MORE IN THE THEORY OF SEQUENCES

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INTRODUCTION

Cauchy gave a necessary condition for the convergence of an infinite series,

$$\sum_{k=1}^{\infty} a(k);$$

namely, that the sequence $(a(n))$ converges to zero as n tends to infinity.

Olivier proved a variation of this theorem, which has, in a sense, generated more interest: Let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers, tending to zero, such that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a(k)$$

exists, then $\lim\limits_{n\to\infty} n \cdot a(n) = 0$.

For one thing, Olivier's theorem allows for extensions in several directions [4]. Niven and Zuckerman, for instance, have proved the following theorem [5]:

Theorem 1: Let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers. Then

$$\lim_{n\to\infty} \sum_{k=n+1}^{[\lambda n]} a(k) \tag{1}$$

exists for each $\lambda > 1$, if and only if $\lim\limits_{n\to\infty} n \cdot a(n)$ exists.

Clearly, Niven and Zuckerman's condition for the convergence of

$$(n \cdot a(n))$$

is weaker than that of Olivier. On the other hand, they have given a necessary and sufficient condition for the convergence of

$$\left(\sum_{k=n+1}^{[\lambda n]} a(k)\right).$$

In this paper, Olivier's theorem will be extended further in this same direction. We consider a sequence of positive numbers $(\emptyset(n))$ (as yet unspecified) and a monotonic nonincreasing sequence of positive numbers $(a(n))$, such that

$$\lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k)$$

exists for every $\lambda > 1$. We will show that $\lim\limits_{n\to\infty} \dfrac{n \cdot a(n)}{\emptyset(n)}$ exists.

When $\emptyset(n) = 1$, $n = 1, 2, 3, \ldots,$ the problem reduces to the case considered by Niven and Zuckerman. But more generally, as we will prove, $(\emptyset(n))$ can be any regularly varying sequence, i.e., any sequence of positive numbers which satisfies

$$\lim_{n\to\infty} \frac{\emptyset([\lambda n])}{\emptyset(n)} = \psi(\lambda) \text{ for every } \lambda > 0, \tag{2}$$

where $\psi(\lambda) = \lambda^{\rho}$, where the index ρ is real.

We summarize this result in Theorem 2.

Theorem 2: Let $(\emptyset(n))$ be a regularly varying sequence and let $(a(n))$ be a monotonic nonincreasing sequence of positive numbers. Then

$$\lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = H(\lambda) \tag{3}$$

exists for each $\lambda > 1$, if and only if $\lim\limits_{n\to\infty} \dfrac{n \cdot a(n)}{\emptyset(n)}$ exists.

Proof: Let

$$H(\lambda) = \lim_{n\to\infty} H_n(\lambda) = \lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k).$$

For each integer $m > \lambda$, let $n = [m/\lambda]$ in $H_n(\lambda)$ and let $r = m - [n\lambda]$. Since $0 = m - m/\lambda \cdot \lambda \le m - n \cdot \lambda$, we have $m \ge n\lambda = [n\lambda]$. Also,

$$0 \le r = m - [n\lambda] < m - (n\lambda - 1) < m - (m/\lambda - 1) = \lambda + 1.$$

Since

$$H_n(\lambda) \ge \frac{([n\lambda] - n) \cdot a([n\lambda])}{\emptyset(n)} \ge \frac{([n\lambda] - n) \cdot a(m)}{\emptyset(n)}$$

and

$$\frac{[n\lambda] + r}{[n\lambda] - n} \le \frac{n\lambda + \lambda + 1}{n\lambda - 1 - n} \le \frac{m + \lambda + 1}{(m/\lambda - 1)\lambda - 1 - n} \le \frac{m + \lambda + 1}{m - \lambda - 1 - m/\lambda},$$

we have

$$\frac{m \cdot a(m)}{\emptyset(n)} = \frac{m \cdot a(m)}{\emptyset(n)} \cdot \frac{\emptyset(n)}{\emptyset(m)} \le \frac{[n\lambda] + r}{[n\lambda] - n} \cdot H_n(\lambda) \cdot \frac{\emptyset[m/\lambda]}{\emptyset(m)}$$

$$\le \frac{m + \lambda + 1}{m - \lambda - 1 - m/\lambda} \cdot H_n(\lambda) \cdot \frac{\emptyset([m/\lambda])}{\emptyset(m)}.$$

Hence, by (2),

$$\lim_{n\to\infty} \sup \frac{m \cdot a(m)}{\emptyset(m)} \le \frac{\lambda}{\lambda - 1} \cdot H(\lambda) \cdot (1/\lambda)^{\rho}. \tag{4}$$

We assert that

$$\lim_{n\to\infty} \frac{A([\lambda\mu n]) - A([\mu n])}{\emptyset([\mu n])} = H(\lambda),$$

where $\lambda > 1$, $\mu > 0$, and

$$A([\lambda n]) = \sum_{k=1}^{[\lambda n]} a(k).$$

It is sufficient to show

$$\lim_{n\to\infty} \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k) = 0,$$

since

$$\frac{A([\lambda\mu n]) - A([\mu n])}{\phi([\mu n])} = H_{[\mu n]}(\lambda) + \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k).$$

Clearly, by (2) and (4),

$$\overline{\lim_{n\to\infty}} \frac{1}{\phi([\mu n])} \sum_{k=(\lambda[\mu n])+1}^{[\lambda\mu n]} a(k)$$

$$= \overline{\lim_{n\to\infty}} \frac{\phi(\lambda[\mu n])}{\phi([\mu n])} \cdot \frac{([\lambda] + 2)}{[\lambda[\mu n]]} \cdot \frac{[\lambda[\mu n]]a\big([\lambda[\mu n]]\big)}{\phi\big([\lambda[\mu n]]\big)} = 0,$$

so our assertion is proved.

Therefore, we have

$$(5) \qquad H(\lambda\mu) = H(\lambda)\mu^{\rho} + H(\mu),$$

since

$$H_n(\lambda\mu) = \frac{A([\lambda\mu n]) - A([\mu n])}{\phi([\mu n])} \cdot \frac{\phi([\mu n])}{\phi(n)} + \frac{A([\mu n]) - A(n)}{\phi(n)}.$$

Interchanging μ with λ in (5) and manipulating the equations simultaneously, we have, if $\rho \neq 0$, $H(\mu)/\mu^{\rho} - 1 = H(\lambda)/\lambda^{\rho} - 1 = A$, A a constant, which implies

$$H'(1) = \lim_{\lambda\to 1^+} \frac{H(\lambda)}{\lambda - 1} = \lim_{\lambda\to 1^+} \frac{H(\lambda)}{\lambda^{\rho} - 1} \cdot \lim_{\lambda\to 1^+} \frac{\lambda^{\rho} - 1}{\lambda - 1} = A \cdot \rho,$$

or

$$(6) \qquad H(\lambda) = \frac{H'(1)}{\rho}(\lambda^{\rho} - 1).$$

If $\rho = 0$, then $H(\lambda\mu) = H(\lambda) + H(\mu)$. Since $H(\cdot)$ is monotonic increasing, $H(\cdot)$ has a point of continuity and it is not hard to show $H(\cdot)$ is continuous on $[1,\infty]$. Hence $H(\cdot)$ is of the form

$$(7) \qquad H(\lambda) = H'(1) \log \lambda.$$

Since

$$H(\lambda) \leq \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)} \cdot \frac{([\lambda n] - n)}{n} = (\lambda - 1) \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)},$$

we have

$$H'(1) = \lim_{\lambda\to 1} \frac{H(\lambda)}{\lambda - 1} = \lim_{n\to\infty} \frac{n \cdot a(n)}{\phi(n)}.$$

On the other hand, as a consequence of (4), we have

$$\frac{H(\lambda)}{\lambda^p} = \lim_{n\to\infty} \frac{A(n) - A([n/\lambda])}{\emptyset(n)} \geq \lim_{n\to\infty} \sup \frac{(n - [n/\lambda])}{n} \cdot \frac{n \cdot a(n)}{\emptyset(n)}.$$

Therefore, from (6) and (7),

$$H'(1) = \lim_{\lambda\to 1^+} \sup \frac{H(\lambda)}{\lambda^\rho(1 - 1/\lambda)} = \lim_{n\to\infty} \sup \frac{n \cdot a(n)}{\emptyset(n)}.$$

Hence,

$$\lim_{n\to\infty} \frac{n \cdot a(n)}{\emptyset(n)} = H'(1).$$

We now prove the converse.

Definition: Let $f(x)$ be a real valued, measurable function which satisties

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho$$

for every $\lambda > 0$. Then $f(x)$ is a regularly varying function of index ρ.

Every regularly varying function $f(x)$ of index ρ can be written as

$$f(x) = \lambda^\rho L(x) \tag{8}$$

where $L(x)$ is regularly varying of index 0 (slowly varying). (See [2].)

Lemma 1: Let $(\emptyset(n))$ be a regularly varying sequence of index ρ, then the function $f(x)$ defined by

$$f(x) = \emptyset([x])$$

is a regularly varying function of index ρ.

Lemma 2: If $L(x)$ is a slowly varying function, then for every $[a,b]$, $0 < a < b < \infty$, the relation

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds uniformly with respect to $x \in [a,b]$.

Lemma 2, known as the Uniform Convergence Theorem for slowly varying functions, has been proved by several persons. A nice proof is given in [1] by Bojanic and Seneta. Lemma 1 is proved by the author in [3].

By hypothesis,

$$\lim_{k\to\infty} \frac{k \cdot a(k)}{\emptyset(k)} = C.$$

Also, by (8), $\emptyset(k)$ can be written as

$$\emptyset(k) = k^\rho L(k),$$

where $L(k)$ is slowly varying. Therefore, $(a(k))$ can be written as

$$a(k) = C(k)k^{\rho-1}L(k),$$

where $\lim_{k\to\infty} C(k) = C$.

Consequently, for n sufficiently large,

$$\frac{(C-\varepsilon)}{n^{\rho}} \min_{n\le k\le[\lambda n]} \frac{L(k)}{L(n)} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1} \le \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k)$$
$$\le \frac{(C-\varepsilon)}{n^{\rho}} \max_{n\le k\le[\lambda n]} \frac{L(k)}{L(n)} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1},$$

where $\varepsilon > 0$.

Clearly,

$$\min_{n\le k\le[\lambda n]} \frac{L(k)}{L(n)} = \min_{1\le k'\le\lambda} \frac{L(k'n)}{L(n)}$$

and

$$\max_{n\le k\le[\lambda n]} \frac{L(k)}{L(n)} = \max_{1\le k'\le\lambda} \frac{L(k'n)}{L(n)}.$$

By Lemmas 1 and 2, we have

$$\lim_{n\to\infty} \min_{1\le k'\le\lambda} \frac{L(k'n)}{L(n)} = 1 = \overline{\lim_{n\to\infty}} \max_{1\le k'\le\lambda} \frac{L(k'n)}{L(n)}.$$

Therefore,

$$\frac{(C-\varepsilon)}{n^{\rho}} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1} = \lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \overline{\lim_{n\to\infty}} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \frac{(C+\varepsilon)}{n^{\rho}} \sum_{k=n+1}^{[\lambda n]} k^{\rho-1}.$$

On the other hand,

$$\sum_{k=n+1}^{[\lambda n]} k^{\rho-1} \simeq \begin{cases} \dfrac{(\lambda^{\rho}-1)n^{\rho}}{\rho} & \text{if } \rho \ne 0 \\ \log\lambda & \text{if } \rho = 0 \end{cases}$$

Hence, letting $\varepsilon \to 0$, we have

$$\lim_{n\to\infty} \frac{1}{\emptyset(n)} \sum_{k=n+1}^{[\lambda n]} a(k) = \begin{cases} \dfrac{C(\lambda^{\rho}-1)}{\rho} & \text{if } \rho \ne 0 \\ C\log\lambda & \text{if } \rho = 0 \end{cases}$$

and the converse is proved.

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A GENERALIZATION OF WYTHOFF'S GAME

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Wythoff's game is a variation of Nim, a two-pile game in which each player removes counters in turn until the winner takes the last counter. The safe-pairs generated in the solution of Wythoff's game have many properties interesting in themselves, and are related to the canonical Zeckendorf representation of an integer using Fibonacci numbers. In Nim, the strategy is related to expressing the numbers in each pile in binary notation, or representing them by powers of 2. Here, the generalized game provides number sequences related to the canonical Zeckendorf representation of integers using Lucas numbers.

1. INTRODUCTION: WYTHOFF'S GAME

Wythoff's game is a two-pile game where each player in turn follows the rules:

(1) At least one counter must be taken;
(2) Any number of counters may be removed from one pile;
(3) An equal number of counters may be removed from each pile;
(4) The winner takes the last counter.

The strategy is to control the number of counters in the two piles to have a safe position, or one in which the other player cannot win. Wythoff devised a set of "out of a hat" safe positions

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ \ldots,\ (a_n,b_n).$$

It was reported by W. W. Rouse Ball [1] that

$$a_n = [n\alpha] \quad \text{and} \quad b_n = [n\alpha^2] = a_n + n,$$

where α is the Golden Section Ratio, $\alpha = (1+\sqrt{5})/2$, and $[n]$ is the greatest integer not exceeding n.

More recently, Nim games have been studied by Whinihan [2] and Schwenk [3], who showed that the safe positions were found from the unique Zeckendorf representation of an integer using Fibonacci numbers, but did not consider properties of the number pairs themselves. Properties of Wythoff pairs have been discussed by Horadam [4], Silber [5], [6], and Hoggatt and Hillman [7].

For completeness, we will list the first forty Wythoff pairs and some of their properties that we will generalize. Also, we will denote the Fibonacci numbers by F_n, where $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, and the Lucas numbers by L_n, $L_n = F_{n-1} + F_{n+1}$. (See Table 1.)

Generation of Wythoff pairs:

I. Begin with $a_1 = 1$. Always take $b_n = a_n + n$, and take a_k as the smallest integer not yet appearing in the table.

II. Let $B = \{b_n\}$ and $A = \{a_n\}$. Then A and B are disjoint sets whose union is the set of positive integers, and A and B are self-generating. B is generated by taking $b_1 = 2$ and $b_{n+1} = b_n + 2$ if $n \in B$ or $b_{n+1} = b_n + 3$ if $n \notin B$. A is generated by taking $a_1 = 1$ and $a_{n+1} = a_n + 2$ if $n \in A$ or $a_{n+1} = a_n + 1$ if $n \notin A$.

TABLE 1

THE FIRST FORTY WYTHOFF PAIRS

n	a_n	b_n	n	a_n	b_n	n	a_n	b_n	n	a_n	b_n
1	1	2	11	17	28	21	33	54	31	50	81
2	3	5	12	19	31	22	35	57	32	51	83
3	4	7	13	21	34	23	37	60	33	53	86
4	6	10	14	22	36	24	38	62	34	55	89
5	8	13	15	24	39	25	40	65	35	56	91
6	9	15	16	25	41	26	42	68	36	58	94
7	11	18	17	27	44	27	43	70	37	59	96
8	12	20	18	29	47	28	45	73	38	61	99
9	14	23	19	30	49	29	46	75	39	63	102
10	16	26	20	32	52	30	48	78	40	64	104

Properties of Wythoff pairs:

(1.1) $a_k + k = b_k$

(1.2) $a_n + b_n = a_{b_n}$

(1.3) $a_{a_n} + 1 = b_n$

(1.4) $a_n = 1 + \delta_3 F_3 + \delta_4 F_4 + \cdots + \delta_k F_k$, where $\delta_i \in \{0, 1\}$

(1.5) $b_n = 2 + \delta_4 F_4 + \delta_5 F_5 + \cdots + \delta_m F_m$, where $\delta_i \in \{0, 1\}$

(1.6) $a_{a_n+1} - a_{a_n} = 2$ and $a_{b_n+1} - a_{b_n} = 1$

(1.7) $b_{a_n+1} - b_{a_n} = 3$ and $b_{b_n+1} - b_{b_n} = 2$

(1.8) $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$

2. GENERALIZED WYTHOFF NUMBERS

First, we construct a table of numbers which are generalizations of the safe Wythoff pair numbers (a_n, b_n) of Section 1. We let $A_n = 1$, and take

$$B_n = A_n + d_n, \text{ where } d_n \neq B_k + 1$$

(that is, $d_{n+1} = d_n + 1$ when $d_n \neq B_k$ or $d_{n+1} = d_n + 2$ when $d_n = B_k$, and $d_1 = 2$). Notice that before, $b_n = a_n + n$; here, we are removing any integer that is expressible by $B_k + 1$. We let $C_n = B_n - 1$. To find successive values of A_n, we take A_n to be the smallest integer not yet used for A_i, B_i, or C_i in the table. We shall find many applications of these numbers, and also show that they are self-generating. In Table 2, we list the first twenty values.

We next derive some properties of the numbers A_n, B_n, and C_n. First, A_n, B_n, and C_n can all be expressed in terms of the numbers a_n and b_n of the Wythoff pairs from Section 1. Note that A_{2k} is even, and A_{2k+1} is odd, an obvious corollary of Theorem 2.1.

Theorem 2.1:

(i) $A_n = 2a_n - n$;

(ii) $B_n = a_n + 2n = b_n + n = a_{a_n} + 1 + n$;

(iii) $C_n = a_n + 2n - 1 = b_n + n - 1 = a_{a_n} + n$.

TABLE 2

THE FIRST TWENTY GENERALIZED WYTHOFF NUMBERS

n	A_n	B_n	d_n	C_n	n	A_n	B_n	d_n	C_n
1	1	3	2	2	11	23	39	16	38
2	4	7	3	6	12	26	43	17	42
3	5	10	5	9	13	29	47	18	46
4	8	14	6	13	14	30	50	20	49
5	11	18	7	17	15	33	54	21	53
6	12	21	9	20	16	34	57	23	56
7	15	25	10	24	17	37	61	24	60
8	16	28	12	27	18	40	65	25	64
9	19	32	13	31	19	41	68	27	67
10	22	36	14	35	20	44	72	28	71

Proof of Theorem 2.1: First, we prove (ii) and (iii). Consider the set of integers $\{1, 2, 3, \ldots, B_n\}$, which contains n Bs and n Cs, since $C_n = B_n - 1$, and j As, where $A_j = B_n - 2$. Since A, B, and C are disjoint sets, B_n is the sum of the number of Bs, the number of Cs, and the number of As, or,

$$B_n = n + n + j = 2n + j,$$

$$C_n = 2n - 1 + j.$$

Note that

$$A_{a_n} = C_n - 1 = B_n - 2, \text{ for } n = 1, 2, 3, 4, 5.$$

Assume that $A_{a_n} = C_n - 1$, or that the number of As less than B_n is $j = a_n$.

$$A_{a_n} = C_n - 1$$

$$A_{a_n} + 1 = C_n \neq A_{a_n+1}$$

$$A_{a_n} + 2 = C_n + 1 = B_n \neq A_{a_n+1}$$

$$A_{a_n} + 3 = C_n + 2$$

but $A_{a_n} + 3 = A_{a_n+1}$, since the As differ by 1 or 3 and they do not differ by 1; and $C_n + 2 = C_{n+1} - 1$ or $C_n + 2 = C_{n+1} - 2$ since the Cs differ by 3 or 4. If $A_{a_n+1} = C_{n+1} - 1$ and $a_n + 1 = a_{n+1}$, we are through; this occurs only when $n = b_k$ by (1.6). If $n \neq b_k$, then $n = a_k$. Note that the As differ by 1 or 3. Since $A_{a_n} + 3 = A_{a_n+1}$, $A_{b_k} + 3 \neq A_{b_k+1}$, since $a_n \neq b_k$ for any k. Thus,

$$A_{b_k} + 1 = A_{b_k+1}.$$

Now, if $A_{a_n+1} = C_{n+1} - 2$, then

$$A_{a_n+1} + 1 = C_{n+1} - 1.$$

But, $n = a_k$, so $a_{a_k} + 1 = b_k = a_n + 1$, so

$$A_{a_n+1} + 1 = A_{b_k} + 1 = A_{b_k+1} = A_{a_n+2} = A_{a_{n+1}}$$

by (1.6), so that again $A_{a_{n+1}} = C_{n+1} - 1$. Thus, by the axiom of mathematical induction, $j = a_n$, and we have established (ii) and (iii).

Now, we prove (i). Either $A_{k+1} = A_k + 1$ or $A_{k+1} = A_k + 3$. From Table 2, observe that $A_1 = 2a_1 - 1$ and $A_2 = 2a_2 - 2$. Assume that $A_k = 2a_k - k$. When $k = b_j$, we have

$$\begin{aligned} A_{k+1} &= A_k + 1 \\ &= (2a_k - k) + 1 \\ &= 2(a_k + 1) - (k + 1) \\ &= 2a_{k+1} - (k + 1) \end{aligned}$$

by (1.6). If $k \neq b_j$, then $k = a_j$, and we have

$$\begin{aligned} A_{k+1} &= A_k + 3 \\ &= (2a_k - k) + 3 \\ &= 2(a_k + 2) - (k + 1) \\ &= 2a_{k+1} - (k + 1) \end{aligned}$$

again by (1.6), establishing (i) by mathematical induction.

Following immediately from Theorem 2.1, and from its proof, we have

Theorem 2.2:

(i) $A_{b_n+1} - A_{b_n} = 1$ and $A_{a_n+1} - A_{a_n} = 3$

(ii) $B_{b_n+1} - B_{b_n} = 3$ and $B_{a_n+1} - B_{a_n} = 4$

(iii) $C_{b_n+1} - C_{b_n} = 3$ and $C_{a_n+1} - C_{a_n} = 4$

(iv) $A_{a_n} = a_n + 2n - 2$

(v) $A_{a_n} = C_n - 1$

Theorem 2.3:

$$B_n = [n\alpha\sqrt{5}],\ C_n = [n\alpha\sqrt{5}] - 1, \text{ and } A_n = 2[n\alpha] - n,$$
where $\alpha = (1 + \sqrt{5})/2$.

Proof: By Theorem 2.1 and property (1.8),

$$\begin{aligned} B_n &= a_n + 2n = [n\alpha] + 2n = [n(\alpha + 2)] \\ &= [n(5 + \sqrt{5})/2] = [n\alpha\sqrt{5}]. \end{aligned}$$

$$A_n = 2a_n - n = 2[n\alpha] - n.$$

Theorem 2.4:

(i) $B_m + B_n \neq A_j$

(ii) $C_m + C_n \neq C_j$

(iii) $B_m + C_n \neq B_j$

(iv) $A_m + B_n \neq C_j$

Proof: (ii) was proved by A. P. Hillman [8] as follows. Let $C_r = a_r + 2r - 1$ and $C_s = a_s + 2s - 1$. Then

$$C_r + C_s = a_r + a_s + 2(r + s) - 2 = C_{r+s} + (a_r + a_s - a_{r+s} - 1),$$

but $a_r = [r\alpha]$ and $a_s = [s\alpha]$ and $a_{r+s} = [(r+s)\alpha]$. However

$$[x] + [y] - [x + y] = 0 \text{ or } -1,$$

so that

$$a_r + a_s - a_{r+s} - 1 = -1 \text{ or } -2,$$

making

$$C_r + C_s = C_{r+s} - 1 \quad \text{or} \quad C_r + C_s = C_{r+s} - 2,$$

but members of the sequence $\{C_k\}$ have differences of 3 or 4 only, so

$$C_{r+s} - 1 \neq C_k, \text{ and } C_{r+s} - 2 \neq C_k, \text{ for any } k.$$

The proof of (i) is similar:

$$\begin{aligned} B_m + B_n &= a_m + 2m + a_n + 2n \\ &= a_m + a_n + 2(m + n) \\ &= a_m + a_n + (b_{m+n} - a_{m+n}) + (m + n) \\ &= (a_m + a_n - a_{m+n}) + \big(b_{m+n} + (m + n)\big) \\ &= (0 \text{ or } -1) + B_{m+n}. \end{aligned}$$

Thus,

$$B_m + B_n = B_{m+n} \quad \text{or} \quad B_m + B_n = B_{m+n} - 1 = C_{m+n},$$

and

$$B_m + B_n \neq A_j, \text{ for any } j.$$

Now, to prove (iii),

$$\begin{aligned} B_m + C_n &= (a_m + 2m) + (a_n + 2n - 1) \\ &= a_m + a_n + (m + n) + (m + n) - 1 \\ &= a_m + a_n + (b_{m+n} - a_{m+n}) + (m + n) - 1 \\ &= (a_m + a_n - a_{m+n}) + \big(b_{m+n} + (m + n) - 1\big) \\ &= (0 \text{ or } -1) + C_{m+n}. \end{aligned}$$

Thus,

$$B_m + C_n = C_{m+n} \quad \text{or} \quad B_m + C_n = C_{m+n} - 1 = B_{m+n} - 2,$$

but consecutive members of B_j differ by 3 or by 4, so

$$B_{m+n} - 2 \neq B_j, \text{ and } B_m + C_n \neq B_j.$$

Lastly, to prove (iv), either

$$C_{j+1} - C_j = 3 \quad \text{or} \quad C_{j+1} - C_j = 4,$$

and

$$A_m \neq C_j, \; A_m \neq B_j, \text{ for any } j.$$

If $A_m \neq C_j$ and $A_m \neq C_j + 1 = B_j$, then either

$$A_m = C_j - 1 \quad \text{or} \quad A_m = C_j + 2 = B_j + 1.$$

If $A_m = C_j - 1$, then

$$A_m + B_n = C_j - 1 + B_n,$$

which equals

$$C_{j+n} - 1 \neq C_k \quad \text{or} \quad C_{j+n} - 2 \neq C_k, \text{ by the proof of (iii).}$$

If $A_m = B_j + 1$, then

$$A_m + B_n = B_j + 1 + B_n$$

which equals either

$$B_{j+n} + 1 = C_{j+n} - 2 \neq C_k \quad \text{or} \quad C_{j+n} + 1 \neq C_k,$$

by the proof of (i).

The proof of Theorem 2.4 gives us immediately three further statements relating A_n, B_n, and C_n, the first three parts of Theorem 2.5.

Theorem 2.5:

(i) $B_m + B_n = B_{m+n}$ or $B_m + B_n = C_{m+n}$

(ii) $C_m + C_n = C_{m+n} - 1$ or $C_m + C_n = C_{m+n} - 2$

(iii) $B_m + C_n = C_{m+n}$ or $B_m + C_n = C_{m+n} - 1$

(iv) $A_m + A_n = A_{m+n}$ or $A_m + A_n = A_{m+n} - 2$

Proof of (iv):

$$\begin{aligned} A_m + A_n &= 2a_m - m + 2a_n - n \\ &= 2b_m - 3m + 2b_n - 3n \\ &= 2b_m + 2b_n - 3(m + n) \\ &= 2b_m + 2b_n - 3(b_{m+n} - a_{m+n}) \\ &= (2b_m + 2b_n - 2b_{m+n}) - b_{m+n} + 3a_{m+n} \\ &= (2b_m + 2b_n - 2b_{m+n}) - (a_{m+n} + (m + n) + 3a_{m+n}) \\ &= (2b_m + 2b_n - 2b_{m+n}) + 2a_{m+n} - (m + n) \\ &= (0 \text{ or } -2) + A_{m+n}, \end{aligned}$$

so that

$$A_m + A_n = A_{m+n} \quad \text{or} \quad A_m + A_n = A_{m+n} - 2.$$

Finally, we can write some relationships between A_n, B_n, and C_n when the subscripts are the same.

Theorem 2.6:

(i) $A_n + B_n = A_{b_n}$

(ii) $A_n + C_n = B_{a_n}$

(iii) $A_n + B_n + C_n = C_{b_n}$

(iv) $A_{b_n} + C_n = C_{b_n}$

(v) $B_{a_n} + B_n = C_{b_n}$

Proof: Proof of (i):

$$\begin{aligned} A_n + B_n &= 2a_n - n + a_n + 2n \\ &= 2a_n + b_n \\ &= a_n + a_{b_n} \\ &= 2a_{b_n} + (a_n - a_{b_n}) = 2a_{b_n} - b_n = A_{b_n}. \end{aligned}$$

Proof of (ii): Using (1.1), (1.3), (1.2), and Theorem 2.1,

$$\begin{aligned} A_n + C_n &= 2a_n - n + a_n + 2n - 1 \\ &= 2b_n + a_n - n - 1 \end{aligned}$$

$$= b_n + a_n - n + a_{a_n}$$
$$= 2a_n + a_{a_n} = B_{a_n}$$

Proof of (iii):

$$\begin{aligned} A_n + B_n + C_n &= 2a_n - n + a_n + 2n + a_n + 2n - 1 \\ &= a_n + 3(a_n + n) - 1 \\ &= a_n + 3b_n - 1 \\ &= (a_n + b_n) + 2b_n - 1 \\ &= a_{b_n} + 2b_n - 1 \\ &= C_{b_n} \end{aligned}$$

by (1.2) and Theorem 2.1.

Note that (iv) and (v) are just combinations of (i) with (iii), and (ii) with (iii).

Notice that there are eighteen possible ways to add two of the As, Bs, or Cs to obtain an A_k, B_j, or C_i. $A_m + A_n = A_{m+n}$ or $A_m + A_n = A_{m+n} - 2$, so that $A_m + A_n = A_k$ or $A_m + A_n = B_j$ or $A_m + A_n = C_i$ for suitable k, j, and i. $A_m + B_n = A_k$ or $A_m + B_n = B_j$ for suitable k and j, but $A_m + B_n \neq C_i$ for any i. $A_m + C_n = A_k$ or $A_m + C_n = B_j$ or $A_m + C_n = C_i$ for suitable values of k, j, and i as readily found in Table 2. Since $B_m + B_n = B_{m+n}$ or $B_m + B_n = C_{m+n}$, solutions exist for $B_m + B_n = B_j$ and $B_m + B_n = C_i$, but $B_m + B_n \neq A_k$ for any k. Since $B_m + C_n = C_{m+n}$ or $B_m + C_n = C_{m+n} - 1$, solutions exist for $B_m + C_n = C_i$ and for $B_m + C_n = A_k$, but $B_m + C_n \neq B_j$ for any j. Lastly, $C_m + C_n = C_{m+n} - 1$ or $C_m + C_n = C_{m+n} - 2$, so it is possible to solve $C_m + C_n = A_k$ and $C_m + C_n = B_j$, but $C_m + C_n \neq C_i$ for any i.

