# AN ALGORITHM FOR PACKING COMPLEMENTS OF FINITE SETS OF INTEGERS <br> GERALD WEINSTEIN <br> The City College of New York, New York, NY 10031 

ABSTRACT
Let $A_{k}=\left\{0=a_{1}<a_{2}<\ldots<a_{k}\right\}$ and $B=\left\{0=b_{1}<b_{2}<\ldots<b_{n} \ldots\right\}$ be sets of $k$ integers and infinitely many integers, respectively. Suppose $B$ has asymptotic density $x: d(B)=x$. If, for every integer $n \geq 0$, there is at most one representation $n=a_{i}+b_{j}$, then we say that $A_{k}$ has a packing complement of density $\geq x$.

Given $A_{k}$ and $x$, there is no known algorithm for determining whether or not $B$ exists.

We define "regular packing complement" and give an algorithm for determining if $B$ exists when packing complement is replaced by regular packing complement. We exemplify with the case $k=5$, i.e., given $A_{5}$ and $x=1 / 10$, we give an algorithm for determining if $A_{5}$ has a regular packing complement $B$ with density $\geq 1 / 10$. We relate this result to the
Conjecture: Every $A_{5}$ has a packing complement of density $\geq 1 / 10$. Let

$$
A_{k}=\left\{0=\alpha_{1}<\alpha_{2}<\ldots<a_{k}\right\}
$$

and

$$
B=\left\{0=b_{1}<b_{2}<\ldots<b_{n}<\ldots\right\}
$$

be sets of $k$ integers and infinitely many integers, respectively. If, for every integer $n \geq 0, n=a_{i}+b_{j}$ has at most one solution, then we call $B$ a packing complement, or $p$-complement, of $A_{k}$.

Let $B(n)$ denote the counting function of $B$ and define $d(B)$, the density of $B$, as follows:

$$
d(B)=\lim _{n \rightarrow \infty} B(n) / n \text { if this limit exists. }
$$

From now on we consider only those sets $B$ for which the density exists.
For a given set $A_{k}$, we wish to find the $p$-complement $B$ with maximum density. More precisely, we define $p\left(A_{k}\right)$, the packing codensity of $A_{k}$, as follows:

$$
p\left(A_{k}\right)=\sup _{B} d(B) \text { where } B \text { ranges over all } p \text {-complements of } A_{k}
$$

Finally, we define $p_{k}$ as the "smallest" $p$-codensity of any $A_{k}$, or, more precisely,

$$
p_{k}=\inf _{A_{k}} p\left(A_{k}\right)
$$

We proved [1] that, for $\varepsilon>0$,

$$
\frac{1}{\binom{k}{2}+1} \leq p_{k} \leq \frac{2.66 \ldots}{k^{2}}+\varepsilon
$$

if $k$ is sufficiently large.
The first four $p_{k}$ are trivial, since we can find sets for which the lower bound is attained. Thus,

$$
A_{1}=\{0\}, A_{2}=\{0,1\}, A_{3}=\{0,1,3\}, A_{4}=\{0,1,4,6\}
$$

give

$$
p_{1}=1, p_{2}=1 / 2, p_{3}=1 / 4, p_{4}=1 / 7 .
$$

But,

$$
\frac{1}{11} \leq p_{5} \leq \frac{1}{10} .
$$

The upper bound is established by $A_{5}=\{0,1,2,6,9\}$ and the lack of certainty in the lower bound is caused by the impossibility of finding $A_{5}$ whose difference set takes on all values $1,2, \ldots, 10$.

Suppose we have a set $A_{k}$, a set $B=\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}$, and a number $N$ such that $a+b \equiv m(\bmod N)$ has at most one solution,

$$
a \varepsilon A_{k}, b \in B \text {, for } 0 \leq m<N .
$$

Then the packing codensity of $A_{k}$ is $\geq n / N$.
If, in the previous paragraph, the $p$-complement $B$ consists entirely of consecutive multiples of $M$, where $(M, N)=1$, i.e., $B=\{M, 2 M, \ldots, n M\}(\bmod$ $N$ ), then we say that $A_{k}$ has a regular $p$-complement of density $\geq n / N$.

As in [2], there is no known algorithm for determining either the packing codensity of $A_{k}$ or even whether $A_{k}$ has a $p$-complement of density $\geq x$.

It is the purpose of this note to give an algorithm for answering the question: does $A_{k}$ have a regular $p$-complement of density $\geq x$ ? We actually give a method for determining whether $A_{5}$ has a regular $p$-complement of density $\geq 1 / 10$, because of its application to the
Conjecture: $\quad p_{5}=1 / 10$.
However, the generalization of our result is obvious.
We adopt the following conventions throughout:
(1) $A_{5}$ represents a set of five integers,

$$
A_{5}=\left\{0=a_{1}<a_{2}<\alpha_{3}<a_{4}<a_{5}\right\} .
$$

(2) $M$ and $N$ are positive integers, with $M<N,(M, N)=1$.
(3) A11 $a_{i}$ are distinct mod $N$.
(4) " $a_{i}$ and $a_{j}$ are adjacent mod $N$ " means that for some $M$ the residues $\bmod N$ of $\alpha_{i}$ and $\alpha_{j}$ occur in the ordered $N$-tuple $\{M, 2 M, \ldots, N M\}(\bmod N)$ with residue $\bmod N$ of no other element $a_{k}$ between them. We illustrate with

$$
A_{5}=\{0,1,24,25,28\}, N=13, M=5 .
$$

The ordered 13 -tuple is

$$
\{5,10,2,7,12,4,9,1,6,11,3,8,0\}
$$

and since

$$
\{0,1,24,25,28\} \equiv\{0,1,2,11,12\}(\bmod 13),
$$

we can write

$$
A_{5} \equiv\{0,1,2,11,12\}(\bmod 13) .
$$

In the ordered 13-tuple, $A_{5}$ has the following adjacent pairs:

$$
\{0,11\},\{11,1\},\{1,12\},\{12,2\},\{2,0\} .
$$

But \{11, 12\} are not adjacent, because 1 is between them in one sense and 0 and 2 are between them in the opposite sense. Similarly,

$$
\{1,2\},\{0,1\},\{2,11\} \text {, and }\{0,12\}
$$

are nonadjacent pairs.
(5) " $A_{5}$ has a regular $p$-complement" will mean that it has a regular $p$ complement of density $\geq 1 / 10$.

Lemma 1: Given $A_{5}$, let $a_{i}$ and $a_{j}$ be adjacent mod $N$ and write

$$
d_{i j} M \equiv a_{i}-a_{j}(\bmod N)
$$

Then $A_{5}$ has a regular $p$-complement if and only if

$$
\frac{N}{10} \leq d_{i j}, d_{j i}<N
$$

for all five adjacent pairs $i, j$.
Proof: Let $C=\{M, 2 M, \ldots, N M\}(\bmod N)$ be an ordered $N$-tuple. Since $a_{1}$, ..., $a_{5}$ will occur in $C$ in some order as distinct residues mod $N$, we assume, without loss of generality, that $0 \leq a_{i}<N, i=1$, ..., 5. Assume that $a_{j}$ is to the left of $\alpha_{i}$ in $C$. (Zero is to the left of the first $a_{k}$ in $C$.) Write

$$
B=\left\{M, 2 M, \ldots, \frac{N}{10} M\right\} \quad(\bmod N)
$$

Suppose now that $N / 10<d_{i j}, d_{j i}<N$. Then $a_{j} \oplus B$ includes the $N / 10$ elements of $C$ immediately to the right of $a_{j}$. Thus, while it may include $\alpha_{i}$, it will not include any element to the right of $\alpha_{i}$ nor, of course, will it include $\alpha_{j}$. Hence, $A_{5} \oplus B$ cannot include any element of $C$ more than once. Since $C$ is a complete residue system mod $N, B$ is a $p$-complement of $A_{5}$. Conversely, if $0<d_{i j}<N / 10$ or $0<d_{j i}<N / 10$, then

$$
\left(a_{j} \oplus B\right) \cap\left(\alpha_{i} \oplus B\right) \neq \phi
$$

and $B$ is not a $p$-complement of $B$.
Lemma 2: Given $A_{5}$, consider the congruence

$$
\begin{equation*}
d_{i j} M \equiv \alpha_{i}-\alpha_{j}(\bmod N) \tag{1}
\end{equation*}
$$

Then $A_{5}$ has a regular $p$-complement if and only if there exists a solution of (1), with $N / 10 \leq d_{i j} \leq 9 N / 10$, for every pair $i, j$, with $1 \leq i, j \leq 5$, $i \neq j$. Proof: If $A_{5}$ has a regular $p$-complement, then Lemma 1 implies that

$$
\frac{N}{10} \leq d_{i j}, d_{j i}<N \text { if } \alpha_{i} \text { and } \alpha_{j} \text { are adjacent } \bmod N
$$

This, in turn, implies that

$$
\frac{N}{10} \leq d_{i j}, d_{j i} \leq \frac{9 N}{10}
$$

Clearly, the inequalities still hold if $a_{i}$ and $a_{j}$ are not adjacent mod $N$. If (1) has the required solution for every pair $i, j$, this implies that adjacent $a^{\prime} s, \bmod N$, are separated by at least $(N / 10) M$, and so, by Lemma $1, A_{5}$ has a regular $p$-complement.

Define $k_{0}$ by $k_{0} M \equiv 1(\bmod N)$ and write $r=k_{0} / N$. Let $D_{i j}=a_{i}-a_{j}$. We have
Lemma 3: The congruence

$$
\begin{equation*}
d_{i j} M \equiv a_{i}-a_{j}(\bmod N) \tag{2}
\end{equation*}
$$

has a solution $N / 10 \leq d_{i j} \leq 9 N / 10$ if and only if $r$ satisfies one of the inequalities:

$$
\frac{10(k-1)+1}{10\left|D_{i j}\right|} \leq r \leq \frac{10(k-1)+9}{10\left|D_{i j}\right|}, k=1,2, \ldots,\left|D_{i j}\right| .
$$

Proo6: Suppose $\frac{N}{10} \leq d_{i j} \leq \frac{9 N}{10}$. We have $d_{i j} M \equiv D_{i j}(\bmod N)$. However, since $k_{0} M \equiv 1(\bmod N)$, we also have

$$
\begin{aligned}
D_{i j} k_{0} M & \equiv D_{i j}(\bmod N), \text { so that } \\
d_{i j} & \equiv D_{i j} k_{0}(\bmod N)
\end{aligned}
$$

Therefore,

$$
D_{i j} r \equiv s(\bmod 1) \quad \text { where } \frac{1}{10} \leq s \leq \frac{9}{10}
$$

This implies that

$$
\frac{10(k-1)+1}{10} \leq\left|D_{i j}\right| r \leq \frac{10(k-1)+9}{10}
$$

or

$$
\frac{10(k-1)+1}{10\left|D_{i j}\right|} \leq r \leq \frac{10(k-1)+9}{10\left|D_{i j}\right|} \text { for some } k, 1 \leq k \leq\left|D_{i j}\right|
$$

The argument can also be read backwards, so this completes the proof.
Since each difference $D_{i j}$ determines a set of intervals $R_{i j}$ on the unit interval:

$$
R_{i j}=\bigcup_{k=1}^{\left|D_{i j}\right|}\left[\frac{10(k-1)+1}{10\left|D_{i j}\right|}, \frac{10(k-1)+9}{10\left|D_{i j}\right|}\right]
$$

our result can be expressed in the following
Theorem: $A_{5}$ does not have a regular $p$-complement if and only if

$$
\begin{equation*}
\bigcap_{1 \leq i<j \leq 5} R_{i j}=\phi \tag{3}
\end{equation*}
$$

Proot: From Lemma 3 we see that every solution, $r=k_{0} / N$, to the congruence

$$
d_{i j} M \equiv a_{i}-a_{j} \quad(\bmod N), \frac{N}{10} \leq d_{i j} \leq \frac{9 N}{10}
$$

must lie in $R_{i j}$. By Lemma 2 we see that for $A_{5}$ to have a regular $p$-complement it is necessary and sufficient that this congruence have a simultaneous solution for every pair $1 \leq i, j \leq 5$. Hence,

$$
\bigcap_{1 \leq i<j \leq 5} R_{i j} \neq \phi
$$

if and only if $A_{5}$ has a regular $p$-complement.
The application of this theorem to a given $A_{5}$ is a tedious procedure without a computer. In [2], we stated that a computer search revealed two sets $A_{4}$, with $a_{4} \leq 100$, that do not have regular (covering) complements of density $\leq 1 / 3$. We have no such computer information on the packing algorithm but still think it likely that at most a finite number of $A_{5}$ 's do not have regular $p$-complements. The obvious attempt to prove this is to assume $a_{5}$ is large and that (3) is satisfied. So far, we have failed to find the desired contradiction.

## REFERENCES

1．G．Weinstein．＂Some Covering and Packing Results in Number Theory．＂J． Number Theory 8 （1976）：193－205．
2．G．Weinstein．＂An Algorithm for Complements of Finite Sets of Integers．＂ Proc．Amer．Math．Soc． 55 （1976）：1－5．

# adDenda to＂pythagorean triples containing <br> FIBONACCI NUMBERS：SOLUTIONS FOR $F_{n}^{2} \pm F_{k}^{2}=K^{2 \prime \prime}$ 

MARJORIE BICKNELL－JOHNSON
A．C．Wilcox High School，Santa Clara，CA 95051
In a recent correspondence from J．H．E．Cohn，it was learned that Ljung－ gren［1］has proved that the only square Pell numbers are 0， 1 ，and 169. （This appears as an unsolved problem， $\mathrm{H}-146$ ，in［2］and as Conjecture 2.3 in ［3］．）Also，if the Fibonacci polynomials $\left\{F_{n}(x)\right\}$ are defined by

$$
F_{0}(x)=0, F_{1}(x)=1, \text { and } F_{n+2}(x)=x F_{n+1}(x)+F_{n}(x)
$$

then the Fibonacci numbers are given by $F_{n}=F_{n}(1)$ ，and the Pell numbers are $P_{n}=F_{n}(2)$ ．Cohn［4］has proved that the only perfect squares among the se－ quences $\left\{F_{n}(\alpha)\right\}, \alpha$ odd，are 0 and 1 ，and whenever $a=k^{2}$ ，$\alpha$ itself．Certain cases are known for $a$ even［5］．

The cited results of Cohn and Ljunggren mean that Conjectures 2．3，3．2， and 4.2 of［3］are true，and that the earlier results can be strengthened as follows．

If $(n, k)=1$ ，there are no solutions in positive integers for

$$
F_{n}^{2}(\alpha)+F_{k}^{2}(\alpha)=K^{2}, n>k>0, \text { when } a \text { is odd and } a \geq 3
$$

This is the same as stating that no two members of $\left\{F_{n}(\alpha)\right\}$ can occur as the lengths of legs in a primitive Pythagorean triangle，for $\alpha$ odd and $\alpha \geq 3$ ．

When $a=1$ ，for Fibonacci numbers，if

$$
F_{n}^{2}+F_{k}^{2}=K^{2}, \quad n>k>0
$$

then $(n, k)=2$ ，and it is conjectured that there is no solution in positive integers．When $a=2$ ，for Pell numbers，$P_{n}^{2}+P_{k}^{2}=K^{2}$ has the unique solu－ tion $n=4, k=3$ ，giving the primitive Pythagorean triple 5－12－13．

## REFERENCES

1．W．Ljunggren．＂Zur Theorie der Gleichung $x^{2}+1=D y^{4} . "$ Avh．Norske Vid． Akad．Oslo I，Nr． 5 （1942）．
2．J．A．H．Hunter．Problem H－146．The Fibonacci Quarterly 6 （1968）： 352.
3．Marjorie Bicknell－Johnson．＂Pythagorean Triples Containing Fibonacci Numbers：Solutions for $F_{n}^{2} \pm F_{k}^{2}=K^{2}$ ．＂The Fibonacci Quarterly 17，No． 1 （1979）：1－12．
4．J．H．E．Cohn．＂Eight Diophantine Equations．＂Proc．London Math．Soc． XVI（1966）：153－166．
5．J．H．E．Cohn．＂Squares in Some Recurrent Sequences．＂Pacific J．Math． 41，No． 3 （1972）：631－646．

# SPECIAL RECURRENCE RELATIONS ASSOCIATED WITH THE SEQUENCE $\left\{w_{n}(a, b ; p, q)\right\}^{*}$ 

A. G. SHANNON

New South Wales Institute of Technology, Sydney, Australia and
A. F. HORADAM

University of New England, Armidale, Australia; University of Reading, England

1. INTRODUCTION

There are three parts to this paper, the link being $\left\{\omega_{n}\right\}$, defined below in (1.1). In the first, a lacunary recurrence relation is developed for $\left\{w_{n}\right\}$ in (2.3) from a multisection of a related series. Then a functional recurrence relation for $\left\{w_{n}\right\}$ is investigated in (3.2). Finally, a $q$-series recurrence relation for $\left\{w_{n}\right\}$ is included in (4.5).

The generalized sequence of numbers $\left\{w_{n}\right\}$ is defined by

$$
\begin{equation*}
w_{n}=p w_{n-2}-q w_{n-2}(n \geq 2), w_{0}=a, w_{1}=b \tag{1.1}
\end{equation*}
$$

where $p, q$ are arbitrary integers. Various properties of $\left\{\omega_{n}\right\}$ have been developed by Horadam in a series of papers [4, 5, 6, 7, and 8].

We shall have occasion to use the "fundamental numbers," $U_{n}(p, q)$, and the "primordial numbers," $V_{n}(p, q)$, of Lucas [10]. These are defined by

$$
\begin{align*}
U_{n}(p, q) & \equiv w_{n}(0,1 ; p, q),  \tag{1.2}\\
V_{n}(p, q) & \equiv w_{n}(2, p ; p, q) .
\end{align*}
$$

For notational convenience, we shall use

$$
\begin{align*}
& U_{n}(p, q) \equiv U_{n} \equiv u_{n-1}=\left(\alpha^{n}-\beta^{n}\right) /(\alpha-\beta),  \tag{1.4}\\
& V_{n}(p, q) \equiv V_{n} \equiv v_{n-1}=\alpha^{n}+\beta^{n}, \tag{1.5}
\end{align*}
$$

where $\alpha, \beta$ are the roots of $x^{2}-p x+q=0$.

## 2. LACUNARY RECURRENCE RELATION

We define the series $w(x)$ by

$$
\begin{equation*}
w(x)=w_{1}(x)=\sum_{n=0}^{\infty} w_{n} x^{n}, \tag{2.1}
\end{equation*}
$$

the properties of which have been examined by Horadam [4].
If $r$ is a primitive $m$ th root of unity, then the $k$ th $m$-section of $w(x)$ can be defined by

$$
\begin{equation*}
w_{k}(x ; m)=m^{-1} \sum_{j=1}^{m} w\left(r^{j} x\right) r^{m-k_{j}} \tag{2.2}
\end{equation*}
$$

It follows that

$$
w_{k}(x ; m)=\frac{1}{m}\left(r^{m-k} w(r x)+r^{m-2 k} w\left(r^{2} x\right)+\cdots+r^{m-m k} w\left(r^{m} x\right)\right)
$$

[^0]\[

$$
\begin{align*}
& =\frac{1}{m}\left(r^{m-k}\left(w_{0}+w_{1} r x+w_{2} r^{2} x^{2}+\cdots\right)+r^{m-2 k}\left(w_{0}+w_{1} r^{2} x+w_{2} r^{4} x^{2}+\cdots\right)\right. \\
& \left.\quad+\cdots+r^{m-m k}\left(w_{0}+w_{1} r^{m} x+w_{2} r^{2 m} x^{2}+\cdots\right)\right) \\
& =\frac{1}{m}\left(w_{0} \sum_{j=1}^{m} r^{m-j k}+w_{1} x \sum_{j=1}^{m} r^{m-j k+j}+\cdots+w_{k} x^{k} \sum_{j=1}^{m} r^{m-j k+j k}+\cdots\right) \\
& =\frac{1}{m}\left(w_{0} \frac{r^{m k}-1}{r^{k}-1}+w_{1} x \frac{r^{m(k-1)}-1}{r^{k-1}-1}+\cdots+w_{k} x^{k} m r^{m}+\cdots\right) \\
& =w_{k} x^{k}+w_{k+2 m} x^{k+2 m}+\cdots \\
& =\sum_{j=0}^{\infty} w_{k+j m^{2}} x^{k+j m}  \tag{i}\\
& =\sum_{j=0}^{\infty}\left(A(\alpha x)^{k+j m}+B(\beta x)^{k+j m}\right) \\
& =A \alpha^{k} x^{k}\left(1-\alpha^{m} x^{m}\right)^{-1}+B \beta^{k} x^{k}\left(1-\beta^{m} x^{m}\right)^{-1} \\
& =x^{k}\left(w_{k}-q^{m} w_{k-m^{2}} x^{m}\right)\left(1-V_{m} x^{m}+q^{m} x^{2 m}\right)^{-1} . \tag{ii}
\end{align*}
$$
\]

Hence, by cancelling the common factor $x^{k}$ and replacing $x^{m}$ by $x$, we get from the lines (i) and (ii)

$$
\left(1-V_{m} x+q^{m} x^{2}\right) \sum_{j=0}^{\infty} w_{k+j m} x^{j}=w_{k}-q^{m} w_{k-m} x .
$$

We then equate the coefficients of $x^{j}$ to get the lacunary recurrence relation for $\left\{\omega_{n}\right\}$ :

$$
\begin{equation*}
w_{k+m j}-V_{m} w_{k+m(j-1)}+q^{m} w_{k+m(j-2)}=\left(w_{k}-V_{m} w_{k-m}+q^{m} w_{k-2 m}\right) \delta_{j 0}, \tag{2.3}
\end{equation*}
$$

where $\delta_{n m}$ is the Kronecker delta:

$$
\delta_{n m}=1 \quad(n=m), \quad \delta_{n m}=0 \quad(n \neq m)
$$

When $j$ is zero, we get the trivial case $w_{k}=w_{k}$. When $j$ is unity, we get

$$
w_{k+m}-V_{m} w_{k}+q^{m} w_{k-m}=0,
$$

which is equation (3.16) of Horadam [5]. It is of interest to rewrite (2.3) as

$$
\begin{equation*}
w_{n m}=V_{n} w_{n(m-1)}+q^{n} w_{n(m-2)} \quad(m \geq 2, n \geq 1) \tag{2.4}
\end{equation*}
$$

Thus

$$
w_{2 n}=V_{n} w_{n}+a q^{n},
$$

and

$$
w_{3 n}=V_{n} w_{2 n}+q^{n} w_{n}
$$

The recurrence relations (2.3) and (2.4) are called lacunary because there are gaps in them. For instance, there are missing numbers between $w_{n(m-1)}$ and $w_{n m}$ in (2.4); when $m=2$ and $n=3$, (2.4) becomes

$$
w_{6}=V_{3} w_{3}+\alpha q^{3},
$$

and the missing numbers are $w_{4}$ and $w_{5}$. A general solution of (2.4), in terms of $w_{n}$, is

$$
\begin{equation*}
w_{m n}=U_{m}\left(V_{n},-q\right) w_{n}+a U_{m-1}\left(V_{n},-q\right) q^{n} \tag{2.5}
\end{equation*}
$$

The proof follows by induction on $m$. For $m=2$ from (1.1) and (1.2),

$$
U_{2}\left(V_{n},-q\right)=V_{n} \quad \text { and } \quad U_{1}\left(V_{n},-q\right)=1
$$

If we assume (2.5) is true for $m=3,4, \ldots, r-1$, then from (2.4)

$$
\begin{aligned}
& w_{r n}= V_{n} w_{n(r-1)}+q^{n} w_{n(r-2)} \\
&= V_{n} U_{r-1}\left(V_{n},-q\right) w_{n}+a V_{n} U_{r-2}\left(V_{n},-q\right) q^{n} \\
&+q^{n} U_{r-2}\left(V_{n},-q\right) w_{n}+a q^{n} U_{r-3}\left(V_{n},-q\right) q^{n} \\
&=\left(V_{n} U_{r-1}\left(V_{n},-q\right)\right. \\
&\left.+q^{n} U_{r-2}\left(V_{n},-q\right)\right) w_{n} \\
&+a\left(V_{n} U_{r-2}\left(V_{n},-q\right)+q^{n} U_{r-3}\left(V_{n},-q\right)\right) q^{n} \\
&= U_{r}\left(V_{n},-q\right) w_{n}+a U_{r-1}\left(V_{n},-q\right) q^{n} .
\end{aligned}
$$

## 3. FUNCTIONAL RECURRENCE RELATION

Following Carlitz [1], we define

$$
\begin{equation*}
w_{n}^{*}(x)=w_{n}^{*}(x, \lambda)=\sum_{k=0}^{\infty} w_{n+k}\binom{x}{k} \lambda^{k} . \tag{3.1}
\end{equation*}
$$

Then,

$$
w_{n}^{*}(0)=w_{n}, \text { and }
$$

$$
\begin{align*}
\omega_{n+1}^{*}(x) & =\sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k} \lambda^{k}  \tag{3.2}\\
& =\sum_{k=0}^{\infty}\left(p w_{n+k}-q w_{n+k+1}\right)\binom{x}{k} \lambda^{k} \\
& =p w_{n}^{*}(x)-q w_{n-1}^{*}(x),
\end{align*}
$$

which is a second-order functional recurrence relation. Moreover, we can show that the power series in (3.1) converges for a sufficiently small $\lambda$ as follows:

$$
\begin{aligned}
w_{n}^{*}(x+1)-w_{n}^{*}(x) & =\sum_{k=0}^{\infty} w_{n+k}\left\{\binom{x+1}{k}-\binom{x}{k}\right\} \lambda^{k} \\
& =\lambda \sum_{k=1}^{\infty} w_{n+k}\binom{x}{k-1} \lambda^{k-1} \\
& =\lambda \sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k} \lambda^{k} \\
& =\lambda w_{n+1}^{*}(x) .
\end{aligned}
$$

If we use $w_{n}=A \alpha^{n}+B \beta^{n}$, where

$$
A=\frac{b-\alpha \beta}{\alpha-\beta} \quad \text { and } \quad B=\frac{\alpha \alpha-b}{\alpha-\beta},
$$

then we get that

$$
\begin{aligned}
\omega_{n}^{*}(x) & =\sum_{k=0}^{\infty}\left\{A \alpha^{n}\binom{x}{k}(\alpha \lambda)^{k}+B \beta^{n}\binom{x}{k}(\beta \lambda)^{k}\right\} \\
& =A \alpha^{n}(1+\lambda \alpha)^{x}+B \beta^{n}(1+\lambda \beta)^{x}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
w_{n}^{*}(x+y) & =A \alpha^{n}(1+\lambda \alpha)^{x+y}+B \beta^{n}(1+\lambda \beta)^{x+y} \\
& =\sum_{k=0}^{\infty}\left\{A \alpha^{n+k}(1+\lambda \alpha)^{x}+B \beta(1+\lambda \beta)^{x}\right\}\binom{y}{k} \lambda^{k} \\
& =\sum_{k=0}^{\infty} w_{n+k}^{*}(x)\binom{y}{k} \lambda^{k} .
\end{aligned}
$$

Similarly, we have for $E=p a b-q \alpha^{2}-b^{2}$, and $E_{\omega}=1+p \lambda+q \lambda^{2}$ :

$$
\begin{aligned}
& w_{n-1}^{*}(x) w_{n+1}^{*}(x)-w_{n}^{* 2}(x) \\
= & \left\{A \alpha^{n-1}(1+\lambda \alpha)^{x}+B \beta^{n-1}(1+\lambda \beta)^{x}\right\}\left\{A \alpha^{n+1}(1+\lambda \alpha)^{x}+B \beta^{n+1}(1+\lambda \beta)^{x}\right\} \\
& -\left\{A \alpha^{n}(1+\lambda \alpha)^{x}+B \beta^{n}(1+\lambda \beta)^{x}\right\} \\
= & E d^{-2}\left(\alpha^{n-1} \beta^{n+1}-2 \alpha^{n} \beta^{n}+\alpha^{n+1} \beta^{n-1}\right)((1+\lambda \alpha)(1+\lambda \beta))^{x} \\
= & q^{n-1} E d^{-2}\left(\beta^{2}-2 \alpha \beta+\alpha^{2}\right) E_{w}^{x} \\
= & q^{n-1} E E_{w}^{x},
\end{aligned}
$$

which is a generalization of equation (4.3) of Horadam [5]:

$$
w_{n-1} w_{n+1}-w_{n}^{2}=q^{n-1} E .
$$

The same type of approach yields

$$
\alpha w_{m+n}^{*}(x+y)+(b-p q) w_{m+n-1}^{*}(x+y)=w_{m}^{*}(x) w_{n}^{*}(y)-q w_{m-1}^{*}(x) w_{n-1}^{*}(y)
$$

as a generalization of Horadam's equation (4.1) [5]:

$$
\alpha w_{m+n}+(b-p q) w_{m+n-1}=w_{m} w_{n}-q w_{m-1} w_{n-1}
$$

4. $q$-SERIES RECURRENCE RELATION
$q$-series are defined by
(4.1) $\quad(q)_{n}=(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{n}\right),\left(q_{0}\right)=1$.

Arising out of these are the so-called $q$-binomial coefficients:

$$
\left[\begin{array}{l}
n  \tag{4.2}\\
k
\end{array}\right]_{q}=(q)_{n} /(q)_{k}(q)_{n-k}
$$

When $q$ is unity, these reduce to the ordinary binomial coefficients. It also follows from (4.1) and (4.2) that

$$
\begin{align*}
{\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\beta / \alpha} } & =\frac{\left(1-(\beta / \alpha)^{n}\right) \cdots\left(1-(\beta / \alpha)^{n-k+1}\right)}{(1-\beta / \alpha)\left(1-(\beta / \alpha)^{2}\right) \cdots\left(1-(\beta / \alpha)^{k}\right)} \\
& =\alpha^{k(n-k) \frac{u_{n-1} u_{n-2} \cdots u_{n-k}}{u_{0} u_{1} \cdots u_{k-1}}} \\
& =U_{n} c_{n k} \alpha^{k(n-k)}, \\
C_{n k} & =\frac{u_{n-2} u_{n-3} \cdots u_{n-k}}{u_{0} u_{1} \cdots u_{k-1}} . \tag{4.3}
\end{align*}
$$

Horadam [5] has shown that

Thus

$$
w_{n+x}=w_{n} u_{r}-q w_{r-1} u_{n-1} .
$$

$$
w_{n+r}=\frac{w_{r} \alpha^{k(k-n-1)}}{C_{n-1, k}}\left[\begin{array}{c}
n+1 \\
k
\end{array}\right]_{\beta / \alpha}-\frac{q w_{r-1} \alpha^{k(k-n)}}{C_{n k}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\beta / \alpha}
$$

which yields

$$
C_{n-1, k} C_{n k} w_{n-r}=\alpha^{k(k-n-1)} C_{n k}\left[\begin{array}{c}
n+1  \tag{4.5}\\
k
\end{array}\right]_{\beta / \alpha} w_{r}-q \alpha^{k(k-n)} C_{n-1, k}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{\beta / \alpha} w_{r-1}
$$

## 5. CONCLUSION

The $q$-series analogue of the binomial coefficient was studied by Gauss, and later developed by Cayley. Carlitz has used the $q$-series in numerous papers. Fairly clearly, other results for $\omega_{n}$ could be obtained with it just as other properties of the functional recurrence relation for $w_{n}$ could be readily produced.

The process of multisection of series is quite an old one, and the interested reader is referred to Riordan [11]. Lehmer [9] discusses lacunary recurrence relations.
$C_{n k}$ was introduced by Hoggatt [3], who used the symbol C. Curiously enough, Gould [2] also used the symbol ' $C$ ' in his generalization of Bernoulli and Euler numbers. Gould's $C=b / a$ ( $\alpha, b$ the roots of $x^{2}-x-1=0$ ) is related to Hoggatt's $C \equiv C_{n k}$ when $p=-q=1$ by

$$
\begin{equation*}
C=b \lim _{k \rightarrow \infty}\left(C_{k+1, k+1} / C_{k k}\right) . \tag{5.1}
\end{equation*}
$$

REFERENCES

1. L. Carlitz. "Some Generalized Fibonacci Identities." The Fibonacci Quarterly 8 (1970):249-254.
2. H. W. Gould. "Generating Functions for Products of Powers of Fibonacci Numbers." The Fibonacci Quarterly 1, No. 2 (1963):1-16.
3. V. E. Hoggatt, Jr. "Fibonacci Numbers and Generalized Binomial Coefficients." The Fibonacci Quarterly 5 (1967):383-400.
4. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." Duke Math. J. 32 (1965):437-446.
5. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." The Fibonacci Quarterly 3 (1965):161-176,
6. A. F. Horadam. "Special Properties of the Sequence $w_{n}(a, b ; p, q)$." The Fibonacci Quarterly 5 (1967):424-434.
7. A. F. Horadam. "Generalization of Two Theorems of K. Subba Rao." BuZZetin of the Calcultta Mathematical Society 58 (1968):23-29.
8. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\left\{\omega_{n}(\alpha, \bar{b} ; p, q)\right\} . "$ The Fibonacci Quarterly 7 (1969):14-22.
9. D. H. Lehmer. "Lacunary Recurrence Formulas for the Numbers of Bernoulli and Euler." Annals of Mathematics 36 (1935):637-649.
10. E. Lucas. Théorie des Nombres. Paris: Gauthier Villars, 1891.
11. J. Riordan. Combinatorial Identities. New York: Wiley, 1968.
\#\#\#**

ON SOME EXTENSIONS OF THE WANG-CARLITZ IDENTITY

M. E. COHEN and H. SUN<br>California State University, Fresno, CA 93740

ABSTRACT
Two theorems are presented which generalize a recent Wang [6]-Carlitz [1] result. In addition, we also obtain its Abel analogue. The method of proof is dependent upon some of our recent work [2].

I
Wang [6] proved the expansion

$$
\begin{equation*}
\sum_{k=1}^{r+1}\binom{r+1}{k} \sum_{\substack{i_{1}+\cdots+i_{k}=n \\ i_{j}>0}} \prod_{m=1}^{k}\left(i_{m}+1\right)=\binom{n+2 r+1}{2 r+1} . \tag{1.1}
\end{equation*}
$$

Recently, Carlitz [1] extended (1.1) to

$$
\begin{equation*}
\sum_{k=0}^{r+1}\binom{r+1}{k} \sum_{\substack{i_{1}+\cdots+i_{i}=n \\ i_{j}>0}} \prod_{m=1}^{k}\binom{i_{m}+a}{i_{m}}=\binom{n+a r+r+a}{n} \tag{1.2}
\end{equation*}
$$

Theorems 1 and 2 in this paper treat a number of different generalizations of (1.2). In particular, a special case of Theorem 1 gives the new expression:

$$
\left.\begin{array}{rl} 
& \sum_{k=0}^{n+1}(r+1  \tag{1.3}\\
k
\end{array}\right) \sum_{\substack{i_{1}+\cdots+i_{k}=n \\
i_{j}>0}} \prod_{m=1}^{k} \frac{(a+1)}{\left(\alpha+1+t i_{m}\right)}\left(a+t i_{m}+i_{m}\right) .
$$

Letting $t=0$ in (1.3) yields (1.2).

