MAXIMUM CARDINALITIES FOR TOPOLOGIES ON FINITE SETS

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If $[n]$ represents the first n natural numbers, D. Stephen showed in [3] that no topology on $[n]$ with the exception of the discrete topology has more than $3(2^{n-2})$ elements and that this number is a maximum. In this article we show that, if k is a nonnegative integer and $k \leq n$, then no topology on $[n]$ with precisely $n - k$ open singletons has more than $(1 + 2^k)2^{n-k-1}$ elements and that this number is attainable over such topologies for $k < n$. We also show that the topology on $[n]$ with no open singletons and the maximum number of elements has cardinality $1 + 2_{n-2}$.

Recently, A. R. Mitchell and R. W. Mitchell have given a much simpler proof of Stephen's result [2]. Their proof consists of showing (1) If $n \geq 2$ and $x,y \in [n]$ with $x \neq y$, then

$$\Gamma(x,y) = \{A \subset [n]: x \in A \text{ or } y \notin A\}$$

is a topology on $[n]$ with precisely $3(2^{n-2})$ elements, and (2) If Γ is a nondiscrete topology on $[n]$, there exist $x,y \in [n]$ with $\Gamma \subset \Gamma(x,y)$. In Section 1, we give proofs of two theorems which in conjunction produce Stephen's result and which dictate what form the nondiscrete topology of maximum cardinality must have.

1. STEPHEN'S RESULT

We let $|A|$ denote the cardinality of a set A. If Γ is a topology on $[n]$ and $x \in [n]$, we let $M(\Gamma,x)$ be the open set about x with minimum cardinality. Evidently, $\Gamma = \{A \subset [n]: M(\Gamma,x) \subset A \text{ whenever } x \in A\}$.

Theorem 1.1: If k is a positive integer and Γ is a topology on $[n]$ with precisely $n - k$ open singletons, there is a topology Δ on $[n]$ with precisely $n - k + 1$ open singletons and $|\Gamma| < |\Delta|$.

Proof: Choose $x \in [n]$ such that $\{x\}$ is not open. Let

$$\Delta = \Big\{A \cup (B \cap \{x\}): A,B \in \Gamma\Big\}.$$

Then Δ is a topology on $[n]$ with precisely $n - k + 1$ open singletons, which satisfies $\Gamma \subset \Delta$ and $\Gamma \neq \Delta$. The proof is complete.

Theorem 1.2: If k is a positive integer and Γ is a topology on $[n]$ with precisely $n - k$ open singletons and for some $x \in [n]$, $\{y\}$ is open for each

$$y \in M(\Gamma,x) - \{x\} \quad \text{and} \quad |M(\Gamma,x)| > 2,$$

there is a topology Γ on $[n]$ with precisely $n - k$ open singletons satisfying $|\Gamma| < |\Delta|$.

Proof: Choose $y \in M(\Gamma,x) - \{x\}$ and let

$$\Delta = \Big\{A \cup (B \cap (M(\Gamma,x) - \{y\})): A,B \in \Gamma\Big\}.$$

Then Δ is a topology on $[n]$ with precisely $n - k$ open singletons, which satisfies $\Gamma \subset \Delta$ and $\Gamma \neq \Delta$. The proof is complete.

Corollary 1.3: Each nondiscrete topology on $[n]$ has at most $3(2^{n-2})$ elements and this number is a maximum.

Proof: If Γ is a nondiscrete topology on $[n]$, then $n \geq 2$. From Theorem 1.1, if Γ has the maximum cardinality over all nondiscrete topologies on $[n]$, then Γ has precisely $n-1$ open singletons; and by Theorem 1.2, if $\{n\}$ is the nonopen singleton, we must have $|M(\Gamma,n)| = 2$. So there is an $x \in [n-1]$ with $M(\Gamma,n) = \{n, x\}$. Thus,

$$\Gamma = \left\{A \subset [n]: n \notin A\right\} \cup \left\{A \subset [n]: \{n, x\} \subset A\right\}.$$

Consequently, $|\Gamma| = 2^{n-1} + 2^{n-2} = 3(2^{n-2})$ and the proof is complete.

Remark 1.4: The topology Δ in the proof of Theorem 1.1 (1.2) is known as the simple extension of Γ through the subset $\{x\}$ $\big(M(\Gamma,x) - \{y\}\big)$ [1].

2. SOME PRELIMINARIES

In this section we present some notation and prove a theorem which will be useful in reaching our main results. If $k \in [n]$, let $\lambda(k)$ be the collection of topologies on $[n]$ which have $\{1\}, \{2\}, \ldots, \{k\}$ as the nonopen singletons. If $1 \leq m \leq k$, let $C(m)$ be the set of increasing functions from $[m]$ to $[k]$; for each $g \in C(m)$, let

$$U(\Gamma,m,g) = \bigcup_{i \in [m]} M(\Gamma,g(i))$$

and

$$\Omega(\Gamma,m,g) = \{A \subset [n]: U(\Gamma,m,g) \subset A \text{ and } |A \cap [k]| = m\}.$$

Lemma 2.1: The following statements hold for each topology $\Gamma \in \lambda(k)$.

(a) $\Gamma = \{A \subset [n]: A \cap [k] = \emptyset\} \cup \bigcup_{m=1}^{k} \bigcup_{g \in C(m)} \Omega(\Gamma,m,g).$

(b) For each $m \in [k]$ and $g \in C(m)$, we have

$$|\Omega(\Gamma,m,g)| = 0 \quad \text{or} \quad |\Omega(\Gamma,m,g)| = 2^{n-k+m-|U(\Gamma,m,g)|}.$$

(c) $(\Gamma,m,g) \cap \Omega(\Gamma,j,h) = \emptyset$ unless $(m,g) = (j,h)$.

Proof of (a): Let Δ represent the set on the right-hand side of the equality sign in (a), and let $W \in \Gamma$. If $W \cap [k] = \emptyset$, then $W \in \Delta$. If $W \cap [k] \neq \emptyset$, then $|W \cap [k]| = m$ for some $m \in [k]$. Let g be the strictly increasing function from $[m]$ to $W \cap [k]$. For each $g(i)$ we have $W \supset M\big(\Gamma,g(i)\big)$, so

$$W \supset U(\Gamma,m,g), \quad W \in \Omega(\Gamma,m,g), \text{ and } \Gamma \subset \Delta.$$

If $W \in \Delta$ and $W \cap [k] = \emptyset$, then $W \in \Gamma$. Otherwise, $W \in \Omega(\Gamma,m,g)$ for some $m \in [k]$ and $g \in C(m)$. For this (m,g) we have

$$g([m]) \subset U(\Gamma,m,g) \subset W;$$

thus, $W \in \Gamma$, since

$$W = U(\Gamma,m,g) \cup \big(W - U(\Gamma,m,g)\big), \quad U(\Gamma,m,g) \in \Gamma,$$

and

$$\big(W - U(\Gamma,m,g)\big) \cap [k] = \emptyset;$$

so $\Delta \subset \Gamma$ and (a) is verified.

Proof of (b): It is easy to verify that $\Omega(\Gamma,m,g)$ is the set of all subsets of $[n] - \big([k] - g([m])\big)$ which contain $U(\Gamma,m,g)$ for each pair (m,g). Consequently (b) holds.

Proof of (c): If $A \in \Omega(\Gamma,m,g) \cap \Omega(\Gamma,j,h)$, then $m = |A \cap [k]| = j$. Also,

$$g([m]) \cup h([m]) \subset A \cap [k],$$

which gives

$$|g([m]) \cup h([m])| = m.$$

Since g and h are strictly increasing, we must have $g = h$, and the proof is complete.

We are now in a position to establish the following useful theorem.

Theorem 2.2: If Γ is an element of $\lambda(k)$, then

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|}$$

with equality if and only if $\Omega(\Gamma,m,g) \neq \emptyset$ for any pair (m,g).

Proof: From Lemma 2.1(a) and (c), we have

$$|\Gamma| = |\{A \subset [n]: A \cap [k] = \emptyset\}| + \sum_{m=1}^{k} \sum_{g \in C(m)} |\Omega(\Gamma,m,g)|.$$

So from Lemma 2.1(b) we get

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|}$$

with equality if and only if $\Omega(\Gamma,m,g) \neq \emptyset$ for any pair (m,g). The proof is complete.

3. THE FIRST TWO OF OUR MAIN RESULTS

The Case $0 \leq k \leq n$: The results are clear for $k = 0$. In the following, we assume that $k \in [n]$.

Theorem 3.1: If n is a positive integer and $\Gamma \in \lambda(k)$, then

$$|\Gamma| \leq (1 + 2^k)2^{n-k-1}.$$

Proof: We proceed by induction on n. The case $n = 1$ is true vacuously. Suppose $n > 1$ and the result holds for all integers $j \in [n - 1]$.

Case 1: $|U(\Gamma,m,g)| = m$ for some pair (m,g). Then we have

$$U(\Gamma,m,g) \subset [k].$$

Let $W \in \Gamma$ with $W \subset [k]$ and $|W|$ a minimum. Then $|W| \geq 2$ and $M(\Gamma,x) = W$ for each $x \in W$. Without loss, assume that $1 \in W$ and if $[n] - W \neq \emptyset$, assume that $[n] - W = \{2, 3, \ldots, n - |W| + 1\}$. Define a topology Δ on $[n - |W| + 1]$ by the following family of minimum-cardinality open sets:

$$M(\Delta,1) = \{1\},\ M(\Delta,x) = \big(M(\Gamma,x) - W\big) \cup \{1\} \text{ if } M(\Gamma,x) \cap W \neq \emptyset$$

and

$$M(\Delta,x) = M(\Gamma,x) \text{ otherwise.}$$

It is not difficult to show that $|\Delta| = |\Gamma|$ and that Δ has $n - k + 1$ open singletons. So by the induction hypothesis, we have

$$|\Gamma| \leq \big(1 + 2^{k-|W|}\big)2^{n-k} \leq (1 + 2^k)2^{n-k-1}.$$

Case 2: $|U(\Gamma,m,g)| > m$ for each pair (m,g). Here we have

$$|U(\Gamma,m,g)| \geq m + 1$$

for each pair (m,g) and, from Theorem 2.2, we get

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|} \leq 2^{n-k} + \left(\sum_{m=1}^{k} \binom{k}{m}\right) 2^{n-k-1};$$

we see easily that

$$2^{n-k} + \left(\sum_{m=1}^{k} \binom{k}{m}\right) 2^{n-k-1} = (1 + 2^k)2^{n-k-1}.$$

The proof is complete.

It is obvious that if $\Gamma \in \lambda(k)$ with $|U(\Gamma,m,g)| = m + 1$ for each pair (m,g) then $|\Gamma|$ will be a maximum over $\lambda(k)$ and we will have

$$|\Gamma| = (1 + 2^k)^{n-k-1}.$$

If such a Γ has $|\Gamma|$ a maximum over $\lambda(k)$, we must have

$$|M(\Gamma,x)| = 2 \quad \text{and} \quad |M(\Gamma,x) \cap [k]| = 1$$

for each $x \in [k]$, since $g \in C(1)$ defined by $g(1) = x$ must satisfy

$$|U(\Gamma,1,g)| = 2 \quad \text{and} \quad \Omega(\Gamma,1,g) \neq \emptyset$$

from Lemma 2.1(b). Moreover, if $x < y$ and $x,y \in [k]$, then

$$|M(\Gamma,x) \cup M(\Gamma,y)| = 3$$

since $g \in C(2)$ defined by $g(1) = x$ and $g(2) = y$ must satisfy

$$|U(\Gamma,2,g)| = 3.$$

Thus,

$$M(\Gamma,x) \cap M(\Gamma,y) \neq \emptyset.$$

This implies that there must be a $j \in [n] - [k]$ with $M(\Gamma,x) = \{x, j\}$ for each $x \in [k]$ and that

$$\Gamma = \{A \subset [n]: A \cap [k] = \emptyset\} \cup \Big\{A \subset [n]: \{x, j\} \subset A \text{ for each } x \in A \cap [k]\Big\}.$$

We have

$$|\Gamma| = (1 + 2^k)2^{n-k-1}$$

from the arguments above and the second of our main results is realized.

Theorem 3.2: For $0 \leq k < n$, there is a topology on $[n]$ with precisely $n - k$ open singletons and $(1 + 2^k)2^{n-k-1}$ elements.

As a by-product of these main results, we obtain Stephen's result.

Corollary 3.3: The only topology on $[n]$ having more than $3(2^{n-2})$ open sets is the discrete topology. Moreover, this upper bound cannot be improved.

Proof: If the topology Γ on $[n]$ is not discrete, then $n > 1$ and there is at least one nonopen singleton. If k is the number of nonopen singletons, we have, from Theorem 3.1, that

$$|\Gamma| \leq 2^{n-1} + 2^{n-k-1} \leq 2^{n-1} + 2^{n-2} = 3(2^{n-2}),$$

and since $n \neq 1$, there is a topology on $[n]$ with precisely $3(2^{n-2})$ elements, from Theorem 3.2. The proof is complete.

4. OUR FINAL TWO MAIN RESULTS

The Case $k = n$: It is obvious that for $k = n$, no topology on $[n]$ has

$$(1 + 2^k)2^{n-k-1}$$

elements. If $\Gamma \varepsilon \lambda(n)$, we let

$$\Phi(\Gamma) = \{A \subset [n]: A = M(\Gamma,x) \text{ for each } x \varepsilon A, \text{ and } \neq \emptyset\}.$$

It is clear from the argument in Case 1 of Theorem 3.1 that $\Phi(\Gamma) \neq \emptyset$.

Theorem 4.1: If Γ is an element of $\lambda(k)$ which has maximum cardinality over $\lambda(k)$, then $|A| = 2$ for each $A \varepsilon \Phi(\Gamma)$.

Proof: If $A \varepsilon \Phi(\Gamma)$ with $|A| > 2$, choose $x,y \varepsilon A$ with $x \neq y$ and let

$$\Delta = \left\{V \cup (B \cap \{x, y\}): V,B \varepsilon \Gamma\right\}.$$

Then $\Delta \varepsilon \lambda(k)$, $\Gamma \subset \Delta$, and $\Gamma \neq \Delta$. The proof is complete.

Theorem 4.2: If Γ is an element of $\lambda(n)$, then $|\Gamma| \leq 1 + 2^{n-2}$.

Proof: Let $\Gamma \varepsilon \lambda(n)$ with $|\Gamma|$ a maximum. Then $|A| = 2$ for each $A \varepsilon \Phi(\Gamma)$. For each $i \varepsilon [|\Phi(\Gamma)|]$, let

$$P(i) = \{n - 2|\Phi(\Gamma)| + i,\ n - i + 1\};$$

without loss, assume that

$$\Phi(\Gamma) = \{P(i): i \varepsilon [|\Phi(\Gamma)|]\}$$

and that

$$[n] - \bigcup_{\Phi(\Gamma)} A = [n - 2|\Phi(\Gamma)|] \quad \text{if} \quad n \neq 2|\Phi(\Gamma)|.$$

Define a topology Δ on $[n - |\Phi(\Gamma)|]$ by specifying its minimum-cardinality open sets for each $x \varepsilon [n - |\Phi(\Gamma)|]$ as

$$M(\Delta,x) = \left(M(\Gamma,x) - \bigcup_{\Phi(\Gamma)} A\right) \cup \{n - 2|\Phi(\Gamma)| + i: P(i) \cap M(\Gamma,x) \neq \emptyset\}.$$

Then Δ has precisely $|\Phi(\Gamma)|$ open singletons and $|\Gamma| = |\Delta|$. By Theorem 3.1,

$$|\Gamma| \leq \left(1 + 2^{n-2|\Phi(\Gamma)|}\right)2^{|\Phi(\Gamma)|-1}$$

where the expression on the right side of the inequality decreases as $|\Phi(\Gamma)|$ increases. Thus, $|\Gamma| \leq 1 + 2^{n-2}$ for all $\Gamma \varepsilon \lambda(n)$ and the proof is complete.

Theorem 4.3: For $n > 1$, there is a topology on $[n]$ with no open singletons and $1 + 2^{n-2}$ elements.

Proof: From Theorem 3.2, there is a topology Γ on $[n-1]$ with $1 + 2^{n-2}$ elements. For this topology, $M(\Gamma,x) = \{x, n - 1\}$ for $x \neq n - 1$ and $M(\Gamma, n - 1) = \{n - 1\}$ may be assumed to be the minimum-cardinality open sets. Let

$$\Delta = \left\{A \subset [n]: M(\Gamma,x) \cup \{n\} \subset A \text{ when } M(\Gamma,x) \subset A\right\}.$$

Then Δ is a topology on $[n]$ with no open singletons and $|\Delta| = |\Gamma|$. The proof is complete.

5. SOME FINAL REMARKS

The following observations may be made from the Theorems and constructions above.

Remark 5.1: It is easy to construct for each $1 \leq j \leq n-k$ a topology $\Gamma \varepsilon \lambda(k)$ with cardinality $(2^k + (-1 + 2^j))2^{n-k-j}$. Let $M(\Gamma,x) = \{x\}$ for each $x \varepsilon [n] - [k]$ and $m(\Gamma,x) = \{x, k+1, k+2, \ldots, k+j\}$ for each $x \varepsilon [k]$. We see from Theorem 2.1 that $|\Gamma|$ is the required number.

Remark 5.2: More generally, if $k \varepsilon [n]$ and for each $x \varepsilon [k]$, $W(x)$ is a nonempty subset of $[n] - [k]$, let Γ be the topology on $[n]$ having minimal cardinality open sets $M(\Gamma,x) = \{x\} \cup W(x)$ for $x \varepsilon [k]$ and $M(\Gamma,x) = \{x\}$ otherwise. Then from Theorem 2.1

$$|\Gamma| = 2^{n-k} + \sum_{m=1}^{k} \sum_{g \varepsilon C(m)} 2^{n-k+m-\left(m+\left|\bigcup_{[m]} W(g(i))\right|\right)}$$

since

$$|U(\Gamma,m,g)| = \left|\bigcup_{[m]} M(\Gamma,g(i))\right| = |g([m])| + \left|\bigcup_{[m]} W(g(i))\right| = m + \left|\bigcup_{[m]} W(g(i))\right|.$$

Remark 5.3: For each $k \varepsilon [n]$, let

$$\mu(k) = \{\Gamma \varepsilon \lambda(k) : \Omega(\Gamma,m,g) \neq \emptyset \text{ for any pair } (m,g)\}.$$

Then $\mu(k) = \{\Gamma \varepsilon \lambda(k) : \text{for each } x \varepsilon [k],\ M(\Gamma,x) = \{x\} \cup W(x) \text{ for some nonempty } W(x) \subset [n] - [k]\}$. Thus $|\mu(k)| = (-1 + 2^{n-k})^k$ for each subset of $[n]$ of cardinality k. Therefore,

$$\binom{n}{k}(-1 + 2^{n-k})^k$$

is the number of topologies, Γ, on $[n]$ such that

$$\Gamma \varepsilon \lambda(k) \quad \text{and} \quad \Omega(\Gamma,m,g) \neq \emptyset \quad \text{for any pair } (m,g).$$

The total number of such topologies is

$$\sum_{k \varepsilon [n]} \binom{n}{k}(-1 + 2^{n-k})^k.$$

REFERENCES

1. N. L. Levine, "Simple Extensions of Topologies," *Amer. Math. Monthly* 71 (1964):22-25.
2. A. R. Mitchell & R. W. Mitchell, "A Note on Topologies on Finite Sets," *The Fibonacci Quarterly* 13, No. 3 (1975):365, 368.
3. D. Stephen, "Topology on Finite Sets," *Amer. Math. Monthly* 75 (1968):739-741.

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A PRIMER ON STERN'S DIATOMIC SEQUENCE

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PART I: HISTORY

1. Eisenstein's Function

In 1850, F. M. G. Eisenstein, a brilliant mathematician and disciple of Gauss, wrote a treatise [1] on number theoretic functions of a reciprocating nature. In this paper he discusses the following sequence as part of another discussion.

For positive integers λ, u, and v:

1) $x_{u,v} = x_{u,u+v} + x_{u+v,v} \pmod{\lambda}$, for $u + v < \lambda$;
2) $x_{u,v} = \emptyset$, for $u + v > \lambda$;
3) $x_{u,v} = v$, for $u + v = \lambda$.

On February 18, 1850, M. A. Stern, who taught theory of equations at the University of Göttingen, attended a conference on Mathematical Physics where Eisenstein mentioned that the function described in his paper was too complex and did not lend itself to elementary study. Within two years of that conference, Eisenstein would die prematurely at the age of 29, but the study of Stern numbers had been born, and research was in progress.

2. Stern's Version

In a paper written in 1858, Stern presented an extensive discussion [2] on what may be characterized as "Generalized Stern Numbers." Many important results were generated in this paper, some of more importance than others. The authors will attempt to present a synopsis of these results, translated from German, as they were presented.

(1) Stern provided the following definition as his specialization of Eisenstein's function. The sequence is a succession of rows, each generated from a previous row starting with two numbers, m and n.

$$\begin{array}{ccccccccc} & & m & & & & n & & \\ & & m & & m+n & & n & & \\ m & & 2m+n & & m+n & & m+2n & & n \\ & & & & \text{etc.} & & & & \end{array}$$

Stern also provided some special terms for the elements of the rows.

Definition: ARGUMENT—The starting terms, m and n, are called ARGUMENTS of the sequence.

Definition: GRUPPE—In each successive row every other term is from the previous row and the terms in between are the sum of the adjacent two. Any three successive elements within a row are called a GRUPPE.

Definition: STAMMGLIED—In each GRUPPE, the two numbers which were from the previous row are termed STAMMGLIED.

Definition: SUMMENGLIED—In each GRUPPE, the middle term, the summed element, is termed SUMMENGLIED.

Some results are immediately obvious. The first SUMMENGLIED, $m+n$, is always the center element in succeeding rows. The arguments m and n always straddle the row. The row is symmetric about the center if $m = n$; even so, if a SUMMENGLIED is of the form $km+ln$, then $lm+kn$ appears reflected about the center element (MITTELGLIED).

(2) If there are k elements in a given row, then there are $2(k-1)+1$ elements in the next row; if the first row has three elements, the pth row has 2^p+1 elements. Also, if we let $S_p(m,n)$ denote the sum of the elements in each row, then

$$S_p(m,n) = \frac{3^p+1}{2}(m+n).$$

Note that $S_p(m,n)$ is reflexive or that $S_p(m,n) = S_p(n,m)$. Stern also observed that

$$\frac{S_p(m',n')}{S_p(m,n)} = \frac{m'+n'}{m+n},$$

and

$$S_p(m+m',n+n') = S_p(m,n) + S_p(m',n').$$

This latter result led to

$$\lim_{n\to\infty} \frac{S_p(F_n,F_{n+1})}{S_p(F_{n-1},F_n)} = 1 + \cfrac{1}{1+\cfrac{1}{1+\ddots}} = \alpha, \text{ the golden ratio,}$$

a nice Fibonacci result.[1]

(3) Stern observed next that some properties concerning odd and even numbers as they occur, or more precisely, Stern numbers mod 2. He noted that, in any three successive rows, the starting sequence of terms is

odd, even, odd
odd, odd, even
odd, even, odd

(4) Given a GRUPPE a, b, c, where b is a SUMMENGLIED in row p, the number will appear also in row $p-k$, where

$$k = \frac{a+b-c}{2b}.$$

Also, if b is in position

$$2^{t-1}(2l-1)+1$$

in row p, then it occurs also in row $p-(t-1)$ in position $2l$. Related to this, Stern noted that with two GRUPPEs a, b, c and d, e, f in different rows, but in the same columns, that

[1] *This is a generalization of what was actually presented. The study of Fibonacci numbers as such was not yet in play. Hoggatt notes that this result is also true for generalized Fibonacci numbers.*

$$\frac{a+c}{b} = \frac{d+f}{e}.^{2}$$

(5) No two successive elements in a given row may have a common factor. Furthermore, in a GRUPPE a, b, c ($b = a + c$), a and c are relatively prime.

(6) Two sequential elements a, b cannot appear together, in the same order, in two different rows or in the same row. When $m = n = 1$ (the starting elements) then a group a, b may never occur again in any successive row.

(7) The GRUPPEs a, b, c and c, b, a may not occur together in the first (or last, because of the symmetry) half of a row.

(8) In the simple Stern sequence using $m = n = 1$, all positive integers will occur and all relatively prime pairs a, c will occur. For all elements of this same sequence that appear as SUMMENGLIED, that same element will be relatively prime to all smaller-valued elements that are STAMMGLIED. Stern pointed out that this is also a result of (6).

(9) The last row in which the number n will occur as a SUMMENGLIED is row $n - 1$. The number n will occur only $n - 1$ more times.

(10) Given a relatively prime pair b, c (or c, b) of a GRUPPE, the row in which that pair of elements will occur may be found by expansion of b/c into a continued fraction. That is, if

$$\frac{b}{c} = (k,\ k',\ k'',\ \ldots,\ k_m,\ r_{m-1}),$$

then b, c occurs in row

$$(k + k' + k'' + \cdots + k_m + r_{m-1} - 1),$$

and the pair $(1, r_{m-1})$ occurs in a row $(k + k' + \cdots + k_m)$.

(11) Let (m,n) denote row p generated by the Generalized Stern Sequence starting with m and n. Then

$$(m,n)_p \pm (m',n')_p = (m \pm m', n \pm n')_p,$$

which says that the element-by-element addition of the same row of two sequences is equal to row p of a sequence generated by the addition, respectively, of the starting elements.

In particular, an analysis of $(\emptyset,1)_p$ generates an interesting result. The first few rows are:

p	(0, 1)
$\emptyset$	0, 1
1	0, 1, 1
2	0, 1, 1, 2, 1
3	0, 1, 1, 2, 1, 3, 2, 3, 1

Interestingly enough, all the nonzero elements in row k appear in the same position in every row thereafter. Stern observed also that in any given column of $(1, 1)_p$ the column was an arithmetic progression whose difference was equal to the value occurring in the same relative column of $(0, 1)_p$.

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[2] *The authors note that $(a+c)/b$ being an integer is not surprising, but the fact that this ratio is the same within columns is not immediately obvious.*

(12) From the last result in (4), we recall that

$$\frac{a+c}{b} = \frac{d+f}{e}$$

where a, b, c; and d, e, f are GRUPPEs and in the same column positions, but perhaps different rows, then

$$|db - ae| = |p_1 - p_2|,$$

where p_1 and p_2 are the row numbers.

(13) The next special case of interest is the examination of row $(1,n)_p$, for $n > 1$. The first noteworthy result is that all elements of the row $(1,n)_p$ appear at the start of the row $(1, 1)_{p+n-1}$. Also, all terms are of the form $k + ln$ or $l + kn$.

(14) Moving right along, the rows $(1,n)_p$ may be written as

$1 + \emptyset n, 1 + 1n, \emptyset + 1n$

$1 + \emptyset n, 2 + 1n, 1 + 1n, 1 + 2n, \emptyset + 1n$

$1 + \emptyset n, 3 + 1n, 2 + 1n, 3 + 2n, 1 + 1n, 2 + 3n, 1 + 2n, 1 + 3n, \emptyset + 1n$

etc.

Notice that the constant coefficients are the elements of $(1, 1)_{p-2}$, and that the coefficients of n are the elements of $(\emptyset, 1)_{p-1}$. Note also that the difference between any two successive elements, $k + ln$ and $k' + l'n$, within a row is

$$|kl' - k'l| = 1,$$

and no element may have the form

$$hk + h'kn.$$

(15) With k and k' in (14), k and k' are relatively prime. Correspondingly, l and l' are also relatively prime.

(16) Given $N > n$ in the sequence $(1,n)$ and of the form $N = K - Ln$, then K and L are relatively prime; N and n are relatively prime; L and N are relatively prime as well as K and N. Numbers between $\emptyset$ and N/n that occur in $(1,n)$ will be relatively prime to all N whenever N is a SUMMENGLIED.

(17) In order to proceed symmetrically, Stern next examined the sequence of rows $(n,1)_p$. The first immediately obvious result is that $(n,1)_p$ is reflexively symmetric to $(1,n)_p$ about the center element. When m and n of (m,n) are relatively prime and p is the largest factor of m or n, then for (m',n'), where $m = pm'$ and $n = pn'$, each element of (m',n') multiplied by p yields the respective element of (m,n). Stern noted at this point that all sequences (m,n) appear as a subset somewhere in $(1, 1)$.

(18) Given that N occurs in (m,n) and

$$N = mk + nl$$

for k and l relatively prime, Stern reported that a theorem of Eisenstein's says that N is relatively prime to elements between $(n_0/n)N$ and $(m_0/m)N$ where m_0 and n_0 are such that $|nm_0 - mn_0| = 1$; N is a SUMMENGLIED. When $m = m_0 = 1$ and $n_0 = n - 1$, N is relatively prime to elements between $(n - 1/n)N$ and N.

(19) Given again that $N = mk + ln$ and N relatively prime to elements between $(n_0/n)N$ and $(m_0/m)N$ and, further, that we are given a GRUPPE

$$k'm + l'n,\ N,\ k''m + l''n, \text{ then } (k' + k'')(k''m + l''n) \equiv n \pmod{N}.$$

Eisenstein stated that for a GRUPPE $\propto, N, \beta$ where

$$\beta = k'm + l'n + sN$$

and

$$\beta \equiv k''m + l''n + TN,$$

then

$$\beta \equiv k''m + l''n \pmod{N}.$$

(20) Eisenstein continued to contribute to Stern's analysis hoping to arrive at the more complex function he had originally proposed. Stern stated that in the analysis of row $(1, 2)_p$ and N that are SUMMENGLIED, that N is relatively prime to elements between $\emptyset$ and $N/2$. Further, since $(1,1)_p$ occurs in the first half of $(1, 2)_p$, that N occurring in the $(1,1)_p$ portion are relatively prime to the rest of the system [not in $(1, 1)_p$] mod N. And last, but not least, Eisenstein commented that if N is relatively prime to numbers between $(n_0/n)N$ and $(m_0/m)N$ then it is also relatively prime to numbers between $(m - m_0/m)N$ and $(n - n_0/n)N$.