3. LUCAS REPRESENTATIONS OF THE NUMBERS A_n, B_n, AND C_n

The numbers A_n, B_n, and C_n can be represented uniquely as sums of Lucas numbers L_n, where $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$. Since the Lucas numbers 2, 1, 3, 4, 7, 11, ..., are complete, one could show that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ cover the positive integers and are disjoint. See [9] and [10]. We write $A = \{A_n\} = \{1, 4, 5, 8, 11, \ldots\}$, numbers in the form

$$A_n = 1 + \delta_2 L_2 + \delta_3 L_3 + \cdots + \delta_m L_m, \ \delta_i \in \{0, 1\}$$

in their natural order; and $B = \{B_n\} = \{3, 7, 10, 14, 18, \ldots\}$,

$$B_n = 3 + \delta_3 L_3 + \delta_4 L_4 + \cdots + \delta_m L_m, \ \delta_i \in \{0, 1\}$$

in their natural order, and $C = \{C_n\} = \{2, 6, 9, 13, 17, \ldots\}$, which are numbers of the form

$$C_n = 2 + \delta_3 L_3 + \delta_4 L_4 + \cdots + \delta_m L_m, \ \delta_i \in \{0, 1\}.$$

The union of A_n, B_n, and C_n is the set of positive integers, and the sets are disjoint. One notes immediately that $B_n = C_n + 1$ because any choice of δs in the set C can be used in the set B so that, for each C_n, there is an element of B which is one greater. Also, each A_n is one greater than a B_j or one less than a C_j. Also, A_ns may be successive integers. A viable approach is to let all the positive integers representable using 1, 3, 4, 7, ... , in Zeckendorf form be classified as having the lowest nonzero binary digit in the even place 3 Lucas Zeckendorf $\boxed{1|0}$, while 4 is in an odd $\boxed{1|0|0}$. This clearly makes A and B distinct sets, since the Zeckendorf representation is unique. Set C consists of the numbers which must use a 2, making C distinct from either A or B, since the positive integers have a distinct and unique representation if no two consecutive Lucas numbers from $\{2, 1, 3, 4, 7, \ldots\}$ are used and $L_0 = 2$ and $L_2 = 3$ are not to be used together in any representation.

$C = \{2, 6, 9, 13, 17, 20, \ldots\}$, are the positive integers that are not representable by $\{1, 3, 4, 7, \ldots\}$, the Lucas numbers when $L_0 = 2$ is deleted. The sequence $\{B_n\} = \{3, 7, 10, 14, \ldots\}$ occurs in the solution to the International Olympiad 1977, problem 2 [11], which states:

> Given a sequence of real numbers such that the sum of seven consecutive terms is negative, and the sum of eleven consecutive terms is positive, show that the sequence has a finite (less than 17) number of terms. A solution with sixteen terms does exist:
>
> 5, -5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5
>
> We note that -13 occurs at positions 3, 7, 10, 14, ...

We know that every positive integer has a Zeckendorf representation in terms of 2, 1, 3, 4, 7, 11, ..., and a second canonical representation such that

$$A \xrightarrow{f^*} B,$$

where f^* merely advances the subscripts on the Zeckendorf representation of A (odd position) to a number from B (even position). One needs a result on lexicographical ordering: If, in comparing the Lucas Zeckendorf representation of M and N from the higher-ordered binary digits, the place where they first differ has a one for M and a zero for N, then $M > N$. Clearly, under f^*, the lexicographical ordering is preserved.

Now, look at the positive integers, and below them write the number obtained by shifting the Lucas subscripts by one upward:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$f^*(n) =$	3	1	4	7	10	8	11	14	12	15	18	21	19
$\Delta f^*(n) =$	-2	3	3	3	-2	3	3	-2	3	3	3	-2	

Note that the -2 occurs between $n = C_k$ and $n = C_k - 1$; all the other differences $\Delta f^*(n) = f^*(n + 1) - f^*(n) = 3$. Now, of course, $1 \to 3$ so that normally the difference of the images of two successive integers is 3, but $2 \to 1$ and $1 \to 3$, so that those integers C_k which require a 2 in their representation always lose 2 in the forward movement of the subscripts.

Now from the positive integers we remove $B_n + 1$, as this is an unpermitted difference; these numbers 4, 8, 11, 15, 19, ..., are A_js immediately after a B_k. Those A_ns remaining in the new set are the second A_n of each adjacent pair. Since $A_{b_n+1} - A_{b_n} = 1$, it follows that

$$A_{b_n+1} \neq B_n + 1,$$

but rather

$$A_{b_n+1} = C_j - 1.$$

The numbers $A_{b_n} = B_m + 1$, $A_{a_n+1} = B_j + 1$, and other A_ns which are of the form $B_s + 1$ are gone. Only $A_{b_n+1} = C_j - 1$ are left in the set, which is {5, 12, 16, 23, 30, ...}, and they are in the set $\{B_n - A_n\}$ and also in $\{A_n\}$. We now wish to look at the fact that each $C_n - 2$ has been removed whenever $C_n - 1$ remains in the set. This opens up an interval difference of six in each case. For instance,

$$\begin{array}{cccccc} 3 & 4 & 5 & 6 = C_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 7 & 10 & 8 \end{array} \qquad \begin{array}{cccc} 10 & 11 & 12 & 13 = C_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 15 & 18 & 21 & 19 \end{array}$$

difference 6 6

Now, without changing anything else, by giving each element of $\{5, 12, 16, 23, 30, \ldots, A_{b_n+1}, \ldots\}$ an image which is five smaller. Each such number uses a one, $1 = L_1$, in the Zeckendorf representation. Replace this L_1 by $-L_{-1}$. Now, regardless of whatever else is present in this Lucas representation, formerly $1 = L_1 \to L_2 = 3$, but now $1 = -L_{-1} \to -2 = -L_0$, so that the difference in images is 5.

$$A_{b_n+1} \xrightarrow{f^*} M$$

$$A_{b_n+1} \xrightarrow{f} M - 5$$

Now, all of the rest of the differences were 3 when the difference in the objects was 1. The differences in the images were -2 only when the objects were C_n and $C_n - 1$. Now, with 1, 4, 8, 11, 15, ..., $B_n + 1$, ..., removed $(B_0 = 0)$, and each image of the object set $\{5, 12, 16, 23, 30, \ldots, A_{b_n+1}, \ldots\}$ replaced by five less, we now find that if the object numbers differ by 1, their images differ by 3, and if the object numbers differ by two, the image numbers differ by one. Thus, if from the object set $M > N$, then the image of M is greater than the image of N under the mapping f of increasing the Lucas number subscripts by one. This shows that the mapping from $\{\Delta_n\} = \{B_n - A_n\}$ into $\{A_j\}$ is such that $\Delta_n \xrightarrow{f} A_n$. Further, under f^*, operating on the Zeckendorf form, $A_n \xrightarrow{f^*} B_n$ because of the lexicographic mapping. Clearly, f is not lexicographic over the positive integers but over the set $\{\Delta_n\}$ where $\{5, 12, 16, 30, \ldots\}$ have been put into special canonical form.

We note that the set $\{\Delta_n\}$ is all numbers of the form

$$\Delta_n = 2 + \delta_1 L_1 + \delta_2 L_2 + \cdots$$

in their natural order, where $\delta_i \in \{0, 1\}$. Since

$$\{C_n\} = \{2, 6, 9, 13, 17, \ldots\}$$

cannot be made using $\{1, 3, 4, 7, 11, 18, \ldots\}$, it follows that

$$\{C_n + 2\} = \{4, 8, 11, 15, \ldots\} = \{B_n + 1\}$$

cannot be so represented. We have thrown these numbers out of the original set of positive integers. What is left is the set so representable. Thus, Δ_n are the numbers of that form in natural order. The number 1 does not appear in $\{\Delta_n\}$. Now, if in $\{\Delta_n\}$ we replace each Lucas number by one with the next higher subscript, then we get all the numbers of the form

$$1 + \delta_1 L_2 + \delta_2 L_3 + \delta_3 L_4 + \cdots$$

in their natural order since we have carefully made the construction so numbers of the image set are out of their natural order. Thus,

$$\Delta_n \xrightarrow{f} A_n \quad \text{and} \quad A_n \xrightarrow{f^*} B_n.$$

4. WYTHOFF'S LUCAS GAME

The Lucas generalization of Wythoff's game is a two-pile game for two players with the following rules:

(1) At least one counter must be taken;
(2) Any number of counters may be taken from one pile;
(3) An equal number of counters may be taken from each pile;
(4) One counter may be taken from the smaller pile, and two from the larger pile;
(5) All counters may be taken if the numbers of counters in the two piles differ by one (hence, a win);
(6) The winner takes the last counter.

Let H be the pile on the left, and G on the right, so that (H_n, G_n) are to be safe pairs.

$$H_{A_n} = A_{a_n} \qquad G_{A_n} = B_{a_n}$$

$$H_{B_n} = A_{b_n} \qquad G_{B_n} = B_{b_n}$$

$$H_{C_n} = C_{a_n} \qquad G_{C_n} = C_{b_n}$$

where (a_n, b_n) is a safe pair for Wythoff's game, and A_n, B_n, and C_n are the numbers of Section 2.

Now remember that (H_{A_n}, G_{A_n}) and (H_{B_n}, G_{B_n}) had all the differences except numbers of the form $B_n + 1$.

$$C_{b_n} - C_{a_n} = B_n + 1 = A_{a_n+1}.$$

The only difference not in $\{G_n - H_n\}$ is one; hence, Rule 5. The differences in the Hs are 1, 2, or 3, and the differences in the Gs are 1, 3, or 4. It is not difficult to see that the rules change a safe position into an unsafe position.

Next, the problem is to prove that using the rules, an unsafe pair can be made into a safe pair. Strategy to win the Wythoff-Lucas game follows.

Suppose you are left with (c, d) which is an unsafe pair. Without loss of generality, take $c < d$.

1. If $c = H_k$ and $d > G_k$, then choose s so that $d - s = G_k$. (Rule 2.)

2. If $c = H_k$ and $d < G_k$ and $d - c = \Delta_m < \Delta_k = G_k - C_k$, where Δ_m appears in the list of differences earlier, then choose s so that

$$d - s = c + \Delta_m - s = c - s + \Delta_m = H_m + \Delta_m = G_m. \quad \text{(Rule 3.)}$$

3. If $c = H_k$ and $d < G_k$ and $d - c = \Delta_m < \Delta_k$ but Δ_m does not appear earlier in the list of differences, that is, $H_m > H_k$, then we need some results before we can proceed.

Lemma 1: $C_{b_n} - C_{a_n} = B_n + 1 = G_{C_n} - H_{C_n}$.

Proof: By Theorem 2.1,

$$\begin{aligned} C_{b_n} - C_{a_n} &= (a_{b_n} + 2b_n - 1) - (a_{a_n} + 2a_n - 1) \\ &= (a_n + b_n + 2b_n - 1) - (a_{a_n} + 1 + 2a_n - 2) \\ &= a_n + 3b_n - 1 - b_n - 2a_n + 2 \\ &= a_n + 2(b_n - a_n) + 1 \\ &= a_n + 2n + 1 \\ &= B_n + 1. \end{aligned}$$

Lemma 2: $G_{B_n} - H_{B_n} = B_{b_n} - A_{b_n} = B_n$.

Proof: By Theorem 2.1,

$$\begin{aligned} B_{b_n} - A_{b_n} &= a_{b_n} + 2b_n - (2a_{b_n} - b_n) \\ &= 3b_n - a_{b_n} \\ &= 3b_n - (a_n + b_n) \\ &= 2b_n - a_n \\ &= 2(a_n + n) - a_n \\ &= a_n + 2n \\ &= B_n \end{aligned}$$

The original list of $\Delta_n = G_n - H_n$ was steadily increasing functions of n. However, with the insertion of $G_{C_n} - H_{C_n} = \Delta_{C_n} = B_n + 1$, the next higher

$$G_{C_n+1} - H_{C_n+1} = G_{B_n} - H_{B_n} = B_{b_n} - A_{b_n} = B_n$$

by Lemma 2. Thus, if $\Delta_m < \Delta_k$ while $H_m > H_k$, this cannot be the case, except when $c = H_k = H_{C_n}$ while $d = G_{C_n} - 1$. The proper response is to subtract one from c and subtract two from d to finish case 3; we also need Lemma 3, which follows from Theorems 2.1 and 2.2:

Lemma 3: $c - 1 = H_{C_n} - 1 = C_{a_n} - 1 = A_{b_n-1}$;

$$d - 2 = G_{C_n} - 3 = C_{b_n} - 3 = B_{b_n-1}.$$

The pair $(H_{C_n-1}, G_{C_n-1}) = (A_{b_n-1}, B_{b_n-1})$ is a safe pair which is obtained by using Rule 4.

4. If $c = G_k$, then if $d > H_k$, choose s so that $d - s = H_k$. (Rule 2.)
5. If $c = G_k$ and $d < H_k$, follow the procedures of cases 2 and 3.
6. If $c = d - 1$, then take all the counters by Rule 5.

Since G_k and H_k cover the integers, cases 1-6 give every possible choice of c and d, $c \neq d$. If $c = d$, then take all the counters and hence win, by Rule 3.

Some comment should be made about why each legal play from a safe pair results in an unsafe pair. We begin with the safe pair (H_k, G_k) and apply each rule.

(a) If from (H_k, G_k) we subtract $s > 0$ from either (Rule 2), then since (H_k, G_k) are a related pair, changing either one without the other results in an unsafe pair.

(b) If from (H_k, G_k) we subtract $s > 0$ from each (Rule 3), then the difference $\Delta_k = G_k - H_k$ is preserved, but the difference Δ_k is unique to the safe pair; hence, changing H_k and G_k but keeping the difference Δ_k the same results in an unsafe pair.

(c) To investigate Rule 4, we need some results on the differences of the sequences H_n and G_n separately.

Lemma 4: The differences of the H_n sequence are 1, 2, or 3:

(i) $H_{C_n+1} - H_{C_n} = 2$;

(ii) $H_{B_n+1} - H_{B_n} = 1$;

(iii) $H_{A_{b_n}+1} - H_{A_{b_n}} = 3$;

(iv) $H_{A_{a_n}+1} - H_{A_{a_n}} = 1$.

Proof: We refer to Theorem 2.1 and the results of Section 1 repeatedly.

(i) $H_{C_n+1} = H_{B_n} = A_{b_n}$ and $H_{C_n} = C_{a_n}$, so

$$\begin{aligned} H_{C_n+1} - H_{C_n} &= A_{b_n} - C_{a_n} = (2a_{b_n} - b_n) - (a_{a_n} + 2a_n - 1) \\ &= (2a + 2b - b_n) - b_n - 2a_n + 2 = 2. \end{aligned}$$

(ii) $H_{B_n} = A_{b_n}$, so

$$H_{B_n+1} = H_{A_{a_n+1}} = A_{a_{a_n+1}} = A_{a_{a_n}+2} = A_{b_n+1} = A_{b_n} + 1,$$

and

$$H_{B_n+1} - H_{B_n} = A_{b_n} + 1 - A_{b_n} = 1.$$

(iii) $H_{A_{b_n}} = A_{a_{b_n}}$ and $H_{A_{b_n}+1} = H_{A_{b_n+1}} = A_{a_{b_n+1}} = A_{a_{b_n}+1} = A_{a_{b_n}} + 3$, so

$$H_{A_{b_n}+1} - H_{A_{b_n}} = 3.$$

(iv)
$$\begin{aligned} H_{A_{a_n}+1} &= H_{C_n} = C_{a_n} = a_{a_n} + 2a_n - 1 \\ &= (b_n - 1) + 2a_n - 1 = a_{b_n} + a_n - 2; \\ H_{A_{a_n}} &= A_{a_{a_n}} = 2a_{a_{a_n}} - a_{a_n} = 2(a_{a_{a_n}+1} - 2) - a_{a_n} \\ &= 2a_{b_n} - 4 - (a_{a_n} + 1) + 1 = 2a_{b_n} - b_n - 3 \\ &= a_{b_n} + (a_n + b_n) - b_n - 3 = a_{b_n} + a_n - 3. \end{aligned}$$

Thus,

$$H_{A_{a_n}+1} - H_{A_{a_n}} = (a_{b_n} + a_n - 2) - (a_{b_n} + a_n - 3) = 1.$$

We now conclude that the G sequence does not have a difference of two; in fact, the differences in the Gs are always 1, 3, or 4.

Lemma 5: $G_{k+1} - G_k \neq 2$ and

(i) $G_{C_n+1} - G_{C_n} = 1;$

(ii) $G_{C_n} - G_{C_n-1} = 3;$

(iii) $G_{C_n+2} - G_{C_n+1} = 3;$

(iv) $G_{A_{b_n}+1} - G_{A_{b_n}} = 4.$

Proof: By construction, the differences $\Delta_n = B_n - A_n$ in natural order cover all the positive integers except numbers of the form $B_n + 1$, but $(H_{C_n}, G_{C_n}) = (C_{a_n}, C_{b_n})$ is such that $\Delta_{C_n} = B_n + 1$.

We now cite some obvious results (see Lemmas 1, 2, and 3).

$$\Delta_{C_n} = B_n + 1,\ \Delta_{C_n+1} = B_n,\ \Delta_{C_n-1} = B_n - 1;$$

therefore,

$$\Delta_{C_n+1} - \Delta_{C_n} = -1, \quad \Delta_{C_n} - \Delta_{C_n-1} = 2, \text{ and } \Delta_{C_n+2} - \Delta_{C_n+1} = 2;$$

but

$$\Delta_{m+1} - \Delta_m = 1, \text{ otherwise.}$$

$$(G_{C_n+1} - H_{C_n+1}) - (G_{C_n} - H_{C_n}) = -1;$$

thus,

(i) $G_{C_n+1} - G_{C_n} = H_{C_n+1} - H_{C_n} - 1 = (2 - 1) = 1.$

Since $H_{C_n} - H_{C_n-1} = 3$, $(G_{C_n} - H_{C_n}) - (G_{C_n-1} - H_{C_n-1}) = 2$, making

(ii) $G_{C_n} - G_{C_n-1} = H_{C_n} - H_{C_n-1} + 2 = 3.$

Next, $H_{B_n+1} - H_{B_n} = H_{C_n+2} - H_{C_n+1} = 1$, and $\Delta_{C_n+2} - \Delta_{C_n+1} = 2$, so that

$$G_{C_n+2} - H_{C_n+2} - (G_{C_n+1} - H_{C_n+1}) = 2,$$

or

$$G_{C_n+2} - G_{C_n+1} - (H_{C_n+2} - H_{C_n+1}) = 2,$$

so that, finally,

(iii) $G_{C_n+2} - G_{C_n+1} = (H_{C_n+2} - H_{C_n+1}) + 2 = 1 + 2 = 3.$

The fourth case has $\Delta_{A_{b_n}+1} - \Delta_{A_{b_n}} = 1$, which means that

$$G_{A_{b_n}+1} - H_{A_{b_n}+1} - (G_{A_{b_n}} - H_{A_{b_n}}) = 1,$$

so that

$$G_{A_{b_n}+1} - G_{A_{b_n}} - (H_{A_{b_n}+1} - H_{A_{b_n}}) = 1,$$

making

(iv) $G_{A_{b_n}+1} - G_{A_{b_n}} = (H_{A_{b_n}+1} - H_{A_{b_n}}) + 1 = 3 + 1 = 4.$

The final conclusion is that no difference of Gs equals two, concluding the proof of Lemma 5.

We now can finish case (c), the investigation of Rule 4. The play of subtracting one from H_k and two from G_k does leave an unsafe pair.

(d) If (H_k, G_k) is a safe pair, then the difference between H_k and G_k is never one, so Rule 5 will not apply.

We have found that applying the rules to a safe pair always leads to an unsafe pair.

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A MODIFICATION OF GOKA'S BINARY SEQUENCE

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ABSTRACT

Goka's binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$, where g_i is a binary has been modified by replacing the binaries (g_i) by matrices of the same order over the binaries. We define, formulate, and discuss the properties of the nth integral from j of G by repeating in succession Melvyn B. Nathanson's formula for I_jG, the integral from j of G. The integral equation $I_jG = G$ has been solved. We investigate the behavior of the decimated sequence, submatrix sequence, sequence of integrals from j, and complementary sequence of the binary matrix sequence (BMS) G in relation to G. An application of the binary sequence has been described.

1. INTRODUCTION

Goka [1] has introduced the binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$, where $g_i = 0$ or 1 and the addition is modulo 2. Nathanson [2] has discussed eventually periodic binary sequences. In his paper he has formulated the nth derivative of G, D^nG, and the integral from j of G, I_jG.

In this paper we present a modification of the binary sequence G to the binary matrix sequence by replacing the binary g_i by a $m \times n$ matrix over the binaries. All arithmetic of the binaries is done modulo 2, and the addition of the binary matrices or binary matrix sequences is done componentwise. Not surprisingly, we will find that all the results established in [2] hold good for our BMS also. We generalize the integration formula for I_jG, where G is a BMS, formulate I_j^nG, the nth integral from j of G, and establish results illustrating its properties. We study certain interesting properties of the decimated sequence of G, the submatrix sequence of G, and the complement of G in relation to their parent BMS G. In the final section, we show how to apply the novel method of binary sequences to represent any sequence of integers. Just indicating whether a member is odd or even and using this method we are able to determine whether

$$\binom{p+r+1}{r},\ \binom{p+r}{r},\ r = 0,\ 1,\ \ldots,\ n-1$$

are odd or even when $p = 2^m q$, where q is odd and $2^{m-1} < n \leq 2^m$.

2. NOTATIONS AND DEFINITIONS

Definition 1: A binary matrix sequence (BMS) is the infinite sequence

$$G = \left(g_i\right)_{i=1}^{\infty},$$

where (g_i) are matrices of the same order over the binaries.

In what follows, we will use the following laws of addition modulo 2:

(i) $1 + 0 = 0 + 1 = 1$;
(ii) $1 + 1 = 0 + 0 = 0$.

It is evident that, if k is a nonnegative integer,

$$k\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\0\end{pmatrix},$$

according as k is odd or even. Addition of the binary matrices and, likewise, the BMSs is done componentwise, i.e., if

$$g_i = (a_{rs}),\ g_j = (b_{rs}),\ g_i + g_j = (a_{rs} + b_{rs}).$$

Similarly, if

$$G = \left(g_i\right)_{i=1}^{\infty} \quad \text{and} \quad H = \left(h_i\right)_{i=1}^{\infty}$$

are two BMSs of the same order, then $G + H = \left(g_i + h_i\right)_{i=1}^{\infty}$.

Notations: $\mathbf{0}$, $\underset{\sim}{\mathbf{0}}$, $\boldsymbol{g}$, $\underset{\sim}{\boldsymbol{g}}$

(i) $\mathbf{0}$ = a binary matrix in which every entry is 0 and is called a binary null matrix.

(ii) $\underset{\sim}{\mathbf{0}}$ = a binary matrix sequence in which every entry is $\mathbf{0}$ and is called a constant null sequence.

(iii) $\boldsymbol{g}$ = a binary matrix in which every entry is 1.

(iv) $\underset{\sim}{\boldsymbol{g}}$ = a binary matrix sequence in which every entry is $\boldsymbol{g}$.

The following results will be useful. If g_i is a binary matrix and k is a nonnegative integer, then

(i) $kg_i = g_i$ or $\mathbf{0}$ according as k is odd or even.

(ii) $g_i + g_j = \mathbf{0}$ means $g_i = g_j$.

Definition 2:

(i) If g_i and h_i are two binary matrices of the same order with $g_i + h_i = \boldsymbol{g}$, then each is said to be the complement of the other.

(ii) If G and H are two BMSs of the same order with $G + H = \underset{\sim}{\boldsymbol{g}}$, each sequence is said to be the complement of the other.

We use the notation $\overline{g}_i$, $\overline{G}$ for the complements of g_i, G, respectively, and to write the complement, we simply change the binaries 0, 1 to 1, 0, respectively.

Example: If $g_i = \begin{pmatrix}1 & 1\\0 & 1\\0 & 0\end{pmatrix}$, $g_j = \begin{pmatrix}1 & 0\\0 & 0\\1 & 1\end{pmatrix}$, then

(i) $g_i + g_j = \begin{pmatrix}0 & 1\\0 & 1\\1 & 1\end{pmatrix}$

(ii) $3g_i = \begin{pmatrix}1 & 1\\0 & 1\\0 & 0\end{pmatrix} = g_i$

(iii) $8g_i = \begin{pmatrix}0 & 0\\0 & 0\\0 & 0\end{pmatrix} = \mathbf{0}$

(iv) $\bar{g}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

Definition 3:

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, its derivative $DG = \left(g_i^1\right)_{i=1}^{\infty}$, where $g_i^1 = g_i + g_{i+1}$.

(ii) If G is a BMS, its nth derivative is defined recursively by $D^n G = D(D^{n-1}G)$.

Definition 4:

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, the integral from j of G is the BMS $I_j G$ whose ith term is

$$g_{i,1} = \begin{cases} \sum_{s=i}^{j-1} g_s & \text{if } i < j \\ \mathbf{0} & \text{if } i = j \\ \sum_{s=j}^{i-1} g_s & \text{if } i > j \end{cases}$$

(ii) If G is a BMS, the nth integral from j of G is the BMS $I_j^n G$ defined recursively by $I_j^n G = I_j(I_j^{n-1}G)$.

Definition 5:

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, the sequence of integrals from j of G is defined as $J = \left(I_j^n G\right)_{n=0}^{\infty}$, where $I_j^0 G = G$.

(ii) The truncated sequence of integrals from j up to p is the infinite sequence J_T whose nth term is the truncated BMS

$$\left(I_j^n\left(g_i\right)\right)_{i=1}^{p}.$$

Definition 6: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS and d is a positive integer, the decimated BMS G^d of G is defined by

$$G^d = \left(g_{kd}\right)_{k=1}^{\infty}.$$

Definition 7: G^* is called a sequence of submatrices of a BMS G, and is obtained by taking the submatrices of the same location from the binary matrices of G.

Example:

$$\text{If } G = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots;$$

$$\left(G^d\right)_{d=2} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots;$$

$$G^* = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \ldots;$$

obtained by taking second and third rows of the binary matrices of G.

Definition 8: Eventual property of a BMS.

A BMS $G = \left(g_i\right)_{i=1}^{\infty}$ is said to have an eventual property from j when it it true for the BMS $\left(g_i\right)_{i=j}^{\infty}$.

SECTION 3

In this section, we establish certain theorems with regard to the integrals from j of a BMS.

Theorem 1: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, its nth integral from j is the BMS $I_j^n G$ whose ith term $g_{i,n}$ is

$$(1.1) \qquad \sum_{s=i}^{j-1} \binom{n-1+s-i}{n-1} g_s \qquad \text{if } i < j$$

$$(1.2) \qquad \mathbf{0} \qquad \text{if } j \le i < j+n$$

$$(1.3) \qquad \sum_{s=j}^{i-n} \binom{i-s-1}{n-1} g_s \qquad \text{if } i \ge j+n$$

Proof:

Case (1.1): Let $i < j$. Then $g_{i,n}$, the ith term in $I_j^n G$ is

$$\sum_{s=i}^{j-1} \left(\sum_{s_{n-1}=1}^{s} \sum_{s_{n-2}=1}^{s_{n-1}} \cdots \sum_{s_2=1}^{s_3} \sum_{s_1=1}^{s_2} 1 \right) g_{i+s-1}.$$

Upon using

$$\sum_{i_n=1}^{p} \sum_{i_{n-1}=1}^{i_n} \cdots \sum_{i_2=1}^{i_3} \sum_{i_1=1}^{i_2} 1 = \binom{n+p-1}{n},$$

we have the ith term as

$$\sum_{s=i}^{j-1} \binom{n-1+s-i}{n-1} g_s.$$

Case (1.2): Let $j \le i < j+n$ and $I_j^n G = \left(g_{i,n}\right)_{i=1}^{\infty}$. As

$$g_{j,1} = \mathbf{0},\ g_{j,2} = g_{j+1,2} = \mathbf{0},\ g_{j,3} = g_{j+1,3} = g_{j+2,3} = \mathbf{0},$$

and finally,

$$g_{j,n} = g_{j+1,n} = g_{j+2,n} = \cdots = g_{j+n-1,n} = \mathbf{0}$$

or, in brief,

$$g_{i,n} = \mathbf{0} \text{ if } j \le i < j+n.$$

Case (1.3): Let $i \ge j+n$. Here the formula can be established on par with case (1.1) and the proof is therefore left to the reader.

Corollary 1: If $G = (g)_{i=1}^{\infty}$ is a constant BMS,

(1.4) (i) $D^n G = \underset{\sim}{\mathbf{0}}$

(ii) $I_j^n G = (g_{i,n})_{i=1}^{\infty}$, where $g_{i,n}$ is:

(1.5) $$\binom{n+j-i-1}{n} g \quad \text{if } i < j$$

(1.6) $$\mathbf{0} \quad \text{if } j \le i < j+n$$

(1.7) $$\binom{i-j}{n} g \quad \text{if } i \ge j+n.$$

Proof: (1.4) follows immediately from the definition of the operator D, and (1.6) is a particular case of (1.2).