We also present the Abel analogue of (1.3):
(1.4) $\sum_{k=0}^{r+1}\binom{r+1}{k} \sum_{\substack{i_{1}+\cdots+i_{k}=n \\ i_{j}>0}} \prod_{m=1}^{k} \frac{(\alpha)\left(\alpha+\ell i_{m}\right)^{i_{m}-1}}{i_{m}!}=\frac{(\alpha)(r+1)(\alpha r+r+\ell n)^{n-1}}{n!}$.
(1.3) and (1.4) are special cases of a number of classes of functions including some well-known orthogonal polynomials. These are considered in the following theorem.
Theorem 1: For $\alpha, \beta, \ell, \ell^{\prime}, t$ complex numbers, $r$ and $n$ nonnegative integers, and $s$ a positive integer

$$
\begin{equation*}
\sum_{k=0}^{r}\binom{r}{k} \sum_{\substack{i_{1}+\cdots+i_{k}=n \\ i_{j}>0}} \prod_{m=1}^{k} \theta_{i_{m}}^{\alpha, \beta}(x)=\theta_{n}^{\alpha r, \beta r}(x) \tag{1.5}
\end{equation*}
$$

where $(\alpha)_{k}=\Gamma(\alpha+k) / \Gamma(\alpha)$, quotient of gamma functions,
[ $n / s$ ] is the greatest integer notation, and
$\theta_{n}^{\alpha, \beta}(x)$ can assume any of the ten functions:
(1.6) $A_{n}^{\alpha, \beta}(x)$
$=\frac{(\alpha)_{t n+n}}{n!(\alpha+1)_{t n}} \sum_{p=0}^{[n / s](-n)_{s p}(\alpha+t n+n)_{\ell p}(\beta)_{\ell^{\prime} p}\{(\alpha+s t p+s p+\ell p) / \alpha\} x^{p}} \underset{p!(\alpha+1+t n)_{s p+\ell p}(\beta+1)_{\ell^{\prime} p-p}}{(\alpha)}$
(1.7) $B_{n}^{\alpha, \beta}(x)$

$$
\begin{aligned}
& \quad=\frac{(\beta)(\alpha)_{t n+n}}{n!(\alpha+1)_{t n}} \sum_{p=0}^{[n / s](-n)_{s p}(\alpha+t n+n)_{\ell p}\left(\beta+\ell^{\prime} p\right)^{p-1}\{(\alpha+s t p+s p+\ell p) / \alpha\} x^{p}} \underset{p!(\alpha+1+t n)_{s p+\ell p}}{(1.8) \quad C_{n}^{\alpha, \beta}(x)}
\end{aligned}
$$

$$
=\frac{(\alpha+\ell n)^{n-1}}{n!} \sum_{p=0}^{[n / s]} \frac{(-1)^{s p}(-n)_{s p}(\beta)_{\ell^{\prime} p}\{\alpha+\ell s p\} x^{p}}{p!(\alpha+\ell n)^{s p}(\beta+1)_{\ell} p-p}
$$

(1.9) $D_{n}^{\alpha, \beta}(x)$

$$
=\frac{(\beta)(\alpha+\ell n)^{n-1}}{n!} \sum_{p=0}^{[n / s]} \frac{(-1)^{s p}(-n)_{s p}\left(\beta+\ell^{\prime} p\right)^{p-1}\{\alpha+\ell s p\} x^{p}}{p!(\alpha+\ell n)^{s p}}
$$

(1.10) $E_{n}^{\alpha, \beta}(x)$

$$
=\frac{1}{(\alpha+\ell n) n!} \sum_{p=0}^{[n / s]} \frac{(-1)^{s p}(-n)_{s p}(\beta)_{n \ell \ell^{\prime}-s p \ell^{\prime}}(\alpha+\ell n)^{p}\{\alpha+\ell n-\ell s p\} x^{n-s p}}{p!(\beta+1)_{n \ell^{\prime}-n-s p \ell^{\prime}+s p}}
$$

(1.11) $F_{n}^{\alpha, \beta}(x)$

$$
=\frac{(\beta)}{(\alpha+\ell n) n!} \sum_{p=0}^{[n / s]} \frac{(-1)^{s p}(-n)_{s p}(\alpha+\ell n)^{p}\left(\beta+\ell \prime n-\ell^{\prime} s p\right)^{n-s p-1}\{\alpha+\ell n-\ell s p\} x^{n-s p}}{p!}
$$

$$
\begin{align*}
& G_{n}^{\alpha, \beta}(x)=\frac{1}{n!} \sum_{p=0}^{\infty} \frac{(\alpha+t p)_{\ell n}(\beta)_{\ell^{\prime} p} x^{p}}{p!(1+\alpha+t p)_{\ell n-n}(\beta+1)_{\ell^{\prime} p-p}}  \tag{1.12}\\
& H_{n}^{\alpha, \beta}(x)=\frac{(\beta)}{n!} \sum_{p=0}^{\infty} \frac{(\alpha+t p)_{\ell n}\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p}}{p!(\alpha+t p+1)_{\ell n-n}}  \tag{1.13}\\
& I_{n}^{\alpha, \beta}(x)=\frac{1}{n!} \sum_{p=0}^{\infty} \frac{(\alpha+t p+\ell n)^{n-1}(\beta)_{\ell^{\prime} p}\{\alpha+t p\} x^{p}}{p!(\beta+1)_{\ell^{\prime} p-p}}  \tag{1.14}\\
& J_{n}^{\alpha, \beta}(x)=\frac{(\beta)}{n!} \sum_{p=0}^{\infty} \frac{(\alpha+t p+\ell n)^{n-1}\left(\beta+\ell^{\prime} p\right)^{p-1}\{\alpha+t p\} x^{p}}{p!} . \tag{1.15}
\end{align*}
$$

Proof of (1.6): From Theorem 4b [2, p. 708],

$$
\begin{equation*}
=(1-z)^{\alpha} \sum_{p=0}^{\infty} \frac{(\beta)_{\ell^{\prime} p} x^{p_{z} s p}(1-z)^{\ell p}}{p!(\beta+1)_{\ell^{\prime} p-p}} \tag{1.16}
\end{equation*}
$$

where $v(1-z)^{t+1}=-z, v(0)=0$.
Hence,

$$
\begin{equation*}
\sum_{k=0}^{x}\left(\frac{p}{k}\right)\left\{\sum_{n=1}^{\infty} v^{n} A_{n}^{\alpha, \beta}(x)\right\}^{k}=(1-z)^{\alpha r} \sum_{p=0}^{\infty} \frac{(\beta r)_{\ell^{\prime} p} x^{p} z^{s p}(1-z)^{\ell_{p}}}{p!(\beta r+1)_{\ell^{\prime} p-p}} . \tag{1.17}
\end{equation*}
$$

(1.17) may be expressed as

$$
\begin{equation*}
\sum_{n=0}^{\infty} v^{n} \sum_{k=0}^{n}\left(\frac{r}{k}\right)_{\substack{i_{1}+\cdots+i_{k} \\ i_{j}>0}} \sum_{m=1}^{k} A_{i_{m}}^{\alpha, \beta}(x)=\sum_{n=0}^{\infty} v^{n} A_{n}^{\alpha r, \beta r}(x) \tag{1.18}
\end{equation*}
$$

Comparing coefficients on both sides gives the required (1.6).
Proof of (1.7): From Theorem 4b [2, p. 708],

$$
\begin{equation*}
\left\{\sum_{n=1}^{\infty} v^{n} B_{n}^{\alpha, \beta}(x)\right\}^{k}=\left\{(1-z)^{\alpha} \sum_{p=0}^{\infty} \frac{\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p} z^{s p}(1-z)^{\ell p}}{p!}-1\right\}^{k} . \tag{1.19}
\end{equation*}
$$

(1.19) is obtained by modifying the arbitrary sequence $\left\{e_{p}\right\}$ to be

$$
\left(\beta+\ell^{\prime} p\right)^{p-1} .
$$

Hence,

$$
\begin{align*}
& \sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty} v^{n} B_{n}^{\alpha, \beta}(x)\right\}^{k}  \tag{1.20}\\
& =\beta r(1-z)^{\alpha r} \sum_{p=0}^{\infty} \frac{(\beta r+\ell \prime p)^{p-1} x^{p} z^{s p}(1-z)^{\ell p}}{p!} \tag{1.21}
\end{align*}
$$

$$
\begin{align*}
& =\sum_{n=0}^{\infty} v^{n_{B}}{ }_{n}^{\alpha r, \beta r}(x)  \tag{1.22}\\
& =\sum_{n=0}^{\infty} v^{n} \sum_{k=0}^{r}\binom{r}{k}_{i_{1}+\cdots+i_{k}=n} \sum_{m=1} \prod_{i_{j}>0}^{k} B_{i_{m}}^{\alpha, \beta}(x) . \tag{1.23}
\end{align*}
$$

Comparing coefficients gives (1.7).
The proofs of (1.6) and (1.7) are the procedures adopted in the above cases with suitable modifications. We are initially required to establish generating functions.
Proof of (1.8): Theorem 2b [2, p. 704] will give

$$
\begin{equation*}
\sum_{n=0}^{\infty} u^{n} C_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{(\beta)_{\ell, p} x^{p} z^{s p}}{p!(\beta+1)_{\ell, p-p}} \tag{1.24}
\end{equation*}
$$

Proof of (1.9): Theorem 2b [2, p. 704] yields

$$
\begin{equation*}
\sum_{n=0}^{\infty} u^{n} D_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p} z^{s p}}{p!} \tag{1.25}
\end{equation*}
$$

Proof of (1.10): Using Theorem 2d [2, p. 704], one may obtain

$$
\begin{equation*}
\sum_{n=0}^{\infty} w^{n} E_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{(\beta)_{\ell} \prime_{p} x^{p_{z}}{ }^{p / s}}{p!(\beta+1)_{\ell}{ }^{\prime} p-p} \tag{1.26}
\end{equation*}
$$

where $w=z^{1 / s} \exp (-\ell z)$.
Proof of (1.11): From Theorem 2d [2, p. 704],

$$
\begin{equation*}
\sum_{n=0}^{\infty} w^{n} F_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p} z^{p / s}}{p!} \tag{1.27}
\end{equation*}
$$

Proof of (1.12): It may be shown that

$$
\begin{equation*}
\sum_{n=0}^{\infty} \xi^{n} G_{n}^{\alpha, \beta}(x)=(1-z)^{\alpha} \sum_{p=0}^{\infty} \frac{(\beta)_{\ell^{\prime} p}(1-z)^{t p} x^{p}}{p!(\beta+1)_{\ell_{p}^{\prime}-p}} \tag{1.28}
\end{equation*}
$$

where $\xi(1-z)^{\ell}=-z$.
Proof of (1.13): One may derive

$$
\begin{equation*}
\sum_{n=0}^{\infty} \xi^{n} H_{n}^{\alpha, \beta}(x)=(1-z)^{\alpha} \sum_{p=0}^{\infty} \frac{\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p}}{p!} \tag{1.29}
\end{equation*}
$$

Proof of (1.14): The generating function of $I_{n}^{\alpha, \beta}(x)$ is

$$
\begin{equation*}
\sum_{n=0}^{\infty} u^{n} I_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{(\beta)_{\ell_{p} p} x^{p} \exp (p t z)}{p!(\beta+1)_{\ell^{\prime} p-p}} \tag{1.30}
\end{equation*}
$$

Proof of (1.15): It may be proved that

$$
\begin{equation*}
\sum_{n=0}^{\infty} u^{n} J_{n}^{\alpha, \beta}(x)=\exp (\alpha z) \sum_{p=0}^{\infty} \frac{\left(\beta+\ell^{\prime} p\right)^{p-1} x^{p} \exp (p t z)}{p!} \tag{1.31}
\end{equation*}
$$

II
A second generalization of the Carlitz result given by our equation (1.2)
is

$$
\begin{equation*}
\sum_{k=0}^{r+1}\binom{r+1}{k} \sum_{\substack{i_{1}+\cdots+i_{2}=n \\ i_{j}>0}} \prod_{m=1}^{k}\binom{\alpha+t i_{m}}{i_{m}} \tag{2.1}
\end{equation*}
$$

$=\frac{(\alpha r+r+\alpha+1)_{t n+n}}{n!(\alpha r+r+a+2)_{t n}} \sum_{p=0}^{n} \frac{(-n)_{p}(r+1)_{p}(-t)^{p}\{(\alpha r+r+\alpha+1+t p+p) /(\alpha r+r+\alpha+1)\}}{p!(\alpha r+r+\alpha+2+t n)_{p}}$.
For $t=0$, the polynomial reduces to unity and (1.2) presents itself. We also have the Abel analogue of (2.1), which assumes the form

$$
\begin{align*}
& \sum_{k=0}^{r+1}\binom{r+1}{k}_{i_{1}+\cdots+i_{k}=n} \sum_{m=1}^{k} \frac{\left(\alpha+\ell i_{m}\right)^{i_{m}}}{i_{m}!}  \tag{2.2}\\
= & \frac{(\alpha r+l n)^{n-1}}{n!} \sum_{p=0}^{n} \frac{(-\ell)^{p}(-n)_{p}(r)_{p}\{\alpha r+p\}}{p!(\alpha r+l n)^{p}} .
\end{align*}
$$

Now both (2.1) and (2.2) are particular cases of Theorem 2.
Theorem 2: For $\alpha, \beta, l, t$ complex numbers, $r$ and $n$ nonnegative integers,
a. $\sum_{k=0}^{x}\binom{r}{k} \sum_{\substack{i_{1}+\cdots+i_{k}=n \\ i_{j}>0}} \prod_{m=1}^{k} R_{i_{m}}^{\alpha, \beta}(t)=R^{\alpha r, \beta r+r-1}(t)$
and

$$
\begin{equation*}
\text { b. } \sum_{k=0}^{r}\binom{r}{k}_{\substack{i_{1}+\ldots+i_{k} \\ i_{j}>0}} \sum_{m=1}^{k} S_{i_{m}}^{\alpha, \beta}(\ell)=S_{n}^{\alpha r, \beta r+r-1}(\ell), \tag{2.4}
\end{equation*}
$$

where

$$
\begin{align*}
& R_{n}^{\alpha, \beta}(t)=\frac{(\alpha)_{t n+n}}{n!(\alpha)_{t n}} \sum_{p=0}^{n} \frac{(-n)_{p}(-t)^{p}(\beta)_{p}}{p!(\alpha+t n)_{p}},  \tag{2.5}\\
& S_{n}^{\alpha, \beta}(\ell)=\frac{(\alpha+\ell n)^{n}}{n!} \sum_{p=0}^{n} \frac{(-\ell)^{p}(-n)_{p}(\beta)_{p}}{p!(\alpha+\ell n)^{p}} . \tag{2.6}
\end{align*}
$$

Proob of Theorem 2(a): With the aid of Theorem 4a [2, p. 708],

$$
\begin{equation*}
\sum_{n=0}^{\infty} R_{n}^{\alpha, \beta}(t) y^{n}=\frac{(1-z)^{\alpha}}{(1+t z)^{\beta+1}}, \tag{2.7}
\end{equation*}
$$

where $y(1-z)^{t+1}=-z$. Note the misprint in equation (4.3) [2, p. 709], in the definition of $C_{n}(x)$. The factor $(\alpha+1+m n-n)_{s k+\ell k}$ should read

$$
(\alpha+m n-n)_{s k+\ell k}
$$

Hence

$$
\begin{align*}
\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty} y^{n} R_{n}^{\alpha, \beta}(t)\right\}^{k} & =\frac{(1-z)^{\alpha r}}{(1+t z)^{\beta r+r}}  \tag{2.8}\\
& =\sum_{n=0}^{\infty} y^{n} R_{n}^{\alpha r, \beta r+r-1}(t)
\end{align*}
$$

Comparing coefficients and simplifying gives (2.3).
Proof of Theorem 2(b): Using Theorem 2a [2, p. 704],

$$
\begin{equation*}
\sum_{n=0}^{\infty} w^{n} S_{n}^{\alpha, \beta}(\ell)=\frac{\exp (\alpha z)}{(1-\ell z)^{\beta+1}} \tag{2.10}
\end{equation*}
$$

where $w=z \exp (-z \ell)$. Thus,

$$
\begin{equation*}
\sum_{k=0}^{r}\binom{r}{k}\left\{\sum_{n=1}^{\infty} w^{n} S_{n}^{\alpha, \beta}(\ell)\right\}^{k}=\frac{\exp (\alpha z r)}{(1-\ell z)^{\beta r+r}} \tag{2.11}
\end{equation*}
$$

Proceeding as in part a gives the required (2.4).

## III. SPECIAL CASES

It is of interest to note that a number of well-known polynomials form special cases of Theorem 1.

1. Putting $x=(1-y) / 2, s=1, \ell=0, \ell^{\prime}=1$, one may express $A_{n}^{\alpha, \beta}(x)$ in (1.6) as a Jacobi polynomial of the form
(3.1) $\frac{\alpha}{\alpha+t n+n}\left\{P_{n}^{\alpha+t n, \beta-1-\alpha-t n-n}(y)+\frac{(t+1) y}{\alpha} \frac{d}{d y} P_{n}^{\alpha+t n, \beta-1-\alpha-t n-n}(y)\right\}$
where the Jacobi polynomial is defined in [4, p. 170].
2. Letting $\ell=0, \ell^{\prime}=0, s=1$, one may express $B_{n}^{\alpha, \beta}(x)$ from (1.7) as (3.2) $\frac{\alpha}{\alpha+t n+n}\left\{L_{n}^{\alpha+t n}(\beta x)+\frac{(t+1) x}{\alpha} \frac{d}{d x} L_{n}^{\alpha+t n}(\beta x)\right\}$,
where the Laguerre polynomial is defined in [4, p. 188]. Hence, one may view $B_{n}^{\alpha, \beta}(x)$ as a generalized Laguerre polynomial.
3. $E_{n}^{\alpha, \beta}(x)$ may be viewed as a generalized Laguerre polynomial with the degree of the polynomial incorporated in the argument. In the special case for $x=1 / y, s=1, \ell^{\prime}=1$,
(3.3) $E_{n}^{\alpha, \beta}(x)=\frac{(-1)^{n}}{\alpha+\ln }\left[\alpha y^{-n} L_{n}^{-\beta-n}[y(\alpha+\ln )]-\operatorname{ly} \frac{d}{d y}\left[y^{-n} L_{n}^{-\beta-n}[y(\alpha+\ln )]\right]\right]$.
4. $F_{n}^{\alpha, \beta}(x)$ may be looked upon as a generalization of the generalized Hermite polynomial defined by Gould et al. [5, p. 58, eqn. (6.2)], and others. See also [2] for properties of this polynomial. The generalized Hermite is defined as

$$
\begin{equation*}
g_{n}^{s}(x, \lambda)=H_{n, s}(x, \lambda)=\sum_{k=0}^{[n / s]} \frac{n!\lambda^{k} x^{n-s k}}{k!(n-s k)!} . \tag{3.4}
\end{equation*}
$$

Letting $\ell^{\prime}=0$ in (1.11), one obtains

$$
\begin{equation*}
F_{n}^{\alpha, \beta}(x)=\frac{1}{n!(\alpha+\ell n)}\left[\alpha H_{n, s}(\beta x, \alpha+\ell n)+\ell x \frac{d}{d x} H_{n, s}(\beta x, \alpha+\ell n)\right] . \tag{3.5}
\end{equation*}
$$

Further, putting $\ell=0$, one obtains a single term on the right-hand side. See also [3] for bilinear generating functions and other expansions for the generalized Hermite polynomial.
5. For the special case $\ell^{\prime}=1, G_{n}^{\alpha, \beta}(x)$ may be expressed as a general polynomial of the type

$$
\begin{equation*}
\frac{(1-x)^{-\beta-n}}{n!} \sum_{k=0}^{n}(-\beta-n)_{k} x^{k} \sum_{p=0}^{k} \frac{(\alpha+t p)_{\ell n}(\beta)_{p}(-1)^{p}}{(k-p)!p!(1+\alpha+t p)_{\ell n-n}(1+\beta+n-k)_{p}} . \tag{3.6}
\end{equation*}
$$

6. $H_{n}^{\alpha, \beta}(x)$ may be considered as a generalization of a polynomial considered by Gould et a1. [5], defined in equation (3.2), p. 53. For the special case $\ell^{\prime}=0$,

$$
\text { (3.7) } \quad H_{n}^{\alpha, \beta}(x)=\frac{\exp (\beta x)}{n!} \sum_{k=0}^{n}(-1)^{k}(\beta x)^{k} \sum_{p=0}^{k} \frac{(-1)^{p}(\alpha+t p)_{\ell n}}{(k-p)!p!(\alpha+t p+1)_{\ell n-n}} \text {. }
$$

Further, for $\ell=0, H_{n}^{\alpha, \beta}(x) \exp (-\beta x)$ gives essentially the polynomial of Gould et al.

The polynomials considered in this paper appear to possess interesting common algebraic properties. One of them is that they all arise from representations of the same group. We shall have occasion to discuss group-theoretical properties of these polynomials elsewhere.

## REFERENCES

1. L. Carlitz. "Note on a Binomial Identity." Bollettino U.M.I. 9 (1974): 644-646.
2. M. E. Cohen. "On Expansion Problems: New Classes of Formulas for the Classical Functions." SIAM J. Math. AnaZ. 7 (1976):702-712.
3. M. E. Cohen. "Extensions of the Mehler-Weisner and Other Results for the Hermite Function." Math. of Comp. 30 (1976):553-564.
4. A. Erdelyi et al. Higher Transcendental Functions, Vo1. II. New York: McGraw-Hill, 1953.
5. H. W. Gould \& A. T. Hopper. "Operational Formulas Connected with Two Generalizations of Hermite Polynomials." Duke Math. J. 29 (1962):51-63.
6. P.C. C. Wang. "A Binomial Identity with an Application to Sequences of Symmetric Bernouille Trials." Rend. Sem. Fac. Sci. Univ. Cagliari 39 (1969):153-155.

# REPRESENTATIONS OF INTEGERS IN TERMS OF GREATEST INTEGER 

 functions and the golden section ratioV. E. HOGGATT, JR.

San Jose State University, San Jose, CA 95192 and
MARJORIE BICKNELL-JOHNSON
A. C. Wilcox High School, Santa Clara, CA 95051

Dedicated to ageless George Pólya.
The first and second powers of the golden section ratio, $\alpha=(1+\sqrt{5}) / 2$, can be used to uniquely represent the positive integers in terms of nested greatest integer functions, relating the compositions of an integer in terms of 1's and 2's with the numbers generated in Wythoff's game. Earlier, Alladi and Hoggatt [1] have shown that there are $F_{n+1}$ compositions of a positive integer $n$ in terms of 1 's and 2's, where $F_{n}$ is the $n$th Fibonacci number, given by $F_{1}=F_{2}=1, F_{n+2}=F_{n+1}+F_{n}$. The numbers generated in Wythoff's game have been discussed recently in [2, 3, 8] and by Silber [4].

Suppose we stack greatest integer functions, using $\alpha$ and $\alpha^{2}$, to represent the integers in yet another way:

$$
\left.\left.\begin{array}{l}
1=[\alpha]=[\alpha[\alpha]]=[\alpha[\alpha[\alpha]]]=[\alpha[\alpha[\alpha[\alpha]]]]=\cdots \\
2=\left[\alpha^{2}\right] \\
3=\left[\alpha\left[\alpha^{2}\right]\right] \\
4=\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right] \\
5=\left[\alpha^{2}\left[\alpha^{2}\right]\right] \\
6=\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right] \\
7
\end{array}\right]\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right\} \text { 8 } 8=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right] .
$$

Essentially, we start out with the compositions of an integer in terms of 1's and 2's. We put in $\alpha^{2}$ wherever there is a 2 , and $\alpha$ wherever there is a one, then collapse any strings of $\alpha$ 's on the right, since $[\alpha]=1$. For example, we write the compositions of 5 and 6:

COMPOSITIONS OF 5:

$$
\begin{array}{ll}
1+1+1+1+1 & {[\alpha[\alpha[\alpha[\alpha[\alpha]]]]]=[\alpha]=1} \\
1+1+1+2 & {\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=6} \\
1+2+1+1 & {\left[\alpha\left[\alpha^{2}[\alpha[\alpha]]\right]\right]=\left[\alpha\left[\alpha^{2}\right]\right]=3} \\
1+1+2+1 & {\left[\alpha\left[\alpha\left[\alpha^{2}[\alpha]\right]\right]\right]=\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]=4} \\
2+1+1+1 & {\left[\alpha^{2}[\alpha[\alpha[\alpha]]]=\left[\alpha^{2}\right]=2\right.} \\
1+2+2 & {\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=8} \\
2+1+2 & {\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]=7} \\
2+2+1 & {\left[\alpha^{2}\left[\alpha^{2}[\alpha]\right]\right]=\left[\alpha^{2}\left[\alpha^{2}\right]\right]=5}
\end{array}
$$

COMPOSITIONS OF 6:

$$
\begin{array}{ll}
1+1+1+1+1+1 & {[\alpha]=1} \\
1+1+1+1+2 & {\left[\alpha\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]\right]=9} \\
1+1+1+2+1 & {\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=6} \\
1+1+2+1+1 & {\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]=4} \\
1+2+1+1+1 & {\left[\alpha\left[\alpha^{2}\right]\right]=3} \\
2+1+1+1+1 & {\left[\alpha^{2}\right]=2} \\
1+1+2+2 & {\left[\alpha\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]\right]=12} \\
1+2+1+2 & {\left[\alpha\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=11} \\
2+1+1+2 & {\left[\alpha^{2}\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=10} \\
1+2+2+1 & {\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=8} \\
2+1+2+1 & {\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]=7} \\
2+2+1+1 & {\left[\alpha^{2}\left[\alpha^{2}\right]\right]=5} \\
2+2+2 & {\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=13}
\end{array}
$$

Notice that the $F_{6}$ compositions of 5 gave the representations of the integers 1 through 8 , and those of 6 , the integers 1 through $F_{7}=13$. We need to systematize; let us arrange the compositions of 5 and 6 so that the representations using $\alpha$ and $\alpha^{2}$ are in natural order.

COMPOSITIONS OF 5:
$1+1+1+1+1$
$2+1+1+1$
$1+2+1+1$
$1+1+2+1$
$2+2+1$
$1+1+1+2$
$2+1+2$
$1+2+2$
COMPOSITIONS OF 6:
$1+1+1+1+1+1$
$2+1+1+1+1$
$1+2+1+1+1$
$1+1+2+1+1$
$2+2+1+1$
$1+1+1+2+1$
$2+1+2+1$
$1+2+2+1$
$1+1+1+1+2$
$2+1+1+2$
$1+2+1+2$
$1+1+2+2$
$2+2+2$

REPRESENTATION:
$[\alpha]=1$
$\left[\alpha^{2}\right]=2$
$\left[\alpha\left[\alpha^{2}\right]\right]=3$
$\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]=4$
$\left[\alpha^{2}\left[\alpha^{2}\right]\right]=5$
$\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=6$
$\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]=7$
$\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=8$
REPRESENTATION:
$[\alpha]=1$
$\left[\alpha^{2}\right]=2$
$\left[\alpha\left[\alpha^{2}\right]\right]=3$
$\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]=4$
$\left[\alpha^{2}\left[\alpha^{2}\right]\right]=5$
$\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=6$
$\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]=7$
$\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=8$
$\left[\alpha\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]\right]=9$
$\left[\alpha^{2}\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=10$
$\left[\alpha\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=11$
$\left[\alpha\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]\right]=12$
$\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=13$

Notice that the representations of the first eight integers using the compositions of 6 agree with the representations using the compositions of 5 .
Theorem 1: Any positive integer $n$ can be represented uniquely in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$, where the exponents match the order of 1's and 2's in a composition in terms of 1's and 2's of an integer $k, n \leq F_{k+1}$, where any $\alpha^{\prime}$ s appearing to the right of the last appearing $\alpha^{2}$ are truncated.
Proof: Arrange all of the $F_{k+1}$ compositions of $k$ so that when $\alpha$ and $\alpha^{2}$ are inserted in the method described, then the results are in natural order. Do the same for the $F_{k+2}$ compositions of ( $k+1$ ) in terms of 1 's and 2's. Notice that the representations agree with the first $F_{k+1}$ from $k$. Now, for the compositions of $k$, tack on the right side $\alpha^{2}$, on the far right of the nested greatest integer functions, and suppress all the excess right $\alpha$ 's. This yields, with the new addition, representation for the numbers

$$
F_{k+1}+1, F_{k+1}+2, \ldots, F_{k+1}+F_{k}=F_{k+2}
$$

Thus, the process may be continued by mathematical induction. The uniqueness also follows as it was part of the inductive hypothesis and carries through. Theorem 1 is proved more formally as Theorems 5 and 6 in what follows.

Next, we write two lemmas.
Lemma 1: $\left[\alpha F_{n}\right]=F_{n+1}, n$ odd, $n \geq 2$;

$$
\left[\alpha F_{n}\right]=F_{n+1}-1, n \text { even, } n \geq 2
$$

Proof: From Hoggatt [5, p. 34], for $\beta=(1-\sqrt{5}) / 2$,

$$
\begin{aligned}
\alpha F_{n} & =F_{n+1}-\beta^{n} ; \\
{\left[\alpha F_{n}\right] } & =\left[F_{n+1}-\beta^{n}\right] .
\end{aligned}
$$

Since $\left|\beta^{n}\right|<1 / 2, n \geq 2$, if $n$ is odd, then $\beta^{n}<0$, and $\left[F_{n+1}-\beta^{n}\right]=F_{n+1}$, while if $n$ is even, $\beta^{n}>0$, making $\left[F_{n+1}-\beta^{n}\right]=F_{n+1}-1$.
Lemma 2: $\left[\alpha^{2} F_{n}\right]=F_{n+2}, n$ odd, $n \geq 2$;

$$
\left[\alpha^{2} F_{n}\right]=F_{n+2}-1, n \text { even, } n \geq 2
$$

Proof: Since $\alpha F_{n}=F_{n+1}-\beta^{n}$,

$$
\alpha^{2} F_{n}=\alpha F_{n+1}-\alpha \beta^{n}
$$

$$
=\left(F_{n+2}-\beta^{n+1}\right)-\alpha \beta^{n}
$$

$$
=F_{n+2}-\beta^{n}(\alpha+\beta)
$$

$$
=F_{n+2}-\beta^{n}
$$

Then, $\left[\alpha^{2} F_{n}\right]=\left[F_{n+2}-\beta^{n}\right]$ is calculated as in Lemma 1 .
Lemma 3: For all integers $k \geq 2$ and $n \geq k$,

$$
\left[\alpha^{k} F_{n}\right]=F_{n+k} \text { if } n \text { is odd; }
$$

$$
\left[\alpha^{k} F_{n}\right]=F_{n+k}-1 \text { if } n \text { is even. }
$$

Proo6: $\alpha^{k} F_{n}=\frac{\alpha^{k}\left(\alpha^{n}-\beta^{n}\right)}{\sqrt{5}}-\frac{\beta^{n+k}}{\sqrt{5}}+\frac{\beta^{n+k}}{\sqrt{5}}=\frac{\alpha^{n+k}-\beta^{n+k}}{\sqrt{5}}-\frac{\beta^{n}\left(\alpha^{k}-\beta^{k}\right)}{\sqrt{5}}$

$$
=F_{k+n}-\beta^{n} F_{k} .
$$

Now, $|\beta|^{n} F_{k}<1$ if and only if $|\beta|^{n}<1 / F_{k}$, which occurs whenever $n \geq k, k \geq 2$,
since

$$
\frac{1}{F_{k}}=\frac{\sqrt{5}}{\alpha^{k}-\beta^{k}}=\frac{\sqrt{5}}{(-1 / \beta)^{k}-\beta^{k}}=\frac{\sqrt{5} \beta^{k}}{(-1)^{k}-\beta^{2 k}} .
$$

If $k$ is even, $k \geq 2, \beta^{k}>0$, and

$$
\begin{gathered}
\frac{\sqrt{5}}{1-\beta^{2 k}}>1 \\
\frac{1}{F_{k}}=\frac{\sqrt{5}}{1-\beta^{2 k}} \cdot \beta^{k}>\beta^{k}
\end{gathered}
$$

Similarly, if $k$ is odd, $k \geq 3, \beta^{k}<0$, and

$$
\begin{gathered}
\frac{\sqrt{5}}{-1-\beta^{2 k}}>-1 \\
\frac{1}{F_{k}}=\frac{\sqrt{5}}{-1-\beta^{2 k}} \cdot \beta^{k}<-\beta^{k}
\end{gathered}
$$

Thus, $|\beta|^{n} F_{k}<1$, and Lemma 2 follows.
Next, observe the form of Fibonacci numbers written with nested greatest integer functions of $\alpha$ and $\alpha^{2}$ :

$$
\begin{array}{ll}
F_{2}=2=[\alpha] & F_{6}=8=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right] \\
F_{3}=2=\left[\alpha^{2}\right] & F_{7}=13=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right] \\
F_{4}=3=\left[\alpha\left[\alpha^{2}\right]\right] & F_{8}=21=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]\right] \\
F_{5}=5=\left[\alpha^{2}\left[\alpha^{2}\right]\right] & F_{9}=34=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]\right]
\end{array}
$$

Theorem 2: $F_{2 n+1}=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}[\cdots]\right]\right]\right]$,
and $\quad F_{2 n+2}=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}[\cdots]\right]\right]\right]\right]$,
both containing $n$ nested $\alpha^{2}$ factors.
Proof: We have illustrated the theorem for $n=1,2, \ldots, 9$. Assume that Theorem 2 holds for all $n \leq k$. By Lemma 1 ,
$F_{2 k+2}=\left[\alpha F_{2 k+1}\right]=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}[\cdots]\right]\right]\right]\right]$
for $k$ nested $\alpha^{2}$ factors; by Lemma 2,
$F_{2 k+3}=\left[\alpha^{2} F_{2 k+1}\right]=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}[\cdots]\right]\right]\right]\right]\right.$
for $(k+1)$ nested $\alpha^{2}$ factors.
Return once again to the listed compositions of 5 and 6 using 1 's and 2's, and let us count the numbers of 1 's and 2's used totally, and the number of $\alpha$ 's and $\alpha^{2}$ 's appearing in the integers represented. We also add the data acquired by listing the compositions of 1, 2, 3, and 4, which appear in the tables if the 1's on the right are truncated carefully.

| $n$ | $1^{\prime \prime} \mathrm{s}$ | $2 ' s$ | $\alpha ' s$ | $\alpha^{2 \prime} \mathrm{~s}$ | Suppressed $\alpha ' s$ |
| ---: | ---: | ---: | ---: | :---: | ---: |
| 1 | 1 | 0 | 1 | 0 | $0=F_{3}-2$ |
| 2 | 2 | 1 | 1 | 1 | $1=F_{4}-2$ |
| 3 | 5 | 2 | 2 | 2 | $3=F_{5}-2$ |
| 4 | 10 | 5 | 4 | 5 | $6=F_{6}-2$ |
| 5 | 20 | 10 | 9 | 10 | $11=F_{7}-2$ |
| 6 | 38 | 20 | 19 | 20 | $19=F_{8}-2$ |

Define $C_{n}$ as the $n$th term in the first Fibonacci convolution [6], [7] sequence $1,2,5,10,20,38, \ldots$, where

$$
C_{n}=\sum_{i=1}^{n} F_{i} F_{n-i}=\frac{n L_{n+1}+2 F_{n}}{5}
$$

and observe where these numbers appear in our table. Note that $L_{n}$ is the $n$th Lucas number defined by $L_{1}=1, L_{2}=3$, and $L_{n+2}=L_{n+1}+L_{n}$.
Theorem 2: Write the compositions of $n$ using 1's and 2's, and represent all integers less than or equal to $F_{n+1}$ in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$ as in Theorem 1 . Then
(i) $C_{n}$ 1's appear;
(ii) $C_{n-1} 2^{\prime}$ s appear;
(iii) $C_{n-1} \alpha^{2 ' s}$ appear;
(iv) $F_{n+2}-2$ a's are truncated;
(v) $\quad\left(C_{n}-F_{n+2}+2\right) \quad \alpha$ 's appear.

Proof: Let the table just given form our inductive basis, since (i) through (v) hold for $n=1,2,3,4,5,6$. Let $t(n)$ and $u(n)$ denote the number of times 2 and 1 respectively appear in a count of all such compositions of $n$. Then, by the rules of formation,

$$
t(n)=t(n-2)+t(n-1)+F_{n-1}
$$

since we will add a 2 on the right to each composition of ( $n-2$ ), giving $t(n-2) 2$ 's already there, and $F_{n-2+1}=F_{n-1}$ new 2's written, and $t(n-1)$ 2's from the compositions of $(n-1)$, each of which will have a 1 added onto the right. Since [6]

$$
C_{n}=F_{n}+C_{n-1}+C_{n-2}
$$

has the same recursion relation and $t(n)$ has the starting values of the table, $t(n)=C_{n-1}$ for positive integers $n$, establishing (ii).

Similarly for (i),

$$
u(n)=u(n-1)+F_{n}+u(n-2)
$$

since 1's are added on the right to the compositions of ( $n-1$ ), keeping $u(n-1) 1$ 's already appearing and adding $F_{n-1+1}=F_{n}$ new 1 's, and all 1's in ( $n-2$ ) will appear, since those compositions have a 2 added on the right. We can again establish $u(n)=C_{n}$ by induction.

Obviously, (ii) and (iii) must have the same count. Since the number of $\alpha$ 's appearing is the difference of the number of l's used and the number of $\alpha$ 's truncated, we have (v) immediately if we prove (iv). But the number of
suppressed $\alpha$ 's for $k$ is the number suppressed in the preceding set of compositions of ( $k-1$ ), each of which had a 1 added on the right, plus the number of new l's on the right, or,

$$
F_{k+1}-2+F_{k}=F_{k+2}-2
$$

so if the formula holds for $1,2,3, \ldots, k-1$, then it also holds for $k$, and the number of suppressed $\alpha$ 's for $n$ is $F_{n+2}-2$ by mathematical induction.

Now, we go on to the numbers $\alpha_{n}$ and $b_{n}$, where $\left(a_{n}, b_{n}\right)$ is a safe-pair in Wythoff's game [2, 4, 8]. We list the first few values for $\alpha_{n}$ and $b_{n}$, and some needed properties:

$$
\begin{array}{lrrrrrrrrrr}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
a_{n} & 1 & 3 & 4 & 6 & 8 & 9 & 11 & 12 & 14 & 16 \\
b_{n} & 2 & 5 & 7 & 10 & 13 & 15 & 18 & 20 & 23 & 26 \\
a_{k}+k=b_{k} & & & & & & & \\
a_{n}+b_{n}=a_{b_{n}} \\
a_{a_{n}}+1=b_{n} \\
a_{a_{n}+1}-a_{a_{n}}=2 & \text { and } & a_{b_{n}+1}-a_{b_{n}}=1  \tag{6}\\
b_{a_{n}+1}-b_{a_{n}}=3 & \text { and } & b_{b_{n}+1}-b_{b_{n}}=2 \\
a_{n}=[n \alpha] \text { and } b_{n}=\left[n \alpha^{2}\right]
\end{array}
$$

We first concentrate on the expressions in (6) for $a_{n}$ and $b_{n}$, using the greatest integer function, and compare to Lemmas 1 and 2 . We can write Lemma 4 immediately, by letting $n=F_{k}$ in (6).
Lemma 4: For all positive integers $k$,

$$
\begin{aligned}
& a_{F_{2 k}}=F_{2 k+1}-1 \quad \text { and } \quad a_{F_{2 k+1}}=F_{2 k+2} \\
& b_{F_{2 k}}=F_{2 k+2}-1 \quad \text { and } \quad b_{F_{2 k+1}}=F_{2 k+3}
\end{aligned}
$$

Next we show that the integer following $F_{n}$ is always a member of $\left\{\alpha_{n}\right\}$.
Theorem 3: $\quad F_{n+1}+1=a_{F_{n}+1}$.
Proof: Part I: $n+1$ is even. Let $F_{n+1}=F_{2 k}=\alpha_{F_{2 k-1}}$ from Lemma 4. Note well that $F_{2 k-1} \varepsilon\left\{b_{n}\right\}$, and by (4), ${ }^{2 k-1}$
so that $\quad a_{F_{2 k-1}+1}-a_{F_{2 k-1}}=1$,

$$
a_{F_{2 k-1}+1}=a_{F_{2 k-1}}+1=F_{2 k}+1
$$

Part II: $\quad n+1$ is odd. Let $F_{n+1}=F_{2 k+1}=b_{F_{2 k-1}} \quad$ from Lemma 4. From (3), we have

$$
b_{F_{2 k-1}}+1=a_{a_{F_{2 k-1}}}+1+1=a_{a_{F_{2 k-1}}+1}
$$

since $a_{a_{n}+1}-a_{a_{n}}=2$. Thus,
$F_{2 k+1}+1=b_{F_{2 k-1}}+1=a_{a_{F_{2 k-1}}+1}=a_{F_{2 k}+1}$
from $F_{2 k}=\alpha_{F_{2 k-1}}$. This concludes Part II and the theorem.
Theorem 4: $\quad a_{F_{n+2}+1}+1=b_{F_{n+1}+1}$.
Proob: Part I: $n$ is even. Let $F_{n+2}=F_{2 k}=\alpha_{F_{2 k-1}}$ and $F_{2 k-1}=b_{F_{2 k-3}}$ by Lemma 4, so that (4) yields

$$
a_{F_{2 k-1}+1}=\alpha_{F_{2 k-1}}+1
$$

From this we get

$$
a_{a_{F_{2 k-1}}+1}+1=a_{a_{F_{2 k-1}+1}}+1=b_{F_{2 k-1}+1}
$$

making use of (3). This concludes Part I.
Part II: $n$ is odd. Let $F_{n+2}=F_{2 k+1}$. Using Theorem 3 and (3),

$$
a_{F_{2 k+1}+1}+1=a_{a_{F_{2 k}+1}}+1=b_{F_{2 k}+1}
$$

which concludes Part II and the proof of the theorem.
Comments: We have seen that

$$
F_{n+2}+1=a_{F_{n+1}+1}
$$

from Theorem 3, and

$$
a_{F_{n+1}+1}+1=b_{F_{n}+1}
$$

from Theorem 4. Thus, the sequence of consecutive $b_{j}^{\prime}$ 's,

$$
b_{F_{n}+1}, b_{F_{n}+2}, b_{F_{n}+3}, \ldots, b_{F_{n+1}},
$$

and consecutive $a_{j}$ 's,

$$
a_{F_{n+1}+1}, a_{F_{n+1}+2}, a_{F_{n+1}+3}, \ldots, a_{F_{n+2}},
$$

cover the sequence

$$
F_{n+2}+1, F_{n+2}+2, F_{n+2}+3, \ldots, F_{n+3},
$$

where, if $F_{n+1}=F_{2 k+1}$, then $b_{F_{2 k-1}}=F_{2 k+1}=F_{n+3}$, and if $F_{n+1}=F_{2 k}$, then $a_{F_{n+2}}=a_{F_{2 k+1}}=F_{2 k+2}=F_{n+3}$. The sequences $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are such that their disjoint union covers the positive integers, and there are $F_{n-1}$ of the $b_{j}$ 's and $F_{n}$ of the $a_{j}$ 's, or collectively, $F_{n+1}$ all together. The interval $\left[F_{n+2}+1, F_{n+3}\right]$ contains precisely $F_{n+1}$ positive integers. We have shown that the union of the two sequences are precisely the integers on this interval. We now are ready to prove Theorem 5 by mathematical induction.
Theorem 5: If $\alpha^{2}$ is added onto the right of the specified function for the compositions of $n$ properly ordered, then we obtain the integers

$$
F_{n+2}+1, F_{n+2}+2, \ldots, F_{n+2}+F_{n+1}=F_{n+3} .
$$

Proof: By our previous discussions, Theorem 5 is true for $n=1,2, \ldots, 6$. Assume it is true for $n=k-1$ and $n=k$. Then, let us add $\alpha^{2}$ on the left to each value of the specified function, making the result be the $F_{k}$ successive $b_{j}$ 's

$$
b_{F_{k+1}+1}, b_{F_{k+1}+2}, \ldots, b_{F_{k+2}}
$$

and let us add $\alpha$ on the left to each value of the specified function, to obtain the $F_{k+1}$ successive $\alpha_{j}$ 's,

$$
a_{F_{k+2}+1}, a_{F_{k+2}+2}, \ldots, a_{F_{k+3}} .
$$

These numbers together give the interpretation of compositions of (k+1) with $\alpha^{2}$ on the right, so we must get $F_{k+3}+1, F_{k+3}+2, \ldots, F_{k+4}$. There are $F_{k}$ consecutive $b_{j}$ 's and $F_{k+1}$ consecutive $\alpha_{j}$ 's which fit together precisely to cover the above interval by the discussion preceding Theorem 5, giving us a proof by mathematical induction.

