6137

RECURRENCES FOR TWO RESTRICTED PARTITION FUNCTIONS

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In this note we shall develop two "pure" recurrences for determination of the functional values $q(n)$ and $q_0(n)$. Accordingly, we recall that for a given natural number n, $q(n)$ denotes the number of partitions of n into distinct parts (or, equivalently, the number of partitions of n into odd parts), and $q_0(n)$ denotes the number of partitions of n into distinct odd parts (or, equivalently, the number of self-conjugate partitions of n). As usual, $p(n)$ denotes the number of unrestricted partitions of n; and, conventionally, we set $p(0) = q(0) = q_0(0) = 1$. Previous tables of values for $q_0(n)$ and $q(n)$ have been constructed on the strength of known tables for $p(n)$; for example, see [1] and [3]. The recurrences of the following two theorems allow us to determine $q_0(n)$ and $q(n)$ without prior knowledge of $p(n)$.

Theorem 1: For each nonnegative integer n,

$$\sum_{k=0} (-1)^{k(k+1)/2} \cdot q_0(n - k(k+1)/2) = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 2: For each nonnegative integer n,

$$q(n) + 2\sum_{k=1} (-1)^k \cdot q(n - k^2) = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1)/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In both theorems, summation is extended over all values of the indices which yield nonnegative integral arguments of q_0 and q.

Our proofs will depend on the following three identities of Euler and Gauss [2, p. 284]:

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left\{x^{(3n^2-n)/2} + x^{(3n^2+n)/2}\right\}. \tag{3}$$

$$\prod_{n=1}^{\infty} (1 - x^{2n}) = \prod_{n=1}^{\infty} (1 + x^{2n-1}) \cdot \sum_{n=0}^{\infty} (-x)^{n(n+1)/2}. \tag{4}$$

$$\prod_{n=1}^{\infty} (1 - x^n) = \prod_{n=1}^{\infty} (1 + x^n) \left\{1 + 2\sum_{n=1}^{\infty} (-1)^n \cdot x^{n^2}\right\}. \tag{5}$$

Proof of Theorem 1: Replace x by x^2 in (3) and eliminate $\Pi(1 - x^{2n})$ between the resulting identity and (4) to obtain

$$\sum_{n=0}^{\infty} q_0(n)x^n \cdot \sum_{n=0}^{\infty} (-x)^{n(n+1)/2} = 1 + \sum_{m=1}^{\infty} (-1)^m \left\{x^{3m^2-m} + x^{3m^2+m}\right\}.$$

[Recall that $\Pi(1 + x^{2n-1})$ generates $q_0(n)$.] The complete expansion of the left side of the foregoing equation is:

$$\sum_{n=0}^{\infty} x^n \sum_{k=0} (-1)^{k(k+1)/2} q_0(n - k(k+1)/2).$$

Equating coefficients of x^n, we obtain the desired conclusion. [Note that $q_0(0) = 1$ is consistent with the statement of our theorem.]

Proof of Theorem 2: In view of the fact that $\Pi(1 + x^n)$ generates $q(n)$, identities (3) and (5) imply

$$\left\{\sum_{n=0}^{\infty} q(n)x^n\right\}\left\{1 + 2\sum_{n=1}^{\infty}(-1)^n x^{n^2}\right\} = 1 + \sum_{m=1}^{\infty}(-1)^m\left\{x^{(3m^2-m)/2} + x^{(3m^2+m)/2}\right\},$$

or, equivalently,

$$\sum_{n=0}^{\infty} x^n\left\{q(n) + \sum_{k=1}(-1)^k \cdot 2q(n - k^2)\right\} = 1 + \sum_{m=1}^{\infty}(-1)^m\left\{x^{(3m^2-m)/2} + x^{(3m^2+m)/2}\right\}.$$

Upon equating coefficients of x^n, we derive the recurrence.

REMARKS

The following table of values for $q_0(n)$, $q(n)$, and $p(n)$, $n = 0(1)25$, is included to show the relative rates of growth of the three functions. For example, $q_0(n)$ grows much more slowly with n than does $p(n)$. So, computing a list of values of $q_0(n)$ by using "large" $p(n)$ values is much less desirable than by use of the recurrence (1).

TABLE 1

n	$q_0(n)$	$q(n)$	$p(n)$	n	$q_0(n)$	$q(n)$	$p(n)$
0	1	1	1	13	3	18	101
1	1	1	1	14	3	22	135
2	0	1	2	15	4	27	176
3	1	2	3	16	5	32	231
4	1	2	5	17	5	38	297
5	1	3	7	18	5	46	385
6	1	4	11	19	6	54	490
7	1	5	15	20	7	64	627
8	2	6	22	21	8	76	792
9	2	7	30	22	8	89	1002
10	2	10	42	23	9	104	1255
11	2	12	56	24	11	122	1575
12	3	15	77	25	12	142	1958

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EXTENSIONS OF A PAPER ON DIAGONAL FUNCTIONS

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INTRODUCTION

Consider the sequences $\{A_n(x)\}$ and $\{B_n(x)\}$ for which

$$A_{n+2}(x) = pxA_{n+1}(x) + qA_n(x), \qquad A_0(x) = 0,\ A_1(x) = 1; \tag{1}$$

$$B_{n+2}(x) = pxB_{n+1}(x) + qB_n(x), \qquad B_0(x) = 2,\ B_1(x) = x. \tag{2}$$

Then, from (1) and (2), we have

$$\left\{\begin{array}{l}
A_0(x) = 0 \\
A_1(x) = 1 \\
A_2(x) = px \\
A_3(x) = p^2x^2 + q \\
A_4(x) = p^3x^3 + 2pqx \\
A_5(x) = p^4x^4 + 3p^2qx^2 + q^2 \\
A_6(x) = p^5x^5 + 4p^3qx^3 + 3pq^2x \\
A_7(x) = p^6x^6 + 5p^4qx^4 + 6p^2q^2x^2 + q^3 \\
A_8(x) = p^7x^7 + 6p^5qx^5 + 10p^3q^2x^3 + 4pq^3x \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right. \tag{3}$$

$$\left\{\begin{array}{l}
B_0(x) = 2 \\
B_1(x) = px \\
B_2(x) = p^2x^2 + 2q \\
B_3(x) = p^3x^3 + 3pqx \\
B_4(x) = p^4x^4 + 4p^2qx^2 + 2q^2 \\
B_5(x) = p^5x^5 + 5p^3qx^3 + 5pq^2x \\
B_6(x) = p^6x^6 + 6p^4qx^4 + 9p^2q^2x^2 + 2q^3 \\
B_7(x) = p^7x^7 + 7p^5qx^5 + 14p^3q^2x^3 + 7pq^3x \\
B_8(x) = p^8x^8 + 8p^6qx^6 + 20p^4q^2x^4 + 16p^2q^3x^2 + 2q^4 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right. \tag{4}$$

In this paper we seek to extend and generalize the results of [1], [2], [3], [4], and Jaiswal [5]. The results hereunder flow on from those in [2], where certain restrictions were imposed on the sequences for the purpose of extending the results of Serkland [6].

DIAGONAL FUNCTIONS FOR $A_n(x)$, $B_n(x)$

Label the rising and descending diagonal functions of x $R_i(x)$ and $D_i(x)$ for $\{A_n(x)\}$, and $r_i(x)$ and $d_i(x)$ for $\{B_n(x)\}$.

From (3) and (4), we readily obtain

$$(5)\qquad \begin{cases} R_1(x) = 1 \\ R_2(x) = px \\ R_3(x) = p^2x^2 \\ R_4(x) = p^3x^3 + q \\ R_5(x) = p^4x^4 + 2pqx \\ R_6(x) = p^5x^5 + 3p^2qx^2 \\ R_7(x) = p^6x^6 + 4p^3qx^3 + q^2 \\ R_8(x) = p^7x^7 + 5p^4qx^4 + 3pq^2x \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(6)\qquad \begin{cases} r_1(x) = 2 \\ r_2(x) = px \\ r_3(x) = p^2x^2 \\ r_4(x) = p^3x^3 + 2q \\ r_5(x) = p^4x^4 + 3pqx \\ r_6(x) = p^5x^5 + 4p^2qx^2 \\ r_7(x) = p^6x^6 + 5p^3qx^3 + 2q^2 \\ r_8(x) = p^7x^7 + 6p^4qx^4 + 5pq^2x \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

with the properties $(n > 3)$

$$(7)\qquad \begin{cases} r_n(x) = R_n(x) + qR_{n-3}(x) \\ R_n(x) = pxR_{n-1}(x) + qR_{n-3}(x) \\ r_n(x) = xr_n(x) + qr_{n-3}(x) \end{cases}$$

Further, we have from (3) and (4), after some simplification in (8),

$$(8)\qquad \begin{cases} D_1(x) = 1 \\ D_2(x) = px + q \\ D_3(x) = (px + q)^2 \\ D_4(x) = (px + q)^3 \\ D_5(x) = (px + q)^4 \\ D_6(x) = (px + q)^5 \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots$$

and

$$(9)\qquad \begin{cases} d_1(x) = 2 \\ d_2(x) = px + 2q \\ d_3(x) = (px + 2q)(px + q) \\ d_4(x) = (px + 2q)(px + q)^2 \\ d_5(x) = (px + 2q)(px + q)^3 \\ d_6(x) = (px + 2q)(px + q)^4 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

whence

$$(10)\qquad D_n(x) = (px + q)^{n-1} \qquad (n \geq 1)$$

$$(11)\qquad d_n(x) = (px + 2q)(px + q)^{n-2} \qquad (n \geq 2)$$

$$(12)\qquad d_n(x) = D_n(x) + qD_{n-1}(x) \qquad (n \geq 2)$$

giving

$$(13)\qquad \frac{D_{n+1}(x)}{D_n(x)} = \frac{d_{n+1}(x)}{d_n(x)} = px + q$$

$$(14)\qquad \frac{d_{n+1}(x)}{D_n(x)} = px + 2q$$

$$(15)\qquad \frac{d_n(x)}{D_n(x)} = \frac{px + 2q}{px + q} \qquad (px + q \neq 0)$$

GENERATING FUNCTIONS FOR THE DIAGONAL FUNCTIONS

Generating functions for the descending diagonal functions are found to be

$$(16)\qquad \sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1 - (px + q)t]^{-1}$$

$$(17)\qquad \sum_{n=2}^{\infty} d_n(x)t^{n-2} = (px + 2q)[1 - (px + q)t]^{-1}$$

while those for the rising diagonal functions are

$$(18)\qquad \sum_{n=1}^{\infty} R_n(x)t^{n-1} = [1 - (pxt + qt^3)]^{-1}$$

$$(19)\qquad \sum_{n=2}^{\infty} r_n(x)t^{n-1} = (1 + qt^3)[1 - (pxt + qt^3)]^{-1}.$$

SOME PROPERTIES INVOLVING DIFFERENTIAL EQUATIONS

Limiting ourselves to the types of results studied by Jaiswal [5], let us write, for convenience,

(20) $$D \equiv D(x,t) = \sum_{n=1}^{\infty} D_n(x)t^{n-1}$$

(21) $$d \equiv d(x,t) = \sum_{n=2}^{\infty} d_n(x)t^{n-2}.$$

Calculations using (16) and (17) and the notation of (20) and (21) then lead to the following differential equations involving the descending diagonal functions:

(22) $$pt\frac{\partial D}{\partial t} - (px + q)\frac{\partial D}{\partial x} = 0$$

(23) $$pt\frac{\partial d}{\partial t} - (px + q)\left[\frac{\partial d}{\partial x} - pD\right] = 0$$

(24) $$(px + q)\frac{d}{dx}D_n(x) = p(n - 1)D_n(x)$$

(25) $$(px + q)\frac{d}{dx}[d_{n+2}(x)] - p(n + 1)d_{n+2}(x) + pq(px + q)D_n(x) = 0.$$

Write

(26) $$R \equiv R(x,t) = \sum_{n=1}^{\infty} R_n(x)t^{n-1}$$

(27) $$r \equiv r(x,t) = \sum_{n=2}^{\infty} r_n(x)t^{n-1}.$$

Corresponding differential equations for the rising diagonal functions are, by (18), (19), (26), and (27):

(28) $$pt\frac{\partial R}{\partial t} - (px + 3qt^2)\frac{\partial R}{\partial x} = 0$$

(29) $$pt\frac{\partial r}{\partial t} - (px + 3qt^2)\frac{\partial r}{\partial x} - 3p(r - R) = 0$$

(30) $$px\frac{d}{dx}R_{n+2}(x) + 3q\frac{d}{dx}R_n(x) - p(n + 1)R_{n+2}(x) = 0$$

(31) $$px\frac{d}{dx}r_{n+2}(x) + 3q\frac{d}{dx}r_n(x) - p(n - 2)r_{n+2}(x) - 3pR_{n+2}(x) = 0.$$

Explicit formulation of expressions for $R_{n+1}(x)$ and $r_{n+1}(x)$ can be obtained by comparison of coefficients of t^n in (18) and (19), respectively.

Computation gives

(32) $$R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i}(px)^{n-3i}q^i$$

(33) $$r_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i}(px)^{n-3i}q^i + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i}(px)^{n-3-3i}q^{i+1},$$

where $[n/3]$ means the integral part of $n/3$.

SOME SPECIAL CASES

Contents of the several papers mentioned in the introduction have thus been generalized, *mutatis mutandis*.

If $p = 1$, $q = 1$, the results of [2] are obtained, including the special cases of the *Fibonacci*, *Lucas*, and *Pell* sequences.

If $p = 2$, $q = -1$, the results of [1] and [4], and of Jaiswal [5], follow for the *Chebyshev* polynomial sequences.

Observe that, for the Chebyshev polynomials of the first kind $U_n(x)$, it is customary (e.g., in [1], [4], and [5]) to define $U_0(x) = 1$, $U_1(x) = 2x$; whereas, from (1), the corresponding generalized forms require

$$A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = px,\ \ldots .$$

For our purposes, this is unimportant. However, suitable adjustments can be made if desired.

If $p = 1$, $q = -2$, the results of [4] for the *Fermat* polynomial sequences follow.

THE FERMAT SEQUENCES

For the record, the following results, which were left to the reader's curiosity in [4], are listed (using the symbolism of [4]).

Differential equation properties

$$t\frac{\partial D}{\partial t} - (x - 2)\frac{\partial D}{\partial x} = 0 \tag{34}$$

$$t\frac{\partial D'}{\partial t} - (x - 2)\left\{\frac{\partial D'}{\partial x} - D\right\} = 0 \tag{35}$$

$$(x - 2)\frac{dD(x)}{dx} = (n - 1)D_n(x) \tag{36}$$

$$(x - 2)\frac{d}{dx}[D'_{n+2}(x)] - (n + 1)D'_{n+2}(x) - 2(x - 2)D_n(x) = 0 \tag{37}$$

with corresponding equations for the rising diagonal functions

$$t\frac{\partial R}{\partial t} - (x - 6t^2)\frac{\partial R}{\partial x} = 0 \tag{38}$$

$$t\frac{\partial R'}{\partial t} - (x - 6t^2)\frac{\partial R'}{\partial x} - 3(R' - R) = 0 \tag{39}$$

$$x\frac{dR_{n+2}(x)}{dx} - 6\frac{dR_n(x)}{dx} - (n + 1)R_{n+2}(n) = 0 \tag{40}$$

$$x\frac{dR'_{n+2}(x)}{dx} - 6\frac{dR'_n(x)}{dx} - (n - 2)R'_{n+2}(x) - 3R_{n+2}(x) = 0 \tag{41}$$

where the primes in $D'_{n+2}(x)$, $R'_{n+2}(x)$, etc., do not indicate derivatives, and where

$$D' \equiv D'(x,t) = \sum_{n=2}^{\infty} D_n'(x)t^{n-2}$$

and

$$R' \equiv R'(x,t) = \sum_{n=2}^{\infty} R_n'(x)t^{n-1}.$$

Explicit formulation

Employing the method used to obtain (32) and (33), we calculate

$$R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i}(-2)^i \tag{42}$$

$$R_{n+1}'(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i}(-2)^i + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i} x^{n-3-3i}(-2)^{i+1}. \tag{43}$$

CONCLUDING REMARKS

Undoubtedly, there are many other facets of this work remaining to be explored. Suffice it for us to comment here that some basic features of many interesting polynomial sequences have been unified.

Finally, it might be noted that our classification here, in (1) and (2) of the sequence, say $\{W_n(x)\}$, for which $W_{n+2}(x) = pxW_{n+1}(x) + qW_n$, into its Fibonacci-type and Lucas-type components (see [2] for the case $p = 1$, $q = 1$) recalls the article by A. J. W. Hilton entitled "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Lucas Sequences" which appeared in this journal, Vol. 12, No. 4 (1974):239-245.

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FACTORS OF THE BINOMIAL CIRCULANT DETERMINANT

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1. INTRODUCTION

Interesting problems and patterns in algebra, number theory, and numerical computation have arisen in the attempt to prove or disprove a conjecture known as Fermat's Last Theorem [7], namely that for odd primes p there are no rational integral solutions x, y, z, with $xyz \neq 0$ to the equation

$$x^p + y^p + z^p = 0. \tag{1.1}$$

Several proofs of special cases involve the prime factors of the determinant D_n of the $n \times n$ binomial circulant matrix B_n with (i, j)-entry

$$\binom{n}{|i - j|}.$$

Thus in 1919 Bachmann [1] proved that (1.1) has no solutions prime to p unless $p^3 | D_{p-1}$, and in 1935 Emma Lehmer [6] proved the stronger requirement, $p^{p-1} | D_{p-1}$, mentioning that $D_n = 0$ iff $n = 6k$, and giving the values of D_{p-1} for $3 \leq p \leq 17$. Later, in 1959-60, L. Carlitz published two papers [2, 3] concerning the residues of D_{p-1} modulo powers of p, including the theorem that (1,1) is solvable with $xyz \neq 0$ only if $D_{p-1} \equiv 0 \pmod{p^{p+43}}$. Our methods give, for example when $p = 47$, the prime factorization

$$-D_{46} = 3 \cdot 47^{45}(139^4 461^2 599^4 691^4 829^2 1151^2 2347^2 3313^2 178481 \cdot 2796203)^3 \tag{1.2}$$

Clearly, a nontrivial solution of (1.1) would require that for all primes q not dividing xyz we should have

$$1 + (y/x)^p \equiv (-z/x)^p \pmod{q}. \tag{1.3}$$

For each such prime p and for all primes $q = 1 + np$ not divisors of xyz, we should have

$$(1 + (y/x)^p)^n \equiv 1 \pmod{q}. \tag{1.4}$$

Thus, all primes $q = 1 + np$ except the finite number that divide xyz must divide the corresponding D_n, which is the resolvent of $v^n - 1$ and $(v + 1)^n - v^n$.

Our concern in this paper is to characterize and compute the rational prime factors of the determinant D_n, an integer of about $0.1403n^2$ digits, when $n \not\equiv 0 \pmod 6$. The 351-digit integer $-D_{50}$ was found to have 127 prime factors, counting multiplicities as high as 24 for the factor 101.

To factor D_n we first note that its $n \times n$ binomial circulant matrix B_n is a polynomial in the $n \times n$ circulant matrix P_n for the permutation (1 2 3 ... n), whose eigenvalues are powers of a primitive nth root of unity, r, and that D_n is the product of the eigenvalues of B_n. Thus, as in [5],

$$B_n = (I_n + P_n)^n - I_n \tag{1.5}$$

$$D_n = \prod_{k=1}^{n} ((1 + r^k)^n - 1), \quad \text{where } r = e^{2\pi i/n}. \tag{1.6}$$

For example, when $n = 4$,

(1.7) $$P_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 4 & 6 & 4 \\ 4 & 1 & 4 & 6 \\ 6 & 4 & 1 & 4 \\ 4 & 6 & 4 & 1 \end{bmatrix} = (I_4 + P_4)^4 - I_4$$

(1.8) $$D_4 = ((1 + i)^4 - 1)(0^4 - 1)((1 - i)^4 - 1)(2^4 - 1) = -3 \cdot 5^3.$$

Factoring the difference of two nth powers in (1.6) yields

(1.9) $$D_n = \prod_{k=1}^{n} \prod_{j=1}^{n} ((1 + r^k)r^j - 1) = (-1)^n \prod_{j=1}^{n} \prod_{k=1}^{n} (1 - r^j - r^k).$$

Theorem 1.1 (E. Lehmer [6]): $D_n = 0$ if and only if $6|n$.

Proof: A factor $(1 - r^j - r^k)$ in (1.9) can vanish if and only if $r^k = r^{-j}$, and $r^{6j} = 1$.

Henceforth we assume $n \not\equiv 0 \pmod 6$.

Experimental evidence indicates that for $n \le 50$,

(1.10) $$|\log_{10}|D_n| - n^2 \log_{10} G| < 0.33, \text{ if } n \not\equiv 0 \pmod 6,$$

where G is the limit as $n \to \infty$ of the geometric mean of the n^2 factors $|1 - r^j - r^k|$ of $(-1)^{n-1}D_n$. If $u - v = \theta$, we have

(1.11) $$\ln G = \pi^{-2} \int_0^{\pi} \int_0^{\pi} \ln|1 - e^{2iu} - e^{2iv}|du\, dv$$
$$= \pi^{-2} \int_0^{\pi} \int_0^{\pi} \ln|2 \cos \theta - e^{-2i\phi}|d\phi\, d\theta.$$

The inner integral vanishes if $|2 \cos \theta| < 1$, and we obtain

(1.12) $$\ln G = (2/\pi) \int_0^{\pi/3} \ln(2 \cos \theta) d\theta = (2/\pi) \int_0^{\pi/6} \theta \cot \theta \, d\theta$$

(1.13) $$\log_{10} G = (0.32306594722\ldots)/\ln(10) = 0.14030575817\ldots .$$

Missing factors in the tables were detected by (1.10), and found.

Our challenge is to assemble the n^2 complex factors of (1.9) into subsets having rational integral products which we call "principal" factors, and then factor these positive integers into their rational prime factors. We find that $(-1)^{n-1}D_n/(2^n - 1)$ is always a square, that $-D_{2n}/3$ is a cube, and that for odd n the sum $F_{n-1} + F_{n+1}$ of two Fibonacci numbers is a double factor of D_n, of about $1 + n/5$ digits, which is frequently prime. For example, D_{47} and D_{53} have respectively as double factors the primes $F_{46} + F_{48} = 6{,}643{,}838{,}879$ and $F_{52} + F_{54} = 119{,}218{,}851{,}371$. Tables 1 and 2 list the prime factors of D_n other than $2^n - 1$ for 16 odd values of n.

TABLE 1

FACTORS $q_p^{(\pm u)}$ OF d_p, WHERE p IS PRIME,
AND UNDERLINED FACTORS ARE $q_p^{(-u)}$

u	d_{19}	d_{23}	d_{29}	d_{31}	d_{37}	d_{41}	d_{43}	d_{47}
2	9349	139 • 461	59 • 19489	3010349	54018521	370248521	969323029	6643838879
3	1483	47 • 139	65657	$5^3 \cdot 1117$	1385429	83 • 77081	431 • 31907	941 • 67399
4	229	1151	9803	27901	132313	83^3	952967	283 • 11939
5	$\underline{761}$	599	59^2	5953	149 • 223	101107	173 • 1033	549149
6	647	$\underline{3313}$	$\underline{24071}$	20089	67489	83^3	516689	1693 • 2351
7	229	47^2	18503	$\underline{16741}$	149 • 1259	$\underline{83 \cdot 3691}$	173 • 6967	6450751
8	419	47^2	59 • 233	$\underline{46439}$	325379	988511	1124107	$\underline{1352191}$
9	191	2347	4931	38069	223 • 1481	821 • 1559	745621	7145599
10		$\underline{599}$	$\underline{18097}$	34721	172717	1335781	173 • 2337	283 • 36943
11		691	59 • 349	5953	146891	83 • 6397	$\underline{2532701}$	1223 • 2663
12			12413	$2^5 \cdot 1489$	262553	791629	1549 • 1721	$\underline{10032151}$
13			$\underline{59^2}$	$\underline{2^5 \cdot 683}$	149 • 223	348911	$\underline{1144919}$	2069 • 5077
14			59^2	$2^5 \cdot 311$	$\underline{332039}$	$\underline{83 \cdot 12301}$	1999243	3462961
15				6263	$\underline{149 \cdot 1999}$	$\underline{206477}$	173 • 1033	1932923
16					$\underline{68821}$	1024099	431 • 5591	$\underline{941 \cdot 8179}$
17					223 • 593	739 • 1723	173 • 10837	4220977
18					32783	$\underline{340793}$	$\underline{173 \cdot 11783}$	5187109
19						$\underline{101107}$	431 • 3613	$\underline{1129 \cdot 6863}$
20						83 • 1231	533459	1754323
21							178021	$\underline{659 \cdot 3761}$
22								$\underline{549149}$
23								549431

TABLE 2

FACTORS $\overline{q}_n^{(u)}$ OF $\overline{d}_n$ FOR COMPOSITE ODD n

u	$\overline{d}_9$	$\overline{d}_{15}$	$\overline{d}_{21}$	$\overline{d}_{25}$	$\overline{d}_{27}$	$\overline{d}_{33}$	$\overline{d}_{35}$	$\overline{d}_{39}$
$3:p$		271	2269			176419		$157 \cdot 10141$
$5:p$							38851	
2	19	31	211	$101 \cdot 151$	5779	9901	$71 \cdot 911$	$79 \cdot 859$
3	37	31	379	1301	811	67^2	7351	22777
-3	19	2^4	43		487	2971		6553
4	1	$2^2 \cdot 1$*	7	3851	919	67	3361	547
5		61	43	1151	109	463	2381	$79 \cdot 3^3$
-5		31		6101			3011	
6		1	463	151	433	331	41*	79^2
-6		61	1		163	3631	29*	1249
7		1	43	251		199	7841	157
-7			547		163		71^2	
8			$1 \cdot 7$*	401	2269	859	71	$\underline{79 \cdot 3^3}$
9			43	1151	19441	2311		1171
-9			43		19927	397	701	3511
10			7^2	5801	1	43*	$71 \cdot 281$	
-10				1951		1*	71^2	1249
11				101	$\underline{757}$	$67 \cdot 661$	71	3121
-11						25411		
12				101	109	1		$79 \cdot 937$
-12					109	$67 \cdot 199$	421	1
13					271	67	5741	$79 \cdot 2887$
-13								398581
14						331	118301	1*
-14							4271	103*
15						397	911	1171
-15						463	211^2	13183
16						67	$\underline{2381}$	157
17							211	1483
18								$313 \cdot 3^3$
-18								$79 \cdot 3^3$
19								157

*If $u^2 \equiv 1 \pmod{n}$, $\left(\overline{q}_n^{(u)}\overline{q}_n^{(-u)}\right)^{1/2}$ replaces $\overline{q}_n^{(u)}$ in $\overline{d}_n$.

2. PRINCIPAL INTEGRAL FACTORS OF D_n

For n odd, we extract from D_n in (1.9) the product $1 - 2^n$ of n factors with $j = k$, the product 1 of the $2(n - 1)$ factors with $j = n \neq k$ or $k = n \neq j$, and the product $q_n^{(-1)}$ of the $n - 1$ real factors with $j + k = n$, and are left with $(n - 1)(n - 3)$ factors whose product d_n^2 is a perfect square because of symmetry in j and k.

Theorem 2.1: For n odd, we have

$$D_n = (2^n - 1)q_n^{(-1)}d_n^2, \tag{2.1}$$

where $q_n^{(-1)} = 4$ if $3|n$, $q_n^{(-1)} = 1$ if $n \equiv \pm 1 \pmod 6$, and d_n is a product of $(n-1)(n-3)/4$ conjugate complex factor pairs, namely

$$d_n = \prod_{0<j<k<n-j} (1 - r^j - r^k)(1 - r^{-j} - r^{-k}),\ r = e^{2\pi i/n}. \tag{2.2}$$

Proof: The product of the $(n-1)$ real factors of (1.9) with $1 \le j \le n-1$ is

$$q_n^{(-1)} = \prod_{j=1}^{n-1} (1 - r^j - r^{-j}) = \prod_{j=1}^{n-1} (-r^{-j})(r^j + \omega)(r^j + \overline{\omega})$$

$$= 1 \cdot (1 + \omega^n)(1 + \omega^{-n}) = (\omega^{n/2} + \omega^{-n/2})^2 \tag{2.3}$$

$$= (2 \cos \pi n/3)^2$$

where $\omega = e^{2\pi i/3}$. This is 4 if $3|n$, or 1 if $n \equiv \pm 1 \pmod 6$. Of the remaining complex factors with $j + k \neq n$, those with $j + k > n$ are the complex conjugates of those with $j + k < n$. Just half the factors of d_n^2 yield d_n, so we take $j < k$ in (2.2).

For even dimension $2n$ we replace $-r^j$ and $-r^k$ in (1.9) by s^{j+n} and s^{k+n}, where $s = e^{\pi i/n}$ and $s^n = -1$. The factor with 3 equal summands is $1 + 1 + 1 = 3$, and the $3(2n - 1)$ factors with 2 equal summands have the product

$$-((4^n - 1)/3)^3.$$

Since $3 \nmid n$, we can divide each of the $(2n - 1)(2n - 2)$ remaining factors by the geometric mean of its 3 summands so the new factors have distinct summands with product 1.

Theorem 2.2: For even dimension $2n$, we have

$$D_{2n} = -3((4^n - 1)/3)^3 g_{2n}^6, \tag{2.4}$$

where g_{2n} is the product of $(n-1)(n-2)/3$ conjugate complex factor pairs

$$g_{2n} = \prod_{0<j<k<n-j/2} |s^j + s^k + s^{-j-k}|^2,\ s = e^{\pi i/n}. \tag{2.5}$$

Proof: Extracting from D_{2n} the factors with repeated summands leaves a product of $(2n - 1)(2n - 2)$ factors with distinct summands

$$-9D_{2n}/(4^n - 1)^3 = \prod_{j,k,i=1}^{2n} (s^j + s^k + s^i),\ s^{j+k+i} = 1, \tag{2.6}$$
$$i, j, k \text{ distinct}.$$

We omit the $3(2n - 2)$ factors with product 1 having i, j, or $k = 2n$. Symmetry in i, j, k shows that each remaining factor is repeated six times, so we call the product g_{2n}^6, where in g_{2n} we assume $1 \le j < k < i < 2n$. Since factors with $j + k + i = 4n$ are the complex conjugates of factors with $j + k + i = 2n$, we replace i by $2n - j - k$ and s^i by s^{-j-k} to obtain (2.5).

