RECURSIVE, SPECTRAL, AND SELF-GENERATING SEQUENCES

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Let p be a fixed integer greater than 1 and define u_n for all integers n by

(1) $$u_0 = 0,\ u_1 = 1,\ u_{n+2} = pu_{n+1} + u_n.$$

Then $u_1, u_2, \ldots$ is an increasing sequence of integers with $u_1 = 1$ and hence a function $\sigma(n)$ is well defined for all n in $N = \{0, 1, 2, \ldots\}$ by

(2) $$\sigma(0) = 0,\ \sigma(n) = u_{j+1} + \sigma(n - u_j) \text{ for } u_j \le n < u_{j+1}.$$

Let $s = (p + \sqrt{p^2 + 4})/2$ and $S_n = [ns]$, where $[x]$ denotes the greatest integer in x.

It is shown below that the spectral sequence $\{S_n\}$ and the *shift function* $\sigma(n)$ are related by the equation

(3) $$S_n = u_2 + \sigma(n - 1)$$

and that $\{S_n\}$ has the self-generating property that

(4) $$S_{n+1} - S_n = \begin{cases} p \text{ if } n \text{ is not in } A = \{S_1, S_2, S_3, \ldots\}; \\ p + 1 \text{ if } n \text{ is in } A. \end{cases}$$

Also investigated are representations of positive integers in terms of $\{u_n\}$, partitions of $Z^+ = \{1, 2, \ldots\}$ into several sequences related to $\sigma(n)$ or S_n, the function counting the number of integers in $A \cap \{1, 2, \ldots, n\}$, and properties of "triangles" of entries $\begin{bmatrix} n \\ k \end{bmatrix}$ defined, for certain fixed x, by

$$\begin{bmatrix} n \\ k \end{bmatrix} = [nx] - [kx] - [(n - k)x] \text{ for } k = 0, 1, \ldots, n.$$

Most of the results presented here are analogous to those given in the authors' paper [4] in which the role of the present u_n is played by h_n satisfying

$$h_i = 2^{i-1} \text{ for } 1 \le i \le d,\ h_{n+d} + h_n = h_{n+1} + \cdots + h_{n+d-1}.$$

The Fibonacci numbers F_{n+1} are the case of the h_n with $d = 2$. The Fibonacci numbers could also be dealt with here by allowing p to equal 1; then the sequence $u_1, u_2, \ldots$ must be replaced by $u_2, u_3, \ldots$ in defining $\sigma(n)$.

For a bibliography on spectra of numbers, see [3].

1. PROPERTIES OF u_n

Here we state the properties of the u_n used below. Proofs are omitted since they are well known or easily derived, or both. Let $r_n = u_{n+1}/u_n$ for n in Z^+.

Lemma 1:

(a) For every k in Z^+, there is exactly one j in Z^+ with $u_j \le k < u_{j+1}$.
(b) $r_1 < r_3 < r_5 < \cdots < s < \cdots < r_6 < r_4 < r_2$.

(c) $u_{n+1}^2 - u_n u_{n+2} = (-1)^n$ for all n in Z.

(d) $r_n - r_{n+1} = (-1)^n/(u_n u_{n+1})$ for n in Z^+.

(e) gcd $(u_n, u_{n+1}) = 1$ for all n in Z.

(f) $u_{2n} = p(u_{2n-1} + u_{2n-3} + \cdots + u_1)$ for n in Z^+.

(g) $u_{2n-1} = p(u_{2n-2} + u_{2n-4} + \cdots + u_2) + u_1$ for n in Z^+.

2. RATIONAL APPROXIMATION

Let x be a positive irrational number. Then, we define a *Farey quadruple* for x to be an ordered quadruple (a, b, c, d) of positive integers, such that $bc - ad = 1$ and $a/b < x < c/d$.

The following result slightly extends some material from the theory of Farey sequences. (See [5] for background.)

Lemma 2: Let (a, b, c, d) be a Farey quadruple for x and let k be a positive integer less than $b + d$. Then:

(a) There is no integer h such that $a/b < h/k < c/d$.
(b) $[kx] = [ka/b]$.
(c) If $d \nmid k$, $[kx] = [kc/d]$.
(d) If $k = de$ with e in $\{1, 2, \ldots, b - 1\}$, $[kx] = [kc/d] - 1$.

The proofs are left to the reader.

We note that parts (b) and (c) of Lemma 1 tell us that

$$(u_{2m+2}, u_{2m+1}, u_{2m+1}, u_{2m}) \quad \text{and} \quad (u_{2m}, u_{2m-1}, u_{2m+1}, u_{2m})$$

are Farey quadruples for s whenever m is a positive integer. This is extended in the following result.

Lemma 3: Let $p \in \{2, 3, \ldots\}$, $s = (p + \sqrt{p^2 + 4})/2$, u be as in (1), and $m \in Z^+$. Then each of

$$(p, 1, 1 + kp, k) \text{ for } k = 1, 2, \ldots, p;$$

$$(u_{2m} + ku_{2m+1}, u_{2m-1} + ku_{2m}, u_{2m+1}, u_{2m}) \text{ for } k = 0, 1, \ldots, p;$$

$$(u_{2m+2}, u_{2m+1}, u_{2m+1} + ku_{2m+2}, u_{2m} + ku_{2m+1}) \text{ for } k = 0, 1, \ldots, p;$$

is a Farey quadruple for s.

Proof: Let (a, b, c, d) represent one of these quadruples. The property

$$bc - ad = 1$$

is easily verified using Lemma 1(c). The property

$$a/b < s < c/d$$

can be shown using Lemma 1(b) and the fact that

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

whenever b and d are positive and $a/b < c/d$.

3. SPECTRA

Let $[x]$ denote the greatest integer in x, that is, the integer such that $[x] \le x < [x] + 1$. The sequence $[x]$, $[2x]$, $[3x]$, ... is called the spectrum of x. It is a well-known result [1] that if y is an irrational number greater than 1 and $(1/x) + (1/y) = 1$ then the spectra $\{[nx]\}$ and $\{[ny]\}$ partition the positive integers Z^+.