To prove (1.5), consider

$$g_{i,n} = \sum_{s=i}^{j-1} \binom{s-i+n-1}{n-1} g = \left(\sum_{s=i}^{j-1} \binom{s-i+n-1}{n-1}\right) g.$$

Upon using

$$\sum_{s=0}^{p} \binom{n+s}{n} = \sum_{s=0}^{p} \binom{n+s}{s} = \binom{n+p+1}{p},$$

$$g_{i,n} = \binom{n+j-i-1}{j-i-1} g = \binom{n+j-i-1}{n} g.$$

Similarly, we prove (1.7).

Corollary 2:

(i) $\underset{n\to\infty}{\mathrm{Lt}}\; I_j^n G$ is eventually null from j. This follows from (1.6).

(ii) $\underset{n\to\infty}{\mathrm{Lt}}\; I_1^n G = \underset{\sim}{\mathbf{0}}$.

Theorem 2: The truncated sequence of integrals J_T from j of a BMS G ($\neq \underset{\sim}{\mathbf{0}}$) up to $j - 1$ has a period 2^m, where $2^{m-1} + 1 < j \le 2^m + 1$.

Proof: If $I^p G = (g_{i,p})_{i=1}^{\infty}$ and $g_{i,p} = g_i$ for $i < j$ with p in its lowest form then

$$g_{i,n+p} = g_{i,n} \text{ for } i < j \text{ and } n = 1, 2, 3, \ldots,$$

and hence it is sufficient to investigate the feasibility of

$$g_{i,p} = g_i,\; i < j.$$

This will happen when

$$\sum_{s=i}^{j-1} \binom{p+s-i-1}{p-1} g_s = g_i,\; i < j,$$

which is true when $\binom{p}{p-1}, \binom{p+1}{p-1}, \ldots, \binom{p+j-3}{p-1}$ are all even. The suitable value of $p = 2^m q$, where q is odd and m is nonnegative with $j \le 2^m + 1$. In order that p is lowest, we set $q = 1$, $j > 2^{m-1} + 1$, and conclude that J_T up to $j - 1$ of J is of period 2^m, where $2^{m-1} + 1 < j \le 2^m + 1$.

Corollary: If a BMS G is eventually null from j, the sequence of integrals from j of G is of period 2^m, where $2^{m-1} + 1 < j \le 2^m + 1$.

Example: Choosing $j = 5$, consider

$$G = \left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \ldots\right)$$

$$I_5^4 G = \left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \ldots\right)$$

From our choice of j, $2^m = 4$ and we find that J_T up to 4 of J has a period 4.

Theorem 3: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS and $I_j G = G$, then $G = \underset{\sim}{\mathbf{0}}$.

Proof: If $I_j G = G$,

$$\sum_{s=i}^{j-1} g_s = g_i \quad \text{if} \quad i < j, \tag{3.1}$$

$$g_j = \mathbf{0} \quad \text{and} \quad \sum_{s=j}^{i-1} g_s = g_i \quad \text{if} \quad i > j. \tag{3.2}$$

Upon setting $i = j - 2, j - 3, \ldots, 1$ in succession in (3.1), we find that

$$g_i = \mathbf{0}, \quad i = 1, 2, \ldots, j - 1;$$

upon setting $i = j + 1, j + 2, \ldots$ in (3.2), we have

$$g_i = \mathbf{0}, \quad i = j + 1, j + 2, \ldots .$$

Thus $G = \underset{\sim}{\mathbf{0}}$.

4. DECIMATED SEQUENCE, COMPLEMENTARY SEQUENCE, AND SUBMATRIX SEQUENCE OF A BMS

Theorem 4: If G is a BMS of eventual period P from j_0 and G^* a sequence of submatrices of G, then G^* is of eventual period p from i_0 where $p|P$ and $i_0 \le j_0$.

Proof: If G is eventually periodic from j_0, then G^* should also be eventually periodic from j_0. As the converse is not true, G^* could have eventual period p, where $p|P$, from $i_0 \le j_0$.

Example: Consider

$$G = \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \ldots\right)$$

Here G has eventual period 2 from $i_0 = 3$.

$$G^* = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \ldots\right)$$

obtained by taking first and second rows has eventual period 1 from $j_0 = 2$.

Corollary: If a BMS G has eventual period p from j_0 where p is a prime, then G^* has eventual period p or 1 from $i_0 \le j_0$.

Theorem 5: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS of eventual period p from i_0, then its decimated sequence $G^d = \left(g_{kd}\right)_{k=1}^{\infty}$ has eventual period $\dfrac{p}{(p,d)}$ from $k_0 \le \left[\dfrac{i_0}{d}\right] + 1$ where $\left[\dfrac{i_0}{d}\right]$ is the integral part of $\dfrac{i_0}{d}$.

Proof: Consider $G^d = \left(g_{kd}\right)_{k=1}^{\infty}$. Here

$$g_i = g_{kd} = g_{(k+l)d} \quad \text{for} \quad i \ge i_0,$$

if

$$(k + l)d = i + mp,$$

where l, k, m are positive integers. Therefore, G^d is periodic for $kd \ge i_0$ if $ld = mp$; i.e., $l = mp/d$. The lowest form of $l = p/(p,d)$. Now, consider the case $d < i_0$. The n-tuple $(g_d, g_{2d}, \ldots, g_{nd})$, where $nd < i_0 \le (n+1)d$ may include a part of the periodic cycle

$$\left(g_{(n+1)d}, g_{(n+2)d}, \ldots, g_{(n+l)d}\right)$$

followed by some full cycles or vice versa. Hence, G^d is of eventual period $l = \dfrac{p}{(p,n)}$ from $k_0 \le \left[\dfrac{i_0}{d}\right] + 1$.

Theorem 6: If $\overline{G}$ is the complement of the BMS G, $D^nG = D^n\overline{G}$.

Proof: As $G + \overline{G} = \underset{\sim}{\mathbf{g}}$, $D^nG + D^n\overline{G} = D^n\underset{\sim}{\mathbf{g}} = \underset{\sim}{\mathbf{0}}$. Hence, $D^nG = D^n\overline{G}$.

5. AN APPLICATION OF THE BINARY SEQUENCE

Here we describe the binary sequence method to show that $\binom{p+r}{r}$ is odd and $\binom{p+r+1}{r}$ is alternately odd and even for $r = 0, 1, 2, \ldots, n-1$ when $p = 2^mq$ and $2^{m-1} < n \le 2^m$. Let $\left(a_i\right)_{i=1}^{n}$ be a sequence of integers, and let $H = \left(h_i\right)_{i=1}^{\infty}$ be an infinite sequence where $h_i = a_i$, if $i \le n$, and $h_i = 0$ if $i > n$. Now we construct the binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$ where g_i = the binary 1 or 0, according as h_i, i.e., a_i, is odd or even. As far as the odd or even nature of the numbers is considered, we are fully justified in the representation of H by the binary sequence G, because the integers strictly obey the laws of addition modulo 2, viz.,

(i) the sum of two odd (or even) numbers is even, i.e.,

$$1 + 1 = 0 + 0 = 0,$$

(ii) the sum of an odd number and an even number is odd, i.e.,

$$1 + 0 = 0 + 1 = 1.$$

Now we prove the following theorem.

Theorem 7: If m is a positive integer and $p = 2^mq$, where q is odd, then

(7.1) (i) $\binom{p+r}{r}$, $r = 0, 1, 2, \ldots, n-1$ are all odd,

and

(7.2) (ii) $\binom{p+r+1}{r}$, $r = 0, 1, 2, \ldots, n-1$ are alternately odd and even, where $2^{m-1} < n \leq 2^m$.

Proof: We represent the sequence of the natural numbers up to n in reverse order in the infinite sequence form

$$H = \left(h_i\right)_{i=1}^{\infty}, \quad \text{where } h_i = n + 1 - i \text{ if } i \leq n \tag{7.3}$$
and $h_i = 0$ if $i > n$.

Now we represent H by the binary sequence

$$G = (101010\ldots101000\ldots) \text{ or } (010101\ldots01000\ldots)$$

according as n is odd or even, where the binary 1 indicates that the corresponding entry h_i in H is odd, the last appearing binary 1 being the nth entry in G.

It is evident that $D^pG \equiv D^pH$ and $I_j^pG \equiv I_j^pH$.* In the usual notation

$$I_{n+1}^pH = \left(h_{i,p}\right)_{i=1}^{\infty}, \quad \text{where } h_{i,p} = \begin{cases} \binom{n-i+p+1}{n-i} & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

The corresponding binary sequence I_{n+1}^pG is identical with G if

$$\binom{p+1}{0}, \binom{p+2}{1}, \ldots, \binom{p+n}{n-1}$$

are alternately odd and even.

We recall that $DI_jG = G$ [2]. Therefore,

$$DI_{n+1}^pG = DI_{n+1}(I_{n+1}G) = I_{n+1}^pG.$$

Now we differentiate I_{n+1}^pG and find that

$$I_{n+1}^{p-1}G = (111\ldots1000\ldots)$$

in order that $I_{n+1}^pG = G$, the last appearing binary 1 being the nth entry. This represents $I_{n+1}^{p-1}H = \left(h_{i,p-1}\right)_{i=1}^{\infty}$, where

$$h_{i,p-1} = \begin{cases} \binom{n-i+p}{n-i} & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

It follows that $\binom{p+r}{r}$, $r = 0, 1, 2, \ldots, n-1$ are all odd. Similarly, we find that

$$I_{n+1}^{p-2}G = (00\ldots01000\ldots)$$

representing $I_{n+1}^{p-2}H$, and we have that $\binom{p+r-1}{r}$, $r = 1, 2, \ldots, n-1$ are all

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**Here the definitions I_j^pG and D^pG for a BMS are extended to any eventually null infinite sequence of numbers.*

even. This is true when $p = 2^m q$ where m is a positive integer and q is odd and $2^{m-1} < n \leq 2^m$. Now we conclude that $\binom{p+r}{r}$ is odd and $\binom{p+r+1}{r}$ is alternately odd and even for $r = 0, 1, 2, \ldots, n-1$ where $p = 2^m q$ and $2^{m-1} < n \leq 2^m$.

Remark 1: Care must be taken not to apply the results of Theorem 2 directly in order to obtain the results of Theorem 5. Similarly, the properties of the derivatives and the integrals of a BMS should not be applied directly to H in (7.3).

Remark 2: The authors earnestly hope that the reader will be able to find further applications of the binary sequences of BMSs.

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RESTRICTED MULTIPARTITE COMPOSITIONS

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1. INTRODUCTION

In [1] the writer discussed the number of compositions

$$n = a_1 + a_2 + \cdots + a_k \tag{1.1}$$

in positive (or nonnegative) integers subject to the restriction

$$a_i \neq a_{i+1} \quad (i = 1, 2, \ldots, k-1). \tag{1.2}$$

In [2] he considered the number of compositions (1.1) in nonnegative integers such that

$$a_i \not\equiv a_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k-1), \tag{1.3}$$

where m is a fixed positive integer.

In the present paper we consider the number of *multipartite compositions*

$$n_j = a_{j1} + a_{j2} + \cdots + a_{jk} \quad (j = 1, 2, \ldots, t) \tag{1.4}$$

in nonnegative a_{js} subject to

$$\mathbf{a}_i \neq \mathbf{a}_{i+1} \quad (i = 1, 2, \ldots, k-1) \tag{1.5}$$

or

$$\mathbf{a}_i \not\equiv \mathbf{a}_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k-1) \tag{1.6}$$

where $\boldsymbol{a}_i$ denotes the vector $(a_{1i}, a_{2i}, \ldots, a_{ri})$ and m is a fixed positive integer.

Let $c(\boldsymbol{n},k)$ denote the number of solutions of (1.4) and (1.5) and let $f(\boldsymbol{n},k)$ denote the number of solutions of (1.4) and (1.6), where $\boldsymbol{n} = (n_1, n_2, n_t)$. We show in particular that

$$\text{(1.7)} \qquad \sum_{\boldsymbol{n}} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \sum_k c(\boldsymbol{n},k) z^k = \left\{1 + \sum_{j=1}^{\infty} \frac{(-1)^j z^j}{(1 - x_1^j)(1 - x_2^j) \cdots (1 - x_t^j)}\right\}^{-1}$$

and

$$\text{(1.8)} \qquad \sum_{\boldsymbol{n}} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \sum_k f(\boldsymbol{n},k) z^k = \left\{1 - \sum_{i_1, \ldots, i_t = 0}^{\infty} \frac{x_1^{i_1} x_2^{i_2} \cdots x_t^{i_t} \lambda}{1 + x_1^{i_1} x_2^{i_2} \cdots x_t^{i_t} \lambda}\right\}^{-1}$$

where

$$\lambda = \frac{z}{(1 - x_1^m)(1 - x_2^m) \cdots (1 - x_t^m)}.$$

For simplicity, proofs are given for the case $t = 2$, but the method applies to the general case.

SECTION 2

To simplify the notation, we consider the case $t = 2$ of (1.4); however, the method applies equally well to the general case. Thus, let $c(n,p,k)$ denote the number of solutions of

$$\text{(2.1)} \qquad \begin{cases} n = a_1 + a_2 + \cdots + a_k \\ p = b_1 + b_2 + \cdots + b_k \end{cases}$$

in nonnegative a_i, b_i such that

$$\text{(2.2)} \qquad (a_i, b_i) \neq (a_{i+1}, b_{i+1}) \quad (i = 1, 2, \ldots, k - 1);$$

let $c(n,p)$ denote the corresponding enumerant when k is unrestricted. For given nonnegative a, b, let $c_{a,b}(n,p,k)$ denote the number of solutions of (2.1) and (2.2) with $a_1 = a$, $b_1 = b$.

Clearly

$$c(n,p,k) = \sum_{a,b} c_{a,b}(n,p,k).$$

It is convenient to define $c(n,p,k)$ and $c_{a,b}(n,p,k)$, $k = 0$, as follows:

$$\text{(2.4)} \qquad c(n,p,0) = \begin{cases} 1 & (n = p = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

and

(2.4)′
$$c_{a,b}(n,p,0) = \begin{cases} 1 & (n = p = a = b = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

It follows at once from the definitions that

(2.5)
$$c_{a,b}(n,p,k) = \sum_{(r,s) \neq (a,b)} c_{r,s}(n - a, p - b, k - 1) \quad (k > 1).$$

Note that (2.5) holds for $k = 1$ except when $n = p = a = b = 0$.

Generating functions $C_{a,b}(x,y,k)$ and $\Phi_k(x,y,u,v)$ are defined by

(2.6)
$$C_{a,b}(x,y,k) = \sum_{n,p=0}^{\infty} c_{a,b}(n,p,k) x^n y^p \quad (k \geq 0)$$

and

(2.7)
$$\Phi_k(x,y,u,v) = \sum_{a,b=0}^{\infty} c_{a,b}(x,y,k) u^a v^b \quad (k \geq 0).$$

It follows from (2.4)′ that

(2.8)
$$C_{a,b}(x,y,0) = \begin{cases} 1 & (a = b = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

and

(2.9)
$$\Phi_0(x,y,u,v) = 1.$$

In the next place, by (2.5) and (2.6), we have for $k > 1$,

$$\begin{aligned} C_{a,b}(x,y,k) &= \sum_{n,p=0}^{\infty} x^n y^p \sum_{(r,s) \neq (a,b)} c_{r,s}(n - a, p - b, k - 1) \\ &= x^a y^b \sum_{n,p=0}^{\infty} x^n y^p \left\{ \sum_{r,s} c_{r,s}(n,p,k-1) - c_{a,b}(n,p,k-1) \right\} \\ &= x^a y^b \sum_{n,p=0}^{\infty} x^n y^p \left\{ c(n,p,k-1) - c_{a,b}(n,p,k-1) \right\}. \end{aligned}$$

Hence,

(2.10)
$$C_{a,b}(x,y,k) = x^a y^b \left\{ C(x,y,k-1) - C_{a,b}(x,y,k-1) \right\} \quad (k > 1),$$

where

(2.11)
$$C(x,y,k) = \sum_{a,b=0}^{\infty} C_{a,b}(x,y,k) = \sum_{n,p=0}^{\infty} c(n,p,k) x^n y^p.$$

Thus, (2.10) yields

$$\sum_{a,b=0}^{\infty} C_{a,b}(x,y,k) u^a v^b = C(x,y,k-1) \sum_{a,b=0}^{\infty} (xu)^a (yv)^b$$

$$- \sum_{a,b=0}^{\infty} F_{a,b}(x,y,k-1)(xu)^a (yv)^b$$

so that, by (2.7), for $k > 1$,

(2.12)
$$\Phi_k(x,y,u,v) = \frac{1}{1 - xu} \frac{1}{1 - yv} \Phi_{k-1}(x,y,1,1) - \Phi_{k-1}(x,y,xu,yv).$$

Iteration gives

$$\Phi_k(x,y,u,v) = \frac{1}{1-xu}\,\frac{1}{1-yv}\Phi_{k-1}(x,y,1,1) - \frac{1}{1-x^2u}\,\frac{1}{1-y^2v}\Phi_{k-2}(x,y,1,1) + \Phi_{k-2}(x,y,x^2u,y^2v) \quad (k > 2),$$

and generally

$$\Phi_k(x,y,u,v) = \sum_{j=1}^{s} \frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) + (-1)^s\Phi_{k-s}(x,y,x^su,y^sv) \quad (k > s).$$

In particular, for $s = k - 1$, this becomes

(2.13) $$\Phi_k(x,y,u,v) = \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) + (-1)^{k-1}\Phi_1(x,y,x^{k-1}u,y^{k-1}v).$$

We have

$$\Phi_1(x,y,u,v) = \sum_{n,p=0}^{\infty}\sum_{a,b} c_{a,b}(n,p,1)x^ny^pu^av^b = \frac{1}{(1-xu)(1-yv)}$$

and (2.13) becomes

(2.14) $$\Phi_k(x,y,u,v) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) \quad (k \geq 1).$$

In particular, for $u = v = 1$, (2.14) reduces to

(2.15) $$\Phi_k(x,y,1,1) + \sum_{j=1}^{k} \frac{(-1)^j}{(1-x^j)(1-y^j)}\Phi_{k-j}(x,y,1,1) = \delta_{k,0}.$$

It follows from (2.15) that

(2.16) $$C(x,y,z) = \left\{1 + \sum_{j=1}^{\infty} \frac{(-1)^jz^j}{(1-x^j)(1-y^j)}\right\}^{-1}$$

where $C(x,y,z)$ is defined by (2.11).

Returning to (2.14), we have

$$\sum_{k=1}^{\infty}\Phi_k(x,y,u,v)z^k = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^j}{(1-x^ju)(1-y^jv)}\sum_{k=0}^{\infty}\Phi_k(x,y,1,1)z^k,$$

and therefore

(2.17) $$\sum_{k=1}^{\infty}\Phi_k(x,y,u,v)z^k = \frac{\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^j}{(1-x^ju)(1-y^jv)}}{1 + \displaystyle\sum_{j=1}^{\infty}\frac{(-1)^jz^j}{(1-x^j)(1-y^j)}}$$

Note that the L.H.S. of (2.16) is

$$\sum_{n,p,k=0}^{\infty} c(n,p,k)x^n y^p z^k; \tag{2.16)'$$

the L.H.S. of (2.17) is

$$\sum_{n,p,a,b,k=0}^{\infty} c_{a,b}\,(n,p,k)x^n y^p u^a v^b z^k. \tag{2.17)'$$

Also, it can be shown (compare [1, §5]) that

$$\sum_{n,p=0}^{\infty} c(n,p)x^n y^p = \left\{1 - \sum_{j=1}^{\infty} \frac{x^{2j-1}(1-x) + y^{2j-1}(1-y) - (xy)^{2p-1}(1-xy)}{(1-x^{2j-1})(1-x^{2j})(1-y^{2j-1})(1-y^{2j})}\right\}^{-1} \tag{2.18}$$

for $|x| < A$, $|y| < A$, where $A \geq \frac{1}{8}$.

SECTION 3

We shall now discuss the problem of enumerating the multipartite compositions that satisfy (1.6). We again take $t = 2$. Let $f(n,p,k)$ denote the number of solutions of

$$\begin{cases} n = a_1 + a_2 + \cdots + a_k \\ p = b_1 + b_2 + \cdots + b_k \end{cases} \tag{3.1}$$

in nonnegative a_s, b_s such that

$$(a_s, b_s) \not\equiv (a_{s+1}, b_{s+1}) \pmod{m} \quad (s = 1, 2, \ldots, k-1). \tag{3.2}$$

Let $f_{i,j}\,(n,p,k)$, for $0 \leq i < m$, $0 \leq j < m$, denote the number of solutions of (3.1) and (3.2) that also satisfy

$$a_1 \equiv i, \; b_1 \equiv j \pmod{m}. \tag{3.3}$$

Finally, let $f_{i,j}\,(n,p,k,a,b)$ denote the number of solutions of (3.1), (3.2), and (3.3) with $a_1 = a$, $b_1 = b$. Thus $f_{i,j}\,(n,p,k,a,b) = 0$ unless $a \equiv i$, $b \equiv j \pmod{m}$.

It is convenient to extend the definitions to include the case $k = 0$. We define

$$f(n,p,0) = \delta_{n0}\delta_{p0}, \; f_{i,j}(n,p,0) = \delta_{i0}\delta_{j0}f(n,p,0) \tag{3.4}$$

and

$$f_{i,j}\,(n,p,0,a,b) = \delta_{a0}\delta_{b0}f_{i,j}\,(n,p,0). \tag{3.5}$$

Thus $f(n,p,0) = 0$ unless $n = p = 0$, $f_{i,j}(n,p,0) = 0$ unless $n = p = i = j = 0$, $f_{i,j}\,(n,p,0,a,b) = 0$ unless $n = p = i = j = a = b = 0$.

It follows from the definition that

$$f(n,p,k) = \sum_{i,j=0}^{m-1} f_{i,j}\,(n,p,k) \tag{3.6}$$

$$= \sum_{i,j=0}^{m-1} \sum_{a=0}^{n} \sum_{b=0}^{p} f_{i,j}(n,p,k,a,b) \quad (n \geq 0,\ p \geq 0,\ k \geq 0).$$

Moreover, we have the recurrence

$$f_{i,j}(n,p,k,a,b) = \sum_{\substack{i',j'=0 \\ (i',j') \neq (i,j)}}^{m-1} \sum_{a=0}^{n} \sum_{b=0}^{p} f_{i',j'}(n,p,k,a,b)$$

$$[k > 0,\ a \equiv i,\ b \equiv j \pmod{m}].$$

This reduces to

(3.7) $$f_{i,j}(n,p,k,a,b) = \sum_{\substack{i',j'=0 \\ (i',j') \neq (i,j)}}^{m-1} f_{i',j'}(n-a,\ p-b,\ k-1)$$

$$[k > 0,\ a \equiv i,\ b \equiv j \pmod{m}].$$

Corresponding to the enumerants, we define a number of generating functions:

$$\begin{cases} F_{i,j}(x,y,z) = \displaystyle\sum_{n,p,k=0}^{\infty} f_{i,j}(n,p,k)x^n y^p z^k \\ F(x,y,z) = \displaystyle\sum_{n,p,k=0}^{\infty} f(n,p,k)x^n y^p z^k \\ F_{i,j}(x,y,z,a,b) = \displaystyle\sum_{n,p,k=0}^{\infty} f_{i,j}(n,p,k,a,b)x^n y^p z^k. \end{cases}$$

Since

$$\begin{cases} f_{0,0}(n,p,1,a,b) = \delta_{na}\delta_{pb} & [a \equiv b \equiv 0 \pmod{m}] \\ f_{0,0}(n,p,0,a,b) = \delta_{na}\delta_{pb}\delta_{n0}\delta_{p0}, \end{cases}$$

it follows that

$$F_{0,0}(x,y,z,a,b) = \delta_{a0}\delta_{b0} + x^a y^b z + x^a y^b z \sum_{(i,j) \neq (0,0)} F_{i,j}(x,y,z)$$

$$[a \equiv b \equiv 0 \pmod{m}].$$

Summing over a and b, we get

(3.8) $$F_{0,0}(x,y,z) = 1 + \frac{z}{(1-x^m)(1-y^m)} + \frac{z}{(1-x^m)(1-y^m)} \sum_{(i,j) \neq (0,0)} F_{i,j}(x,y,z)$$

On the other hand, for $(i,j) \neq (0,1)$ and $a \equiv i$, $b \equiv j \pmod{m}$, it follows from (3.7) that

$$F_{i,j}(x,y,z,a,b) = \sum_{n,p,k} x^n y^p z^k \sum_{(i',j') \neq (i,j)} f_{i',j'}(n-a,\ p-b,\ k-1)$$

$$= x^a y^b z \sum_{(i',j') \neq (i,j)} F_{i',j'}(x,y,z).$$

Hence, summing over a and b, we get

$$(3.9)\qquad F_{i,j}(x,y,z) = \frac{x^i y^j z}{(1-x^m)(1-y^m)} \sum_{(i',j')\neq(i,j)} F_{i,j}(x,y,z) \qquad [(i,j)\neq(0,0)].$$

Since

$$\sum_{(i',j')\neq(i,j)} F_{i',j'}(x,y,z) = F(x,y,z) - F_{i,j}(x,y,z),$$

(3.8) and (3.9) become

$$\left(1 + \frac{z}{(1-x^m)(1-y^m)}\right) F_{0,0}(x,y,z) = 1 + \frac{z}{(1-x^m)(1-y^m)} + \frac{z}{(1-x^m)(1-y^m)} F(x,y,z)$$

and

$$\left(1 + \frac{x^i y^j z}{(1-x^m)(1-y^m)}\right) F_{i,j}(x,y,z) = \frac{x^i y^j z}{(1-x^m)(1-y^m)} F(x,y,z) \qquad (i,j)\neq(0,0),$$

respectively. Hence,

$$(3.10)\qquad \begin{cases} F_{0,0}(x,y,z) = 1 + \dfrac{\dfrac{z}{(1-x^m)(1-y^m)}}{1 + \dfrac{z}{(1-x^m)(1-y^m)}} F(x,y,z) \\[2ex] F_{i,j}(x,y,z) = \dfrac{\dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}{1 + \dfrac{x^i y^j z}{(1-x^m)(1-y^m)}} F(x,y,z) \end{cases} \qquad [(i,j)\neq(0,0)].$$

Summing over the m^2 equations in (3.10), we get

$$(3.11)\qquad \left\{1 - \sum_{i,j=0}^{m-1} \frac{\dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}{1 + \dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}\right\} F(x,y,z) = 1.$$

For brevity, put

$$\lambda = \frac{z}{(1-x^m)(1-y^m)}$$

so that (3.11) becomes

$$\left\{1 - \sum_{i,j=0}^{m-1} \frac{x^i y^j \lambda}{1 + x^i y^j \lambda}\right\} F(x,y,z) = 1. \tag{3.12}$$

Let

$$P_m(\lambda) = P_m(\lambda,x,y) = \prod_{i,j=0}^{m-1} (1 + x^i y^j \lambda); \tag{3.13}$$

clearly $P_m(\lambda)$ is a polynomial in λ of degree m^2. By logarithmic differentiation

$$\frac{\lambda P_m'(\lambda)}{P_m(\lambda)} = \sum_{i,j=0}^{m-1} \frac{x^i y^j \lambda}{1 + x^i y^j \lambda}.$$

Thus (3.12) becomes

$$F(x,y,z) = \frac{P_m(\lambda)}{Q_m(\lambda)} \qquad \lambda = \frac{z}{(1 - x^m)(1 - y^m)}, \tag{3.14}$$

where

$$Q_m(\lambda) = P_m(\lambda) - P_m'(\lambda). \tag{3.15}$$

For example, for $m = 2$,

$$\begin{cases} P_2(\lambda) = 1 + (1 + x)(1 + y)\lambda + (x + y + 2xy + x^2y + xy^2)\lambda^2 \\ \qquad\qquad + xy(1 + x)(1 + y)\lambda^3 + x^2y^2\lambda^4 \\ Q_2(\lambda) = 1 - (x + y + 2xy + x^2y + xy^2)\lambda^2 \\ \qquad\qquad - 2xy(1 + x)(1 + y)\lambda^3 - 3\lambda^4. \end{cases}$$

SECTION 4

As in [2], the limiting case, $m = \infty$ of $f(n,p,k)$, is closely related to $c(n,p,k)$. We assume $|x| < 1$, $|y| < 1$, so that

$$\lambda = \frac{z}{(1 - x^m)(1 - y^m)} \to z \quad (m \to \infty).$$

Thus, (3.12) becomes

$$\left\{1 - \sum_{i,j=0}^{\infty} \frac{x^i y^j z}{1 + x^i y^j z}\right\} F^*(x,y,z) = 1, \tag{4.1}$$

where

$$F^*(x,y,z) = \lim F(x,y,z).$$

Now

$$\sum_{i,j=0}^{\infty} \frac{x^i y^j z}{1 + x^i y^j z} = \sum_{i,j=0}^{\infty} \sum_{s=1}^{\infty} (-1)^{s-1} x^{is} y^{js} z^s$$

$$= \sum_{s=1}^{\infty} (-1)^{s-1} \frac{z^s}{(1 - x^s)(1 - y^s)}.$$

Hence, we may replace (4.1) by

$$\left\{1 + \sum_{s-1}^{\infty} (-1)^s \frac{z^s}{(1 - x^s)(1 - y^s)}\right\} F^*(x,y,z) = 1. \tag{4.2}$$

Comparing (4.2) with (2.16) and (2.16)′, it follows at once that

$$f^*(n,p,k) = c(n,p,k), \tag{4.3}$$

where $f^*(n,p,k)$ is the limiting case ($m = \infty$) of $f(n,p,k)$; (4.3) is of course to be expected from the definitions.