Theorem 2.3: For odd $n = 2m + 1$ not divisible by 3, $g_{2n} = d_n h_n$ where h_n is the product of $m(m - 2)/3$ factor pairs

$$(2.7) \qquad h_n = g_{2n}/d_n = \prod_{0<j<k<(n-j)/2} |r^j + r^k + r^{-j-k}|^2, \; r = e^{2\pi i/n}.$$

Proof: The $m(m-2)/3$ factor pairs in (2.5) with j and k both even yield the factor pairs of h_n in (2.7). We next delete the m factor pairs in (2.5) for which j or k equals $n - j - k$, since $s^n = -1$ and these factors have the product 1. In the remaining $m(m-1)$ factor pairs having two summands with odd exponents, we multiply these two summands by $-s^n = 1$ to create even exponents, divide the factor by the third summand, set $s^2 = r$, and obtain precisely the factors of d_n in (2.2).

Note that (2.4) and (2.7) imply that for $n \equiv \pm 1 \pmod 6$

$$(2.8) \qquad -D_{2n}/D_n^3 = 3^{-2}(2^n + 1)^3 h_n^6, \text{ if } n \equiv \pm 1 \pmod 6.$$

Theorem 2.4: For $n = 2m$ not divisible by 6, $g_{2n} = g_n k_n$, where k_n is the product of $m(m-1)$ factor pairs:

$$(2.9) \qquad k_n = g_{2n}/g_n = \prod_{0<j<k<2n-j} |1 + s^j + s^k|^2, \; j, k \text{ odd}, \; s = e^{\pi i/n}.$$

Proof: The $(m-1)(m-2)/3$ factor pairs in (2.5) having j and k both even yield the factor pairs of g_n for even n. We obtain the remaining $m(m-1)$ factor pairs for k_n in (2.9) by dividing each of the remaining factors of g_{2n} by its summand with even exponent.

If desired, we can remove the $[m/2]$ factor pairs with product 1 in (2.9) for which $k = n + j$. For example, when $m = 2$, one of the two factor pairs in $k_4 = g_8/g_4$ can be removed, leaving

$$(2.10) \qquad k_4 = g_8/g_4 = |1 + s + s^3|^2 = |1 + i\sqrt{2}|^2 = 3, \; s = e^{\pi i/4}.$$

Since $g_4 = g_2 = d_1 = 1$, we have $D_8 = -3(85)^3 \cdot 3^6 = -3^7 \cdot 5^3 \cdot 17^3$. The reduced integral factors $\overline{d}_n$ of d_n and $\overline{h}_n$ of h_n are products of those complex factors of (2.2) or (2.7) in which j, k, n have no common factor.

The extended principal factors of d_n, h_n, and k_{2n} are products of those complex factors of d_n, h_n, or k_{2n} in which the exponent ratios $k:j$ are constant (mod n). They are rational integers, since they are symmetric functions of roots of unity. In such an extended principal factor $q_n^{(v:u)}$, we assume u, v relatively prime and replace (j,k) by (vj, uj) where $0 < j < n$. For $\overline{d}_n$ and $\overline{h}_n$ we restrict j to a reduced set of residues (mod n) denoted R_n, in which $(j,n) = 1$. We define the extended principal factors $q_n^{(v:u)}$ and the principal factors $\overline{q}_n^{(v:u)}$ by

$$(2.11) \qquad q_n^{(v:u)} = \pm \prod_{j=1}^{n-1} (1 - r^{vj} - r^{uj}) > 0, \; q_n^{(u)} = q_n^{(1:u)} = q_n^{(u:1)}$$

$$(2.12) \qquad \overline{q}_n^{(v:u)} = \pm \prod_{j \in R_n} (1 - r^{vj} - r^{uj}) > 0, \; \overline{q}_n^{(u)} = \overline{q}_n^{(1:u)} = \overline{q}_n^{(u:1)}$$

where $r = e^{2\pi i/n}$. The corresponding integral factors of k_n or h_n with complex factors $(1 + r^{vj} + r^{uj})$ are denoted by $q_{n+}^{(v:u)}$, etc. Factors of $q_{n\pm}^{(v:u)}$ for which $(j,n) = n/f$ divide $q_{f\pm}^{(v:u)}$ for divisors f of n.

For calculations with a calculator that computes cosine functions, the following factors are useful. We set

$$(2.13) \qquad \overline{f}_n^{(y;x)} = \pm \prod_{j \in R_n} (c_{yj} + c_{yj}^{-1} - c_{xj}) > 0, \; (x,y) = 1$$

where $c_k = r^k + r^{-k} = 2\cos 2\pi k/n$, and where R_n' denotes the set of $\varphi(n)/2$ residues $j \in R_n$ with $j < n/2$.

Theorem 2.5: If $2x = (u + v)$, $2y = u - v$, then

$$\overline{f}_n^{(y;x)} = \overline{q}_n^{(v:u)},\ \overline{f}_n^{(v;u)} = \overline{q}_n^{(y;x)},\ n \text{ odd}. \tag{2.14}$$

Proof:

$$\begin{aligned}\overline{q}_n^{(v:u)} &= \prod_{j \in R_n'} |1 - r^{vj} - r^{uj}|^2 = \prod_{j \in R_n'} (3 + c_{2yj} - c_{vj} - c_{uj}) \\ &= \prod_{j \in R_n'} (1 + c_{yj}^2 - c_{yj}c_{xj}) = \pm \prod_{j \in R_n'} (c_{yj} + c_{yj}^{-1} - c_{xj})\end{aligned} \tag{2.15}$$

since the product of the c_{yj} is ± 1. Solving for u,v in terms of x,y yields the second part of (2.14)

Theorem 2.6: If $n = 2m + 1$ is a prime $p > 3$, then

$$d_p = \prod_{u=2}^{m} q_p^{(\varepsilon u)},\ \varepsilon = \pm 1 \tag{2.16}$$

where $\varepsilon = 1$ if $u < u' \equiv 1/u \pmod p$ or $\varepsilon = -1$ if $u' < u < p/2$.

Proof: The product of the $p - 3$ integers $q_p^{(u)}$ for $2 \le u \le p - 2$ is d_p^2. Since $q^{(u')} = q^{(u)}$ if $uu' \equiv 1 \pmod p$, we multiply together one factor from each of these pairs to obtain d_p.

For example

$$\begin{aligned} d_5 &= q_5^{(2)} = f_5^{(3)} = 11;\ d_7 = q_7^{(2)} q_7^{(3)} = f_7^{(3)} f_7^{(2)} = 29 \cdot 8 \\ d_{11} &= q_{11}^{(2)} q_{11}^{(3)} q_{11}^{(-4)} q_{11}^{(5)} = f_{11}^{(3)} f_{11}^{(2)} f_{11}^{(5)} f_{11}^{(-4)} = 199 \cdot 67 \cdot \underline{23} \cdot 23 \\ d_{13} &= \prod_{u=2}^{6} q_{13}^{(u)} = 521 \cdot 131 \cdot 79 \cdot 27 \cdot 53 \\ d_{17} &= 3571 \cdot 613 \cdot 409 \cdot 137 \cdot \underline{307} \cdot \underline{137} \cdot 103. \end{aligned} \tag{2.17}$$

Theorem 2.7: If p^b is a maximal prime power divisor of $q_n^{(u)}$ for prime $n > u > 0$, then $p^b \equiv 1 \pmod n$.

Proof: If $p | q_n^{(u)}$, there is a smallest field $GF(p^e)$ of characteristic p that contains a mark $\overline{r}$ such that $\overline{r}^n \equiv 1 \equiv \overline{r} + \overline{r}^u \pmod p$. Raising to pth powers we see that $\overline{r}^{p^k}$ is a solution for $k = 0, 1, \ldots, e - 1$. Since b factors $1 - \overline{r}^j - \overline{r}^{uj}$ vanish $\pmod p$, e divides b. Since the order of $\overline{r} \not\equiv 1$ is a factor of the prime n, it is n. Hence n divides the order $p^e - 1$ of the multiplicative group of $GF(p^e)$, which divides $p^b - 1$.

We find, for example, that $q_7^{(3)} = 2^3$, $q_{13}^{(4)} = 3^3$, and 2^5 divides $q_{31}^{(u)}$ for $u = 12, -13$, and 14. Factors of $q_p^{(u)}$ for primes 19 to 47 are listed in Table 1 above.

When, for composite n, we have $u^2 \equiv 1 \pmod n$ but $u \not\equiv \pm 1 \pmod n$, the factors $q_n^{(u)}$ and $q_n^{(-u)}$ of $\overline{d}_n^2$ are squares without reciprocal mates, so we must include only their square roots in $\overline{d}_n$. Also, $\overline{d}_n$ may include factors $q^{(v:u)}$ where u and v are relatively prime divisors of n. For example, the

$(n-1)(n-3)/2 = 84$ complex factors of d_{15} include $4 \cdot 2/2 = 4$ from d_5 and $2 \cdot 0/2 = 0$ from d_3, leaving 40 complex conjugate pairs in $\overline{d}_{15}$. The latter include four pairs each from $\overline{q}_{15}^{(u)}$ for $u = 2, 3, 5, 6, 7, 9, 10$, and 12, four from $\overline{q}_{15}^{(3:5)}$, but only two pairs each from $\overline{q}_{15}^{(4)} = 16$ and $\overline{q}_{15}^{(-4)} = 1$.

$$\overline{d}_{15} = 31 \cdot 31 \cdot 61 \cdot 1 \cdot 1 \cdot 61 \cdot 31 \cdot 2^4 \cdot 271 \cdot (2^4 \cdot 1)^{1/2}. \tag{2.18}$$

The factor $q_{15}^{(4)}$ was found by (2.13) to be

$$q_{15}^{(4)} = f_{15}^{(3;5)} = (\sqrt{5}+1)^2(-\sqrt{5}+1)^2 = 2^4. \tag{2.19}$$

To evaluate the principal factor $\overline{q}_{3p}^{(3:p)}$ for primes $p \geq 5$, we set

$$r^p = \omega = e^{2\pi i/3}$$

and obtain

$$\begin{aligned} \overline{q}_{3p}^{(3;p)} &= \prod_{j \in R_{3p}} (1 - r^{pj} - r^{3j}) = |(1-\omega^j)^p - 1|^2 \\ &= 3^p - (\omega^{-p} - \omega^p)(\omega - \omega^2)^p + 1 = 3 \quad - \sigma 3^{(p+1)/2} + 1 \end{aligned} \tag{2.20}$$

where $\sigma = (-3/p) = \pm 1$ is the quadratic character of $-3 \pmod p$. In particular, $\overline{q}_{15}^{(3;5)} = 3^5 + 3^3 + 1 = 271$ (see Table 2), and

$$\overline{q}_{21}^{(3;7)} = 2269, \quad \overline{q}_{33}^{(3:11)} = 176419, \quad q_{39}^{(3;13)} = 157 \cdot 10141. \tag{2.21}$$

To compute $q_{27}^{(\pm 9)}$, we note that the ninth roots of ω are r^{1+3k}. Hence,

$$\begin{aligned} q_{27}^{(\pm 9)} &= \prod_{k=1}^{9} |1 - r^9 - r^{\pm 1 + 3k}|^2 = |(1-\omega)^9 - \omega^{\pm 1}|^2 \\ &= 3^9 \pm 3^5 + 1 = 19684 \pm 243. \end{aligned} \tag{2.22}$$

3. THE FIBONACCI FACTORS OF d_n AND g_{2n}

Several extended principal factors of D_n are expressible as sums or ratios of Fibonacci numbers.

Theorem 3.1: For n odd, the factor $q_n^{(2)}$ of D_n is given by

$$q_n^{(2)} = F_{2n}/F_n = F_{n-1} + F_{n+1} = [\tau^n], \quad \tau = (\sqrt{5}+1)/2 \tag{3.1}$$

where [ ] denotes the greatest integer function, and F_k denotes the kth Fibonacci number, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}. \tag{3.2}$$

Proof: The roots of $z^2 - z - 1 = 0$ are $\tau = (\sqrt{5}+1)/2$ and $\overline{\tau} = -1/\tau$. Factorization of (2.11) for $u = 2$ and n odd yields

$$q_n^{(2)} = -\prod_{j=1}^{n} (1 - r^j\tau)(1 - r^j\overline{\tau}) = -(1-\tau^n)(1-\overline{\tau}^n) = \tau^n + \overline{\tau}^n = [\tau^n]. \tag{3.3}$$

It is known, and can be shown by induction, that

$$F_k = (\tau^k - \overline{\tau}^k)/(\tau - \overline{\tau}), \quad F_{2k}/F_k = \tau^k + \overline{\tau}^k \tag{3.4a}$$

(3.4b) $\qquad F_{k-1} + F_{k+1} = (\tau^{k-1} + \tau^{k+1} - \overline{\tau}^{k-1} - \overline{\tau}^{k+1})/(\tau - \overline{\tau}) = \tau^k + \overline{\tau}^k.$

Hence (3.3) and (3.4) imply (3.1).

The Fibonacci factors $[\tau^n] = q_n^{(2)}$ for the first 25 odd numbers $n = 10t + d$ follow, with factors underlined which are omitted from $\overline{q}_n^{(2)}$.

(3.5)

	$10t$				
d	0	10	20	30	40
1	1	199	$\underline{2^2\cdot 29}\cdot 211$	3010349	370248451
3	2^2	521	$139\cdot 461$	$\underline{2^2\cdot 199}\cdot 9901$	969323029
5	11	$\underline{2^2\cdot 11}\cdot 31$	$\underline{11}\cdot 101\cdot 151$	$\underline{11\cdot 29}\cdot 71\cdot 911$	$\underline{2^2\cdot 11\cdot 19\cdot 31}\cdot 181\cdot 541$
7	29	3591	$\underline{2^2\cdot 19}\cdot 5779$	54018521	6643838879
9	$\underline{2^2}\cdot 19$	9349	$59\cdot 19489$	$\underline{2^2\cdot 521}\cdot 79\cdot 859$	$29\cdot 599786069$

Note that each prime factor of $\overline{q}_n^{(2)}$ (not underlined) is congruent to 1 (mod n).

Since d_n divides g_{2n} for odd n, so does F_{2n}/F_n.

Theorem 3.2: The integer g_{2n} is divisible by F_n for even n and by F_{2n}/F_n for odd n.

Proof: The product of the $[n/2] - 1$ factor pairs in (2.5) for which $j + k = n$ and $s = -1$ is expressible as

$$\prod_{0<2j<n} |s^j - s^{-j} - 1|^2 = \prod_{0<2j<n} (3 - s^{2j} - s^{-2j})$$

$$(3.6) \qquad = \prod_{0<2j<n} (\tau + s^{2j}\overline{\tau})(\tau + s^{-2j}\overline{\tau})$$

$$= (\tau^n - (-\overline{\tau})^n)/(\tau - (-1)^n\overline{\tau})$$

where $\tau + \overline{\tau} = -\tau\overline{\tau} = 1$. This is F_n for n even, and F_{2n}/F_n for n odd.

For $n = 2m$, the factors of (3.6) with j odd have product

$$(\tau^m + (-\overline{\tau})^m)/(\tau + (-1)^m\overline{\tau})$$

which divides k_{2m}. This product is F_m for m odd and F_{2m}/F_m for m even. So

(3.7) $\qquad 3|k_4,\ 7|k_8,\ 5|k_{10},\ 13|k_{14},\ 47|k_{16},\ 123|k_{20},\ 89|k_{22}.$

Theorem 3.3: If p is a prime > 5, then d_{5p} has the factor

(3.8) $\qquad \overline{q}_{5p}^{(5h)} = 1 + 5F_p(F_p - \sigma),\ \sigma = (p/5) = \pm 1,\ 5h \equiv 1 \pmod p$

where F_p is the pth Fibonacci number and $\sigma = \pm 1$ is the quadratic character of p (mod 5).

Proof: Taking $r = e^{2\pi i/5p}$, $z = r^p$, $\tau^{-1} = z + z^{-1}$,

$$q_{5p}^{(5h)} = \prod_{j \in R_{5p}} (1 - r^j - r^{5hj}) = \prod_{j \in R_{5p}} (r^{-5hj} - r^{(1-5h)j} - 1)$$

$$(3.9) \qquad = \prod_{j=1}^{4} (1 - (z^{2j} + 1)^p) = |1 - z^p\tau^{-p}|^2 |1 - z^{2p}(-\tau)^p|^2$$

$$= (\tau^p + \tau^{-p} - z^p - z^{-p})(\tau^p + \tau^{-p} + z^{2p} + z^{-2p})$$

$$= 5F_p(F_p - \sigma) + 1$$

since $\tau^p + \tau^{-p} = \sqrt{5}F_p$, $(z^1 + z^{-1})(z^2 + z^{-2}) = -1$, and

$$(z^p + z^{-p} - z^{2p} - z^{-2p})/\sqrt{5} = \sigma$$

is 1 if $p^2 \equiv 1 \pmod 5$ or -1 if $p^2 \equiv -1 \pmod 5$. The following such factors $q_{5p}^{(5h)}$ are prime except when $p = 13$

(3.10)

$5p$	15	35	55	65	85	95	115
$\overline{q}_{5p}^{(5h)}$	31	911	39161	131 • 2081	12360031	87382901	4106261531

Similarly, $181 | d_{45}$ and $21211 | d_{105}$.

4. POWER SUM FORMULAS FOR PRINCIPAL FACTORS OF D_n

The extended principal factors of $q_n^{(-1)} d_n$ in (2.2) or the corresponding factors $q_{n,c}^{(v:u)}$ of h_n in (2.7) may be treated together by defining

$$(4.1) \qquad (c + 2)q_{n,c}^{(v:u)} = \prod_{j=1}^{n} |c + r^{vj} + r^{uj}|, \quad c = \pm 1, \quad r = e^{2\pi i/n}$$

when u,v are integers with $(u,v) = 1$ and $u > |v| > 0$.

Theorem 4.1: If z_k are the m roots of the equation

$$(4.2) \qquad z^u + z^v + c = 0, \quad c = \pm 1, \quad u > |v| > 0$$

where $m = u$ for $v > 0$ or $m = u - v$ for $v < 0$, then

$$(4.3) \qquad \prod_{j=1}^{n} |c + r^{vj} + r^{uj}| = \prod_{k=1}^{m} |1 - z_k^n|.$$

Proof: Both sides of (4.3) equal the double product

$$(4.4) \qquad \prod_{j=1}^{n} \prod_{k=1}^{m} |r^j - z_k|.$$

When $m = 2$, the two cases $(u,v) = (1,-1)$ and $(2,1)$ were involved in computing $q_n^{(-1)}$ in (2.3) with $z_k = -\omega, -\overline{\omega}$ and $q_n^{(2)}$ in (3.3) with $z_k = -\tau, -\overline{\tau}$. The factor $q_{n+}^{(2)}$ of h_n is 0 if $3|n$ or 1 otherwise, and may be omitted, since $3 \nmid n$.

The unexpected identities

$$(4.5a) \qquad (z^5 + z - 1) = (z + z^{-1} - 1)z(z^3 + z^2 - 1)$$

$$(4.5b) \qquad (z^5 + z + 1) = (z^2 + z + 1)z(z^2 + z^{-1} - 1)$$

enable us to write

(4.6) $$q_n^{(5)} = q_n^{(-1)}q_n^{(2:3)},\; q_{n+}^{(5)} = q_{n+}^{(2)}q_n^{(-2)} = q_n^{(-2)},$$

so the cubic cases $m = 3$ in (4.2) yield not only $q_{n+}^{(3)}$ and $q_n^{(3)}$ but also the two pairs of equal integral factors

$$q_n^{(5)}/q_n^{(-1)} = q_n^{(2:3)} \quad\text{and}\quad q_{n+}^{(5)} = q_n^{(-2)}.$$

Combining (4.1) and (4.3) for $m = 3$ yields

(4.7) $$(2 + c)\cdot q_{n,c}^{(v:u)} = |1 - s_{n,c}^{(v:u)} - \delta^n(1 - s_{-n,c}^{(v:u)})|,\qquad \delta = \Pi z_k$$

where

(4.8) $$s_{n,c}^{(v:u)} = \sum_{k=1}^{m} z_k^n \quad\text{for}\quad z_k^u + z_k^v + c = 0.$$

The product $\delta = \Pi z_k$ is 1 for $q_n^{(3)}$ and $q_n^{(2:3)}$ and -1 for $q_{n+}^{(3)}$ or $q_n^{(-2)}$. We omit the subscript c when $c = -1$ and omit v when $v = 1$.

Replacement of z_k by $-1/z_k$ converts the roots z_k of $z^2 + z^{-1} - 1 = 0$ to those of $z^3 + z^2 - 1 = 0$, and replacement of z_k by $-z_k$ converts $z^3 + z + 1 = 0$ to $z^3 + z - 1 = 0$. Hence

(4.9) $$s_n^{(-2)} = (-1)\, s_{-n}^{(2:3)},\; s_{n+}^{(3)} = (-1)\, s_n^{(3)}.$$

Thus all six extended principal factors for $m = 3$ can be computed from the values of $s_n^{(2:3)}$ and $s_n^{(3)}$ for positive and negative n.

<u>*Theorem 4.2*</u>: The power sums $s_{n,c}^{(v:u)}$ satisfy the recurrence relations

(4.10) $$s_{n+u,c}^{(v:u)} + s_{n+v,c}^{(v:u)} + cs_{n,c}^{(v:u)} = 0.$$

<u>*Proof*</u>: Multiply $z_k^u + z_k^v + c = 0$ by z_k^n and sum over k.

Starting with the value $m = 3$ for $n = 0$, and the values $s_n^{(v:3)}$ for $n = \pm 1$, we obtain values where $v = 2$ or 1 as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$s_n^{(2:3)}$	-1	1	2	-3	4	-2	-1	5	-7	6	-1	-6	12
$s_{-n}^{(2:3)}$	0	2	3	2	5	5	7	10	12	17	22	29	39
$s_n^{(3)}$	0	-2	3	2	-5	1	7	-6	-6	13	0	-19	13
$s_{-n}^{(3)}$	1	1	4	5	6	10	15	21	31	46	67	98	144

Using (4.7) and (4.9) we can then compute the three extended principal factors $q_n^{(-2)}$, $q_n^{(2:3)}$, and $q_n^{(3)}$ of d_n and the factor $q_{n+}^{(3)}$ of h_n or $k_{n/2}$. We use (4.6) to compute the additional factors $q_n^{(5)}$ and $q_{n+}^{(5)}$. We compute

$$\overline{q}_{n+}^{(v:u)} = \overline{f}_{n+}^{(y:x)}$$

by replacing $-c_{xj}$ by c_{xj} in Theorem 2.5. By (4.6) we write $\overline{q}_{n+}^{(5)} = \overline{q}_n^{(-2)}$. Then

$$h_7 = (\overline{q}_{7+}^{(3)})^{1/3} = 2,\; h_{11} = \overline{q}^{(-2)} = 23,$$

$$h_{13} = (\overline{q}_{13+}^{(-3)})^{1/3} = 53\cdot 3,$$ *(continued)*

$$
\begin{aligned}
h_{17} &= \overline{q}_{17}^{(-2)}\overline{q}_{17+}^{(3)} = 103 \cdot 239 \\
h_{19} &= \overline{q}_{19}^{(-2)}\overline{q}_{19+}^{(3)}\left(\overline{q}_{19+}^{(3)}\right)^{1/3} = 191 \cdot 47 \cdot 7 \qquad (4.14)\\
h_{23} &= \overline{q}_{23}^{(-2)}\overline{q}_{23+}^{(3)}\overline{q}_{23+}^{(-3)} = 691 \cdot 47^2 \cdot 829
\end{aligned}
$$

Similarly, since $(2m - 1)^2 \equiv 1 \pmod{4m}$, the factor of k_n in (2.9) is not $\overline{q}_{n+}^{(n-1)}$ but its square root. Using $\overline{f}_{n+}^{(y;x)}$ as before, the factors k_n of D_{2n} for $2n < 44$ are

(4.15)

k_n	k_4	k_8	k_{10}	k_{14}	k_{16}	k_{20}
$(\overline{q}_{2n+}^{(n-1)})^{1/2}$	3	7	5	13	47	41
$\overline{q}_{2n}^{(-2)}$		17	5	2^3	97	281
$\overline{q}_{2n+}^{(3)}$		17	61	337	449	241
$\overline{q}_{2n+}^{(-3)}$			5	29	193	881
$\overline{q}_{2n+}^{(-5)}$			41	197	97	41
$\overline{q}_{2n+}^{(7)}$				113	353	281
$q_{2n+}^{(-7)}$				29	257	41

The remaining factors of k_{20} are

$$(\overline{q}_{40+}^{(9)}\overline{q}_{40+}^{(-9)}\overline{q}_{40+}^{(11)}\overline{q}_{40+}^{(-11)})^{1/2}\,\overline{q}_{40}^{(15)}\overline{q}_{40}^{(-15)} = 3^2 \cdot 31 \cdot 11 \cdot 41 \cdot 641 \cdot 41 \qquad (4.16)$$

Note that the factors $\overline{q}_{2n+}^{(u)}$ in (4.15) are congruent to their squares (mod $2n$). Factors of k_{22} are

$$
\begin{aligned}
k_{22} = {} & 67 \cdot 89 \cdot 353 \cdot 397 \cdot 419 \cdot 617 \cdot 661 \cdot 1013 \cdot 2113 \qquad (4.17)\\
& 2333 \cdot 3257 \cdot 4357
\end{aligned}
$$

The complete factorization of D_{44} is

$$D_{44} = -3(23 \cdot 89 \cdot 683)^3(5 \cdot 397 \cdot 2113)^3(d_{11}h_{11}k_{22})^6. \qquad (4.18)$$

5. FINITE BINOMIAL SERIES FOR THE POWER SERIES OF ROOTS

The two sums $s_{n,b,c}^{(v:u)}$ and $s_{-n,b,c}^{(v:u)}$ of the nth and $-n$th powers of the u roots z of the trinomial equation

$$z^u + bz^v + bc = 0,\ b^2 = c^2 = 1,\ u > v > 0 \qquad (5.2)$$

can both be expressed as sums of a total of at most $2 + |n|/v(u - v)$ integers that involve binomial coefficients.

Theorem 6.1: The sum of the nth powers of the roots z_k of (5.1) is

$$s_{n,b,c}^{(v:u)} = \sum_{0 \le j} \frac{n}{i}\binom{i}{j}(-b)^i c^{i-j}, \text{ where } ui - vj = n \qquad (5.2a)$$

$$= \sum_{0 \le j} u\binom{i}{j} - v\binom{i-1}{j-1}(-b)^i c^{i-j}, \text{ where } ui - vj = n. \qquad (5.2b)$$

Proof: If we set $w_k = -bc$, then Equation (5.1) for z_k becomes

$$w_k^{-u} = (-bc)^{-1} = z_k^{-u}(1 + z_k^v/c), \tag{5.3}$$

which can be solved for z_k in terms of w_k by applying formula (3.5c) of [4], replacing the letters λ, μ, ν, c, q, k in [4] by $v' = u - v$, v, $-u$, w_k, n, j, respectively. Thus

$$z_k^n = \sum_{j=0}^{\infty} \frac{n}{jv + n}\binom{(jv + n)/u}{j} w_k^{jv+n} c^{-j}. \tag{5.4}$$

The sum of the u values of w_k^{jv+n} is $u(-bc)^i$ if $jv + n$ is an integral multiple ui of u, but is 0 otherwise. We obtain (5.2a) from (5.4) by setting $jv + n = ui$ and summing over j subject to this condition and $j \geq 0$. The equivalent form (5.2b) obtained by setting $n = ui - vj$ is clearly a sum of integers when $b^2 = c^2 = 1$. It also serves to assign the value $(-1)^j v$ to $\frac{n}{i}\binom{i}{j}$ when $i = 0$, $j = -n/v > 0$.

The conditions $j \geq 0$ and $(u - v)i/n + v(i - j)/n = 1$ in (5.2) imply $i/n \geq 0$, since $\binom{i}{j}$ vanishes for $0 < i < j$. Hence, $0 \leq j \leq i \leq n/(u - v)$ for $n > 0$, and $0 \leq j \leq j - i \leq -n/v$ for $n < 0$. Since successive j's differ in (6.2a) by u, there are at most $1 + n/u(u - v)$ terms for $n > 0$ and at most $1 + |n|/uv$ for $n < 0$. Both sums can be computed with at most $2 + |n|/v(u - v)$ terms.

The four sums in (4.11) and corresponding sums when $v = 1$ or $u - 1$ and $u > 3$ are expressible in terms of the following 4 simple nonnegative sums:

$$\sigma_0 = 1 + \sum_{0<k\leq n/u}'' \frac{n}{n - vk}\binom{n - vk}{k}, \quad \sigma_1 = \sum_{0<k\leq n/u}' \frac{n}{n - vk}\binom{n - vk}{k} \tag{5.5a}$$

$$\sigma_2 = \sum_{n/u\leq k\leq n/v}'' \frac{n}{k}\binom{k}{n - vk}, \qquad \sigma_3 = \sum_{n/u\leq k\leq n/v}' \frac{n}{k}\binom{k}{n - vk} \tag{5.5b}$$

where Σ'' and Σ' denote, respectively, the sums over even and odd k, and $u = v + 1$. Note that $\sigma_0 - 1$, σ_1, σ_2, and σ_3 are divisible by n when n is a prime.

Theorem 5.2: The 16 power sums $s_{m,b,c}^{(v:v+1)}$ and $s_{m,b,c}^{(v+1)}$ for $b^2 = c^2 = 1$, $m = \pm n$, are expressible for $n > 0$ in terms of the 4 binomial sums (5.5) as follows:

$$s_{n,b,c}^{(v:v+1)} = (-b)^n(\sigma_0 + (-b)^v c\sigma_1) \tag{5.6a}$$

$$s_{-n,b,c}^{(v:v+1)} = b^n(\sigma_2 - b^v c\sigma_3) \tag{5.6b}$$

$$s_{n,b,c}^{(v+1)} = c^n(\sigma_2 - c^v b\sigma_3) \tag{5.6c}$$

$$s_{-n,b,c}^{(v+1)} = (-c)^n(\sigma_0 - c^v b\sigma_1) \tag{5.6d}$$

Proof: For $n > 0$ and $u = v + 1$, we set $i - j = k$, $i = n - kv$ in (5.2a) and obtain

$$s_{n,b,c}^{(v:v+1)} = \sum_{0\leq k\leq n/u} \frac{n}{n - kv}\binom{n - kv}{k}(-b)^{n-kv}c^k. \tag{5.7}$$

Separating the sums for even and odd k, as in (5.5a), yields (5.6a). To obtain (5.6c), we replace v by 1 and u by $v+1$, in (5.2a), and apply (5.5b). Then set $i = k$, $i - j = n - vk$, and separate terms for even and odd k. Replacing z_k by $1/z_k$ interchanges n and $-n$, b and c, v and $u - v$, taking $z^u + bz^b + bc = 0$ into $z^u + cz^{u-v} + bc = 0$, (5.6a) into (5.6d), and (5.6c) into (5.6b).