Let p be in $\{2, 3, 4, \ldots\}$, $s = (p + \sqrt{p^2 + 4})/2$, $x = s - p + 1$, and $y = s + 1$. Also let $S_n = [ns]$, $X_n = [nx]$, and $Y_n = [ny]$. It is easily seen that y is irrational, $y > 1$, and $(1/x) + (1/y) = 1$; hence the spectra $\{X_n\}$ and $\{Y_n\}$ partition Z^+. It is also clear that $Y_n = X_n + np$ and that each of X_n and Y_n is an increasing function of n. It follows that $\{X_n\}$ and $\{Y_n\}$ may be self-generated using the following algorithm.

(5) $X_1 = 1$, $Y_1 = 1 + p$, X_k for $k > 1$ is the smallest positive integer not in the set $\{X_1, Y_1, X_2, Y_2, \ldots, X_{k-1}, Y_{k-1}\}$, and $Y_k = X_k + kp$.

Then $\{S_n\}$ is easily obtained from $S_n = Y_n - n = X_n + n(p - 1)$. It is shown below that $\{S_n\}$ can be self-generated from the initial condition $S_1 = p$ and the difference property (4) above.

The following result gives symmetry properties of finite segments $[x]$, $\ldots$, $[ex]$ of a spectrum for the cases in which e is the b or d of a Farey quadruple (a, b, c, d) for x.

Lemma 4: Let (a, b, c, d) be a Farey quadruple for x. Then:

(a) $[bx] = [kx] + [(b - k)x] + 1$ for $k = 1, 2, \ldots, b - 1$;
(b) $[dx] = [kx] + [(d - k)x]$ for $k = 0, 1, \ldots, d$.

Proof of (a): We have $[bx] = a$ from Lemma 2(b). Let $0 < k < b$, $j = b - k$, $h = [kx]$, and $i = [jx]$. Since x is irrational, $h < kx$ and so $h/k < x$. This, $x < c/d$, $k < b$, and Lemma 2(a) imply that $h/k < a/b$. Similarly, $i/j < a/b$. Since $(h + i)/(k + j)$ is in the closed interval with endpoints h/k and i/j, we have $(h + i)/(k + j) < a/b$. As $k + j = b$, this means that $h + i < a$ or $[kx] + [jx] < [bx]$. Then the desired result follows from the fact that, for all real y and z,

$$[y + z] - [y] - [z] \in \{0, 1\}. \tag{6}$$

Proof of (b): Lemma 2(d) tells us that $[dx] = c - 1$. We only need consider the k with $0 < k < d$. Let $j = d - k$, $[kx] = h$, and $[jx] = i$. Then $h + 1 > kx$ and so $(h + 1)/k > x$. This, $x > a/b$, $k < d$, and Lemma 2(a) then imply that $(h + 1)/k > c/d$. Similarly, $(i + 1)/j > c/d$, and hence $(h + 1 + i + 1)/(k + j) > c/d$. As $k + j = d$, one has $h + i + 2 > c$, which implies

$$[kx] + [(d - k)x] + 1 > [dx].$$

Again, the desired result follows from (6).

4. THE SHIFT PROPERTY

When convenient, $S_n = [ns]$ will also be denoted by $S(n)$. Also, we recall that $\sigma(n)$ is defined in (2) and u_j is defined in (1).

Theorem 1: If $u_j < n < u_j + u_{j+1}$ and $j \in Z^+$, then $S(n) = u_{j+1} + S(n - u_j)$.

Proof: Let (a, b, c, d) be the Farey quadruple $(u_{2m}, u_{2m-1}, u_{2m+1}, u_{2m})$ for s. Then Lemma 2(b) tells us that $S(n) = [ns] = [nr_{2m-1}]$ for $0 < n < u_{2m-1} + u_{2m}$. Hence

$$S(n) = [nu_{2m}/u_{2m-1}] = \left[\frac{u_{2m-1}u_{2m} + (n - u_{2m-1})u_{2m}}{u_{2m-1}}\right] = u_{2m} + S(n - u_{2m-1}) \tag{7}$$

for $u_{2m-1} < n < u_{2m-1} + u_{2m}$.

Next we use the Farey quadruple $(u_{2m+2}, u_{2m+1}, u_{2m+1}, u_{2m})$ for s and we find, from Lemma 2(c) and (d), that

$$S(n) = [nr_{2m}] \text{ if } 0 < n < u_{2m} + u_{2m+1} \text{ and } u_{2m} \nmid n,$$
$$S(n) = [nr_{2m}] - 1 \text{ if } n = ku_{2m} \text{ with } k \text{ in } \{1, 2, \ldots, u_{2m+1} - 1\}.$$

Using these facts, one can verify that

$$S(n) = u_{2m+1} + S(n - u_{2m}) \text{ for } u_{2m} < n < u_{2m} + u_{2m+1}. \tag{8}$$

The desired result follows from (7) when j is odd and from (8) when j is even.

Theorem 2: $S_n = u_2 + \sigma(n - 1)$ for n in Z^+.

Proof: Since $S_1 = p = u_2$ and $\sigma(0) = 0$, the result holds for $n = 1$. Then a strong induction establishes it for all positive integers n using the consequence

$$S(n) = u_{j+1} + S(n - u_j) \text{ for } u_j < n \leq u_{j+1}$$

of Theorem 1 and the consequence

$$\sigma(n - 1) = u_{j+1} + \sigma(n - 1 - u_j) \text{ for } u_j < n \leq u_{j+1}$$

of the definition (2).

5. SEQUENCES OF COEFFICIENTS

Let V be the set of all sequences $E = [e_1, e_2, \ldots]$ with each e_i in $\{0, 1, \ldots, p\}$, with an i_0 such that $e_i = 0$ for $i > i_0$, and with $e_i = p$ implying that both $i > 1$ and $e_{i-1} = 0$. For such E, the sum

$$e_1u_{n+1} + e_2u_{n+2} + e_3u_{n+3} + \cdots$$

is actually a finite sum which we denote by $E \cdot U_n$. Also, we let $E \cdot U$ stand for $E \cdot U_0$.

Lemma 4: If E and E' are in V and $E \cdot U = E' \cdot U$, then $E = E'$.

This is shown using parts (f) and (g) of Lemma 1.

