now use equation (1) and first suppose n even. Then case (b) reduces to

$$z^3 - 20y^3 = 1,$$

with y and z squares. By Theorem 2, the only solutions of this equation are $z = 1$, $y = 0$, and $z = -19$, $y = -7$. Of these, only the first yields a sqaure for z and we get $n = 0$. Case (c) reduces to solving

$$4z^3 - 5y^3 = 1,$$

with y and z squares. Again, by Theorem 2, the only solution of this equation is $z = -1$, $y = -1$, which does not yield a square value for z. If n is odd, we get the two equations

$$z^3 - 20y^3 = -1 \quad \text{and} \quad 4z^3 - 5y^3 = -1$$

with z and y squares. By the results above, the only solutions of these two equations are $(z, y) = (-1, 0)$, $(19, 7)$, and $(1, 1)$. Of these, only the last yields a square value of z and we get $n = 3$. Thus, the only cubes in the sequence F_{2n} are $F_0 = 0$, $F_2 = 1$, and $F_6 = 8$.

If m is odd, and $F_m = x^3$, F_m cannot be even since then (1) yields

$$L_m^2 - 5x^6 = -4,$$

which is impossible (mod 32). Thus, the problem reduces to

$$10y^3 - 8 = 2x^2, \tag{6}$$

with x and y odd. But (6) was solved completely in [4, pp. 107-110]. We outline the solution here. Let $\theta^3 = 10$, where θ is real. We use the fact that $Z[\theta]$ is a unique factorization domain and apply ideal factorization theory to reduce this problem to

$$y\theta - 2 = (-2 + \theta)\left(\frac{a + b\theta + c\theta^2}{3}\right) \tag{7}$$

and three other equations, all of which may be proved impossible by congruence conditions.

To solve (7), we equate coefficients of θ and θ^2 to get

$$a^2 + 20bc - 5b^2 - 10ac = 9, \tag{8}$$

and

$$b^2 + 2ac = ab + 5c^2. \tag{9}$$

Equation (9) may be written

$$(b + 2c - a)(b - 2c) = c^2,$$

and (8) and (9) yield a odd, b even, and c even. Thus, we conclude

$$\begin{aligned} b - a + 2c &= dh_1^2 \\ b - 2c &= 4dh_2^2 \\ c &= 2dh_1h_2\varepsilon \end{aligned} \tag{10}$$

where $\varepsilon = \pm 1$, d, h_1, h_2 are rational integers, and $d > 0$. If we solve (10) for a, b, c and substitute in (8) we get

$$h^4 + 4h_1^3h_2\varepsilon - 24h_1^2h_2^2 - 16h_1h_2\varepsilon - 64h_2^4 = 9/d^2,$$

which reduces to

$$u^4 - 30u^2v^2 + 40uv^3 - 75v^4 = 9/d^2. \tag{11}$$

If $d = 1$, (11) is impossible (mod 5). If $d = 3$, (11) may be written

$$(u - 5v)(u^3 + 5u^2v - 5uv^2 + 15v^3) = 1,$$

and from this it follows easily that the only integer solutions of (11) are $(u, v) = (\pm 1, 0)$. Thus, we have:

Theorem 8: The only cubes in the Fibonacci sequence are

$$F_0 = 0,\ F_1 = F_2 = 1, \text{ and } F_6 = 8.$$

D. HIGHER POWERS IN THE LUCAS AND FIBONACCI SEQUENCES

In this section, we investigate the problem of finding all pth powers in the Fibonacci and Lucas sequences, where p is a prime and $p \geq 5$. We show that L_{3n} and L_{2n} are never pth powers, and that if F_{2^tn} is a pth power, then $t = 0$ or 1. We conclude by finding all the 5th powers in the sequence F_{2n}.

Suppose $L_n = x^p$; then equation (1) yields

$$x^{2p} - 5F_n^2 = 4(-1)^n.$$

If $n \equiv 0 \pmod 3$, x and F_{2n} are even, and this equation is impossible (mod 32) regardless of the parity of n.

Suppose further that $n = 2m$, $m \not\equiv 0 \pmod 3$; then (1) yields

$$5F_m^2 + 4 = x^{2p},$$

with x and F_m odd. Since the class number of $Q(\sqrt{-5})$ is 2, this equation has no solutions with F_m odd. Thus, we have:

Theorem 9: L_{3k} and L_{2k} are never pth powers for any k. Finally, if n is odd, $n \not\equiv 0 \pmod 3$, we have to solve

$$5y^2 - 4 = x^{2p}.$$

Unfortunately, known methods of treating this equation lead to the solution of irreducible equations. Thus, it is quite difficult to solve.

Now let us suppose $F_n = x^p$. If $n \equiv 3 \pmod 6$, F_n is even and Equation (1) becomes

$$L_n^2 - 5x^{2p} = -4,$$

which is impossible (mod 32). Thus, F_{6k+3} is never a pth power for any k.

Now we prove:

Theorem 10: If F_{6k} is a pth power, then $k = 0$.

Proof: Let $m = 3k$, then $(F_m, L_m) = 2$, and since L_{3k} is not a pth power, we conclude:

$$\text{a.} \quad \begin{cases} L_m = 2u^p \\ F_m = 2^{rp-1}v^p, \end{cases}$$

or

$$\text{b.} \quad \begin{cases} L_m = 2^{rp-1}u^p \\ F_m = 2v^p, \end{cases}$$

with u and v odd, m even, and $r \geq 1$.

Now note that m cannot be odd in either case, since then (1) yields

$$L_m^2 - 5F_m^2 = -4 = L_m^2 - 5 \cdot 2^t v^{2p} \text{ for some integer } t,$$

which is impossible (mod 32).

If we substitute $F_m = 2v^p$ in (1), case (b) reduces to

$$x^2 - 1 = 5v^{2p}, \tag{12}$$

since m is even. If v is odd, (12) is impossible (mod 8). Thus, x is odd, v is even, and (12) yields

$$\begin{cases} \dfrac{x+1}{2} = u^{2p} \\ \dfrac{x-1}{2} = 5v^{2p}, \end{cases}$$

i.e.,

$$u^{2p} - 5v^{2p} = 1.$$

Since the fundamental unit of $Z[1, \sqrt{5}]$ is $2 + 5$ and $N(2 + \sqrt{5}) = -1$, this equation has no solution for $v \neq 0$, by Theorem 5. The solution $v = 0$ yields $L_0 = 2$ and $p = 2$, which is impossible. Thus, $F_{6k} \neq 2v^p$ for any $k \neq 0$.

To solve case (a), suppose $m \neq 0$ and $m = 2^t\ell$, ℓ odd, $\ell \equiv 0 \pmod 3$. Then

$$F_{2^t\ell} = F_{2^{t-1}\ell} I_{2^{t-1}\ell} = 2^{rp-1}v^p.$$

Since $F_{2^{t-1}\ell} \neq 2v^p$ and $L_{2^{t-1}\ell}$ is not a pth power, we conclude

$$L_{2^{t-1}\ell} = 2u_1^p,$$
$$F_{2^{t-1}\ell} = 2^{rp-1}v_1^p,$$

with u_1 and v_1 odd. By continuing this process, we eventually get

$$F_{2^s\ell} = y^p \quad \text{for some } s \text{ and } y \text{ odd},$$
$$F_\ell = y^p \quad \text{for } y \text{ odd, or}$$
$$F_\ell = 2^s y^p \quad \text{for some } s \text{ and } y \text{ odd}.$$

Since $F_{2^s \ell}$ and F_ℓ are even if $\ell \equiv 0 \pmod 3$, the first two of these equations are impossible.

To settle the third, note that we have

$$L_\ell = 2x^p$$
$$F_\ell = 2^s y^p$$

with ℓ odd, $\ell \equiv 0 \pmod 3$ and x and y odd. Then (1) yields

$$x^{2p} - 5 \cdot 4^{s-1} y^{2p} = -1.$$

If $s > 1$, this equation is impossible mod 20; if $s = 1$, we get

$$x^{2p} - 5y^{2p} = -1,$$

which is impossible mod 4, since x and y are odd. Thus, $m = 0$.

Next, suppose $m \not\equiv 0 \pmod 3$, and F_{2m} is a pth power. Then

$$F_{2m} = F_m L_m, \ (F_m, L_m) = 1$$

and F_m and L_m are both pth powers. This enables us to prove the following result:

Theorem 11: If $F_{2^t m}$ is a pth power, m odd, $m \not\equiv 0 \pmod 3$, then $t \leq 1$.

Proof: Suppose $F_{2^t m}$ is a pth power with m odd, $m \not\equiv 0 \pmod 3$ and $t > 1$. Then

$$F_{2^t m} = F_{2^{t-1} m} L_{2^{t-1} m}$$

and both $L_{2^{t-1} m}$ and $F_{2^{t-1} m}$ are pth powers. Further,

$$F_{2^{t-1} m} = F_{2^{t-2} m} L_{2^{t-2} m},$$

and both $F_{2^{t-2} m}$ and $L_{2^{t-2} m}$ are powers. By continuing this process, we eventually get

$$F_{4m} = F_{2m} L_{2m},$$

and both F_{2m} and L_{2m} are pth powers. This is impossible by Theorem 9, and thus $t \leq 1$.

