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CONTENTS

On the Density of the Image Sets of Certain Arithmetic Functions—III .... *Rosalind Guaraldo* 481
Evaluation of Sums of Convolved Powers Using Stirling and Eulerian Numbers .... *H. W. Gould* 488
b-Adic Numbers in Pascal's Triangle Modulo b .... *Heiko Harborth* 497
Divisibility Properties of Polynomials in Pascal's Triangle .... *V. E. Hoggatt, Jr., and Marjorie Bicknell-Johnson* 501
The Golden Section in the Earliest Notated Western Music .... *Paul Larson* 513
On Fibonacci Numbers Which Are Powers .... *Neville Robbins* 515
Primes, Powers, and Partitions .... *B. de La Rosa* 518
On Odd Perfect Numbers .... *G. L. Cohen* 523
A Simple Continued Fraction Represents a Mediant Nest of Intervals .... *Irving Adler* 527
Folded Sequences and Bode's Problem .... *W. E. Greig* 530
Fibonacci Numbers in Coin Tossing Sequences .... *Mark Finkelstein and Robert Whitley* 539
Strong Divisibility Sequences with Nonzero Initial Term .... *Clark Kimberling* 541
Minimum Periods Modulo n for Bernoulli Numbers .... *W. Herget* 544
The Rank-Vector of a Partition .... *Hansraj Gupta* 548
The Andrews Formula for Fibonacci Numbers .... *Hansraj Gupta* 552
Operational Formulas for Unusual Fibonacci Series .... *H. W. Gould* 555
A Figurate Number Curiosity: Every Integer is a Quadratic Function of a Figurate Number .... *Harvey J. Hindin* 561
Elementary Problems and Solutions .... *Edited by A. P. Hillman* 562
Advanced Problems and Solutions .... *Edited by Raymond E. Whitney* 566
Volume Index 570

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ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—III

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1. INTRODUCTION

Let n be a fixed but arbitrary nonnegative integer. It is known (see [1], for example) that n may be uniquely represented in the form $n = d_1 1! + d_2 2! + \cdots + d_k k!$, $0 \le d_j \le j$. Suppose that $f(d, j)$ is a nonnegative integer-valued function of j for each "digit" d, $0 \le d \le j$, $j = 1, 2, \ldots$, and define

$$S(n) = \sum_{j=1}^{k} f(d_j, j),$$

$$T(n) = n + S(n),$$

$$\Omega(k, r) = \{T(x) \mid k \le x \le r\},$$

$$D(k, r) = |\Omega(k, r)|$$

$$\Omega(r) = \Omega(0, r)$$

$$D(r) = D(0, r)$$

$$\mathcal{R} = \{x \mid x = T(n) \text{ for some } n\}, \text{ and}$$

$$\mathcal{C} = \{x \mid x \ne T(n) \text{ for any } n\}.$$

Our objective here is to prove some results concerning the asymptotic density of the sets $\mathcal{R}$ and $\mathcal{C}$ analogous to those which we proved when we considered the representation of n as an integer in base b (see [2] and [3]).

2. EXISTENCE AND COMPUTABILITY OF THE DENSITY

Theorem 2.1: Let $f(d, j)$, $0 \le d \le j$ be as described above. If

(a) $f(0, j) = 0$, $j = 1, 2, \ldots$
(b) $f(d, j) = o(j!)$ uniformly in j, i.e.,
$\sup\{f(d, j),\ 0 \le d \le j\} = o(j!)$

then the density of $\mathcal{R}$ exists.

Proof: We first show that

(2.2) $$D(dk!, dk! + r) = D(r),\ 0 \le r \le k! - 1.$$

To prove 2.2, let us suppose that

$$x = dk! + \sum_{j=1}^{k-1} d_j j! \quad \text{and} \quad y = dk! + \sum_{j=1}^{k-1} d_j' j!.$$

Clearly, $T(x) = T(y)$ if and only if

$$T\left(\sum_{j=1}^{k-1} d_j j!\right) = T\left(\sum_{j=1}^{k-1} d_j' j!\right).$$

Suppose that $d_{k-1} = d_{k-2} = \cdots = d_{k-t} = 0$ (or that $d'_{k-1} = d'_{k-2} = \cdots = d'_{k-t} = 0$). Since $f(0, j) = 0$, it must be the case that

$$T\left(\sum_{j=0}^{k+t-1} d_j j!\right) = T\left(\sum_{j=0}^{k-1} d_j j!\right) = T\left(\sum_{j=0}^{k-1} d'_j j!\right).$$

We have therefore exhibited a one-one correspondence between the elements of $\Omega(dk!, dk! + r)$ and $\Omega(r)$, $0 \leq r \leq k! - 1$, and hence 2.2 follows. In particular, if $r = k! - 1$, we obtain

(2.3) $$D(dk!, (d+1)k! - 1) = D(k! - 1).$$

Our next result will enable us to find a relationship between

$$D((k+1)! - 1) \quad \text{and} \quad \sum_{d=0}^{k+1} D(dk! - 1).$$

Lemma 2.4: There exists an integer k_0 such that for all $k \geq k_0$ the sets $\Omega(0, k! - 1)$, $\Omega(k!, 2k! - 1)$, ..., $\Omega(kk!, (k+1)! - 1)$ are pairwise disjoint, except possibly for adjacent pairs.

Proof: The maximum value in $\Omega(dk!, (d+1)k! - 1)$ is at most $(d+1)k! - 1 + kM_k$, where $M_k = \max\{f(d, j), 1 \leq j \leq k\}$, and the minimum value in $\Omega((d+2)k!, (d+3)k! - 1)$ is at least $(d+2)k!$. By assumption (b), there exists k'_0 such that $f(d, j) < j!/2$, for all $j \geq k'_0$, and there exists $k_0 \geq k'_0$ such that $f(d, j) < j!/2 - k'_0 M_{k'_0}$, for all $j \geq k_0$, where $M_{k'_0} = \max\{f(d, j) \mid 1 \leq j \leq k'_0\}$. Therefore, if $k \geq k_0$, we have

$$\sum_{j=1}^{k} f(d_j, j) = \sum_{j=1}^{k'_0} f(d_j, j) + \sum_{j=k'_0+1}^{k_0} f(d_j, j) + \sum_{j=k_0+1}^{k} f(d_j, j) < k'_0 M_{k'_0} + \sum_{j=k'_0+1}^{k} j!/2 - k'_0 M_{k'_0}(k - k_0) \leq \sum_{j=k'_0+1}^{k} j!/2 < k!.$$

In particular, $kM_k < k!$ if $k \geq k_0$. Hence, we certainly have $(d+1)k! - 1 + kM_k < (d+2)k!$ if $k \geq k_0$, so the result is proved.

Now let $\lambda_{d,k} = |\Omega(dk!, (d+1)k! - 1) \cap \Omega((d+1)k!, (d+2)k! - 1)|$, $0 \leq d \leq k - 1$. Using 2.3 and 2.4 and the fact that

$$D((k+1)! - 1) = \sum_{d=0}^{k} D(dk!, (d+1)k! - 1) - Q,$$

where Q depends on the number of elements that the sets $\Omega(0, k! - 1)$, $\Omega(k!, 2k! - 1)$, ..., $\Omega(kk!, (k+1)! - 1)$ have in common, we obtain

(2.5) $$D((k+1)! - 1) = (k+1)D(k! - 1) - \sum_{d=0}^{k-1} \lambda_{d,k}.$$

Let $A_k = D(k! - 1)/k!$ and $\varepsilon_k = \sum_{d=0}^{k-1} \lambda_{d,k}/(k+1)!$, $k \geq k_0$. Then 2.5 becomes

$$A_{k+1} - A_k = -\varepsilon_k.$$

Therefore,

$$\begin{aligned}
A_{k+1} - A_k &= -\varepsilon_k \\
A_k - A_{k-1} &= -\varepsilon_{k-1} \\
&\vdots \\
A_{k_0} - A_k &= -\varepsilon_{k_0}
\end{aligned}$$

so $A_{k+1} - A_{k_0} = -\sum_{j=k_0}^{k} \varepsilon_j$, i.e., $A_{k+1} = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j$. Replacing $k+1$ by k, we obtain

$$A_k = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j. \tag{2.6}$$

Clearly, $1/k! \leq A_k \leq 1$ and $\sum_{j=k_0}^{k-1} \varepsilon_j = A_{k_0} - A_k \leq A_{k_0} \leq 1$. Thus, $\sum_{j=k_0}^{\infty} \varepsilon_j$ is a series of nonnegative terms bounded by A_{k_0}, hence is convergent. Let

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j. \tag{2.7}$$

Note that we have just shown that $0 \leq L \leq 1$. Then, 2.6 yields

$$A_k = L + \sum_{j=k}^{\infty} \varepsilon_j, \quad k \geq k_0. \tag{2.8}$$

Since $\sum_{j=k}^{\infty} \varepsilon_j = 0(1)$ as $k \to \infty$, we have

$$A_k = L + o(1).$$

Multiplying both sides of this equation by $k!$ and using the definition of the A_k, we obtain

$$D(k! - 1) = Lk! + o(k!). \tag{2.9}$$

Using 2.3, 2.4, 2.9, and the definition of the λ's and the ε's, we have

$$\begin{aligned}
D(dk! - 1) &= \sum_{c=0}^{d-1} D(ck!, (c+1)k! - 1) - \sum_{c=0}^{d-2} \lambda_{c,k} \\
&= \sum_{c=0}^{d-1} (Lk! + o(k!)) + o((k+1)!\varepsilon_k) \\
&= dk!L + o((k+1)!) + o((k+1)!),
\end{aligned}$$

i.e.,

$$D(dk! - 1) = dk!L + o((k+1)!). \tag{2.10}$$

Now let $n = \sum_{j=0}^{k} d_j j!$ be any nonnegative integer. Then $D(n) = D(d_k k! - 1) + D(d_k k!, d_k k! + d_{k-1}(k-1)! + \cdots) - Q$, where Q is the number of elements that the sets $\Omega(0, d_k k! - 1)$ and $\Omega(d_k k!, d_k k! + d_{k-1}(k-1)! + \cdots)$ have in common. Hence, if n is sufficiently large, then, by using 2.2, 2.10, and the definition of the λ's, we obtain

$$\begin{aligned}
D(n) &= d_k k!L + D(d_{k-1}(k-1)! + \cdots) + o((k+1)!) + o((k+1)!) \\
&= d_k k!L + D(d_{k-1}(k-1)! + \cdots) + o((k+1)!).
\end{aligned}$$

Applying the same type of reasoning yields

$$D(d_{k-1}(k-1)! + \cdots) = d_{k-1}(k-1)! + o(k!)$$
$$= d_{k-1}(k-1)!L + o((k+1)!).$$

Continuing in this manner, we obtain

$$D(n) = L\left(n - \sum_{j=1}^{k_0-1} d_j j!\right) + D\left(\sum_{j=1}^{k_0-1} d_j j!\right) + (k - k_0),$$

errors of size $o((k+1)!)$.

Therefore,

$$D(n) = L\left(n - \sum_{j=1}^{k_0-1} d_j j!\right) + D\left(\sum_{j=1}^{k_0-1} d_j j!\right) + o(k!),$$

so

$$D(n)/n = L - L \cdot o(1) + o(1) + o(1),$$

which implies that the density of $\mathcal{R}$ is L, so the proof is complete.

Our next result is an immediate consequence of Theorem 2.1.

Corollary 2.11: If $f(d, j) = f(d)$ depends only on d, where $f(0) = 0$ and $f(d) = o(j!)$ uniformly in j for all other "digits" d, then the density of $\mathcal{R}$ is L, where L is defined as in equation 2.7.

Corollary 2.12: We have $L < 1$ if and only if the function $T(n)$ is not one-one.

Proof: We have $L = A_{k_0} - \sum_{j=k}^{\infty} \varepsilon_j = A_k - \sum_{j=k_0}^{\infty} \varepsilon_j$, for all $k \geq k_0$, where k_0 is defined as in Lemma 2.4. Therefore, $L \leq A_k$ if $k \geq k$. if $T(x) = T(y)$, $x \neq y$, and k is such that $k \geq k_0$ and $x \leq k! - 1$, $y \leq k! - 1$; then, since

$$A_k = D(k! - 1)/k!,$$

it follows that $L \leq A_k \leq 1$. If T is one-one, then it follows from the definition of the A's and the ε's that $A_k = 1$ and $\varepsilon_k = 0$ for all k, so $L = 1$.

It seems to be true, although possibly difficult to prove, that $L < 1$ if each $f(d, j) = f(d)$ depends only on d and f satisfies the hypotheses of Theoren 2.1. It also seems to be the case that we should always have $L > 0$ under these hypotheses; this result again will be left to conjecture.

3. EXISTENCE OF THE DENSITY WHEN $f(d, j) = O(j!/j^2 \log^2 j)$

The main drawback to Theorem 2.1 is the condition $f(0, j) = 0$. If we assume that $f(d, j) = O(j!)$ uniformly in j for all "digits" d, it seems to be difficult to find a workable relationship between the quantities A_k, but on the other hand, it also seems to be difficult to find an example of an image set $\mathcal{R}$ which does not have density under this assumption. However, we do have the following result.

Theorem 3.1: If $f(d, j) = O(j!/j^2 \log^2 j)$ uniformly in j, then the density of $\mathcal{R}$ exists.

Proof: Let D and Ω be as before. If $n = \sum_{j=1}^{k} d\, j!$, then $S(n) = \sum_{j=1}^{k} O(j!/j^2 \log^2 j) = O(k!/k^2 \log^2 k)$.

Suppose that $r \le s \le t (r < t)$ and $s < (k + 1)!$; then,

$$D(r, t) = D(r, s) + D(s + 1, t) - |\Omega(r, s) \cap \Omega(s + 1, t)|.$$

Since $S(n) = 0(k!/k^2 \log^2 k)$, we have

$$D(r, t) = D(r, s) + D(s + 1, t) + 0(k!/k^2 \log^2 k). \tag{3.2}$$

In particular, if $r = 0$, $s = (k - 1)! - 1$, and $t = k! - 1$, we obtain

$$\begin{aligned} D(k! - 1) = D(0, (k - 1)! - 1) &+ D((k - 1)!, k! - 1)) \\ &+ 0((k - 1)!/(k - 1)^2 \log^2 (k - 1)). \end{aligned}$$

Applying the same reasoning to compute the quantities $D(0, j! - 1)$, $2 \le j \le k - 1$, we see that

$$\begin{aligned} D(k! - 1) = D(0) &+ D(1!, 2! - 1) + D(2!, 3! - 1) + \cdots \\ &+ D((k - 1)!, k! - 1) \\ &+ 0((k - 1)!/(k - 1)^2 \log^2 (k - 1)) \\ &+ 0((k - 2)!/(k - 2)^2 \log^2 (k - 2)) + \cdots \end{aligned}$$

so we finally obtain

$$D(k! - 1) = D(0) + \sum_{q=1}^{k-1} D(q!, (q + 1)! - 1) + 0(k!/k^2 \log^2 k). \tag{3.3}$$

Now, by 3.2, we have

$$\begin{aligned} D(dk!, (d + 1)k! - 1) = D(dk!, dk!) &+ D(dk! + 1, (dk+ 1) ! - 1) \\ &+ 0(k!/k^2 \log^2 k) \end{aligned}$$

and by repeated application of 3.2, we obtain

$$\begin{aligned} D(dk!, (d + 1)k! - 1) = D(dk!, dk!) &+ D(dk! + 1, dk! + \ ! - 1) \\ &+ \cdots + D(dk! + (k - 1)!, (d + 1)k! - 1) + k \\ &\text{errors of size } 0(k!/k^2 \log^2 k), \end{aligned}$$

i.e.,

$$\begin{aligned} D(dk!, (d + 1)k! - 1) = D(dk!, dk!) &+ \sum_{q=1}^{k-1} D(dk! + q!, dk! \\ &+ (q + 1)! - 1) + 0(k!/k \log^2 k). \end{aligned} \tag{3.4}$$

Since all integers x which satisfy $dk! + q! \le x \le dk! + (q + 1)! - 1$ have the same number of leading zeros, we have

$$D(dk! + q!, dk! + (q + 1)! - 1) = D(q!, (q + 1)! - 1), \quad 1 \le q \le k - 1$$

(cf. the argument used to prove 2.2).

Using this fact, 3.4 becomes

$$D(dk!, (d + 1) ! - 1) = D(0) + \sum_{q=1}^{k-1} D(q!, (q + 1)! - 1) + 0(k!/k \log^2 k) \tag{3.5}$$

and 3.3 and 3.5 imply that

$$D(dk!, (d + 1)k! - 1) = D(k! - 1) + 0(k!/k \log^2 k). \tag{3.6}$$

Now, using 3.6, we obtain

$$\begin{aligned} D((k+1)!-1) &= D(k!-1) + D(k!,\ (k+1)!-1 + 0(k!/k^2\log^2 k) \\ &= D(k!-1) + D(k!,\ 2k!-1) + D(2k!,\ (k+1)!-1) \\ &\qquad + 0(k!/k^2\log^2 k) + 0(k!/k^2\log^2 k) \\ &= 2D(k!-1) + D(2k!,\ (k+1)!-1) + 0(k!/k\log^2 k). \end{aligned}$$

By repeated application of 3.6, we finally obtain

$$D((k+1)!-1) = (k+1)D(k!-1) + k + 1,$$

errors of size $0(k!/k\log^2 k)$;

thus,

(3.7) $$D((k+1)!-1) = (k+1)D(k!-1) + 0((k+1)!/k\log^2 k).$$

Define $A_k = D(k!-1)/k!$. Then 3.7 becomes

$$(k+1)!A_{k+1} - (k+1)!A_k = 0((k+1)!/k\log^2 k);$$

and by telescoping, we see that

$$A_{k+1} = A_0 + \sum_{j=1}^{k} 0(1/j\log^2 j).$$

It is not difficult to verify that $\sum_{j=1}^{k} 0(1/j\log^2 j) = O(1/\log^2 k)$. Therefore, using the above equation, we may conclude that there exists a constant L such that

(3.8) $$A_k = L + 0(1/\log k).$$

Now let $n = \sum_{j=1}^{m} d_{k_j} k!_j$ be any nonnegative integer, where each $d_{k_j} \neq 0$. Then

$$D(n) = D(d_{k_m}k!_m - 1) + D(d_{k_m}k! + d_{m-1}k!_{m-1} + \cdots) + 0(k!_m/k_m^2\log^2 k_m).$$

By the same type of reasoning employed to get 3.4 and 3.7, we see that

$$D(d_{k_m}k!_m - 1) = d_{k_m}D(k!_m - 1) + 0(k!_m/k\ \log^2 k_m) + D(d_{k_m}k!_m,\ d_{k_m}k!_m + \cdots).$$

Since $d_{k_m} \neq 0$ for any j, we have

$$D\left(d_{k_m}k!_m,\ \sum_{j=1}^{m} d_{k_j}k!_j\right) = D\left(\sum_{j=1}^{m-1} d_{k_j}k!_j\right).$$

Therefore,

$$D(n) = d_{k_m}k!_m(L + 0(1/\log k_m)) + 0(k!_m/k_m\log^2 k_m) + D\left(\sum_{j=1}^{m-1} d_{k_j}k!_j\right).$$

Continuing in this manner yields

$$D(n) = nL + 0(k!_m/k_m\log^2 k_m) + \sum_{j=1}^{k_m} 0(j!/\log j).$$

Hence, $D(n) = nL + 0(k!_m/\log k_m)$,

so $D(n)/n = L + 0(1/\log k_m) = L + o(1)$,

which proves that the density of $\mathcal{R}$ is L.

Remark 1: Theorem 3.1 has the drawback that the computability of the density has been lost.

Remark 2: If we assume that $f(d, j) = o(j!)$ uniformly in j, then there exists an image set $\mathcal{R}$ which does not have density. For example, let $f(d, j) = 0$ when j is even and $f(d, j) = j!$ when j is odd. Then,

$$T\left(k! + \sum_{j=1}^{k-1} d_j j!\right) = k! + \sum_{j=1}^{k-1} d_j j! + k! + (k - 2) + \cdots + 1! \geq 2k!$$

if k is odd, and

$$T\left(k! + \sum_{j=1}^{k-1} d_j j!\right) = k! + \sum_{j=1}^{k-1} d_j j! + (k - 1)! + (k - 3)! + \cdots + 1!$$

if k is even. Therefore, the number of integers between $k!$ and $2k!$ that belong to $\mathcal{R}$ if k is odd is at most $1 + (k - 2)! + (k - 4)! + \cdots + 1$, and the number of integers between $k!$ and $2k!$ that belong to $\mathcal{R}$ if k is even is at $k! - (k - 1)! - (k - 3)! - \cdots - 1!$. Hence, if we let δ and Δ denote the lower and upper density of $\mathcal{R}$, respectively, we see that

$$\delta \leq 0 + o(1) \quad \text{and} \quad \Delta \geq 1 + o(1),$$

so $\delta = 0$ and $\Delta = 1$.

It is also interesting to note that, if we let $f(d, j) = o(j!)$ uniformly in j, there do exist image sets $\mathcal{R}$ of density 0. For example, if $f(d, j) = 0$ when $d \neq 1$ or $j = 1$ and $f(d, j) = 2j!$ if $d = 1$ and $j > 1$, then no member of (except 1) has the "digit" 1 anywhere in its factorial representation, and the set

$$\left\{ n \,\middle|\, n = \sum_{j=1}^{k} d_j j!,\ d_j \neq 1,\ 1 \leq j \leq k \right\} \tag{3.8}$$

is easily seen to be the set of density 0.

Our next result is an immediate corollary of Theorem 3.1.

Corollary 3.9: If $f(d, j) = f(d)$ depends only on d and

$$f(d) = 0(j!/j^2 \log^2 j)$$

uniformly in j, then the density of $\mathcal{R}$ exists.

Finally, just as in [2] and [3], we wish to consider the special case that arises when we assume that $f(d, j) = f(d) = d$ for all "digits" d [so that $T(n)$ is the function n + the sum of the "digits" of n]. Clearly, $f(d)$ satisfies the assumptions of Corollary 2.11, so we know that the density of $\mathcal{R}$ is L, where L is defined as in 2.7. In this case, it is easy to verify that $k_0 = 0$ and that the value of $\lambda_{d,k}$ does not depend on d. Let us therefore set $\lambda_{d,k} = \lambda_k$, $0 \leq d \leq k$. In the following table, we give the values of λ_k and ε_k to the nearest 6 decimal places; it appears to be difficult to develop an algorithm to calculate the λ_k in general.

Using this table together with Taylor's formula and Lagrange's form for the remainder, we obtain the following result.

Theorem 3.10: When $T(n)$ is the function n + the sum of the "digits" of n, the density of $\mathcal{R}$ is 0.879888. The error made using this figure is less than $e/2 \cdot 9!$. Therefore, $\mathcal{R}$ has positive density in this case.

The Values of λ_k and ε_k, $1 \le k \le 10$

k	λ_k	ε_k
1	0	0
2	0	0
3	0	0
4	2	0.066667
5	6	0.041667
6	8	0.008929
7	14	0.002401
8	17	0.000375
9	26	0.000064
10	39	0.000009

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EVALUATION OF SUMS OF CONVOLVED POWERS USING STIRLING AND EULERIAN NUMBERS

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ABSTRACT

It is shown here how the method of generating functions leads quickly to compact formulas for sums of the type

$$S(i,j;n) = \sum_{0 \le k \le n} k^i (n-k)^j$$

using Stirling numbers of the second kind and also using Eulerian numbers. The formulas are, for the most part, much simpler than corresponding results using Bernoulli numbers.

1. INTRODUCTION

Neuman and Schonbach [9] have obtained a formula for the series of convolved powers

$$S(i,j;n) = \sum_{k=0}^{n} k^i (n-k)^j \tag{1.1}$$

using Bernoulli numbers. Although the formula expresses $S(i,j;n)$ as a polynomial of degree $i + j + 1$ in n, and this mode of expression is useful, still the formula is rather clumsy and hard to recall. Below we shall show how the method of generating functions can be used to obtain elegant closed forms for (1.1) very quickly. The first of these uses the Stirling numbers of the second kind, and the second uses the Eulerian numbers. Both results give (1.1) as series of binomial coefficients in n, rather than directly as polynomials expressed explicitly in powers of n. For many purposes of computation and number theoretic study, such expressions are desirable. The significant results below are formulas (3.6), (3.8), (5.3), and (7.3).

Glaisher [4] and [5] was the first to sum (1.1) using Bernoulli numbers. Carlitz [3] has shown some extensions of [9] and connections with Eulerian numbers. Our results overlap some of those of Carlitz, but were obtained in August 1974 before [3] was written.

2. A GENERATING FUNCTION

$$G(t;i,j) = \sum_{n=0}^{\infty} t^n S(i,j;n). \tag{2.1}$$

Then

$$G(t;i,j) = \sum_{k=0}^{\infty} k^i \sum_{n=k}^{\infty} t^n (n-k)^j = \sum_{k=0}^{\infty} k^i \sum_{n=0}^{\infty} t^{n+k} n^j,$$

so that we have at once the elegant generating function

$$G(t;i,j) = \sum_{k=0}^{\infty} k^i t^k \cdot \sum_{n=0}^{\infty} n^j t^n. \tag{2.2}$$

The generalized power series

$$\sum_k k^p t^k$$

may be summed in a variety of ways. We shall use the methods of (i) Stirling numbers of the second kind and (ii) Eulerian numbers. Our (2.2) is (3.4) in Carlitz [3].

3. METHOD OF STIRLING NUMBERS OF THE SECOND KIND

It is an old fact that

$$(tD)^p f(t) = \sum_{k=0}^{p} S(p,k) t^k D^k f(t), \tag{3.1}$$

where $D = d/dt$ and $S(p,k)$ is a Stirling number of the second kind. Explicitly,

$$k!S(p,k) = \Delta^k 0^p = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} k^p. \tag{3.2}$$

The formula dates back more than 150 years, but, for a recent example, see Riordan [10, p. 45, ex. 18]. Riordan gives a full account of the properties of Stirling numbers of both first and second kinds. Other historical remarks and variant notations are discussed in [6]. Applying the formula is easy because $(tD)^p t^k = k^p t^k$, whence we have

$$\sum_{k=0}^{\infty} k^p t^k = \sum_{k=0}^{p} k!S(p,k)\frac{t^k}{(1-t)^{k+1}}. \tag{3.3}$$

This, too, is a very old formula. It converges for $|t| < 1$, but we treat it as a formal power series. Carlitz [2] gives a good discussion of formal power series techniques.

Using (3.3) in (2.2), we find

$$G(t;i,j) = \sum_{r=0}^{i+j} \frac{t^r}{(1-t)^{r+2}} \sum_{k=0}^{r} k!(r-k)!S(i,k)S(j,r-k). \tag{3.4}$$

Throughout the rest of the paper, we shall write, for brevity,

$$\mathcal{S}_r(i,j) = \sum_{k=0}^{r} k!(r-k)!S(i,k)S(j,r-k). \tag{3.5}$$

Applying the binomial theorem, we find next

$$\begin{aligned} G(t;i,j) &= \sum_{r=0}^{i+j} \sum_{n=0}^{\infty} \binom{n+r+1}{r+1} t^{n+r} \mathcal{S}_r(i,j) \\ &= \sum_{r=0}^{i+j} \sum_{n=r}^{\infty} \binom{n+1}{r+1} t^n \mathcal{S}_r(i,j) \\ &= \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} \binom{n+1}{r+1} \mathcal{S}_r(i,j). \end{aligned}$$

In the next-to-last step here, the upper limit $r = i + j$ might as well have been $r = \infty$ because of zero terms involved, since $S(p,k) = 0$ when $k > p$. This makes manipulation easier. Equating coefficients of t^n and dropping some zero terms, we find finally then our desired formula

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{n+1}{r+1} \mathcal{S}_r(i,j). \tag{3.6}$$

This simple expression may be compared with the bulky form of expression given in [9] using Bernoulli numbers.

Having found our desired formula, we can next offer a much quicker proof. Recall [10, p. 33] that

$$x^n = \sum_{r=0}^{n} \binom{x}{r} r!S(n,r). \tag{3.7}$$

This gives at once

$$k^i(n-k)^j = \sum_{r=0}^{i} r!S(i,r) \sum_{s=0}^{j} s!S(j,s) \binom{k}{r}\binom{n-k}{s},$$

whence, using formula (3.3) in [8], a modified Vandermonde addition formula, we get on summing from $k = 0$ to $k = n$,

$$S(i,j;n) = \sum_{r=0}^{i} r!S(i,r) \sum_{s=0}^{j} s!S(j,s) \binom{n+1}{r+s+1}. \tag{3.8}$$

By simply putting $s - r$ for s and interchanging the summation order, we see that this is nothing other than our former result (3.6).

4. EXAMPLES OF THE STIRLING NUMBER METHOD

For the sake of completeness, we recall [10, p. 48] some of the values of $S(n,k)$:

	0	1	2	3	4	5	6	7	$\cdots$	k
0	1									
1		1								
2		1	1							
3		1	3	1						
4		1	7	6	1					
5		1	15	25	10	1				
6		1	31	90	65	15	1			
7		1	63	301	350	140	21	1		
$\vdots$										
n										

Here, $S(n,k) = 0$ when $k > n$ and $S(n,0) = 0$ for $n \geq 1$.

For $j = 0$, formula (3.6) becomes the well known

$$S(i,0;n) = \sum_{r=0}^{i} \binom{n+1}{r+1} r! S(i,r), \quad n \geq 0, \quad i \geq 0. \tag{4.1}$$

Incidentally, in some places in the vast literature $r!S(i,r)$ has been called a Stirling number, and both arrays turn up very often in odd places with new notations. There are at least 50 notations for Stirling numbers. Here are a few examples of (4.1):

$$\begin{aligned}
S(1,0;n) &= \binom{n+1}{2}, \\
S(2,0;n) &= \binom{n+1}{2} + 2\binom{n+1}{3}, \\
S(3,0;n) &= \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}, \\
S(4,0;n) &= \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}.
\end{aligned}$$

For $j = 1$ we shall obtain substantially the same coefficients, the difference being that the lower indices are each increased by 1. Thus:

$$\begin{aligned}
S(2,1;n) &= \binom{n+1}{3} + 2\binom{n+1}{4} = \frac{n^4 - n^2}{12}, \\
S(3,1;n) &= \binom{n+1}{3} + 6\binom{n+1}{4} + 6\binom{n+1}{5} = \frac{3n^5 - 5n^3 + 2n}{60}, \\
S(4,1;n) &= \binom{n+1}{3} + 14\binom{n+1}{4} + 36\binom{n+1}{5} + 24\binom{n+1}{6} \\
&= \frac{2n^6 - 5n^4 + 3n^2}{60},
\end{aligned}$$

where we have indicated, for comparison, the values obtained in [9].

For $j = 1$, the following is a brief table of the coefficients in the array:

$i = 2$:	1	2				
$i = 3$:	1	6	6			
$i = 4$:	1	14	36	24		
$i = 5$:	1	30	150	240	120	
$i = 6$:	1	62	450	1560	1800	720

For $j = 3$, we find the following formulas:

$$S(0,3;n) = \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4},$$

$$S(1,3;n) = \binom{n+1}{3} + 6\binom{n+1}{4} + 6\binom{n+1}{5},$$

$$S(2,3;n) = \binom{n+1}{3} + 8\binom{n+1}{4} + 18\binom{n+1}{5} + 12\binom{n+1}{6},$$

$$S(3,3;n) = \binom{n+1}{3} + 12\binom{n+1}{4} + 48\binom{n+1}{5} + 72\binom{n+1}{6} + 36\binom{n+1}{7},$$

and so forth.

5. METHOD OF EULERIAN NUMBERS

The Eulerian numbers [1], [10, pp. 39, 215] are given by

$$A_{n,j} = \sum_{k=0}^{j} (-1)^k \binom{n+1}{k} (j-k)^n. \tag{5.1}$$

These must not be confused with Euler numbers appearing in the power series expansion of the secant function. The Eulerian numbers satisfy

$$A_{n,j} = A_{n,n-j+1}, \text{ row symmetry, } n \geq 1,$$

$$A_{n,j} = jA_{n-1,j} + (n - j + 1)A_{n-1,j-1},$$

and

$$\sum_{j=1}^{n} A_{n,j} = n!.$$

Again, for completeness, here is a brief table of $A_{n,j}$:

	0	1	2	3	4	5	6	7	$\cdots$ j
0	1								
1		1							
2		1	1						
3		1	4	1					
4		1	11	11	1				
5		1	26	66	26	1			
6		1	57	302	302	57	1		
7		1	120	1191	2416	1191	120	1	
$\vdots$									
n									

These numbers are frequently rediscovered, for example, recently by Voelker [11] and [12], where no mention is made of the vast literature dealing with these numbers and tracing back to Euler. For our purposes, we need the well-known expansion

$$\sum_{k=0}^{\infty} k^n t^k = (1-t)^{-n-1} \sum_{k=0}^{n} t^k A_{n,k}. \tag{5.2}$$

This expansion is known to be valid for $|t| < 1$, but again we treat all series here as formal power series since we do not use the sums of any infinite series. We never assign t a value, but equate coefficients only.

Applying this to (2.2), we find

$$\begin{aligned} G(t;i,j) &= (1-t)^{-i-j-2} \sum_{r=0}^{i} t^r A_{i,r} \sum_{s=0}^{j} t^s A_{j,s} \\ &= \sum_{k=0}^{\infty} \binom{i+j+k+1}{k} t^k \sum_{r=0}^{i+j} t^r \sum_{s=0}^{r} A_{i,s} A_{j,r-s} \\ &= \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} \binom{i+j+n-r+1}{n-r} \sum_{s=0}^{r} A_{i,s} A_{j,r-s} \end{aligned}$$

and by comparison of coefficients of t^n we have our desired formula

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{i+j+n-r+1}{i+j+1} \sum_{s=0}^{r} A_{i,s} A_{j,r-s}. \tag{5.3}$$

Here we have again dropped some of the terms that are zero by noting that $A_{n,j} = 0$ whenever $j > n$. Formula (5.3) is (3.6) in Carlitz [3].