For $n = 7$, $v = 2$, we have

$$(5.8)\quad \begin{aligned} \sigma_0(17) &= 1 + \frac{17}{13}\binom{13}{2} + \frac{17}{9}\binom{9}{4} = 341; \quad \sigma_1(17) = \frac{17}{15}\binom{15}{1} + \frac{17}{11}\binom{11}{3} = 323; \\ \sigma_2(17) &= \frac{17}{6}\binom{6}{5} + \frac{17}{8}\binom{8}{1} = 34; \qquad \sigma_3(17) = \frac{17}{7}\binom{7}{3} = 85. \end{aligned}$$

To obtain the extended principal factors $q_n^{(-3)}$, $q_n^{(3:4)}$, $q_n^{(4)}$, and $q_{n+}^{(4)}$ related to quartic equations (4.2) or the 6 factors other than $q_n^{(5)}$ and $q_{n+}^{(5)}$ of (4.6) related to quintic equations, we apply Theorem 4.2 and express the sums $\Sigma(z_j z_k)^n$ for positive or negative n by $(s_n^2 - s_{2n})/2$. For the equation $z^4 + z^v + c = 0$ with $v = 1$ or 3 and $c = \pm 1$, we have $(z^4 + c)^2 = z^{2v}$, so g_{2n} satisfies the recurrence

$$(5.9)\qquad s_{8+2n} + 2cs_{4+2n} + s_{2n} = s_{2n+2v}.$$

We omit the details concerning the computation of these 10 extended factors —some of which may coincide with the two "quadratic" and six "cubic" factors described above. For higher degree than 5, the factors listed in Section 7 were computed by pocket calculator using (2.5).

6. THE MULTIPLICITY OF $p = 2n + 1$ IN D_n

The multiplicity of factors 23 in d_{11}, 59 in d_{29}, 83 in d_{41}, etc., as seen in Table 1, is clarified by the following theorem.

Theorem 6.1: If $p = 2n + 1$ is prime, then p^e divides D_n for some exponent $e \geq [(n - 1)/2]$.

Proof: If $\bar{s}$ is a primitive root (mod p), $1 < \bar{s} < 2n$, then $\bar{s}^{2n} \equiv 1$ (mod p) and the even powers $\bar{s}^{2j} = \bar{r}^j$ are quadratic residues which are nth roots of unity (mod p). A principal factor $\bar{q}_n^{(v:u)}$ of d_n will vanish (mod p) if and only if the congruence $s^{2jv} + s^{2ju} \equiv 1$ (mod p) holds for some j relatively prime to n. If $(v,u) = 1$, parametric solutions of this congruence are

$$(6.1)\qquad s^{jv} \equiv 2/(h' + h),\ s^{ju} \equiv (h' - h)/(h' + h) \text{ where } hh' \equiv 1 \pmod p.$$

There are $4[(n - 1)/2]$ admissible values of h, excluding $h^2 = \pm 1$ or 0, of which the four distinct values $\pm h$, $\pm h'$ yield the same ordered pair (s^{2jv}, s^{2ju}). Hence, there are $[(n - 1)/2]$ distinct ordered pairs of squares with sum 1 (mod p) and at least $[(n - 1)/2]$ factors p in D_n.

Note that the substitution of $(h \pm 1)/(h \mp 1)$ for h interchanges the squares s^{2jv} and s^{2ju}. If these squares are equal (mod p), each is 1/2, so 2 is a quadratic residue of p, p divides $2^n - 1$, $p \equiv \pm 1$ (mod 8), and $[(n - 1)/2]$ is odd. For example, 7 divides $2^3 - 1$, 17 divides $2^8 - 1$, 23 divides $2^{11} - 1$, etc. In any case, $[(n - 1)/4]$ factors p divide d_n. For example,

$$(6.2)\qquad 23^2 | d_{11},\quad 47^5 | d_{23},\quad 59^9 | d_{29},\quad 83^{10} | d_{41}$$

and the inequality $e \geq [(n - 1)/2]$ is exact except for $p = 59$ where

$$[(n - 1)/4] = 7 < e/2 = 9.$$

In this case we have

$$(6.3)\qquad \begin{aligned} 1 &\equiv 25 + 25^2 \equiv 15 + 15^5 \equiv 19 + 19^8 \equiv 3 + 3^{-11} \equiv 16^{-1} + 16^{13} \\ &\equiv 9 + 9^{-2} \equiv 17^{-1} + 17^2 \pmod{59} \end{aligned}$$

but three factors $q_{29}^{(u)}$ are 59^2, for $u = 5$ and -13 (or 3/2) as well as -2.

7. SUMMARY

We list all the principal factors $q_p^{(u)}$ of d_p for prime p in Table 1, defining u' so that $uu' \equiv 1 \pmod{p}$, and taking all u from 2 to $(p - 1)/2$, except when $0 < u' < u$. We then replace $q_p^{(u)}$ by $q_p^{(-u)}$ on the list, and indicate by underlining that this has been done. However, in computing, we take $u = -2$ instead of $(p - 1)/2$, and $u/v = 3/2$ instead of $u = (3 - p)/2$, $(2 \pm p)/3$ or 5. Similarly, we can use the "quartic" factors with $u/v = -3$ or 4/3 instead of higher degree product formulas requiring more complicated calculations.

To find the prime factors of a large principal factor like

$$q_{47}^{(13)} = 10504313,$$

we assume a factorization $(1 + 94j)(1 + 94k)$ by Theorem 2.6, subtract 1, divide by 94, and get

$$(7.1)\qquad (1188)(94) + 76 = 94jk + j + k.$$

This implies $j + k = 76 + 282m$, and $jk = 1188 - 3m$ for some m. The only prime for $j < 7$ is 283, which does not divide $q_{47}^{(13)}$. Hence $j \geq 7$, and

$$j + k < 1188/7 + 7 < 177,$$

so $m = 0$. Thus, $j = 22$, $k = 54$, and $2069 \cdot 5077$ is the factorization.

For odd composite n, both $q_n^{(u)}$ and $q_n^{(-u)}$ may be listed as in (2.18) if u and n have a common factor, so we list them together in (7.3). Factors $q_{3p}^{(3:p)}$ in (2.21) must also be included in d_{3p} and factors like (3.8) in d_{5p}.

Factors of D_{4n+2} were given in (2.4), (2.7), and (4.14), whereas those of D_{4n} are obtained from (2.4), (2.9), and (4.15).

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A NOTE ON TILING RECTANGLES WITH DOMINOES

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INTRODUCTION

In how many ways can an $m \times n$ chessboard be covered by dominoes, each of which covers two adjacent squares? For general m and n this is the "dimer problem" which is known to be difficult (see [2] for details). However, when one of the dimensions, say m, is small, some results can be obtained, and will be given in this paper. The method used has some similarities with that used for the cell-growth problem in [3], although there are differences.

1. THE METHOD

We shall illustrate the general procedure by referring to the case $m = 3$. Any covering of a $3 \times n$ rectangle with dominoes can be regarded as having been built up, domino by domino, in a standard way, starting at the left-hand edge of the rectangle. Each domino is placed so that it covers an uncovered square furthest to the left, and, if there is more than one such square, it covers the one nearest the "top" of the board. Thus if the construction of a covering has proceeded as far as the stage shown in Figure 1, the next domino must be placed so as to cover the position marked with an asterisk. There may be two ways of placing the new domino (as in Figure 1), but there will be only one way if the space below the asterisk is already covered.

In the course of constructing $3 \times n$ rectangles, the figures produced will have irregular right-hand ends—their "profiles." We start by listing the possible profiles and the ways in which one profile can be converted to another by adding an extra domino. This information is given in Figure 2, in which the profiles have been labelled A to I.

Let A_r, B_r, etc., denote the numbers of ways of obtaining figures ending in profiles A, B, etc., by assembling r dominoes. Then, by reference to Figure 2, we obtain the equations:

$$(1.1)\qquad \begin{cases} A_{r+1} = D_r + E_r + F_r \\ B_{r+1} = A_r \\ C_{r+1} = A_r + H_r \\ D_{r+1} = B_r \\ E_{r+1} = B_r + I_r \\ F_{r+1} = C_r \\ G_{r+1} = E_r \\ H_{r+1} = F_r \\ I_{r+1} = G_r \end{cases}$$

Since $A_0 = 1$ and all other values are 0 when $r = 0$, we can use (1.1) to calculate these numbers, and in particular A_r, for $r = 1, 2$, etc. Equations (1.1) can also be transformed in an obvious way to an equation which expresses the vector $(A_{r+1}, B_{r+1}, \ldots, I_{r+1})$ as a 9×9 matrix times the vector $(A_r, B_r, \ldots, I_r)$, but this is not very useful.

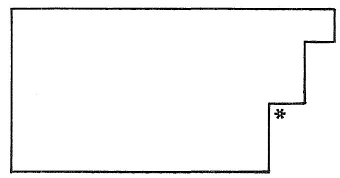

FIGURE 1

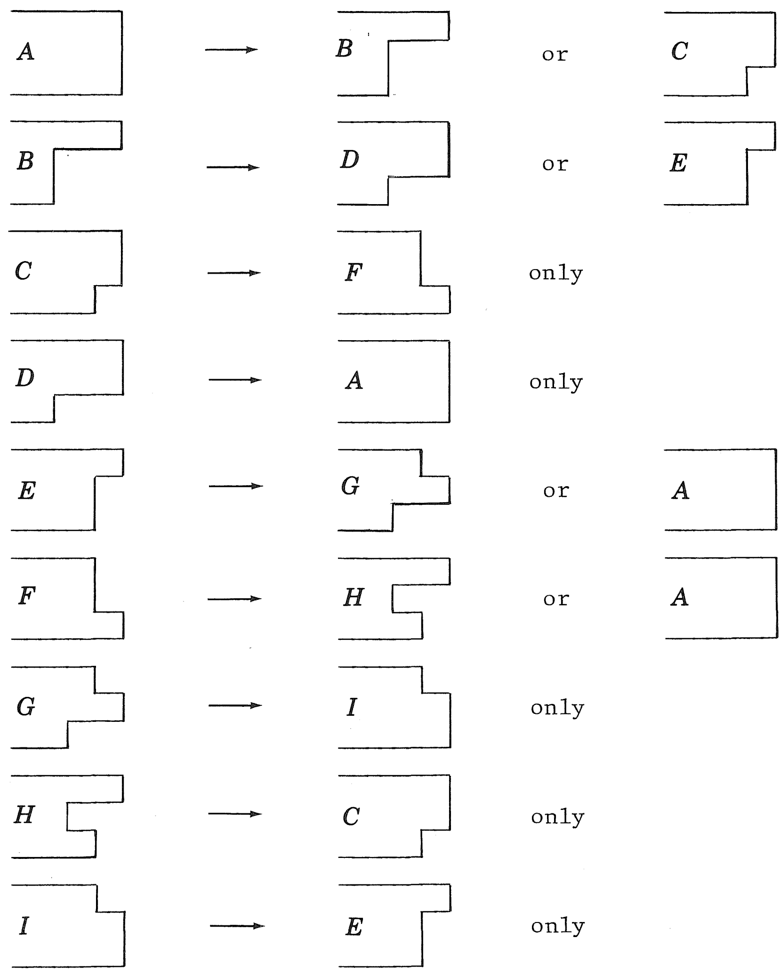

FIGURE 2

A better approach is to define generating functions

$$A(t) = \sum_{r=0}^{\infty} A_r t^r, \text{ etc.}$$

Remembering that $A(t)$ will be the only one of these functions having a constant term, we obtain the relations

$$(1.2)\qquad \begin{cases} A(t) = 1 + tD(t) + tE(t) + tF(t) \\ B(t) = tA(t) \\ C(t) = tA(t) + tH(t) \\ D(t) = tB(t) \\ E(t) = tB(t) + tI(t) \\ F(t) = tC(t) \\ G(t) = tE(t) \\ H(t) = tF(t) \\ I(t) = tG(t) \end{cases}$$

Solving these equations for $A(t)$ we obtain

$$(1 - 4t^3 + t^6)A(t) = 1 - t^3,$$

which can be more conveniently expressed as

$$(1.3)\qquad (1 - 4x + x^2)A(x) = 1 - x,$$

writing $A(x) = \sum_{r=0}^{\infty} a_r x^r$ where $a_r = A_{3r}$. (Clearly $A_k = 0$ if k is not a multiple of 3.)

From (1.3), we find that

$$a_r = 4a_{r-1} - a_{r-2}.$$

2. RESULTS

When $m = 2$, there are two profiles (A and B of Figure 2, with the bottom row omitted) and the corresponding equations are

$$A(t) = 1 + tA(t) + tB(t)$$

$$B(t) = tA(t)$$

whence $A(t) = (1 - t - t^2)^{-1}$. The numbers of tilings are therefore the Fibonacci numbers.

When $m = 4$, the profiles are as shown in Figure 3 and by following the method of Section 1, we obtain the equations

$$\begin{aligned} A(t) &= 1 + tC(t) + tG(t) + tH(t) + tI(t) \\ B(t) &= tA(t);\ C(t) = tA(t) + tD(t) + tK(t) \\ D(t) &= tB(t);\ E(t) = tB(t) + tL(t) \\ F(t) &= tC(t);\ G(t) = tD(t);\ H(t) = tE(t) \\ I(t) &= tF(t);\ J(t) = tH(t);\ K(t) = tI(t);\ L(t) = tJ(t) \end{aligned}$$

from which, on solving for $A(t)$, we obtain

$$A(x) = (1 - x^2)/(1 - x - 5x^2 - x^3 + x^4)$$

and the corresponding recursive formula

$$a_{r+1} = a_r + 5a_{r-1} + a_{r-2} - a_{r-3}.$$

For $m > 4$, the method becomes tedious by hand, but I found it quite easy to write a program (in APL) which would first generate the relations between the profiles (as in Figure 2) and then calculate the required numbers from the equations analogous to (1.1). In this way, results were obtained for m = 5, 6, 7, 8, and 9. They are given in Table 1 below. Note that Kasteleyn [1] has given results for $m = n$ = 2, 4, 6, and 8, with which the entries in the table agree.

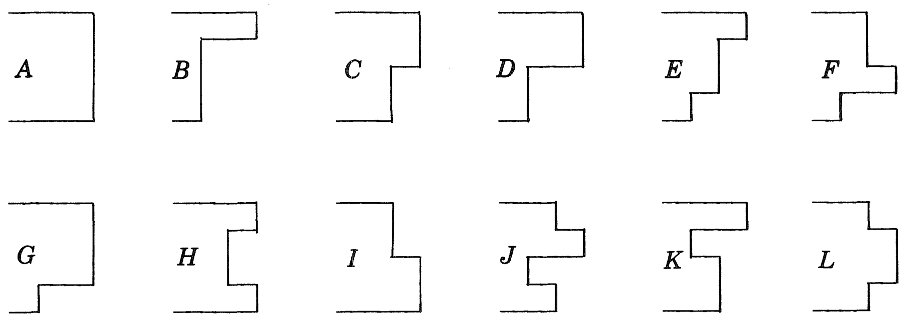

FIGURE 3

TABLE 1

$n \backslash m$	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	2	3	5	8	13	21	34	55
3	3	0	11	0	41	0	153	0
4	5	11	36	95	281	781	2245	6336
5	8	0	95	0	1183	0	14824	0
6	13	41	281	1183	6728	31529	167089	817991
7	21	0	781	0	31529	0	1292697	0
8	34	153	2245	14824	167089	1292697	12988816	108435745
9	55	0	6336	0	817991	0	108435745	0
10	89	571	10861	185921	4213133	53175517	1031151241	14479521761
11	144	0	51205	0	21001799	0	8940739824	0
12	233	2131	145601	2332097	106912793	2188978117	82741005829	1937528668711
13	377	0	413351	0	536948224	0	731164253833	0
14	610	7953	1174500	29253160	2720246633	90124167441	6675498237130	259423766712000
15	987	0	3335651	0	13704300553	0	59554200469113	
16	1597	29681	9475901	366944287	69289288909	3710708201969	540061286536921	
17	2584	0	26915305	0	349519610713	0	4841110033666048	

$n \backslash m$	2	3	4	5	6	7
18	4181	110771	76455961	4602858719	1765711581057	152783289861989
19	6765	0	217172736	0	8911652846951	0
20	10946	413403	616891945	57737128904	45005025662792	6290652543875133
21	17711	0	1752296281	0	227191499132401	0
22	28657	1542841	4977472781	724240365697	1147185247901449	
23	46368	0	14138673395	0	5791672851807479	
24	75025	5757961	40161441636	9084693297025		
25	121393	0	114079985111	0		
26	196418	21489003	324048393905	113956161827912		
27	317811	0	920471087701	0		
28	514229	80198051	2614631600701	1429438110270431		
29	832040	0	7426955448000	0		
30	1346269	299303201	21096536145301			

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SOME EXTENSIONS OF WYTHOFF PAIR SEQUENCES

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In [1] it is shown that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ then

$$[[n\alpha]\alpha] = [n\alpha] + n - 1 \tag{1}$$

for all positive integers n. Out first purpose in this paper is to give an alternate proof of (1) and also show that (1) holds even if n is negative. Next, we prove that the converse of (1) holds even if (1) is true for all negative integers. In conclusion, we derive an additional identity using the greatest integer function together with the golden ratio, and we discuss two sets of sequences related to these results.

First we show

Theorem 1: If $\delta = \dfrac{1+\sqrt{5}}{2}$ then $[[n\delta]\ ] = [n\delta] + n - 1$ for all integers $n \neq 0$.

Before proving Theorem 1, let us recall a theorem of Skolem and Bang which can be found in [2].

Theorem 2: Let ε and t be positive real numbers. Denote the set of all positive integers by Z and the null set by $\emptyset$. Let $N_\gamma = \{[n\gamma]\}_{n=1}^{\infty}$. Then $N_\varepsilon \cap N_t = \emptyset$ and $N_\varepsilon \cup N_t = Z$ if and only if ε and t are irrational and $\varepsilon^{-1} + t^{-1} = 1$.

Proof of Theorem 1: Let us assume that n is positive. Since $n\delta$ is not an integer for any $n \neq 0$, we have $[n\delta] < n\delta < [n\delta] + 1$ provided $n \neq 0$, so that

$$[n\delta]\delta < n\delta^2 < ([n\delta] + 1)\delta. \tag{2}$$

In Theorem 2, let $\varepsilon = \delta$ and $t = \delta^2$, then $\varepsilon^{-1} + t^{-1} = 1$, so that $N_\delta \cap N_{\delta^2} = \emptyset$ and $N_\delta \cup N_{\delta^2} = Z$. Because $[[n\delta]\delta]$ and $[([n\delta]+1)\delta]$ are elements of N_δ, while $[n\delta^2]$ belongs to N_{δ^2}, we know from (2) that

$$[[n\delta]\delta] < [n\delta^2] < [([n\delta] + 1)\delta]. \tag{3}$$

Using the well-known fact that $[a + b] = [a] + [b] + \gamma$ where $\gamma = 0$ or 1, we see that $[[n\delta]\delta + \delta] = [[n\delta]\delta] + [\delta] + \gamma = [[n\delta]\delta] + 1 + \gamma$ where $\gamma = 0$ or 1. Since $[n\delta^2] - [[n\delta]\delta]$ is an integer, we conclude from (3) that

$$[n\delta^2] - [[n\delta]\delta] = 1 \tag{4}$$

and $\gamma = 1$. Recalling that $\delta^2 = \delta + 1$, we obtain

$$[[n\delta]\delta] = [n\delta] + n - 1 \tag{5}$$

and the theorem is proved if $n > 0$.

Let us now assume that $n < 0$ and recall that since $n\delta$ is not an integer then $[n\delta] = -[-n\delta] - 1$. Using this fact together with the results above for $n > 0$, we have

$$\begin{aligned}[[n\delta]\delta] &= -[-[n\delta]\delta] - 1\\ &= -[([-n\delta] + 1)\delta] - 1\end{aligned}$$

$$\begin{aligned}&= -([[-n\delta]\delta] + [\delta]) - 2\\ &= -([-n\delta] - n) - 2\\ &= -(-[n\delta] - n) - 1\\ &= [n\delta] + n - 1\end{aligned}$$

and the theorem is proved.

We now show

Theorem 3: If $[[n\delta]\delta] = [n\delta] + n - 1$ for all integers $n \neq 0$, then $\delta = \frac{1+\sqrt{5}}{2}$.

Proof: Since $[[n\delta]\delta] = [n\delta] + n - 1$, we have $[n\delta] + n - 1 \leq [n\delta]\delta < [n\delta] + n$. Therefore, $1 < \frac{[n\delta](\delta - 1)}{n} \leq 1 - \frac{1}{n}$ when $n < 0$, while $1 - \frac{1}{n} \leq \frac{[n\delta](\delta - 1)}{n} < 1$ if $n > 0$.

Hence,

$$\lim_{n \to 0} \frac{[n\delta]}{n} = \frac{1}{\delta - 1}, \tag{6}$$

provided $\delta \neq 1$, which is obviously true.

By definition of the greatest integer, we know that $[n\delta] \leq n\delta < [n\delta] + 1$ for any integer n and any δ so that

$$\delta - \frac{1}{n} < \frac{[n\delta]}{n} \leq \delta \text{ if } n > 0, \text{ while } \delta \leq \frac{[n\delta]}{n} < \delta - \frac{1}{n} \text{ when } n < 0.$$

In both cases,

$$\lim_{n \to 0} \frac{[n\delta]}{n} = \delta. \tag{7}$$

Equating (6) and (7), we see that $\delta^2 - \delta - 1 = 0$. If $\delta = \frac{1-\sqrt{5}}{2}$ and $n = 1$ then $[[n\delta]\delta] = [-\delta] = 0$, while $[n\delta] + n - 1 = -1$. Hence (1) is false. Therefore, $\delta = \frac{1+\sqrt{5}}{2}$ and we are done.

Another identity which arose while investigating (1) is

Theorem 4: If $\delta = \frac{1+\sqrt{5}}{2}$, then $[[n\delta]\delta + n\delta] = 2[n\delta] + n$ for all integers n and conversely.

The proof of Theorem 4 is omitted, since it is essentially the same as the proof of Theorems 1 and 3, with the only difficulty arising when trying to prove the result for $n < 0$. This difficulty is overcome by using the fact that

$$[[n\delta]\delta + n\delta + \delta] = [[n\delta]\delta + n\delta] + 1 \text{ when } n > 0 \text{ and } \delta = \frac{1+\sqrt{5}}{2}.$$

The argument for the validity of the last statement can be found in [3].

Let us now illustrate some interesting applications of Theorems 1, 3, and 4. To do so we introduce two special sets of sequences. For any integer $n \neq 0$, define $\{S_m(n)\}_{m=1}^{\infty}$ by

$$S_m(n) = S_{m-1}(n) + S_{m-2}(n),\ m \geq 3,\ S_1(n) = n,\ S_2(n) = [n\alpha],\ \alpha = \frac{1+\sqrt{5}}{2} \tag{8}$$

and for any integer n, define $\{S_m^*(n)\}_{m=1}^{\infty}$ by

(9) $$S_m^*(n) = S_{m-1}^*(n) + S_{m-2}^*(n) - 1,\ m \geq 3,$$

$$S_1^*(n) = n,\ S_2^*(n) = [n\alpha],\ \alpha = \frac{1+\sqrt{5}}{2}.$$

Since (8) is a generalized Fibonacci sequence, it is easy to show that

(10) $$S_m(n) = F_{m-1}[n\alpha] + nF_{m-2},\ n \neq 0 \text{ and } m \geq 1.$$

The terms of $\{S_m(n)\}_{m=1}^{\infty}$ for $1 \leq m \leq 7$ and $-10 \leq n \leq 10$, $n \neq 0$, are presented in Table 1.

TABLE 1

$S_m(n)$ \ n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$S_1(n)$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$S_2(n)$	-17	-15	-13	-12	-10	-9	-7	-5	-4	-2
$S_3(n)$	-27	-24	-21	-19	-16	-14	-11	-8	-6	-3
$S_4(n)$	-44	-39	-34	-31	-26	-23	-18	-13	-10	-5
$S_5(n)$	-71	-63	-55	-50	-42	-37	-29	-21	-16	-8
$S_6(n)$	-115	-102	-89	-81	-68	-60	-47	-34	-26	-13
$S_7(n)$	-186	-165	-144	-131	-110	-97	-76	-55	-42	-21
$S_m(n)$ \ n	1	2	3	4	5	6	7	8	9	10
$S_1(n)$	1	2	3	4	5	6	7	8	9	10
$S_2(n)$	1	3	4	6	8	9	11	12	14	16
$S_3(n)$	2	5	7	10	13	15	18	20	23	26
$S_4(n)$	3	8	11	16	21	24	29	32	37	42
$S_5(n)$	5	13	18	26	34	39	47	52	60	68
$S_6(n)$	8	21	29	42	55	63	76	84	97	110
$S_7(n)$	13	31	47	68	89	102	123	136	157	178

One of the first observations made was that some of the rows appear to be subsets of previous rows. A more careful examination implies that for a specific n the positive and negative values for a given row are related by the Fibonacci numbers. The latter result is stated as Theorem 5, while the former is Theorem 6.

Theorem 5: For all integers $n \neq 0$, $S_m(n) + S_m(-n) = -F_{m-1}$, $m \geq 1$.

The proof of Theorem 5 is a direct result of (10) and is thus omitted.

Theorem 6: For any integer $m \geq 3$ and any integer $n \neq 0$,

$$S_m(n) = S_{m-2}(S_3(n)).$$

Proof: By definition, $S_1(S_3(n)) = S_3(n)$, so the theorem is true if $m = 3$. By Theorem 4 we have

$$S_4(n) = S_3(n) + S_2(n) = 2[n\alpha] + n = [[n\alpha]\alpha + n\alpha]$$
$$= S_2([n\alpha] + n) = S_2(S_3(n)),$$

so that the result is true for $m = 4$.

Assuming the theorem true for all positive integers $m \leq k$ where $k \geq 4$, we have

$$S_{k+1}(n) = S_k(n) + S_{k-1}(n) = S_{k-2}(S_3(n)) + S_{k-3}(S_3(n)) = S_{k-1}(S_3(n))$$

and the theorem is proved.

An immediate consequence of Theorem 6 is

$$\{S_2(n)\} \supseteq \{S_4(n)\} \supseteq \{S_6(n)\} \supseteq \{S_8(n)\} \supseteq \cdots \tag{11}$$

and

$$\{S_3(n)\} \supseteq \{S_5(n)\} \supseteq \{S_7(n)\} \supseteq \{S_9(n)\} \supseteq \cdots . \tag{12}$$

By the theorem of Skolem and Bang, we have

$$\{S_2(n)\}_{n=1}^{\infty} \cap \{S_3(n)\}_{n=1}^{\infty} = \emptyset.$$

Using this result and Theorem 5, it is easy to see that

$$\{S_2(-n)\}_{n=1}^{\infty} \cap \{S_3(-n)\}_{n=1}^{\infty} = \emptyset.$$

Hence $\{S_2(n)\} \cap \{S_3(n)\} = \emptyset$ and $\{S_m(n)\} \cap \{S_{m-1}(n)\} = \emptyset$ for all $m \geq 3$. That is, no row has any elements in common with the row immediately preceding it.

We now turn our attention to an investigation of the columns of Table 1. To do this, we use C_i to represent the ith column. You will, after extending the number of columns, see that

$$C_1 \supseteq C_2 \supseteq C_5 \supseteq C_{13} \supseteq C_{34} \cdots$$
$$C_3 \supseteq C_7 \supseteq C_{18} \supseteq C_{47} \supseteq C_{123} \cdots$$

and

$$C_4 \supseteq C_{10} \supseteq C_{26} \supseteq C_{68} \supseteq C_{178} \cdots .$$

Analyzing the subscripts, we are led to conjecture that, for all integers $n \neq 0$,

$$C_{S_1(n)} \supseteq C_{S_3(n)} \supseteq C_{S_5(n)} \supseteq C_{S_7(n)} \supseteq C_{S_9(n)} \supseteq \cdots . \tag{13}$$

In proving this, we arrived at what we believe is an interesting commutative property of this set of sequences.

Theorem 7: If $m \geq 1$, then $S_3(S_{2m-1}(n)) = S_{2m-1}(S_3(n))$

Proof: The theorem is obviously true for $m = 1$ and $m = 2$. Furthermore, by Theorem 6 and the induction hypothesis, we have

$$S_3(S_{2m+1}(n)) = S_3\big(S_{2m-1}(S_3(n))\big) = S_{2m-1}\big(S_3(S_3(n))\big) = S_{2m+1}(S_3(n)),$$

and the theorem is proved.

Since $S_{2m-1}(S_3(n)) = S_1(S_{2m+1}(n))$, we have

$$S_3(S_{2m-1}(n)) = S_1(S_{2m+1}(n)). \tag{14}$$

Furthermore, by Theorems 6 and 7,

$$S_4(S_{2m-1}(n)) = S_2\big(S_3(S_{2m-1}(n))\big) = S_2(S_{2m+1}(n)). \tag{15}$$

Together, (14) and (15) tell us that

(16) $$C_{S_{2m+1}(n)} \subseteq C_{S_{2m-1}(n)}$$

for all $m \geq 1$. This result proves the validity of (13).

We now turn our attention to the sequences $\{S_m^*(n)\}_{m=1}^{\infty}$. The elements for the first seven sequences are given for $-10 \leq n \leq 10$ in Table 2. An examination of this table leads to results that are very similar to those associated with $\{S_m(n)\}_{m=1}^{\infty}$. A number of these proofs are omitted, since they are similar to the proofs of their counterpart theorems.

TABLE 2

$S_m^*(n)$ \ n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$S_1^*(n)$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$S_2^*(n)$	-17	-15	-13	-12	-10	-9	-7	-5	-4	-2
$S_3^*(n)$	-28	-25	-22	-20	-17	-15	-12	-9	-7	-4
$S_4^*(n)$	-46	-41	-36	-33	-28	-25	-20	-15	-12	-7
$S_5^*(n)$	-75	-67	-59	-54	-46	-41	-33	-25	-20	-12
$S_6^*(n)$	-122	-109	-96	-88	-75	-67	-54	-41	-33	-20
$S_7^*(n)$	-198	-177	-156	-143	-122	-109	-88	-67	-54	-33

$S_m^*(n)$ \ n	0	1	2	3	4	5	6	7	8	9	10
$S_1^*(n)$	0	1	2	3	4	5	6	7	8	9	10
$S_2^*(n)$	0	1	3	4	6	8	9	11	12	14	16
$S_3^*(n)$	-1	1	4	6	9	12	14	17	19	22	25
$S_4^*(n)$	-2	1	6	9	14	19	22	27	30	35	40
$S_5^*(n)$	-4	1	9	14	22	30	35	43	48	56	64
$S_6^*(n)$	-7	1	14	22	35	48	56	69	77	90	103
$S_7^*(n)$	-12	1	22	35	56	77	90	111	124	145	166

Theorem 8: If m is an integer and $m \geq 1$, then

$$S_m^*(n) = [n\alpha]F_{m-1} + nF_{m-2} - F_m + 1.$$

Theorem 9: If m is an integer and $m \geq 1$, $n \neq 0$, then

$$S_m^*(n) + S_m^*(-n) = -F_{m+2} + 2.$$

Theorem 10: If $m \geq 2$ is an integer and $n \neq 0$, then

$$S_m^*(n) = S_{m-1}^*(S_2^*(n)).$$

The proof of Theorem 10 is similar to the proof of Theorem 6, except that one needs Theorem 1 to show that $S_3^*(n) = S_2^*(S_2^*(n))$. The rest of the proof is omitted.