If m is odd, $m \not\equiv 0 \pmod 3$ and F_{2m} is a pth power, then

$$F_{2m} = F_m L_m$$

and both F_m and L_m are pth powers. Thus, we must solve

(13) $$x^{2p} - 5y^{2p} = -4.$$

Unfortunately, it seems quite difficult to solve this equation for arbitrary p. We shall give the solution for $p = 5$ presently, and shall return to (13) in a future paper.

Finally, if m is odd and $F_m = x^p$, we have to solve

$$x^{2p} + 4 = 5y^{2p},$$

with y odd. Again, the solution of this equation leads to irreducible equations and is thus quite difficult to solve. To conclude this section, we prove

Theorem 12: The only 5th power in the sequence F_{2m} is $F_2 = 1$.

Proof: if we substitute $p = 5$ in (13), it reduces to

(14) $$x^5 + 5y^5 = 4,$$

and we must prove that the only integer solution of this equation is $x = -1$,

$y = 1$. To this end, we consider the field $Q(\theta)$, where $\theta^5 = 5$. We find that an integral basis is $(1, \theta, \theta^2, \theta^3, \theta^4)$ and that a pair of fundamental units is given by

$$\varepsilon_1 = 1 - 10 - 5\theta^2 + 3\theta^3 + 4\theta^4,$$

$$\varepsilon_2 = -24 + 15\theta - 5\theta^2 - 2\theta^3 + 5\theta^4.$$

Since $(2, -1 + \theta)^2 = (-1 + \theta)$, the only ideal of norm 4 is $(-1 + \theta)$. Thus, (14) reduces to

$$u + v\theta = (-1 + \theta)\varepsilon_1^m \varepsilon_2^n. \tag{15}$$

We now use Skolem's method [8]. We find

$$\varepsilon_1^5 = 1 + 5\xi_1$$

with

$$\xi_1 = 3\theta^3 + 4\theta^4 + 5A$$

and

$$\varepsilon_2^5 = 1 + 5\xi_2$$

with

$$\xi_2 = 3\theta^3 + 5B,$$

where A and B are elements of $Z[\theta]$.

If we write $m = 5u + r$, $n = 5v + s$ and treat (15) as a congruence (mod 5), we find that it holds only for $r = s = 0$. Thus, (15) may be written

$$\begin{aligned} u + v\theta &= (-1 + \theta)(1 + 5\xi_1)^u (1 + 5\xi_2)^v \\ &= (-1 + \theta)[1 + 5(u\xi_1 + v\xi_2) + \cdots]. \end{aligned}$$

Now we equate the coefficients of θ^3 and θ^4 to 0 and get

$$-3u + 3v + 0(5) = 0,$$

$$-u + 3v + 0(5) = 0.$$

Since $\begin{vmatrix} -3 & 3 \\ -1 & 3 \end{vmatrix} \not\equiv 0 \pmod 5,$

the equation

$$u + v\theta + w\theta^2 = (-1 + \theta)\varepsilon_1^m \varepsilon_2^n$$

has no solution except $m = n = 0$, when $m \equiv n \equiv 0 \pmod 5$, by a result of Skolem [8] and Avanesov [2]. Thus, the only solution of (15) is $m = n = 0$, and the result follows.

E. LUCAS NUMBERS OF THE FORM $y^p + 1$

For our final result, we prove

Theorem 13: Let p be an odd prime. If $L_{2m} = y^p + 1$, then $m = 0$.

Proof: Again, we use (1). We set $L_{2m} = y^p + 1$, $F_{2m} = x$, and get

$$(y^p + 1)^2 - 4 = 5x^2,$$

i.e.,

$$y^{2p} + 2y^p - 3 = 5x^2,$$

i.e.,

$$(y^p + 3)(y^p - 1) = 5x^2.$$

The GDC of $y^p + 3$ and $y^p - 1$ divides 4. If y is even, both these numbers are relatively prime, and we get

$$\begin{cases} y^p + 3 = u^2, \\ y^p - 1 = 5v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 5u^2, \\ y^p - 1 = v^2. \end{cases}$$

Since y is even, both these systems are impossible (mod 8).

Suppose next that $y \equiv 3 \pmod 4$. Then $(y^p + 3,\ y^p - 1) = 2$, and we get

$$\begin{cases} y^p + 3 = 2u^2, \\ y^p - 1 = 10v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 10u^2, \\ y^p - 1 = 2v^2. \end{cases}$$

By Theorem 3, the equation $y \quad - 1 = 10v^2$ has no solution with y odd for $y \neq 1$ since the class number of $Q(\sqrt{-10})$ is 2. But $y = 1$ contradicts $y = 3 \pmod 4$. Further, $y^p - 1 = 2v$ has no solution with y odd except $y = 1$, $v = 0$, and $y = 3$, $v = \pm 11$, $p = 5$. But $y = 3$ does not yield a value for u.

Finally, if $y \equiv 1 \pmod 4$, $(y^p + 3,\ y^p - 1) = 4$, and we get

$$\begin{cases} y^p + 3 = 20u^2, \\ y^p - 1 = 4v^2, \end{cases}$$

or

$$\begin{cases} y^p + 3 = 4u^2, \\ y^p - 1 = 20v^2. \end{cases}$$

By Theorem 6, $y^p - 1 = 4v^2$ has no integer solution except $y = 1$, $v = 0$. However, this does not yield a value for u. By Theorem 3, the only solution of $y^p - 1 = 20v^2$ with y odd is $y = 1$, $v = 0$, and we get

$$y = 1,\ u = 1, \text{ and } x = 0.$$

The result follows.

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GENERALIZED TWO-PILE FIBONACCI NIM

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1. INTRODUCTION

Consider a take-away game with one pile of chips. Two players alternately remove a positive number of chips from the pile. A player may remove from 1 to $f(t)$ chips on his move, t being the number removed by his opponent on the previous move. The last player able to move wins.

In 1963, Whinihan [3] revealed winning strategies for the case when $f(t) = 2t$, the so-called *Fibonacci Nim*. In 1970, Schwenk [2] solved all games for f nondecreasing and $f(t) \geq t \; \forall t$. In 1977, Epp & Ferguson [1] extended the solution to the class where f is nondecreasing and $f(1) \geq 1$.

Recently, Ferguson solved a *two-pile analogue of Fibonacci Nim*. This motivated the author to investigate take-away games with more than one pile of chips. In this paper, winning strategies are presented for a class of two-pile take-away games which *generalize* two-pile Fibonacci Nim.

2. THE TWO-PILE GAME

Play begins with two piles containing m and m' chips and a positive integer w. Player I selects a pile and removes from 1 to w chips. Suppose t chips are taken. Player II responds by taking from 1 to $f(t)$ chips from one of the piles. We assume f *is nondecreasing* and $f(t) \geq t \; \forall t$. The two players alternate moves in this fashion. The player who leaves both piles empty is the winner. If $m = m'$, Player II is assured a win.

Set $d = m' - m$. For $d \geq 1$, define $L(m, d)$ to be the least value of w for which Player I can win. Set $L(m, 0) = \infty \; \forall m \geq 0$. One can systematically generate a *tableau* of values for $L(m, d)$. Given the position (m, d, w), the player about to move can win iff he can:

(1) take t chips, $1 \leq t \leq w$, from the large pile, leaving the next player in position $(m, d - t, f(t))$ with $f(t) < L(m, d - t)$; or

(2) take t chips, $1 \leq t \leq w$, from the small pile, leaving the next player in position $(m - t, d + t, f(t))$ with $f(t) < L(m - t, d + t)$.

(See Fig. 2.1.) Consequently, the tableau is governed by the functional equation

$$L(m, d) = \min\left\{t > 0 \,\middle|\, f(t) < L(m, d - t) \text{ or } f(t) < L(m - t, d + t)\right\}$$

subject to $L(m, 0) = +\infty \; \forall m \geq 0$. Note that $L(m, d) \leq d \; \forall d \geq 1$. Dr. Ferguson has written a computer program which can quickly furnish the players with a 60×40 tableau. As an illustration, Figure 2.2 gives a tableau for the two-pile game with $f(t) = 2t$, two-pile Fibonacci Nim.