(21) Let us now examine rows $(m,n)_p$ and SUMMENGLIED of the form $kn + ln$. Let the GRUPPE be

$$k'm + l'n,\ km + ln,\ k''m + l''n,$$

then

$$1)\quad k'l - kl' = 1$$

and

$$2)\quad k''l - kl'' = -1.$$

Now presume that the continued fraction

$$\frac{k}{l} = (a,\ a_1,\ a_2,\ \ldots,\ a_m)$$

and that $k' = k_0$ and $l' = l_0$ or $k'' = k_0$ and $l'' = l_0$ (at the reader's option). Eisenstein states that the following is true:

$$\frac{k'}{k''} = a_m + (-1,\ a_{m-1},\ \ldots,\ a_1,\ a);$$

and, consequently, that

$$p = a + a, + \cdots + a_m - 1.$$

This result is, of course, similar to the result (1) observed by Stern.

(22) Now with some of Stern's sequence theory under our belts, we can analyze Eisenstein's function:

(a) $f(m,n) = f(m,m + n) + f(m + n,n)$ when $m + n < \lambda$;

(b) $f(m,n) = n$ when $m + n = \lambda$;

(c) $f(m,n) = \emptyset$ when $m + n > \lambda$;

where m and n are positive numbers and λ is prime.

Note the relationship to Stern numbers when expanding $f(m,n)$:

$$\begin{aligned} f(m,n) &= f(m,m+n) + f(m+n,n) \\ &= f(m,2m+n) + f(2m+n,m+n) \\ &\qquad + f(m+n,m+2n) + f(m+2n,n) \end{aligned}$$

The arguments of the function are generalized Stern numbers. The following conclusion can now be drawn concerning Eisenstein's function.

1. For any given $f(km + ln, k'm + l'n)$, that $(k + k')m + (l + l')n = \lambda$.
2. If $m = 1$ and $n = 2$, then (16) implies that $f(1, 2)$ can be composed of elements of the form $f(\propto, \lambda - \propto)$ and that
$$f(1,\,2) = \lambda - \propto + \lambda - \propto' + \lambda - \propto'' + \cdots$$
3. For whole numbers "r" such that $\frac{\lambda + 1}{2} \le r \le \lambda - 1$,
$$f(1,\,2) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$
4. For whole numbers "r" such that, as in (18),
$$\frac{n_0 \lambda}{n} \le r \le \frac{m_0 \lambda}{m}$$
then
$$f(m,n) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$

REFERENCES

1. Dr. G. Eisenstein, "Überein einfaches Mittel zur Auffindung der höheren Reciprocitätsgesetze und der mit ihnen zu verbindenden Ergänzungssätze," *Journal für die reine und Angewandte Mathematik in zwanglosen heften* (University of Berlin), Vol. 39, p. 356.
2. Herrn Stern, "Ueber eine zahlentheoretische Funktion," *Journal für die reine und Angewandte Mathematik in zwanglosen heften* (Göttingen), Vol. 55, pp. 193-220.

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A MULTINOMIAL GENERALIZATION OF A BINOMIAL IDENTITY

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1. The binomial identity which we wish to generalize is the following:

$$(x + y)^n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} \left(x^k + y^k\right) \left(\frac{xy}{x+y}\right)^{n-k}. \tag{1}$$

It can be found and is proved in [2]. Let us begin by giving a demonstration suitable to a generalization to more than two variables. Symbolizing $C_{t^n} f(t)$ for the coefficient a_n of t^n in any power series $f(t) = \sum_{n \ge 0} a_n t^n$, it is easily shown that the second number of (1) is:

$$C_{t^{n-1}}\left(\frac{x}{1 - tx} + \frac{y}{1 - ty}\right)\left(1 - t\frac{xy}{x + y}\right)^{-n}. \tag{2}$$

Indeed, it is sufficient to carry out the Cauchy product of the two following power series (in t):

$$\frac{x}{1 - tx} + \frac{y}{1 - ty} = \sum_{k \geq 1} (x^k + y^k)t^{k-1}$$

$$\left(1 - t\frac{xy}{x + y}\right)^{-n} = \sum_{l \geq 0} \binom{n + l - 1}{n - 1} t^l \left(\frac{xy}{x + y}\right)^l.$$

To calculate (2) otherwise, let us apply the Lagrange reversion formula under the following form [1, I, p. 160 (8c)]: let $f(t) = \sum_{n \geq 0} a_n t^n$ be a formal series $a_0 = 0$, $a_1 \neq 0$, of which the reciprocal series is $f^{\langle -1 \rangle}(t)$ $\left[\text{that is to say, } f(f^{\langle -1 \rangle}(t)) = f^{\langle -1 \rangle}(f(t)) = t\right]$, and let $\Phi(t)$ be any other formal series with derivative $\Phi'(t)$; then we have:

$$nC_{t^n}\,\Phi(f^{\langle -1 \rangle}(t)) = C_{t^{n-1}}\,\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}. \tag{3}$$

In view of demonstrating (1), let us put in (3),

$$f(t) = t - t^2\frac{xy}{x + y}, \quad \Phi'(t) = \frac{x}{1 - tx} + \frac{y}{1 - ty},$$

which guarantees that the second member of (3) is effectively (2) in this case. But then,

$$\begin{aligned}\Phi(t) &= -\log(1 - tx) - \log(1 - ty)\\ &= -\log\{1 - t(x + y) + t^2xy\}\\ &= -\log\{1 - (x + y)f(t)\},\end{aligned}$$

that is to say, thanks to the well-known expansion $-\log(1 - \tau) = \sum_{n \geq 1} \tau^n/n$ for (*):

$$\begin{aligned}nC_{t^n}\,\Phi(f^{\langle -1 \rangle}(t)) &= nC_{t^n}\, - \log\left\{1 - (x + y)f(f^{\langle -1 \rangle}(t))\right\}\\ &= nC_{t^n}\, - \log(1 - (x + y)t) \overset{(*)}{=} (x + y)^n;\end{aligned}$$

Consequently, we have equality (1) as a result of (3).

2. To generalize formula (1), let us call $\sigma_1, \sigma_2, \sigma_3, \ldots$ the *elementary* symmetric of the variables $x_1, x_2, \ldots, x_m$, and $S_1, S_2, S_3, \ldots$ the symmetric functions which are *sums of the powers*; in other words,

$$\sigma_1 = \sum_{1 \leq i \leq m} x_i,\ \sigma_2 = \sum_{1 \leq i_1 < i_2 \leq m} x_{i_1}x_{i_2},\ \sigma_3 = \sum_{1 \leq i_1 < i_2 < i_3 \leq m} x_{i_1}x_{i_2}x_{i_3},\ \ldots \tag{4}$$

$$S_1\ (=\sigma_1) = \sum_{1 \leq i \leq m} x_i,\ S_2 = \sum_{1 \leq i \leq m} x_i^2,\ S_3 = \sum_{1 \leq i \leq m} x_i^3,\ \ldots\,. \tag{5}$$

Let us apply the Lagrange formula (3), this time with

$$f(t) = t - t^2\frac{\sigma_2}{\sigma_1} + t^3\frac{\sigma_3}{\sigma_1} - \cdots + (-1)^{m-1}\frac{\sigma_m}{\sigma_1}t^m$$

$$= \frac{1}{\sigma_1}\left\{1 - (1 - tx_1)(1 - tx_2)\cdots(1 - tx_m)\right\},$$

$$\Phi(t) = -\log\bigl(1 - \sigma_1 f(t)\bigr) = -\log(1 - tx_1)(1 - tx_2)\cdots(1 - tx_m)$$

$$= -\sum_{j=1}^{m}\log(1 - tx_j),$$

$$\Phi'(t) = \frac{x_1}{1 - tx_1} + \frac{x_2}{1 - tx_2} + \cdots + \frac{x_m}{1 - tx_m}.$$

Now, the first member of (3) equals:

(6) $$nC_{t^n} - \log\left\{1 - \sigma_1 f(f^{\langle -1\rangle}(t))\right\} = nC_{t^n} - \log(1 - \sigma_1 t) = \sigma_1^n,$$

and the second member of (3) may be written

(7) $$C_{t^{n-1}}\,\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}$$
$$= C_{t^{n-1}}\left(\frac{x_1}{1 - tx_1} + \cdots + \frac{x_m}{1 - tx_m}\right)\left(1 - t\frac{\sigma_2}{\sigma_1} + t^2\frac{\sigma_3}{\sigma_1} - \cdots\right)^{-n}.$$

Let us introduce the simplified writing for the multinomial coefficients

$$(n-1,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_{m-1}) = \frac{(n-1+\nu_1+\nu_2+\cdots+\nu_{m-1})!}{(n-1)!\,\nu_1!\,\nu_2!\,\ldots\,\nu_{m-1}!}$$

$\left[\text{in particular, } (a, b-a) = \binom{b}{a}\right]$, and in expanding (7) as a multiple series of order $(m - 1)$, [1, I, p. 53 ($12m'$)], there comes:

(8) $$C_{t^{n-1}}\left\{\sum_{k\geq 1} S_k t^{k-1}\right\}\left\{\sum_{\nu_1,\nu_2,\ldots,\nu_{m-1}\geq 0}(n-1,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_{m-1})\,t^{\nu_1+2\nu_2+3\nu_3+\cdots}\left(\frac{\sigma_2}{\sigma_1}\right)^{\nu_1}\left(-\frac{\sigma_3}{\sigma_1}\right)^{\nu_2}\cdots\right\}.$$

Finally, by comparing (3), (6), and (8), we find:

Theorem: With the notations (4) and (5), we have the multinomial identity:

(9) $$\sigma_1^n = \sum_{k=1}^{n}\left\{S_k\sum_{\nu_1+2\nu_2+\cdots+(m-1)\nu_{m-1}=n-k}(-1)^{\nu_2+\nu_4+\nu_6\cdots}(n-1,\ \nu_1,\ \nu_2,\ \ldots)\left(\frac{\sigma_2}{\sigma_1}\right)^{\nu_1}\left(\frac{\sigma_3}{\sigma_1}\right)^{\nu_2}\cdots\left(\frac{\sigma_m}{\sigma_1}\right)^{\nu_{m-1}}\right\}.$$

For example, by $m = 2$, we find again formula (1) under the term

$$(x_1 + x_2)^n = \sum_{k=1}^{n} S_k \binom{2n-k-1}{n-1} \left(\frac{\sigma_2}{\sigma_1}\right)^{n-k}.$$

For three variables, x_1, x_2, x_3, $m = 3$, we have ($\nu = \nu_2$):

$$(x_1 + x_2 + x_3)^n = \sum_{k=1}^{n} \left\{ S_k \sum_{0 \le \nu \le \frac{n-k}{2}} (-1)^\nu \frac{(2n-k-1-\nu)!}{(n-1)!\nu!(n-k-2\nu)!} \left(\frac{\sigma_2}{\sigma_1}\right)^{n-k-2\nu} \left(\frac{\sigma_3}{\sigma_1}\right)^{\nu} \right\}.$$

REFERENCES

1. Comtet, *Analyse Combinatoire* (Presses Universitaires de France, 1970).
2. Toscano, "Tu due sviluppi della potenza di un bionomio," *Bol. della Soc. Mathm. Calabrese* 16 (1965):1-8.

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WHICH SECOND-ORDER LINEAR INTEGRAL RECURRENCES HAVE ALMOST ALL PRIMES AS DIVISORS?

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This paper will prove that essentially only the obvious recurrences have almost all primes as divisors. An integer n is a divisor of a recurrence if n divides some term of the recurrence. In this paper, "almost all primes" will be taken interchangeably to mean either all but finitely many primes or all but for a set of Dirichlet density zero in the set of primes. In the context of this paper, the two concepts become synonymous due to the Frobenius density theorem. Our paper relies on a result of A. Schinzel [2], whose paper uses "almost all" in the same sense.

Let $\{w_n\}$ be a recurrence defined by the recursion relation

(1) $$w_{n+2} = aw_{n+1} + bw_n$$

where a, b, and the initial terms w_0, w_1 are all integers. We will call a and b the parameters of the recurrence. Associated with the recurrence (1) is its characteristic polynomial

(2) $$x^2 - ax - b = 0,$$

with roots α and β, where $\alpha + \beta = a$ and $\alpha\beta = -b$.

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of this polynomial.

In general, if $D \neq 0$,

(3) $$w_n = c_1\alpha^n + c_2\beta^n,$$

where

(4) $$c_1 = (w_1 - w_0\beta)/(\alpha - \beta)$$

and

(5) $$c_2 = (w_0\alpha - w_1)/(\alpha - \beta).$$

We allow n to be negative in (3), though then w_n is rational but not necessarily integral.

There are two special recurrences with parameters a and b which we will refer to later. They are the Primary Recurrence (PR) $\{u_n\}$ with initial terms $u_0 = 0$, $u_1 = 1$ and the Lucas sequence $\{v_n\}$ with initial terms $v_0 = 2$ and $v_1 = a$. By (4) and (5) we see that the nth term of the PR is

(6) $$u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$$

and the nth term of the Lucas sequence is

(7) $$v_n = \alpha^n + \beta^n.$$

The following lemma will help give us a partial answer to the problem of determining those recurrences which have almost all primes as divisors.

Lemma 1: Let $\{w_n\}$ be a recurrence with parameters a and b. Let p be a prime. If $b \not\equiv 0 \pmod{p}$, then $\{w_n\}$ is purely periodic modulo p.

Proof: First, if a pair of consecutive terms (w_n, w_{n+1}) is given, the recurrence $\{w_n\}$ is completely determined from that point on by the recursion relation. Now, a pair of consecutive terms (w_m, w_{m+1}) must repeat (mod p) since only p^2 pairs of terms are possible (mod p). Suppose (w_m, w_{m+1}) is the first pair of terms to repeat (mod p) with $m \neq 0$. But then

$$bw_{m-1} = w_{m+1} - aw_m$$

by the recursion relation. Hence,

$$w_{m-1} \equiv b^{-1}(w_{m+1} - aw_m) \pmod{p}.$$

Thus, w_{m-1} is now determined uniquely (mod p) and the pair $(w_{m-1}, w\ )$ repeats (mod p) which is a contradiction. Therefore, $m = 0$ and the sequence is purely periodic modulo p.

Thus, we now have at least a partial answer to the question of our title. The PR $\{u_n\}$ clearly satisfies our problem since any prime divides the initial term $u_0 = 0$. Further, any multiple of a translation of this sequence also works. The sequence $\{w_n\}$, where $w_0 = ru_{-n}$, $w_1 = ru_{-n+1}$ with r rational and $n \geq 0$ clearly has 0 as a term. Moreover, by our previous result, Lemma 1, if $p \nmid b$, then p divides some term of $\{w_n\}$, where $w_0 = ru_n$, $w_1 = ru_{n+1}$ with r rational and $n \geq 0$. Clearly, there are only finitely many primes p dividing b. We shall show that these are essentially the only such recurrences satisfying our problem. This is expressed in the following main theorem of our paper.

Theorem 1: Consider the recurrence $\{w_n\}$ with parameters a and b. Suppose

$$b \neq 0,\ D \neq 0,\ w_1 \neq \alpha w_0, \text{ and } w_1 \neq \beta w_0.$$

Then almost all primes are divisors of the recurrence $\{w_n\}$ if and only if

$$w_0 = ru_n,\ w_1 = ru_{n+1}$$

for some rational r and integer n, not necessarily positive.

We will now explore how far we can go towards proving our main theorem using just elementary and well-known results of number theory.

Theorem 2: Consider the recurrence $\{w_n\}$ with parameters a and b. Suppose that neither $w_1^2 - w_0w_2$ nor $(-b)(w_1^2 - w_0w_2)$ is a perfect square. Then, there exists a set of primes of positive density that does not contain any divisors of $\{w_n\}$.

Proof: It can be proved by induction that

$$w_n^2 - w_{n-1}w_{n+1} = (w_1^2 - w_0w_2)(-b)^{n-1}. \tag{8}$$

By the law of quadratic reciprocity, the Chinese remainder theorem, and Dirichlet's theorem on the infinitude of primes in arithmetic progressions, it can be shown that there exists a set of primes p of positive density such that

$$(-b/p) = 1 \quad \text{and} \quad (w_1^2 - w_0w_2/p) = -1.$$

We suppress the details. Now suppose that p divides some term w_{n-1}. Then

$$w_n^2 - 0 \equiv (w_1^2 - w_0w_2)(-b)^{n-1} \pmod{p}.$$

But

$$(w_n^2/p) = 1$$

and

$$\left((w_1^2 - w_0w_2)(-b)^{n-1}/p\right) = (1)(-1) = -1.$$

This is a contradiction and the theorem follows.

Unfortunately, there are recurrences which are not multiples of translations of PRs and which do not satisfy the hypothesis of Theorem 2. For example, consider the recurrence $\{w_n\}$ with parameters $a = 3$, $b = 5$, and initial terms 5, 21, 88, 369. Then

$$w_1^2 - w_0w_2 = 1$$

and the conditions of Theorem 2 are not met. However, it is easily seen that this recurrence is not a multiple of a translation of the PR with parameters 3 and 5.

To prove our main theorem, we will need a more powerful result.

Lemma 2: Let L be an algebraic number field. If λ and θ are nonzero elements of L and the congruence

$$\lambda^x \equiv \theta \pmod{P}$$

is solvable in rational integers for almost all prime ideals P of L, then the corresponding equation

$$\lambda^x = \theta$$

is solvable for a fixed rational integer.

Proof: This is a special case of Theorem 2 of A. Schinzel's paper [2].

Before going on, we will need three technical lemmas.

Lemma 3: In the PR $\{u_n\}$ with parameters a and b, suppose that $b \neq 0$. Then

$$u_{-n} = (-1)^{n+1}(u_n/b^n) \quad \text{for } n \geq 0.$$

Proof: Use induction on n.

Lemma 4: Consider the PR $\{u_n\}$ with parameters a and b. Then

$$\alpha_n = bu_{n-1} + u_n\alpha$$

and

$$\beta_n = bu_{n-1} + u_n\beta$$

where $n \geq 0$.

Proof: Notice that

(9) $$\alpha^{n+2} = a\alpha^{n+1} + b\alpha^n$$

and

(10) $$\beta^{n+2} = a\beta^{n+1} + b\beta^n.$$

Now use induction on n and the recursion relations (9) and (10).

Lemma 5: In the recurrence $\{w_n\}$ with parameters a and b, suppose that

$$D \neq 0,\ b \neq 0,\ w_1 \neq \alpha w_0,\ \text{and}\ w_1 \neq \beta w_0.$$

Let $\gamma = w_1 - w_0\alpha$ and $\delta = w_1 - w_0\beta$ be the roots of the quadratic equation

$$x^2 - (2w_1 - aw_0)x - (bw_0^2 + aw_0w_1 - w_1^2) = 0.$$

Then

$$\gamma/\delta = (\alpha/\beta)^n$$

for some rational integer n, not necessarily positive, if and only if

$$w_0 = ru_{-n},\ w_1 = ru_{-n+1}$$

for some rational number r.

Proof: First we will prove necessity. Suppose that

$$\gamma/\delta = (\alpha/\beta)^n.$$

By hypothesis none of α, β, γ, or δ is equal to 0. Then $\gamma = m\alpha^n$ and $\delta = m\beta^n$ for some element m of the algebraic number field $K = Q(\sqrt{D})$. We now claim that m is a rational number. Let t_k be the kth term of the PR with parameters $2w_1 - aw_0$ and $bw_0^2 + aw_0w_1 - w_1^2$. Then

$$t_k = (\gamma^k - \delta^k)/(\gamma - \delta).$$

In particular,

$$\begin{aligned} t_2 = 2w_1 - aw_0 &= (m^2\alpha^{2n} - m^2\beta^{2n})/(m\alpha^n - m\beta^n) \\ &= m(\alpha^n + \beta^n) = mv_n, \end{aligned}$$

where v_n is the nth term of the Lucas sequence with parameters a and b. Hence

$$m = (2w_1 - aw_0)/v_n$$

is a rational number. Now remember that

$$\gamma = w_1 - w_0\alpha = m\alpha^n \quad\text{and}\quad \delta = w_1 - w_0\beta = m\beta^n.$$

By Lemma 4, we can express α^n and β^n in terms of u_{n-1}, u_n, α, and β. Now γ and δ are already expressed in terms of w_0, w_1, α, and β. We can thus solve for w_0, w_1 in terms of α, β, u_{n-1}, and u_n. We now use Lemma 3 to express u_{-n} in terms of u_n. If n is positive, we obtain

(11) $$w_0 = [(-1)^n mb^n]u_{-n},\quad w_1 = [(-1)^n mb^n]u_{-n+1}.$$

If n is negative or zero, we obtain

(12) $$w_0 = mu_{-n}, \quad w_1 = mu_{-n+1}$$

as required. We have now proved necessity. To prove sufficiency, we simply reverse our steps in the proof so far.

We are now ready for the proof of our main theorem.

Proof of Theorem 1: We have already shown the sufficiency of the theorem in our remarks following Lemma 1. To prove necessity, suppose that for almost all primes p there exists a rational integer n such that $p|w_n$. Then by (3),

$$w_n = c_1\alpha^n + c_2\beta^n \equiv 0 \pmod{p}$$

is satisfiable for some integral n for almost all rational primes p. In the algebraic number field $K = Q(\sqrt{D})$, we thus have

$$c_1\alpha^n + c_2\beta^n \equiv 0 \pmod{P}$$

for the prime ideals P dividing (p) in K. Thus,

$$(\alpha/\beta)^n \equiv -c_2/c_1 \equiv \gamma/\delta \pmod{P}$$

by the definition c_1, c_2, γ, and δ. Consequently,

$$\gamma/\delta \equiv (\alpha/\beta)^x \pmod{P}$$

is solvable for almost all prime ideals P in K. Hence, by Lemma 2,

$$\gamma/\delta = (\alpha/\beta)^n$$

for some rational integer n. Therefore, by Lemma 5,

$$w_0 = ru_{-n}, \quad w_1 = ru_{-n+1}$$

for some rational number r and we are done.

For completeness, the next theorem will answer the question of the title for those recurrences excluded by the hypothesis of Theorem 1.

Theorem 3: In the recurrence $\{w_n\}$ with parameters a and b, suppose that

$$(w_0,w_1) = (0,0),\ b = 0,\ D = 0,\ w_1 = \alpha w_0, \text{ or } w_1 = \beta w_0.$$

Let p denote a rational prime.

(i) If $w_0 = 0$ and $w_1 = 0$, then $p|w_n$ for all n regardless of a and b. Note that in this case, the recurrence $\{w_n\}$ is a multiple of the PR $\{u_n\}$.

(ii) If $b = 0$ and $(w_0,w_1) \neq (0,0)$, then the recurrence $\{w_n\}$ has almost all primes as divisors only in the following cases:

(a) $b = 0$, $a \neq 0$, $w_0 = 0$, and $w_1 \neq 0$. Then $p|w_0$ for all primes p and $p\nmid w_n$, $n \geq 1$, if $p\nmid aw_1$. Clearly, in this case the recurrence is a multiple of the PR $\{u_n\}$.

(b) $b = 0$, $a \neq 0$, $w_0 \neq 0$, and $w_1 = 0$. Then $w_n = 0$ for $n \geq 1$ and $p|w_n$ for all p if $n \geq 1$.

(c) $b = 0$, $a = 0$. Then $p|w_n$ for all p if $n \geq 2$.

(iii) Suppose $b = 0$, $(w_0,w_1) \neq (0,0)$, $a \neq 0$, and $b \neq 0$. Then the recurrence $\{w_n\}$ has almost all primes p as divisors if and only if $w_1 \neq (a/2)w_0$.

(iv) Suppose that $w_1 = \alpha w_0$ or $w_1 = \beta w_0$. Further, suppose that D is a perfect square, $w_0 \neq 0$, and $b \neq 0$. Then almost all primes are *not* divisors of the recurrence $\{w_n\}$. Moreover, $p\nmid w_n$ for any n if $p\nmid w_1$.

Proof: (i) and (ii) can be proved by direct verification.

(iii) Let $a' = a/2$. It can be shown by induction that

(13) $$w_n = (a')^{n-1}(a'w_0 + (w_1 - aw_0)n).$$

We can assume that $a' \not\equiv 0 \pmod{p}$ since, by hypothesis, $a' \equiv 0 \pmod{p}$ holds only for finitely many primes p. Then if $w_1 - a'w_0 \not\equiv 0 \pmod{p}$, $w_n \equiv 0$ when

$$n \equiv -a'w_0/(w_1 - a'w_0) \pmod{p}.$$

If $w_1 - a'w_0 \equiv 0 \pmod{p}$ for almost all primes p, then $w_1 = a'w_0$. Hence, by (13),

$$w_n = (a')^n w_0 = \alpha^n w_0.$$

In this case, the only primes which are divisors of the recurrence are those primes which divide $a'w_0$. Note that if the hypotheses of (iii) hold, then the only recurrences not having almost all primes as divisors are those that are multiples of translations of the Lucas sequence $\{v_n\}$.

(iv) Since

$$\alpha^{n+2} = a\alpha^{n+1} + b\alpha^n$$

and

$$\beta^{n+2} = a\beta^{n+1} + b\beta^n,$$

it follows that either the terms of the recurrence $\{w_n\}$ are of the form $\{\alpha^n w_0\}$ or they are of the form $\{\beta^n w_0\}$. The result is now easily obtained.

To conclude, we note that as a counterpoise to Theorem 1, which states that essentially only one class of recurrences has almost all primes as divisors, there is the following theorem by Morgan Ward [3]. It states that, in general, every recurrence has an infinite number of prime divisors.

Theorem 3 (Ward): In the recurrence $\{w_n\}$ with parameters a and b, suppose that $b \neq 0$, $w_1 \neq \alpha w_0$, and $w_1 \neq \beta w_0$. Then if α/β is not a root of unity, the recurrence $\{w_n\}$ has an infinite number of prime divisors.

REFERENCES

1. Marshall Hall, "Divisors of Second-Order Sequences," *Bull. Amer. Math. Soc.* 43 (1937):78-80.
2. A. Schinzel, "On Power Residues and Exponential Congruences," *Acta Arith.* 27 (1975):397-420.
3. Morgan Ward, "Prime Divisors of Second-Order Recurring Sequences," *Duke Math. J.* 21 (1954):607-614.

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NOTE ON A TETRANACCI ALTERNATIVE TO BODE'S LAW

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Bode's law is an empirical approximation to the mean distances of the planets from the Sun; it arises from a simply-generated sequence of integers. Announced in 1772 by Titius and later appropriated by Bode, it has played an important role in the exploration of the Solar System [1].

The Bode numbers are defined by

$$B_1 = 4$$

$$B_n = 2^{n-2} \times 3 + 4, \ n = 2, \ldots$$

Then the quantities $0.1B_n$, $n = 1, \ldots, 10$ represent the mean distances of the nine planets and the asteroid belt from the Sun in terms of the Earth's distance.

In view of the numerical explorations reported in [2], [3], and [4], it seems plausible to look for improvements to Bode's law among the Multinacci sequences and, indeed, the Tribonacci, Tetranacci, Pentanacci, and Hexanacci numbers are suited to this task. The Tetranacci numbers provide the best fit, slightly superior to the original Bode solution.

The Tetranacci numbers are defined by the recurrence

$$T_1, \ldots, T_4 = 1$$

$$T_n = \sum_{i=1}^{4} T_{n-i}, \; n = 5, \ldots$$

The alternative Bode numbers are then given by

$$\tilde{B}_n = T_{n+3} + 3, \; n = 1, \ldots$$

The quantities $0.1\tilde{B}_n$ can then be compared with their Bode counterparts. See the accompanying table.

Planet	Actual Distance	Bode	Tetranacci
Mercury	0.39	0.40	0.40
Venus	0.72	0.70	0.70
Earth	1.00	1.00	1.00
Mars	1.52	1.60	1.60
(asteriods)	2.70	2.80	2.80
Jupiter	5.20	5.20	5.20
Saturn	9.54	10.00	9.70
Uranus	19.18	19.60	18.40
Neptune	30.06	38.80	35.20
Pluto	39.44	77.20	67.60

It can be seen that the fits are poor for Neptune and bad for Pluto. However, the Tetranacci alternative is somewhat better in both cases.

No rigorous dynamical explanation is apparent for the Bode or Tetranacci representations. They are either numerical coincidences, as the result in [5] indicates, or, if they contain physical information, may simply illustrate that the period of revolution of a planet is strongly a function of the periods of nearby planets. This conjecture arises from the Kepler relation $(\text{distance})^3 \propto (\text{period})^2$ and the fact that period relationships are often important in determining the state of a dynamical system.

REFERENCES

1. S. L. Jaki, "The Titius-Bode Law: A Strange Bicentenary," *Sky and Telescope* 43, No. 5 (1972):280-281.
2. B. A. Read, "Fibonacci-Series in the Solar System," *The Fibonacci Quarterly* 8, No. 4 (1970):428-438.
3. B. Davis, "Fibonacci Numbers in Physics," *The Fibonacci Quarterly* 10, No. 6 (1972):659-660, 662.