An immediate consequence of Theorem 10 is that if we omit the column when $n = 0$, then every row is a subset of every row preceding it. That is,

$$\{S_1^*(n)\} \supseteq \{S_2^*(n)\} \supseteq \{S_3^*(n)\} \supseteq \{S_4^*(n)\} \supseteq \{S_5^*(n)\} \;\ldots, \tag{17}$$

provided $n \neq 0$.

Using an inductive argument similar to that of Theorem 7, one can show

Theorem 11: If $m \geq 1$ is an integer and $n \neq 0$,

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)).$$

Combining Theorems 10 and 11, we have

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)) = S_{m+1}^*(n) = S_1^*(S_{m+1}^*(n)),\; n \neq 0, \tag{18}$$

and

$$S_3^*(S_m^*(n)) = S_2^*\big(S_2^*(S_m^*(n))\big) = S_2^*\big(S_m^*(S_2^*(n))\big) = S_2^*(S_{m+1}^*(n)),\; n \neq 0. \tag{19}$$

Together, (18) and (19) yield

$$C^*_{S^*_{m+1}(n)} \subseteq C^*_{S^*_m(n)} \tag{20}$$

for all integers $m \geq 1$, $n \neq 0$, where C_i^* is the ith column of Table 2.

The next result, whose proof we omit, since it is by mathematical induction, establishes a relationship between Table 1 and Table 2.

Theorem 12: If m is an integer, $m \geq 1$, $n \neq 0$, then $S_m^*(n) = S_m(n) - F_m + 1$.

Using the fact that $S_2^*(n) = S_2(n)$ in Theorem 10 and applying Theorem 12, we have

$$S_{m+1}(n) + 1 - F_{m+1} = S_{m+1}^*(n) = S_m^*(S_2^*(n)) = S_m(S_2(n)) - F_m + 1$$

or

$$S_{m+1}(n) = S_m(S_2(n)) + F_{m-1},\; n \neq 0. \tag{21}$$

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THE APOLLONIUS PROBLEM

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On p. 326 of *The Fibonacci Quarterly* 12, No. 4 (1974), Charles W. Trigg gave a formula for the radius of a circle which touches three given circles which, in turn, touch each other externally.

The following is a more general formula:

Given triangle ABC with AB = α, BC = β, CA = γ, and circles with centres A, B, and C having radii a, b, and c, respectively.

Let $\ell = a + b + \alpha$; $m = b + c + \beta$; $n = \alpha + b - a$; $p = \beta + b - c$;
$q = a + b - \alpha$; $t = b + c - \beta$; $u = \alpha + a - b$; $v = \beta + c - b$;
$s = (\alpha + \beta + \gamma)/2$.

Then, if x is the radius of a circle touching the three given ones:

$$4(x + b)\sqrt{s(s - \gamma)} = \sqrt{np(2x + \ell)(2x + m)} \pm \sqrt{uv(2x + q)(2x + t)}$$

the positive sign being taken if the centre of the required circle falls outside angle ABC, and the negative sign if it falls inside angle ABC.

The formula applies to *external* contact. If a given circle of radius a, say, is to make *internal* contact with the required one, then $-a$ must replace $+a$ in the formula. If a given circle of radius a, say, becomes a point, put $a = 0$.

When the three given circles touch each other externally,

$$\alpha = a + b,\ \beta = b + c, \text{ and } \gamma = a + c,$$

and the above formula yields the solution mentioned by Trigg, viz.

$$x = abc/[2\sqrt{abc(a + b + c)} \pm (ab + bc + ca)].$$

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LETTER TO THE EDITOR

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Dear Professor Hoggatt,

In a recent article with Claudia Smith [*Fibonacci Quarterly* 14 (1976): 343], you referred to the question whether a prime p and its square p^2 can have the same rank of apparition in the Fibonacci sequence, and mentioned that Wall (1960) had tested primes up to 10,000 and not found any with this property.

I have recently extended this search and found that no prime up to one million (1,000,000) has this property.

My computations in fact test the Lucas sequence for the property

(1) $$L_p \equiv 1 \pmod{p^2} \qquad p = \text{prime}.$$

For $p > 5$, this is easily shown to be a necessary and sufficient condition for p and p^2 to have the same rank of apparition in the Fibonacci sequence, because of the identity

(2) $$(L_p - 1)(L_p + 1) = 5F_{p-1}F_{p+1}.$$

So far, I have shown that the congruence (1) does not hold for any prime less than one million; I hope to extend the search further at a later date.

You may wish to publish these results in *The Fibonacci Quarterly*.

Yours sincerely,

[*Dr L. A. G. Dresel*]

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SUMMATION OF THE SERIES $y^n + (y+1)^n + \cdots + x^n$

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In 1970, Levy [1] published a number of results concerning the sum of the series $1^n + 2^n + \cdots + x^n$, which is known to be an $n+1$-degree polynomial $P_n(x)$ whenever x is a positive integer. However, there is a natural generalization that will also hold for negative integers and zero as well. This is given in the following theorem.

Theorem 1: For each positive integer n there is exactly one polynomial such that

$$\sum_{k=y+1}^{x} k^n = P_n(x) - P_n(y)$$

for all integral values of x and y, where $y < x$.

This theorem also holds for $n = 0$ if 0^0 is interpreted as 1. The proof follows easily from two lemmas.

Lemma 1: For each integer value of $x \geq 0$,

$$\sum_{k=1}^{x} k^n = P_n(x) - P_n(0).$$

This is true because $P_n(0) = 0$ for all n.

Lemma 2: For each integer value of $y < 0$,

$$\sum_{k=y+1}^{0} k^n = P_n(0) - P_n(y).$$

Proof:

$$\sum_{k=y+1}^{0} k^n = \sum_{j=0}^{-y-1} (-j)^n = (-1)^n P_n(-y-1) = -P_n(y),$$

where the last equality follows from Theorem 3 in the paper by Levy. When x is a positive integer, $P_n(x)$ is the sum of the series from 1 to n, and when x is a negative integer, then $-P_n(x)$ is the sum of the series from $x+1$ to 0.

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ROOTS OF $(H - L)/15$ RECURRENCE EQUATIONS IN GENERALIZED PASCAL TRIANGLES

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1. INTRODUCTION

In this paper, we shall examine the roots of recurrence equations for $(H - L)/15$ sequences in Pascal's binomial, trinomial, quadrinomial, pentanomial, hexanomial, and heptanomial triangles.

Recall that the regular Lucas and Fibonacci sequences have the recurrence equation

$$x^2 - x - 1 = 0,$$

with roots

$$\alpha = (1 + \sqrt{5})/2 \quad \text{and} \quad \beta = (1 - \sqrt{5})/2.$$

As the roots of the $(H - L)/15$ sequences are examined, α and β appear frequently.

Generalized Pascal triangles arise from the multinomial coefficients obtained by the expansion of

$$(1 + x + x^2 + \cdots + x^{j-1})^n, \; j \geq 2, \; n \geq 0,$$

where n denotes the row in each triangle. For $j = 6$, the hexanomial coefficients give rise to the following triangle:

1															
1	1	1	1	1	1										
1	2	3	4	5	6	5	4	3	2	1					
1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1
etc.															

In order to explain the $(H - L)/15$ sequences, we shall first define sums of partition sets in the rows of Pascal triangles. The partition sums are defined

$$S(n,j,k,r) = \sum_{i=0}^{M} \left[\!\!\left[\begin{matrix} n \\ r + ik \end{matrix} \right]\!\!\right]_j; \; 0 \leq r \leq k - 1, \quad M = \left[\frac{(j - 1)n - r}{k} \right],$$

the brackets denoting the greatest integer function. To clarify, we give a numerical example. Consider $S(3,6,15,0)$. This denotes the partition sums in the third row of the hexanomial triangle in which every fifteenth element is added, beginning with the zeroth column. The $S(3,6,15,0) = 1 + 3 = 4$. (Conventionally, the column of 1's at the far left is the zeroth column and the top row is the zeroth row.)

In the nth row of the j-nomial triangle the sum of the elements is j^n. This is expressed by

$$S(n,j,k,0) + S(n,j,k,1) + \cdots + S(n,j,k,k - 1) = j^n.$$

$$S(n,j,k,0) = (j^n + A_n)/k$$

$$S(n,j,k,1) = (j^n + B_n)/k \; \ldots$$

$$S(n,j,k,k - 1) = (j^n + Z_n)/k.$$

Since $S(0,j,k,0) = 1$,

$$S(0,j,k,1) = 0 \; \ldots \; S(0,j,k,k - 1) = 0,$$

we can solve for A_0, B_0, ..., Z_0 to get $A_0 = k - 1$, $B_0 = -1$, ..., $Z_0 = -1$.

Now a departure table can be formed with A_0, B_0, ..., Z_0 as the zeroth row. The term *departure* refers to the quantities, A_n, B_n, ..., Z_n that depart from the average value j^n/k. Pascal's rule of addition is the simple method for finding the successive rows in each departure table. The departure table for 15 partitions in the hexanomial triangle appears below. Each row has 15 elements which have been spread out by the computer into 3 *rows*.

TABLE 1

SUMS OF FIFTEEN PARTITIONS IN THE HEXANOMIAL TRIANGLE

14.	-1.	-1.	-1.	-1.
-1.	-1.	-1.	-1.	-1.
-1.	-1.	-1.	-1.	-1.
9.	9.	9.	9.	9.
9.	-6.	-6.	-6.	-6.
-6.	-6.	-6.	-6.	-6.
-21.	-6.	9.	24.	39.
54.	39.	24.	9.	-6.
-21.	-36.	-36.	-36.	-36.
-186.	-171.	-126.	-66.	9.
99.	159.	189.	189.	159.
99.	9.	-66.	-126.	-171.
-441.	-711.	-846.	-846.	-711.
-441.	-96.	264.	579.	804.
894.	804.	579.	264.	-96.
2004.	399.	-1251.	-2676.	-3651.
-3996.	-3651.	-2676.	-1251.	399.
2004.	3249.	3924.	3924.	3249.
18354.	16749.	12249.	5649.	-1926.
-9171.	-14826.	-17901.	-17901.	-14826.
-9171.	-1926.	5649.	12249.	16749.

The $(H - L)/15$ sequences are obtained from the difference of the maximum and minimum value sequences in a departure table, divided by 15, where 15 is the number of partitions. A table of $(H - L)/15$ sequences follows.

TABLE 2

$(H - L)/15$ SEQUENCES IN j-NOMIAL TRIANGLES

j =	2	3	4	5	6	7
	1	1	1	1	1	1
	1	1	1	1	1	1
	2	3	4	5	6	7
	3	7	12	19	25	28
	6	19	44	80	116	140
	10	51	153	331	528	658
	20	141	553	1379	2417	3164
	35	392	1960	5740	11053	15106
	70	1098	7042	23906	50562	72302

(continued)

j =	2	3	4	5	6	7
	126	3085	25080	99565	231283	345775
	252	8688	89861	414704	1057967	1654092
	462	24498	320661	1727341	4839483	7911790
	924	69136	1147444	7194890	22137392	37846314
	1716	195209	4098172	29969004	101263708	
	3431	551370		124831190	463213542	

2. BINOMIAL TRIANGLE

The pivotal element method was used to derive the $(H - L)/15$ recurrence equation in the binomial triangle,

$$x^7 - x^6 - 6x^5 + 5x^4 + 10x^3 - 6x^2 - 4x + 1 = 0.$$

We factor out $x - 1$ to get

$$x^6 - 6x^4 - x^3 + 9x^2 + 3x - 1 = 0,$$

which can be written as

$$x^2(x^2 - 3)^2 - x(x^2 - 3) - 1 = 0.$$

Let $y = x(x^2 - 3)$, then the equation above becomes

$$y^2 - y - 1 = 0,$$

with the roots α and β.

Solve $x(x^2 - 3) = \alpha$ and $x(x^2 - 3) = \beta$. β is the root of the first because

$$\beta(\beta^2 - 3) = \left(\frac{1 - \sqrt{5}}{2}\right)\left(\frac{6 - 2\sqrt{5}}{2} - 3\right) = \left(\frac{1 - \sqrt{5}}{2}\right)\left(\frac{-6 - 2\sqrt{5}}{4}\right)$$

$$= \frac{-6 + 4\sqrt{5} + 10}{8} = \frac{4 + 4\sqrt{5}}{8} = \frac{1 + \sqrt{5}}{2} = \alpha.$$

α is a root of the second because $\alpha(\alpha^2 - 3) = \beta$. We factor out $x - \beta$ from the first to get

$$x^2 + \beta x + (-3 + \beta^2) = 0,$$

and factor out $x - \alpha$ from the second to get

$$x^2 + \alpha x + (-3 + \alpha^2) = 0.$$

These quadratic equations have roots

$$\frac{-\beta \pm \sqrt{-3\beta^2 + 12}}{2} \quad \text{and} \quad \frac{-\alpha \pm \sqrt{-3\alpha^2 + 12}}{2}.$$

Thus the roots of the recurrence equation are

$$1, \alpha, \beta, \frac{-\beta \pm \sqrt{-3\beta^2 + 12}}{2}, \frac{-\alpha \pm \sqrt{-3\alpha^2 + 12}}{2}.$$

3. TRINOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^6 - 6x^5 + 9x^4 + 5x^3 - 15x^2 + 5 = 0.$$

We rewrite as

$$(x^2(x - 3))^2 + 5x^2(x - 3) + 5 = 0.$$

Let $y = x^2(x - 3)$, then the equation above becomes

$$y^2 + 5y + 5 = 0,$$

with the roots $-\sqrt{5}\alpha$ and $\sqrt{5}\beta$.

Solve $x^2(x - 3) = -\sqrt{5}\alpha$ and $x^2(x - 3) = \sqrt{5}\beta$. α is a root of the first, since

$$\alpha^2(\alpha - 3) = \left(\frac{6 + 2\sqrt{5}}{4}\right)\left(\frac{1 + \sqrt{5}}{2} - 3\right) = \left(\frac{6 + 2\sqrt{5}}{4}\right)\left(\frac{-5 + \sqrt{5}}{2}\right)$$

$$= \frac{-20 - 4\sqrt{5}}{8} = -5\left(\frac{\sqrt{5} + 1}{2}\right) = -\sqrt{5}\alpha.$$

β is a root of the second, since $\beta^2(\beta - 3) = \sqrt{5}\beta$. We factor out $x - \alpha$ from the first to get

$$x^2 + (-3 + \alpha)x + (-3 + \alpha^2) = 0,$$

and factor out $x - \beta$ from the second to get

$$x^2 + (-3 + \beta)x + (-3 + \beta^2) = 0.$$

Since $-3 + \alpha = \sqrt{5}\beta$, and $-3 + \beta = -\sqrt{5}\alpha$, the roots to these quadratic equations may be simplified to

$$\frac{-\sqrt{5}\beta \pm \sqrt{3\sqrt{5}\alpha}}{2} \quad \text{and} \quad \frac{\sqrt{5}\alpha \pm \sqrt{3\sqrt{5}(-\beta)}}{2}.$$

Thus the roots of the recurrence equation again include α and β.

4. QUADRINOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^6 - x^5 - 10x^4 + 10x^2 + x - 1 = 0.$$

We factor out $(x - 1)(x + 1)$ to get

$$x^4 - x^3 - 9x^2 - x + 1 = 0.$$

Divide through by x^2, then let $y = x + 1/x$. We obtain

$$(2 + x^2 + 1/x^2) - (x + 1/x) - 11 = 0.$$

Then, after substituting y, the equation above becomes

$$y^2 - y - 11 = 0,$$

with the roots

$$\frac{1 \pm 3\sqrt{5}}{2}.$$

Now we solve

$$x + 1/x = \frac{1 \pm 3\sqrt{5}}{2}.$$

Multiply this equation by x to obtain

$$x^2 - \left(\frac{1 \pm 3\sqrt{5}}{2}\right)x + 1 = 0.$$

The roots of this pair of quadratic equations are found to be

$$\frac{1 \pm 3\sqrt{5}}{2} \pm \sqrt{\frac{1 + 45 \pm 6\sqrt{5}}{4} - 4} = \frac{1 \pm 3\sqrt{5} \pm 2\sqrt{3\sqrt{5}\alpha}}{4}.$$

Now $\dfrac{1 + 3\sqrt{5}}{2} = 2\alpha - \beta$ and $\dfrac{1 - 3\sqrt{5}}{2} = 2\beta - \alpha$, thus we may simplify. The roots of the recurrence equation are, therefore,

$$+1,\ -1,\ \frac{2\beta - \alpha \pm \sqrt{3\sqrt{5}(-\beta)}}{2},\ \frac{2\alpha - \beta \pm \sqrt{3\sqrt{5}\alpha}}{2}.$$

5. PENTANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^5 - 5x^4 + 15x^2 - 9 = 0.$$

We factor out $x + 1$ to get

$$x^4 - 6x^3 + 6x^2 + 9x - 9 = 0.$$

Let $y = x - 3/2$. Then $y^2 = x \quad - 3x + 9/4$ and

$$y^4 = x^4 - 6x^3 + (27/2)x^2 - (27/2)x + 81/16,$$

so the recurrence equation may be written

$$y^4 - (15/2)y^2 + (45/16) = 0.$$

Letting $z = y^2$ produces a quadratic equation in z with roots

$$\frac{\frac{15}{2} \pm \sqrt{\frac{225}{4} - 4\left(\frac{45}{16}\right)}}{2} = \frac{15 \pm 6\sqrt{5}}{4}.$$

Thus,

$$y = \frac{\pm\sqrt{15 \pm 6\sqrt{5}}}{2} \quad \text{and} \quad x = \frac{3 \pm \sqrt{15 \pm 6\sqrt{5}}}{2}.$$

We rewrite these last four roots as follows:

$$\frac{3 \pm \sqrt{15 + 6\sqrt{5}}}{2} = \frac{3 \pm \sqrt{3\sqrt{5}\alpha^3}}{2} \quad \text{and} \quad \frac{3 \pm \sqrt{15 - 6\sqrt{5}}}{2} = \frac{3 \pm \sqrt{3\sqrt{5}(-\beta^3)}}{2}.$$

Thus, the five roots to the recurrence equation are the four just above and -1.

6. HEXANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^5 - 4x^4 - 5x^3 + 10x^2 + 5x - 5 = 0.$$

We factor out $x + 1$ to get

$$x^4 - 5x^3 + 10x - 5 = 0.$$

To use Ferrari's solution of the quartic equation, we must determine a, b, and k such that

$$x^4 - 5x^3 + 10x - 5 + (ax + b)^2 = (x^2 - (5/2)x + k)^2.$$

The determination of a, b, and k is accomplished by equating the coefficients of like powers of x in the equations above. This leads to the relations

$$a^2 = 2k + 25/4;\ 2ab + 10 = -5k;\ b^2 - 5 = k^2$$

which gives rise to the resolvent cubic equation in k:

$$8k^3 - 60k + 25 = 0.$$

A root of this cubic equation is $k = 5/2$. We substitute this value of k in the relations above to solve for a and b. We find

$$a = \frac{3\sqrt{5}}{2} \quad \text{and} \quad b = \frac{-3\sqrt{5}}{2}.$$

Now we solve an equation in which both members are perfect squares:

$$(x^2 - (5/2)x + (5/2))^2 = (ax + b)^2.$$

Therefore,

$$x^2 - (5/2)x + (5/2) = ax + b$$

and

$$x^2 - (5/2)x + (5/2) = -ax - b.$$

The four roots of the quartic equation can be found by solving these two quadratic equations. We substitute the values of a and b in these quadratic equations to obtain

$$x^2 - \left(\frac{5 + 3\sqrt{5}}{2}\right)x + \frac{5 + 3\sqrt{5}}{2} = 0$$

and

$$x^2 - \left(\frac{5 - 3\sqrt{5}}{2}\right)x + \frac{5 - 3\sqrt{5}}{2} = 0.$$

Hence

$$x = \frac{\dfrac{5 \pm 3\sqrt{5}}{2} \pm \sqrt{\dfrac{70 \pm 30\sqrt{5}}{4} - 4\left(\dfrac{5 \pm 3\sqrt{5}}{2}\right)}}{2} = \frac{5 \pm 3\sqrt{5} \pm \sqrt{30 \pm 6\sqrt{5}}}{4}.$$

Thus, $x = \dfrac{\sqrt{5}\alpha^2 \pm \sqrt{3}\sqrt{\sqrt{5}\sqrt{\alpha}}}{2}$ or $x = \dfrac{-\sqrt{5}\beta^2 \pm \sqrt{3}\sqrt{\sqrt{5}\sqrt{-\beta}}}{2}$.

These four roots together with $x = -1$ comprise the five roots to the recurrence equation.

7. HEPTANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^7 - 4x^6 - 6x^5 + 10x^4 + 5x^3 - 6x^2 - x + 1 = 0.$$

Let $y = 1/x$ to obtain

$$1 - 4y - 6y^2 + 10y^3 + 5y^4 - 6y^5 - y^6 + y^7 = 0.$$

This equation in y is precisely the recurrence equation of the $(H - L)/15$ sequence in the binomial triangle. Hence, the roots we are seeking are the reciprocals of the roots that were derived in the binomial triangle. These reciprocal roots are

$$1,\ 1/\alpha,\ 1/\beta,\ \frac{\beta^3 \pm \sqrt{3\sqrt{5}(-\beta^3)}}{2},\ \frac{\alpha^3 \pm \sqrt{3\sqrt{5}\alpha^3}}{2}.$$

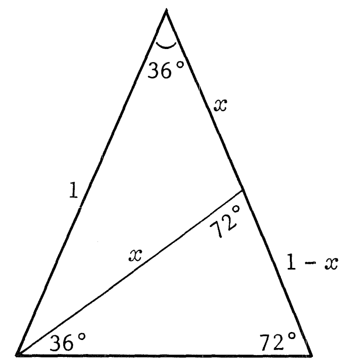
8. CONCLUSION

How surprising to see α and β appear with such frequency in the roots to all the cases with 15 partitions!

Another unexpected result was the reciprocal relationship that occurred between the recurrence equations of the binomial and heptanomial triangles. More study into the successive j-nomial triangles could certainly surface more interesting results.

Theorem A: $\cos 12° = \dfrac{-\beta + \sqrt{3}\sqrt{\sqrt{5}}\sqrt{\alpha}}{4}$.

Proof: Consider the Golden Triangle:



$x = 2 \sin 18°$

Therefore $\dfrac{1 - x}{x} = \dfrac{x}{1}$, which implies $x = \dfrac{\sqrt{5} - 1}{2} = -\beta$.

$$\cos 18° = 1 - \left(\frac{3 - \sqrt{5}}{8}\right) = \frac{\sqrt{5}}{4}\left(\frac{1 + \sqrt{5}}{2}\right)$$

$$\cos 18° = \frac{\sqrt{\sqrt{5}}}{2}\sqrt{\alpha}$$

$$\sin 18° = \frac{\sqrt{5} - 1}{4}$$

$$\begin{aligned}\cos 12° &= \cos 18° \cos 30° + \sin 18° \sin 30°\\ &= \frac{\sqrt{\sqrt{5}}}{2}\sqrt{\alpha}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{\beta}{2}\right) = \frac{-\beta + \sqrt{3}\sqrt{\sqrt{5}}\sqrt{\alpha}}{4}.\end{aligned}$$

We note this occurs often in the roots.

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A NOTE ON THE MULTIPLICATION OF TWO 3 X 3 FIBONACCI-ROWED MATRICES

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A Fibonacci-rowed matrix is defined to be a matrix in which each row consists of consecutive Fibonacci numbers in increasing order.

Laderman [1] presented a noncommutative algorithm for multiplying two 3 x 3 matrices using 23 multiplications. It still needs 18 multiplications if Laderman's algorithm is applied to the product of two 3 x 3 Fibonacci-rowed matrices. In this short note, an algorithm is developed in which only 17 multiplications are needed. This algorithm is mainly based on Strassen's result [2] and the fact that the third column of a Fibonacci-rowed matrix is equal to the sum of the other two columns.

Let $C = AB$ be the matrix of the multiplication of two 3 x 3 Fibonacci-rowed matrices. Define

$$\begin{aligned}
\mathrm{I} &= (a_{11} + a_{22})(b_{11} + b_{22}) \\
\mathrm{II} &= a_{23}b_{11} \\
\mathrm{III} &= a_{11}(b_{12} - b_{22}) \\
\mathrm{IV} &= a_{22}(-b_{11} + b_{21}) \\
\mathrm{V} &= a_{13}b_{22} \\
\mathrm{VI} &= (-a_{11} + a_{21})b_{13} \\
\mathrm{VII} &= (a_{12} - a_{22})b_{23}
\end{aligned}$$

Then

$$C = \begin{bmatrix} \mathrm{I} + \mathrm{IV} - \mathrm{V} + \mathrm{VII} + a_{13}b_{31} & \mathrm{III} + \mathrm{V} + a_{13}b_{32} & c_{11} + c_{12} \\ \mathrm{II} + \mathrm{IV} + a_{23}b_{31} & \mathrm{I} + \mathrm{III} - \mathrm{II} + \mathrm{VI} + a_{23}b_{32} & c_{21} + c_{22} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & c_{31} + c_{32} \end{bmatrix}.$$

There are only 17 multiplications involved in calculating . However, 18 multiplications are needed if Laderman's algorithm [1] is applied, namely

$$m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ m_7,\ m_8,\ m_{11},\ m_{12},$$
$$m_{13},\ m_{14},\ m_{15},\ m_{16},\ m_{17},\ m_{19},\ m_{20},\ m_{22}$$

(see [1]). In fact, only 18 multiplications are needed if the usual process of multiplication is applied.

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POWERS OF THE PERIOD FUNCTION FOR THE SEQUENCE OF FIBONACCI NUMBERS

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If m is an integer greater than or equal to 2, we write $\phi(m)$ for the length of the period of the sequence of Fibonacci numbers reduced to least nonnegative residues modulo m. The function ϕ has been studied quite extensively (see, for example, [1], [2], and [3]). It is easy to discover that for small values of m there exists a positive integer k such that

$$\phi^k(m) = \phi^{k+1}(m),$$

i.e., that the sequence

$$\phi(m),\ \phi(\phi(m)),\ \phi(\phi(\phi(m))),\ \ldots$$

eventually becomes stationary. The purpose of this note is to prove this fact in general.

We start by observing that it is sufficient to consider m to be of the form $2^a3^b5^c$ for nonnegative integers a, b, and c. For, if $\psi(m)$ denotes the rank of apparition of m in the Fibonacci sequence modulo m, then by Lemma 12 of [1], if $p \neq 5$ is an odd prime we have $\psi(p) \mid (p \pm 1)$, while $\psi(5) = 5$. Thus, for an odd prime $q \neq 5$ with $q \geq p$ such that $q \mid \psi(p)$, we have that $q \mid (p \pm 1)$, which is impossible. Consequently, the primes occurring in the prime decomposition of $\psi(p)$ are all less than p or, as we shall say, $\psi(p)$ "involves" only primes less than p. Now, by a Theorem of Vinson [2], we know that

$$\phi(p) = 2^r\psi(p) \text{ where } r = 0,\ 1, \text{ or } 2,$$

so that $\phi(p)$ also involves only primes less than p.

Suppose $\phi(m) = dp^\beta$, where p is a prime greater than 5, and d involves only primes less than p and $\beta \neq 0$. Then using Lemma 14 of [1] and Theorem 5 of [3] we have that

$$\phi^2(m) = \begin{cases} [\phi(d),\ p^{\beta-1}\phi(p)] & \text{if } \phi(p^2) \neq \phi(p) \\ [\phi(d),\ p^{\beta-2}\phi(p)] & \text{if } \phi(p^2) = \phi(p) \text{ and } \beta \neq 1 \end{cases}$$

where square brackets with integers inside denote the lowest common multiple of those integers. Now, $\phi(d)$ and $\phi(p)$ involve only primes less than p, so that $\phi^2(m) = d_1p^\gamma$, say, where $0 \leq \gamma < \beta$ and d_1 involves only primes less than p. Carrying on in this way, we eventually find an integer s such that $\phi^s(m)$ does not involve p and so, continuing, we may find an integer t such that $\phi^t(m)$ involves only 2, 3, and 5. Thus

$$\phi^t(m) = 2^a3^b4^c \text{ for some } a,\ b,\ c \geq 0.$$

This justifies the assertion that we need consider only integers of the stated form.

We now define a sequence $\{\alpha_n\}$ by $\alpha_1 = a - 1$, where $a > 1$, and $\alpha_{n+1} = \max(\alpha_n - 1,\ 3)$ if $n \geq 1$. Then it is easy to see that $\{\alpha_n\}$ eventually takes the constant value 3: in fact, $\alpha_{a-3} = 3$ if $a \geq 5$ and $\alpha_2 = 3$ if $a < 5$. Now $\phi^n(2^a) = 2^{\alpha_n} \cdot 3$, so that if $a \geq 5$ we have $\phi^{a-3}(2^a) = 2^3 \cdot 3$, and if $a < 5$ we have $\phi^2(2^a) = 2^3 \cdot 3$. Thus, we see that there exists an integer $u \geq 2$ such that $\phi^u(2^a) = 2^3 \cdot 3$ if $a > 1$. Similarly, if we define the sequence $\{\beta_n\}$ by

$\beta_1 = b - 1$, where $b > 1$, and $\beta_{n+1} = \max(\beta_n - 1, 1)$ if $n \geq 1$, we have that $\beta_{b-1} = 1$ if $b \geq 3$, $\beta_2 = 1$ if $b < 3$, and that $\phi^n(3^b) = 2^3 \cdot 3^{\beta_n}$. Thus, there exists an integer $v \geq 2$ such that $\phi^v(3^b) = 2^3 \cdot 3$ if $b > 1$.