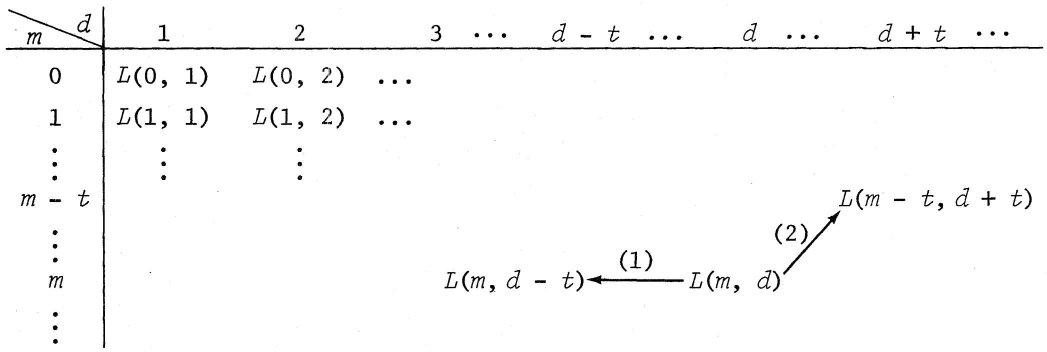

Fig. 2.1 The Tableau

Given f, one can construct a strictly-increasing infinite sequence $\langle H_k \rangle_1^\infty$ as follows: $H_1 = 1$ and for $k \geq 1$, $H_{k+1} = H_k + H_j$ where j is the least integer such that $f(H_j) \geq H_k$. For example, $\langle H_k \rangle_1^\infty$ is the Fibonacci sequence when $f(t) = 2t$, and $H_k = 2^{k-1}$, $k \geq 1$ when $f(t) = t$. Schwenk [2] showed that each positive integer d can be represented as a *unique sum* of the H_k's

(2.1) $$d = \sum_{i=1}^{s} H_{n_i} \text{ such that } f\left(H_{n_i}\right) < H_{n_{i+1}} \text{ for } i = 1, 2, \ldots, s - 1.$$

Moreover, for the take-away game with a single pile of d (= $m' - 0$) chips, Player I can win iff he can remove H_{n_1} chips from the pile (i.e., iff $H_{n_1} \leq w$). So for the two-pile game with one pile exhausted,

(2.2) $$L(0, d) = H_{n_1}.$$

For the one-pile game with $d = H_{n_1} + \cdots + H_{n_s}$, $s \geq 1$, chips, H_{n_1} is the key term. It turns out that for the two-pile game where $d = m' - m = H_{n_1} + H_{n_2} + \cdots + H_{n_s}$, $s \geq 1$, H_{n_2} *(when it exists) as well as* H_{n_1} *plays a decisive role*. Denote $n_1 = n$ and n_2 (when it exists) $= n + r$. Thus, we shall write

$$d = H_n + H_{n+r} + \cdots + H_{n_s}, \quad s \geq 1.$$

For each positive integer k, define $\ell(k)$ to be the greatest integer such that

(2.3) $$f\left(H_{k-\ell(k)}\right) \geq H_k.$$

Note that $\ell(1) = 0$, $\ell(k) \geq 0$, and $H_{k+1} = H_k + H_{k-\ell(k)}$ $\forall k \geq 1$.

In the sequel, we present winning strategies for the class of two-pile games for which $\ell(k) \in \{0, 1\}$ $\forall k$. We refer to such games as *generalized two-pile Fibonacci Nim*.

It would be nice if one could find some NASC on f such that $\ell(k) \in \{0, 1\}$ $\forall k$. The following partial results have been obtained:

(1) If $f(t) < (5/2)t$ $\forall t$, then $\ell(k) \in \{0, 1\}$ $\forall k$.
In particular, for $f(t) = ct$,
(a) if $1 \leq c < 2$, then $\ell(k) = 0$ $\forall k \geq 1$;
(b) if $2 \leq c < 5/2$, then $\ell(k) = 1$ $\forall k \geq 2$;
(c) if $c \geq 5/2$, then $\ell(3) = 2$ or $\ell(4) = 2$.

(2) If $\ell(k) \in \{0, 1\}$ $\forall k$, then $f(t) < 6t$ $\forall t$.

(3) A NASC such that $\ell(k) = 0$ $\forall k$ is $f(2^k) < 2^{k+1}$ $\forall k \geq 0$.

(4) A NASC such that $\ell(k) = 1 \ \forall k \geq 2$ is $F_k \leq f(F_{k-1}) < F_{k+1} \ \forall k \geq 2$, where $\langle F_k \rangle_1^\infty$ is the Fibonacci sequence 1, 2, 3, 5, 8, 13, ...

m \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
0	1	2	3	1	5	1	2	8	1	2	3	1	13	1	2	3	1	5	1	2	21	1	2	3	1	5	1	2	8	1	2	3	1	34
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38	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
39	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
40	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
41	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
42	1	2	2	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21

Fig. 2.2 A Tableau for Two-Pile Fibonacci Nim $\left(f(t) = 2t\right)$*

*Computer program supplied by T. S. Ferguson.

3. SOME GOOD AND BAD MOVES

Lemma: For the position (m, d), $d = H_n + \cdots + H_{n_s}$, $s \geq 1$, it is *never* a winning move to take

(1) t chips from the large pile if $0 < t < H_n$.

(2) t chips from the small pile if $0 < t < H_n$, $t \neq H_{n-\ell(n)}$.

It is *always* a winning move to take

(3) H_n chips from the large pile with the possible exception of the *special case*: $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+1) = \ell(n+2) = 1$, $m \geq H_{n+1}$.

(4) $H_{n-\ell(n)}$ chips from the small pile when $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+1) = \ell(n+2) = 1$, $m \geq H_{n-\ell(n)}$. (This contains the special case.)

Proof: The statements (1)-(4) imply that $L(m, d) \in \{H_n, H_{n-\ell(n)}\}$ $\forall m \geq 0$. We shall use this observation and double induction in our argument.

Schwenk [2] proved the assertions for the positions $(0, d)$, $\forall d \geq 1$ (see equation 2.2). Suppose they hold for the positions (m, d) $\forall m \leq M - 1$, $\forall d \geq 1$ for some $M \geq 1$. We must show that (1)-(4) hold for the positions (M, d) $\forall d \geq 1$.

The claim is trivial for position $(M, 1)$. Suppose it is true for (M, d) $\forall d \leq D - 1$ for some $D \geq 2$. Consider the two types of moves which can be made from position (M, D), $D = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \geq 1$.

A. *Taking from the big pile*:

Take t chips, $0 < t < H_n$, from the big pile. Then $D - t = H_k + \cdots + H_{n_s}$ where $k < n$. $t \geq H_{k-1}$ if $\ell(k) = 1$, and $t \geq H_k$ if $\ell(k) = 0$. By the inductive assumption $L(M, D - t) \leq H_k$. Hence,

$$f(t) \geq f(H_{k-1}) \geq H_k \geq L(M, D - t) \quad \text{if} \quad \ell(k) = 1,$$

and

$$f(t) \geq f(H_k) \geq H_k \geq L(M, D - t) \quad \text{if} \quad \ell(k) = 0.$$

Statement (1) follows.

Suppose you take $t = H_n$ chips from the big pile. Consider the following cases.

(1) $D = H_n$. Taking H_n chips from the large pile is obviously a winning move.

(2) $D > H_n$. Write $D - H_n = H_{n+r} + \cdots + H_s$.

(a) $r = 1$. Necessarily, $\ell(n+1) = 0$. By the inductive assumption, $L(M, D - H_n) = H_{n+1}$. Thus, $f(H_n) < L(M, D - H_n)$ and it is a good move to take H_n chips from the large pile.

(b) $r \geq 3$. By the inductive assumption, $L(M, D - H_n) \geq H_{n+2}$. Thus, $f(H_n) < H_{n+2} \leq L(M, D - H_n)$ and it is a good move to take H_n chips from the large pile.

(c) $r = 2$.

(i) $\ell(n+1) = 0$. $f(H_n) < H_{n+1}$ and, by the inductive assumption, $L(M, D - H_n) \geq H_{n+1}$. A good move is to take H_n chips from the big pile.

(ii) $\ell(n+1) = 1$ and $\ell(n+2) = 0$. By the second equation and the inductive assumption, $L(M, D - H_n) = H_{n+2}$. Thus, $f(H_n) < H_{n+2} \leq L(M, D - H_n)$, so taking H_n chips from the large pile wins.

(iii) $\ell(n+1) = 1$ and $\ell(n+2) = 1$. Here $f(H_n) \geq H_{n+1}$. By the inductive assumption, it is possible that $L(M, D - H_n) = H_{n+1}$. If $L(M, D - H_n) = H_{n+1}$, then $M \geq H_{n+1}$ follows from (1) of the Lemma. The possibility of $f(H_n) \geq L(M, D - H_n)$ signifies that taking H_n chips from the large pile might be a bad move. Thus, (3) holds.

B. Taking from the small pile:

If t chips, $0 < t < H_n$, $t \neq H_{n-\ell(n)}$, are removed from the small pile, the resulting position is $(M - t, D + t)$, $D + t = H_{n-k} + \cdots + H_{n_s}$ for some $k \geq 1$ and $t \geq H_{n-k}$. But $L(M - t, D + t) \leq H_{n-k}$ by assumption. Since

$$f(t) \geq t \geq H_{n-k} \geq L(M - t, D + t),$$

this is a bad move. Thus, (2) holds.

C. Case A2.c.(iii) *revisited*:

Here $D = H_n + H_{n+2} + \cdots + H_{n_s}$, $\ell(n + 1) = \ell(n + 2) = 1$. Suppose taking H_n chips from the large pile is *not* a good move. Then, $L(M, D - H_n) = H_{n+1}$.

For position (M, D), $M \geq H_{n-\ell(n)}$, take $H_{n-\ell(n)}$ chips from the small pile to get $\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right)$. $D + H_{n-\ell(n)} = H_{n-\ell(n)} + H_n + H_{n+2} + \cdots + H_{n_s} = H_{n+1} + H_{n+2} + \cdots + H_{n_s'} = H_{n+k} + \cdots + H_{n_{s'}}$, for some $k \geq 3$ and $n_{s'} \geq n_s$, since $\ell(n + 2) = 1$. By the inductive assumption, $L\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right) \geq H_{n+2}$. But $f\left(H_{n-\ell(n)}\right) \leq f(H_n) < H_{n+2}$. Thus,

$$f\left(H_{n-\ell(n)}\right) < L\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right).$$

Taking $H_{n-\ell(n)}$ chips from the small pile is a good move, so (4) holds.