As with our previous Stirling number argument, we could obtain (5.3) by another method. We recall that in fact

$$x^n = \sum_{j=0}^{n} \binom{x+j-1}{n} A_{n,j} \tag{5.4}$$

and form the product $k^i(n-k)^j$ and sum from $k = 0$ to $k = n$ to obtain a formula for (5.3) analogous to (3.8). We omit the details.

6. EXAMPLES OF THE EULERIAN NUMBER METHOD

When $j = 0$, formula (5.3) becomes, of course, the familiar relation

$$S(i,0;n) = \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,r}, \quad n \geq 0, \ i \geq 1. \tag{6.1}$$

To see that this is so, we proceed as follows. By (5.3),

$$\begin{aligned} S(i,0;n) &= \sum_{r=0}^{i} \binom{i+n-r+1}{i+1} \sum_{s=0}^{r} A_{i,s} A_{0,r-s} \\ &= \sum_{r=0}^{i} \binom{i+n-r+1}{i+1} A_{i,r}, \text{ since } A_{0,r-s} = 0 \text{ for } r \neq s, \end{aligned}$$

$$= \sum_{r=1}^{i} \binom{i+n-r+1}{i+1} A_{i,r}, \text{ since } A_{i,0} = 0 \text{ for } i \geq 1,$$

$$= \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,i-r+1}, \text{ by putting } i - r + 1 \text{ for } r,$$

$$= \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,r}, \text{ by the symmetry relation.}$$

For $j = 0$, then, we have the following formulas:

$$S(1,0;n) = \binom{n+1}{2},$$

$$S(2,0;n) = \binom{n+1}{3} + \binom{n+2}{3},$$

$$S(3,0;n) = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4}$$

$$S(4,0;n) = \binom{n+1}{5} + 11\binom{n+2}{5} + 11\binom{n+3}{5} + \binom{n+4}{5},$$

etc.

For $j = 1$, we find

$$S(2,1;n) = \binom{n+1}{4} + \binom{n+2}{4}$$

$$S(3,1;n) = \binom{n+1}{5} + 4\binom{n+2}{5} + \binom{n+3}{5}$$

$$S(4,1;n) = \binom{n+1}{6} + 11\binom{n+2}{6} + 11\binom{n+3}{6} + \binom{n+4}{6}$$

and so on. These again are a different way of saying what was found in [9].

7. ALTERNATIVE EXPRESSION OF THE STIRLING NUMBER EXPANSION

Formula (3.6) uses the values of $\binom{n+1}{r+1}$. We wish to show now that we can transform this result easily into a formula using just $\binom{n}{r+1}$, i.e., directly as a series of binomial coefficients in n rather than $n + 1$. We will need to recall, see [10], the recurrence relation for Stirling numbers of the second kind

$$S(m,k) = kS(m-1,k) + S(m-1,k-1). \tag{7.1}$$

In this, set $m = j + 1$ and replace k by $r - k$. We get

$$S(j+1,r-k) = (r-k)S(j,r-k) + S(j,r-k-1). \tag{7.2}$$

Now, by (3.6) and the usual recurrence for binomial coefficients, we have

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{n+1}{r+1} \mathcal{S}_r(i,j) = \sum_{r=0}^{i+j} \left\{ \binom{n}{r} + \binom{n}{r+1} \right\} \mathcal{S}_r(i,j)$$

$$= \sum_{r=0}^{i+j} \binom{n}{r} \mathcal{S}_r(i,j) + \sum_{r=1}^{i+j+1} \binom{n}{r} \mathcal{S}_{r-1}(i,j)$$

$$= \sum_{r=0}^{i+j+1} \binom{n}{r} \left\{ \mathcal{S}_r(i,j) + \mathcal{S}_{r-1}(i,j) \right\}.$$

However, $\mathcal{S}_r(i,j) + \mathcal{S}_{r-1}(i,j)$

$$= \sum_{k=0}^{r} k!\,(r-k)!\,S(i,k)S(j,r-k) + \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r} k!\,(r-k-1)!\,S(i,k)(r-k)S(j,r-k) + \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r} k!\,(r-1-k)!\left\{S(i,k)S(j+1,r-k) - S(i,k)S(j,r-k-1)\right\}, \quad \text{by (7.2)}$$

$$+ \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r-1} k!\,(r-k-1)!\,S(i,k)S(j+1,r-k) + r!\,S(i,r)S(j,0).$$

The extra term here may be dropped when we consider $j \geq 1$. Therefore, we have the new result that

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{n}{r+1} \sum_{k=0}^{r} k!\,(r-k)!\,S(i,k)S(j+1,r+1-k), \quad j \geq 1,\ i \geq 0. \tag{7.3}$$

Examples: Let $j = 1$ again. We find

$$S(0,1;n) = \binom{n}{1} + \binom{n}{2},$$

$$S(1,1;n) = \binom{n}{2} + \binom{n}{3},$$

$$S(2,1;n) = \binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4},$$

For $j = 1$, the general pattern of these coefficients begins as follows:

	0	1	2	3	4	5	6	7 ··· r
0	1	1						
1		1	1					
2		1	3	2				
3		1	7	12	6			
4		1	15	50	60	24		
5		1	31	180	390	360	120	
6		1	63	602	2100	3360	3520	720
⋮								
i								

It is interesting to note that these coefficients appear in another old formula:

$$(7.4)\qquad S(i,0;n) = \sum_{k=0}^{i} (-1)^k \binom{n}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^i,$$

valid for $i \geq 1$, $n \geq 1$.

Examples:

$$S(1,0;n) = \binom{n}{1} + \binom{n}{2},$$

$$S(2,0;n) = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3},$$

$$S(3,0;n) = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$

and so forth.

There is yet another old formula involving Stirling numbers of the second kind which we should mention. It is

$$(7.5)\qquad S(i,0;n) = \sum_{r=0}^{i} (-1)^{i-r} \binom{n+r}{r+1} r!\,S(i,r), \quad n \geq 0,\ i \geq 1.$$

This occurs, for example, as the solution to a problem [13] in the *American Mathematical Monthly*.

Examples:

$$S(1,0;n) = \binom{n+1}{2},$$

$$S(2,0;n) = -\binom{n+1}{2} + 2\binom{n+2}{3},$$

$$S(3,0;n) = \binom{n+1}{2} - 6\binom{n+2}{3} + 6\binom{n+3}{4},$$

$$S(4,0;n) = -\binom{n+1}{2} + 14\binom{n+2}{3} - 36\binom{n+3}{4} + 24\binom{n+4}{5},$$

and so forth.

8. FINAL REMARKS

It is interesting to note that the original sum (1.1) is a type of convolution. So also formulas (3.6), (5.3), and (7.3) involve convolutions of the Stirling and Eulerian numbers. The formula found in [9] is not of this type. This is so because of the way in which the binomial theorem was first used. It would evidently be possible to obtain convolutions of the Bernoulli numbers. To get such a formula using Bernoulli polynomials is easy. Let us recall that

$$(8.1)\qquad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x), \quad |t| < 2\pi,$$

defines the Bernoulli polynomial $B_n(x)$. Then $B_n(0) = B_n$ are the Bernoulli numbers. It is also a well-known old formula that then for all real x,

$$(8.2)\qquad x^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x), \quad n \geq 0.$$

Form the product $k^i(n-k)^j$ by using this formula to expand k^i and $(n-k)^j$. Sum both sides and we get

$$S(i,j;n) = \frac{1}{i+1}\sum_{r=0}^{i}\binom{i+1}{r}\frac{1}{j+1}\sum_{s=0}^{j}\binom{j+1}{s}\sum_{k=0}^{n}B_r(k)B_s(n-k), \tag{8.3}$$

which brings in a convolution of Bernoulli polynomials. Since the Bernoulli polynomials may be expressed in terms of Bernoulli numbers by the further formula

$$B_n(x) = \sum_{m=0}^{n}\binom{n}{m}x^{n-m}B_m, \tag{8.4}$$

it would be possible to secure a convolution of the Bernoulli numbers. However, the author has not reduced this to any interesting or useful formula that appears to offer any advantages over those we have derived here or those in [9]. We leave this as a project for the reader.

It is also possible to obtain a mixed formula by proceeding first as in [9] to get

$$S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{n}k^{i+r},$$

apply one of our Stirling number expansions to the inner sum and get, e.g.,

$$S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{i+r}\binom{n+1}{k+1}k!S(i+r,k), \tag{8.5}$$

but the writer sees no remarkable advantages to be gained.

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(continues on page 560)

b-ADIC NUMBERS IN PASCAL'S TRIANGLE MODULO b

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For the binomial coefficients in Pascal's triangle we write their smallest nonnegative residues modulo a base b. Then blocks of consecutive integers within the rows may be interpreted as b-adic numbers. What b-adic numbers can occur in the Pascal triangle modulo b? In this article we will give the density of such numbers and determine the smallest positive integer $h(b)$, such that its b-adic representation does not occur (see [3] for $b = 2$).

We use the notation

$$t = \sum_{i=0}^{m}a_ib^i = (a_ma_{m-1}\,\ldots\,a_1a_0)_b\,,\ 0 \le a_i \le b-1,\ a_m \ne 0,$$

for positive integers t. First we will prove the existence of b-adic numbers which do not occur.

Lemma 1: $(1011)_2$ is not to be found within any row of the Pascal triangle modulo 2.

Proof: We assume that there are integers n and k with

$$\binom{n}{k} \equiv \binom{n}{k+2} \equiv \binom{n}{k+3} \equiv 1 \quad \text{and} \quad \binom{n}{k+1} \equiv 0 \pmod 2.$$

These congruences substituted in

$$(1) \qquad (k + 1 + i)\binom{n}{k + 1 + i} = (n - k - i)\binom{n}{k + i}$$

for $i = 0, 1, 2$, gives $n \equiv k \pmod 2$, $k \equiv 0 \pmod 2$, and $n \equiv 1 \pmod 2$, respectively, which is a contradiction.

Lemma 2: $(111)_b$ is not to be found within any row of Pascal's triangle modulo b with $b > 2$.

Proof: We assume that

$$\binom{n}{k} \equiv \binom{n}{k+1} \equiv \binom{n}{k+2} \equiv 1 \pmod b.$$

Together with (1), for $i = 0$ and $i = 1$, we conclude that $n \equiv 2k + 1 \pmod b$, and $n \equiv 2k + 3 \pmod b$, respectively. However, both congruences are possible only if $b = 2$.

We are now able to determine the density.

Theorem 1: Almost all b-adic numbers cannot occur within the rows of Pascal's triangle modulo b.

Proof: As noted in [4], it is well known that the density of those b-adic integers not containing a given sequence of digits is 0 (see [2], p. 120). Thus, the proof is given by Lemmas 1 and 2.

Theorem 2: Let $h(b)$ be the smallest b-adic number not being found within any row of Pascal's triangle modulo b. Then, $h(b) = b^2 + b + 1 = (111)_b$ for $b > 2$, and $h(2) = 11 = (1011)_2$.

We first prove two lemmas.

Lemma 3: Let $b = b_1 b_2$ with $(b_1, b_2) = 1$. Then $(a_m \ldots a_0)_b$ occurs in the Pascal triangle modulo b if and only if $(a_{im} \ldots a_{i0})_{b_i}$ for $i = 1, 2$ occur in the triangles modulo b_i with $a_{ij} \equiv a_j \pmod{b_i}$, $j = 0, 1, \ldots, m$.

Proof: One direction of the proof is trivial.

In the following, we use the result of [1] and [6], that $\binom{n}{k} \pmod b$ is periodic for fixed k with the minimal period N being the product of all prime powers $p^{\alpha+\beta}$ with p^α from the canonical factorization of b and β from $p^\beta \le k < p^{\beta+1}$. Thus, N depends only on the prime factors of b and on k (see [5] for further references). By reasons of symmetry, a corresponding periodicity of length L holds for $\binom{n+\ell}{k+\ell}$ with fixed n and k.

From this and by the assumption, we are able to find n_i and k_i such that for $i = 1, 2$,

$$\binom{n_i}{k_i + j} \equiv \binom{n_i + x_i L_i}{k_i + x_i L_i + j} \equiv a_{i(m-j)} \pmod{b_i},\ j = 0, 1, \ldots, m,$$

with minimal periods L_i each being the lowest common multiple of $m + 1$ minimal periods. From $(b_1, b_2) = 1$ we have $(L_1, L_2) = 1$. Thus, the diophantine equation,

$$k_1 + x_1L_1 = k_2 + x_2L_2,$$

has solutions x_1, x_2. For fixed values x_1, x_2, we then have minimal periods N with

$$\binom{n_i + x_iL_i + y_iN_i}{k_i + x_iL_i + j} \equiv a_{i(m-j)} \pmod{b_i},\ j = 0, 1, \ldots, m.$$

Finally, $(N_1, N_2) = 1$ guarantees solutions y_1, y_2 of

$$n_1 + x_1L_1 + y_1N_1 = n_2 + x_2L_2 + y_2N_2,$$

which completes the proof.

Lemma 4: In Pascal's triangle modulo p^α, p being a prime, there are arbitrarily large partial triangles with

$$\binom{n + n_r}{k + k_r} \equiv r\binom{n}{k} \pmod{p^\alpha},\ n \geq 0,\ k \geq 0,$$

for every r from 1 to p^α.

Proof: We first show

(2) $$\binom{rp^{\alpha\beta}}{k} \equiv 0 \pmod{p^\alpha} \text{ for } p^{\alpha\beta} - p^{\alpha\beta-\alpha+1} < k < p^{\alpha\beta} + p^{\alpha\beta-\alpha+1},\quad k \neq p^{\alpha\beta}.$$

Let γ be the exponent of p in the canonical factorization of the binomial coefficient in (2). Then, by a theorem of Legendre ([7], p. 13), we have,

$$\begin{aligned}\gamma &= \sum_{i\geq 1}\left\{\left[\frac{rp^{\alpha\beta}}{p^i}\right] - \left[\frac{k}{p^i}\right] - \left[\frac{rp^{\alpha\beta} - k}{p^i}\right]\right\}\\ &\geq \sum_{i=1}^{\alpha\beta}\left\{-\left[\frac{k}{p^i}\right] - \left[\frac{-k}{p^i}\right]\right\} \geq \sum_{i=\alpha\beta-\alpha+1}^{\alpha\beta} 1 = \alpha,\end{aligned}$$

where $[x]$ means the greatest integer not exceeding x.

We further show by induction on α that $\binom{rp^{\alpha\beta}}{p^{\alpha\beta}}$, for $r = 1, 2, \ldots, p^\alpha$, is a complete system of residues modulo p^α. Let

(3) $$P_j(r) = \prod_{\substack{i=1\\(i,p)=1}}^{p^j-1} \frac{(r-1)p^j + i}{i}.$$

Then for $\alpha = 1$ we can write

$$\binom{rp^\beta}{p^\beta} = r\prod_{j=1}^{\beta} P_j(r) \equiv r \pmod{p}.$$

In general, with $r = vp^{\alpha-1} + \rho$, $1 \leq \rho \leq p^{\alpha-1}$, $0 \leq v \leq p-1$, we get

$$\binom{rp^{\alpha\beta}}{p^{\alpha\beta}} = r\prod_{j=1}^{\alpha\beta} P_j(r) \equiv r\prod_{j=1}^{\alpha-1} P_j(r) \equiv r\prod_{j=1}^{\alpha-1} P_j(\rho) \equiv vp^{\alpha-1} + \rho\prod_{j=1}^{\alpha-1} P_j(\rho) \pmod{p^\alpha}.$$

If we assume $\rho\pi P_j(\rho)$ to take all residues modulo $p^{\alpha-1}$, then the induction is complete.

As β may be chosen arbitrarily large, Lemma 4 follows with $n_r = rp^{\alpha\beta}$ and $k_r = p^{\alpha\beta}$.

Proof of Theorem 2: Lemmas 1 and 2 yield $h(b) \leq \ldots$. Because of Lemma 3, we need to consider only prime powers as moduli. Trivially,

$$(a_0)_{p^\alpha}, \quad 1 \leq a_0 < p^\alpha,$$

occur as $\binom{n}{k}$ in the Pascal triangle modulo p^α (let $n = a_0$ and $k = 1$), and so do $(1a_o)_{p^\alpha}$ (let $n = a_o$ and $k = 0, 1$), with $1 \leq a_0 \leq p^\alpha$. We then multiply the digits of $(1a_0)_{p^\alpha}$ by r, $1 \leq r < p^\alpha$, and obtain all numbers $(a_1a_0)_{p^\alpha}$, including those with $(a_1, p^\alpha) > (a_0, p^\alpha)$. This is because of Lemma 4 and the symmetry of binomial coefficients. Further, $(100)_{p^\alpha}$ occurs if $n = 2p^\alpha$, $k = 0$, 1, 2, and $(110)_{p^\alpha}$ if $n = 2p^\alpha + 1$, $k = 0$, 1, 2.

Now

$$\sum_{i \geq 1}\left\{\left[\frac{rp^\alpha - 2}{p^i}\right] - \left[\frac{p^\alpha - 1}{p^i}\right] - \left[\frac{(r-1)p^\alpha - 1}{p^i}\right]\right\} \geq \sum_{i=1}^{\alpha}\{\cdots\} = \alpha,$$

so that $\binom{n}{k} \equiv 0 \pmod{p^\alpha}$, if $n = rp^\alpha - 2$ and $k = p^\alpha - 1$. Using (3), and with v being an integer, we have

$$\binom{rp^\alpha - 2}{p^\alpha - 2} = \frac{p^\alpha - 1}{rp^\alpha - 1}\prod_{j=1}^{\alpha} P_j(r) \equiv 1 + vp \pmod{p^\alpha}, \tag{4}$$

$$\binom{rp^\alpha - 2}{p^\alpha} = (r-1)\frac{(r-1)p^\alpha - 1}{p^\alpha - 1}\binom{rp^\alpha - 2}{p^\alpha - 2} \equiv (r-1)\binom{rp^\alpha - 2}{p^\alpha - 2} \pmod{p^\alpha}. \tag{5}$$

As $(1 + vp, p^\alpha) = 1$, we can find an integer x such that multiplying (4) and (5) by x yields the residues 1 and $r - 1$. Because of Lemma 4, corresponding binomial coefficients occur in the Pascal triangle, so that the existence of all numbers $(10(r-1))_{p^\alpha}$, $2 \leq r \leq p^\alpha$, is proved.

Thus, we have shown $h(b) \geq (111)_b$ for $b \geq 2$. The remaining binary numbers $(111)_2$, $(1000)_2$, $(1001)_2$, and $(1010)_2$ are to be found within the rows 3, 4, 5, and 6, respectively.

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DIVISIBILITY PROPERTIES OF POLYNOMIALS IN PASCAL'S TRIANGLE

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Divisibility properties of the Fibonacci sequence $\{F_n\}$ are well known, including the property of greatest common divisors,

$$(F_m, F_n) = F_{(m,n)}.$$

Here the derivation of the greatest common divisor of a sequence pair is extended to the Fibonacci polynomials, the Morgan-Voyce polynomials, the Chebyshev polynomials, and more general polynomials from a problem of Schechter [1]. Moreover, all of these polynomials have coefficients which lie along rising diagonals of Pascal's triangle, and all of these polynomials satisfy $(u_m(x), u_n(x)) = u_{(m,n)}(x)$ with suitable adjustment of subscripts.

1. INTRODUCTION

The Morgan-Voyce polynomials in [2], [3], and [4] are defined by

$$B_0(x) = 1,\ B_1(x) = x + 2;\ b_0(x) = 1,\ b_1(x) = x + 1,$$

and

$$\begin{aligned} &B_n(x) = b_{n-1}(x) + (1 + x)B_{n-1}(x), \\ (1.1)\qquad &b_n(x) = xB_{n-1}(x) + b_{n-1}(x), \\ &B_n(x) = B_{n-1}(x) + b_n(x). \end{aligned}$$

It is easy to show that $B_{-1}(x) = 0$, and $b_{-1}(x) = 1$. These mixed recurrences could be solved for pure recurrences as each separately satisfies

$$(1.2)\qquad u_{n+2}(x) = (x + 2)u_{n+1}(x) - u_n(x),$$

with $u_0 = 1$ and $u_1 = x + 2$, and $u_0 = 1$ and $u_1 = x + 1$, respectively.

If one lists these polynomials,

$$\begin{aligned} b_0(x) &= 1 \\ B_0(x) &= 1 \\ b_1(x) &= x + 1 \\ B_1(x) &= x + 2 \\ b_2(x) &= x^2 + 3x + 1 \\ B_2(x) &= x^2 + 4x + 3 \\ b_3(x) &= x^3 + 5x^2 + 6x + 1 \\ B_3(x) &= x^3 + 6x^2 + 10x + 4 \\ &\vdots \end{aligned}$$

Clearly, we see that the coefficients of this double sequence lie along the rising diagonals of Pascal's triangle.

The Fibonacci polynomials are

$$(1.3)\qquad f_0(x) = 0,\ f_1(x) = 1,\ f_{n+2}(x) = xf_{n+1}(x) + f_n(x),$$

and we list the first few of these polynomials:

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= x \\ f_3(x) &= x^2 + 1 \\ f_4(x) &= x^3 + 2x \end{aligned}$$

$$\begin{aligned}
f_5(x) &= x^4 + 3x^2 + 1\\
f_6(x) &= x^5 + 4x^2 + 3x\\
f_7(x) &= x^6 + 5x^4 + 6x^2 + 1\\
f_8(x) &= x^7 + 6x^5 + 10x^3 + 4x\\
&\vdots
\end{aligned}$$

Once again, we see that the coefficients lie along the rising diagonals of Pascal's triangle.

It can be shown that [3], [4]

$$(1.4)\qquad \begin{aligned} b_n(x^2) &= f_{2n+1}(x)\\ xB_n(x^2) &= f_{2n+2}(x), \end{aligned}$$

and the fact that coefficients lie on the rising diagonals of Pascal's triangle follows from that property for the Fibonacci polynomials. The Fibonacci polynomials obey

$$(1.5)\qquad f_{n+4}(x) = (x^2 + 2)f_{n+2}(x) - f_n(x),$$

which agrees with (1.2) when x is replaced by x^2 throughout.

Next, we are interested in finding the greatest common divisor of a pair of Fibonacci polynomials.

Theorem 1.1: For Fibonacci polynomials,

$$(f_m(x),\ f_n(x)) = f_{(m,n)}(x).$$

Proof: Rewrite the recursion (1.3) for the Fibonacci polynomials,

$$f_{m+1}(x) - xf_m(x) = f_{m-1}(x),$$

and set $(f_m(x),\ f_{m+1}(x)) = d(x)$. Then, since $d(x)|f_m(x)$ and $d(x)|f_{m+1}(x)$, we must have $d(x)|f_{m-1}(x)$. In turn, $f_m(x) - xf_{m-1}(x) = f_{m-2}(x)$ implies that $d(x)|f_{m-2}(x)$, and, continuing, finally $d(x)|f_1(x) = 1$. Therefore, $d(x) = 1$, and Theorem 1.1 holds for $n = m + 1$, or,

$$(1.6)\qquad (f_m(x),\ f_{m+1}(x)) = 1.$$

From [5], we also have

$$(1.7)\qquad f_{p+r}(x) = f_{p-1}(x)f_r(x) + f_p(x)f_{r+1}(x),$$

and

$$(1.8)\qquad f_m(x)|f_n(x) \text{ if and only if } m|n.$$

Next, let $c = (m, n)$, and let $d(x) = (f_m(x),\ f_n(x))$. Since $c|m$ and $c|n$, by (1.8), $f_c(x)|f_m(x)$ and $f_c(x)|f_n(x)$ implies that $f_c(x)|d(x)$. Since $c = (m, n)$, by the Euclidean algorithm, there exist integers a and b such that $c = am + bn$. Since $c \le m$, $m, n > 0$, $a \le 0$ or $b \le 0$. Suppose $a \le 0$ and let $k = -a$. Then $bn = c + km$ applied to (1.7) gives

$$f_{bn}(x) = f_{c+km}(x) = f_{c-1}(x)f_{km}(x) + f_c(x)f_{km+1}(x).$$

By (1.8), $f_n(x)|f_{bn}(x)$ and $f_m(x)|f_{km}(x)$, and since $d(x)|f_n(x)$ and $d(x)|f_m(x)$, we have $d(x)|f_c(x)f_{km+1}(x)$. But $(f_{km}(x),\ f_{km+1}(x)) = 1$ by (1.6), which implies that $(d(x), f_{km+1}(x)) = 1$, and $d(x) | f_c(x)$. Also, since $f_c(x) | d(x)$, $d(x) = f_c(x)$, or $(f_m(x), f_n(x)) = f_{(m,n)}(x)$, concluding the proof, which is similar to that by Michael [6] for Fibonacci numbers. Also see [7] and [8].

2. POLYNOMIALS FROM A PROBLEM BY SCHECHTER

Next, we consider some polynomials arising from a problem by Schechter [1] and their relationships to the Fibonacci polynomials and the Morgan-Voyce polynomials. Consider the sequence defined by $S_1 = 1$, $S_2 = m$, and

$$(2.1)\qquad \begin{cases} S_k = mS_{k-1} + S_{k-2}, & k \text{ even}, \\ S_k = nS_{k-1} + S_{k-2}, & k \text{ odd}. \end{cases}$$

We now list the first few polynomials in m and n, and compare to the Morgan-Voyce polynomials.

$$\begin{aligned} S_1(m, n) & & &= b_0(mn) \\ S_2(m, n) &= m & &= mB_0(mn) \\ S_3(m, n) &= mn + 1 & &= b_1(mn) \\ S_4(m, n) &= m(mn + 2) & &= mB_1(mn) \\ S_5(m, n) &= (mn)^2 + 3mn + 1 & &= b_2(mn) \\ S_6(m, n) &= m[(mn)^2 + 4mn + 3] & &= mB_2(mn) \end{aligned}$$

Thus, it appears that

$$(2.2)\qquad \begin{cases} S_{2k+2}(m, n) = mB_k(mn), \\ S_{2k+1}(m, n) = b_k(mn). \end{cases}$$

Now, from (1.4), we have $mnB_k(m^2n^2) = f_{2k+2}(mn)$; thus,

$$(2.3)\qquad S_{2k+2}(m^2, n^2) = m^2B_k(m^2n^2) = \frac{m}{n}f_{2k+2}(mn).$$

For example, $S_4(m^2, n^2) = m^2(m^2n^2 + 2)$, $B_1(m^2n^2) = m^2n^2 + 2$, and $f_4(mn) = (mn)^3 + 2mn$, and we see that

$$\begin{aligned} S_4(m^2, n^2) &= m^2B_1(m^2n^2) = \frac{m}{n}(mn)(m^2n^2 + 2) \\ &= \frac{m}{n}(m^3n^3 + 2mn) = \frac{m}{n}f_4(mn). \end{aligned}$$

Next, we state and prove a matrix theorem in order to derive further results for the polynomials $S_k(m, n)$.

Theorem 2.1: Let $A = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$. Then,

$$(AB)^k = \begin{pmatrix} b_k(xy) & xB_{k-1}(xy) \\ yB_{k-1}(xy) & b_{k-1}(xy) \end{pmatrix},$$

where $b_k(x)$ and $B_k(x)$ are the Morgan-Voyce polynomials.

Proof:

$$(AB)^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_0(xy) & xB_{-1}(xy) \\ yB_{-1}(xy) & b_{-1}(xy) \end{pmatrix}$$

$$(AB)^1 = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} b_1(xy) & xB_0(xy) \\ yB_0(xy) & b_0(xy) \end{pmatrix}$$

Assume that $(AB)^k$ has the form of the theorem. Then,

$$(AB)(AB)^k = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix} \cdot \begin{pmatrix} b_k(xy) & xB_{k-1}(xy) \\ yB_{k-1}(xy) & b_{k-1}(xy) \end{pmatrix}$$

$$= \begin{pmatrix} xyb_k(xy) + xyB_{k-1}(xy) + b_k(xy) & x[(xy+1)B_{k-1}(xy) + b_{k-1}(xy)] \\ yb_k(xy) + yB_{k-1}(xy) & xyB_{k-1}(xy) + b_{k-1}(xy) \end{pmatrix}$$

$$= \begin{pmatrix} b_{k-1}(xy) & xB_k(xy) \\ yB_k(xy) & b_k(xy) \end{pmatrix},$$

by applying the mixed recurrences of (1.1), completing a proof by induction.

Now, returning to the matrices of Theorem 2.1, since the determinant of AB is 1, it follows that

$$b_k(xy)b_{k-1}(xy) - xyB_{k-1}^2 = 1. \tag{2.4}$$

Returning to the polynomials $S_k(m, n)$, we have also that

$$(AB)^k = \begin{pmatrix} S_{2k+1} & \frac{n}{m}S_{2k} \\ S_{2k} & S_{2k-1} \end{pmatrix},$$

so that, taking determinants,

$$S_{2k-1}S_{2k+1} - \frac{n}{m}S_{2k}^2 = 1. \tag{2.5}$$

The polynomials $S_k(m, n)$ are related to the Morgan-Voyce polynomials by

$$\begin{cases} S_{2k+1}(m, n) = b_k(mn), \\ \frac{n}{m}S_{2k}(m, n) = nB_{k-1}(mn), \\ S_{2k}(m, n) = mB_{k-1}(mn). \end{cases} \tag{2.6}$$

Since the polynomials $S_k(m, n)$, the Morgan-Voyce polynomials, and the Fibonacci polynomials are interrelated by (1.4) and (2.3), which can be rewritten as

$$\begin{cases} S_{2k+1}(m, n) = f_{2k+1}(\sqrt{mn}), \\ S_{2k}(m, n) = \frac{m}{\sqrt{mn}}f_{2k}(\sqrt{mn}), \end{cases} \tag{2.7}$$

and since the coefficients of the Fibonacci polynomials lie along the rising diagonals of Pascal's triangle, we can write the following theorem.

Theorem 2.2: The coefficients of $f_k(x)$, $b_k(x)$, $B_k(x)$, and $S_k(m, n)$ are all coefficients which lie along the rising diagonals of Pascal's triangle.

3. DIVISIBILITY PROPERTIES OF POLYNOMIALS IN PASCAL'S TRIANGLE

Using the relationships of §2, we can expand upon Theorem 1.1 to write a greatest common divisor property for Morgan-Voyce polynomials.

Theorem 3.1: For the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$,

(i) $(B_m(x), B_n(x)) = B_{(m+1,\, n+1)-1}(x)$,

(ii) $(b_m(x), b_n(x)) = b_{((2m+1,\, 2n+1)-1)/2}(x)$,

(iii) $(B_m(x), b_n(x)) = b_{((2m+2,\, 2n+1)-1)/2}(x)$.

Proof:

(i) $$x(B_m(x^2), B_n(x^2)) = (f_{2m+2}(x), f_{2n+2}(x)) = f_{2(m+1,\, n+1)}(x) = xB_{(m+1,\, n+1)-1}(x^2)$$

by applying (1.4), Theorem 1.1, and returning to (1.4). For $x \neq 0$, (i) is immediate by replacing x^2 with x after dividing both sides by x. If $x = 0$, $B_n = n + 1$, making (i) become $(m + 1, n + 1) = (m + 1, n + 1) - 1 + 1$.

Applying (1.4) and Theorem 1.1 to (ii),

$$(b_m(x^2), b_n(x^2)) = (f_{2m+1}(x), f_{2n+1}(x)) = f_{(2m+1,\, 2n+1)}(x) = f_{2k+1}(x)$$

since the greatest common divisor of $2m + 1$ and $2n + 1$ is odd. Thus,

$$(b_m(x^2), b_n(x^2)) = b_k(x^2)$$

by (1.4), where $2k + 1 = (2m + 1, 2n + 1)$, so that

$$k = ((2m + 1, 2n + 1) - 1)/2.$$

Replacing x^2 by x yields (ii).

Finally, we observe that $b_n(0) = 1$, so that $x \nmid b_n(x)$, and again use (1.4) and Theorem 1.1:

$$(B_m(x^2), b_n(x^2)) = (xB_m(x^2), b_n(x^2)) = (f_{2m+2}(x), f_{2n+1}(x)) = f_{(2m+2,\, 2n+1)}(x).$$

Next, set $(2m + 2, 2n + 1) = 2k + 1$, since it must be odd, and

$$(B_m(x^2), b_n(x^2)) = f_{2k+1}(x) = b_k(x^2)$$

where

$$k = ((2m + 2, 2n + 1) - 1)/2.$$

Replacing x^2 by x establishes (iii), finishing the proof of Theorem 3.1.

Returning to the polynomials $S_k(m, n)$, and using (2.7) with Theorem 1.1, gives us

Theorem 3.2: $(S_i(m, n), S_j(m, n)) = S_{(i,\, j)}(m, n)$.