Now we note that $\phi^4(2) = \phi^3(3) = 2^3 \cdot 3$ and that $\phi^3(5^c) = 2^3 \cdot 3 \cdot 5^c$ for any $c \geq 1$ and that $\phi(2^3 \cdot 3 \cdot 5^c) = 2^3 \cdot 3 \cdot 5^c$ holds even for $c = 0$. Again using Lemma 14 of [1] we have for a, $b > 1$ that

$$\begin{aligned}\phi^{u+v}(2^a 3^b) &= [\phi^{u+v}(2^a),\ \phi^{u+v}(3^b)]\\ &= [\phi^v(2^3 \cdot 3),\ \phi^u(2^3 \cdot 3)]\\ &= 2^3 \cdot 3,\end{aligned}$$

so that

$$\phi^{u+v}(2^a 3^b 5^c) = [2^3 \cdot 3,\ 2^3 \cdot 3 \cdot 5^c] = 2^3 \cdot 3 \cdot 5^c$$

since $u + v > 3$. Consequently

$$\phi^{u+v+1}(2^a 3^b 5^c) = \phi^{u+v}(2^a 3^b 5^c).$$

The remaining cases are when $a \leq 1$ or $b \leq 1$, and it is easy to check that $\phi^{v+3}(2^a 3^b 5^c) = \phi^{v+2}(2^a 3^b 5^c)$ if $a \leq 1$ and $\phi^{u+3}(2^a 3^b 5^c) = \phi^{u+2}(2^a 3^b 5^c)$ if $b \leq 1$.

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SOME REMARKS ON THE PERIODICITY OF THE SEQUENCE OF FIBONACCI NUMBERS—II

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The Fibonacci sequence $\{F_n\}$ is defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \quad (n \geq 1).$$

If t is an integer greater than 2 and $\phi(t)$ is the length of the period of the sequence reduced to least nonnegative residues modulo t, it was shown in [2] that $\phi(F_{m-1} + F_{m+1}) = 4m$ if m is even and $\phi(F_{m-1} + F_{m+1}) = 2m$ if m is odd. It follows for $m > 4$ that

$$\phi(F_{m-1} + F_{m+1}) = \frac{1}{2}(\phi(F_{m-1}) + \phi(F_{m+1})).$$

I conjectured in the same paper that if $m - k > 3$ then

$$\phi(F_{m-k} + F_{m+k}) = \frac{k}{2}(\phi(F_{m-k}) + \phi(F_{m+k})).$$

The object of this note is to show that this conjecture is false and to give the correct answer in some special cases.

That the conjecture is false may be seen by taking $m = 12$ and $k = 4$, for example, because in this case

$$\phi(F_8 + F_{16}) = \phi(1008) = 48,$$

whereas

$$2(\phi(F_8) + \phi(F_{16})) = 96.$$

In what follows, we write $[x,y]$ and (x,y) for the lowest common multiple and the greatest common divisor of the integers x and y, respectively, and let x_2 denote the largest number e for which $2^e | x$. Also we define

$$H_a = F_{a-1} + F_{a+1} \quad (a \geq 1).$$

Theorem: Suppose that k and m are integers with $3 < k \leq m$. Then

(i) if k is even and $(H_m, F_k) = 1$, we have

$$\phi(F_{m-k} + F_{m+k}) = \begin{cases} 2[k,m] \text{ if } m \text{ is even and } k_2 < m_2 \\ 4[k,m] \text{ otherwise,} \end{cases}$$

(ii) if k is odd and $(H_k, F_m) = 1$, we have

$$\phi(F_{m-k} + F_{m+k}) = 4[k,m].$$

The proof of this requires the fact that if $n = \alpha\beta$ and $(\alpha,\beta) = 1$, then $\phi(n) = [\phi(\alpha), \phi(\beta)]$, essentially proved in Theorem 2 of [3]. Now it is well known that

$$F_{m-k} = (-1)^k (F_{k-1}F_m - F_k F_{m-1})$$
$$F_{m+k} = F_{k+1}F_m + F_k F_{m-1},$$

so that

$$F_{m-k} + F_{m+k} = \begin{cases} H_k F_m \text{ if } k \text{ is even} \\ H_m F_k \text{ if } k \text{ is odd.} \end{cases}$$

Consequently, if k is even and $(H_k, F_m) = 1$, then

$$\phi(F_{m-k} + F_{m+k}) = [\phi(H_m), \phi(F_k)] = \begin{cases} [4k,2m] \text{ if } m \text{ is even} \\ [4k,4m] \text{ if } m \text{ is odd,} \end{cases}$$

using results proved in [1] and [2]. Similarly, if k is odd and $(H_m, F_k) = 1$, we have that

$$\phi(F_{m-k} + F_{m+k}) = [\phi(H_k), \phi(F_m)] = \begin{cases} [4m,4k] \text{ if } m \text{ is even} \\ [2m,4k] \text{ if } m \text{ is odd.} \end{cases}$$

The result now follows by noting that if k and m are even then $[4k,2m]$ equals $2[k,m]$ or $4[k,m]$ depending on whether $k_2 < m_2$ or $k_2 \geq m_2$, respectively; if k is even and m is odd then $[4k,4m] = 4[k,m]$, and if k and m are both odd then $[4k,2m] = 4[k,m]$.

The cases not covered by the Theorem are when $k \leq 3$. The case $k = 1$ was dealt with in [2]. When $k = 2$, we have $F_{m-2} + F_{m+2} = 3F_m$. Now $3 | F_m$ if and only if $4 | m$, from which we see that if $(3, F_m) = 1$ and $m > 3$ then

$$\phi(F_{m-2} + F_{m+2}) = \begin{cases} 4m \text{ if } m \text{ is even} \\ 8m \text{ if } m \text{ is odd.} \end{cases}$$

When $k = 3$, then $F_{m-3} + F_{m+3} = 2H_m$. Now $2|H_m$ if and only if $3|m$. Thus, if $(2,H_m) = 1$ we have that

$$\phi(F_{m+3} + F_{m-3}) = \begin{cases} 12m \text{ if } m \text{ is even} \\ 6m \text{ if } m \text{ is odd.} \end{cases}$$

Finally, it may be worthwhile commenting on the conditions of the form $(H_a, F_b) = 1$ which have been necessary for our computations. $(H_a, F_b) > 1$ is not a rare phenomenon because, for instance, given a it is easy to determine an infinite number of values of b for which $H_a|F_b$. In fact, as we now show, $H_a|F_b$ if and only if b is a positive integral multiple of $2a$. For, $H_a|F_{2a}$ because $F_{2a} = F_aH_a$. Thus, $H_a|F_{2ac}$ for any positive integer c. Actually, $2a$ is the least suffix b for which $H_a|F_b$, as shown by the proof of Theorem B in [2]. Let B denote the set of all positive integers b for which $H_a|F_b$. Then B is nonempty, and if $b_1, b_2 \in B$ since

$$F_{b_1+b_2} = F_{b_1+1}F_{b_2} + F_{b_1}F_{b_2-1}$$
$$F_{b_1-b_2} = (-1)^{b_2}(F_{b_2-1}F_{b_1} - F_{b_2}F_{b_1-1}),$$

we see that $b_1 + b_2$, $b_1 - b_2 \in B$. This means that B consists of all multiples of some least element which, as already pointed out, is $2a$ (see Theorem 6 in Chapter I of [4]).

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MUTUALLY COUNTING SEQUENCES

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ABSTRACT

Let n and m be positive integers with $n \leq m$. Let A be the sequence of n nonnegative integers $a(0)$, $a(1)$, ..., $a(n - 1)$, and let B be the sequence of m nonnegative integers $b(0)$, $b(1)$, ..., $b(m - 1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. We prove that for $n > 7$, there are exactly 3 ways to generate such pairs of sequences.

***

Let n and m be positive integers with $n \leq m$. Let A be the sequence of n nonnegative integers $a(0)$, $a(1)$, ..., $a(n - 1)$, and let B be the sequence of m nonnegative integers $b(0)$, $b(1)$, ..., $b(m - 1)$, where $a(i)$ is the multiplicity of i in B and $b(j)$ is the multiplicity of j in A. Then A and B

are said to form a pair of *mutually counting sequences*. Observe at the outset that

$$S(A) = \sum_{i=0}^{n-1} a(i) = m \quad \text{and} \quad S(B) = \sum_{j=0}^{m-1} b(j) = n.$$

In this paper, we prove the following result:

Theorem: For $n > 7$, a pair of mutually counting sequences A and B can be formed in exactly 3 ways:

(I) $a(0) = m - 3$, $a(1) = a(3) = a(n - 4) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 4$, $b(1) = 3$, $b(m - 3) = 1$,
$b(j) = 0$ for all remaining j.

(II) $a(0) = m - 4$, $a(1) = 3$, $a(n - 3) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 3$, $b(1) = b(3) = b(m - 4) = 1$,
$b(j) = 0$ for all remaining j.

(III) $a(0) = m - 4$, $a(1) = 2$, $a(2) = a(n - 4) = 1$,
$a(i) = 0$ for all remaining i;
$b(0) = n - 4$, $b(1) = 2$, $b(2) = b(m - 4) = 1$,
$b(j) = 0$ for all remaining j.

Proof: Let A and B be a pair of mutually counting sequences. Then clearly, $b(m - 2) + b(m - 1) \leq 1$. Suppose that $b(m - 1) = 1$. Then $m - 1$ has multiplicity 1 in A, and since $S(A) = m$, one of the remaining entries of A must be 1, while the other $n - 2$ entries are 0. Therefore,

$$b(0) = n - 2,\ b(1) = b(m - 1) = 1, \text{ and } b(j) = 0 \text{ for all remaining } j,$$

which implies that $a(1) = 2$, a contradiction. Now suppose that $b(m - 2) = 1$. Then $m - 2$ has multiplicity 1 in A. Again, from $S(A) = m$, we may conclude that either (i) one of the remaining entries of A is 2 and the other $n - 2$ entries are 0 or (ii) two of the remaining entries of A are 1 and the other $n - 3$ entries are 0. In (i), we get

$$b(0) = n - 2,\ b(2) = b(m - 2) = 1,$$

while (ii) yields

$$b(0) = n - 3,\ b(1) = 2,\ b(m - 2) = 1.$$

In both instances it follows from $S(B) = n$ that the remaining $m - 3$ entries of B are 0. But this implies that $a(0) = m - 3$, a contradiction. Thus, we may conclude that the initial inequality must be strict, which immediately gives $b(m - 2) = b(m - 1) = 0$. By an analogous argument, it follows that $a(n - 2) = a(n - 1) = 0$ as well.

Note next that $b(m - 3) \leq 1$. If equality holds, then $m - 3$ has multiplicity 1 in A, and since $S(A) = m$, the sum of the remaining entries of A must be 3. Three possibilities exist for these remaining entries: (i) one is 3 and the other $n - 2$ are 0; (ii) one is 2, another is 1, and the other $n - 3$ are 0; (iii) three are 1 and the other $n - 4$ are 0. In (i), we have $b(0) = n - 2$, contradicting $a(n - 2) = 0$; in (ii), we have

$$b(0) = n - 3,\ b(1) = b(2) = b(m - 3) = 1,$$

and $b(j) = 0$ for all remaining j,

implying that $a(0) = m - 4$, a contradiction; from (iii), we obtain

$$b(0) = n - 4,\ b(1) = 3,\ b(m - 3) = 1,$$
$$\text{and } b(j) = 0 \text{ for all remaining } j,$$

which implies that

$$a(0) = m - 3,\ a(1) = a(3) = a(n - 4) = 1,$$
$$\text{and } a(i) = 0 \text{ for all remaining } i.$$

That is, if $b(m - 3) = 1$, we get a pair of mutually counting sequences of type (I). Similarly, observe that $a(n - 3) \leq 1$. If we assume that $a(n - 3) = 1$, then the same kind of procedure as above will produce a pair of mutually counting sequences of type (II). In what follows, therefore, we will assume without loss of generality that $a(n - 3) = b(m - 3) = 0$.

Now note that $b(m - 4) \leq 1$ [only when $n = m = 8$ is it possible for $b(m - 4) = 2$, and a simple calculation leads to a quick contradiction]. If $b(m - 4) = 1$, then $m - 4$ has multiplicity 1 in A, so that the sum of the remaining entries of A must be 4. There are five possibilities here for these remaining entries: (i) one is 4 and the other $n - 2$ are 0; (ii) one is 3, another is 1, and the other $n - 3$ are 0; (iii) two are 2 and the other $n - 3$ are 0; (iv) four are 1 and the other $n - 5$ are 0; and (v) two are 1, another is 2, and the other $n - 4$ are 0. In (i), we have $b(0) = n - 2$, contradicting $a(n - 2) = 0$; in (ii) and (iii), we get $b(0) = n - 3$, which contradicts $a(n - 3) = 0$; in (iv), we find $b(0) = n - 5$, $b(1) = 4$, and $b(j) = 0$ for all remaining j, which implies that $a(0) = m - 3$, again a contradiction; finally in (v), we have $b(0) = n - 4$, $b(1) = 2$, $b(2) = 1$, and $b(j) = 0$ for all remaining j. This yields

$$a(0) = m - 4,\ a(1) = 2,\ a(2) = 1,\ a(n - 4) = 1,$$
$$\text{and } a(i) = 0 \text{ for all remaining } i.$$

That is, under the stated hypotheses, we have produced a pair of mutually counting sequences of type (III).

It remains to show that if $b(m - 4) = 0$, then no other pair of mutually counting sequences can be constructed. This result is easily verified for $n = 8$ and $n = 9$, so for $n \geq 0$ we will now assume that another such pair exists and will deduce an eventual contradiction.

If $b(j) = 0$ for all $j \geq [m/2]$, then the multiplicity of 0 in B is at least $m - [m/2]$, i.e., $a(0) \geq m - [m/2] \geq [m/2]$. But this implies that some integer $j \geq [m/2]$ appears in A, contradicting the initial assumption. Thus $b(j^*) > 0$ for at least one integer $j^* \geq [m/2]$, where $j^* < m - 4$. If j_1^* and j_2^* are distinct integers with this property, then both appear at least once in A, so that $m = S(A) \geq j_1^* + j_2^* > 2[m/2]$. If m is even, then we obtain $m > m$, which is an obvious contradiction; if m is odd, then $2[m/2] = m - 1$, which gives $j_1^* + j_2^* = S(A)$. It then follows that all remaining entries of A must be 0, so $b(0) = n - 2$. But this contradicts $a(n - 2) = 0$. Therefore, j^* is unique.

Next, it is apparent that $b(j^*) = 1$ or 2. If $b(j^*) = 2$, then we easily conclude that $j^* = [m/2]$, from which it follows that $m = S(A) \geq 2j^* = 2[m/2]$. This again leads to contradictory statements whether m is odd or even, so we may assert that $b(j^*) = 1$.

Suppose that $a(i) = j^*$ for some $i > 2$. Then since the multiplicity of i in B is j^* and the multiplicity of 1 in B is at least 1 [since $b(j^*) = 1$], it follows that $n = S(B) \geq ij^* + 1 > 2[m/2] + 1 \geq m$, a contradiction. If $a(2) = j^*$, then 2 has multiplicity j^* in B, and since $b(0) \geq 3$, we get $n \geq 2j^* + 3 > m$, again a contradiction. Suppose next that $a(1) = j^*$. Since $b(j) = 0$ for all $j \geq [m/2]$, $j \neq j^*$, it follows that $a(0) \geq m - [m/2] - 1$. Therefore,

$$m = S(A) \geq a(0) + a(1) \geq m - [m/2] - 1 + j^* \geq m - 1,$$

which implies that either one of the remaining entries of A is 1 and all others are 0 or all remaining entries of A are 0. So the multiplicity of 0 in A is either $n - 3$ or $n - 2$, implying that either $b(n - 3)$ or $b(n - 2)$ is nonzero, both contradictions. Hence, $a(0) = j^*$.

Now consider the case in which $j^* = [m/2]$. Then $b(j) = 0$ for all $j > [m/2]$, accounting for $m - [m/2] - 1$ entries of 0 in B. Since $a(0) = j^* = [m/2]$, the number of remaining zero entries of B, denoted by r, is given by

$$r = [m/2] - (m - [m/2] - 1) = 2[m/2] - m + 1.$$

If m is odd, then $r = 0$, so in particular, $b([m/2] - k)$, $k = 1, 2, 3$ are all nonzero. This means that in addition to $[m/2]$, the integers $[m/2] - k$, $k = 1, 2, 3$ all appear at least once in A. Then

$$m = S(A) \geq 4[m/2] - 6 = 2m - 8,$$

which yields $m \leq 8$, a contradiction. If m is even, then $r = 1$, so only one of the remaining entries of B is 0. Then at least three of the four entries $b([m/2] - k)$, $k = 1, 2, 3, 4$ are nonzero, which implies that in addition to $[m/2]$, at least three of the integers between $[m/2] - 4$ and $[m/2] - 1$ appear at least once in A. Then

$$m = S(A) \geq [m/2] + ([m/2] - 2) + ([m/2] - 3) + ([m/2] - 4),$$

i.e.,

$$m \geq 4[m/2] - 9 = 2m - 9.$$

But this implies that $m \leq 9$, a contradiction. We conclude that $j^* > [m/2]$.

At this point, we may improve our results concerning the zero entries of B. For, suppose that $b(j) \neq 0$ for some $j \geq m - j^* - 1$, $j \neq j^*$. Then, j and j^* both have multiplicity at least 1 in A, so that

$$m = S(A) \geq j + j^* \geq m - 1.$$

Therefore, either one of the remaining entries of A is 1 and the other $n - 3$ entries are 0, or each of the $n - 2$ remaining entries of A is 0. Then the multiplicity of 0 in A is either $n - 3$ or $n - 2$, implying that either $b(0) = n - 3$ or $b(0) = n - 2$. But this means that either $a(n - 3)$ or $a(n - 2)$ is nonzero, both contradictions. So, $b(j) = 0$ for all $j \geq m - j^* - 1$, $j \neq j^*$, which accounts for precisely j^* entries of 0 in B. Since $a(0) = j^*$, it follows that all remaining entries of B must be nonzero. In particular, $b(m - j^* - 2)$, $b(m - j^* - 3)$, and $b(1)$ are all nonzero, which means that in addition to j^*, the integers $m - j^* - 2$, $m - j^* - 3$, and 1 all appear in A. So

$$m = S(A) \geq (m - j^* - 2) + (m - j^* - 3) + j^* + 1 = 2m - j^* - 4,$$

which implies that $j^* \geq m - 4$, the desired contradiction. Consequently, the assumption that another pair of mutually counting sequences can be generated must be false, and the theorem is proved.

It is left to the interested reader to generate the mutually counting sequences that exist for $n \leq 7$.

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MULTISECTION OF THE FIBONACCI CONVOLUTION ARRAY AND GENERALIZED LUCAS SEQUENCE

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1. INTRODUCTION

The general problem of multisecting a general sequence rapidly becomes very complicated. In this paper we multisect the convolutions of the Fibonacci sequence and certain generalized Lucas sequences.

When we m-sect a sequence, we write a generating function for every mth term of the sequence. To illustrate, we recall [1], [2],

$$\sum_{k=0}^{\infty} F_{mk+r}x^k = \frac{F_r + (-1)^r F_{m-r}x}{1 - L_m x + (-1)^m x^2}, \tag{1.1}$$

which m-sects the Fibonacci sequence $\{F_n\}$, where

$$F_1 = F_2 = 1,\ F_{n+1} = F_n + F_{n-1},$$

and where L_m is the mth term of the Lucas sequence $\{L_n\}$,

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}.$$

For later comparison, it is well known that the Fibonacci and Lucas sequences enjoy the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \tag{1.2}$$

where α and β are the roots of $x^2 - x - 1 = 0$,

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Also, the generating functions for F_n and L_n are

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n, \quad \frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^n. \tag{1.3}$$

The Fibonacci convolution array, written in rectangular form, is

1	1	1	1	1	...
1	2	3	4	5	...
2	5	9	14	20	...
3	10	22	40	65	...
5	20	51	105	190	...
8	38	111	256	511	...
...	...	...	...	...	...

where each column is the convolution of the succeeding column with the Fibonacci sequence. The convolution sequence $\{c_n\}$ of two sequences $\{a_n\}$ and $\{b_n\}$ is formed by

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1}.$$

Also, it is known that the generating functions of successive convolutions of the Fibonacce sequence are given by $(1 - x - x^2)^{-k-1}$, $k = 0, 1, 2, \ldots$, where $k = 0$ gives the Fibonacci sequence itself.

2. MULTISECTION OF THE FIBONACCI CONVOLUTION ARRAY

We now proceed to multisect the Fibonacci convolution array. Recalling (1.1), we let

$$G_r = \frac{F_r + (-1)\ F^r_{k-r}x}{1 - L_k x + (-1)\ x^2}, \qquad G_r^* = \frac{F_r + (-1)\ F^r_{k-r}x}{1 - L_k x^k + (-1)^k x^{2k}}.$$

Clearly,

$$\sum_{r=0}^{k-1} G_r^* x^r = \frac{1}{1 - x \quad x^2}.$$

Thus,

$$\sum_{r=0}^{k-1} (F_r + (-1)^r F_{k-r} x^k) x^r = Q_k(x),$$

where

$$Q_k(x) = \frac{1 - L_k x^k + (-1)^k x^{2k}}{1 - x - x^2}.$$

To multisect the general convolution sequence for the Fibonacci numbers, let us work on column s, where $s = 1$ is the Fibonacci sequence itself. Then

$$Q_k^s(x) = \left(\frac{1 - L_k x^k + (-1)^k x^{2k}}{1 - x - x^2}\right)^s.$$

Now there are k separate k-sectors. The coefficients of the numerator polynomial of the jth generator are given by every kth coefficient of $Q_k^s(x)$, beginning with $1 \le j \le k$, while the denominator is $(1 - L_k x^k + (-1)^k x^{2k})^s$.

It is now simple to see how to multisect the columns of Pascal's triangle (see [2]) by taking

$$Q^s(x) = \left(\frac{1 - x^k}{1 - x}\right)^s.$$

We can even multisect the negative powers, which in the Fibonacci case is just a finite polynomial $(1 - x - x^2)^s$ from which we take every kth coefficient.

3. THE TRIBONACCI AND HIGHER CONVOLUTION ARRAYS

Define the Tribonacci numbers $\{T_n\}$ by

(3.1) $$T_0 = 0,\ T_1 = T_2 = 1,\ T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

The Tribonacci convolution triangle, with the Tribonacci numbers appearing in the leftmost column, is

1	1	1	1	1	...
1	2	3	4	5	...
2	5	9	14	20	...
4	12	25	44	70	...
7	26	63	135	...	...

(continued)

13	56	153	...	...	...
24	...	...	...	...	...
...	...	...	...	...	...

Since

$$\frac{x}{1 - x - x^2 - x^3} = \sum_{n=0}^{\infty} T_n x^n, \tag{3.2}$$

the generating functions for the Tribonacci convolution sequences are given successively by

$$[x/(1 - x - x^2 - x^3)]^{k+1}, \; k = 0, 1, 2, \ldots,$$

where $k = 0$ gives the Tribonacci sequence itself.

Let

$$S_k = \alpha^k + \beta^k + \gamma^k$$

where α, β, and γ are the roots of $x^3 - x^2 - x - 1 = 0$. Then the multisecting generating functions are obtained from

$$Q_k(x) = \frac{1 - S_k x^k + S_{-k} x^{2k} - x^{3k}}{1 - x - x^2 - x^3}, \tag{3.3}$$

where the coefficients of $Q_k^s(x)$ used are

$$T_1, T_2, T_3, \ldots, T_k, (T_{k+1} - S_k), \ldots, (T_{k+s} - S_k T_s), T_{-k-1}, T_{-k}, \ldots, T_{-2}.$$

The coefficients of the numerator polynomial of the jth generator are given by every kth coefficient of $Q_k^s(x)$, beginning with $1 \le j \le k$, while the denominator is $(1 - S_k x^k + S_{-k} x^{2k} - x^{3k})^s$.

From the auxiliary polynomial $x^3 - x^2 - x - 1 = 0$,

$$T_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma T_{n-1} = \frac{\beta^n - \gamma^n}{\beta - \gamma} + \alpha T_{n-1} = \frac{\gamma^n - \alpha^n}{\gamma - \alpha} + \beta T_{n-1}$$

or

$$3T_n - T_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \frac{\gamma^n - \alpha^n}{\gamma - \alpha}. \tag{3.4}$$

Also,

$$T_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \gamma^2\frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} + \cdots + \gamma^n. \tag{3.5}$$

For the Quadranacci numbers $\{Q_n\}$ defined by

$$Q_0 = 0,\; Q_1 = Q_2 = 1,\; Q_3 = 2,\; Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n \tag{3.6}$$

we get similar results. If we let α, β, γ, and δ be the roots of $x^4 - x^3 - x^2 - x - 1 = 0$, then

$$Q_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \gamma\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \cdots + \gamma^n + \delta Q_{n-1}. \tag{3.7}$$

In multisecting the Quadranacci convolution array,

$$G_k(x) = \frac{(1 - \alpha^k x^k)(1 - \beta^k x^k)(1 - \gamma^k x^k)(1 - \delta^k x^k)}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)},$$

where $G_k(x)$ is the numerator polynomial from which the generating functions can be derived for multisecting the Quadranacci convolution sequences.

We can derive the following from (3.7):

$$6Q_n - 3Q_{n-1} - Q_{n-2} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \frac{\gamma^n - \delta^n}{\gamma - \delta} + \frac{\delta^n - \alpha^n}{\delta - \alpha} + \frac{\beta^n - \delta^n}{\beta - \delta} + \frac{\alpha^n - \gamma^n}{\alpha - \gamma}. \tag{3.8}$$

4. GENERALIZED FIBONACCI AND LUCAS NUMBERS

Start with

$$f(x) = \prod_{i=1}^{m} (x - \alpha_i);$$

then if

$$f(x) = x^m - x^{m-1} - x^{m-2} - \cdots - 1,$$

in particular, then

$$\frac{1}{s!} \cdot \frac{f^{(s)}(x)}{f(x)} = \prod \frac{1}{(x - \alpha_{i_1})(x - \alpha_{i_2}) \ldots (x - \alpha_{i_s})}$$

$$1 \le i_1 < i_2 < i_3 < \cdots < i_s \le m$$

over all subscripts restrained above.

If $s = m$, then we get, after some effort,

$$\frac{x}{1 - x - x^2 - \cdots - x^m} = \sum_{n=0}^{\infty} F_n^* x^n, \tag{4.1}$$

where F_n^* are the generalized Fibonacci numbers of the preceding section.

If $s = m$, we get the corresponding Lucas numbers

$$\mathcal{L}_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n.$$

But, for those $1 < s < m$ we get other generalized Fibonacci sequences with some interesting properties studies by Chow [3]. We note two quick theorems.

Theorem 4.1:

Let

$$f(x) = \prod_{i=1}^{m} (x - \alpha_i), \; m \ge 2.$$

Then $\{\mathcal{L}_n\} = \{m, 1, 3, 7, 15, 32, \ldots\}$ for m terms. That is,

$$\mathcal{L}_0 = m, \; \mathcal{L}_1 = 2^1 - 1, \; \mathcal{L}_2 = 2^2 - 1, \; \ldots, \; \mathcal{L}_s = 2^s - 1, \; \ldots, \; \mathcal{L}_m = 2^m - 1.$$

After m terms, the recurrence takes over. In fact, $\mathcal{L}_m$ is the first term yielded by the recurrence. Further,

Theorem 4.2: The generating function for $\{\mathcal{L}_n\}$ is

$$\frac{mx - (m-1)x^2 - (m-2)x^3 - \cdots - x^m}{1 - x - x^2 - \cdots - x^m} = \sum_{n=0}^{\infty} \mathcal{L}_n x^n. \tag{4.2}$$

Using the observation that

$$G_m(x) + x \approx G_{m+1}(x)$$

For $(m+1)$ terms, one can then get an inductive proof for the starting values theorem. Of course, one has a starting values theorem for the regular generalized Fibonacci numbers in generalized Pascal triangles, and these are $1, 1, 2, 2^2, 2^3, \ldots,$ until we reach the full length of the recurrence. Of great interest, of course, are those of the form

$$\frac{kx - x^2}{1 - x - x^2 - \cdots - x^m},$$

which starts off k, $k-1$, $2k-1$, ..., which now double until the recurrence takes over.

For $s = 2$,

$$\frac{1}{2!} \cdot \frac{f^{(2)}(x)}{f(x)} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^n - \gamma^n}{\alpha - \gamma} + \cdots + \frac{\beta^n - \gamma^n}{\beta - \gamma} + \cdots$$
$$= \frac{T_{m-1}x - T_{m-2}x^2 - \cdots - x^{m-1}}{1 - x - x^2 - x^3 - \cdots - x^m},$$

where the T_m are the triangular numbers.

If one attempts to multisect the generalized Fibonacci numbers, one needs, of course, the generalized Lucas numbers in the recurrence relation. Recapping our results so far, we list each auxiliary polynomial:

$m = 2$ Fibonacci $\quad L_k = \alpha^k + \beta^k$
$$x^2 - L_k x + (-1)^k$$

$m = 3$ Tribonacci $\quad S_k = \alpha^k + \beta^k + \gamma^k$
$$x^3 - S_k x^2 + S_{-k}x - 1$$

$m = 4$ Quadranacci $\quad S_k = \alpha^k + \beta^k + \gamma^k + \delta^k$
$$x^4 - S_k x^3 + \frac{1}{2}(S_k^2 - S_{2k})x^2 - S_{-k}x + 1$$

What is involved, then, are the elementary symmetric functions for the original polynomial but for the kth powers of the roots.

5. GENERALIZED LUCAS NUMBERS AND SYMMETRIC FUNCTIONS OF kTH POWERS

If
$$x^m + c_1x^{m-1} + c_2x^{m-2} + \cdots + c_m = 0$$
has roots $\alpha_1, \alpha_2, \ldots, \alpha_m$, and $S_k = \alpha_1^k + \alpha_2^k + \cdots + \alpha_m^k$, then

$$c_k = \frac{(-1)^k}{k!}\begin{vmatrix} S_1 & 1 & 0 & 0 & \cdots \\ S_2 & S_1 & 2 & 0 & \cdots \\ S_3 & S_2 & S_1 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & k-1 \\ S_k & S_{k-1} & \cdots & \cdots & S_1 \end{vmatrix} \tag{5.1}$$

which stems from the system of equations

$$
\begin{aligned}
&S_1 + c_1 &&= 0\\
(5.2)\qquad &S_2 + c_1S_1 + 2c_2 &&= 0\\
&S_3 + c_1S_2 + c_2S_1 + 3c_3 &&= 0\\
&S_4 + c_1S_3 + c_2S_2 + c_3S_1 + 4c_4 &&= 0\\
&\qquad\cdots &&\ \ \cdots
\end{aligned}
$$

which are Newton's Identities as given by Conkwright [4].