In *A*, *B*, and *C* we established that (1)-(4) hold for the position (M, D), which completes the induction on d. Hence, they hold for (M, d) $\forall d \geq 1$. This in turn completes the induction on m. Thus, (1)-(4) hold for (m, d) $\forall m \geq 0$, $\forall d \geq 1$. Q.E.D.

Corollary 1: $L(m, d) \in \left\{H_n, H_{n-\ell(n)}\right\}$ $\forall m \geq 0$.

Observe that if $\ell(n) = 0$, then $L(m, d) = H_n$ $\forall m \geq 0$. But when $\ell(n) = 1$, there are two possible values $L(m, d)$ might assume. However, if $m < H_{n-1}$, then $L(m, d) = H_n$.

Corollary 2—How to win (if you can) when you know $L(m, d)$:

(1) If $L(m, d) = H_{n-1}$, take H_{n-1} chips from the small pile to win.
(2) If $L(m, d) = H_n$, a winning move is to take H_n chips from the large pile, except possibly for the special case cited in the Lemma. In the special case, take H_n chips from the small pile to win.

4. HOW TO WIN IF YOU CAN

Knowing $L(m, d)$ at the beginning of play reveals whether Player I has a winning strategy. Compare $L(m, d)$ and w. If Player I knows the value of $L(m, d)$ and $w \geq L(m, d)$, he can use Corollary 2 to determine a winning move.

Which of the two possible values $L(m, d)$ assumes is not obvious under certain circumstances. The position (m, d, w) defies immediate classification when $L(m, d)$ is unknown and $H_{n-1} \leq w < H_n$.

Fortunately, not knowing whether one can win at the beginning of play does not prevent one from describing a winning strategy, provided such a strategy exists. A strategy of play, constructed from the Corollaries, is presented in Table 4.1. This table tells how to move optimally in all situations in which there exists a possibility of winning. An $N(P)$ represents a position for which there exists a winning move for Player I (II).

The only case in which the status of a position is now known at the start of play arises in 2(b) of the table. There, the player about to move is an *optimist* and *pretends* $L(m, d) = H_{n-1}$. This dictates taking H_{n-1} chips from the small pile. The outcome of the game will reveal the value of $L(m, d)$ depending on who wins.

Table 4.1. How To Win (If You Can) Without Knowing $L(m, d)$

(1) If $\ell(n) = 0$ [so necessarily $L(m, d) = H_n$] and

(a) $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+2) = \ell(n+1) = 1$

	$m \geq H_n$	$m < H_n$
$w \geq H_n$	N, Take H_n from s.p.	N, Take H_n from l.p.
$w < H_n$	P	P

(b) not as in (a)

	$m \geq H_n$	$m < H_n$
$w \geq H_n$	N, Take H_n from l.p.	N, Take H_n from l.p.
$w < H_n$	P	P

(2) If $\ell(n) = 1$ and

(a) $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+2) = \ell(n+1) = 1$

	$m \geq H_{n-1}\left(L(m, d) = H_{n-1}\right)$	$m < H_{n-1}\left(L(m, d) = H_n\right)$
$w \geq H_n$	N, Take H_{n-1} from s.p.	N, Take H_n from l.p.
$H_n > w \geq H_{n-1}$	N, Take H_{n-1} from s.p.	P
$w < H_{n-1}$	P	P

(b) not as in (a)

	$m \geq H_{n-1}\left(L(m, d) = ??\right)$	$m < H_{n-1}\left(L(m, d) = H_n\right)$
$w \geq H_n$	N, Take H_n from l.p.	N, Take H_n from l.p.
$H_n > w \geq H_{n-1}$	??, Take H_{n-1} from s.p.	P
$w < H_{n-1}$	P	P

(Note: s.p. = small pile; l.p. = large pile.)

As an illustration, consider two-pile Fibonacci Nim. It was first solved by Ferguson in the form of Table 4.1. For $f(t) = 2t$, the sequence $\langle H_k \rangle_1^\infty$ is the Fibonacci sequence. The first few values are

k	1	2	3	4	5	6	7	8	9	10
H_k	1	2	3	5	8	13	21	34	55	89

$\ell(1) = 0$ and $\ell(k) = 1\ \forall k \geq 2$, since $H_{k+1} = H_k + H_{k-1}\ \forall k \geq 2$. What is the status of position $m = 20$, $d = 42$, $w = 6$? $d = 34 + 8 = H_8 + H_5$. Player I is an optimist and *assumes* that $L(20, 42) = 5$, not 8. 2(b) in the table tells him to take 5 chips from the small pile.

Player II is left in position $m = 20 - 5 = 15$, $d = 42 + 5 = 47$, $w = f(5) = 10$. $d = 34 + 13 = H_8 + H_6$, $\ell(6) = 1$, $r = 2$, $\ell(8) = \ell(7) = 1$. $H_5 = 8 \le w < H_6 = 13$. By 2(a) of the above table, this is a winning position ($L(15, 47) = 8$). Player II takes 8 chips from the small pile to win. We conclude that Player I has no winning strategy for the position (20, 42, 6). Consequently, $L(20, 42) = 8$, not 5.

Only after playing the game for a while were we able to determine who could win.

5. ELIMINATING SUSPENSE

It turns out that the suspense which can arise when $L(m, d)$ is unknown can be eliminated. The Theorem of this section presents a *simple method for computing* $L(m, d)$. If $d = H_n + \cdots + H_{n_s}$, then the entries in the dth column of the tableau can assume only the values H_n and H_{n-1}. We say that the dth column of the tableau makes k *flips*, $0 \le k \le \infty$, if it has the form in Figure 5.1. If $k < \infty$, the kth flip is followed by an infinite string of

$$\begin{cases} H_n\text{'s} & \text{if } k \text{ is even} \\ H_{n-1}\text{'s} & \text{if } k \text{ is odd.} \end{cases}$$

$$\begin{array}{c|ccccl}
m \diagdown d & 1 & 2 & 3 & \cdots & d = H_n + \cdots + H_{n_s} \\ \hline
0 & & & & & \left.\begin{matrix} H_n \\ \vdots \\ H_n \end{matrix}\right\} H_{n-1} \text{ entries} \\
1 & & & & & \longleftarrow \text{first flip} \\
2 & & & & & \left.\begin{matrix} H_{n-1} \\ \vdots \\ H_{n-1} \end{matrix}\right\} H_n \text{ entries} \\
\vdots & & & & & \longleftarrow \text{second flip} \\
 & & & & & \left.\begin{matrix} H_n \\ \vdots \\ H_n \end{matrix}\right\} H_{n+1} \text{ entries} \\
 & & & & & \longleftarrow \text{third flip} \\
 & & & & & \left.\begin{matrix} H_{n-1} \\ \vdots \\ H_{n-1} \end{matrix}\right\} H_{n+2} \text{ entries}
\end{array}$$

Fig. 5.1 The dth Column Makes k "flips"

Theorem: For $n \ge 1$, set $A_n = \{z \mid z \ge 0,\ \ell(n + z) = 0\}$. Then:

A. *Simple Case*: $d = H_n$. The dth column makes k flips, where $k = \min A_n$. (Convention: $\min \emptyset = \infty$.)

B. *Compound Case*: $d = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \ge 2$. The dth column makes k flips, where

$$k = \begin{cases} r - 1 & \text{if } \min A_n > r, \\ \min A_n & \text{if } \min A_n \le r. \end{cases}$$

Proof:

A. *Simple Case*:

(1) $A_n \neq \emptyset$. We proceed by induction on $k = \min A_n$. For $k = 0$, $L(m, H_n) = H_n \; \forall m \ge 0$, since $\ell(n) = 0$. There are zero flips.

Suppose the result holds $\forall k \le K - 1$ for some $K \ge 1$. (That is, if $d = H_m$ and $\min A_m \le K - 1$, then the column for $d = H_m$ makes $\min A_m$ flips.)

By the Lemma, each entry of the column $d = H_n$ is H_n, unless a good move can be made by taking H_{n-1} from the small pile. Removing H_{n-1} chips from the small pile is a winning move for position (M, H_n), $M \ge H_{n-1}$ iff $f(H_{n-1}) < L(M - H_{n-1}, H_n + H_{n-1})$. Since $\ell(n) = 1$, $H_n + H_{n-1} = H_{n+1}$ and $L(M - H_{n-1}, H_{n+1}) = H_{n+1}$ or H_n. Moreover, $H_{n+1} > f(H_{n-1}) \ge H_n$. This can be a good move iff $L(M - H_{n-1}, H_{n+1}) = H_{n+1}$. The column $d = H_{n+1}$ makes $K - 1$ flips. Thus, the column $d = H_n$ makes K flips. (See Fig. 5.2). This completes the induction on k.

(2) $A_n = \emptyset$, $\ell(n + k) = 1$ and $A_{n+k} = \emptyset \; \forall k \ge 0$. We show that each column $d = H_{n+k}$, $k \ge 0$, makes infinitely many flips. Let us proceed by induction on m.