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THE RECURRENCE RELATION $(r + 1)f_{r+1} = xf_r' + (K - r + 1)x^2 f_{r-1}$

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1. INTRODUCTION

In a recent note, in [3], Worster conjectured, on the basis of computer calculations, that for each positive integer k there exists an odd polynomial $Q_{2k-1}(x)$ of degree $2k - 1$ such that, for every zero a of the Bessel function $J_0(x)$

$$\int_0^a Q_{2k-1}(x)[J_0(x)]^{2k}\,dx = [aJ_1(a)]^{2k}.$$

The conjecture was extended and proved in [1] the extended result being: for each positive k there exists an odd polynomial $Q(x)$, with nonnegative integer coefficients and of degree k or $k - 1$ according to whether k is odd or even, such that for every zero a of $J_0(x)$

$$\int_0^a Q(x)[J_0(x)]^k\,dx = (k - 1)![aJ_1(a)]^k. \tag{1.1}$$

If the factor $(k - 1)!$ on the right-hand side is omitted, then the coefficients in $Q(x)$ are no longer integers. In addition, [1] also contained the following generalization due to Hammersley: if F_0, F_1, G_0, and G_1 are four functions of x such that

$$G_0\frac{dF_0}{dx} = -F_1, \qquad \frac{dF_1}{dx} = G_1F_0,$$

$$\text{and } F_0(a) = G_0(0) = 0, \text{ so that } F_1(0) = 0,$$

then there exists $Q(x)$ depending only on G_0, G_1, and K with the property

$$(1.2)\qquad (k-1)![F_1(a)]^k = \int_0^a Q(x)[F_0(x)]^k dx.$$

As is observed in [1], Worster's extended conjecture corresponds to the case $G_0(x) = G_1(x) = x$.

Subsequently there has been some interest (see [2]) in the determination of the coefficients occurring in the Worster polynomial $Q(x)$. In this paper we show that by considering a certain recurrence relation, namely that given in the title, the coefficients can be expressed as multiple sums. Also, we show how to determine these multiple sums analytically and numerically. To obtain the recurrence relation, which is central to the work, we first consider an alternative proof to that given in [1] of Hammersley's generalization of Worster's conjecture.

SECTION 2

We begin by defining the function $\phi(x)$ by

$$\phi(x) = \sum_{r=0}^{k} f_r(x) F_0^r(x) F_1^{k-r}(x),$$

where $f_0(x), f_1(x), \ldots, f_k(x)$ is some sequence of functions which, for the moment we leave unspecified. Differentiating the expression for $\phi(x)$, and omitting the argument x occurring in the various functions, we have

$$\phi' = \sum_{r=0}^{k}\left\{f_r' F_0^r F_1^{k-r} + f_r(rF_0^{r-1}F_0'F_1^{k-r} + (k-r)F_0^r F_1^{k-r-1}F_1')\right\}.$$

Since $G_0F_0' = -F_1$ and $F_1' = G_1F_0$, we obtain

$$\phi' = \sum_{r=0}^{k}\left\{f_r' F_0^r F_1^{k-r} - \frac{rf_r}{G_0}F_0^{r-1}F_1^{k-r-1} + (k-r)f_rG_1F_0^{r+1}F_1^{k-r+1}\right\}.$$

This can be put in the alternative and more convenient form

$$\phi' = \left\{f_0' - \frac{f_1}{G_0}\right\}F_1^k + \sum_{r=1}^{k-1}\left[f_r' - \frac{(r+1)}{G_0}f_{r+1} + (k-r+1)f_{r-1}G_1\right]F_0^rF_1^{k-r}$$
$$+ (f_k' + f_{k-1}G_1)F_0^k.$$

We put $f_0 = (k-1)!$ and choose the functions $f_1, f_2, \ldots, f_k$ so that the coefficients of $F_0^rF_1^{k-r}$, $r = 0, 1, 2, \ldots, k-1$ vanish. It immediately follows that $f_1 = 0$, while

$$(2.1)\qquad (r+1)f_{r+1} = G_0\left\{f_r' + (k-r+1)f_{r-1}G_1\right\},\quad r = 1, 2, \ldots, k-1.$$

The sequence of functions $f_0, f_1, \ldots, f_k$ is now completely defined, and it clearly depends only on k, G_0, and G_1. For $r \geq 2$, $f_r(0) = 0$ since $G_0(0) = 0$.

The expression for ϕ' reduces to

$$(2.2)\qquad \phi' = (f_k' + f_{k-1}G_1)F_0^k.$$

Integrating (2.2) with respect to x between 0 and a, we obtain, reinserting arguments where appropriate,

$$\left[\sum_{r=0}^{k} f_r(x)F_0^r(x)F_1^{k-r}(x)\right]_0^a = \int_0^a (f_k' + f_{k-1}G_1)F_0^k dx.$$

Using the properties of the various functions on the left-hand side of this equation, we deduce

$$(k-1)!F_1^k(a) = \int_0^a (f_k' + f_{k-1}G_1)F_0^k dx.$$

Hence, the generalization stated in (2.2) follows immediately if we take

$$Q(x) = f_k' + f_{k-1}G_1.$$

If we define f_{k+1} by putting $r = k$ in (2.1), then

$$Q(x) = \frac{(k+1)}{G_0} f_{k+1}.$$

Omitting the factor $(k-1)!$ occurring in (1.1) we see that the determination of $Q(x)$ for the Worster problem is achieved by solving

$$\begin{aligned} & f_0 = 1, \quad f_1 = 0 \\ (2.3) \qquad & (r+1)f_{r+1} = xf_r' + (k - r + 1)x^2 f_{r-1}, \; r = 1, 2, \ldots, k \\ & xQ(x) = (k+1)f_{k+1} \end{aligned}$$

The following are readily deduced:

$$\begin{aligned} & f_2 = \frac{kx^2}{2!}, \; f_3 = \frac{2kx^2}{3!}, \; f_4 = \frac{2^2kx^2}{4!} + 3k(k-2)\frac{x^4}{4!} \\ (2.4) \qquad & f_5 = \frac{2^3kx^2}{5!} + \left\{3\cdot 4k(k-2) + 2\cdot 4k(k-3)\right\}\frac{x^4}{5!} \\ & f_6 = \frac{2^4kx^2}{6!} + \left\{3\cdot 4^2k(k-2) + 2\cdot 4^2k(k-3) + 2^2 5k(k-4)\right\}\frac{x^4}{6!} \\ & \qquad + 3\cdot 5k(k-2)(k-4)\frac{x^6}{6!} \end{aligned}$$

Thus, we can find the first four of the polynomials $Q(x)$. These correspond to $k = 2, 3, 4$, and 5, respectively. We now proceed to establish a number of results concerning the functions f_r. From these, we deduce expressions for the coefficients of the powers of x in $Q(x)$.

SECTION 3

It is first convenient to prove the following results for multiple sums

$$(3.1) \qquad \sum_{q=3}^{n-1}\sum_{p=q+2}^{n+1} a_{qp} = \sum_{q=3}^{n-2}\sum_{p=q+2}^{n} a_{qp} + \sum_{q=3}^{n-1} a_{q,n+1}$$

and

$$(3.2) \qquad \sum_{q=3}^{n-3}\sum_{p=q+2}^{n-1}\sum_{\ell=p+2}^{n+1} a_{qp\ell} = \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2}\sum_{\ell=p+2}^{n} a_{qp\ell} + \sum_{q=3}^{n-3}\sum_{p=q+2}^{n-1} a_{qp,n+1}.$$

We have

$$\sum_{q=3}^{n-2}\sum_{p=q+2}^{n} a_{qp} = \sum_{q=3}^{n-2}\left\{\sum_{p=q+2}^{n+1} a_{qp} - a_{q,n+1}\right\} = \left\{\sum_{q=3}^{n-1} - \sum_{q=n-1}^{n-1}\right\}\left\{\sum_{p=q+2}^{n+1} a_{qp}\right\} - \sum_{q=3}^{n-2} a_{q,n+1}.$$

When $q = n - 1$, p can only take the value $n + 1$, so that the above expression reduces to

$$\sum_{q=3}^{n-1}\sum_{p=q+2}^{n+1} a_{qp} - a_{n-1,n+1} - \sum_{q=3}^{n-2} a_{q,n+1} = \sum_{q=3}^{n-1}\sum_{p=q+2}^{n+1} a_{qp} - \sum_{q=3}^{n-1} a_{q,n+1}.$$

Thus the result given in (3.1) now follows. To prove (3.2) we proceed similarly.

$$\begin{aligned}\sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2}\sum_{\ell=p+2}^{n} a_{qp\ell} &= \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2}\left\{\sum_{\ell=p+2}^{n+1} - \sum_{\ell=n+1}^{n+1}\right\} a_{qp\ell}\\ &= \sum_{q=3}^{n-4}\left\{\sum_{p=q+2}^{n-1} - \sum_{p=n-1}^{n-1}\right\}\sum_{\ell=p+2}^{n+1} a_{qp\ell} - \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2} a_{qp,n+1}\\ &= \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-1}\sum_{\ell=p+2}^{n+1} a_{qp\ell} - \sum_{q=3}^{n-4} a_{q,n-1,n+1} - \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2} a_{qp,n+1}\end{aligned}$$

since ℓ can only take the value $n + 1$ when $p = n - 1$. Continuing, we have

$$\begin{aligned}\sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2}\sum_{\ell=p+2}^{n} a_{qp\ell} &= \sum_{q=3}^{n-3}\sum_{p=q+2}^{n-1}\sum_{\ell=p+2}^{n+1} a_{qp\ell} - a_{n-3,n-1,n+1} - \sum_{q=3}^{n-4} a_{q,n-1,n+1}\\ &\qquad - \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2} a_{qp,n+1}\\ &= \sum_{q=3}^{n-3}\sum_{p=q+2}^{n-1}\sum_{\ell=p+2}^{n+1} a_{qp\ell} - \sum_{q=3}^{n-3} a_{q,n-1,n+1} - \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-2} a_{qp,n+1}.\end{aligned} \tag{3.3}$$

Using (3.1) with $a_{qp,n+1}$ instead of a_{qp} and n replaced by $n - 2$ now leads us directly to (3.2). The results given in (3.1) and (3.2) can be extended to quadruple and higher-tuple sums. Thus, for quadruple sums the analogous result to (3.3) is

$$\begin{aligned}&\sum_{q=3}^{n-6}\sum_{p=q+2}^{n-4}\sum_{\ell=p+2}^{n-2}\sum_{j=\ell+2}^{n} a_{qp\ell j}\\ &\quad= \sum_{q=3}^{n-5}\sum_{p=q+2}^{n-3}\sum_{\ell=p+2}^{n-1}\sum_{j=\ell+2}^{n+1} a_{qp\ell j} - a_{n-5,n-3,n-1,n+1} - \sum_{q=3}^{n-6} a_{q,n-3,n-1,n+1}\\ &\qquad - \sum_{q=3}^{n-4}\sum_{p=q+2}^{n-6} a_{qp,n-1,n+1} - \sum_{q=3}^{n-6}\sum_{p=q+2}^{n-4}\sum_{\ell=p+2}^{n-2} a_{qp\ell,n+1}.\end{aligned}$$

If we now apply (3.1) and (3.2) to this equation, we obtain the result for the quadruple sum. The general result for p-tuple sums can be written as follows:

$$\begin{aligned}&\sum_{q_1=3}^{n-2p+3}\sum_{q_2=q_1+2}^{n-2p+5}\cdots\sum_{q_i=q_{i-1}+2}^{n-2p+2i+1}\cdots\sum_{q_p=q_{p-1}+2}^{n+1} a_{q_1q_2\,\cdots\,q_p} = \sum_{q_1=3}^{n-2p+2}\sum_{q_2=q_1+2}^{n-2p+4}\cdots\sum_{q_i=q_{i-1}+2}^{n-2p+2i}\\ &\qquad\cdots\sum_{q_p=q_{p-1}+2}^{n} a_{q_1q_2\,\cdots\,q_p} + \sum_{q_1=3}^{n-2p+3}\sum_{q_2=q_1+2}^{n-2p+5}\cdots\sum_{q_{p-1}=q_{p-2}+2}^{n-1} a_{q_1q_2,q_{p-1},n+1}.\end{aligned} \tag{3.4}$$

The first of our results concerning the sequence of functions f_r is

(i) f_{2r}, f_{2r+1}, where $r \geq 1$, are even polynomials of degree $2r$, the least power in each being that of x^2. This can be readily established using the recurrence relation in (2.3), the expressions in (2.4), and induction. Next, we prove:

(ii) the coefficient of x^2 in f_{r+1} is $\dfrac{2^{r-1}k}{(r+1)!}$, $r = 1, 2, 3, \ldots$.

From the recurrence relation (2.3), we have that

$$f_{r+2} = \frac{x}{r+2} f'_{r+1} + \frac{x^2(k-r)}{r+2} f_r$$

Hence we see, with the help of (i), that the term in x^2 in f_{r+2} will arise from differentiating the term in x^2 in f_{r+1} and multiplying by

$$\frac{x}{r+2}.$$

Assuming the result stated in (ii) is true for a specific r, then we have that the coefficient of x^2 in f_{r+2} is

$$\frac{2^r k}{(r+2)!}.$$

Thus, induction with the aid of (2.4) completes the proof.

(iii) The coefficient of x^4 in f_{r+1} is

$$\frac{k}{(r+1)!}\sum_{q=3}^{r} q(k-q+1)4^{r-q}2^{q-3}, \quad r \geq 3.$$

From the recurrence relation, we observe that the term in x^4 in f_{r+2} arises from the term in x^2 in f_r and the differentiation of the term in x^4 in f_{r+1}. Assuming that (iii) is true for fixed r, then we have with the aid of (ii) that the coefficient of x^4 in f_{r+2} is

$$\frac{(k-r)2^{r-2}k}{(r+2)r!} + \frac{4k}{(r+2)!}\sum_{q=3}^{r} q(k-q+1)4^{r-q}2^{q-3}$$

which reduces to

$$\frac{k}{(r+2)!}\sum_{q=3}^{r+1} q(k-q+1)4^{r+1-q}2^{q-3}.$$

Noting the expression for f_4 in (2.4) we see that induction completes our proof.

(iv) The coefficient of x^6 in f_{r+1} for $r \geq 5$ is

$$\frac{k}{(r+1)!}\sum_{q=3}^{r-2}\sum_{p=q+2}^{r} q(k-q+1)p(k-p+1)6^{r-p}4^{p-q-2}2^{q-3}.$$

The recurrence formula shows that to obtain the term in x^6 in f_{r+2} we must consider the term in x^4 in f_r and the result of differentiating the term in x^6 in f_{r+1}. If (iv) holds for a definite r then the coefficient of x^6 in f_{r+2} is seen, with the help of (iii), to be

$$\begin{aligned}
&\frac{k-r}{r+2}\,\frac{k}{r!}\sum_{q=3}^{r-1} q(k-q+1)4^{r-1-q}2^{q-3} \\
&\quad + \frac{6k}{(r+2)!}\sum_{q=3}^{r-2}\sum_{p=q+2}^{r} q(k-q+1)p(k-p+1)6^{r-p}4^{p-q-2}2^{q-3} \\
&= \frac{k}{(r+2)!}\sum_{q=3}^{r-1}(k-r)(r+1)q(k-q+1)4^{r-1-q}2^{q-3} \\
&\quad + \sum_{q=3}^{r-2}\sum_{p=q+2}^{r} q(k-q+1)p(k-p+1)6^{r+1-p}4^{p-q-2}2^{q-3}.
\end{aligned}$$

If we take

$$a_{qp} = q(k-q+1)p(k-p+1)6^{r+1-p}4^{p-q-2}2^{q-3},$$

we find

$$a_{q,r+1} = q(k-r)(r+1)q(k-q+1)4^{r-1-q}2^{q-3},$$

so that applying (3.1) with r instead of n we have the required coefficient of x^6 in f_{r+2}:

$$\frac{k}{(r+2)!}\sum_{q=3}^{r-1}\sum_{p=q+2}^{r+1} q(k-q+1)p(k-p+1)6^{r+1-p}4^{p-q-2}2^{q-3}.$$

Induction now completes our proof.

(v) The coefficient of x^{2r} in f_{2r}, $r \geq 3$, is

$$\frac{\left(\frac{k}{2}\right)!}{r!\left(\frac{k}{2}-r\right)!}$$

When k is odd, we take $\left(\frac{k}{2}\right)!$ and $\left(\frac{k}{2}-r\right)!$ to be generalized factorial functions. Use of the recurrence relation (2.3) yields

$$f_{2r+2} = \frac{x}{2r+2}f_{2r+1}' + x^2\,\frac{(k-2r)}{2r+2}f_{2r}.$$

Noting (i), we see that it is the term

$$\frac{x^2(k-2r)}{2r+2}f_{2r}$$

which gives rise to the power x^{2r+2} in f_{2r+2}. Thus if (v) is correct for fixed r, then the coefficient of x^{2r+2} in f_{2r+2} is

$$\frac{(k-2r)\left(\frac{k}{2}\right)!}{(2r+2)r!\left(\frac{k}{2}-r\right)!} = \frac{\left(\frac{k}{2}\right)!}{(r+1)!\left(\frac{k}{2}-r-1\right)!}$$

Once more induction, with the help of the expression for f_6 in (2.4), completes our proof.

(vi) The coefficient of x^{2t} in f_{r+1}, $3 \le t \le \left[\frac{r+1}{2}\right]$, $r \ge 5$, ..., is given by $S(r,t)$ where

$$S(r,t) = \frac{k}{(r+1)!} \sum_{q_1=3}^{r-2t+4} \sum_{q_2=q_1+2}^{r-2t+6} \cdots \sum_{q_i=q_{i-1}+2}^{r-2t+2i+2} \cdots \sum_{q_{t-1}=q_{t-2}+2}^{r} a_{q_1 q_2 \cdots q_{t-1}}(r,t)$$

and

$$a_{q_1 q_2 \cdots q_{t-1}}(r,t) = (2t)^{r-q_{t-1}} 2^{q_1-3} \prod_{j=1}^{t-1} q_j (k - q_j + 1) \prod_{j=2}^{t-1} (2j)^{q_j - q_{j-1} - 2}.$$

From the given expression, it is evident that $S(r,t)$ is a $(t-1)$-tuple sum. It is readily verified that (vi) reduces to (iv) when $t = 3$. Further, some elementary manipulation shows that:

$$S(2r-1,r) = \frac{\left(\frac{k}{2}\right)!}{r!\left(\frac{k}{2} - r\right)!}$$

so that (vi) also agrees with the result in (v). It is perhaps worth noting that the q_i in this latter case each take just one value, viz. $q_i = 1 + 2i$ ($i = 1, 2, \ldots, r-1$). To prove (vi) we first show that if for fixed r and t the coefficients of x^{2t} in f_r and x^{2t-2} in f_{r-1} are given, respectively, by $S(r,t)$ and $S(r-1,t-1)$ then $S(r+1,t)$ is the coefficient of x^{2t} in f_{r+2}. Using the recurrence relation (2.3) in the form

$$f_{r+2} = \frac{x}{r+2} f'_{r+1} + \frac{k - r + 1}{r + 2} x^2 f_r$$

we have that the coefficient of x^{2t} in f_{r+2} is

$$\frac{2t}{r+2} S(r,t) + \frac{k - r + 1}{r+2} S(r-1,t-1)$$

which is equal to

$$\frac{k}{(r+2)!} \left\{ \sum_{q_1=3}^{r-2t+4} \cdots \sum_{q_{t-1}=q_{t-2}+2}^{r} a_{q_1 q_2 \cdots q_{t-1}}(r+1,t) + \right. \tag{3.5}$$
$$\left. \sum_{q_1=3}^{r-2t+5} \cdots \sum_{q_{t-2}=q_{t-3}+2}^{r-1} \{a_{q_1 q_2 \cdots q_{t-2}}(r-1,t-1)\}(k-1+1)(r+1) \right\}$$

Now

$$\begin{aligned} &a_{q_1 q_2 \cdots q_{t-2}, r+1}(r+1,t) \\ &= 2^{q_1-3}(r+1)(k-r+1) \prod_{j=1}^{t-2} q_j (k - q_j + 1) \\ &\qquad \times \prod_{j=2}^{t-2} (2j)^{q_j - q_{j-1} - 2} (2t-2)^{r-1-q_{t-2}} \\ &= (k-r+1)(r+1) a_{q_1, q_2, \ldots, q_{t-2}}(r-1,t-1). \end{aligned}$$

Hence, using (3.4) with n replaced by r and p by $t-1$ we can see that (3.5) reduces to $S(r+1,t)$. As already observed, the formula shown in (vi) correctly gives the coefficient of x^6 in $f_6, f_7, f_8, \ldots$, and also the coefficients of x^8 in f_8, x^{10} in f_{10}, x^{12} in f_{12}, etc. Hence by the result just proved with $2t = r = 8$ (vi) correctly gives the coefficient of x^8 in f_9. Applying the result again with $2t = r - 1 = 8$, we see that formula (vi) correctly gives the coefficient of x^8 in f_{10}. Thus, continuing the process, we prove that formula (vi) is also correct for the coefficient of x^8 in $f_{11}, f_{12}, \ldots$. The process is now repeated, starting with $2t = r = 10$. By this means, we successively establish the formula for the coefficients of x^8, x^{10}, x^{12}, etc.

From (2.3) we have $xQ(x) = (k+1)f_{k+1}$, so that it is now possible to deduce a number of results concerning $Q(x)$. These are:

the coefficient of x is $\dfrac{2^{k-1}}{(k-1)!}$,

that of x^3 is $\dfrac{1}{(k-1)!}\displaystyle\sum_{q=3}^{k} q(k-q+1)4^{k-q}2^{q-3}$, and

that of x^{2t-1} $(t \geq 3)$ is the $(t-1)$-tuple sum

$$\frac{1}{(k-1)!}\sum_{q_1=3}^{k-2t+4}\ \sum_{q_2=q_1+2}^{k-2t+6}\cdots\sum_{q_{t-1}=q_{t-2}+2}^{k} a_{q_1q_2\cdots q_{t-1}}(k,t)$$

where

$$a_{q_1q_2\cdots q_{t-1}}(k,t) = (2t)^{k-q_{t-1}}2^{q_1-3}\prod_{j=1}^{t-1} q_j(k-q_j+1) \times \prod_{j=2}^{t-1}(2j)^{q_j-q_{j-1}-2}.$$

In the next section we show how the multiple sums can be determined and find them in certain cases.

SECTION 4

Referring to the end of the last section we see that the coefficient of x^3 in $Q(x)$ can be written as

$$\frac{2^{k-3}}{(k-1)!}S(k)$$

where

(4.1) $$S(k) = \sum_{q=3}^{k} q(k-q+1)2^{k-q}.$$

We now put

(4.2) $$S(k) = kS_1(k) - S_2(k)$$

where

(4.3) $$S_1(k) = \sum_{q=3}^{k} q2^{k-q}$$

and

(4.4) $$S_2(k) = \sum_{q=3}^{k} q(q - 1)2^{k-q}.$$

These series have the sums

(4.5) $$S_1(k) = 2^k - k - 2$$

and

(4.6) $$S_2(k) = 72^{k-1} - k^2 - 3k - 4.$$

Hence

$$S(k) = (2k - 7)2^{k-1} + k + 4,$$

giving the coefficient of x^3 as

$$\frac{2^{k-3}}{(k - 1)!}\{2^{k-1}(2k - 7) + k + 4\}.$$

It is perhaps worth noting that this expression vanishes for $k = 1$ and 2.

Again referring to the end of Section 3, we see that the coefficient of x^5 in $Q(x)$ can be written as

$$\frac{2^{k-5}}{(k - 1)!}T(k),$$

where

$$T(k) = \sum_{q=3}^{k-2} \sum_{p=q+2}^{k} \{kq - q(q - 1)\}\{kp - p(p - 1)\}3^{k-p}2^{p-q-2}.$$

Putting

(4.7) $$T(k) = k^2 T_1(k) - kT_2(k) + T_3(k),$$

then

(4.8) $$T_1(k) = \sum_{q=3}^{k-2} \sum_{p=q+2}^{k} pq3^{k-p}2^{p-q-2}$$

(4.9) $$T_2(k) = \sum_{q=3}^{k-2} \sum_{p=q+2}^{k} \{pq(q - 1) + qp(p - 1)\}3^{k-p}2^{p-q-2},$$

and

(4.10) $$T_3(k) = \sum_{q=3}^{k-2} \sum_{p=q+2}^{k} q(q - 1)p(p - 1)3^{k-p}2^{p-q-2}.$$

With the help of (3.1), (4.3), (4.4), and (4.8) to (4.10), we deduce

(4.11) $$\begin{aligned} T_1(k) &= 3T_1(k - 1) + kS_1(k - 2) \\ T_2(k) &= 3T_2(k - 1) + k(k - 1)S_1(k - 2) + kS_2(k - 2) \\ T_3(k) &= 3T_3(k - 1) + k(k - 1)S_2(k - 2). \end{aligned}$$

Since $T_1(5) = 15$, $T_2(5) = 90$, and $T_3(5) = 120$, these recurrence relations enable us, with the help of (4.5) and (4.6), to find $T_1(k)$, $T_2(k)$, and $T_3(k)$ numerically, and hence, from (4.7), we can determine $T(k)$. We can also use the recurrence relations to find analytical expressions for the $T_i(k)$, $i = 1, 2, 3$. The method is the same in each instance. Therefore, we illustrate it by considering $T_1(k)$, then stating corresponding results for $T_2(k)$ and $T_3(k)$. The method depends on recognizing that the recurrence relation (4.11) and the condition $T_1(5) = 15$ can be satisfied by taking $T_1(k)$ in the form

(4.12) $$T_1(k) = f_3(k)3^k + f_2(k)2^k + f_1(k),$$

where $f_1(k)$, $f_2(k)$, and $f_3(k)$ are polynomials in k. It is perhaps worth emphasizing that once we have a solution for $T_1(k)$ it will be the solution. Inspection suggests we write

$$T_1(k) = a_0 3^k + (b_0 + b_1 k)2^k + c_0 + c_1 k + c_2 k^2. \tag{4.13}$$

From (4.11) and (4.5), we have

$$\begin{aligned} &a_0 3^k + (b_0 + b_1 k)2^k + c_0 + c_1 k + c_2 k^2 \\ &= a_0 3^k + \frac{3}{2}(b_0 + b_1(k-1))2^k + 3(c_0 + c_1(k-1) \\ &\qquad + c_2(k-1))^2 + k(2^{k-2} - k). \end{aligned}$$

Comparing coefficients, we obtain

$$b_1 = -\frac{1}{2},\ b_0 = -\frac{3}{2},\ c_2 = \frac{1}{2},\ c_1 = \frac{3}{2},\ \text{and } c_0 = \frac{3}{2}$$

while a_0 is indeterminate. To obtain a_0 we can proceed in two ways. First, we calculate a_0 from (4.13) by putting $k = 5$ and noting that $T_1(5) = 15$. This gives $a_0 = 1/2$. Second, we observe that we can regard $T_1(k)$ as being defined for all k by (4.5), (4.11), and $T_1(5) = 15$; thus, determine $T_1(0)$ and so obtain a_0 by putting $k = 0$ in (4.13). This is a somewhat easier procedure to carry out computationally than the first. It is readily found that $T_1(4) = T_1(3) = 0$, $T_1(2) = T_1(1) = 1$, and $T_1(0) = 1/2$, again giving us $a_0 = 1/2$. So,

$$T_1(k) = \frac{1}{2}3^k - (k+3)2^{k-1} + \frac{1}{2}(k^2 + 3k + 3). \tag{4.14}$$

Likewise, we find $T_2(4) = T_2(3) = T_3(4) = T_3(3) = 0$, $T_2(2) = 3$, $T_2(1) = 2$, $T_2(0) = 3/4$, $T_3(2) = 2$, $T_3(1) = 1$, and $T_3(0) = 1/3$. Assuming appropriate forms for $T_2(k)$ and $T_3(k)$, we obtain

$$T_2(k) = \frac{21}{4}3^k - (2k^2 + 17k + 45)2^{k-2} + k^3 + \frac{7k^2}{2} + 7k + \frac{27}{4} \tag{4.15}$$

and

$$\begin{aligned} T_3(k) = \frac{139}{4}3^{k-1} - 7(k^2 + 5k + 12)2^{k-2} + \frac{k^4}{2} + 2k^3 \\ + 6k^2 + 11k + \frac{39}{4} \end{aligned} \tag{4.16}$$

so that the coefficient of x^5 is

$$\begin{aligned} \frac{2^{k-7}}{(k-1)!}\Big\{3^{k-1}(6k^2 - 63k + 139) + 2^{k+1}(2k-7)(k+6) \\ + 2k^2 + 17k + 39\Big\}. \end{aligned}$$

We note that this last expression vanishes for $k = 1, 2, 3$, and 4.