If you look at these equations, you have four unknowns c_1, c_2, c_3, and c_4 if S_1, S_2, S_3, and S_4 are given. Thus, you can treat this as a nonhomogeneous system and hence solve for c_1, c_2, c_3, or c_4, but strangely enough, while working, this does not yield the clever expression first given.

Consider instead

$$
\begin{aligned}
c_0S_1 + c_1 &= 0\\
c_0S_2 + c_1S_1 + 2c_2 &= 0\\
c_0S_3 + c_1S_2 + c_2S_1 &= -3c_3
\end{aligned}
$$

where $c_0 = 1$. Solve the system for c_0 by Cramer's rule:

$$
c_0 = 1 = \frac{\begin{vmatrix} 0 & 1 & 0\\ 0 & S_1 & 2\\ -3c_3 & S_2 & S_1 \end{vmatrix}}{\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}} = \frac{-3!c}{\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}}
$$

or

$$
c_3 = \frac{(-1)^3}{3!}\begin{vmatrix} S_1 & 1 & 0\\ S_2 & S_1 & 2\\ S_3 & S_2 & S_1 \end{vmatrix}.
$$

From (5.1) one can sequentially find $c_1, c_2, \ldots, c_k$ given $S_1, S_2, \ldots, S_k$, but this soon becomes untractable in practice.

However, we can make a new representation of the generalized Lucas sequences by using the set of equations (5.2) to derive

$$
(5.3)\qquad S_k = (-1)^k \begin{vmatrix}
1c_1 & 1 & 0 & 0 & 0 & \cdots & 0\\
2c_2 & c_1 & 1 & 0 & 0 & \cdots & 0\\
3c_3 & c_2 & c_1 & 1 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
kc_k & c_{k-1} & c_{k-2} & \cdots & \cdots & \cdots & c_1
\end{vmatrix}.
$$

We rewrite (5.2) as

$$
\begin{aligned}
(1)c_1 + S_1 &= 0\\
(1)2c_2 + S_1c_1 + S_2 &= 0\\
(1)3c_3 + S_1c_2 + S_2c_1 + S_3 &= 0\\
(1)4c_4 + S_1c_3 + S_2c_2 + S_3c_1 + S_4 &= 0
\end{aligned}
$$

Here, again, we have a known variable (1) which we solve for using Cramer's rule for the nonhomogeneous set of equations, as

$$1 = \frac{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & c_1 & 1 & 0 \\ 0 & c_2 & c_1 & 1 \\ -S_4 & c_3 & c_2 & c_1 \end{vmatrix}}{\begin{vmatrix} 1c_1 & 1 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}} \quad \frac{S_4}{\begin{vmatrix} 1c_1 & 0 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}}$$

or

$$S_4 = (-1)^4 \begin{vmatrix} c_1 & 1 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}.$$

Considering where these problems came from, if $c_1 = c_2 = -1$, $c_k = 0$ for $k > 2$, then $S_k = L_k$, the familiar Lucas numbers, which are then given by a tri-diagonal continuant,

$$L_k = (-1)^k \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -2 & -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{vmatrix},$$

while the generalized Lucas sequence related to the Tribonacci numbers is given by the quadradiagonal continuant,

$$\mathscr{L}_k = (-1)^k \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -2 & -1 & 1 & 0 & 0 & \cdots & 0 \\ -3 & -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & -1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{vmatrix}.$$

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SOME RESTRICTED MULTIPLE SUMS

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1. INTRODUCTION

Let a, b be positive integers, $(a,b) = 1$. Consider the sum

$$S = \sum_{br+as<ab} x^{br+as} \equiv \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} x^{br+as}. \tag{1.1}$$

We will show that

$$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}. \tag{1.2}$$

As an application of (1.2), let $B_n(x)$ denote the Bernoulli polynomial of degree n defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!},\ B_n = B_n(0).$$

Then we have

$$\sum_{br+as<ab} B_n\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(ab) + abB(x))^n - (aB + bB(ax))^n, \tag{1.3}$$

where

$$(uB(x) + vB(y))^n \equiv \sum_{k=0}^{n}\binom{n}{k} u^k v^{n-k} B_k(x) B_{n-k}(y).$$

We also evaluate the sum

$$\sum_{br+as<ab} (x + br + as)^n \tag{1.4}$$

in terms of Bernoulli polynomials; see (3.8) below.

Let a, b, c be positive integers such that $(b,c) = (c,a) = (a,b) = 1$. The sum (1.1) suggests the consideration of the sums

$$S_1 = \sum_{bcr+cas+abt<abc} x^{bcr+cas+abt}$$

and

$$S_2 = \sum_{bcr+cas+abt<2abc} x^{bcr+cas+abt}$$

where $0 \le r < a$, $0 \le s < b$, $0 \le t < c$. We are unable to evaluate S_1 and S_2 separately. However, we show that

(1.5) $$x^{abc}S_1 + S_2 = \frac{(1 - x^{abc})^2}{(1 - x^{bc})(1 - x^{ca})(1 - x^{ab})} - \frac{x^{2abc}}{1 - x}.$$

For applications to triple sums analogous to (1.3) and (1.4) see (5.5), (5.6), and (5.7) below.

We remark that the case $x = 0$ of (1.3) is implicit in the proof of Theorem 1 of [1].

2. PROOF OF (1.2)

We have

$$S = \sum_{br+as<ab} x^{br+as} = \sum_{r=0}^{a-1} x^{br} \sum_{as<b(a-r)} x^{as} = \sum_{r=0}^{a-1} x^{br} \sum_{s=0}^{[b(a-r)/a]} x^{as}.$$

Since

$$[b(a - r)/a] = b - [br/a] - 1,$$

it follows that

(2.1) $$S = \sum_{r=0}^{a-1} x^{br} \frac{1 - x^{ab-a[br/a]}}{1 - x^a} = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x^a} \sum_{r=0}^{a-1} x^{br-a[br/a]}.$$

Clearly the exponent

(2.2) $$br - a[br/a] \qquad (r = 0, 1, \ldots, a - 1)$$

is the remainder obtained in dividing br by a. Since $(a,b) = 1$, it follows that the numbers (2.2) are a permutation of 0, 1, ..., $a - 1$. Hence, (2.1) becomes

$$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x^a} \frac{1 - x^a}{1 - x},$$

so that

(2.3) $$S = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}.$$

This proves (1.2). Note that the complementary sum

(2.4) $$\overline{S} = \sum_{\substack{r=0 \\ br+as>ab}}^{a-1} \sum_{s=0}^{b-1} x^{br+as}$$

satisfies

$$S + \overline{S} = \frac{1 - x^{ab}}{1 - x^a} \frac{1 - x^{ab}}{1 - x^b}.$$

Hence, by (2.3),

(2.5) $$\overline{S} = \frac{x^{ab}}{1 - x} - \frac{x^{ab}(1 - x^{ab})}{(1 - x^a)(1 - x^b)}.$$

3. SOME APPLICATIONS

In (1.1) and (1.2), replace x by $e^{z/ab}$:

(3.1) $$\sum_{br+as<ab} e^{(br+as)z/ab} = \frac{e^z}{e^{z/ab} - 1} - \frac{e^z - 1}{(e^{z/a} - 1)(e^{z/b} - 1)}.$$

Multiplying by $z^2e^{xz}/(e^z - 1)$, we get

$$\frac{z^2}{e^z - 1}\sum_{br+as<ab} e^{(br+as)z/ab+xz} \tag{3.2}$$
$$= \frac{z^2e^{(x+1)z}}{(e^z - 1)(e^{z/ab} - 1)} - \frac{z^2e^{xz}}{(e^{z/a} - 1)(e^{z/b} - 1)}.$$

Since

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!},$$

(3.2) becomes

$$z\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{br+as<ab} B_n\left(x + \frac{r}{a} + \frac{s}{b}\right) = \frac{z^2e^{(x+1)z}}{(e^z - 1)(e^{z/ab} - 1)} - \frac{z^2e^{xz}}{(e^{z/a} - 1)(e^{z/b} - 1)}$$
$$= ab\sum_{j,k=0}^{\infty} B_j(x)B_k(ab)\frac{z^j(z/ab)^k}{j!k!} \tag{3.3}$$
$$- ab\sum_{j,k=0}^{\infty} B_j(ax)B_k\frac{(z/a)^j(a/b)^k}{j!k!}.$$

Equating coefficients of z^n, we get

$$n\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (ab)^{1-n}\sum_{k=0}^{n}\binom{n}{k}(ab)^{n-k}B_{n-k}(x)B_k(ab) \tag{3.4}$$
$$- (ab)^{1-n}\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}B_{n-k}(ax)B_k.$$

This can be written more compactly in the form

$$n(ab)^{n-1}\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(ab) + abB(x))^n - (aB + bB(ax))^n, \tag{3.5}$$

where it is understood that

$$(uB(x) + vB(y))^n = \sum_{k=0}^{n}\binom{n}{k}u^kv^{n-k}B_k(x)B_{n-k}(y).$$

Alternatively, (3.3) can be replaced by

$$ab\sum_{j,k=0}^{\infty} B_j(1)B_k(abx)\frac{z^j(z/ab)^k}{j!k!} - ab\sum_{j,k=0}^{\infty} B_j(ax)B_k\frac{(z/a)^j(z/b)^k}{j!k!}.$$

Hence, we now get

$$n(ab)^{n-1}\sum_{br+as<ab} B_{n-1}\left(x + \frac{r}{a} + \frac{s}{b}\right) = (B(abx) + abB(1))^n - (aB + bB(ax))^n. \tag{3.6}$$

Note that comparison of (3.6) with (3.5) gives

(3.7) $$(B(ab) + abB(x))^n = (B(abx) + abB(1))^n,$$

which indeed holds for arbitrary a, b. We also recall that

$$B_n(1) = B_n, \ (n \neq 1); \ B_1(1) = B_1 + 1 = \frac{1}{2}.$$

For $n = 1$, since $B_1 = -\frac{1}{2}$ and

$$\sum_{br+as<ab} 1 = \sum_{r=0}^{a-1} \sum_{s<\frac{b}{a}(a-r)} 1 = \sum_{r=0}^{a-1}\left(b - \left[\frac{br}{a}\right]\right)$$
$$= ab - \frac{1}{2}(a-1)(b-1), \ ((a,b) = 1),$$

(3.5) reduces to

$$ab - \frac{1}{2}(a-1)(b-1) = (B(ab) + abB(x))' - (aB + bB(ax))'$$
$$= \left(ab - \frac{1}{2}\right) + ab\left(x - \frac{1}{2}\right) + \frac{1}{2}a - b\left(ax - \frac{1}{2}\right),$$

which is correct.

In place of (3.2) we now take

$$z^2 \sum_{br+as<ab} e^{(br+as)z+xz} = \frac{z^2 e^{(x+ab)z}}{e^z - 1} - \frac{z^2(e^{abz} - 1)e^{xz}}{(e^{az} - 1)(e^{bz} - 1)}$$
$$= z\sum_{n=0}^{\infty} B_n(x + ab)\frac{z^n}{n!} - (ab)^{-1}\sum_{j,k=0}^{\infty}(B_j(b) - B_j)B_k\left(\frac{x}{b}\right)\frac{(az)^j(bz)^k}{j!k!}.$$

It follows that

(3.8) $$n(n-1)ab\sum_{br+as<ab}(x + br + as)^{n-2}$$
$$= nabB_{n-1}(x + ab) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k(B_{n-k}(b) - B_{n-k})B_k\left(\frac{x}{b}\right).$$

For example, for $n = 2$, since $B_2(x) = x^2 - x + \frac{1}{6}$, we have

$$2ab\left(ab - \frac{1}{2}(a-1)(b-1)\right) = 2ab\left(x + ab - \frac{1}{2}\right) - a^2(b^2 - b) - 2ab^2\left(\frac{x}{b} - \frac{1}{2}\right),$$

which is correct.

Note that, for $b = 1$, (3.8) becomes

$$n(n-1)a\sum_{r=0}^{a-1}(x + r)^{n-2} = naB_{n-1}(x + a) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}(B_{n-k}(1) - B_{n-k})B_k(x).$$

Since

$$B_n(1) = B_n, \ (n \neq 1); \ B_1(1) = \frac{1}{2}, \ B_1 = -\frac{1}{2},$$

we get

$$n(n-1)a\sum_{r=0}^{a-1}(x + a)^{n-2} = na\left(B_{n-1}(x + a) - B_{n-1}(a)\right),$$

that is, the familiar formula (replacing $n - 1$ by n)

$$\sum_{r=0}^{a-1}(x + a)^{n-1} = \frac{1}{n}\{B_n(x + a) - B_n(x)\}.$$

Similarly, for $b = 1$, (3.5) reduces to

$$na^{n-1}\sum_{r=0}^{a-1}B_{n-1}\left(x + \frac{r}{a}\right) = (B(a) + aB(x))^n - (ab + B(ax))^n. \tag{3.9}$$

We recall [2, p. 21] that

$$B_{n-1}(ax) = a^{n-2}\sum_{r=0}^{a-1}B_{n-1}\left(x + \frac{r}{a}\right). \tag{3.10}$$

Comparison of (3.10) with (3.9) yields

$$(B(a) + aB(x))^n - (aB + B(ax))^n = naB_{n-1}(ax). \tag{3.11}$$

To give a direct proof of (3.11), let R_n denote the left-hand side of (3.11). Then,

$$\begin{aligned}\sum_{n=0}^{\infty}R_n\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}a^kB_k(x)B_{n-k}(a) - \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}a^kB_kB_{n-k}(ax)\\ &= \frac{aze^{axz}}{e^{az} - 1}\,\frac{ze^{az}}{e^z - 1} - \frac{az}{e^{az} - 1}\,\frac{ze^{axz}}{e^z - 1}\\ &= \frac{az^2e^{axz}}{e^z - 1} = az\sum_{n=0}^{\infty}B_n(ax)\frac{z^n}{n!},\end{aligned}$$

and (3.11) follows at once.

4. PROOF OF (1.5)

Put

$$S_1 = \sum_{bcr + cas + abt < abc} x^{bcr + cas + abt} \tag{4.1}$$

and

$$S_2 = \sum_{bcr + cas + abt < 2abc} x^{bcr + cas + abt}. \tag{4.2}$$

It is understood that in all such sums

$$0 \le r < a,\ 0 \le s < b,\ 0 \le t < c. \tag{4.3}$$

As for S_1, we have

$$\begin{aligned}S_1 &= \sum_{br + as < ab} x^{c(br + as)}\sum_{t < c - \frac{c}{ab}(br + as)} x^{abt}\\ &= \sum_{br + as < ab} x^{c(br + as)}\frac{1 - x^{ab(c - [c(br + as)/ab])}}{1 - x^{ab}}\\ &= \frac{1}{1 - x^{ab}}\sum_{br + as < ab} x^{c(br + as)} - \frac{x^{ab}}{1 - x^{ab}}\sum_{br + as < ab} x^{R(c(br + ab)/ab)}\end{aligned} \tag{4.4}$$

where $R(m/ab)$ denotes the remainder obtained in dividing m by ab. It will be convenient to put

$$U = \{u | u = c(br + as),\ br + as < ab\}$$
$$V = \{v | v = c(br + as),\ br + as > ab\}.$$

Thus (4.4) becomes

$$S_1 = \frac{1}{1 - x^{ab}} \sum_{u \in U} x^u - \frac{x^{ab}}{1 - x^{ab}} \sum_{u \in U} x^{R(u/ab)}. \tag{4.5}$$

Next put

$$S_2' = \sum_{\substack{bcr + cas + abt < 2abc \\ br + as > ab}} x^{c(br+as)+abt}$$

$$S_2'' = \sum_{\substack{bcr + cas + abt < 2abc \\ br + as > ab}} x^{c(br+as)+abt},$$

so that $S_2 = S_2' + S_2''$. Clearly

$$S_2' = \sum_{br + as < ab} x^{c(br+as)} \sum_{t=0}^{c-1} x^{abt}$$

$$= \frac{1 - x^{abc}}{1 - x^{ab}} \sum_{u \in U} x^u \tag{4.6}$$

The evaluation of S_2'' is less simple. We have

$$S_2'' = \sum_{br + as > ab} x^{c(br+as)} \sum_{t < 2c - \frac{c}{ab}(br+as)} x^{abt} = \sum_{v \in V} x \sum_{t < 2c - (v/ab)} x^{abt}$$

$$= \sum_{v \in V} \frac{x^v 1 - x^{ab(2c - [v/ab])}}{1 - x^{ab}}$$

$$= \frac{1}{1 - x^{ab}} \sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x^{ab}} \sum_{v \in V} x^{r(v/ab)}. \tag{4.7}$$

It follows from (4.5) and (4.7) that

$$x^{abc} S_1 + S_2'' = \frac{x^{abc}}{1 - x^{ab}} \sum_{u \in U} x^u + \frac{1}{1 - x^{ab}} \sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x^{ab}} \left\{ \sum_{u \in U} x^{R(u/ab)} + \sum_{v \in V} x^{R(v/ab)} \right\}.$$

Since

$$\sum_{u \in U} x^{R(u/ab)} + \sum_{v \in V} x^{R(v/ab)} = \sum_{t=0}^{ab-1} x^t = \frac{1 - x^{ab}}{1 - x},$$

we get

$$x^{abc}S_1 + S_2'' = \frac{x^{abc}}{1 - x^{ab}}\sum_{u \in U} x^u + \frac{1}{1 - x^{ab}}\sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x}. \tag{4.8}$$

Hence, by (4.6) and (4.8), we have

$$\begin{aligned} x^{abc}S_1 + S_2' + S_2'' &= \left(\frac{x^{abc}}{1 - x^{ab}} + \frac{1 - x^{abc}}{1 - x^{ab}}\right)\sum_{u \in U} x^u + \frac{1}{1 - x^{ab}}\sum_{v \in V} x^v - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\left\{\sum_{u \in U} x^u + \sum_{v \in V} x^v\right\} - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1} x^{c(br+as)} - \frac{x^{2abc}}{1 - x} \\ &= \frac{1}{1 - x^{ab}}\,\frac{1 - x^{abc}}{1 - x^{bc}}\,\frac{1 - x^{abc}}{1 - x^{ac}} - \frac{x^{2abc}}{1 - x}. \end{aligned}$$

Therefore,

$$x^{abc}S_1 + S_2 = \frac{(1 - x^{abc})^2}{(1 - x^{bc})(1 - x^{ca})(1 - x^{ab})} - \frac{x^{2abc}}{1 - x}. \tag{4.9}$$

5. SOME RESTRICTED TRIPLE SUMS

It follows from (4.9) with x replaced by $e^{z/abc}$ that

$$e^z\sum_{\sigma < 1} e^{\sigma z} + \sum_{\sigma < 2} e^{\sigma z} = \frac{(1 - e^z)^2}{(1 - e^{z/a})(1 - e^{z/b})(1 - e^{z/c})} - \frac{e^{2z}}{1 - e^{z/abc}}, \tag{5.1}$$

where for brevity we put

$$\sigma = \frac{r}{a} + \frac{s}{b} + \frac{t}{c}. \tag{5.2}$$

Multiplying both sides of (5.1) by $z^3e^{xz}/(e^z - 1)^2$, we get

$$\frac{z^3}{(e^z - 1)^2}\sum_{\sigma < 1} e^{(x+\sigma+1)} + \frac{z^3}{(e^z - 1)^2}\sum_{\sigma < 2} e^{(x+\sigma)}$$

$$= \frac{z^3e^{(x+2)z}}{(e^z - 1)^2(e^{z/abc} - 1)} - \frac{z^3e^{xz}}{(e^{z/a})(e^{z/b} - 1)(e^{z/c} - 1)}. \tag{5.3}$$

In order to obtain a compact result we make use of Nörlund's definition of Bernoulli numbers of higher order [2, Chapter 6]. Let $\omega_1, \omega_2, \ldots, \omega_k$ denote parameters and define the polynomial $B_n(x|\omega_1, \ldots, \omega_k)$ by means of

$$\frac{\omega_1\omega_2 \ldots \omega_k z^k e^{xz}}{(e^{\omega_1 z} - 1)(e^{\omega_2 z} - 1) \ldots (e^{\omega_k z} - 1)} = \sum_{n=0}^{\infty} B_n^{(k)}(x|\omega_1, \ldots, \omega_k)\frac{z^n}{n!}. \tag{5.4}$$

With this notation, (5.3) becomes

$$z\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{\sigma<1}B_n^{(2)}(x+\sigma+1|1,\ 1)+\sum_{\sigma<2}B_n^{(2)}(x+\sigma|1,\ 1)\right\}$$

$$=abc\sum_{n=0}^{\infty}\frac{z}{n!}\left\{B_n^{(3)}(x+2|1,\ 1,\ (abc)^{-1})-B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1})\right\}.$$

Hence, equating coefficients of z , we get

$$n\left\{\sum_{\sigma<1}B_{n-1}^{(2)}(x+\sigma+1|1,\ 1)+\sum_{\sigma<2}B_{n-1}^{(2)}(x+\sigma|1,\ 1)\right\} \tag{5.5}$$

$$=abc\left\{B_n^{(3)}(x+2|1,\ 1,\ (abc)^{-1})-B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1})\right\}.$$

Similarly, it follows from (5.1) that

$$n(n-1)(n-2)\left\{\sum_{\sigma<1}(x+\sigma+1)^{n-3}+\sum_{\sigma<2}(x+\sigma)^{n-3}\right\} \tag{5.6}$$

$$=n(n-1)(abc)^{-n+1}B_{n-2}(abc(x+2))-\Delta_x^2B_n^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1}),$$

where Δ_x^2 is the familiar difference operator:

$$\Delta_x^2 f(x) = f(x+2) - 2f(x+1) + f(x).$$

Finally, multiplying both sides of (5.1) by $z^3e^{xz}/(e^z-1)$, we get

$$n(n-1)\left\{\sum_{\sigma<1}B_{n-2}(x+\sigma+1)+\sum_{\sigma<2}B_{n-2}(x+\sigma)\right\} \tag{5.7}$$

$$=nabcB_{n-1}^{(2)}(x+2|1,\ (abc)^{-1})-nabc\Delta_xB_{n-1}^{(3)}(x|a^{-1},\ b^{-1},\ c^{-1}).$$

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SOME REMARKS ON THE BELL NUMBERS

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1. The Bell numbers A_n can be defined by means of the generating function,

$$e^{e^x - 1} = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}. \tag{1.1}$$

This is equivalent to

$$A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k. \tag{1.2}$$

Another familiar representation is

$$A_n = \sum_{k=0}^{n} S(n,k), \tag{1.3}$$

where $S(n,k)$ denotes a Stirling number of the second kind [3, Ch. 2].

The definition (1.1) suggests putting

$$e^{a(e^x - 1)} = \sum_{n=0}^{\infty} A_n(a) \frac{x^n}{n!}; \tag{1.4}$$

$A_n(a)$ is called the *single-variable Bell polynomial*. It satisfies the relations

$$A_{n+1}(a) = a \sum_{k=0}^{n} \binom{n}{k} A_k(a) \tag{1.5}$$

and

$$A_n(a) = \sum_{k=0}^{n} a^k S(n,k). \tag{1.6}$$

(We have used A_n and $A_n(a)$ to denote the Bell numbers and polynomials rather than B_n and $B_n(a)$ to avoid possible confusion with Bernoulli numbers and polynomials [2, Ch. 2].)

Cohn, Ever, Menger, and Hooper [1] have introduced a scheme to facilitate the computation of the A_n. See also [5] for a variant of the method. Consider the following array, which is taken from [1].

$A_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6
0	1	1	2	5	15	52	203
1	2	3	7	20	67	255	1080
2	5	10	27	87	322	1335	
3	15	37	114	409	1657		
4	52	151	523	2066			
5	203	674	2589				
6	877	3263					

The $A_{n,k}$ are defined by means of the recurrence

(1.7) $$A_{n+1,k} = A_{n,k} + A_{n,k+1} \qquad (n \geq 0)$$

together with $A_{00} = 1$, $A_{01} = 1$. It follows that

(1.8) $$A_{0,k} = A_k,\ A_{n,0} = A_{k+1}.$$

The definition of $A_n(a)$ suggests that we define the polynomial $A_{n,k}(a)$ by means of

(1.9) $$A_{n+1,k}(a) = A_{n,k}(a) + A_{n,k+1}(a) \qquad (n \geq 0)$$

together with

$$A_{00}(a) = 1,\ A_{01}(a) = a.$$

We then have

(1.10) $$A_{0,k}(0) = A_k(a),\ aA_{n,0}(a) = A_{n+1}(a).$$

For $a = 1$, (1.10) evidently reduces to (1.8).

2. Put

(2.1) $$F_n(z) = \sum_{k=0}^{\infty} A_{n,k}\frac{z}{k!}$$

and

(2.2) $$F(x,z) = \sum_{n=0}^{\infty} F_n(z)\frac{x^n}{n!} = \sum_{n,k=0}^{\infty} A_{n,k}\frac{x^n z^k}{n!k!}.$$

It follows from (2.1) and the recurrence (1.7) that

(2.3) $$F_{n+1}(z) = F_n(z) + F_n'(z).$$

It is convenient to write (2.3) in the operational form

(2.4) $$F_{n+1}(z) = (1 + D_z)F_n(z) \qquad \left(D_z \equiv \frac{d}{dz}\right).$$

Iteration leads to

(2.5) $$F_n(z) = (1 + D_z)^n F_0(z) \qquad (n \geq 0).$$

Since, by (1.1) and (1.8), $F_0(z) = e^{e^z - 1}$, we get

(2.6) $$F_0(z) = (1 + D_z)^n e^{e^z - 1}.$$

Incidentally, (2.5) is equivalent to

(2.7) $$A_{n,k} = \sum_{j=0}^{n}\binom{n}{j}A_{j+k} = \sum_{j=0}^{n}\binom{n}{j}A_{k+n-j}.$$

The inverse of (2.7) may be noted:

(2.8) $$A_{n+k} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}A_{j,k}.$$

Making use of (2.5), we are led to a definition of $A_{n,k}$ for negative n. Replacing n by $-n$, (2.5) becomes

$$(1 + D_z)^n F_{-n}(z) = F_0(z).$$

Thus, if we put

(2.9) $$F_{-n}(z) = \sum_{k=n}^{\infty} A_{-n,k}\frac{z^k}{k!},$$

we have

(2.10) $$\sum_{j=0}^{n}\binom{n}{j} A_{-n,j+k} = A_k \qquad (k = 0, 1, 2, \ldots).$$

It can be verified that (2.10) is satisfied by

(2.11) $$A_{-n,k} = \sum_{j=0}^{k-n} (-1)^j \binom{j+n-1}{j} A_{k-n-j} = \sum_{j=0}^{k-n} \binom{-n}{j} A_{k-n-j}.$$

Indeed, it is enough to take

$$\begin{aligned} A_{-n,k} + A_{-n,k+1} &= \sum_{j=0}^{k-n} (-1)^j \binom{j+n-1}{j} A_{k-n-j} + \sum_{j=0}^{k-n+1} (-1)^j \binom{j+n-1}{j} A_{k-n-j+1} \\ &= \sum_{j} (-1)^j A_{j-n-j+1} \left\{ \binom{j+n-1}{j} - \binom{j+n-2}{j-1} \right\} \\ &= \sum_{j=0}^{k-n+1} (-1)^j \binom{j+n-2}{j} A_{k-n-j+1}, \end{aligned}$$

so that

(2.12) $$A_{-n,k} + A_{-n,k+1} = A_{-n+1,k}$$

and (2.10) follows by induction on n.

Note that by (2.9)

(2.13) $$A_{-n,k} = 0 \qquad (0 \le k < n).$$

The following table of values of $A_{-n,k}$ is computed by means of (2.12) and (2.13).

Put

$n \backslash k$	0	1	2	3	4	5	6	7
6	0	0	0	0	0	0	1	-5
5	0	0	0	0	0	1	-4	12
4	0	0	0	0	1	-3	8	-13
3	0	0	0	1	-2	5	-5	54
2	0	0	1	-1	3	0	49	105
1	0	1	0	2	3	49	154	723
0	1	1	2	5	52	203	877	4140

Clearly,

(2.14) $$A_{-n,n} = 1 \qquad (n = 0, 1, 2, \ldots).$$

Put

$$G \equiv G(x,z) = \sum_{n=0}^{\infty} F \quad (z)x \quad = \sum_{k=0}^{\infty} \frac{z}{n!} \sum_{n=0}^{k} A_{-n,k} x^n.$$

Then, since by (2.12),

$$(1 + D_z)F_{-n}(z) = F_{-n-1}(z) \qquad (n > 0),$$

we have

$$(1 + D_z)G = xG + F_1(z);$$

that is,

$$D\ G + (-x)G = F_1(z) = (1 + e^z)e^{e^z - 1}.$$

This differential equation has the solution

$$e^{(1-x)z}G = \int_0^z e^{(1-x)}\ (1 + e^t)e^{e^t - 1}\,dt + \phi(x), \tag{2.15}$$

where $\phi(x)$ is independent of z.

For $z = 0$, (2.15) reduces to

$$G(x,0) = \phi(x).$$

By (2.15)

$$G(x,0) = A_{0,0} = 1$$

and, therefore

$$G(x,z) = e^{(-1-x)z}\int_0^z e^{(1-x)t}(1 + e^t)e^{e^t - 1}dt + e^{-(1-x)z}. \tag{2.16}$$

In the next place, by (2.2) and (2.5),

$$F(x,z) = \sum_{n=0}^{\infty} \frac{x^n(1 + D_z)^n}{n!}F_0(z) = e^{x(1+D_z)}F_0(z).$$

Since

$$e^{xD_z}F_0(z) = F_0(x + z),$$

we get

$$F(x,z) = e^x F_0(x + z) = e^x e^{e^{x+z} - 1}. \tag{2.17}$$

It follows from (2.5) that

$$e^z F(x,z) = e^x F(z,x), \tag{2.18}$$

which is equivalent to

$$\sum_{j=0}^{k}\binom{k}{j}A_{n,j} = \sum_{j=0}^{n}\binom{n}{j}A_{k,j}. \tag{2.19}$$

Using (2.7), it is easy to give a direct proof of (2.10).