Theorem 6: The $F_{n+2}$ compositions of $(n+1)$ using 1 's and 2 's when put into the nested greatest integer function with 1 and 2 the exponents on $\alpha$ can be arranged so that the results are the integers $1,2, \ldots, F_{n+2}$ in sequence.
Proof: We have illustrated Theorem 6 for $n=1,2, \ldots, 5$. Assume that the $F_{n}$ compositions for ( $n-1$ ) have been so arranged in the nested greatest integer function representations. By Theorem 5, the results of putting 2 on the right of the compositions, or an $\alpha^{2}$ on the right of each representation, yields the numbers $F_{n+2}+1, F_{n+2}+2, \ldots, F_{n+3}$. The adding of a one to the right of compositions of $(n+1)$ yields a composition of $(n+2)$ but it does not change the results of the nested greatest integer representations. Thus the list now goes for compositions of $(n+2)$, the first $F_{n+2}$ coming from the one added on the right of those for $(n+1)$ and the $F_{n+1}$ more coming from the two added on the right of those for $n$. Thus, by mathematical induction, we complete the proof of the theorem for all $n \geq 1$.

The above proof is constructive, as it yields the proper listing of the composition for $(n+2)$ if we have them for $n$ and for $(n+1)$.

Notice the pattern of our representations if we simply record them in a different way:

$$
\begin{aligned}
& 1=[\alpha]=\alpha_{1} \\
& 2=\left[\alpha^{2}\right]=b_{1} \\
& 3=\left[\alpha\left[\alpha^{2}\right]\right]=a_{b_{1}} \\
& 4=\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]=\alpha_{a_{b_{1}}} \\
& 5=\left[\alpha^{2}\left[\alpha^{2}\right]\right]=b_{b_{1}} \\
& 6=\left[\alpha\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]=\alpha_{a_{a_{b_{1}}}} \\
& 7=\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]=b_{a_{b_{1}}} \\
& 8=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=\alpha_{b_{b_{1}}}
\end{aligned}
$$

In other words, Theorems 3 through 6 and Lemma 4 will allow us to write a representation of an integer such that each $\alpha$ in its nested greatest integer function becomes a subscripted $a$, and each $\alpha^{2}$ a subscripted $b$, in a continued subscript form.

Next, we present a simple scheme for writing the representations of the integers in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$, as in Theorems 1 and 6. We use the difference of the subscripts of Fibonacci numbers to obtain the exponents 1 and 2 , or the compositions of $n$ in terms of 1's and 2's, by using $F_{n+1}$ in the rightmost column. We illustrate for $n=6$, using $F_{7}$. Notice that every other column in the table is the subscript difference of the two adjacent Fibonacci numbers, and compare with the compositions of 6 and the representations of the integers $1,2, \ldots, 13$ in natural order given just before Theorem 1. We use the Fibonacci numbers as place holders. One first writes the column of $13 F_{7}$ 's, which is broken into $8 F_{6}$ 's and $5 F_{5}^{\prime \prime} \mathrm{s}$. The $8 F_{6}^{\prime} \mathrm{s}$ are broken into $5 F_{5}^{\prime} \mathrm{s}$ and $3 F_{4}^{\prime} \mathrm{s}$, and the $5 F_{5}^{\prime}$ 's into $3 F_{4}^{\prime}$ s and $2 F_{3}^{\prime}$ 's. The pattern continues in each column, until each $F_{2}$ is broken into $F_{1}$ and $F_{0}$, so ending with $F_{1}$. In each new column, 1 always replaces $F_{n} F_{n}^{\prime}$ 's with $F_{n-1} F_{n-1}$ 's and $F_{n-2} F_{n-2}$ 's. Notice that the next level, representing all integers through $F_{8}=21$, would be formed by writing $21 F_{8}^{\prime}$ s in the right column, and the present array as the top $13=F_{7}$ rows, and the array ending in $8 F_{6}^{\prime}$ 's now in the top $8=F_{6}$ rows would appear in the bottom
eight rows. Notice further that, just as in the proofs of Theorems 1 and 6, this scheme puts a 1 on the right of all compositions of ( $n-1$ ) and a 2 on the right of all compositions of $(n-2)$.

SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS $n \leq F_{7}$

$$
\begin{array}{lllllllllllll}
F_{1} & 1 & F_{2} & 1 & F_{3} & 1 & F_{4} & 1 & F_{5} & 1 & F_{6} & 1 & F_{7} \\
& F_{1} & 2 & F_{3} & 1 & F_{4} & 1 & F_{5} & 1 & F_{6} & 1 & F_{7} & n=1 \\
& F_{1} & 1 & F_{2} & 2 & F_{4} & 1 & F_{5} & 1 & F_{6} & 1 & F_{7} & n=3 \\
& F_{1} & 1 & F_{2} & 1 & F_{3} & 2 & F_{5} & 1 & F_{6} & 1 & F_{7} & n=4 \\
& & & F_{1} & 2 & F_{3} & 2 & F_{5} & 1 & F_{6} & 1 & F_{7} & n=5 \\
& F_{1} & 1 & F_{2} & 1 & F_{3} & 1 & F_{4} & 2 & F_{6} & 1 & F_{7} & n=6 \\
& & F_{1} & 2 & F_{3} & 1 & F_{4} & 2 & F_{6} & 1 & F_{7} & n=7 \\
& & & F_{1} & 1 & F_{2} & 2 & F_{4} & 2 & F_{6} & 1 & F_{7} & n=8 \\
& & F_{1} & 1 & F_{2} & 1 & F_{3} & 1 & F_{4} & 1 & F_{5} & 2 & F_{7} \\
& & & F_{1} & 2 & F_{3} & 1 & F_{4} & 1 & F_{5} & 2 & F_{7} & n=9 \\
& & & F_{1} & 1 & F_{2} & 2 & F_{4} & 1 & F_{5} & 2 & F_{7} & n=11 \\
& & & F_{1} & 1 & F_{2} & 1 & F_{3} & 2 & F_{5} & 2 & F_{7} & n=12 \\
& & & & F_{7} & 2 & F_{3} & 2 & F_{5} & 2 & F_{7} & n=13
\end{array}
$$

Within the array just given, we have used $8 F_{6}{ }^{\prime} \mathrm{s}, 5 F_{5}$ 's, $6 F_{4}{ }^{\prime} \mathrm{s}, 6 F_{3}{ }^{\prime} \mathrm{s}$, $5 F_{2}^{\prime}$ 's, and $8 F_{1}^{\prime}$ 's, where $8+5+6+6+5+8=38=C_{6}$, where again $C_{n}$ is the $n$th element in the Fibonacci convolution sequence. These coefficients appear in the array:

> row sum

| 1 |  |  |  |  |  | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | ---: |
| 1 | 1 |  |  |  |  | 2 |
| 2 | 1 | 2 |  |  |  | 5 |
| 3 | 2 | 2 | 3 |  |  | 10 |
| 5 | 3 | 4 | 3 | 5 | 8 | 38 |
| 8 | 5 | 6 | 6 | 5 | 8 | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $F_{n}$ | $1 F_{n-1}$ | $2 F_{n-2}$ | $3 F_{n-3}$ | $5 F_{n-4}$ | $8 F_{n-5}$ | $\cdots$ |$C_{n}$

The rows give the number of $F_{n}$ 's, $F_{n-1}$ 's, $F_{n-2}$ 's, .... , used in the special array to write the compositions of $n$ in natural order. Properties of the array itself will be considered later.

Now we turn to the Lucas numbers. We observe

$$
\begin{aligned}
& L_{2}=3=\left[\alpha\left[\alpha^{2}\right]\right] \\
& L_{3}=4=\left[\alpha\left[\alpha\left[\alpha^{2}\right]\right]\right] \\
& L_{4}=7=\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right] \\
& L_{5}=11=\left[\alpha\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right] \\
& L_{6}=18=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right] \\
& L_{7}=29=\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]\right] \\
& L_{8}=47=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right]\right]\right.
\end{aligned}
$$

Thus, it appears that $\left[\alpha L_{n}\right]=L_{n+1}$ if $n$ is odd, and that $\left[\alpha^{2} L_{n}\right]=L_{n+2}$ if $n$ is even. Also, we see the form of Lucas numbers, and can compare them with the representation of Fibonacci numbers. We first need a lemma.
Lemma 5: $-1<\beta^{n} \sqrt{5}<1$ for $n \geq 2$.
Proof: $\beta^{2}=(3-\sqrt{5}) / 2$, and $\beta^{2} \sqrt{5}=(3 \sqrt{5}-5) / 2<.85<1$. Thus,

$$
0<\beta^{2 n} \sqrt{5} \leq \beta^{2} \sqrt{5}<1 \text { for } n \geq 1
$$

If $0<\beta^{2} \sqrt{5}<1$, then $0>\beta^{3} \sqrt{5}>-1$, so that

$$
-1<\beta^{2 n+1} \sqrt{5}<0 \text { for } n \geq 1
$$

establishing Lemma 5.
Lemma 6: $\left[\alpha L_{n}\right]=L_{n+1}$ for $n$ even, if $n \geq 2$;

$$
\left[\alpha L_{n}\right]=L_{n+1}-1 \text { for } n \text { odd, if } n \geq 3
$$

Proof: Apply Lemma 5 to the expansion of $\alpha L_{n}$ :

$$
\begin{aligned}
\alpha L_{n} & =\alpha\left(\alpha^{n}+\beta^{n}\right)=\alpha^{n+1}+\beta^{n+1}+\alpha \beta^{n}-\beta^{n+1} \\
& =L_{n+1}+\beta^{n}(\alpha-\beta)=L_{n+1}+\beta^{n} \sqrt{5} . \\
\text { Lemma 7: }\left[\alpha^{2} L_{n}\right] & =L_{n+2} \text { if } n \text { is even and } n \geq 2 ; \\
{\left[\alpha^{2} L_{n}\right] } & =L_{n+2}-1, \text { if } n \text { is odd and } n \geq 1 .
\end{aligned}
$$

Proof: We apply Lemma 5 to

$$
\begin{aligned}
\alpha^{2} L_{n} & =\alpha^{2}\left(\alpha^{n}+\beta^{n}\right)=\alpha^{n+2}+\beta^{n+2}+\beta^{n}\left(\alpha^{2}-\beta^{2}\right) \\
& =L_{n+2}+\beta^{n} \sqrt{5} .
\end{aligned}
$$

Theorem 7: The Lucas numbers $L_{n}$ are representable uniquely in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$ in the forms

$$
\begin{aligned}
L_{2 n+1} & =\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha\left[\alpha^{2}\right]\right] \ldots\right]\right]\right]\right] \\
L_{2 n} & =\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha\left[\alpha^{2}\right]\right] \ldots\right]\right]\right]
\end{aligned}
$$

where the number of $\alpha^{2}$ consecutively is $(n-1), n \geq 1$.
Proof: Theorem 7 has already been illustrated for $n=1,2, \ldots, 8$. A proof by mathematical induction follows easily from Lemmas 6 and 7.

Comparing Theorems 2 and 7 , we notice that the representations of $F_{k}$ and $L_{k+1}$ are very similar, with the representation of $L_{k+1}$ duplicating that of $F_{k}$ with $\left[\alpha\left[\alpha^{2}\right]\right]$ added on the far right. We write

$$
\begin{aligned}
& \text { Theorem 8: } F_{2 n+2}=\left[\alpha\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha^{2}\right] \ldots\right]\right]\right] \text { and } \\
& L_{2 n+3}=\left[\alpha\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right] \ldots\right]\right]\right] ; \\
& F_{2 n+1}=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha^{2}\right] \ldots\right]\right]\right] \text { and } \\
& L_{2 n+2}=\left[\alpha^{2}\left[\alpha^{2}\left[\alpha^{2} \ldots\left[\alpha^{2}\left[\alpha\left[\alpha^{2}\right]\right]\right] \ldots\right]\right]\right] \text {, }
\end{aligned}
$$

where there are $n$ consecutive $\alpha^{2}$ 's.
Theorem 8, restated, shows that if a 1 and a 2 is added on the right to the composition of ( $k-1$ ) in terms of $1^{\prime}$ s and 2 's that gave rise to $F_{k}$, one obtains $L_{k+1}$. If we add a 1 and a 2 on the right of the compositions of $n$, we observe:


Theorem 9: If to the compositions of $n$ in terms of 1 's and 2's, written in the order producing representations of $1,2, \ldots, F_{n+1}$ in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$ in natural order, we add a 1 and a 2 on the right, then the resulting nested greatest integer functions of $\alpha$ and $\alpha^{2}$ have values

$$
F_{n+3}+1, F_{n+3}+2, \ldots, F_{n+3}+F_{n+1}=L_{n+2} .
$$

Now, notice that, since the representation giving rise to a Lucas number in the nested greatest integer representation ends with a 1 and a 2 , the next representation, taken in natural order, will end in a 2 and a 2 . Consider the compositions of $n$, where we add two 2's on the right:

$$
\begin{aligned}
& n=1: \quad \overline{1} 22 \quad\left[\alpha\left[\alpha^{2}\left[\alpha^{2}\right]\right]\right]=8=L_{4}+1=F_{6}
\end{aligned}
$$

$$
\begin{aligned}
& n=3
\end{aligned}
$$

Theorem 10: If to the compositions of $n$ in terms of 1's and 2's, written in the order that produces representations of $1,2, \ldots, F_{n+1}$ in natural order in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$, we add two 2 's on
the right, then the resulting nested greatest integer functions of $\alpha$ and $\alpha^{2}$ have the consecutive values

$$
L_{n+3}+1, L_{n+3}+2, \ldots, L_{n+3}+F_{n+1}=F_{n+5}
$$

We are now in a position to count in two different ways all the $\alpha$ 's and $\alpha^{2}$ 's appearing in the display of all integers from 1 through $L_{n}$ simultaneously. Of the $F_{n+1}$ compositions of $n$, there are $F_{n}$ which end in a 1 , and $F_{n-1}$ which end in a 2 . Those ending in a 1 are the compositions of $(\bar{n}-1)$ with our $\frac{1}{2}$ added, while those ending in a $\frac{2}{2}$ are the compositions of ( $n-2$ ) with our $\underline{2}$ added. Now, if we add $\underline{2}$ to each of these $F_{n+1}$ compositions, by Theorem 5, we get the numbers

$$
F_{n+2}+1, F_{n+2}+2, \ldots, F_{n+2}+F_{n+1}=F_{n+3} .
$$

Of these, there were $F_{n}$ ending in a 1 , which now end in a $1-2$ and cover the numbers

$$
F_{n+2}+1, F_{n+2}+2, \ldots, F_{n+2}+F_{n}=L_{n+1}
$$

and those that end in a $\underline{2-2}$ cover the numbers

$$
L_{n+1}+1, L_{n+1}+2, \ldots, L_{n+1}+F_{n+1}=F_{n+3}
$$

when used in the nested greatest integer functions of $\alpha$ and $\alpha^{2}$ in natural order. We can now count the number of $\alpha$ 's and $\alpha^{2 ' s}$ used to display all the representations of the integers from 1 to $L_{n+1}$. We count all of those up to and including $F_{n+3}$ by Theorem 2, and subtract the total $\alpha$ and $\alpha^{2}$ content of the compositions of ( $n-2$ ), which is $C_{n-2} \alpha^{\prime} s$ and $C_{n-3} \alpha^{\prime \prime}$ s, and subtract $2 F_{n-1} \alpha^{2}$ 's, or, we can count all of those up to and including $F_{n+2}$, and add on the $F_{n} \alpha^{\prime} s$ and $F_{n} \alpha^{2 ' s}$, and add the number of 1 's in the compositions of ( $n-1$ ), which all become $\alpha$ 's in counting from $F_{n+2}+1$ through $F_{n+2}+F_{n}=$ $L_{n+1}$. The first method gives us, for the number of $\alpha$ 's,

$$
\left(C_{n+2}-F_{n+4}+2\right)-C_{n-2},
$$

and for the number of $\alpha^{2} \mathrm{~s}$,

$$
C_{n+1}-C_{n+3}-2 F_{n-1} .
$$

The second method gives the number of $\alpha^{\prime} s$ as

$$
\left(C_{n+1}-F_{n+3}+2\right)+C_{n-2}+F_{n},
$$

which simplifies to

$$
C_{n+1}+C_{n-1}-2 F_{n+1}+2
$$

and the number of $\alpha^{2}$ 's as

$$
C_{n}+C_{n-2}+F_{n},
$$

finishing a proof of Theorem 11.
Theorem 11: Write the compositions of $(n+2)$ using 1 's and 2's, and represent all integers less than or equal to $L_{n+1}$ in terms of nested greatest integer functions of $\alpha$ and $\alpha^{2}$ in natural order as in Theorem 1. Then,
(i) $\left(C_{n+2}-C_{n-2}-F_{n+4}+2\right)=\left(C_{n+1}+C_{n-1}-2 F_{n+1}+2\right)$
is the number of $\alpha$ 's appearing, and

$$
\text { (ii) }\left(C_{n+1}+C_{n-2}+F_{n}\right)=\left(C_{n+1}-C_{n-3}-2 F_{n-1}\right)
$$

is the number of $\alpha^{2 '}$ s appearing.

## REFERENCES

1. Krishnaswami Alladi \& V. E. Hoggatt, Jr. "Compositions with Ones and Twos." The Fibonacci Quarterty 13, No. 3 (]975):233-239.
2. V. E. Hoggatt, Jr., \& Marjorie Bicknell-Johnson. "A Generalization of Wythoff's Game." The Fibonacci Quarterly 17, No. 3 (1979):198-211.
3. V. E. Hoggatt, Jr., \& Marjorie Bickne11-Johnson. "Lexicographic Ordering and Fibonacci Representations." The Fibonacci Quarterly, to appear.
4. R. Silber. "Wythoff's Nim and Fibonacci Representations." The Fibonacci Quarterly 15, No. 1 (1977):85-88.
5. Verner E. Hoggatt, Jr. Fibonacci and Lucas Numbers. Houghton Miffin Mathematics Enrichment Series. Boston: Houghton Mifflin, 1969.
6. Verner E. Hoggatt, Jr., \& Marjorie Bicknell. "Convolution Triangles." The Fibonacci Quarterly 10, No. 6 (1972):599-608.
7. V. E. Hoggatt, Jr., \& Marjorie Bicknell. "Fibonacci Convolution Sequences." The Fibonacci Quarterly 15, No. 2 (1977):117-121.
8. V. E. Hoggatt, Jr., \& A. P. Hillman. "A Property of Wythoff Pairs." The Fibonacci Quarterly 16, No. 5 (1978):472.
9. A. F. Horadam. "Wythoff Pairs." The Fibonacci Quarterly 16, No. 2 (1978):147-151.
*****
```
A PRIMER ON STERN'S DIATOMIC SEQUENCE
CHRISTINE GIULI
University of Santa Clara, Santa Clara, CA 95053
and
ROBERT GIULI
University of California, Santa Cruz, CA 96050
```

PART III: ADDITIONAL RESULTS
An examination of the sequence yields corollaries to some of the previously known results. Being fundamentally Fibonacci minded, and at the onset not aware of the works of Stern, Eisenstein, Lehmer and Lind, we noticed the following results not already mentioned-some may even seem trivial.
(1) $s(n, 1)=n$
$s(n, 2)=n-1$
$s(n, 4)=n-2$
:
$s\left(n, 2^{m}\right)=n-m$
(2) $s\left(n, \alpha 2^{m}\right)=s(n-m, \alpha)$
(3) Another statement of symmetry is $s\left(n, 2^{n-2}-a\right)=s\left(n, 2^{n-2}+\alpha\right)$
(4)
$s\left(n, 2^{n-1}\right)=1$
$s\left(n, 2^{n-2}\right)=2$
$s\left(n, 2^{n-2}\right)=2$
:
$s\left(n, 2^{n-k}\right)=k$
(5) $s\left(n+k-1, \frac{2^{k}-(-1)^{k}}{3}\right)=F_{k-1}+n F_{k}$, or

$$
s\left(N, \frac{2^{k}-(-1)^{k}}{3}\right)=F_{k-1}+(N-k+1) F_{r}
$$

(6) $s\left(n, \frac{2^{n-2}+(-1)^{n-1}}{3}\right)=L_{n-1}$ when $L_{n}$ are Lucas numbers
(7) $s\left(n, 2^{k}\right)+s\left(n, 2^{k+1}\right)=s\left(n, 3 \cdot 2^{k-1}\right)$
(8) $s\left(n, K \cdot 2^{k-1}\right)=s\left(n, K \cdot 2^{k}\right)+1$
(9) $s\left(n, 3 \cdot 2^{m-1}\right)=2(n-m)+1, \quad n>0$
$s\left(n, 5 \cdot 2^{m-1}\right)=3(n-m)-1, n>1$
$s\left(n, 7 \cdot 2^{m-1}\right)=3(n-m)-2, n>1$
$s\left(n, 9 \cdot 2^{m-1}\right)=4(n-m)-5, n>2$
$s\left(n, 11 \cdot 2^{m-1}\right)=5(n-m)-7, \quad n>2$
$s\left(n, 13 \cdot 2^{m-1}\right)=5(n-m)-8, \quad n>2$
$s\left(n, 15 \cdot 2^{m-1}\right)=4(n-m)-7, \quad n>2$
etc.
where $m=1,2,3, \ldots$
(10) The table on page 320 is the sequence of combinatorial coefficients mod 2. Hoggatt informed us that he suspected the sums of the rising diagonals were Stern numbers-he was right.

The formal statement of the problem is that

$$
\sum_{i-0}^{\left[\frac{j-1}{2}\right]}\binom{j-i-1}{i} \bmod 2=s\left(k+1, j-2^{k}\right)
$$

where $k=\left[\log _{2} j\right]$ and $2^{k} \leq j \leq 2^{k+1}$.
The proof is by induction relying essentially on the following theorem.
Theorem: Given the binomial coefficient mod $2, \begin{aligned} & n_{2} \\ & k_{2}\end{aligned}$, then

$$
\binom{n}{k}_{2} \equiv\binom{2 n+1}{2 k}+\binom{2 n}{2 k+1} \bmod 2
$$

the right-hand side (after some reduction) may be rewritten as

$$
\frac{(2 n)(2 n-2) \ldots(2 n-2 k+2)}{(2 k)(2 k-2) \ldots 4 \cdot 2} \cdot \frac{(2 n-1)(2 n-3) \ldots(2 n-2 k+1)}{(2 k-1) \ldots 5 \cdot 3 \cdot 1} .
$$

The right-hand factor is $\equiv 1 \bmod 2$; therefore, this is congruent to

$$
\frac{2^{k} \cdot n(n-1) \ldots(n-k+1)}{2^{k} \cdot k(k-1) \ldots 2 \cdot 1}
$$

which is congruent to

$$
\binom{n}{k} .
$$

STERN NUMBERS VERSUS SUMS OF RISING DIAGONALS OF BINOMIAL NUMBERS MOD 2

|  | Column |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Row | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |  | 11 | 12 | 13 | 14 |  |  |  | 18 |  |
| 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | $1=S[1,0]$ |
| 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | $1=S[2,0]$ |
| 2 | 1 | 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | $2=S[2,1]$ |
| 3 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | $1=S[3,0]$ |
| 4 | 1 | 0 | 0 | 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  | $3=S[3,1]$ |
| 5 | 1 | 1 | 0 | 0 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  | $2=S[3,2]$ |
| 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  | $3=S[3,3]$ |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |  | $1=S[4,0]$ |
| 8 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |  |  |  |  |  |  |  |  |  | $4=S[4,1]$ |
| 9 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |  |  |  |  |  |  |  |  |  | $3=S[4,2]$ |
| 10 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |  |  |  |  |  |  |  |  | $5=S[4,3]$ |
| 11 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  | $2=S[4,4]$ |
| 12 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |  |  |  |  |  |  | $5=S[4,5]$ |
| 13 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |  |  |  |  |  | $3=S[4,6]$ |
| 14 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |  |  |  |  | $4=S[4,7]$ |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  |  | $1=S[5,0]$ |
| 16 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |  | $5=S[5,1]$ |
| 17 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |  | $4=S[5,2]$ |
| 18 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | $7=S[5,3]$ |

Many thanks go to Dudley [1] and to Hoggatt for sponsoring the authors to write this series of articles. At this writing, the authors still do not know the general form of

$$
s\left(n,(2 r+1) 2^{m}\right)
$$

and suggest that some ambitious reader show the relationship to the Fibonacci numbers.

## REFERENCES

1. Underwood Dudley. Elementary Number Theory. San Francisco: Freeman \& Co., 1969. P. 163.
2. Christine \& Robert Giuli. "A Primer on Stern's Diatomic Sequence, Part II. The Fibonacci Quarterly 17, No. 3 (1979): 246-248.

# RESTRICTED COMPOSITIONS II 

## L. CARLITZ

Duke University, Durham, NC 27706

## 1. INTRODUCTION

In [1], the writer considered the number of compositions

$$
\begin{equation*}
n=a_{1}+a_{2}+\cdots+a_{k}, \tag{1.1}
\end{equation*}
$$

where the $\alpha_{i}$ are either nonnegative or strictly positive and in addition (1.2) $\quad a_{i} \neq a_{i+1} \quad(i=1,2, \ldots, k-1)$.

In the present paper, we consider the number of compositions (1.1) in nonnegative $a_{j}$ that satisfy
(1.3) $\quad a_{i} \not \equiv a_{i+1}(\bmod m) \quad(i=1,2, \ldots, k-1)$,
where $m$ is a fixed positive integer.
For $n \geq 0, k \geq 1$, let $f_{m}(n, k)$ denote the number of solutions of (1.1) and (1.3) and let

$$
\begin{equation*}
f_{m}(n)=\sum_{k=1}^{\infty} f_{m}(n, k) \tag{1.4}
\end{equation*}
$$

denote the corresponding enumerant when the number of parts in (1.1) is unrestricted. Also, for $0 \leq j<m$, let $f_{m, j}(n, k)$ denote the number of solutions of (1.1) and (1.3) with $\alpha_{1} \equiv j(\bmod m)$.

For $m=2$ explicit results are obtained, in particular,

$$
\begin{equation*}
f_{2, i}(n, k)=\binom{k+s-1}{s} \quad(i=0,1), \tag{1.5}
\end{equation*}
$$

where

$$
\begin{equation*}
s=\frac{1}{2}\left(n-\frac{1}{2}(k+i)\right) \tag{1.6}
\end{equation*}
$$

and $[x]$ is the greatest integer $\leq x$.
For arbitrary $m \geq 1$, we show in particular that

$$
\begin{equation*}
\sum_{n, k=0}^{\infty} f_{m}(n, k) x^{n} y^{k}=\frac{P_{m}(z)}{Q_{m}(z)} \quad\left(z=\frac{y}{1-x^{m}}\right), \tag{1.7}
\end{equation*}
$$

$$
P_{m}(z)=\prod_{j=0}^{m-1}\left(1+x^{j} z\right)
$$

and

$$
Q_{m}(z)=P_{m}(z)-z P_{m}^{\prime}(z) .
$$

For additional results, see Section 4 below.
SECTION 2
In order to evaluate $f_{m}(n, k)$, we define the following functions. Let $f_{m, j}(n, k)$, where $n \geq 0, k \geq 1,0 \leq j<m$, denote the number of solutions in nonnegative integers of

$$
\begin{equation*}
n=a_{1}+a_{2}+\cdots+a_{k}, \tag{2.1}
\end{equation*}
$$

where
(2.2)

$$
a_{i} \not \equiv a_{i+1}(\bmod m) \quad(i=1,2, \ldots, k-1)
$$

and
(2.3)

$$
a_{1} \equiv j(\bmod m)
$$

Also let $f_{m, j}(n, k, \alpha)$ denote the number of solutions of (2.1), (2,2), (2.3), with $a_{1}=a$. Thus $f_{m, j}(n, k, a)=0$ if $a \not \equiv j(\bmod m)$.

It is convenient to extend the above definitions to include the case $k=0$. We put
(2.4)

$$
f_{m}(n, 0)=\delta_{n 0},
$$

where $\delta_{i j}$ is the Kronecker delta:

$$
\delta_{i j}= \begin{cases}1 & (i=j) \\ 0 & (i \neq j)\end{cases}
$$

We also define

$$
\begin{equation*}
f_{m, j}(n, 0)=\delta_{j 0} \delta_{n 0} \tag{2.5}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{m, j}(n, 0, a)=\delta_{j 0} \delta_{n 0} \delta_{a 0}, \tag{2.6}
\end{equation*}
$$

that is, $f_{m, j}(n, 0)=0$ unless $n=j=0$ and $f_{m, j}(n, 0, \alpha)=0$ unless $n=j=\alpha=0$. It follows from the definitions that

$$
\begin{align*}
f_{m}(n, k) & =\sum_{j=0}^{m-1} f_{m, j}(n, k)  \tag{2.7}\\
& =\sum_{j=0}^{m-1} \sum_{a=0}^{n} f_{m, j}(n, k, \alpha) \quad(n \geq 0, k \geq 0) .
\end{align*}
$$

Moreover, we have the recurrence

$$
\begin{aligned}
& f_{m, j}(n, k, a)=\sum_{\substack{i=0 \\
i \neq j}}^{m-1} \sum_{b=0}^{n-a} f_{m, i}(n-a, k-1, b) \\
& \quad[k>0, a \equiv j(\bmod m)],
\end{aligned}
$$

which reduces to

$$
\begin{equation*}
f_{m, j}(n, k, a)=\sum_{\substack{i=0 \\ i \neq j}}^{m-1} f_{m, i}(n-a, k-1) \quad[k>0, a \equiv j(\bmod m)] \tag{2.8}
\end{equation*}
$$

Corresponding to the various enumerants we define a number of generating functions:

$$
\begin{aligned}
F_{m, j}(x, y) & =\sum_{n, k=0}^{\infty} f_{m, j}(n, k) x^{n} y^{k} \\
F_{m}(x, y) & =\sum_{n, k=0}^{\infty} f_{m}(n, k) x^{n} y^{k} \\
F_{m, j}(x, y, a) & =\sum_{n, k=0}^{\infty} f_{m, j}(n, k, a) x^{n} y^{k} .
\end{aligned}
$$

## SECTION 3

We first discuss the case $m=2$. The recurrence (2.8) reduces to

$$
\left\{\begin{align*}
& f_{2,0}(n, k, 2 a)=f_{2,1}(n-2 a, k-1)(k>1)  \tag{3.1}\\
& f_{0}(n, 1,2 a)=\delta_{n, 2 a} \\
& f_{2,1}(n, k, 2 a+1)=f_{2,0}(n-2 a-1, k-1) \quad(k \geq 1)
\end{align*}\right.
$$

Hence,

$$
\left\{\begin{array}{l}
F_{2,0}(x, y, 2 a)=\delta_{a, 0}+x^{2 a} y+x^{2 a} y F_{2,1}(x, y) \\
F_{2,1}(x, y, 2 a+1)=x^{2 a+1} y F_{2,0}(x, y)
\end{array}\right.
$$

Summing over $a$, we get

$$
\left\{\begin{array}{l}
F_{2,0}(x, y)=1+\frac{y}{1-x^{2}}+\frac{y}{1-x^{2}} F_{2,1}(x, y) \\
F_{2,1}(x, y)=\frac{x y}{1-x^{2}} F_{2,0}(x, y)
\end{array}\right.
$$

It follows that

$$
\begin{equation*}
F_{2,0}(x, y)=\frac{1+\frac{y}{1-x^{2}}}{1-\frac{x y^{2}}{\left(1-x^{2}\right)^{2}}}, F_{2,1}(x, y)=\frac{\frac{x y}{1-x^{2}}\left(1+\frac{y}{1-x^{2}}\right)}{1-\frac{x y^{2}}{\left(1-x^{2}\right)^{2}}} \tag{3.2}
\end{equation*}
$$

so that

$$
\begin{equation*}
F_{2}(x, y)=F_{2,0}(x, y)+F_{2,1}(x, y)=\frac{\left(1+\frac{y}{1-x^{2}}\right)\left(1+\frac{x y}{1-x^{2}}\right)}{1-\frac{x y^{2}}{\left(1-x^{2}\right)^{2}}} \tag{3.3}
\end{equation*}
$$

From the first of (3.2), we get

$$
\begin{aligned}
F_{2,0}(x, y)= & \left(1+\frac{y}{1-x^{2}}\right) \sum_{r=0}^{\infty} \frac{x^{r} y^{2 r}}{\left(1-x^{2}\right)^{2 r}} \\
= & \sum_{r=0}^{\infty} x^{r} y^{2 r} \sum_{s=0}^{\infty}\binom{2 r+s-1}{s} x^{2 s} \\
& +\sum_{r=0}^{\infty} x^{r} y^{2 r+1} \sum_{s=0}^{\infty}\binom{2 r+s}{s} x^{2 s}
\end{aligned}
$$

Since

$$
\begin{equation*}
=\sum_{\substack{n=0 \\ r+2 s=n}}^{\infty}(2 r+s-1) x^{n} y^{2 r}+\sum_{\substack{n=0 \\ s+2 s=n}}^{\infty}\binom{2 r+s}{s} x^{n} y^{2 r+1} \tag{3.4}
\end{equation*}
$$

$$
F_{2,0}(x, y)=\sum_{n, k=0}^{\infty} f_{2,0}(n, k) x^{n} y^{k}
$$

it follows from (3.4) that

$$
\begin{equation*}
f_{2,0}(n, k)=\binom{k+s-1}{s} \tag{3.5}
\end{equation*}
$$

where
that is,

$$
s= \begin{cases}\frac{1}{2}\left(n-\frac{1}{2}(k)\right) & (k \text { even }) \\ \frac{1}{2}\left(n-\frac{1}{2}(k-1)\right) & (k \text { odd })\end{cases}
$$

$$
\begin{equation*}
s=\frac{1}{2}\left(n-\left[\frac{1}{2}(k)\right]\right) \tag{3.6}
\end{equation*}
$$

Similarly,

Since

$$
\begin{align*}
F_{2,1}(x, y)= & \sum_{r=0}^{\infty} x^{r+1} y^{2 r+1} \sum_{s=0}^{\infty}\binom{2 r+s}{s} x^{2 s} \\
& +\sum_{r=0}^{\infty} x^{r+1} y^{2 r+2} \sum_{s=0}^{\infty}\binom{2 r+s+1}{s} x^{2 s} \\
= & \sum_{\substack{n=1 \\
r+2 s+1=n}}^{\infty}\binom{2 r+s}{s} x^{n} y^{2 r+1}+\sum_{\substack{n=1 \\
r+2 s+1=n}}^{\infty}\binom{2 r+s+1}{s} x^{n} y^{2 r+2} \tag{3.7}
\end{align*}
$$

$$
F_{2,1}(x, y)=\sum_{n, k=1}^{\infty} f_{2,1}(n, k) x^{n} y^{k}
$$

it follows from (3.7) that

$$
\begin{equation*}
f_{2,1}(n, k)=\binom{k+s-1}{s} \tag{3.8}
\end{equation*}
$$

where

$$
s= \begin{cases}\frac{1}{2}\left(n-\frac{1}{2}(k+1)\right) & (k \text { odd }) \\ \frac{1}{2}\left(n-\frac{1}{2}(k)\right) & (k \text { even })\end{cases}
$$

that is

$$
\begin{equation*}
s=\frac{1}{2}\left(n-\left[\frac{1}{2}(k+1)\right]\right) \tag{3.9}
\end{equation*}
$$

Hence, we can combine (3.5), (3.6), (3.8), (3.9) in the formula

$$
\begin{equation*}
f_{2, i}(n, k)=\binom{k+s-1}{s} \quad(i=0,1) \tag{3.10}
\end{equation*}
$$

where

$$
\begin{equation*}
s=\frac{1}{2}\left(n-\left[\frac{1}{2}(k+i)\right]\right) . \tag{3.11}
\end{equation*}
$$

For $y=1$, (3.4) reduces to
so that

$$
F_{2,0}(x, 1)=\sum_{n=0}^{\infty} x^{n} \sum_{2 s \leq n}\left\{\binom{2 r+s-1}{s}+\binom{2 r+s}{s}\right\},
$$

$f_{2,0}(n)=\sum_{2 s \leq n}\left\{\binom{2 n-3 s-1}{s}+\binom{2 n-3 s}{s}\right\}$.