We now proceed to find the coefficient of x^7 in $Q(x)$. Since the procedure is similar to that for finding the coefficient of x^5, we merely state the essential results. Suffix notation employed in the expression for the coefficient of x^{2t-1} ($t \geq 3$) is not used here; it is sufficient to write the coefficient of x^7 as

$$\frac{2^{k-7}}{(k-1)!}R(k)$$

where

$$R(k) = \sum_{q=3}^{k-4} \sum_{p=q+2}^{k-2} \sum_{r=p+2}^{k} \{kq - q(q - 1)\}\{kp - p(p - 1)\}\{kr - r(r - 1)\} 4^{k-r} 3^{r-p-2} 2^{p-q-2}$$

$$= k^3 R_1(k) - k^2 R_2(k) + kR_3(k) - R_4(k).$$

Now

$$R_1(k) = 4R_1(k - 1) + kT_1(k - 2)$$
$$R_2(k) = 4R_2(k - 1) + k(k - 1)T_1(k - 2) + kT_2(k - 2)$$
$$R_3(k) = 4R_3(k - 1) + k(k - 1)T_2(k - 2) + KT_3(k - 2)$$
$$R_4(k) = 4R_4(k - 1) + k(k - 1)T_3(k - 2).$$

We deduce, with the help of the results for $T_i(k)$,

$$R_1(0) = -\frac{1}{6},\ R_2(0) = -\frac{1}{2},\ R_3(0) = -\frac{41}{72},\ R_4(0) = -\frac{11}{48}.$$

Again, making appropriate choice of forms, we obtain

$$R_1(k) = \frac{4^k}{6} - \frac{3^{k-1}}{2}(k + 4) + 2^{k-3}(k^2 + 5k + 8) - \frac{k^3}{6} - \frac{k^2}{2} - \frac{5k}{6} - \frac{2}{3}$$

$$R_2(k) = \frac{7}{2}4^k - \frac{3^{k-1}}{4}\{2k^2 + 35k + 132\} + 2^{k-4}\{4k^3 + 33k^2 + 125k + 192\} - \frac{k^4}{2} - \frac{3k^3}{2} - 4k^2 - \frac{27k}{4} - 5$$

$$R_3(k) = \frac{1553}{72}4^k - 3^k\left\{\frac{7k^2}{9} + \frac{145k}{9} + \frac{517}{9}\right\} + 2^{k-4}\{2k^4 + 30k^3 + 173k^2 + 551k + 812\} - \frac{k^5}{2} - \frac{3k^4}{2} - \frac{35k^3}{6} - \frac{57k^2}{4} - 21k - \frac{139}{9}$$

$$R_4(k) = \frac{16277}{432}4^k - \frac{139}{4}3^{k-2}(k^2 + 7k + 24) + 72^{k-4}(k^4 + 8k^3 + 41k^2 + 118k + 168) - \frac{k^6}{6} - \frac{k^5}{2} - \frac{8k^4}{3} - \frac{25k^3}{3} - \frac{73k^2}{4} - \frac{947k}{36} - \frac{506}{27}$$

so that the coefficient of x^7 is

$$\frac{2^{k-9}}{3(k - 1)!}\left[4^k\left\{2k^3 - 42k^2 + \frac{1553k}{6} - \frac{16277}{36}\right\} + 3^k(k + 8)(6k^2 - 63k + 139 + 32^{k-1}(2k - 7)(2k^2 + 25k + 84) + 2k^3 + 27k^2 + \frac{391k}{3} + \frac{2024}{9}\right]$$

This expression vanishes when $k = 1, 2, 3, 4, 5,$ and 6. We could now proceed, in a similar manner, to find the coefficient of x^9 and that of higher powers in $Q(x)$. It is now evident that the details become increasingly complicated. Hence, it is preferable to calculate the coefficient for a given power by means of the appropriate recurrence relations. However, using the last of the three results occurring at the end of Section 3, it is possible to deduce the coefficient of x^{k-1} when k is even. The coefficient is

$$\frac{k}{k-1}\left\{1 + \frac{3(k-2)}{2(k-3)} + \frac{3 \cdot 5(k-2)(k-4)}{2 \cdot 4(k-3)(k-5)} + \frac{3 \cdot 5 \cdot 7(k-2)(k-4)(k-6)}{2 \cdot 4 \cdot 6(k-3)(k-5)(k-7)} + \cdots\right\},$$

the expression within the brackets terminating, since k is even.

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FIBONACCI RATIO IN A THERMODYNAMICAL CASE

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Consider the thermodynamics of an infinite chain of alternately spaced $2N$ molecules of donors and acceptors ($N \to \infty$), and assume there is an average of one mobile electron per molecule (as is quite common for some one-dimensional organic crystals [1, 2]).

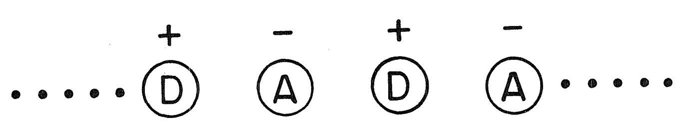

FIGURE 1

Each molecule may contain a maximum of two such electrons and as the temperature is raised two electrons may jump onto the same molecule. Because electrons repel each other, it costs an energy U_D or U_A to put two electrons on a molecule type D or type A, respectively; a common situation is that

$$U_D >> U_A.$$

Under these conditions, it can cost almost no energy to have sites A doubly occupied, while double occupancy of sites D is effectively eliminated.

In the grand-canonical ensemble, the partition function Z of the electrons can then be approximated by

(1) $$Z = \lambda^N,$$

where $\lambda = z_A z_D$; z_A and z_D being the partition functions "per molecule" of type A and D, respectively. In terms of the fugacity [3], z and λ can be obtained easily, in fact,

(2) $$z_A \cong 1 + 2z + z^2 = (1 + z)^2$$

and

(3) $$z_D \cong 1 + 2z.$$

The three terms in (2) (in ascending powers of z) correspond to zero occupancy, single occupancy (with spin up or down), and double occupancy (respectively) of sites A. In (3) there is no z^2 term, because double occupancy of sites D is effectively eliminated.

In the grand-canonical ensemble, the positive quantity z is determined [3] by fixing the "average" number of particles (in this case, electrons). Since we have an average of one electron per site, z will be determined by the condition [3]

(4) $$z\frac{\partial\lambda}{\partial z} = 2\lambda.$$

Substituting for λ in terms of (2) and (3) and simplifying, (4) gives the cubic equation

(5) $$(z + 1)(z^2 - z - 1) = 0$$

for z. Finally, the positive

(6) $$z_+ = \frac{1 + \sqrt{5}}{2}$$

The Fibonacci ratio is the only appropriate physical solution of (5) for the fugacity z. From the grand-partition function Z and the numerical value of λ,

(7) $$\lambda = (1 + z)^2(1 + 2z) = z_+^7,$$

the thermodynamics [3] then easily follows.

In particular, the entropy S that arises from the number of possible arrangements of the electrons in the chain is given by

(8) $$\frac{S}{k_B} = 5N \ln z\ ,$$

where k_B is Boltzmann's constant.

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ON GROUPS GENERATED BY THE SQUARES

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1. INTRODUCTION

It was known that the quaternion group and the octic group could not be generated by the squares of any group [5, pp. 193-194]. A natural question is which groups are generated by the squares of some groups. Clearly, groups of odd order and simple groups are generated by their own squares. In this paper, we show in a concrete manner that abelian groups are generated by the squares of some groups, and we show that every group is contained in the set of squares of some group. We give conditions for the dihedral and dicyclic groups to be generated by the squares of some groups. Also we show that several classes of nonabelian 2-groups cannot be generated by the squares of any group.

2. NOTATIONS AND DEFINITIONS

Throughout this paper, all groups considered are assumed to be finite. For a group G, we let G^2 denote the set of squares, $I(G)$ the group of innerautomorphisms, $A(G)$ the group of automorphisms, $Z(G)$ the center, $|G|$ the order of G, G^1 the commutator subgroup. For any subset S of G, $\langle S \rangle$ denotes the subgroup generated by S. G is called an <u>S-group</u> if it is generated by the squares of some group L; to be more precise, there is a group L such that $\langle L^2 \rangle$ is isomorphic to G.

3. CLASSES OF S-GROUPS

In a group of odd order, every element is a square; therefore, it is an S-group. A simple group is also an S-group since it is generated by its own squares; for, if the set of squares generates a proper subgroup, it would be a normal subgroup with abelian quotient. We next show that an abelian group is an S-group.

Theorem 3.1: An abelian group is an S-group.

Proof: Let G be an abelian group. Then

$$G = H_1 \times H_2 \times \cdots \times H_n,$$

where the H_i are cyclic groups. Let $|H_i| = k_i$. The permutation group generated by the n circular permutations

$$(a_{11}a_{12} \ldots a_{2k_2}), \ldots, (a_{n1}a_{n2} \ldots a_{nk_n}),$$

where the a_{ij} are $|G|$ distinct symbols, is isomorphic to G. Let L be the permutation group generated by the n circular permutations

$$(a_{11}a_{12} \ldots a_{1k_1} b_{11}b_{12} \ldots b_{1k_1}),$$
$$(a_{21}a_{22} \ldots a_{2k_2} b_{21}b_{22} \ldots b_{2k_2}), \ldots,$$
$$(a_{n1}a_{n2} \ldots a_{nk_n} b_{n1}b_{n2} \ldots b_{nk_n}),$$

where the b_{ij}'s are $|G|$ distinct symbols all different from the a_{ij}'s. Then clearly $L^2 \cong G$, and G is an S-group.

Using the same technique, we can prove the following:

Theorem 3.2: Every group is contained in the set of squares of some group. (See also [9].)

Proof: Let G be a group, and let P be a permutation group on n symbols isomorphic to G. We will construct a permutation group L such that P is isomorphic to a subgroup in L^2.

Let Q be a permutation group isomorphic to P on n symbols distinct from those of P. Let i be the isomorphism of P onto Q. If each element x in P is multiplied to $i(x)$ in Q, we obtain a group

$$R = \{xi(x) \,|\, x \in P\}$$

isomorphic to P. Clearly, each permutation in R is the square of a permutation in $2n$ symbols. Let L be the permutation group generated by the permutations whose squares are in R. Then $R \subset L^2$.

Unfortunately, homomorphic images of S-groups need not be S-groups. If, however, the kernel of the homomorphism is a characteristic subgroup of the S-group, then the homomorphic image is also an S-group. To prove this, we need the following lemma, which can be proved by straightforward set-inclusion.

Lemma 3.1: Let N be a normal subgroup of G which is contained in $\langle G^2 \rangle$. Then

$$\langle (G/N)^2 \rangle = \langle G^2 \rangle / N.$$

Theorem 3.3: Let G be an S-group, and let θ be a homomorphism from G onto $\overline{G}$ such that the kernel of θ is a characteristic subgroup of G. Then, $\overline{G}$ is an S-group.

Proof: Let L be a group such that $\langle L^2 \rangle = G$. Then, the kernel of θ, being a characteristic subgroup of G, is normal in L. By the lemma,

$$\langle (L/\text{kernel } \theta)^2 \rangle = \langle L^2 \rangle / \text{kernel } \theta = G/\text{kernel } \theta,$$

which is isomorphic to $\overline{G}$. Hence, $\overline{G}$ is an S-group.

As corollaries to Theorem 3.3, if G is an S-group, the quotient groups of its center, i.e., its group of inner-automorphisms, its Frattini subgroup, and its Fitting subgroup, are all S-groups.

Theorem 3.4: A nilpotent group is an S-group if and only if its Sylow 2-subgroup is an S-group.

Proof: Let G be a nilpotent group. Then $G = T \times H$, where T is a 2-group and H is a group of odd order. If T or H is trivial, then the Theorem is evident. Suppose T is an S-group, say $\langle F^2\rangle \cong T$, letting $L = F \times H$, we have

$$\langle L^2\rangle = G.$$

Conversely, let G be an S-group. T is a homomorphic image of G, with kernel of the homomorphism being H. Since H is a characteristic subgroup, by Theorem 3.3, T is an S-group.

4. DIHEDRAL AND DICYCLIC GROUPS

Theorem 4.1: A dihedral group D_m of order $2m$ is an S-group if and only if the congruence $t^2 \equiv -1 \pmod{m}$ has a solution.

Proof: D_m has presentation

$$a^m = b^2 = 1, \; b^{-1}ab = a^{-1}.$$

If there were a group L such that $\langle L^2\rangle = D_m$, there would have to be elements c in L such that $c^2 = a^ib$, for some i. For $m = 2$, D_m is abelian, hence is an S-group. For $m = 4$, D_m is not an S-group. For $m \neq 1, 2, 4$, $\langle a\rangle$ is a characteristic subgroup of D_m, hence normal in L. Therefore,

$$c^{-1}ac = a^t,$$

but

$$(a^ib)^{-1}a(a^ib) = a^{-1},$$

so

$$a^{-1} = (a^ib)^{-1}a(a^ib) = c^{-1}(c^{-1}ac)c = c^{-1}(a^t)c = a^{t^2}.$$

$t^2 \equiv -1 \pmod{m}$ must have a solution.

Conversely, if $t^2 \equiv -1 \pmod{m}$ has a solution t_0, we define the group $L = \langle c,d\rangle$ as follows:

$$c^{2m} = d^4 = 1, \; d^{-1}cd = c^{t_0}.$$

Then clearly $\langle L^2\rangle$ is isomorphic to D_m.

G. A. Miller stated [4, p. 152] that no dicyclic group can be generated by the squares of any group. The following theorem gives counterexamples to his statement [7]:

Theorem 4.2: A dicyclic group $D(m)$ of order $4m$ is an S-group if and only if $t^2 \equiv -1 \pmod{2m}$ has a solution.

Proof: For $m = 2$, $D(m)$ is not an S-group. For $m > 2$, let $D(m)$ have presentation

$$a^{2m} = b^4 = 1, \; b^2 = a^m, \; b^{-1}ab = a^{-1}.$$

If there were a group L such that $\langle L^2\rangle = D(m)$, there would have to be an element c in L with $c^2 = a^ib$ for some $i = 0, 1, 2, \ldots, 2m - 1$. $\langle a\rangle$ is a characteristic subgroup of $D(m)$, hence normal in L. $c^{-1}ac = a^t$, for some t, but $(a^ib)^{-1}a(a^ib) = a^{-1}$; therefore,

$$a^{-1} = (a^ib)^{-1}a(a^ib) = c^{-1}(c^{-1}ac)c = c^{-1}a^tc = a^{t^2}.$$

Thus, $t^2 \equiv -1 \pmod{2m}$ must have a solution.

Conversely, if $t^2 \equiv -1 \pmod{2m}$ has a solution t_0, we define the group $L = \langle c,d \rangle$ by

$$d^4 = c^{2m},\ c^{4m} = d^8 = 1,\ d^{-1}cd = c^{t_0}.$$

Then clearly $\langle L^2 \rangle$ is isomorphic to $D(m)$.

5. 2-GROUPS

Since a nilpotent group is an S-group if and only if its Sylow 2-subgroup is an S-group, 2-groups are particularly important in the determination of S-groups.

Lemma 5.1: Let G be a 2-group, and let N be a normal subgroup of order 4. Then the index of the centralizer of N, $[G:C(N)]$, is at most 2.

Proof: Since N is normal, for a in N, every conjugate of a is also in N. The number of conjugates is either 1 or 2, because at least two of the elements of N are in $Z(G)$. This means that, for every a in N, the index of its centralizer, $[G:C(a)]$, is at most 2. If N is cyclic, let a be its generator, then $C(a) = C(N)$. If N is not cyclic,

$$N = \langle a \rangle \times \langle b \rangle, \text{ where } |a| = |b| = 2.$$

Let $a \in Z(G)$. If $b \notin Z(G)$, then $C(N) = C(b)$, so $[G:C(N)]$ is at most 2. If $b \in Z(G)$ also, then $C(N) = G$.

Lemma 5.2: Let G be a 2-group, let N be an abelian normal subgroup of order 8 contained in $\langle G^2 \rangle$. If $N = \langle a \rangle \times \langle b \rangle$, where a is an element of order 4 in $Z(\langle G^2 \rangle)$, then $N \subset Z(\langle G^2 \rangle)$.

Proof: Let M be a subgroup of N of order 2 contained in $Z(G)$. If M is not contained in $\langle a \rangle$, then

$$N = \langle a,M \rangle \subset Z(G) \cap \langle G^2 \rangle \subset Z(\langle G^2 \rangle).$$

If $M = \langle a^2 \rangle$, then b, an element of order 2 in N, can only be conjugate to b and ba^2, and the index of $C(b)$ is equal to the number of conjugates of b, so $[G:C(b)]$ is at most 2. Since $C(b)$ contains $\langle G^2 \rangle$, b is in $Z(\langle G^2 \rangle)$.

Theorem 5.1: A nonabelian 2-group with cyclic center is not an S-group.

Proof: By induction on the order of G; it is true for $|G| = 2^3$ [5, pp. 193-194]. Suppose that G is a group of lowest order with cyclic center and that there exists a 2-group L such that $\langle L^2 \rangle = G$. Let $\langle c \rangle$ be a subgroup of order 2 contained in $G \cap Z(L)$. Then, by Lemma 3.1, $\langle (L/\langle c \rangle)^2 \rangle = G/\langle c \rangle$. $Z(G/\langle c \rangle)$ cannot be cyclic if $G/\langle c \rangle$ is nonabelian. If $G/\langle c \rangle$ is abelian, then $G/\langle c \rangle = Z(G/\langle c \rangle)$. Since $\langle c \rangle$ is contained in $Z(G)$, $G/Z(G)$ is a homomorphic image of $G/\langle c \rangle$. $G/Z(G)$ is never cyclic, so $G/\langle c \rangle$ is not cyclic. Thus, in any case, $Z(G/\langle c \rangle)$ is not cyclic.

Let E be the largest elementary abelian 2-group contained in $Z(G/\langle c \rangle)$. Since $Z(G/\langle c \rangle)$ is not cyclic, $|E|$ is at least 4. E is a characteristic subgroup of $G/\langle c \rangle$, therefore normal in $L/\langle c \rangle$. There exist normal subgroups $\overline{M}$, $\overline{N}$ of $L/\langle c \rangle$ of orders 2 and 4, respectively, such that $\overline{M} \subset \overline{N} \subset E$. Let M and N be the normal subgroups of L which are the preimages of $\overline{M}$ and $\overline{N}$ under the natural homorphism of L onto $L/\langle c \rangle$. Then,

$$|M| = 4,\ |N| = 8, \text{ and } \langle c \rangle \subset M \subset N.$$

By Lemma 5.1, $[L:C(M)]$ is at most 2, which means

$$G = \langle L^2 \rangle \subset C(M), \text{ or } M \subset Z(G),$$

which is cylcic. Now N is abelian, $N \subset G\overline{N} \subset E$, which is noncyclic, so N is noncyclic; $M \subset N$, if M is cyclic, by Lemma 5.2, $N \subset Z(G)$, which contradicts the assumption that $Z(G)$ is cyclic.

Theorem 5.2: Let G be a nonabelian 2-group with commutator subgroup of index 4. Then G is not an S-group.

Proof: Suppose L is a 2-group with $\langle L^2 \rangle = G$, G' nontrivial, and $[G:G'] = 4$. Let N be a normal subgroup of L contained in G', with $[G':N] = 2$ [3, p. 127]. Then L/N is a 2-group such that $\langle (L/N)^2 \rangle = G/N$, by Lemma 3.1. But, $(G/N)' = G'/N$ is nontrivial, and the order of G/N,

$$[G:N] = [G:G'][G':N] = 8.$$

Thus, G/N is a nonabelian group of order 8 which cannot be an S-group. This contradiction shows that G is not an S-group.

Theorem 5.3: Let G be a nonabelian 2-group with $\langle G^2 \rangle$ cyclic and $[G:\langle G^2 \rangle] = 4$. Then G is not an S-group.

Proof: Use induction on the order of G. It is true for $|G| = 2^3$. Assuming the theorem for all 2-groups of order less than 2 , let G be a nonabelian group of order 2^n, and let $[G:\langle G^2 \rangle] = 4$ with $\langle G^2 \rangle$ cyclic. Suppose there is an L with $\langle L^2 \rangle = G$. We consider two cases with $|G'| = 2$ and $|G'| > 2$.

Let $|G'| = 2$. Then every noncentral element has just two conjugates, i.e., for every x in G, $[G:C(x)] \leq 2$. Hence,

$$\bigcap_{x \in G} C(x) = Z(G) \supseteq \langle G^2 \rangle.$$

Since $[G:Z(G)] \geq 4$, $Z(G) = \langle G^2 \rangle$. By Theorem 5.1, G is not an S-group.

Now suppose $|G'| > 2$. Since $\langle G^2 \rangle$ is cyclic, let $\langle G^2 \rangle = \langle c \rangle$. Then $|c| = 2^{n-2}$. Let a be the 2^{n-1}th power of c. Then $\langle a \rangle$ is a characteristic subgroup of order 2 in G, thus normal in L. Now $\langle (L/\langle a \rangle)^2 \rangle = G/\langle a \rangle$. Since $|G'| > 2$, G' is not contained in $\langle a \rangle$, so $G/\langle a \rangle$ is nonabelian. Moreover,

$$[G/\langle a \rangle : \langle (G/\langle a \rangle)^2 \rangle] = [G:\langle G^2 \rangle] = 4.$$

Therefore, $G/\langle a \rangle$ is a nonabelian 2-group of order 2^{n-1} with cyclic $\langle (G/\langle a \rangle)^2 \rangle$ of index 4. This contradicts the induction hypothesis.

Applying Theorems 5.1-5.3, we obtain the following theorems.

Theorem 5.4: Let G be a nonabelian 2-group whose center

$$Z(G) = \langle a \rangle \times \langle b \rangle, \text{ where } |a| = 2^n, \ |b| = 2.$$

If $Z(G)$ contains exactly one element which is not a square and is not in the commutator subgroup, then G is not an S-group.

Proof: Let c be the central element which is neither a square nor a commutator. Then $c = b$ or $a^i b$ for some i, so $Z(G)/\langle c \rangle = Z(G/\langle c \rangle)$ is cyclic. $\langle c \rangle$ is a characteristic subgroup of G. Since $c \notin G'$, $G/\langle c \rangle$ is nonabelian. By Theorem 5.1 $G/\langle c \rangle$ is not an S-group; by Theorem 3.3 G is not an S-group.

An example of this is the group of order 16 with presentation $a^4 = b^4 = 1$, $b^{-1}ab = a^{-1}$. Here, a^2b^2 is a central element which is not in G' and is not a square, so the group is not an S-group [1, p. 146].

Theorem 5.5: Let G be a nonabelian 2-group with

$$\langle G^2 \rangle = \langle a \rangle \times \langle b \rangle, \text{ where } |a| = {}^n, \ |b| = 2.$$

Suppose $\langle G^2 \rangle$ contains exactly one element c which is not a square; also suppose that either $c \notin G'$ or $|G'| > 2$, and $[G:G'] = 4$. G is not an S-group.

The proof of this theorem is similar to that for Theorem 5.4. An example is the group G of order 3_2 with presentation

$$a^4 = b^2 = c^2 = d^2 = 1, \ d^{-1}ad = a,$$

$$d^{-1}cd = cb, \ c^{-1}ac = a^{-1},$$

where a^2 and b are central elements. Here

$$G' = \langle G^2 \rangle = \langle a^2, b \rangle,$$

and the element a^2b is not a square. By Theorem 5.5 G is not an S-group.

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A PRIMER ON STERN'S DIATOMIC SEQUENCE—II

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PART II: SPECIAL PROPERTIES

In 1929, D. H. Lehmer, at Brown University, presented a summary [1] of discovered results concerning Stern's sequence. Also, in July 1967, some additional results were reported by D. A. Lind [2]. In order to standardize the results, we will define Stern's sequence to be $s(i,j)$ where

(1) $s(i,0) = 1$, for $i = 0, 1, 2, \ldots$
(2) $s(0,j) = 0$, for $j = 1, 2, 3, \ldots$

(3) $s(n,2k) = s(n,k)$, for $n,k = 1, 2, 3, \ldots$
(4) $s(n,2k + 1) = s(n - 1,k) + s(n - 1,k + 1)$.

A table follows:

STERN NUMBER TABLE

	Column																		
Row	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	3	2	3	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0
4	1	4	2	5	1	5	3	4	1	3	2	3	1	2	1	1	0	0	0
5	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1	4	3
6	1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7
7	1	7	6	11	5	14	9	13	4	15	11	18	7	17	10	13	3	14	11
8	1	8	7	13	6	17	11	16	5	19	14	23	9	22	13	17	4	19	15
9	1	9	8	15	7	20	13	19	6	23	17	28	11	27	16	21	5	24	19
10	1	10	9	17	8	23	15	22	7	27	20	33	13	32	19	25	6	29	23
11	1	11	10	19	9	26	17	25	8	31	23	38	15	37	22	29	7	34	27
12	1	12	11	21	10	29	19	28	9	35	26	43	17	42	25	33	8	39	31
13	1	13	12	23	11	32	21	31	10	39	29	48	19	47	28	37	9	44	35
14	1	14	13	25	12	35	23	34	11	43	32	53	21	52	31	41	10	49	39
15	1	15	14	27	13	38	25	37	12	47	35	58	23	57	34	45	11	54	43
16	1	16	15	29	14	41	27	40	13	51	38	63	25	62	37	49	12	59	47
17	1	17	16	31	15	44	29	43	14	55	41	68	27	67	40	53	13	64	51
18	1	18	17	33	16	47	31	46	15	59	44	73	29	72	43	57	14	69	55

The authors will attempt to move quickly through the properties of these numbers without proof.

(1) The number of terms in row n is $2^n + 1$.
(2) The sum of all terms in row n is $3^n + 1$.
(3) The average value of all terms approaches $(3/2)^n$.
(4) The table is symmetric:
$s(n,k) = s(n,2^n + 2 - k)$ for $2^n + 2 - k \geq 0$.
(5) In three successive terms a, b, c, $(a + c)/b$ is an integer. (See Part I [3], Sections 4 and 11.)
(6) Given a, b, and c again, then b occurs at
$s(n - k, (a + c - b)/2b)$. (See [3], Section 4.)
(7) Any two consecutive terms are relatively prime. (See [3], Section 5.)
(8) Any ordered pair can only appear once in the table. (See [3], Section 6.)
(9) If $a/b = (k, k_1, k_2, \ldots, k_m, r_{m-1})$, then a and b appear together in line $(k + k_1 + k_2 + \cdots + k_m + r_{m-1} - 1)$. (See [3], Section 10.)
(10) The number of times that an element k can appear in the row $k - 1$, and all succeeding rows, is Euler's function $0(k)$.

(11) "p" is a prime if and only if it appears exactly $(p-1)$ times in line $(p-1)$.

(12) $s(n,r)$ will appear again at locations $s(n+k,\ 2^k(r-1)+1)$ for $k = 1, 2, 3, \ldots$.

(13) If the sequence r_1, r_2 occurs in row n, $r_1 > r_2$, the smallest element in row $n+k$ positioned between r_1 and r_2 is
$s(n+k,\ 2^k r) = r_1 + kr_2$.

(14) In any row, there are two equal terms greater than all others in the row.

(15) For Fibonacci followers:
$s(n,r) = F_{n+1}$, for $r = (2^{n-1} + 2 + \{1 + (-1)^n\})/3 - 1$,
and it is the largest element in the row.
(See [3], p. 65; notation changed to standard form.)

Not all of the discovered results are considered here, since there are remote connections to so many areas of number theory.

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SUMS OF PRODUCTS: AN EXTENSION

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The purpose of this note is to extend the results of Berzsenyi [1] and Zeilberger [3] on sums of products by using the generalized sequence

$$\{W_n(a,b;p,q)\}$$

described by the author in [2], the notation of which will be assumed.

Equation (4.18) of [2, p. 173] tells us that

$$W_{n-r}W_{n+r+t} - W_nW_{n+t} = eq^{n-r}U_{r-1}U_{r+t-1}. \tag{1}$$

Putting $n - r = k$ and summing appropriately, we obtain

$$\sum_{k=0}^{n} W_kW_{k+2r+t} = \sum_{k=0}^{n} W_{k+r}W_{k+r+t} + eU_{r-1}U_{r+t-1}\sum_{k=0}^{n} q^k. \tag{2}$$

Values $t = 1$, $t = 0$ give, respectively,

$$\sum_{k=0}^{n} W_kW_{k+2r+1} = \sum_{k=0}^{n} W_{k+r}W_{k+r+1} + eU_{r-1}U_r\sum_{k=0}^{n} q^k, \tag{3}$$

and

(4) $$\sum_{k=0}^{n} W_k W_{k+2r} = \sum_{k=0}^{n} W_{k+r}^2 + eU_{r-1}^2 \sum_{k=0}^{n} q^k.$$

If $q = -1$, then

(5) $$\sum_{k=0}^{n} q^k = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Using the Binet form for W_n and U_n, we find after calculation that (3) and (4), under the restrictions (5), become, respectively,

(6) $$\sum_{k=0}^{n} W_k W_{k+2r+1} = \begin{cases} \frac{1}{p}(W_{r+n+1}^2 - W_{r+1}^2) - W_0 W_{2r+1} & \text{if } n \text{ is even} \\ \frac{1}{p}(W_{r+n+1}^2 - W_r^2) & \text{if } n \text{ is odd,} \end{cases}$$

and

(7) $$\sum_{k=0}^{n} W_k W_{k+2r} = \begin{cases} \frac{1}{p}(W_{r+n} W_{r+n+1} - W_r W_{r+1}) + W_0 W_{2r} & \text{if } n \text{ is even} \\ \frac{1}{p}(W_{r+n} W_{r+n+1} - W_{r-1} W_r) & \text{if } n \text{ is odd.} \end{cases}$$

When $p = 1$, so that $W_n = H_n$ (and $U_n = F_n$), (6) and (7) reduce to the four formulas given by Berzsenyi [1]. That is, Berzsenyi's four formulas are special cases of (1), i.e., of equation (4.18) of [2].

Zeilberger's theorem [3] then generalizes as follows:

Theorem: If $\{Z_n\}$ and $\{W_n\}$ are two generalized Fibonacci sequences, in which $q = -1$, then

$$\sum_{i,j=0}^{n} a_{ij} Z_i W_j = 0$$

if and only if

$$P(z,\omega) = \sum_{i,j=0}^{n} a_{ij} z^i \omega^j$$

vanishes on $\{(\alpha,\alpha), (\alpha,\beta), (\beta,\alpha), (\beta,\beta)\}$ where α,β are the roots of

$$x^2 - px - 1 = 0.$$

Zeilberger's example [3[now refers to

(8) $$\sum_{k=0}^{n} Z_k W_{k+2r+1} = \frac{1}{p}(Z_{r+n+1} W_{r+n+1} - Z_{r+1} W_{r+1}) + Z_0 W_{2r+1}.$$

(In both [1] and [3], m is used instead of our r.)

Verification of the above results involves routine calculation. Difficulties arise when $q \neq -1$.