Theorem 3: The sequences of V form a sequence $E_0, E_1, E_2, \ldots$ such that

$$E_m \cdot U = m.$$

Proof: The only E in V with $E \cdot U = 0$ is $[0, 0, \ldots]$, which we denote by E_0. Now we assume that $k > 0$, and that there is a unique E_m in V with $E_m \cdot U = m$ for $m = 0, 1, \ldots, k - 1$. By Lemma 1(a), $u_j \leq k < u_{j+1}$ for some j in Z^+. Let $h = k - u_j$; then we can let $[e_{h1}, e_{h2}, \ldots]$ be the unique E_h in V with $E_h \cdot U = h$. Then let $e_{kj} = 1 + e_{hj}$, $e_{ki} = e_{hi}$ for $i \neq j$, and $E_k = [e_{k1}, e_{k2}, \ldots]$. Since

$$k < u_{j+1} = pu_j + u_{j-1} < (p + 1)u_j,$$

one sees that $e_{kj} \leq p$ and that if $e_{kj} = p$, then $j > 1$ and $e_{k,j-1} = 0$. Thus, E_k is in V. Clearly,

$$E_k \cdot U = E_h \cdot U + u_j = h + u_j = k.$$

Finally, there is no other E in V with $E \cdot U = k$ by Lemma 4.

The case with $p = 2$ of Theorem 3 was shown in [2].

6. PARTITIONING V

We now partition V into subsets V_1, V_2, V_3 and use these subsets to indicate the relationship of E_{m+1} to E_m. Let $E = [e_1, e_2, \ldots]$ be in V; then, E is in V_1 if $e_1 = p - 1$, E is in V_2 if $e_1 = 0$ and $e_2 = p$, and E is in V_3 if $e_1 < p - 1$ and $e_2 < p$. Since $e_1 > 0$ implies $e_2 < p$, one sees that each E of V is in one and only one of the V .

Lemma 5: Let $E_m = [e_1, e_2, \ldots]$ and $E_{m+1} = [f_1, f_2, \ldots]$. Then:

(a) If E_m is in V_1, let j be the smallest positive integer such that $e_{2j+1} < p$; then $f_i = 0$ for $i < 2j$, $f_{2j} = 1 + e_{2j}$, and $f_i = e_i$ for $i > 2j$.

(b) If E_m is in V_2, let h be the smallest positive integer such that $e_{2h} < p$; then $f_i = 0$ for $1 \le i \le 2h - 2$, $f_{2h-1} = 1 + e_{2h-1}$, and $f_i = e_i$ for $i \ge 2h$.

(c) If E_m is in V_3, $f_1 = 1 + e_1$ and $f_i = e_i$ for $i > 1$.

Proof: If we let $F = [f_1, f_2, \ldots]$ with the f_i as in (a), (b), and (c), it is easily seen that F is in V and $F \cdot U = 1 + E_m \cdot U = 1 + m$. This and Theorem 3 establish the present result.

Lemma 6: Let $\Delta_n(m) = E_{m+1} \cdot U_n - E_m \cdot U_n$. Then:

(a) $\Delta_n(m) = u_n + u_{n+1}$ if E_m is in V_1.

(b) $\Delta_n(m) = u_{n+1}$ if E_m is in V_2 or V_3.

Proof: These statements are easily verified using the parts of Lemma 5.

7. POWERS OF σ

Let $E_m = [e_{m1}, e_{m2}, \ldots]$ and let h be the largest i with $e_{mi} \ne 0$, then one can use the definition of σ in (2) to show that

$$\sigma(m) = \sigma(e_{m1}u_1 + \cdots + e_{mh}u_h) = e_{m1}u_2 + \cdots + e_{mh}u_{h+1} = E_m \cdot U_1.$$

Hence, there is no contradiction in defining σ^n for all integers n to be the function from N to Z given by

$$\sigma^n(m) = E_m \cdot U_n = e_{m1}u_{n+1} + e_{m2}u_{n+2} + \cdots . \tag{9}$$

Also let a_n be the function from Z^+ to Z defined by

$$a_n(k) = u_{n+1} + \sigma^n(k - 1). \tag{10}$$

We note that $a_0(k) = k$, that $a_1(k) = S_k$, and that, for fixed k, the $a_n(k)$ satisfy the same recurrence as the u_n, i.e.,

$$a_{n+2}(k) = pa_{n+1}(k) + a_n(k).$$

We also let A_n be the image set of a_n, i.e.,

$$A_n = \{a_n(k) : k \in Z^+\}.$$

Lemma 7: For n in $\{1, 2\}$, $A_n = \{i + 1 : E_i \in V_n\}$.

Proof: Using (10) and (9), one sees that

$$a_n(m + 1) = (1 + e_{m1})u_{n+1} + e_{m2}u_{n+2} + e_{m3}u_{n+3} + \cdots . \tag{11}$$

As m takes on all values in N, $F_m = [p - 1, e_{m1}, e_{m2}, \ldots]$ ranges through all the E_j in V_1 and $G_m = [0, p, e_{m1}, e_{m2}, \ldots]$ ranges through all the E_h in V_2. It follows from (11), Lemma 5, and the recursion in (1) that if $F_m = E_j$ then

$$j + 1 = E_{j+1} \cdot U = a_1(m + 1)$$

and, similarly, that if $G_m = E_h$ then

$$h + 1 = E_{h+1} \cdot U = a_2(m + 1).$$

These facts establish the lemma.

8. SELF-GENERATING SEQUENCES

Clearly, $a_n(1) = u_{n+1}$. This, and the following result, provide an easy self-generating rule for obtaining the sequence $\{a_1(k)\}$ and a similar easy rule for using $\{a_1(k)\}$ to obtain any $\{a_n(k)\}$.

Theorem 4: For n in Z and j in Z^+, $a_n(j + 1) - a_n(j)$ equals $u_n + u_{n+1}$ if j is in $A_1 = \{a_1(k) : k \in Z^+\}$ and equals u_{n+1} otherwise.

Proof: Lemma 7 tells us that $A_1 = \{j : E_{j-1} \in V_1\}$. Also,

$$a_n(j + 1) - a_n(j) = \sigma^n(j) - \sigma^n(j - 1) = E_j \cdot U_n - E_{j-1} \cdot U_n.$$

Hence, the desired result follows from Lemma 6.

Theorem 5: The number of integers in $A_1 \cap \{1, 2, \ldots, m\}$ is $a_{-1}(m + 1)$.