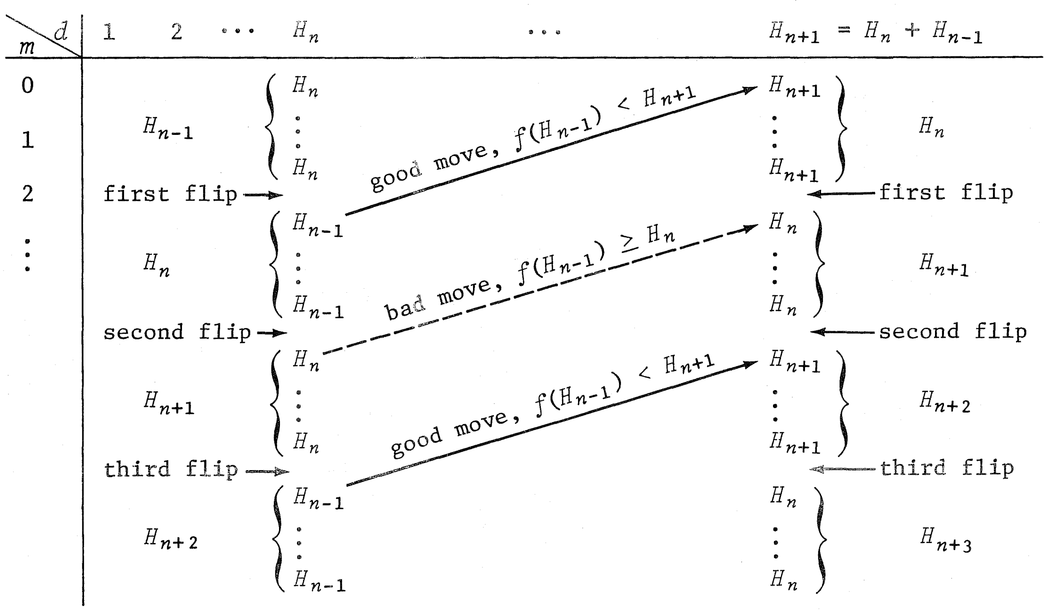

Fig. 5.2 Case A(1)

By the remark to Corollary 1, $L(m, H_{n+k}) = H_{n+k} \; \forall m < H_{n-1}$, $\forall k \ge 0$. The tableau has the desired values for the first H_{n-1} entries in columns $d = H_{n+k}$, $k \ge 0$.

Suppose that the tableau assumes the desired values in the entries $m = 0, 1, \ldots, M - 1$ in the columns $d = H_{n+k}$, $k \ge 0$, for some $M \ge H_{n-1}$. One can find $k_0 \ge 0$ such that

$$H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \ge M > H_{n-1} + H_n + \cdots + H_{n+k_0-1} - 1.$$

Equivalently, $H_n + \cdots + H_{n+k_0} - 1 \ge M - H_{n-1} > \begin{cases} -1 & \text{if } k_0 = 0, \\ H_n + \cdots + H_{n+k_0-1} - 1 & \text{if } k_0 \ge 1. \end{cases}$

By the inductive assumption,

$$L(M - H_{n-1},\ H_{n+1}) = \begin{cases} H_{n+1} & \text{if } k_0 \text{ is even,} \\ H_n & \text{if } k_0 \text{ is odd.} \end{cases}$$

(See Fig. 5.3.) Thus, for the position (M, H_n),

(a) if k_0 is even, taking H_{n-1} chips from the small pile is a good move since $f(H_{n-1}) < H_{n+1} = L(M - H_{n-1}, H_{n+1})$;

(b) if k_0 is odd, taking H_{n-1} chips from the small pile is a bad move since $f(H_{n-1}) \geq H_n = L(M - H_{n-1}, H_{n+1})$.

As desired, we conclude

$$L(M, H_n) = \begin{cases} H_n & \text{if } k_0 \text{ is odd,} \\ H_{n-1} & \text{if } k_0 \text{ is even.} \end{cases}$$

An identical argument reveals that the entries $L(M, H_{n+k})$, $k > 0$, have the desired values. Thus, the row $m = M$ assumes the desired values in the entries corresponding to columns $d = H_{n+k}$, $k \geq 0$. This completes the induction on m.

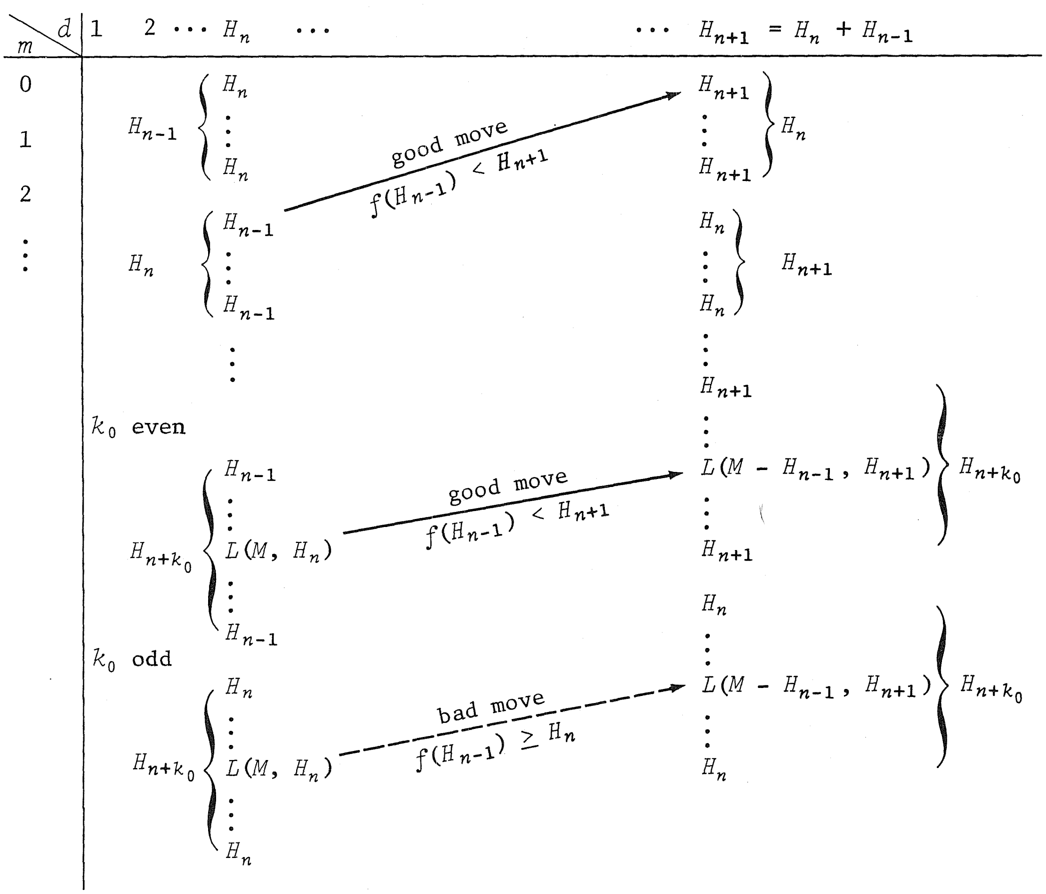

Fig. 5.3 Case A(2)

B. *Compound Case*:

Suppose $\underline{\ell(n) = 0}$. Then $L(m,\ d) = H_n\ \forall m \geq 0$. There are no flips in the dth column. Note that $\min A_n = 0$.

If $\underline{\ell(n) = 1}$, we consider two cases:

(1) $k = \min A_n \leq r$. By Corollary 1, $L(m,\ d - H_n + H_{n+k}) \geq H_{n+k}\ \forall m \geq 0$. The tableau from column $d - H_n + 1$ to column $d - H_n + H_{n+k} - 1$, inclusive, is a copy of the tableau from column 1 to column $H_{n+k} - 1$, inclusive. The dth column is identical to the H_nth column. By Part A, the latter column makes k flips, $k = \min A_n$.

(2) $\min A_n > r$. Here $\ell(n) = \ell(n+1) = \cdots = \ell(n+r) = 1$. Necessarily, $r > 1$. Let $d' = d - H_n + H_{n+r-1}$. Since $\ell(n+r) = 1$, d' has the form $d' = H_{n+r+u} + \cdots + H_{n_{s'}}$, for some $u \geq 1$ and $n_{s'} \geq n_s$. By Corollary 1, $L(m,\ d') \geq H_{n+r+u-1}$. Consider the position $(m,\ d'')$, $m \geq H_{n+r-3}$, where $d'' = d - H_n + H_{n+r-2}$. Note that $d'' + H_{n+r-3} = d'$. It is a good move to take H_{n+r-3} chips from the small pile, since $f(H_{n+r-3}) < H_{n+r-1} < L(m,\ d')$. Thus, $L(m, d'') = H_{n+r-3}\ \forall m \geq H_{n+r-3}$. The column $d'' = d - H_n + H_{n+r-2}$ makes one flip. Since the column d'' makes one flip, argue as in Part A(1) of the proof that column $d''' = d - H_n + H_{n+r-3}$ makes two flips. Similarly, column $d^{iv} = d - H_n + H_{n+r-4}$ makes three flips. Continue and argue that column $d = d - H_n + H_n$ makes $r - 1$ flips. (See Fig. 5.4). Q.E.D.