4. W. E. Greig, "Bode's Rule and Folded Sequences," *The Fibonacci Quarterly* 14, No. 2 (1976):129-134.
5. R. Kurth, *Dimensional Analysis and Group Theory in Astrophysics* (New York: Pergamon Press, 1972), p. 203.

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REFLECTIONS ACROSS TWO AND THREE GLASS PLATES

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1. INTRODUCTION

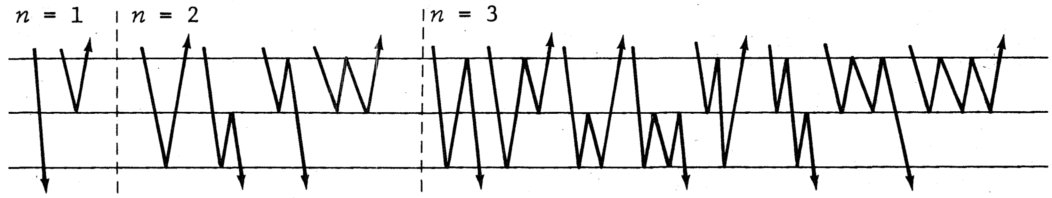
That reflections of light rays within two glass plates can be expressed in terms of the Fibonacci numbers is well known [Moser, 1]. In fact, if one starts with a single light ray and if the surfaces of the glass plates are half-mirrors such that they both transmit and reflect light, the number of possible paths through the glass plates with n reflections is F_{n+2}. Hoggatt and Junge [2] have increased the number of glass plates, deriving matrix equations to relate the number of distinct reflected paths to the number of reflections and examining sequences of polynomials arising from the characteristic equations of these matrices.

Here, we have arranged the counting of the reflections across the two glass plates in a fresh manner, fixing our attention upon the number of paths of a fixed length. One result is a physical interpretation of the compositions of an integer using 1's and 2's (see [3], [4], [5]). The problem is extended to three glass plates with geometric and matrix derivations for counting reflection paths of different types as well as analyses of the numerical arrays themselves which arise in the counting processes. We have counted reflections in paths of fixed length for regular and for bent reflections, finding powers of two, Fibonacci numbers and convolutions, and Pell numbers.

2. PROBLEM I

Consider the compositions of an even integer $2n$ into ones and twos as represented by the possible paths of length $2n$ taken in reflections of a light ray in two glass plates.

REFLECTIONS OF A LIGHT RAY IN PATHS OF LENGTH $2n$



For a path length of 2, there are 2 possible paths and one reflection; for a path length of 4, 4 possible paths and 8 reflections; for a path length of 6, 8 possible paths and 28 reflections. Notice that an odd path length would end at the middle surface rather than exiting.

First, the number of paths possible for a path length of $2n$ is easily derived if one notes that each path of length $2(n - 1)$ becomes a path of length $2n$ by adding a segment of length 2 which either passes through the center plate or reflects on the center plate, so that there are twice as many paths of length $2n$ as there were of length $2(n - 1)$.

Result 1: There are 2^n paths of length $2n$.

Continuing the same geometric approach yields the number of reflections for a path length $2n$. Each path of length $2(n - 1)$ gives one more reflection when a length 2 segment is added which passes through the center plate, and two more reflections when a length 2 segment is added which reflects on the center plate, or, the paths of length $2n$ have $3 \cdot 2^{n-1}$ new reflections coming from the 2^{n-1} paths of length $2(n - 1)$ as well as twice as many reflections as were in the paths of length $2(n - 1)$. Note that the number of reflections for path lengths $2n$ is $2^{n-1}(3n - 2)$ for $n = 1, 2, 3$. If there are

$$2^{n-2}\big(3(n - 1) - 2\big)$$

reflections in a path of length $2(n - 1)$, then there are

$$2 \cdot 2^{n-2}\big(3(n - 1) - 2\big) + 3 \cdot 2^{n-1} = 2^{n-1}(3n - 2)$$

reflections in a path of length $2n$, which proves the result following by mathematical induction.

Result 2: There are $2^{n-1}(3n - 2)$ reflections in each of the paths of length $2n$.

Proofs: Let A represent a reflection down $\bigwedge$ or up $\bigvee$, and B represent a straight path down $\downarrow$ or up $\uparrow$, where both A and B have length two. Note that it is impossible for the two types of A to follow each other consecutively. Now, each path of length $2n$ is made up of A's and B's in some arrangement. Thus, the expansion of $(A + B)^n$ gives these arrangements counted properly, and $N = 2n$, so that the number of distinct paths is 2^n.

Now, in counting reflections, there is a built-in reflection for each A and a reflection between A and B, A and A, and B and B. Consider

$$f(x) = x^{n-1}(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-1+j}.$$

Each term in $(A + B)^n$ has degree n and there are $(n - 1)$ spaces between factors. The x^{n-1} counts the $(n - 1)$ spaces between factors, since each A has a built-in reflection. The exponents of x count reflections from A; there are no reflections from B. Since we wish to count the reflections, we differentiate $f(x)$ and set $x = 1$.

$$\begin{aligned} f'(x) &= \left\{(n - 1)x^{n-2}(1 + x)^n + nx^{n-1}(1 + x)^{n-1}\right\}\Big|_{x=1} \\ &= (n - 1)2^n + n \cdot 2^{n-1} = 2^{n-1}(3n - 2). \end{aligned}$$

Interpretation as a composition using ones and twos: All the even integers have compositions in which, whenever strings of ones appear, there are an even number of them. Each A is a $1 + 1$ (taken as a pair) and each B is a 2, and each reflection is a plus sign. From $f(x)$, let $s = n - 1 + j$ so that $j = s - n + 1$, and we get $\binom{n}{s-n+1}$ compositions of $2n$, each with exactly s plus signs. Note that $s \geq n - 1$, with equality when all twos are used.

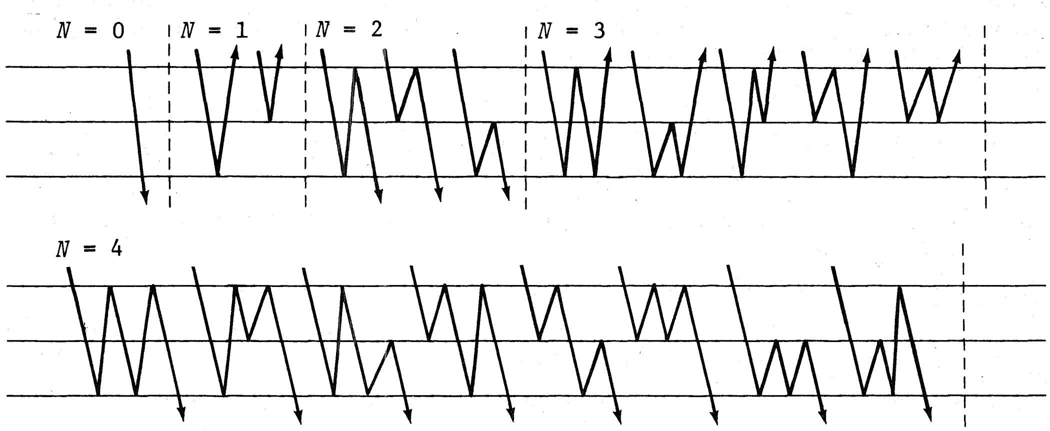
We note in passing that the number of possible paths through the two plates with n reflections is F_{n+2}, while the number of compositions of n using all ones and twos is F_{n+1} [3].

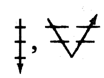
3. PROBLEM II

Given a particular configuration (path), how many times does it appear as a subconfiguration in all other paths with a larger but fixed number of reflections?

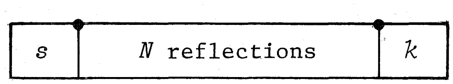
This leads to convolutions of the Fibonacci numbers.

PATHS WITH A FIXED NUMBER OF REFLECTIONS



Note that the subconfigurations  each occur 1, 2, 5, 10, 20, ... times in successive collections of all possible paths with a larger but fixed number of reflections. The same sequence occurs for any subconfiguration chosen.

Consider a subconfiguration that contains N reflections. It could be preceded by s reflections and followed by k reflections. Clearly, since each path starts at the upper left, the configurations in the front must start in the upper left and end up in the upper right, which demands an odd number of reflections. Thus, s is odd, but conceivably there are no configurations in



the part on the front. Now, the part on the end could join up at the top or the bottom, depending on whether N is odd or even. In case N is even, then the regular configurations may be turned over to match. Thus, if the total number of reflections is specified, the allowable numbers will be determined.

4. RESULTS OF SEPARATING THE REFLECTION PATHS

In Sections 2 and 3, the reflection paths $\searrow$ and $\nearrow$, and $\wedge$ and $\vee$, were counted together. If one separates them, then, with the right side up, one obtains {1, 1, 4, 5, 14, 19, 46, 65, ...} which splits into two convolution sequences:

$$\{A_1, A_3, A_5, \ldots\} = \{1, 2, 5, 13, \ldots\} * \{1, 2, 5, 13, \ldots\};$$

$$\{A_2, A_4, A_6, \ldots\} = \{1, 2, 5, 13, \ldots\} * \{1, 3, 8, 21, \ldots\}.$$

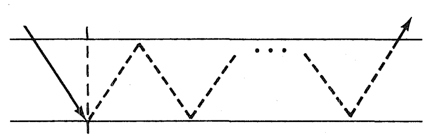
This second set agrees with the upside-down case {0, 1, 1, 5, 6, 19, 25, 65, ...} which splits into two convolution sequences:

$$\{B_1, B_3, B_5, \ldots\} = \{0, 1, 3, 8, \ldots\} * \{1, 3, 8, 21, \ldots\};$$

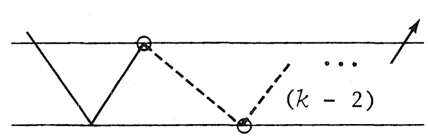
$$\{B_2, B_4, B_6, \ldots\} = \{A_2, A_4, A_6, \ldots\}.$$

Clearly, there are only two cases, $\searrow$, $\vee$, where we assume that the configurations in which these appear start at the left top and end at either right top or right bottom.

First we discuss the number of occurrences of $\searrow$. Here we consider only those patterns which start in the upper left. If there are no prepatterns, then we consider odd and even numbers of reflections separately. We get one free reflection by joining $\searrow$ to a pattern which begins on the bottom left.



Let us assume that the added-on piece has k (even) internal reflections. There are F_{k+2} such right-end pieces and $F_{0+2} = F_2 = 1$ left-end pieces. Next, let the piece on the right have $k - 2$ internal reflections and the one on the left have one internal reflection:



$$F_{1+2} \cdot F_{k-2+2}$$

Generally,

$$F_1F_{k+2} + F_3F_k + F_5F_{k-2} + \cdots .$$

Specifically,

$$k = 0: \quad F_1F_2 = 1$$

$$k = 2: \quad F_1F_4 + F_3F_2 = 1 \cdot 3 + 2 \cdot 1 = 5$$

$k = 4$: $F_1F_6 + F_3F_4 + F_5F_2 = 1 \cdot 8 + 2 \cdot 3 + 5 \cdot 1 = 19$

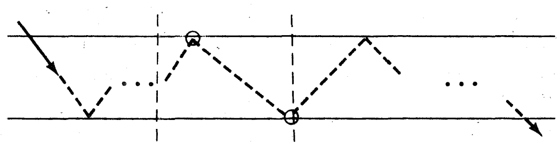
$k = 6$: $F_1F_8 + F_3F_6 + F_5F_4 + F_7F_2 = 1 \cdot 21 + 2 \cdot 8 + 5 \cdot 3 + 13 \cdot 1 = 65$

If k is odd, the same basic plan holds, so that for no pieces front or back, $F_2F_2 = 1$,

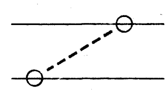
$k = 1$: $F_1F_3 + F_3F_1 = 1 \cdot 2 + 2 \cdot 1 = 4$

$k = 3$: $F_1F_5 + F_3F_3 + F_5F_1 = 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 = 14$

This is precisely the same as the other case except that it must start at the top left, have a free reflection where it joins a section at the top, a free reflection where it joins the right section at the bottom, and the right section must end at the bottom.



Any of our subconfigurations can appear complete by itself first. Our sample, of course, holds for any block with an even number of reflections. The foregoing depends on the final configuration starting on the upper left and the subconfiguration (the one we are watching) also starting on the upper left. However, if we "turn over" our subconfiguration then we get a different situation



which must fit into a standard configuration which starts in the upper left. Hence, this particular one cannot appear normally by itself, nor can any one with an even number of reflections. Here we must have a pre-configuration with an even number of reflections.

Let k be even again.

$F_2F_2 = 1 \cdot 1 = 1$

$F_2F_4 + F_4F_2 = 1 \cdot 3 + 3 \cdot 1 = 6$

$F_2F_6 + F_4F_4 + F_6F_2 = 1 \cdot 8 + 3 \cdot 3 + 8 \cdot 1 = 25$

...

Let k be odd.

$F_2F_1 = 1 \cdot 1 = 1$

$F_2F_3 + F_4F_1 = 1 \cdot 2 + 3 \cdot 1 = 5$

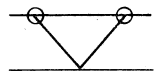
$F_2F_5 + F_4F_3 + F_6F_1 = 1 \cdot 5 + 3 \cdot 2 + 8 \cdot 1 = 19$

...

These sequences are $\{1, \underline{1}, 4, \underline{5}, 14, \underline{19}, \ldots\}$ (right side up) and $\{0, \underline{1}, 1, \underline{5}, 6, \underline{19}, \ldots\}$ (upside down), and added together, they produce the first Fibonacci convolution $\{1, 2, 5, 10, 20, 38, \ldots\}$.

Each subconfiguration which starts at the upper left and comes out at the lower right can be put in place of the configuration which makes a straight through crossing with the same results, of course.

For the results dealing with



the restrictions on the left are exactly the same as just described, and the endings on the right are merely those for the earlier case endings turned upside down to match the proper connection.

Reconsidering the four sequences of this section gives some interesting results. In the sequences $\{A_n\}$ (right side up) and $\{B_n\}$ (upside down), adding A_i and B_i gives successive terms of the first Fibonacci convolution sequence. Taking differences of odd terms gives $1-0=1$, $4-1=3$, $14-6=8$, ..., which is clearly $1, 3, 8, 21, \ldots, F_{2k}, \ldots$, the Fibonacci numbers with even subscripts.

Further, for $\{A_n\}$,

$$\begin{array}{ll} 1+1+2=4 & 1+4=5 \\ 4+5+5=14 & 5+14=19 \\ 14+19+13=46 & 19+46=65 \\ \ldots & \ldots \\ A_n+A_{n+1}+F_{n+2}=A_{n+2},\ n \text{ odd} & A_n+A_{n+1}=A_{n+2},\ n \text{ even} \end{array}$$

while for $\{B_n\}$,

$$\begin{array}{ll} 1+1+3=5 & 0+1=1 \\ 5+6+8=19 & 1+5=6 \\ 19+25+21=65 & 6+19=25 \\ \ldots & \ldots \\ B_n+B_{n+1}+F_{n+2}=B_{n+2},\ n \text{ even} & B_n+B_{n+1}=B_{n+2},\ n \text{ odd} \end{array}$$

The results of this section can be verified using generating functions as follows. (See, for example, [6].) The generating function for the first convolution of the Fibonacci sequence, which sequence we denote by $\{F_n^{(2)}\}$, is

$$\left(\frac{1}{1-x-x^2}\right)^2 = \sum_{n=0}^{\infty} F_{n+1}^{(2)} x$$

while the sequence of odd terms of $\{A_n\}$ is the first convolution of Fibonacci numbers with odd subscripts, or,

$$\left(\frac{1-x^2}{1-3x^2+x^4}\right)^2 = \sum_{n=0}^{\infty} A_{2n+1}x^n$$

and the sequence of odd terms of $\{B_n\}$ is the first convolution of Fibonacci numbers with even subscripts, or,

$$\left(\frac{x}{1-3x^2+x^4}\right)^2 = \sum_{n=0}^{\infty} B_{2n+1}x^n$$

and the even terms of $\{A_n\}$ as well as of $\{B_n\}$ are the convolution of the sequence of Fibonacci numbers with even subscripts with the sequence of Fibonacci numbers with odd subscripts, or,

$$\left(\frac{x}{1-3x^2+x^4}\right)\cdot\left(\frac{1-x^2}{1-3x^2+x^4}\right) = \sum_{n=0}^{\infty} A_{2n}x^n = \sum_{n=0}^{\infty} B_{2n}x^n.$$

That $\{F_n^{(2)}\}$ is given by the term-wise sum of $\{A_n\}$ and $\{B_n\}$ is then simply shown by adding the generating functions, since

$$\frac{(1-x^2)^2}{(1-3x^2+x^4)^2} + \frac{2x(1-x^2)}{(1-3x^2+x^4)^2} + \frac{x^2}{(1-3x^2+x^4)^2}$$

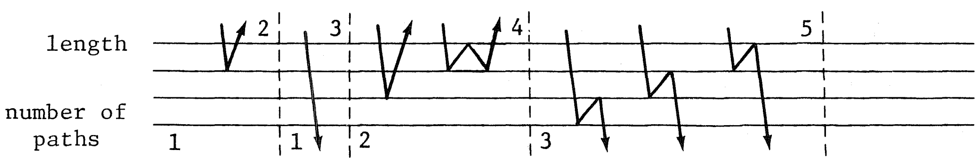
$$= \frac{(1+x-x^2)^2}{(1-3x^2+x^4)^2} = \frac{(1+x-x^2)^2}{(1-2x^2+x^4-x^2)^2}$$

$$= \frac{(1+x-x^2)^2}{(1-x^2+x)^2(1-x^2-x)^2} = \frac{1}{(1-x-x^2)^2}$$

Quite a few identities for the four sequences of this section could be derived by the same method.

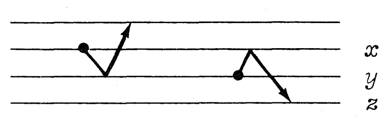
5. THREE STACKED PLATES

Theorem A: In reflective paths in three stacked glass plates, there are F_{n-1} paths of length n that enter at the top plate and exist at the top or bottom plate.



Discussion: Note that the paths end in lengths 3, 2 + 2, 1 + 1, or 1 + 2. We therefore assume of the paths of length n, that there are F_{n-4} which end in 3, F_{n-3} which end in 1 + 1, F_{n-5} which end in 2 + 2, F_{n-4} which end in 1 + 2, where $n \geq 5$. This is the same as saying that there are F_{n-4} paths of length $n - 3$ reflecting inwardly at an inside surface.

Proof: We proceed by induction. Thus the paths of length $k + 1$ are made up of paths which end in 3, 1 + 1, 2 + 2, or 1 + 2. We assume that there are F_{k-3} paths which end in 3, F_{k-2} paths which end in 1 + 1, F_{k-4} which end in 2 + 2, and F_{k-3} which end in 1 + 2. Since $F_{k-3} + F_{k-2} + F_{k-4} + F_{k-3} = F_k$, we will have a proof by induction if we can establish the assumption about path lengths. The first three are straightforward, but that F_{k-3} paths end in 1 + 2 needs further elaboration. In order to be on an outside edge after 1 + 2, the ray must have been on plate x or y with a reflection at the beginning:



How can the paths get to the x-dot for n even or the y-dot for n odd? Assume that there are F_{k-6} paths of length $k - 5$ which come from the upper surface, go to plate y, and then to the x-dot (note that the total path would then have length $k + 1$, since a path of 2 + 1 would be needed to reach the x-dot and a path of 1 + 2 to leave the x-dot). There are F_{k-5} paths which reflect from plate x, go to plate y and return to the x-dot, and F_{k-5} paths which relect from the bottom surface upward to the x-dot. Thus, there are

$$F_{k-6} + F_{k-5} + F_{k-5} = F_{k-3}$$

paths of length $k - 2$ coming upward to a reflection x-dot if k is even and downward to a y-dot if k is odd.

By careful counting, one can establish several other results involving Fibonacci numbers.

Theorem B: There are F_n paths of length n in three stacked plates that enter at the top plate and terminate on one of the internal surfaces.

Theorem C: There are F_{n+1} paths of length n which enter at the top plate and terminate on one of the four surfaces, and F_{n-1} that terminate on outside surfaces.

Theorem D: Of paths of length n terminating on any one of the four surfaces, there are F_n paths that end in a unit jump. There are $2F_{n-3}$ paths that end in a two unit jump, and there are F_{n-4} paths that end in a three unit jump.

Theorem E: There are nF_{n-3} ones used in all paths of length n which terminate on outside plates.

Theorem F: For $n \geq 3$, the number of threes in paths of length n which terminate on outside plates is a convolution of 1, 0, 1, 1, 2, 3, ..., F_{n-2}, ..., with itself. The convolution sequence is given by $2F_{n-4} + C_{n-6}$, where $C_n = (nL_{n+1} + F_n)/5$.

Theorem G: Let T_n' be the number of threes in all paths of length n that end on an inside line. Then the number of twos used in all paths of length n which terminate on outside faces is $2T_{n+1}' = 2F_{n-3} + 2C_{n-5}$.

Theorem H: $T_n' = T_n - F_{n-4}$, where T_n is the number of threes used totally in all paths of length n which terminate on outside faces, and T_n' is the number of threes in all paths of length n which end on an inside plate.

Corollary: The number of twos used in all paths of length n which terminate on outside surfaces is

$$2(T_{n+1} - F_{n-3}) = 2(2F_{n-3} + C_{n-5} - F_{n-3}) = 2[5F_{n-3} + (n - 5)L_{n-4} + 2F_{n-5}]/5.$$

From this, of course, we can now discuss the numbers of ones, twos, and threes used in the reflections. We will let U_n be the number of ones used, D_n the number of twos, T_n the number of threes used in all paths of length n terminating on outside faces, while we will prime these to designate paths that only terminate on inside plates.

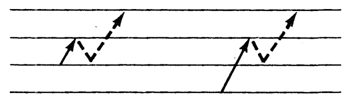
We return to the proof of Theorem A, that there are F_{n-1} paths of length n in three stacked glass plates, to glean more results. Recall that the plate paths end in 3, 1 + 1, 2 + 2, and 1 + 2.

Let P_n be the number of paths of length n. Then

$$P_n = P_{n-3} + P_{n-2} + P_{n-4} + \emptyset_{n-3},$$

where P_{n-3} paths end in 3, P_{n-2} in 1 + 1, P_{n-4} in 2 + 2, and $\emptyset_{n-3}$ is the number of paths terminating on an inside plate and of length n, but the last path segment was from the inside (i.e., from plate y to x). Suppose we approach x from below and the path is $n - 3$ units long; then we add the dotted portion. However, we can get to x from y or we can get to x from z. The number of paths from z is F_{n-6} by induction since there are F_{n-1} paths. The number of paths from y is $\emptyset_{n-4}$. Assume $\emptyset_n = F_{n-1}$ also so that

$$\emptyset_{n+1} = \emptyset_n + F_{n-2} = F_{n-1} + F_{n-2} = F_n.$$



Now

$$P_{n+1} = P_{n-2} + P_{n-1} + P_{n-3} + \emptyset_{n-2}$$
$$= (F_{n-3} + F_{n-2}) + (F_{n-4} + F_{n-3})$$
$$= F_{n-1} + F_{n-2} = F_n.$$

If we display all F_{n-1} paths of length n, the number of ones used is nF_{n-3}.

We need some further results. Earlier we saw that there were F_{n-1} paths from the inside approaching one of the inside plates. We now need to know how many paths approach the inside lines from outside (a unit step from an outside line). Clearly, it is F_{n-2}; since the path length to the inside line is n, then the path length to the outside line is $n - 1$, making F_{n-2} paths. Let U_n be the number of ones used:

$$U_{n+1} = (U_{n-2} + 2U_{n-3}) + (U_{n-3}) + (U_{n-4}) + (U_{n-3} + U_{n-4}),$$

considering paths ending in 1 + 1, 3, 2 + 2, and 1 + 2.

Let us look at T_n, the number of threes used in paths of length n. By taking paths ending in 3, then 1 + 1, 2 + 2, and 1 + 2, we have

(A) $$T_n = (T_{n-3} + F_{n-4}) + T_{n-2} + T_{n-4} + T'_{n-3}$$

(B) $$T'_n = T'_{n-1} + T_{n-2}$$

Writing (A) for T_{n+1} and subtracting the expression above for T_n gives

$$T_{n+1} - T_n = T_{n-1} - T_{n-4} + T'_{n-2} - T'_{n-3} + F_{n-3} - F_{n-4}$$
$$= T_{n-1} + F_{n-5} + (T'_{n-2} - T'_{n-3} - T_{n-4})$$
$$= T_{n-1} + F_{n-5} + 0.$$

Therefore,

$$T_{n+1} = T_n + T_{n-1} + F_{n-5},$$

which shows that $\{T_n\}$ is a Fibonacci convolution (first) sequence. It is easy to verify that

$$T_n = 2F_{n-4} + C_{n-6},\ T_1 = 0,\ T_2 = 0,\ T_3 = 1,$$
$$T_4 = 0,\ T_5 = 2,\ T_6 = 2,$$

where $\{C_n\}$ is *the* first Fibonacci convolution sequence.

Also,

$$T'_n = T_n - F_{n-4}$$

Next, consider D_n, the number of twos used in paths of length n. Again taking paths ending in 3, then in 1 + 1, 2 + 2, and 1 + 2, we have

(C) $$D_n = (D_{n-4} + 2F_{n-5}) + D_{n-3} + D_{n-2} + (D'_{n-3} + F_{n-4})$$

(D) $$D'_n = D'_{n-1} + D_{n-2} + F_{n-3}$$

Proceeding exactly as before, writing (C) for D_{n+1} and subtracting the expression for D_n, and then using identity (D), one derives

$$D_{n+1} = D_n + D_{n-1} + 2F_{n-4}.$$

We now show that $D_n = 2T'_{n+1}$. From $T'_n = T_n - F_{n-4}$, then

$$\begin{aligned} 2T'_{n+2} &= 2T_{n+2} - 2F_{n-2} \\ &= 2T_{n+1} - 2F_{n-3} + 2T_n - 2F_{n-4} + 2F_{n-4} \end{aligned}$$

by taking advantage of $T_n = T_{n-1} + T_{n-2} + F_{n-6}$. Therefore,

$$T_{n+2} = T_{n+1} + T_n + F_{n-2} - F_{n-3} = T_{n+1} + T_n + F_{n-4}.$$

From the total length of F_{n-1} paths of length n, we know that

$$U_n + 2D_n + 3T_n = nF_{n-1},$$

so that

$$U_n = nF_{n-1} - 3T_n - 2D_n.$$

On the right-hand side, each term will satisfy a recurrence of the form

$$H_n = H_{n-1} + H_{n-2} + K_n,$$

where K_n is a generalized Fibonacci sequence. In this case, by looking at

$$U_1 = 0,\ U_2 = 2,\ U_3 = 0,\ U_4 = 4,$$

$$U_n = U_{n-1} + U_{n-2} + L_{n-4}.$$

This is precisely satisfied by $U_n = nF_{n-3}$.

If U_n is the number of ones used, D_n the number of twos, and T_n the number of threes, then clearly every number is followed by a reflection except the last one. Thus, if there are F_n total paths, then the number of reflections in paths of length n which terminate on outside faces is

$$\begin{aligned} R_n &= U_n + D_n + T_n - F_{n-1} \\ &= (nF_{n-3}) + \left(\frac{2}{5}[5F_{n-3} + (n-5)L_{n-4} + 2F_{n-5}]\right) \\ &\qquad + (2F_{n-4} + [(n-6)L_{n-5} + 2F_{n-6}]/5) - F_{n-1} \\ &= [(5n-3)F_{n-3} + (n-3)L_{n-2}]/5,\ n \geq 1. \end{aligned}$$

In summary, we write

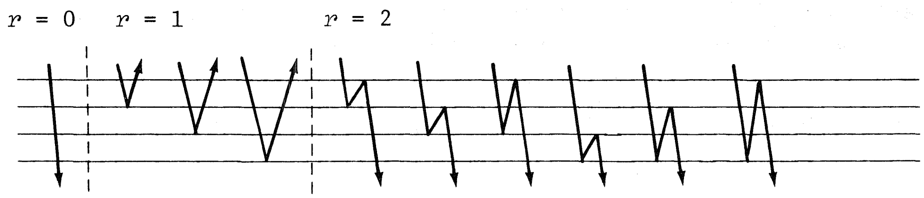
Theorem 1: In the total paths of length n which exit at outside plates, the number of paths is F_{n-1}, and the number of reflections R_n is

$$U_n + D_n + T_n - F_{n-1},$$

where

$$\begin{aligned} U_n &= nF_{n-3} \\ D_n &= \left(\frac{2}{5}[5F_{n-3} + (n-5)L_{n-4} + 2F_{n-5}]\right) \\ T_n &= 2F_{n-4} + [(n-6)L_{n-5} + 2F_{n-6}]/5 \\ R_n &= [(5n-3)F_{n-3} + (n-3)L_{n-2}]/5. \end{aligned}$$

To conclude our discussion of paths and reflections in three glass plates, we consider a fixed number of reflections for paths which exit through either outside surface.



When there are $r = 0$ reflections, there is 1 path possible; for $r = 1$, 3 paths, and for $r = 2$, 6 paths. The number of paths P_r for r reflections yields the sequence 1, 3, 6, 14, 31, 70, 157, ...