Proof: If i and j are both odd, (2,7) and Theorem 1.1 give the above result immediately. If i and j are both even,

$$\begin{aligned}(S_i(m,\,n),\,S_j(m,\,n)) &= (S_{2k}(m,\,n),\,S_{2h}(m,\,n))\\ &= \left(\frac{m}{\sqrt{mn}}f_{2k}\,(\sqrt{mn}),\,\frac{m}{\sqrt{mn}}f_{2h}\,(\sqrt{mn})\right)\\ &= \frac{m}{\sqrt{mn}}(f_{2k}\,(\sqrt{mn}),\,f_{2h}\,(\sqrt{mn})) = \frac{m}{\sqrt{mn}}f_{2(k,\,h)}(\sqrt{mn})\\ &= S_{2(k,h)}\,(m,\,n) = S_{(2k,\,2h)}\,(m,\,n) = S_{(i,\,j)}(m,\,n).\end{aligned}$$

If i is odd and j is even, since $S_{2k+1}(m,\,n)$ always ends in the constant 1 so that $\sqrt{mn}\nmid S_{2k+1}(m,\,n)$, and since $f_{2k+1}(x)$ also ends in 1,

$$\begin{aligned}(S_i(m,\,n),\,S_j(m,\,n)) &= (S_{2k+1}(m,\,n),\,S_{2h}(m,\,n))\\ &= (S_{2k+1}(m,\,n),\,\sqrt{mn}S_{2h}(m,\,n))\\ &= (f_{2k+1}(\sqrt{mn}),\,mf_{2h}(\sqrt{mn})) = (f_{2k+1}(\sqrt{mn}),\,f_{2h}(\sqrt{mn}))\\ &= f_{(2k+1,\,2h)}(\sqrt{mn}) = S_{(2k+1,\,2h)}(m,\,n) = S_{(i,\,j)}(m,\,n),\end{aligned}$$

where we can again use (2.7) because $(2k+1,\,2h)$ is odd, concluding the proof of Theorem 3.2.

We quickly have divisibility properties for the polynomials $S_k(m,\,n)$.

Theorem 3.3: $S_i(m,\,n)\,|\,S_j(m,\,n)$ if and only if $i\,|\,j$.

Proof: If $i\,|\,j$, then $(i,\,j)=i$, and $S_i(m,\,n)\,|\,S_j(m,\,n)$ by Theorem 3.2. If $S_i(m,\,n)\,|\,S_j(m,\,n)$ with $i\nmid j$, then $f_i(x)\,|\,f_j(x)$ where $i\nmid j$, a contradiction of (1.8).

From all of this, we can also write divisibility properties for Morgan-Voyce polynomials.

Theorem 3.4: For the Morgan-Voyce polynomials,

$B_m(x)\,|\,B_n(x)$ if and only if $(m+1)\,|\,(n+1)$;

$b_m(x)\,|\,b_n(x)$ if and only if $(2m+1)\,|\,(2n+1)$;

$b_m(x)\,|\,B_n(x)$ if and only if $(2m+1)\,|\,(n+1)$.

Proof: $B_m(x)\,|\,B_n(x)$ if and only if $(B_m(x),\,B_n(x)) = B_m(x)$, but

$$(B_m(x),\,B_n(x)) = B_{(m+1,\,n+1)-1}(x)$$

by Theorem 3.1. Setting the subscripts equal, $m = (m+1,\,n+1)-1$, or, $m+1 = (m+1,\,n+1)$, which forces $(m+1)\,|\,(n+1)$. The case for $b_m(x)$ and $b_n(x)$ is entirely similar.

In the case of $b_m(x)$ and $B_n(x)$, $B_n(x)$ cannot divide $b_m(x)$ for $n>0$ because $b_m(x)$ always ends in the constant 1, while the constant for $B_n(x)$ is greater than 1, $n>0$. Since $b_m(x)\,|\,B_n(x)$ if and only if

$$(b_m(x),\,B_n(x)) = b_m(x),$$

and since

$$(b_m(x),\,B_n(x)) = b_{((2n+2,\,2m+1)-1)/2}(x)$$

by carefully rearranging (iii) in Theorem 3.1, equating the subscripts leads to

$$m = ((2n+2,\,2m+1)-1)/2,$$

or

$$2m+1 = (2m+1,\,2n+2).$$

Thus, $(2m + 1)|(2n + 2)$, but since $(2m + 1)$ is odd, we must have

$$(2m + 1)|(n + 1),$$

concluding the proof.

Returning to the greatest common divisor property of the Fibonacci polynomials, $(f_m(x),\ f_n(x)) = f_{(m,\,n)}(x)$, we make some observations from Theorem 3.1(i) regarding the Morgan-Voyce polynomials $B_n(x)$. From

$$(B_n(x),\ B_m(x)) = B_{(n+1,\,m+1)-1}(x),$$

it would follow that if $B_n^*(x) = B_{n-1}(x)$ and $B_m^*(x) = B_{m-1}(x)$, then

$$(3.1)\qquad (B_n^*(x),\ B_m^*(x)) = B_{(n,\,m)}^*$$

which sequence $\{B_n^*(x)\} = \{0,\ 1,\ x + 2,\ \ldots\}$ obeys

$$(3.2)\qquad B_n^*(x) = (x + 2)B_{n-1}^*(x) - B_{n-2}^*(x)$$

and is in fact the Fibonacci polynomial, so to speak, for the auxiliary polynomial $\lambda^2 - (x + 2)\lambda + 1 = 0$, since

$$B_n^*(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

where λ_1 and λ_2 are the roots. But (3.2) can also be expressed as

$$u_n = xu_{n-1} - u_{n-2}$$

where x is replaced by $(x + 2)$. Thus one set of polynomials with coefficients on diagonals of Pascal's triangle transforms into another set with the same property.

This property of transforming one set of polynomials whose coefficients are on diagonals of Pascal's triangle to another set of polynomials with coefficients also on diagonals of Pascal's triangle is shared by the Chebyshev polynomials $\{T_n(x)\}$ [9] of the first kind, defined by $T_0(x) = 1$, $T_1(x) = x$, and

$$(3.3)\qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

since

$$(3.4)\qquad T_n(T_m(x)) = T_m(T_n(x)) = T_{mn}(x).$$

The property (3.4) is easy to prove from the Binet form associated with the auxiliary polynomial

$$(3.5)\qquad \lambda^2 - 2x\lambda + 1 = 0,$$

with roots λ_1 and λ_2.

The Chebyshev polynomials $\{U_n(x)\}$ of the second kind are $U_0(x) = 1$, and $U_1(x) = 2x$,

$$(3.6)\qquad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

First, to establish (3.4), we prove by induction that

$$(3.7)\qquad \begin{cases} \lambda_1^n = T_n(x) + \sqrt{(x^2 - 1)}\,U_{n-1}(x), \\ \lambda_2^n = T_n(x) - \sqrt{(x^2 - 1)}\,U_{n-1}(x). \end{cases}$$

We prove only one part, since the second part is entirely similar. Since, $U_{-1}(x) = 0$, and $T_0(x) = 1$, $\lambda_1^n = T_n(x) + \sqrt{(x^2 - 1)}\,U_{n-1}(x)$ for $n = 0$. Assume that $\lambda_1^k = T_k(x) + \sqrt{(x^2 - 1)}U_{k-1}(x)$ and $\lambda_1^{k+1} = T_{k+1}(x) + \sqrt{(x^2 - 1)}\,U\ (x)$. Then, by (3.5),

$$\lambda_1^{k+2} = 2x\lambda_1^{k+1} - \lambda_1^k = (2xT_{k+1}(x) - T_k(x)) + \sqrt{(x^2 - 1)}\,(2xU_{k+1}(x) - U_k(x))$$
$$= T_{k+2}(x) + \sqrt{(x^2 - 1)}U_{k+1}(x),$$

using (3.4) and (3.6), establishing the form of λ_1^n in (3.7) by mathematical induction.

Notice that, since $\lambda_1\lambda_2 = 1$, by multiplying the forms of λ_1^n and λ_2^n from (3.7), we can derive

(3.8) $$T_n^2(x) - 1 = (x^2 - 1)U_{n-1}^2(x).$$

Also, by adding in (3.7), we can establish

(3.9) $$T_n(x) = (\lambda_1^n + \lambda_2^n)/2.$$

Now, $\lambda_1(x) = x + \sqrt{x^2 - 1}$. Replace x by $T_m(x)$, and the root becomes

$$\lambda_1(T_m(x)) = T_m(x) + \sqrt{T_m^2(x) - 1},$$

satisfying the auxiliary polynomial (3.5), so that

$$\lambda_1^2(T_m(x)) - 2T_m(x)\lambda_1(T_m(x)) + 1 = 0.$$

That is,

$$T_m(x) = \frac{\lambda_1^2(T_m(x)) + 1}{2\lambda_1(T_m(x))} = [\lambda_1(T_m(x)) + 1/\lambda_1(T_m(x))]/2.$$

But $\lambda_1\lambda_2 = 1$, so

$$T_m(x) = [\lambda_1(T_m(x)) + \lambda_2(T_m(x))]/2.$$

Referring back to (3.9), we write

$$\lambda_1 = \lambda_1^m(T_m(x)) \quad\text{and}\quad \lambda_2^m = \lambda_2(T_m(x)).$$

Now,

$$T_{mn}(x) = [\lambda_1^{mn} + \lambda_2^{mn}]/2 = [(\lambda_1^m)^n + (\lambda_2^m)^n]/2 = [\lambda_1^n(T_m(x)) + \lambda_2^n(T_m(x))]/2,$$

so that $T_{mn}(x) = T_n(T_m(x))$ and similarly, $T_{mn}(x) = T_m(T_n(x))$, finishing the proof of (3.4).

Returning to divisibility properties, observe that the Chebyshev polynomials of the second kind are the polynomials with the Fibonacci-like property

$$U_{n-1}(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

where λ_1 and λ_2 are the roots of $\lambda^2 - 2x\lambda + 1 = 0$. We now list the first few polynomials and let

$$U_n^*(x) = U_{n-1}(x).$$

$$\begin{aligned}
U_{-1}(x) &= 0 && = U_0^*(x)\\
U_0(x) &= 1 && = U_1^*(x)\\
U_1(x) &= 2x && = U_2^*(x)\\
U_2(x) &= 4x^2 - 1 && = U_3^*(x)\\
U_3(x) &= 8x^3 - 4x = 4x(2x^2 - 1) && = U_4^*(x)\\
U_4(x) &= 16x^4 - 12x^2 + 1 && = U_5^*(x)
\end{aligned}$$

$$\begin{aligned}
U_5(x) &= 32x^5 - 32x^3 + 6x = 2x(8x^4 - 8x^2 + 3) &&= U_6^*(x)\\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1 &&= U_7^*(x)\\
&\vdots &&\vdots
\end{aligned}$$

It would appear that

$$U_m^*(x),\ U_n^*(x) = U_{(m,n)}^*(x). \tag{3.10}$$

That this is indeed the case can be established very simply. Since $U_n^*(x)$ satisfies

$$U_{n+1}^*(x) = 2xU_n^*(x) - U_{n-1}^*(x),$$

$\{U_n(x)\}$ is a special case of the polynomial sequence $\{U_n(x, y)\}$ defined by Hoggatt and Long [7] as

$$U_{n+2}(x, y) = xU_{n+1}(x, y) + yU_n(x, y), \tag{3.11}$$

where $U_0(x, y) = 0$ and $U_1(x, y) = 1$. Note that $\{U_n^*(x)\}$ is the special case $x = 2x$ and $y = -1$. Since

$$\big(U_m(x, y),\ U_n(x, y)\big) = U_{(m,n)}(x, y), \tag{3.12}$$

we see that (3.10) is immediate.

We summarize as

Theorem 3.4: By suitable shifting of subscripts in the original definitions, the Fibonacci Polynomials, the Morgan-Voyce polynomials $B_n(x)$, the Chebyshev polynomials $U_n(x)$, and the polynomials $S_k(m, n)$ all satisfy

$$(u_m,\ u_n) = u_{(m,n)}.$$

4. A MORE GENERAL POLYNOMIAL SEQUENCE

Define $S_k(a, b, c, d)$ by taking $S_1 = 1$, $S_2 = a$,

$$\begin{cases} S_k = aS_{k-1} + bS_{k-2}, & k \text{ even},\\ S_k = cS_{k-1} + dS_{k-2}, & k \text{ odd}. \end{cases} \tag{4.1}$$

Let $S_1^* = 1$, $S_2^* = c$, and define $S_k^*(a, b, c, d)$ by taking

$$\begin{cases} S_k^* = cS_{k-1}^* + dS_{k-2}^*, & k \text{ even},\\ S_k^* = aS_{k-1}^* + bS_{k-2}^*, & k \text{ odd}. \end{cases} \tag{4.2}$$

Let $K_0 = 0$, $K_1 = 1$, $K_n = (ac + b + d)K_{n-1} - bdK_{n-2}$.

Let $Q = \begin{pmatrix} a & b\\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c & d\\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ac + b & ad\\ c & d \end{pmatrix}$; then,

$$Q^k = \begin{pmatrix} S_{2k+1}^* & dS_{2k}\\ S_{2k}^* & dS_{2k-1} \end{pmatrix} = \begin{pmatrix} K_{k+1} - dK_k & daK_k\\ cK_k & d(K_k - bK_{k-1}) \end{pmatrix}$$

Now, $\{K_n\}$ is the "Fibonacci sequence,"

$$K_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

for the quadratic $\lambda^2 - (ac + b + d)^\lambda + bd = 0$, with roots λ_1, λ_2. Applying results [7] for $\{U_n(x, y)\}$ from (3.11) and (3.12) to $\{K_n\}$, we have immediately that

$$(K_m, K_n) = K_{(m,n)}.$$

To continue, we write the first few terms of $\{S_k(a, b, c, d)\}$.

$$S_1 = 1$$
$$S_2 = a$$
$$S_3 = ac + d$$
$$S_4 = a^2c + ad + ab$$
$$S_5 = a^2c^2 + 2acd + abc + d^2$$
$$S_6 = a^3c^2 + 2a^2cd + 2a^2bc + ad^2 + abd + ab^2$$
$$S_7 = a^3c^3 + 3a^2c^2d + 2a^2bc^2 + 3acd^2 + 2abcd + ab^2c + d^3$$

We consider some special cases. If $a = 0$, then $S_{2k+2} = 0$, and $S_{2k+1} = d^k$, $k \geq 0$. If $b = 0$, $S_{2k+2} = a(ac + d)^k$ and $S_{2k+1} = (ac + d)^k$, $k \geq 0$. If $c = 0$, then $S_{2k-1} = d^{k-1}$ and $S_{2k} = a[(d^k - b^k)/(d - b)]$, $k \geq 1$. If $d = 0$, then $S_{2k} = a(ac + b)^{k-1}$ and $S_{2k+1} = ac(ac + b)^{k-1}$, $k \geq 1$. The expansions of $S_k^*(a, b, c, d)$ are not very interesting, since they are the same as those of $S_k(a, b, c, d)$ with the roles of a and c exchanged.

The special case of $S_k(a, b, c, d)$ where $b = d$ proves fruitful. We list the first few terms of $\{S_k(a, b, c)\}$ below:

$$S_1 = 1$$
$$S_2 = a$$
$$S_3 = ac + b$$
$$S_4 = a^2c + 2ab$$
$$S_5 = a^2c^2 + 3abc + b^2$$
$$S_6 = a^3c^2 + 4a^2bc + 3ab^2 = a(ac + b)(ac + 3b) = S_2S_3(ac + 3b)$$

We are interested in the case $b = d$, or, taking $S_k(a, b, c)$ and $S_k^*(a, b, c)$, so that S_3 will divide S_6. It is not difficult to prove by induction that

$$S_{2k+j} = S_{j+1}^*S_{2k} + bS_jS_{2k-1}, \tag{4.3}$$

$$S_{2k+1+j} = S_{j+1}S_{2k+1} + bS^*S_{2k}. \tag{4.4}$$

It is not hard to see that

$$S_{2k+1} = S_{2k+1}^* \quad \text{and} \quad aS_{2k} = cS_{2k}^*. \tag{4.5}$$

We now prove $S_j | S_{jm}$ for j odd and m odd, or, $jm = 2k + 1$. From (4.4),

$$S_{j(m+1)} = S_{j+1}S_{jm} + bS_j^*S_{2k} = S_{j+1}S_{jm} + bS_jS_{2k},$$

since $S_j = S_j^*$ for j odd. So, if $S_j | S_j$ and $S_j | S_{jm}$, then $S_j | S_{j(m+1)}$ for j odd. Thus, for j and m both odd, we see that $S_j | S_{jm}$ for all odd m.

Next, suppose that j is odd and m is even; then, from (4.3),

$$S_{2m'j+j} = S_{j+1}^*S_{2m'j} + bS_jS_{2m'j-1}, \quad m = 2m'.$$

Now, if $S_j | S_j$ and $S_j | S_{2m'j}$, then $S_j | S_{(2m'+1)j} = S_{j(m+1)}$.

Next, let j be even;

$$S_{2k+2j} = S^*_{2j'+1} S_{2k} + bS_{2j'} S_{2k-1} \quad \text{and} \quad 2k = 2j'm.$$

Since $S_j | S_{2j'}$ and $S_j | S_{2j'm} = S_{2k}$, we have $S_j | S_{2j'm+2j'} = S_{j(m+1)}$. This completes the proof that if $i|j$, then $S_i | S_j$. Since, algebraically, $\{S_i\}$ are of increasing degree in the two variables a and c collectively, $S_j \nmid S_i$ for $i < j$. Last, using (4.3) and (4.4), it is now straightforward to show

Theorem 4.1: $S_j(a, b, c) | S_i(a, b, c)$ if and only if $j|i$.

We can also now prove

Theorem 4.2: $(S_i(a, b, c), S_j(a, b, c)) = S_{(i,j)}(a, b, c)$.

Proof: Let $P(x)$ be a monic polynomial of degree $r + s$ with integral coefficients with two factors $Q(x)$ and $R(x)$ of degree r and s, respectively. Then,

$$b^{r+s}P(x/b) = b^r Q(x/b) b^s R(x/b)$$

$$P^*(x, b) = Q^*(x, b) R^*(x, b).$$

In particular, if $P(x)$ is of degree p, $T(x)$ of degree t, $W(x)$ of degree w, and $(P(x), T(x)) = W(x)$, then

$$(b^p P(x/b), b^t T(x/b)) = b^w W(x/b).$$

For application to Theorem 4.2:

$$c^2 S_{2m}(a^2, b^2, c^2) = acb^{2m-1} f_{2m}(ac/b); \tag{4.6}$$

$$S_{2m+1}(a^2, b^2, c^2) = b^{2m} f_{2m+1}(ac/b). \tag{4.7}$$

Case 1: Both subscripts even.

$$\begin{aligned}
&(c^2 S_{2m}(a^2, b^2, c^2), c^2 S_{2n}(a^2, b^2, c^2)) \\
&= (acb^{2m-1} f_{2m}(ac/b), acb^{2n-1} f_{2n}(ac/b)) \\
&= acb^{(2m, 2n)-1} f_{(2m, 2n)}(ac/b) \\
&= c^2 S_{(2m, 2n)}(a^2, b^2, c^2).
\end{aligned}$$

Therefore,

$$(S_{2m}(a^2, b^2, c^2), S_{2n}(a^2, b^2, c^2)) = S_{(2m, 2n)}(a^2, b^2, c^2).$$

Case 2: Both subscripts odd.

$$\begin{aligned}
&(S_{2m+1}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\
&= (b^{2m} f_{2m+1}(ac/b), b^{2n} f_{2n+1}(ac/b)) \\
&= b^{(2m+1, 2n+1)-1} f_{(2m+1, 2n+1)}(ac/b) \\
&= S_{(2m+1, 2n+1)}(a^2, b^2, c^2).
\end{aligned}$$

Case 3: One subscript odd, one subscript even.

$$\begin{aligned}
&(c^2 S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\
&= (acb^{2m-1} f_{2m}(ac/b), b^{2n} f_{2n+1}(ac/b)) \\
&= b^{(2m, 2n+1)-1} f_{(2m, 2n+1)}(ac/b) \\
&= S_{(2m, 2n+1)}(a^2, b^2, c^2),
\end{aligned}$$

since $(ac, b) = 1$. Also, since $(c^2, S_{2n+1}) = 1$,

$$\begin{aligned}
&(c^2 S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\
&= (S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\
&= S_{(2m, 2n+1)}(a^2, b^2, c^2),
\end{aligned}$$

finishing the proof of Theorem 4.2 by replacing a^2 with a, b^2 with b, and c^2 with c.

Let $f_n^*(x)$ be a modified Fibonacci polynomial, with

$$\begin{cases} f_n^*(x) = f_n(x), & n \text{ odd}, \\ f_n^*(x) = \dfrac{f_n(x)}{x}, & n \text{ even}. \end{cases}$$

Listing the first few values,

$$\begin{aligned}
f_1^*(x) &= 1 \\
f_2^*(x) &= 1 \\
f_3^*(x) &= x^2 + 1 \\
f_4^*(x) &= x^2 + 2 \\
f_5^*(x) &= x^4 + 3x^2 + 1 \\
f_6^*(x) &= x^4 + 4x^2 + 3 \\
f_7^*(x) &= x^6 + 5x^4 + 6x^2 + 1 \\
f_8^*(x) &= x^6 + 6x^4 + 10x^2 + 4.
\end{aligned}$$

Here,

$$\begin{cases} f_{n+2}^*(x) = f_{n+1}^*(x) + f_n^*(x), & n \text{ even}, \\ f_{n+2}^*(x) = x^2 f_{n+1}^*(x) + f_n^*(x), & n \text{ odd}. \end{cases}$$

This is $\{S_k(a, b, c, d)\}$ with $a = b = d = 1$, $c = x^2$. Thus, by Theorem 4.2,

$$(f_m^*(x), f_n^*(x)) = f_{(m,n)}^*(x).$$

Let $v_k(x)$ be a modified Morgan-Voyce polynomial defined by

$$v_{2n+2}(x) = B_n(x), \quad v_{2n+1}(x) = b_n(x).$$

The first few values for $\{v_k(x)\}$ are

$$\begin{aligned}
v_1(x) &= 1 &&= b_0(x) \\
v_2(x) &= 1 &&= B_0(x) \\
v_3(x) &= x + 1 &&= b_1(x) \\
v_4(x) &= x + 3 &&= B_1(x) \\
v_5(x) &= x^2 + 3x + 1 &&= b_2(x) \\
v_6(x) &= x^2 + 4x + 3 &&= B_2(x) \\
v_7(x) &= x^3 + 5x^2 + 6x + 1 &&= b_3(x) \\
v_8(x) &= x^3 + 6x^2 + 10x + 4 = (x+2)(x^2+4x+2) &&= B_3(x)
\end{aligned}$$

Since $v_k(x)$ satisfies

$$\begin{cases} v_n(x) = v_{n-1}(x) + v_{n-2}(x), & n \text{ even}, \\ v_n(x) = xv_{n-1}(x) + v_{n-2}(x), & n \text{ odd}, \end{cases}$$

this is $\{S_k(a, b, c, d)\}$ with $a = b = d = 1$ and $c = x$. Then, by Theorem 4.2,

$$(v_n(x), v_m(x)) = v_{(m,n)}(x).$$

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THE GOLDEN SECTION IN THE EARLIEST NOTATED WESTERN MUSIC

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The persistent use of the golden section as a proportion in Western Art is well recognized. Architecture, the visual arts, sculpture, drama, and poetry provide examples of its use from ancient Greece to the present day. No similar persistence has been established in music. One possible reason is that what ancient Greek music has survived is of such a fragmentary nature that it is not possible to make reliable musical deductions from it. However, beginning with the early Middle Ages a large body of music has survived in manuscripts that from ca. 10th century can be read and the music can be performed. This body of music is known as Roman liturgical chant or, more commonly, as Gregorian chant. These chants have not previously been analyzed from the standpoint of the golden section. Acknowledging the probability of the presence of a number of structural designs and proportions in these chants, it is the author's intention to establish the musical use of the golden section as an organizing principle in them.

The official collection of Roman liturgical chant is the *Liber Usualis*. The chants selected for the present study are "Kyrie" chants of which there are 30 in the collection. The chants span at least 600 years, having been written beginning with the 10th century.

The basic structure of a "Kyrie" is determined by the text, as shown in Diagram 1:

Diagram 1

unit	*Kyrie eleison.*	section	total chant
	Kyrie eleison.		
	Kyrie eleison.		
unit	*Christe eleison.*		
	Christe eleison.		
	Christe eleison.		
unit	*Kyrie eleison.*		
	Kyrie eleison.		
	Kyrie eleison.		

Each chant falls into nine separate sections. The three repetitions of the sections form three larger units which, in turn, make up the complete chant. While there is considerable variety in the melodic treatment of the text, the text itself had remained constant in the above form since ca. 900.

The actual nature of the rhythm of these chants is still open to question. Because music is a time art, any analysis that does not account for the proportional movement of the pitches in time cannot pretend to be a statement about the total nature of the music. In this sense, the following findings, though factual, remain theoretical to the degree that while pitches in succession imply time, exact temporal proportions are not deducible from that succession alone. In addition, the reader should be advised that there are more than 200 "Kyrie" melodies known to exist. In this light, the chants analyzed for this study represent a sampling of the repertory.

METHOD OF ANALYSIS

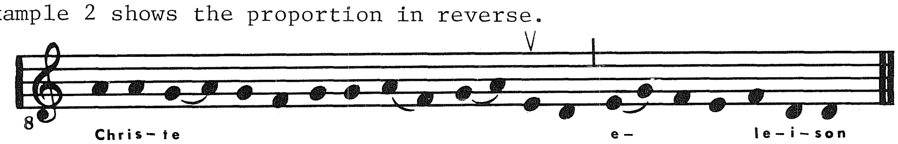
Because different treatments of the same text are usually set to different pitches, 146 distinct musical sections are present in the 30 chants, the remaining being exact repetitions of other sections. The pitches in each section were totaled, and ϕ was determined for each section. A section was examined to determine if any significant musical event occurred at either the major or minor mean. A significant event was defined as the beginning or ending of a musical phrase. The three statements of the "Kyrie," the "Christe," and the "Kyrie" tend to form larger units; these were analyzed according to the same procedure. Finally, the pitches in the complete chant were totaled, i.e., nine separate sections of text.

THE FINDINGS

Applying the analytical method described above revealed the presence of the golden section in 105 of the 146 individual sections of the "Kyries" in the *Liber Usualis*. These 105 sections make up .72 of the cases. The major mean precedes the minor mean twice as often as the minor mean precedes the major mean. Example 1 is a section of chant conforming to the M:m proportion.

Example 1*

Example 2 shows the proportion in reverse.



Example 2*

Twenty-one sections have phrase divisions occurring at the arithmetic mean.

The same method was applied to the next larger formal unit, i.e., the three repetitions of each exclamation. In 30 chants there are 90 such units. ϕ is found in 53 (.59) of these units. Where the musical phrase either falls short of the exact mean or extends beyond it, a tolerance of .02 of the total number of pitches was maintained in defining the unit as a golden section.

A performance of an entire chant includes nine sections as shown in Diagram 1. An analysis of the 30 chants revealed that 20 (.66) exhibit the golden section proportion. In more than half of the cases, the mean occurs at the end of the first or at the beginning of the second "Christe eleison."

CONCLUSION

At this stage, these findings tend to establish the presence of the golden section in one of the earliest notated forms of Western music, i.e., the "Kyrie" chants. To establish the presence of the golden section in chants other than the "Kyrie," requires further analysis of the general body of Gregorian chant.

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ON FIBONACCI NUMBERS WHICH ARE POWERS

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INTRODUCTION

Let $F(n)$, $L(n)$ denote the nth Fibonacci and Lucas numbers, respectively. (This slightly unconventional notation is used to avoid the need for second-order subscripts.) Consider the equation

(0) $$F(m) = c^p,$$

*Source: *Liber Usualis* (Desclee & Co., Tournai [Belb.], 1953), p. 25.

where p is prime and $m > 2$, so that $c > 1$. (The restriction on m eliminates from consideration the trivial solutions which arise because $c^p = c$ if $c = 1$, $c = 0$, or $c = -1$ and p is odd.)

The complete solution of (0) was given for $p = 2$ by J. H. E. Cohn [1] and by O. Wyler [4], and for $p = 3$ by H. London and R. Finkelstein [3]. In this article, we consider (0) for $p \geq 5$. It follows from Theorem 1 that if a nontrivial solution exists, then one exists such that m is odd. In Theorem 2, we give some necessary conditions for the existence of such a solution.

PRELIMINARIES

We will need the following definitions and formulas; r, s denote odd integers such that $(r, s) = 1$.

Definition 1: If q is a prime, then $z(q)$ is the Fibonacci entry point of q, i.e., $z(q) = \min\{m: q|F(m)\}$.

Definition 2: If q is a prime, then $y(q)$ is the least prime divisor of $z(q)$.

(1) If $(x, y) = 1$ and $xy = z^n$, then $x = u^n$ and $y = v^n$, where $(u, v) = 1$ and $uv = z$.

(2) $F(2n) = F(n)L(n)$.

(3) $\big(F(n), L(n)\big) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$

(4) $F(n) = 2r \leftrightarrow n \equiv 3 \pmod 6 \leftrightarrow L(n) = 4s$.

(5) If $(x, y) = 1 < x$, and $x^m y = z^n$, then $n|m$.

(6) $F(n) = 2^k r,\ k > 1 \leftrightarrow k \geq 3,\ 3*2^{k-2}|n \leftrightarrow L(n) = 2s$.

(7) $2|F(n) \leftrightarrow 3|n$.

(8) $3|F(n) \leftrightarrow 4|n$.

(9) $\big(F(n), F(kn)/F(n)\big)|k$.

(10) t odd $\rightarrow \big(F(t), F(3t)/F(t)\big) = 1$.

(11) $t > 0 \rightarrow F(t) < F(6t)$.

(12) $q|F(m) \rightarrow z(q)|m$.

(13) $F(2n + 1) = F(n)^2 + F(n + 1)^2$.

(14) c, n odd $\rightarrow c^n \equiv c \pmod 8$.

Remarks: (1) through (8) and (11) through (14) are elementary and/or well-known; for proof of (9), see [2], Lemma 16; (10) follows from (8) and (9).

THE MAIN THEOREMS

For a given prime, p, let $m = m(p) > 2$ be the least integer such that, by assumption, (0) has a nontrivial solution. By inspection,

$$m(2) = 12 \quad \text{and} \quad m(3) = 6.$$

Theorem 1: If $m = 2n > 2$ is the least integer such that $F(m) = c^p$, where p is prime, then either (i) $m = 6$, $p = 3$, or (ii) $m = 12$, $p = 2$.

Proof:

Case 1—If $n \not\equiv 0 \pmod 3$, then by hypothesis, (1), (2), and (3), we have

$F(n) = b^p$. If $b > 1$, we have a contradiction, since $n < m$. If $b = 1$, then hypothesis $\rightarrow n = 2 \rightarrow m = 4 \rightarrow F(m) = 3$, a contradiction.

Case 2—If $n \equiv 3 \pmod 6$, then $(4) \rightarrow F(n) = 2r$, $L(n) = 4s$, with rs odd. Now hypothesis and $(2) \rightarrow F(m) = 8rs = c^p$, so that $(5) \rightarrow p|3 \rightarrow p = 3$. By [3], we must have $c = 2$, $n = 3$, $m = 6$.

Case 3—If $n \equiv 0 \pmod 6$, let $n = n_0 = 2^j 3^k t$, where $j, k \geq 1$ and $(6, t) = 1$. Let $n_i = 2^{-i} n_0$ for each i such that $1 \leq i \leq j$. Let $h_0 = n_j = 3^k t$, and let $h_i = 3^{-i} h_0$ for each i such that $1 \leq i \leq k$, so that $t = h_k$. By (6), we have $F(n) = 2^{2+j} r$, $L(n) = 2s$, where rs is odd and $(r, s) = 1$. Now hypothesis, (1), and (2) imply $r = r_0^p$, $s = s_0^p$, with $r_0 s_0$ odd and $(r_0, s_0) = 1$. Therefore, $F(n) = F(n_0) = 2^{2+j} r_0^p$, $L(n) = L(n_0) = 2s_0^p$, $r_0 s_0 = c$. Since $n_i = 2n_{i+1}$, we may repeat our reasoning to obtain $F(n_i) = F(n_{i+1}) L(n_{i+1}) = 2^{2+j-i} r_i^p$, $L(n_i) = 2s_i^p$ for $i = 0, 1, 2, \ldots, j - 1$. By (4) we have $F(h_0) = F(n_j) = 2r_j^p$, $L(n_j) = 4s_j^p$; moreover, $r_i s_i = r_{i-1}$ is odd and $(r_i, s_i) = 1$ for $i = 1, 2, 3, \ldots, j$. Now, let $r_j = u_0$, so that $F(h_0) = 2u_0^p$. We have $F(h_{i-1}) = F(h_i) {*} F(h_{i-1})/F(h_i)$ for $i = 1, 2, 3, \ldots, k$. By (7), (10), and (1), if $i < k$, we have $F(h_i) = 2u_i^p$, $F(h_{i-1})/F(h_i) = v_i^p$; if $i = k$, we have $F(t) = F(h_k) = u_k^p$, $F(h_{k-1})/F(h_k) = 2v_k^p$; moreover, $(u_i, v_i) = 1$ and $u_i v_i = u_{i-1}$ is odd for $i = 1, 2, 3, \ldots, k$.

But $(11) \rightarrow F(t) < F(6t) \leq F(n) < F(m) = c^p \rightarrow u_k = 1 \rightarrow t = 1$. If $k \geq 2$, then $F(h_{k-2})/F(h_{k-1}) = F(9)/F(3) = 17 = v_{k-1}^p \rightarrow p = 1$, a contradiction. Hence, $k = 1$, $h_0 = n_j = 3$. If $j \geq 2$, then $L(n_{j-2}) = L(12) = 322 = 2s_{j-2}^p \rightarrow s_{j-2}^p = 161 \rightarrow p = 1$, a contradiction. Therefore, $k = j = 1$, $n = 6$, $m = 12$, $p = 2$.