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A CONJECTURE IN GAME THEORY

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We consider a team composed of n players, with each member playing the same r games, $G_1, G_2, \ldots, G_r$. We assume that each game G_j has two possible outcomes, success and failure, and that the probability of success in game G_j is equal to p_j for each player. We let X_{ij} be equal to one (1) if player i has a success in game j and let X_{ij} be equal to zero (0) if player i has a failure in game j. We assume throughout this paper that the random variables X_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, r$ are independent.

Let S_{jn} denote the total number of successes in the jth game. We define the point-value of a team to be

$$\Psi_n = \min_{1 \le j \le r} S_{jn}.$$

This means that the point-value of a team is equal to the minimum number of successes in any particular game. Clearly,

$$P\{S_{jn} = m\} = \binom{n}{m} p_j^m (1 - p_j)^{n-m}, \; m = 0, 1, 2, \ldots, n,$$

and

$$\begin{aligned} E[\Psi_n] &= \sum_{k=0}^{n} kP\{\Psi_n = k\} = \sum_{k=0}^{n-1} P\{\Psi_n > k\} \\ &= \sum_{k=0}^{n-1} P\{S_{1n} > k, S_{2n} > k, \ldots, S_{rn} > k\} \\ &= \sum_{k=0}^{n-1} \prod_{j=1}^{r} P\{S_{jn} > k\} \\ &= \sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}. \end{aligned} \tag{1}$$

It follows from the definition of Ψ_n that the expected point-value for a team is an increasing function of n, i.e.,

$$E[\Psi_n] \le E[\Psi_{n+1}], \; n = 1, 2, 3, \ldots .$$

Since a team can add players in order to increase its expected point-value, it seems reasonable to define the score to be the expected point-value per player. Namely, we denote the score by

$$W_n = \frac{1}{n} E[\Psi_n].$$

Thus, from (1), we obtain

$$W_n = \frac{1}{n}\sum_{k=0}^{n-1} \prod_{j=1}^{r} \sum_{m=k+1}^{n} \binom{n}{m} p_j^m (1 - p_j)^{n-m}. \tag{2}$$

It is not obvious from (2) how the score varies as the number of players increases. We now prove that W_n is a strictly increasing function of n in the special case $r = 2$ and $p_1 = p_2$. We first prove three lemmas, which are also of independent interest.

Lemma 1: Let a team be composed of j players, with each member playing the same two games, G_1 and G_2. Let the probability of success for each player in both games G_1 and G_2 be equal and be denoted by p. Let $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j. Then

$$\frac{1}{2\Pi}\int_0^{2\Pi} |p + qe^{i\theta}|^{2j} d\theta = u_j,$$

where $q = 1 - p$.

Proof: Using the fact that

$$P\{S_{ij} = m\} = \binom{j}{m} p^m (1 - p)^{j-m}, \; m = 0, 1, 2, \ldots, j, \; i = 1, 2,$$

and the independence of the random variables S_{1j} and S_{2j}, we obtain

$$u_j = \sum_{m=0}^{j} \left[\binom{j}{m}\right]^2 p^{2m} (1 - p)^{2(j-m)}, \; j = 1, 2, 3, \ldots . \tag{3}$$

We note that if f is the polynomial $f(z) = \sum_{m=0}^{j} a_m z^m$, then

$$\frac{1}{2\Pi}\int_0^{2\Pi} |f(e^{i\theta})|^2 d\theta = \sum_{m=0}^{j} a_m^2. \tag{4}$$

We now apply the binomial expansion and (4) to the function $f(z) = (p + qz)^j$, where j is a positive integer. The binomial expansion yields

$$f(z) = (p + qz)^j = \sum_{m=0}^{j} \left[\binom{j}{m} p^m q^{j-m}\right] z^{j-m},$$

and using (3) and (4), we obtain

$$\frac{1}{2\Pi}\int_0^{2\Pi} |p + qe^{i\theta}|^{2j} d\theta = \sum_{m=0}^{j} \left[\binom{j}{m}\right]^2 p^{2m} q^{2(j-m)} = u_j. \tag{5}$$

Lemma 2: Let $r = 2$, $p_1 = p_2$, and $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j. Then $u_j < u_{j-1}$.

Proof: Since

$$|p + qe^{i\theta}|^2 \le 1, \text{ for } 0 \le \theta \le 2\Pi$$

and

$$|p + qe^{i\theta}|^2 < 1, \text{ for } 0 < \theta < 2\Pi,$$

the desired result follows from (5).

Lemma 3: Let $u_j = P\{S_{1j} = S_{2j}\}$, for all positive integers j and let $u_0 = 1$. Let $d_j = \Psi_{j+1} - \Psi_j$, $j = 0, 1, 2, \ldots$, and let $\Psi_0 = 0$. Then

$$E[d_j] = u_j p^2 + (1 - u_j)p. \tag{6}$$

Proof: Clearly, d_j can assume only the values 0 and 1 with the following probabilities:

$$P\{d_j = 0\} = 1 - [u_j p^2 + (1 - u_j)p],$$
$$P\{d_j = 1\} = u_j p^2 + (1 - u_j)p.$$

Since $E[d_j] = 0 \cdot P\{d_j = 0\} + 1 \cdot P\{d_j = 1\}$, we obtain the desired result.

Theorem: Let a team be composed of n players, with each member playing the same two games, G_1 and G_2. Let the probability of success for each player in both games G_1 and G_2 be equal and be denoted by p. Then

$$W_n < W_{n+1},\ n = 1, 2, 3, \ldots .$$

Proof: Using the definition of W_n, we obtain

$$W_{n+1} - W_n = E\left[\frac{\Psi_{n+1}}{n+1} - \frac{\Psi_n}{n}\right] = \frac{1}{n(n+1)} E[n(\Psi_{n+1} - \Psi_n) - \Psi_n]. \tag{7}$$

Using d_j, as defined in Lemma 3, and noting that $\Psi_n = \sum_{j=0}^{n-1} d_j$, (7) reduces to

$$W_{n+1} - W_n = \frac{1}{n(n+1)} E\left[nd_n - \sum_{j=0}^{n-1} d_j\right].$$

Using (6), we obtain

$$W_{n+1} - W_n = \frac{1}{n(n+1)}\left[n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p)\right].$$

Thus, to prove that $W_n < W_{n+1}$, it suffices to show that

$$n(u_n p^2 + (1 - u_n)p) - \sum_{j=0}^{n-1} (u_j p^2 + (1 - u_j)p) > 0. \tag{8}$$

Proving inequality (8) is equivalent to showing that

$$nu_n - \sum_{j=0}^{n-1} u_j = \sum_{j=1}^{n} j(u_j - u_{j-1}) < 0. \tag{9}$$

Since (9) follows from Lemma 2, we conclude that

$$W_n < W_{n+1},\ n = 1, 2, 3, \ldots .$$

It is the author's conjecture that in the general case discussed in the beginning of this paper ($r > 2$ and p_1 not necessarily equal to p_2) that W_n is a strictly increasing function of n, too. The above proven theorem and some elementary numerical computations suggest the truth of this statement, but the author has not been able to supply a complete proof.

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THE CYCLE OF SIX

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ABSTRACT

The purpose of this paper is to show that a certain automorphism has order six when restricted to compositions considered as plane trees.

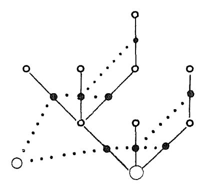
Part I is devoted to the proof of this and in Part II some applications are given. In particular, a duality between various Fibonacci families is discussed which also yields some interesting new settings for the Fibonacci families. Some open questions are mentioned in Part III.

The author would like to thank both Bertrand Harper and Robert Donaghey for helpful conversations.

PART I

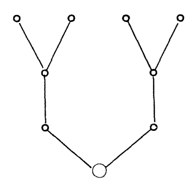
It is well known that plane trees with n edges are equinumerous with binary plane trees with $n+1$ end points. This correspondence was given in a paper by DeBruijn and Morselt [1] in 1967. A modification yields an automorphism on the set of plane trees. Throughout this paper, plane trees will be called trees.

We illustrate this automorphism, which we will denote A, as follows:



, T.

Straightening out the dotted lines yields another plane tree:



, $A(T)$.

Since both T and $A(T)$ have the same number of edges it follows that the number of distinct trees in the sequence T, $A(T)$, $A^2(T)$, $A^3(T)$, is at most

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

since there are C_n trees with n edges.

We give another illustration in Figure I-1. This particular example is not chosen at random; in fact, it illustrates the cycle of six. In general, it is extremely difficult, given a tree T, to predict the order n such that $A^n(T) = T$. Some work has been done on this problem (see [2]) but the central problem remains untouched. This paper represents the first interesting special case.

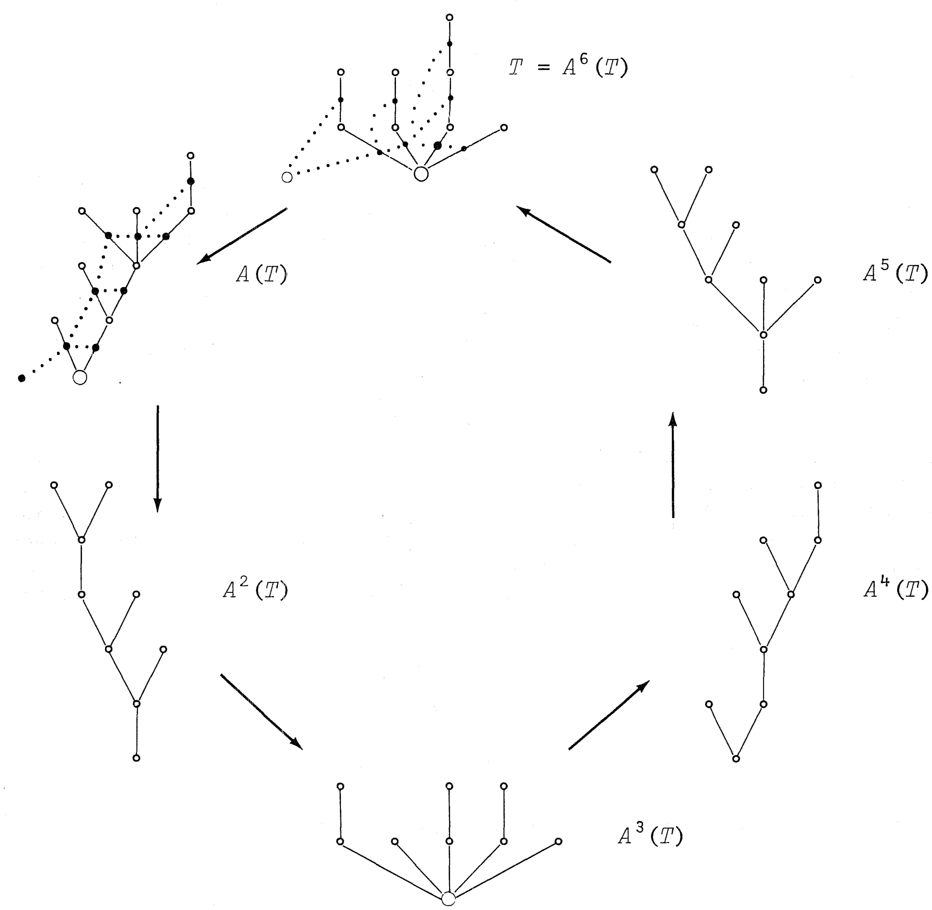
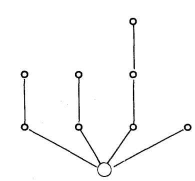


FIGURE I-1

Any composition of a number can easily be represented by a plane tree as follows. If $n = n_1 + n_2 + \cdots + n_k$, then the corresponding tree has only the root as a branch point and the lengths of the branches from the root, going left to right, are $n_1, n_2, \ldots, n_k$. For example,

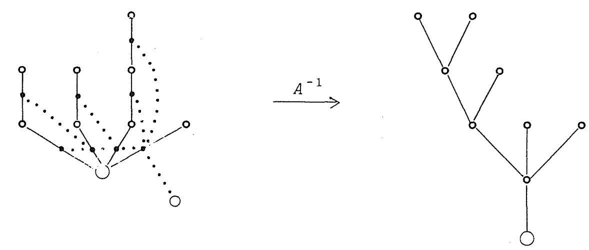
$2 + 2 + 3 + 1 \longleftrightarrow$ 

Theorem: If T represents a composition, then $A^3(T)$ also represents a composition and $A^6(T) = T$.

Proof: We will just trace through the six steps. The illustration is vital for following this proof

Let T be a composition. Since the only branch point is the root, we see that as we are constructing $A(T)$, all of the edges up from a vertex are terminal except the rightmost.

Note next that A^{-1} is defined as is A but from the right. For instance



This shows that this 'terminal-edges-except-for-the-rightmost-edge' condition precisely yields the set $A(T)$ where T is a composition.

Next we have that $A^2(T)$ consists of all trees such that all edges except the leftmost up from a vertex are terminal.

From here it is not hard to see that $A^3(T)$ is again a composition. So, $A^6(T)$ must again be a composition and we only need show $A^6(T) = T$.

Let us define $A^3(T)$ as the dual composition of T.

Suppose $n = n_1 + n_2 + \cdots + n_k$ is the composition that T represents. Then, $A(T)$ has

n_1 edges at the root

n_2 edges at height 2

...

n_k edges at height k.

We construct $A^2(T)$ by first taking a path of length n_1 starting at the root and going up taking the rightmost branch at each node.

Eliminate these n_1 edges and repeat the procedure to get n_2. If elimination disconnects the tree then operate on the upper component first. Continue this procedure to find paths of lengths n_3, n_4, ...

When computing $A^3(T)$, these paths each overlap by 1.

We wish to define a matrix D that will specify the A^3 automorphism exactly. We illustrate this before giving the precise definition:

$$T \leftrightarrow n = 2 + 2 + 3 + 1 \leftrightarrow D_T = \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \leftarrow 2 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 1 \end{array} \end{array}$$

$A^3(T)$ is given by the column sums read in reverse, here $2 + 1 + 2 + 2 + 1$.

Let $n = n_1 + n_2 + \cdots + n_k$ be a composition T. Then, D_T is a $k \times n - k + 1$ matrix with

$$d_{ij} \quad \begin{cases} 1 & \text{if } \sum_{k=1}^{i-1} n \ - i + 1 \le j \le \sum_{k=1}^{i} n \ - i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$D(2 + 1 + 2 + 2 + 1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is $D(T)$ reflected about the 45° line passing through the middle of the matrix. This situation holds in general.

Repeating this reflection twice yields the original matrix and thus

$$A^6(T) = A^3(A^3(T)) = T,$$

concluding the proof of the theorem.

PART II: SOME APPLICATIONS TO FIBONACCI NUMBERS

The following results were contained in an exercise in a set of lecture notes of R. Stanley.

The following sets are enumerated by the Fibonacci numbers.

A. All compositions of n where all parts are ≥ 2.
B. All compositions of n where all parts are equal to 1 or 2.
C. All compositions of n into odd parts.

These assertions are all easily verified by induction. We will add the following:

D. All compositions, $n = n_1 + n_2 + \cdots + n_{2k+1}$ where all $n_{2j} = 1$.
E. All compositions, $n = n_1 + n_2 + \cdots + n_{2k+1}$ where all $n_j = 1$ for $k + 1 < j \leq 2k$.
F. All compositions, $n = n_1 + n_2 + \cdots + n_m$ where $n_1 \geq n_j$ for $2 \leq j < \ell$, $(-1)^n = (-1)^m$, and $2n_1 + m \geq n + 2$.

Of these, F is perhaps the most interesting. It also seems to be less trivial to prove directly.

For the sake of brevity, we will ignore $A(T)$ and $A^2(T)$ in this discussion and go directly by way of the matrices from T to $A^3(T)$ leaving $A(T)$ and $A^2(T)$ to the diligent reader.

Proposition 1: A and B are dual Fibonacci families (except for a subscript shift).

Let $n = n_1 + n_2 + \cdots + n_k$ where all $n \geq 2$. Then we obtain

$$\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \quad & n_k \end{array} \\ \begin{pmatrix} 11\ldots1 & & & 0 \\ & 11\ldots1 & & \\ & & \ddots & \\ 0 & & & 11\ldots1 \\ & & & & 11\ldots1 \end{pmatrix} \end{array}$$

The column sums are either 1 or 2 with the first and last column sums always equal to 1. Obviously the compositions of n with first and last parts equal to 1 are bijective with all compositions of $n - 2$. Thus, A and B are essentially dual families, one enumerated by $\{F_n\}$ and the other by $\{F_{n-2}\}$.

We next want to consider the dual of family C. We have

$$n = n_1 + n_2 + \cdots + n_k \text{ where each } n_j \text{ is odd.}$$

For instance

$$3 + 1 + 5 + 1 + 1 + 7 \leftrightarrow \begin{pmatrix} 1 & 1 & 1 & & & & & & & & & & & \\ & & 1 & & & & & & & & & & 0 & \\ & & 1 & 1 & 1 & 1 & 1 & & & & & & & \\ & & & & & & 1 & & & & & & & \\ 0 & & & & & & 1 & & & & & & & \\ & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{pmatrix}$$

The column sums can be larger than 1 only in columns 1, 3, 5, 7, ... This is family D. This time C and D are exact duals and we have proved:

Proposition 2: C and D are dual Fibonacci families.

Proposition 3: E and F are dual Fibonacci families.

Since D and E are equinumerous, E is enumerated by the Fibonacci numbers. We need only show duality. Again we start by looking at an example:

$$11 = n = 1 + 1 + 3 + 2 + 1 + 1 + 1 \quad \text{so that} \quad k = 3,\ 2k + 1 = 7$$

$$\leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The last column sum is at least $k + 1$, and this must be as large as any other column sum because each of $n_1, n_2, \ldots, n_{k+1}$ can contribute at most 1 to each column.

Note that the matrix D has $2k + 1$ rows and $n - 2k$ columns. Thus if the dual composition is

$$n = n_1^* + n_2^* + \cdots + n_m^*$$

we have $n_1^* \geq k + 1 = \dfrac{n - m}{2} + 1$, or

$$2n_1^* + m \geq n + 2, \text{ the specification for F.}$$

PART III

To conclude, we mention some open problems and include some related remarks.

1. For a tree T, what is the smallest positive integer k such that $A^k(T) = T$? Even such simple questions as what information about T will guarantee that k is even are unsolved.
2. How many compositions of n with

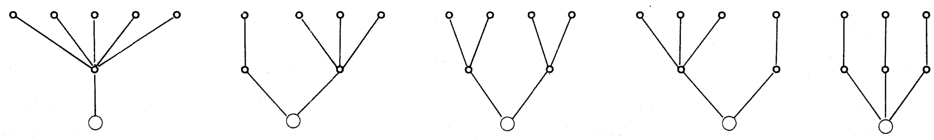
$$n = n_1 + n_2 + \cdots + n_k \text{ have } n_1 \geq n_i \text{ for all } i?$$

A related question would specify also that n and k have the same parity. The first few values are shown in the following table.

	$n =$ 1	2	3	4	5	6	7	8	9	10
	1	2	3	5	8	14	24	43	77	130
with $(-1)^{n-k} = 1$	1	1	2	3	5	8	13	22	39	65
with $(-1)^{n-k} = -1$	0	1	1	2	3	6	11	21	38	65

An answer to this question would be of interest in studying partitions.

3. If we specify that all end points of a tree be at height 2 then another Fibonacci family is obtained. For instance, for $n = 6$, we obtain the following five trees:



If we specify height 3 instead of height 2, we obtain the Tribonacci numbers 1, 1, 1, 2, 4, 7, 13, 24, ... If we specify height 3 or less we obtain the sequence 1, 2, 5, 13, 34, 89, ... = $\left\{F_{2n}\right\}_{n=0}^{\infty}$. If we knew more about Question 1, we could do more with each of these families. Each of these statements translates into statements about permutations achievable with push down stacks. See Knuth [4] for definitions and explanation.

How many permutations are achievable with a push down stack that holds two elements where each time the stack is empty two elements are put in (or the run ends)? The answer is F_{n-2}, and is equivalent to our first remark in this subsection.

4. What alterations can we make to get reasonably natural settings for the Lucas numbers, the Tribonacci numbers, and the Pell numbers?

One way to obtain the Lucas numbers is to specify compositions

$$n = n_1 + n_2 + \cdots + n_k \text{ where each } n_j \text{ is odd and } n_1 \text{ is 1 or 3.}$$

The dual of this yields the compositions

$$n = n_1 + n_2 + \cdots + n_{2k+1} \text{ with all } n_{2j} = 1 \text{ and } n_1 n_3 \neq 1.$$

5. We have ignored $A(T)$ and $A^2(T)$ throughout. However all the interpretations available for plane trees can be used. See for instance Gardner [3] and the references there. As one example, consider elections where votes are cast one at a time for candidates P and Q. There are $2n$ voters, P never trails Q, and at the end they tie. There are

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

such elections possible. Let us add the condition that the last K votes are for Q but that until then the election was almost monotonic in that if P's lead was ℓ votes, his lead would never be less than $\ell - 1$ thereafter, except for the last K votes. This is just the interpretation of $A(T)$ in Part I. Thus, we see that there are 2^{n-1} such elections, since an integer n has a total of 2^{n-1} compositions.

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PROFILE NUMBERS

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ABSTRACT

We describe a family of numbers that arises in the study of balanced search trees and that enjoys several properties similar to those of the binomial coefficients.

1. INTRODUCTION

In the course of a recent investigation [4] concerning balanced search trees [2, Section 6.2.3], the following combinatorial problem arose. We encountered in the investigation a family $\{T_L\}$ of $(2L + 1)$-level binary trees, $L = 1, 2, \ldots$; the problem was to determine, as a function of L and $l \in \{0, 1, \ldots, 2L\}$, the number of nonleaf nodes at level l of the $(2L + 1)$-level tree T_L. (By convention, the root of T_L is at level 0, the root's two sons are at level 1, and so on.) The numbers solving this problem, which we call *profile numbers* since, fixing L, the numbers yield the *profile* of the tree T_L [3], that is, the number of nodes at each level of T_L, enjoy a number of features that are strikingly similar to properties of binomial coefficients. Foremost among these similarities are the generating recurrences and summation formulas of the two families of numbers. Let us denote by $P(n,k)$, $n \geq 1$ and $k \geq 0$, the number of nonleaf nodes at level k of the tree T_n, conventionally letting $P(n,k) = 0$ for all $k > 2n$; and let us denote by $C(n,k)$, $n \geq 1$ and $k \geq 0$, the binomial coefficient, conventionally letting $C(n,k) = 0$ for $k > n$. The well-known generating recurrence

$$C(n + 1,k + 1) = C(n,k + 1) + C(n,k), \quad k \geq 0$$

for the binomial coefficients is quite similar to the generating recurrence

$$P(n + 1,k + 1) = P(n,k) + 2P(n,k - 1), \quad k > 0 \tag{1}$$

for profile numbers. Further, the simple closed-form solution of the well-known summation

$$\sum_{0 \leq k < n} C(n,k) = 2^n - 1$$

for binomial coefficients corresponds to the equally simple solution of the summation

$$\sum_{0 \le k < 2n} P(n,k) = 3^n - 1 \tag{2}$$

for our new family of numbers. Further examples of relations between these two families of numbers will manifest themselves in the course of the development. As an aid to the reader, we close this introductory section with a portion of the triangle of numbers defined by the recurrence (1) with the boundary conditions

$$\begin{aligned} P(n,0) &= 1 && \text{for all } n \ge 1 \\ P(1,1) &= 1 && \\ P(n,1) &= 2 && \text{for all } n > 1 \\ P(1,k) &= 0 && \text{for all } k > 1 \end{aligned} \tag{3}$$

(4)

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2
2		3	4	4	4	4	4	4	4	4
3		2	7	8	8	8	8	8	8	8
4			8	15	16	16	16	16	16	16
5			4	22	31	32	32	32	32	32
6				20	52	63	64	64	64	64
7				8	64	114	127	128	128	128
8					48	168	240	255	256	256
9					16	176	396	494	511	512
10						112	512	876	1004	1023

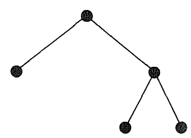
2. THE NUMBERS $P(n,k)$

Our goal in this section is to derive the basic properties of the profile numbers. By describing the tree-oriented origins of the numbers, we verify in Subsection A that they do indeed obey recurrence (1) with boundary conditions (3). We then proceed in Subsection B to solve recurrence (1), obtaining an explicit expression for $P(n,k)$ in terms of exponentials and binomial coefficients. In Subsection C, we derive the generating recurrences for individual rows and columns of the triangular array (4). These recurrences permit us in Subsection D to derive the summation formula (2) for profile numbers. Finally, in Subsection E, we use the summation formula to determine the so-called *internal path length* of the trees $\{T_L\}$, which determination was one of the motivations for studying the profile numbers. Our investigation will then have gone full circle.

In what follows, we shall refer often to binomial coefficients. These references will be very much facilitated by the convention $C(n,i) = 0$ whenever $i < 0$ or $i > n$, which should always be understood.

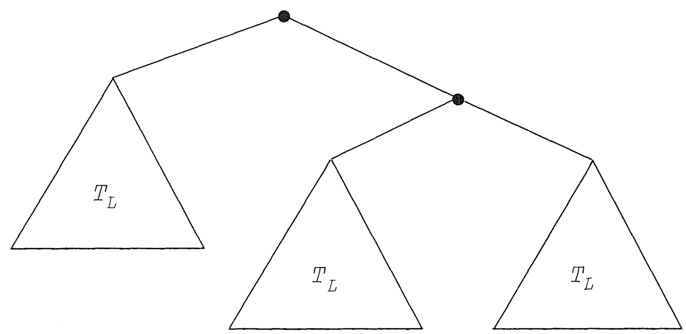
A. The Family $\{T_L\}$ of Trees: The trees T_L are specified recursively as follows. T_1 is the 3-leaf binary tree

T_1:



and, for each $L \geq 1$, the tree T_{L+1} is obtained by appending a copy of the tree T_L to each of the three leaves of T_1, as in

T_{L+1}:



The fact that $P(n,k)$ denotes, when $k \in \{0, \ldots, 2n\}$, the number of nonleaf nodes at level k of the tree T_n renders obvious the validity of recurrence (1) and boundary conditions (3) in addition to verifying the reasonableness of the convention

$$P(n,k) = 0 \quad \text{whenever} \quad n \geq 1 \quad \text{and} \quad k > 2n.$$

B. The Solution of Recurrence (1):

Theorem 1: For all $n \geq 1$ and all $k \geq 0$,

$$P(n,k) = 2^{k-n} \sum_{0 \leq i < 2n-k} C(n,i).$$

The theorem asserts, in particular, that $P(n,k) = 2^k$ for all $k < n$, and $P(n,k) = 0$ for all $k > 2n$.

Proof: We proceed by induction on n. The case $n = 1$ being validated by the boundary conditions (3), we assume for induction that the theorem holds for all $n < m$, and we consider an arbitrary number $P(m,k)$.

If $k \in \{0, 1\}$, then the boundary conditions (3) assure us that

$$P(m,k) = 2^k = 2^{k-n} \cdot 2^n = 2^{k-n} \sum_{0 \leq i < 2n-k} C(n,i),$$

which agrees with the theorem's assertion.

If $k > 1$, then recurrence (1) and the inductive hypothesis yield

$$\begin{aligned} P(m,k) &= P(m-1,k-1) + 2P(m-1,k-2) \\ &= 2^{k-m} \sum_{0 \leq i < 2m-k-1} C(m-1,i) + 2^{k-m} \sum_{0 \leq j < 2m-k} C(m-1,j) \end{aligned}$$

$$= 2^{k-m}C(m,0) + 2^{k-m}\sum_{0<i<2m-k} C(m-1,i) + C(m-1,i-1)$$

$$= 2^{k-m}\sum_{0\le i<2m-k} C(m,i),$$

which agrees with the theorem's assertion.

Since k was arbitrary, the induction is extended, and the theorem is proved. □

C. The Triangle of Profile Numbers: Yet more of the relation between profile numbers and binomial coefficients is discernible in the recurrences that generate individual rows and columns of the triangle (4).

Theorem 2: For all $n \geq 1$ and all $k \geq 0$,

(a) $P(n,k+1) = 2P(n,k) - 2^{k-n+1}C(n,k-n+1);$

(b) $P(n+1,k) = P(n,k) + 2^{k-n-1}\{C(n,k-n) + C(n+1,k-n)\}.$

Proof: Recurrence (1) translates to the three recurrences

(5) $$P(n,k) = P(n-1,k-1) + 2P(n-1,k-2).$$

(6) $$P(n,k+1) = P(n-1,k) + 2P(n-1,k-1).$$

(7) $$P(n+1,k) = P(n,k-1) + 2P(n,k-2).$$

Combining (5) and (6) leads, via Theorem 1, to the chain of equalities

$$\begin{aligned} P(n,k+1) - 2P(n,k) &= P(n-1,k) - 4P(n-1,k-2) \\ &= 2^{k-n+1}\left\{\sum_{0\le i<2n-k-2} C(n-1,i) - \sum_{0\le i<2n-k} C(n-1,i)\right\} \\ &= -2^{k-n+1}\{C(n-1,2n-k-2) + C(n-1,2n-k-1)\} \\ &= -2^{k-n+1}C(n,k-n+1), \end{aligned}$$

whence part (a) of the theorem.