Theorem J: Let P_r be the number of paths which exit through either outside face in three glass plates and contain r reflections. Then

$$P_{r+1} = 2P_r + P_{r-1} - P_{r-2}$$

where

$$P_0 = 1,\ P_1 = 3,\ P_2 = 6,\ P_3 = 14.$$

It is easy to derive the sequence $\{P_r\}$. P_{r+1} is formed by adding a reflection at the outside face for each P_r path, and by adding a reflection at surface 1 or 2, which is the number of paths in P_r that end in a two unit jump plus twice the number ending in a three unit jump, which is P_{r-1}. The number ending in a unit jump in P_r paths is P_{r-2}. The number ending in a two unit jump in P_r paths is $P_r - P_{r-2} - P_{r-1}$. Thus,

$$\begin{aligned} P_{r+1} &= P \ + (P_r - P_{r-2} - P_{r-1}) + 2P_{r-1} \\ &= 2P_r + P_{r-1} - P_{r-2}. \end{aligned}$$

Fults [7] has given an explicit expression for P_r as well as its generating function.

6. A MATRIX APPROACH TO REFLECTIONS IN TWO AND THREE STACKED PLATES

Besides counting paths of constant length or paths of a constant number of reflections, there are many other problems one could consider. Here, matrices give a nice method for solving such counting problems.

We return to two glass plates and the paths of length n, where we consider paths that go from line zero to lines one and two, one step at a time. Let u_n, v_n, and w_n be the paths of length n to lines 0, 1, and 2, respectively, and consider the matrix Q defined in the matrix equation below, where we note that $QV_n = V_{n+1}$ and $Q^nV_1 = V_{n+1}$, as below:

$$QV_n = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = V_{n+1}$$

It is easy to see that $u_{n+1} = v_n$, since a path to line zero could have come only from line 1; therefore, each path to line zero was first a path of length n to line 1, then one more step to line zero. Paths to line 1 could have come from line zero or line two, so that $v_{n+1} = u_n + w_n$. Paths to line 2 came from line 1, or, $w_{n+1} = v_n$. This sets up the matrix Q whose characteristic polynomial is $x^2 - 2x = 0$ with solutions $x = 0$ or $x^2 = 2$, so that

$$u_{n+2} = 2u_n,\quad v_{n+2} = 2v_n,\quad \text{and}\quad w_{n+2} = 2w_n.$$

All paths of length zero start on line zero, and in one step of unit length one obtains only one path to line 1, or, using matrix Q,

$$QV_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = V_1$$

Sequentially, we see $Q^nV_0 = V_n$, or,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \to \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} \to \cdots \begin{pmatrix} 2^{n-1} \\ 0 \\ 2^{n-1} \end{pmatrix} \to \begin{pmatrix} 0 \\ 2^n \\ 0 \end{pmatrix} \cdots$$

Now, notice that there are 2^{n-1} paths coming out of the top line and 2^{n-1} paths coming out of the bottom line, each of length $2n$, so that there are 2^n such paths.

If one lets u_n^*, v_n^*, and w_n^* be the number of *regular reflections* on the paths of length n beginning on the top plate and terminating on the top, middle, or bottom plate, respectively, then it can be shown that, from the geometry of the paths,

$$\begin{aligned} u_{n+1}^* &= v_n^* + u_{n-1} \\ v_{n+1}^* &= u_n^* + w_n^* + 2v_{n-1} \\ w_{n+1}^* &= v_n^* + w_{n-1}. \end{aligned}$$

We can write both systems of equations in a 6 x 6 matrix

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}.$$

The method of solution now can be through solving the system of equations directly and, once the recurrence relations are obtained, recognize them. Or one can work with the characteristic polynomial $[x(x^2 - 2)]^2$ via the Hamilton-Cayley theorem and go directly for the generating functions. The recurrence relations yield the general form of the generating function

$$\frac{r(x)}{P_n(x)} = A_0 + A_1x + A_2x^2 + \cdots,$$

whence one can get as many values as needed from the matrix application repeated to a starting column vector, as

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

to use the method of undetermined coefficients for $r(x)$.

The regular reflections are or , while the *bends* look like . These occur in paths which permit horizontal moves as well as jumps between surfaces. These are necessarily more complicated. The matrix Q^* yields paths of length n where "bend" reflections are allowed. That is,

$$Q^* V_n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = V_{n+1}$$

allows paths to move along the lines themselves as well as between the lines. The same reasoning prevails. The characteristic polynomial $(1-x)(x^2-2x-1)$ yields Pell numbers for the paths of length n, sequentially, as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \to \begin{pmatrix} 9 \\ 12 \\ 8 \end{pmatrix} \to \begin{pmatrix} 21 \\ 29 \\ 20 \end{pmatrix} \to \begin{pmatrix} 50 \\ 70 \\ 49 \end{pmatrix} \to \cdots$$

The formation of the number sequences themselves is easy, since

$$u_{n+1} = v_n + u_n, \quad w_{n+1} = u_{n+1} - 1, \quad \text{and} \quad v_{n+1} = 2v_n + v_{n-1}.$$

We see that paths of length n to line 1 are the Pell numbers P_n,

$$P_{n+1} = 2P_n + P_{n-1}, \; P_0 = 0, \; P_1 = 1,$$

while the paths to lines 0 and 2 have sums 1, 3, 7, 17, ..., the sum of two consecutive Pell numbers. In terms of Pell numbers P_n, we can write

$$u_n + w_n = P_n + P_{n-1} \quad \text{and} \quad u_n - w_n = 1,$$

so that

$$\begin{aligned} u_n &= (P_n + P_{n-1} + 1)/2 \\ v_n &= P_n \\ w_n &= (P_n + P_{n-1} - 1)/2. \end{aligned}$$

This means that u_n and w_n separately obey the recurrence

$$U_{n+3} = 3U_{n+2} - U_{n+1} - U_n,$$

whose characteristic polynomial is

$$x^3 - 3x^2 + x + 1 = (x-1)(x^2-2x-1).$$

The corresponding matrix for the system with bend reflections is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}$$

Now, there are, of course, *regular reflections* along these paths, too, as well as bends, and the corresponding matrix for these is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}$$

with starting vector $u_1^* = v_1^* = w_1^* = 0$, $u_0 = 1$, $v_0 = w_0 = 0$.

One can verify that the generating functions for u_n^*, v_n^*, and w_n^* are

$$u_n^*: \quad \frac{(1-x)^4 + 2x^2}{(1-x)^2(1-2x-x^2)^2}$$

$$v_n^*: \quad \frac{3(1-x)^3x}{(1-x)^2(1-2x-x^2)^2}$$

$$w_n^*: \quad \frac{4(1-x)^2 - 2x^2}{(1-x)^2(1-2x-x^2)^2}$$

while their sum, $u_n^* + v_n^* + w_n^*$, yields the generating function

$$\frac{1+x+2x^2}{(1-2x-x^2)^2}$$

all clearly related to the Pell sequence, Pell first convolution, and partial sum of the Pell first convolution sequence.

In three stacked plates, these three systems of matrices generalize nicely. For *regular reflections* in paths of equal length n without horizontal moves,

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix},$$

while the *bend reflections* have the system

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix},$$

and the *regular reflections in bent paths* are given by

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix}.$$

7. REFLECTIONS ALONG BEND PATHS IN THREE STACKED PLATES

Here we count bend reflections and regular reflections in paths where bends are allowed. We begin with bend reflections in bend paths. Let U_n, V_n, W_n, and Y_n be the number of paths of length n terminating on lines 0, 1, 2, and 3, respectively. Let U_n^*, V_n^*, W_n^*, and Y_n^* be the number of bend reflections for those paths, and let a *bend* be a horizontal segment in a path. We shall show the following:

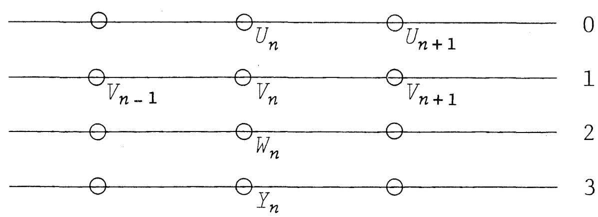
(A) $$U_{n+1}^* = V_n^* + U_n^* + 2V_{n-1}$$

(B) $$V_{n+1}^* = V_n^* + U_n^* + W_n^* + 2(U_{n-1} + W_{n-1})$$

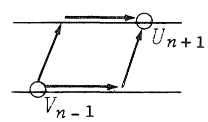
(C) $$W_{n+1}^* = W_n^* + Y_n^* + V_n^* + 2(Y_{n-1} + V_{n-1})$$

(D) $$Y_{n+1}^* = Y_n^* + W_n^* + 2W_{n-1}$$

We need a geometric derivation for the bends.



The paths to the point marked U_n contain U_n^* bends, and there are U_n such paths. We can go to U_{n+1} from V_{n-1} by either the upper or lower path, but we have added a bend at the upper path and a bend at the lower path;



thus, $2V_{n-1}$ merely counts the extra bends by these end moves. We can reach U_{n+1} from U_n and from V_n and each of these path bundles contains by declaration U_n^* and V_n^* bends, respectively. Thus,

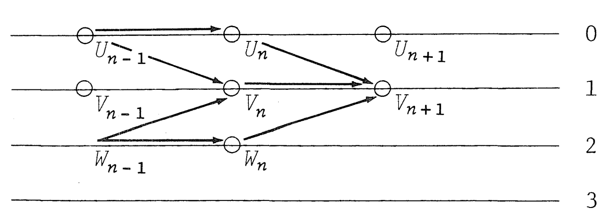
$$U_{n+1}^* = U_n^* + V_n^* + 2V_{n-1},$$

establishing (A). The derivation for (D) is similar.

We now tackle (B). Notice that we can reach V_{n+1} in a unit step from U_n, V_n, or W_n, so that we must count all bends in each of those previously counted paths, with no new bends added. We cannot use V_{n-1}, but paths routed through W_{n-1} and U_n or W_{n-1} and W_n as well as those through U_{n-1} and U_n or through U_{n-1} and V_n each collect one new bend, so that the number of added bends is $2(U_{n-1} + W_{n-1})$, making

$$V_{n+1}^* = U_n^* + V_n^* + W_n^* + 2(U_{n-1} + W_{n-1}),$$

which is identity (B). Similarly, we could establish (C).



To solve the system of equations (A), (B), (C), (D), let

$$A_n^* = U_n^* + Y_n^* \qquad A_n = U_n + Y_n$$

and

$$B_n^* = V_n^* + W_n^* \qquad\qquad B_n = V_n + W_n$$

Then (A) added to (D) yields

(E*) $$A_{n+1}^* = A_n^* + B_n^* + 2B_{n-1}$$

while (B) plus (C) yields

(G*) $$B_{n+1}^* = A_n^* + 2B_n^* + 2(A_{n-1} + B_{n-1}).$$

Let

$$C_n^* = U_n^* - Y_n^* \qquad\qquad C_n = U_n - Y_n$$

and

$$D_n^* = V_n^* - W_n^* \qquad\qquad D_n = V_n - W_n$$

Then subtracting (D) from (A) and (C) from (B) yields, respectively,

(F*) $$C_{n+1}^* = C_n^* + D_n^* + 2D_{n-1}$$

and

(H*) $$D_{n+1}^* = C_n^* + 2(C_{n-1} + D_{n-1}).$$

Now, A_n, B_n, C_n, and D_n are easily found. Returning to the first diagram of this section, from $U_{n+1} = U_n + V_n$ and $Y_{n+1} = Y_n + W_n$, we have

(E) $$A_{n+1} = A_n + B_n$$

(F) $$C_{n+1} = C_n + D_n$$

while $V_{n+1} = U_n + V_n + W_n$ and $W_{n+1} = W_n + V_n + Y_n$ yield

(G) $$B_{n+1} = 2B_n + A_n$$

(H) $$D_{n+1} = C_n.$$

From (E), we get $B_n = A_{n+1} - A_n$, which we use in (G) to obtain

$$(A_{n+2} - A_{n+1}) = 2(A_{n+1} - A_n) + A_n,$$

so that

$$A_{n+2} - 3A_{n+1} + A_n = 0.$$

From the starting data, $A_1 = 1$, $A_2 = 2$, so that A_n is a Fibonacci number with odd subscript, and

$$A_n = F_{2n-1}$$

$$B_n = A_{n+1} - A_n = F_{2n+1} - F_{2n-1} = F_{2n}.$$

From (F) and (H), in a similar manner, one finds that

$$C_n = F_{n+1} \quad \text{and} \quad D_n = F_n.$$

From these, we can find U_n, V_n, W_n, and Y_n by simultaneous linear equations, using

$$\begin{cases} U_n + Y_n = F_{2n-1} \\ U_n - Y_n = F_{n+1} \end{cases} \qquad \begin{cases} V_n + W_n = F_{2n} \\ V_n - W_n = F_n \end{cases}$$

The solutions are

$$\begin{cases} U_n = (F_{2n-1} + F_{n+1})/2 \\ Y_n = (F_{2n-1} - F_{n+1})/2 \end{cases} \qquad \begin{cases} V_n = (F_{2n} + F_n)/2 \\ W_n = (F_{2n} - F_n)/2 \end{cases}$$

Notice that

$$U_n + V_n + W_n + Y_n = F_{2n+1}.$$

Next, we can solve the full system for A_n^*, B_n^*, C_n^*, and D_n^*, since we now know A_n, B_n, C_n, and D_n. From (E*),

$$B_n^* = A_{n+1}^* - A_n^* - 2B_{n-1},$$

which substituted into (G*) gives us

$$(A_{n+2}^* - A_{n+1}^* - 2B_n) = A_n^* + 2(A_{n+1}^* - A_n^* - 2B_{n-1}) + 2(A_{n-1} + B_{n-1})$$

which simplifies to

$$A_{n+2}^* - 3A_{n+1}^* + A_n^* = 2B_n + 2A_{n-1} - 2B_{n-1} = 2L_{2n-2}$$

where we recognize the recursion relation for alternate Fibonacci numbers on the left while, as seen above, B_n and A_{n-1} are alternate Fibonacci numbers. It can be verified directly that if

$$A_n^* = 2(n-1)F_{2n-4},$$

then $A_{n+2}^* - 3A_{n+1}^* + A_n^* = 2L_{2n-2}$. From $B_n^* = A_{n+1}^* - A_n^* - 2B_{n-1}$ and $B_n = F_{2n}$ we get

$$B_n^* = 2nF_{2n-3} - 2F_{2n-3} = 2(n-1)F_{2n-3}.$$

In a similar fashion, we can verify that

$$C_{n+2}^* - C_{n+1}^* - C_n^* = 2L_n$$

is satisfied by

$$C_n^* = 2(n-1)F_{n-2}$$

and from

$$D_n^* = C_{n-1}^* + 2(C_{n-2} + D_{n-2})$$

where $C_n = F_{n+1}$ and $D_n = F_n$, we obtain

$$D_n^* = 2(n-1)F_{n-3}.$$

From these, we get

$$U_n^* = (n-1)(F_{2n-4} + F_{n-2})$$

$$V_n^* = (n-1)(F_{2n-3} + F_{n-3})$$

$$W_n^* = (n-1)(F_{2n-3} - F_{n-3})$$

$$Y_n^* = (n-1)(F_{2n-4} - F_{n-2})$$

This completes our solution for bend reflections in bend paths in three glass plates.

It is instructive, however, to consider a matrix approach to counting bend reflections in bend paths. A matrix which corresponds to the system of equations just given, counting the number of paths of length n and the number of bend reflections for those paths, is

$$\left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}$$

Expanding the characteristic polynomial,

$$\begin{aligned} &[(x-1)^4 - 3(x-1)^2 + 1]^2 \\ &= \left[((x-1)^2 - 1)^2 - (x-1)^2\right]^2 \\ &= [x^2 - 2x + 1 - 1 - (x-1)]^2[x^2 - 2x + 1 - 1 + (x-1)]^2 \\ &= (x^2 - 3x + 1)^2(x^2 - x - 1)^2 = 0 \end{aligned}$$

Notice that $(x^2 - 3x + 1) = 0$ yields the recurrence relation for the alternate Fibonacci numbers, while $(x^2 - x - 1) = 0$ gives the regular Fibonacci recurrence. A generating function derivation could be made for all formulas given in this section.

Values of the vector elements generated by the matrix equation for

$$n = 1, \ldots, 7$$

are given in the table below.

BEND REFLECTIONS

n	1	2	3	4	5	6	7
U_n^*	0	0	4	12	40	120	360
V_n^*	0	2	4	18	56	180	552
W_n^*	0	0	4	12	48	160	516
Y_n^*	0	0	0	6	24	90	300
U_{n-1}	1	1	2	4	9	21	51
V_{n-1}	0	1	2	5	12	30	76
W_{n-1}	0	0	1	3	9	25	68
Y_{n-1}	0	0	0	1	4	13	38

Finally, we list values for A_n^*, B_n^*, C_n^*, and D_n^*:

n	1	2	3	4	5	6	7	n
$A_n^* = U_n^* + Y_n^*$	0	0	4	18	64	210	660	$2(n-1)F_{2n-4}$
$B_n^* = V_n^* + W_n^*$	0	2	8	30	104	340	1068	$2(n-1)F_{2n-3}$
$C_n^* = U_n^* - Y_n^*$	0	0	4	6	16	30	60	$2(n-1)F_{n-2}$
$D_n^* = V_n^* - W_n^*$	0	2	0	6	8	20	36	$2(n-1)F_{n-3}$

We now shift our attention to the problem of counting regular reflections which occur in paths of length n in which bends are allowed. The matrix which solves the system of equations in that case follows, where starred entries denote regular reflections; otherwise, the definitions are as before. Notice that the characteristic polynomial is the same as that of the preceding matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}$$

Values of successive vector elements for $n = 1, \ldots, 8$ are given in the table following:

REGULAR REFLECTIONS IN BEND PATHS

n	1	2	3	4	5	6	7	8
U_n^*	0	1	2	7	20	60	176	517
V_n^*	0	0	3	9	31	95	290	868
W_n^*	0	0	0	5	20	75	250	794
Y_n^*	0	0	0	0	6	30	118	406
U_{n-1}	1	1	2	4	9	21	51	127
V_{n-1}	0	1	2	5	12	30	76	195
W_{n-1}	0	0	1	3	9	25	68	182
Y_{n-1}	0	0	0	1	4	13	38	106

The system of regular reflections in bend paths is not solved explicitly here, but generating functions for successive values are not difficult to obtain by using the characteristic polynomial of the matrix just given. Generating functions for A_n^*, B_n^*, C_n^*, and D_n^* are:

$$A_n^* = U_n^* + Y_n^*: \quad \frac{x^2(1 - 4x + 6x^2)}{(1 - 3x + x^2)^2}$$

$$B_n^* = V_n^* + W_n^*: \quad \frac{x^3(3 - 4x)}{(1 - 3x + x^2)^2}$$

$$C_n^* = U_n^* - Y_n^*: \quad \frac{x^2(1 + 2x^2)}{(1 - x - x^2)^2}$$

$$D_n^* = V_n^* - W_n^*: \quad \frac{x^3(3 - 2x)}{(1 - x - x^2)^2}$$

Since $A_n^* + B_n^* = U_n^* + V_n^* + W_n^* + Y_n^*$, the generating function for regular reflections in bend paths terminating on all four surfaces is

$$\frac{(x^2 - x^3 + 2x^4)}{(1 - 3x + x^2)^2}.$$

8. REGULAR REFLECTIONS IN THREE STACKED PLATES

If one wishes equations for the number of paths ending upon certain lines and the number of regular reflections, the procedure is the same as when "bends" are allowed, as in the last section. Let U_n, V_n, W_n, and Y_n be the number of paths of length n from line 0 to lines 0, 1, 2, and 3. Let U_n^*, V_n^*, W_n^*, and Y_n^* be the number of regular reflections counted for those paths.

The system of equations to solve is

$$\begin{aligned} U_{n+1}^* &= V_n^* + U_{n-1} & \qquad U_{n+1} &= V_n \\ V_{n+1}^* &= U_n^* + W_n^* + 2V_{n-1} & V_{n+1} &= U_n + W_n \\ W_{n+1}^* &= Y_n^* + V_n^* + 2W_{n-1} & W_{n+1} &= V_n + Y_n \\ Y_{n+1}^* &= W_n^* + Y_{n-1} & Y_{n+1} &= W_n \end{aligned}$$

These differ from the equations used in Section 7 only in that no horizontal moves along the lines are allowed, so that one represses terms that correspond to that same line. The method of solution is exactly the same.

One finds that

$$\begin{aligned} U_{2k} &= F_{2k-1} & \qquad U_{2k+1} &= 0 \\ Y_{2k+1} &= F_{2k} & Y_{2k} &= 0 \\ V_{2k+1} &= F_{2k+1} & V_{2k} &= 0 \\ W_{2k} &= F_{2k} & W_{2k+1} &= 0 \end{aligned}$$

which agrees with Theorems A and B of Section 5, since $U_n + Y_n = F_{n-1}$ is the number of paths ending at outside lines, while $V_n + W_n = F_n$ is the number of paths ending on inside surfaces. Notice that $U_n + V_n + W_n + Y_n = F_{n+1}$, which agrees with Theorem C.

As for the number of reflections to paths ending on outside surfaces,

$$U_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3},\ n \text{ even};\ U_n^* = 0,\ n \text{ odd};$$

$$Y_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3},\ n \text{ odd};\ \ Y_n^* = 0,\ n \text{ even};$$

where $\{C_n\}$ is the first Fibonacci convolution, $C_n = (nL_{n+1} + F_n)/5$. One can verify that the total number of reflections for paths of length n which exit at either outside surface is $U_n^* + Y_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3}$, which is equivalent to the formula given for R_n in Theorem I of Section 5.

Finally, we write, again for the first Fibonacci convolution $\{C_n\}$,

$$V_n^* = 3C_{n-2} - C_{n-3},\ n \text{ odd};\ \ V_n^* = 0,\ n \text{ even};$$

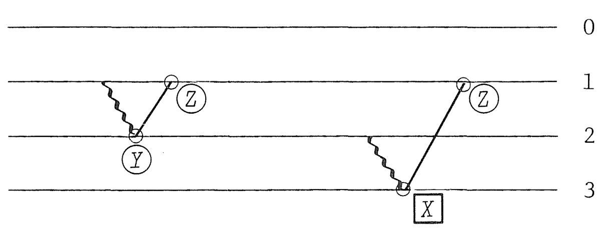
$$W_n^* = 3C_{n-2} - C_{n-3},\ n \text{ even};\ W_n^* = 0,\ n \text{ odd}.$$

Here, the matrix solution for the number of regular reflections in paths without bends follows from

$$\left[\begin{array}{cccc:cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}.$$

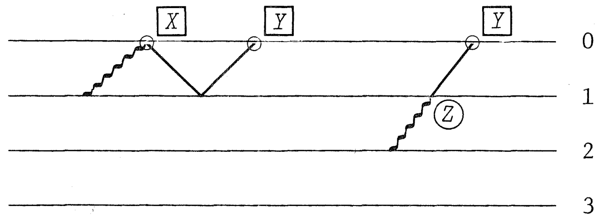
9. NUMERICAL ARRAYS ARISING FROM REGULAR REFLECTIONS IN THREE STACKED PLATES

Let circled numbers denote reflections on paths coming to the inside lines from the inside. Let boxed numbers denote reflections in paths to the outside lines.



$$Z = Y + \boxed{X}$$

Note that Z is one longer and one reflection more than Y, while it is two longer and one reflection more than X. Since the paths under discussion are to the inside lines from the inside, paths going from 2 to 1 imply a reflection as indicated. Since the paths from 3 must have come from 2, this also implies a reflection as shown. Thus, $Z = Y + \boxed{X}$. Secondly, the two types of reflections are related by $\boxed{Y^*} = Z^* + \boxed{X^*}$ from considering the following:



$\boxed{Y} = Z + \boxed{X}$

Paths indicated which come through from the inside are extended to Y by one but do not add a reflection. The paths coming through which have one added reflection at the inside line imply a reflection at X since paths to the top line can come only from the middle line.

The geometric considerations just made give the recursive patterns in the following array. The circled numbers are the number of reflections for paths of length n which enter from the top and terminate on inside lines by segments crossing the center space only (*not* immediately reflected from either outside face), while the boxed numbers are regular paths from the top line to either outside line.

Reflections

Path Length	0	1	2	3	4	5	6	7	...
1									
2	1	$\boxed{1}$							
3	$\boxed{1}$	1							
4		$\boxed{1}$	2	$\boxed{1}$					
5		1	$\boxed{3}$	2					
6		$\boxed{1}$	2	$\boxed{3}$	3	$\boxed{1}$			
7			$\boxed{2}$	5	$\boxed{6}$	3			
8			1	$\boxed{6}$	8	$\boxed{6}$	4	$\boxed{1}$	
...									
	$\boxed{A}$			$\boxed{A^*}$					
	B	$\boxed{C}$		B^*					
	$\boxed{D}$				C^*				
			$\boxed{E}$						

$\boxed{E} = \boxed{A} + B + \boxed{C} + \boxed{D}$ $\qquad C^* = B^* + \boxed{A^*}$

Please note that each row sum is $2F_{n-1}$, where the sum of the circled numbers as well as the sum of the boxed numbers is in each case F_{n-1}. Also note that the row sum is not the total number of paths of length n, since, for example, when $n = 5$, there is one path with two reflections which terminates inside, and one path with four reflections which terminates inside. Also note that the circled diagonal numbers in the table are partial sums of the boxed diagonal numbers in the diagonal above.

Let $D_n(x)$ be the generating function for the nth diagonal sequence going downward to the right in the table. That is, $D_0(x)$ generates the boxed sequence 1, 0, 1, 0, 1, 0, 1, ... and $D_1(x)$ generates the circled sequence 1, 1, 2, 2, 3, 3, 4, 4, ..., while $D_2(x)$ generates the boxed sequence 1, 1, 3, 3, 6, 6, ... From the table recurrence, $C^* = B^* + A^*$, since C^* and B^* are on the same falling diagonal,

$$D_1(x) = x^2 D_1(x) + D_0(x),$$

so that

$$D_1(x) = [D_0(x)]/(1 - x^2).$$

We write

$$D_0(x) = \frac{x}{1 - x^2}$$

$$D_1(x) = \frac{1 + x}{(1 - x^2)^2}$$

$$D_2(x) = \frac{1 + x}{(1 - x^2)^3}$$

$$D_3(x) = \frac{(1 + x)^2}{(1 - x^2)^4}$$

$$D_4(x) = \frac{(1 + x)^2}{(1 - x^2)^5}$$

Notice that $D_n(x)$ generates boxed numbers for n even and circled numbers for n odd. Summing $D_n(x)$ for n even gives the row sum for the boxed numbers by producing the generating function for the Fibonacci sequence and, similarly, for taking n odd and circled numbers. The column sums of circled or boxed numbers each obey the recurrence $u_n = 2u_{n-1} + u_{n-2} - u_{n-3}$.

Notice that

$$D_{2n+1}(x) = [D_1(x)]^{n+1}$$

$$D_{2n}(x) = (1 - x)D_{2n+1}(x) = (1 - x)[D_1(x)]^{n+1},$$

so we see once again the pleasantry of a convolution array intimately related to Pascal's triangle.

REFERENCES

1. L. Moser & M. Wyman, Problem B-6, *The Fibonacci Quarterly* 1, No. 1 (1963): 74.
2. Bjarne Junge & V. E. Hoggatt, Jr., "Polynomials Arising from Reflections Across Multiple Plates," *The Fibonacci Quarterly* 11, No. 3 (1973):285-291.

3. Krishnaswami Alladi & V. E. Hoggatt, Jr., "Compositions with Ones and Twos," *The Fibonacci Quarterly* 13, No. 3 (1975):233-239.
4. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Palindromic Compositions," *The Fibonacci Quarterly* 13, No. 4 (1975):350-356.
5. V. E. Hoggatt, Jr., & Marjorie Bicknell, "Special Partitions," *The Fibonacci Quarterly* 13, No. 3 (1975):278 f.
6. V. E. Hoggatt, Jr., & D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI," *The Fibonacci Quarterly* 5, No. 5 (1967):445-463.
7. Douglas A. Fults, "Solution of a Certain Recurrence Relation" (unpublished paper written while a student at Saratoga High School, Saratoga, California).
8. Reuben C. Drake, Problem B-180, *The Fibonacci Quarterly* 8, No. 1 (1970): 106.
9. L. Carlitz, Solution to B-180, *The Fibonacci Quarterly* 8, No. 5 (1970): 547-548.
10. V. E. Hoggatt, Jr., & Joseph Arkin, "A Bouquet of Convolutions," *Proceedings of the Washington State University Conference on Number Theory*, March 1971, pp. 68-79.
11. Verner E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," *The Fibonacci Quarterly* 16, No. 2 (1970):158-171.
12. V. E. Hoggatt, Jr., Problem H-281, *The Fibonacci Quarterly* 16, No. 2 (1978):188.
13. Dorothy Denevan, "Reflections in Glass Plates and Other Counting Problems" (unpublished Master's thesis, San Jose State University, August 1977).
14. Ronald Garibaldi, "Counting Problems and Properties of Hexagon Lattice Sequence" (unpublished Master's thesis, San Jose State University, May 1978).
15. R. J. Weinshenk, "Convolutions and Difference Equations Associated with N-Reflections of Light in Two Glass Plates" (unpublished Master's thesis, San Jose State University, June 1965).

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ON PSEUDO-FIBONACCI NUMBERS OF THE FORM $2S^2$, WHERE S IS AN INTEGER

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If the pseudo-Fibonacci numbers are defined by

$$u_1 = 1,\ u_2 = 4,\ u_{n+2} = u_{n+1} + u_n,\ n > 0, \tag{1}$$

then we can show that $u_1 = 1$, $u_2 = 4$, and $u_4 = 9$ are the only square pseudo-Fibonacci numbers.