Proof: Let $\Delta_{-1}(i) = a_{-1}(i + 1) - a_{-1}(i)$. Clearly,

$$a_{-1}(m + 1) = a_{-1}(1) + \Delta_{-1}(1) + \Delta_{-1}(2) + \cdots + \Delta_{-1}(m). \tag{12}$$

Now $a_{-1}(1) = u_0 + \sigma^{-1}(0) = 0 + 0 = 0$. Also, Theorem 4 tells us that $\Delta_{-1}(i) = u_0 = 0$ when i is not in A_1 and $\Delta_{-1}(i) = u_0 + u_{-1} = 1$ when i is in A_1. Thus, the sum on the right side of (12) is the number of i that are in both $\{1, 2, \ldots, m\}$ and A_1, as desired.

9. PARTITIONING Z^+

We saw in Lemma 7 that $A_n = \{i + 1 : E_i \in V_n\}$ for n in $\{1, 2\}$. Let $B = \{j + 1 : E_j \in V_3\}$. Since V_1, V_2, V_3 is a partitioning of $V = \{E_0, E_1, \ldots\}$, it follows that A_1, A_2, B is a partitioning of $Z^+ = \{1, 2, \ldots\}$.

For $k = 1, 2, \ldots, p - 1$, we let

$$b_k(n) = a_1(n) + k - p = k + \sigma(n - 1)$$

and let

$$B_k = \{b_k(n) : n \in Z^+\}.$$

It is easily seen that

$$B_k = \{m : e_{m1} = k,\ e_{m2} < p\} \text{ for } 1 \le k < p$$

and that $B_1, B_2, \ldots, B_{p-1}$ is a partitioning of B. Hence, the sequences

$$\{b_1(n)\}, \{b_2(n)\}, \ldots, \{b_{p-1}(n)\}, \{a_1(n)\}, \{a_2(n)\}$$

partition the positive integers.

10. SPECTRUM TRIANGLES

Let x be irrational and greater than 1 and let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote $[nx] - [nk] - [(n - k)x]$ for integers n and k with $0 \le k \le n$. It now follows from (6) that

$\begin{bmatrix}n\\k\end{bmatrix}$ is always in $\{0, 1\}$. The fact that $\begin{bmatrix}n\\0\end{bmatrix} = 0 = \begin{bmatrix}n\\n\end{bmatrix}$ and the symmetry property $\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\n-k\end{bmatrix}$ are obvious. Part (c) of the following result implies other symmetries for certain finite subtriangles of the infinite triangle of values of $\begin{bmatrix}n\\k\end{bmatrix}$.

Theorem 6: Let (a, b, c, d) be a Farey quadruple for x. Then:

(a) $\begin{bmatrix}b\\k\end{bmatrix} = 1$ for $0 < k < b$.

(b) $\begin{bmatrix}d\\k\end{bmatrix} = 0$ for $0 \le k \le d$.

(c) $\begin{bmatrix}d-s+t\\t\end{bmatrix} = \begin{bmatrix}s\\t\end{bmatrix}$ for $0 \le t \le s \le d$.

Proof: Parts (a) and (b) are a restatement of Lemma 4. For (c) we use Lemma 4(b), or the present part (b), to see that

$$[dx] = [(s-t)x] + [(d-s+t)x] = [sx] + [(d-s)x].$$

Hence $[(d-s+t)x] - [(d-s)x] = [sx] - [(s-t)x]$, and so

$$\begin{aligned}\begin{bmatrix}d-s+t\\t\end{bmatrix} &= [(d-s+t)x] - [tx] - [(d-s)x]\\ &= [sx] - [tx] - [(s-t)x] = \begin{bmatrix}s\\t\end{bmatrix}\end{aligned}$$

as desired.

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LOCAL PERMUTATION POLYNOMIALS OVER Z_p

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1. INTRODUCTION

If p is a prime, let Z_p denote the integers modulo p and Z_p^* the set of nonzero elements of Z_p. It is well known that every function from $Z_p \times Z_p$ into Z_p can be represented as a polynomial of degree $<p$ in each variable. We say that a polynomial $f(x_1, x_2)$ with coefficients in Z_p is a *local permutation polynomial* over Z_p if $f(x_1, a)$ and $f(b, x_2)$ are permutations in x_1 and x_2 for all $a, b \in Z_p$.

In Section 2, we obtain a set of necessary and sufficient conditions on the coefficients of a polynomial $f(x_1, x_2)$ over Z_p, p an odd prime, in order that $f(x_1, x_2)$ be a local permutation polynomial. Clearly the number of local permutation polynomials over Z_p equals the number of Latin squares of order p. Thus, the number of Latin squares of order p equals the number of sets of coefficients satisfying the set of conditions given in Section 2. Finally, in Section 3, we use our theory to show that there are twelve local permutation polynomials over Z which are given by

$$f(x_1, x_2) = a_{10}x_1 + a_{01}x_2 + a_{00}$$

where $a_{10} = 1$ or 2, $a_{01} = 1$ or 2, and $a_{00} = 0, 1,$ or 2.

2. A NECESSARY AND SUFFICIENT CONDITION

Clearly, the only local permutation polynomials over Z_2 are $x_1 + x_2$ and $x_1 + x_2 + 1$ so that we may assume p to be an odd prime. We will make use of the following well-known formula

$$\sum_{m=1}^{p-1} j^k = \begin{cases} 0 \text{ if } k \not\equiv 0 \pmod{p-1}, \\ -1 \text{ if } k \equiv 0 \pmod{p-1}. \end{cases} \tag{2.1}$$

Suppose

$$f(x_1, x_2) = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} a_{mn}x_1^m x_2^n$$

is a local permutation polynomial. Let $f(i, j) = k_{ij}$ for $0 \le i, j \le p - 1$. Since no permutation over Z_p can have degree $p - 1$, we have

$$\text{(C1)} \qquad \begin{cases} a_{0,p-1} = 0, \\ \displaystyle\sum_{m=1}^{p-1} k^m a_{m,p-1} = 0, \; k = 1, \ldots, p-1. \end{cases}$$

Suppose $i = 0$ so that

$$f(0, j) = a_{00} + a_{01}j + \cdots + a_{0,p-1}j^{p-1} = k_{0j}.$$

Let $k_{0j}' = k_{0j} - k_{00}$ for $j = 1, \ldots, p - 1$. The set $\{k_{0j}'\} = Z_p^*$ and, moreover,

$$a_{01}j + a_{02}j^2 + \cdots + a_{0,p-1}j^{p-1} = k_{0j}' \text{ for } j = 1, \ldots, p-1.$$

Raising each of the $p - 1$ equations to the kth power, summing by columns and using (2.1), we obtain

$$\text{(C2)} \qquad \sum \frac{k!}{i_{01}! \cdots i_{0,p-1}!} a_{01}^{i_{01}} \cdots a_{0,p-1}^{i_{0,p-1}} = \begin{cases} 0 \text{ if } k = 2, \ldots, p-2 \\ 1 \text{ if } k = p-1 \end{cases}$$

where the sum is over all $(p - 1)$-tuples $(i_{01}, \ldots, i_{0,p-1})$ with

(a) $0 \le i_{01}, \ldots, i_{0,p-1} \le k$,

(b) $i_{01} + \cdots + i_{0,p-1} = k$,

(c) $i_{01} + 2i_{02} + \cdots + (p - 1)i_{0,p-1} \equiv 0 \pmod{p - 1}$.