Part (b) follows by direct calculation from recurrence (7) and Theorem 1:

$$\begin{aligned} P(n+1,k) - P(n,k) &= P(n,k-1) + 2P(n,k-2) - P(n,k) \\ &= 2^{k-n-1}\left\{\sum_{0\le i<2n-k+1} C(n,i) + \sum_{0\le i<2n-k+2} C(n,i) - 2\sum_{0\le i<2n-k} C(n,i)\right\} \\ &= 2^{k-n-1}\{2C(n,2n-k) + C(n,2n-k+1)\} \\ &= 2^{k-n-1}\{C(n,k-n) + C(n+1,k-n)\}. \ \square \end{aligned}$$

D. The Summation Formula (2): Theorem 2(a) permits easy verification of the summation formula for profile numbers.

Theorem 3: For all $n > 0$, $\displaystyle\sum_{0\le k<2n} P(n,k) = 3^n - 1.$

Proof: Theorems 1 and 2(a) justify the individual equalities in the following chain.

$$\sum_{0 \le k < 2n} P(n,k) = 1 + \sum_{0 \le k < 2n} P(n,k+1)$$

$$= 1 + 2 \sum_{0 \le k < 2n} (P(n,k) - 2^{k-n}C(n,k-n+1)).$$

Thus, we have

$$\sum_{0 \le k < 2n} P(n,k) = \sum_{0 \le j \le 2n-1} 2^{n-j}C(n,j) - 1$$

$$= 2^n \cdot (3/2)^n - 1$$

$$= 3^n - 1. \square$$

E. The Internal Path Lengths of the Trees $\{T_L\}$: Theorems 2(a) and 3 greatly facilitate the determination of the *internal path length* [1, Section 2.3.4.5] $I(L)$ of the tree T_L of Section 2A, which is given by

$$I(L) = \sum_{0 \le k < 2L} kP(L,k).$$

Theorem 4: For all $L > 0$, $I(L) = \frac{5}{3}L3^L - 2(3^L - 1)$.

Proof: $I(L) = \sum_{1 \le k < 2L+1} kP(L,k) = \sum_{0 \le k < 2L} (k+1)P(L,k+1)$

$= A + B + C + D$

where

$$A = 2 \sum_{0 \le k < 2L} kP(L,k) = 2I(L);$$

$$B = 2 \sum_{0 \le k < 2L} P(L,k) = 2(3^L - 1);$$

$$C = -\sum_{0 \le k < 2L} 2^{k-L+1}C(L,k-L+1) = -3^L;$$

$$D = -\sum_{0 \le k < 2L} k2^{k-L+1}C(L,k-L+1)$$

$$= -\sum_{0 \le j \le L} (j+L-1)2^jC(L,j) = -5L3^{L-1} + 3^L.$$

Combining terms yields the theorem. $\square$

We close by remarking that the quantity $I(L)$ can be determined just as easily from the recurrence

$$I(L) = 3I(L-1) + \frac{5}{3}3^L - 4,$$

which is derived easily from the form of the trees $\{T_L\}$.

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A STUDY OF THE MAXIMAL VALUES IN PASCAL'S QUADRINOMIAL TRIANGLE

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1. INTRODUCTION

In this paper we search for the generating function of the maximal values in Pascal's quadrinomial triangle. We challenge the reader to find this function as well as a general formula for obtaining all generating functions of the $(H - L)/k$ sequences obtained from partition sums in Pascal's quadrinomial triangle.

Generalized Pascal triangles arise from the multinomial coefficients obtained by the expansion of

$$(1 + x + x^2 + \cdots + x^{j-1})^n,\ j \geq 2,\ n \geq 0,$$

where n denotes the row in each triangle. For $j = 4$, the quadrinomial coefficients produce the following triangle:

1									
1	1	1	1						
1	2	3	4	3	2	1			
1	3	6	10	12	12	10	6	3	1

The partition sums are defined by

$$S(n,j,k,r) = \sum_{i=0}^{M} \left[\!\left[\begin{matrix} n \\ r + ik \end{matrix} \right]\!\right]_j;\ 0 \leq r \leq k - 1,$$

where

$$M = \left[\frac{(j-1)n - r}{k}\right];$$

the brackets [ ] denote the greatest integer function. To clarify, we give a numerical example. Consider $S(3,4,5,1)$. This denotes the partition sums in the third row of the quadrinomial triangle, in which every fifth element is added, beginning with the first column. Thus,

$$S(3,4,5,1) = 3 + 10 = 13.$$

(Conventionally, the column of 1's at the far left is the 0th column and the top row is the 0th row.)

In the nth row of the j-nomial triangle, the sum of the elements is j^n. This is expressed by

$$S(n,j,k,0) + S(n,j,k,1) + \cdots + S(n,j,k,k-1) = j^n.$$

Let

$$S(n,j,k,0) = (j^n + A_n)/k$$

$$S(n,j,k,1) = (j^n + B_n)/k \ \ldots$$

$$S(n,j,k,k-1) = (j^n + Z_n)/k.$$

Since $S(0,j,k,0) = 1$,

$$S(0,j,k,1) = 0 \ \ldots \ S(0,j,k,k-1) = 0,$$

we can solve for A_0, B_0, ..., Z_0 to get $A_0 = k - 1$, $B_0 = -1$, ..., $Z_0 = -1$.

Now a departure table can be formed with A_0, B_0, ..., Z_0 as the 0th row. Pascal's rule of addition is the simplest method for finding the successive rows in each departure table. The departure table for six partitions in the quadrinomial triangle appears below.

TABLE 1. SUMS OF SIX PARTITIONS IN THE QUADRINOMIAL TRIANGLE

5	-1	-1	-1	-1	-1
2	2	2	2	-4	-4
-4	-4	2	8	2	-4
2	-10	-10	2	8	8
20	8	-10	-16	-10	8

In particular, the $(H - L)/k$ sequences defined as the difference of the maximum and minimum value sequences in a departure table, divided by k partitions will be of prime importance. Table 2 is a table of the $(H - L)/k$ sequences for $k = 5$ to $k = 15$ partitions.

TABLE 2. $(H - L)/k$ SEQUENCES FOR $k = 5$ TO $k = 15$

$k = 5$	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
1	3	3	4	4	4	4	4	4	4	4
1	6	6	8	10	11	12	12	12	12	12
1	9	14	24	30	36	39	42	43	44	44
1	18	31	56	85	105	125	135	145	149	153
1	27	70	160	246	340	404	468	503	538	553

The primary purpose of this paper is to share the progress that has been made toward finding a generating function for the maximal values in Pascal's quadrinomial triangle. The generating functions for the maximal values in the binomial and trinomial triangles are known. In the February 1979 issue of *The Fibonacci Quarterly*, we showed that the limit of the generating functions for the $(H - L)/k$ sequences was precisely the generating function for the maximal values in the rows of the binomial and trinomial triangles. We would like to establish this for the quadrinomial triangle as well.

2. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN THE QUADRINOMIAL TRIANGLE

As k increases, one sees the $(H - L)/k$ sequences obtain more of the values of the sequence of central (maximal) values in the quadrinomial triangle. For $k = 14$, we observe from Table 2 that the $(H - L)/14$ sequence contains the first five values. We examined all even values of k, up to $k = 52$. The $(H - L)/50$ sequence has its first 17 values coinciding with the central values in the quadrinomial triangle. The $(H - L)/50$ sequence is

1, 1, 4, 12, 44, 155, 580, 2128, 8092, 30276, 116304, 440484, 1703636, 6506786, 25288120, 97181760, 379061020, 1463609338... .

We observed that $k = 3m + 2$ has $m + 1$ of the central values in the quadrinomial triangle.

In an attempt to discover a pattern for predicting all recurrence relations of the $(H - L)/k$ sequences, we examined the recurrence relations for the even partitions up to $k = 48$. These equations are displayed in Table 3.

As the reader can see, the size (both degree and coefficient) of the equations grows rapidly. For example, in finding the recurrence equation in the case with 48 partitions, we used the first 30 elements in the $(H - L)/48$ sequence. The last element has 17 digits, too large for accuracy in most computers and calculators; thus, much computation was done by hand using the pivotal element method. Even after examining so many cases, we were unable to derive a general formula for predicting successive recurrence equations. However, we discovered several patterns that enabled us to make accurate conjectures about most of the coefficients and the degree of the recurrence equation.

We predict that for $N = 8m$, $8m + 2$, or $8m + 4$, the degree of the recurrence equation is $4m$. For $N = 8m + 6$, the degree is $4m + 2$.

The first coefficient is 1.

The second coefficient we predict to have the form $-N$ for $N \geq 8$. A difference of 2 is observed between successive elements in the sequence of all second coefficients for even partitions.

The third coefficient we predict to have the form $\frac{1}{2}N(N - 11)$ for $N \geq 14$. A second difference of 4 is observed between successive elements in the sequence of third coefficients for even partitions. We show this below:

$$\begin{array}{cccccccccccccccccccccccc}
21 & & 40 & & 63 & & 90 & & 121 & & 156 & & 195 & & 238 & & 285 & & 336 & & 391 & & 450 & \ldots \\
& 19 & & 12 & & 23 & & 31 & & 35 & & 39 & & 43 & & 47 & & 51 & & 55 & & 59 & \ldots \\
& & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & \ldots &
\end{array}$$

The fourth coefficient can be found by making a table of third differences between successive elements in the sequence of fourth coefficients for even partitions. This appears to be valid for $N \geq 20$. The third difference is 8. We show this below:

TABLE 3. RECURRENCE EQUATIONS FOR $(H - L)/k$ SEQUENCES FOR EVEN PARTITIONS IN THE QUADRINOMIAL TRIANGLE

N	Recurrence Equation
4	$x - 1 = 0$
6	$x^2 - 3 = 0$
8	$x^4 - 8x^2 + 8 = 0$
10	$x^4 - 10x^2 + 5 = 0$
12	$x^4 - 12x^2 + 9 = 0$
14	$x^6 - 14x^4 + 21x^2 - 7 = 0$
16	$x^8 - 16x^6 + 40x^4 - 32x^2 + 8 = 0$
18	$x^8 - 18x^6 + 63x^4 - 57x^2 + 9 = 0$
20	$x^8 - 20x^6 + 90x^4 - 100x^2 + 25 = 0$
22	$x^{10} - 22x^8 + 121x^6 - 176x^4 + 88x^2 - 11 = 0$
24	$x^{12} - 24x^{10} + 156x^8 - 296x^6 + 225x^4 - 72x^2 + 8 = 0$
26	$x^{12} - 26x^{10} + 195x^8 - 468x^6 + 455x^4 - 169x^2 + 13 = 0$
28	$x^{12} - 28x^{10} + 238x^8 - 700x^6 + 833x^4 - 392x^2 + 49 = 0$
30	$x^{14} - 30x^{12} + 285x - 1000x + 1440x - 903x + 230x^2 - 15 = 0$
32	$x^{16} - 32x^{14} + 336x^{12} - 1376x^{10} + 2376x^8 - 1920x^6 + 736x^4 - 128x^2 + 8 = 0$
34	$x^{16} - 34x^{14} + 391x^{12} - 1836x^{10} + 3757x^8 - 3740x^6 + 1819x^4 - 374x^2 + 17 = 0$
36	$x^{16} - 36x^{14} + 450x^{12} - 2388x^{10} + 5715x^8 - 6804x^6 + 4059x^4 - 1080x^2 + 81 = 0$
38	$x^{18} - 38x^{16} + 613x^{14} - 3040x^{12} + 8398x^{10} - 11742x^8 + 8512x^6 - 3059x^4 + 475x^2 - 19 = 0$
40	$x^{20} - 40x^{18} + 580x^{16} - 3800x^{14} + 11970x^{12} - 19408x^{10} + 16860x^8 - 7800x^6 + 1825x^4 - 200x^2 + 8 = 0$
42	$x^{20} - 42x^{18} + 651x^{16} - 4676x^{14} + 16611x^{12} - 30912x^{10} + 31647x^8 - 17937x^6 + 5334x^4 - 700x^2 + 21 = 0$
44	$x^{20} - 44x^{18} + 726x^{16} - 5676x^{14} + 22517x^{12} - 47652x^{10} + 56628x^8 - 38236x^6 + 14036x^4 - 2420x^2 + 121 = 0$
46	$x^{22} - 46x^{20} + 805x^{18} - 6808x^{16} + 29900x^{14} - 71346x^{12} + 97198x^{10} - 76912x^8 + 34500x^6 - 8119x^4 + 851x^2 - 23 = 0$
48	$x^{24} - 48x^{22} + 888x^{20} - 8080x^{18} + 38988x^{16} - 104064x^{14} + 160888x^{12} - 147360x^{10} + 79329x^8 - 24080x^6 + 3816x^4 - 288x^2 + 8 = 0$

100	176	296	468	700	1000	1376	1836	2388	...
76	120	172	232	300	376	460	552	...	
44	52	60	68	76	84	92	...		
8	8	8	8	8	8	...			

The fifth coefficient can be found by making a table of fourth differences between successive elements in the sequence of fifth coefficients for even partitions. This appears to be valid for $N \geq 26$. The fourth difference is 16. See Table 4.

TABLE 4. PREDICTING 5th, 6th, AND 7th COEFFICIENTS

455	833	1440	2376	3757	5715	8398	11970	16611	22517	29900	...
378	607	936	1381	1958	2683	3572	4641	5906	7383	...	
229	329	445	577	725	889	1069	1265	1477	...		
100	116	132	148	164	180	196	212	...			
16	16	16	16	16	16	16	...				
1920	3740	6804	11742	19408	30912	47652	71346	104064	...		
1820	3064	4938	7666	11504	16740	23964	32718	...			
1244	1874	2728	3838	5236	6954	9024	...				
630	854	1110	1398	1718	2070	...					
224	256	288	320	352	...						
32	32	32	32	...							
8512	16860	31647	56628	97198	160888	257925	...				
8348	14787	24981	40570	63690	97037	...					
6439	10194	15589	23120	33347	...						
3755	5395	7531	10227	...							
1640	2136	2696	...								
496	560	...									
64	...										

3. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN A MULTINOMIAL TRIANGLE

We challenge the reader to finish this problem: to find the generating functions of the $(H - L)/k$ sequences for all k in the quadrinomial triangle. Then perhaps we could find the generating function of maximal values in Pascal's quadrinomial triangle, and show that the limits of the $(H - L)/k$ generating functions are precisely the generating function of maximal values.

This problem can be extended to examine the maximal values in Pascal's pentanomial triangle and larger multinomial triangles. Again, the pursuer of such an adventure will encounter numbers with up to 20 digits, in which the accuracy of each digit matters in order to find recurrence equations.

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THE STUDY OF POSITIVE INTEGERS (a,b) SUCH THAT $ab+1$ IS A SQUARE

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1. INTRODUCTION

A P-set will be defined as a set of positive integers such that if a and b are two distinct elements of this set, $ab+1$ is a square.

There are many examples of P-sets such as [2, 12] or [1, 3, 8, 120] and even formulas such as

$$[n-1,\ n+1,\ 4n,\ 4n(4n^2-1)]$$

or

$$[m,\ n^2-1+(m-1)(n-1)^2,\ n(mn+2),\ 4m(mn^2-mn+2n-1)^2 + 4(mn^2-mn+2n-1)].$$

(See Cross [1].) However, none of these formulas are general.

More recently, there has been considerable work on P-sets with polynomials (by Jones [2, 3]) and in connection with Fibonacci numbers (by Hoggatt and Bergum [4]).

It is of interest to find out how much these sets can be extended by adding new positive integers to the set; for example [2, 12] can be extended to [2, 12, 420]. A P-set which cannot be extended will be called nonextendible. One purpose of this article is to show that a nonextendible set must have at least four members. Then it will be demonstrated that the number of members of a P-set is finite. Finally, it will be shown that certain types of five-member P-sets will be impossible.

2. EXTENDING P-SETS TO FOUR ELEMENTS

The proof that sets of one or two elements are extendible is very simple, for $[N]$ can always be extended to $[N, N+2]$ and $[a, b]$ can be extended to $[a, b, a+b+2x]$ where $x^2 = ab+1$. (See Euler [5].)

Let $[a, b, N]$ be members of a P-set. Then,

(1) $$ab + 1 = x^2,$$

(2) $$aN + 1 = y^2,$$

(3) $$bN + 1 = z^2.$$

Therefore,

$$by^2 - az^2 = b - a. \tag{4}$$

Let $\overline{y} = by$. Then,

$$\overline{y} - abz^2 = b(b - a). \tag{5}$$

Let the auxiliary Pell equation of (5) be

$$m^2 - abn^2 = 1 \tag{6}$$

or

$$m^2 - (x^2 - 1)n^2 = 1. \tag{7}$$

The minimal positive solution of (7) is $(x,1)$. Hence all the solutions of (7) are given by

$$m_i + \sqrt{x^2 - 1}n_i = (x + \sqrt{x^2 - 1})^i, \quad i = 1, 2, 3, \ldots,$$

and all solutions of (5) are given by

$$\overline{y}_i + \sqrt{x^2 - 1}z_i = (\overline{y}_0 + \sqrt{x^2 - 1}z_0)(x + \sqrt{x^2 - 1})^i, \quad i = 0, 1, 2, \ldots, \tag{8}$$

where $(\overline{y}_0, z_0)$ can take only a finite number of values, one of which must be $(b,1)$. (See Nagell [6].)

There is a one-to-one correspondence between the solutions (y_i, z_i) of (4) and $(\overline{y}_i, z_i)$ of (5) where $\overline{y}_i = by_i$ because $\overline{y}_i^2 = b(b - a + az_i^2)$, and hence y_i is always an integer.

Theorem 1: Let

$$N_k = \frac{y_k^2 - 1}{a}. \tag{9}$$

Then

$$N_i N_{i+j} + 1 = (m_j N_i + n_j y_i z_i)^2 + 1 - n_j^2. \tag{10}$$

Proof: From (8),

$$\overline{y}_{i+j} + \sqrt{x^2 - 1}z_{i+j} = (\overline{y}_i + \sqrt{x^2 - 1}z_i)(m_j + \sqrt{x^2 - 1}n_j).$$

Then $\quad \overline{y}_{i+j} = m_j \overline{y}_i + n_j(x^2 - 1)z_i.$

Hence $\quad by_{i+j} = bm_j y_i + abn_j z_i.$

Therefore $\quad y_{i+j} = m_j y_i + an_j z_i.$

Hence $\quad N_{i+j} = \frac{1}{a}(m_j^2 y_i^2 + 2am_j n_j y_i z_i + a^2 n_j^2 z_i^2 - 1)$, using (9).

Therefore

$$\begin{aligned} N_i N_{i+j} + 1 &= \frac{1}{a^2}(y_i^2 - 1)(m_j^2 y_i^2 + 2am_j n_j y_i z_i + a^2 n_j^2 z_i^2 - 1) + 1 \\ &= \frac{1}{a^2}(m_j^2 y_i^4 + 2am_j n_j z_i y_i^3 + [a^2 n_j^2 z_i^2 - 1 - m_j^2]y_i^2 \\ &\quad - 2am_j n_j z_i y_i - an_j^2[by_i^2 - b + a] + 1 + a^2), \\ &\qquad \text{using (4)}, \\ &= \frac{1}{a^2}(m_j^2 y_i^4 + 2am\, n_j z_i y_i^3 + [a^2 n_j^2 z_i^2 - 1 - m_j^2 - abn_j^2]y_i^2 \end{aligned}$$

(continued)

$$- 2am_jn_jz_iy_i + abn_j^2 - a^2n_j^2 + 1 + a^2)$$

$$= \frac{1}{a^2}(m_j^2y_i^4 + 2am_jn_jz_iy_i^3 + [a^2n_j^2z_i^2 - 2m_j^2]y_i^2$$

$$- 2am_jn_jz_iy_i + m_j^2) + 1 - n_j^2, \text{ using (6)},$$

$$= (m_jN_i + n_jy_iz_i)^2 + 1 - n_j^2, \text{ using (9)}.$$

Theorem 2: The P-set $[a, b, N_i]$ can be extended to $[a, b, N_i, N_{i+1}]$.

Proof: Now $y_{i+1} = m_1y_i + an_1z_i > y_i$.

Therefore $N_{i+1} > N_i$, using (9).

Therefore N_{i+1} is positive if N_i is positive.

Also, if N_i is an integer, $y_i^2 \equiv 1 \bmod a$. Now,

$$\begin{aligned} y_{i+1}^2 &= m_1^2y_i^2 + 2am_1n_1y_iz_i + a^2n_1^2z_i^2 \\ &\equiv (ab + 1)y_i^2 \bmod a \text{ as } m_1 = x = \sqrt{ab + 1} \\ &\equiv y_i^2 \bmod a. \end{aligned}$$

Therefore N_{i+1} is an integer.

In fact, it can be shown by induction that if N_i is a positive integer, then so must be N_{i+j}.

Now as $(m_1, n_1) = (x, 1)$, then

$$N_iN_{i+1} + 1 = (xN_i + y_iz_i)^2$$

and therefore $[a, b, N_i]$ can be extended to $[a, b, N_i, N_{i+1}]$.

A formula can be developed for N_{i+1} from a, b, and N_i; that is,

$$N_{i+1} = a + b + N_i + 2abN_i + 2\sqrt{(ab + 1)(aN_i + 1)(bN_i + 1)}.$$

3. FINITENESS OF P-SETS

There are no known P-sets of more than four members. However, it can be proved that there are no infinite sets. In fact, given three members of the set a, b, and c, it can be shown that all other members are bounded, for if

$$aN + 1 = x^2,\ bN + 1 = y^2,\ cN + 1 = z^2, \text{ and } t = xyz,$$

then $abcN^3 + (ab + bc + ca)N^2 + (a + b + c)N + 1 = t^2$.

Let $H = \max\{abc, ab + bc + ca, a + b + c\}$.

Now, as $abcN^3 + (ab + bc + ca)N^2 + (a + b + c)N + 1$ has no squared linear factor in N, by Baker [7],

$$N < \exp\{(10^6H)^{10^6}\}.$$

Until recently there was no way of knowing if a P-set was nonextendible if it had four elements. However, in Baker and Davenport [8], it has been proved that [1, 3, 8, 120] cannot be extended. In fact, it has been shown that [1, 3, 8] can only be extended to [1, 3, 8, 120]. There were calculations done to prove this that needed the aid of a computer. The method in Baker and Davenport [8] would seem workable for checking if there are other sets of four which are nonextendible.

A recent adaptation of this method was given by Grinstead [9].

4. RESTRICTIONS ON EXTENDING FOUR-MEMBER P-SETS

First it should be noted that Baker and Davenport [8] (from their relationship (20) and Section 5) seem to indicate that any fifth member of a P-set that is very large compared to the first four would have to satisfy some very unusual conditions.

The following lemma and theorem give some limitations in the reverse direction.

Lemma: $x < a + b$ if $a > 0$ and $b > 0$.

Proof: If $x \geq a + b$, then $a^2 + ab + b \leq 1$, using (1). Therefore, $a = 0$ or $b = 0$, which is not true.

Theorem 3: If $2i \geq j - 1$, then with the exception of $j = 1$, $N_i N_{i+1} + 1$ is not a square. (N_i is not equal to zero, as members of P-sets are defined to be positive.)

Proof: Let $L = m_j N_i + n_j y_i z_i$. Suppose $N_i N_{i+j} + 1$ is a square. Now

$$L^2 - (L - 1)^2 = 2L - 1.$$

Then

$$n_j^2 - 1 \geq 2L - 1 \quad \text{from (10)} \quad \text{if } j \neq 1.$$

Therefore

$$\frac{m_j N_i + n_j y_i z_i}{\frac{n_j^2}{2}} \leq 1. \tag{11}$$

Now

$$by_i + \sqrt{ab}z_i = (by_0 + \sqrt{ab}z_0)(m_i + \sqrt{ab}n_i)$$

implies

$$y_i = m_i y_0 + a n_i z_0$$

and

$$z_i = b n_i y_0 + m_i z_0.$$

Let $M = x + \sqrt{x^2 - 1}$. Then it can be shown that

$$m_j = \frac{1}{2}(M^j + M^{-j})$$

and

$$n_j = \frac{1}{2\sqrt{x^2 - 1}}(M^j - M^{-j}).$$

Therefore

$$\frac{n_j^2}{2} = \frac{1}{8(x^2 - 1)}(M^j - M^{-j})^2.$$

Now

$$y_i \geq \frac{1}{2}(M^i + M^{-i}) + \frac{a}{2\sqrt{x^2 - 1}}(M^i - M^{-i})$$

and

$$z_i \geq \frac{b}{2\sqrt{x^2 - 1}}(M^i - M^{-i}) + \frac{1}{2}(M^i + M^{-i}).$$

Then

$$y_i z_i \geq \frac{b}{4\sqrt{x^2-1}}(M^{2i} - M^{-2i}) + \frac{ab}{4(x^2-1)}(M^i - M^{-i})^2$$
$$+ \frac{1}{4}(M^i + M^{-i})^2 + \frac{a}{4\sqrt{x^2-1}}(M^{2i} - M^{-2i})$$
$$= \frac{a+b}{4\sqrt{x^2-1}}(M^{2i} - M^{-2i}) + \frac{1}{2}(M^{2i} + M^{-2i}).$$

Therefore

$$\frac{n_j y_i z_i}{\frac{n_j^2}{2}} \geq \frac{\frac{a+b}{8(x^2-1)}(M^{2i} - M^{-2i}) + \frac{1}{4\sqrt{x^2-1}}(M^{2i} + M^{-2i})}{\frac{1}{8(x^2-1)}(M^j - M^{-j})}$$

$$= (a+b)\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})} + 2\sqrt{x^2-1}\frac{(M^2 + M^{-2})}{(M^j - M^{-j})} > x\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})}$$
$$+ 2\sqrt{x^2-1}\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})}$$

(as $x < a + b$ from the Lemma)

$$= \frac{M^{2i+1} - M^{-2i+1} + M^{2i}\sqrt{x^2-1} - M^{-2i}\sqrt{x^2-1}}{M^j - M^{-j}}.$$

Now, if $i > 0$, it can be easily shown that

$$M^{-4i} + \frac{M^{-4i+1}}{\sqrt{x^2-1}} < 1 \quad \text{as} \quad x > 1.$$

Therefore $\quad -M^{-2i+1} + M^{2i}\sqrt{x^2-1} - M^{-2i}\sqrt{x^2-1} > 0 > -M^{-2i-1}.$

Hence

$$\frac{n_j y_i z_i}{\frac{n_j^2}{2}} > \frac{M^{2i+1} - M^{-(2i+1)}}{M^j - M^{-j}} \geq 1 \quad \text{if} \quad 2i + 1 \geq j \text{ or } j - 1 \leq 2i.$$

Thus, Theorem 3 is proved, from (11), as $j > 1$; therefore, i must be greater than zero.

5. A PARTICULAR RATIONAL FIVE-MEMBER P-SET BY EULER CANNOT BE INTEGER

It will now be shown what will happen if rationals are allowed. Suppose the P-set $[a, b, c, d]$ is extended to

$$\left[a, b, c, d, \frac{4r + 2p(s+1)}{(s-1)^2}\right]$$

where a, b, c, and d are positive integers,

$$c = 2x + a + b,\ d = 4x(x + a)(x + b),$$

$$p = a + b + c + d,\ r = abc + abd + acd + bcd,$$

and $$s = abcd.$$

Here, s can have a positive or negative value. This was first given by Euler [5].

Theorem 4: $\dfrac{4r + 2p(s + 1)}{(s - 1)^2}$ is never a positive integer. In fact, it is always less than 1.

The following proof has been considerably shortened due to some suggestions from Professor Jones.

Proof: Re-order a, b, c, and d such that $a < b < c < d$. If $a = 1$ and $b = 2$, then $ab + 1 = 3$, which is not a square. Therefore $b \geq 3$, $c \geq 4$, and $d \geq 5$. Now

$$\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd} \leq \frac{13}{60} < \frac{1}{4}.$$

Therefore $4p < s$. Also,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 2.$$

Therefore $abc + abd + acd + bcd < 2abcd$ or $r < 2s$.

$$\text{Hence } \frac{4r + 2p(s + 1)}{(s - 1)^2} < \frac{8s + \dfrac{s(s + 1)}{2}}{(s - 1)^2} = \frac{s^2 + 17s}{2(s - 1)^2}$$

$$= \frac{1}{2} + \frac{19}{2(s - 1)} + \frac{9}{(s - 1)^2}$$

$$< \frac{1}{2} + \frac{19}{116} + \frac{9}{3364} \quad \text{as} \quad s > 59$$

$$< 1.$$

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ABSORPTION SEQUENCES

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1. INTRODUCTION

In the classical gambler's ruin problem, a gambler beginning with i dollars, either wins or loses one dollar each play. The game ends when he has lost all his initial money or has accumulated $a(\geq i)$ dollars. The situation can also be described as a simple random walk on the integers beginning at with absorbing barriers at 0 and a. Let $F_a(i,n)$ represent the number of different paths of exactly n steps which begin at i ($i = 0, 1, 2, \ldots, a$) and end with absorption at either 0 or a. For fixed values of a and i, $F_a(i,n)$ is a sequence of nonnegative integers called an "absorption sequence." In other words, $F_a(i,n)$ represents the number of different ways a gambler who begins with i dollars can end his play using n one dollar bets.

2. A RECURRENCE RELATION WITH BOUNDARY CONDITIONS

Appropriate boundary conditions, suggested by the condition that the random walk stops when it first hits either 0 or a are

$$F_a(0,0) = F_a(a,0) = 1$$

$$F_a(i,0) = 0,\ i = 1, 2, \ldots, a - 1$$

$$F_a(0,n) = F_a(a,n) = 0,\ n \quad 0.$$

A path which begins at $0 < i < a$ must in one step go to either $i - 1$ or $i + 1$. For this reason, we have a recurrence relation for the number of paths:

$$F_a(i,n) = F_a(i - 1,n - 1) + F_a(i + 1,n - 1),\ n > 0,\ 0 < i < a.$$

3. EXAMPLES OF RECURRENCE RELATIONS AND ABSORPTION SEQUENCES

TABLE 1. $F_5(i,n)$

$i \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
5	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	1	2	3	5	8	13	21	34	55
3	0	0	1	1	2	3	5	8	13	21	34	55	89
2	0	0	1	1	2	3	5	8	13	21	34	55	89
1	0	1	0	1	1	2	3	5	8	13	21	34	55
0	1	0	0	0	0	0	0	0	0	0	0	0	0

The entries in each row are the beginning of an absorption sequence. Absorption at 0 or 5.