In this paper we will describe a method to show that none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

Even if we remove the restriction $n > 0$, we do not obtain any number of the form $2S^2$, where S is an integer.

It can be easily shown that the general solution of the difference equation (1) is given by

$$u_n = \frac{7}{5.2^n}(\alpha^n + \beta^n) - \frac{1}{5.2^{n-1}}(\alpha^{n-1} + \beta^{n-1}), \tag{2}$$

where

$$\alpha = 1 + \sqrt{5},\ \beta = 1 - \sqrt{5}, \text{ and } n \text{ is an integer.}$$

Let

$$\eta_r = \frac{\alpha^r + \beta^r}{2^r}; \quad \xi_r = \frac{\alpha^r - \beta^r}{2^r\sqrt{5}}$$

Then we easily obtain the following relations:

(3) $$u_n = \frac{1}{5}(7\eta_n - \eta_{n-1}),$$

(4) $$\eta_r = \eta_{r-1} + \eta_{r-2},\ \eta_1 = 1,\ \eta_2 = 3$$

(5) $$\xi_r = \xi_{r-1} + \xi_{r-2},\ \xi_1 = 1,\ \xi_2 = 1$$

(6) $$\eta_r^2 - 5\xi_r^2 = (-1)^r 4,$$

(7) $$\eta_{2r} = \eta_r^2 + (-1)^{r+1}2,$$

(8) $$2\eta_{m+n} = 5\xi_m\xi_n + \eta_m\eta_n,$$

(9) $$2\xi_{m+n} = \eta_n\xi_m + \eta_n\xi_m,$$

(10) $$\xi_{2r} = \eta_r\xi_r$$

The following congruences hold:

(11) $$u_{n+2r} \equiv (-1)^{r+1}u_n \pmod{\eta_r 2^{-s}},$$

(12) $$u_{n+2r} \equiv (-1)^r u_n \pmod{\xi_r 2^{-s}},$$

where $S = 0$ or 1.

We also have the following table of values:

n	0	1	2	3	4	5	6	7	8	9	12	14	16
u_n	3	1	4	5	9	14	23	37	60	97	411	1076	2817

t	4	5	8	10	t	5
ξ_t	3	5	$3\cdot 7$	$5\cdot 11$	η_t	11

Let

(13) $$2x^2 = u_n, \text{ where } x \text{ is an integer.}$$

The proof is now accomplished in eighteen stages.

(a) (13) is impossible if $n \equiv 0 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_0 \pmod{\xi_8} \\ &\equiv 3 \pmod{7}, \text{ since } 7/\xi_8 \\ &\equiv 10 \pmod{7}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 5 \pmod{7}, \text{ since } (2,7) = 1,$$

and since $\left(\frac{5}{7}\right) = -1$, (13) is impossible.

(b) (13) is impossible if $n \equiv 1 \pmod 8$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_1 \pmod{\xi_4} \\ &\equiv 1 \pmod 3 \\ &\equiv 4 \pmod 3. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 2 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{2}{3}\right) = -1$, (13) is impossible.

(c) (13) is impossible if $n \equiv 2 \pmod 8$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_2 \pmod{\xi_4} \\ &\equiv 4 \pmod 3. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 2 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{2}{3}\right) = -1$, (13) is impossible.

(d) (13) is impossible if $n \equiv 3 \pmod{16}$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_3 \pmod{\xi_8} \\ &\equiv 5 \pmod 7, \text{ since } 7/\xi_8 \\ &\equiv 12 \pmod 7. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 6 \pmod 7, \text{ since } (2,7) = 1,$$

and since $\left(\frac{6}{7}\right) = -1$, (13) is impossible.

(e) (13) is impossible if $n \equiv 4 \pmod{10}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv \pm u_4 \pmod{\xi_5} \\ &\equiv \pm 9 \pmod 5 \\ &\equiv \pm 4 \pmod 5. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv \pm 2 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(f) (13) is impossible if $n \equiv 5 \pmod{10}$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_5 \pmod{\xi_5} \\ &\equiv \pm 14 \pmod 5. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv \pm 7 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-7}{5}\right) = \left(\frac{7}{5}\right) = -1$, (13) is impossible.

(g) (13) is impossible if $n \equiv 6 \pmod{20}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_6 \pmod{\xi_{10}} \\ &\equiv 23 \pmod{11}, \text{ since } 11/\xi_{10} \\ &\equiv 12 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 6 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{6}{11}\right) = -1$, (13) is impossible.

(h) (13) is impossible if $n \equiv 7 \pmod 8$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{\xi_4} \\ &\equiv 37 \pmod 3 \\ &\equiv 34 \pmod 3. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 17 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{17}{3}\right) = -1$, (13) is impossible.

(i) (13) is impossible if $n \equiv 8 \pmod{10}$, for, using (11) we find that

$$\begin{aligned} u_n &\equiv u_8 \pmod{\eta_5} \\ &\equiv 60 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 30 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{30}{11}\right) = -1$, (13) is impossible.

(j) (13) is impossible if $n \equiv 1 \pmod{10}$, for, using (12) in this case

$$\begin{aligned} u_n \equiv\ &\pm u_1 \pmod{\xi_5} \\ &\pm 1 \pmod 5 \\ &\pm 4 \pmod 5. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv \pm 2 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(k) (13) is impossible if $n \equiv 12 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n \quad &u_{12} \pmod{\xi_8} \\ &411 \pmod 7, \text{ since } 7/\xi_8 \\ &404 \pmod 7. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 202 \pmod 7, \text{ since } (2,7) = 1,$$

and since $\left(\frac{202}{7}\right) = -1$, (13) is impossible.

(l) (13) is impossible if $n \equiv 3 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_3 \pmod{\eta_5} \\ &\equiv 5 \pmod{11} \\ &\equiv 16 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 8 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{8}{11}\right) = -1$, (13) is impossible.

(m) (13) is impossible if $n \equiv 14 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_{14} \pmod{\xi_8} \\ &\equiv 1076 \pmod{7}, \text{ since } 7/\xi_8. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 538 \pmod{7}, \text{ since } (2,7) = 1,$$

and since $\left(\frac{538}{7}\right) = -1$, (13) is impossible.

(n) (13) is impossible if $n \equiv 0 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_0 \pmod{\eta_5} \\ &\equiv 3 \pmod{11} \\ &\equiv 14 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 7 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{7}{11}\right) = -1$, (13) is impossible.

(o) (13) is impossible if $n \equiv 16 \pmod{20}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_{16} \pmod{\xi_{10}} \\ &\equiv 2817 \pmod{11}, \text{ since } 11/\xi_{10} \\ &\equiv 2806 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 1403 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{1403}{11}\right) = -1$, (13) is impossible.

(p) (13) is impossible if $n \equiv 2 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv \pm u_2 \pmod{\xi_5} \\ &\equiv \pm 4 \pmod{5}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 2 \pmod{5}, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(q) (13) is impossible if $n \equiv 7 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{\eta_5} \\ &\equiv 37 \pmod{11} \\ &\equiv 26 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 13 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{13}{11}\right) = -1$, (13) is impossible.

(r) (13) is impossible if $n \equiv 9 \pmod{10}$, for, using (11) we find that

$$\begin{aligned} u_n &\equiv u_9 \pmod{\eta_5} \\ &\equiv 97 \pmod{11} \\ &\equiv 86 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 43 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{43}{11}\right) = -1$, (13) is impossible.

Hence, none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

REFERENCE

A. Eswarathasan, "On Square Pseudo-Fibonacci Numbers," *The Fibonacci Quarterly* 16, No. 4 (1978):310-314.

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INFINITE SERIES WITH FIBONACCI AND LUCAS POLYNOMIALS

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In [7], D. A. Millin poses the problem of showing that

$$\sum_{n=0}^{\infty} F_{2^n}^{-1} = \frac{7 - \sqrt{5}}{2} \tag{1}$$

where F_k is the kth Fibonacci number. A proof of (1) by I. J. Good is given in [5], while in [3], Hoggatt and Bicknell demonstrate ten different methods of finding the same sum. Furthermore, the result of (1) is extended by Hoggatt and Bicknell in [4], where they show that

(2) $$\sum_{n=0}^{\infty} F_{2^n k}^{-1} = \frac{1}{F_k} + \frac{\alpha^2 + 1}{\alpha(\alpha^{2k} - 1)}.$$

The main purpose of this paper is to lift the results of (1) and (2) to the sequence of Fibonacci polynomials $\{F_k(x)\}_{k=1}^{\infty}$ defined recursively by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{k+2}(x) = xF_{k+1}(x) + F_k(x),\ k \geq 1.$$

Furthermore, we will examine several infinite series containing products of Fibonacci and Lucas polynomials where the Lucas polynomials are defined by

$$L_k(x) = F_{k+1}(x) + F_{k-1}(x).$$

If we let $\alpha(x) = (x + \sqrt{x^2 + 4})/2$ and $\beta(x) = (x - \sqrt{x^2 + 4})/2$, then it is a well-known fact that

(3) $$F_k(x) = [\alpha^k(x) - \beta^k(x)]/[\alpha(x) - \beta(x)]$$

and

(4) $$L_k(x) = \alpha^k(x) + \beta^k(x).$$

When $x > 0$, we have $-1 < \beta(x) < 1$ and $\alpha(x) > 1$ so that $|\beta(x)/\alpha(x)| < 1$ and $\lim_{n\to\infty}[\beta(x)/\alpha(x)]^n = 0$. But, from (3), we obtain

(5) $$\frac{F_{n+1}(x)}{F_n(x)} = \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha^n(x) - \beta^n(x)} = \frac{\alpha(x) - \beta(x)}{1 - [\beta(x)\alpha^{-1}(x)]^n} + \beta(x).$$

Therefore,

(6) $$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \alpha(x), \text{ if } x > 0.$$

When $x < 0$, we have $0 < \alpha(x) < 1$ and $\beta(x) < -1$ so that $\beta(x)/\alpha(x) < -1$. From (5), we see that

(7) $$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \beta(x), \text{ if } x < 0.$$

Using (3) and (4), it is easy to show that

$$L_{n+k}(x) + L_{n-k}(x) = L_n(x)L_k(x),\ k \text{ even}$$

and

$$F_{2n}(x) = L_n(x)F_n(x).$$

Letting S_n be the nth partial sum of

$$\sum_{n=1}^{\infty} xF_{2^n k}^{-1}(x)$$

and using the two preceding equations with induction, it can be shown that

$$S_n = \frac{x}{F_{2^n k}(x)}\left[\sum_{t=1}^{2^{n-1}-1} L_{2^n k - 2kt}(x) + 1\right].$$

The definition of $L_k(x)$ together with (6) enables us to show for $x > 0$ that

$$\lim_{n\to\infty} S_n = x\sum_{t=1}^{\infty} \frac{\alpha^2(x) + 1}{\alpha^{2kt+1}(x)} = \frac{[\alpha^2(x) + 1]x}{\alpha(x)[\alpha^{2k}(x) - 1]}$$

while for $x < 0$ we use (7) to obtain

$$\lim_{n\to\infty} S_n = x\sum_{t=1}^{\infty}\frac{\beta^2(x)+1}{\beta^{2kt+1}(x)} = \frac{[\beta^2(x)+1]x}{\beta(x)[\beta^{2k}(x)-1]}$$

Hence,

$$\text{(8)} \qquad \sum_{n=0}^{\infty} xF^{-1}_{2^n k}(x) = \frac{x}{F(x)} + \begin{cases} \left[(\alpha^2(x)+1)x\right]\Big/\left[\alpha(x)(\alpha^{2k}(x)-1)\right], & x > 0 \\ \left[(\beta^2(x)+1)x\right]\Big/\left[\beta(x)(\beta^{2k}(x)-1)\right], & x < 0 \end{cases}.$$

We now examine the infinite series

$$\text{(9)} \qquad U(q,a,b,x) = \sum_{n=1}^{\infty}\frac{(-1)^{qn+a-k}F_{b-a+k}(x)F_k(x)}{F_{qn+a-k}(x)F_{qn+b}(x)}, \quad q = b - a + k.$$

First observe that, by using (3) and (4), we can show

$$\text{(10)} \qquad F_{qn+a}(x)F_{qn+b}(x) - F_{qn+a-k}(x)F_{qn+b+k}(x) = (-1)^{qn+a-k}F_k(x)F_{b-a+k}(x).$$

Letting S_n be the nth partial sum of (9) and using (10), we notice that there is a telescoping effect so that

$$S_n = \frac{F_{b+k}(x)}{F_b(x)} - \frac{F_{qn+b+k}(x)}{F_{qn+b}(x)}.$$

Hence, by (6) and (7), we have

$$\text{(11)} \qquad U(q,a,b,x) = \frac{F_{b+k}(x)}{F_b(x)} - \begin{cases} \alpha^k(x), & x > 0 \\ \beta^k(x), & x < 0 \end{cases},$$

where $q = b - a + k$. In particular, we see that

$$\text{(12)} \qquad U(a,a,a,x) = \sum_{n=1}^{\infty}\frac{(-1)^{an}F_a^2(x)}{F_{an}(x)F_{a(n+1)}(x)} = L_a(x) - \begin{cases} \alpha^a(x), & x > 0 \\ \beta^a(x), & x < 0 \end{cases},$$

$$\text{(13)} \qquad U(1,1,1,x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{F_n(x)F_{n+1}(x)} = \begin{cases} \beta(x), & x > 0 \\ \alpha(x), & x < 0 \end{cases},$$

$$\text{(14)} \qquad U(2,2,2,x) = \sum_{n=1}^{\infty}\frac{x^2}{F_{2n}(x)F_{2(n+1)}(x)} = \begin{cases} x^2 - x\alpha(x) + 1, & x > 0 \\ x^2 - x\beta(x) + 1, & x < 0 \end{cases},$$

and

$$\text{(15)} \qquad U(b,1,b,x) = \sum_{n=1}^{\infty}\frac{(-1)^{bn}F_b(x)}{F_{bn}(x)F_{b(n+1)}(x)} = \frac{F_{b+1}(x)}{F_b(x)} - \begin{cases} \alpha(x), & x > 0 \\ \beta(x), & x < 0 \end{cases}.$$

If we combine (13) and (14) with the identity

$$L_{2n+1}(x) = L_n(x)L_{n+1}(x) + (-1)^{n+1}x$$

we obtain the very interesting result

$$\text{(16)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} L_{2n+1}(x)}{F_{2n}(x) F_{2(n+1)}(x)} = \frac{1}{x}.$$

Next, we examine the infinite series

$$\text{(17)} \qquad V(q,a,b,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{qn+a-k} F_k(x) F_{b-a+k}(x)}{L_{qn+a-k}(x) L_{qn+b}(x)}, \quad q = b - a + k.$$

To do this, we first use (3) and (4) to show that

$$\text{(18)} \qquad \begin{aligned} &L_{qn+a}(x) L_{qn+b}(x) - L_{qn+a-k}(x) L_{qn+b+k}(x) \\ &= -(x^2+4)(-1)^{qn+a-k} F_k(x) F_{b-a+k}(x). \end{aligned}$$

Letting S_n be the nth partial sum of (17) and using (18), we notice that there is a telescoping effect so that

$$S_n = \frac{L_{b+k}(x)}{L_b(x)} - \frac{L_{qn+b+k}(x)}{L_{qn+b}(x)}.$$

Using the definition of $L_m(x)$ together with (6) and (7), we obtain

$$\text{(19)} \qquad V(q,a,b,x) = \frac{L_{b+k}(x)}{L_b(x)} - \begin{cases} \alpha^k(x), & x > 0 \\ \beta^k(x), & x < 0 \end{cases}$$

where $q = b - a + k$. In particular, we note that

$$\text{(20)} \qquad V(a,a,a,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{an} F_a^2(x)}{L_{an}(x) L_{a(n+1)}(x)} = \frac{L_{2a}(x)}{L_a(x)} - \begin{cases} \alpha^a(x), & x > 0 \\ \beta^a(x), & x < 0 \end{cases},$$

$$\text{(21)} \qquad V(b,1,b,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{bn} F_b(x)}{L_{bn}(x) L_{b(n+1)}(x)} = \frac{L_{b+1}(x)}{L_b(x)} - \begin{cases} \alpha(x), & x > 0 \\ \beta(x), & x < 0 \end{cases}.$$

In conclusion, we observe that

$$\text{(22)} \qquad F_{n-1}(x) F_{n+1}(x) - F_{n+2}(x) F_{n-2}(x) = (-1)^n (x^2+1).$$

Letting S_n be the nth partial sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+1)}{F_{n+1}(x) F_{n+2}(x)}$$

and using (22), we see that

$$S_n = -\frac{F_{-1}(x)}{F_2(x)} + \frac{F_{n-1}(x)}{F_{n+2}(x)} = -\frac{1}{x} + \frac{F_{n-1}(x)}{F_{n+2}(x)}$$

so that

$$\text{(23)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (x^2+1)}{F_{n+1}(x) F_{n+2}(x)} = -\frac{1}{x} - \begin{cases} \beta^3(x), & x > 0 \\ \alpha^3(x), & x < 0 \end{cases}.$$

REFERENCES

1. G. E. Bergum & A. W. Kranzler, "Linear Recurrences, Identities and Divisibility Properties" (unpublished paper).
2. Marjorie Bicknell & V. E. Hoggatt, Jr., "A Primer for the Fibonacci Numbers," The Fibonacci Association, San Jose State University, San Jose, California.
3. Marjorie Bicknell-Johnson & V. E. Hoggatt, Jr., "Variations on Summing a Series of Reciprocals of Fibonacci Numbers," *The Fibonacci Quarterly* (to appear).
4. Marjorie Bicknell-Johnson & V. E. Hoggatt, Jr., "A Reciprocal Series of Fibonacci Numbers with Subscripts $2^n k$," *The Fibonacci Quarterly* (to appear).
5. I. J. Good, "A Reciprocal Series of Fibonacci Numbers," *The Fibonacci Quarterly* 12, No. 4 (1974):346.
6. I. J. Good & Paul S. Bruckman, "A Generalization of a Series of De Morgan with Applications of Fibonacci Type," *The Fibonacci Quarterly* (to appear).
7. D. A. Millin, Problem H-237, *The Fibonacci Quarterly* 12, No. 3 (1974): 309.

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A NOTE ON 3-2 TREES*

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ABSTRACT

Under the assumption that all of the 3-2 trees of height h are equally probable, it is shown that in a 3-2 tree of height h the expected number of keys is $(.72162)3^h$ and the expected number of internal nodes is $(.48061)3^h$.

INTRODUCTION

One approach to the organization of large files is the use of "balanced" trees (see Section 6.2.3 of [3]). In particular, one such class of trees, suggested by J. E. Hopcroft (unpublished), is known as 3-2 trees. A 3-2 tree is a tree in which each internal node contains either 1 or 2 keys and is hence either a 2-way or 3-way branch, respectively. Furthermore, all external nodes (i.e., leaves) are at the same level. Figure 1 shows some examples of 3-2 trees.

Insertion of a new key into a 3-2 tree is done as follows to preserve the 3-2 property: To add a new key into a node containing one key, simply insert it as the second key; if the node already contains two keys, split it into two one-key nodes and insert (recursively) the middle key into the parent node. This may cause the parent node to be split in a similar way, if it already contains two keys. For more details about 3-2 trees see [1] and [3].

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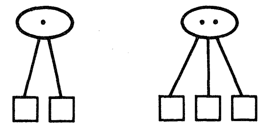
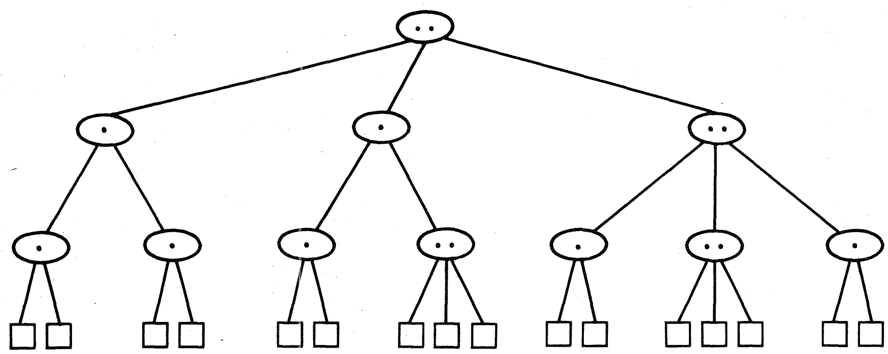
(a) The unique 3-2 tree of height 0



(b) The only two 3-2 trees of height 1



(c) A 3-2 tree of height 3 with 15 keys, 11 internal nodes, and 16 external nodes (leaves)

FIGURE 1.—SOME EXAMPLES OF 3-2 TREES. THE SQUARES ARE EXTERNAL NODES (LEAVES), THE OVALS ARE INTERNAL NODES, AND THE DOTS ARE KEYS.

Yao [4] has studied the average number of internal nodes in a 3-2 tree with k keys, assuming that the tree was built by a sequence of k random insertions done by the insertion algorithm outlined above. He found the expected number of internal nodes to be between $.70k$ and $.79k$ for large k. Unfortunately, the distribution of 3-2 trees induced by the insertion algorithm is not well understood and Yao's techniques will probably not be extended to provide sharper bounds.

Using techniques like those in Khizder [2], some results can be obtained, however, for the (simpler) distribution in which all 3-2 trees of height are equally probable. In this paper we show that, under this simpler distribution, in a 3-2 tree of height h the expected number of keys and internal nodes are, respectively, $(.72162)3^h$ and $(.48061)3^h$.

ANALYSIS

Let $a_{n,k,h}$ be the number of 3-2 trees of height h with n nodes and k keys. Since there is a unique tree of height 0 (consisting of a single leaf—see Figure 1), and since a 3-2 tree of height $h > 0$ is formed from either two or three 3-2 trees of height $h - 1$, we have

$$a_{n,k,0} = \begin{cases} 1 & \text{if } n = k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{n,k,h} = \sum_{\substack{i+j=n-1 \\ u+v=k-1}} a_{i,u,h-1}\, a_{j,v,h-1} + \sum_{\substack{i+j+l=n-1 \\ u+v+w=k-2}} a_{i,u,h-1}\, a_{j,v,h-1}\, a_{l,w,h-1} \tag{1}$$

Let

$$A_h(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k,h}\, x^n y^k$$

be the generating function for $a_{n,k,h}$. From (1) we have

$$\begin{aligned} A_0(x,y) &= 1 \\ A_h(x,y) &= xyA_{h-1}^2(x,y) + xy^2A_{h-1}^3(x,y) \end{aligned} \tag{2}$$

and thus the number of 3-2 trees of height h is $A_h = A_h(1,1)$, the total number of keys in all 3-2 trees of height h is

$$K_h = \left.\frac{\partial A_h(x,y)}{\partial y}\right|_{x=y=1},$$

and the total number of internal nodes in all 3-2 trees of height h is

$$N_h = \left.\frac{\partial A_h(x,y)}{\partial x}\right|_{x=y=1}.$$

The table gives the first few values for A_h, K_h, and N_h as calculated from the recurrence relations arising from (2).

THE FIRST FEW VALUES FOR A_h, K_h, AND N_h

h	$A_h = A_h(1,1)$	$K_h = \left.\frac{\partial A_h(x,y)}{\partial y}\right\|_{x=y=1}$	$N_h = \left.\frac{\partial A_h(x,y)}{\partial x}\right\|_{x=y=1}$
0	1	0	0
1	2	3	2
2	12	68	44
3	1872	34608	21936
4	6563711232	377092654848	237180213504

Assuming that all of the 3-2 trees of height h are equally probable, the average number of keys in a 3-2 tree of height h is given by

$$\kappa_h = \frac{K_h}{A_h} = \left.\frac{\dfrac{\partial A_h(x,y)}{\partial y}}{A_h(x,y)}\right|_{x=y=1}$$

and the average number of internal nodes in a 3-2 tree of height h is given by

$$\nu_h = \frac{N_h}{A_h} = \left.\frac{\dfrac{\partial A_h(x,y)}{\partial x}}{A_h(x,y)}\right|_{x=y=1}$$

To determine κ_h, we use the recurrence relations for A_h and K_h arising from (2):

$$A_0 = 1$$
$$A_h = A^2_{h-1} + A^3_{h-1}$$

and

$$K_0 = 0$$
$$K_h = 2A_{h-1}K_{h-1} + A^2_{h-1} + 2A^3_{h-1} + 3A^2_{h-1}K_{h-1}.$$

Rewriting the equation for K_h in terms of κ_h gives

$$K_h = \kappa_{h-1}(3A_h - A^2_{h-1}) + 2A_h - A^2_{h-1}$$

and so

$$\kappa_h = \frac{K_h}{A_h} = \kappa_{h-1}\left(3 - \frac{A^2_{h-1}}{A_h}\right) + 2 - \frac{A^2_{h-1}}{A_h}$$
$$= 3\kappa_{h-1} + 2 - \frac{A^2_{h-1}}{A_h}(\kappa_{h-1} + 1)$$

giving

$$(\kappa_h + 1) = 3(\kappa_{h-1} + 1) - \frac{K_{h-1} + A_{h-1}}{A^2_{h-1} + A_{h-1}}.$$

Letting $\varepsilon_h = \dfrac{K_h + A_h}{A^2_h + A_h}$, we get

$$(\kappa_h + 1) = 3^h(\kappa_0 + 1) - \sum_{i=1}^{h} 3^{i-1}\varepsilon_{h-i}.$$

But $\kappa_0 + 1 = \dfrac{K_0}{A_0} + 1 = \dfrac{0}{1} + 1 = 1$, and so

$$\frac{K_h}{A_h} + 1 = \kappa_h + 1 = 3^h\left(1 - \sum_{i=1}^{h} \frac{\varepsilon_{h-i}}{3^{h-i+1}}\right) = 3^h\left(1 - \sum_{i=0}^{h-1} \frac{\varepsilon_i}{3^{i+1}}\right) \tag{3}$$

i.e.,

$$\lim_{h\to\infty} \frac{1}{3^h}\left(\frac{K_h}{A_h} + 1\right) = 1 - \sum_{i=0}^{\infty} \frac{\varepsilon_i}{3^{i+1}}.$$

What is $\displaystyle\sum_{i=0}^{\infty} \frac{\varepsilon_i}{3^{i+1}}$? It is easy to show by induction that $A^2_i > K_i$ and so

$$\varepsilon_i = \frac{K_i + A_i}{A^2_i + A_i} < 1.$$

The comparison test thus insures that the summation converges:

$$\sum_{i=0}^{\infty} \frac{\varepsilon_i}{3^{i+1}} < \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} = \frac{1}{2}.$$

Now, in order to use $\displaystyle\sum_{i=0}^{h} \frac{\varepsilon_i}{3^{i+1}}$ as an approximation to $\displaystyle\sum_{i=0}^{\infty} \frac{\varepsilon_i}{3^{i+1}}$ we need an upper bound on $\sum_{i=h+1}^{\infty} \frac{\varepsilon_i}{3^{i+1}}$. From the definition of ε_i, we have

$$\text{(4)} \qquad \sum_{i=h+1}^{\infty} \frac{\varepsilon_i}{3^{i+1}} = \frac{1}{3} \sum_{i=h+1}^{\infty} \frac{1}{3^i} \frac{K_i + A_i}{A_i^2 + A_i} = \frac{1}{3} \sum_{i=h+1}^{\infty} \frac{\frac{1}{3^i}\frac{K_i}{A_i} + \frac{1}{3^i}}{A_i + 1}.$$

From (3) and the fact that $0 < \varepsilon_i < 1$, we know that

$$\frac{1}{3^h}\frac{K_h}{A_h} + \frac{1}{3^h} = 1 - \sum_{i=0}^{h-1} \frac{\varepsilon_i}{3^{i+1}} < 1,$$

and so (4) becomes

$$\sum_{i=h+1}^{\infty} \frac{\varepsilon_i}{3^{i+1}} < \frac{1}{3} \sum_{i=h+1}^{\infty} \frac{1}{A_i + 1} < \frac{1}{3} \sum_{i=h+1}^{\infty} \frac{1}{A_i}.$$

But since $A_h = A_{h-1}^2 + A_{h-1}^3 > 2A_{h-1}^2$, we have by induction that $A_h > \frac{1}{2}2^{2^h}$, and so

$$\sum_{i=h+1}^{\infty} \frac{\varepsilon_i}{3^{i+1}} < \frac{2}{3} \sum_{i=h+1}^{\infty} 2^{-2^i} < \frac{2}{3}\left(2^{-2^{h+1}} + 2^{-2^{h+1}-1}\right) = 2^{-2^{h+1}}.$$

Using the values in the table, we find that

$$\sum_{i=0}^{4} \frac{\varepsilon_i}{3^{i+1}} = .2783810593,$$

and thus

$$0 < \sum_{i=0}^{\infty} \frac{\varepsilon_i}{3^{i+1}} - .2783810593 < 2^{-2^5} < 3 \times 10^{-10}.$$

We conclude that

$$0 < \lim_{h\to\infty} \frac{1}{3^h}\left(\frac{K_h}{A_h} + 1\right) - .7216189407 < 3 \times 10^{-10}.$$