Corollary: If (0) has a nontrivial solution for $p \geq 5$, then it has a nontrivial solution such that m is odd.

Proof: The proof follows directly from Theorem 1.

Theorem 2: If $F(m) = c^p > 1$, where the prime $p \geq 5$, and m is odd, then either (i) $m \equiv \pm 1 \pmod{12}$ and $c \equiv 1 \pmod 8$ or (ii) $m \equiv \pm 5 \pmod{12}$ and $c \equiv 5$ (mod 8; furthermore, if q is any prime factor of c, then $y(q) \geq 5$, so that

$$q \in \{5, 13, 37, 73, 89, 97, 113, 149, 157, \ldots\}.$$

Proof: If $2|c$, then $2^p | c^p \rightarrow 2^p | F(m)$, so that by (6), $3{*}2^{p-2} | m$, contradicting hypothesis. Now c is odd, so that $F(m)$ is odd, and by (7), $3 \nmid m$. Therefore, $m \equiv \pm 1$ or $\pm 5 \pmod{12}$. If q is any prime factor of c, then

$$(12) \rightarrow y(q) | z(q) | m.$$

Since $(6, m) = 1$, we must have $y(q) \geq 5$.

Case 1—If $m = 12t \pm 1$, then $(13) \rightarrow F(m) = F(6t)^2 + F(6t \pm 1)^2 = c^p$. Now, $F(6t) \equiv 0 \pmod 8$ and $F(6t \pm 1)$ is odd, so $F(6t \pm 1)^2 \equiv 1 \pmod 8$. Therefore, $c^p \equiv 0 + 1 \equiv 1 \pmod 8$, and (14) implies $c \equiv 1 \pmod 8$.

Case 2—If $m = 12t \pm 5$, then $(13) \rightarrow F(m) = F(6t \pm 3)^2 + F(6t \pm 2)^2 = c^p$. Now, $F(6t \pm 3) \equiv 2 \pmod 8$ and $F(6t \pm 2)$ is odd, so $F(6t \pm 2)^2 \equiv 1 \pmod 8$. Therefore, $c^p \equiv 4 + 1 \equiv 5 \pmod 8$, and (14) implies $c \equiv 5 \pmod 8$.

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PRIMES, POWERS, AND PARTITIONS

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Elementary arguments are employed in this paper to give a characterization of the set of primes and to extend this set to a larger one whose elements are defined by a single property: we show that a positive integer is either a prime or a power of 2 if and only if such an integer cannot be expressed as a sum of at least three consecutive positive integers. This fact provides an easy sieve to isolate the primes (and if one prefers, the immediately recognizable powers of 2) less than or equal to any preassigned positive integer. We also describe the possible ways in which a given composite number may be expressed as a sum of at least three consecutive positive integers; such representations for the sake of brevity shall be termed σ-partitions of the given integers. Furthermore, "number" shall mean "positive integer" and the set of all these numbers will, as usual, be denoted by $\mathbb{N}$.

Lemma 1: An odd number m admits a σ-partition if and only if m is a composite number.

Proof ($\Rightarrow$): Let $m = n + (n + 1) + \cdots + (n + k)$, $n \in \mathbb{N}$, $k \geq 2$. Then,

$$m = \frac{k+1}{2}(2n + k).$$

If $k + 1$ is even, then $(k + 1)/2$ is an odd number ≥ 3, since $k \geq 3$ in this case; obviously, therefore, $2n + k$ is an odd number ≥ 5. Hence, m is a composite number. If $k + 1$ is odd, then k is even, and since $k \geq 2$, one must have that $2n + k$ is an even number ≥ 4. Since m is an odd number, it follows that $(2n + k)/2$ is an odd number > 2. The fact that $k + 1 \geq 3$ now shows that m is a composite number.

Proof ($\Leftarrow$): Consider an arbitrary factorization $m = kg$, $(3 \leq k \leq g)$. Then,

$$m = kg = \frac{k}{2}(2g) = \frac{k}{2}\left[2\left(g - \frac{k-1}{2}\right) + k - 1\right] = \frac{k}{2}(2a + k - 1)$$

where

$$a = g - \frac{k-1}{2} \in \mathbb{N}.$$

Hence, we have

$$m = \sum_{r=1}^{k} (a + r - 1),$$

which is a σ-partition, since $a \in \mathbb{N}$ and $k \geq 3$.

Corollary 1: An odd number is prime if and only if it admits no σ-partition.

Lemma 2: An even number m admits a σ-partition if and only if m is not a power of 2. (Cf. [1], p. 17.)

Proof ($\Rightarrow$): Let $m = n + (n + 1) + \cdots + (n + k)$; $n \in \mathbb{N}$, $k \geq 2$. Then,

$$m = \frac{k+1}{2}(2n + k).$$

If $m = 2^s$, then, since $k \geq 2$, we must have that $k + 1 = 2^t$, $t \geq 2$, and $2n + k = 2^u$, $u \geq 2$. This is a contradiction, since $k + 1 = 2^t$ would imply that k is odd, so that $2n + k$ would also be odd. Hence, we have that m is not a power of 2.

Proof ($\Leftarrow$): Suppose that m is not a power of 2. Set $m = 2^n v$, $n \geq 1$, v an odd number ≥ 3.

Case (*i*): $v < 2^n$. Write $k = v$ and $g = 2^n$. Then,

$$m = kg = \frac{k}{2}(2g) = \frac{k}{2}\left[2\left(g - \frac{k-1}{2}\right) + k - 1\right] = \frac{k}{2}(2a + k - 1),$$

where

$$a = g - \frac{k-1}{2} \in \mathbb{N}.$$

Thus we have

$$m = \sum_{r=1}^{k} (a + r - 1)$$

which is clearly a σ-partition.

Case (*ii*): $v > 2^n$. Write $k = 2^n$ and $g = v$. Now,

$$g \leq 2k - 1 \Rightarrow k \geq \frac{g+1}{2} \Rightarrow k > \frac{g-1}{2} \Rightarrow k - \frac{g-1}{2} \in \mathbb{N},$$

and we have

$$m = gk = \frac{g}{2}(2k) = \frac{g}{2}\left[2\left(k - \frac{g-1}{2}\right) + g - 1\right] = \frac{g}{2}(2a + g - 1)$$

where

$$a = k - \frac{g-1}{2}.$$

Hence,

$$m = \sum_{r=1}^{g} (a + r - 1),$$

a σ-partition. On the other hand,

$$g > 2k - 1 \Rightarrow \frac{g+1}{2} > k \Rightarrow \frac{g+1}{2} - k \in \mathbb{N},$$

and now

$$m = \frac{2k}{2}g = \frac{2k}{2}\left[2\left(\frac{g+1}{2} - k\right) + 2k - 1\right] = \sum_{r=1}^{2k} (a + r - 1)$$

where

$$a = \frac{g+1}{2} - k \in \mathbb{N}, \text{ and clearly } 2k \geq 4.$$

This completes the proof.

Corollary 2: An even number is a power of 2 if and only if it admits no σ-partition.

We now have a natural extension of the sequence of primes, defined by a single property, in the following direct consequence of our two corollaries.

Theorem 1: A number m is either a prime or a power of 2 if and only if m admits no σ-partition.

This theorem provides an easy sieve to isolate the set of primes (and immediately recognizable powers of 2) less than or equal to any preassigned number x of moderate size: one simply writes down the segment $1, 2, 3, \ldots, x$ and crosses out the σ-partitions less than or equal to x, starting with those with leading term $a = 1$, then those with $a = 2$, etc. The least upper bound for the set of these a's is the number $\left[\frac{x-3}{3}\right]$, for clearly $a \lesseqgtr \frac{x-3}{3}$ if and only if $3a + 3 \lesseqgtr x$, which is equivalent to $a + (a + 1) + (a + 2) \lesseqgtr x$.

A simple example of "sieving" out the primes, for instance with $x = 15$, shows that a given composite number may admit more than one σ-partition. Our next problem is to give an account of the different possible σ-partitions of a given composite number m. First, we deal with the case where m is an odd number. Consider an arbitrary factorization $m = kg$ $(3 \le k \le g)$. Corresponding with this factorization, one always has the σ-partition

$$(1) \qquad \sum_{r=1}^{k}\left[\left(g - \frac{k-1}{2}\right) + r - 1\right] \text{ of } m.$$

If $g < 2k + 1$, then $k - \frac{g-1}{2} \in \mathbb{N}$, and once again direct computation shows that

$$(2) \qquad \sum_{r=1}^{g}\left[\left(k - \frac{g-1}{2}\right) + r - 1\right]$$

is a σ-partition of m. Clearly, (2) coincides with the fixed partition (1) if and only if $k = g$, i.e., the case where m is a square and is factored as such.

If $g \ge 2k + 1$, then clearly $\frac{g+1}{2} - k \in \mathbb{N}$, and we obtain the σ-partition

$$(3) \qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2} - k\right) + r - 1\right] \text{ of } m.$$

The partitions (1) and (3), having different lengths, can never coincide.

Conversely, the indicated possible σ-partitions corresponding to the particular type of factorization of m are the only possible ones m can have. For, consider an arbitrary σ-partition

$$m = \sum_{r=1}^{n}(a + r - 1) = \frac{n}{2}(2a + n - 1), \; n \ge 3.$$

If n is even, then $n/2$ is an odd divisor of m and so is $2a + n - 1$. Clearly, $2a + n - 1 > n > n/2$ so that we may write $k = n/2$ and $g = 2a + n - 1$, $k < g$. In this notation, we have that $g = 2a + 2k - 1 \ge 2k + 1$, since $2a \ge 2$, and $a = (g + 1)/2 - k$. Hence, the given partition is of the form (3). If n is an odd number, we have that $2a + n - 1$ is even and that $w = (2a + n - 1)/2$ is an odd divisor of m. If $n \le w$, we put $k = n$ and $g = w$. Then, $2g = 2a + k - 1$, so that $a = g - (k - 1)/2$, and the given partition has the form (1). If $n > w$, we write $k = w$ and $g = n$. Then, $2k = 2a + g - 1$, so that $g = 2k + 1 - 2a < 2k + 1$, since $a \ge 1$, and $a = k - (g - 1)/2$. This shows that the given partition has the form (2).

Summarizing these observations, we obtain the following characterization of the σ-partitions of a given composite odd number.

Theorem 2: The σ-partitions of a composite odd number m are precisely those determined by the factorizations of the form $m = kg$ $(3 \le k \le g)$, namely

$$(1) \qquad \sum_{r=1}^{k}\left[\left(g - \frac{k-1}{2}\right) + r - 1\right];$$

and exactly one of

$$(2) \qquad \sum_{=1}^{g}\left[\left(k - \frac{g-1}{2}\right) + r - 1\right] \quad (\text{if } g < 2k + 1)$$

and

$$(\ ) \qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2}-k\right)+r-1\right] \quad (\text{if } g \geq 2k+1).$$

If $g < 2k + 1$, the two valid partitions (*1*) are (*2*) are different, except in the case where $g = k$, and if $g \geq 2k + 1$, the valid partitions (*1*) and (*3*) are always different.

Now consider any two different factorizations (if they exist) of m into two factors: $m = kg$ $(3 \leq k \leq g)$ and $m = k'g'$ $(3 \leq k' \leq g')$. Then, comparing the lengths of the resulting σ-partitions, one sees that every possible partition corresponding with $m = kg$ differs from every possible one corresponding with $m = k'g'$. Hence, if m admits t different factorizations of the form $m = kg$ $(3 \leq k \leq g)$, then m admits $2t$ different σ-partitions, except in the case where m is a square in which case the number is $2t - 1$.

Finally, we consider the nature and number of σ-partitions of even numbers other than powers of 2.

Theorem 3: Let m be an even number other than a power of 2. Then, there exists at least one factorization of the form $m = kg$ $(k < g)$, where one of the factors is an even number and the other an odd number ≥ 3. For each such factorization, exactly one of the following three conditions holds:

(*1*) k is even, g is odd and $g < 2k + 1$;
(*2*) k is even, g is odd and $g \geq 2k + 1$;
(*3*) k is odd, g is even;

and only that sum in the list

$$(1') \qquad \sum_{r=1}^{g}\left[\left(k-\frac{g-1}{2}\right)+r-1\right],$$

$$(2') \qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2}-k\right)+r-1\right],$$

$$(3') \qquad \sum_{r=1}^{k}\left[\left(g-\frac{k-1}{2}\right)+r-1\right],$$

which corresponds with the valid condition is a σ-partition of m. Finally, these are the only possible types of σ-partitions of m.

Proof: First we observe that m may be written in the form $m = 2^n u$, where $n \geq 1$ and u is an odd number ≥ 3. Now, consider an arbitrary factorization, $m = kg$ $(k < g)$, into an even and an odd factor. If k is even, then the possibility (*3*) is ruled out and the ordering axiom ensures that exactly one of (*1*) and/or (*2*) holds. If k is odd, then the first two possibilities are excluded and (*3*) obviously holds.

Concerning the next part of Theorem 3, we note that each of the indicated sums is of the form

$$\sum_{r=1}^{n}(a+r-1),$$

and that $n \geq 3$ and $a \in \mathbb{N}$, providing that the condition to which the particular sum corresponds holds. Moreover, in each of these cases the indicated summation results in the product kg. On the other hand, the validity of any given condition (i) clearly results in $a \notin \mathbb{N}$ in the sums (j'), $i \neq j$. This concludes this part of the proof.

Finally, we consider an arbitrary σ-partition

$$m = \sum_{r=1}^{n} (a + r - 1) = \frac{n}{2}(2a + n - 1).$$

If n is even, then $2a + n - 1$ is an odd divisor of m, and since m is even, one must have that n contains a factor 4. Since $2a + n - 1 > n > n/2$, we may write $k = n/2$ and $g = 2a + n - 1$. Then $m = kg$, $k < g$, k is even and g is odd. Moreover, $g > 2k$, so $g \geq 2k + 1$. Finally, from $g = 2a + 2k - 1$, we obtain

$$a = \frac{g + 1}{2} - k.$$

Hence, the given partition has the form ($2'$). If n is odd, then n is a divisor of m and $2a + n - 1$ is an even number. Since m is even, we must have that $(2a + n - 1)/2$ is an even divisor of m and we may write $(2a + n = 1)/2 = 2w$. Considering first the case where $n > 2w$ and using the notation $g = n$ and $k = 2w$, one easily checks that k and g satisfy the requirements of condition (*1*) and that the given partition has the form ($1'$). A similar straightforward analysis of the case $n < 2w$ shows that $k = n$ and $g = 2w$ satisfy condition (*3*) and that the given partition has the form ($3'$). This completes the proof.

In conclusion, we want to determine the number of different σ-partitions of an even number m other than a power of 2. We once again consider two different factorizations as specified in the theorem (if they exist):

(α) $m = kg$,
(β) $m = k'g'$.

Let condition (i) in the theorem be satisfied in (α), and let condition (j) be satisfied in (β). We consider two possibilities:

$i = j$: Here k and k' are both even or both odd. Since $k \neq k'$ and $g \neq g'$ we must have that the σ-partition (i') relative to (α) is different from the σ-partition (j') = (i') relative to (β).

$i \neq j$: Suppose that k and k' are both even. Then g and g' are both odd and of the resulting σ-partitions one is of the form ($1'$) and the other of the form ($2'$). Noting that one of the lengths here is odd and the other one even, we conclude that the two σ-partitions are different. Suppose now that one of the factors k and k' is even and the other one odd. Without loss of generality we may assume that k is even. Then $j = 3$ and the factorization (β) yields the σ-partition ($3'$) of odd length. If $i = 1$, then the equality of ($1'$) relative to (α) and ($3'$) relative to (β) would imply that $g = k'$, so that $k = g'$. This, however, would imply that $k > g$, a contradiction. Hence, we have that the two resulting σ-partitions are different in this case as well. If $i = 2$, then the partition ($2'$) relative to (α) has even length, while ($3'$) relative to (β) has odd length, so they do not coincide. Therefore, we may conclude that if m admits t factorizations $m = kg$ where one of the factors is an even number and the other one an odd number ≥ 3, then m admits t different σ-partitions.

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ON ODD PERFECT NUMBERS

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If $\sigma(n)$ denotes the sum of the positive divisors of a natural number n, and $\sigma(n) = 2n$, then n is said to be perfect. Elementary textbooks give a necessary and sufficient condition for an even number to be perfect, and to date 24 such numbers, 6, 28, 496, ..., have been found. (The 24th is

$$2^{19936}(2^{19937} - 1),$$

discovered by Bryant Tuckerman in 1971 and reported in the *Guiness Book of Records* [3]. The three preceding ones were given by Gillies [2].)

It is not known whether there are any odd perfect numbers, though many necessary conditions for their existence have been established. The most interesting of recent conditions are that such a number must have at least eight distinct prime factors (Hagis [4]) and must exceed 100^{200} (Buxton and Elmore [1]).

Suppose $p_1, \ldots, p_t$ are the distinct prime factors of an odd perfect number. In this note we will give a new and simple proof that

$$(1) \qquad \sum_{i=1}^{t} \frac{1}{p_i} < \log 2,$$

a result due to Suryanarayana [5], who also gave upper and lower bounds for

$$\sum_{i=1}^{t} \frac{1}{p_i}$$

when either or both of 3 and 5 are included in $\{p_1, \ldots, p_t\}$.

Most of these bounds were improved in a subsequent paper with Hagis [6], but no improvement was given for the upper bound in the case when both 3 and 5 are factors. We will prove here that in that case

$$\sum_{i=1}^{t} \frac{1}{p_i} < .673634,$$

the upper bound in [5] being .673770. We will also give a further improvement in the upper bound when 5 is a factor and 3 is not; namely,

$$\sum_{i=1}^{t} \frac{1}{p_i} < .677637,$$

the upper bound in [6] being .678036. (These are six-decimal-place approximations to the bounds obtained.)

We assume henceforth that n is an odd perfect number.

An old result, due to Euler, states that we may write

$$n = \prod_{i=1}^{t} p_i^{\alpha_i},$$

where $p_1, \ldots, p_t$ are distinct primes and $p_k \equiv \alpha_k \equiv 1 \pmod 4$ for just one k in $\{1, \ldots, t\}$ and $\alpha_i \equiv 0 \pmod 2$ when $i \neq k$. We will assume further that $p_1 < \ldots < p_t$, and later will commonly write $\alpha_{(r)}$ for α_i when $p_i = r$. The subscript k will always have the significance just given and Π' and Σ' will denote that $i = k$ is to be excluded from the product or sum.

We will need the well-known result

$$\frac{1}{2}(p_k + 1)\,|\,n, \tag{2}$$

which is easily proved (see [6]). It follows that

$$p_1 \le \frac{1}{2}(p_k + 1). \tag{3}$$

We also use the inequality

$$1 + x + x^2 > \exp\left(x + \frac{1}{4}x^2\right), \quad 0 < x \le \frac{1}{3}. \tag{4}$$

To prove this, note that

$$\begin{aligned}\exp\left(x + \frac{1}{4}x^2\right) - (1 + x + x^2) &= 1 + x + \frac{x^2}{4} + \frac{1}{2!}\left(x + \frac{x^2}{4}\right)^2 + \cdots - (1 + x + x^2)\\ &= -\frac{1}{4}x^2 + \frac{x^3}{4} + \frac{x^4}{32} + \frac{1}{3!}\left(x + \frac{x^2}{4}\right)^3 + \cdots,\end{aligned}$$

so we wish to prove that

$$\frac{x}{4} + \frac{x^2}{32} + \frac{1}{3!x^2}\left(x + \frac{x^2}{4}\right)^3 + \frac{1}{4!x^2}\left(x + \frac{x^2}{4}\right)^4 + \cdots < \frac{1}{4}, \quad 0 < x \le \frac{1}{3}.$$

Now,

$$\frac{x}{4} + \frac{x^2}{32} \le \frac{1}{12} + \frac{1}{288} < .09$$

and

$$\begin{aligned}&\frac{1}{3!x^2}\left(x + \frac{x^2}{4}\right)^3 + \frac{1}{4!x^2}\left(x + \frac{x^2}{4}\right)^4 + \cdots\\ &\qquad < \frac{1}{6x^2}\left(x + \frac{x^2}{4}\right)^3 1 + \left(x + \frac{x^2}{4}\right) + \left(x + \frac{x^2}{4}\right)^2 + \cdots\\ &\qquad \le \frac{1}{18}\left(\frac{13}{12}\right)^3 \frac{36}{23} < .12.\end{aligned}$$

Hence (4) is true. Other and better inequalities of this type can be established but the above is sufficient for our present purposes.

Now we prove (1). Since n is perfect,

$$2n = \sigma(n) = \prod_{i=1}^{t}(1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i})$$

so

$$2 = \prod_{i=1}^{t}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}\right)$$

By Euler's result, $\alpha_k \ge 1$ and $\alpha_i \ge 2$ $(i \ne k)$, so

$$2 \ge \left(1 + \frac{1}{p_k}\right)\prod_{i=1}^{t}{}'\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2}\right) > \left(1 + \frac{1}{p_k}\right)\prod_{i=1}^{t}{}'\exp\left(\frac{1}{p_i} + \frac{1}{4p_i^2}\right),$$

by (4). Hence,

$$\log 2 > \log\left(1 + \frac{1}{p_k}\right) + \sum_{i=1}^{t}{}'\left(\frac{1}{p_i} + \frac{1}{4p_i^2}\right)$$

$$> \frac{1}{p_k} - \frac{1}{2p_k^2} + {\sum_{i=1}^{t}}' \frac{1}{p_i} + \frac{1}{4}{\sum_{i=1}^{t}}' \frac{1}{p_i^2} > \sum_{i=1}^{t} \frac{1}{p_i} + \frac{1}{4p_1^2} - \frac{1}{2p_k^2}$$

$$\geq \sum_{i=1}^{t} \frac{1}{p_i} + \frac{1}{(p_k+1)^2} - \frac{1}{2p_k^2} > \sum_{i=1}^{t} \frac{1}{p_i}$$

using (3).

We end with the

Theorem: (i) If $15|n$, then

$$\sum_{i=1}^{t} \frac{1}{p_i} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{61} + \log \frac{2950753}{2815321} = a, \text{ say.}$$

(ii) If $5|n$ and $3\nmid n$, then

$$\sum_{i=1}^{t} \frac{1}{p_i} < \frac{1}{5} + \frac{1}{31} + \frac{1}{61} + \log \frac{293105}{190861} = b, \text{ say.}$$

Proof: The proofs consist of considering a number of cases which are mutually exclusive and exhaustive.

(i) We are given that $p_1 = 3$ and $p_2 = 5$. Suppose first that $\alpha_1 = 2$ and $\alpha_2 = 1$ (so that we are assuming, until the last paragraph of this proof, that $k = 2$). Since $\sigma(3^2) = 13$, we have $13|n$.

Suppose $\alpha_{(13)} = 2$, so that, since $\sigma(13^2) = 183 = 3 \cdot 61$, $61|n$. Since also $\sigma(5) = 6 = 2 \cdot 3$, we cannot have $\alpha_{(61)} = 2$, for $\sigma(61^2) = 3783 = 3 \cdot 13 \cdot 97$ and we would have $3^3|n$ (i.e., $\alpha_1 > 2$). Hence, $\alpha_{(61)} \geq 4$. Then, using a simple consequence of (4),

$$2 = \prod_{i=1}^{t}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}\right)$$
$$> \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{13} + \frac{1}{13^2}\right)\left(1 + \frac{1}{61} + \frac{1}{61^2}\right.$$
$$\left. + \frac{1}{61^3} + \frac{1}{61^4}\right) \times \prod_{\substack{i=3 \\ p_i \neq 13, 61}}^{t} \exp\left(\frac{1}{p_i}\right),$$

so, taking logarithms and rearranging,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log \frac{13}{9} - \log \frac{6}{5} - \log \frac{183}{169} - \log \frac{14076605}{13845841}$$
$$+ \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{61} = a.$$

If $\alpha_{(13)} \geq 4$, then we similarly obtain

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right) - \log\left(1 + \frac{1}{5}\right)$$
$$- \log\left(1 + \frac{1}{13} + \frac{1}{13^2} + \frac{1}{13^3} + \frac{1}{13^4}\right) + \frac{1}{3} + \frac{1}{5} + \frac{1}{13} < a.$$

Suppose now that $\alpha_1 \geq 4$ and $\alpha_2 = 1$. Then,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) - \log\left(1 + \frac{1}{5}\right) + \frac{1}{3} + \frac{1}{5} < a.$$

Next, suppose that $\alpha_2 \geq 5$. Then,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right)$$

$$- \log\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5}\right) + \frac{1}{3} + \frac{1}{5} < a.$$

Finally, suppose $k > 2$, so $\alpha_2 \geq 2$. Since $\alpha_k \geq 1$, we obtain, proceeding as above,

$$\log 2 > \log\left(1 + \frac{1}{p_k}\right) + \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right) + \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) + \sum_{i=3}^{t}{}' \frac{1}{p_i}$$

$$> \sum_{i=1}^{t} \frac{1}{p_i} + \log \frac{13}{9} + \log \frac{31}{25} - \frac{1}{3} - \frac{1}{5} - \frac{1}{2p_k^2}.$$

But $p_k \geq 13$ (though we can easily demonstrate that in fact $p_k \geq 17$), so,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log \frac{13}{9} - \log \frac{31}{25} + \frac{1}{3} + \frac{1}{5} + \frac{1}{338} < a.$$

This completes the proof of (i).

(ii) We are given that $p_1 = 5$. The details in the following are similar to those above. Suppose, until the last paragraph of this proof, that $\alpha_1 = 2$. Since $\sigma(5^2) = 31$, we have $31|n$. Now, $\sigma(31^2) = 993 = 3 \cdot 331$ and $3 \nmid n$, so we must have $\alpha_{(31)} \geq 4$. It follows from (2) and from the fact that $3 \nmid n$, that if $p_k < 73$, then p_k must be either 13, 37, or 61 (so we cannot have $\alpha_1 = 1$).

Suppose first that $p_k = 61$. Then $\alpha_{(61)} \geq 1$ and

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$

$$- \log\left(1 + \frac{1}{61}\right) + \frac{1}{5} + \frac{1}{31} + \frac{1}{61} = b.$$

If $p_k = 13$, then, by (2), $p_2 = 7$. $\sigma(7^2) = 57 = 3 \cdot 19$, so $\alpha_2 \geq 4$, since $3 \nmid n$. Also, $\alpha_{(13)} \geq 1$, so

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4}\right)$$

$$- \log\left(1 + \frac{1}{13}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$

$$+ \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{31} < b.$$

If $p_k = 37$, then, by (2), $19|n$. $\sigma(19^2) = 381 = 3 \cdot 127$, so $\alpha_{(19)} \geq 4$. Since $\alpha_k \geq 1$,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{19} + \frac{1}{19^2} + \frac{1}{19^3} + \frac{1}{19^4}\right)$$

$$- \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$

$$- \log\left(1 + \frac{1}{37}\right) + \frac{1}{5} + \frac{1}{19} + \frac{1}{31} + \frac{1}{37} < b.$$

If $p_k \geq 73$, then, as in the last paragraph of the proof of (i), we have

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right) + \frac{1}{5} + \frac{1}{31} + \frac{1}{2 \cdot 73^2} < b.$$

Finally, suppose $\alpha_1 \geq 4$. Then $p_k \geq 13$ and, as in the preceding paragraph,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) + \frac{1}{5} + \frac{1}{2 \cdot 13^2} < b.$$

This completes the proof of (ii).

I am grateful to Professor H. Halberstam for suggesting a simplification of this work through more explicit use of the inequality (4).

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A SIMPLE CONTINUED FRACTION REPRESENTS A MEDIANT NEST OF INTERVALS

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1. While working on some mathematical aspects of the botanical problem of phyllotaxis, I came upon a property of simple continued fractions that is simple, pretty, useful, and easy to prove, but seems to have been overlooked in the literature. I present it here in the hope that it will be of interest to people who have occasion to teach continued fractions. The property is stated below as a theorem after some necessary terms are defined.

2. *Terminology*: For any positive integer n, let $n/0$ represent ∞. Let us designate as a "fraction" any positive rational number, or 0, or ∞, in the form a/b, where a and b are nonnegative integers, and either a or b is not zero. We say the fraction is in lowest terms if $(a, b) = 1$. Thus, 0 in lowest terms is 0/1, and ∞ in lowest terms is 1/0.

If inequality of fractions is defined in the usual way, that is

$$a/b < c/d \text{ if } ad < bc,$$

it follows that $x < \infty$ for $x = 0$ or any positive rational number.

3. *The Mediant*: If a/b and c/d are fractions in lowest terms, and $a/b < c/d$, the mediant between a/b and c/d is defined as $(a + c)/(b + d)$. Note that $a/b < (a + c)/(b + d) < c/d$.

Examples—The mediant between 1/2 and 1/3 is 2/5. If n is a nonnegative integer, the mediant between n and ∞ is $n + 1$. If n is a nonnegative integer and m is a positive integer, the mediant between n and $n+1/m$ is $n+1/(m+1)$.

4. *A Mediant Nest*: A mediant nest is a nest of closed intervals $I_0, I_1, \ldots, I_n, \ldots$ defined inductively as follows:

$$I_0 = [0, \infty].$$

For $n \geq 0$, if $I_n = [r, s]$, then I_{n+1} = either $[r, m]$ or $[m, s]$, where m is the mediant between r and s.

It is easily shown that if at least one I_n for $n \geq 1$ has for form $[r, m]$, then the length of I_n approaches 0 as $n \to \infty$, so that such a mediant nest is truly a nest of intervals, and it determines a unique number x that is contained in every interval of the nest. For the case where every I_n for $n \geq 1$ has the form $[m, s]$, let us say that the nest determines and "contains" the number ∞. Mediant nests are obviously related to Farey sequences.

5. *Long Notation for a Mediant Nest*: A mediant nest and the number it determines can be represented by a sequence of bits $b_1 b_2 b_3 \ldots b_i \ldots$, where, for $i > 0$, if $I_{i-1} = [r, s]$ and m is the mediant between r and s, $b_i = 0$ if $I_i = [r, m]$, and $b_i = 1$ if $I_i = [m, s]$.

Examples—$\dot{0} = 0$; $\dot{1} = \infty$; $\dot{1}\dot{0} = \tau$, the golden section; where each of these three examples is periodic, and the recurrent bits are indicated by the dots above them.

6. *Abbreviated Notation for a Mediant Nest*: The sequence of bits representing a mediant nest is a sequence of clusters of ones and zeros,

$$b_1 b_2 b_3 \ldots b_i \ldots = \overbrace{1 \ldots 1}^{a_1}\overbrace{0 \ldots 0}^{a_2}\overbrace{1 \ldots 1}^{a_3} \ldots$$

where the a_i indicate the number of bits in each cluster; $0 \leq a_1 \leq \infty$; $0 < a \leq \infty$ for $n > 1$; and the sequence (a_i) terminates with a_n if $a_n = \infty$. As an abbreviated notation for a mediant nest and the number x that it determines we shall write $x = (a_1, a_2, \ldots)$. Then $a_1 \leq x < a_1 + 1$. The sequence (a_i) terminates if and only if x is rational or ∞. Every positive rational number is represented by exactly two terminating sequences (a_i).

Examples—$(\infty) = \dot{1} = \infty$; $(0, \infty) = \dot{0} = 0$; $(0, 2, \infty) = 00\dot{1} = \frac{1}{2}$; $(0, 1, 1, \infty) = 01\dot{0} = \frac{1}{2}$. In general, if $x = (a_1, \ldots, a_{n-1}, a_n, \infty)$ where $a_n > 1$, then $x = (a_1, \ldots, a_{n-1}, a_n - 1, 1, \infty)$, and vice versa.

7. *Theorem*: If $x = (a_1, a_2, \ldots, a_n, \ldots)$, then $x = a_1 + 1/a_2 + \cdots + 1/a_n + \cdots$ and conversely. If $x = (a_1, \ldots, a_n, \infty)$, then $x = a_1 + 1/a_2 + \cdots + 1/a_n$ and conversely.

Proof of the Theorem:

I. The nonterminating case, $x = (a_1, a_2, \ldots, a_i, \ldots)$. Thus, x is irrational. Let p_i/q_i, for $i \geq 1$, be the principal convergents of $a_1 + 1/a_2 + \ldots$. Then a straightforward proof by induction establishes that for all even $i \geq 2$,

$$I_{a_1 + \cdots + a_i} = \left[p_{i-1}/q_{i-1},\ p_i/q_i\right],$$

and for all odd $i \geq 1$,

$$I_{a_1+\cdots+a_i} = \left[p_i/q_i,\ p_{i-1}/q_{i-1}\right].$$

Consequently, the nest determined by successive pairs of consecutive principal convergents of $a_1 + 1/a_2 + \cdots + 1/a_n + \cdots$ defines the same number as the mediant nest $(a_1, a_2, \ldots, a_n, \ldots)$.

II. The terminating case, $x = (a_1, \ldots, a_n, \infty)$. It follows from I that

$$I_{a_1+\cdots+a_{n+1}} = \left[p_n/q_n,\ p_{n+1}/q_{n+1}\right] \text{ or } \left[p_{n+1}/q_{n+1},\ p_n/q_n\right],$$

where

$$p_{n+1}/q_{n+1} = (p_{n-1} + a_{n+1}p_n)/(q_{n-1} + a_{n+1}q_n).$$

Since

$$\lim_{a_{n+1}\to\infty} p_{n+1}/q_{n+1} = p_n/q_n,$$

it follows that

$$x = \lim_{a_{n+1}\to\infty} I_{a_1+\cdots+a_{n+1}} = p_n/q_n = a_1 + 1/a_2 + \cdots + 1/a_n.$$

III. The "conversely" in the theorem follows from the fact that the mapping of the set of mediant nests into the set of simple continued fractions established in I and II is one-to-one and onto.