TABLE 2. $F_9(i,n)$

i \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
9	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	1	0	1	0	2	0	5	1	14	7	42	34
7	0	0	1	0	2	0	5	1	14	7	42	34	132
6	0	0	0	1	0	3	1	9	6	28	27	90	109
5	0	0	0	0	1	1	4	5	14	20	48	75	165
4	0	0	0	0	1	1	4	5	14	20	48	75	165
3	0	0	0	1	0	3	1	9	6	28	27	90	109
2	0	0	1	0	2	0	5	1	14	7	42	34	132
1	0	1	0	1	0	2	0	5	1	14	7	42	34
0	1	0	0	0	0	0	0	0	0	0	0	0	0

The entries in eqch row are the beginning of an absorption sequence. Absorption at 0 or 9.

(a) $F_3(1,n) = F_3(2,n) = 1,\ n > 0.$

(b) $F_4(1,2m) = 0,\ F_4(1,2m+1) = 2^m,\ m \geq 0;$

$F_4(2,2m) = 2^m,\ m > 0,\ F_4(2,2m+1) = 0,\ m \geq 0.$

(c) Let F_n represent the well-known Fibonacci number sequence [1]:

$$F_1 = 1,\ F_2 = 1,\ \ldots,\ F_{n+1} = F_n + F_{n-1}$$

in general. We have

$$F_5(1,n+2) = F_5(2,n+1) = F_n \quad \text{(see Table 1)}$$

$$F_5(1,n) = F_5(4,n),\ F_5(2,n) = F_5(3,n)$$

by symmetry.

By enumerating, see Table 1, it is easy to show that (assuming $a = 5$ and omitting the subscript)

$$F(2,2) = F(2,3) = 1$$

$$\begin{aligned} F(2,n+1) &= F(1,n) + F(3,n) \quad \text{(recurrence relation)} \\ &= F(1,n) + F(2,n) \quad \text{(symmetry)} \\ &= F(2,n-1) + F(2,n) \quad \text{(boundary condition for } n > 1). \end{aligned}$$

The sequence $F(2,n)$ thus satisfies the initial conditions and recurrence relation for the Fibonacci numbers. In the case of $F_3(1,n)$, the argument is similar.

(d) $F_6(1,2m) = 0,\ F_6(1,2m+1) = 3^{m-1},\ m \geq 1$, and $F_6(1,1) = 1;$

$F_6(2,2m) = 3^{m-1},\ m \geq 1,\ F_6(2,2m+1) = 0;$

$F_6(3,2m) = 0,\ F_6(3,2m+1) = 2 \cdot 3^{m-1},\ m \geq 1$, and $F_6(3,1) = 0.$

(e) Let $a = 9$ and omit the subscript.

$$F(1,1) = 1,\ F(1,2) = 0,\ F(1,3) = 1$$

and

$$F(1,n) = 3F(1,n-2) + F(1,n-3) - 1,\ n > 3.$$

$$F(2,1) = 0,\ F(2,2) = 1,\ F(2,3) = 0$$

and

$$F(2,n) = 3F(2,n-2) + F(2,n-3) - 1,\ n > 3.$$

$$F(3,1) = 0,\ F(3,2) = 0,\ F(3,3) = 1$$

and

$$F(3,n) = 3F(3,n-2) + F(3,n-3),\ n > 3.$$

$$F(4,1) = 0,\ F(4,2) = 0,\ F(4,3) = 0$$

and

$$F(4,n) = 3F(4,n-2) + F(4,n-3) + 1,\ n > 3.$$

$F(9 - i,n) = F(i,n)$ by symmetry.

By enumeration, see Table 2, the initial conditions can be seen to hold as well as the fact that (assuming $a = 9$ and omitting the subscript)

$$F(1,4) = 0,\ F(2,4) = 2,\ F(3,4) = 0, \text{ and } F(4,4) = 1.$$

The recurrence relations therefore hold if $n = 4$. For an induction argument assume they all hold for a general value of n.

$$\begin{aligned} F(1,n+1) &= F(0,n) + F(2,n) = F(2,n) \\ &= 3F(2,n-2) + F(2,n-3) - 1 \quad \text{(the induction hypothesis)} \\ &= 3F(1,n-1) + F(1,n-2) - 1 \end{aligned}$$

[for $i > 0$, $F(0,i) = 0$.] Similarly,

$$\begin{aligned} F(2,n+1) &= F(1,n) + F(3,n) \\ &= 3[F(1,n-2) + F(3,n-2)] + F(1,n-3) \\ &\qquad + F(3,n-3) - 1 \quad \text{(the induction hypothesis)} \\ &= 3F(2,n-1) + F(2,n-2) - 1. \end{aligned}$$

In just the same way, it is easy to show that both $F(3,n+1)$ and $F(4,n+1)$ satisfy, respectively, the stated recurrence relation.

(f) Assume $a = 10$ and omit the subscript.

$F(1,2m) = 0$, $F(1,1) = 1$, $F(1,3) = 1$, and

$$F(1,2m+1) = 4F(1,2m-1) - \sum_{k=1}^{m-1} F(1,2k-1) - 1,\ m > 1.$$

$F(2,2m-1) = 0$, $m \geq 1$, $F(2,2) = 1$, $F(2,4) = 2$, and

$$F(2,2m+2) = 4F(2,2m) - \sum_{k=1}^{m-1} F(2,2k) - 2.$$

$F(3,2m) = 0,\ m \geq 0,\ F(3,1) = 0,\ F(3,3) = 1$, and

$$F(3,2m+1) = 4F(3,2m-1) - \sum_{k=1}^{m-1} F(3,2k-1) - 1,\ m > 1.$$

$F(4,2m+1) = 0,\ m \geq 0,\ F(4,2) = 0,\ F(4,4) = 1$, and

$$F(4,2m+2) = 4F(4,2m) - \sum_{k=1}^{m-1} F(4,2k) + 1.$$

$F(5,2m) = 0,\ m \geq 0,\ F(5,1) = 0,\ F(5,3) = 0$, and

$$F(5,2m+1) = 4F(5,2m-1) - \sum_{k=1}^{m-1} F(5,2k-1) + 2.$$

$F_{10}(10 - i, n) = F_{10}(i,n),\ i = 1, 2, 3, 4$, by symmetry.

In the manner shown in example (e), all of these statements can be verified easily. Because of their length and repetitive nature, this discussion is omitted.

A referee has noted that if $A = (a_{ij})$ is the square matrix of order a defined by $a_{ij} = 1$ if $|i - j| = 1,\ i \neq 1,\ i \neq a$; $a_{ij} = 0$ otherwise, then the nth column X_n in the array of absorption sequences is given by

$$A^n X_0 = X_n \quad \text{where} \quad X_0 = (1, 0, 0, \ldots, 0, 1)^T.$$

This approach, as it has been applied to the related problem of counting paths in reflections in glass plates [2], might be used to codify and expand many of the current results. The referee has also made a (apparently correct) conjecture: if p is a prime and $a = 2p$, then p divides $F_{2p}(i,n)$ for $n \geq (p + 1)$ and $0 \leq 1 \leq 2p$.

4. RESULTS FOR SEQUENCES USING PROBABILISTIC REASONING

To illustrate what results follow from the connection between absorption sequences and probability, let us use the Fibonacci number sequence, F_n, which appears in example 3(c). Similar results can be found for any absorption sequence.

(a) The probability that absorption at one of the boundaries will take place is one [2, p. 345]. In the case where zero and five are the boundaries, $F_5(2,n)$ represents the number of paths that begin at two, and end at zero or five in n steps. If a "win" or a "loss" is equally likely, then the probability that the game is over in n steps is $2^{-n}F_5(2,n)$. Hence,

$$\sum_{n=1}^{\infty} 2^{-n}F_5(2,n) = 1 \quad \text{or} \quad \sum_{n=2}^{\infty} 2^{-n}F_{n-1} = 1.$$

(b) The expected duration of play in the equally likely case is given, in general, by the formula $i(a - i)$ [2, p. 349]. It is also given in this example by

$$\sum_{n=2}^{\infty} n2^{-n}F_5(2,n)$$

from the definition of expected value. We have then, with $a = 5$ and $i = 2$, that for the Fibonacci sequence

$$\sum_{n=2}^{\infty} n2^{-n}F_{n-1} = 6.$$

(c) In a formula attributed to Lagrange [2, p. 353] for the equally likely case with absorptions at 0 or 5, the probability of ruin (or absorption at zero) on the nth step is given as

$$u(i,n) = \frac{1}{5}\sum_{\nu=1}^{4}\left(\cos\frac{\pi\nu}{5}\right)^{n-1}\sin\frac{\pi\nu}{5}\sin\frac{\pi i\nu}{5}$$

$$i = 1,\ 2,\ 3,\ 4, \text{ and } n > 0.$$

In this formula, if $(n - i)$ is odd, $u(i,n) = 0$, as seems logical in terms of the random walk formulation as well as in light of trigonometric identities. If $(n - i)$ is even,

$$u(i,n) = \frac{2}{5}\left[\left(\cos\frac{\pi}{5}\right)^{n-1}\sin\frac{\pi}{5}\sin\frac{\pi i}{5} + \left(\cos\frac{2\pi}{5}\right)^{n-1}\sin\frac{2\pi}{5}\sin\frac{2\pi i}{5}\right].$$

Since, furthermore, each path of length n has probability 2^{-n}, the *number* of paths of length n involved is $2^n u(i,n)$. In particular, if $i = 3$, $n = 2m + 1$, then $2^{2m+1}u(3,2m + 1)$, which, as shown above, is the Fibonacci number F_{2m}. We obtain a trigonometric representation for "one-half" the Fibonacci numbers:

$$F_{2m} = \frac{2^{2m+2}}{5}\left[\left(\cos\frac{\pi}{5}\right)^{2m}\sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \left(\cos\frac{2\pi}{5}\right)^{2m}\sin\frac{2\pi}{5}\sin\frac{6\pi}{5}\right],$$

$$m = 1,\ 2,\ 3,\ \ldots\ .$$

To use Lagrange's probability of ruin formula for the rest of the Fibonacci numbers, the number of paths that begin at 2 and are absorbed at 0 in $2m$ steps for $m > 0$ is, as indicated above, $F(2,2m)$ or F_{2m-1}. Therefore, we have $2^{2m}u(2,2m) = F_{2m-1}$ or

$$F_{2m-1} = \frac{2^{2m+1}}{5}\left[\left(\cos\frac{\pi}{5}\right)^{2m-1}\sin\frac{\pi}{5}\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5}\right)^{2m-1}\sin\frac{2\pi}{5}\sin\frac{4\pi}{5}\right],$$

$$m = 1,\ 2,\ 3,\ \ldots\ .$$

Using trigonometric identities, these two formulas combine into one new trigonometric representation of the Fibonacci numbers.

$$F_n = \frac{2^{n+2}}{5}\left(\cos\frac{\pi}{5}\right)^n\sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \left(-\cos\frac{2\pi}{5}\right)^n\sin\frac{2\pi}{5}\sin\frac{6\pi}{5}\ ,\ n > 0.$$

(d) By using the method of images, repeatedly reflecting the path from the end points [2, p. 96], it is possible to show that in the random walk beginning at 3 with absorption at 0 or 5, the number of paths that arrive at 1 in $(n - 1)$ steps hitting neither 0 nor 5 is given by

$$\sum_k\left[\binom{n-1}{\frac{n+10k+1}{2}} - \binom{n-1}{\frac{n+10k+3}{2}}\right]$$

where the sum extends over the positive and negative integers k with the convention that the "binomial coefficient" $\binom{n}{x}$ is zero whenever x does not equal an integer between 0 and n. (This sum has a finite number of n n-zero terms.) With $n = 2m + 1$, it follows that the number of paths which are absorbed at 0 in $2m + 1$ steps is

$$F(3, 2m + 1) = F_{2m} = \sum_k \left[\binom{2m}{m + 5k + 1} - \binom{2m}{m + 5k + 2}\right].$$

To obtain the "other half" of the Fibonacci numbers, we count

$$F_{2m-1} = F(2, 2m),$$

the number of paths that begin at 2 and are absorbed at 0 in $2m$ steps. The method of repeated reflections gives us

$$F_{2m-1} = \sum_k \left[\binom{2m - 1}{m + 5k} - \binom{2m - 1}{m + 5k + 1}\right]$$

the sum extending over all positive and negative integers.

Two slightly different representations of the Fibonacci numbers can now be obtained through use of the easily verified relations

$$\binom{2m}{m + 5k + 1} - \binom{2m}{m + 5k + 2} = \frac{10k + 3}{2m + 1}\binom{2m + 1}{m + 5k + 2}$$

and

$$\binom{2m - 1}{m + 5k} - \binom{2m - 1}{m + 5k + 1} = \frac{5k + 1}{m}\binom{2m}{m + 5k + 1}$$

where k is any integer, m is a positive integer, and the conventions for the binomial coefficients introduced above continue to apply. By direct substitution, we obtain

$$F_{2m} = \sum_k \frac{10k + 3}{2m + 1}\binom{2m + 1}{m + 5k + 2} \quad \text{and} \quad F_{2m-1} = \sum_k \frac{5k + 1}{m}\binom{2m}{m + 5k + 1}.$$

Finally, by treating the terms with positive k separately from those with negative k, we obtain

$$F_{2m} = \frac{1}{2m + 1}\left\{\sum_{k=0}^{r}(10k + 3)\binom{2m + 1}{m + 5k + 2} - \sum_{k=1}^{s}(10k - 3)\binom{2m + 1}{m + 5k - 1}\right\},$$

$$F_{2m+1} = \frac{1}{m}\left\{\sum_{k=0}^{r}(5k + 1)\binom{2m}{m + 5k + 1} - \sum_{k=1}^{t}(5k - 1)\binom{2m}{m + 5k - 1}\right\},$$

$$r = \left[\frac{m - 1}{5}\right], \quad s = \left[\frac{m + 2}{5}\right], \quad t = \left[\frac{m + 1}{5}\right],$$

with [ ] the greatest integer in x, and the convention that a sum is zero if its lower limit exceeds its upper limit.

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1. R. A. Brualdi. *Introductory Combinatorics*. New York: North-Holland, 1977. Pp. 90-96.
2. W. Feller. *An Introduction to Probability Theory and Its Applications*. Vol. I; 3rd ed., rev. New York: Wiley, 1968.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR. S.E., ALBUQUERQUE, NEW MEXICO 87108. Each solution (or problem) should be submitted on a separate sheet of paper. Preference will be given to solutions (or problems) typed, double-spaced, in the format used below. Solutions (or problems) should be received no later than four months following (or prior to) the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-406 *Proposed by Wray G. Brady, Slippery Rock State College, PA.*

Let $x_n = 4L_{3n} - L_n^3$ and find the greatest common divisor of the terms of the sequence $x_1, x_2, x_3, \ldots$.

B-407 *Proposed by Robert M. Giuli, Univ. of California, Santa Cruz, CA.*

Given that

$$\frac{1}{1 - x - xy} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} x^n y^k$$

is a double ordinary generating function for a_{nk}, determine a_{nk}.

B-408 *Proposed by Lawrence Somer, Washington, D.C.*

Let $d \in \{2, 3, \ldots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. For $d = 2$ and $z(p)$ an even integer $2k$, it was shown in B-386 that

$$F_{n+1} G_{n+k} \equiv F_n G_{n+k+1} \pmod{p}.$$

Establish a generalization for $d \geq 2$.

B-409 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Let $P_n = F_n F_{n+a}$.

Must $P_{n+6r} - P_n$ be an integral multiple of $P_{n+4r} - P_{n+2r}$ for all nonnegative integers a and r?

B-410 *Proposed by M. Wachtel, Zürich, Switzerland.*

Some of the solutions of

$$5(x^2 + x) + 2 = y^2 + y$$

in positive integers x and y are:

$$(x,y) = (0,1),\ (1,3),\ (10,23),\ (27,61).$$

Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n, y_n). Also find $\lim(x_{n+1}/x_n)$ and $\lim(x_{n+2}/x_n)$ as $n \to \infty$ in term of

$$\alpha = (1 + \sqrt{5})/2.$$

B-411 *Proposed by Bart Rice, Crofton, MD.*

Tridiagonal n by n matrices $A_n = (a_{ij})$ of the form

$$a_{ij} = \begin{cases} 2a \ (a \text{ real}) \text{ for } j = i \\ 1 \quad \text{for } j = i \pm 1 \\ 0 \quad \text{otherwise} \end{cases}$$

occur in numerical analysis. Let $d_n = \det A_n$.

(i) Show that $\{d_n\}$ satisfies a second-order homogeneous linear recursion.
(ii) Find closed-form and asymptotic expressions for d_n.
(iii) Derive the combinatorial identity

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1}(-x)^k = (x+1)^{(n-1)/2}\,\frac{\sin rn}{\sin r},$$

$$x > 0,\ r = \tan^{-1}\sqrt{x}.$$

SOLUTIONS

Lucky L Units Digit

B-382 *Proposed by A. G. Shannon, N.S.W. Inst. of Technology, Australia.*

Prove that L_n has the same last digit (i.e., units digit) for all n in the infinite geometric progression 4, 8, 16, 32, ...

Note: Several solvers pointed out that the subscript n was missing from the L_n.

Solution by Lawrence Somer, Washington, D.C.

I present two solutions, the first of which is more direct.

First Solution: Note that

$$L_n^2 = (a^n + b^n)^2 = a^{2n} + b^{2n} + 2(ab)^n = L_{2n} + 2(-1)^n.$$

We now proceed by induction. $L_4 = 7$. Now assume

$$L_{2^n} \equiv 7 \pmod{10},\ n \geq 2.$$

Then

$$L_{2^{n+1}} + 2(-1)^{2^n} = L_{2^{n+1}} + 2 = L_{2^n}^2 \equiv 7^2 \equiv 9 \pmod{10}.$$

Thus,

$$L_{2^{n+1}} \equiv 9 - 2 \equiv 7 \pmod{10}$$

and we are done.

Second Solution: Note that the Lucas sequence has a period modulo 10 of 12. Now $\{2^n\}_{n=2}^{\infty}$ (mod 12) is of the form 4, 8, 4, 8, ... But L_4 and L_8 both end in 7. Thus, we are done.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, Bob Prielipp, Sahib Singh, Charles W. Trigg, Gregory Wulczyn, and the proposer.

Reappearance

B-383 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.*

Solve the difference equation

$$U_{n+2} - 5U_{n+1} + 6U_n = F_n.$$

Note: Bob Prielipp and Sahib Singh point out that B-383 is a rerun of B-370.

Solvers in addition to those of B-370 are Ralph Garfield, Lawrence Somer, and Gregory Wulczyn.

A Recursion for F_{2n}^4 or F_{2n+1}^4

B-384 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.*

Establish the identity

$$F_{n+10}^4 = 55(F_{n+8}^4 - F_{n+2}^4) - 385(F_{n+6}^4 - F_{n+4}^4) + F_n^4.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA.

It suffices to prove that

$$F_{n+10}^4 - F_n^4 = 55(F_{n+8}^4 - F_{n+2}^4) - 385(F_{n+6}^4 - F_{n+4}^4). \tag{1}$$

Factoring the difference of squares, one sees that (1) follows from the two formulas:

$$(F_{n+10}^2 - F_n^2)/55 = (F_{n+8}^2 - F_{n+2}^2)/8 = F_{n+6}^2 - F_{n+4}^2; \tag{2}$$

$$F_{n+10}^2 + F_n^2 = 8(F_{n+8}^2 + F_{n+2}^2) - 7(F_{n+6}^2 + F_{n+4}^2). \tag{3}$$

Each of (2) and (3) can be established using the Binet formulas, $a^2 = a + 1$, $a^4 = 3a + 2$, etc., and the corresponding formulas for powers of b.

Also solved by Paul S. Bruckman and the proposer.

Counting Some Triangular Numbers

B-385 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let $T_n = n(n+1)/2$. For how many positive integers n does one have both $10^6 < T_n < 2 \cdot 10^6$ and $T_n \equiv 8 \pmod{10}$?

Solution by Lawrence Somer, Washington, D.C.

By inspection, $T_n \equiv 8 \pmod{10}$ if and only if

$$n \equiv 7 \pmod{20} \text{ or } n \equiv 12 \pmod{20}. \tag{1}$$

Now, $10^6 < T < 2 \cdot 10^6$ if and only if

(2) $$-1/2 + \sqrt{2{,}000{,}000.25} < n < -1/2 + \sqrt{4{,}000{,}000.25}$$

or $1414 \le n \le 1999$. There are 58 integers satisfying conditions (1) and (2). The answer is thus 58.

Also solved by Paul S. Bruckman, Sahib Singh, Charles W. Trigg, Gregory Wulczyn, and the proposer.

Elusive Generalization

B-386 *Proposed by Lawrence Somer, Washington, D.C.*

Let p be a prime and let the least positive integer m with $F_m \equiv 0 \pmod{p}$ be an even integer $2k$. Prove that

$$F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.$$

Generalize to other sequences, if possible.

Solution by Paul S. Bruckman, Concord, CA.

The following formula may be readily verified from the Binet definitions:

(1) $$F_{n+1}L_{n+k} - F_nL_{n+k+1} = (-1)^nL_k.$$

Since $F_{2k} = F_kL_k \equiv 0 \pmod{p}$ and $2k$ is the least positive integer m such that $F_m \equiv 0 \pmod{p}$, thus $F_k \not\equiv 0 \pmod{p}$, which implies $L_k \equiv 0 \pmod{p}$. From (1), we see that this, in turn, implies

(2) $$F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.$$

The desired generalization to other sequences appears to be elusive.

Editor's note: For one generalization, see B-408, proposed in this issue.

Also solved by Sahib Singh, Gregory Wulczyn, and the proposer.

One's Own Infinitude

B-387 *Proposed by George Berzsenyi, Lamar Univ., Beaumont, TX.*

Prove that there are infinitely many ordered triples of positive integers (x,y,z) such that

$$3x^2 - y^2 - z^2 = 1.$$

Editor's note: An infinite number of solutions were produced with $y = z + 2$ by Paul. S. Bruckman, with $y = z$ and with $z = 1$ by Bob Prielipp, with $x = z$ by Sahib Singh, with $z \in \{1, 5, 11, 25\}$ by Gregory Wulczyn, and with $(x^2, y^2, z^2) = (F_{2n+2}w + 1, F_{2n}w + 1, F_{2n+4}w + 1)$, where $w = 4F_{2n+1}F_{2n+2}F_{2n+3}$, by the proposer. Of these, the following was chosen for publication because of its bibliographic and historical references.

Solution by Bob Prielipp, Univ. Of Wisconsin-Oshkosh, WI.

We will show that there are infinitely many ordered triples of positive integers $(x,y,1)$ such that

$$3x^2 - y^2 - 1^2 = 1.$$

The preceding equation is equivalent to $y^2 - 3x^2 = -2$. To assist us in finding all of its solutions, we will employ the following results:

(1) If D and N are positive integers, and if D is not a perfect square, the equation $u^2 - Dv^2 = -N$ has a finite number of classes of solutions. If (u^*, v^*) is the fundamental solution of the class K, we obtain all the solutions (u,v) of K by the formula

$$u + v\sqrt{D} = (u^* + v^*\sqrt{D})(s + t\sqrt{D}),$$

where (s,t) runs through all solutions of $s^2 - Dt^2 = 1$, including $(\pm 1, 0)$.

(2) If p is a prime, and if the equation $u^2 - Dv^2 = -p$ is solvable, it has one or two classes of solutions, according as the prime p divides $2D$ or not. [See Nagell, *Introduction to Number Theory* (2nd ed.; New York: Chelsea Publishing Company, 1964), pp. 204-208.]

By (2), there is only one class of solutions of the equation

$$y^2 - 3x^2 = -2.$$

The fundamental solution is (1,1). The fundamental solution of the equation $y^2 - 3x^2 = 1$ is (2,1). So, all positive integer solutions of $y^2 - 3x^2 = -2$ are given by the formula

$$y + x\sqrt{3} = (1 + \sqrt{3})(2 + \sqrt{3})^n,\ n = 0, 1, 2, 3, \ldots .$$

Thus, the first six positive integer solutions (y,x) of $y^2 - 3x^2 = -2$ are

(1,1), (5,3), (19,11), (71,41), (265,153), and (989,571).

The corresponding six positive integer solutions (x,y,z), with $z = 1$, of the equation $3x^2 - y^2 - z^2 = 1$ are

(1,1,1), (3,5,1), (11,19,1), (41,71,1), (153,165,1), and (571,989,1).

It may be of interest to note that, in his famous *Measurement of a Circle*, Archimedes determines that $3\frac{1}{7} > \pi > 3\frac{10}{71}$ and in deducing these inequalities he uses $\frac{1351}{780} > \sqrt{3} > \frac{265}{153}$. It can be shown that these good approximations to $\sqrt{3}$ satisfy the equations $a^2 - 3b^2 = 1$ and $a^2 - 3b^2 = -2$, respectively, so that Archimedes knew at least some solutions of these equations.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months of publication of the problems.

H-302 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX*

Let c be a constant and define the sequence $\langle a_n \rangle$ by $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} + ca_{n-2}$ for $n \geq 2$. Determine the sequence $\langle b_n \rangle$ for which

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k.$$

H-303 *Proposed by Paul Bruckman, Concord, CA*

If $0 < s < 1$, and n is any positive integer, let

$$(1) \quad H_n(s) = \sum_{k=1}^{n} k^{-s},$$

and

$$(2) \quad \theta_n(s) = \frac{n^{1-s}}{1-s} - H_n(s).$$

Prove that $\lim\limits_{n\to\infty} \theta_n(s)$ exists, and find this limit.

H-304 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

(a) Show that there is a unique partition of the positive integers, N, into two sets, A_1 and A_2, such that

$$A_1 \cup A_2 = n, \qquad A_1 \cap A_2 = \emptyset,$$

and no two distinct elements from the same set add up to a Lucas number.

(b) Show that every positive integer, M, which is not a Lucas number is the sum of two distinct elements of the same set.

H-305 *Proposed by Martin Schechter, Swarthmore College, Swarthmore, PA*

For fixed positive integers, m, n, define a Fibonacci-like sequence as follows:

$$S_1 = 1,\ S_2 = m,\ S_k = \begin{cases} mS_{k-1} + S_{k-2} & \text{if } k \text{ is even} \\ nS_{k-1} + S_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

(Note that for $m = n = 1$, one obtains the Fibonacci numbers.)

(a) Show the Fibonacci-like property holds that if j divides k then S_j divides S_k and in fact that $(S_q, S_r) = S_{(q,r)}$ where $(\ ,\ )$ = g.c.d.

(b) Show that the sequence obtained when

$$[m = 1, n = 4] \text{ and when } [m = 1, n = 8],$$

respectively, have only the element 1 in common.

H-306 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

(a) Prove that the system S,

$$a + b = F_p, \quad b + c = F_q, \quad c + a = F_r,$$

cannot be solved in positive integers if F_p, F_q, F_r are positive Fibonacci numbers.

(b) Likewise, show that the system T,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + e = F_s,\ e + a = F_t,$$

has no solution under the same conditions.

(c) Show that if F_p is replaced by any positive non-Fibonacci integer, then S and T have solutions.

If possible, find necessary and sufficient conditions for the system U,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + a = F_s,$$

to be solvable in positive integers.

SOLUTIONS

Indifferent

H-276 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Show that the sequence of Bell numbers, $\{B_i\}_{i=0}^{\infty}$ is invariant under repeated differencing.

$$B_0 = 1,\ B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad (n \geq 0).$$

Solution by Paul S. Bruckman, Concord CA

The following exponential inversion formula is well known:

$$(1) \quad f(n) = \sum_{k=0}^{n} \binom{n}{k} g(k) \quad \text{iff} \quad g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k).$$

Setting $g(n) = B_n$ and $f(n) = B_{n+1}$ in (1), we obtain the result:

$$(2) \quad B_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_{k+1}, \quad n = 0,\ 1,\ 2,\ \ldots\ .$$

However, the right member of (2) is precisely $\Delta^n B_1$. Hence,

(3) $\Delta^n B_1 = B_n, \quad n = 0, 1, 2, \ldots .$

Also solved by N. Johnson and the proposer.

OLDER STUMPERS!

H-254 *Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA*

Consider the Fibonacci-Pascal-type triangle given below.

$$\begin{array}{ccccccccc}
 & & & & F_1 & & & & \\
 & & & F_1 & & F_1 & & & \\
 & & F_1 & & F_2 & & F_1 & & \\
 & F_1 & & F_3 & & F_3 & & F_1 & \\
F_1 & & F_4 & & F_6 & & F_4 & & F_1 \\
- & - & - & - & - & - & - & - & -
\end{array}$$

Find a formula for the row sums of this array.

H-260 *Proposed by H. Edgar, San Jose State University, San Jose, CA*

Are there infinitely many subscripts, n, for which F_n or L_n are prime?

H-271 *Proposed by R. Whitney, Lock Haven State College, Lock Haven, PA*

Define the binary dual, D, as follows:

$$D = \left\{ t \,\middle|\, t = \prod_{i=0}^{n} (a_i + 2i); \quad a_i \in \{0, 1\}; \quad n \geq 0 \right\}.$$

Let $\overline{D}$ denote the complement of D with respect to the set of positive integers. Form a sequence, $\left\{S_n\right\}_{n=1}^{\infty}$, by arranging $\overline{D}$ in increasing order. Find a formula for S_n.

(Note: The elements of D result from interchanging + and × in a binary number. D contains items like $2^n \cdot n!$, $3 \cdot 2^{n-1} n!$, ...)

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