Thus, under the assumption that all the 3-2 trees of height h are equally probable, the expected number of keys in a 3-2 tree of height h is

$$\kappa_h = \frac{K_h}{A_h} \approx (.7216189407)3^h.$$

A similar analysis works for ν_h, the average number of internal nodes in a 3-2 tree of height h. We again use the recurrence relations arising from (2):

$$A_0 = 1$$

$$A_h = A_{h-1}^2 + A_{h-1}^3$$

as before, and

$$N_0 = 0$$

$$\begin{aligned} N_h &= 2A_{h-1}N_{h-1} + A_{h-1}^2 + A_{h-1}^3 + 3A_{h-1}^2 N_{h-1} \\ &= 2A_{h-1}N_{h-1} + 3A_{h-1}^2 N_{h-1} + A_h. \end{aligned}$$

Rewriting this last equation in terms of $\nu_h = N_h/A_h$ gives

$$N_h = \nu_{h-1}(3A_h - A_{h-1}^2) + A_h,$$

and so

$$\nu_h = \frac{N_h}{A_h} = \nu_{h-1}\left(3 - \frac{A_{h-1}}{A_h}\right) + 1 = 3\nu_{h-1} + 1 - \frac{N_{h-1}}{A_h},$$

giving

$$\left(\nu_h + \frac{1}{2}\right) = 3\left(\nu_{h-1} + \frac{1}{2}\right) - \frac{N_{h-1}}{A_h}.$$

Letting $\delta_h = \dfrac{N_h}{A_h}$, we get

$$\left(\nu_h + \frac{1}{2}\right) = 3^h\left(\nu_0 + \frac{1}{2}\right) - \sum_{i=1}^{h} 3^{i-1}\delta_{h-i}.$$

But $\nu_0 + \dfrac{1}{2} = \dfrac{N_0}{A_0} + \dfrac{1}{2} = \dfrac{0}{1} + \dfrac{1}{2} = \dfrac{1}{2}$, and so

$$\frac{N_h}{A_h} + \frac{1}{2} = \nu_h + \frac{1}{2} = 3^h\left(\frac{1}{2} - \sum_{i=1}^{h} \frac{\delta_{h-i}}{3^{h-i+1}}\right) = 3^h\left(\frac{1}{2} - \sum_{i=0}^{h-i} \frac{\delta_i}{3^{i+1}}\right), \tag{5}$$

i.e.,

$$\lim_{h\to\infty} \frac{1}{3^h}\left(\frac{N_h}{A_h} + \frac{1}{2}\right) = \frac{1}{2} - \sum_{i=0}^{\infty} \frac{\delta_i}{3^{i+1}}.$$

What is $\displaystyle\sum_{i=0}^{\infty} \frac{\delta_i}{3^{i+1}}$? It is easy to show by induction that $A_{i+1} > N_i$ and so $\delta_i = N_i/A_{i+1} < 1$; hence, the comparison test insures that the summation converges:

$$\sum_{i=0}^{\infty} \frac{\delta_i}{3^{i+1}} < \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} = \frac{1}{2}.$$

In order to use $\displaystyle\sum_{i=0}^{h} \frac{\delta_i}{3^{i+1}}$ as an approximation to $\displaystyle\sum_{i=0}^{\infty} \frac{\delta_i}{3^{i+1}}$ we need an upper bound on $\displaystyle\sum_{i=h+1}^{\infty} \frac{\delta_i}{3^{i+1}}$. From the definition of δ_i, we have

$$\sum_{i=h+1}^{\infty} \frac{\delta_i}{3^{i+1}} = \frac{1}{3}\sum_{i=h+1}^{\infty} \frac{\frac{1}{3^i}\frac{N_i}{A_i}}{A_i + A_i^2}. \tag{6}$$

Since $0 < \delta_i < 1$, (5) tells us that

$$\frac{1}{3^h}\frac{N_h}{A_h} = \frac{1}{2}\left(1 - \frac{1}{3^h}\right) - \sum_{i=0}^{h-1} \frac{\delta_i}{3^{i+1}} < \frac{1}{2},$$

and so (6) becomes

$$\sum_{i=h+1}^{\infty} \frac{\delta_i}{3^{i+1}} < \frac{1}{6}\sum_{i=h+1}^{\infty} \frac{1}{A_i + A_i^2} < \frac{1}{6}\sum_{i=h+1}^{\infty} \frac{1}{A_i^2}.$$

Recalling that $A_i > \frac{1}{2}2^{2^i}$, this becomes

$$\sum_{i=h+1}^{\infty} \frac{\delta_i}{3^{i+1}} < \frac{1}{6}\sum_{i=h+1}^{\infty} 4 \cdot 2^{-2^{i+1}} = \frac{2}{3}\sum_{i=h+2}^{\infty} 2^{-2^i} < \frac{2}{3}\left(2^{-2^{k+2}} + 2^{-2^{k+2}-1}\right) = 2^{-2^{k+2}}.$$

Using the values in the table, we find that

$$\sum_{i=0}^{3} \frac{\delta_i}{3^{i+1}} = .0193890884,$$

and thus

$$0 < \sum_{i=0}^{\infty} \frac{\delta_i}{3^{i+1}} - .0193890884 < 2^{-2^5} < 3 \times 10^{-10}.$$

We conclude that

$$0 < \lim_{h\to\infty} \frac{1}{3^h}\left(\frac{N_h}{A_h} + \frac{1}{2}\right) - .4806109116 < 3 \times 10^{-10}.$$

Thus, under the assumption that all 3-2 trees of height h are equally probable, the expected number of internal nodes in a 3-2 tree of height h is

$$\nu_h = \frac{N_h}{A_h} \approx (.4806109116)3^h.$$

REFERENCES

1. A. V. Aho, J. E. Hopcroft, & J. D. Ullman, *The Design and Analysis of Computer Algorithms* (Reading, Mass.: Addison-Wesley, 1974).
2. L. A. Khizder, "Some Combinatorial Properties of Dyadic Trees," *Zh. Vychisl. Matem. i Matem. Fiziki* (Russian) 6, No. 2 (1966):389-394. English translation in *USSR Comput. Math. and Math. Phys.* 6, No. 2 (1966): 283-290.
3. D. E. Knuth, *The Art of Computer Programming*, Vol. III: *Sorting and Searching* (Reading, Mass.: Addison-Wesley, 1973).
4. A. C. Yao, "On Random 3-2 Trees," *Acta Informatica* 9 (1978):159-170.

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CONCAVITY PROPERTY AND A RECURRENCE RELATION FOR ASSOCIATED LAH NUMBERS

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ABSTRACT

A recurrence relation is obtained for the associated Lah numbers,

$L_k(m,n)$,

via their generating function. Using this result, it is shown that $L_k(m,n)$ is a strong logarithmic concave function of n for fixed k and m.

1. INTRODUCTION

The Lah numbers $L(m,n)$ (see Riordan [4, p. 44]) with arguments m and n are given by the relation

$$L(m,n) = (-1)^n (m!/n!)\binom{m-1}{n-1}, \tag{1}$$

where $L(m,n) = 0$ for $n > m$. Since the sign of $L(m,n)$ is the same as that of $(-1)^n$, we may write (1) in absolute value as

$$|L(m,n)| = (m!/n!)\binom{m-1}{n-1}. \tag{2}$$

We define the associated Lah numbers $L_k(m,n)$ for integral $k > 0$ as

$$L_k(m,n) = (m!/n!)\sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\binom{m+rk-1}{m} \tag{3}$$

where $L_k(m,n) = 0$ for $n > m$. Using the binomial coefficient identity (12.13) in Feller [2, p. 64], it can be easily seen that

$$L_1(m,n) = |L(m,n)|. \tag{4}$$

The use of the associated Lah numbers $L_k(m,n)$ has recently arisen in a paper by the author [1], where the n-fold convolution of independent random variables having the decapitated negative binomial distribution is derived in terms of the numbers $L_k(m,n)$. In this paper, we first provide a recurrence relation for the numbers $L_k(m,n)$. This result is then utilized to show that $L_k(m,n)$ is a strong logarithmic concave (SLC) function of n for fixed k and m, that is, $L_k(m,n)$ satisfies the inequality

$$[L_k(m,n)]^2 > L_k(m,n+1)L_k(m,n-1) \tag{5}$$

for $k = 1, 2, \ldots, m = 3, 4, \ldots$, and $n = 2, 3, \ldots, m-1$.

2. RECURRENCE RELATION FOR $L_k(m,n)$

The author [1] has provided a generating function for the numbers $L_k(m,n)$ in the form

$$[(1-\theta)^{-k} - 1]^n = \sum_{m=n}^{\infty} n!L_k(m,n)\theta^m/m!. \tag{6}$$

Differentiating both sides of (6) with respect to θ, then multiplying both sides by $(1 - \theta)$, gives

$$(7) \qquad nk[(1 - \theta)^{-k} - 1]^{n-1}(1 - \theta)^{-k} = (1 - \theta)\Sigma n!L_k(m,n)\theta^{m-1}/(m - 1)!$$

which, using (6), becomes

$$(8) \qquad nk\Sigma n!L_k(m,n)\theta^m/m! + nk\Sigma(n - 1)!L_k(m,n - 1)\theta^m/m!$$
$$= (1 - \theta)\Sigma n!L_k(m,n)\theta^{m-1}/(m - 1)!.$$

Now, equating the coefficient of θ^m in (8), we obtain the recurrence formula for $L_k(m,n)$ as

$$(9) \qquad L_k(m + 1,n) = (nk + m)L_k(m,n) + kL_k(m,n - 1).$$

The recurrence relation (9) is used to obtain Table I for the associated Lah numbers $L_k(m,n)$ for $n = 1(1)5$ and $m = 1(1)5$. It may be remarked that, for $k = 1$, Table I reduces to the one for the absolute Lah numbers given in Riordan [4, p. 44].

3. CONCAVITY OF $L_k(m,n)$

The proof of the SLC property of the numbers $L_k(m,n)$ is based on the following result of Newton's inequality given in Hardy, Littlewood, and Polya [3, p. 52]: If the polynomial

$$P(x) = \sum_{n=1}^{m} c_n x^n$$

has only real roots, then

$$(10) \qquad c_n^2 > c_{n+1}c_{n-1}$$

for $n = 2, 3, \ldots, m - 1$. To establish the SLC property, we need the following:

Lemma: If

$$P_m(x) = \sum_{n=1}^{m} L_k(m,n)x^n$$

then the m roots of $P_m(x)$ are real, distinct, and nonpositive for all $m = 1, 2, \ldots$.

Proof: It can be easily seen that $P_m(x)$, using (9), may be expressed as

$$(11) \qquad P_m(x) = \sum_{n=1}^{m} L_k(m,n)x^n$$
$$= \sum_{n=1}^{m} [(nk + m - 1)L_k(m - 1,n) + kL_k(m - 1,n - 1)]x^n$$
$$= (kx + m - 1)P_{m-1}(x) + kx[dP_{m-1}(x)/dx].$$

TABLE I

ASSOCIATED LAH NUMBERS, $L_k(m,n)$

$m \backslash n$	1	2	3	4	5
1	k				
2	$k(k+1)$	k^2			
3	$k(k+1)(k+2)$	$3k^2(k+1)$	k^3		
4	$k(k+1)(k+2)(k+3)$	$k^2(k+1)(7k+11)$	$6k^3(k+1)$	k^4	
5	$k(k+1)(k+2)(k+3)(k+4)$	$5k^2(k+1)(k+2)(3k+5)$	$5k^3(k+1)(5k+7)$	$10k^4(k+1)$	k^5

By induction, we find that

$$P_1(x) = kx, \; P_2(x) = kx(kx + k + 1),$$

and

$$P_3(x) = kx[k^2x^2 + 3k(k + 1)x + (k + 1)(k + 2],$$

so that the statement is true for $m = 1$, 2, and 3. For $m > 3$, assume that $P_{m-1}(x)$ has $m - 1$ real, distinct, and nonpositive roots. If we define

$$T_m(x) = e^x x^{m/k} P_m(x), \tag{12}$$

then, since

$$P_m(0) = 0,$$

$T_m(x)$ has exactly the same finite roots as $P_m(x)$, and the identity (11) for $P_m(x)$ gives

$$T_m(x) = kx^{(k+1)/k} dT_{m-1}(x)/dx. \tag{13}$$

By hypothesis, $P_{m-1}(x)$, and hence $T_{m-1}(x)$, has $m - 1$ real, distinct, and nonpositive roots. $T_{m-1}(x)$ also has a root at $-\infty$, and, by Rolle's theorem, between any two roots of $T_{m-1}(x)$, $dT_{m-1}(x)/dx$ will have a root. This places $m - 1$ distinct roots of $T_{m-1}(x)$ on the negative real axis; $x = 0$ is obviously another one, making m altogether. This proves the result by induction.

Thus the above lemma, together with the inequality (10), provides us the following:

Theorem: For $m \geq 3$, $k = 1, 2, \ldots$, and $n = 2, 3, \ldots, m - 1$, the associated Lah numbers $L_k(m,n)$ satisfy the inequality (5).

It may be remarked that, as a consequence of the above result and relation (4), we have the following:

Corollary: For $m \geq 3$, and $n = 2, 3, \ldots, m - 1$, the Lah numbers $L(m,n)$ satisfy the inequality

$$[L(m,n)]^2 > L(m,n + 1)L(m,n - 1). \tag{14}$$

REFERENCES

1. J. C. Ahuja, "Distribution of the Sum of Independent Decapitated Negative Binomial Variables," *Ann. Math. Statist.* 42 (1971):383-384.
2. W. Feller, *An Intorduction to Probability Theory and Its Applications* (New York: Wiley, 1968).
3. G. H. Hardy, J. E. Littlewood, & G. Ploya, *Inequalities* (Cambridge: The University Press, 1952).
4. J. Riordan, *An Introduction to Combinatorial Analysis* (New York: Wiley, 1958).

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FIBONACCI NUMBERS

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The purpose of this paper is to derive a few relations involving Fibonacci numbers; these numbers are defined according to the expressions

$$f_{n+1} = f_n + f_{n-1},\ f_0 = 0,\ f_1 = 1$$

due to Girard [1]. They can also be obtained from a known [2] matrix representation that we rederive in Part II. In Part III we obtain the sum of two infinite series and some recurrence relations.

PART I: HISTORICAL NOTE

The sequence of integers $\{f_n\}$ was discovered by Leonardo Pisano [3, 4], in his *Liber Abacci*, as the solution to a hypothetical problem concerning the breeding of rabbits; in this problem, Pisano admitted that the rabbits never die, that each month every pair begets a new pair that becomes productive at the age of two months. The experiment begins in the first month with a newborn pair. Fibonacci numbers occur in many different areas. In geometry, for instance, in Euclid's golden section problem where the number $\frac{1}{2}(\sqrt{5} - 1)$ appears. In the botanical phenomenon called phyllotaxis, where it is well known that in some trees the leaves are disposed in the spirals according to the Fibonacci sequence

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \ldots,\ \frac{f_n}{f_{n+1}}$$

that results from the expansion of $\frac{1}{2}(\sqrt{5} - 1)$ in continued fractions. It is also known that in the sunflower the number of spirals usually present are the Fibonacci numbers 34 and 55; in the giant sunflower they are 55 and 89, and recent experiments have reported that sunflowers of 89 and 144 as well as 144 and 233 spirals also exist. These are all Fibonacci numbers.

PART II: THEORY

Consider the numbers f_1', $k = 0, 1, 2, \ldots$, defined by

$$\begin{pmatrix} f'_{k+1} & f'_k \\ f'_k & f'_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k. \tag{2.1}$$

For $k = 1$, we have $f_0' = f_0$, $f_1' = f_1$, and $f_2' = f_2$. Let us suppose that $f_n' = f_n$ is valid for arbitrary n. It is easily seen from (2.1) that $f_n' = f_n$ is also valid for n $n + 1$, since we have from (2.1) that

$$f'_{n+2} = f_{n+1} + f_n = f_{n+2};\ f'_{n+1} = f_n + f_{n-1} = f_{n+1}.$$

We see then that (2.1) defines the Fibonacci numbers f_n.

Define the matrices $F(n)$ and A according to the following expressions:

$$F(n) = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = A^n; \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

It is easily proved that the above equation contains Lucas' definition of Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]; \tag{2.3}$$

in fact, the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$. We see therefore that the matrix that diagonalizes A is given by

$$U = \begin{pmatrix} \alpha_1\lambda_1 & \alpha_2\lambda_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \text{ where } \alpha_i = (1+\lambda_i^2)^{-1/2}, \tag{2.4}$$

$$U^{-1}AU = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We have then, from (2.2),

$$F(n) = U\Lambda^n U^{-1}, \tag{2.5}$$

which explicitly reads as:

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1^{n-1} - \lambda_2^{n-1} \end{pmatrix}$$

PART III: SERIES AND RECURRENCE RELATIONS

From (2.2), we write the following expression:

$$\sum_1^\infty \frac{1}{n!}F(n) = e^A - 1, \tag{3.1}$$

from which we infer that

$$\sum_1^\infty \frac{1}{n!}U^{-1}F(n)U = e^\Lambda - 1. \tag{3.2}$$

The matrix elements are given by:

$$[U^{-1}F(n)U]_{11} = \frac{1}{2}(f_{n+1} + f_{n-1}) + \frac{\sqrt{5}}{2}f_n = \alpha^n; \tag{3.3}$$

$$[U^{-1}F(n)U]_{12} = -[U^{-1}F(n)U]_{21} = \frac{\sqrt{5}}{2}(f_{n+1} - f_n - f_{n-1}) = 0;$$

$$[U^{-1}F(n)U]_{22} = \frac{1}{2}(f_{n+1} + f_{n-1}) - \frac{\sqrt{5}}{2}f_n = \beta^n.$$

From (3.1), the following series are derived:

$$\sum_0^\infty \frac{1}{n!} f_n = \frac{2e^{1/2}}{\sqrt{5}} \sinh\left(\frac{\sqrt{5}}{2}\right) \tag{3.4}$$

$$\sum_0^\infty \frac{1}{n!}(f_{n+1} + f_{n-1}) = 2e^{1/2} \cosh\left(\frac{\sqrt{5}}{2}\right),$$

where we extended Fibonacci numbers to negative values according to

$$f_{-n} = (-1)^{n+1} f_n.$$

We now set $A = 1 + B$ in (2.2) to obtain

$$F(n) = \sum_0^n \binom{n}{k} B^k. \tag{3.5}$$

B^k can be easily evaluated if we use Cauchy's integral

$$B^k = (2\pi i)^{-1} \int (dZ) Z^k (Z - B)^{-1}.$$

B^k is given by

$$B^k = F(k)^{-1} = \begin{pmatrix} f_{k-1} & -f_k \\ -f_k & f_{k+1} \end{pmatrix} (-1)^k. \tag{3.6}$$

Therefore, we have the following recurrence relations that also define Fibonacci numbers if we add to them the appropriate boundary conditions

$$f_0 = 0,\ f_1 = 1:$$

$$f_{n\pm 1} = \sum_0^n (-1)^k \binom{n}{k} f_{k\pm 1} \tag{3.7}$$

$$f_n = \sum_0^n (-1)^{k+1} \binom{n}{k} f_k.$$

If we multiply (2.2) by $(-1)^n F(n)^{-1}$, we obtain the following orthogonality relations:

$$\sum_0^n (-1)^k \binom{n}{k} f_{n+k\pm 1} = (-1)^n \tag{3.8}$$

$$\sum_0^n (-1)^k \binom{n}{k} f_{n+k} = 0.$$

Many important relations can be easily obtained from (2.2), and we just list a few of them.

The determinant of (2.2) gives

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n.$$

Setting $n = j + k$ and $A^n = A^j A^k$ in (2.2) gives the following well-known recurrence relations:

$$f_{j+k\pm 1} = f_{j\pm 1} f_{k\pm 1} + f_j f_k; \qquad (3.9)$$
$$f_{j+k} = f_{j+1} f_k + f_j f_{k+1}.$$

From the above, or from $F(np) = F(n)^p$, we are also able to obtain other familiar expressions such as:

$$f_{2n\pm 1} = f_n^2 + f_{n\pm 1}^2; \qquad (3.10)$$
$$\frac{f_{2n}}{f_n} = f_{n+1} + f_{n-1}$$
$$f_{3n} = f_{n+1}^3 + f_n^3 - f_{n-1}^3;$$
$$\frac{f_{3n}}{f_n} = 2f_{n+1}^2 + f_n^2 + f_{n+1}f_{n-1}.$$

REFERENCES

1. A. Girard, *L'Arithmétique de Simon Steven de Bruges* (Leiden, 1634).
2. D. E. Knuth, *Fundamental Algorithms: The Art of Computer Programming*, I (1969).
3. H. S. M. Coxeter, "The Golden Section, Phyllotaxis and Wythoff's Game," *Scripta Mathematica* 19 (1953).
4. R. J. Webster, "The Legend of Leonardo of Pisa," *Mathematical Spectrum* 3, No. 2 (1970/71).

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A NOTE ON BASIC M-TUPLES

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Definition 1: A set of integers $\{b_i\}_{i\geq 1}$ will be called a base for the set of all integers, whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} a_i b_i, \text{ where } a_i = 0 \text{ or } 1 \quad \text{and} \quad \sum_{i=1}^{\infty} a_i < \infty.$$

Now, a sequence $\{d_i\}_{i\geq 1}$ of odd numbers will be called basic whenever the sequence $\{d_i 2^{i-1}\}_{i\geq 1}$ is a base. If the sequence $\{d_i\}_{i\geq 1}$ of odd integers is such that $d_{i+s} = d_i$ for all i's, then the sequence is said to be periodic mod s and is denoted by $\{d_1, d_2, d_3, \ldots, d_s\}$. In reference [2], I have obtained some results concerning nonbasic sequence with periodicity mod 3 or nonbasic triples. In this paper, we are concerned with basic sequence.

Theorem 1: A necessary and sufficient condition for the sequence $\{d_i\}_{i\geq 1}$ of odd integers, which is periodic mod s, to be basic is that

$$0 = \sum_{i=1}^{m} a_i 2^{i-1} d_i \equiv 0 \pmod{2^{ns} - 1}$$

is impossible for $n \geq 1$ and $a_i = 0$ or 1 for all $i \geq 1$.

Proof: This is proved in reference [1].

Theorem 2: The m-tuple

$$\left\{2^{mk+1} - 1, -1, -1, \ldots, -1\right\}$$

is a basic sequence where k and m are integers with $k \geq 1$ and $m \geq 2$.

Proof: Suppose that the given m-tuple is not basic. Then (1) of Theorem 1.8 holds for some integers $n \geq 1$. Then there exist integers $a_i, b_i, \ldots, r_i$ for $0 \leq k \leq n - 1$ such that

$$\begin{aligned}(1)\quad & (2^{mk+1} - 1)a_0 - 2b_0 - 2^2c_0 - \cdots - 2^{m-1}r_0 + (2^{mk+1} - 1)2^m a_1 \\ & -2^{m+1}b_1 - 2^{m+2}c_1 - \cdots - 2^{2m-1}r_1 + (2^{mk+1} - 1)2^{2m}a_2 - 2^{2m+1}b_2 \\ & -2^{2m+2}c_2 - \cdots - 2^{3m-1}r_2 + \cdots + (2^{mk+1} - 1)2^{mn-m}a_{n-1} \\ & -2^{mn-m+1}b_{n-1} - 2^{mn-m+2}c_{n-1} - \cdots - 2^{mn-1}r_{n-1} \equiv 0 \pmod{2^{mn} - 1}.\end{aligned}$$

Collecting terms in the above congruence, we obtain

$$\begin{aligned}& (2 \cdot 2^{mk} - 1)(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) \\ & -2(b_0 + 2^m b_1 + \cdots + 2^{mn-m}b_{n-1}) - 2^2(c_0 - 2^m c_1 + 2^{2m}c_2 + \cdots \\ & + 2^{mn-m}c_{n-1}) - \cdots - 2^{m-1}(r_0 + 2^m r_1 + \cdots + 2^{mn-m}r_{n-1}) \equiv 0 \pmod{2^m - 1} \\ & \qquad - (a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) \\ & + 2(2^{mk}a_0 + 2^{mk+m}a_1 + 2^{mk+2m}a_2 + \cdots + 2^{mk+mn-m}a_{n-1} \\ & -b_0 - 2^m b_1 - \cdots - 2^{mn-m}b_{n-1} - 2c_0 - 2^{m+1}c_1 - \cdots \\ & - 2^{mn-m+1}c_{n-1} - \cdots - 2^{m-2}r_0 - 2^{2m-2}r_1 - \cdots - 2^{mn-2}r_{n-1}) \\ & \equiv 0 \pmod{2^m - 1}\end{aligned}$$

which can be put in the form

$$\begin{aligned}& -(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) + 2\{2^{mk}(a_0 - b_k - 2c_k - \cdots \\ & - 2^{m-2}r_k) + 2^{m(k+1)}(a_1 - b_{k+1} - 2c_{k+1} - \cdots - 2^{m-2}r_{k+1}) + \cdots \\ & + 2^{m(n-1)}(a_{n-1-k} - b_{n-1} - 2c_{n-1} - \cdots - 2^{m-2}r_{n-1}) \\ & + 2(2^{mn}a_{n-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots \\ & + 2^{m(k-1)}(2^{mn}a_{n-1} - b_{k-1} - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1}) \\ & + (2^{mn}a_{n-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots\end{aligned}$$

$$+ 2^{m(k-1)}(2^{mn}a_{n-1} - b_{k-1} - 2c_{k-1} - \cdots - 2^{m-2}r_{n-1})\}$$
$$\equiv 0 \pmod{2^{mn}-1}.$$

Now define $a_{n-i} = a_{-i}$ for $1 \le i \le k$, and let

$$\begin{aligned} Q = {} & -(a_0 + 2^m a_1 + 2^{2m}a_2 + \cdots + 2^{mn-m}a_{n-1}) + 2\{2^{mk}(a_0 - b_k - 2c_k \\ & - \cdots - 2^{m-2}r_k) + 2^{m(k+1)}(a_1 - b_{k+1} - 2c_{k+1} - \cdots - 2^{m-2}r_{k+1}) \\ & + \cdots + 2^{m(n-1)}(a_{n-1-k} - b_{n-1} - 2c_{n-1} - \cdots - 2^{m-2}r_{n-1}) \\ & + (a_{-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0) + \cdots + 2^{m(k-1)}(a_{-1} - b_{k-1} \\ & - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1})\} \equiv 0 \pmod{2^{mn}-1}. \end{aligned} \tag{3}$$

Rearranging terms in (3), we obtain

$$\begin{aligned} Q = {} & \{-a_0 + 2(a_{-k} - b_0 - 2c_0 - \cdots - 2^{m-2}r_0)\} + 2^m\{-a_1 + 2(a_{-k+1} \\ & - b_1 - 2c_1 - \cdots - 2^{m-2}r_1) + \cdots + 2^{m(k-1)} - a_{k-1} + 2(a_{-1} - b_{k-1} \\ & - 2c_{k-1} - \cdots - 2^{m-2}r_{k-1}) + 2^{mk} - a_k + 2(a_0 - b_k - 2c_k \\ & - \cdots - 2^{m-2}r_k)\} + \cdots + 2^{m(n-1)}\{-a_{n-1} + 2(a_{n-1-k} - b_{n-1} - 2c_{n-1} \\ & - \cdots - 2^{m-2}r_{n-1})\} \equiv 0 \pmod{2^{mn}-1}. \end{aligned} \tag{4}$$

Taking absolute values and using the triangle inequality, we obtain

$$\begin{aligned} |Q| & \le (2^m - 1) + 2^m(2^m - 1) + 2^{2m}(2^m - 1) + \cdots + 2^{m(n-1)}(2^m - 1) \\ & = (2^m - 1) + (2^{2m} - 2^m) + (2^{3m} - 2^{2m}) + \cdots + (2^{mn} - 2^{m(n-1)}) \\ & = 2^{mn} - 1. \end{aligned}$$

Now, $|Q| = 2^{mn} - 1$, provided

$$-a_i + 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i) = 2^m - 1$$

for all i with $0 \le i \le n - 1$. But this clearly implies that

$$a_i = 1,\ a_{-k+i} = 0, \text{ and } b_i = c_i = \cdots = r_i = 1 \quad \text{for all } i.$$

Since the first two equalities are clearly contradictory, it follows that we must have $Q = 0$ and hence

$$-a_i = 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i), \tag{5}$$

and yet $a_i = 0$ or $a_i = 1$ for all i. Since the right-hand side of (5) is divisible by 2, it follows that $r_i = 0$ for all i. Thus,

$$0 = 2(a_{-k+i} - b_i - 2c_i - \cdots - 2^{m-2}r_i) \tag{5$'$}$$

or

(6) $$a_{-k+i} - b_i = 2c_i + \cdots + 2^{m-2}r_i \text{ for all } i.$$

Possibilities for $a_{-k+i} - b_i$ are 0, 1, and -1. But the right-hand side of (6) is divisible by 2. Hence, we must have that $a_{-k+i} - b_i = 0$ for all i. Since $a_{-k+i} = 0$ for all i, this implies that $b_i = 0$ for all i and hence that $c_i = 0, \ldots, r_i = 0$ for all i. But since this contradicts Theorem 1.8, it follows that the m-tuple $2^{mk+1} - 1, -1, -1, \ldots, -1$ is basic as claimed.

REFERENCES

1. N. G. de Druijn, "On Bases for the Set of Integers," *Publ. Math. Debrecen* 1 (1950):232-242.
2. Norman Woo, "On Nonbasic Triples," *The Fibonacci Quarterly* 13, No. 1 (1975):56-58.