Similarly, (3.7) yields

$$
F_{2,1}(x, 1)=\sum_{n=1} x^{n} \sum_{r+2 s+1=n}\left\{\binom{2 r+s}{s}+\binom{2 r+s+1}{s}\right\},
$$

which implies

$$
\begin{equation*}
f_{2,1}(n)=\sum_{2 s \leq n-1}\left\{\binom{2 n-3 s-2}{s}+\binom{2 n-3 s-1}{s}\right\} \tag{3.13}
\end{equation*}
$$

We can combine (3.12) and (3.13) in the single formula

$$
\begin{equation*}
f_{2, i}(n)=\sum_{2 s \leq n-i}\left\{\binom{2 n-3 s-i-1}{s}+\binom{2 n-3 s-i}{s}\right\} \quad(i=0,1) \tag{3.14}
\end{equation*}
$$

It follows from (3.14) that

$$
\begin{equation*}
f_{2}(n)=\sum_{2 s \leq n}\left\{\binom{2 n-3 s}{s}+2\binom{2 n-3 s-1}{s}+\binom{2 n-3 s-2}{s}\right\} \tag{3.15}
\end{equation*}
$$

## SECTION 4

For arbitrary $m \geq 1$, we have, by (2.8),

$$
f_{m, j}(n, k, a)=\sum_{\substack{i=0 \\ i \neq j}}^{m-1} f_{m, i}(n-\alpha, k-1) \quad[k>0, a \equiv j(\bmod m)]
$$

together with

$$
\begin{cases}f_{m, 0}(n, 1, \alpha)=\delta_{n, a} & {[a \equiv 0(\bmod m)]} \\ f_{m, 0}(n, 0, \alpha)=\delta_{n 0} \delta_{a 0} . & \end{cases}
$$

It follows that

$$
\left\{\begin{array}{l}
F_{m, 0}(x, y, \alpha)=\delta_{a, 0}+x^{a} y+x^{a} y \sum_{i=1}^{m-1} F_{m, i}(x, y) \quad[\alpha \equiv 0(\bmod m)] \\
F_{m, j}(x, y, \alpha)=x^{a} y \sum_{\substack{i=0 \\
i \neq j}}^{m-1} F_{m, i}(x, y) \quad[1 \leq j<m ; a \equiv j(\bmod m)]
\end{array}\right.
$$

Summing over $a$ we get

$$
\left\{\begin{array}{l}
F_{m, 0}(x, y)=1+\frac{y}{1-x^{m}}+\frac{y}{1-x^{m}} \sum_{i=1}^{m-1} F_{m, i}(x, y)  \tag{4.1}\\
F_{m, j}(x, y)=\frac{x^{j} y}{1-x^{m}} \sum_{\substack{i=0 \\
i \neq j}}^{m-1} F_{m, i}(x, y) \quad(1 \leq j<m) .
\end{array}\right.
$$

$$
\sum_{\substack{i=0 \\ i \neq j}}^{m-1} F_{m, i}(x, y)=F_{m}(x, y)-F_{m, j}(x, y),
$$

(4.1) becomes

$$
\left\{\begin{array}{l}
\left(1+\frac{y}{1-x^{m}}\right) F_{m, 0}(x, y)=1+\frac{y}{1-x^{m}}+\frac{y}{1-x^{m}} F_{m}(x, y) \\
\left(1+\frac{x^{j} y}{1-x^{m}}\right) F_{m, j}(x, y)=\frac{x^{j} y}{1-x^{m}} F_{m}(x, y) \quad(1 \leq j<m) \tag{4.2}
\end{array}\right.
$$

This in turn gives

$$
\left\{\begin{array}{l}
F_{m, 0}(x, y)=1+\frac{\frac{y}{1-x^{m}}}{1+\frac{y}{1-x^{m}}} F_{m}(x, y) \\
F_{m, j}(x, y)=\frac{\frac{x^{j} y}{1-x^{m}}}{1+\frac{x^{j} y}{1-x^{m}}} F_{m}(x, y) \quad(1 \leq j<m) .
\end{array}\right.
$$

Hence, by adding together these equations, we get

$$
\begin{equation*}
\left\{1-\sum_{j=0}^{m-1} \frac{\frac{x^{j} y}{1-x^{m}}}{1+\frac{x^{j} y}{1-x^{m}}}\right\} F_{m}(x, y)=1 \tag{4.3}
\end{equation*}
$$

For brevity, put $z=y /\left(1-x^{m}\right)$, so that (4.3) reduces to

$$
\left\{1-\sum_{j=0}^{m-1} \frac{x^{j} z}{1+x^{j} z}\right\} F_{m}(x, y)=1
$$

Put

$$
\begin{equation*}
P_{m}(z)=P_{m}(z, x)=\prod_{j=0}^{m-1}\left(1+x^{j} z\right) \tag{4.5}
\end{equation*}
$$

It is well-known that

$$
P_{m}(z)=\sum_{j=0}^{m}\left[\begin{array}{c}
m  \tag{4.6}\\
j
\end{array}\right] x^{\frac{1}{2} j(j-1)} z^{j}
$$

$$
\left[\begin{array}{l}
m \\
j
\end{array}\right]=\frac{\left(1-x^{m}\right)\left(1-x^{m-1}\right) \ldots\left(1-x^{m-j+1}\right)}{(1-x)\left(1-x^{2}\right) \ldots\left(1-x^{j}\right)}
$$

Moreover, it follows from (4.5) that

$$
\frac{z P_{m}^{\prime}(z)}{P_{m}(z)}=\sum_{j=0}^{m-1} \frac{x^{j} z}{1+x^{j} z} .
$$

Thus (4.4) becomes
and therefore $\left\{1-\frac{z P_{m}^{\prime}(z)}{P_{m}(z)}\right\} F_{m}(x, y)=1$,

$$
\begin{equation*}
F_{m}(x, y)=\frac{P_{m}(z)}{Q_{m}(z)} \quad\left(z=\frac{y}{1-x^{m}}\right) \tag{4.7}
\end{equation*}
$$

where

$$
Q_{m}(z)=P_{m}(z)-z P_{m}^{\prime}(z)=\sum_{j=0}^{m}(1-j)\left[\begin{array}{c}
m  \tag{4.8}\\
j
\end{array}\right] x^{\frac{1}{2} j(j-1)} z^{j}
$$

For example, for $m=2$, (4.7) gives

$$
\begin{equation*}
F_{2}(x, y)=\frac{(1+z)(1+x z)}{1-x z^{2}} \quad\left(z=\frac{y}{1-x^{2}}\right) \tag{4.9}
\end{equation*}
$$

while, for $m=3$, we get

$$
\begin{equation*}
F_{3}(x, y)=\frac{(1+z)(1+x z)(1+y z)}{1-\left(x+x^{2}+x^{3}\right) z^{2}-2 x^{3} z^{3}} \quad\left(z=\frac{y}{1-x^{3}}\right) . \tag{4.10}
\end{equation*}
$$

## SECTION 5

A few words may be added about the limiting case $m=\infty$. We take $|x|<1$ so that $x^{m} \rightarrow 0$ and

$$
z=\frac{y}{1-x^{m}} \rightarrow y
$$

Thus (4.3) becomes

$$
\begin{equation*}
\left\{1-\sum_{j=0}^{\infty} \frac{x^{j} y}{1+x^{j} y}\right\}\left\{1+\sum_{n, k=1}^{\infty} f_{\infty}(n, k) x^{n} y^{k}\right\}=1 \tag{5.1}
\end{equation*}
$$

On the other hand, the condition

$$
a_{i} \not \equiv a_{i+1}(\bmod m) \quad(i=1,2, \ldots, k-1)
$$

becomes

$$
\begin{equation*}
a_{i} \neq a_{i+1} \quad(i=1,2, \ldots, k-1) \tag{5.2}
\end{equation*}
$$

In the notation of [1], the number of solutions in nonnegative integers of $n=a_{1}+\cdots+a_{k}$ and (5.2) is denoted by $\bar{c}(n, k)$ and it is proved that

$$
\begin{equation*}
1+\sum_{n, k=1}^{\infty} \bar{c}(n, k) x^{n} y^{k}=\left\{1+\sum_{j=1}^{\infty}(-1)^{j} \frac{y^{j}}{1-x^{j}}\right\}^{-1} . \tag{5.3}
\end{equation*}
$$

Clearly,

$$
\begin{equation*}
f_{\infty}(n, k)=\bar{c}(n, k) . \tag{5.4}
\end{equation*}
$$

To verify that (5.1) and (5.3) are equivalent, we take

$$
\begin{aligned}
1-\sum_{j=0}^{\infty} \frac{x^{j} y}{1+x^{j} y} & =1-\sum_{j=0}^{\infty} x^{j} y \sum_{s=0}^{\infty}(-1)^{s} x^{s j} y^{s}=1+\sum_{j=0}^{\infty} \sum_{k=1}^{\infty}(-1)^{k} x^{j k} y^{k} \\
& =1+\sum_{k=1}^{\infty}(-1)^{k} \frac{y^{k}}{1-x^{k}} .
\end{aligned}
$$

REFERENCE
1．L．Carlitz．＂Restricted Compositions．＂The Fibonacci quarterly 14，No． 3 （1976）：254－264．

米米米米

## CHEBYSHEV AND FERMAT POLYNOMIALS FOR DIAGONAL FUNCTIONS

## A．F．HORADAM

University of New England，Armidale，N．S．W．，Australia
INTRODUCTION
Jaiswal［3］and the author［1］examined rising diagonal functions of Chebyshev polynomials of the second and first kinds，respectively．Also，in ［2］，the author investigated rising and descending functions of a wide class of sequences satisfying certain criteria．Excluded from consideration in［2］ were the Chebyshev and Fermat polynomials that did not satisfy the restrict－ ing criteria．

The object of this paper is to complete the above articles by studying descending diagonal functions for the Chebyshev polynomials in Part I，and both rising and descending diagonal functions for the Fermat polynomials in Part II．

Chebyshev polynomials $T_{n}(x)$ of the second kind are defined by

$$
\begin{equation*}
T_{n+2}(x)=2 x T_{n+1}(x)-T_{n}(x) \quad T_{0}(x)=2, T_{1}(x)=2 x \quad(n \geq 0), \tag{1}
\end{equation*}
$$

while Chebyshev polynomials $U_{n}(x)$ of the first kind are defined by

$$
\begin{equation*}
U_{n+2}(x)=2 x U_{n+1}(x)-U_{n}(x) \quad U_{0}(x)=1, U_{1}(x)=2 x \quad(n \geq 0) \tag{2}
\end{equation*}
$$

Often we write $x=\cos \theta$ to obtain trigonometrical sequences．
PART I
DESCENDING DIAGONAL FUNCTIONS FOR $T_{n}(x)$
From（1），we obtain

$$
\left\{\begin{array}{l}
T_{0}(x)=2  \tag{3}\\
T_{1}(x)=2 x \\
T_{2}(x)=4 x^{2}-2 \\
T_{3}(x)=8 x^{3}-6 x \\
T_{4}(x)=16 x^{4}-16 x^{2}+2 \\
T_{5}(x)=32 x^{5}-40 x^{3}+10 x \\
T_{6}(x)=64 x^{6}-96 x^{4}+36 x^{3}-2 \\
T_{7}(x)=128 x^{7}-224 x^{5}+112 x^{3}-14 x \\
T_{8}(x)=256 x^{8}-512 x^{6}+320 x^{4}-64 x^{2}+2 \\
T_{9}(x)=512 x^{9}-1152 x^{7}+864 x^{5}-240 x^{3}+18 x \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots . .
\end{array}\right.
$$

Descending diagonal functions of $x, a_{i}(x) \quad(i=1,2,3, \ldots)$, for $T_{n}(x)$ are, from (3) [taking $\left.a_{0}(x)=0\right]$,

$$
\left\{\begin{array}{l}
a_{1}(x)=2  \tag{4}\\
a_{2}(x)=2 x-2 \\
a_{3}(x)=4 x^{2}-6 x+2 \\
a_{4}(x)=8 x^{3}-16 x^{2}+10 x-2 \\
a_{5}(x)=16 x^{4}-40 x^{3}+36 x^{2}-14 x+2 \\
a_{6}(x)=32 x^{5}-96 x^{4}+112 x^{3}-64 x^{2}+18 x-2 \\
a_{7}(x)=64 x^{6}-224 x^{5}+320 x^{4}-240 x^{3}+100 x^{2}-22 x+2 \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{array}\right.
$$

These yield

$$
\begin{equation*}
a_{n+1}(x)=(2 x-1) a_{n}(x)=(2 x-2)(2 x-1)^{n-1} \quad(n \geq 1) \tag{5}
\end{equation*}
$$

DESCENDING DIAGONAL FUNCTIONS FOR $U_{n}(x)$
From (2), we obtain
(6)

$$
\begin{aligned}
& \left\{\begin{array}{l}
U_{0}(x)=1 \\
U_{1}(x)=2 x \\
U_{2}(x)=4 x^{2}-1 \\
U_{3}(x)=8 x^{3}-4 x \\
U_{4}(x)=16 x^{4}-12 x^{2}+1 \\
U_{5}(x)=32 x^{5}-32 x^{3}+6 x \\
U_{6}(x)=64 x^{6}-80 x^{4}+24 x^{2}-1 \\
U_{7}(x)=128 x^{7}-192 x^{5}+80 x^{3}-8 x \\
U_{8}(x)=256 x^{8}-448 x^{6}+240 x^{4}-40 x^{2}+1
\end{array}\right. \\
& \text {. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . }
\end{aligned}
$$

Descending diagonal functions of $x, b_{i}(x) \quad(i=1,2,3, \ldots)$, for $U_{n}(x)$ are, from (6) [taking $b_{0}(x)=0$,
(7)

$$
\left\{\begin{array}{l}
b_{1}(x)=1 \\
b_{2}(x)=2 x-1 \\
b_{3}(x)=4 x^{2}-4 x+1=(2 x-1)^{2} \\
b_{4}(x)=8 x^{3}-12 x^{2}+6 x-1=(2 x-1)^{3} \\
b_{5}(x)=16 x^{4}-32 x^{3}+24 x^{2}-8 x+1=(2 x-1)^{4} \\
b_{6}(x)=32 x^{5}-80 x^{4}+80 x^{3}-40 x^{2}+10 x-1=(2 x-1)^{5} \\
b_{7}(x)=64 x^{6}-192 x^{5}+240 x^{4}-160 x^{3}+60 x^{2}-12 x+1=(2 x-1)^{6} \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{array}\right.
$$

These yield

$$
\begin{aligned}
b_{n+1}(x)= & (2 x-1) b_{n}(x)=(2 x-1)^{n} \\
& \text { PROPERTIES OF } a_{i}(x), b_{i}(x)
\end{aligned}
$$

Notice that

$$
\begin{equation*}
a_{n}(x)=b_{n}(x)-b_{n-1}(x) \quad(n \geq 2) \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{a_{n}(x)}{a_{n-1}(x)}=\frac{b_{n}(x)}{b_{n-1}(x)}=(2 x-1) \quad(n>2) \tag{10}
\end{equation*}
$$

Write

$$
\begin{align*}
& b \equiv b(x, t)=[1-(2 x-1) t]^{-1}=\sum_{n=1}^{\infty} b_{n}(x) t^{n-1}  \tag{11}\\
& a \equiv a(x, t)=(2 x-2)[1-(2 x-1) t]^{-1}=\sum_{n=2}^{\infty} a_{n}(x) t^{n-2} \tag{12}
\end{align*}
$$

Calculations yield

$$
\begin{equation*}
2 t \frac{\partial b}{\partial t}-(2 x-1) \frac{\partial b}{\partial x}=0 \tag{13}
\end{equation*}
$$

$$
\begin{equation*}
2 t \frac{\partial a}{\partial t}-(2 x-1) \frac{\partial a}{\partial x}+2(2 x-1) b=0 \tag{14}
\end{equation*}
$$

Also

$$
\begin{align*}
& (2 x-1) b_{n}^{\prime}(x)-2(n-1) b_{n}(x)=0  \tag{15}\\
& (2 x-1) a_{n+2}^{\prime}(x)-2(n+1) a_{n+2}(x)-2(2 x-1) b_{n}(x)=0 \tag{16}
\end{align*}
$$

where the prime (dash) represents the first derivative w.r.t. $x$.
Results (9), (10), and (13)-(16) should be compared with corresponding results in [2] for the class of sequences studied there.

## PART II

RISING AND DESCENDING DIAGONAL FUNCTIONS FOR FERMAT POLYNOMIALS
The First Fermat Polynomials $\phi_{n}(x)$; The Second Fermat Polynomials $\theta_{n}(x)$
The sequence $\left\{\phi_{n}\right\}=\{0,1,3,7,15, \ldots\}$ for which
( $17^{\prime}$ ) $\quad \phi_{n+2}=3 \phi_{n+1}-2 \phi_{n} \quad \phi_{0}=0, \phi_{1}=1 \quad(n \geq 0)$
is generalized to the first Fermat polynomial sequence $\left\{\phi_{n}(x)\right\}$ for which

$$
\begin{equation*}
\phi_{n+2}(x)=x \phi_{n+1}(x)-2 \phi_{n}(x) \quad \phi_{0}(x)=0, \phi_{1}(x)=1 \quad(n \geq 0) \tag{17}
\end{equation*}
$$

Similarly, the sequence $\left\{\theta_{n}\right\}=\{2,3,5,9, \ldots\}$ for which
(18') $\quad \theta_{n+2}=3 \theta_{n+1}-2 \theta_{n} \quad \theta_{0}=2, \theta_{1}=3 \quad(n \geq 0)$
is generalized to the second Fermat polynomial sequence $\left\{\theta_{n}(x)\right\}$ for which

$$
\begin{equation*}
\theta_{n+2}(x)=x \theta_{n+1}(x)-2 \theta_{n}(x) \quad \theta_{0}(x)=2, \theta_{1}(x)=x \quad(n \geq 0) \tag{18}
\end{equation*}
$$

Terms of these sequences are as follows:


RISING AND DESCENDING DIAGONAL FUNCTIONS FOR $\phi_{n}(x), \theta_{n}(x)$
Label the rising and descending diagonal functions

$$
R_{i}(x), D_{i}(x) \text { for }\left\{\phi_{n}(x)\right\}
$$

and

$$
R_{i}^{\prime}(x), D_{i}^{\prime}(x) \text { for }\left\{\theta_{n}(x)\right\}
$$

Of course, in this context the primes do not represent derivatives. Reading from the listed information in (19) and (20),

$$
\text { if } D_{1}(x)=1, D_{1}^{\prime}(x)=2,
$$

we have,

$$
\begin{align*}
& D_{n}(x)=(x-2)^{n-1}  \tag{21}\\
& D_{n}^{\prime}(x)=(x-4)(x-2)^{n-2} \quad(n \geq 2) \tag{22}
\end{align*}
$$

whence

$$
\begin{cases}\frac{D_{n+1}(x)}{D_{n}(x)}=\frac{D_{n+1}^{\prime}(x)}{D_{n}^{\prime}(x)}=x-2 & (n \geq 2) \\ \frac{D_{n}^{\prime}(x)}{D_{n}(x)}=\frac{x-4}{x-2} & (n \geq 2 ; x \neq 2) \\ \frac{D_{n+1}^{\prime}(x)}{D_{n}(x)}=x-4 & \end{cases}
$$

A1so
(24)

$$
D_{n}(x)-2 D_{n-1}(x)=D_{n}^{\prime}(x) .
$$

Rising diagonal functions may be tabulated thus:
(25)

| $i=$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\ldots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $R_{i}(x)$ | 1 | $x$ | $x^{2}$ | $x^{3}-2$ | $x^{4}-4 x$ | $x^{5}-6 x^{2}$ | $x^{6}-8 x^{3}+4$ | $x^{7}-10 x^{4}+12 x$ | $\ldots$ |
| $R_{i}^{\prime}(x)$ | 2 | $x$ | $x^{2}$ | $x^{3}-4$ | $x^{4}-6 x$ | $x^{5}-8 x^{2}$ | $x^{6}-10 x^{3}+8$ | $x^{7}-12 x^{4}+20 x$ | $\ldots$ |

with the properties ( $n>3$ ),
(27)

$$
\left\{\begin{array}{l}
R_{n}^{\prime}(x)=R_{n}(x)-2 R_{n-3}(x) \\
R_{n}(x)=x R_{n-1}(x)-2 R_{n-3}(x) \\
R_{n}^{\prime}(x)=x R_{n-1}^{\prime}(x)-2 R_{n-3}^{\prime}(x) .
\end{array}\right.
$$

Calculations of results similar to those in (13)-(16) follow as a matter of course for both rising and descending diagonal functions, but these are left for the curious reader. (A comparison with corresponding results in [2] is desirable.)

However, it is worthwhile to record the generating functions for the diagonal functions associated with the two Fermat sequences. These are, for $D_{i}(x), D_{i}^{\prime}(x), R_{i}(x), R_{i}^{\prime}(x)$, respectively:

$$
\begin{align*}
& \sum_{n=1}^{\infty} D_{n}(x) t^{n-1}=[1-(x-2) t]^{-1} ;  \tag{28}\\
& \sum_{n=2}^{\infty} D_{n}^{\prime}(x) t^{n-2}=(x-4)[1-(x-2) t]^{-1} ;  \tag{29}\\
& \sum_{n=1}^{\infty} R_{n}(x) t^{n-1}=\left[1-\left(x t-2 t^{3}\right)\right]^{-1} ; \\
& \sum_{n=2}^{\infty} R_{n}^{\prime}(x) t^{n-1}=\left(1-2 t^{3}\right)\left[1-\left(x t-2 t^{3}\right)\right]^{-1}
\end{align*}
$$

It is expected that the results of [1], [2], and [3] will be generalized in a subsequent paper.

## REFERENCES

1．A．F．Horadam．＂Polynomials Associated with Chebyshev Polynomials of the First Kind．＂The Fibonacci Quarterly 15，No．3（1977）：255－257．
2．A．F．Horadam．＂Diagonal Functions．＂The Fibonacci Quarterly 16，No． 1 （1978）：33－36．
3．D．V．Jaiswal．＂On Polynomials Related to Tchebichef Polynomials of the Second Kind．＂The Fibonacci quarterly 12，No．3（1974）：263－265．
＊がが＊

## ON EULER＇S SOLUTION OF A PROBLEM OF DIOPHANTUS <br> JOSEPH ARKIN <br> 197 Old Nyack Turnpike，Spring Valley，NY 10977 <br> V．E．HOGGATT，JR． <br> San Jose State University，San Jose，CA 95192 <br> and <br> E．G．STRAUS＊ <br> University of California，Los Angeles，CA 90024

1．The four numbers $1,3,8,120$ have the property that the product of any two of them is one less than a square．This fact was apparently discovered by Fermat．As one of the first applications of Baker＇s method in Diophantine approximations，Baker and Davenport［2］showed that there is no fifth posi－ tive integer $n$ ，so that

$$
n+1,3 n+1,8 n+1, \text { and } 120 n+1
$$

are all squares．It is not known how large a set of positive integers $\left\{x_{1}\right.$ ， $\left.x_{2}, \ldots, x_{n}\right\}$ can be found so that all $x_{i} x_{j}+1$ are squares for all $1 \leq i<j$ $\leq n$ ．

A solution attributed to Euler［1］shows that for every triple of inte－ gers $x_{1}, x_{2}, y$ for which $x_{1} x_{2}+1=y^{2}$ it is possible to find two further in－ tegers $x_{3}, x_{4}$ expressed as polynomials in $x_{1}, x_{2}, y$ and a rational number $x_{5}$ ， expressed as a rational function in $x_{1}, x_{2}, y$ ；so that $x_{i} x_{j}+1$ is the square of a rational expression $x_{1}, x_{2}$ ，$y$ for all $1 \leq i<j \leq 5$ ．

In this note we analyze Euler＇s solution from a more abstract algebraic point of view．That is，we start from a field $k$ of characteristic $\neq 2$ and ad－ join independent transcendentals $x_{1}, x_{2}, \ldots, x_{m}$ ．We then set $x_{i} x_{j}+1=y_{i j}^{2}$ and pose two problems：

I．Find nonzero elements $x_{1}, x_{2}, \ldots, x_{m}, x_{m+1}, \ldots, x_{n}$ in the ring
$R=k\left[x_{1}, \ldots, x_{m} ; y_{12}, \ldots, y_{m-1, m}\right]$ so that $x_{i} x_{j}+1=y_{i j}^{2}$ ；and
$y_{i j} \varepsilon R$ for $1 \leq i<j \leq n$ ．
II．Find nonzero elements $x_{1}, x_{2}, \ldots, x_{m}, x_{m+1}, \ldots, x_{n}$ in the field
$K=k\left(x_{1}, \ldots, x_{m} ; y_{12}, \ldots, y_{m-1, m}\right)$ so that $x_{i} x_{j}+1=y_{i j}^{2}$ ；and
$y_{i j} \in K$ for all $1 \leq i<j \leq n$ ．
In Section 2 we give a complete solution to Problem I for $m=2, n=3$ ．
In Section 3 we give solutions for $m=2$ ，$n=4$ which include both Euler＇s

[^1]solution and a solution for $m=3, n=4$ which generalize the solutions mentioned above.

In Section 4 we present a solution for $m=2$ or 3 , $n=5$ of Problem II, which again contains Euler's solution as a special case. Finally, in Section 5 we apply the results of Section 4 to Problem II for $m=2, n=3$.

The case char $k=2$ leads to trivial solutions, $x=x_{1}=x_{2}=\ldots=x_{n}$, $y_{i j}=x+1$.

Many of the ideas in this paper arose from conversations between Straus and John H. E. Cohn.

$$
\begin{align*}
& \text { 2. Solutions for } x_{1} x_{3}+1=y_{13}^{2}, x_{2} x_{3}+1=y_{23}^{2} \text { with } \\
& \qquad x_{3} y_{13}, y_{23} \in R=k\left[x_{1}, x_{2}, \sqrt{x_{1} x_{2}+1}\right] . \\
& \text { We set } \sqrt{x_{1} x_{2}+1}=y_{12} \text { and note that the simultaneous equations } \\
& x_{1} x_{3}+1=y_{13}^{2} \\
& \text { (1) } \quad x_{2} x_{3}+1=y_{23}^{2} \tag{1}
\end{align*}
$$

lead to a Pell's equation

$$
\begin{equation*}
x_{1} y_{23}^{2}-x_{2} y_{13}^{2}=x_{1}-x_{2} \tag{2}
\end{equation*}
$$

In $R\left[\sqrt{x_{1}}, \sqrt{x_{2}}\right]$ we have the fundamental unit $y_{12}+\sqrt{x_{1} x_{2}}$ which, together with the trivial solution $y_{13}=y_{23}=1$ of (2), leads to the infinite class of solutions of (2) which we can express as follows:
(3) $y_{23} \sqrt{x_{1}}+y_{13} \sqrt{x_{2}}= \pm\left(\sqrt{x_{1}} \pm \sqrt{x_{2}}\right)\left(y_{12}+\sqrt{x_{1} x_{2}}\right)^{n} ; n=0, \pm 1, \pm 2, \ldots$.

In other words,

$$
\begin{aligned}
& \pm y_{23}(n)=\frac{1}{2 \sqrt{x_{1}}}\left[\left(\sqrt{x_{1}} \pm \sqrt{x_{2}}\right)\left(y_{12}+\sqrt{x_{1} x_{2}}\right)^{n}+\left(\sqrt{x_{1}} \mp \sqrt{x_{2}}\right)\left(y_{12}-\sqrt{x_{1} x_{2}}\right)^{n}\right] \\
& \pm y_{13}(n)=\frac{1}{2 \sqrt{x_{2}}}\left[\left(\sqrt{x_{1}} \pm \sqrt{x_{2}}\right)\left(y_{12}+\sqrt{x_{1} x_{2}}\right)^{n}-\left(\sqrt{x_{1}} \mp \sqrt{x_{2}}\right)\left(y_{12}-\sqrt{x_{1} x_{2}}\right)^{n}\right]
\end{aligned}
$$

Once $y_{13}, y_{23}$ are determined, then $x_{3}$ is determined by (1).
The cases $n=1,2$ give Euler's solutions:

$$
\begin{aligned}
y_{13}(1) & =x_{1}+y_{12}, y_{23}(1)=x_{2}+y_{12}, x_{3}(1)=x_{1}+x_{2}+2 y_{12} ; \\
y_{13}(2) & =1+2 x_{1} x_{2}+2 x_{1} y_{12}, y_{23}(2)=1+2 x_{1} x_{2}+2 x_{2} y_{12} ; \\
x_{3}(2) & =4 y_{12}\left[1+2 x_{1} x_{2}+\left(x_{1}+x_{2}\right) y_{12}\right] .
\end{aligned}
$$

The interesting fact is that
$x_{3}(1) x_{3}(2)+1=\left[3+4 x_{1} x_{2}+2\left(x_{1}+x_{2}\right) y_{12}\right]^{2}$;
and in general

$$
x_{3}(n) x_{3}(n+1)+1=\left[x_{3}(n) y_{12}+y_{13}(n) y_{23}(n)\right]^{2} .
$$

The main theorem of this section is the following (see [3] for a similar result).
Theorem 1: The general solution of (1) and (2) in $R$ is given by (3).
We first need two lemmas.
Lemma 1: If $y_{13}, y_{23} \varepsilon R$ are solutions of (2), then, for a proper choice of the sign of $y_{23}$, we have

$$
\eta=\frac{\sqrt{x_{2} y_{13}}-\sqrt{x_{1}} y_{23}}{\sqrt{x_{2}}-\sqrt{x_{1}}} \varepsilon R\left[\sqrt{x_{1} x_{2}}\right],
$$

where $\eta$ is a unit of $R\left[\sqrt{x_{1} x_{2}}\right]$.
Proof: Write $y_{13}=A+B y_{12}, y_{23}=C+D y_{12}$, where $A, B, C, D \varepsilon k\left[x_{1}, x_{2}\right]$. Then equation (2) yields

$$
\begin{equation*}
x_{2}-x_{1}=x_{2}\left(A+B y_{12}\right)^{2}-x_{1}\left(C+D y_{12}\right)^{2} \tag{4}
\end{equation*}
$$

Under the homomorphism of $R$ which maps $x_{1} \rightarrow x, x_{2} \rightarrow x$, we get

$$
y_{12} \rightarrow \sqrt{x^{2}}+1 A\left(x_{1}, x_{2}\right) \rightarrow A(x, x)=A(x), \text { etc. }
$$

and (4) becomes

$$
\begin{equation*}
0=x\left[(A+C)+(B+D) y_{12}\right]\left[(A-C)+(B-D) y_{12}\right] \tag{5}
\end{equation*}
$$

Thus, one of the factors on the right vanishes and by proper choice of sign, we may assume $A(x)=C(x), B(x)=D(x)$, which is the same as saying that

$$
\frac{A\left(x_{1}, x_{2}\right)-C\left(x_{1}, x_{2}\right)}{x_{2}-x_{1}}=P, \quad \frac{B\left(x_{1}, x_{2}\right)-D\left(x_{1}, x_{2}\right)}{x_{2}-x_{1}}=Q
$$

with $P, Q \in \mathbb{K}\left[x_{1}, x_{2}\right]$. Thus,

$$
\begin{aligned}
\eta & =\frac{\sqrt{x_{2} y_{13}}-\sqrt{x_{1} y_{23}}}{\sqrt{x_{2}}-\sqrt{x_{1}}}=y_{13}+\sqrt{x_{1}}\left(\sqrt{x_{2}}+\sqrt{x_{1}}\right)\left(P+Q y_{12}\right) \\
& =y_{13}+\left(x_{1}+\sqrt{x_{1} x_{2}}\right)\left(P+Q y_{12}\right) \varepsilon R\left[\sqrt{x_{1} x_{2}}\right]
\end{aligned}
$$

and, if we set

$$
\bar{n}=\frac{\sqrt{x_{2} y_{13}}+\sqrt{x_{1} y_{23}}}{\sqrt{x_{2}}+\sqrt{x_{1}}}=y_{13}+\left(x_{1}-\sqrt{x_{1} x_{2}}\right)\left(P+Q y_{12}\right)
$$

we get $\eta \bar{\eta}=1$.
Lemma 2: A11 units $\eta$ of $R\left[\sqrt{x_{1} x_{2}}\right]$ are of the form

$$
\eta=\kappa\left(y_{12}+\sqrt{x_{1} x_{2}}\right)^{n} ; \kappa \varepsilon k^{*} ; n=0, \pm 1, \ldots .
$$

Proof: Write $x_{1} x_{2}=s, x_{1}=x, x_{2}=s / x, t=\sqrt{s+1}$. Then,

$$
R=k[x, s / x, \sqrt{s+1}] \subset k[x, 1 / x, t]=R^{*}
$$

We now consider the units, $\eta^{*}$, of $R^{*}[\sqrt{s}]$ and show that they are of the form:

$$
\begin{equation*}
\eta^{*}=k x\left(t+\sqrt{t^{2}-1}\right)^{n}, \kappa \varepsilon k^{*} ; m, n \varepsilon Z . \tag{6}
\end{equation*}
$$

Write $\eta^{*}=A+B \sqrt{t^{2}-1}$, where $A$ and $B$ are polynomials in $t$ with coefficients in $k[x, 1 / x]$ and proceed by induction on $\operatorname{deg} A$ as a polynomial in $t$.

If $\operatorname{deg} A=0$, then $B=0$ and $A$ is a unit of $k[x, 1 / x]$, that is, $\eta=k x^{m}$, $\kappa \varepsilon k^{*}, m \in \mathbb{Z}$.

Now assume the lemma true for $\operatorname{deg} A<n$ and write

$$
A=a_{n} t^{n}+a_{n-1} t^{n-1}+\cdots, B=b_{n-1} t^{n-1}+b_{n-2} t^{n-2}+\cdots
$$

Since $\eta^{*}$ is a unit, we get that

$$
\eta^{*} \eta^{-*}=A^{2}-\left(t^{2}-1\right) B^{2}
$$

is a unit of $k[x, 1 / x]$. So, comparing coefficients of $t^{2 n}$ and $t^{2 n-1}$, we get:
or

$$
a_{n}^{2}=b_{n-1}^{2}, a_{n} a_{n-1}=b_{n-1} b_{n-2}
$$

Thus,

$$
a_{n}= \pm b_{n-1}, a_{n-1}= \pm b_{n-2}
$$

$$
\begin{aligned}
n^{* *} & =n^{*}\left(t \mp \sqrt{t^{2}-1}\right)=\left[t A \mp\left(t^{2}\right) B\right]+(t B \pm A) \sqrt{t^{2}=1} \\
& =A_{1}+B_{1} \sqrt{t^{2}-1},
\end{aligned}
$$

where $A_{1}=a_{n} t^{n+1}+a_{n-1} t^{n}+\cdots \mp\left(t^{2}-1\right)\left(a_{n} t^{n-1}+a_{n-1} t^{n-2} \ldots\right)$, so that $\operatorname{deg} A_{1}<n$ and $\eta^{* *}$ is of the form (6) by the induction hypothesis. Therefore $\eta^{*}=n^{* *}\left(t \pm \sqrt{t^{2}-1}\right)$ is also of the form (6).

Now $n^{*}$ is a unit of $R\left[\sqrt{t^{2}-1}\right]$ if and only if $k x^{m}$ is a unit of $R$; that is, if and only if $m=0$.

Theorem 1 now follows directly from Lemmas 1 and 2 if we write

$$
\sqrt{x_{2} y_{13}}+\sqrt{x_{1} y_{23}}=\kappa\left(\sqrt{x_{2}} \pm \sqrt{x_{1}}\right)\left(y_{12}+\sqrt{x_{1} x_{2}}\right)^{n}
$$

and get

$$
x_{2} y_{13}^{2}-x_{1} y_{23}^{2}=\kappa^{2}=1
$$

so that $k= \pm 1$.
Note that Theorem 1 does not show that, for any two integers $x_{1}, x_{2}$ for which $x_{1} x_{2}+1$ is a square, all integers $x_{3}$ for which $x_{i} x_{3}+1$ are squares; $i=1,2$; are of the given forms. But these forms are the only ones that can be expressed as polynomials in $x_{1}, x_{2}, \sqrt{x_{1} x_{2}+1}$ and work for all such triples.

As mentioned above, we have the recursion relations

$$
\begin{aligned}
y_{13}(n+1) & =x_{1} y_{23}(n)+y_{12} y_{13}(n), \\
y_{23}(n+1) & =x_{2} y_{13}(n)+y_{12} y_{23}(n), \\
x_{3}(n+1) & =x_{1}+x_{2}+x_{3}(n)+2 x_{1} x_{2} x_{3}(n)+2 y_{12} y_{13}(n) y_{23}(n),
\end{aligned}
$$

and therefore

$$
\begin{equation*}
x_{3}(n) x_{3}(n+1)+1=\left[y_{12} x_{3}(n)+y_{13}(n) y_{23}(n)\right]^{2}, \tag{7}
\end{equation*}
$$

so that the quadruple $x_{1}, x_{2}, x_{3}(n)=x_{3}, x_{3}(n+1)+x_{4}$ has the property that $x_{i} x_{j}+1$ is a square for $1 \leq i<j \leq 4$.

From [3, Theorem 3], we get the following.
Theorem 2: $\quad x_{3}(m) x_{3}(n)+1$ is a square in $R$ if any only if $|m-n|=1$.
Note that while the proof in [3] is restricted to a more limited class of solutions, the solutions there are obtained by specialization from the solutions presented here.
3. Solutions for $x_{i} x_{4}+1=y_{i_{4}}^{2}$; $i=1,2,3$ with $x_{4}, y_{i 4} \varepsilon R=k\left[x_{1}, x_{2}, x_{3}\right.$, $\left.y_{12}, y_{13}, y_{23}\right]$ where $y_{i j}=\sqrt{x_{i} x_{j}+1} ; 1 \leq i<j \leq 3$.

The solution (7) using $x_{3}=x_{3}(n), x_{4}=x_{4}(n)$ as polynomials in $x_{1}, x_{2}, y_{12}$ can be generalized as follows.
Theorem 3: For $x_{4}=x_{1}+x_{2}+x_{3}+2 x_{1} x_{2} x_{3}+2 y_{12} y_{13} y_{23}$, we have

$$
x_{i} x_{4}+1=y_{i 4}^{2}, y_{i 4}=x_{i} y_{j k}+y_{i j} y_{i k} ;\{i, j, k\}=\{1,2,3\}
$$

Proof: We have

$$
\begin{aligned}
y_{i 4}^{2}-1 & =-1+x_{i}^{2}\left(x_{j} x_{k}+1\right)+\left(x_{i} x_{j}+1\right)\left(x_{i} x_{k}+1\right)+2 x_{i} y_{12} y_{13} y_{23} \\
& =x_{i}\left(x_{1}+x_{2}+x_{3}+2 x_{1} x_{2} x_{3}+2 y_{12} y_{13} y_{23}\right) \\
& =x_{i} x_{4} .
\end{aligned}
$$

Note that since the choice of the sign of $y_{i j}$ is arbitrary, we always get two conjugate solutions for $x_{4} \varepsilon R$. This corresponds to the choices

$$
x_{4}=x_{3}(n \pm 1)
$$

in the previous section.
Theorem 4: The values $x_{4}$ in Theorem 3 are the only nonzero elements of $R$ with $\overline{x_{i} x_{4}+1}$ squares in $R$ for $i=1,2,3$.
Proo6: Let $x_{4}=P\left(x_{1}, x_{2}, x_{3}, y_{12}, y_{13}, y_{23}\right) \varepsilon R$ where, in order to normalize the expression we assume that $P$ is linear in the $y_{i j}$ and $P \neq 0$. By Theorem 2, we have

$$
P\left[x_{1}, x_{2}, x_{3}(n), y_{12}, y_{13}(n), y_{23}(n)\right]=x_{3}(n+1)
$$

for each $n=0, \pm 1, \pm 2, \ldots$. Without loss of generality we may assume that $P=x_{3}(n+1)$ for infinitely many choices of $n$. Then the algebraic function of $x_{3}$

$$
P\left(x_{1}, x_{2}, x_{3}, y_{12}, y_{13}, y_{23}\right)-x_{1}-x_{2}-x_{3}-2 x_{1} x_{2} x_{3}-2 y_{12} y_{13} y_{23}
$$

has infinitely many zeros $x_{3}=x_{3}(n)$ and hence is identically 0 .
The values $x_{4}$ in Theorem 3 can be characterized in the following symmetric way.
Lemma 3: Let $\sigma_{i} ; i=1,2,3,4$ be the elementary symmetric functions of $x_{1}$, $\overline{x_{2}, x_{3},} x_{4}$. Then $x_{4}$ is the value given by Theorem 3 if and only, if

$$
\begin{equation*}
\sigma_{1}^{2}=4\left(\sigma_{2}+\sigma_{4}+1\right) \tag{8}
\end{equation*}
$$

Proob: If we write $\Sigma_{1}, \Sigma_{2}, \Sigma_{3}$ for the elementary symmetric functions of $x_{1}$, $\bar{x}_{2}, x_{3}$, then $x_{4}=\Sigma_{1}+2 \Sigma_{3}+2 Y$ where

$$
Y=y_{12} y_{13} y_{23}=\sqrt{\Sigma_{3}^{2}+\Sigma_{1} \Sigma_{3}+\Sigma_{2}+1}
$$

Hence

$$
\begin{aligned}
& \sigma_{1}=2\left(\Sigma_{1}+\Sigma_{3}+Y\right) \\
& \sigma_{2}=\Sigma_{2}+x_{4} \Sigma_{1}=\Sigma_{2}+\Sigma_{1}^{2}+2 \Sigma_{1} \Sigma_{3}+2 \Sigma_{1} Y \\
& \sigma_{4}=x_{4} \Sigma_{3}=\Sigma_{1} \Sigma_{3}+2 \Sigma_{3}^{2}+2 \Sigma_{3} Y .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sigma_{1}^{2} & =4\left[\Sigma_{1}^{2}+2 \Sigma_{1} \Sigma_{3}+\Sigma_{3}^{2}+2 \Sigma_{1} y+2 \Sigma_{3} y+Y^{2}\right] \\
& =4\left[\sigma_{2}+\sigma_{4}-\Sigma_{2}-\Sigma_{1} \Sigma_{3}-\Sigma_{3}^{2}+\left(x_{1} x_{2}+1\right)\left(x_{1} x_{3}+1\right)\left(x_{2} x_{3}+1\right)\right] \\
& =4\left(\sigma_{2}+\sigma_{4}+1\right) .
\end{aligned}
$$

Conversely, if we solve the quadratic equation (8) for $x_{4}$, we get the two values in Theorem 3.
4. Solutions for $x_{i} x_{5}=y_{i 5}^{2} ; i=1,2,3,4$ with $x_{4}, y_{i 5} \varepsilon K=k\left(x_{1}, x_{2}, x_{3}\right.$, $y_{12}, y_{13}, y_{23}$ ) where $x_{4}$ is given by Theorem 3.