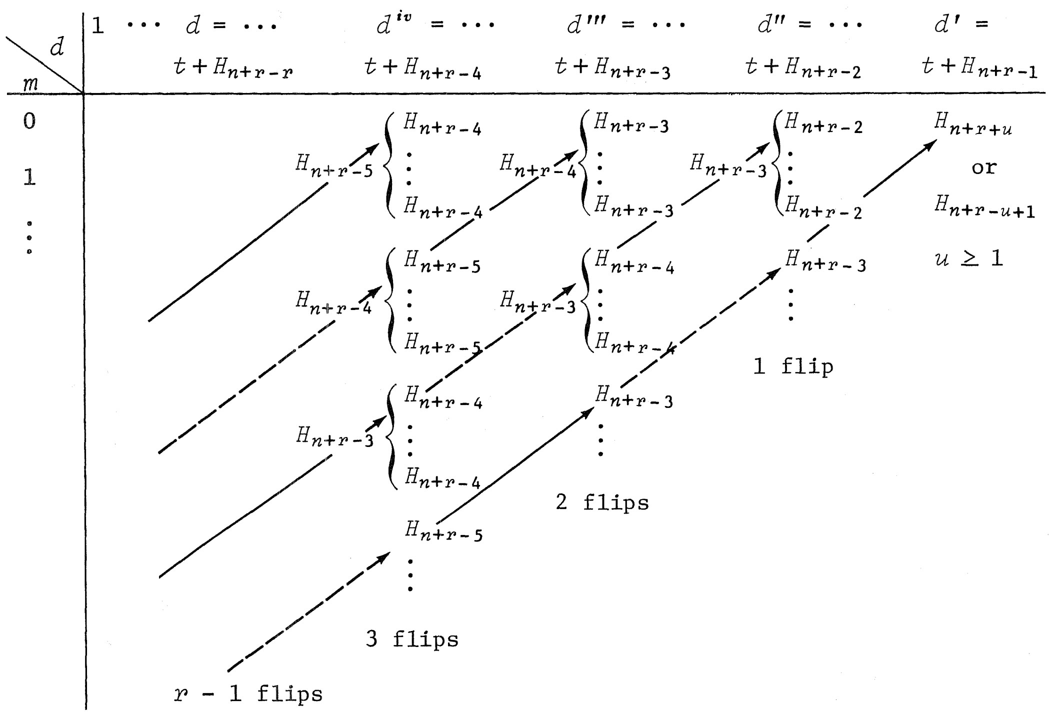

Fig. 5.4 Case B(2)

Notation: $t = d - H_n$.

$\longrightarrow$ = a good move.

$-\,-\,-\,-\!\!\rightarrow$ = a bad move.

6. TWO-PILE FIBONACCI NIM REVISITED

Ferguson's solution for two-pile Fibonacci Nim was in the form of Table 4.1. His solution does not necessarily reveal which player can win at the beginning of play, because $L(m, d)$ might not be known then. The Theorem tells us the value of $L(m, d)$ be revealing the behavior of the columns of the tableau. Knowing $L(m, d)$ at the start of play leaves no uncertainty as to who can win. As an illustration of the Theorem, we compute $L(m, d)$ for two-pile Fibonacci Nim.

Suppose $d = H_n$ for some n. If $d = H_1$, then $L(m, d) = H_1 = 1 \; \forall m \geq 0$, since $\ell(1) = 0$. If $n \geq 2$, the dth column makes infinitely many flips, since $A_n = \emptyset$. For a particular value of m, find the least integer $k_0 \geq -1$ such that $H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \geq m$. Then,

$$L(m, d) = \begin{cases} H_n & \text{if } k_0 \text{ is odd,} \\ H_{n-1} & \text{if } k_0 \text{ is even.} \end{cases}$$

Suppose d has compound form $d = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \geq 2$. Note that $r > 1$. If $n = 1$, the dth column of the tableau has each entry equal to 1. If $n \geq 2$, the dth column makes $r - 1$ flips. If k_0 is the least integer such that $k_0 \geq -1$ and $H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \geq m$, then

$$L(m, d) = \begin{cases} H_n & \text{if } k_0 \text{ is odd and } k_0 \leq r - 2, \text{ or} \\ & \quad r \text{ is odd and } k_0 > r - 2. \\ H_{n-1} & \text{if } k_0 \text{ is even and } k_0 \leq r - 2, \text{ or} \\ & \quad r \text{ is even and } k_0 > r - 2. \end{cases}$$

7. CONCLUSION

The function $\ell(k)$ was defined by (2.3). In Table 4.1, a winning strategy (provided one exists) is given for the class of two-pile take-away games in which $\ell(k) \in \{0, 1\} \; \forall k \geq 1$. By revealing $L(m, d)$, the Theorem enables us to determine at the beginning of play whether such a strategy exists for the player about to move.

The author has considered several particular two-pile take-away games in which $\ell(k)$ assumes values other than 0 and 1. For example, when $f(t) = 3t$, then $\ell(k) = 3 \; \forall k \geq 5$. I have found no general solution for any such game. Can we find solutions for the general class of games which impose no restrictions on $\ell(k)$? Can we extend to games beginning with arbitrarily many piles of chips? Let me know if you can.

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ON *K*TH-POWER NUMERICAL CENTERS

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1. INTRODUCTION

In a previous paper [2], we considered the problem of determining all positive integers which possess kth-power numerical centers, and proved that there are infinitely many positive integers possessing first-power numerical centers and that the only positive integer possessing a second-power numerical center is 1. In the present paper, we treat the cases $k = 3$, 4, and 5.

2. THE CASE $k = 3$

Let us begin by recalling the following

Definition: Given the positive integer n, we call the positive integer N, $(N \leq n)$, a kth-power numerical center for n in case the sum of the kth powers of the integers from 1 to N equals the sum of the kth powers from N to n.

In this section, we prove the following

Theorem 1: The only positive integer possessing a third-power numerical center is 1.

Proof: Let N be any third-power numerical center for the positive integer n. Since the sum of the cubes of the first N positive integers is given by

$$\sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4},$$

the condition that N be a third-power numerical center for n requires that

$$\frac{N^2(N+1)^2}{4} = \frac{n^2(n+1)^2}{4} - \frac{N^2(N-1)^2}{4}.$$

On setting $X = 2N^2 + 1$, we obtain

$$X^2 - 2n^2(n+1)^2 = 1. \tag{1}$$

Let us now consider the following

Problem: To find all triangular numbers whose square is also triangular. This requires

$$\left(\frac{n(n+1)}{2}\right)^2 = \frac{N(N+1)}{2}. \tag{2}$$

and, on setting $X = 2n + 1$, we again obtain equation (1). But equation (2) was solved by Ljunggren [3] and Cassels [1], who showed that its only positive integer solutions are $(n, N) = (1, 1)$ and $(3, 8)$. Thus, the only positive integer solutions of (1) with X odd are $(x, n) = (3, 1)$ and $(17, 3)$.

From this, it follows that the only positive integer solution of (1) satisfying $X = 2n^2 + 1$ is $(X, n) = (3, 1)$, and our result is proved.

3. THE CASES $k = 4$ AND 5

Since

$$\sum_{i=1}^{N} i^4 = \frac{N(N+1)(6N^3 + 9N^2 + N - 1)}{30},$$

the condition that N be a fourth-power numerical center for n requires

$$\begin{aligned} &N(N+1)(6N^3 + 9N^2 + N - 1) \\ &= n(n+1)(6n^3 + 9n^2 + n - 1) - N(N-1)(6N^3 - 9N^2 + N + 1), \end{aligned}$$

and on setting $X = 2n + 1$, $Y = 2N$, we obtain

(3) $$3X^5 - 10X^3 + 7X = 6Y^5 + 40Y^3 - 16Y$$

subject to the conditions

(4) X positive and odd, Y positive and even.

Further, since

$$\sum_{i=1}^{N} i^5 = \frac{N^2(N^2 + 2N + 1)(2N^2 + 2N - 1)}{12}$$

the condition that N be a fifth-power numerical center for n requires

$$\begin{aligned} &N^2(N^2 + 2N + 1)(2N^2 + 2N - 1) \\ &= n^2(n^2 + 2n + 1)(2n^2 + 2n - 1) - N^2(N^2 - 2N + 1)(2N^2 - 2N - 1), \end{aligned}$$

and, on setting

$$X = (2n + 1)^2, \quad Y = (2N)^2$$

it reduces to

(5) $$X^3 - 5X^2 + 7X - 3 = 2Y^3 + 20Y^2 - 16Y$$

subject to the conditions

(6) X a positive odd square, Y a positive even square.

Unfortunately, we have been unable to discover a method of solving equations (3) and (5) completely, although we have used a computer to verify that the only integer solution of (3), subject to (4), with $X < 205$ is $(X, Y) = (3, 2)$ and that the only integer solution of (5), subject to (6), with $X < 411$ is $(X, Y) = (9, 4)$.