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PYTHAGOREAN TRIPLES AND TRIANGULAR NUMBERS

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1. INTRODUCTION

In [4] W. Sierpiński proves that there are an infinite number of Pythagorean triples in which two members are triangular and the hypotenuse is an integer. [A number T_n is triangular if T_n is of the form $T_n = n(n+1)/2$ for some integer n. A Pythagorean triple is a set of three integers x, y, z such that $x^2 + y^2 = z^2$.] Further, Sierpinski gives an example due to Zarankiewicz,

$$T_{132} = 8778, \quad T_{143} = 10296, \quad \text{and} \quad T_{164} = 13530,$$

in which every member of the Pythagorean triple is triangular. He states that this is the only known nontrivial example of this phenomenon, and that it is not known whether the number of such triples is finite or infinite.

This paper will give some partial results related to the above problem. In particular, we will give necessary and sufficient conditions for the existence of Pythagorean triples in which all members are triangular. We will extend these conditions to discuss the problem of triangulars being represented as sums of powers.

2. PYTHAGOREAN TRIPLES WITH TRIANGULAR SOLUTIONS

By a triangular solution to a Diophantine equation $f(x, \ldots, x_n) = 0$, we mean a solution in which every variable is triangular.

Theorem 1: The Pythagorean equation $x^2 + y^2 = z^2$ has a triangular solution $x = T_a$, $y = T_b$, $z = T_c$ if and only if there exist integers m and k such that

$$T_b^2 = m^3 + (m+1)^3 + \cdots + (m+k)^3;$$

that is, T_b^2 is a sum of $k + 1$ consecutive cubes.

Proof: It is a known formula that

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} = T_n^2.$$

So if

$$T_a^2 + T_b^2 = T_c^2$$

with $a \leq b$, then

$$T_b^2 = T_c^2 - T_a^2 = \sum_{k=a+1}^{c} k^3.$$

To show the converse, we need only reverse the steps. Q.E.D.

Using Zarankiewicz's example, we can note that T_{143}^2 is a sum of 31 cubes; i.e.,

$$T_{143}^2 = \sum_{k=133}^{164} k^3.$$

3. TRIANGULARS AS CUBES AND SUMS OF CUBES

We first show that a triangular cannot be a cube. This is an old result, first proved by Euler in 1738 [2]. However, it is so closely related to our work that we will include a proof here.

Lemma 2: The triangular T_n is a kth power if and only if T_n^2 is a kth power.

Proof: This is an easy exercise using the fact that every integer has a unique decomposition into primes.

Lemma 3: The equality $T_n = m^k$ holds nontrivially if and only if the equations $x^k - 2y^k = \pm 1$ have nontrivial solutions. Take the plus sign if n is even and the minus sign if n is odd.

Proof: Let

$$T_n = \frac{n(n+1)}{2} = m_k.$$

Clearly $(n, n+1) = 1$. Let $n = 2j$; then

$$(2j)(2j+1)/2 = m^k.$$

Thus there are integers x and y such that $j = y$ and $2j + 1 = x^k$; whence

$$x^k - 2y^k = 1.$$

Now let $n = 2j - 1$. In the same way as above, there are integers y, x such that $j = y^k$, $2j - 1 = x^k$, and $x^k - 2y^k = -1$.

Since the steps are reversible, the converse is easily proved. Q.E.D.

Theorem 4: There is no triangular number greater than 1 which is a cube.

Proof: If $T_n = m^3$, then by Lemma 3, $x^3 - 2y^3 = -1$ has a solution. However, by [1, p. 72], $x^3 - 2y^3 = 1$ has only $x = -1$, $y = 0$ as solutions. Hence, by the construction in Lemma 3, $n = 1$ or 0. Q.E.D.

We will now state, without proof, a theorem due to Siegel which will be of utmost importance in that which follows.

Theorem 5: (Siegel [3, p. 264]) The equation

$$y^2 = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

has only a finite number of integer solutions if the right-hand side has at least three different linear factors.

We can immediately apply this theorem in the proof of the following result.

Theorem 6: For a fixed k, there are only a finite number of sums of k consecutive cubes which can be the square of a triangular number. For every k, there is at least one such sum which is the square of a triangular.

Proof: The last statement follows from the identity

$$\sum_{n=0}^{k} n^3 = T_k^2.$$

To prove the first statement we consider two cases. Assume $k = 2l + 1$. Consider the equation

$$T_n^2 = \sum_{j=-l}^{l} (m + j)^3. \tag{1}$$

We want to show that this equation has only a finite number of solutions in n and m. We have

$$\begin{aligned} T_n^2 = \sum_{j=-l}^{l} (m + j)^3 &= Am^3 + Bm \qquad AB \neq 0 \\ &= m(Am^2 + B). \end{aligned} \tag{2}$$

Now $Am^2 + B$ is never a square since $(am + b)^2$ always has a first-degree term. Thus, equation (2) has no squared linear factors on its right-hand side, and by Theorem 5 it has only a finite number of solutions.

If $k = 2l$, we consider

$$\begin{aligned} T^2 &= \sum_{-l}^{l+1} (m + j)^3 \\ &= (2L + 1)m^3 + L(L - 1)(2L + 1)m + (m + L - 1)^3 \\ &= (L + 1)\left(2m^3 + 3m^2 + (2L^2 + 4L + 3)m + (L + 1)^2\right). \end{aligned} \tag{3}$$

To show that the right-hand side does not have a square linear factor, we show that it and its derivative,

$$6m^2 + 6m + (2L^2 + 4L + 3)$$

have a greatest common divisor of 1. This is an easy application of the Euclidean algorithm. Hence, using Theorem 5, equation (3) has only a finite number of integral solutions. Q.E.D.

Combining Theorems 1 and 6, we have a type of finiteness condition for all members of a Pythagorean triple to be triangular. Of course, the k can vary, so we do not have the condition that only a finite number of such triples exist, but that for a fixed k, only a finite number exist.

4. TRIANGULARS AND SUMS OF HIGHER POWERS

We can prove theorems similar to Theorems 4 and 6 for higher powers.

Theorem 7: The equations $T_n = m^4$ and $T_n = m^5$ are impossible for $n > 1$.

Proof: This follows from Lemma 3 and the fact that the equations

$$x^4 - 2y^4 = \pm 1 \quad \text{and} \quad x^5 - 2y^5 = \pm 1$$

have no nontrivial solutions [1]. Q.E.D.

Theorem 7 was first stated by Fermat in 1658, but he apparently gave no proof; at least none has been found. The first proof was given by Euler [3].

Theorem 8: For a fixed k, the equations

$$T_n^2 = \sum_{i=0}^{k} (m + i)^4$$

and

$$T_n^2 = \sum_{i=0}^{k} (m + i)^5$$

have only a finite number of solutions.

Proof: These statements are proven using techniques completely similar to the proof of Theorem 6. Greatest common divisor calculations are extremely complicated and are therefore omitted. Q.E.D.

The techniques of Theorem 6 appear to apply to even higher powers. However, there does not appear to be a general method of handling all such cases simultaneously because of the differences of the equations and the derivatives.

5. THE EQUATION $T_{(n+1)^2} = k^2$

The theorems of this section digress from the main topics of this paper, but they are included as nice illustrations of the use of Theorem 5.

Theorem 9: The equation $T_{(n+1)^2} = k^2$ has only a finite number of solutions.

Proof: If $(n + 1)^2((n + 1)^2 + 1)/2 = k^2$, then

$$2k^2 = n^4 + 4n^3 + 7n^2 + 6n + 2. \tag{4}$$

The derivative of the right-hand side is

$$4n^3 + 12n^2 + 14n + 6 = 2(n + 1)(2n^2 + 4n + 3).$$

It is easy to check that no root of the derivative is a root of equation (4), so equation (4) has no squared factor. Hence, by Theorem 5, there are only a finite number of solutions to the equation of the theorem. Q.E.D.

Note that $T_{(1)} = (1)^2$ and $T_{(7)^2} = (35)^2$.

In [4] Sierpiński shows that the equation

$$(T_{2u})^2 + (T_{2u+1})^2 = [(2u + 1)v]^2$$

with $v^2 = u^2 + (u + 1)^2$ has only a finite number of solutions. Since we have that the identity

$$(T_{2u+1})^2 + (T_{2u})^2 = T_{(2u+1)^2}$$

holds, we have the following theorem.

Theorem 10: The equation

$$T_{(2u+1)^2} = [(2u + 1)v]^2$$

with $v^2 = u^2 + (u + 1)^2$ has only a finite number of solutions.

Proof: Use Theorem 9.

REFERENCES

1. R. D. Carmichael, *Diophantine Analysis* (New York: Dover, 1959).
2. L. Dickson, *History of the Theory of Numbers*, Vl III (Washington, D.C.: Carnegie Institution, 1923).
3. L. Mordell, *Diophantine Equations* (New York: Academic Press, 1969).
4. W. Sierpiński, "Sur les nombres triangulaires carrés," *Bull. Soc. Royale Sciences Liège*, 30ᵉ année, n° 5-6 (1961):189-194.

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EXTENSIONS OF THE W. MNICH PROBLEM

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ABSTRACT

W. Sierpiński publicized the following problem proposed by Werner Mnich in 1956: *Are there three rational numbers whose sum and product are both one*? In 1960, J. W. S. Cassels proved that there are no rationals that meet the Mnich condition. This paper extends the Mnich problem to k-tuples of rationals whose sum and product are one by providing infinite solutions for all $k > 3$. It also provides generating forms that yield infinite solutions to the original Mnich problem in real and complex numbers, as well as providing infinite solutions for rational sums and products other than one.

HISTORICAL OVERVIEW

Sierpiński [6] cited a question posed by Werner Mnich as a most interesting problem, and one that at that time was unsolved. The Mnich question concerned the existence of three rational numbers whose sum and product are both one:

$$(1) \qquad x + y + z = xyz = 1 \qquad (x,\ y,\ z \text{ rational}).$$

Cassels [1] proved that there are no rationals that satisfy the conditions of (1). Cassels also shows that this problem was expressed by Mordell [3], in equivalent, if not exact form. Additionally, Cassels has compiled an excellent bibliography that demonstrates that the "Mnich" problem has its roots in the work of Sylvester [13] who in turn obtained some results from the 1870 work of the Reverend Father Pépin. Sierpiński [9] provides a more elementary proof of the impossibility of a weaker version of (1), along with an excellent summary of some of the equivalent forms of the "Mnich" problem. Later, Sansone and Cassels [4] provided another proof of the impossibility of (1).

EXTENSIONS TO k-TUPLES

It is natural to consider the generalization of the "Mnich" problem. Do there exist k-tuples of rational numbers such that their sums and products are both one for a given natural number k:

$$x_1 + x_2 + x_3 + \cdots + x_k = x_1x_2x_3 \ldots x_k = 1, \quad \text{for } k > 3? \tag{2}$$

Sierpiński [6, p. 127] states that Andrew Schinzel has proven that there are an infinite number of solutions for every k-tuple in (2). However, in the source cited, Trost [14] only appears to credit Schinzel with the proof of infinite k-tuples in (2) when k is of the form $4n$ or $4n + 1$, where n is a natural number (i.e., $k = 4, 5, 8, 9, 12, 13$, etc.). Schinzel provided a general form for generating an infinite number of solutions to (2) when $k = 4$. He provided one case for $k = 5$ (viz., $1, 1, 1, -1, -1$), but failed to demonstrate any solutions at all for (2) when the values of k are of the form $4n + 2$ or $4n + 3$ (viz., $6, 7, 10, 11, 14, 15$, etc.). Explicit generating functions will now be given which prove that there are indeed an infinite number of rational k-tuples for all $k > 3$ that satisfy (2).

It is quite obvious that for $k = 2$ there are no real solutions, since $xy = x + y = 1$ yields the quadratic equation $x^2 - x + 1 = 0$ whose discriminant is -3. For $k = 4$, a general form was given by Schinzel [14]:

$$\{n^2/(n^2 - 1),\ 1/(1 - n^2),\ (n^2 - 1)/n,\ (1 - n^2)/n\},\ n \neq \pm 1, 0, \tag{3}$$

e.g., for $n = 2$,

$$4/3 - 1/3 + 3/2 - 3/2 = (4/3)(-1/3)(3/2)(-3/2) = 1.$$

I derived the following general generating functions for all k-tuples greater than 4. Beyond the restrictions cited, they yield an infinite set of solutions for (2) by using any rational value of n.

$$\text{For } k = 5,\ \{n,\ -1/n,\ -n,\ 1/n,\ 1\},\ n \neq 0, \tag{4}$$

e.g., for $n = 2$,

$$2 - 1/2 - 2 + 1/2 + 1 = (2)(-1/2)(-2)(1/2)(1) = 1.$$

$$\text{For } k = 6,\ \{1/n^2(n + 1),\ -1/n^2(n + 1),\ (n + 1)^2,\ -n^2,\ -n,\ -n\},\quad n \neq 0, -1, \tag{5}$$

e.g., for $n = 2$,

$$1/12 - 1/12 + 9 - 4 - 2 - 2$$
$$= (1/12)(-1/12)(9)(-4)(-2)(-2) = 1.$$

$$\text{For } k = 7,\ \{(n - 1)^2,\ (n - 1/2),\ (n - 1/2),\ 1,\ -n^2,\ 1/n(n - 1)(n - 1/2),\ -1/n(n - 1)(n - 1/2)\},\quad n \neq 0, 1, 1/2, \tag{6}$$

e.g., for $n = 2$,

$$1 + 3/2 + 3/2 + 1 - 4 + 1/3 - 1/3$$
$$= (1)(3/2)(3/2)(1)(-4)(1/3)(-1/3) = 1.$$

Since the elements of the set $U = (1, -1, 1, -1)$ have a sum of 0 and a produce of 1, U forms the basis for generating all remaining explicit expressions beyond $k = 7$ by adjoining the elements of U onto the k-tuple results for $k = 4, 5, 6$, and 7. The process is then repeated as often as is necessary as a 4-cycle. For example:

(7) For $k = 8$,

$$\{n^2/n^2 - 1,\ n^2 - 1/n,\ 1 - n^2/n,\ 1/1 - n^2,\ 1, -1, 1, -1\}$$
$$= \{k = 4, U\},\ n \neq 0, -1, +1, \text{ or } \{k = 4, k = 4\}.$$

(8) For $k = 9$,

$$\{n, -1/n, -n, 1/n, 1, 1, -1, 1, -1\} = \{k = 5, U\},\ n \neq 0.$$

(9) For $k = 10$, $\{k = 6, U\}$; for $k = 11$, $\{k = 7, U\}$;
for $k = 12$, $\{k = 4, k = 4, k = 4\}$ or $\{k = 6, k = 6\}$
or $\{k = 8, U\}$ or $\{k = 8, k = 4\}$ or $\{k = 7, k = 5\}$
or $\{k = 4, U, U\}$.
Etc.

No claim is made here that the k-tuple form of the generating functions in (4) through (9) are unique.

EXTENSIONS TO OTHER NUMBER SYSTEMS

Although the conditions for generating rational roots for equation (1) have been demonstrated to be impossible, it is clear that rational roots approximating the Mnich criterion can be generated with any degree of accuracy required. Consider the example:

(10) $$(7/3)(-5/9)(-27/35) = 1, \text{ but } 7/3 - 5/9 - 27/35 = 951/845$$

and

(11) $$-.726547 - .540786 + 2.333333 = 1, \text{ but}$$
$$(-.726547)(-.5406786)(2.333333) = 0.9999999.$$

If the solution domain for equation (1) is expanded from the rationals to the reals, then there are an infinite number of solutions of the form $(a \pm \sqrt{b}, c)$ which can be derived from the Mnich conditions

$$2a + c = 1 \quad\text{and}\quad (a^2 - b)c = 1.$$

One form of the solution in reals yields the following infinite set in which a is real:

(12) $$(a + \sqrt{a^2 + 1/(2a - 1)},\ a - \sqrt{a^2 + 1/(2a - 1)},\ 1 - 2a),\ a \neq 1/2,$$

e.g., for $a = 2$,

$$2 + \sqrt{13/3} + 2 - \sqrt{13/3} - 3 = (2 + \sqrt{13/3})(2 - \sqrt{13/3})(-3) = 1.$$

The generating form in (12) remains real as long as $a + 1/(2a - 1) \geq 0$, which is the case provided that $a > 1/2$ or $a \leq -.6572981$.

This latter condition for a makes the discriminant in (12) zero and yields the only solution with two equal elements:

(13)
$$A = \left\{\sqrt[3]{-53/216 + \sqrt{13/216}} + \sqrt[3]{-53/216 - \sqrt{13/216}} + 1/6\right\};$$
$$B = 2\left\{1/3 - \sqrt[3]{-53/216 + \sqrt{13/216}} - \sqrt[3]{-53/216 - \sqrt{13/216}}\right\};$$
$$A + A + B = (A)(A)(B) = 1.$$

For values of a in the interval $-.6572981 < a < 1/2$, the generating form in (12) yields complex conjugate results, e.g.,

(14)
$$a = 0,\ \{\pm i,\ 1\},\ \text{and}\ a = -1/2,\ \{(-1 \pm i)/2,\ 2\}.$$

Solutions of the Mnich problem in reals have not appeared in the literature, although Sierpiński [7, p. 176] does cite the first example in (14).

Also absent from the literature is a discussion of the Mnich problem in the complex plane. Assuming that the solution for (1) is of the form

$$(a \pm i\sqrt{b},\ c)$$

yields the infinite generating form with n real as follows:

(15)
$$\left\{\frac{1 \pm i\sqrt{(n^3 - n + 2)/(n - 2)}}{n},\ \frac{n - 2}{n}\right\},\quad n \neq 0,\ 2,$$

e.g., for $a = 4$ $\{(1 \pm i\sqrt{31})/4,\ 1/2\}$.

The generating form in (15) remains complex as long as

$$(n^3 - n + 2)/(n - 2) \geq 0,$$

which is the case provided that $n > 2$ or $n < -1.5213797$. Note that these limits are the reciprocals of those for (12). When n is in the interval $-1.5213797 \leq n < 2$, (15) generates real solutions. Clearly, the generating forms (12) and (15) presented here for yielding real and complex solutions to (1) are not unique.

EXTENSIONS TO OTHER CONSTANT SUMS AND PRODUCTS

If the restriction in (2) that the product and sum must be equal to one is replaced by some rational number c, then a more general Mnich problem develops for rational x_i:

(16)
$$x_1 + x_2 + x_3 + \cdots + x_k = x_1x_2x_3 \ldots x_k = c,\ \text{for}\ k \geq 2.$$

When $k = 2$, the infinite generating set is of the form:

(17)
$$\{x,\ x/(x - 1)\}\ \text{where the product and sum} = x^2/(x - 1),\ x \neq 1,$$

e.g., $2 + 2 = (2)(2) = 4$, and $3 + 3/2 = (3)(3/2) = 9/2$.

When $k = 3$, then $x + y + z = xyz = c$. If we assume that $y = x/(x - 1)$ as in (17), then solving for z yields $z = (x + y)/(xy - 1) = x^2/(x^2 - x + 1)$. The infinite set is:

(18) $\left\{x, \dfrac{x}{x-1}, \dfrac{x^2}{x^2-x+1}\right\}$, $x \neq 1$, and the product and sum equal $x^4/(x^3 - 2x^2 + 2x - 1)$,

e.g., $2 + 2 + 4/3 = (2)(2)(4/3) = 16/3$.

When $k = 4$, using the previous results yields the infinite set:

(19) $\left\{x, \dfrac{x}{x-1}, \dfrac{x^2}{x^2-x+1}, \dfrac{x^4}{x^4-x^3-2x^2-2x+1}\right\}$, $x \neq 1$,

e.g., $2 + 2 + 4/3 + 16/13 = (2)(2)(4/3)(16/13) = 256/39$.

It is obvious that this process can be generalized in a recursive way to generate infinite rational solutions for any k-tuple.

Sierpiński [6, p. 127] credits Schinzel with demonstrating that the elements in (16) can be restricted to integers by the following substitutions for all $k \geq 2$:

(20) $x_{k-1} = 2$ and $x_k = k$ (fulfilled first), and

$x_1, x_2, x_3, \ldots, x_{k-2} = 1$.

The following table presents the results of (20).

k-Tuple	Solution Set	Product = Sum = C
2	(2, 2)	4
3	(1, 2, 3)	6
4	(1, 1, 2, 4)	8
5	(1, 1, 1, 2, 5)	10
⋮	⋮	⋮
k	$(\underbrace{1, 1, \ldots, 1,}_{k-2} 2, k)$	$k + 2 + \sum_{k=1}^{k-2} 1 = 2k$

It is worth noting that the number of integer solutions for sufficiently large k in (16) is still an open question. Also worth noting is that the result for $k = 3$ in the above table can be derived from assuming that the integers are of the form $x - p$, x, $x + p$, from which it follows that

$$x = \pm\sqrt{p^2 + 3}.$$

The only rational results generated are for $p = 1$ (1, 2, 3), and for $p = -1$ (-1, -2, -3).

SUMMARY

This paper traced the "Mnich" problem back to the work of Father Pépin in the 1870s, and identified the proofs of Cassels, Sansone, and Sierpiński as having decided the question in (1) in the negative. This restatement is needed because their results are not widely known, and sources such as (10) and (12) continue to cite the "Mnich" problem as unsolved.

Infinite generating forms for the extension of the Mnich conditions to all k-tuples greater than three are provided in (3)-(9). Infinite generating forms for the "Mnich" problem in the real and complex plane are provided in (12) and (15), respectively; also, approximate rational solutions are given in (10) and (11). Finally, the "Mnich" problem is extended to rational sums and products other than 1, and the recursive generating forms are provided for an infinite number of rational solutions for $k \geq 2$, with $k = 2$, $k = 3$, and $k = 4$ given explicitly, in (17)-(19).

REFERENCES

1. J. W. S. Cassels, "On a Diophantine Equation," *Acta Arithmetica* 6 (1960): 47-52.
2. A. Hurwitz, "Über ternäre diophantische Gleichungen dritten Grades," *Vierteljahrschrift d. Naturf. Ges.* (Zürich) 62 (1917):207-229.
3. L. J. Mordell, "The Diophantine Equation $x^3 + y^3 + z^3 + kxyz = 0$," in *Colloque sur la théorie des nombres* (Bruxelles, 1955), pp. 67-76.
4. G. Sansone & J. W. S. Cassels, "Sur le problème de M. Werner Mnich," *Acta Arithmetica* 7 (1962):187-190.
5. A. Schinzel, "Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières," *L'Enseignement Mathématique* 4 (1958): 71-72.
6. W. Sierpiński, "On Some Unsolved Problems of Arithmetic," *Scripta Mathematica* 25 (1959):125-126.
7. W. Sierpiński, *Teoria Liczb*, Part II (Warszawa, 1959).
8. W. Sierpiński, "Sur quelques problèmes non résolus d'arithmétique," *L'Enseignement Mathématique* 5 (1959):221-222.
9. W. Sierpiński, "Remarques sur le travail de M. J. W. S. Cassels 'On a Diophantine Equation,'" *Acta Arithmetica* 6 (1961):469-471.
10. W. Sierpiński, "Some Unsolved Problems of Arithmetic," in *Enrichment Mathematics for High Schools—28th Yearbook* (Washington, D.C.: National Council of Teachers of Mathematics, 1967), pp. 205-217. Reprinted first from [6] in 1963 without updating.
11. W. Sierpiński, *A Selection of Problems in the Theory of Numbers* (New York: The Macmillan Company, Inc., 1964).
12. S. K. Stein, *Mathematics: The Man-Made Universe* (3rd ed.; San Francisco: W. H. Freeman & Co., 1976).
13. J. J. Sylvester, "On Certain Ternary Cubic-Form Equations," *American Journal of Mathematics* 2 (1878):280-285.
14. E. Trost, "Ungeloste Probleme Nr. 14" (Schinzel's Extension of the W. Mnich Problem), *Elemente der Mathematik* 11 (1956):134-135.
15. A. Weil, "Sur un théorème de Mordell," *Bull. des Sci. Math.* 54 (1930): 182-191.

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GROWTH TYPES OF FIBONACCI AND MARKOFF*

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1. PRELIMINARY REMARKS

The pattern of Fibonaccian growth in pure and applied mathematics is well known and seemingly ubiquitous. In recent work of the author (see [1]), a generalization of this pattern emerged where the "linear" growth of Fibonacci type is replaced by a "tree" growth which might appropriately be called the "Markoff type." There are many instances where tree-growth is used for number-theoretic functions (for a recent example, see [4]). What is different here is the application of the tree to (noncommutative) strings of symbols. This, paradoxically, makes for a simpler device but one with applications to many different fields.

The use of the "Markoff" designation requires some clarification. We refer to A. A. Markoff (1856-1922), the number-theorist. He was also the probabilitist (with the name customarily spelled "Markov" in this context), but the growth type we desire is *nonrandom* and strictly a consequence of his number-theoretic work. To compound the confusion, he had a lesser known brother, V. A. Markoff (also a number-theorist), and a very famous son, the logician A. A. Markov (still alive today).

2. SEMIGROUP

We consider S_2 a free semigroup consisting of strings of symbols in A and B (including "$_1$" the null symbol) to form *words* $w = w(A,B)$. If the word w has a symbols A and b symbols B (for $a \geq 0$, $b \geq 0$), then we say word w has *coordinates* $\{a, b\}$. For instance, some coordinates and words are

$$\begin{array}{cccccc} \{0,0\}, & \{1,0\}, & \{0,1\}, & \{1,1\}, & \{1,1\}, & \{4,2\}, \\ 1, & A, & B, & AB, & BA, & AAABAB, \text{ etc.} \end{array}$$

Of course, distinct words (e.g., AB and BA) may have the same coordinates. Naturally, we abbreviate $AAABAB$ as A^3BAB, etc.

We also introduce the concept of *equivalence*. Two words of S_2 are said to be equivalent if they are cyclic permutations of one another including the trivial (identity) permutation. This is denoted by "$\sim$". Thus,

$$w_1(A,B)w_2(A,B) \sim w_2(A,B)w_1(A,B).$$

$$ABAA \sim ABAA \sim BAAA \sim AAAB \sim \cdots$$

Equivalent words have the same coordinates, of course (but not conversely, $ABAB$ and $AABB$ have coordinates $\{2, 2\}$).

Actually $w_1 \sim w_2$ means $Tw_1 = w_2T$ (for $T \in S_2$), and for computational purposes it might be convenient to do computations inside the *free group* by writing $w_1 = T^{-1}w_2T$. In principle, however, growth requires only a semigroup. We also need the symbol when we have multiple equivalence

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$$(w_1, w_2, \ldots) \sim (w_1', w_2', \ldots) \Longleftrightarrow Tw_1 = w_1'T,\ Tw_2 = w_2'T,\ \ldots$$

for the same T in each case.

3. TYPES OF GROWTH

Fibonaccian growth suggests the sequence

$$(f_{-2} = 1,\ f_{-1} = 0),\ f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ \ldots,\ f_{n+1} = f_{n-1} + f_n.$$

If we start with A and B instead of f_0 and f_1 we have a sequence of strings, $w_n(A,B)$

$$w_0 = A,\ w_1 = B,\ w_2 = AB,\ \ldots,\ w_{n+1} = w_{n-1}w_n.$$

To list a few strings with coordinates

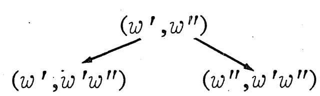
$$\begin{array}{ccccc} \{1,\ 0\}, & \{0,\ 1\}, & \{1,\ 1\}, & \{1,\ 2\}, & \{2,\ 3\}, \\ A, & B, & AB, & BAB, & ABBAB, \ \ldots \end{array}$$

Clearly $w_n(A,B)$ has the coordinates $\{f_{n-2},\ f_{n-1}\}$.

Here we have used the strings $w_n(A,B)$ instead of f_n but the progression is still linearly ordered:

$$\cdots \to (w_{n-1}, w_n) \to (w_n, w_{n-1}w_n) \to \cdots \quad \text{(Fibonacci type)}.$$

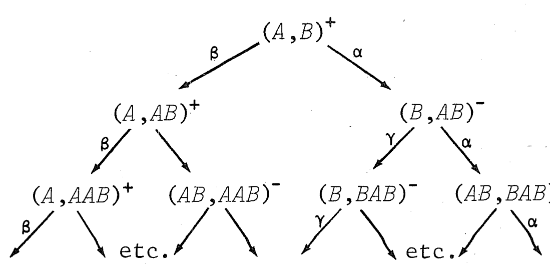
We now consider a generalization of this growth where the ordering is not linear but tree-like,



(Markoff type).

Thus, once $w(=w'w'')$ is formed, we have the choice of dropping w' (Fibonacci again) or dropping w''.

We illustrate the *Markoff tree* generated by starting with the pair (A,B). (The "+" and "-" signs are explained in Section 4 below).



There are 2^{n-1} possible pairs on the nth level.