If $i > 0$ is fixed, consider

$$\text{(2.2)} \qquad f(i, j) - k_{i0} = \sum_{m=0}^{p-1} \sum_{n=1}^{p-1} a_{mn} i^m j^n = k'_{ij}, \; j = 1, \ldots, p - 1,$$

so that $\{k'_{ij}\} = Z_p^*$. For each $k = 2, \ldots, p - 1$ raise each of the $p - 1$ equations in (2.2) to the kth power, sum by columns, and use (2.1) to obtain

$$\text{(C3)} \qquad \sum \prod_{m=0}^{p-1} \prod_{n=1}^{p-1} \frac{k! a_{mn}^{i_{mn}} i^{\Sigma m}}{i_{mn}!} = \begin{cases} 0 \text{ if } k = 2, \ldots, p-2 \\ 1 \text{ if } k = p-1 \end{cases}$$

for each $i = 1, \ldots, p - 1$, where the sum is over all $(p^2 - p)$-tuples

$$(i_{01}, \ldots, i_{mn}, \ldots, i_{p-1,p-1})$$

which satisfy

(d) $0 \le i_{mn} \le k$,

(e) $\displaystyle\sum_{m=0}^{p-1} \sum_{n=1}^{p-1} i_{mn} = k$,

(f) $\displaystyle\sum_{m=0}^{p-1} i_{m1} + 2\sum_{m=0}^{p-1} i_{m2} + \cdots + (p - 1)\sum_{m=0}^{p-1} i_{m,p-1} \equiv 0 \pmod{p - 1}$.

A further word of explanation about the sum in (C3) may be helpful at this time. Conditions (d) and (e) arise because of the multinomial coefficients, while (f) determines which terms appear in the given condition. Moreover, the Σm appearing in (C3) is understood to mean the sum, counting multiplicities, of all the first subscripts of the a_{mn}'s which appear in a given term. Finally, we note that condition (C3) actually involves a total of $(p - 1)(p - 2)$ conditions.

If we now fix j and proceed as above, we obtain another set of necessary conditions. For brevity, we simply state these as

$$\text{(C1')} \qquad \begin{cases} a_{p-1,0} = 0, \\ \displaystyle\sum_{n=1}^{p-1} k^n a_{p-1,n} = 0, \; k = 1, \ldots, p - 1. \end{cases}$$

When $j = 0$, we have

$$\text{(C2}'\text{)} \qquad \sum \frac{k!}{i_{10}! \cdots i_{p-1,0}!} a_{10}^{i_{10}} \cdots a_{p-1,0}^{i_{p-1,0}} = \begin{cases} 0 \text{ if } k = 2, \ldots, p-2 \\ 1 \text{ if } k = p-1 \end{cases}$$

where the sum is over all $(p-1)$-tuples $(i_{10}, \ldots, i_{p-1,0})$ with

(a$'$) $0 \le i_{10}, \ldots, i_{p-1,0} \le k$,

(b$'$) $i_{10} + \cdots + i_{p-1,0} = k$,

(c$'$) $i_{10} + 2i_{20} + \cdots + (p-1)i_{p-1,0} \equiv 0 \pmod{p-1}$.

When $j = 1, \ldots, p-1$, we obtain

$$\text{(C3}'\text{)} \qquad \sum \prod_{m=1}^{p-1} \prod_{n=0}^{p-1} \frac{k! a_{mn}^{i_{mn}} j^{\Sigma n}}{i_{mn}!} = \begin{cases} 0 \text{ if } k = 2, \ldots, p-2 \\ 1 \text{ if } k = p-1 \end{cases}$$

where the sum is over all $(p^2 - p)$-tuples $(i_{10}, \ldots, i_{mn}, \ldots, i_{p-1,p-1})$ that satisfy

(d$'$) $0 \le i_{mn} \le k$,

(e$'$) $\displaystyle\sum_{m=1}^{p-1} \sum_{n=0}^{p-1} i_{mn} = k$,

(f$'$) $\displaystyle\sum_{m=0}^{p-1} i_{1n} + 2\sum_{n=0}^{p-1} i_{2n} + \cdots + (p-1)\sum_{n=0}^{p-1} i_{p-1,n} \equiv 0 \pmod{p-1}$.

We now proceed to show that if the coefficients of a polynomial $f(x_1, x_2)$ satisfy the above conditions, then $f(x_1, x_2)$ is a local permutation polynomial. Suppose the coefficients of $f(x_1, x_2)$ satisfy (C1), (C2), (C3), (C1$'$), (C2$'$), and (C3$'$). For each fixed i, let $t_{ij} = f(i, j) - f(i, 0)$ for $j = 1, \ldots, p-1$. The above conditions imply that for fixed $i = 0, 1, \ldots, p-1$ the t_{ij} satisfy

$$\text{(2.3)} \qquad \sum_{j=1}^{p-1} t_{ij}^k = \begin{cases} 0 \text{ if } k = 1, \ldots, p-2, \\ -1 \text{ if } k = p-1. \end{cases}$$

Let V be the matrix

$$V = \begin{bmatrix} 1 & \cdots & 1 \\ t_{i1} & \cdots & t_{i,p-1} \\ \vdots & & \vdots \\ t_{i1}^{p-2} & \cdots & t_{i,p-1}^{p-2} \end{bmatrix}$$

Using (2.3), we see that

$$\det(V^2) = \det(V)\det(V)' = \det\begin{bmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \end{bmatrix} = \pm 1.$$

Since $\det(V)$ is the Van der Monde determinant, we have, for fixed i,

$$\det(V) = \prod_{j>k} (t_{ij} - t_{ik}) \neq 0$$

so that the t_{ij} for $j = 1, \dots, p-1$ are distinct. Hence,

$$f(i, 0) \text{ and } f(i, j) = t_{ij} + f(i, 0) \text{ for } j = 1, \dots, p-1$$

constitute all of Z_p.