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2. R. P. Finkelstein, "The House Problem," *American Math. Monthly*, Vol. 72 (1965), pp. 1082-1088.
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A PROPERTY OF WYTHOFF PAIRS

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The Wythoff pairs A_n and B_n are the ordered safe-pairs in the game. See for example [1].

$$A = \{A_n\} = \{[n\alpha]\} = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, \ldots\}$$

$$B = \{B_n\} = \{[n\alpha^2]\} = \{2, 5, 7, 10, 13, 15, 18, 20, 23, \ldots\}$$

where $\alpha = (1 + \sqrt{5})/2$. $\alpha^2 = \alpha + 1$. The following properties will be assumed:

(i) The sets A and B are disjoint sets whose union is the set of positive integers.

(ii) $B_n = A_n + n$.

Lemma 1: $A_{A_n} + 1 = B_n$.

Proof: Consider the set of integers 1, 2, 3, ..., B_n. Of these, n are B's, and the rest are $A_1, A_2, A_3, \ldots, A_j = B_n - 1$. Thus, $j + n = B_n$, but $A_n + n = B_n$, so that $A_{A_n} + 1 = B_n$.

If we consider the set of integers 1, 2, 3, ..., A_n, there are n A's and $B_1, B_2, \ldots, B_j \leq A_n - 1$; thus,

Lemma 2: There are $A_n - n$ B's less than A_n.

Theorem: $A_{A_n+1} - A_{A_n} = 2, \qquad A_{B_n+1} - A_{B_n} = 1;$

$B_{A_n+1} - B_{A_n} = 3, \qquad B_{B_n+1} - B_{B_n} = 2.$

Proof: It is easy to see that no two B's are adjacent. Consider $A_n + 1 = A_{n+1}$ or $A_n + 1 = B_j$, then

$$A_{n+1} - (n + 1) - (A_n - n) = 1 \text{ iff } A_n + 1 = B_j.$$

Fix j, then since $A_n + 1$ is a strictly increasing sequence in n, there is at most one solution to $A_n + 1 = B_j$, and from $A_{A_n} + 1 = B_n$, we see $n = A_j$, so

$$A_{A_j+1} - A_{A_j} = 2 \text{ and } A_{B_j+1} - A_{B_j} = 1.$$

From $A_n + n = B_n$, it easily follows that

$$B_{A_j+1} - B_{A_j} = 3 \text{ and } B_{B_j+1} - B_{B_j} = 2.$$

We now show that $\{A_n\}$ and $\{B_n\}$ are self-generating sequences. We illustrate only with $B_n = [n\alpha^2] = \{2, 5, 7, 10, 13, \ldots\}$: $B_1 = 2$ and $B_2 - B_1 = 3$, so $B_2 = 5$; $B_3 - B_2 = 2$, so $B_3 = 7$; $B_4 - B_3 = 3$, so $B_4 = 10$; $B_5 - B_4 = 3$, so $B_5 = 13$. Now, knowing that

$$B_{n+1} - B_n \text{ is } 2 \text{ if } n \in B \text{ and } B_{n+1} - B_n = 3 \text{ if } n \notin B,$$

we can generate as many terms of the $\{B_n\}$ sequence as one would want only by knowing the earlier terms and which difference to add to these to obtain the next term.

REFERENCE

1. L. Carlitz, Richard Scoville, & V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly*, Vol. 10, No. 1 (January 1972), pp. 1-28.

ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F and Lucas numbers L satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-382 *Proposed by A. G. Shannon, N.S.W. Institute of Technology, Australia.*

Prove that L has the same last digit (i.e., units digit) for all n in the infinite geometric progression

$$4, 8, 16, 32, \ldots .$$

B-383 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Solve the difference equation

$$U_{n+2} - 5U_{n+1} + 6U_n = F_n$$

B-384 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identity

$$F^4_{n+10} = 55(F^4_{n+8} - F^4_{n+2}) - 385(F^4_{n+6} - F^4_{n+4}) + F^4_n.$$

B-385 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let $T_n = n(n+1)/2$. For how many positive integers n does one have both $10^6 < T_n < 2 \cdot 10^6$ and $T_n \equiv 8 \pmod{10}$?

B-386 *Proposed by Lawrence Somer, Washington, D.C.*

Let p be a prime and let the least positive integer m with $F_m \equiv 0 \pmod{p}$ be an even integer $2k$. Prove that $F_{n+1}L_{n+k} \equiv F_n L_{n+k+1} \pmod{p}$. Generalize to other sequences, if possible.

B-387 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

Prove that there are infinitely many ordered triples of positive integers (x, y, z) such that

$$3x^2 - y^2 - z^2 = 1.$$

SOLUTIONS

ALMOST ALWAYS COMPOSITE

B-358 *Proposed by Phil Mana, Albuquerque, New Mexico.*

Prove that the integer u_n such that $u_n \le n^2/3 < u + 1$ is a prime for only a finite number of positive integers n. (Note that $u_n = [n^2/3]$, where $[x]$ is the greatest integer in x and $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $u_4 = 5$, and $u_5 = 8$.)

Solution by Graham Lord, Université Laval, Québec.

If $n = 3m$, $3m + 1$, or $3m + 2$, where $m = 0, 1, 2, \ldots,$ then, $u_n = 3m^2$, $m(3m + 2)$ or $(m + 1)(3m + 1)$, respectively. Thus, the only values of u_n that are prime are 3 and 5.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Herta T. Freitag, Bob Prielipp, and the proposer.

TRIBONACCI SEQUENCE

B-359 *Proposed by R. S. Field, Santa Monica, CA.*

Find the first three terms T_1, T_2, and T_3 of a Tribonacci sequence of positive integers $\{T_n\}$ for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{and} \quad \sum_{n=1}^{\infty}(T_n/10^n) = 1/T_4.$$

Solution by Graham Lord, Université Laval, Québec.

If $S(x) = \sum_{n=1}^{\infty} T_n x^n$, then

$$S(x) = [T_1(x - x^2 - x^3) + T_2(x^2 - x^3) + T_3 x\ ]/(1 - x - x^2 - x^3),$$

and, in particular,

$$S(1/10) = (89T_1 + 9T_2 + T_3)/889.$$

Hence,

$$T_4(89T_1 + 9T_2 + T_3) = 889 = 7 \cdot 127.$$

Since $T_4 = T_3 + T_2 + T_1 \ge 3$, it must be the smaller prime factor, 7, and

$$89T_1 + 9T_2 + T_3 = 127.$$

Thus, $T_1 = 1$, $T_2 = 4$, and $T_3 = 2$.

Also solved by George Berzsenyi, Michael Brozinski, Paul S. Bruckman, Roger Engle & Benjamin Freed & Sahib Singh, Charles B. Shields, and the proposer.

APPLYING QUATERNION NORMS

B-360 *Proposed by T. O'Callahan, Aerojet Manufacturing Co., Fullerton, CA.*

Show that for all integers a, b, c, d, e, f, g, h there exist integers w, x, y, z such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (w^2 + 2x^2 + 3y^2 + 6z^2).$$

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.

Defining the real quaternions A and B as

$$A = a + (\sqrt{2}b)i + (\sqrt{3}c)j + (\sqrt{6}d)k,$$

$$B = e + (\sqrt{2}f)i + (\sqrt{3}g)j + (\sqrt{6}h)k$$

and using the multiplicative property of norm N, namely $N(AB) = N(A)N(B)$, we conclude by comparison that

$$w = ae - 2bf - 3cg - 6dh, \quad x = af + be + 3ch - 3dg,$$

$$y = ag - 2bh + ce + 2df, \quad z = ah + bg - cf + de.$$

Also solved by Paul S. Bruckman, Bob Prielipp, Gregory Wulczyn, and the proposer.

A RATIONAL FUNCTION

B-361 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)}$$

is a rational function of x, y, u, and v when these four variables are less than 1 in absolute value.

Solution by Roger Engle & Sahib Singh, Clarion State College, Clarion, PA.

If S denotes the required sum, then

$$S = \sum_{i=0}^{\infty}(xv)^i + \sum_{i=1}^{\infty}(yv)^i + xyuvS$$

$$\therefore S(1 - xyuv) = \frac{1}{1 - xv} + \frac{yv}{1 - yv}$$

$$\therefore S = \frac{1 - xyv^2}{(1 - xv)(1 - yv)(1 - xyuv)}$$

Also solved by Paul S. Bruckman, Robert M. Giuli, Graham Lord, and proposer.

TRIANGULAR NUMBER RESIDUES

B-362 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let m be an integer greater than one (1) and let R_n be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by m. Show that the sequence $R_0, R_1, R_2, \ldots$ repeats in a block $R_0, R_1, \ldots, R_t$ which reads the same from right to left as it does from left to right. (For example, if $m = 7$ then the smallest repeating block is 0, 1, 3, 6, 3, 1, 0.)

Solution by Graham Lord, Université Laval, Québec.

Since $T_{n+2m} = T_n + m(2n + 1 + 2m)$ then $R_n = R_{n+2m}$: the sequence repeats in blocks. And for $0 \le n < m$, as $T_{2m-n-1} = T_n + m(2m - 2n - 1)$ it follows that $R_n = R_{2m-n-1}$, which implies the reflecting property.