Example—The mediant nest $(0, 2, 3, \infty)$ and the continued fraction $0 + 1/2 + 1/3$ represent the same number. Verification:

a. $(0, 2, 3, \infty)$ is the abbreviated notation for the sequence of bits $00111\dot{0}$.

The intervals I_n defined by this sequence of bits are:

Bit	*Interval*	*Mediant between Endpoints of Interval*
	$I_0 = [0/1, 1/0]$	$(0+1)/(1+0) = 1/1$
0	$I_1 = [0/1, 1/1]$	$(0+1)/(1+1) = 1/2$
0	$I_2 = [0/1, 1/2]$	$(0+1)/(1+2) = 1/3$
1	$I_3 = [1/3, 1/2]$	$(1+1)/(3+2) = 2/5$
1	$I_4 = [2/5, 1/2]$	$(2+1)/(5+2) = 3/7$
1	$I_5 = [3/7, 1/2]$	$(3+1)/(7+2) = 4/9$
0	$I_6 = [3/7, 4/9]$	$(3+4)/(7+9) = 7/16$
	$\vdots$	
0	$I_n = [3/7, m_{n-1}]$	$n \geq 6$, m_{n-1} = the mediant between the endpoints of I_{n-1}.

Since $\lim_{n\to\infty} m_{n-1} = 3/7$, the number defined by this mediant nest is 3/7.

b. The continued fraction

$$0 + \cfrac{1}{2 + \cfrac{1}{3}} = 3/7.$$

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FOLDED SEQUENCES AND BODE'S PROBLEM

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Readers of this journal have long been interested in Bode's Rule, see, e.g., [26] and [15]. Indeed attempts to solve it have ranged from those devoid of science but fairly accurate to those based on some physical principle(s) but rather inaccurate, such as Berlage's and O. Schmidt's theories. The problem of planetary motions was first tackled by Eudoxus, who proposed rotating tilted concentric spheres, rather like a gyroscope, to explain each planet. When Kepler solved the problem of their motions with the concept of areal velocity, the area swept out in an invariable plane divided by the time is a constant, the problem of a law for their spacing remained. Indeed, I think the unit of angular momentun should be named after Kepler for his contribution of area as a vector. Bode's problem is of great value to the history and especially the philosophy of science. The qualities that distinguish pure mathematics are succinctness, elegance, fertility, and relevance to the unsolved problems. But to a scientist the first criterion is reproducibility. The multifarious, variegated, and at times loquacious and mellifluous monographs on this aspect of cosmogony attest to man's persistent and insistent attempts, at times based on specious assumptions, to find order in a theatre of nature that may have no reason to be other than nearly random. Such is one view. But while science corrects its mistakes (so far) it must be remembered that Boltzmann committed suicide because his contemporaries would not accept his counting of molecules, Wagoner's 1911 theory of continental drift was not believed until the 'sixties, and Newton's theory was not believed on the continent until Clairaut's prediction of the return of Halley's comet in 1759 ($P = 76.75 \pm 1.5$ yr) came true. Such is the lag between prediction and proof. The final answer to Bode's problem will be known within twenty years when sophisticated computer simulations are finished. My work will probably remain the most accurate, namely 1-percent with a few exceptions either way. In any case, my work has led to some interesting mathematics, especially the Self-Lucas property (see Section 2). May it be that Urania and Euterpe have recessed a part of Nirvana to sequester all who have slaved over this vexing problem. The impetus for this paper comes from Kowal's [17] recent discovery of an object between Saturn and Uranus that *prima faciae*, see Table 1, fits my rule [6, 7] and does not fit any other rule published! Sequentially, I present an overview of the history of Bode's Rule, Kowal's discovery and then generalized folded sequences.

1. BODE'S PROBLEM AND KOWAL'S DISCOVERY

Historically, the first offered solution to Bode's problem was Kepler's [1] perfect solids, which model, in fact, antedates Bode by two centuries. Gingerich [11] has discussed the accuracy of Kepler's youthful proposal. At first, the Titius-Bode mnemonic was successful with the Asteroids and Uranus but it fails badly for Neptune and Mercury. Attempting to save it, Miss Blagg [3] introduced two more parameters into it. Nieto, see [9] of paper [7], supports her work. The literature is full of algebraic rules of the form, distance $\propto b^n$ and indeed of more complicated rules. It is interesting to look at the range of b values. A partial list is:

Dermott	$b = 2^{1/3} = 1.2599$ (Saturnian moons)
von Weizsacker	$b = 1.370889$ (10 eddies, inner planets),
Greig	$b = 1.378241$ (Saturn's inner moons)
Quadranacci	$b = 1.380$ (see below)
Dermott	$b = 3^{1/3} = 1.4422$ (Uranian moons)
Cale	$b = \phi/\sqrt{3} = 1.5115$ (see [15])
Pierucci & Dermott	$b = 4^{1/3} = 1.5874$ (Jovian moons)
Gaussin	$b = 1.72$
Blagg	$b = 1.73$
Dermott	$b = 6^{1/3} = 1.817121$
Belot	$b = 1.886$
von Weizsacker	$b = 1.894427$
Greig	$b = 1.8995476$
Quadranacci	$b = 1.905$ (see below)
Titius	$b = 2.00$

See Gould [15] for references I have not cited. Dermott (see [2] and [10] of paper [7]) was forced to take Earth and Venus together to retain his period factor of $\sqrt{6}$. Dermott's arbitrary period factor of $\sqrt{2}$ for Saturn's moons misses both Rhea and Janus.

Indeed, I have myself happened upon some rather well-fitting arbitrary rules. One such is a bisection of the Quadranacci recurrence $Q_{n+1} = Q_n + Q_{n-3}$, namely: 1, 1, 1, 1, 2, N = 3, 4, U = 5, 7, A = 10, 14, J = 19, ..., E = 95, ..., which is very good at representing reciprocal distances. Another one for the distances begins: 4, 7, 10, 15, ... The distance factor, b, is $(1.380278)^2 = 1.905166$. The most complex rule of which I am aware is Rothman's (see [9]), $d = n(5.5 + F_n^2)/9(1 + F_n)$. It has seven parameters: n, 5.5, 2, 9, and 1, and two to determine the Fibonacci sequence, and since only 9 planets are fitted then only 2 degrees of freedom are left which is unscientific. All of the above rules are arbitrary, except von Weizsacker's and my own. My own view is that any rule with more than *two* parameters violates Occkham's razor, "*Essentia non sunt multiplicanda praeter necessitatem.*"

Dermott (*ibid.*) proposed different period factors for each satellite system, whereas my theory is simpler since the same limiting ratio, ϕ^2, applies to all. He also ignored the outer Jovian and Saturnian moons. I chose to emphasize them. This suggests the principle of Contrary Ignorability: whatever earlier researchers ignore—that is the path to pursue. My work indicates that outer Jovian moons should cluster well within 10 percent of 97, 257, 730, and 608 days (Table 2 of [6]). The announcement of Jupiter's XIII moon [21] at 239 day and $i = 27°$ [25] came after my initial work [4], [22] and satisfies the above sequence. I also studied the relevance, if any, of rotation periods and grazing periods of parent bodies in [4], using:

$$(35) \qquad P_g^2 \rho_M = 3\pi/G \quad \text{where} \quad G = 498 \text{ day}^{-2}(\text{g/cc})^{-1}.$$

The period of a satellite just grazing the surface of Saturn, the Sun, Jupiter, and Uranus would be 0.167, 0.116, 0.12, and 0.11 day. The criterion for coalescence against tidal forces may be written

$$(36) \qquad \rho_m > fM/\text{distance}^3 \quad \text{or} \quad P_m > P_g\sqrt{(4\pi f \rho_M / 3\rho_m)}$$

where m, M are the satellite and parent masses, ρ_m and ρ_M their densities, P_m the period of a satellite at this Roche limit, and f a factor between 2 and 10 [24, p. 18]. When (36) is not satisfied "rings" result. In [4] an inner Uranian moon was suggested which would have a period of 1/1.3292 = 0.752 day according to (17) of [6]. It has even been proposed that the separation, a, between binary stars satisfies Bode's rule [27], but I am very skeptical of that.

My own interest in Fibonacci numbers dates back at least to 1966 when I obtained a copy of Vorobyev's book. I tried these numbers on the planets with what seemed good accuracy and communicated this to Gould [13], pointing out the relevance of bisected Fibonacci sequences. After long arduous efforts, I thought I had put the problem to rest when news of Kowal's discovery [17] of a planetoid between Saturn and Uranus, too big to be a comet nucleus and too small to be a large planet, was announced. He calls the object Chiron after one of the Greek half-man/half-horse animals. Is this discovery to prognosticate that this Chinese year 4676 (see [18]), beginning 7 Feb. 1978, should not be the year of Earth-Horse, but rather the year of the Centaur!? One can see in Table 1 that Chiron fits very neatly into the bisected half-integer sequence. I could have predicted this object three years ago [5, 6] from the folded sequences I had discovered but it would have been considered wildly delusionary at the time. The major body that should occur before Neptune in my sequence given in [5] and Table 2 of [6] is easily calculated to have a reciprocal period corresponding to -1974 - 4558 = -6532. Its period should then be 317816/6532 = 48.66 yrs. The agreement with Chiron's period of 47 to 51 yrs is quite good. The ellipsis (...) in [5] and [6] clearly indicated that the sequence continued in both directions so that a body at Chiron's position was implied. In an earlier work [4] I had stated, ". . . one should really ask why don't Jupiter, Uranus and Neptune have a plane of particulate matter [rings] inside Roche's limit since that is natural considering the pervasiveness of grains and cometesimals . . ." (p. 16). As we now know, rings have been found around Uranus [16], [28]. The way to test my theory would be a computer simulation using reciprocal periods given by (23a) or (23b) or [7] (or, perhaps, using part of a bisected odd-N folded sequence in [6]) to see if the broad maxima in [10] can be sharpened.

The agreement of Chiron's period with Folded Fibonacci sequences is reassuring but not perfect. I have been able to represent Neptune and outer satellites in general more accurately than *any* other rule simply because I worked with recursive sequences rather than naive power laws. Also, mine is the only work to represent the several comet groups (see Table 1). So that, in terms of completeness and goodness of fit, my hypothesis is the best. It remains for a computer simulation to test whether my proposal gives maximum stability. Such a simulation may solve the following question: Why are some period ratios nearly but not quite small integers? Saturn:Jupiter is not 5:2 but is 6551:2638 to seven significant digits. Accurate to only five digits is 149:30. Neptune:Uranus is not 2:1 but is 51:26 to five digits. And Uranus:Saturn is 77:27 to nearly five digits. Similarly, Earth:Venus is not F_7: F_6 but 1172:721 to eight-digit accuracy. These ratios suggest that low-order commensurabilities (LOC) are avoided, except for the ratio 2 among the Galilean satellites. The Kirkwood gaps indicate that LOC are unstable if the ratio ≥ 2, such as 11/5, 9/4, 7/3, 5/2, 8/3, and 3/1.

The problem is ancient. The Pythagoreans believed in orbits in arithmetic progression and added a Central Fire and a Counter-Earth [23] to obscure that Fire so that the total number of moving bodies be the "magic" number $1 + 2 + 3 + 4 = 10$. Yet Aristarchus placed the Sun in the center for reasons of simplicity 18 centuries before Copernicus. Later, Ptolemy and others confounded the picture with equants and epicycles, until Kepler discovered that blemished curve—the ellipse.

Further back in time, the concepts become anthropocentric and folklorish as in the mural of Ra and Noot found in the tomb of Rameses VI.

There is still the possibility that Bode's problem has no solution or that the distribution of planets is random on a logarithmic axis, save that they cannot be too close to each other. But my work has led to some interesting sequences that I will discuss in the future.

TABLE 1

The Correspondence between the Half-Integer Sequence and the Planets

Reciprocal Period	Period (yrs)	Solar System
-55∅ - 233	0.999969	
34∅ + 144	1.618	null
-21∅ - 89	2.6183	Hungaria #434 (991 da = 2.71 yrs), etc. (Average = 2.75 yrs)
13∅ + 55	4.235	null
-8∅ - 34	6.859	Faye (7.35 yrs); Brooks II (6.72 yrs); d'Arrest (6.67 yrs), Finlay (6.90 yrs), etc.
5∅ + 21	11.07	null
-3∅ - 13	18.035	Neujmin (17.97 yrs)
2∅ + 8	28.66	null
-∅ - 5	48.66	Chiron (47 to 51 yrs), other Centaurs
∅ + 3	69.73	Olbers (69.6 yrs); Brorson-Metcalf (69.1 yrs); Pons-Brooks (71 yrs); Halley
- 2	161.00	Neptune (164.79 yrs); $N + P$ (168.4 yrs)
∅ + 1	123.0	Swift-Tuttle (119.6 yrs); Barnard II (128.3 yrs)
∅ - 1	521.0	Planet X (464.? yrs)
2∅	99.5	null
3∅ - 1	83.55	Uranus (84.01 yrs)
5∅ - 1	45.42	null
8∅ - 2	29.42	Saturn (29.46 yrs)
13∅ - 3	17.85	null
21∅ - 5	11.11	(Jupiter 11.86 yrs) not meant to fit Jupiter.
34∅ - 8	6.849	null
55∅ - 13	4.237	Astrea (4.13 yrs); asteroids (0.23 yr^{-1})
89∅ - 21	2.6177	null
144∅ - 34	1.618	(Mars)
233∅ - 55	1.0	(Earth)
377∅ - 89	0.618	(Venus, 0.615 yr)
610∅ - 144	0.382	null
987∅ - 233	0.236	(Mercury, 0.241 yr)

2. GENERALIZED FOLDED SEQUENCES

The obvious generalization of the definition of folded sequences, (4) of [6], is

$$\{\mathcal{P}_{j,N}\}_k = P_{j,k} + (-1)^{N+1} P_{j,k-N} \quad \text{with} \quad 0 \le k \le N - 1 \tag{37}$$

where a *script* letter denotes a folded sequence, $\{P_j\}$ is the jth coprime sequence as in (1) of [20], and $\{\mathcal{P}_{j,N}\}$ is finite if N is finite. This latter point differs from (1) of [6] wherein folded sequences were made infinite by repeating the cycle *ad infinitum*. It will help to display the Folded Pell array $\{\mathcal{P}_2\}$ in Table 2:

TABLE 2

Folded Pell Sequences for N Odd

N													Sum
1						1							1
3					5	-1	3						7
5				29	-11	7	3	13					41
7			169	-69	31	-7	17	27	71				239
9		985	-407	171	-65	41	17	75	167	409			1393
11	5741	-2377	987	-403	181	-41	99	157	413	983	2379		8119
13	-13859	5743	-2373	997	-379	239	99	437	973	2383	5739	13861	47321
15	...	-13855	5753	-2349	1055	-239	577	915	2407	5729	13865	...	
17			...	5811	-2209	1393	577	2547	5671	13889	...		
∞			...	$6+5\sqrt{8}$	$-2-2\sqrt{8}$	$2+\sqrt{8}$	$2+0$	$6+\sqrt{8}$	...				
∞				...	$6+\sqrt{8}$	$-2-0$	$2+\sqrt{8}$	$2+2\sqrt{8}$	$6+5\sqrt{8}$	...			
r		...	$-7h$	$-5h$	$-3h$	$-h$	h	$3h$	$5h$	$7h$	...		

The last row is the half-integer subscript r defined by $2k = 2r + N$. Both infinite folded sequences (which I often call half-integer sequences) are given, namely for $N = 1$ (mod 4) and $N = 3$ (mod 4). Subscripts within braces are part of the *name* of the array/sequence, whereas those outside the braces indicate the value of the row/element.

In order to find the jth Half-Integer sequence the theorem in [6] is generalized.

Theorem: $\{\mathcal{P}_{j,N}\}_{k+1} / \{\mathcal{P}_{j,N}\}_k$ approaches a limit as $N \to \infty$ for each k dependent only upon the value of N modulo 4.

Proof: Define $\theta = N$ modulo 4. Since $r = (k - N/2)$ then $r = -h$ gives

$$k = (N - 1)/2 = [N/2].$$

We will need $P_{-k}P_{-k-1}/(P_{k+1}P_k) = -1$, from which one finds

$$P_{-k}/P_{k+1} = -z = -P_k/P_{-k-1}.$$

Thus $z \to \pm\beta$ as $N \to \infty$, while $\theta = 3$ or 1, respectively. Then,

$$\{\mathcal{P}_{j,N}\}_h / \{\mathcal{P}_{j,N}\}_{-h} = (P_{k+1} + P_{-k})/(P_k + P_{-k-1})$$
$$= i^{\theta-1}(1 - z)/(1 + z)$$

which $\to \pm((\alpha - 1)/(\alpha + 1))^{\pm 1}$ as $N \to \infty$, while $N = 1$ or 3 (mod 4), respectively. Hence, where the limit of $\mathcal{P}$ is $\mathcal{S}$,

$$\{\mathcal{S}_j\}_{-h}/\{\mathcal{S}_j\}_h = (\alpha + 1)/(\alpha - 1) = -\{\mathcal{S}_j^*\}_h/\{\mathcal{S}_j^*\}_{-h}. \tag{38}$$

In the following the first subscript of $\mathcal{S}_{j,r}$ and $P_{j,n}$ is suppressed.

A number of relations follow from the elegant

$$\mathcal{S}_r = P^*_{r+h} + dP_{r-h} \quad \text{and} \quad \mathcal{S}^* = P^*_{r-h} + dP_{r+h}; \tag{39}$$

these are (40), (41), (42), (43), and (45).

(40) $jP^*_m = (\mathcal{S}_{m+h} - \mathcal{S}^*_{m-h})$ and $P_n = (\mathcal{S}^*_{n+h} - \mathcal{S}_{n-h})/jd$,

(41) $(2 + d)P^*_{r+h} = \mathcal{S}^*_{r+1} + \mathcal{S}_r$ and $d(d + 2)P_{r-h} = (\mathcal{S}_r + \mathcal{S}^*_{r-1})$,

(42) $(\mathcal{S}_r + \mathcal{S}^*_r) = 2\zeta\alpha^r$ and $(\mathcal{S}_r - \mathcal{S}^*_r) = 2\zeta i\beta^r$,

and the Binet-like formulas

(43) $\mathcal{S}_r = \zeta(\alpha^r + i\beta^r)$ and $\mathcal{S}^*_r = \zeta(\alpha^r - i\beta^r)$

where

(44) $\zeta = (\alpha^h + i\beta^{-h}) = (\alpha^h + \alpha^{-h}) = -i(\beta^h - \beta^{-h})$, where $i = \sqrt{-1}$,

and analogous to $F_n = (-1)^{n+1}F_{-n}$, we have

(45) $\mathcal{S}^*_r + \mathcal{S}_r = (\mathcal{S}_{-r} - \mathcal{S}^*_{-r})i^{r+h}$.

Now the initial values are determined by (37) or (38) and are

(46) $\mathcal{S}_{j,-h} = (2 + d) = \mathcal{S}^*_{j,h}$ and $\mathcal{S}_{j,h} = j = -\mathcal{S}^*_{j,-h}$ for all j.

The bisections of the general Half-Integer sequence for $N = 1 \pmod 4$ appear in (47) and (48), where t instead of j is used from now on for the parameter. Compare with the penultimate rows of Table 2. Also (49) is the subscript r. In Table 3 note that $(2 + d) = \zeta^2 = (\alpha^h + \alpha^{-h})^2$ and $v = \sqrt{5}$.

(47) $(t^4+dt^4+4t^2+3dt^2+d+2)$ $\quad (t^2+dt^2+d+2)$ $\quad 2+d$ $\quad (t^2+2+d)$

(48) $-(t^3+dt^3+3t+2dt)$ $\quad -(t+dt)$ $\quad t$ $\quad (t^3+3t+dt)$

(49) $-\frac{9}{2}$ $\quad -\frac{7}{2}$ $\quad -\frac{5}{2}$ $\quad -\frac{3}{2}$ $\quad -\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad \frac{3}{2}$ $\quad \frac{5}{2}$

TABLE 3

Parameters of Half-Integer t-Fib Sequences

t	d	α	I_{-h}/I_h	$(\alpha^h + \alpha^{-h})$	$(\alpha^h - \alpha^{-h})$	td
$\sqrt{v - 2}$	$\sqrt{v + 2}$	1.272020	8.352410	2.014490	0.241187	1.000
$1/\sqrt{2}$	$3/\sqrt{2}$	1.414213	5.828427	2.030104	0.348311	1.500
1	$\sqrt{5}$	1.618034	4.236068	2.058171	0.485868	2.236
1.5	5/2	2.0	3.0	2.121320	0.707107	3.750
2	$\sqrt{8}$	2.414213	2.414213	2.197368	0.910180	5.657
$\sqrt{5}$	3	2.618034	2.236068	2.236068	1.0	6.708
8/3	10/3	3.0	2.0	2.309401	1.154700	8.888
3	$\sqrt{13}$	3.302776	1.868517	2.367605	1.267103	10.817
$2\sqrt{3}$	4	3.732050	1.732050	2.449490	1.414213	13.856
4	$\sqrt{20}$	4.236068	1.618034	2.544039	1.572303	17.888
$\sqrt{32}$	6	5.828427	1.414214	2.828427	2.000000	33.941

Table 3 shows us that the Half-Integer Pell sequence is the only one in which the ratio of the central pair, S_{-h}/S_h equals the characteristic root. This is clear from $(\alpha + 1)/(\alpha - 1) = \alpha$ whose solution is $1 + \sqrt{2}$. Sequences given by $(\alpha^h + \alpha^{-h}) = \alpha$ have a very simple Binet-like formula but the larger root $\alpha = 2.1478990$ is the solution to a cubic. Are there Half-Integer sequences with integer terms? Consider t-Fib sequences,

$$P_{t,n+1} = tP_{t,n} + P_{t,n-1},$$

for which d is an integer, $d^2 = t^2 + 4$. Consider the Root-Five sequences, $t = \sqrt{5}$, which have the FL-types: 1, 0, 1, v, 6, $7v$, 41, $48v$, 281, $(7 \cdot 47)v$, $(18 \cdot 107)$, $(55 \cdot 41)$, ... and 2, v, 7, $8v$, 47, $55v$, $(23 \cdot 14)$, $377v$, 2207, ... where $v = \sqrt{5}$. In the corresponding Half-Integer sequence, from $r = -11h$ to $+11h$,

(50) $\quad \ldots, -199v, 170, -29v, 25, -4v, 5, v, 10, 11v, 65, 76v, 445, \ldots$

we see that both bisections are integers (after dividing by common factors). Furthermore, one bisection consists of every fourth Fibonacci number including F_5 and the other consists of every fourth Lucas number including L_5. Can this be generalized? As a little algebra shows, yes,

(51) $\quad d_i = d_4^2 - 2 \quad \text{or} \quad d_i = t_4^2 + 2 \quad \text{or} \quad t_i = t_4 d_4$

where t_4, d_4 refer to the sequence from which every fourth term is extracted and d_i, t_i refer to the chosen sequence. To illustrate this, the Root-32 Half-Integer sequence, see Table 3, has from (46)

$$S_h = t = 4\sqrt{2} \quad \text{and} \quad S_{-h} = 2 + d = 8.$$

The bisections of this reduced by common factors are:

(52) $\quad \ldots, 33461, 985, 29, 1, 5, 169, 5741, \ldots$

(53) $\quad \ldots, -8119, -239, -7, 1, 41, 1393, 47321, \ldots$

which are every fourth Pell number as expected beginning with $P_5 = 29$ and $P_5^*/2 = 41$. Proofs of statements above follow easily from (39) or (43). So (23a, b) are every fourth term of the $t = \sqrt{(\sqrt{5} - 2)}$ sequences. The "F" sequence to 3 decimals is ..., 1.236, -0.486, 1, 0, 1, 0.485868, 1.236, 1.086, 1.764, 1.943, ... and clearly the ratios

$$1.236:1:1.764 = 3 + v:2 + v:3 + 2v = 2\emptyset:\emptyset + 1:\emptyset + 3$$

show that every fourth term of "F" gives (23a) of [7] and equivalently a bisection of Table 1. The general recurrence of these bisections is

$$B_{t,n+1} = P^*_{t,4}B_{j,n} - B_{t,n-1}$$

where $P^*_{2,4} = 34$ for (52) and (53). A bisected t-Fib sequence has the recurrence

$$P_{n+2} = (t^2 + 2)P_n - P_{n-2} \quad \text{or} \quad \delta^2 P_n = t^2 P_n.$$

Indeed the recurrence's middle term for m-sectioning has a coefficient given by the mth rising diagonal of the Lucas triangle. The bi-bisection case is $P_{n+4} = (a^4 + 4a^2b + 2b^2)P_n - b^4P_{n-4}$, and so on.

We come now to what I regard as the most important property of these sequences. The Self-Lucas property, (14) of [6], remains unchanged in this generalization, namely

(54) $\quad (\mathcal{S}_{r+1} + \mathcal{S}_{r-1})/d = (-1)^{r-h}\mathcal{S}_{-r} \quad \text{and} \quad (\mathcal{S}^*_{r+1} + \mathcal{S}^*_{r-1})/d = (-1)^{r+h}\mathcal{S}^*_{-r},$

where $d = (\alpha - \beta)$ and $(\alpha + \beta) = j = t$. Now (54) may be proven from (43). Note that terms with subscripts $(r + 1)$, $(r - 1)$ and $-r$ all belong to the *same* bisection of $\mathcal{S}_t$ or $\mathcal{S}_t^*$. This is obvious since $(r + 1) - (r - 1) = 2$ and $(r - 1) - (-r) = 2k - (N + 1)$ which is also even. Taking ratios of (54) one may form the triplet rule, for both $\mathcal{S}_t$ and $\mathcal{S}_t^*$,

$$(\mathcal{S}_r + \mathcal{S}_{r+2})/(\mathcal{S}_r + \mathcal{S}_{r-2}) = \mathcal{S}_{-r-1}/\mathcal{S}_{-r+1}. \tag{55}$$

Now this is readily illustrated when d is an integer so consider (50). Obviously $(10 + 65)/3 = 25$ and $(11v + 76v)/3 = 29v$ and both are members of (50). Again from (52) one has $(985 + 29)/6 = 169$. From Table 1, I illustrate (55) by $\big((8\emptyset - 2) + (3\emptyset - 1)\big)/\big((3\emptyset - 1) + (\emptyset - 1)\big) = (\emptyset + 5)/2$ by using the trick $(\emptyset + 2) = \emptyset v$. But this last ratio, $(\emptyset + 5):2$ is Chiron:Neptune.

I introduce a new operator capital lambda, Λ:

$$\Lambda \equiv I + E, \tag{56}$$

where E and I are the forward shift and identity operators and, therefore, $(\Lambda - \nabla) \equiv (E + E^{-1})$. Then the Self-Lucas property may be written

$$(\Lambda - \nabla)\mathcal{S}^{\pm}_{t,r} = d(-1)^{r \mp h}\mathcal{S}^{\pm}_{t,-r}, \tag{57}$$

where $\mathcal{S}^{\pm}$ is another notation for $\mathcal{S}$ and $\mathcal{S}^*$, respectively. We may also write $\Lambda B_n = dB_{-n}$ or $-dB_{-n}$ depending upon which bisection of $\mathcal{S}_t$ or $\mathcal{S}_t^*$ is being considered. The Self-Lucas property does not hold for F-like sequences: $P_{t,1} = 1 = P_{t,-1}$, or L-like sequences: $P^*_{t,1} = 1 = -P^*_{t,-1}$. In this sense, the Half-Integer sequences are more important.

How are (54) and (57) to be interpreted geometrically? Let alternate terms of the bisection be made negative, then the recurrence is

$$\delta^2 N_n = -(t^2 + 4)N_n. \tag{58}$$

Further let the terms of N_n be reciprocal periods of planets and let a minus sign mean retrograde (backward) motion. Then the Self-Lucas property may be written

$$\Delta N_n = -dN_{-n}, \tag{59}$$

which in words says that the set of synodic (apparent) frequencies of a collection of alternately pro- and retrograde planets are simply proportional to the negative of the sidereal (real) frequencies in reverse order.

3. COMMENTS ON THE RECIPROCAL PERIOD RULE

Why are the planetary frequencies not Folded or Half-Integer *Pell* sequences? The limiting distance ratio would be 3.2386766. One solution to this is point (x) of [7], namely to bring the planets closer to each other, thereby minimizing their potential energy which is negative; that is,

$$\max \sum_{i \neq j} \left(GM_J m_i / d_{av}^2\right), \tag{60}$$

where M_J is Jupiter's mass, m_i the mass of any planet except Jupiter, and d_{av} is a time-weighted distance from Jupiter. The Pell and, indeed, all t-Fib sequences satisfy point (ix), the avoidance of low-order commensurabilities, since $\gcd(P_{t,n+1}, P_{t,n}) = 1$ for all integers t and n. This can also be seen by noting that the continued fraction of the roots of any t-Fib recurrence consists of repeated $(1/t)$'s, so no one convergent is a great deal better than another. The sequence, 11, 12, 16, 24, 38, ..., is an example where gcd(12, 16) $\neq$ 1. Given that the total number of planets is a constant, then minimization of the cumulative perturbation frequencies (synodic) occurs as t becomes small (point xii). Of course, as t becomes small the average distance becomes smaller and the average perturbation force becomes very large.

4. FINALE

The logic [7] of my rule suggests that other civilizations may be signaling us in binary code with $1/\phi^2 = 0.01100,00111,00100,01000,01100, \ldots$. But let me assure you that my getting into Bode's Rule was not a matter of choice. Its rewards, though, have been a large number of empyreal highs, some over ideas I later rejected; but now I am glad to be through with this whirlpool. Finally, we all know that the idea of the "music of the spheres" which dates back to Eudoxus is poetic license, nonetheless I could not help noting that though most of the "notes" in my scale are cacophpnous, the first note, $2+v$, corresponds to C-sharp two octaves high, since $(2+v) \cong 2^{25/12}$.

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FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

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The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a p-coin until k consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let O_n be the set of all sequences of H and T of length n which terminate in HH and have no other occurrence of two consecutive heads. Let S_n be the number of sequences in O_n. Any sequence in O_n either begins with T, followed by a sequence in O_{n-1}, or begins with HT followed by a sequence in O_{n-2}. Thus,

$$S_n = S_{n-1} + S_{n-2},\ S_1 = 0,\ S_2 = 1. \tag{1}$$

Consequently, $S_{n-2} = F_n$, the nth Fibonacci number. The probability of termination in n trials is $S_n/2^n$. Letting

$$g(x) = \sum_2^\infty S_n x^n,$$

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

$$\sum_{n=1}^\infty nS_n/2^n = (1/2)g'(1/2) = 6.$$

We generalize this result to the following

Theorem: Consider tossing a p-coin, $Pr(H) = p$, repeatedly until k consecutive heads appear. If P_n is the probability of terminating in exactly n trials (tosses), then the generating function

$$G(x) = \sum_k^\infty P_n x^n \text{ is given by } G(x) = \frac{(px)^k(1 - px)}{1 - x + \frac{(1-p)}{p}(px)^{k+1}} \tag{2}$$

The expected number of trials, $G'(1)$ is

$$1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1-p}\left[\frac{1}{p^k} - 1\right]. \tag{3}$$

Proof: Let O_n be the set of all sequences of H and T of length n which terminate in k heads and have no other occurrence of k consecutive heads. Let S_n be the number of sequences in O_n and $P_n = Pr(O_n)$ be the probability of the event O_n. One possibility is that a sequence in O_n begins wi h a T, followed by a sequence in O_{n-1}; the probability of this is

$$Pr(T)Pr(O_{n-1}) = qP_{n-1}, \quad q = 1 - p.$$

The next possibility to consider is that a sequence in O_n begins with HT, followed by a sequence in O_{n-2}; this has probability

$$Pr(HT)Pr(O_{n-2}) = qpP_{n-2}.$$

Continuing in this way, the last possibility to be considered is that a sequence in O_n begins with $k - 1$ H's followed by a T and then by a sequence in O_{n-k}, the probability of which is $qp^{k-1}P_{n-k}$. Hence, the recursion:

(4) $$P_n = qP_{n-1} + qpP_{n-2} + \cdots + qp^{k-1}P_{n-k},$$
$$P_1 = P_2 = \cdots = P_{k-1} = 0, \; P_k = p^k.$$

(Note that the probability of achieving k heads with k tosses is p^k, while with less than k tosses it is impossible.) The technique to find the generating function for the Fibonacci numbers applies to finding

$$G(x) = \sum_k^{\infty} P_n x^n.$$

Consider

$$H(x) = \sum_{n=k}^{\infty} P_{n+1} x^n;$$

then

$$xH(x) = \sum_k^{\infty} P_{n+1} x^{n+1} = \sum_k^{\infty} P_n x^n - P_k x^k = G(x) - (px)^k.$$

Hence,

$$H(x) = [G(x) - (px)^k]/x.$$

On the other hand,

$$\begin{aligned} H(x) &= \sum_k^{\infty} P_{n+1} x^n = \sum_k^{\infty} (qP_n + qpP_{n-1} + \cdots + qp^{k-1}P_{n-k+1})x^n \\ &= q\sum_k P_n x^n + qpx\sum_k P_{n-1}x^{n-1} + \cdots + q(px)^{k-1}\sum_k P_{n-k+1}x^{n-k+1}, \end{aligned}$$

and recalling that $P_j = 0$ for $j < k$,

$$\begin{aligned} &= q\sum_k P_n x^n + qpx\sum_k P_n x^n + \cdots + q(px)^{k-1}\sum_k P_n x^n \\ &= qG[1 + px + \cdots + (px)^{k-1}] = qG\left[\frac{1 - (px)^k}{1 - px}\right]. \end{aligned}$$

Solving for G yields (2).