A similar argument shows that if for each fixed j,

$$s_{ij} = f(i, j) - f(0, j) \text{ for } i = 1, \dots, p-1,$$

then

$$f(0, j) \text{ and } f(i, j) = s_{ij} + f(0, j) \text{ for } i = 1, \dots, p-1$$

run through the elements of Z_p. Hence, we have

Theorem 1: If $f(x_1, x_2)$ is a polynomial over Z_p, p an odd prime, then f is a local permutation polynomial over Z_p if and only if the coefficients of f satisfy (C1), (C2), (C3), (C1′), (C2′), and (C3′).

Corollary 2: The number of Latin squares of order p an odd prime equals the number of sets of coefficients $\{a_{mn}\}$ satisfying the above conditions.

We note from condition (C1) that $a_{0,p-1} = a_{1,p-1} = \cdots = a_{p-1,p-1} = 0$, since the determinant of the coefficient matrix in (C1) is the Van der Monde determinant. Similarly, (C1′) implies that $a_{p-1,0} = a_{p-1,1} = \cdots = a_{p-1,p-1} = 0$. We further note that we have a total of $2p(p-1)$ conditions so that, in general, the conditions are not independent.

3. ILLUSTRATIONS

As a simple illustration of the above theory, we determine all local permutation polynomials over Z_3. If

$$f(x_1, x_2) = \sum_{m=0}^{2} \sum_{n=0}^{2} a_{mn} x_1^m x_2^n$$

then the set of necessary and sufficient conditions becomes

(2.4) $$a_{02} = a_{12} = a_{22} = a_{21} = a_{20} = 0,$$

(2.5) $$a_{01}^2 + a_{02}^2 = a_{10}^2 + a_{20}^2 = 1,$$

(2.6) $$a_{01}^2 + a_{11}^2 + 2a_{01}a_{11} = a_{10}^2 + a_{11}^2 + 2a_{10}a_{11} = 1,$$

(2.7) $$a_{01}^2 + a_{11}^2 + a_{01}a_{11} = a_{10}^2 + a_{11}^2 + a_{10}a_{11} = 1.$$

Using (2.4) and (2.5), we see that $a_{01} = 1$ or 2 and $a_{10} = 1$ or 2. From (2.6) and (2.7), we have $a_{11} = 0$. Since a_{00} is arbitrary, we see that there are a total of twelve local permutation polynomials over Z_3, given by

$$f(x_1, x_2) = a_{10}x_1 + a_{01}x_2 + a_{00},$$

where $a_{10} = 1$ or 2, $a_{01} = 1$ or 2, and $a_{00} = 0$, 1, or 2.

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GENERALIZED CYCLOTOMIC POLYNOMIALS, FIBONACCI CYCLOTOMIC POLYNOMIALS, AND LUCAS CYCLOTOMIC POLYNOMIALS*

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1. INTRODUCTION AND MAIN THEOREM

In [6], Hoggatt and Long ask what polynomials in $I[x]$ are divisors of the Fibonacci polynomials, which are defined by the recursion

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \text{ for } n \geq 2.$$

In this paper, we answer this question in terms of cyclotomic polynomials. We prove that each Fibonacci polynomial $F_n(x)$, for $n \geq 2$, has one and only one irreducible factor which is not a factor of any $F_k(x)$ for any positive k less than n. We call this irreducible factor the nth *Fibonacci cyclotomic polynomial* and denote it $\mathcal{F}_n(x)$.

The method applied to F_n's to produce $\mathcal{F}_n$'s applies naturally to the more general polynomials $\ell_n(x, y, z)$ which were introduced in [7] and are defined just below. Accordingly, in Section 2, we shall apply the method at this more general level rather than directly to the F_n's. The polynomials $C_n(x, y, z)$ so obtained from the $\ell_n(x, y, z)$'s we call *generalized cyclotomic polynomials*. Special cases of the C_n's are the ordinary cyclotomic polynomials $C_n(x, 1, 0)$, the Fibonacci cyclotomic polynomials $\mathcal{F}_n$ already mentioned, and a sequence

$$\mathcal{L}_n(x) = C_n(x, 0, 1)$$

which we call the *Lucas cyclotomic polynomials*. Section 3 is devoted to the $\mathcal{F}_n$'s and Section 4 to the $\mathcal{L}_n$'s. In Sections 3, 4, and 5, we determine all the irreducible factors of the Fibonacci polynomials, the modified Lucas polynomials defined in [7] as $\ell_n(x, 0, 1)$, and the Lucas polynomials.

In Section 6, we transform the generalized Fibonacci and Lucas polynomials into sequences $U_n(x, z)$ and $V_n(x, z)$ having the same divisibility properties as the F_n's and L_n's, respectively. The coefficients of these polynomials are all binomial coefficients, in accord with the identity

$$zU_n(x, z) + V_n(x, z) = (x + z)^n.$$

The polynomials $\ell_n(x, y, z)$ may be defined as follows:

$$\ell_n(x, y, z) = \frac{L_n(x, z) - L_n(y, z)}{x - y} \quad \text{for } n \geq 0,$$

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where $L_n(x, z)$ is the nth generalized Lucas polynomial, defined by the recursion

$$L_0(x, z) = 2,\ L_1(x, z) = x,\ L_n(x, z) = xL_{n-1}(x, z) + zL_{n-2}(x, z) \text{ for } n \geq 2.$$

The two special cases of particular interest are the generalized Fibonacci polynomials, namely

$$\ell_n\left(\frac{x + \sqrt{x^2 + 4z}}{2}, \frac{x - \sqrt{x^2 + 4z}}{2}, 0\right), \tag{1}$$

and the generalized modified Lucas polynomials, namely $\ell_n(x, 0, z)$. Other special cases, to be treated briefly in Section 5, are the Chebyshev polynomials of the first and second kinds.

Following the method of Hoggatt and Bicknell in [5], we now determine the roots of the polynomials $\ell_n(x, y, z)$. The first theorem is basic to all subsequent developments in this paper.