The reader can easily recognize Fibonaccian growth on the extreme right diagonal (α)

$$A,\ B,\ AB,\ BAB,\ ABBAB,\ \ldots$$

On the extreme left diagonal (β), we see the simpler growth

$$B,\ AB,\ AAB,\ AAAB,\ \ldots$$

This may seem asymmetrical, but a parallel diagonal (γ) gives

$$B,\ AB,\ BAB,\ BBAB,\ \ldots$$

which is equivalent (with the same "T" = B) to

$$B,\ BA,\ BBA,\ BBBA,\ \ldots$$

4. EUCLIDEAN PARTITION

If we look at the words in the Markoff tree (in Section 3), we see that they have coordinates as follows:

$$\begin{array}{cccc}
 & (\{1,0\},\{0,1\})^{+} & & \\
(\{1,0\},\{1,1\})^{+} & & (\{0,1\},\{1,1\})^{-} & \\
(\{1,0\},\{2,1\})^{+}, & (\{1,1\},\{2,1\})^{-} & (\{0,1\},\{1,2\})^{-} & (\{1,1\},\{1,2\})^{+}
\end{array}$$

In general, a pair (w_1, w_2) has the coordinates

$$(\{a_1,\ b_1\},\ \{a_2,\ b_2\}) \quad \text{where} \quad a_1b_2 - a_2b_1 = \pm 1.$$

(The "+" and "−" designations give this sign in Section 3 and above.) We can prove an even stronger result if we introduce a definition:

Let a, a', a'', b, b', b'' all be ≥ 0, then we say

$$(a,b) = (a',b') + (a'',b'')$$

is a *euclidean partition* exactly when $a'b'' - b'a'' = +1$. Then every such (a,b) has a euclidean partition if $ab > 0$ by virtue of the *euclidean algorithm* by the solvability of

$$ax - by = \pm 1, \quad (0 \leq x < b,\ 0 \leq y < a).$$

For +1, $(x,y) = (a'',b'')$; for −1, $(x,y) = (a',b')$. Clearly any (a,b) can be ultimately partitioned to (0,1) and (1,0). For instance, if we start with (5,7), we have:

$$\begin{aligned}
&(5,7) = (3,4) + (2,3), \quad && (3,4) = (1,1) + (2,3),\\
&(2,3) = (1,1) + (1,2), \quad && (1,2) = (1,1) + (0,1),\\
&(1,1) = (1,0) + (0,1).
\end{aligned}$$

We now see, generally, that if (w',w'') is in the Markoff tree and $w = w'w''$ with $\{a',\ b'\}$, $\{a'',\ b''\}$, and $\{a,\ b\}$ the coordinates of w', w'', and w (respectively), then we write

$$(w',w'')^{+} \Rightarrow (a,b) = (a',b') + (a'',b'')$$

$$(w',w'')^{-} \Rightarrow (a,b) = (a'',b'') + (a',b'),$$

as euclidean partitions in each case. The property is preserved in the Markoff tree, *so every* $\{a,\ b\}$ with gcd $(a,b) = 1$ (*and* $a \geq 0$, $b \geq 0$) *is represented as the coordinate of some word in the Markoff tree.*

We shall next see how words in the Markoff tree are composed by euclidean partitions.

5. STEP-WORD

The symbol we introduce to explain words in the Markoff tree is called the *step-word*

$$(A,B)^{a,b} = \prod_{s=1}^{a} AB^{e_s}, \quad e_s = [sb/a] - [(s-1)b/a]$$

where $n = [\xi]$ is the integral part of ξ (satisfying $n \le \xi < n + 1$). Here we assume $a > 0$, $b > 0$, and gcd $(a,b) = 1$. The further definition "by fiat" includes $a = 0$ $(b = 1)$,

$$(A,B)^{0,1} = B.$$

In any case, $(A,B)^{a,b}$ has coordinates $\{a,\ b\}$, $\left(\text{i.e., } \sum_{s=1}^{a} e_s = b\right)$.

Some of the simple cases are:

$$(A,B)^{1,0} = A, \quad (A,B)^{0,1} = B, \quad (A,B)^{1,1} = AB$$

$$(A,B)^{n,1} = A^nB, \quad (A,B)^{1,n} = AB^n, \quad (A,B)^{2m+1,2} = A^mBA^{m+1}B,$$

$$(A,B)^{2,2m+1} = AB^mAB^{m+1}, \quad (A,B)^{3,3m+2} = AB^mAB^{m+1}AB^{m+1}, \text{ etc.}$$

Note that the values of e_s (if more than one occurs) are chosen from *two* consecutive integers, $[b/a]$ and $[b/a] + 1$.

The symbol can be extended to an arbitrary integral pair (a,b) but this is not relevant to present work.

To see why the symbol is called a "step-word" let us note that the values of $e_1, \ldots, e_a$ are found by differencing the sequence $[bs/a]$ for $s = 0$, $1, 2, \ldots, a$, in other words, by differencing the integral values of the *step-function* $y = [bx/a]$ lying just below the line $y = bx/a$ for $0 \le x \le a$.

6. NIELSEN PARTITION

We now construct a partition of step-words $w = (A,B)^{a,b}$ based on the euclidean partition of (a,b). (It is called a "Nielsen partition" for reasons explained in [1].) *The idea is that if*

$$(a,b) = (a',b') + (a'',b'')$$

is a euclidean partition, then the step-word has a (Nielsen) partition

$$(A,B)^{a,b} = (A,B)^{a',b'} \cdot (A,B)^{a'',b''}.$$

For example, since $(5,7) = (3,4) + (2,3)$, we obtain the partition:

$$ABABAB^2ABAB^2 = ABABAB^2 \cdot ABAB^2.$$

The justification is that the triangle bounded by the (integral) lattice points $(0,0)$, (a',b'), (a,b) has no lattice points in its interior and lies below the line $y = bx/a$ (since $ab' - ba' = -1$). Hence the step-function for $y = bx/a$ agrees with that of $y = b'x/a'$ for $0 \le x \le a'$ and agrees with that of the segment from (a',b') to (a,b) (of slope b''/a''), for

$$a' \le x \le a' + a'' = a.$$

Inductive property of Nielsen partituons. Let (w_0', w_0'') be a pair of words in the Markoff free. Assume that if $(w_0', w_0'')^+$ occurs, then $(w_0', w_0'') \sim (w', w'')$ with $w = w'w''$ a Nielsen partition (of step-words), and also assume that if $(w_0', w_0'')^-$ occurs, then $(w_0', w_0'') \sim (w', w'')$ with $w = w'w''$ a Nielsen partition (of step-words). *Then, the same property is hereditary to the next stage of the tree.*

The property is almost immediate, the only difficulty is in the order of the words. If we have $(w_0', w_0'')^+$ then if $(w_0', w_0'') \sim (w', w'')$ then $(w_0', w_0'', w_0'w_0'') \sim (w', w'', w'w'')$, so the property passes on to $(w_0', w_0'w_0'')^+$. On the other hand, $(w_0'', w_0', w_0'')^- \sim (w_0'', w_0''w_0')^+$, (using "$T$" = w_0''). Hence the property passes on to $(w_0'', w_0', w_0'w_0'')^-$ as well! The rest of the details are left to the reader.

7. MAIN THEOREM

If $w(A,B)$ is a word in the Markoff tree (with the coordinates $\{a, b\}$), then $a \geq 0$, $b \geq 0$, gcd $(a,b) = 1$, and

$$w(A,B) \sim (A,B)^{a,b}.$$

Conversely, for every pair (a,b) satisfying the above conditions, a representative $w(A,B)$ occurs in the Markoff tree.

The proof is a direct consequence of the inductive property of the euclidean partition and the Nielsen partition. Clearly, the first stage (A,B) gives a Nielsen partition $AB = A.B$!

A strange consequence of this result is that the same proof would hold if we used the step-word as $(B,A)^{b,a}$ instead. (Basically, this is a consequence of the relation $AB \sim BA$.) Thus, since the main theorem is now very clear on obtaining *both* $(A,B)^{a,b}$ and $(B,A)^{b,a}$, we have

$$(A,B)^{a,b} \sim (B,A)^{b,a}.$$

This is an elementary fact to verify but it is *not* trivial. For instance, if $(a,b) = (5,7)$, we have

$$ABABA.B^2ABAB^2 \sim B^2ABAB^2.ABABA$$

The dot indicates the point at which cyclic permutations would begin. The reader will find it amusing to explicitly write the T for which

$$(A,B)^{a,b}\,T = T(B,A)^{b,a}.$$

[It involves the congruence $bx \equiv -1 \pmod{a}$.]

8. MARKOFF TRIPLES

In conclusion, we shall indicate (without proofs) how some basic number-theoretic work of Markoff [2] leads to Markoff trees of words of a semigroup. The central device is the equation in positive integers defining a so-called *Markoff triple* (m_1, m_2, m_3)

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3, \quad (m_i > 0).$$

This so-called *Markoff equation* is discussed in [1] in terms of its connections with many branches of mathematics.

The important fact about the Markoff triple is that if $m_1^* = 3m_2m_3 - m_1$, $m_2^* = 3m_3m_1 - m_2$, $m_3^* = 3m_1m_2 - m_3$ then additional Markoff triples are verifiable as

$$(m_1^*, m_2, m_3),\ (m_1, m_2^*, m_3),\ (m_1, m_2, m_3^*).$$

The presence of three *neighbors* is exactly the property of the Markoff tree, one neighbor is the *ancestor* of (m_1, m_2, m_3) and two neighbors are *descendents*. The point is that all solutions can be obtained from (1,1,1) by neighbor formation, and if we consider only solutions which have *unequal* m_1, m_2, m_3, they can be obtained from (1,2,5). [Its neighbors are (29,2,5), (1,13,5) and (1, 2,1), which is excluded, see the tree below.]

The connection with the semigroup S_2 arises as follows: If $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, then every word on the Markoff tree consists of a pair of matrices (w', w''). Then a general Markoff triple (of unequal m_i) is given (in some order) by

$$m_1 = \frac{1}{3} \text{ trace } w',\ m_2 = \frac{1}{3} \text{ trace } w'',\ m_3 = \frac{1}{3} \text{ trace } w'w''.$$

Since traces are equal for equivalent words, then, by the main theorem, the Markoff triple is given by step-words in a Nielsen partition $w'w'' = w$. Since the partition is unique, each triple is given by the coordinates $\{a,\ b\}$ of (say) w. The reader can verify that for $\{1,\ 1\}$, $(w', w'') = (A, B)$ and the triple (1,2,5) comes from 1/3 of the traces of A, B, and AB.

More generally, the Markoff tree of Section 3 leads to three solutions (rearranging the order so $m_1 < m_2 < m_3$):

$$\begin{tikzcd}[column sep=tiny]
 & & & \{1,\ 1\}(1,2,5) \arrow[dll] \arrow[drr] & & & \\
 & \{2,\ 1\}(1,5,13) \arrow[dl] \arrow[dr] & & & & \{1,\ 2\}(2,5,29) \arrow[dl] \arrow[dr] & \\
\{3,\ 1\}(1,13,34) & & \{3,\ 2\}(5,13,194) & & \{1,\ 3\}(2,29,169) & & \{2,\ 3\}(5,29,433)
\end{tikzcd}$$

A result which is still a troublesome conjecture (see [3]), is that there exists a unique nonnegative pair (a, b) for which the matrix

$$M = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \right)^{a,b}$$

has a given trace. Thus, m_3 (= 1/3 trace M) determines m_1, m_2 completely (if we keep $m_1 < m_2 < m_3$ as before).

REFERENCES

1. H. Cohn, "Markoff Forms and Primitive Words," *Math. Ann.* 196 (1972):8-22.
2. A. A. Markoff, "Sur les formes binaires indéfinies," *Math. Ann.* 15 (1879): 381-409.
3. D. Rosen & G. S. Patterson, Jr., "Some Numerical Evidence Concerning the Uniqueness of the Markoff Numbers," *Math. of Comput.* 25 (1971):919-921.
4. R. P. Stanley, "The Fibonacci Lattice," *The Fibonacci Quarterly* 13, No. 3 (1975):215-233.
5. G. T. Herman & P. M. B. Vitányi, "Growth Functions Associated With Biological Development," *Amer. Math. Monthly* 83 (1976):1-15. (The author thanks Mike Anshel for calling attention to production-theory.)

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S. E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-400 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the nth triangular number $n(n + 1)/2$. For which positive integers n is $T_1^2 + T_2^2 + \cdots + T_n^2$ an integral multiple of T_n?

B-401 *Proposed by Gary L. Mullen, Pennsylvania State University, Sharon, PA*

Show that $\lim\limits_{n\to\infty} [(n!)^{2n}/(n^2)!] = 0$.

B-402 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Show that $(L_nL_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$ is a Pythagorean triple.

B-403 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let $m = 5^n$. Show that $L_{2m} \equiv -2 \pmod{5m^2}$.

B-404 *Proposed by Phil Mana, Albuquerque, NM*

Let x be a positive irrational number. Let a, b, c, and d be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with r rational, implies that the denominator of r exceeds b, we call a/b a good lower approximation (GLA) for x. If $x < r < c/d$, with r rational, implies that the denominator of r exceeds d, c/d is a good upper approximation (GUA) for x. Find all the GLAs and all the GUAs for $(1 + \sqrt{5})/2$.

B-405 *Proposed by Phil Mana, Albuquerque, NM*

Prove that for every positive irrational x, the GLAs and GUAs for x (as defined in B-404) can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_nq_{n+1} = \pm 1 \quad \text{for all } n.$$

SOLUTIONS

Complementary Primes

B-376 *Proposed by Frank Kocher and Gary L. Mullen, Pennsylvania State University, University Park and Sharon, PA*

Find all integers $n > 3$ such that $n - p$ is an odd prime for all odd primes p less than n.

Solution by Paul S. Bruckman, Concord, CA

Let n be a solution to the problem, and p any odd prime less than n. Since p and $n - p$ are odd, clearly n must be even. Hence, $n \equiv 0, 2, 4 \pmod 6$. Since $4 - 3 = 6 - 5 = 8 - 7 = 1$ and 1 is not a prime, it follows that $n \neq 4$, $n \neq 6$, $n \neq 8$. Hence, $n \geq 10$.

If $n \equiv 0 \pmod 6$, then $n - 3 \equiv 3 \pmod 6$, which shows that $n - 3$ is composite and ≥ 9. Likewise, if $n \equiv 2 \pmod 6$, then $n - 5 \equiv 3 \pmod 6$, which shows that $n - 5$ is composite and ≥ 9. Finally, if $n \equiv 4 \pmod 6$, then $n - 7 \equiv 3 \pmod 6$, which is composite, *unless* $n = 10$, in which case $n - 7 = 3$, a prime. Hence, $n = 10$ is the only possible solution. Since $10 - 3 = 7$, $10 - 5 = 5$, $10 - 7 = 3$, which are all primes, $n = 10$ is indeed the only solution to the problem.

Also solved by Heiko Harborth (W. Germany), Charles Joscelyne, Graham Lord, J. M. Metzger, Bob Prielipp, E. Schmutz & M. Wachtel (Switzerland), Sahib Singh, Rolf Sonntag (W. Germany), Charles W. Trigg, Gregory Wulczyn, and the proposer.

Counting Lattice Points

B-377 *Proposed by Paul S. Bruckman, Concord, CA*

For all real numbers $a \geq 1$ and $b \geq 1$, prove that

$$\sum_{k=1}^{[a]} \left[b\sqrt{1 - (k/a)^2}\right] = \sum_{k=1}^{[b]} \left[a\sqrt{1 - (k/b)^2}\right],$$

where $[x]$ is the greatest integer in x.

Solution by J. M. Metzger, University of North Dakota, Grand Forks, ND

Each sum counts the number of lattice points in the first quadrant of

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1,$$

the first along the vertical lines, $x = 1$, $x = 2$, ..., $x = [a]$, the second along the horizontal lines, $y = 1$, $y = 2$, ..., $y = [b]$. The two counts must agree.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

Congruence Mod 3

B-378 *Proposed by George Berzsenyi, Laram University, Beaumont, TX*

Prove that $F_{3n+1} + 4^n F_{n+3} \equiv 0 \pmod 3$ for $n = 0, 1, 2, \ldots$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oskosh, WI

We shall establish that $F_{3n+1} + F_{n+3} \equiv 0 \pmod 3$ for $n = 0, 1, 2, \ldots,$ which is equivalent to the stated result because $4^n \equiv 1 \pmod 3$ for each nonnegative integer n. Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{3k+1} + F_{k+3} \equiv 0 \pmod 3$ and $F_{3k+4} + F_{k+4} \equiv 0 \pmod 3$, where k is an arbitrary nonnegative integer. Then, by addition,

$$F_{3k+1} + F_{3k+4} + F_{k+5} \equiv 0 \pmod 3.$$

But

$$6F_{3k+2} + 4F_{3k+1} + F_{3k+4} = F_{3k+7}$$

so

$$F_{3k+1} + F_{3k+4} \equiv F_{3k+7} \pmod 3.$$

Hence

$$F_{3k+7} + F_{k+5} \equiv 0 \pmod 3$$

and our proof is complete by mathematical induction.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Congruence Mod 5

B-379 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for all nonnegative integers n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, WI

Clearly the desired result holds when $n = 0$ and when $n = 1$. Assume that $F_{2k} \equiv k(-1)^{k+1} \pmod 5$ and $F_{2k+2} \equiv (k + 1)(-1)^{k+2} \pmod 5$, where k is an arbitrary nonnegative integer. Then, since

$$\begin{aligned} F_{2k+4} &= 3F_{2k+2} - F_{2k}, \\ F_{2k+4} &\equiv (3k + 3)(-1)^{k+2} - k(-1)^{k+1} \pmod 5 \\ &\equiv (-1)^{k+2}(4k + 3) \pmod 5 \\ &\equiv (k + 2)(-1)^{k+3} \pmod 5. \end{aligned}$$

Our solution is now complete by mathematical induction.

Also solved by Paul S. Bruckman, Charles Joscelyne, Graham Lord, Sahib Singh, Gregory Wulczyn, and the proposer.

Binomial Convolution

B-380 *Proposed by Dan Zwillinger, Cambridge, MA*

Let a, b, and c be nonnegative integers. Prove that

$$\sum_{k=1}^{n} \binom{k+a-1}{a}\binom{n-k+b-c}{b} = \binom{n+a+b-c}{a+b+1}.$$

Here $\binom{m}{r} = 0$ if $m < r$.

Solution by Phil Mana, Albuquerque, NM

For every nonnegative integer d, the Maclaurin series for $(1 - x)^{-d-1}$ is

$$\sum_{n=0}^{\infty} \binom{n+d}{d} x^n.$$

Then

$$(1 - x)^{-a-1}(1 - x)^{-b-1} = (1 - x)^{-a-b-2},$$

$$\sum_{i=0}^{\infty} \binom{i+a}{a} x^i \cdot \sum_{j=0}^{\infty} \binom{j+b}{b} x^j = \sum_{n=0}^{\infty} \binom{n+a+b+1}{a+b+1} x^n.$$

Equating coefficients of x^{n-c-1} on both sides, one has

$$\sum_{k=1}^{n-c} \binom{k-1+a}{a}\binom{n-c-k+b}{b} = \binom{n-c+a+b}{a+b+1}$$

The upper limit $n - c$ for the sum here can be replaced by n, since any terms for $n - c < k \leq n$ will vanish using the convention that $\binom{m}{r} = 0$ for $m < r$. This gives the desired result.

Also solved by Paul S. Bruckman, Bob Prielipp & N. J. Kuenzi, A. G. Shannon, and the proposer.

Generating Function

B-381 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $a_{2n} = F_{n+1}^2$ and $a_{2n+1} = F_{n+1}F_{n+2}$. Find the rational function that has

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

as its Maclaurin series.

Solution by Sahib Singh, Clarion State College, Clarion, PA

By the result $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$, we get the Mclaurin series as:

$$F_1^2 + F_1^2x(1 + x^2 + x^4 + \cdots) + F_2^2X_2^2 + F^2X^3(1 + x^2 + x^4 + \cdots) + \cdots$$

$$= F_1^2\left(1 + \frac{x}{1 - x^2}\right) + F_2^2X^2\left(1 + \frac{x}{1 - x^2}\right) + F_3^2X^4\left(1 + \frac{x}{1 - x^2}\right) + \cdots$$

$$= \frac{1 + x - x^2}{1 - x^2}\left[F_1^2 + F_2^2X^2 + F_3^2X^4 + F_4^2X^6 + \cdots\right].$$

Using $F_n^2 = \left(\frac{a^n - b^n}{a - b}\right)^2$, the above becomes

$$\left(\frac{1 + x - x^2}{1 - x^2}\right) \cdot \frac{1}{(a - b)^2}\Big[(a^2 + a^4x^2 + a^6x^4 + \cdots)$$
$$+ (b^2 + b^4x^2 + b^6x^4 + \cdots) - 2ab(1 + abx^2 + a^2b^2x^4 + \cdots)\Big]$$
$$= \left(\frac{1 + x - x^2}{1 - x^2}\right) \cdot \frac{1}{(a - b)^2}\left[\frac{a^2}{1 - a^2x^2} + \frac{b^2}{1 - b^2x^2} - \frac{2ab}{1 - abx^2}\right],$$

which simplifies to

$$\left(\frac{1 + x - x^2}{1 - x^2}\right)\left(\frac{(1 - x^2)}{(1 + x^2)(1 - 3x^2 + x^4)}\right) = \frac{1 + x - x^2}{(1 + x^2)(1 - 3x^2 + x^4)}.$$

Also solved by Paul S. Bruckman, R. Garfield, John W. Vogel, and the proposer.

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ERRATA

The following errors have been noted:

Volume 16, No. 5 (October 1978), p. 407 [J. A. H. Hunter's "Congruent Primes of Form $(8r+1)$"]. The equations presented in the second line of the article should read

$$X^2 - eY^2 = Z^2, \text{ and } X^2 + eY^2 = W^2.$$

Volume 17, No. 1 (February 1979), p. 84 (A. P. Hillman & V. E. Hoggatt, Jr.'s "Nearly Linear Functions"). Equation (1) should read

$$C' \cdot H - C \cdot H = \sum_{i=1}^{k} (c_i' - c_i)h_i \geq h_k - \sum_{i=1}^{k-1} c_i h_i. \tag{1}$$

The second line of the proof of Lemma 7 should read

The hypothesis $E \cdot E' = 0$ implies . . . .

In the proof of Theorem 1, Equation (10) should read

$$b_j(m) = C^*_{m-1} \cdot H_j - C_{m-1} \cdot H_j. \tag{10}$$

(Kindness of Margaret Owens)

ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-299 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Vandermonde determinants:

(A) Evaluate

$$\Delta = \begin{vmatrix} F_{2r} & F_{6r} & F_{10r} & F_{14r} & F_{18r} \\ F_{4r} & F_{12r} & F_{20r} & F_{28r} & F_{36r} \\ F_{6r} & F_{18r} & F_{30r} & F_{42r} & F_{54r} \\ F_{8r} & F_{28r} & F_{40r} & F_{56r} & F_{72r} \\ F_{10r} & F_{30r} & F_{50r} & F_{70r} & F_{90r} \end{vmatrix}$$

(B) Evaluate

$$D = \begin{vmatrix} 1 & L_{2r+1} & L_{4r+2} & L_{6r+3} & L_{8r+4} \\ 1 & -L_{6r+3} & L_{12r+6} & L_{18r+9} & L_{24r+12} \\ 1 & L_{10r+5} & L_{20r+10} & L_{30r+15} & L_{40r+20} \\ 1 & -L_{14r+7} & L_{28r+14} & -L_{42r+21} & L_{56r+28} \\ 1 & L_{18r+9} & L_{36r+18} & L_{54r+27} & L_{72r+36} \end{vmatrix}$$

(C) Evaluate

$$D_1 = \begin{vmatrix} 1 & L_{2r} & L_{4r} & L_{6r} & L_{8r} \\ 1 & L_{6r} & L_{12r} & L_{18r} & L_{24r} \\ 1 & L_{10r} & L_{20r} & L_{30r} & L_{40r} \\ 1 & L_{18r} & L_{36r} & L_{54r} & L_{72r} \end{vmatrix}$$

H-300 *Proposed by James L. Murphy, California State College, San Bernardino, CA*

Given two positive integers A and B relatively prime, form a "multiplicative" Fibonacci sequence $\{A_i\}$ with $A_1 = A$, $A_2 = B$, and $A_{i+2} = A^*A_{i+1}$. Now form the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{i=1}^{n} A_i.$$

$\{S_n\}$ is a subsequence of the arithmetic sequence $\{T_n\}$ where

$$T_n = A + nB,$$

and by Dirichlet's theorem we know that infinitely many of the T_n are prime. The question is: Does such a sparse subsequence $\{S_n\}$ of the arithmetic sequence $A + nB$ also contain infinitely many primes?

Notes:

$S_1 = A$, $S_2 = A + B$, $S_3 = A + B + AB$,

$S_4 = A + B + AB + AB^2$, $S_5 = A + B + AB + AB^2 + A^2B^3$, etc.

Some examples:

For $A = 2$ and $B = 3$, the first few S_i are:

2, 5, 11, 29, 137, 2081, all prime, and

$S_7 = 212033 = 43*4931$.

For $A = 3$ and $B = 14$, the first few S_i are:

3, 17, 59, 647, 25343, 14546591, all prime, and

$S_7 = 358631287199 = 43*8340262493$.

For $A = 2$ and $B = 21$, the first few S are:

2, 23, prime; $S_3 = 65$, a composite; but

$S_4 = 947$ and $S_5 = 37881$, both prime.

Looking at the first six terms of the sequence $\{S_i\}$ for 68 different choices of A and B, I found the following distribution:

Number of Primes in the First Six Terms	Number of Sequences Having This Number of Primes
1	2
2	19
3	21
4	22
5	2
6	2
	68

H-301 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $A_0, A_1, A_2, \ldots, A_n, \ldots$ be a sequence such that the nth differences are zero (that is, the Diagonal Sequence terminates). Show that, if

$$A(x) = \sum_{i=0}^{\infty} A_i x^i,$$

then

$$A(x) = \frac{1}{1-x} \cdot D\left(\frac{x}{1-x}\right), \quad \text{where} \quad D_n(x) = \sum_{i=0}^{\infty} d_i x^i.$$

SOLUTIONS

Pell Mell

H-275 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let P_n denote the Pell Sequence defined as follows:

$$P_1 = 1,\ P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n \quad (n \geq 1).$$

Consider the array below:

$$\begin{array}{ccccccccccc}
1 & & 2 & & 5 & & 12 & & 29 & & 70\ \ldots\ (P_n) \\
& 1 & & 3 & & 7 & & 17 & & 41\ \ldots & \\
& & 2 & & 4 & & 10 & & 24\ \ldots & & \\
& & & 2 & & 6 & & 14\ \ldots & & & \\
& & & & 4 & & 8\ \ldots & & & & \\
& & & & & 4 & & & & &
\end{array}$$

Each row is obtained by taking differences in the row above.

Let D_n denote the left diagonal sequence in this array; i.s.,

$$D_1 = D_2 = 1,\ D_3 = D_4 = 2,\ D_5 = D_6 = 4,\ D_7 = D_8 = 8,\ \ldots\ .$$

(i) Show $D_{2n-1} = D_{2n} = 2^{n-1}$ $(n \geq 1)$.

(ii) Show that if $F(x)$ represents the generating function for $\left\{P_n\right\}_{n=1}^{\infty}$ and $D(x)$ represents the generating function for $\left\{D_n\right\}_{n=1}^{\infty}$, then

$$D(x) = F\left(\frac{x}{1+x}\right).$$

Solution by George Berzsenyi, Lamar University, Beaumont, TX

First observe that each row in the array inherits the recursive relation of the Pell numbers. This is true more generally, for if $\{x_n\}$ is a sequence defined recursively by

$$x_{n+2} = \alpha x_{n+1} + \beta x_n$$

and if $\{y_n\}$ is defined by

$$y_n = x_{n+1} - x_n,$$

then

$$\begin{aligned} y_{n+2} &= x_{n+3} - x_{n+2} = \alpha(x_{n+2} - x_{n+1}) + \beta(x_{n+1} - x_n) \\ &= \alpha y_{n+1} + \beta y_n. \end{aligned}$$

Let E_n be the second diagonal sequence in the array; i.e.,

$$E_1 = 2,\ E_2 = 3,\ E_3 = 4,\ E_4 = 6,\ E_5 = 8,\ \ldots\ .$$

We shall prove by induction that for each $n = 1, 2, \ldots, D_{2n-1} = D_{2n} = 2^{n-1}$, while $E_{2n-1} = 2 \cdot 2^{n-1}$ and $E_{2n} = 3 \cdot 2^{n-1}$. The portion of the array shown exhibits this fact for $n = 1$; assume it for $n = k$. Then the first few members of the $2k - 1$st and $2k$th rows can be obtained by using the recursion formula and upon taking differences one obtains the first two members of the next two rows as follows:

$$\begin{array}{ccccccccc}
2^{k-1} & & 2\cdot 2^{k-1} & & 5\cdot 2^{k-1} & & 12\cdot 2^{k-1} & & 29\cdot 2^{k-1} \\
& 2^{k-1} & & 3\cdot 2^{k-1} & & 7\cdot 2^{k-1} & & 17\cdot 2^{k-1} & \\
& & 2^k & & 2\cdot 2^k & & 5\cdot 2^k & & \\
& & & 2^k & & 3\cdot 2^k & & &
\end{array}$$

This completes the induction and establishes part (i).

To prove part (ii), recall that

$$F(x) = \frac{x}{1 - 2x - x^2},$$

and therefore,

$$F\left(\frac{x}{1+x}\right) = \frac{x + x^2}{1 - 2x^2}.$$

On the other hand, if

$$D(x) = \sum_{n=1}^{\infty} D_n x^n,$$

then

$$D(x) = (x + x^2) + 2(x^3 + x^4) + 2^2(x^5 + x^6) + \cdots$$

while

$$-2x^2 D(x) = -2(x^3 + x^4) - 2^2(x^5 + x^6) - \cdots .$$

Hence, $(1 - 2x^2)D(x) = x + x^2$, and

$$D(x) = \frac{x + x^2}{1 - 2x^2}.$$

Consequently the desired relationship, $D(x) = F\left(\frac{x}{1+x}\right)$ follows.

Also solved by V. E. Hoggatt, Jr., P. Bruckman, G. Wulczyn, and A. Shannon.

Late Acknowledgment: P. Bruckman solved H-274.

HELP WANTED

S E N D I N

P R O P O S A L S

A N D

S O L U T I O N S!

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