In the case $p = 1/2$, the combinatorial numbers $S_n = 2^n P_n$ satisfy the recursion $S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k}$. For these numbers, the generating function $(1 - x - x^2 - \cdots - x^k)^{-1}$ was found by V. Schlegel in 1894. See [1, Chap. XVII] for this and other classical references.

An alternate solution to the problem can be obtained as follows. Consider a sequence of experiments: Toss a p-coin X_1 times, until a sequence of $k - 1$ heads occurs. Then toss the p-coin once more and if it comes up heads, set $Y = 1$. If not, toss the p-coin X_2 times until a sequence of $k - 1$ heads occurs again, and then toss the p-coin once more and if it comes up heads, set $Y = 2$. If not, continue on in this fashion until finally the value of Y is set. At this time, we have observed a sequence of k heads in a row for the first time, and we have tossed the coin $Y + X_1 + X_2 + \cdots + X_Y$ times. The X_i are independent, identically distributed random variables and Y is independent of all of the X_i. Let E_k = the expected number of tosses to observe k heads in a row. Let $Z = X_1 + \cdots + X_Y$. Then,

$$\begin{aligned} E_k &= E(Y + Z) = E(Y) + E(Z) \\ &= E(Y) + E(Z|Y = 1)Pr(Y = 1) + E(Z|Y = 2)Pr(Y = 2) + \cdots \\ &= E(Y) + \sum_{n=1}^{\infty} E(Z|Y = n)Pr(Y = n) = E(Y) + \sum_{n=1}^{\infty} nE(X_1)Pr(Y = n) \\ &= E(Y) + E(X_1)E(Y). \end{aligned}$$

But $E(Y)$ = the expected number of tosses to observe a head $= 1/p$, and $E(X_1) = E_{k-1}$. Thus $E_k = 1/p + (1/p)E_{k-1}$, which yields (3).

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STRONG DIVISIBILITY SEQUENCES WITH NONZERO INITIAL TERM

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In 1936, Marshall Hall [1] introduced the notion of a kth *order linear divisibility sequence* as a sequence of rational integers $u_0, u_1, \ldots, u_n, \ldots$ satisfying a linear recurrence relation

(1) $$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n,$$

where $a_1, a_2, \ldots, a_k$ are rational integers and $u_m | u_n$ whenever $m|n$. Some divisibility sequences satisfy a stronger divisibility property, expressible in terms of greatest common divisors as follows:

$$(u_m, u_n) = u_{(m,n)}$$

for all positive integers m and n. We call such a sequence a *strong divisibility sequence*. An example is the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$.

It is well known that for any positive integer m, a linear recurrence sequence $\{u_n\}$ is periodic modulo m. That is, there exists a positive integer M depending on m and $a_1, a_2, \ldots, a_k$ such that

(2) $$u_{n+M} \equiv u_n \pmod{m}$$

for all $n \geq n_0[m, a_1, a_2, \ldots, a_k]$; in particular, $n_0 = 0$ if $(a_k, m) = 1$.

Hall [1] proved that a linear divisibility sequence $\{u_n\}$ with $u_0 \neq 0$ is *degenerate* in the sense that the totality of primes dividing the terms of $\{u_n\}$ is finite. One should expect a stronger conclusion for a linear strong divisibility sequence having $u_0 \neq 0$. The purpose of this note is to prove that such a sequence must be, in the strictest sense, periodic. That is, there must exist a positive integer M depending on $a_1, a_2, \ldots, a_k$ such that

$$u_{n+M} = u_n, \quad n = 0, 1, \ldots .$$

Suppose $\{u_n\}$ is a kth order linear strong divisibility sequence. In terms of a generating function for $\{u_n\}$, we write

(3) $$u_0 + u_1 t + u_2 t^2 + \cdots = \frac{H(t)}{K(t)} = \frac{H(t)}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_k t)},$$

where $H(t)$ and $K(t)$ are polynomials with integer coefficients. Let $q = x_1 x_2 \ldots x_k$ $(=a_k)$. We assume that $q \neq 0$.

Lemma 1: $u_m | q^m u_0$ for $m = 1, 2, \ldots$.

Proof: The 0th m-multisection of (3) (e.g., Riordan [2]) gives

$$u_{jm} = M_1 u_{(j-1)m} - M_2 u_{(j-2)m} + \cdots + (-1)^{k-1} M_k u_0,$$

where the M_i are integers. Since $u_m | u_{cm}$ for $c = 1, 2, \ldots$, we have

$$u_m | (-1)^{k+1} M_k u_0,$$

and this finishes the proof, because $M_k = q^m$.

Another proof of Lemma 1, depending on the periodicities (2), may be found in Hall [1].

Henceforth, we assume $u_0 \neq 0$. Let $p_1, p_2, \ldots, p_v$ be all the prime divisors of qu_0, so that we may write

$$q = p_1^{s_1} p_2^{s_2} \ldots p_v^{s_v} \quad \text{and} \quad u_0 = p_1^{i_{1,0}} p_2^{i_{2,0}} \ldots p_v^{i_{v,0}}.$$

Then, since $u_m | q^m u_0$ for $m = 0, 1, 2, \ldots$, we can write

$$u_m = p_1^{i_{1,m}} p_2^{i_{2,m}} \ldots p_v^{i_{v,m}}, \quad m = 0, 1, 2, \ldots .$$

Consider the set $\sigma_\ell = \{i_{\ell,1}, i_{\ell,2}, \ldots\}$, $\ell = 1, 2, \ldots, v$. Let $|\sigma_\ell|$ be the number of elements in σ_ℓ, with $|\sigma_\ell| = \infty$ if σ_ℓ is an infinite set. Define $a_\ell(j)$ for $j = 1, 2, \ldots$ inductively as follows:

$$a_\ell(1) = 1$$

$$a_\ell(2) = \begin{cases} 1 \text{ if } |\sigma_\ell| = 1 \\ \text{least } w \text{ such that } i_{\ell,w} \neq i_{\ell,1}, \text{ if } |\sigma_\ell| > 1 \end{cases}$$

$$\vdots$$

$$a_\ell(j) = \begin{cases} a_\ell(j-1) \text{ if } |\sigma_\ell| \leq j - 1 \\ \text{least } w \text{ such that } i_{\ell,w} \notin \{i_{\ell,a_\ell(r)} : 1 \leq r < j - 1\} \\ \text{if } |\sigma_\ell| > j - 1. \end{cases}$$

Thus, either the sequence $a_\ell(1), a_\ell(2), a_\ell(3), \ldots$ is strictly increasing and unbounded, or else it is strictly increasing up to some point and constant thereafter, or else it is the constant sequence $1, 1, \ldots$.

Lemma 2: Suppose $1 \leq \ell < v$. Then $a_\ell(j) | a_\ell(j + 1)$ for $j = 1, 2, \ldots$.

Proof: To simplify notation, let $a = a_\ell(j)$, $b = a_\ell(j + 1)$, and $c = (a, b)$. Without loss we assume $a \neq b$. Clearly $c \leq a$. Suppose $1 \leq c < a$. Then $i_{\ell,c} = i_{\ell,a_\ell(r)}$ for some $r < j$, so that $i_{\ell,c} \neq i_{\ell,a}$ and $i_{\ell,c} \neq i_{\ell,b}$. From $u_c = (u_a, u_b)$ follows $i_{\ell,c} = \min\{i_{\ell,a}, i_{\ell,b}\}$. This contradiction shows that $c = a$, as required.

Lemma 3: Suppose $1 \leq \ell < v$ and $j \geq 1$. If $1 \leq w \leq a_\ell(j) = a$, then

$$i_{\ell,w} \leq i_{\ell,a}.$$

Proof: If $1 \le w \le a$, then $i_{\ell,w} = i_{\ell,a_\ell(r)}$ for some $r < j$. Since $a_\ell(r)|a$, by Lemma 2, we have $u_{a_\ell(r)}|u_a$, so that $i_{\ell,a_\ell(r)} \le i_{\ell,a}$.

Lemma 4: Suppose $1 \le \ell \le v$ and $j \ge 1$. If $1 \le w \le a_\ell(j) = a$, then

$$i_{\ell,(w,a)} = i_{\ell,w}.$$

Proof: $(u_w, u_a) = u_{(w,a)}$, so $\min\{i_{\ell,w}, i_{\ell,a}\} = i_{\ell,(w,a)}$. Now $i_{\ell,w} \le i_{\ell,a}$, by Lemma 3, so $i_{\ell,(w,a)} = i_{\ell,w}$.

Lemma 5: Suppose $1 \le \ell \le v$ and $j \ge 2$. Suppose $a = a_\ell(j) \ge 2$ and b is a positive integer. Then

$$(i_{\ell,ba+1}, i_{\ell,ba+2}, \ldots, i_{\ell,ba+a-1}) = (i_{\ell,1}, i_{\ell,2}, \ldots, i_{\ell,a-1}).$$

Proof: Suppose $1 \le w \le a - 1$. Then $(u_{ba+w}, u_a) = u_{(ba+w,a)} = u_{(w,a)}$, so $\min\{i_{\ell,ba+w}, i_{\ell,a}\} = i_{\ell,(w,a)} = i_{\ell,w}$ by Lemma 4. Since $i_{\ell,w} < i_{\ell,a}$ by definition of a, we conclude $i_{\ell,ba+w} = i_{\ell,w}$.

Lemma 6: Suppose $1 \le \ell \le v$ and $2 \le |\sigma_\ell| < \infty$. Let $L = a_\ell(|\sigma_\ell|)$, and let b be a positive integer. Then

$$(i_{\ell,bL+1}, i_{\ell,bL+2} \ldots, i_{\ell,2bL-1}) = (i_{\ell,1}, i_{\ell,2}, \ldots, i_{\ell,bL-1}).$$

Proof: By Lemma 5, we already know

$$\begin{aligned}(i_{\ell,1}, \ldots, i_{\ell,L-1}) &= (i_{\ell,L+1}, \ldots, i_{\ell,2L-1})\\ &= (i_{\ell,2L+1}, \ldots, i_{\ell,3L-1})\\ &\ \ \vdots\\ &= (i_{\ell,(b-1)L+1}, \ldots, i_{\ell,bL-1}),\end{aligned}$$

so it remains only to see that $i_{\ell,L} = i_{\ell,2L} = \cdots = i_{\ell,(b-1)L}$. For $1 \le c \le b - 1$, we have $(u_{cL}, u_L) = u_L$, so that $\min\{i_{\ell,cL}, i_{\ell,L}\} = i_{\ell,L}$. Since $i_{\ell,cL} < i_{\ell,L}$, we conclude $i_{\ell,cL} = i_{\ell,L}$.

Lemma 7: There exists a positive integer M such that $u_{M+j} = u_j$ for $j = 1, 2, \ldots, k$.

Proof: For $1 \le \ell \le v$, if $|\sigma_\ell| = \infty$, choose j_ℓ so large that $a_\ell(j_\ell) > k$, and if $|\sigma_\ell| < \infty$, let $a_\ell(j_\ell) = a_\ell(|\sigma_\ell|)$. Let M be the least common multiple of the numbers $a_1(j_1), a_2(j_2), \ldots, a_v(j_v), 2k$. (We include $2k$ to ensure that $M > k$ in case $|\sigma_\ell| < \infty$ for all ℓ.)

Now, by Lemma 5, for each ℓ with $|\sigma_\ell| = \infty$, we have

$$(i_{\ell,M+1}, \ldots, i_{\ell,M+k}) = (i_{\ell,1}, \ldots, i_{\ell,k}).$$

This same equation holds, by Lemma 6, for each ℓ with $2 \le |\sigma_\ell| < \infty$, and clearly holds also for $\sigma_\ell = 1$. Therefore, for $1 \le j \le k$, we have $i_{\ell,M+j} = i_{\ell,j}$ for $1 \le \ell \le v$, so that $u_{M+j} = u_j$ for $1 \le j \le k$.

Theorem: Suppose $\{u_n\}$, $n = 0, 1, \ldots,$ is a kth order strong divisibility sequence with $u_0 \ne 0$. Then the sequence $\{u_n\}$ is periodic and has a generating function of the form $H(t)/(1 - t^\rho)$, where ρ is the fundamental period of $\{u_n\}$. If $H(t)$ has no linear factor of the form $1 - rt$, where $r^\rho = 1$, then ρ is the least possible recurrence order of $\{u_n\}$. If

$$\rho = \rho_1^{e_1}\rho_2^{e_2}\ \ldots\ \rho_t^{e_t}$$

if the prime factorization of ρ, then

$$u_\rho = Uu_{\rho_1^{e_1}}\, u_{\rho_2^{e_2}}\ \ldots\ u_{\rho_t^{e_t}}$$

for some nonzero integer U. Finally, $u_0 = u_\rho$, and $u_n | u_0$ for $n = 0, 1, \ldots$.

Proof: By Lemma 7 and the fact that $\{u_n\}$ is a kth order recurrent sequence, the sequence $\{u_n\}$ is periodic with period M. Letting ρ be the fundamental period, we now show that the denominator of the generating function $H(t)/K(t)$ must be of the form $1 - t^\rho$:

$$\begin{aligned}\frac{H(t)}{K(t)} &= u_0 + u_1 t + \cdots + u_{\rho-1}t^{\rho-1} + u_0 t^\rho + u_1 t^{\rho+1} + \cdots \\ &= u_0(1 + t^\rho + t^{2\rho} + \cdots) + u_1 t(1 + t^\rho + t^{2\rho} + \cdots) + \cdots \\ &= (u_0 + u_1 t + \cdots + u_{\rho-1}t^{\rho-1})(1 + t^\rho + t^{2\rho} + \cdots) \\ &= (u_0 + u_1 t + \cdots + u_{\rho-1}t^{\rho-1})\frac{1}{1 - t^\rho}.\end{aligned}$$

If $H(t)$ has no linear factors $1 - rt$ with $r^\rho = 1$, then $H(t)$ has no linear factors in common with $K(t)$. This means that no recurrence order for $\{u_n\}$ can be less than ρ.

We see that $\rho_i^{e_i} | \rho$ and $(\rho_i^{e_i}, \rho_j^{e_j}) = 1$ for $1 \leq i < j \leq t$, so that

$$u_\rho = U u_{\rho_1^{e_1}} u_{\rho_2^{e_2}} \cdots u_{\rho_t^{e_t}}$$

for some integer U. For $n \geq 1$, we have $u_{n\rho} = u_\rho$ and $u_n | u_{n\rho}$, so that $u_n | u_\rho$. That $u_0 = u_\rho$, so that $u_n | u_0$ for all n, follows from

$$\begin{aligned}a_k u_0 &= u_k - a_2 u_{k-1} - \cdots - a_k u_1 \\ &= u_{\rho+k} - a_2 u_{\rho+k-1} - \cdots - a_k u_{\rho+1} \\ &= a_k u_\rho.\end{aligned}$$

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MINIMUM PERIODS MODULO n FOR BERNOULLI NUMBERS

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The Bernoulli numbers B_m may be defined by

$$B_0 = 1 \tag{1}$$
$$B_m = \frac{1}{m+1}\sum_{i=0}^{m-1}\binom{m+1}{i}B_i \quad (m > 0).$$

By the Kummer congruence, we have [2, p. 78 (3.3)],

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i}\frac{B_{m+iw}}{m+iw} \equiv 0 \bmod p^{re}, \tag{2}$$

with $w := p^{e-1}(p - 1)$, where $r \geq 1$, $e \geq 1$, $m > re$, p prime such that $p - 1 \nmid m$. With $r = 1$ we get, in particular

(3) $$\frac{B_{m+p^{e-1}(p-1)}}{m+p^{e-1}(p-1)} \equiv \frac{B_m}{m} \mod p^e,$$

where $m > e$, $p - 1 \nmid m$.

Therefore, the sequence of the Bernoulli numbers is periodic after being reduced modulo n (where n is any integer) in the following sense. A rational a/b with $a, b \in Z$, $\gcd(a, b) = 1$, may be interpreted as an element of Z_n, the ring of integers modulo n, if and only if the congruence relation $yb \equiv a \mod n$ has a unique solution $y \in \{0, 1, 2, \ldots, n - 1\}$, i.e., if and only if $\gcd(b, n) = 1$. In this case, a/b is said to be *n-integral*.

By the famous von Staudt-Clausen theorem we have for integer i and prime p (cf. [1] and [2]),

$$B_{2i} \text{ } p\text{-integral} \Longleftrightarrow p - 1 \nmid 2i.$$

Since $B_0 = 1$, $B_1 = -1/2$ and $B_{2i+1} = 0$ for $i \in N$, we get

(4) $$B_m \text{ } p\text{-integral} \Longleftrightarrow p - 1 \nmid m \vee m = 0 \vee m \in \{3, 5, 7, \ldots\}.$$

Now let $L(n)$ be the smallest integer greater than 1 with the following property:

$$\exists m_0 \forall k, m \geq m_0:$$

(5) $$\left(B_k \text{ } n\text{-integral} \wedge k \equiv m \mod L(n) \Rightarrow B_m \text{ } n\text{-integral} \wedge B_k \equiv B_m \mod n\right).$$

$L(n)$ is called the *period-length* of the sequence $\{B_k \mod n\}$.

The smallest possible integer m_0 in (5) is then called the *preperiod* of $\{B_k \mod n\}$ and will be denoted by $V(n)$.

If $n = n_1 n_2$, where n_1, n_2 are coprime, then clearly

$$L(n) = \operatorname{lcm}\left(L(n_1), L(n_2)\right) \quad \text{and} \quad V(n) = \max\left(V(n_1), V(n_2)\right).$$

Hence, it suffices to discuss the case $n = p^e$, p a prime. We will prove

Theorem 1: (a) $L(2^e) = L(3^e) = 2$

(b) $V(2^e) = V(3^e) = 2$

(c) $L(p^e) = p^e(p - 1)$, where $p > 3$

(d) $V(p^e) \leq e + 1$.

Proof: If $2|n$ or $3|n$, none of the B_{2i} is n-integral by (4); since $B_2 = 0$, this proves (a) and $V(2^e)$, $V(3^e) \leq 2$. But $V(2^e) = 1$ and $V(3^e) = 1$, respectively, is impossible because $B_1 = -1/2$ is not 2-integral and $B_1 \not\equiv 0 \mod 3^e$. So we get (b) too.

Now let $p > 3$. From (3) we have, for $m > e$, $p - 1 \nmid m$, $t \geq 0$,

$$\frac{B_{m+tp^{e-1}(p-1)}}{m+tp^{e-1}(p-1)} \equiv \frac{B_m}{m} \mod p^e; \text{ hence,}$$

(6) $$k = m + sp^e(p - 1) \wedge p - 1 \nmid m \wedge m > e \Rightarrow B_k \equiv B_m \mod p^e.$$

Consequently, $L(p^e) | p^e(p - 1)$. On the other hand, we first prove $p - 1 | L(p^e)$: suppose $p - 1 \nmid L(p^e)$; we may choose $m \geq V(p^e) + L(p^e)$ such that $p - 1 | m$ (and therefore $m \neq 0$ and $m \notin \{3, 5, 7, \ldots\}$), hence by (4) B_m is not p-integral. For $k := m - L(p^e)$, we have $k \equiv m \mod p^e$, $k \geq V(p^e)$ and $p - 1 \nmid k$, hence by (4) B_k is p-integral. But this is a contradiction to (5). So $L(p^e) = p^i(p - 1)$ where $i \in \{0, \ldots, e\}$. It remains to show $i = e$. For this, we choose $q \in N$ such that $s := \left(qp(p - 1) + 2\right)p^e > V(p^e)$. Because $p^e | s$ and $p - 1 \nmid s$, we have

$B_s \equiv 0 \bmod p^e$ [2, p. 78, Theorem 5]. Now suppose $i < e$. Then, $B_k \equiv B_s \equiv 0 \bmod p^e$ if $k \equiv s \bmod p_i(p-1)$. Take

$$k := s + (p-1)p^i = \big(2 + (qp^2 + 3)(p-1)\big)p^i = 2 + t(p-1),$$

where

$$t := 2\frac{p^i - 1}{p - 1} + (qp^2 + 3)p^i \in N;$$

then by (3) with $e = 1$ and $m = 2$,

$$\frac{B_2}{2} \equiv \frac{B_{2+(p-1)}}{2 + (p-1)} \equiv \cdots \equiv \frac{B_k}{k} \bmod p,$$

where $B_k \equiv 0 \bmod p^e$. But, $p^e | s$ and $p^e \nmid (p-1)p^i$ gives $p^e \nmid k$ and, therefore, $B_2/2 \equiv 0 \bmod p$, contradictory to $B_2 = 1/6$. Hence, $i = e$ holds, and thus

$$L(p^e) = p^e(p-1) \quad \text{and} \quad V(p^e) \le e + 1$$

by (6).

Now we may improve this last inequality as follows:

Theorem 2:

1. $V(p) = 2$ for p prime.
2. Let p be a prime, $p > 3$ and $e \in \{2, 4, 6, \ldots\}$. Then,
   (a) $B_e \not\equiv 0 \bmod p \wedge p - 1 \nmid e \Rightarrow V(p^e) = e + 1$.
   (b) k maximal such that
   $$\forall 0 \le i \le k\colon (B_{e-2i} \equiv 0 \bmod p^{2i+1} \vee p - 1 | e - 2i)$$
   $$\Rightarrow V(p^e) = e - 1 - 2k.$$
3. Let p be a prime, $p > 3$ and $e \in \{3, 5, 7, \ldots\}$. Then,
   (a) $B_{e-1} \not\equiv 0 \bmod p^2 \wedge p - 1 \nmid e - 1 \Rightarrow V(p^e) = e$.
   (b) k maximal such that
   $$\forall 0 \le i \le k\colon (B_{e-1-2i} \equiv 0 \bmod p^{2i+2} \vee p - 1 | e - 1 - 2i)$$
   $$\Rightarrow V(p^e) = e - 2 - 2k.$$

Proof: By Theorem 1(d), we have $V(p) \le 2$. But $V(p) < 2$ is impossible since $B_1 = -1/2 \not\equiv 0 \bmod p$ and $B_{1+L(p)} = 0$, thus $V(p) = 2$.

For the proof of the other assertions we note that [4, p. 321, Cor.]:

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i} B_{m+i\nu}(1 - p^{m-1+i\nu}) \equiv 0 \bmod p^{r(\omega+1)-1},$$

where p prime, $p \ne 2$, $p - 1 | \nu$, and p^ω is the highest power of p contained in ν.

Setting $r := 1$ and $\nu := k - m$, we get

$$B_m(1 - p^{m-1}) - B_k(1 - p^{k-1}) \equiv 0 \bmod p^e,$$

where $p^e(p-1) | k - m$ and $k \ge m \ge 1$. Because

$$k - 1 \ge m + p^e(p-1) - 1 \ge p^e(p-1) \ge 3^e \cdot 2 \ge e,$$

we have, for $k > m \ge 1$, $p - 1 \nmid m$:

$$k \equiv m \bmod p^e(p-1) \Rightarrow B_k - B_m \equiv p^{m-1}B_m \bmod p^e. \tag{7}$$

Now it is easy to verify the assertions.

It is not very difficult to derive the following corollary, which gives the value of $V(p^e)$ "explicitly" for regular p (a prime p is said to be *regular* if and only if $B_k \not\equiv 0 \bmod p$ for each $k \in \{2, 4, \ldots, p-3\}$.

Corollary 1: Let p be regular, $p > 3$ and $e > 0$.

(a) If $2|e$ then

$$V(p^e) = e + 1 \Longleftrightarrow p\nmid e \wedge p - 1\nmid e$$

$$V(p^e) \le e - 1 \Longleftrightarrow p|e \vee p - 1|e$$

$$V(p^e) \le e - 3 \Longleftrightarrow (p|e \wedge p - 1|e - 2) \vee (p - 1|e \wedge p^3|e - 2)$$

$$\Longleftrightarrow e \equiv 2p \bmod p(p - 1) \vee e \equiv 2 - 2p^3 \bmod p^3(p - 1)$$

$$V(p^e) = e - 5 \Longleftrightarrow p = 5 \wedge e \equiv 252 \bmod 500$$

$$V(p^e) \ge e - 5.$$

(b) If $2\nmid e$ then

$$V(p^e) = e \Longleftrightarrow p^2\nmid e - 1 \wedge p - 1\nmid e - 1$$

$$V(p^e) \le e - 2 \Longleftrightarrow p^2|e - 1 \vee p - 1|e - 1$$

$$V(p^e) \le e - 4 \Longleftrightarrow (p^2|e - 1 \wedge p - 1|e - 3) \vee (p - 1|e - 1 \wedge p^4|e - 3)$$

$$\Longleftrightarrow e \equiv 2p^2 + 1 \bmod p^2(p - 1) \vee e$$

$$\equiv -2p^4 + 3 \bmod p^4(p - 1)$$

$$V(p^e) = e - 6 \Longleftrightarrow p = 5 \wedge e \equiv 1253 \bmod 2500$$

$$V(p^e) \ge e - 6.$$

For the proof, note that $2\nmid V(p^e)$ holds for $e > 1$ and that in case of regular p and $p - 1\nmid 2i$, we have

$$B_{2i} \equiv 0 \bmod p^e \Longleftrightarrow p^e|2i.$$

The assertions of Corollary 1 with "$\Leftarrow$" are also valid for any irregular prime p.

By Corollary 1, you may see that only for greater integers p^e, the value $V(p^e)$ differs from e and $e + 1$, respectively. We get

Corollary 2: For prime p, $p > 3$, let $e_1 = p - 1$, $e_2 = p$, $e_3 = 2p$, $e_4 = 2p^2 + 1$, $e_5 = 252$, $e_6 = 1253$. Then we have

(a) $V(p^{e_i}) \le e_i - i$, $i \in \{1, \ldots, 4\}$.

If p is regular, then $V(p^{e_i}) = e_i - i$, $i \in \{1, \ldots, 4\}$, and there is no smaller power of p such that $V(p^e) = e - i$.

(b) $V(5^{e_i}) = e_i - i$, $i \in \{5, 6\}$, and there is no smaller power of 5 such that $V(5^e) = e - i$.

(c) If p is regular and $p > 5$, then $V(p^e) \ge e - 4$.

For irregular primes, it is naturally somewhat more difficult to derive similar results about the smallest power of p such that $V(p^e) = e - i$, where $i \ge 1$. By Theorem 2, we get

$$B_e \equiv 0 \bmod p \wedge 2|e \Rightarrow V(p^e) \le e - 1;$$

hence, for each irregular prime p, we have $V(p^e) \le e - 1$ for at least one e such that $e \le e_1 = p - 1$.

Considering the table of irregular primes in [1] we may compute that $n = 691^{12}$ is the smallest power of an irregular prime such that $V(p^e) = e - 1$.

There are still some open questions:

1. Are there powers $n = p^e$ of some (necessarily irregular) prime p such

that $e < e_i$ and $V(p^e) \leq e - i$, where $i \in \{2, 3, 4\}$? (By the computational results in [5] we may conclude that this does not happen when $p < 30{,}000$.)

2. Is there a power $n = p^e$ of some irregular prime such that

$$V(p^e) \leq e - 5?$$

Final Remark: Professor L. Carlitz and Jack Levine in [3] asked similar questions about Euler numbers and polynomials. Analogous results about the periodicity of the sequence of the Bernoulli polynomials reduced modulo n and the polynomial functions over Z generated by the Bernoulli polynomials will be derived in a later paper.

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THE RANK-VECTOR OF A PARTITION

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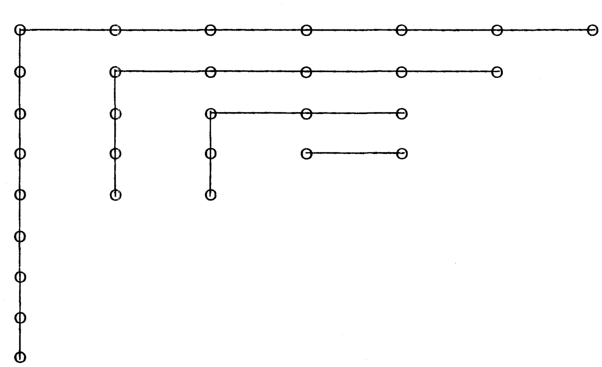
1. INTRODUCTION

The Ferrars graph of a partition may be regarded as a set of nested right angles of nodes. The depth of a graph is the number of right angles it has. For example, the graph



is four deep or is of depth four. It is clear that a graph of depth k cannot have less than k^2 nodes.

Denote by x_i the number of nodes on the horizontal, and by y_i the number of those on the vertical section of the ith right angle, starting with the outermost right angle as the first. Then, the partition can be very conveniently represented by the $2 \times k$ matrix:

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_k \\ y_1 & y_2 & y_3 & \cdots & y_k \end{bmatrix}$$

or simply by

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k .$$

Evidently, we must have

(1.1) $$x_i \geq x_{i+1} + 1, \quad y_i \geq y_{i+1} + 1, \quad i \leq k - 1.$$

It must be remembered that x's and y's are positive integers. The Atkin-ranks of the graph [1] are given by

(1.2) $$R_k = [x_1 - y_1,\ x_2 - y_2,\ \ldots,\ x_k - y_k] = [x_i - y_i]_k,$$

which we shall call the rank-vector both of the graph and of the partition it represents.

The number of nodes in the graph is given by

(1.3) $$n = \sum_{i=1}^{k} (x_i + y_i - 1).$$

In our graph, the matrix

$$\begin{bmatrix} 7 & 5 & 3 & 2 \\ 9 & 4 & 3 & 1 \end{bmatrix}$$

represents a partition of 30 and its rank-vector is

$$[-2 \quad 1 \quad 0 \quad 1].$$

Obviously, if R_k is the rank-vector of a partition, then the rank-vector of its conjugate partition is $-R_k$. Hence, the rank-vector of a self-conjugate partition of depth k must be $[0]_k$.

Again, if $[r_i]_k$ is the rank-vector of the partition given by $\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k$, then we have

(1.4) $$y_i = x_i - r_i.$$

2. SOME CONSEQUENCES OF (1.1)

Since $y_i \geq y_{i+1} + 1$, we must have $x_i - r_i \geq x_{i+1} - r_{i+1} + 1$. Hence, for each $i \leq k - 1$,

(2.1) $$x_i \geq \max(x_{i+1} + 1,\ x_{i+1} + r_i - r_{i+1} + 1).$$

Since y_k is a positive integer, we conclude that

(2.2) $$x_k \geq \max(r_k + 1,\ 1).$$

From (1.3) and (1.4), we further have

(2.3) $$\sum_{i=1}^{k} x_i = \frac{1}{2}\left(n + k + \sum_{i=1}^{k} r_i\right).$$

Hence a partition of n with a given rank-vector $[r_i]_k$ can exist only if n has the same parity as

$$k + \sum_{i=1}^{k} r_i .$$

In what follows, we assume that our n's satisfy this condition. Moreover, i shall invariably run over the integers from 1 to k.

3. THE BASIS OF A GIVEN RANK-VECTOR

There are an infinite number of Ferrars graphs which have the same rank-vector. All such graphs have the same depth but not the same number of nodes necessarily.

Theorem: Among the graphs with the same rank-vector, there is just one with the least number of nodes.

Proof: Using the equality sign in place of the sign $\geq$ in (2.2) and (2.1), we obtain the least value of each of the x_i's, $i \leq k$. (1.3) and (1.4) then give n_0 that is the least n for which a graph with the given rank-vector exists. This proves the theorem.

Incidentally, we also get the unique partition with the given rank-vector and the least number of nodes. We call this unique partition the basis of the given rank-vector.

Example: Let us find the basis of the rank-vector $[-2 \;\; 3 \;\; 0 \;\; 1]$. With the equality sign in place of the of the inequality sign, (2.2) gives $x_4 = 2$. With the equality sign in place of $\geq$, (2.1) now gives, in succession,

$$x_3 = 3,\ x_2 = 7, \text{ and } x_1 = 8.$$

From (4) of Section 1, we now have

$$y_4 = 1,\ y_3 = 3,\ y_2 = 4, \text{ and } y_1 = 10.$$

Hence, the required basis is

$$\begin{bmatrix} 8 & 7 & 3 & 2 \\ 10 & 4 & 3 & 1 \end{bmatrix}.$$

This represents a partition of 34.

We leave the reader to verify the following two trivial-looking but very useful observations:

(a) If $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ is the basis of $[r_i]$ and h is an integer, then the basis of the vector $[r_i + h]$ is given by

$$\begin{bmatrix} x_i + h \\ \\ y_i \end{bmatrix} \text{ or } \begin{bmatrix} x_i \\ \\ y_i - h \end{bmatrix}$$

according as h is positive or negative.

(b) If $h_1 \geq h_2 \geq \cdots \geq h_k \geq 0$ are integers, then the graphs of

$$\begin{bmatrix} x_i \\ \\ y_i \end{bmatrix} \text{ and } \begin{bmatrix} x_i + h_i \\ \\ y_i + h_i \end{bmatrix}$$

have the same rank-vector.

4. PARTITIONS OF n WITH A GIVEN RANK-VECTOR

Let $\begin{bmatrix} x_i \\ y_i \end{bmatrix}_k$ be the basis of the given rank-vector and n_0 the number of nodes in the basis. For our n to have any partitions with the given rank-vector, it is necessary that n has the same parity as n_0 and $n \geq n_0$. Assume that this is so. Write

$$m = \frac{1}{2}(n - n_0).$$

List all the partitions of m into at most k parts. Let

$$m = h_1 + h_2 + \cdots + h_k,$$

with $h_1 \geq h_2 \geq \ldots \geq h_k \geq 0$, be any such partition of m. Then the matrix

$$\begin{bmatrix} x_i + h_i \\ y_i + h_i \end{bmatrix} \tag{4.1}$$

provides a partition of n with the given rank-vector.