Theorem 1: For $n \geq 2$, the roots of $\ell_n(x, y, z)$ are

$$2\sqrt{z} \sinh (\sinh^{-1} y/2\sqrt{z} + 2k\pi i/n), \text{ where } k = 1, 2, \ldots, n - 1. \tag{2}$$

Proof: We have $(x - y)\ell_n(x, y, z) = t_1^n + t_2^n - (t_3^n + t_4^n)$, where

$$t_1 = \frac{x + \sqrt{x^2 + 4z}}{2},\ t_2 = \frac{x - \sqrt{x^2 + 4z}}{2},\ t_3 = \frac{y + \sqrt{y^2 + 4z}}{2},\ t_4 = \frac{y - \sqrt{y^2 + 4z}}{2}.$$

Let $x = 2\sqrt{z} \sinh u$, so that $\sqrt{x^2 + 4z} = 2\sqrt{z} \cosh u$, and

$$t_1 = \sqrt{z}e^u \quad\text{and}\quad t_2 = -\sqrt{z}e^{-u}.$$

Let $y = 2\sqrt{z} \sinh v$, so that $\sqrt{y^2 + 4z} = 2\sqrt{z} \cosh v$, and

$$t_3 = \sqrt{z}e^v \quad\text{and}\quad t_4 = -\sqrt{z}e^{-v}.$$

Then

$$\begin{aligned}(x - y)\ell_n(x, y, z) &= z^{\frac{n}{2}}[e^{nu} + (-1)^n e^{-nu}] - z^{\frac{n}{2}}[e^{nv} + (-1)^n e^{-nv}] \\ &= \begin{cases} 2z^{\frac{n}{2}}(\sinh nu - \sinh nv) \text{ for odd } n, \\ 2z^{\frac{n}{2}}(\cosh nu - \cosh nv) \text{ for even } n.\end{cases}\end{aligned}$$

Dividing by $x - y = 2\sqrt{z}(\sinh u - \sinh v)$, we find

$$\ell_n(x, y, z) = \begin{cases} z^{\frac{n-1}{2}}\dfrac{\sinh nu - \sinh nv}{\sinh u - \sinh v} \text{ for odd } n, \\[2ex] z^{\frac{n-1}{2}}\dfrac{\cosh nu - \cosh nv}{\sinh u - \sinh v} \text{ for even } n.\end{cases}$$

Now suppose n is odd. Then $\ell_n(x, y, z) = 0$ when

$$\sinh nu = \sinh nv \text{ and } \sinh u \neq \sinh v;$$

i.e., when $nu = nv + 2k\pi i$ and k is not an integral multiple of n. Thus,

$$\ell_n(x, y, z) = 0 \text{ when } u = v + 2k\pi i/n \text{ for } k = 1, 2, \ldots, n - 1.$$

For even n we similarly reach the same result. Substitution for u and v now completes the proof.

2. GENERALIZED CYCLOTOMIC POLYNOMIALS

Following the treatment of cyclotomic polynomials in Nagell [9, p. 158], for $n \geq 2$ let $p_1, p_2, \ldots, p_r$ be the distinct prime factors of n; let

$$\Pi_0 = \ell_n,$$

and for $1 \leq k \leq r$, let

$$\Pi_k = \Pi \ell_{n/p_{i_1} p_{i_2} \cdots p_{i_k}}$$

the product extending over all the k indices i_j which satisfy the conditions

$$1 \leq i_1 < i_2 < \cdots < i_k \leq r.$$

Lemma 2: Let $C_1(x, y, z) = 1$, and for $n \geq 2$, let

$$C_n(x, y, z) = \frac{\Pi_0 \Pi_2 \ldots}{\Pi_1 \Pi_3 \ldots}. \tag{3}$$

The number of factors ℓ_q in the numerator equals the number of factors ℓ_q in the denominator.

Proof: First consider the number of ℓ_q's in the numerator: for $0 \leq j \leq [r/2]$ there are $\binom{r}{2j}$ of the ℓ_q's in Π_{2j}, so that the number we seek is

$$\sum_{j=0}^{[r/2]} \binom{r}{2j}.$$

Similarly, we count $\displaystyle\sum_{j=0}^{[(r-1)/2]} \binom{r}{2j+1}$ factors ℓ_q in the denominator. That these two sums are equal for any $r \geq 1$ follows from the identity

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} = (1 - 1)^r = 0.$$

Let us recall now some facts about cyclotomic polynomials (e.g., [9]): In case $\ell_n = x^n - 1$, the quotient C_n in (3) defines, for $n \geq 2$, the nth cyclotomic polynomial, which is irreducible over the ring of integers. (The first cyclotomic polynomial is defined to be $x - 1$). Thus, for $n \geq 1$, the roots of the nth cyclotomic polynomial are the primitive nth roots of unity: $e^{2k\pi i/n}$ where $(k, n) = 1$. Writing $\phi(n)$ for Euler's phi-function, the nth cyclotomic polynomial therefore has degree $\phi(n)$.

Referring to (2), let us call the root

$$2\sqrt{z} \sinh(\sinh^{-1} y/2\sqrt{z} + 2k\pi i/n)$$

a *primitive nth root* of $\ell_n(x, y, z)$ if $(k, n) = 1$.

Theorem 2: For $n \geq 2$, the quotient $C_n(x, y, z)$ in (3) is a polynomial with integer coefficients, having degree $\phi(n)$ in x. Moreover, for $n \geq 2$, $C_n(x, 1, 0)$ is the nth cyclotomic polynomial.

Proof: Suppose $n \geq 2$. By Lemma 2, if the quotient in (3) is formed with the polynomials $(x - 1)\ell_n(x, 1, 0)$ in the products Π_k instead of $\ell_n(x, 1, 0)$, then the result is $C_n(x, 1, 0)$. But

$$(x - 1)\ell_n(x, 1, 0) = x^n - 1,$$

so that $C_n(x, 1, 0)$ is the nth cyclotomic polynomial, which has degree $\phi(n)$ in x.

It remains to be proved that $C_n(x, y, z)$ is a polynomial for $n \geq 2$; i.e., that the polynomial $D = \Pi_1\Pi_3 \ldots$ divides the polynomial $N = \Pi_0\Pi_2 \ldots$ over the ring of integers. Since this is the case for $(x, 1, 0)$, each linear factor $x - r$ of D is a factor of N and must occur at least as many times in N as in D. But each such r is an nth root of unity, $r = e^{2k\pi i/n}$ for some k and n. So in the general case (x, y, z), each linear factor $x - 2\sqrt{z}\ \sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)$ of D occurs at least as many times in N as in D. Thus, D divides N. Since all the coefficients of N and D have only integer coefficients, the same must be true of the quotient $C_n(x, y, z)$, by the division algorithm for polynomials in x over the ring $I[y, z]$ of bivariate polynomials with integer coefficients.