If we use the $x_{4}$ of the previous section and define

$$
\begin{equation*}
x_{5}=\frac{4 \sigma_{3}+2 \sigma_{1}+2 \sigma_{1} \sigma_{4}}{\left(\sigma_{4}-1\right)^{2}} \tag{10}
\end{equation*}
$$

we get the following.
Theorem 5: We have

$$
x_{i} x_{5}+1=\left(\frac{2 x_{i}^{2}-\sigma_{1} x_{i}-\sigma_{4}-1}{\sigma_{4}-1}\right)^{2} ; i=1,2,3,4 .
$$

Proo 6: The $x_{i}$ are the roots of the equation
(11)

$$
x_{i}^{4}-\sigma_{1} x_{i}^{3}+\sigma_{2} x_{i}^{2}-\sigma_{3} x_{i}+\sigma_{4}=0
$$

Hence

$$
\begin{equation*}
\left(\sigma_{4}-1\right)^{2}\left(x_{i} x_{5}+1\right)=4 \sigma_{3} x_{i}+2 \sigma_{1} x_{i}+2 \sigma_{1} \sigma_{4} x_{i}+\left(\sigma_{4}-1\right)^{2} \tag{12}
\end{equation*}
$$

If we substitute $4 \sigma_{3} x_{i}=4\left(x_{i}^{4}-\sigma_{1} x_{i}^{3}+\sigma_{2} x_{i}^{2}+\sigma_{4}\right)$ from (11), we get

$$
\begin{align*}
\left(\sigma_{4}-1\right)^{2}\left(x_{i} x_{5}+1\right) & =4 x_{i}^{4}-4 \sigma_{1} x_{i}^{3}+4 \sigma_{2} x_{i}^{2}+2 \sigma_{1}\left(\sigma_{4}+1\right) x_{i}+\left(\sigma_{4}+1\right)^{2}  \tag{13}\\
& =\left(2 x_{i}^{2}-\sigma_{1} x_{i}-\sigma_{4}-1\right)^{2}-\left(\sigma_{1}^{2}-4 \sigma_{4}-4-4 \sigma_{2}\right) x_{i}^{2} \\
& =\left(2 x_{i}^{2}-\sigma_{1} x_{i}-\sigma_{4}-1\right)^{2}
\end{align*}
$$

since the last bracket vanishes by Lemma 3 .
Thus, the famous quadruple $1,3,8,120$ can be augmented by

$$
x_{5}=\frac{777480}{2879^{2}}
$$

We conjecture that the quintuple given by Theorem 5 is the only pair of quintuples in which $x_{4}$ is a polynomial in $x_{1}, x_{2}, x_{3} ; y_{12}, y_{13}, y_{23}$ and $x_{5}$ is rational in these quantities.

Finally, we show that the value $x_{5}$ given by Theorem 5 is never an integer when $x_{1}, x_{2}, x_{3}, y_{12}, y_{13}, y_{23}$ and, hence, $x_{4}$ and $y_{14}, y_{24}, y_{34}$ are positive integers.
Theorem 6: If the quantities $x_{1}, x_{2}, x_{3}, y_{12}, y_{13}, y_{23}$ in Theorem 5 are positive integers, then $0<x_{5}<1$.
Proof: Since we have already verified the theorem for the case $x_{1}=1, x_{2}=$ $\overline{3, x_{3}}=8$, we may assume that

$$
\frac{\Sigma_{1}}{\Sigma_{3}}=\frac{1}{x_{1} x_{2}}+\frac{1}{x_{1} x_{3}}+\frac{1}{x_{2} x_{3}}<\frac{1}{3}+\frac{1}{8}+\frac{1}{24}=\frac{1}{2}
$$

and the smallest $\Sigma_{1}$ is obtained for the triple 2, 4, 12. Thus,

$$
\begin{equation*}
18 \leq \Sigma_{1}<\frac{1}{2} \Sigma_{3} . \tag{14}
\end{equation*}
$$

Similarly

$$
\frac{\Sigma_{2}}{\Sigma_{3}}<1+\frac{1}{3}+\frac{1}{8}<\frac{3}{2}
$$

and

$$
\begin{equation*}
80 \leq \Sigma_{2}<\frac{3}{2} \Sigma_{3} . \tag{15}
\end{equation*}
$$

Next, $Y=y_{12} y_{13} y_{23}$ satisfies $Y=\sqrt{\Sigma_{3}^{2}+\Sigma_{1} \Sigma_{3}+\Sigma_{2}+1}$, so that from (14) and (15) we get

$$
\begin{equation*}
\Sigma_{3}+9 \leq Y<\frac{3}{2}\left(\Sigma_{3}+1\right) \tag{16}
\end{equation*}
$$

Thus, the numerator of $1-x_{5}$ is

$$
\begin{align*}
\left(\sigma_{4}-1\right)^{2}-2 \sigma_{1} \sigma_{4}-4 \sigma_{3}-2 \sigma_{1} & =\left(\sigma_{4}-\sigma_{1}-1\right)^{2}-\sigma_{1}^{2}-4 \sigma_{3}-4 \sigma_{1}  \tag{17}\\
& =\left(\sigma_{4}-\sigma_{1}-1\right)^{2}-4\left(\sigma_{4}+\sigma_{2}+1\right) \\
& -4 \sigma_{3}-4 \sigma_{1} \\
& =\left(\sigma_{4}-\sigma_{1}-3\right)^{2}-4 \sigma_{3}-4 \sigma_{2}-8 \sigma_{1}-8
\end{align*}
$$

$$
\begin{aligned}
=\left(2 \Sigma_{3}^{2}\right. & \left.+2 \Sigma_{3} Y+\Sigma_{1} \Sigma_{3}-2 \Sigma_{3}-2 Y-2 \Sigma_{1}-3\right)^{2}-8 \Sigma_{2} \Sigma_{3}-8 \Sigma_{2} Y \\
& -4 \Sigma_{1} \Sigma_{2}-4 \Sigma_{3}-8 \Sigma_{1} \Sigma_{3}-8 \Sigma_{1} Y-4 \Sigma_{1}^{2}-4 \Sigma_{2}-16 \Sigma_{3}-16 Y \\
& -16 \Sigma_{1}-8 \\
>\left(4 \Sigma_{3}^{2}\right. & \left.+30 \Sigma_{3}-6\right)^{2}-12 \Sigma_{3}^{2}-18 \Sigma_{3}\left(\Sigma_{3}+1\right)-3 \Sigma_{3}^{2}-4 \Sigma_{3}-4 \Sigma_{3}^{2} \\
& -6 \Sigma_{3}\left(\Sigma_{3}+1\right)-\Sigma_{3}^{2}-6 \Sigma_{3}-16 \Sigma_{3}-24\left(\Sigma_{3}+1\right)-8 \Sigma_{3}-8 \\
=\left(4 \Sigma_{3}^{2}\right. & \left.+30 \Sigma_{3}-6\right)^{2}-44 \Sigma_{3}^{2}-82 \Sigma_{3}-32 \\
>\left(4 \Sigma_{3}^{2}\right. & \left.+30 \Sigma_{3}-12\right)^{2}>0 .
\end{aligned}
$$

Thus，our algebraic method has the result that for every three positive integers $x_{1}, x_{2}, x_{3}$ so that $x_{i} x_{j}+1$ is a square for $1 \leq i<j \leq 3$ there always exists a fourth positive integer（and usually two distinct fourth integers） $x_{4}$ so that $x_{i} x_{4}+1$ ；$i=1,2$ ， 3 ，is a square．Finally，there always exists a fifth rational number，$x_{5}$ ，always a proper fraction，so that $x_{i} x_{5}+1$ ；$i=$ 1，2，3， 4 is a square．

The question of finding more than four positive integers remains open．
5．Solutions of $x_{i} x_{3}^{\prime}+1=y_{i 3}^{\prime 2} ; i=1,2$ with $x_{3}^{\prime}, y_{i 3}^{\prime} \in K=k\left(x_{1}, x_{2}, y_{12}\right)$ ．The field $K=k\left(x_{1}, x_{2}, y_{12}\right)$ is，of course，the pure transcendental extension $k\left(x_{1}\right.$ ， $y_{12}$ ）．Sections 4 and 5 show that $K$ contains many solutions $x_{3}^{\prime}, y_{i 3}^{\prime}$ of equa－ tion（1）that are not in $R=k\left[x_{1}, x_{2}, y_{12}\right]$ and，therefore，are not given in Theorem 1.

For example，we may define a quadruple $x_{1}, x_{2}, x_{3}=x_{3}(n), x_{4}=x_{3}(n+1)$ which satisfies Theorem 3 and then define

$$
x_{3}^{\prime}(n)=x_{5}=\frac{1}{\left(\sigma_{4}-1\right)^{2}}\left[2 \sigma_{1}+4 \sigma_{3}+2 \sigma_{1} \sigma_{4}\right]
$$

as in（10）to get an infinite sequence of triples $x_{1}, x_{2}, x_{3}^{\prime}(n) \varepsilon K$ which sat－ isfy（1）．The triple $x_{1}, x_{2}, x_{3}^{\prime}(n)$ can be augmented，by Theorem 3，to a quad－ ruple $x_{1}, x_{2}, x_{3}(n), x_{4}^{\prime}(n)$ ，where $x_{4}^{\prime}(n)$ has the same denominator

$$
\left[\sigma_{4}(n)-1\right)^{2}=\left[x_{1} x_{2} x_{3}(n) x_{3}(n+1)-1\right]^{2}
$$

as $x_{3}^{\prime}(n)$ ．By Theorem 5，this quadruple can be augmented to a quintuple

$$
x_{1}, x_{2}, x_{3}^{\prime}(n), x_{4}^{\prime}(n), x_{5}^{\prime}(n) .
$$

Once this process is completed we can start anew，beginning with the triples $x_{1}, x_{2}, x_{4}^{\prime}(n)$ or $x_{1}, x_{2}, x_{5}^{\prime}(n)$ ．Each of the triples can be augmented to quadru－ ples and quintuples，etc．In short，the family of solutions of（1）with $x_{3}$ ， $y_{13}, y_{23} \varepsilon K$ appears to be very large，and is quite difficult to characterize completely．

## REFERENCES

1．American Math．Monthly 6 （1899）：86－87．
2．A．Baker \＆H．Davenport．＂The Equations $3 x^{2}-2=y^{2}$ and $8 x^{2}-7=z^{2}$ ．＂ Quarterly J．Math．，Oxford Series（2）， 20 （1969）：129－137．
3．B．W．Jones．＂A Variation on a Problem of Davenport and Diophantus．＂ Quarterly J．Math．，Oxford Series（2）， 27 （1976）：349－353．

# golden mean of the human body* 

T. ANTONY DAVIS<br>Indian Statistical Institute, Calcutta 700-035, India<br>RUDOLF ALTEVOGT<br>Zoologisches Institut der Universität, Münster, West Germany

## ABSTRACT

The value of $\phi=(\sqrt{5}+1) / 2$, or $1.61803 \ldots$ is referred to as the Golden Ratio or Divine Proportion. Such a ratio is sometimes discovered in nature, one instance being the mean between lengths of some organs of the human body. Leonardo da Vinci found that the total height of the body and the height from the toes to the navel depression are in Golden Ratio. We have confirmed this by measuring 207 students at the Pascal Gymnasium in Münster, where the almost perfect value of $1.618 .$. was obtained. This value held for both girls and boys of similar ages. However, similar measurements of 252 young men at Calcutta gave a slightly different value-1.615... . The tallest and shortest subjects in the German sample differed in body proportions, but no such difference was noted among the Indians in the Calcutta sample.

## INTRODUCTION

Marcus Vitruvius Pollio, Roman architect and author of De Architecture (c. 25 B.C.), remarked on a similarity between the human body and a perfect building: "Nature has designed the human body so that its members are duly proportioned to the frame as a whole." He inscribed the human body into a circle and a square, the two figures considered images of perfection. Later (in 1946) Le Corbusier gave a further dimension to the subject by depicting a proportionate human nude (Fig. 1A). In the sketch, he clearly adopted the Fibonacci system and Golden Mean to depict the proportion in a good-looking human body [7]. As shown in the sketch, the figure of a 1.75 -meter man with his left hand raised is drawn so that the distance from the foot to the navel measures 108 cm ; from the navel to the top of the head measures 66.5 cm ; and from the head to the tip of the upraised hand measures 41.5 cm . The ratio between 175 (height of man) and 108 is 1.62 , as is the ratio between 108 and 66.5, while the ratio between 66.5 and 41.5 is 1.6 . All these means are very close to the Golden Ratio, i.e., $\phi=(\sqrt{5}+1) / 2=1.61803 \ldots$. In order to verify this fascinating exposition, we set about taking measurements of boys and girls in two remote centers. The experimental subjects showed no visible signs of physical deformity.

## MATERIALS AND METHOD

During the last week of October 1973, a group of 207 students (175 boys and 32 girls) at the Pascal Gymnasium in Münster were chosen as subjects for measurement. Also, in early 1974, 252 young men (aged 16-32), most of whom were students at the Indian Statistical Institute in Calcutta, were measured.

The following measurements were taken of bare-footed boys and girls who were asked to stand erect, but without stretching their bodies abnormally,
*The authors wish to thank the Director of the Pascal Gymnasium in Münster, W. Germany, for allowing them to record measurements of his pupils and Mr. S. K. De, artist at the Indian Statistical Institute in Calcutta, for making the drawing. Davis is grateful to the German Academic Exchange Service for financial assistance that enabled him to visit Münster in 1973.
against a strong, vertically held pole which was marked in centimeters. With the help of a set-square, three measurements were taken: total height; distance from feet to level of nipples; and distance from feet to navel depression. The following five values were computed from the above three recorded measurements: (A) distance between navel and nipples; (B) distance between nipples and top of head; (C) $A+B$ (navel to top of head); (D) distance from navel to bottom of feet; and (E) total height of subject. Figure 1B illustrates these demarcations. No measurement was made of the distance between the head and the tip of the upraised hand indicated in Le Corbusier's drawing (Fig. 1A).

## RESULTS

The German and Indian data were rearranged, separately, in regular descending order, always keeping the tallest subject as first and the shortest subject as last. These data are summarized in Tables 1 and 2.

TABLE 1. BODY MEASUREMENTS OF GERMAN SCHOOL CHILDREN

| Particulars | A | B | C | D | E |
| :--- | ---: | ---: | ---: | ---: | ---: |
| Total, tallest 50 observations | 1127 | 2136 | 3263 | 5335 | 8618 |
| Total, shortest 50 observations | 1010 | 1757 | 2767 | 4354 | 7121 |
|  | Grand total (for 207) | $\frac{4313}{}$ | 8009 | 12322 | 19900 |
| Grand Mean | $\frac{20.836}{}$ | 38.690 | 59.526 | 96.135 | 155.622 |
| Total, girls only | 600 | 1206 | 1806 | 2885 | 4691 |
| Total, boys only | 3713 | 6803 | 10516 | 17015 | 27531 |

TABLE 2. BODY MEASUREMENTS OF YOUNG MEN FROM CALCUTTA

| Particulars | A | B | C | D | E |
| :--- | :---: | :---: | :---: | ---: | ---: |
| Total, tallest 63 observations | 1496 | 2729 | 4225 | 6678 | 10903 |
| Total, shortest 63 observations | 1314 | 2348 | 3662 | 5885 | 9547 |
| All men (for 252) | 5645 | 10166 | 15811 | 25239 | 41050 |
| mean | 22.40 | 40.34 | 62.74 | 100.15 | 162.90 |

Calculated ratios between $A \& B, B \& C, C \& D$, and $D \& E$ are presented in Tables 3 and 4.

TABLE 3: GERMAN STUDENTS: PROPORTION BETWEEN BODY LENGTHS

| Population | A/B | B/C | C/D | D/E |
| :--- | :---: | :---: | :---: | :---: |
| Tallest 25\% (approximately) | 0.528 | 0.655 | 0.609 | 0.621 |
| Shortest 25\% (approximately) | 0.575 | 0.635 | 0.636 | 0.611 |
| Girls only | 0.498 | 0.668 | 0.626 | 0.615 |
| Boys only | 0.544 | 0.647 | 0.618 | 0.618 |
| All students (207) | 0.537 | 0.650 | 0.619 | 0.618 |

TABLE 4. CALCUTTA YOUNG MEN: PROPORTION BETWEEN BODY LENGTHS

| Population | A/B | B/C | C/D | D/E |
| :--- | :---: | :---: | :---: | :---: |
| Tallest 25\% | 0.548 | 0.646 | 0.633 | 0.612 |
| Shortest 25\% | 0.560 | 0.641 | 0.622 | 0.616 |
| A11 men (252) | 0.555 | 0.643 | 0.627 | 0.615 |

FIGURE 1


Some differences were found to exist between the proportions of corresponding body lengths of the tallest and the shortest subjects. Statistical tests were performed to determine: (1) the extent of the difference; (2) if boys and girls differed in body proportions; and (3) if the Germans differed structurally from the Indians.

## STATISTICAL ANALYSIS

For the set of 207 observations on German boys and girls from different age groups, the following statistical hypotheses were tested.

Let $U=\mathrm{A} / \mathrm{B}, V=\mathrm{B} / \mathrm{C}, W=\mathrm{C} / \mathrm{D}, X=\mathrm{D} / \mathrm{E}$ and let $\bar{u}, \bar{v}, \bar{w}, \bar{x}$ represent the corresponding sample means and the corresponding population means.

There were 27 boys and 32 girls in the same age group in the German sample. Based on their measurements, $H_{0}: \mu G=\mu B$ was tested. Here

$$
\mu G=(\mu u G, \mu v G, \mu v G, \mu x G) ; \mu B=(\mu u B, \mu v B, \mu w B, \mu x B) .
$$

It is assumed that $(U, V, W, X) N(\mu, \Sigma)$. The test statistic used was

$$
F=\frac{n_{1}+n_{2}-5}{4} \cdot \frac{1}{\left(1 / n_{1}+1 / n_{2}\right)} \quad\left({\underset{\sim}{\underset{Y}{V}}}_{G}-{\underset{\sim}{\underset{Y}{Y}}}_{B}\right)^{\prime} A^{-1}\left({\underset{\sim}{\underset{Y}{Y}}}_{G}-{\underset{\sim}{\underset{Y}{P}}}_{B}\right),
$$

which is distributed as an $F$ statistic with $4, n_{1}+n_{2}-5$, d.f.

$$
\bar{Y}_{G}=\left(\bar{U}_{G}, \bar{V}_{G}, \bar{W}_{G}, \bar{X}_{G}\right) ; \bar{Y}_{B}=\left(\bar{U}_{B}, \bar{V}_{B}, \bar{W}_{B}, \bar{X}_{B}\right) .
$$

$A=A_{1}+A_{2}, A_{i}=$ sum of squares and products matrix for the $i$ th population, $i=1,2$.

$$
\begin{aligned}
\bar{Y}_{G} & =(.5962, .6300, .6303, .6137) \\
\bar{Y}_{B} & =(.4883, .6460, .7883, .5769) \\
F & =4.49, F .05 ; 4.50=5.70
\end{aligned}
$$

So, $H_{0}$ is accepted at the $5 \%$ level of significance, i.e., there is no significant difference between measurements of girls and boys. However, the test for $H_{0}: \mu u^{\prime}=\mu u^{2}$ gave an insignificant value for the $t$ statistic, which was less than 1.

Again, $H_{0}: \mu=\mu^{2}$ was tested for the 50 tallest and the 50 shortest individuals, where $\mu i=$ ( $\mu u i, \mu v i, \mu w i, \mu x i)$.

$$
\begin{aligned}
& {\underset{\sim}{Y}}_{1}=(.5196, .6417, .5979, .6089) \text { for } 50 \text { tallest; } \\
& {\underset{\sim}{\bar{Y}}}_{2}=(.5817, .6350, .6368, .6114) \text { for } 50 \text { shortest. }
\end{aligned}
$$

The computed $F=10.1574$ and $F .05 ; 4.95=5.66, F_{.01} ; 4.95=14.57$, so $H_{0}$ is rejected at the $5 \%$ level of significance.

Next, $H_{0}: \mu u^{\prime}=\mu u^{2}$ was rejected at both the $5 \%$ and $1 \%$ levels of significance because $t=-2.93$ with 98 d.f. Also, $H_{0}: \mu w^{\prime}=\mu w^{2}$ was rejected (at both levels) because $t=-3.3$ with 98 d.f.

For the Indian data, $H_{0}: u^{\prime}=u^{2}$ was tested for the 63 tallest and the 63 shortest subjects ( $25 \%$ of the total). For this both $H_{0}: \mu u^{\prime}=\mu u^{2}$ and $H_{0}$ : $\mu \omega^{\prime}=\mu \omega^{2}$ were accepted because the corresponding $t$ statistics were $<1$.

Again for the Indian data we did not find any significant difference between measurements of the tallest and shortest subjects. This might have been due to the short range of heights among the Indian sample.

$$
\frac{\text { Indian Data }}{\bar{E}_{1}=173.38, \bar{E}_{2}=152.95} \quad \bar{E}_{1}=\frac{\text { German Data }}{172.36, \bar{E}_{2}=}
$$

$H_{0}: \mu E_{1}=\mu E_{2}$ was rejected because $t \simeq 5$ with 111 d.f.; i.e., the heights of the shortest individuals in the Indian sample and those in the German sample differed significantly. The variance in mean ages of the two sample groups might also be an important reason for the difference.

DISCUSSION
The data on German students presented in Tables 1 and 3 confirm La Corbusier's definition of a good-looking human body.

The Parthenon at Athens is considered one of the most perfect buildings ever constructed by man and one that has survived centuries of neglect. The secret lies in the fact that the Parthenon was constructed according to the principle of Divine Proportion [4]. The width of the building and its height are in Golden Sections. Hoggatt [3] has cited further examples in which the Golden Section has been used.

Also, it is now known [see 5] that the Great Pyramid of Giza, Egypt, was built in accordance with Divine Proportion; its vertical height and the width of any of its sides are in Golden Sections.

These examples confirm Vitruvius' statement that perfect buildings and proportionate human bodies have something in common.

According to available data, the navel of the human body is a key point that divides the entire length of the body into Golden Sections (their ratio is the Golden Ratio). This point is also vitally important for the developing fetus, since the umbilical cord-the life-line between mother and fetus-is connected through the navel. Compared to the position of the navel, the line of the nipples is not particularly important, because it does not divide the body (above the nave1) into Golden Sections. Data from both Germany and India confirm this fact.

There is a close connection between the Golden Ratio and the Fibonacci Sequence-1, 1, 2, 3, 5, 8, 13, 21, ... . Each number is obtained by adding the two numbers just previous to it. This numerical sequence is named after the thirteenth-century Italian mathematician Leonardo Pisano, who discovered it while solving a problem on the breeding of rabbits. Ratios of successive pairs of some initial numbers give the following values:

```
1/1 = 1.000; 1/2 = 0.500; 2/3 = 0.666...; 3/5 = 0.600; 5/8 = 0.625;
8/13 = 0.615...; 13/21 = 0.619...; 21/34 = 0.617...; 34/55 = 0.618...;
```

55/89 = 0.618... .

Thereafter, the ratio reaches a constant that is almost equivalent to the Golden Ratio. Such a ratio has been detected in most plants with alternate (spiral) phyllotaxis, because any two consecutive leaves subtend a Fibonacci angle approximating 317.5 degrees. Thus, many investigators of phyllotaxis identify the involvement of Fibonacci series on foliar arrangement, the most recent being Mitchison [6].
(please turn to page 384)

# A RECURRENCE RELATION FOR GENERALIZED <br> MULTINOMIAL COEFFICIENTS 

## A. G. SHANNON

The New South wales Institute of Technology, Sydney, Australia

## 1. INTRODUCTION

Gould [2] has defined Fontené-Ward multinomial coefficients by

$$
\left\{\begin{array}{c}
n \\
s_{1}, s_{2}, \ldots, s_{r}
\end{array}\right\}=u_{n}!/ u_{s_{1}}!u_{s_{2}}!\ldots u_{s_{r}}!
$$

where $\left\{u_{n}\right\}$ is an arbitrary sequence of real or complex numbers such that

$$
\begin{aligned}
& u_{n} \neq 0 \text { for } n \geq 1, \\
& u_{0}=0 \\
& u_{1}=1,
\end{aligned}
$$

and

$$
u_{n}!=u_{n} u_{n-1} \ldots u_{1},
$$

with $\quad u_{0}!=1$.
These are a generalization of ordinary multinomial coefficients for which there is a recurrence relation

$$
\binom{n}{s_{1}, \ldots, s_{r}}=\sum_{j=1}^{r}\left(s_{1}-\delta_{1_{j}}, \ldots, s_{r}-\delta_{r j}\right)
$$

as in Hoggatt and Alexanderson [4].
Hoggatt [3] has also studied Fontené-Ward coefficients when $r=2$ and $\left\{u_{n}\right\}=\left\{F_{n}\right\}$, the sequence of Fibonacci numbers. We propose to consider the case where the $u_{n}$ are elements which satisfy a linear homogeneous recurrence relation of order $r$.

## 2. THE COEFFICIENTS

We consider $r$ basic sequences, $\left\{u_{s, n}^{(r)}\right\}$, which satisfy a recurrence relation of order $r$ :

$$
U_{s, n}^{(r)}=\sum_{j=1}^{r}(-1)^{j+1} P_{r j} U_{s, n-j}^{(r)}, n>r ; U_{s, n}^{(r)}=\delta_{s n}, 1 \leq n \leq r,
$$

where $\delta_{s n}$ is the Kronecker delta and the $P_{r, j}$ are arbitrary integers. We designate $\left\{U_{r, n}^{(r)}\right\}$ as the fundamental sequence by analogy with Lucas' second-order fundamental sequence $\left\{U_{2, n}^{(2)}\right\}$. Since this sequence is used frequently, we let

$$
\left\{U_{r, n+r}^{(r)}\right\}=\left\{u_{n}^{(r)}\right\}
$$

for convenience of notation.
Note that the terms "fundamental" and "basic" follow from the nature of these sequences as expounded in Jarden [5] and Bell [1], respectively.

Let $M_{n}^{(r)}$ denote the square matrix of order $r$ :

$$
M_{n}^{(r)}=\left[U_{j, n+i}^{(r)}\right], 1 \leq i, j \leq r
$$

wherein $i$ refers to the rows and $j$ to the columns.
Lemma: $u_{n}^{(r)}=\sum_{j=0}^{r-1} U_{r-j, r+m}^{(r)} u_{n-m-j}^{(r)}$.
Proof: It is easily proved by induction that

$$
M_{n}^{(r)}=M_{m}^{(r)} M_{n-m}^{(r)},
$$

and so from equating the elements in the last row and last column we get

$$
\begin{aligned}
u_{n}^{(r)} & =U_{r, r+n}^{(r)}=\sum_{j=0}^{r-1} U_{r-j, r+m}^{(r)} U_{r, r+n-m-j}^{(r)} \\
& =\sum_{j=0}^{r-1} U_{r-j, r+m}^{(r)} u_{n-m-j}^{(r)}
\end{aligned}
$$

We now define Fibonacci multinomial coefficients by

$$
\left\{\begin{array}{c}
n \\
s_{1}, \ldots, s_{r}
\end{array}\right\}_{u}=u_{n}^{(r)}!/ u_{s_{1}}^{(r)} \ldots u_{s_{r}}^{(r)}
$$

such that $n=\sum_{i=1}^{r} s_{i}$.
Thus, when $r=2$, we have the Fibonacci binomial coefficients

$$
\left\{\begin{array}{c}
n \\
s_{1}, s_{2}
\end{array}\right\}_{u}=\frac{u_{n}^{(r)}!}{u_{s_{1}}^{(2)}!u_{s_{2}}^{(2)}!}=\frac{u_{n}^{(2)}!}{u_{s_{1}}^{(2)}!u_{n-s_{1}}^{(2)}!}=\left\{\begin{array}{c}
n \\
s_{1}
\end{array}\right\} .
$$

We next seek the recurrence relation for these Fibonacci multinomial coefficients.

## 3. THE RELATION

Theorem: The recurrence relation for the Fibonacci multinomial coefficients is given by

$$
\left\{\begin{array}{c}
n \\
s_{1}, \ldots, s_{r}
\end{array}\right\}_{u}=\sum_{j=1}^{r}\left\{\begin{array}{c}
n-1 \\
s_{1}-\delta_{1 j}, \ldots, s_{r}-\delta_{r j}
\end{array}\right\}_{u} U_{r-j+1,2+m}^{(r)}
$$

in which $s_{i}=n-m-i+1$
and $\quad m=n(1-1 / r)+\frac{1}{2}(1-r)$.
Proof: We note first that

$$
\begin{aligned}
\sum_{i=1}^{r} s_{i} & =\sum_{i=1}^{r}(n-m-i+1)=r(n-m+1)-\sum_{i=1}^{r} i \\
& =r n-r m+r-\frac{1}{2} r(r+1) \\
& =r n+\frac{1}{2} r-\frac{1}{2} r^{2}-r m=n .
\end{aligned}
$$

Using the lemma, we have that

$$
\begin{aligned}
& \sum_{j=1}^{r}\left\{s_{1}-\delta_{1 j}, \ldots, s_{r}-\delta_{r j}\right\}_{n} U_{r-j+1, r+m}^{(r)} \\
& =\frac{u_{n-1}^{(r)}!\left\{u_{s_{1}}^{(r)} U_{r, r+m}^{(r)}+\cdots+u_{s_{r}}^{(r)} U_{1, r+m}^{(r)}\right\}}{u_{s_{1}}^{(r)}!\cdots u_{s_{r}}^{(r)}!} \\
& =\frac{u_{n-1}^{(r)}!\left\{u_{n-m}^{(r)} U_{r, r+m}^{(r)}+\cdots+u_{n-m-r+1}^{(r)} U_{1, r+m}^{(r)}\right\}}{u_{s_{1}}^{(r)}!\cdots u_{s_{r}}^{(r)}!} \\
& =u_{n-1}^{(r)}!u_{n}^{(r)} / u_{s_{1}}^{(r)}!\ldots u_{s_{2}}^{(r)}!\text { (from the 1emma) } \\
& =\left\{\begin{array}{c}
n \\
s_{1}, \ldots, s_{r}
\end{array}\right\}_{u} \text { as required. } \\
& \text { 4. CONCLUSION }
\end{aligned}
$$

As an example, suppose $r=2, n=2 k+1$; then $m=k, s_{1}=k+1$, and $s_{2}=k$, and the theorem becomes

$$
\begin{aligned}
\left\{\begin{array}{c}
2 k+1 \\
k
\end{array}\right\}_{u} & =\frac{u_{2 k}^{(2)}!U_{2, k+2}^{(2)}}{u_{k}^{(2)}!u_{k}^{(2)}!}+\frac{u_{2 k}^{(2)}!U_{1, k+2}^{(2)}}{u_{k+1}^{(2)}!u_{k-1}^{(2)}!} \\
& =U_{2, k+2}^{(2)}\left\{\begin{array}{c}
2 k \\
k
\end{array}\right\}_{u}+U_{1, k+2}^{(2)}\left\{\begin{array}{c}
2 k \\
k-1
\end{array}\right\}_{u}
\end{aligned}
$$

This is the same as the equivalent result (F) in Hoggatt [3] (in our notation):

$$
\left\{\begin{array}{c}
2 k+1 \\
k
\end{array}\right\}_{u}=U_{2, k+2}^{(2)}\left\{\begin{array}{c}
2 k \\
k
\end{array}\right\}_{u}-P_{22} U_{2, k+1}^{(2)}\left\{\begin{array}{c}
2 k \\
k-1
\end{array}\right\}_{u}
$$

as it can be readily shown that

$$
U_{1, k+2}^{(2)}=-P_{22} U_{2, k+1}^{(2)}
$$

The first five values of $U_{j, n}^{(2)}, j=1,2$, are:

| $U_{j, n}^{(2)}$ | $n=1$ | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $j=1$ | 1 | 0 | $-P_{22}$ | $-P_{21} P_{22}$ | $-P_{21}^{2} P_{22}+P_{22}^{2}$ |
| 2 | 0 | 1 | $P_{21}$ | $P_{21}^{2}-P_{22}$ | $P_{21}^{3}-2 P_{21} P_{22}$ |

REFERENCES

1. E. T. Bell. "Notes on Recurring Series of the Third Order." Tôhoku Mathematical Journal 24 (1924):168-184
2. H. W. Gould. 'The Bracket-Function and Fontené-Ward Generalized Binomial Coefficients with Applications to Fibonomial Coefficients." The Fibonacci Quarterly 7 (1969):23-40, 55.
3. V. E. Hoggatt, Jr. "Fibonacci Numbers and Generalized Binomial Coefficients." The Fibonacci Quarterly 5 (1967):383-400.
4. V. E. Hoggatt, Jr., \& G. L. Alexanderson. "A Property of Multinomial Coefficients." The Fibonacci Quarterly 9 (1971), 351-356, 420-421.
5. D. Jarden. Recurring Sequences. Jerusalem: Riveon Lematematika, 1966, p. 107.
******

## generalized fibonacci numbers As ELEMENTS OF IDEALS

## A. G. SHANNON

The New South Wales Institute of Technology, Sydney, Australia
Wyler [3] has looked at the structure of second-order recurrences by considering them as elements of a commutative ring with the Lucas recurrence as unit element.

It is possible to supplement Wyler's results and to gain further insight into the structure of recurrences by looking at ideals in this commutative ring.

The purpose of this note is to look briefly at the structure of Horadam's generalized sequence of numbers [2] defined recursively by

$$
\begin{equation*}
w_{n}=p w_{n-1}-q w_{n-2} \quad(n \geq 2) \tag{1}
\end{equation*}
$$

with $w_{0}=a, w_{1}=b$, and where $p, q$ are arbitrary integers.
DeCarli [1] has examined a similarly generalized sequence over an arbitrary ring. It is proposed here to assume that the sequence $\left\{\omega_{n}\right\}$ of numbers are elements of a commutative ring $R$ and to examine $\left\{w_{n}\right\}$ in terms of ideals of $R$. To this end, suppose that $p, q$ are elements of an ideal of $R$.
$\langle\langle p\rangle,\langle q\rangle$ are then the ideals generated by $p$ and $q$, respectively, and $(\langle p\rangle,\langle q\rangle)$ is the sum of the ideals generated by $p$ and $q$.

Theorem 1: $\quad w_{n} \varepsilon(\langle p\rangle,\langle q\rangle)$.
Proof: $a, b \varepsilon R, p \varepsilon\langle p\rangle, q \varepsilon\langle q\rangle$ implies $p b \varepsilon\langle p\rangle$, and $-q \alpha \varepsilon\langle q\rangle$.

$$
\therefore w_{2}=p b-q a \varepsilon\langle p\rangle+\langle q\rangle .
$$

$$
p b \varepsilon\langle p\rangle \text { and so } p b \varepsilon R .
$$

Hence

$$
-q(p b) \varepsilon\langle q\rangle
$$

$$
w_{2} \varepsilon R, p \varepsilon\langle p\rangle \text { implies } p w_{2} \varepsilon\langle p\rangle
$$

$$
w_{3}=p w_{2}-q b \varepsilon\langle p\rangle+\langle q\rangle .
$$

It follows by induction that $w_{n} \varepsilon\langle p\rangle+\langle q\rangle$ : that is,

$$
w_{n} \varepsilon(\langle p\rangle,\langle q\rangle) .
$$

The general term of $\left\{w_{n}\right\}$ can be expressed in terms of

$$
\alpha=\frac{1}{2}\left(p+\sqrt{\left(p^{2}-4 q\right)}\right) \quad \text { and } \quad \beta=\frac{1}{2}\left(p-\sqrt{\left(p^{2}-4 q\right)}\right)
$$

as follows:

$$
\begin{equation*}
w_{n}=A \alpha^{n}+B \beta^{n} \tag{2}
\end{equation*}
$$

where $A=(b-\alpha \beta) /(\alpha-\beta)$ and $B=(\alpha \alpha-b) /(\alpha-\beta)$.
Suppose $A, B$ are elements of a commutative ring $Q$, and $\alpha, \beta$ are elements of an ideal of $Q$. It follows that $w_{n}$ belongs to $Q$ and is also a member of the sum of the ideals generated by $\alpha$ and $\beta$ (from Theorem 1).
Theorem 2: $\exists S \subset Q$ such that $S=\langle\alpha\rangle \oplus\langle\beta\rangle$ if $p^{2}-4 q \neq 0$.
Proof: $\alpha^{i}=\beta^{j}$ iff $p^{2}-4 q=0$. Hence, $\langle\alpha\rangle \cap\langle\beta\rangle=\langle 0\rangle$,
and the result follows.
Let $\left\{v_{n}\right\}$ and $\left\{w_{n}\right\}$ be two sequences of elements of $Q$ such that

$$
\begin{equation*}
v_{n}=p^{\prime} v_{n-1}-q^{\prime} v_{n-2} \tag{3}
\end{equation*}
$$

(with suitable initial values and $p^{\prime}, q^{\prime}$ arbitrary integers) and $w_{n}$ is defined as before.

Define $\left\{v_{m}\right\} \equiv\left\{w_{n}\right\}$ when $w_{n}-v_{m} \varepsilon S$ for small $n$, $m$ where $S=\langle\alpha\rangle \oplus\langle\beta\rangle$
as before.
Note that if $a, \bar{b}, c \varepsilon S$, then (i) $a-a \varepsilon S$; (ii) $a-b \varepsilon S$ implies $b-a \varepsilon S$;
(iii) $a-b \varepsilon S$ and $b-c \varepsilon S$ imply that $a-c \varepsilon S$.

Theorem 3: If $v_{m}-w_{n} \varepsilon\langle\alpha\rangle \oplus\langle\beta\rangle$ for small $n, m$, then

$$
v_{m}-w_{n} \varepsilon\langle\alpha\rangle \oplus\langle\beta\rangle \text { for all } n, m
$$

Proof: $w_{n} \varepsilon S=\alpha \oplus \beta$ for all $n$ from Theorem 2. It is known that $v_{m} \in S$ for $m \leq N$, say. Now,

$$
p^{\prime} v_{N} \in S \quad \text { and } \quad q^{\prime} v_{N-1} \in S
$$

Hence, $\quad v_{N+1}=p^{\prime} v_{N}-q^{\prime} v_{N-1} \varepsilon S$, and the result follows.

To prove the stronger result that if

$$
\left\{v_{m}\right\} \equiv\left\{w_{n}\right\} \text { for any } n, m \text {, then }\left\{v_{m}\right\} \equiv\left\{w_{n}\right\} \text { for all } n, m,
$$

it would be necessary to replace "small" with "large" in the enunciation of Theorem 3. This would require $S$ to be a prime ideal which could be achieved by embedding $S$ in a maximal ideal $\mu \alpha \beta$ which could be proved prime. However, this would then require restrictions on $p^{\prime}$ and $q^{\prime}$ as it would be easy to show that $q^{\prime} v_{N-1} \varepsilon S$ but it would not automatically follow that $v_{N-1} \varepsilon S$.

Another problem that might be worth investigating is to look for commutators for relations like

$$
w_{n+1}^{p}-w_{n}^{p}-w_{n-1}^{p}, \text { where } p \text { is a prime. }
$$

These could be useful in Lie algebras.

## REFERENCES

1. D. J. DeCarli. "A Generalized Fibonacci Sequence over an Arbitrary Ring." The Fibonacci Quarterly 8, No. 2 (1970):182-184.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." The Fibonacci Quarterly 3, No. 3 (1965):161-176.
3. 0. Wyler. "On Second Order Recurrences." American Mathematical Monthly 72, No. 5 (1965):500-506.

## \#

## A GENERALIZATION OF HILTON'S PARTITION OF HORADAM's SEQUENCES

A. G. SHANNON

The New South Wales Institute of Technology, Sydney, Australia

## 1. INTRODUCTION

If $P_{r 1}, P_{r 2}, \ldots, P_{r r}$ are distinct integers for positive $r$, let

$$
\omega=\omega\left(P_{r 1}, \ldots, P_{r p}\right)
$$

be the set of integer sequences

$$
\left\{W_{s n}^{(r)}\right\}=\left\{W_{s 0}^{(r)}, W_{s 1}^{(r)}, W_{s 2}^{(r)}, \ldots\right\}
$$

which satisfy the recurrence relation of order $r$,

$$
\begin{equation*}
W_{s, n+r}^{(r)}=\sum_{j=1}^{r}(-1)^{j+1} P_{r j} W_{s, n+r-j}^{(r)}, \quad(s=1,2, \ldots, r), n \geq 1 . \tag{1.1}
\end{equation*}
$$

This is a generalization of of $\left\{W_{s n}^{(2)}\right\}$ studied in detail by Horadam [1, 2, 3, 4, 5].