3. The results of §2 are easily carried over to the polynomial $A_n(a)$. Put

$$F_n(z|a) = \sum_{k=0}^{\infty} A_k(a)\frac{z^k}{k!}, \tag{3.1}$$

and

$$F(x,z|a) = \sum_{n=0}^{\infty} F_n(z|a)\frac{x^n}{n!}. \tag{3.2}$$

It follows from (1.9) and (3.1) that

$$F_{n+1}(z|a) = (1 + D_z)F_n(z|a), \tag{3.3}$$

so that

$$F_n(z|a) = (1 + D_z)^n F_0(z|a) = (1 + D_z)^n e^{a(e^z - 1)}. \tag{3.4}$$

Thus,

(3.5) $$A_{n,k}(a) = \sum_{j=0}^{n}\binom{n}{j}A_{j+k}(a).$$

As in §2, we find that

(3.6) $$F(x,z|a) = e^{x}F_0(x+z|a),$$

so that

(3.7) $$e^{z}F(x,z|a) = e^{x}F(z,x|a),$$

which is equivalent to

(3.8) $$\sum_{j=0}^{n}\binom{k}{j}A_{n,j} = \sum_{j=0}^{n}\binom{n}{j}A_{j,k}.$$

By (1.4),

$$\sum_{k=0}^{\infty}A_k(a)\frac{x^k}{k!} = e^{a(e^x-1)}.$$

Thus (3.6) becomes

(3.9) $$F(x,z|a) = e^{x}e^{a(e^{x+z}-1)}.$$

Differentiation with respect to a yields

$$\sum_{n,k=0}^{\infty}A'_{n,k}(a)\frac{x^n z^k}{n!k!} = (e^{x+z}-1)\sum_{n,k=0}^{\infty}A_{n,k}(a)\frac{x^n z^k}{n!k!}$$

and therefore

(3.10) $$A'_{n,k}(a) = \sum_{\substack{i=0\\ i+j<n+k}}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}A_{i,j}(a).$$

Similarly, differentiation with respect to z gives

$$\sum_{n,k=0}^{\infty}A_{n,k+1}(a)\frac{x^n z^k}{n!k!} = ae^{x+y}\sum_{n,k=0}^{\infty}A_{n,k}(a)\frac{x^n z^k}{n!k!},$$

so that

(3.11) $$A_{n,k+1}(a) = a\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}A_{i,j}(a).$$

Comparing (3.11) with (3.10), we get

(3.12) $$A_{n,k+1}(a) = aA_{n,k}(a) + A'_{n,k}(a).$$

Differentiation of (3.9) with respect to x leads again to (1.9).

4. It follows from (1.3) and (2.7) that

(4.1) $$A_{n,k} = \sum_{i=0}^{n}\binom{n}{i}A_{k+i} = \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k+i}S(k+i,j).$$

Since

$$S(n,j) = \frac{1}{j!}\sum_{t=0}^{j}(-1)^{j-t}\binom{j}{t}t^{k+i},$$

it follows from (4.1) that

$$A_{n,k} = \sum_{j=0}^{k+n} S(n,k,j), \tag{4.2}$$

where

$$S(n,k,j) = \frac{1}{j!}\sum_{t=0}^{j}(-1)^{j-t}\binom{j}{t}t^k(t+1)^n. \tag{4.3}$$

Clearly, $S(0,k,j) = S(k,j)$.

In the next place, by (4.1) or (4.3), we have

$$\sum_{k,n=0}^{\infty} S(n,k,j)\frac{x^k y^n}{k!n!} = \frac{e^y}{j!}(e^{x+y} - 1) \ . \tag{4.4}$$

Differentiation with respect to x gives

$$\begin{aligned}\sum_{k,n=0}^{\infty} S(n,k+1,j)\frac{x^k y^n}{k!n!} &= e^{x+y} \cdot \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1} \\ &= \frac{e^y}{(j-1)!}(e^{x+y} - 1)^j + \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1},\end{aligned}$$

so that

$$S(n,k+1,j) = S(n,k,j-1) + jS(n,k,j), \tag{4.5}$$

generalizing the familiar formula

$$S(k+1,j) = S(k,j-1) + jS(k,j).$$

Differentiation of (4.4) with respect to x gives

$$\sum_{k,n=0}^{\infty} S(n+1,k,j) = \frac{e^y}{j!}(e^{x+y} - 1)^j + e^{x+y} \cdot \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1}$$

and, therefore

$$S(n+1,k,j) = S(n,k,j) + S(n,k+1,j). \tag{4.6}$$

This result can be expressed in the form

$$\Delta_n S(n,k,j) = S(n,k+1,j), \tag{4.7}$$

where Δ_n is the partial difference operator. We can also view (4.6) as the analog of (1.7) for $S(k,n,j)$.

Since $S(0,k,j) = S(k,j)$, iteration of (4.6) yields

$$S(n,k,j) = \sum_{i=0}^{n}\binom{n}{i}S(k+i,j). \tag{4.8}$$

We recall that

$$x^k = \sum_{j=0}^{k} S(k,j)x(x-1) \ \ldots \ (x-j+1).$$

Hence, it follows from (4.8) that

$$(x+1)^n x^k = \sum_{j=0}^{n+k} S(n,k,j)x(x-1) \ \ldots \ (x-j+1). \tag{4.9}$$

Replacing x by $-x$, (4.9) becomes

$$(4.10) \qquad (x - 1)^n x^k = \sum_{j=0}^{n+k} (-1)^{n+k-j} S(n,k,j) x(x+1) \ldots (x+j-1).$$

5. To get a combinatorial interpretation of $A_{n,k}$, we recall [4] that A_k is equal to the number of partitions of a set of cardinality n. It is helpful to sketch the proof of this result.

Let $\overline{A}_k$ denote the number of partitions of the set $S_k = \{1, 2, \ldots, k\}$, $k = 1, 2, 3, \ldots$, and put $\overline{A}_0 = 1$. Then $\overline{A}_{k+1}$ satisfies

$$(5.1) \qquad \overline{A}_{k+1} = \sum_{j=0} \binom{k}{j} \overline{A}_j,$$

since the right member enumerates the number of partitions of the set S_{k+1}, as the element $k + 1$ is in a block with $0, 1, 2, \ldots, k$ additional elements. Hence, by (1.2),

$$\overline{A}_k = A_k \qquad (k = 0, 1, 2, \ldots).$$

For $A_{n,k}$ we have the following combinatorial interpretation.

Theorem 1: Put $S = \{1, 2, \ldots, n\}$, $T = \{n+1, n+2, \ldots, n+k\}$. Then, $A_{n,k}$ is equal to the number of partitions of all sets $R \cup T$ as R runs through the subsets (the null set included) of S.

The proof is similar to the proof of (5.1), but makes use of (2.7), that is

$$(5.2) \qquad A_{n,k} = \sum_{j=0}^{n} \binom{n}{j} A_{j+k}.$$

It suffices to observe that the right-hand side of (5.2) enumerates the partitions of all sets obtained as union of T and the various subsets of S.

For $n = 0$, it is clear that (5.2) gives A_k; for $k = 0$, we get A_{n+1}.

The Stirling number $S(k,j)$ is equal to the number of partitions of the set $1, 2, \ldots, k$ into j nonempty sets. The result for $S(n,k,j)$ that corresponds to Theorem 1 is the following.

Theorem 2: Put $S = \{1, 2, \ldots, n\}$, $T = \{n+1, n+2, \ldots, n+k\}$. Then, $S(n,k,j)$ is equal to the number of partitions into j blocks of all sets $R \cup T$ as R runs through the subsets (the null set included) of S.

The proof is similar to the proof of Theorem 1, but makes use of (4.8), that is,

$$(5.3) \qquad S(n,k,j) = \sum_{i=0}^{n} \binom{n}{i} S(k+i,j).$$

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SOME LACUNARY RECURRENCE RELATIONS

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1. INTRODUCTION

Kirkpatrick [4] has discussed aspects of linear recurrence relations which skip terms in a Fibonacci context. Such recurrence relations are called "lacunary" because there are gaps in them where they skip terms. In the same issue of this journal, Berzsenyi [1] posed a problem, a solution of which is also a lacunary recurrence relation. These are two instances of a not infrequent occurrence.

We consider here some lacunary recurrence relations associated with sequences $\{w_n^{(r)}\}$, the elements of which satisfy the linear homogeneous recurrence relation of order r:

$$w_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n-j}^{(r)}, \quad n > r,$$

with suitable initial conditions, where the P_{rj} are arbitrary integers. The sequence, $\{v_n^{(r)}\}$, with initial conditions given by

$$v_n^{(r)} = \begin{cases} 0 & n < 0, \\ \displaystyle\sum_{j=1}^{r} \alpha_{rj}^n & 0 \le n < r \end{cases}$$

is called the "primordial" sequence, because when $r = 2$, it becomes the primordial sequence of Lucas [6]. The α_{rj} are the roots, assumed distinct, of the auxiliary equation

$$x^r = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, x^{r-j}.$$

We need an arithmetical function $\delta(m,s)$ defined by

$$\delta(m,s) = \begin{cases} 1 & \text{if } m \mid s, \\ 0 & \text{if } m \nmid s. \end{cases}$$

We also need $s(r,m,j)$, the symmetric functions of the α_{ri}^m, $i = 1, 2, \ldots, r$, taken j at a time, as in Macmahon [5]:

$$s(r,m,j) = \Sigma \alpha_{ri_1}^m \alpha_{ri_2}^m \ldots \alpha_{ri_j}^m,$$

in which the sum is over a distinct cycle of α_{ri}^m taken j at a time and where we set $s(r,m,0) = 1$.

For example,

$$\begin{aligned}
s(3,m,1) &= \alpha_{31}^m + \alpha_{32}^m + \alpha_{33}^m,\\
s(3,m,2) &= (\alpha_{31}\alpha_{32})^m + (\alpha_{32}\alpha_{33})^m + (\alpha_{33}\alpha_{31})^m,\\
s(3,m,3) &= (\alpha_{31}\alpha_{32}\alpha_{33})^m;\\
s(r,m,1) &= v_m^{(r)},\\
s(r,1,j) &= P_{rj}\\
s(r,m,r) &= P_{rr}^m.
\end{aligned}$$

2. PRIMORDIAL SEQUENCE

Lemma 1: For $m \geq 0$,

$$\sum_{n=0}^{\infty} v_{(n+1)m}^{(r)} x^n = \left(\sum_{j=1}^{n+1} js(r,m,j)(-x)^{j-1}\right) \Big/ \left(\sum_{j=0}^{r} (-1)^j s(r,m,j)x^j\right).$$

Proof: $\displaystyle\sum_{n=0}^{\infty} v_{(n+1)m}^{(r)} x^n = \sum_{n=0}^{\infty}\sum_{i=1}^{r} \alpha_{ri}^{nm+m} x^n$

$$= \sum_{i=1}^{r} \alpha_{ri}^m \sum_{n=0}^{\infty} (\alpha_{ri}^{mn} x)^n = \sum_{i=1}^{r} \alpha_{ri}^m (1 - \alpha_{ri}^m x)^{-1}$$

$$= \sum_{i=1}^{r} \alpha_{ri}^m \prod_{\substack{j=1\\ j\neq 1}}^{r} (1 - \alpha_{rj}^m x) \Big/ \prod_{j=1}^{r} (1 - \alpha_{rj}^m x)$$

$$= \frac{\displaystyle\sum_{i=1}^{r} \alpha_{ri} - \sum_{\substack{j=1\\ j\neq 1}}^{r} \alpha_{ri}^m \alpha_{rj}^m x + \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{r} \alpha_{ri}^m \alpha_{rj}^m \alpha_{rk}^m x^2 - \cdots}{\displaystyle\prod^{r} (1 - \alpha_{rj}^m x)}$$

$$= \frac{s(r,m,1) - 2s(r,m,2)x + 3s(r,m,3)x^2 - \cdots}{\displaystyle\sum_{j=0}^{r} (-1)^j s(r,m,j)x^j}$$

because each α_{ri}, $i = 1, 2, \ldots, j \leq r$ moves through j positions in a complete cycle.

Examples of the lemma when $r = 2$ are obtained by comparing the coefficients of x^n in

$$\sum_{n=0}^{\infty} (-1)^n s(r,m,n)x^n \sum_{i=0}^{\infty} v_{(i+1)m}^{(r)} x^i = \sum_{j=1}^{r+1} js(r,m,j)(-x)^{j-1}$$

x^0: on the left, $s(2,m,0)v_m^{(2)} = v_m^{(2)}$ = right-hand side;

x^1: on the left, $-s(2,m,1)v_m^{(2)} + s(2,m,0)v_{2m}^{(2)} = \alpha_{21}^{2m} + \alpha_{22}^{2m} - (\alpha_{21}^m + \alpha_{22}^m)^2$
$= -2(\alpha_{21}\alpha_{22})^m$,
$= -2s(2,m,2)$
= right-hand side.

We note that

$$[(r + 2)/(j + 2)] = 0 \quad \text{for} \quad j > r \geq 0$$

and

$$r > [(r + 2)/(j + 2)] \quad \text{for} \quad 0 \leq j < r \quad \text{if} \quad r > 2,$$

where [•] represents the greatest integer function.

Theorem 1: The lacunary recurrence relation for $v_n^{(r)}$ for $r \geq 2$ is given by

$$\sum_{n=0}^{\min(r,j)} (-1)^n s(r,m,n) v_{(j-n+1)}^{(r)}$$

$$= (-1)^j (j + 1) s(r,m,j + 1)\ 1 - \delta\ r, [(r + 2)/(j + 2)] \quad \text{for positive } j.$$

Proof: We have from the lemma that

$$\sum_{n=0}^{\infty} (-1)^n s(r,m,n) x^n \sum_{i=0} v_{(i+1)m}^{(r)} x^i = \sum_{j=1}^{r+1} j s(r,m,j)(-x)^{j-1}$$

which can be rearranged to give

$$\sum_{j=0}^{\infty} \sum_{n=0}^{j} (-1)^n s(r,m,n) v_{(j-n+1)m}^{(r)} x^j = \sum_{j=0}^{r} (j + 1) s(r,m,j + 1)(-x)^j.$$

On equating coefficients of x^j, we get

$$\sum_{n=0}^{j} (-1)^n s(r,m,n) v_{(j-n+1)m}^{(r)} = \begin{cases} 0 & \text{if } j > r, \\ (-1)^j (j + 1) s(r,m,j + 1) & \text{if } 0 \leq j \leq r. \end{cases}$$

But

$$\big(1 - \delta(r, [(r + 2)/(j + 2)])\big) = \begin{cases} 0 & \text{for } j > r \\ 1 & \text{for } 0 \leq j < r,\ r > 2, \end{cases}$$

and $0 \leq n < r$ in $s(r,m,n)$ from which we get the required result when $r > 2$, as we exclude negative subscripts for $v_n^{(r)}$.

We next discuss the case for $r = 2$.

When j is unity, we get

$$s(r,m,0) v_{2m}^{(r)} - s(r,m,1) v_m^{(r)} = 2s(r,m,2)$$

which can be reorganized as

$$v_{2m}^{(r)} - (v_m^{(r)})^2 + 2s(r,m,2) = 0.$$

When $r = 2$, this becomes

$$v_{2m}^{(2)} - (v_m^{(2)})^2 + 2P_{22}^m = 0,$$

which is in agreement with Equation (3.16) of Horadam [2].

Similarly, when $j = 2$, we find that for arbitrary r,

$$s(r,m,0) v_{3m}^{(r)} - s(r,m,1) v_{2m}^{(r)} + s(r,m,2) v_m^{(r)} = 3s(r,m,4)$$

or

$$v_{3m}^{(r)} - v_m^{(r)} v_{2m}^{(r)} + s(r,m,2) v_m^{(r)} = 3s(r,m,4),$$

which, when $r = 2$, becomes

$$v_{3m}^{(2)} - v_m^{(2)} v_{2m}^{(2)} + P_{22}^m v_m^{(2)} = 0,$$

and this also agrees with Equation (3.16) of Horadam if we put $n = 2m$ and $w_m^{(2)} = v_m^{(2)}$ there. Thus, the theorem also applies when $r = 2$ if $j \geq 1$. If j were zero, and $r = 2$, since $\delta(2,[4/2]) = 1$, the theorem would reduce to

$$s(r,m,0)v_m^{(2)} = 0,$$

which is false.

Corollary 1: $v_{km}^{(r)} = \sum_{n=1}^{r} (-1)^{n+1} s(r,m,n) v_{(k-n)m}^{(r)}.$

Proof: Put $j = k - 1 > r$ in the theorem and we get

$$\sum_{n=0}^{r} (-1)^n s(r,m,n) v_{(k-n)m}^{(r)} = 0$$

which gives

$$\sum_{n=1}^{r} (-1)^{n+1} s(r,m,n) v_{(k-n)m}^{(r)} = v_{km}^{(r)} .$$

A particular case of the corollary occurs when $m = 1$, namely

$$\begin{aligned} v_k^{(r)} &= \sum_{n=1}^{r} (-1)^{n+1} s(r,1,n) v_{k-n}^{(r)} \\ &= \sum_{n=1}^{r} (-1)^{n+1} P_{rn} v_{k-n}^{(r)}, \end{aligned}$$

as we would expect.

The recurrence relation in Theorem 1 has gaps; for instance, there are missing numbers between $v_{(j+1)m}^{(r)}$ and $v_{jm}^{(r)}$. When $j = m = 2$, the lacunary recurrence relation becomes

$$\begin{aligned} &v_6^{(r)} - s(r,2,1)v_4^{(r)} + s(r,2,2)v_2^{(r)} - s(r,2,3)v_0^{(r)} \\ &= 3s(r,2,3)\bigl(1 - \delta(r,[(r+2)/4])\bigr), \end{aligned}$$

and the numbers $v_1^{(r)}$, $v_3^{(r)}$, and $v_5^{(r)}$ are missing. For further discussion of lacunary recurrence relations, see Lehmer [5]. The lacunary recurrence relations can be used to develop formulas for $v_n^{(r)}$.

3. GENERALIZED SEQUENCE

In this section we consider the more generalized sequence $\{w_n^{(r)}\}$.

Theorem 2: $w_{tn}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} s(r,t,j) w_{t(n-j)}^{(r)}, \quad n > r.$

Proof: Put

$$w_n^{(r)} = \sum_{j=1}^{r} A_j \alpha_{rj}^n$$

in which the A_j will be determined by the initial values of $\{w_{rj}^{(r)}\}$.

$$\sum_{j=1}^{r}(-1)^{j+1}s(r,t,j)w_{t(n-j)}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}s(r,t,j)\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-t}$$

$$= \sum_{j=1}^{r}\alpha_{rj}^{t}\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-t} - \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}\alpha_{rj}^{t}\alpha_{rk}^{t}\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-2t}$$

$$+ \cdots + (-1)^{r+1}(\alpha_{r1}^{t}\alpha_{r2}^{t}\ \ldots\ \alpha_{rr}^{t})\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-rt}$$

$$= \sum_{j=1}^{r}A_j\alpha_{rj}^{tn} + \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}A_j\alpha_{rj}^{tn-t}\alpha_{rk}^{t} - \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}A_j\alpha_{rj}^{tn-t}\alpha_{rk}^{t}$$

$$- \sum_{\substack{i,\,j,\,k=1\\ i\neq j\neq k}}^{r}A_i\alpha_{ri}^{tn-2t}\alpha_{rj}^{t}\alpha_{rk}^{t} + \cdots$$

$$= \sum_{j=1}^{r}A_j\alpha_{rj}^{tn} = w_{tn}^{(r)},$$

as required.

When $t = r = 2$, we have $s(2,2,1) = 3$ and $s(2,2,2) = 1$, so that if $w_n^{(2)} = F_n$, the nth Fibonacci

$$F_{2n} = 3F_{2n-2} - F_{2n-4},$$

which result has been used by Rebman [8] and Hilton [2] in their combinatorial studies. There, too, the result

$$n = \sum_{\gamma(n)}(-1)^{k-1}F_{2a_1}F_{2a_2}\ \ldots\ F_{2a_k}$$

was useful.

[$\gamma(n)$ indicates summation over all compositions $(a_1, \ldots, a_k)$ of n, the number of components being variable.] The lacunary generalization of this result can be expressed as

Theorem 3: $W_n^{(r)} = \sum_{\gamma(n)}(-1)^{k-1}w_{ta_1}^{(r)}\ \ldots\ w_{ta_k}^{(r)}$, in which

$$W_n^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}\{s(r,t,j) + h_j\}W_{j-n}^{(r)},\ n > r,$$

where

$$h_j = \sum_{m=1}^{j}(-1)^m s(r,t,j-m)w_{tm}^{(r)}.$$

That the theorem generalizes the result can be seen if we let $r = 2$, $t = 1$, and $w_n^{(2)} = F_n$ again. Then, as before,

$$F_{2n} = 3F_{2n-2} - F_{2n-4}$$

and

$$W_n^{(2)} = \sum_{j=1}^{2}(-1)^{j+1}\{s(2,2,j) + h_j\}W_{n-j}^{(2)}$$

$$= \{s(2,2,2) + h_1\}W_{2-1}^{(2)} - \{s(2,2,2) + h_2\}W_{n-2}^{(2)}$$

$$= \{s(2,2,1) - s(2,2\ 0)F_2\}W_{n-1}^{(2)} - \{s(2,2,2) - s(2,2,1) + s(2,2,0)F_4\}W_{n-2}^{(2)}$$
$$= (3 - 1)_{n-1}^{(2)} - (1 - 3 + 3)W_{n-2}^{(2)} = 2W_{n-1}^{(2)} - W_{n-2}^{(2)};$$

i.e., $W^{(2)} = n$ as in the result.

To prove Theorem 3, we need the following lemmas.

Lemma 3.1: $W(x) = w(x)/(1 + w(x))$, where

$$W(x) = \sum_{n=1}^{\infty} W_n^{(r)} x^n \quad \text{and} \quad w(x) = \sum_{n=1}^{\infty} w_{tn}^{(r)} x^n.$$

Proof:

$$W(x) = \sum_{n=1}^{\infty} W_n^{(r)} x^n$$
$$= \sum_{n=1}^{\infty}\left(\sum_{\Upsilon(x)} (-1)^{k-1} w_{ta_1}^{(r)} \ldots w_{ta_k}^{(r)}\right) x^n$$
$$= \sum_{k=1}^{\infty} - \left(-\sum_{n=1}^{\infty} w_{tn}^{(r)} x^n\right)^k$$
$$= \sum_{k=1}^{\infty} - (-w(x))^k$$
$$= w(x)/(1 + w(x)).$$

Lemma 3.2: If $f(x) = \sum_{j=0}^{r} (-1)^{r-j} s(r,t,j)x^j$,

and

$$h(x) = \sum_{j=1}^{r} (-1)^{r-j} h_j x^j,$$

where

$$h(x) = f(x)w(x),$$

then

$$h_j = \sum_{m=1}^{j} (-1)^m s(r,t,j - m)w_{tm}^{(r)}.$$

Proof: If $h(x) = f(x)w(x)$,

then

$$W(x) = f(x)w(x)/(f(x) + f(x)w(x)) = h(x)/(f(x) + h(x)),$$

so that

$$h(x) = (f(x) + h(x))W(x).$$

Now

$$h(x) = \sum_{m=1}^{\infty} w_{tn}^{(r)} x^n \sum_{j=0}^{r} (-1)^{r-j} s(r,t,j)x^j$$
$$= \sum_{j=1}^{r}\left(\sum_{m=1}^{j} (-1)^{r-j+m} s(r,t;j - m)w_{tm}^{(r)}\right)x^j$$
$$+ \sum_{j=1}^{\infty}\left(\sum_{m=0}^{r} (-1)^m s(r,t,r - m)w_{(j+m)}^{(r)}\right)x^{r+j}$$

$$= \sum_{j=1}^{r} (-1)^{r-j}\left(\sum_{m=1}^{j}(-1) \quad s(r,t,j-m)w^{(r)}\right)x^{j}$$

from Theorem 2. The result follows when the coefficients of x are equated. Thus,

$$f(x) + h(x) = \sum_{j=1}^{r}(-1)^{r-j}\{s(r,t,j) + h_j\}x^j + 1.$$

And since

$$h(x) = \big(f(x) + h(x)\big)w(x),$$

Theorem 3 follows.

Shannon and Horadam [10] have looked at the development of second-order lacunary recurrence relations by using the process of multisection of series. The same approach could be used here. Riordan [9] treats the process in more detail.

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ANOTHER PROOF THAT $\phi(F_n) \equiv 0$ MOD 4 FOR ALL $n > 4$

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1. INTRODUCTION AND DISCUSSION

The problem, as originally proposed by Douglas Lind [1], was as follows:

If F_n is the nth Fibonacci number, then show that

$$\phi(F_n) \equiv 0 \pmod 4,\ n > 4, \text{ where } \phi(n) \text{ is Euler's } \phi\text{-function.}$$

An incomplete solution due to John L. Brown, Jr., appeared in [2]. The problem resurfaced in Problem E 2581, proposed by Clark Kimberling [3]. An extremely elegant solution was given by Peter Montgomery [4].

The main object of this note is to provide another solution to the original problem cited and some generalizations [5]. However, before giving our solution, we cannot resist redocumenting Montgomery's simple and beautiful solution:

Consider the set $H = \{-F_{n-1}, -1, +1, F_{n-1}\}$. The first observation is that the elements of this set are pairwise incongruent modulo F_n. Only four of the $\binom{4}{2}$ incongruences to be checked are distinct, and three of these four are trivialities. The most interesting of these is $F_{n-1} \not\equiv -F_{n-1} \pmod{F_n}$, which can easily be done by showing that $F_n < 2F_{n-1} < 2F_n$ so that $F_n \mid 2F_{n-1}$ is impossible. Second, since $(F_n, F_{n-1}) = 1$, the set H is a subset of $(Z/F_nZ)^*$, the multiplicative group (under multiplication modulo F_n) of units of the ring Z/F_nZ (see S. Lang [6]). Finally, since $F_{n-1}^2 - F_{n-2}F_n = (-1)^n$, it follows that H is closed under multiplication and hence (being finite) is a subgroup of $(Z/F_nZ)^*$. However, the order of $(Z/F_nZ)^*$ is $\phi(F_n)$, and the order of subgroup H is 4, so that the conclusion follows from Lagrange's Theorem: "The order of a subgroup of a finite group divides the order of the group." The basic ideas of Montgomery's proof have been extended to generalized Fibonacci numbers satisfying $u_{n+1}u_{n-1} - u_n^2 = \pm 1$ in [5].

2. ANOTHER PROOF

Our proof breaks up into two parts. The first part characterizes those positive integers m for which $4 \nmid \phi(m)$. The second part shows that $F_n \neq m$, whenever $n > 4$. $\phi(1) = \phi(2) = 1$, and $2 \mid \phi(m)$ for all positive integers $m \geq 3$, so that the first part of our proof amounts to characterizing those positive integers m for which $2 \parallel \phi(m)$ [i.e., $2 \mid \phi(m)$ but $2^2 \nmid \phi(m)$]. If the canonical decomposition of m is

$$m = p_1^{e_1} p_2^{e_2} \ldots p_g^{e_g},$$

then

$$\phi(m) = p_1^{e_1-1} p_2^{e_2-1} \ldots p_g^{e_g-1}(p_1 - 1)(p_2 - 1) \ldots (p_g - 1),$$

where $2 \leq p_1 < p_2 < \ldots < p_g$ and $p_1, p_2, \ldots, p_g$ are primes.

If $p_1 = 2$, then $m = 2^{e_1} p_2^{e_2} \ldots p_g^{e_g}$, and

$$\phi(m) = 2^{e_1-1} p_2^{e_2-1} p_3^{e_3-1} \ldots p_g^{e_g-1}(2 - 1)(p_2 - 1)(p_3 - 1) \ldots (p_g - 1).$$

This requirement forces $1 \leq e_1 < 2$. If $e_1 = 2$, then $g = 1$ is forced and m must be 4. If $e_1 = 1$, then

$$\phi(m) = p_2^{e_2-1}p_3^{e_3-1} \ldots p_g^{e_g-1}(p_2 - 1)(p_3 - 1) \ldots (p_g - 1)$$

so that $g = 2$ is forced, and $m = 2p^e$ for some odd prime p and some positive integer e. Furthermore, $p \equiv 3 \pmod 4$ must obtain. If $p_1 > 2$, we must have $g = 1$ so that $m = p^e$, where the conditions on p and e are precisely as above. Summarizing, we have shown that $4 \nmid \phi(m)$ if and only if $m = 1, 2, 4p^e$, or $2p^e$, where p is any prime satisfying $p \equiv 3 \pmod 4$ and e is any positive integer.

If now suffices to prove that $F_n \neq 1, 2, 4p^e$, or $2p^e$ whenever $n > 4$, where p is a prime such that $p \equiv 3 \pmod 4$ and e is a positive integer.

Case 1: $F_n = p \equiv 3 \pmod 4$, p a prime, is impossible if $n > 4$.

If n is even, then $n \geq 6$ and $F_n = F_{2k} = F_kL_k$, where $k \geq 3$. Since $F_k > 1$ and $L_k > 1$ whenever $k \geq 3$, it follows that F_n is composite.

If n is odd, then $F_n = F_{2k+1} = F_k^2 + F_{k+1}^2 \not\equiv 3 \pmod 4$.

Case 2: $F_n = 2p$ with $p \equiv 3 \pmod 4$ and p a prime is impossible.

If $n > 4$, $F_6 = 8$ is not of the prescribed form. If n is even and $n \geq 8$, then $F_n = F_{2k}L_k = 2p$ is impossible since $k \geq 4$ forces $F_k > 2$ and $L_k > 2$. If n is odd, then $F_n = 2p = F_{2k+1} = F_{6r+3}$ because $2|F_n$ if and only if $3|n$. Hence, $F_{2r+1}|F_{6r+3} = 2p$ since $2r + 1|6r + 3$. $F_9 = 34 = 2 \cdot 17$, but $17 \not\equiv 3 \pmod 4$. Otherwise, $2 < F_{2r+1} < F_{6r+3}$ and $F_{2r+1} \neq p$ by Case 1, and so Case 2 is complete.

Case 3: $F_n = p^e$ with $p \equiv 3 \pmod 4$ and p a prime is impossible.

If $n > 4$, then we may assume that the positive integer e is greater than one, because of Case 1. If n is even, then $F_n = F_{2k} = F_kL_k$ with $(F_k, L_k) = 1$ or 2, a contradiction. If n is odd, then $F_n = F_{2k+1}$ and $2k + 1 \equiv 3 \pmod 6$, since we cannot tolerate $2|F_n$. Hence, $2k + 1 \equiv \pm 1 \pmod 6$ must obtain, which forces $F_n \equiv 1 \pmod 4$, and so $2|e$. However, the only Fibonacci squares are $F_1 = F_2 = 1$ and $F_{12} = 144$, and so Case 3 is complete.

Case 4: $F_n = 2p^e$ with $p \equiv 3 \pmod 4$, p a prime, is impossible.