The one-one correspondence between the partitions of m and the matrices (4.1) establishes the following

Theorem: The number of partitions of n with the given rank-vector is the same as the number of partitions of m into at most k parts where m is as defined above.

Example: Let the given rank-vector be $[-3 \quad 2 \quad 1 \quad -1]$ and $n = 43$. Then the basis of the vector is readily seen to be

$$\begin{bmatrix} 7 & 6 & 4 & 1 \\ 10 & 4 & 3 & 2 \end{bmatrix}$$

so that $n_0 = 33$ and $m = 5$.

The partitions of 5 into at most 4 parts are:

$$5;\ 4 + 1,\ 3 + 2;\ 3 + 1 + 1,\ 2 + 2 + 1;\ 2 + 1 + 1 + 1.$$

Therefore, the required partitions of 43 are provided by the matrices:

$$\begin{bmatrix} 12 & 6 & 4 & 1 \\ 15 & 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 11 & 7 & 4 & 1 \\ 14 & 5 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 8 & 4 & 1 \\ 13 & 6 & 3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 10 & 7 & 5 & 1 \\ 13 & 5 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 8 & 5 & 1 \\ 12 & 6 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 7 & 5 & 2 \\ 12 & 5 & 4 & 3 \end{bmatrix}.$$

We leave it to the reader to see how the graphs of partitions of n can be constructed directly from that of the basis. As an exercise, he/she might also find a formula for the number of self-conjugate partitions of n.

As a corollary to the theorem of this section, we have

Corollary: The number of partitions of $n + hk$, $h > 0$, with rank-vector $[r_i + h]$ is the same as the number of partitions of n with rank-vector $[r_i]$.

This follows immediately from observation (a) in the preceding section.

5. THE BOUNDS FOR THE ATKIN-RANKS

What can be said concerning the Atkin-ranks of partitions of n for which $x_1 \leq a$, $y_1 \leq b$?

We show that these ranks are bounded both above and below. Since $x_1 \le a$, the number of rows a partition of n can occupy is not less than u, where

$$u - 1 < n/a \le u.$$

Hence, none of the ranks can exceed $(a - u)$.

Similarly, none of the ranks can fall short of $(v - b)$, where

$$v - 1 < n/b \le v.$$

Of course, for n to have a partition of said type, it is necessary to have

$$n \le ab.$$

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THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

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1. In what follows: small letters denote integers; $n > 0$; p denotes an odd prime other than 5; [ ] is the greatest integer function; and for convenience, we write

$$(n;r) \text{ for } \binom{n}{r}.$$

The two relations

(1.1) $$(n;r) = (n;n - r), \text{ and}$$

(1.2) $$(n;r - 1) + (n;r) = (n + 1;r)$$

are freely used, and we take, as usual,

$$(t;0) = 1 \text{ for all integers } t, \text{ and}$$

$$(n;r) = 0 \text{ if } r > n, \text{ and also when } r \text{ is negative.}$$

We further define

(1.3) $$S(n,r) = \sum_j (n;j),$$

where j runs over all nonnegative integers which are $\equiv r \pmod 5$.

As a consequence of this definition and the relations (1.1) and (1.2) we have

(1.4) $$S(n,r) = S(n,n - r), \text{ and}$$

(1.5) $$S(n,r - 1) + S(n,r) = S(n + 1,r).$$

2. The Fibonacci numbers F_n are defined by the relations

(2.1) $$F_1 = 1 = F_2, \text{ and}$$

(2.2) $$F_n + F_{n+1} = F_{n+2} \text{ for each } n \ge 1.$$

G. E. Andrews [1] has given the following formulas for F_n:

(2.3) $$F_n = \sum_j (-1)^j \big(n - 1; [(n - 1 - 5j)/2]\big);$$

(2.4) $$F_n = \sum_j (-1)^j \big(n; [(n - 1 - 5j)/2]\big);$$

where j runs over the set of integers.

The object of this note is to provide a simple proof of these formulas and to obtain some congruence properties of F_n. Let

$$[(n - 1)/2] \equiv m \pmod 5$$

so that

$$n - 1 = 2m \quad \text{or} \quad 2m + 1 \pmod{10}$$

according as n is odd or even. Then (2.3) and (2.4) can be written as:

(2.5) $$F_n = S(n - 1, m) - S(n - 1, m - 2);$$

(2.6) $$F_n = S(n, m) - S(n, m - 1).$$

We first assert that (2.5) and (2.6) are equivalent and prove the assertion as follows:

For any integer j, we have

$$(n; m + 5j) - (n - 1; m + 5j) = (n - 1; m + 5j - 1).$$

Also

$$\begin{aligned}(n; m - 1 + 5j) &- (n - 1; m - 2 + 5j)\\ &= (n; n - m + 1 - 5j) - (n - 1; n - m + 1 - 5j)\\ &= (n - 1; n - m - 5j)\\ &= (n - 1; m + 5j - 1).\end{aligned}$$

Hence, letting j vary suitably, we get

$$S(n, m) - S(n - 1, m) = S(n, m - 1) - S(n - 1, m - 2),$$

and our assertion follows immediately.

3. Proof of (2.5) is by induction. It is easy to verify that (2.5) and (2.6) hold for $n = 1$ and $n = 2$. Assume that they hold for each $n \leq t + 1$. Then, from (2.6), we have

(3.1) $$F_t = S(t, m) - S(t, m - 1)$$

with $m \equiv [(t - 1)/2] \pmod 5$. For the same m, (2.5) gives

(3.2) $$F_{t+1} = S(t, m) - S(t, m - 2) \text{ for } t \text{ odd},$$

(3.3) $$= S(t, m + 1) - S(t, m - 1) \text{ for } t \text{ even}.$$

If t is odd, let $t = 10k + 2m + 1$; then

(3.4) $$S(t, m) = S(t, t - m) = S(t, 10k + m + 1) = S(t, m + 1).$$

If t is even, let $t = 10k + 2m + 2$; then

(3.5) $$S(t, m - 1) = S(t, t - m + 1) = S(t, 10k + m + 3) = S(t, m - 2);$$

so that

(3.6) $$F_{t+1} = S(t, m + 1) - S(t, m - 2) \text{ for } t \text{ odd as well as } t \text{ even}.$$

From (3.1) and (3.6), we get

$$F_t + F_{t+1} = \{S(t,m) + S(t,m+1)\} - \{S(t,m-1) + S(t,m-2)\}$$
$$= S(t+1,m+1) - S(t+1,m-1).$$

Thus,

$$F_{t+2} = S(t+1,m+1) - S(t+1,m-1).$$

Inductive reasoning now proves (2.5) for all $n > 0$.

4. From (2.5) and (2.6), we can derive not only the well-known congruences modulo p for F_p, F_{p+1}, and F_{p-1} (in the manner of Andrews), but also some congruences modulo p^2.
We first give the expressions for F_{p^2}, F_{p^2+1}, and F_{p^2-1}.

(i) If p is a prime of the form $10k \pm 1$, then we have

$$[(p^2 - 1)/2] \equiv 0 \pmod 5,$$

and so also

$$[p^2/2] \equiv 0 \pmod 5.$$

Hence,

$$F_{p^2} = S(p^2, 0) - S(p^2, 4),$$
$$F_{p^2+1} = S(p^2, 0) - S(p^2, 3);$$

and therefore,

$$F_{p^2-1} = S(p^2, 4) - S(p^2, 3).$$

(ii) If p is a prime of the form $10k \pm 3$, then

$$[(p^2 - 1)/2] \equiv 4 \pmod 5,$$

and so also is

$$[p^2/2] \equiv 4 \pmod 5.$$

Hence,

$$F_{p^2} = S(p^2, 4) - S(p^2, 3),$$
$$F_{p^2+1} = S(p^2, 4) - S(p^2, 2);$$

and therefore,

$$F_{p^2-1} = S(p^2, 3) - S(p^2, 2).$$

All that we need now for our purpose is the

Lemma: For $1 \le h \le p^2 - 1$,

$$(p^2; h) \equiv (-1)^{h-1}p^2/h \pmod{p^2}.$$

Proof: We have

$$(p^2; h) = \frac{p^2}{h} \cdot \frac{p^2 - 1}{1} \cdot \frac{p^2 - 2}{2} \cdot \ldots \cdot \frac{p^2 - h + 1}{h - 1}.$$

Since for $1 \le r \le h - 1$,

$$\frac{p^2 - r}{r} \equiv -1 \pmod{p^2}$$

the lemma follows immediately.

Evidently, if $p \nmid h$, then

(4.1) $$(p^2;h) \equiv 0 \pmod{p^2};$$

otherwise,

(4.2) $$(p^2;h)/p \equiv (-1)^{h-1}p/h \pmod{p}.$$

We have, of course,

(4.3) $$(p^2;0) = 1 = (p^2;p^2).$$

As an application of the lemma, we have, for example:

(i) when $1 \leq m \leq 4$,

(4.4) $$S(p^2,m) \equiv \sum_{j \geq 0} (p^2;m + 5j) \pmod{p^2}.$$

On the right of the sigma in (4.4), we need consider only those nonnegative values of j for which

$$m + 5j \leq p^2 \text{ and } m + 5j \equiv 0 \pmod{p};$$

(ii) when $m = 0$, we have,

(4.5) $$S(p^2,0) - 1 \equiv \sum_{j \geq 1} (p^2;5j) \pmod{p^2},$$

so that

(4.6) $$\frac{S(p^2,0) - 1}{p} \equiv \sum_{j} (-1)^{j-1}/5j \pmod{p},$$

where $1 \leq j < p/5$. Thus

$$\frac{F_{121} - 1}{11} \equiv \frac{1}{5} - \frac{1}{10} + \frac{1}{4} - \frac{1}{9} \equiv 9 - 10 + 3 - 5 \equiv 8 \pmod{11}.$$

Therefore,

$$F_{121} \equiv 89 \pmod{121}.$$

REFERENCE

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OPERATIONAL FORMULAS FOR UNUSUAL FIBONACCI SERIES

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Operational formulas can play a fascinating role in finding transformations and sums of series. For instance, by using the differential operator D $(=d/dx)$ we can transform

(1) $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \quad |x| < 1,$$

into

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}, \quad |x| < 1.$$

The operator $\theta = xD$ is even more interesting. It has the basic property that $\theta^p x^k = k^p x^k$, so that (1) can be transformed into

$$\sum_{k=0}^{\infty} k^p x^k = \theta^p \left\{\frac{1}{1-x}\right\}, \tag{2}$$

and since it can also be shown (and is well known) that

$$\theta^p f(x) = \sum_{k=0}^{p} S(p, k) x^k D^k f(x), \tag{3}$$

where $S(p, k)$ are Stirling numbers of the second kind, explicitly

$$k! S(p, k) = \Delta^k 0^p = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^p, \tag{4}$$

then series (2) can be found in closed form for it is trivial to find the higher derivatives needed in (3). The result is a very old and well-known formula. In [7] is given an extension of (3) applied to generalized Hermite polynomials. There are numerous similar generalized expansions involving the D operator. Here we propose to examine some rather unusual variations that are not too well known, and which have applications to Fibonacci numbers among other things.

We shall need several other well-known operational formulas whose proofs involve some calculus and/or mathematical induction, and we tabulate these below:

$$\theta = D_z, \text{ where } x = e^z, \tag{5}$$

$$\theta^n = D_z^n, \tag{6}$$

$$x^n D_x^n = n! \binom{D_z}{n}, \tag{7}$$

where the binomial coefficient is defined as usual by $\binom{x}{n} = x(x-1) \ldots (x-n+1)/n!$, with $\binom{x}{0} = 1$.

$$e^D = 1 + \Delta = E, \tag{8}$$

where

$$\Delta f(x) = f(x+1) - f(x) \quad \text{and} \quad Ef(x) = f(x+1).$$

More generally

$$e^{tD_x} = f(x+t) = \underset{x,h}{E} f(x). \tag{9}$$

The q-operator

$$f(qx) = Qf(x), \text{ where } Q = q^{\theta}. \tag{10}$$

This was used, e.g., in [10], and is very convenient when working with basic hypergeometric series.

In the references at the end are several papers, viz. [1], [2], [4], [5], from the older literature where properties of a great number of familiar and unfamiliar operators were developed. The master calculator was almost certainly George Boole. The English literature for the period from about 1830 to 1890 is especially rich in papers on unusual operators.

In [1], Boole gave the pair of very remarkable operational expansions

$$f(x + \phi'(D))u(x) = e^{\phi(D)} f(x) e^{-\phi(D)} u(x), \tag{11}$$

and

$$f(D + \phi'(x))u(x) = e^{-\phi(x)} f(D) e^{\phi(x)} u(x), \tag{12}$$

which hold for arbitrary functions f, $\emptyset$, and u. The formulas are certainly true for polynomials, and in order to avoid matters of convergence of any series we shall explain that we interpret these as statements about formal power series. In that context there is no difficulty and we use formal power series definitions of all operators. Thus, if L is a linear operator, we should like to define e^L by

$$e^L = \sum_{k=0}^{\infty} \frac{1}{k!} L^k. \tag{13}$$

Boole's formulas (11)-(12) have a bearing on expansions in [7]. They are representative of some of the most unusual operational formulas.

But stranger still, we shall consider the operator L^L, which we define as follows:

$$\begin{aligned} L^L f(x) &= \left\{(L - 1) + 1\right\}^L f(x) = \sum_{n=0}^{\infty} \binom{L}{n} (L - 1)^n f(x) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} C_j^n L^j \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L^k f(x) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_j^n L^{j+k} f(x), \end{aligned} \tag{14}$$

where C_j^n are Stirling numbers of the first kind, i.e., coefficients in the expansion of a binomial coefficient:

$$\binom{x}{n} = \sum_{j=0}^{n} C_j^n x^j. \tag{15}$$

In the familiar notation of Riordan, $n! C_j^n = s(n, j)$.

For a particular choice of L we may be able to give a more compact definition. Thus, with $f = f(x)$,

$$\begin{aligned} D^D f &= \left\{(D - 1) + 1\right\}^D f = \sum_{n=0}^{\infty} \binom{D}{n} (D - 1)^n f \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} D_z^n (D_x - 1)^n f, \text{ by (7), with } z = e^x, \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_z^n D_x^k f(x). \end{aligned} \tag{16}$$

For an example of this expansion, let $f(x) = e^{ax}$. Then

$$D_x^k e^{ax} = a^k e^{ax} = a^k z^a,$$

whence

$$\begin{aligned} D^D e^{ax} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_z^n (a^k z^a) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k \binom{a}{n} n! z^{a-n} \\ &= z^a \sum_{n=0}^{\infty} \binom{a}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k \end{aligned}$$

$$= z^a \sum_{n=0}^{\infty} \binom{a}{n} (a - 1)^n = z^a a^a = a^a e^{ax},$$

so that we have the attractive formula

$$D^D e^{ax} = a^a e^{ax}. \tag{17}$$

It is instructive to compare this with $\theta^p x^k = k^p x^k$, and to recall a little terminology from vector analysis. A characteristic vector for a linear transformation L is a non-zero vector f such that $Lf = cf$ for some scalar c. With each operator we like to find a natural function or characteristic function. For the operator θ it is x^k, for D^D it is e^{ax}, etc.

Formula (17) allows us to write down symbolic sums for various peculiar series. Thus

$$\sum_{k=0}^{n-1} k^k e^{kx} t^k = \sum_{k=0}^{n-1} t^k D^D e^{kx} = D^D \sum_{k=0}^{n-1} (te^x)^k = D^D \left\{ \frac{t^n e^{nx} - 1}{te^x - 1} \right\}. \tag{18}$$

In particular,

$$\sum_{k=0}^{\infty} k^k t^k = D^D \left\{ \frac{1}{1 - te^x} \right\} \Bigg|_{x=0} . \tag{19}$$

All that would be necessary to sum (19) would be to find a different method of attaching a meaning to the right-hand member.

For a Fibonacci-Lucas application, recall the general Lucas function

$$L_n = L_n(a, b) = a^n + b^n.$$

Then

$$\sum_{k=0}^{n-1} k^k e^{kx} L_{pk} = D^D \left\{ \frac{a^{pn} e^{nx} - 1}{a^p e^x - 1} + \frac{b^{pn} e^{nx} - 1}{b^p e^x - 1} \right\}, \tag{20}$$

and for the generalized Fibonacci function

$$F_n = F_n(a, b) = (a^n - b^n)/(a - b),$$

then

$$(a - b) \sum_{k=0}^{n-1} k^k e^{kx} F_{pk} = D^D \left\{ \frac{a^{pn} e^{nx} - 1}{a^p e^x - 1} - \frac{b^{pn} e^{nx} - 1}{b^p e^x - 1} \right\} \tag{21}$$

Following the methods outlined in [3], [6], [8], [9], or [11], we could set down complicated symbolic formulas for the general series

$$\sum_{k=0}^{n-1} k^k e^{kx} u^k F_{pk}^r L_{qk}^s, \tag{22}$$

but we shall not take the space to exhibit the result.

For another application, let us rewrite (17) as $a^a = e^{-x} D^D e^{ax}$, so that we have an obvious application in the two forms

$$L_n^{L_n} = e^{-x} D^D e^{L_n x}, \quad F_n^{F_n} = e^{-x} D^D e^{F_n x}, \tag{23}$$

which allow us to introduce Fibonacci powers of Fibonacci (and Lucas powers of Lucas) numbers into known series. In particular,

$$\sum_{n=0}^{\infty} t^n L_n^{L_n} = e^{-x} D^D \sum_{n=0}^{\infty} t^n e^{L_n x}, \tag{24}$$

and a similar formula with F in place of L.

In principle then we could sum such series if we could sum the series

$$S(t, u) = \sum_{n=0}^{\infty} t^n u^{L_n}, \tag{25}$$

and

$$T(t,\ u) = \sum_{n=0}^{\infty} t^n u^{F_n}. \tag{26}$$

These are offered as research projects; the author would be interested in hearing of any success by others. $D_u S|_{u=1}$ and $D_u T|_{u=1}$ are known.

The operator θ^θ may be considered finally. We find

$$\begin{aligned}\theta^\theta f = \{(\theta - 1) + 1\}^\theta f &= \sum_{n=0}^{\infty} \binom{\theta}{n}(\theta - 1)^n f \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n (\theta - 1)^n f, \text{ by (7)}, \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \theta^k f,\end{aligned}$$

so we have

$$\theta^\theta f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \theta^k f(x). \tag{27}$$

Let $f(x) = x^p$, then

$$\begin{aligned}\theta^\theta x^p &= \sum_{n=0}^{\infty} \frac{x^n}{n!} D^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} p^k x^p = \sum_{n=0}^{\infty} \frac{x^n}{n!} (p - 1)^n \binom{p}{n} n! x^{p-n} \\ &= x^p \sum_{n=0}^{\infty} \binom{p}{n} (p - 1)^n = p^p x^p,\end{aligned}$$

or therefore, another formula analogous to (17),

$$\theta^\theta x^p = p^p x^p. \tag{28}$$

As an application we can get a different version of formula (19) as

$$\sum_{k=0}^{\infty} k^k x^k = \theta^\theta \sum_{k=0}^{\infty} x^k = \theta^\theta \left\{\frac{1}{1 - x}\right\}. \tag{29}$$

We wish to remark that even stranger formulas have been published. Cayley [4], [5] expressed the Lagrange series inversion formula in the most curious operational form

$$F(x) = (D_u)^{hD_h - 1}\left\{F'(u) e^{hf(u)}\right\}, \tag{30}$$

where $x = u + hf(x)$ and $F(x)$ is an arbitrary function. By differentiation, he expressed the second form of this expansion as

$$\frac{F(x)}{1 - hf'(x)} = (D_u)^{hD_h}\left\{F(u) e^{hf(u)}\right\}. \tag{31}$$

Cayley says these are well known, and goes on to write similar formulas for functions of several variables.

Bronwin [2] writes

$$f(a + x) = D_a^\theta \{f(a) e^x\} \tag{32}$$

as a symbolic form of Taylor's expansion. This is, of course, a special case of the Lagrange expansion.

In conclusion, we wish to emphasize that the formulas presented here are offered more for further research than as final answers to any of the questions raised. It certainly is possible to introduce unusual terms into generating functions by the use of unusual operators.

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(continued from page 497)

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A FIGURATE NUMBER CURIOSITY: EVERY INTEGER IS A QUADRATIC FUNCTION OF A FIGURATE NUMBER

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In this note we prove the following: Every positive integer n can be expressed in an infinite number of ways as a quadratic function for each of the infinite number of figurate number types.

The nth figurate r-sided number p_n^r is given by

$$p_n^r = n\big((r-2)n - r + 4\big)/2, \tag{1}$$

where $n = 1, 2, 3, \ldots$ and $r = 3, 4, 5, \ldots$. Therefore, the snth figurate number is given by

$$p_{sn}^r = sn\big((r-2)sn - r + 4\big)/2. \tag{2}$$

However, (2) is a quadratic in n. Solving for n and taking the positive root yields

$$n = \frac{(r-4) + \sqrt{(r-4)^2 + 8(r-2)p_{sn}^r}}{2(r-2)s}, \tag{3}$$

which allows us to express n as stated above. A special case of (3) for pentagonal numbers ($r = 5$) was obtained by Hansen [1].

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-388 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let T_n be the triangular number $n(n + 1)/2$. Show that

$$T_1 + T_2 + T_3 + \cdots + T_{2n-1} = 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2$$

and express these equal sums as a binomial coefficient.

B-389 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Find the complete solution, with two arbitrary constants, of the difference equation

$$(n^2 + 3n + 3)U_{n+2} - 2(n^2 + n + 1)U_{n+1} + (n^2 - n + 1)U_n = 0.$$

B-390 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Find, as a rational function of x, the generating function

$$G_k(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots + \binom{k+n}{k}x^n + \cdots, \quad |x| < 1.$$

B-391 *Proposed by M. Wachtel, Zurich, Switzerland.*

Some of the solutions of $5x^2 + 1 = y^2$ in positive integers x and y are $(x, y) = (4, 9)$, $(72, 161)$, $(1292, 2889)$, $(23184, 51841)$, and $(416020, 930249)$. Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n, y_n) and find $\lim_{n\to\infty}(x_{n+1}/x_n)$ in terms of $a = (1 + \sqrt{5})/2$.

B-392 *Proposed by Phil Mana, Albuquerque, NM.*

Let $Y_n = (2 + 3n)F_n + (4 + 5n)L_n$. Find constants h and k such that

$$Y_{n+2} - Y_{n+1} - Y_n = hF_n + kL_n.$$

B-393 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $T_n = \binom{n+1}{2}$, $P_0 = 1$, $P_n = T_1T_2 \cdots T_n$ for $n > 0$, and $\begin{bmatrix} n \\ k \end{bmatrix} = P_n/P_kP_{n-k}$ for integers k and n with $0 \leq k \leq n$. Show that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{n - k + 1}\binom{n}{k}\binom{n+1}{k+1}.$$

SOLUTIONS

INCONTIGUOUS ZERO DIGITS

B-364 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

Find and prove a formula for the number $R(n)$ of positive integers less than 2^n whose base 2 representations contain no consecutive 0's. (Here n is a positive integer.)

Solution by C. B. A. Peck, State College, PA.

Let S_n be the number of integers m with $2^{n-1} \leq m < 2^n$ and having a binary representation $B(m)$ with no consecutive pair of 0's. Clearly $S_n = R_n - R_{n-1}$ for $n > 1$ and $S_1 = R_1$. Also,

$$S_n = S_{n-1} + S_{n-2} \text{ for } n > 2,$$

since S_{n-1} counts the desired m for which $B(m)$ starts with 11 and S_{n-2} counts the desired m for which $B(m)$ starts with 101. It follows inductively that $S_n = F_{n+1}$, and then

$$R_n = S_1 + S_2 + \cdots + S_n = F_2 + F_3 + \cdots + F_{n+1} = F_{n+3} - 2.$$

Also solved by Michael Brozinsky, Paul S. Bruckman, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, Rolf Sonntag, Gregory Wulczyn, and the proposer.

CONGRUENT TO A G.P.

B-365 *Proposed by Phil Mana, Albuquerque, NM*

Show that there is a unique integer $m > 1$ for which integers a and r exist with $L_n \equiv ar^n \pmod{m}$ for all integers $n \geq 0$. Also, show that no such m exists for the Fibonacci numbers.

Solution by Graham Lord, Université Laval, Québec.

Since $7 = L_4L_1 \equiv a^2r^5 \equiv L_2L_3 = 12 \pmod{m}$, then m divides 5, hence $m = 5$. Furthermore, $a = ar^0 \equiv L_0 = 2 \pmod{5}$. And finally, $ar^2 \equiv L_2 = L_1 + L_0 \equiv ar + a \pmod{5}$ together with $a \equiv 2 \pmod{5}$ implies $r^2 \equiv r + 1 \pmod{5}$, i.e., $r \equiv 3 \pmod{5}$. In all, $m = 5$, and a and r can be taken equal to 2 and 3, respectively. Note for any $n \geq 1$, $L_{n+1} = L_n + L_{n-1} \equiv ar^n + ar^{n-1} \equiv ar^{n+1} \pmod{5}$.

For the Fibonacci numbers, if m were to exist, then

$$3 = F_1F_4 \equiv a^2r^5 \equiv F_2F_3 = 2 \pmod{m},$$

i.e., $1 \equiv 0 \pmod{m}$, which is impossible if $m > 1$.

Also solved by George Berzsenyi, Paul S. Bruckman, Bob Prielipp, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and the proposer.

LUCAS CONGRUENCE

B-366 *Proposed by Wray G. Brady, University of Tennessee, Knoxville, TN and Slippery Rock State College, Slippery Rock, PA.*

Prove that $L_iL_j \equiv L_hL_k \pmod{5}$ when $i + j = h + k$.

Solution by Paul S. Bruckman, Concord, CA and Sahib Singh, Clarion State College, Clarion, PA (independently).

Using the result of B-365,

$$L_iL_j - L_hL_k \equiv 2 \cdot 3^{i+j} - 2 \cdot 3^{h+k} \equiv 0 \pmod 5,$$

since $i + j = h + k$.

Also solved by George Berzsenyi, Herta T. Freitag, Graham Lord, T. Ponnudurai, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the proposer.

ROUNDING DOWN

B-367 *Proposed by Gerald E. Bergum, Sr., Dakota State University, Brookings, SD.*

Let $[x]$ be the greatest integer in x, $a = (1 + \sqrt{5})/2$ and $n \geq 1$. Prove that

(a) $F_{2n} = [aF_{2n-1}]$

and

(b) $F_{2n+1} = [a^2F_{2n-1}]$.

Solution by George Berzsenyi, Lamar University, Beaumont, TX.

In view of Binet's formula,

$$aF_{2n-1} - F_{2n} = a\frac{a^{2n-1} - b^{2n-1}}{a - b} - \frac{a^{2n} - b^{2n}}{a - b} = -b^{2n-1}.$$

Similarly,

$$a^2F_{2n-1} - F_{2n+1} = a^2\frac{a^{2n-1} - b^{2n-1}}{a - b} - \frac{a^{2n+1} - b^{2n+1}}{a - b} = -b^{2n-1}.$$

Since $-1 < b = \dfrac{1 - \sqrt{5}}{2} < 0$ implies that $0 < -b^{2n-1} < 1$, the desired results follow.

Also solved by J. L. Brown, Jr., Paul S. Bruckman, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

CONVOLUTING FOR CONGRUENCES

B-368 *Proposed by Herta T. Freitag, Roanoke, VA.*

Obtain functions $g(n)$ and $h(n)$ such that

$$\sum_{i=1}^{n} iF_iL_{n-i} = g(n)F_n + h(n)L_n$$

and use the results to obtain congruences modulo 5 and 10.

Solution by Sahib Singh, Clarion State College, Clarion, PA.

Let $A_n = \sum_{i=1}^{n} iF_iL_{n-i}$. Then the generating function $A_1 + A_2x + A_3x^2 + \cdots$ is a rational function with $(1 - x - x^2)^3$ as the denominator. It follows that $g(n)$ and $h(n)$ are quadratic functions of n. Then, solving simultaneous equations for the coefficients of these quadratics leads to

$$g(n) = (5n^2 + 10n + 4)/10 \quad \text{and} \quad h(n) = n/10$$

so that

$$(5n^2 + 4)F_n + nL_n \equiv 0 \pmod{10}.$$

This also gives us $nL_n \equiv F_n \pmod 5$.

Also solved by Paul S. Bruckman, Graham Lord, Gregory Wulczyn, and the proposer.

NO LONGER UNSOLVED

B-369 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

For all integers $n \geq 0$, prove that the set

$$S_n = \{L_{2n+1}, L_{2n+3}, L_{2n+5}\}$$

has the property that if $x, y \in S_n$ and $x \neq y$ then $xy + 5$ is a perfect square. For $n = 0$, verify that there is no integer z that is not in S_n and for which $\{z, L_{2n+1}, L_{2n+3}, L_{2n+5}\}$ has this property. (For $n > 0$, the problem is unsolved.)

Solution by Graham Lord, Université Laval, Québec.

That S_n has the property follows from the identities:

$$L_{2n+1}L_{2n+3} + 5 = L^2_{2n+2},$$

and

$$L_{2n+1}L_{2n+5} + 5 = L^2_{2n+3}.$$

In the second part of this solution use is made of the results:

(1) $2 \nmid L_{6k+1}$ and $2 \nmid L_{6k+5}$

(2) $4 = L_3 | L_{6k+3}$

(3) $4 \nmid L_{2k}$

(4) $4 \nmid F_{6k+3}$

Of these, (1) is somewhat well known and the latter three are consequences of the results in "A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February 1964), pp. 15-28, by L. Carlitz.

By (1) and (2) there is exactly one even number, L_{6k+3}, in the set S_n, $n \geq 0$. So if $\{z\} \cup S_n$ has the desired property, then $zL_{6k+3} + 5$ will be an odd square and thus congruent to 1 modulo 8. This implies that z, if it exists, is odd.

Now the other two members of S_n are either:

(a) L_{6k-1}, L_{6k+1}; (b) L_{6k+5}, L_{6k+7}; or (c) L_{6k+1}, L_{6k+5}.

Each of these is odd by (1), and hence the sum of 5 and any one of them multiplied by z will equal an even square. Thus, in case (a) [and similarly in case (b)]:

$$zL_{6k-1} + 5 \equiv 0 \pmod 4, \text{ and } zL_{6k+1} + 5 \equiv 0 \pmod 4;$$

i.e.,

$$zL_{6k} = z(L_{6k+1} - L_{6k-1}) \equiv 0 \pmod 4.$$

But this is impossible by (3) and the fact that z is odd.

And in case (c),

$$z \cdot 5F_{6k+3} = (zL_{6k+5} + 5) - (zL_{6k+1} + 5) \equiv 0 \pmod 4,$$

which is also impossible by (4).

Consequently, no z exists such that the set $\{z\} \cup S_n$ has the desired property. Note that it was not assumed that $n = 0$.

Also solved by Paul S. Bruckman, Herta T. Freitag, T. Ponnudurai, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvanis 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-290 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Show that:

(a) $F_k F^2_{k+6r+3} - F^3_{k+4r+2} = (-1)^{k+1} F^2_{2r+1}(F_{k+8r+4} - 2F_{k+4r+2})$;

(b) $F_k F^2_{k+6r} - F^3_{k+4r} = (-1)^{k+1} F^2_{2r}(F_{k+8r} + 2F_{k+4r})$.

H-291 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX*

Prove that there are infinitely many squares which are differences of consecutive cubes.

H-292 *Proposed by F. S. Cater and J. Daily, Portland State University, Portland, OR.*

Find all real numbers $r \in (0,1)$ for which there exists a one-to-one function f_r mapping $(0,1)$ onto $(0,1)$ such that

(1) f_r and f_r^{-1} are infinitely many times differentiable on $(0,1)$, and
(2) the sequence of functions $f_r,\ f_r \circ f_r,\ f_r \circ f_r \circ f_r,\ f_r \circ f_r \circ f_r \circ f_r,\ \ldots$ converges pointwise to r on $(0,1)$.

H-293 *Proposed by Leonard Carlitz, Duke University, Durham, NC.*

It is known that the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ defined by

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz - z^2}$$

satisfy the relation

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{z^n}{n!} = e^{2xz - z^2} H_k(x - z) \quad (k = 0, 1, 2, \ldots).$$

Show that conversely if a set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ satisfy

$$(1) \qquad \sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!} f_k(x - z) \quad (k = 0, 1, 2, \ldots),$$

where $f_0(x) = 1$, $f_1(x) = 2x$, then

$$f_n(x) = H_n(x) \quad (n = 0, 1, 2, \ldots).$$

H-294 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Evaluate

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & F_{12r+6} & F_{20r+10} & F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{36r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{20r+15} & F_{50r+25} & F_{70r+36} & F_{50r+45} \end{vmatrix}$$

SOLUTIONS

SYMMETRIC SUM

H-272 (Corrected) *Proposed by Leonard Carlitz, Duke University, Durham, NC.*

Show that

$$\sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \Big/ \binom{m}{j} \equiv C_m(p,\ q,\ r)$$

is symmetric in p, q, r.

Solution by Paul Bruckman, Concord, CA.