Hilton [6] partitioned Horadam's sequence into a set $F$ of generalized Fibonacci sequences and a set $L$ of generalized Lucas sequences. We extend this to show that $\omega$ can be partitioned naturally into $r$ sets of generalized sequences.

## 2. NOTATION

We define $r$ sequences of order $r,\left\{V_{s n}^{(r)}\right\}(s=1,2, \ldots, r)$ by

$$
\begin{equation*}
V_{s n}^{(r)}=d^{1-s} \sum_{j=1}^{r} A_{s j}^{(r)} \alpha_{r j}^{n}, \quad n \geq 1, \tag{2.1}
\end{equation*}
$$

where the $\alpha_{r j}$ are the distinct roots of

$$
\begin{equation*}
x^{r}=\sum_{k=1}^{r}(-1)^{k+1} P_{r k} x^{r-k} \tag{2.2}
\end{equation*}
$$

and

$$
d=\operatorname{det} D
$$

where $D$ is the Vandermonde matrix

$$
D=\left[\begin{array}{llll}
1 & 1 & \ldots & 1 \\
\alpha_{r 1} & \alpha_{r 2} & \ldots & \alpha_{r r} \\
\alpha_{r 1}^{2} & \alpha_{r 2}^{2} & \ldots & \alpha_{r r}^{2} \\
\alpha_{r 1}^{r-1} & \alpha_{r 2}^{r-1} & \ldots & \alpha_{r r}^{r-1}
\end{array}\right]
$$

and the $A_{s j}^{(r)}$ are suitable constants that depend on the initial values of the sequence:

$$
\left\{V_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, P_{r 2}, \ldots, P_{r p}\right)
$$

Pro06: $\quad V_{s, n+r}^{(r)}=d^{1-s} \sum_{j=1}^{r} A_{s j}^{(r)} \alpha_{r j}^{n+r}$

$$
=d^{1-s} \sum_{j=1}^{r} A_{s j}^{(r)} \alpha_{r j}^{n} \sum_{k=1}^{r}(-1)^{k+1} P_{r k} \alpha_{r j}^{r-k}
$$

$$
=\sum_{k=1}^{r}(-1)^{k+1} P_{1 p k}\left(d^{1-s} \sum_{j=1}^{r} A_{s j}^{(r)} \alpha_{r j}^{n+r-k}\right)
$$

$$
=\sum_{k=1}^{r}(-1)^{k+1} P_{r c k} V_{s, n+r-k}^{(r)} \text {, as required. }
$$

3. THE PARTITION OF $\omega\left(P_{r 1}, \ldots, P_{r p}\right)$

It follows from (2.1) that we can represent $V_{r m}^{(x)}$ by

$$
V_{r n}^{(r)}=d^{t-r} \sum_{j=1}^{r} B_{t j}^{(r)} \alpha_{r j}^{n} \quad(t=1,2, \ldots, r)
$$

so that

$$
B_{1 j}^{(r)} \equiv A_{r j}^{(r)}
$$

and $V_{r n}^{(r)}$ can be put in the form of any of the $V_{s n}^{(r)}$. For example, when $t=3$,

$$
V_{r n}^{(r)}=d^{3-r} \sum_{j=1}^{r} B_{3 j}^{(r)} \alpha_{r j}^{n}
$$

has the form of

$$
V_{r-2, n}^{(r)}=d^{3-r} \sum_{j=1}^{r} A_{r-2, j}^{(r)} \alpha_{r j}^{n} .
$$

We shall now consider the derivation of one sequence from another, so that in what follows the results hold for any of the $r$ sequences. Thus there are $r$ such partitions.

We say that $W_{s n}^{(r)}$ is in Fibonacci form when it is represented as in

$$
\begin{equation*}
W_{s n}^{(r)}=\frac{1}{d} \sum_{j=1}^{r} A_{r j} \alpha_{r j}^{n} \quad|d| \neq 1 \tag{3.1}
\end{equation*}
$$

and in Lucas form when it is represented as in

$$
\begin{equation*}
W_{s n}^{(r)}=\sum_{j=1}^{r} B_{r j} \alpha_{r j}^{n} \quad|a| \neq 1 \tag{3.2}
\end{equation*}
$$

where the $B_{r j}$ are different constants from the $A_{r j}$. This is analogous to Hilton. To continue the analogy, one can see from (2.1) that there are $r$ such forms which correspond to the distinct values of $s$. When $W_{s n}^{(r)}$ is in Fibonacci form we may perform an operation (') to obtain a number

$$
W_{s n}^{(r)^{\prime}}
$$

where

$$
W_{s n}^{(r)^{\prime}}=\sum_{j=1}^{r} B_{r j} \alpha_{r j}^{n} .
$$

We say (1ike Hilton) that the sequence $\left\{W_{s n}^{(r)^{\prime}}\right\}$ is derived from the sequence $\left\{W_{s n}^{(r)}\right\}$. Throughout this paper we assume that $|d|$ is not unity, because when $d$ is unity the essential distinction between (3.1) and (3.2) breaks down. There would still be $r$ partitions, provided the $A_{s j}^{(r)}$ of Equation (2.1) are distinct for all values of $s$, but the groups of sequences would have the basic Lucas form. Now

$$
W_{s n}^{(r)^{\prime}}=\sum_{j=1}^{r} A_{r j} \alpha_{r j}^{n}=d\left(\frac{1}{d} \sum_{j=1}^{r} A_{r j} \alpha_{r j}^{n}\right)=d W_{s n}^{(r)},
$$

and so $W_{s n}^{(r) "}=d^{2} W_{s n}^{(r)}$, which corresponds to Hilton's Theorem 1.
It follows from (3.1) and Jarden [7] that

$$
D \underset{\sim}{a}=d \underset{\sim}{w}
$$

where

$$
\underset{\sim}{\sim} \underset{\sim}{\sim}=\left[\tilde{A}_{r 1}, A_{r 2}, \ldots, A_{r r}\right]^{T}
$$

and

$$
\underset{\sim}{w}=\left[W_{s 0}^{(r)}, W_{s 1}^{(r)}, \ldots, W_{s, r-1}^{(r)}\right]^{T}
$$

So

$$
\begin{aligned}
& \underset{\sim}{a}=d D^{-1} \underset{\sim}{w} \\
& d D^{-1} \equiv\left[d_{\rho K}\right]
\end{aligned}
$$

is the matrix with $d_{\rho \kappa}$ in row $\rho$ and column $\kappa$, where

$$
d_{\rho \kappa}=(-1)^{\rho+\kappa} \prod_{\substack{\rho \neq m, n \\ m>n}}\left(\alpha_{r m}-\alpha_{r n}\right) \sum_{m \neq \rho} \alpha_{r m_{1}} \alpha_{r m_{2}} \ldots \alpha_{r m_{r-k}}
$$

For $r=2$,

$$
D=\left[\begin{array}{ll}
1 & 1 \\
\alpha_{21} & \alpha_{22}
\end{array}\right], \quad D^{-1}=\frac{1}{d}\left[\begin{array}{cc}
\alpha_{22} & -1 \\
-\alpha_{21} & 1
\end{array}\right]
$$

For $r=3$,

$$
\text { where } d=\alpha_{22}-\alpha_{21}
$$

$$
\begin{aligned}
& D=\left[\begin{array}{lll}
1 & 1 & 1 \\
\alpha_{31} & \alpha_{32} & \alpha_{33} \\
\alpha_{31}^{2} & \alpha_{32}^{2} & \alpha_{33}^{2}
\end{array}\right], \\
& D^{-1}=\frac{1}{d}\left[\alpha_{33} \alpha_{32}\left(\alpha_{33}-\alpha_{32}\right),-\left(\alpha_{33}+\alpha_{32}\right)\left(\alpha_{33}-\alpha_{32}\right),\left(\alpha_{33}-\alpha_{32}\right)\right] \\
& \text { etc. }
\end{aligned}
$$

where $d=\left(\alpha_{32}-\alpha_{31}\right)\left(\alpha_{33}-\alpha_{31}\right)\left(\alpha_{33}-\alpha_{32}\right)$.
For $r=4$,

$$
D=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
\alpha_{41} & & & \\
\alpha_{41}^{2} & & \text { etc. } & \\
\alpha_{41}^{3} & & &
\end{array}\right], \quad D^{-1}=\frac{1}{d}\left[\begin{array}{cccc}
d_{11} & d_{12} & d_{13} & d_{14} \\
& & & \\
& \text { etc. }
\end{array}\right]
$$

where $d=\left(\alpha_{42}-\alpha_{41}\right)\left(\alpha_{43}-\alpha_{41}\right)\left(\alpha_{43}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{41}\right)\left(\alpha_{44}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{43}\right)$

$$
\begin{aligned}
& d_{11}=\alpha_{42} \alpha_{43} \alpha_{44}\left(\alpha_{43}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{43}\right) \\
& d_{12}=-\left(\alpha_{42} \alpha_{43}+\alpha_{43} \alpha_{44}+\alpha_{44} \alpha_{42}\right)\left(\alpha_{43}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{43}\right) \\
& d_{13}=\left(\alpha_{42}+\alpha_{43}+\alpha_{44}\right)\left(\alpha_{43}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{43}\right) \\
& d_{14}=-\left(\alpha_{43}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{42}\right)\left(\alpha_{44}-\alpha_{43}\right) .
\end{aligned}
$$

From $\underset{\sim}{a}=d D^{-1} \underset{\sim}{w}$, we have

$$
A_{r \rho}=\sum_{K=1}^{r} d_{\rho K} W_{s, k-1}^{(r)} \quad \text { and } \quad W_{s n}^{(r)^{\prime}}=\sum_{\rho=1}^{r} A_{r \rho} \alpha_{r \rho}^{n},
$$

so

$$
\begin{equation*}
W_{s n}^{(r)^{\prime}}=\sum_{\rho=1}^{r} \sum_{\kappa=1}^{r} d_{\rho K} \alpha_{r \rho}^{n} W_{s, k-1}^{(r)} \tag{3.3}
\end{equation*}
$$

which is effectively a generalization of Equations (2) and (3) of Hilton. Suppose

$$
\left\{X_{s n}^{(r)}\right\} \text { and }\left\{Y_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, \ldots, P_{r p}\right)
$$

and that

$$
X_{s n}^{(r)^{\prime}}=Y_{s n}^{(r)} \quad(n=0,1,2, \ldots)
$$

Since

$$
d \underset{\sim}{x}(r)={\underset{\sim}{y}}^{(r)},
$$

$$
\begin{equation*}
Y_{s n}^{(r)}=\sum_{\rho=1}^{r} \sum_{k=1}^{r} d_{\rho \kappa} \alpha_{r \rho}^{n} X_{s, k-1}^{(r)} \text {, from (3.3) } \tag{3.4}
\end{equation*}
$$

and

$$
\begin{align*}
d^{2} X_{s n}^{(r)} & =d Y_{s n}^{(r)} \\
& =\sum_{\rho=1}^{r} \sum_{k=1}^{r} d_{\rho K} \alpha_{\rho \rho}^{n} X_{s, K-1}^{(r)}, \text { from (3.4). } \tag{3.5}
\end{align*}
$$

(3.5) is a generalization of Theorem 2(i) of Hilton.

The analogue of Theorem 2 (ii) of Hilton can be stated as:
if $\left\{W_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, \ldots, P_{r r}\right), d^{2} \mid \sum_{\rho} \sum_{\kappa} d_{\rho K} \alpha_{r \rho}^{n} W_{s, K-1}^{(r)}, n<r$,
then $\left\{W_{s n}^{(r)}\right\}=\left\{X_{s n}^{(r)^{\prime}}\right\}$ for some $\left\{X_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, \ldots, P_{r r}\right)$.
Proof: If $\left\{X_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, \ldots, P_{r r}\right)$,
then

$$
X_{s n}^{(r)^{\prime}}=\sum_{\rho} \sum_{\kappa} d_{\rho \kappa} \alpha_{r \rho}^{n} X_{s, \kappa-1}^{(r)}, \text { from (3.3). }
$$

If

$$
X_{s n}^{(r)}=d^{-2} \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} W_{s, K-1}^{(r)}, n<r,
$$

then

$$
X_{s n}^{(r)^{\prime}}=\frac{1}{d} \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{\rho \rho}^{n} W_{s, K-1}^{(r)}, \quad n<r,
$$

but

$$
W_{s n}^{(r)}=\frac{1}{d} \sum_{\rho} \sum_{\kappa} a_{\rho K} \alpha_{r \rho}^{n} W_{s, K-1}^{(r)}, \quad n<r
$$

So

$$
d X_{s n}^{(r)}=W_{s n}^{(r)} \text { for } n<r
$$

from which the result follows.

## 4. THEOREMS

The basic linear relationships between $\left\{X_{s n}^{(r)}\right\}$ and $\left\{Y_{s n}^{(r)}\right\}$ are described
in the following theorem.
Theorem A: The following are equivalent:

$$
\begin{align*}
\left\{X_{s n}^{(r)}\right\} & =\left\{Y_{s n}^{(r)}\right\}  \tag{4.1}\\
Y_{s, n+m}^{(r)} & =\sum_{\rho=1}^{r} \sum_{K=1}^{r} d_{\rho K} \alpha_{r \rho}^{m} X_{s, n+K-1}^{(r)}, \text { for all } n \geq 0,  \tag{4.2}\\
X_{s, n+m}^{(r)} & =\frac{1}{d^{2}} \sum_{\rho=1}^{r} \sum_{K=1}^{r} d_{\rho K} \alpha_{r \rho}^{m} Y_{s, n+K-1}^{(r)}, \text { for all } n \geq 0 . \tag{4.3}
\end{align*}
$$

Proof: For each of (4.2) and (4.3) we need only require that the expression is true for $r$ adjacent values of $n$.
if

$$
\begin{aligned}
(4.1) & \Longrightarrow(4.2) ; \\
\left\{X_{s n}^{(r)^{\prime}}\right\} & =\left\{Y_{s n}^{(r)}\right\} \\
Y_{s, n}^{(r)} & =\sum_{\rho=1}^{r} \sum_{K=1}^{r} d_{\rho K} \alpha_{r \rho}^{n} X_{s, K-1}^{(r)}, \text { from (3.3). }
\end{aligned}
$$

Thus (4.2) is true for $n=0$. Let $t \geq r$ and assume (4.2) is true for $0 \leq n<t$.

$$
\begin{aligned}
Y_{s, t+m}^{(r)} & =\sum_{j=1}^{r}(-1)^{j+1} P_{r j} Y_{s, t+m-j}^{(r)}, \text { from (1.1), } \\
& =\sum_{j=1}^{r} \sum_{\rho=1}^{r} \sum_{K=1}^{r} d_{\rho \kappa} \alpha_{\rho \rho}^{m}(-1)^{j+1} P_{r j} X_{s, t-j+\kappa-1}^{(r)} \\
& =\sum_{\rho=1}^{r} \sum_{K=1}^{r} d_{\rho \kappa} \alpha_{r \rho}^{m} \sum_{j=1}^{r}(-1)^{j+1} P_{r j} X_{s, t-j+K-1}^{(r)} \\
& =\sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} X_{s, t+K-1}^{(r)}, \text { as required. }
\end{aligned}
$$

Similarly, (4.3) follows if we use (3.3) and induction.

$$
(4.3) \Longrightarrow(4.1) \text {; }
$$

since

$$
X_{s n}^{(r)}=\frac{1}{d^{2}} \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} Y_{s, K-1}^{(r)}, \text { for } n<r,
$$

it follows from the generalization of Hilton's Theorem 2(ii) that
$\left\{X_{s n}^{(r)}\right\}=\left\{Y_{s n}^{(r)}\right\}$.
Similarly, it can be shown that $(4.2) \Longrightarrow(4.1)$. This completes the proof of Theorem A.

We now describe a partition of $\omega\left(P_{r 1}, \ldots, P_{r r}\right)$. If $\left\{W_{s n}^{(r)}\right\} \varepsilon \omega\left(P_{r 1}, \ldots, P_{r p}\right)$,
let $W_{s n}^{(r)}=d^{2} \omega_{8 n}^{(r)}$ for all $n \geq 0$ where $m \geq 0$ is an integer, $\left\{\omega_{s n}^{(r)}\right\} \varepsilon \omega$ and $d^{2} \nmid \omega_{s n}^{(r)}$
for at least one $n \geq 0$. Then, for $\left|d_{i}\right| \neq 1$,

$$
\left\{W_{s n}^{(r)}\right\} \varepsilon L
$$

if $d^{2} \mid \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} \omega_{s, K-1}^{(r)}$, for all $n, 0 \leq n<r$;

$$
\left\{W_{s n}^{(r)}\right\} \varepsilon F
$$

if $d^{2} \nmid \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} \omega_{s, K-1}^{(r)}$, for at least one $n, 0 \leq n<r$.
In view of Theorem A, if $\left\{W_{s n}^{(r)}\right\}$ is a member of $F$ (or $L$ ), then any "tail" of $\left\{W_{s n}^{(r)}\right\}$ is also a member of $F$ (or $L$ ), respectively. Note that this partition of $\left\{W_{s n}^{(r)}\right\}$ is not unique, since in terms of (2.1) $L$ corresponds to $s=1$ and $F$ corresponds to $s=2$. We could proced with similar partitions for $s=3, \ldots, r$, but they do not tell us anything essentially new.
Theorem B: $\left\{X_{s n}^{(r)}\right\} \varepsilon F$ iff $\left\{Y_{s n}^{(r)}\right\} \varepsilon L$.
Proof: (i) If $\left\{X_{s n}^{(r)}\right\} \varepsilon F$, suppose that

$$
X_{s n}^{(r)}=d^{2 m} x_{s n}^{(r)} \quad \text { for all } n \geq 0
$$

where $m \geq 0$ is an integer, and

$$
\left\{x_{s n}^{(r)}\right\} \varepsilon F \quad \text { and } \quad d^{2} \mid x_{s n}^{(r)} \text { for at least one } n \geq 0
$$

Clearly $d^{2} \nmid x_{s 0}^{(r)}$, or $d^{2} \nmid x_{s 1}^{(r)}, \ldots$, or $d^{2} \nmid x_{s, r-1}^{(r)}$. By Theorem $A$,

$$
Y_{s n}^{(r)}=\sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} X_{s, K-1}^{(r)} \quad \text { for } 0 \leq n<r
$$

Let $\quad Y_{s n}^{(r)}=d^{2 m} y_{s n}^{(r)}$ for all $n \geq 0$.

Then

$$
y_{s n}^{(r)}=\sum_{\rho} \sum_{K} d_{\rho \mathrm{K}} \alpha_{\rho \rho}^{n} x_{s, k-1}^{(r)}, 0 \leq n<r .
$$

Since $x_{s n}^{(r)} \varepsilon F, d^{2} \nmid \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{r \rho}^{n} x_{s, k-1}^{(r)}$ for at least one $n, 0 \leq n<r$.
Therefore, $\quad d^{2} \nmid y_{s n}^{(r)} \quad$ for at least one $n, 0 \leq n<r$.

But it follows from Theorem $A$ that for all $n, 0 \leq n<r$,

$$
d^{2} x_{s n}^{(r)}=\sum_{\rho} \sum_{\kappa} d_{\rho \kappa} \alpha_{r \rho}^{n} y_{s, k-1}^{(r)}
$$

Therefore $\left\{y_{s n}^{(r)}\right\} \varepsilon L$, and so $\left\{y_{s n}^{(r)}\right\} \varepsilon L$.
(i.i) If $\left\{Y \begin{array}{l}(r) \\ s n\end{array}\right\} \varepsilon L$, suppose that

$$
Y_{s n}^{(r)}=d^{2 m} y_{s n}^{(r)} \text { for all } n \geq 0
$$

where $m \geq 0$ is an integer, and

$$
\left\{y_{s n}^{(r)}\right\} \varepsilon L \text { and } d^{2} \chi_{s n}^{(r)} \text { for at least one } n \geq 0
$$

Clearly $d^{2} \chi_{s 0}^{(r)}$, or $d^{2} \chi_{s 1}^{(r)}, \ldots$, or $d^{2} \chi_{s 1}^{(r)} y_{s, r-1}^{( }$. By Theorem $A$,

$$
X_{s n}^{(r)}=\frac{1}{d^{2}} \sum_{\rho} \sum_{\kappa} d_{\rho K} \alpha_{r \rho}^{n} y_{s, \kappa-1}^{(r)} \text { for } 0 \leq n<r
$$

Let

$$
X_{s n}^{(r)}=d^{2 m} x_{s n}^{(r)} \text { for a11 } n \geq 0
$$

Then

$$
x_{s n}^{(r)}=\frac{1}{d^{2}} \sum_{\rho} \sum_{\kappa} d_{\rho K} \alpha_{r \rho}^{n} y_{s, \kappa-1}^{(r)}, 0 \leq n<r
$$

Since $\left\{y_{s n}^{(r)}\right\} \varepsilon L, d^{2} \mid \sum_{\rho} \sum_{K} d_{\rho K} \alpha_{\rho \rho}^{n} y_{s, k-1}^{(r)}$ for all $n, 0 \leq n<r$.
So $x_{\boldsymbol{s} 0}^{(r)}, x_{\boldsymbol{s} 1}^{(r)}, \ldots, x_{\boldsymbol{s}, r-1}^{(r)}$ are integers and so $\left\{x_{s n}^{(r)}\right\} \varepsilon \omega$. But

$$
y_{s n}^{(r)}=\sum_{\rho} \sum_{k} d_{\rho \kappa} \alpha_{r \rho}^{n} x_{s, k-1}^{(r)} \text { for all } n, 0 \leq n<r,
$$

and since $d^{2} X_{s 0}^{(r)}$, or $d^{2} \chi_{y_{s 1}}^{(r)}, \ldots$, or $d^{2} X_{s, r-1}^{(r)}$, it follows that

$$
d^{2} \nmid \sum_{\rho} \sum_{\kappa} d_{\rho \kappa} \alpha_{r \rho}^{n} x_{s, \kappa-1}^{(r)} \text { for al least one } n, 0 \leq n<r
$$

Therefore $\left\{x_{s n}^{(r)}\right\} \in F$, and so $\left\{X_{s n}^{(r)}\right\} \varepsilon F$. This completes the proof of Theorem
B.
At this point, Hilton considered identities obtained from the binomial theorem. The corresponding application of the multinomial theorem to the roots $\alpha_{r j}$ of the auxiliary equation seems too complicated to pursue, though it is possible.

Another approach is to modify the method of Williams [10]: let

$$
\varepsilon=\exp (2 i \pi / r), \text { where } i^{2}=-1
$$

and as before

$$
d=\prod_{\substack{j, k=1 \\ j>k}}^{r}\left(\alpha_{r j}-\alpha_{r k}\right)
$$

If we let

$$
\alpha_{r j}=\frac{1}{r} \sum_{k=0}^{r-1} W_{k, r^{n}+1}^{(r)} d^{k} \varepsilon^{-j k} \quad(j=1,2, \ldots, r),
$$

then it is shown in Shannon [9] that

$$
\alpha_{r j}^{m}=r^{-1} \sum_{k=0}^{r-1} W_{k, n+r}^{(r)} d^{k} \varepsilon^{-j k},
$$

which seems to be a more useful form than the corresponding multinomial expression.

## REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." The Fibonacci Quarterly 3 (1965):161-176.
2. A. F. Horadam. "Generating Functions for Powers of a Certain Genera1ized Sequence of Numbers." Duke Math. J. 32 (1965):437-446.
3. A. F. Horadam. "Generalizations of Two Theorems of K. Subba Rao." Bull. Calcutta Math. Soc. 58, Nos. $1 \& 2$ (1966):23-29.
4. A. F. Horadam. "Special Properties of the Sequence $W(a, b ; p, q) . "$ The Fibonacci Quarterly 5 (1967):424-434.
5. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\left\{W_{n}(a, b ; p, q)\right\} . "$ The Fibonacci Quarterly 7 (1969):14-22.
6. A. J. W. Hilton. "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Generalized Lucas Sequences." The Fibonacei Quarterly 12 (1974):339-345.
7. Dov Jarden. Recurring Sequences. Jerusalem: Riveon Lematematika, 1966, p. 107.
8. Edouard Lucas. "Théorie des fonctions numériques périodiques." American J. Math. 1 (1878):184-240, 289-321.
9. A. G. Shannon. "A Generalization of the Hilton-Ferns Theorem on the Expansion of Fibonacci and Lucas Numbers." The Fibonacci Quarterly 12 (1974):237-240.
10. H. C. Williams. "On a Generalization of the Lucas Functions." Acta Arithmetica 20 (1972):33-51.

# DEGENERACY OF TRANSFORMED COMPLETE SEQUENCES 

GRAHAM LORD*
and
HERVE G. MORIN**
Université Laval, Québec, Canada

## 1. INTRODUCTION

A sequence $S=\left\{s_{i}\right\}_{i=1,2, \ldots}$ of natural numbers is said to be complete (see [4]) if every positive integer can be represented as the sum of distinct terms of $S$. If, furthermore, the sequence is nondecreasing and begins with $s_{1}=1$, then a necessary and sufficient condition in order that $S$ be complete is:

$$
s_{n+1} \leq 1+\sum_{i=1}^{n} s_{i}, \quad \text { for } n>1 \quad \text { (see [2]). }
$$

Note that this condition includes the possibility that some members of $S$ may be equal, thus corresponding to a representation in which certain terms may be repeated, as has been considered [1].

It is shown in [3] that completeness is preserved under certain transformations of $S$, an example of which is $x \rightarrow\langle\ln x\rangle$, where $\ln x$ is the natural logarithm of $x$ and $\langle\ln x\rangle$ is the smallest integer greater than $\ln x$. Two complete sequences to which this transformation is applied are the Fibonacci sequence and the sequence of prime numbers (with 1 included).

We will develop here conditions, on $S$ and the transformation considered, which guarantee that the transformed sequence is degenerate in the sense that it includes all natural numbers (and hence is complete). The above two sequences, discussed in [3], satisfy our conditions.

## 2. DEGENERACY

To ensure that a transformed sequence be a sequence of natural numbers, as well as complete, we use the following operation:

Definition: For $x$ a real number $\langle x\rangle$ is the smallest integer strictly greater than $x$.

Lemma: If $x, y$, and $z$ are real numbers such that $0 \leq x-y \leq z$, then

$$
0 \leq\langle x\rangle-\langle y\rangle<z+1 .
$$

Proof: By the definition of $\langle\cdot\rangle$,

$$
\langle x\rangle-1 \leq x\langle\langle x\rangle \text { and }\langle y\rangle-1 \leq y<\langle y\rangle,
$$

hence, $\quad z \geq x-y\rangle\langle x\rangle-1-\langle y\rangle$.
Clearly, $\quad x \geq y$ implies $\langle x\rangle \geq\langle y\rangle$.
We now use this property in the proof of our fundamental result. Note that the sequence $S$ need not be complete.

[^2] a real-valued nondecreasing function defined on $S$ such that
$$
\lim _{i \rightarrow \infty} f\left(s_{i}\right)=\infty .
$$

If there exists an $I$ such that for all $i \geq I, f\left(s_{i+1}\right)-f\left(s_{i}\right) \leq 1$, then the set $A=\left\{\left\langle f\left(s_{i}\right)\right\rangle \mid i=1,2, \ldots\right\}$ contains $\{M, M+1, \ldots\}$, the set of all integers greater than $M-1$, where $M=\left\langle f\left(s_{I}\right)\right\rangle$.
Proof: For any $i \geq I$, since $f$ is nondecreasing,

$$
0 \leq f\left(s_{i+1}\right)-f\left(s_{i}\right) \leq 1
$$

Thus, by the lemma

$$
0 \leq\left\langle f\left(s_{i+1}\right)\right\rangle-\left\langle f\left(s_{i}\right)\right\rangle<2 ;
$$

that is,

$$
\left\langle f\left(s_{i+1}\right)\right\rangle=\left\{\begin{array}{l}
\left\langle f\left(s_{i}\right)\right\rangle \\
\left\langle f\left(s_{i}\right)\right\rangle+1
\end{array}\right.
$$

But since $\lim _{i \rightarrow \infty} f\left(s_{i}\right)=\infty$ the transformed sequence

$$
\left\{\left\langle f\left(s_{i}\right)\right\rangle\right\}_{i=1,2}, \ldots
$$

cannot remain eventually constant and so $A \supset\{M, M+1, \ldots\}$.
Of particular interest is the case when $0 \leq f\left(s_{I}\right)<1$ : the set $A$ becomes the set $\mathbb{N}$ of all natural numbers. For example, if

$$
S=\{1,2,3,4,11,12,13, \ldots, n, \ldots\}
$$

and $f=1 n$, the transformed sequence is $\mathbb{N}$. Note that in this example

$$
s_{5}=2.75 s_{4}>e \cdot s_{4}
$$

Corollary 1: Let $S=\left\{s_{i}\right\}_{i=1,2, \ldots}$ be a nondecreasing sequence of integers with $s_{1}=1$ and $\lim _{i \rightarrow \infty} s_{i}=\infty$. If $a$ is a real number $>1$ such that $s_{i+1} \leq a s$, for all $i \geq 1$, then:

$$
\left\{\left\langle\ln _{b} s_{i}\right\rangle \mid i=1,2,3, \ldots\right\}=\mathbb{N}, \text { for all } b \geq a
$$

Proof: The conditions of the theorem are met with

$$
\begin{aligned}
M & =\left\langle\ln _{b} 1\right\rangle=1 \\
\text { and } \quad 0 & \leq \ln _{b} s_{i+1}-\ln _{b} s_{i} \leq \ln _{b} a \leq 1, \text { for all } i \geq 1
\end{aligned}
$$

Example 1: If $a$ is an integer $\geq 2$ and

$$
s_{i}=a^{i-1} \text { for } i=1,2, \ldots,
$$

then clearly $\left\{\left\langle\ln _{a} a^{i-1}\right\rangle\right\}=\mathbb{N}$.
Example 2: If $a=e$ (so $\ln _{a}$ becomes the natural logarithm 1 n ) and

$$
S=\{1,2,3,5,7,11, \ldots\}
$$

the prime numbers with 1 included, the transformed sequence passes through all natural numbers. The inequality condition of the corollary is satisfied by virtue of Bertrand's postulate $p_{n+1}<2 p_{n}$, where $p_{n}$ is the $n$th prime [5].

Example 3: Again, if $a=e$ but with the sequence $S$ now the sequence of Ficonacci numbers, the image is still $\mathbb{N}$, for

$$
F_{n+1}=\frac{(1+\sqrt{5})}{2} \cdot F_{n}+\left(\frac{1-\sqrt{5}}{2}\right)^{n}<\left(\frac{1+\sqrt{5}}{2}+1\right) F_{n}<e F_{n} .
$$

Example 4: $\left\{\left\langle\ln L_{n}\right\rangle \mid n=1,2, \ldots\right\}=I N$ where $L_{n}$ are the Lucas numbers.
The last three examples in fact satisfy the stronger conditions of:
Corollary 2: Let $S=\left\{s_{i}\right\}_{i=1,2, \ldots}$ be a nondecreasing sequence of integers starting $s_{1} \leq 2, s_{2} \leq 7$, with $\lim _{i \rightarrow \infty} s_{i}=\infty$, and satisfying the inequality

$$
s_{n+2} \leq s_{n+1}+s_{n}, n \geq 1
$$

Then $\quad\left\{\left\langle\ln s_{i}\right\rangle \mid i=1,2, \ldots\right\}=\mathbb{N}$.
Proof: If $s_{n+2} \leq s_{n+1}+s_{n}$, then $s_{n+2} \leq 2 \cdot s_{n+1}<e \cdot s_{n+1}$.
The theorem guarantees

$$
\left\{\left\langle\ln s_{n}\right\rangle \mid n=2, \ldots\right\}=\{M, M+1, \ldots\}
$$

where $\quad M=\left\langle\ln s_{2}\right\rangle \leq\langle\ln 7\rangle=2$.
And the condition on $s_{1}$ ensures $\left\langle\ln s_{1}\right\rangle=1$.
Note that Example 2 satisfies the conditions of Corollary 2 since, for the primes $p_{i+2} \leq p_{i+1}+p_{i}$. (See, for example [5, p. 139].)
Corollary 3: Let $S=\left\{s_{i}\right\}$, $i=1,2,3, \ldots$ be a nondecreasing sequence of integers with $s_{1}=1$ and $\lim _{i \rightarrow \infty} s_{i}=\infty$. Let $a$ and $b$ be two positive integers such that $s_{i+2} \leq a s_{i+1}+b s_{i}$ for all $i \geq 1$. Then

$$
\left\{\left\langle\ln _{c} s_{i}\right\rangle \mid i=1,2,3, \ldots\right\}=I N \text { for all } c \geq a+b
$$

Proof: Since $s_{i+2} \leq(a+b) s_{i+1}$ for all $i \geq 1$, we have

$$
0 \leq 1 n_{c} s_{i+2}-1 n_{c} s_{i+1} \leq 1 n_{c}(a+b) \leq 1
$$

Furthermore, $M=\left\langle\ln _{c} 1\right\rangle=1$. Hence, all the conditions of the theorem are met.

## 3. CONCLUDING REMARKS

As can be found in [3], there exist transformations which do not degenerate complete sequences, e.g., the Lucas transformation and the function

$$
f(x)=\alpha x \text {, where } 0<\alpha<1
$$

We note that even the quantized logarithmic transformation, $x \rightarrow\langle\ln x\rangle$, does not itself produce degeneracy as is shown by a complete sequence that begins $1,2,3,4,5,6,22, \ldots$.

Another example of an explicit function which sometimes degenerates sequences is $\Pi(x)$, the number of primes not exceeding the real number $x$. It is clear that this function degenerates the sequence of primes itself; as would the counting function of $S$, an arbitrary (countable) sequence, degenerate $S$. However, the image of the Fibonacci numbers under $\Pi$ is not $\mathbb{N}$, since $\Pi(8)=4$ but $\Pi(13)=6$.

A11 the sequences above are complete（although repetitions must be per－ mitted in Example 1 if $a>2$ ），but the theorem does not assume completeness． We conclude with an example of a sequence which is not complete but by an im－ mediate application of Corollary 1 is seen to be transformed into $\mathbb{N}$ under $f(x)=\ln x$. The sequence in question is $s_{1}=1, s_{2}=2$ ，and for $n \geq 3$ ， $s_{n}=5 \cdot 2^{n-3}$ ．To see that this sequence is not complete，observe that

```
            5}\cdot\mp@subsup{2}{}{n-3}-1, for n\geq3
```

can never be expressed as the sum of distinct terms of the sequence．
Finally，we would like to sincerely thank Professor Gerald E．Bergum for suggesting many improvements in the content and presentation of this article．

REFERENCES
1．H．L．Alder．＂The Number System in More General Scales．＂Mathematics Magazine 35 （1962）：145－151．
2．John L．Brown，Jr．＂Note on Complete Sequences of Integers．＂The Amer－ ican Math．Monthly 68 （1961）：557－560．
3．John L．Brown，Jr．＂Some Sequence－to－Sequence Transformations Which Pre－ serve Completeness．＂The Fibonacci Quarterly 16 （1978）：19－22．
4．V．E．Hoggatt，Jr．\＆C．H．King．＂Problem E－1424．＂The American Math． Monthly 67 （1960）：593．
5．Waclaw Sierpinski．Elementary Theory of Numbers．Warszawa， 1964.

## 事水为为

## Finding the general solution of a

LINEAR DIOPHANTINE EQUATION
SUSUMU MORITO and HARVEY M．SALKIN＊
Case Western Reserve University，Cleveland，OH 44106

## ABSTRACT

A new procedure for finding the general solution of a linear diophantine equation is given．As a byproduct，the algorithm finds the greatest common divisor（gcd）of a set of integers．Related results and discussion concern－ ing existing procedures are also given．

## 1．INTRODUCTION

This note presents an alternative procedure for computing the greatest common divisor of a set of $n$ integers $a_{1}, a_{2}, \ldots, a_{n}$ ，denoted by

$$
\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)
$$

[^3]and for finding the general solution of a linear diophantine equation in which these integers appear as coefficients. A classical procedure for finding the gcd of integers is based on the repeated application of the standard Euclidean Algorithm for finding the gcd of two integers. More specifically, it repeatedly uses the argument:
$$
\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\operatorname{gcd}\left(\operatorname{gcd}\left(a_{1}, a_{2}\right), a_{3}, \ldots, a_{n}\right)
$$

A more efficient algorithm, which is related to the procedure presented here for computing the gcd was given by Blankinship [1]. Weinstock [2] developed a procedure for finding a solution of a linear diophantine equation, and Bond [3] later showed that the Weinstock Algorithm can be applied repeatedly to find the general solution of a linear diophantine equation.

In this note, we present an alternative approach to finding the general solution, and show that the algorithm produces ( $n-1$ ) $n$-dimensional vectors with integer components whose integer linear combination generates all solutions which satisfy the linear diophantine equation with the right-hand side 0 . We call a set of these ( $n-1$ ) "generating" vectors a generator. It is easy to show that the generator is not unique for $n \geq 3$. In fact, for $n \geq 3$ there exist infinitely many generators. The proposed algorithm has certain desirable characteristics for computer implementation compared to the Bond Algorithm. Specifically, the Bond Algorithm generally produces generating vectors whose (integer) components are mostly huge numbers (in absolute values). This often makes computer implementation unwieldy [5]. The approach, presented here, was initially suggested by Walter Chase of the Naval Ocean Systems Center, San Diego, California, in a slightly different form for solving the radio frequency intermodulation problem [4].

For illustrative purposes, we will continuously use the following example with $n=3$ :

$$
\left(a_{1}, a_{2}, a_{3}\right)=(8913,5677,4378)
$$

Or, we are interested in the generator of:

$$
8913 x_{1}+5677 x_{2}+4378 x_{3}=0
$$

It turns out that the Bond Algorithm [3] produces the two generating vectors $(5677,-8913,0)$ and $(2219646,3484888,-1)$, whereas the procedure we propose gives (cf. Section 3) ( $-57,17,94$ ) and ( $61,-95,-1$ ).

Three obvious results are given without proof. Throughout this paper, we assume that the right-hand side of a linear diophantine equation $a_{0}$, if it is nonzero, is an integer multiple of $d=\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$. This is because of the well-known result [6] which says that a linear diophantine equation has a solution if and only if $\alpha_{0}$ is divisible by $d$, and if $d$ divides $\alpha_{0}$ there are an infinite number of solutions.
Lemma 1: Consider the following two equations:

$$
\begin{align*}
& a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}=0  \tag{1}\\
& a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}=a_{0} \tag{1}
\end{align*}
$$

Assume that $\left(x_{F_{1}}, \ldots, x_{F_{n-1}}\right.$ ) is the generator of (1). Then, all solutions $x=\left(x_{i}\right)$ of (2) can be expressed in the form

$$
\begin{equation*}
x=x^{0}+k_{1} x_{F_{1}}+k_{2} x_{F_{2}}+\cdots+k_{n-1} x_{F_{n-1}}, \tag{3}
\end{equation*}
$$

where $x^{0}$ is any solution satisfying (2) and $k_{1}, k_{2}, \ldots, k_{n-1}$ are any integers.

Lemma 2: If $a_{1}^{\prime}=a_{1}+\ell_{2} a_{2}+\ell_{3} a_{3}+\cdots+\ell_{n} a_{n}$ for some integers $\ell_{2}, \ell_{3}, \ldots, \ell_{n}$, then gcd $\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\operatorname{gcd}\left(a_{1}^{\prime}, a_{2}, \ldots, a_{n}\right)$.
Lemma 3: If $a_{1}+\ell_{2} a_{2}+\ell_{3} a_{3}+\ldots+\ell_{n} a_{n}=0$ for some integers $\ell_{2}, \ell_{3}, \ldots, \ell_{n}$, then gcd $\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\operatorname{gcd}\left(a_{2}, \ldots, a_{n}\right)$.