By Case 2, we can assume $e > 1$. Since $2|F_n$, we must have $3|n$, and so $F_n = F_{3k} = 2p^e$. If $2|k$, then $6|n$, and hence $8 = F_6|F_n$, a contradiction, so $k = 2r + 1$, and $F_{2r+1}|F_{6r+3} = F_{3k} = F_n = 2p^e \equiv 2 \pmod 4$. $F_{2r+1} \neq 2$, once $r > 1$. $F_{2r+1} \neq p$, by Case 1; $F_{2r+1} \neq 2p$, by Case 2; and $F_{2r+1} \neq p^t$ for any integer t such that $0 \leq t \leq e$, by Case 3; so $F_{2r+1} = 2p^s$ is forced for some positive integer $s < r$. Let r be the least subscript for which F_{2r+1} is of this form. Since $2|F_{2r+1}$, $F_{2r+1} = F_{6n+3}$ for some suitable positive integer n. Thus, $F_{2r+1} = F_{6n+3} = 2p^s$, and $F_{2n+1}|F_{6n+3} = 2p^s$. But now $F_{2n+1} = 2p^t$ for suitable positive integral t is forced, contradicting the minimal nature of subscript r. The proof of Case 4, and with it the solution to the original problem, is complete.

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LETTER TO THE EDITOR

DAVID L. RUSSELL
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Dear Professor Hoggatt:

. . . In response to your request for me to point out the errors in your article "A Note on the Summation of Squares," *The Fibonacci Quarterly* 15, No. 4 (1977):367-369, . . . I have enclosed a xerox copy of your paper with corrections marked. The substantive errors occur in the top two equations of p. 369, where an incorrect sign and some minor errors result in an incorrect denominator for the RHS. As an example, consider the case $p = 1$, $q = 2$, $n = 4$; your formula evaluates to 0, which is clearly incorrect:

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1,\ P_3 = 3,\ P_4 = 5,\ P_5 = 11,\ P_6 = 21;$$
$$8P_5P_4 - (P_6^2 - 1) = (8)(11)(5) - 440 = 0.$$

Only if the denominator is also zero does a numerator of zero make sense.

Sincerely yours,
[*David L. Russell*]

CORRECTIONS TO "A NOTE ON THE SUMMATION OF SQUARES" BY VERNER E. HOGGATT, JR.

The following corrections to the above article were noted by Prof. David L. Russell.

<u>Page 368</u>: The equation on line 19, $q^{n-1}P_2P_1 = q^{n-1}P_1^2 + q^nP_1P_0$, should be:

$$q^nP_2P_1 = q^nP_1^2 + q^{n+1}P_1P_0$$

The equation on line 27, $P_{j+2}^2 = P^2P_{j+1}^2 + q^2P_j^2 + 2pqP_jP_{j+1}$, should be:

$$P_{j+2}^2 = p^2P_{j+1}^2 + q^2P_j^2 + 2pqP_jP_{j+1}$$

In the partial equation on line 32 (last line) the = sign should be a - (minus) sign.

<u>Page 369</u>: Lines 1-11 should read:

$$pP_{n+1}^2 + \left(\sum_{j=1}^{n} P_j^2\right)\left(p + \frac{(1-q)(p^2+q^2-1)}{2pq}\right)$$
$$= P_{n+2}P_{n+1} + \frac{1-q}{2pq}[P_{n+2}^2 + P_{n+1}^2 - 1 - p^2P_{n+1}^2]$$

$$\sum_{j=1}^{n} P_j^2 = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2 + \frac{(1-q)}{2pq}[P_{n+2}^2 + P_{n+1}^2(1-p^2) - 1]}{(2p^2q + p^2 + q^2 - 1 - qp^2 - q^3 + q)/2pq}$$

Testing $p = 1$, $q = 1$,

$$\sum_{i=1}^{n} F_i^2 = \frac{2F_{n+2}F_{n+1} - 2F_{n+1}^2}{2} = F_{n+1}F_n.$$

For $q = 1$ only,

$$\sum_{i=1}^{n} P_i^2 = \frac{2pP_{n+2}P_{n+1} - 2p^2P_{n+1}^2}{2p^2} = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2}{p} = \frac{P_{n+1}P_n}{p}$$

so that

$$\sum_{i=1}^{n} P_i^2 = P_{n+1}P_n/p.$$

Thus,

$$\sum_{j=1}^{n} P_j^2 = \frac{p\left[2qP_{n+2}P_{n+1} - 2pqP_{n+1}^2 + \frac{(1-q)}{p}[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]\right]}{(q+1)(p^2 - (q-1)^2)}$$

$$= \frac{p\left[2q^2\left(P_{n+1}P_n\right) + \frac{(1-q)}{p}[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]\right]}{(q+1)(p^2 - (q-1)^2)}.$$

According to Prof. Russell, this last equation can also be written as

$$\sum_{j=1}^{n} P_j^2 = \frac{2pq^2P_{n+1}P_n + (1-q)[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{(q+1)(p^2 - (q-1)^2)}$$

$$= \left[\frac{2pqP_nP_{n+1} + (1-q)P_{n+1}^2 + q^2(1-q)P_n^2}{(q+1)(p^2 - (q-1)^2)}\right]_0^n,$$

since $P_{n+2}^2 = p^2P_{n+1}^2 + 2pqP_nP_{n+1} + q^2P_n^2$.

The author is grateful to Prof. Russell for the above corrections.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-418 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $n^{15} - n^3$ is an integral multiple of $2^{15} - 2^3$ for all integers n.

B-419 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For i in $\{1, 2, 3, 4\}$, establish a congruence

$$F_n L_{5k+i} \equiv a_i n L_n F_{5k+i} \pmod 5$$

with each a_i in $\{1, 2, 3, 4\}$.

B-420 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let

$$g(n, k) = F^4_{n+10k} + F^4_n - (L_{4k} + 1)(F^4_{n+8k} + F^4_{n+2k}) + L_{4k}(F^4_{n+6k} + F^4_{n+4k}).$$

Can one express $g(n, k)$ in the form $L_r F_s F_t F_u F_v$ with each of r, s, t, u, and v linear in n and k?

B-421 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $\{u_n\}$ be defined by the recursion $u_{n+3} = u_{n+2} + u_n$ and the initial conditions $u_1 = 1$, $u_2 = 2$, and $u_3 = 3$. Prove that every positive integer N has a unique representation

$$N = \sum_{i=1}^{n} c_i u_i, \text{ with } c_n = 1, \text{ each } c_i \in \{0, 1\},$$

$$c_i c_{i+1} = 0 = c_i c_{i+2} \quad \text{if} \quad 1 \le i \le n - 2.$$

B-422 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

With representations as in B-421, let

$$N = \sum_{i=1}^{n} c_i u_i, \quad N + 1 = \sum_{i=1}^{m} d_i u_i.$$

Show that $m \geq n$ and that if $m = n$ then $d_k > c_k$ for the largest k with $c_k \neq d_k$.

B-423 *Proposed by Jeffery Shallit, Palo Alto, CA*

Here let F_n be denoted by $F(n)$. Evaluate the infinite product

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{13}\right)\left(1 + \frac{1}{610}\right) \cdots = \prod_{n=1}^{\infty}\left[1 + \frac{1}{F(2^{n+1} - 1)}\right].$$

SOLUTIONS

Note by Paul S. Bruckman, Concord, CA:

There is an omission in the published solution to B-371 (Feb. 1979, p. 91). The set of residues (mod 60) should include 55 and consists of 24 elements.

Triple Products and Binomial Coefficients

B-394 *Proposed by Phil Mana, Albuquerque, NM*

Let $P(x) = x(x - 1)(x - 2)/6$. Simplify the following expression:

$$P(x + y + z) - P(y + z) - P(x + z) - P(x + y) + P(x) + P(y) + P(z).$$

I. *Solution by C. B. A. Peck, State College, PA*

Let $G(x, y, z)$ denote the given expression. Clearly,

$$G(0, y, z) = G(x, 0, z) = G(x, y, 0) = 0.$$

Since the total degree of G in x, y, z is at most 3, this means that

$$G = kxyz, \text{ with } k \text{ constant.}$$

Then $G(1, 1, 1) = 1$ implies that $k = 1$ and $G = xyz$.

II. *Generalization by L. Carlitz, Duke University, Durham, NC*

We shall prove the following more general result. Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

be an arbitrary polynomial of degree $\leq n$ and put

$$\begin{aligned} S_n &= S_n(x_1, x_2, \ldots, x_n) \\ &= f(x_1 + x_2 + \cdots + x_n) - \Sigma f(x_1 + \cdots + x_{n-1}) \\ &\qquad + \Sigma f(x_1 + \cdots + x_{n-2}) - \cdots + (-1)^n f(0), \end{aligned}$$

where the first sum is over all sums of $n - 1$ of the x_j, the second over all sums of $n - 2$ of the x_j, etc. Then

$$(*) \qquad S_n = a_0 n! x_1 x_2 \cdots x_n.$$

Proof: Put

$$S(n,k;x_1, \ldots, x_k) = (x_1 + \cdots + x_k)^n - \Sigma(x_1 + \cdots + x_{k-1})^n + \Sigma(x_1 + \cdots + x_{k-2})^n - \cdots,$$

where the summations have the same meaning as in the definition of S_n, except that we now have k indeterminates.

It follows from the definition that

$$(**) \qquad \sum_{n=0}^{\infty} S(n,k;x_1, \ldots, x_k)\frac{z}{n!}$$

$$= e^{(x_1 + \cdots + x_k)z} - \Sigma e^{(x_1 + \cdots + x_{k-1})z} + \Sigma e^{(x_1 + \cdots + x_{k-2})z}$$

$$= (e^{x_1 z} - 1)(e^{x_2 z} - 1) \cdots (e^{x_k z} - 1).$$

Hence, comparing coefficients of z^n, we get

$$S(n,k;x_1, \ldots, x_k) = \begin{cases} 0 & (0 \le n < k) \\ k!x_1 \cdots x_k & (n = k) \end{cases}$$

The assertion (*) is an immediate consequence.

For $n = 3$, $a = \frac{1}{6}$, (*) reduces to the required result.

Remark: For $x_1 = \cdots = x_k = 1$, it is clear that (**) reduces to

$$\sum_{n=0}^{\infty} S(n,k;1, \ldots, 1)\frac{z^n}{n!} = (e^z - 1)^k.$$

Hence,

$$S(n,k;1, \ldots, 1) = k!S(n,k),$$

where

$$S(n,k) = \frac{1}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

a Stirling number of the second kind.

Also solved by Mangho Ahuja, Paul S. Bruckman, Herta T. Freitag, Graham Lord, John W. Milsom, Charles B. Shields, Sahib Singh, Gregory Wulczyn, and the proposer.

Reciprocals of Golden Powers

B-395 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $c = (\sqrt{5} - 1)/2$. For $n = 1, 2, \ldots,$ prove that

$$1/F_{n+2} < c^n < 1/F_{n+1}.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

Since $c = \frac{1}{a}$, it suffices to show that $F_{n+2} > a^n > F_{n+1}$. Consider

$$F_{n+2} - a^n = \frac{a^{n+2} - b^{n+2}}{a - b} - a^n(a + b) = b^2 F_n.$$

Similarly, $a^n - F_{n+1} = -bF_n$.

Since b is negative, the conclusion follows.

Also solved by Mangho Ahuja, Clyde A. Bridger, Paul S. Bruckman, Herta T. Freitag, Graham Lord, C. B. A. Peck, Bob Prielipp, E. D. Robinson, Charles B. Shields, Lawrence Somer, and the proposer.

Multiples of Ten

B-396 *Based on the solution to B-371 by Paul S. Bruckman, Concord, CA*

Let $G_n = F_n(F_n + 1)(F_n + 2)(F_n + 3)/24$. Prove that 60 is the smallest positive integer m such that $10|G_n$ implies $10|G_{n+m}$.

Solution by Paul S. Bruckman, Concord, CA

In B-371, it was shown that $10|G_n$ iff $n \equiv r \pmod{60}$, where r is any of 24 possible given residues (mod 60). Thus,

$$n \equiv r \pmod{60} \iff 10|G_n \Rightarrow 10|G_n \iff n + m \equiv r \pmod{60},$$

or, equivalently,

$$n \equiv r \pmod{60} \Rightarrow n + m \equiv r \pmod{60} \Rightarrow m \equiv 0 \pmod{60} \Rightarrow 60\ m.$$

Clearly, the smallest m with this property is $m = 60$, since *any* multiple of 60 (including 60 itself) has the property. See note after B-423.

Also solved by C. B. A. Peck, Sahib Singh, Lawrence Somer, & Gregory Wulczyn.

Semi-Closed Form

B-397 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Find a closed form for the sum

$$\sum_{k=0}^{2s} \binom{2s}{k} F_{n+kt}^2.$$

Note: The proposer intended t to be odd but this condition was inadvertently omitted by the elementary problems editor. The solution which follows gives a closed form for t even and for t odd.

Solution by Paul S. Bruckman, Concord, CA

Let

$$\theta_{2s,n,t} = \sum_{k=0}^{2s} \binom{2s}{k} F_{n+kt}^2.$$

Then

$$\theta_{2s,n,t} = \frac{1}{5} \sum_{k=0}^{2s} \binom{2s}{k} \left\{a^{2n+2kt} - 2(-1)^{n+kt} + b^{2n+2kt}\right\}$$

(continued)

$$= \frac{1}{5}\left\{a^{2n}(1 + a^{2t})^{2s} - 2(-1)^n\{1 + (-1)^t\}^{2s} + b^{2n}(1 + b^{2t})^{2s}\right\},$$

or

$$\text{(1)} \qquad \theta_{2s,n,t} = \frac{1}{5}\left\{a^{2n+2st}(a^t + a^{-t})^{2s} + b^{2n+2st}(b^t + b^{-t})^{2s} - 2^{2s+1}(-1)^n\left(\frac{1 + (-1)^t}{2}\right)\right\}$$

We may distinguish two cases, in order to further simplify (1):

$$\theta_{2s,n,2u} = \frac{1}{5}\{(a^{2n+4su} + b^{2n+4su})(a^{2u} + b^{2u})^{2s} - (-1)^n 2^{2s+1}\},$$

or

$$\text{(2)} \qquad \theta_{2s,n,2u} = \frac{1}{5}(L_{2n+4su}L_{2u}^{2s} - (-1)^n 2^{2s+1});$$

also

$$\theta_{2s,n,2u+1} = \frac{1}{5}(a^{2n+2s(2u+1)} + b^{2n+2s(2u+1)})(a^{2u+1} - b^{2u+1})^{2s},$$

or

$$\text{(3)} \qquad \theta_{2s,n,2u+1} = 5^{s-1}L_{2n+2s(2u+1)}F_{2u+1}^{2s}.$$

Also solved for t odd by the proposer.

The Added Ingredient

B-398 *Proposed by Herta T. Freitag, Roanoke, VA*

Is there an integer K such that

$$K - F_{n+6} + \sum_{j=1}^{n} j^2 F_j$$

is an integral multiple of n for all positive integers n?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

According to (17) on p. 215 of the October 1965 issue of this journal,

$$\sum_{k=0}^{n} k^2 F_k = (n^2 + 2)F_{n+2} - (2n - 3)F_{n+3} - 8.$$

Since $F_{n+6} = 3F_{n+3} + 2F_{n+2}$, it follows that

$$8 - F_{n+6} + \sum_{j=1}^{n} j^2 F_j = n(nF_{n+2} - 2F_{n+3})$$

where n is an arbitrary positive integer.

Also solved by Paul S. Bruckman, Sahib Singh, Gregory Wulczyn, and the proposer.

Not Quite Tribonacci

B-399 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $f(x) = u_1 + u_2x + u_3x^2 + \cdots$ and $g(x) = v_1 + v_2x + v_3x^2 + \cdots$ where $u_1 = u_2 = 1$, $u_3 = 2$, $u_{n+3} = u_{n+2} + u_{n+1} + u_n$, and $v_{n+3} = v_{n+2} + v_{n+1} + v_n$. Find initial values v_1, v_2, and v_3 so that $e^{g(x)} = f(x)$.

I. *No such series exists.*
Demonstration by Jonathan Weitsman, College Station, TX

The equation $e^{g(x)} = f(x)$ leads to $v_1 = 0$, $v_2 = 1$, $v_3 = 3/2$, and $v_4 = 7/3$. These values contradict the given recursion for the v's.

II. *Correction and solution by Paul S. Bruckman, Concord, CA*

There is an error in the statement of the problem. One correct rewording would be to replace "$g(x)$" where it *first* occurs by "$g'(x)$".

Note that $u_n = T_{n+2}$, where $(T_n)_{n=0}^{\infty} = (0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots)$ is the Tribonacci sequence; also $f(x) = (1 - x - x^2 - x^3)^{-1}$, the well-known generating function for the Tribonacci numbers.

Since the v_n's satisfy the same recursion, it follows that

$$g'(x) = p(x)f(x),$$

where p is some quadratic polynomial. But, if we are to have

$$f(x) = \exp(g(x)),$$

then $g(x) = \log(f(x))$, and $g'(x) = f'(x)/f(x)$. Hence,

$$p(x) = f'(x)/f^2(x) = -\frac{d}{dx}(1/f(x)) = -\frac{d}{dx}(1 - x - x^2 - x^3) = 1 + 2x + 3x^2.$$

Therefore,

$$g'(x) = \frac{1 + 2x + 3x^2}{1 - x - x^2 - x^3} \quad \text{and} \quad g(x) = c - \log(1 - x - x^2 - x^3),$$

for some constant c. Since $g(0) = \log(f(0)) = \log 1 = 0 = c - \log 1 = c - 0 = c$, thus $g(x) = -\log(1 - x - x^2 - x^3)$.

Now $g'(x) = p(x)f(x) = (1 + 2x + 3x^2)\sum_{n=0}^{\infty} T_{n+2}x^n$; hence,

$$g'(x) = \sum_{n=0}^{\infty} T_{n+2}x^n + 2\sum_{n=1}^{\infty} T_{n+1}x^n + 3\sum_{n=2}^{\infty} T_n x^n$$
$$= \sum_{n=0}^{\infty} (T_{n+2} + 2T_{n+1} + 3T_n)x^n = \sum_{n=0}^{\infty} v_{n+1}x^n,$$

which implies $v_{n+1} = T_{n+2} + 2T_{n+1} + 3T_n$, $n = 0, 1, 2, \ldots$. Thus, $v_1 = 1 + 0 + 0 = 1$; $v_2 = 1 + 2 + 0 = 3$; and $v_3 = 2 + 2 + 3 = 7$. The first few terms of the series for $g(x)$ are as follows:

$$g(x) = -\log(1 - x - x^2 - x^3) = \frac{1x}{1} + \frac{3x^2}{2} + \frac{7x^3}{3} + \frac{11x^4}{4} + \frac{21x^5}{5} + \frac{39x^6}{6} + \cdots .$$

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS

H-311 *Proposed by Paul Bruckman, Corcord, CA*

Let a and b be relatively prime positive integers such that ab is not a perfect square. Let $\theta_0 = \sqrt{b/a}$ have the continued fraction expansion

$$[u_1, u_2, u_3, \ldots],$$

with convergents p_n/q_n $(n = 1, 2, \ldots)$; also, define $p_0 = 1$, $q_0 = 0$, $p_{-1} = 0$. The process of finding the sequence $(u_n)_{n=1}^{\infty}$ may be described by the recursions:

$$\theta_n = u_{n+1} + 1/\theta \quad = \frac{\sqrt{ab} + r_n}{d_n}, \tag{1}$$

where $r_0 = 0$, $d_0 = a$, $0 < \theta_n < 1$,

r_n and d_n are positive integers, $n = 1, 2, \ldots$.

Prove:

$$r_n = (-1)^{n-1}(ap_n p_{n-1} - bq_n q_{n-1}); \tag{2}$$

$$d_n = (-1)^n (ap_n^2 - bq_n^2), \; n = 0, 1, 2, \ldots . \tag{3}$$

H-312 *Proposed by L. Carlitz, Duke University, Durham, NC*

Let m, r, and s be nonnegative integers. Show that

$$\sum_{j,k} (-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s}\frac{m!}{(m-j)!\,(m-k)!\,(j+k-m)!} = (-1)^{m-r}\binom{m}{r}\delta_{rs}, \tag{*}$$

where

$$\delta_{rs} = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s). \end{cases}$$

SOLUTIONS

Who's Who?

H-281 *Proposed by V. E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 16, No. 2, April 1978)*

Consider the matrix equation:

(a) $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & G_n \\ H_n & I_n & J_n \end{pmatrix} \quad (n \geq 1).$$

Identify A_n, B_n, C_n, ..., J_n.

Consider the matrix equation:

(b) $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} A'_n & B'_n & C'_n \\ D'_n & E'_n & G'_n \\ H'_n & I'_n & J'_n \end{pmatrix} \quad (n \geq 1).$$

Identify A'_n, B'_n, C'_n, ..., J'_n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

(a) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

From the symmetry and other properties of A, it follows that A^n has the form

$$\begin{pmatrix} a+1 & b & a \\ b & 2a+1 & b \\ a & b & a+1 \end{pmatrix}.$$

Hence, A^n is determined when the 1st row of A^n is known; also if the 1st row of A^j is (x,y,z), then the 1st row of A^{j+1} will be $(x+y, x+y+z, y+z)$.

For the first five *odd* positive integers, we have the following entries in the 1st row:

Left-hand Entry $u_k + 1$	Middle Entry v_k	Right-hand Entry u_k
1	1	0
4	5	3
21	29	20
120	169	119
697	985	696

We observe that in each case

$$u_k^2 + (u_k + 1)^2 = v_k^2,$$

so $(u_k, u_k + 1, v_k)$ is a (primitive) Pythagorean triple. It is known that Pythagorean triples of the type indicated above are given by

$$u_{k+1} = 6u_k - u_{k-1} + 2 \quad \text{where } u_1 = 0 \text{ and } u_2 = 3$$

and

$$v_{k+1} = 6v_k - v_{k-1} \quad \text{where } v_1 = 1 \text{ and } v_2 = 5.$$

[See Osborne, "A Problem in Number Theory," *Amer. Math. Monthly* (May 1914): 148-150.]

It follows that

$$u_k = \frac{(2 + \sqrt{2})(3 + 2\sqrt{2})^{k-1} - (2 - \sqrt{2})(3 - 2\sqrt{2})^{k-1}}{4\sqrt{2}} - \frac{1}{2}, \quad k = 1, 2, 3, \ldots.$$

and

$$v_k = \frac{(1 + \sqrt{2})(3 + 2\sqrt{2})^{k-1} - (1 - \sqrt{2})(3 - 2\sqrt{2})^{k-1}}{2\sqrt{2}}, \quad k = 1, 2, 3, \ldots.$$

[See Example 3-5, pp. 66-67, of Liu, *Introduction to Combinatorial Mathematics* (New York: McGraw-Hill Book Company, 1968), for the procedure used to obtain the above formulas.]

Therefore, for $n \geq 1$, if $n = 2k - 1$, then

$$\begin{array}{lll} A_n = u_k + 1 & B_n = v_k & C_n = u_k \\ D_n = v_k & E_n = 2u_k + 1 & G_n = v_k \\ H_n = u_k & I_n = v_k & J_n = u_k + 1 \end{array}$$

while, if $n = 2k$, then

$$\begin{array}{lll} A_n = u_k + v_k + 1 & B_n = u_k + v_k + 1 & C_n = u_k + v_k \\ D_n = 2u_k + v_k + 1 & E_n = 2u_k + 2v_k + 1 & G_n = 2u_k + v_k + 1 \\ H_n = u_k + v_k & I_n = 2u_k + v_k + 1 & J_n = u_k + v_k + 1 \end{array}$$

It is interesting to note that, for n even, the entry in the upper right-hand corner of A^n is the subscript of a triangular number that is a perfect square. [Recall that $t_1 = 1^2$, $t_8 = 6^2$, $t_{49} = 35^2$, $t_{288} = 204^2$, $t_{1681} = 1189^2$, $t_{9800} = 6930^2$, etc.]

(b) Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From the symmetry and other properties of B, it follows that:

$$B^{2k-1} = \begin{pmatrix} 0 & 2^{k-1} & 0 \\ 2^{k-1} & 0 & 2^{k-1} \\ 0 & 2^{k-1} & 0 \end{pmatrix}, \quad k = 1, 2, 3, \ldots,$$

and

$$B^{2k} = \begin{pmatrix} 2^{k-1} & 0 & 2^{k-1} \\ 0 & 2 & 0 \\ 2^{k-1} & 0 & 2^{k-1} \end{pmatrix}, \quad k = 1, 2, 3, \ldots .$$

This can easily be verified in each of the two cases indicated above using induction and the fact that

$$B^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

[Just multiply B^{2k-1} by B^2 and multiply B^{2k} by B^2.]

Therefore, for $n \geq 1$, if $n = 2k - 1$, then

$$\begin{array}{lll} A'_n = 0 & B'_n = 2^{k-1} & C'_n = 0 \\ D'_n = 2^{k-1} & E'_n = 0 & G'_n = 2^{k-1} \\ H'_n = 0 & I'_n = 2^{k-1} & J'_n = 0 \end{array}$$

while, if $n = 2k$, then

$$\begin{array}{lll} A'_n = 2^{k-1} & B'_n = 0 & C'_n = 2^{k-1} \\ D'_n = 0 & E'_n = 2^{k-1} & G'_n = 0 \\ H'_n = 2^{k-1} & I'_n = 0 & J'_n = 2^{k-1} \end{array}$$

Also solved by P. Bruckman, G. Wulczyn, R. Giuli, and P. Russell.

Speedy Series

H-282 *Proposed by H. W. Gould and W. E. Greig, West Virginia University (Vol. 16, No. 2, April 1978)*

Prove

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k} - 1},$$

where $a = (1 + \sqrt{5})/2$, and determine which series converges the faster.

Solution by Robert M. Giuli, San Jose State University, San Jose, CA

The equivalence of the two relations is easily established algebraically if $a^{4n} \neq 1$, and disregarding convergence,

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{n=1}^{\infty} \frac{-a^{2n}}{1 - a^{4n}} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} - a^{2n}a^{4nr} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} - a^{(2n)(2r+1)}$$

Or if $k = 2r + 1$ ($k = 1, 3, 5, \ldots$) and $a^{2n} \neq 1$,

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = -\sum_{k} \sum_{n=1}^{\infty} a^{2nk} = \sum_{k} \frac{-1}{1 - a^{2k}} = \sum_{k} \frac{1}{a^{2k} - 1}.$$

To show "speed" of convergence, the relation may be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{a^{2n} - a^{-2n}} = \sum_{n=1}^{\infty} \frac{1}{a^{4n-2} - 1}$$

where $k = 2n - 1$. The series whose terms decrease in magnitude the fastest will converge the fastest (noting that all terms are positive for $a = 1.618$). For $n = 1, 2, 3, \ldots$, we conjecture then that

$$\frac{1}{a^{2n} - a^{-2n}} > \frac{1}{a^{4n-2} - 1} \quad \text{or that } a^{2n} - a^{-2n} < a^{4n-2} - 1,$$

which is easily established, by induction, to be true for $n = 2, 3, 4, \ldots$. Therefore, the right-hand side series of odd terms converges the fastest.

Also solved by L. Carlitz, P. Bruckman, and E. Robinson.

Close Ranks!

H-283 *Proposed by D. Beverage, San Diego Evening College, San Diego, CA (Vol. 16, No. 2, April 1978)*

Define $f(n)$ as follows:

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k} \qquad (n \geq 0).$$

Express $f(n)$ in closed form.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

We shall show that $f(n) = 1$. Since $f(0) = 1$, to complete our proof, it suffices to show that $f(n + 1) - f(n) = 0$ for $n = 0, 1, 2, \ldots$. Now,

$$\begin{aligned} f(n+1) - f(n) &= \sum_{k=0}^{n+1} \binom{n+1+k}{n+1} \left(\frac{1}{2}\right)^{n+1+k} - \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k} \\ &= \binom{2n+2}{n+1} \left(\frac{1}{2}\right)^{2n+2} + \sum_{k=0}^{n} \left[\binom{n+1+k}{n+1} \left(\frac{1}{2}\right)^{n+1+k} - \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k}\right] \\ &= \binom{2n+2}{n+1} \left(\frac{1}{2}\right)^{2n+2} + \sum_{k=0}^{n} \left[\binom{2n+1-k}{n+1} \left(\frac{1}{2}\right)^{2n+1-k} \right. \\ &\qquad \left. - \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}\right]. \end{aligned}$$

Let

$$S_j = \sum_{k=0}^{j} \left[\binom{2n+1-k}{n+1} \left(\frac{1}{2}\right)^{2n+1-k} - \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k} \right]$$

and

$$s_j = \binom{2n+2}{n+1} \left(\frac{1}{2}\right)^{2n+2} + S_j .$$

Claim:

$$s_j = \binom{2n-j}{n+1} \left(\frac{1}{2}\right)^{2n-j}.$$

We have,

$$\begin{aligned}
s_0 &= \binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2} + \left[\binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \\
&= \binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} + \left[\binom{2n+1}{n+1}\left(\frac{1}{2}\right)^{2n+1} - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \\
&= \left[\binom{2n+1}{n+1} - \binom{2n}{n}\right]\left(\frac{1}{2}\right)^{2n} \\
&= \left[\binom{2n}{n+1} + \binom{2n}{n} - \binom{2n}{n}\right]\left(\frac{1}{2}\right)^{2n} \\
&= \binom{2n}{n+1}\left(\frac{1}{2}\right)^{2n}
\end{aligned}$$

so the desired result holds when $j = 0$. Assume that

$$s_t = \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t} .$$

Then

$$\begin{aligned}
s_{t+1} &= s_t + \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t} - \binom{2n-t-1}{n}\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \binom{2n-t}{n+1}\left(\frac{1}{2}\right)^{2n-t-1} - \binom{2n-t-1}{n}\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \left[\binom{2n-t-1}{n+1} + \binom{2n-t-1}{n} - \binom{2n-t-1}{n}\right]\left(\frac{1}{2}\right)^{2n-t-1} \\
&= \binom{2n-t-1}{n+1}\left(\frac{1}{2}\right)^{2n-t-1}.
\end{aligned}$$

The claimed result now follows.

Finally,

$$f(n+1) - f(n) = s_{n-1} + \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} - \binom{n}{n}\left(\frac{1}{2}\right)^{n}$$

$$= \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} + \binom{n+1}{n+1}\left(\frac{1}{2}\right)^{n+1} - \binom{n}{n}\left(\frac{1}{2}\right)^{n}$$

$$= \left(\frac{1}{2}\right)^{n} - \left(\frac{1}{2}\right)^{n} = 0.$$

An interesting corollary to the result of this problem is that

$$\sum_{k=0}^{n} \binom{n+k}{n}\left(\frac{1}{2}\right)^{k} = 2^n.$$

NOTE: It is also true that

$$f(n+1) - f(n) = s_n = \binom{n}{n+1}\left(\frac{1}{2}\right)^{n} = 0$$

by the usual conventions employed with binomial coefficients because $n < n + 1$.

Also solved by L. Carlitz, W. Moser, P. Bruckman, and P. Russell.

Late Acknowledgments

H-278 *Also solved by J. Shallit.*

H-279 *Also solved by G. Lord.*

H-280 *Also solved by G. Lord, L, Carlitz, and B. Prielipp.*

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