Define

$$C_m(p,\ q,\ r) = \sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \Big/ \binom{m}{j}. \tag{1}$$

Clearly, $C_m(p,\ q,\ r) = C_m(q,\ p,\ r)$. A moment's reflection reveals that it therefore suffices to show that $C_m(p,\ q,\ r) = C_m(q,\ r,\ p)$. Replacing j by $m - j$ in (1) and applying Vandermonde's convolution theorem on the term involving p and q yields:

$$\begin{aligned} C_m(p,\ q,\ r) &= \sum_{j=0}^{m} \binom{r}{m-j}\binom{p}{j}\binom{q}{j} \Big/ \binom{m}{j} \sum_{k=0}^{m-j} \binom{p-j}{m-j-k}\binom{q}{k} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{r}{m-j}\binom{q}{j}\binom{q}{k}\binom{p}{m-k}\binom{m-k}{j} \Big/ \binom{m}{j}. \end{aligned}$$

Replacing k by $m - k$ in the last expression yields:

$$\begin{aligned} C_m(p,\ q,\ r) &= \sum_{j=0}^{m} \sum_{k=j}^{m} \binom{r}{m-j}\binom{q}{j}\binom{q}{m-k}\binom{p}{k}\binom{k}{j} \Big/ \binom{m}{j} \\ &= \sum \binom{p}{k}\binom{q}{m-k} \sum \binom{r}{m-j}\binom{q}{j}\binom{k}{j} \Big/ \binom{m}{j}. \end{aligned}$$

However, it is easy to verify that

$$\binom{r}{m-j} \Big/ \binom{m}{j} = \binom{r}{m} \Big/ \binom{r-m+j}{j} = \binom{r}{m}(-1)^j \Big/ \binom{m-r-1}{j}. \tag{2}$$

Therefore,

$$C_m(p,\ q,\ r) = \binom{r}{m} \sum_{k=0}^{m} \binom{p}{k}\binom{q}{m-k} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\binom{q}{j} \Big/ \binom{m-r-1}{j}. \tag{3}$$

Now, formula (7.1) in *Combinatorial Identities* (H. W. Gould, Morgantown, 1972), is as follows:

$$(4) \qquad \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{z}{k} \Big/ \binom{y-z}{n} \Big/ \binom{y}{n}.$$

Letting $k = j$, $n = k$, $z = q$, $y = m - r - 1$ in (4), we therefore simplify (3) as follows:

$$C_m(p,\ q,\ r) = \binom{r}{m}\sum_{k=0}^{m}\binom{p}{k}\binom{q}{m-k}\binom{m-r-q-1}{k}\Big/\binom{m-r-1}{k}$$

$$= \binom{r}{m}\sum_{k=0}^{m}\binom{p}{k}\binom{q}{m-k}\binom{q+r-m+k}{k}\Big/\binom{r-m+k}{k};$$

now using (2) once again and replacing k by j yields:

$$C_m(p,\ q,\ r) = \sum_{j=0}^{m}\binom{p}{j}\binom{q}{m-j}\binom{r}{m-j}\binom{q+r-m+j}{j}\Big/\binom{m}{j}$$

$$= C_m(q,\ r,\ p). \quad \text{Q.E.D.}$$

Also solved by the proposer.

A RAY OF LUCAS

H-273 *Proposed by W. G. Brady, Slippery Rock State College, Slippery Rock, PA.*

Consider, after Hoggatt and H-257, the array D, indicated below, in which L_{2n+1} (n = 0, 1, 2, ...) is written in staggered columns:

1				
4	1			
11	4	1		
29	11	4	1	
76	29	11	4	1

i. Show that the row sums are $L_{2n+2} - 2$;
ii. Show that the rising diagonal sums are $F_{2n+3} - 1$ where L_{2n+1} is the largest element in the sum.
iii. Show that if the columns are multiplied by 1, 2, 3, ... sequentially to the right then the row sums are $L_{2n+3} - (2n + 3)$.

Solution by A. G. Shannon, The N.S.W. Institute of Technology, Australia.

In effect we are asked to prove:

i. $$\sum_{j=0}^{n} L_{2n-2j+1} = L_{2n+2} - 2;$$

ii. $$\sum_{j=0}^{[n/2]} L_{2n-4j+1} = F_{2n+3} - 1;$$

iii. $$\sum_{j=0}^{n} (j+1)L_{2n-2j+1} = L_{2n+3} - (2n+3).$$

(i) $$\sum_{j=0}^{n} L_{2n-2j+1} = \sum_{j=0}^{n}\left(L_{2n-2(j-1)} - L_{2n-2j}\right) = \sum_{j=0}^{n} L_{2n-2(j-1)} - \sum_{j=1}^{n+1} L_{2n-2(j-1)}$$

$$= L_{2n+2} - L_0, \text{ as required.}$$

(ii) $$\sum_{j=0}^{[n/2]} L_{2n-4j+1} = \sum_{j=0}^{[n/2]} \left(F_{2n-4j+3} - F_{2n-4j-1}\right) = \sum_{j=0}^{[n/2]} F_{2n-4j+3} - \sum_{j=1}^{[n/2]+1} F_{2n-4j+3}$$

$$= F_{2n+3} - F_1\sigma(2, n) - F_{-1}\sigma(2, n+1) = F_{2n+3} - 1$$

in which

$$\sigma(n, m) = \begin{cases} 1 & \text{if } n|m, \\ 0 & \text{if } n\nmid m. \end{cases}$$

(iii) $$\sum_{j=0}^{n} (j+1)L_{2n-2j+1} = \sum_{i=0}^{n}\sum_{j=1}^{n} L_{2n-2j+1} = \sum_{i=0}^{n}\left(L_{2n-2i+2} - 2\right)$$

[from (i)]

$$= \sum_{i=0}^{n}\left(L_{2n-2i+3} - L_{2n-2i+1} - 2\right)$$

$$= \sum_{i=0}^{n+1} L_{2(n+1)-2i+1} - L_1 - \left(L_{2n+2} - 2\right) - 2(n+1)$$

$$= L_{2n+4} - 2 - 1 - L_{2n+2} + 2 - 2(n+1)$$

$$= L_{2n+3} - (2n+3), \text{ as required.}$$

Also solved by P. Bruckman, G. Wulczyn, H. Freitag, B. Prielipp, Dinh Thế'Hūng, and the proposer.

Late Acknowledgments: F. T. Howard solved H-268 and M. Klamkin solved H-270.

A CORRECTED OLDIE

H-225 *Proposed by G. A. R. Guillotte, Quebec, Canada.*

Let p denote an odd prime and $x^p + y^p = z^p$ for positive integers, x, y, and z. Show that

(A) $p < x/(z-x) + y/(z-y)$, and

(B) $z/2(z-x) < p < y/(z-y)$.

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VOLUME INDEX

ADLER, IRVING. "A Simple Continued Fraction Represents a Mediant Nest of Intervals," Vol. 16, No. 6, pp. 527-529.

ANDERSON, O. D. "Some More Patterns from Pascal's Triangle," Vol. 16, No. 4, pp. 296-301.

ANDERSON, PETER G. "On the Formula $\pi = 2\Sigma$ arcot f_{2k+1}," Vol. 16, No. 2, p. 118.

AUSTIN, RICHARD. "Binary Sequences without Isolated Ones," Vol. 16, No. 1, pp. 84-86 (co-author, Richard Guy).

BEITER, MARION. "Coefficients of the Cyclotomic Polynomials $F_{3qr}(x)$," Vol. 16, No. 4, pp. 302-306.

BERGUM, G. E. "A Combinatorial Problem Involving Recursive Sequences and Tridiagonal Matrices," Vol. 16, No. 2, pp. 113-118 (co-author, V. E. Hoggatt, Jr.); "A Family of Tridiagonal Matrices," Vol. 16, No. 3, pp. 285-288 (co-author, V. E. Hoggatt, Jr). Problem proposed: B-367, Vol. 16, No. 6, p. 564. Problem solved: B-367, Vol. 16, No. 6, p. 564.

BERNSTEIN, LEON. "An Invariant for Combinatorial Identities," Vol. 16, No. 4, pp. 354-369.

BERZSENYI, GEORGE. Problems proposed: B-351, Vol. 16, No. 1, p. 91; H-266, Vol. 16, No. 1, p. 94; B-378, Vol. 16, No. 2, p. 184; B-387, Vol. 16, No. 5, p. 473, H-269, Vol. 16, No. 5, p. 478; B-364, B-369, Vol. 16, No. 6, pp. 563, 565, H-291, Vol. 16, No. 6, p. 566. Problems solved: B-346, B-347, B-248, B-349, B-350, B-351, Vol. 16, No. 1, pp. 89-91, H-266, Vol. 16, No. 1, p. 94; B-352, B-356, Vol. 16, No. 2, pp. 185, 186; B-358, B-359, B-362, Vol. 16, No. 5, pp. 474, 475, H-269, Vol. 16, No. 5, p. 478; B-364, B-365, B-366, B-367, B-369, Vol. 16, No. 6, pp. 563-565.

BEVERAGE, D. Problems Solved: H-264, H-265, Vol. 16, No. 1, p. 93. Problem Proposed: H-283, Vol. 16, No. 2, p. 188.

BICKNELL-JOHNSON, MARJORIE. "A Golden Double Crostic," Vol. 16, No. 1, pp. 67-69; "A Golden Double Crostic Solution," Vol. 16, No. 1, p. 83; "A Primer for the Fibonacci Numbers XVII: Generalized Fibonacci Numbers Satisfying $u_{n+1}u_{n-1} - u_n^2 = \pm 1$," Vol. 16, No. 2, pp. 130-137 (co-author, V. E. Hoggatt, Jr.); "Properties of Generating Functions of a Convolution Array," Vol. 16, No. 4, pp. 289-295 (co-author, V. E. Hoggatt, Jr.); "Convolution Arryas for Jacobsthal and Fibonacci Polynomials," Vol. 16, No. 5, pp. 385-402 (co-author, V. E. Hoggatt, Jr.); "Fibonacci Chromotology or How To Paint Your Rabbit," Vol. 16, No. 5, pp. 426-428; "Divisibility Properties of Polynomials in Pascal's Triangle," Vol. 16, No. 6, pp. 501-513 (co-author, V. E. Hoggatt, Jr.).

BRADY, WRAY G. "More on Benford's Law," Vol. 16, No. 1, pp. 51-52. Problems proposed: B-366, Vol. 16, No. 6, p. 563, H-273, Vol. 16, No. 6, p. 568. Problems solved: B-347, B-348, B-349, Vol. 16, No. 1, pp. 89-91; B-352, B-354, B-355, Vol. 16, No. 2, pp. 185-186; H-270, Vol. 16, No. 5, p. 479; B-366, Vol. 16, No. 6, p. 563, H-273, Vol. 16, No. 6, p. 568.

BRIDGER, CLYDE A. Problem solved: B-347, Vol. 16, No. 1, p. 90.

BROUSSEAU, BROTHER ALFRED. "Formula Development through Finite Differences," Vol. 16, No. 1, pp. 53-67.

BROWN, J. L., JR. "Some Sequence-to-Sequence Transformations which Preserve Completeness," Vol. 16, No. 1, pp. 19-22; "A Primer for the Fibonacci Numbers, Part XVI, The Central Column Sequence," Vol. 16, No. 1, pp. 41-46 (co-author, V. E. Hoggatt, Jr.); "Simplified Proof of a Greatest Integer Function Theorem," Vol. 16, No. 4, pp. 307-309. Problem Solved: B-367, Vol. 16, No. 6, p. 564.

BROZINSKY, MICHAEL. Problems Solved: B-351, Vol. 16, No. 1, p. 91; B-359, Vol. 16, No. 5, p. 474; B-364, Vol. 16, No. 6, p. 563.

BRUCKMAN, PAUL S. Problems proposed: H-280, Vol. 16, No. 1, p. 92; B-377, Vol. 16, No. 2, p. 184; H-268, Vol. 16, No. 5, p. 477. Problems Solved: B-346, B-347, B-348, B-349, B-350, B-351, Vol. 16, No. 1, pp. 89-91, H-264, H-265, H-266, Vol. 16, No. 1, pp. 94-95; B-352, B-353, B-354, B-355, B-356, B-357, Vol. 16, No. 2, pp. 185-186, H-267, H-268, Vol. 16, No. 2, pp. 191-192; B-358, B-359, B-360, B-361, B-362, B-363, Vol. 16, No. 5, pp. 474-476, H-269, H-270, Vol. 16, No. 5, pp. 478-479; B-364, B-365, B-366, B-367, B-368, B-369, Vol. 16, No. 6, pp. 563-565.

BUNDER, M. W. "More Fibonacci Functions," Vol. 16, No. 2, pp. 94-98.

BUTCHER, J. C. "On a Conjecture Concerning a Set of Sequences Satisfying the Fibonacci Difference Equation," Vol. 16, No. 1, pp. 80-83.

CARLITZ, LEONARD. "Recurrence of the Third-Order and Related Combinatorial Identities," Vol. 16, No. 1, pp. 11-18; "Generalized Eulerian Numbers and Polynomials," Vol. 16, No. 2, pp. 138-146, 151 (co-author, V. E. Hoggatt, Jr.); "Some Polynomials Related to Fibonacci and Eulerian Numbers," Vol. 16, No. 3, pp. 216-226; "A Recurrence Suggested by a Combinatorial Problem," Vol. 16, No. 3, pp. 227-242; "Some Remarks on a Combinatorial Identity," Vol. 16, No. 3, pp. 243-248; "Enumeration of Certain Weighted Sequences," Vol. 16, No. 3, pp. 249-254; "The Number of Derangements of a Sequence with Given Specification," Vol. 16, No. 3, pp. 255-258; "Enumeration of Permutations by Sequence," Vol. 16, No. 3, pp. 259-268; "Some Classes of Fibonacci Sums," Vol. 16, No. 5, pp. 411-426. Problems Solved: H-264, H-265, Vol. 16, No. 1, pp. 93-94; H-268, Vol. 16, No. 2, p. 191; B-361, Vol. 16, No. 5, p. 475, H-270, Vol. 16, No. 5, p. 479; H-272, Vol. 16, No. 6, p. 567. Problems Proposed: H-264, Vol. 16, No. 1, p. 92; H-268, Vol. 16, No. 2, p. 191; B-361, Vol. 16, No. 5, p. 475; H-289, H-270, Vol. 16, No. 5, pp. 477, 479; H-293, Vol. 16, No. 6, p. 566.

CATER, F. S. Problem Proposed: H-292, Vol. 16, No. 6, p. 566 (co-proposer, J. Daily).

COHEN, G. L. "On Odd Perfect Numbers," Vol. 16, No. 6, pp. 523-527.

CULL, PAUL. "Knight's Tour Revisited," Vol. 16, No. 3, pp. 276-284 (co-author, Jeffery De Curtins).

DAILY, J. Problem Proposed: H-292, Vol. 16, No. 6, p. 566 (co-proposer, F.S. Cater).

DE CURTINS, JEFFERY. "Knight's Tour Revisited," Vol. 16, No, 3, pp. 276-284 (co-author, Paul Cull).

DESMOND, JAMES E. "On the Existence of the Rank of Apparition of m in the Lucas Sequence," Vol. 16, No. 1, pp. 7-10; "on the Equality of Periods of Different Moduli in the Fibonacci Sequence." Vol. 16, No. 1, pp. 86-87.

DE VITA, JOSEPH. "Fibonacci, Insects, and Flowers," Vol. 16, No. 4, pp. 315-317.

DRESSLER, ROBERT E. "Interpolation of Fourier Transforms on Sums of Fibonacci Numbers," Vol. 16, No. 3, pp. 193-194 (co-author, Louis Pigno); "Topological, Measure Theoretic and Analytic Properties of the Fibonacci Numbers," Vol. 16, No. 3, pp. 195-197 (co-author, Louis Pigno).

ENGLE, ROGER. Problems Solved: B-358, B-360, B-361, B-362, B-363, Vol. 16, No. 5, pp. 474-476 (co-solver, Sahib Singh); B-359, Vol. 16, No. 5, p. 474 (co-solvers, Benjamin Freed & Sahib Singh).

ESWARATHASAN, A. "On Square Pseudo-Fibonacci Numbers," Vol. 16, No. 4, pp. 310-314.

FERGUSON, HELAMAN ROLFE PRATT. "The Fibonacci Pseudogroup, Characteristic Polynomials and Eigenvalues of Tridiagonal Matrices, Periodic Linear Recurrence Systems and Application to Quantum Mechanics," Vol. 16, No. 5, pp. 435-447.

FIELD, R. S. Problem Proposed: B-359, Vol. 16, No. 5, p. 474. Problem solved: B-359, Vol. 16, No. 5, p. 474.

FLANIGAN, JIM. "Generalized Two-Pile Fibonacci Nim," Vol. 16, No. 5, pp. 459-469.

FREED, BENJAMIN. Problem Solved: B-359, Vol. 16, No. 5, p. 474 (co-solvers, Roger Engle & Sahib Singh).

FREITAG, HERTA T. Problems Proposed: B-371, B-372, Vol. 16, No. 1, p. 88; B-379, B-356, Vol. 16, No. 2, pp. 184, 186; B-385, B-362, B-363, Vol. 16, No. 5, pp. 473, 475-476; B-388, B-368, Vol. 16, No. 6, pp. 562, 564. Problems Solved: B-346, B-348, B-350, B-351, Vol. 16, No. 1, pp. 89-91; B-352, B-353, B-355, B-356, Vol. 16, No. 2, pp. 185-186; B-358, B-362, B-363, Vol. 16, No. 5, pp. 474-476; B-366, B-368, B-369, Vol. 16, No. 6, pp. 563-565, H-273, Vol. 16, No. 6, p. 568.

FULLER, LEONARD E. "Vectors Whose Elements Belong to a Generalized Fibonacci Sequence," Vol. 16, No. 5, pp. 447-450.

GARFIELD, RALPH. Problems Solved: B-346, B-348, B-349, Vol. 16, No. 1, pp. 89-91; B-352, B-353, B-354, B-355, Vol. 16, No. 2, pp. 185-186.

GERDES, WALTER. "Convergent Sequences and Generalized Fibonacci Numbers," Vol. 16, No. 3, pp. 269-275.

GIULI, ROBERT M. Problem Solved: B-361, Vol. 16, No. 5, p. 475.

GLADWIN, A. S. "Expansion of the Fibonacci Numbers F_{nm} in nth Powers of Fibonacci or Lucas Numbers," Vol. 16, No. 3, pp. 213-215.

GOULD, H. W. "Remark on Problem H-123," Vol. 16, No. 2, p. 189; "Evaluation of Sums of Convolved Powers Using Stirling and Eulerian Numbers," Vol. 16, No. 6, pp. 488-497, 560-561; "Operational Formulas for Unusual Fibonacci Series," Vol. 16, No. 6, pp. 555-560. Problem Proposed: H-282, Vol. 16, No. 2, p. 188 (co-proposer, W. E. Greig).

GRASSL, RICHARD M. Problems Proposed: B-349, B-350, Vol. 16, No. 1, pp. 90-91. Problems Solved: B-349-B-350, Vol. 16, No. 1, pp. 90-91.

GREGORY, M. B. "Fibonacci Sine Sequences," Vol. 16, No. 2, pp. 119-120 (co-author, J. M. Metzger).

GREIG, W. E. "On Generalized $G_{j,k}$ Numbers," Vol. 16, No. 2, pp. 166-170; "Folded Sequences and Bode's Problem," Vol. 16, No. 6, pp. 530-539. Problem Proposed: H-282, Vol. 16, No. 2, p. 188 (co-proposer, H. W. Gould).

GUARALDO, ROSALIND. "On the Density of the Image Sets of Certain Arithmetic Functions—I," Vol. 16, No. 4, pp. 318-326; "On the Density of the Image Sets of Certain Arithmetic Functions—II," Vol. 16, No. 5, pp. 428-434; "On the Density of the Image Sets of Certain Arithmetic Functions—III," Vol. 16, No. 6, pp. 481-488.

GUERIN, E. E. "Matrices and Convolutions of Arithmetic Functions," Vol. 16, No. 4, pp. 327-334.

GUILLOTTE, G. A. R. Problem Proposed: H-225, Vol. 16, No. 6, p. 569. Problems Solved: H-272(corrected), H-273, Vol. 16, No. 6, pp. 567-568.

GUPTA, HANSRAJ. "The Rank-Vector of a Partition," Vol. 16, No. 6, pp. 548-552; "The Andrews Formula for Fibonacci Numbers," Vol. 16, No. 6, pp. 552-555.

GUY, RICHARD. "Binary Sequences without Isolated Ones," Vol. 16, No. 1, pp. 84-86 (co-author, Richard Austin).

HANSEN, RODNEY T. "General Identities for Linear Fibonacci and Lucas Summations," Vol. 16, No. 2, pp. 121-128.

HARBORTH, HEIKO. "b-Adic Numbers in Pascal's Triangle Modulo b," Vol. 16, No. 6, pp. 497-500.

HEED, JOSEPH L. "Entry Points of the Fibonacci Sequence and the Euler ϕ Function," Vol. 16, No. 1, pp. 47-50 (co-author, Lucille Kelly).

HENDY, M. D. "Stolarsky's Distribution of Positive Integers," Vol. 16, No. 1, pp. 70-80.

HENSLEY, DOUGLAS. "Fibonacci Tiling and Hyperbolas," Vol. 16, No. 1, pp. 37-40.

HERGET, W. "Minimum Periods Modulo n for Bernoulli Numbers," Vol. 16, No. 6, pp. 544-548.

HICKERSON, DEAN R. "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," Vol. 16, No. 1, pp. 5-6; "An Identity Relating Compositions and Partitions," Vol. 16, No. 1, pp. 23-26.

HIGGINS, FRANK. Problem Proposed: B-357, Vol. 16, No. 2, p. 186. Problem Solved: B-357, Vol. 16, No. 2, p. 186.

HILLMAN, A. P. "A Property of Wythoff Pairs," Vol. 16, No. 5, p. 472 (co-author, V. E. Hoggatt, Jr.). Editor: "Elementary Problems and Solutions," Vol. 16, No. 1, pp. 88-91, 96; Vol. 16, No. 2, pp. 184-187; Vol. 16, No. 5, pp. 473-476; Vol. 16, No. 6, pp. 562-565.

HOGGATT, V. E., JR. "A Primer for the Fibonacci Numbers, Part XVI, The Central Column Sequence," Vol. 16, No. 1, pp. 41-46 (co-author, J. L. Brown, Jr.); "A Combinatorial Problem Involving Recursive Sequences and Tridiagonal Matrices," Vol. 16, No. 2, pp. 113-118 (co-author, G. E. Bergum); "A Primer for the Fibonacci Numbers XVII: Generalized Fibonacci Numbers

Satisfying $u_{n+1}u_{n-1} - u_n^2 = \pm 1$," Vol. 16, No. 2, pp. 130-137 (co-author, Marjorie Bicknell-Johnson); "Generalized Eulerian Numbers and Polynomials," Vol. 16, No. 2, pp. 138-146, 151 (co-author Leonard Carlitz); "A Family of Tridiagonal Matrices," Vol. 16, No. 3, pp. 285-288 (co-author, G. E. Bergum); "Properties of Generating Functions of a Convolution Array," Vol. 16, No. 4, pp. 289-295 (co-author Marjorie Bicknell-Johnson); "Convolution Arrays for Jacobsthal and Fibonacci Polynomials," Vol. 16, No. 5, pp. 385-402 (co-author, Marjorie Bicknell-Johnson); "A Property of Wythoff Pairs," Vol. 16, No. 5, p. 472 (co-author, A. P. Hillman); "Divisibility Properties of Polynomials in Pascal's Triangle," Vol. 16, No. 6, pp. 501-513 (co-author, Marjorie Bicknell-Johnson). Problems Proposed: B-375, Vol. 16, No. 1, p. 88, B-346, B-347, Vol. 16, No. 1, p. 89; H-278, H-265, Vol. 16, No. 1, pp. 91, 94; B-381, B-352, B-353, Vol. 16, No. 2, pp. 184-185, H-281, Vol. 16, No. 2, p. 188, H-267(corrected), Vol. 16, No. 2, p. 190; H-285, Vol. 16, No. 5, p. 477; B-390, B-393, Vol. 16, No. 6, p. 562. Problems Solved: B-346, Vol. 16, No. 1, p. 89; B-352, B-353, Vol. 16, No. 2, p. 185; H-267, Vol. 16, No. 2, p. 190.

HORADAM, A. F. "Diagonal Functions," Vol. 16, No. 1, pp. 33-36; "Wythoff Pairs," Vol. 16, No. 2, pp. 147-151.

HORADAM, E. M. "Solved, Semi-Solved, and Unsolved Problems in Generalized Integers: A Survey," Vol. 16, No. 4, pp. 370-381.

HOWARD, F. Problems Solved: H-268, Vol. 16, No. 2, pp. 191-192.

HUNG, DINH THÊ'. Problems Solved: B-348, B-351, Vol. 16, No. 1, pp. 90, 91; B-352, B-353, B-354, Vol. 16, No. 2, pp. 185-186; H-273, Vol. 16, No. 6, p. 568.

HUNTER, J. A. H. "Congruent Primes of Form $(8r + 1)$," Vol. 16, No. 5, pp. 407-411.

IVIE, JOHN. Problems Solved: B-352, B-353, Vol. 16, No. 2, p. 185.

JONES, BURTON W. "A Second Variation on a Problem of Diophantus and Davenport," Vol. 16, No. 2, pp. 155-165.

KELLY, LUCILLE. "Entry Points of the Fibonacci Sequence and the Euler ϕ Function," Vol. 16, No. 1, pp. 47-50 (co-author, Joseph J. Heed).

KIMBERLING, CLARK. "Strong Divisibility Sequences with Nonzero Initial Term," Vol. 16, No. 6, pp. 541-544.

KLAMIN, M. Problem Solved: H-270, Vol. 16, No. 5, p. 479.

KNUTH, DONALD E. "Identities from Partition Involutions," Vol. 16, No. 3, pp. 198-212 (co-author, Michael S. Paterson).

KOCHER, FRANK. Problem Proposed: B-376, Vol. 16, No. 2, p. 184 (co-proposer, Gary L. Mullen).

KRAVITZ, SIDNEY. Problem Proposed: B-348, Vol. 16, No. 1, p. 90. Problem Solved: B-348, Vol. 16, No. 1, p. 90.

LA ROSA, B. DE. "Primes, Powers, and Partitions," Vol. 16, No. 6, pp. 518-522.

LARSON, PAUL. "The Golden Section in the Earliest Notated Western Music," Vol. 16, No. 6, pp. 513-515.

LORD, GRAHAM. Problems Solved: B-346, B-347, B-349, B-350, B-351, Vol. 16, No. 1, pp. 89-91; B-352, B-353, B-354, B-355, B-356, Vol. 16, No. 2, pp. 185-186; B-358, B-359, B-361, B-362, B-363, Vol. 16, No. 5, pp. 474-476; B-365, B-366, B-367, B-368, B-369, Vol. 16, No. 6, pp. 563-565.

MANA, P. L. Problems Proposed: B-373, Vol. 16, No. 1, p. 88; B-354, Vol. 16, No. 2, p. 185; B-358, Vol. 16, No. 5, p. 474; B-392, B-365, Vol. 16, No. 6, pp. 562, 563.

METZGER, J. M. "Fibonacci Sine Sequences," Vol. 16, No. 2, pp. 119-120 (co-author, M. B. Gregory).

MILSOM, JOHN W. Problems Solved: B-352, B-353, B-354, B-355, Vol. 16, No. 2, pp. 185-186.

MOHANTY, S. P. "The Number of Primes Is Infinite," Vol. 16, No. 4, pp. 381-384.

MULLEN, GARY L. Problem Proposed: B-376, Vol. 16, No. 2, p. 184 (co-proposer, Frank Kocher).

MULLIN, A. Problem Proposed: H-287, Vol. 16, No. 5, p. 477.

O'CALLAHAN, T. Problem Proposed: B-360, Vol. 16, No. 5, pp. 474-475. Problem Solved: B-360, Vol. 16, No. 5, pp. 474-475.

PATERSON, MUCHAEL S. "Identities from Partition Involutions," Vol. 16, No. 3, pp. 198-212 (co-author, Donald E. Knuth).

PECK, C. B. A. Problems Solved: B-346, B-348, Vol. 16, No. 1, pp. 89-90; B-352, B-353, B-354, B-355, Vol. 16, No. 2, pp. 185-186; B-364, Vol. 16, No. 6, p. 563.

PIGNO, LOUIS. "Interpolation of Fourier Transforms on Sums of Fibonacci Numbers," Vol. 16, No. 3, pp. 193-194 (co-author, Robert E. Dressler); "Topological, Measure Theoretic and Analytic Properties of the Fibonacci Numbers," Vol. 16, No. 3, pp. 195-197 (co-author, Robert E. Dressler).

PONNUDURAI, T. Problems Solved: B-366, B-369, Vol. 16, No. 6, pp. 563, 565.

PRIELIPP, A. G. Problems Solved: B-346, B-347, B-351, Vol. 16, No. 1, pp. 89-91; B-352, B-353, B-354, B-355, B-356, Vol. 16, No. 2, pp. 185-186; B-364, B-365, B-366, B-367, B-369, Vol. 16, No. 6, pp. 563-565. H-273, Vol. 16, No. 6, p. 568.

ROBBINS, NEVILLE. "On Fibonacci Numbers Which Are Powers," Vol. 16, No. 6, pp. 515-517.

SHANNON, A. G. "Fibonacci and Lucas Numbers and the Complexity of a Graph," Vol. 16, No. 1, pp. 1-4; "On the Multiplication of Recursive Sequences," Vol. 16, No. 1, pp. 27-32; "Pellian Diophantine Sequences," Vol. 16, No. 2, pp. 99-102. Problem Proposed: B-382, Vol. 16, No. 5, p. 473. Problems Solved: B-346, B-347, Vol. 16, No. 1, pp. 89-90; B-352, B-353, B-354, B-355, B-356, B-357, Vol. 16, No. 2, pp. 185-187; H-269, Vol. 16, No. 5, p. 478; B-364, B-365, B-366, B-367, B-369, Vol. 16, No. 6, pp. 563-565.

SHIELDS, CHARLES. Problem Solved: B-359, Vol. 16, No. 5, p. 474.

SINGH, SAHIB. Problems Solved: B-358, B-360, B-361, B-362, B-363, Vol. 16, No. 5, pp. 474-476 (co-solver, Roger Engle); B-359, Vol. 16, No. 5, p. 474 (co-solvers, Benjamin Freed & Roger Engle).

SINNAMON, GORDON. Problems Solved: B-351, Vol. 16, No. 1, p. 91.

SMITH, PAMELA GRAVES. "Expansion," Vol. 16, No. 2, p. 112.

SOMER, LAWRENCE. Problem Proposed: B-386, Vol. 16, No. 5, p. 473. Problem Solved: B-351, Vol. 16, No. 1, p. 91.

SONNTAG, ROLF. Problem Solved: B-364, Vol. 16, No. 6, p.

STEINER, RAY. "On Nth Powers in the Lucas and Fibonacci Series," Vol. 16, No. 5, pp. 451-469; "On kth Numerical Centers," Vol. 16, No. 5, pp. 470-471.

STERN, FREDRICK. Problem Proposed: B-374, Vol. 16, No. 1, p. 88.

SUN, HUGO S. "Embedding a Group in the pth Powers," Vol. 16, No. 1, p. 4.

SUTTENFIELD, JAMES M., JR. "A New Series," Vol. 16, No. 4, pp. 335-343.

TOSCANO, L. "Some Results for Generalized Bernoulli, Euler, Stirling Numbers," Vol. 16, No. 2, pp. 103-112.

TRIGG, CHARLES W. Problem Solved: B-348, Vol. 16, No. 1, p. 90.

VINCE, ANDREW. "The Fibonacci Sequence Modulo N," Vol. 16, No. 5, pp. 403-407.

WACHTEL, M. Problem Proposed: B-391, Vol. 16, No. 6, p. 562.

WADDILL, MARCELLUS E. "Some Properties of a Generalized Fibonacci Sequence Modulo m," Vol. 16, No. 4, pp. 344-353.

WHITLEY, ROBERT. "Fibonacci Numbers in Coin Tossing Sequences," Vol. 16, No. 6, pp. 539-541 (co-author, Mark Finkelstein).

WHITNEY, RAYMOND E. "Geometric Sequences and the Initial Digit Problem," Vol. 16, No. 2, pp. 152-154. Problem Proposed: H-271(corrected), Vol. 16, No. 5, p. 480. Editor: "Advanced Problems and Solutions," Vol. 16, No. 1, pp. 92-96; Vol. 16, No. 2, pp. 188-192; Vol. 16, No. 5, pp. 477-480; Vol. 16, No. 6, pp. 566-569.

WULCZYN, GREGORY. Problems Proposed: B-370, Vol. 16, No. 1, p. 88, H-279, Vol. 16, No. 1, p. 92; B-353, Vol. 16, No. 2, p. 186, H-284, Vol. 16, No. 2, p. 188; B-383, B-384, Vol. 16, No. 5, p. 473, H-288, Vol. 16, No. 5, p. 477; B-389, Vol. 16, No. 6, p. , H-290, H-294, Vol. 16, No. 6, pp. 566-567. Problems Solved: B-346, B-347, Vol. 16, No. 1, pp. 89-90; B-352, B-354, B-355, B-356, Vol. 16, No. 2, pp. 185-186; B-360, B-362, B-363, Vol. 16, No. 5, pp. 474-476; B-364, B-365, B-366, B-368, Vol. 16, No. 6, pp. 563-565, H-273, Vol. 16, No. 6, p. 568.

ZEITLIN, DAVID. "An Inequality for a Class of Polynomials," Vol. 16, No. 2, pp. 128-129, 146. Problems Solved: B-237, B-349, Vol. 16, No. 1, pp. 90, 91; B-352, B-353, B-356, Vol. 16, No. 2, pp. 185, 186.

ZENZ, F. A. "The Fluid Mechanics of Bubbling Beds," Vol. 16, No. 2, pp. 171-183.

ZWILLINGER, DAN. Problem Proposed: B-380, Vol. 16, No. 2, p. 184.

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