Notice, for example, Lemma 3 is true because if

$$
d=\operatorname{gcd}\left(a_{2}, \ldots, a_{n}\right)
$$

then

$$
a_{1}=\left(\sum_{i=2}^{n} \ell_{i}^{\prime}\right) \cdot d \text {, for some integers } \ell_{i}^{\prime}(2 \leq i \leq n) .
$$

Thus,

$$
\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\operatorname{gcd}\left(a_{1}, d\right)=d
$$

Finding the general solution of a linear diophantine equation having a right-hand side different from zero (say $a_{0} \neq 0$ ) is straightforward, because of Lemma 1, if the generator and one solution for (2) is known. The algorithm we propose first finds a solution, say $x^{d}$, for the linear diophantine equation with right-hand side $d=\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ as well as the generator. Then a solution for (2) can be found as $\left(\alpha_{0} / d\right) x^{d}$.

## 2. THE ALGORITHM

We now present the algorithm to find the general solution of the linear diophantine equation (2). The method is based on Lemma 1 , namely, it finds the generator ( $x_{F_{1}}, x_{F_{2}}, \ldots, x_{F_{n-1}}$ ) of (1) as well as any one solution $x^{0}$ of (2), so that any solution of (2) can be expressed as in (3). A solution $x^{0}$ of (2) is found as a by-product of finding the generator. We list the steps:

$$
\text { Step 0. Set } k=1, b_{1}^{(1)}=a_{1}, b_{2}^{(1)}=a_{2}, \ldots, b_{n}^{(1)}=a_{n} \text {, and } N=n \text {. }
$$

Also let
$x\left(b_{1}^{(1)}\right)=(1,0, \ldots, 0), x\left(b_{2}^{(1)}\right)=(0,1,0, \ldots, 0), \ldots$,
$x\left(b_{n}^{(1)}\right)=(0, \ldots, 0,1)$,
where $x(b)$ denotes the solution of (2) with right-hand side $a_{0}=b$ 。
Step 1. Find integers $\ell_{2}, \ell_{3}, \ldots, \ell_{N}$ so that they satisfy

$$
\begin{aligned}
b_{1}^{(k)} & =\ell_{2} b_{2}^{(k)}+r_{2}, 0 \leq r_{2}<b_{2}^{(k)} \\
r_{2} & =\ell_{3} b_{3}^{(k)}+r_{3}, 0 \leq r_{3}<b_{3}^{(k)} \\
\vdots & \\
r_{N-1} & =\ell_{N} b_{N}^{(k)}+r_{N}, 0<r_{N}=b_{1}^{\prime}<b_{N}^{(k)},
\end{aligned}
$$

and thus

$$
b_{1}^{(k)}=\ell_{2} b_{2}^{(k)}+\ell_{3} b_{3}^{(k)}+\cdots+\ell_{N} b_{N}^{(k)}+b_{1}^{\prime} .
$$

Step 2. Find a solution $x\left(b_{1}^{\prime}\right)$ for $a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}=b_{1}^{\prime}$ as follows:
$x\left(b_{1}^{\prime}\right)=x\left(b_{1}^{(k)}\right)-\ell_{2} x\left(b_{2}^{(k)}\right)-\ell_{3} x\left(b_{3}^{(k)}\right)-\cdots-\ell_{N} x\left(b_{N}^{(k)}\right)$.

Step 3. If $b_{1}^{\prime}=0, x\left(b_{1}^{\prime}\right)$ is one of the generating vectors. Eliminate one variable, i.e., $N=N-1$, and set

$$
b_{1}^{(k+1)}=b_{2}^{(k)}, b_{2}^{(k+1)}=b_{3}^{(k)}, \ldots, b_{N}^{(k+1)}=b_{N+1}^{(k)}
$$

If $N=1$, go to Step 4 (termination). If $N>1$, increment the iteration count (i.e., $k=k+1$ ) and return to Step 1. If $b^{\prime} \neq 0$, set $b_{1}^{(k+1)}=b_{2}^{(k)}, b_{2}^{(k+1)}=b_{3}^{(k)}, \ldots, b_{N}^{(k+1)}=b_{1}^{\prime}$, $k=k+1$ and return to Step 1 .
Step 4. We now have ( $n-1$ ) generating vectors for (1), and $b_{1}^{(k+1)}$ is the gcd $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$. A solution for (2) can be found as

$$
\frac{a_{0}}{b_{1}^{(k+1)}} x\left(b_{1}^{(k+1)}\right)
$$

Stop.
We now give three results which show the validity of the algorithm.
Theorem 1: There is a one-to-one correspondence between the solutions of (4) and (5):

$$
\begin{align*}
& a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}=0  \tag{4}\\
& b_{1}^{(k)} y_{1}+b_{2}^{(k)} y_{2}+\cdots+b_{n}^{(k)} y_{n}=0 \tag{5}
\end{align*}
$$

Here $b_{1}^{(k)} \neq 0, b_{2}^{(k)} \neq 0, \ldots, b_{n}^{(k)} \neq 0$ correspond to the values obtained for $b_{i}$ in the $k$ th iteration of Step 1 , as far as $N=n$.

Proo6: Consider the following two equations corresponding to any two consecutive iterations of the algorithm:

$$
\begin{aligned}
\text { (kth iteration }) & b_{1}^{(k)} y_{1}^{(k)}+b_{2}^{(k)} y_{2}^{(k)}+\cdots+b_{n}^{(k)} y_{n}^{(k)}=0 ; \\
(k+1 \text { st iteration }) & \left(b_{1}^{(k)}-\ell_{2} b_{2}^{(k)}\right. \\
& \left.-\cdots-\ell_{n} b_{n}^{(k)}\right) y_{1}^{(k+1)}+b_{2}^{(k)} y_{2}^{(k+1)} \\
& +\cdots+b_{n}^{(k)} y_{n}^{(k+1)}=0 .
\end{aligned}
$$

The second equation can be written as

$$
b_{1}^{(k)} y_{1}^{(k+1)}+b_{2}^{(k)}\left(y_{2}^{(k+1)}-\ell_{2} y_{1}^{(k+1)}\right)+\cdots+b_{n}^{(k)}\left(y_{n}^{(k+1)}-\ell_{n} y_{1}^{(k+1)}\right)=0 .
$$

This means

$$
y_{1}^{(k)}=y_{1}^{(k+1)}, y_{2}^{(k)}=y_{2}^{(k+1)}-\ell_{2} y_{1}^{(k+1)}, \ldots, y_{n}^{(k)}=y_{n}^{(k+1)}-\ell_{n} y_{1}^{(k+1)}
$$

Using vector-matrix notation, we have

$$
y^{(k)}=\left(\begin{array}{c}
y_{1}^{(k)} \\
y_{2}^{(k)} \\
\vdots \\
y_{n}^{(k)}
\end{array}\right)=\left(\begin{array}{cccc}
-\ell_{2} & 1 & 0 & 0 \\
-\ell_{3} & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\ell_{n} & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{array}\right)\left(\begin{array}{c}
y_{1}^{(k+1)} \\
y_{2}^{(k+1)} \\
\vdots \\
y_{n}^{(k+1)}
\end{array}\right)=T y^{(k+1)} .
$$

Notice that $|\operatorname{det} T|$ (i.e., the absolute value of the determinant of $T$ ) $=1$. We now show inductively on $k$ that there exists a matrix $M$ such that

$$
x=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\vdots \\
x_{n}
\end{array}\right)=M y^{(k)}
$$

which satisfies $\mid$ det $M \mid=1$. Clearly, for the first iteration, $T=M$ and
$|\operatorname{det} T|=|\operatorname{det} M|=1$.
Assume that there exists a matrix $M^{\prime}$ with $\mid$ det $M^{\prime} \mid=1$ such that $x=M^{\prime} y^{(k)}$. Substituting $y^{(k)}=T y^{(k+1)}$, we get $x=M^{\prime} T y^{(k+1)}$. As
$\left|\operatorname{det}\left(M^{\prime} T\right)\right|=\left|\operatorname{det} M^{\prime}\right| \times|\operatorname{det} T|=1$.
Thus, $x=M y{ }^{(k+1)}$ where $M=M^{\prime} T$ and $\mid$ det $M \mid=1$.
It is well known (e.g., see [7]) that, if there exists a matrix $M$ such that $x=M y$ with $|\operatorname{det} M|=1$, there is a one-to-one correspondence between the solutions $x$ and $y$. Thus, the theorem is proved. Q.E.D.
Theorem 2: If $\left(y_{F_{1}}, y_{F_{2}}, \ldots, y_{F_{n-1}}\right)$ is the generator of (5), the corresponding $\left(x_{F_{1}}, x_{F_{2}}, \ldots, x_{F_{n-1}}\right)$ is the generator of (4).
Proo6: Assume that ( $x_{F_{1}}, x_{F_{2}}, \ldots, x_{F_{n-1}}$ ) is not the generator of (4). Then there exists a solution vector $x$ satisfying (4) such that it cannot be expressed as an integer linear combination of $x_{F_{1}}, x_{F_{2}}, \ldots, x_{F_{n-1}}$. However, because of the one-to-one correspondence (Theorem 1), there exists a unique $y$ which corresponds to $x$ (i.e., $M y=x$ ), and there are integers $\beta_{1}, \beta_{2}, \ldots$, $\beta_{n-1}$ such that $y=\beta_{1} y_{F_{1}}+\beta_{2} y_{F_{2}}+\cdots+\beta_{n-1} y_{F_{n-1}}$ as $\left(y_{F_{1}}, y_{F_{2}}, \ldots, y_{F_{n-1}}\right)$ is the generator. However,

$$
\begin{aligned}
x=M y & =M\left(\beta_{1} y_{F_{1}}+\beta_{2} y_{F_{2}}+\cdots+\beta_{n-1} y_{F_{n-1}}\right) \\
& =\beta_{1} M y_{F_{1}}+\beta_{2} M y_{F_{2}}+\cdots+\beta_{n-1} M y_{F_{n-1}} \\
& =\beta_{1} x_{F_{1}}+\beta_{2} x_{F_{2}}+\cdots+\beta_{n-1} x_{F_{n-1}},
\end{aligned}
$$

and thus a contradiction. Q.E.D.
Theorem 3: Assume that $d=\operatorname{gcd}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\operatorname{gcd}\left(a_{2}, \ldots, a_{n}\right)$. Then the general solution of (6) can be expressed as $x=k x^{0}+x^{\prime}$, where $k$ is an integer, $x^{0}$ any solution of (7), and $x^{\prime}$ the general solution of (8).

$$
\begin{align*}
a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n} & =0 ;  \tag{6}\\
a_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n} & =0 ;  \tag{7}\\
a_{2} x_{2}+\cdots+a_{n} x_{n} & =0 \tag{8}
\end{align*}
$$

Proof: Since $d$ divides $a_{1}$, we have, for $\ell$ integer, $\alpha_{1}=l d$, and thus there are solutions $x_{2}, x_{3}, \ldots, x_{n}$ to (7). This means (6) has solutions when $x_{1}$ is fixed to any integer. Clearly, $x^{0}$ is any such solution to (6) in which $x_{1}=$ 1. Observe that all solutions for (6) can be characterized by fixing $x_{1}$ to any integer $k$ and solving (6) in the remaining variables, $x_{2}, x_{3}, \ldots, x_{n}$. More specifically, for $x_{1}$ fixed to $k$, we want all solutions which satisfy

$$
\begin{equation*}
a_{2} x_{2}+\cdots+a_{n} x_{n}=-a_{1} k \tag{6}
\end{equation*}
$$

From Lemma 1, however, solutions for (6)' can be expressed as a sum of a solution for (6)' and the general solution for (8). Thus,

$$
x=k x^{0}+\left(k_{1} x_{F_{1}}+k_{2} x_{F_{2}}+\cdots+k_{n-2} x_{F_{n-2}}\right)
$$

is the general solution for (6) for integer, $k_{1}, k_{2}, \ldots, k_{n-2}$, where $k x^{0}$ is a solution satisfying (6)' and ( $x_{F_{1}}, x_{F_{2}}, \ldots, x_{F_{n-2}}$ ) is the generator of (8) with $x_{1}=0$. Setting

$$
x^{\prime}=\sum_{i=1}^{n-2} k_{i} x_{F_{i}},
$$

means that $x^{\prime}$ is any solution to (8), and hence the result. Q.E.D.

## 3. EXAMPLE AND DISCUSSION

Table 1 lists the computational process for finding the generator ( $x_{F_{1}}$, $x_{F_{2}}$ ) for a 3-variable diophantine equation with the right-hand side equal to zero. The two vectors

$$
\left(\begin{array}{r}
-57 \\
17 \\
94
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{r}
61 \\
-95 \\
-1
\end{array}\right)
$$

form the generator.
From Theorem 1, there is a one-to-one relationship between (9) and (10):

$$
\begin{align*}
8913 x_{1}+5677 x_{2}+4378 x_{3} & =0  \tag{9}\\
10 y_{1}+5 y_{2}+3 y_{3} & =0 \tag{10}
\end{align*}
$$

The relationship is $x=M y$, where

$$
M=\left(\begin{array}{rrr}
-3 & 27 & 4 \\
-3 & -10 & 13 \\
10 & -42 & -25
\end{array}\right),|\operatorname{det} M|=1
$$

From Theorem 2, the generator ( $y_{F_{1}}, y_{F_{2}}$ ) of (10), if found, will be translated to the generator

$$
\left(x_{F_{1}}, x_{F_{2}}\right)=\left(M y_{F_{1}}, M y_{F_{2}}\right)
$$

of (9).
Iteration 10 of the algorithm (cf. Table 1) finds a solution

$$
y=\left(\begin{array}{r}
1 \\
-2 \\
0
\end{array}\right)
$$

for (10), and from Theorem 3, the general solution for (10) can be found as

$$
k\left(\begin{array}{r}
1 \\
-2 \\
0
\end{array}\right)+y^{\prime}, \text { where } y^{\prime}=\left(\begin{array}{l}
0 \\
y_{2} \\
y_{3}
\end{array}\right)
$$

is the general solution for (10) with $y_{1}=0$. Iterations 11 through 13 are performed to find the general solution for

$$
\begin{equation*}
5 y_{2}+3 y_{3}=0 \tag{11}
\end{equation*}
$$

It can easily be checked that the general solution for (11) is

$$
y^{\prime}=\ell\left(\begin{array}{r}
0 \\
3 \\
-5
\end{array}\right) \text { for } \ell \text { integer. }
$$

Thus,

$$
\left(\begin{array}{r}
1 \\
-2 \\
0
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{r}
0 \\
3 \\
-5
\end{array}\right)
$$

form a generator for (10).
TABLE 1. ALGORITHM COMPUTATIONS


In general, whenever the final remainder (i.e., $b^{\prime}$ ) of Step 1 in each iteration becomes 0 , we obtain a vector which is one of the $n-1$ generating vectors, and the size of problem (i.e., the number of variables) is reduced by 1 .

Theorem 3 shows that this elimination of one variable at a time guarantees the generating characteristic. After the problem is reduced, the same arguments (i.e., Theorem 1-Theorem 3) will be applied to the reduced problem, sequentially. Eventually, a 2-variable problem will be solved which yields the ( $n-1$ )st or last generating vector, and the process terminates.

From Lemmas 2 and 3, the last nonzero remainder in the algorithm gives the greatest common divisor of $a_{1}, a_{2}, \ldots, a_{n}$. In the example, detailed in Table 1, the last nonzero remainder is 1 and is the gcd of 8913,5677 , and 4378. To see this, note that

```
    gcd (8913, 5677, 4378) = gcd (10, 5, 3)
```

by Lemma 2 which, in turn, is equal to gcd $(5,3)$ by Lemma 3. Repeating the same argument gives

$$
\operatorname{gcd}(5,3)=\operatorname{gcd}(3,2)=\operatorname{gcd}(2,1)=\operatorname{gcd}(1)=1,
$$

or
gcd $(8913,5677,4378)=1$.

Finally, Table 1 displays a solution for the equation with the righthand side equal to $1=\operatorname{gcd}(8913,5677,4378)$. The general solution for the equation with the right-hand side $\alpha_{0}$ can then be expressed as:

$$
a_{0}\left(\begin{array}{r}
-19 \\
36 \\
-8
\end{array}\right)+k_{1}\left(\begin{array}{r}
-57 \\
17 \\
94
\end{array}\right)+k_{2}\left(\begin{array}{r}
61 \\
-95 \\
-1
\end{array}\right)
$$

where $k_{1}$ and $k_{2}$ are integers.

## REMARKS

1. An examination of the algorithm indicates that the divisions in Step 1 can be made computationally more efficient by using the least absolute remainder rather than the positive remainder. Specifically, we find $\ell_{i}(i=2, \ldots$, $N$ ) such that $\left|r_{i}\right|$ is minimized ( $0 \leq\left|r_{i}\right| \leq b_{i}^{(k)}$ ) in Step 1 , rather than using $r_{i}$, where $0 \leq r_{i} \leq b_{i}$. This change allows the proofs of the theorems to go through essentially unchanged.
2. The preceding discussion can be used to show that the Blankinship Alsorithm [1] for finding the gcd of $n$ integers will also find the general solution of a linear diophantine equation. Specifically, the algorithm presented here can be regarded as a modified Blankinship Algorithm where the modification is in selecting the operators (according to Blankinship's terminology). The Blankinship Algorithm, on the other hand, can be regarded as a special case of our method where $\ell_{2} \equiv \ell_{3} \equiv \ldots \equiv \ell_{n-1} \equiv 0$ in Step 2 of the algorithm presented here.

## REFERENCES

1. W. A. Blankinship. "A New Version of the Euclidean Algorithm." American Math. Monthly 70, No. 7 (1963):742-745.
2. R. Weinstock. "Greatest Common Divisor of Several Integers and an Associated Linear Diophantine Equation." American Math. Monthly 67, No. 7 (1960):664-667.
3. J. Bond. "Calculating the General Solution of a Linear Diophantine Equation." American Math. Monthly 74, No. 8 (1967):955-957.
4. W. Chase. "The Indirect Threat Algorithm." Technical Memorandum, Naval Electronics Laboratory Center, San Diego, California, November 1975.
5. S. Morito \& H. M. Salkin. "A Comparison of Two Heuristic Algorithms for a Radio Frequency Intermodulation Problem." Technical Memorandum, Case Western Reserve University, Department of Operations Research, Cleveland, Ohio, October 1977.
6. T. Saaty. Optimization in Integers and ReZated Extremal Problems. New York: McGraw-Hill, 1970.
7. H. M. Salkin. Integer Programming. Reading, Mass.: Addison-Wesley, 1975.

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>A. P. HILLMAN<br>University of New Mexico, Albuquerque, NM 87131

Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR. S.E., ALBUQUERQUE, NEW MEXICO 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy

$$
F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1
$$

and

$$
L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1
$$

Also, $a$ and $b$ designate the roots $(1+\sqrt{5}) / 2$ and $(1-\sqrt{5}) / 2$, respectively, of $x^{2}-x-1=0$.

PROBLEMS PROPOSED IN THIS ISSUE
B-412 Proposed by Phil Mana, Albuquerque, $N M$
Find the least common multiple of the integers in the infinite set

$$
\left\{2^{9}-2,3^{9}-3,4^{9}-4, \ldots, n^{9}-n, \ldots\right\}
$$

B-413 Proposed by Herta T. Freitag, Roanoke, VA
For every positive integer $n$, let $U_{n}$ consist of the points $j+k e^{2 \pi i / 3}$ in the Argand plane with $j \in\{0,1,2, \ldots, n\}$ and $k \in\{0,1, \ldots, j\}$. Let $T(n)$ be the number of equilateral triangles whose vertices are subsets of $U_{n}$. For example, $T(1)=1, T(2)=5$, and $T(3)=13$.
a. Obtain a formula for $T(n)$;
b. Find all $n$ for which $T(n)$ is an integral multiple of $2 n+1$.

B-414 Proposed by Herta T. Freitag, Roanoke, VA
Let $S_{n}=L_{n+5}+\binom{n}{2} L_{n+2}-\sum_{i=2}^{n}\binom{i}{2} I_{i}-11$. Determine all $n$ in $\{2,3,4$, ...\} for which $S_{n}$ is (a) prime; (b) odd.

B-415 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA
The circumference of a circle in a fixed plane is partitioned into $n$ arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings which can be rotated into one another should be considered to be the same.

B-416 Proposed by Gene Jakubowski and V.E. Hoggatt, Jr. San Jose State University, San Jose, CA

Let $F_{n}$ be defined for all integers (positive, negative, and zero) by
and hence

$$
F_{0}=0, F_{1}=1, F_{n+2}=F_{n+1}+F_{n},
$$

$$
F_{n}=F_{n+2}-F_{n+1} .
$$

Prove that every positive integer $m$ has at least one representation of the form

$$
m=\sum_{-N}^{N} \alpha_{j} F_{j},
$$

with each $\alpha_{j}$ in $\{0,1\}$ and $\alpha_{j}=0$ when $j$ is an integral multiple of 3 .

## B-417 Proposed by R. M. Grassl and P. L. Mana

 University of New Mexico, Albuquerque, $N M$Here let $[x]$ be the greatest integer in $x$. Also, let $f(n)$ be defined by $f(0)=1=f(1), f(2)=2, f(3)=3$, and

$$
f(n)=f(n-4)+\left[1+(n / 2)+\left(n^{2} / 12\right)\right]
$$

for $n \varepsilon\{4,5,6, \ldots\}$. Do there exist rational numbers $a, b, c, d$ such that

$$
f(n)=\left[a+b n+c n^{2}+d n^{3}\right] ?
$$

## SOLUTIONS

## Partitioning Squares Near the Diagonals

B-388 Proposed by Herta T. Freitag, Roanoke, VA
Let $T_{n}$ be the triangular number $n(n+1) / 2$. Show that

$$
T_{1}+T_{2}+T_{3}+\cdots+T_{2 n-1}=1^{2}+3^{2}+5^{2}+\cdots+(2 n-1)^{2}
$$

and express these equal sums as a binomial coefficient.
Solution by Phil Mana, Albuquerque, NM
It is readily seen that $T_{1}=1=1^{2}$ and $T_{2 k}+T_{2 k+1}=(2 k+1)^{2}$ for $k=$ $1,2, \ldots$. . The displayed equation then follows. Next one notes that

$$
\begin{aligned}
T_{1}+T_{2}+\cdots+T_{2 n-1} & =\binom{2}{2}+\binom{3}{2}+\cdots+\binom{2 n}{2} \\
& =\binom{3}{3}+\left[\binom{4}{3}-\binom{3}{3}\right]+\cdots+\left[\binom{2 n+1}{3}-\binom{2 n}{3}\right] \\
& =\binom{2 n+1}{3} .
\end{aligned}
$$

Also solved by Paul Bracken, Wray G. Brady, Paul S. Bruckman, R. Garfield, Hans Klauser (Switzerland), Peter A. Lindstrom, Graham Lord, Ellen R. Miller, C. B. A. Peck, Bob Prielipp, A. G. Shannon (Australia), Sahib Singh, Paul Smith, Lawrence Somer, Rolf Sonntag (W. Germany), Gregory Wulczyn, and proposer.

## Transformed Arithmetic Progression

B-389 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA
Find the complete solution, with two arbitrary constants, of the difference equation

$$
\left(n^{2}+3 n+3\right) U_{n+2}-2\left(n^{2}+n+1\right) U_{n+1}+\left(n^{2}-n+1\right) U_{n}=0
$$

Solution by Paul S. Bruckman, Concord, CA
Let

$$
\begin{align*}
& V_{n}=\left(n^{2}-n+1\right) U_{n} .  \tag{1}\\
& \text { Then, } \\
& V_{n+1}=\left(n^{2}+n+1\right) U_{n+1}, V_{n+2}=\left(n^{2}+3 n+3\right) U_{n+2}, \text { and so }  \tag{2}\\
& V_{n+2}-2 V_{n+1}+V_{n}=0, \\
& \Delta^{2} V_{n}=0 .
\end{align*}
$$

It follows that $V_{n}=a n+b$, for some constants $a$ and $b$. Note that

$$
V_{0}=b=U_{0}, \text { and } V_{1}=a+b=U_{1}
$$

Hence, $b=U_{0}$ and $a=U_{1}-U_{0}$, which implies

$$
\begin{equation*}
U_{n}=\frac{\left(U_{1}-U_{0}\right) n+U_{0}}{n^{2}-n+1} \tag{4}
\end{equation*}
$$

Also solved by Wray G. Brady, R. Garfield, C. B. A. Peck, Sahib Singh, Paul Smith, and proposer.

## Generating Diagonals of Pascal's Triangle

B-390 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA
Find, as a rational function of $x$, the generating function

$$
G_{k}(x)=\binom{k}{k}+\binom{k+1}{k} x+\binom{k+2}{k} x^{2}+\cdots+\binom{k+n}{k} x^{n}+\cdots, \quad|x|<1 .
$$

I. Solution by Ralph Garfield, College of Insurance; Graham Lord, Université Laval; and Paul Smith, University of Victoria (independently).
$G_{k}(x)$ is well known to be $(1-x)^{-k-1}$. (Consider the Taylor series or Newton binomial expansion of this latter function.)
II. Solution by Wray G. Brady, Slippery Rock State College; Robert M. Giuli, University of California, Santa Cruz, and Herta T. Freitag, Roanoke, VA (independently).

First we show the identity $G_{k}(x)=G_{k-1}(x) /(1-x)$ or

$$
G_{k-1}(x)=(1-x) G_{k}(x)
$$

$(1-x) G(x)=\binom{k}{k}+\left[\binom{k+1}{k}-\binom{k}{k}\right] x+\left[\binom{k+2}{k}-\binom{k+1}{k}\right] x^{2}+\cdots$
(continued)

$$
\begin{aligned}
& =\binom{k-1}{k-1}+\binom{k}{k-1} x+\binom{k+1}{k-1} x^{2}+\cdots \\
& =G_{k-1}(x)
\end{aligned}
$$

Now an induction will show $G_{k}(x)=1 /(1-x)^{k+1}$ since $G_{0}(x)=1 /(1-x)$.
III. Solution by Paul Bracken (Toronto); Phil Mana; and C. B. A. Peck (independently).

Let $F_{k}(x)=(1-x)^{-k-1}$. Then one readily sees that $F_{0}(x)=(1-x)^{-1}=G_{0}(x), F_{k}(0)=1=G_{k}(0)$,

$$
d F_{k}(x) / d x=(k+1) F_{k+1}(x), d G_{k}(x) / d x=(k+1) G_{k+1}(x)
$$

Using integration and induction, one establishes that

$$
G_{k}(x)=F_{k}(x)=(1-x)^{-k-1} \text { for } k=0,1,2, \ldots .
$$

Also solved by Paul S. Bruckman, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and proposer.

## Approximations to Root Five

B-391 Proposed by M. Wachtel, Zurich, Switzerland
Some of the solutions of $5 x^{2}+1=y^{2}$ in positive integers $x$ and $y$ are $(x, y)=(4,9),(72,161),(1292,2889),(23184,51841)$, and $(416020,930249)$. Find a recurrence formula for the $x_{n}$ and $y_{n}$ of a sequence of solutions ( $x_{n}, y_{n}$ ) and find $\lim _{n \rightarrow \infty}\left(x_{n+1} / x_{n}\right)$ in terms of $\alpha=(1+\sqrt{5}) / 2$.

Solution by Paul S. Bruckman, Concord, California
The Diophantine equation

$$
\begin{equation*}
y^{2}-5 x^{2}=1 \tag{1}
\end{equation*}
$$

is a special case of the general Pell equation: $y^{2}-m x^{2}=1$, where $m$ is not a square. From the theory of the Pell equation, it is known that (1) possesses infinitely many solutions, and indeed that all of the solutions ( $x_{n}, y_{n}$ ) in positive integers are given by the relation:
$y_{n}+x_{n} \sqrt{5}=\left(y_{1}+x_{1} \sqrt{5}\right)^{n}, n=1,2,3, \ldots$,
where $\left(x_{1}, y_{1}\right)$ is the minimal solution.
We readily find that $\left(x_{1}, y_{1}\right)=(4,9)$. Let $A=9+4 \sqrt{5}$ and $B=9-4 \sqrt{5}$. Note that $A=(2+\sqrt{5})^{2}=a^{6}$ and $B=A^{-1}=b^{6}$. Since $y_{n}-x_{n} \sqrt{5}=B^{n}$, it follows that $y_{n}=\left(A^{n}+B^{n}\right) / 2=\left(a^{6 n}+b^{6 n}\right) / 2$, and

$$
\begin{align*}
& x_{n}=\frac{1}{2 \sqrt{5}}\left(A^{n}-B^{n}\right)=\frac{a^{6 n}-b^{6 n}}{2(a-b)}, \text { or } \\
& \left(x_{n}, y_{n}\right)=\left(\frac{1}{2} F_{6 n}, \frac{1}{2} L_{6 n}\right), n=1,2,3, \ldots . \tag{3}
\end{align*}
$$

Since $(z-A)(z-B)=z^{2}-18 z+1$, it follows that $x_{n}$ and $y_{n}$ satisfy the common recursion:
(4)

$$
z_{n+2}-18 z_{n+1}+z_{n}=0
$$

Moreover, $L \equiv \lim _{n \rightarrow \infty}\left(x \quad \mid x_{n}\right)=\lim _{n \rightarrow \infty}\left(\frac{A^{n+1}-B^{n+1}}{A^{n}-B^{n}}\right)=A$, since $A>1,0<B<1$,
i.e.,
(5)

$$
L=a^{6}
$$

Also solved by Wray G. Brady, C. B. A. Peck, A. G. Shannon, Sahib Singh, Paul Smith, and proposer.

$$
\text { Half-Way Application of }\left(E^{2}-E-1\right)^{2}
$$

B-392 Proposed by Phil Mana, Albuquerque, NM
Let $Y_{n}=(2+3 n) F_{n}+(4+5 n) L_{n}$. Find constants $h$ and $k$ such that

$$
Y_{n+2}-Y_{n+1}-Y_{n}=h F_{n}+k L_{n}
$$

Solution by Graham Lord, Université Laval, Québec

$$
\begin{aligned}
& Y_{n+2}-Y_{n+1}-Y_{n}=(2+3 n+6) F_{n+2}+(4+5 n+10) L_{n+2}-(2+3 n+3) F_{n+1} \\
&-(4+5 n+5) L_{n+1}-(2+3 n) F_{n}-(4+5 n) L_{n} \\
&= 6 F_{n+2}-3 F_{n+1}+10 L_{n+2}-5 L_{n+1}=20 F_{n}+14 L_{n} .
\end{aligned}
$$

Thus $h=20$ and $k=14$.
Also solved by Paul Bracken, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, John W. Milsom, C. B.A. Peck, Bob Prielipp, A. G. Shannon, Sahib Singh, Paul Smith, Rolf Sonntag, Gregory Wulczyn, and proposer.
Triangle of Triangular Factorials

B-393 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA Let $T_{n}=\binom{n+1}{2}, P_{0}=1, P_{n}=T_{1} T_{2} \ldots T_{n}$ for $n>0$ and $\left[\begin{array}{l}n \\ k\end{array}\right]=P_{n} / P_{k} P_{n-k}$ for integers $k$ and $n$ with $0 \leq k \leq n$. Show that

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]=\frac{1}{n-k+1}\binom{n}{k}\binom{n+1}{k+1} .
$$

Solution by Paul S. Bruckman, Concord, CA

$$
\begin{aligned}
P_{n}=\prod_{k=1}^{n} T_{k} & =\prod_{k=1}^{n} k(k+1) / 2=2^{-n} n!(n+1)!\text { Therefore } \\
{\left[\begin{array}{l}
n \\
k
\end{array}\right] } & =\frac{P_{n}}{P_{k} P_{n-k}}=\frac{n!(n+1)!2^{k} 2^{n-k}}{2^{n} k!(k+1)!(n-k)!(n+1-k)!} \\
& =\frac{n!}{k!(n-k)!} \cdot \frac{(n+1)!}{(k+1)!(n+1-k)!}=\frac{\binom{n}{k}\binom{n+1}{k+1}}{n-k+1}
\end{aligned}
$$

Also solved by Herta T. Freitag, Ralph Garfield, Peter A. Lindstrom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Sahib Singh, Paul Smith, Gregory Wulczyn, and proposer.

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, PA 17745
Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months of publication of the problems.

H-307 Proposed by Larry Taylor, Briarwood, NY
(A) If $p \equiv \pm 1(\bmod 10)$ is prime, $x \equiv \sqrt{5}$, and $\alpha \equiv \frac{2(x-5)}{x+7}(\bmod p)$, prove that $a, a+1, a+2, a+3$, and $a+4$ have the same quadratic character modulo $p$ if and only if $11<p \equiv 1$ or $11(\bmod 60)$ and $(-2 x / p)=1$.
(B) If $p \equiv 1(\bmod 60),(2 x / p)=1$, and $b \equiv \frac{-2(x+5)}{7-x}(\bmod p)$, then $b$, $b+2, b+3$, and $b+4$ have the same quadratic character modulo $p$. Prove that $(11 a b / p)=1$.

H-308 Proposed by Paul Bruckman, Concord, CA

Let $\left[\alpha_{1}, a_{2}, \ldots, a_{n}\right]=\frac{p_{n}}{q_{n}} \frac{p_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right)}{q_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right)}$ denote the $n$th convergent of the infinite simple continued fraction $\left[\alpha_{1}, \alpha_{2}, \ldots\right], n=1,2, \ldots$. Also, define $p_{0}=1, q_{0}=0$. Further, define

$$
\begin{align*}
W_{n, k}= & p_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right) q_{k}\left(a_{1}, a_{2}, \ldots, a_{k}\right)  \tag{1}\\
& -p_{k}\left(a_{1}, a_{2}, \ldots, a_{k}\right) q_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
& p_{n} q_{k}-p_{k} q_{n}, 0 \leq k \leq n .
\end{align*}
$$

Find a general formula for $W_{n, k}$.
H-309 Proposed by David Singmaster, Polytechnic of the South Bank, London, England

Let $f$ be a permutation of $\{1,2, \ldots, m-1\}$ such that the terms $i+f(i)$ are all distinct (mod $m$ ). Characterize and/or enumerate such $f$. [Each such $f$ gives a decomposition of the $m(m+1) m$-nomial coefficients, which are the nearest neighbors of a given $m$-nomial coefficient, into $m$ sets of $m+1$ coefficients which have equal products and are congruent by rotation-see Hoggatt \& Alexanderson, "A Property of Multinomial Coefficients," The Fibonacci Quarterly 9, No. 4 (1971):351-356, 420-421.]

I have run a simple program to generate and enumerate such $f$, but can see no pattern. The number $N$ of such permutations is given below for $m \leq 10$. The ratio $N /(m-1)$ ! is decreasing steadily leading to the conjecture that it converges to 0 .

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $N$ | 1 | 1 | 2 | 3 | 8 | 19 | 64 | 225 | 928 |

H-310 Proposed by V.E. Hoggatt, Jr., San Jose State University, San Jose, CA Let $\alpha=(1+\sqrt{5}) / 2,[n \alpha]=a_{n}$, and $\left[n \alpha^{2}\right]=b_{n}$. Clearly, $a_{n}+n=b_{n}$.
a) Show that if $n=F_{2 m+1}$, then $a_{n}=F_{2 m+2}$ and $b_{n}=F_{2 m+3}$.
b) Show that if $n=F_{2 m}$, then $a_{n}=F_{2 m+1}-1$ and $b_{n}=F_{2 m+2}-1$.
c) Show that if $n=L_{2 m}$, then $a_{n}=L_{2 m+1}$ and $b_{n}=L_{2 m+2}$.
d) Show that if $n=L_{2 m+1}$, then $a_{n}=L_{2 m+2}-1$ and $b_{n}=L_{2 m+3}-1$.

## SOLUTIONS

Editorial Nate: Starting with this issue, we shall indicate the issue and date when each problem was proposed.

Continue
H-278 Proposed by V.E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 16, No. 1, Feb. 1978)

Show

$$
\sqrt{\frac{5 F_{n+2}}{F_{n}}}=\langle 3, \underbrace{1,1, \ldots, 1,6}_{n-1}\rangle
$$

(Continued fraction notation, cyclic part under bar.)
Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

$$
\begin{aligned}
D=\frac{F_{n+2}}{F_{n}} & =L_{2} \quad \text { and remainder }\left(-\alpha^{n-2}-\beta^{n-2}\right) \\
2 & \leq D \leq 3 \\
10 & \leq 5 D \leq 15 \\
{[\sqrt{5 D}] } & =3 .
\end{aligned}
$$

$\left(F_{n}, F_{n+1}\right)=1$ implies $\left(F_{n}, F_{n+2}\right)=1$. $\sqrt{5 D}$ has a unique periodic C.F. expansion with first element 3 and terminal element 6.

$$
\begin{aligned}
L_{n+1}^{2}-\frac{5 F_{n+2}}{F_{n}} F_{n}^{2}=L_{2 n+2}+2(-1)^{n+1}-L_{2 n+2}+(-1)^{n} L_{2} & =(-1)^{n} \\
x & =L_{n+1}, y=F_{n}
\end{aligned}
$$

is a solution of $x^{2}-5 D y^{2}= \pm 1$.
For the $p_{i}$ and $q_{i}$ convergents formed from the C.F. expansion of $\sqrt{5 D}$ to terminate with $p_{n}=L_{n+1}$ and $q_{n}=F_{n}$, the middle elements must be ( $n-1$ ) ones. Also solved by the proposer.