Note that if m is even the period is $2m$, since neither T_m nor T_2 is congruent to 0 modulo m. And if m is odd the period is m. The latter is proven thus: As $T_{n+m} \equiv T_n \pmod{m}$, the period, d, must divide m. But, by the reflecting property and the periodicity $T_0 \equiv T_{d-1} \equiv T_d \pmod{m}$, that is, m divides $T_d - T_{d-1} = d$. Hence, $d = m$.

Also solved by George Berzsenyi, Paul S. Bruckman, Roger Engle & Sahib Singh, Bob Prielipp, Gregory Wulczyn, and the propeser.

OVERLAPPING PALINDROMIC BLOCKS

B-363 *Proposed by Herta T. Freitag, Roanoke, VA.*

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n - 1)/2$ also have the symmetry property stated in B-362 for their residues modulo m?

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA.

For this symmetry property, it is necessary that two consecutive members of S_n or P_n be congruent to zero modulo m.

(a) $S_n = n^2$, $S_{n+1} = (n + 1)^2$.

Since $(n, n + 1) = 1$, S_n does not have the symmetry property of B-362.

(b) $P_n = \frac{n}{2}(3n - 1)$, $P_{n+1} = \frac{n+1}{2}(3n + 2)$, $P_n = 1, 5, 12, 22, 35, \ldots$.

For any factor m of n, $(n, n + 1) = 1$, $(n, 3n + 2) = 1, 2$.
For any factor m of $3n - 1$, $(3n - 1, 3n + 2) = 1$, $(3n - 1, n + 1) = 1, 2, 4$.

Since the only common factor to P_n and P_{n+1} is 2, P_n does not have the symmetry property of B-362.

Also solved by Paul S. Bruckman, Roger Engle & Sahib Singh, Graham Lord, Bob Prielipp, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-285 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Consider two sequences $\left\{H_n\right\}_{n=1}^{\infty}$ and $\left\{G_n\right\}_{n=1}^{\infty}$ such that

(a) $(H_n, H_{n+1}) = 1$,
(b) $(G_n, G_{n+1}) = 1$,
(c) $H_{n+2} = H_{n+1} + H_n \quad (n \geq 1)$, and
(d) $H_{n+1} + H_{n-1} = sG_n \quad (n \geq 1)$,
where s is independent of n.

Show $s = 1$ or $s = 5$.

H-286 *Proposed by P. Bruckman, Concord, CA.*

Prove the following congruences:

(1) $F_{5^n} \equiv 5^n \pmod{5^{n+3}}$;

(2) $F_{5^n} \equiv L_{5^{n+1}} \pmod{5^{2n+1}}$, $n = 0, 1, 2, \ldots$.

H-287 *Proposed by A. Mullin, Ft. Hood, Texas.*

Suppose $g(\cdot)$ is *any* strictly-positive, real-valued arithmetic function satisfying the functional equation:

$$\left(g(n+1)/(n+1)\right) + n = (n+1)g(n)/g(n+1)$$

for every integer n exceeding some prescribed positive integer m. Then $g(n)$ is *necessarily* asymptotic to $\pi(n)$, the number of prime numbers not exceeding n; i.e., $g(n) \sim \pi(n)$.

H-288 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identities:

(a) $F_k L_{k+6r+3}^2 - F_{k+8r+4} L_{k+2r+1}^2 = (-1)^{k+1} L_{2r+1}^3 F_{2r+1} L_{k+4r+2}$.

(b) $F_k L_{k+6r}^2 - F_{k+8r} L_{k+2r}^2 = (-1)^{k+1} L_{2r}^3 F_{2r} L_{k+4r}$.

H-289 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Put the multinomial coefficient

$$(m_1, m_2, \ldots, m_k) = \frac{(m_1 + m_2 + \cdots + m_k)!}{m_1! m_2! \ldots m_k!}.$$

Show that

$$(*) \sum_{r+s+t=\lambda} (r, s, t)(m - 2r, n - 2s, p - 2t)$$

$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)(m-j-k, n-k-i, p-i-j)(m+n+p \geq 2\lambda).$$

SOLUTIONS

A PAIR OF SUM SEQUENCES

H-269 *Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.*

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, defined by

$$a_n = \sum_{k=0}^{[n/3]} \binom{n-2k}{k} \text{ and } b_{2n} = \sum_{k=0}^{[n/2]} \begin{bmatrix} n-2k \\ 2k \end{bmatrix},$$

$$b_{2n+1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix},$$

are obtained as diagonal sums from Pascal's triangle and from a similar triangular array of numbers formed by the coefficients of powers of x in the expansion of $(x^2 + x + 1)^n$, respectively.

(More precisely, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the coefficient of x^k in $(x^2 + x + 1)^n$.)
Verify that $a_n = b_{n-1} + b_n$ for each $n = 1, 2, \ldots$.

Solution by A. Shannon, School of Math Sciences, New South Wales Institute of Technology, Broadway, Australia.

It follows from Equation (4.1) of Shannon [2] with $P = R = 1$, $Q = 0$, that $a_n = a_{n+1} + a_{n+3}$.

A Pascal triangle for $\begin{bmatrix} n \\ k \end{bmatrix}$ can be set up as follows,

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1	1								
2	1	2	3	2	1						
3	1	3	6	7	6	3	1				
4	1	4	10	16	19	16	10	4	1		
5	1	5	15	30	45	51	45	30	15	5	1

and it can be observed, and readily proved by induction that,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}.$$

By an extension of the methods of Carlitz [1] we can establish with somewhat tedious detail that

$$b_n = b_{n-2} + b_{n-3} + b_{n-4}.$$

Then, again with inductive methods, we get

$$a_n = a_{n-1} + a_{n-3} = b_{n-1} + b_{n-2} + b_{n-3} + b_{n-4} = b_{n-1} + b_n,$$

as required.

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1. L. Carlitz, "Some Multiple Sums and Binomial Identities," *S.I.A.M. J. Appl. Math.*, Vol. 13 (1965), pp. 469-486.
2. A. G. Shannon, "Iterative Formulas Associated with Generalized Third-Order Recurrence Relations," *S.I.A.M. J. Appl. Math.*, Vol. 23 (1972), pp. 364-368.

Also solved by P. Bruckman and the proposer.

IT'S A SINH
(Corrected)

H-270 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Sum the series

$$S \equiv \sum_{a,b,c} \frac{x^a y^b z^c}{(b+c-a)!(c+a-b)!(a+b-c)!}$$

where the summation is over all nonnegative a, b, c such that

$$a \leq b + c,\ b \leq a + c,\ c \leq a + b.$$

Solution by P. Bruckman, Concord, CA.

Let $r = b + c - a$, $s = a + c - b$, and $t = a + b - c$. Then, $r + s = 2c$, $s + t = 2a$, $r + t = 2b$; this implies that r, s, and t are either all even or all odd. Hence,

$$(1)\quad S = \sum_{\substack{r,s,t\geq 0\\ r\equiv s\equiv t \pmod 2}} \frac{x^{\frac{1}{2}(s+t)}}{r!}\,\frac{y^{\frac{1}{2}(t+r)}}{s!}\,\frac{z^{\frac{1}{2}(r+s)}}{t!}$$

Thus, $S = S_1 + S_2$, where

$$(2)\quad S_1 = \sum_{r,s,t\geq 0} \frac{x^{s+t}}{(2r)!}\,\frac{y^{t+r}}{(2s)!}\,\frac{z^{r+s}}{(2t)!},$$

$$(3)\quad S_2 = \sum_{r,s,t\geq 0} \frac{x^{s+t+1}}{(2r+1)!}\,\frac{y^{t+r+1}}{(2s+1)!}\,\frac{z^{r+s+1}}{(2t+1)!}$$

But S_1 and S_2 are readily evaluated, namely:

$$S_1 = \sum_{r,s,t\geq 0} \frac{(\sqrt{yz})^{2r}}{(2r)!}\,\frac{(\sqrt{xz})^{2s}}{(2s)!}\,\frac{(\sqrt{xy})^{2t}}{(2t)!} = \cosh\sqrt{yz}\cdot\cosh\sqrt{xz}\cdot\cosh\sqrt{xy},$$

and

$$S_2 = \sum_{r,s,t\geq 0} \frac{(\sqrt{yz})^{2r+1}}{(2r+1)!}\frac{(\sqrt{xz})^{2s+1}}{(2s+1)!}\frac{(\sqrt{xy})^{2t+1}}{(2t+1)!} = \sinh\sqrt{yz}\cdot\sinh\sqrt{xz}\cdot\sinh\sqrt{xy}.$$

Therefore,

$$(4)\quad S = \cosh\sqrt{xy}\cdot\cosh\sqrt{yz}\cdot\cosh\sqrt{zx} + \sinh\sqrt{xy}\cdot\sinh\sqrt{yz}\cdot\sinh\sqrt{zx}.$$

Also solved by W. Brady and the proposer.

H-271 (corrected)

Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA.

Define the binary dual, D, as follows:

$$D = \left\{t \,\middle|\, t = \prod_{i=0}(a_i + 2i);\ a_i \in \{0, 1\};\ n \geq 0\right\}.$$

Let $\overline{D}$ denote the complement of D with respect to the set of positive integers. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\overline{D}$ in increasing order. Find a formula for S_n.

(Note: The elements of D result from interchanging + and × in a binary number.)

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