## A Rare Mixture

H-279 Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA (Vol. 16, No. 1, Feb. 1978)

Establish the $F-L$ identities:
(a) $F_{n+6 r}^{4}-\left(L_{4 r}+1\right)\left(F_{n+4 r}^{4}-F_{n+2 r}^{4}\right)-F^{4}=F_{2 r} F_{4 r} F_{6 r} F_{4 n+12 r}$
(b) $F_{n+6 r+3}^{4}+\left(L_{4 r+2}-1\right)\left(F_{n+4 r+2}^{4}-F_{n+2 r+1}^{4}\right)-F_{n}^{4}$

$$
=F_{2 r+1} F_{4 r+2} F_{6 r+3} F_{4 n+12 r+6^{\circ}}
$$

Solution by Paul Bruckman, Concord, CA
Lemma 1: $\quad L_{3 m}-(-1)^{m} L_{m}=5 F_{m} F_{2 m}$.
Proof: $L_{3 m}-(-1)^{m} L_{m}=a^{3 m}+b^{3 m}-(a b)^{m}\left(a^{m}+b^{m}\right)$

$$
=\left(a^{m}-b^{m}\right)\left(a^{2 m}-b^{2 m}\right)=5 F_{m} F_{2 m}
$$

Lemma 2: $5\left(F_{u}^{4}-F_{v}^{4}\right)=F_{u-v} F_{u+v}\left(L_{u-v^{\prime}} I_{u+v}-4(-1)^{u}\right)$.
Proob: $\quad 25 F_{u}^{4}=\left(a^{u}-b^{u}\right)^{4}=a^{4 u}+b^{4 u}-4(-1)^{u}\left(a^{2 u}+b^{2 u}\right)+6$.
Therefore,

$$
\begin{aligned}
& 25\left(F_{u}^{4}-F_{v}^{4}\right)=a^{4 u}-a^{4 v}+b^{4 u}-b^{4 v}-4(-1)^{u} a^{2 u}+4(-1) a^{2} \\
&-4(-1)^{u} b^{2 u}+4(-1)^{v} b^{2 v} \\
&=\left(a^{2 u+2 v}-b^{2 u+2 v}\right)\left(a^{2 u-2 v}-b^{2 u-2 v}\right) \\
&-4(-1)^{u} a^{u+v}\left(a^{u-v}-(-1)^{v-u} a^{v-u}\right) \\
&-4(-1)^{u} b^{u+v}\left(b^{u-v}-(-1)^{v-u} b^{v-u}\right) \\
&= 5 F_{2 u+2 v} F_{2 u-2 v}-4(-1)^{u}\left(a^{u+v}-b^{u+v}\right)\left(a^{u-v}-b^{u-v}\right) \\
&= 5 F_{2 u+2 v} F_{2 u-2 v}-20(-1)^{u} F_{u+v} F_{u-v} \\
&= 5 F_{u+v} F_{u-v}\left(L_{u+v^{2}} L_{u-v}-4(-1)^{u}\right),
\end{aligned}
$$

which implies the statement of the lemma.
Lemma 3: $(-1)^{m} L_{2 m}+1=(-1)^{m} F_{3 m} / F_{m}$.
Proob: $(-1)^{m} L_{2 m}+1=(-1)^{m}\left(L_{2 m}+(-1)^{m}\right)=(-1)^{m}\left(a^{2 m}+a^{m} b^{m}+b^{2 m}\right)$

$$
=(-1)^{m}\left\{\frac{a^{3 m}-b^{3 m}}{a^{m}-b^{m}}\right\}=(-1)^{m} \frac{F_{3 m}}{F_{m}} .
$$

Now

$$
F_{n+3 m}^{4}-\left((-1)^{m} L_{2 m}+1\right)\left(F_{n+2 m}^{4}-F_{n+m}^{4}\right)-F_{n}^{4}
$$

$$
=\frac{1}{5} F_{3 m} F_{2 n+3 m}\left(L_{3 m} L_{2 n+3 m}-4(-1)^{n+3 m}\right)
$$

$$
-\left((-1)^{m} L_{2 m}+1\right) \frac{1}{5} F_{m} F_{2 n+3 m}\left(L_{m} L_{2 n+3 m}-4(-1)^{n+2 m}\right)
$$

(applying Lemma 2 twice, with $u=n+3 m, v=n$ and $u=n+2 m, v=n+m$ )

$$
\begin{aligned}
=\frac{1}{5} F_{2 n+3 m} L_{2 n+3 m} F_{3 m} L_{3 m} & -\left((-1)^{m} L_{2 m}+1\right) F_{m} L_{m} \\
& -\frac{4}{5}(-1)^{n+2 m} F_{2 n+3 m}\left((-1)^{m} F_{3 m}-\left((-1)^{m} L_{2 m}+1\right) F_{m}\right)
\end{aligned}
$$

$$
\begin{aligned}
=\frac{1}{5} F_{2 n+3 m} L_{2 n+3 m}\left(F_{3 m} L_{3 m}\right. & \left.-(-1)^{m} F_{3 m} L_{m}\right) \\
& -\frac{4}{5}(-1)^{n} F_{2 n+3 m}\left((-1)^{m} F_{3 m}-(-1)^{m} F_{3 m}\right)
\end{aligned}
$$

(applying Lemma 3)

$$
\begin{aligned}
& =\frac{1}{5} F_{3 m} F_{4 n+6 m}\left(L_{3 m}-(-1)^{m} L_{m}\right) \\
& =F_{m} F_{2 m} F_{3 m} F_{4 n+6 m}(\text { by Lemma } 1) .
\end{aligned}
$$

Therefore:

$$
F_{n+3 m}^{4}-\left((-1)^{m} L_{2 m}+1\right)\left(F_{n+2 m}^{4}-F_{n+m}^{4}\right)-F^{4}=F_{m} F_{2 m} F_{3 m} F_{4 n+6 m} .
$$

Setting $m=2 r$ and $m=2 r+1$ yields (a) and (b), respectively.
Also solved by the proposer.

## Mod Ern

H-280 Proposed by P. Bruckman, Concord, CA (Vol. 16, No. 1, Feb. 1978)
Prove the congruences
(1) $F_{3 \cdot 2^{n}} \equiv 2^{n+2}\left(\bmod 2^{n+3}\right)$;
(2) $I_{3 \cdot 2^{n}} \equiv 2+2^{2 n+2}\left(\bmod 2^{2 n+4}\right), n=1,2,3, \ldots$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA
(1) $n=1, F_{6}=8 \equiv 8(\bmod 16)$.

Since $L_{3 k}^{2}-5 F_{3 k}^{2}= \pm 4,\left(L_{3 k}, F_{3 k}\right)=2$,
$2^{r} \mid F_{3 \cdot 2^{n}}, r>1, n \geq 1 ;$
$2^{t} \mid L_{3} \cdot 2^{n}$ if and only if $t=1$.
Assume $F_{3 \cdot 2^{n}} \equiv 2^{n+2}\left(\bmod 2^{3}\right) \equiv 2^{n+2}\left(\bmod 2^{n+4}\right)$
$F_{3 \cdot 2^{n+1}}=F_{3 \cdot 2^{n} L_{3} \cdot 2^{n} \equiv 2^{n+3}\left(\bmod 2^{n+4}\right) . ~}^{\text {. }}$
(2) $n=1, L_{6}=18 \equiv 2+2^{4}\left(\bmod 2^{6}\right), L_{2 k}^{2}=L_{4 k}+2$ 。

Assume $L_{3} \cdot 2^{n} \equiv 2+2^{2 n+2}\left(\bmod 2^{2 n+4}\right)$

$$
\begin{aligned}
& L_{3 \cdot 2^{n+1}}=L_{3 \cdot 2^{n}}^{2}-2 \equiv 2+2^{2 n+4}+2^{4 n+4}\left(\bmod 2^{4 n+5}\right) \\
& L_{3 \cdot 2^{n+1}} \equiv 2+2^{2 n+4}\left(\bmod 2^{2 n+6}\right), n \geq 1 .
\end{aligned}
$$

Also solved by the proposer, who noted that this is Corollary 6 in "Periodic Continued Fraction Representations of Fibonacci-type Irrationals," by V. E. Hoggatt, Jr. \& Paul S. Bruckman, in The Fibonacci Quarterly 15, No. 3 (1977): 225-230.

## INDEX OF ADVANCED PROBLEMS

| No. | Volume Proposed | Volume Solved | No. | Volume Proposed | Volume Solved | No. | Volume Proposed | Volume <br> d Solved |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| P-1 | 1-1 |  | H-46 | 2-4 | Article | H-97 | 4-4 | 6-4 |
| P-2 | 1-1 | 2-2 | H-47 | 2-4,4-3 |  | H-98 | 4-4 | 7-2 |
| P-3 | 1-1 | 3-1 | H-48 | 2-4 | 3-4 | H-99 | 4-4 | 6-4 |
| P-4 | 1-1 |  | H-49 | 2-4 | 3-4 | H-100 | 4-4 |  |
| P-5 | 1-1 |  | H-50 | 2-4 | 4-3 | H-101 | 4-4 | 6-4 |
| H-1 | 1-1 | Article | H-51 | 2-4 | 4-3 | H-102 | 4-4 | 9-2 |
| H-2 | 1-1 | Article | H-52 | 3-1 | 4-3 | H-103 | 5-1 | 6-6 |
| H-3 | 1-1 | 1-3 | H-53 | 3-1 | Article | H-104 | 5-1 | 6-6 |
| H-4 | 1-1 | 1-3 | H-54 | 3-1 | 4-4 | H-105 | 5-1 | 6-6 |
| H-5 | 1-1 | 1-3 | H-55 | 3-1 | Article | H-106 | 5-1 | 6-6 |
| H-6 | 1-1 | 1-3 | H-56 | 3-1 | 4-4 | H-107 | 5-1 | 6-6 |
| H-7 | 1-1 | 1-3 | H-57 | 3-1 | 4-4 | H-108 | 5-1 | 11-1 |
| H-8 | 1-1 | 1-3 | H-58 | 3-1 | 4-4 | H-109 | 5-1 | 7-1 |
| H-9 | 1-2 | 1-4 | H-59 | 3-2 | 5-5 | H-110 | 5-1 |  |
| H-10 | 1-2 | 1-4 | H-60 | 3-2 |  | H-111 | 5-1 | 7-1 |
| H-11 | 1-2 | 1-4 | H-61 | 3-2 |  | H-112 | 5-1 | 7-1 |
| H-12 | 1-2 | 1-4 | H-62 | 3-2 |  | H-113 | 5-2 |  |
| H-13 | 1-2 | 1-4 | H-63 | 3-2 |  | H-114 | 5-2 |  |
| H-14 | 1-2 | 1-4 | H-64 | 3-2 |  | H-115 | 5-2 |  |
| H-15 | 1-2 |  | H-65 | 3-3 | 5-1 | H-116 | 5-2 |  |
| H-16 | 1-2 | 1-4,2-1 | H-66 | 3-3 | 5-1 | H-117 | 5-2 | 7-1 |
| H-17 | 1-2 |  | H-67 | 3-3 | 5-1 | H-118 | 5-2 | 11-1 |
| H-18 | 1-2 | 2-2 | H-68 | 3-3 | 5-1 | H-119 | 5-3 | 7-1 |
| H-19 | 1-3 | 2-2 | H-69 | 3-3 | 5-2 | H-120 | 5-3 | 7-2 |
| H-20 | 1-3 | 2-2 | H-70 | 3-4 | 5-3 | H-121 | 5-3 | 7-2 |
| H-21 | 1-3 | 2-2 | H-71 | 3-4 | 5-2 | H-122 | 5-3 |  |
| H-22 | 1-3 |  | H-72 | 3-4 | Article | H-123 | 5-5 | 7-2 |
| H-23 | 1-3 |  | H-73 | 3-4 | 5-3 | H-124 | 5-5 | 7-2 |
| H-24 | 1-3 | 2-3 | H-74 | 3-4 | 4-1 | H-125 | 5-5 | Partial |
| H-25 | 1-4 | 2-3 | H-75 | 3-4 | 5-5 |  |  | 11-1,11-2 |
| H-26 | 1-4 | 3-3 | H-76 | 3-4 |  | H-126 | 6-1 | 7-3 |
| H-27 | 1-4 | 2-3 | H-77 | 3-4 | 5-3 | H-127 | 6-1 | 7-2 |
| H-28 | 1-4 | 2-3 | H-78 | 4-1 | 5-3 | H-128 | 6-1 | 7-3 |
| H-29 | 2-1 | 2-4 | H-79 | 4-1 | 5-3 | H-129 | 6-1 | 7-3 |
| H-30 | 2-1 | 3-2 | H-80 | 4-1 | 5-3 | H-130 | Not p | published |
| H-31 | 2-1 | 2-4 | H-81 | 4-1 | 6-1 | H-131 | 6-2 | 7-3 |
| H-32 | 2-1 | 2-4 | H-82 | 4-1 | 6-1 | H-132 | 6-2 | 7-5 |
| H-33 | 2-1 | 2-4 | H-83 | 4-1 | 6-1 | H-133 | 6-2 | 7-5 |
| H-34 | 2-2 | Article | H-84 | 4-2 |  | H-134 | 6-2 | 7-5 |
| H-35 | 2-2 | 3-1 | H-85 | 4-2 | 6-1 | H-135 | 6-2 | 7-5 |
| H-36 | 2-2 | 3-1 | H-86 | 4-2 | 11-1 | H-136 | 6-4 | 7-5 |
| H-37 | 2-2 | Article | H-87 | 4-2 | 12-1 | H-137 | 6-4 | 8-1 |
| H-38 | 2-2 | 3-3 | H-88 | 4-2 | 6-4 | H-138 | 6-4 | 8-1 |
| H-39 | 2-2 | 3-1 | H-89 | 4-3 | 6-2 | H-139 | 6-4 | 8-3 |
| H-40 | 2-2 |  | H-90 | 4-3 |  | H-140 | 6-4 | 8-1 |
| H-41 | 2-3 | 3-2 | H-91 | 4-3 |  | H-141 | 6-4 | 8-3 |
| H-42 | 2-3 | 3-2,3-3 | H-92 | 4-3 |  | H-142 | 6-4 | 8-3 |
|  |  | 4-2 | H-93 | 4-3,4-4 |  | H-143 | 6-6 | 8-3 |
| H-43 | 2-3 |  | H-94 | 4-3 |  | H-144 | 6-6 | 8-4 |
| H-44 | 2-3 |  | H-95 | 4-3 |  | H-145 | 6-6 | 8-4 |
| H-45 | 2-3 |  | H-96 | 4-3 | 6-2 | H-146 | 6-6 |  |
|  |  |  |  | 378 |  |  |  |  |


| No. | Volume Proposed | Volume <br> d <br> Solved | No. | Volume Proposed | Volume Solved | No. | Volume Proposed | Volume Solved |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| H-147 | 6-6 | 8-4 | H-197 | 10-4 | 12-1 | H-247 | 13-1 | 15-1 |
| H-148 | 7-1 |  | H-198 | 10-6 | 12-1 | H-248 | 13-1 | 15-1 |
| H-149 | 7-1 | 8-4 | H-199 | 10-6 | 12-2 | H-249 | 13-2 | 15-1 |
| H-150 | 7-1 | 8-4 | H-200 | 10-6 | 12-2 | H-250 | 13-2 | 15-1 |
| H-151 | 7-1 | 8-5 | H-201 | 10-6 | 12-2 | H-251 | 13-2 | 15-2 |
| H-152 | 7-1 |  | H-202 | 10-6 | 12-3 | H-252 | 13-3 | 15-2 |
| H-153 | 7-2 | 8-5 | H-203 | 10-6 |  | H-253 | 13-3 | 15-2 |
| H-154 | 7-2 | 8-5 | H-204 | 10-6 |  | H-254 | 13-3 | 17-3 |
| H-155 | 7-2 | 8-5 | H-205 | 11-1 | 12-3 | H-255 | 13-4 | 15-3 |
| H-156 | 7-2 | 9-1 | H-206 | 11-1 | 12-3 | H-256 | 13-4 | 15-4 |
| H-157 | 7-2 | 9-1 | H-207 | 11-1 | 12-4 | H-257 | 13-4 | 15-3 |
| H-158 | 7-3 | 9-1 | H-208 | 11-1 | 12-4 | H-258 | 14-1 | 15-3 |
| H-159 | 7-3 | 9-1 | H-209 | 11-1 | 12-4 | H-259 | 14-1 | 15-3 |
| H-160 | 7-3 | 9-1 | H-210 | 11-1 | 12-4 | H-260 | 14-1 | 17-3 |
| H-161 | 7-3 | 9-2 | H-211 | 11-1 |  | H-261 | 14-2 | 15-4 |
| H-162 | 7-5 | 9-1 | H-212 | 11-1 |  | H-262 | 14-2 | 15-4 |
| H-163 | 7-5 | 9-4 | H-213 | 11-1 |  | H-263 | 14-2 | 15-4 |
| H-164 | 7-5 | 9-1 | H-214 | 11-1 |  | H-264 | 14-2 | 16-1 |
| H-165 | 7-5 | 9-4 | H-215 | 11-2 |  | H-265 | 14-3 | 16-1 |
| H-166 | 8-1 | 9-4 | H-216 | 11-2 | 13-1 | H-266 | 14-3 | 16-1 |
| H-167 | 8-1 | 9-5 | H-217 | 11-2 | 13-1 | H-267 | 14-5 | 16-2 |
| H-168 | 8-1 | 9-5 | H-218 | 11-2 | 13-1 | H-268 | 14-5 | 16-2 |
| H-169 | 8-3 | 9-5 | H-219 | 11-2 | 13-2 | H-269 | 15-1 | 16-5 |
| H-170 | 8-3 |  | H-220 | 11-2 | 13-2 | H-270 | 15-1 | 16-5 |
| H-171 | 8-3 | 9-5 | H-221 | 11-3 | 13-2 | H-271 | 15-1 | 17-3 |
| H-172 | 8-4 | 9-5 | H-222 | 11-3 |  | H-272 | 15-2 | 16-6 |
| H-173 | 8-4 | 9-5 | H-223 | 11-3 | 13-4 | H-273 | 15-2 | 16-6 |
| H-174 | 8-4 Pa | Partial 11-2 | H-224 | 11-3 |  | H-274 | 15-3 | 17-1 |
| H-175 | 8-5 | 9-5 | H-225 | 11-3 | 16-6,17-1 | H-275 | 15-3 | 17-2 |
| H-176 | 8-5 | 10-2 | H-226 | 11-3 | 13-3 | H-276 | 15-4 | 17-3 |
| H-177 | 8-5 | 10-2 | H-227 | 11-5 | 13-4 | H-277 | 15-4 |  |
| H-178 | 9-1 | 10-2 | H-228 | 11-5 |  | H-278 | 16-1 |  |
| H-179 | 9-1 | 10-2 | H-229 | 11-5 | 13-4 | H-279 | 16-1 |  |
| H-180 | 9-1 | 10-3 | H-230 | 12-1 | 14-1 | H-280 | 16-1 |  |
| H-181 | 9-2 | 10-3 | H-231 | 12-1 | 14-1 | H-281 | 16-2 |  |
| H-182 | 9-2 Re | Remark 11-2 | H-232 | 12-1 | 14-1 | H-282 | 16-2 |  |
| H-183 | 9-4 | 10-3 | H-233 | 12-1 | 14-1 | H-283 | 16-2 |  |
| H-184 | 9-4 | 10-3 | H-234 | 12-2 | 14-2 | H-284 | 16-2 |  |
| H-185 | 9-4 | 10-4 | H-235 | 12-2 | 14-2 | H-285 | 16-5 |  |
| H-186 | 9-5 | 10-4 | H-236 | 12-2 | 14-2 | H-286 | 16-5 |  |
| H-187 | 9-5 | 10-4 | H-237 | 12-3 | 14-2 | H-287 | 16-5 |  |
| H-188 | 9-5 | 10-6 | H-238 | 12-3 | 14-3 | H-288 | 16-5 |  |
| H-189 | 10-2 | 10-4 | H-239 | 12-4 | 14-3 | H-289 | 16-5 |  |
| H-190 | 10-2 | 10-4 | H-240 | 12-4 | 14-3 | H-290 | 16-6 |  |
| H-191 | 10-2 | 10-6 | H-241 | 12-4 | 14-3 | H-291 | 16-6 |  |
| H-192 | 10-3 | 10-4 | H-242 | Not pub | lished | H-292 | 16-6 |  |
| H-193 | 10-3 | 11-2 | H-243 | Not pub | . 14-3 | H-293 | 16-6 |  |
| H-194 | 10-3 | 10-6 | H-244 | Not pub | . 14-5 | H-294 | 16-6 |  |
| H-195 | 10-4 | 11-5 | H-245 | 13-1 | 14-5 | H-295 | 17-1 |  |
| H-196 | 10-4 | 11-5 | H-246 | 13-1 | 14-5 | H-296 | 17-1 |  |

## *****

## VOLUME INDEX

ADI KESAVAN, A. S. "A Modification of Goka's Binary Sequence," 17(3):212-220 (copauthor, S. Narayanaswami).
AHUJA, J. C. "Concavity Property and a Recurrence Relation for Associated Lah Numbers," 17(2):158-161 (co-author, E. A. Enneking).
ALLARD, A. "Periods and Entry in Fibonacci Sequence," 17(1):51-57 (co-author, P. Lecomte)

ALMEIDA AZEVEDO, J. E. de. "Fibonacci Numbers," 17(2):162-164.
ALTEVOGT, RUDOLF. "Golden Mean of the Human Body," 17(4):340-344 (co-author, T. Antony Davis).

ANDERSON, DAVID A. "The Diophantine Equation $N b^{2}=c^{2}+N+1, " 17(1): 69-70$ (co-author, Milton W. Loyer).
ARKIN, JOSEPH. "On Euler's Solution to a Problem of Diophantus," 17(4):333339 (co-authors, V. E. Hoggatt, Jr., and E. G. Straus).
BALLEW, DAVID W. "Pythagorean Triples and Triangular Numbers," 17(2):168-172 (co-author, Ronald C. Weger).
BERGUM, GERALD E. "Infinite Series with Fibonacci and Lucas Polynomials," 17(2):147-151 (co-author, V. E. Hoggatt, Jr.).
BERZSENYI, GEORGE. Problems Proposed: H-274, 17(1):95; H-302, 17(3):286. Problems Solved: B-378, 17(2):186; H-275, 17(2):191.
BICKNELL-JOHNSON, MARJORIE. "Py.thagorean Triples Containing Fibonacci Numbers: Solutions for $F_{n}^{2} \pm F_{k}^{2}=K^{2}, " 17(1): 1-12$. "Reflections Across Two and Three Glass P1ates," 17(2):118-142 (co-author, V. E. Hoggatt, Jr.). "A Generalization of Wythoff's Game," 17(3):198-211 (co-authors, V. E. Hoggatt, Jr., and R. Sarsfield). "Addenda to 'Pythagorean Triples Containing Fibonacci Numbers: Solutions for $F_{n}^{2} \pm F_{k}^{2}=K^{2}, ' " 17(4): 293$. "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio," 17(4):306-318 (co-author, V. E. Hoggatt, Jr.).
BOARDMAN, JOHN. "The Normal Modes of a Hanging Oscillator of Order $N$, " 17(1): 37-40.
BOHIGAN, HAIG E. "Extensions of the W. Mnich Problem," 17(2):172-177.
BRACKEN, PAUL. Problems Solved: B-388, 17(4):370, B-390, 17(4):372, B-392, 17(4):373.
BRADY, WRAY G. Problem Proposed: B-406, 17(3):281. Problems Solved: B-388, 17(4):370; B-389, B-390, 17(4):371; B-391, 17(4):373.
BRUCKMAN, PAUL S. "Some Divisibility Properties of Generalized Fibonacci Sequences," 17(1):42-49. Problems Proposed: B-396 (based on the solution to $B-371), 17(1): 90 ; H-303,17(3): 286 ; H-308,17(4): 374$. Problems Solved: $\mathrm{B}-370, \mathrm{~B}-371,17(1) ; 91 ; \mathrm{B}-372, \mathrm{~B}-373,17(1): 92 ; \mathrm{B}-374, \mathrm{~B}-375,17(1): 93 ;$ Н-274, 17(1):96; B-376, B-377, 17(2):185; B-378, B-379, 17(2):186; B-380, $17(2): 187 ;$ B-381, $17(2): 188 ; \mathrm{H}-275,19(2): 192 ; \mathrm{B}-382, \mathrm{~B}-384,17(3): 283$; B-385, B-386, 17(3):284; H-275, 17(3):287-288; B-388, 17(4):370; B-389, 17(4):371; B-390, $17(4): 372$; B-391, $17(4): 372-373$; B-392, B-393, $17(4):$ 373; H-279, H-280, 17 (4):376-377.
CARLITZ, LEONARD. "Restricted Multipartite Compositions," 17(3):220-228. "Restricted Compositions II," 17(4):321-328.
COHEN, M. E. "On Some Extensions of the Wang-Carlitz Identity," 17(4):299305 (co-author, H. Sun.)
COHN, HARVEY. "Growth Types of Fibonacci and Markoff," 17(2):178-183.

COMTET, LOUIS. "A Multinomial Generalization of a Binomial Identity," 17(2): 108-111.
DAVIS, T. ANTONY. "Golden Mean of the Human Body," 17(4):340-344 (co-author, Rudolf Altevogt).
EDGAR, H. Problem Proposed: H-260, 17(3):288.
ENNEKING, E. A. "Concavity Property and a Recurrence Relation for Associated Lah Numbers," 17(2):158-161 (co-author, J. C. Ahuja).
ERCOLANO, JOSEPH. "Matrix Generators of Pell Sequences," 17(1):71-77.
ESWARATHASAN, A. "On Pseudo-Fibonacci Numbers of the Form $2 S^{2}$, Where $S$ Is an Integer," 17(2):142-147.
FREITAG, HERTA T. Prob1ems Proposed: B-398, 17(1):90; B-400, 17(2):184; B413, B-414, 17(4):369. Problems Solved: B-371, 17(1):91; B-372, 17(1):92; B-374, B-375, $17(1): 93 ; \mathrm{B}-378, \mathrm{~B}-379,17(2): 186 ; \mathrm{B}-382,17(3): 283 ; \mathrm{B}-385$, $17(3): 283-284 ; B-388,17(4): 370 ; B-390,17(4): 371 ; B-392, B-393,17(4): 373$.
FULTS, DOUGLAS A. Problems Solved: B-374, B-375, 17(1):93.
GALLINAR, JEAN-PIERRE. "Fibonacci Ratio in a Thermodynamical Case," 17(3): 239-241.
GARFIELD, R. Problems Solved: B-381, 17(2):188; B-383, 17(3):283; B-388, $17(4): 370 ; B-389, B-390,17(4): 371 ; B-392, B-393,17(4): 373$.
GIULI, CHRISTINE. "A Primer on Stern's Diatomic Sequence," Part I, 17(2): 103-108; Part II, $17(3): 246-248$; Part III, $17(4): 318-320$ (co-author, Robert Giuli).
GIULI, ROBERT. "A Primer on Stern's Diatomic Sequence," Part I, 17(2):103108; Part II, 17(3):246-248; Part III, 17(4):318-320 (co-author, Christine Giuli). Problem Proposed: B-407, 17(3):281. Problem Solved: B-390, 17 (4):371.

GRASSL, R. M. Problem Proposed: B-417, 17(4):370 (co-proposer, P. L. Mana).
GUILLOTTE, G. A. R. Problem Proposed and Solved: H-225, 17(1):95.
HARBORTH, HEIKO. Problem Solved: B-376, 17(2):185.
HEICHELHEIM, PETER. "The Study of Positive Integers $(a, b)$ Such that $a b+1$ Is a Square," 17(3):269-274.
HIGGINS, RADA. "More in the Theory of Sequences," 17(3):193-197.
HILLMAN, A. P. "Nearly Linear Functions," 17(1):84-89 (co-author, V. E. Hoggatt, Jr.).
HILLMAN, A. P. Ed. "Elementary Problems and Solutions," 17(1):90-93; 17(2): 184-188; 17(3):281-285; 17(4):369-373.
HOCHBERG, MURRAY. "A Conjecture in Game Theory," 17(3):250-252.
HOGGATT, V. E. Jr. "Generating Functions of Central Values in Generalized Pascal Triangles," 17(1):58-67 (co-author, Claudia Smith). "Nearly Linear Functions," 17(1):84-89 (co-author, A. P. Hillman). "Reflections Across Two and Three Glass Plates," 17(2):118-142 (co-author, Marjorie Bickne11Johnson). "Infinite Series with Fibonacci and Lucas Polynomials," 17(2): 147-151 (co-author, Gerald E. Bergum). "A Generalization of Wythoff's Game," 17(3):198-211 (co-authors, Marjorie Bicknel1-Johnson and Richard Sarsfield). "A Study of the Maximal Values in Pascal's Quadrinomial Triang1e," 17(3):264-269 (co-author, Claudia Smith). "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio," 17(4):306-318 (co-author, Marjorie Bickne11-Johnson). "On Euler's Solution to a Problem of Diophantus," 17(4):333-339 (co-authors, Joseph Arkin and E. G. Straus). Problems proposed: B-395, B-399, 17(1):90; H-301, 17 (2): 190; H-304, 17(3):286; H-306, 17(3):287; B-390, 17(4):371; B-393, 17 (4):373; H-310, 17(4):375. Problems Solved: B-373, B-375, 17(1):92-93;

HOGGATT, V. E. Jr. (continued)
$\mathrm{B}-381,17(2): 187 ; \mathrm{H}-275,17(2): 192 ; \mathrm{H}-276,17(3): 287-288 ; \mathrm{B}-415,17(4): 369$;
B-416, $17(4): 370$ (co-proposer, G. Jakubowski); H-278, H-310, 17 (4):375.
HORADAM, A. F. "Sums of Products: An Extension," 17(3):248-250. "Chebyshev and Fermat Polynomials for Diagonal Functions," 17(4):328-333.
INDEX OF ADVANCED PROBLEMS, 17(4):378-379.
JAKUBOWSKI, GENE. Problem Proposed: B-416, 17(4):370. (co-proposer, V. E. Hoggatt, Jr.).
JOHNSON, N. Problem Solved: H-276, 17(3):287-288.
JOSCELYNE, CHARLES. Problems Solved: B-376, 17(2):185; B-379, 17(2):186.
JOSEPH, JAMES E. "Maximum Cardinalities for Topologies on Finite Sets," 17 (2):97-102.

KIMBERLING, CLARK. "Strong Divisibility Sequences and Some Conjectures," 17 (1):13-17. "Greatest Common Divisors of Sums and Differences of Fibonacci, Lucas, and Chebyshev Polynomials," 17(1):18-22.
KLAUSER, HANS. Problem Solved: B-388, $17(4): 370$.
KOCHER, FRANK. Problem Solved: B-376, 17(2):185.
KUENZI, N. J. Problem Solved: B-380, $17(2): 187$ (co-solver, Bob Prielipp).
KUIPERS, L. Problem Proposed: H-298, 17 (1):94.
LECOMTE, P. "Periods and Entry Points in Fibonacci Sequence," 17(1):51-57 (co-author, A. Allard).
LIGHT, F. W. Jr. "Enumeration of Truncated Latin Rectangles," 17(1):34-36.
LINDSTROM, PETER A. Problems Solved: B-388, 17(4):370; B-393, 17(4):373.
LORD, GRAHAM. "Degeneracy of Transformed Complete Sequences," 17(4):358-361 (co-author, Herve G. Morin). Problems Solved: B-376, 17(2):186; B-378, B$379,17(2): 186 ; \mathrm{B}-388,17(4): 370 ; \mathrm{B}-390,17(4): 371, \mathrm{~B}-392,17(4): 373$.
LOYER, MILTON W. "The Diophantine Equation $N b^{2}=c^{2}+N+1, " 17(1): 69-70$ (co-author, David A. Anderson).
MANA, PHIL. Problems Proposed: B-394, $17(1): 90 ; B-404, B-405,17(2): 184$; B-412, $17(4): 369 ; B-417,17(4): 370$ (co-proposer, R. M. Grass1). Problems Solved: $B-370,17(1): 91 ; B-380,17(2): 187 ; B-388,17(4): 370 ; B-390,17(4):$ 372; B-392, 17(4):373.
McLAUGHLIN, WILLIAM I. "Note on a Tetranacci Alternative to Bode's Law," 17 (2):116-118.

METZGER, J. M. Problems Solved: B-376, B-377, 17(2):185.
MIDTTUN, NORVALD. "Congruences for Certain Fibonacci Numbers," 17(1):40-41.
MILLER, ELLEN R. Problem Solved: B-388, 17(4):370.
MORIN, HERVE G. "Degeneracy of Transformed Complete Sequences," 17(4):358361 (co-author, Graham Lord).
MORITO, SUSUMU. "Finding the General Solution of a Linear Diophantine Equation," 17(4):361-368 (co-author, Harvey M. Salkin).
MULLEN, GARY L. Problem Proposed: B-401, 17(2):184. Problem Solved: B-376, 17(2):185.
MURPHY, JAMES L. Problem Proposed: $\mathrm{H}-300,17(2): 189$.
NARAYANSWAMI, S. "A Modification of Goka's Binary Sequence," 17(3):212-220 (co-author, A. S. Adikesavan).
NEUMANN, B. H. "Some Sequences Like Fibonacci's," 17(1):80-83 (co-author, L. G. Wilson).

O'DONNELL, WILLIAM J. "A Note on a Pe11-Type Sequence," $17(1): 49-50$. "Two Theorems Concerning Hexagonal Numbers," 17(1):77-79.
OWINGS, JAMES C., Jr. "Solution of $\binom{y+1}{x}=\binom{y}{x+1}$ in Terms of Fibonacci
Numbers," $17(1): 67-69$.

PECK, C. B. A. Problem Solved: B-370, 17(1):91; B-388, 17(4):370; B-390, 17 (4):372; B-391, B-393, 17(4):373.

PRIELIPP, BOB. Problems Solved: B-370, B-371, 17(1):91; B-372, B-373, 17(1): 92; B-374, B-375, $17(1): 93 ; \mathrm{B}-376, \mathrm{~B}-377,17(2): 185$; B-379, $17(2): 186 ; \mathrm{B}-$ 380, 17(2):187 (co-solver, N. J. Kuenzi) ; B-382, 17(3):283; B-387, 17(3): 284-285; B-388, 17(4):370; B-389, 17(4):371; B-393, 17(4):373.
REINGOLD, EDWARD M. "A Note on 3-2 Trees," 17(2):151-157.
RICE, BART. Problem Proposed: B-411, 17(3):282.
ROSENBERG, ARNOLD L. "Profile Numbers," 17(3):259-264.
SALKIN, HARVEY M. "Finding the General Solution of a Linear Diophantine Equation," 17(4):361-368 (co-author, Susumu Morito).
SARSFIELD, RICHARD. "A Generalization of Wythoff's Game," 17(3):198-211 (coauthors, V. E. Hoggatt, jr., and Marjorie Bicknell-Johnson).
SAYER, F. P. "The Recurrence Relation $(r+1) f_{r-1}=x f^{\prime}+(K-r+1) x^{2} f_{r-1}$," 17(3):228-239.
SCHMUTZ, E. Problem Solved: B-376, 17(2):185 (co-solver, M. Wachte1).
SHANNON, A. G. "Special Recurrence Relations Associated with the Sequence $\left\{w_{n}(a, b ; p, q)\right\}, " 17(4): 294-299$. "A Recurrence Relation for Generalized Multinomial Coefficients," 17(4):344-347. "Generalized Fibonacci Numbers as Elements of Ideals," 17(4):347-349. "A Generalization of Hilton's Partition of Horadam's Sequences," 17(4):349-357. Problems Solved: B-373, 17 (1):92; B-374, B-375, $17(1): 93 ; \mathrm{B}-380,17(2): 187 ; \mathrm{H}-275,17(2): 192 ; \mathrm{B}-382$, 17(3):283; B-388, $17(4): 370 ; B-390,17(4): 372 ; B-391, B-392, B-393,17(4):$ 373.

SHAPIRO, LOUIS W. "The Cycle of Six," 17(3):286-287.
SHECHTER, MARTIN. Problem Proposed: H-305, 17(3):286-287.
SINGH, SAHIB. Problems Solved: B-370, B-371, 17(1):91; B-372, B-373, 17(1): 92; B-374, B-375, $17(1): 93 ; \mathrm{B}-376, \mathrm{~B}-377,17(2): 185 ; \mathrm{B}-378, \mathrm{~B}-379,17(2):$ 186; B-381, $17(2): 187 ;$ B-382, B-384, $17(3): 283 ; \mathrm{B}-385,17(3): 283-284 ; \mathrm{B}-386$, $17(3): 284 ; \mathrm{B}-388$, $17(4): 370$; B-389, $17(4): 371 ; \mathrm{B}-390$, $17(4): 372$; B-391, B392, B-393, 17(4):373.
SINGMASTER, DAVID. Problem Proposed: H-309, 17(4):374.
SMITH, CLAUDIA. "Generating Functions of Central Values in Generalized Pascal Triangles," 17(1):58-67 (co-author, V. E. Hoggatt, Jr.). "A Study of the Maximal Values in Pascal's Quadrinomial Triangle," 17(3):264-267 (coauthor, V. E. Hoggatt, Jr.).
SMITH, PAUL. Problems Solved: B-388, 17(4):370; B-389, B-390, 17(4):371; B-391, B-392, B-393, 17(4):373.
SOMER, LAWRENCE. "Which Second-Order Linear Integral Recurrences Have Almost A11 Primes as Divisors?" 17(2):111-116. Problems Proposed: B-408, 17(3): 281. Problems Solved: B-382, 17(3):282-283; B-383, 17(3):283; B-385, 17 (3):283-284; B-386, $17(3): 284 ; \mathrm{B}-388,17(4): 370$.

SONNTAG, ROLF. Problems Solved: B-376, 17(2):185; B-388, 17(4):370; B-392, 17(4):373.
STERN, FREDERICK. "Absorption Sequences," 17(3):275-280. Problem Proposed: B-374, 17(1):93. Prob1em Solved: B-374, 17(1):93.
STRAUS. E. G. "On Euler's Solution to a Problem of Diophantus," 17(4):333339 (co-authors, Joseph Arkin and V. E. Hoggatt, Jr.).
SUN, H. S. "On Groups Generated by the Squares," 17(3):241-246. "On Some Extensions of the Wang-Carlitz Identity, 17(4):299-305 (co-author, M. Cohen).
TAYLOR, LARRY. Problem Proposed: H-307, 17(4):374.

TRIGG, CHARLES W. Problems Solved: B-376, 17(2):185; B-382, 17(3):283; B385, 17(3):283-284.
TURNER, STEPHEN JOHN. "Probability Via the Nth Order Fibonacci-T Sequence," 17(1):23-28.
VOGEL, JOHN W. Problem Solved: B-381, 17(2):188.
WACHTEL, M. Problem Proposed: B-410, 17(3):282. Problems Solved: B-376, 17(2):185 (co-solver, E. Schmutz); B-391, 17(4):372-373.
WALL, CHARLES R. "Some Congruences Involving Generalized Fibonacci Numbers," 17(1):29-33.
WEGER, RONALD C. "Pythagorean Triples and Triangular Numbers," 17(2):168-172 (co-author, David W. Ballew).
WEINSTEIN, GERALD. "An Algorithm for Packing Complements of Finite Sets of Integers," 17(4):289-293.
WHITNEY, RAYMOND E. Problems Proposed: H-254, H-271, 17(3):288.
WHITNEY, RAYMOND E., Ed. "Advanced Problems and Solutions," 17(1):94-96; $17(2): 189-192 ; 17(3): 286-288 ; 17(4): 374-377$.
WILSON, L. G. "Some Sequences Like Fibonacci's," 17(1):80-83 (co-author, B. H. Neumann).

WOO, NORMAN. "A Note on Basic M-Tuples," 17(2):165-168.
WULCZYN, GREGORY. Problems Proposed: B-397, 17(1):90; H-295, 17(1):94; B402, B-403, 17(2):184; H-299, 17(2):189; B-409, 17(3):281. Problems Solved: $\mathrm{B}-370,17(1): 91 ; \mathrm{B}-376,17(2): 185$; B-378, B-379, $17(2): 186$; H-275, $17(2):$ 191; B-382, B-383, B-384, 17(3):283; B-385, 17(3):283-284; B-386, $17(3):$ 284; B-388 17(4):370; B-389, 17(4):371; B-390, 17(4):372; B-392, B-394, 17(4):373.
ZWILLINGER, DAN. Problem Solved: B-380, 17(2):186.
*
(continued from page 344)
Fibonacci numbers have assumed great importance since the formation of The Fibonacci Association. Some introductory books (e.g., [1] and [8]), and popular articles (e.g., [2] and [3]), have brought the Fibonacci concept to those who are endowed with a thirst for serious mathematical knowledge.

## REFERENCES

1. Brother U. Alfred [Brousseau]. An Introduction to Fibonacci Discovery. San Jose, Calif: The Fibonacci Association, 1965.
2. M. Gardner. "The Multiple Fascinations of the Fibonacci Sequence." Scientific American 220, No. 3 (1969):116-120.
3. V. E. Hoggatt, Jr. "The Number Theory: The Fibonacci Sequence." Yearbook of Science and the Future, Encyclopedia Britannica. Chicago, 1977, pp. 179-191.
4. H. E. Huntley. The Divine Proportion. New York: Dover, 1970.
5. J. Michell. "Is the Fibonacci Sequence or Golden Ratio Represented in the Dimensions of the Great Pyramid of Egypt?" The Fibonacci NewsZetter (July 1974).
6. G. J. Mitchison. "Phyllotaxis and the Fibonacci Series." Science 196 (1977):270-275.
7. J. Reichardt. "Art at Large." New Scientist 56, No. 823 (1972):601.
8. N. N. Vorob'ev. Fibonacci Numbers. Oxford: Pergamon Press, 1961.

[^0]:    *Submitted ca 1972.

[^1]:    ＊Research was supported in part by Grant MCS79－03162 from the National Science Foundation．

[^2]:    *Partially supported by Grant $A 4025$ of the Conseil National de Recherches Canada.
    **Partially supported by Grant 5072 of Laval University.

[^3]:    ＊The authors would like to express their appreciation to Professor Dong Hoon Lee（Department of Mathematics，Case Western Reserve University）for his time and helpful discussions．
    Part of this work was supported by the Office of Naval Research under con－ tract number NOOO14－67－A－0404－0010．