Theorem 3: For $n \geq 2$,

$$C_n(x, y, z) = \prod_{\substack{(k,n)=1\\ 0\leq k\leq n}} [x - 2\sqrt{z}\ \sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)].$$

Proof: This is an obvious consequence of the one-to-one correspondence between roots of $C_n(x, y, z)$ and roots of the nth cyclotomic polynomial

$$C_n(x, 1, 0) = \prod_{\substack{(k,n)=1\\ 0\leq k\leq n}} (x - e^{2k\pi i/n}).$$

Theorem 4: For $n \geq 1$,

$$\ell_n(x, y, z) = \prod_{d|n} C_d(x, y, z).$$

Proof: First, $\ell_1(x, y, z) = C_1(x, y, z) = 1$. Now suppose $n \geq 2$. Then

$$C_d(x, y, z) = (x - r_1) \ldots (x - r_{\phi(d)}),$$

where the r_i's range through the roots $2\sqrt{z}\ \sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)$ of $\ell_d(x, y, z)$ for which $(k, d) = 1$. Each root of $\ell_n(x, y, z)$ is a primitive dth root of one and only one $C_d(x, y, z)$ where $d|n$. Thus each linear factor of $\ell_n(x, y, z)$ occurs in one and only one $C_d(x, y, z)$.

Lemma 5: For $n \geq 1$, the polynomial $C_n(x, y, 0)$ is irreducible over the ring of integers.

Proof: The statement is clearly true for $n = 1$. For $n \geq 2$, suppose

$$C_n(x, y, 0) = d(x, y)q(x, y).$$

Then

$$C_n(x, 1, 0) = d(x, 1)q(x, 1).$$

Since the cyclotomic polynomial $C_n(x, 1, 0)$ is irreducible, one of the polynomials $d(x, 1)$ and $q(x, 1)$ must be the constant 1 polynomial. Without any loss, we may suppose this one to be $d(x, 1)$ and thus have

$$d(x, y) = 1 + (y - 1)e(x, y)$$

for some polynomial $e(x, y)$. Then

$$C_n(x, y, 0) = q(x, y) + (y - 1)e(x, y)q(x, y).$$

Now $q(x, y)$ includes the term $x^{\phi(n)}$, which cannot appear in

$$(y - 1)e(x, y)q(x, y).$$

Therefore, $e(x, y) = 0$, so that $d(x, y) = 1$.

Theorem 5: For $n \geq 1$, the polynomial $C_n(x, y, z)$ is irreducible over the ring of integers.

Proof: Suppose

$$C_n(x, y, z) = d(x, y, z)q(x, y, z).$$

Then

$$C_n(x, y, 0) = d(x, y, 0)q(x, y, 0).$$

By Lemma 5, one of the polynomials $d(x, y, 0)$ and $q(x, y, 0)$ is the constant 1 polynomial. Consequently, as in the proof of Lemma 5, we have

$$d(x, y, z) = 1 + ze(x, y, z)$$

for some polynomial $e(x, y, z)$. Then

$$C_n(x, y, z) = q(x, y, z) + ze(x, y, z)q(x, y, z).$$

Now $q(x, y, z)$ includes the term $x^{\phi(n)}$, which cannot appear in

$$ze(x, y, z)q(x, y, z).$$

Therefore, $e(x, y, z) = 0$, so that $d(x, y, z) = 1$.

TABLE 1

Generalized Cyclotomic Polynomials $C_n = C_n(x, y, z)$

$$\begin{aligned}
C_1 &= 1\\
C_2 &= x + y\\
C_3 &= x^2 + xy + y^2 + 3z\\
C_4 &= x^2 + y^2 + 4z\\
C_5 &= x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 5z(x^2 + xy + y^2) + 5z^2\\
C_6 &= x^2 - xy + y^2 + 3z\\
C_8 &= x^4 + y^4 + 4z(x^2 + y^2) + 4z^2\\
C_9 &= x^6 + x^3y^3 + y^6 + 3z(2x^4 + x^3y + xy^3 + 2y^4)\\
&\quad + 9z^2(x^2 + xy + y^2) + 3z^3\\
C_{10} &= (x^5 + y^5)/(x + y) + 5z(x^3 + y^3)/(x + y) + 5z^2\\
C_{12} &= x^4 - x^2y^2 + y^4 + 2z(x^2 + y^2) + z^2
\end{aligned}$$

Abbreviating $C_n(x, y, 0)$ as c_n, we note that

$C_3 = c_3 + 3z$, $C_4 = c_4 + 4z$, $C_6 = c_6 + 3z$, $C_8 = c_8 + 4zc_4 + 4z^2$,

$$C_{10} = c_{10} + 5zc_6 + 5z^2,\ C_{12} = c_{12} + 2zc_4 + z^2,$$

and

$$C_9 = c_9 + 3z(c_5 + c_{12}) + 9z^2c_3 + 3z^3.$$

One wonders if all the coefficients of powers of z are linear combinations of c_i's.

3. THE CASE $z = 0$: FIBONACCI CYCLOTOMIC POLYNOMIALS

Here we will determine the irreducible factors of the generalized Fibonacci polynomials. In Section 1, the (not generalized) irreducible factors were named the Fibonacci cyclotomic polynomials and denoted $\mathcal{F}_n(x)$. Here, however, we shall deal with the natural generalization: the *generalized Fibonacci cyclotomic polynomials*, denoted $\mathcal{F}_n(x, y)$. Theorem 6 will show that

$$\mathcal{F}_n(x, y) = C_n\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right) \text{ for } n \geq 1,$$

and Corollary 7 will show that the $\mathcal{F}_n(x)$'s can be expressed as linear combinations of generalized (unmodified) Lucas polynomials.

Theorem 6: For $n \geq 1$, let $F_n(x, y)$ be the nth generalized Fibonacci polynomial. Then

$$F_n(x, y) = \prod_{d|n} C_d\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right).$$

Moreover, the polynomials $C_d\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right)$, as polynomials in x and y, are irreducible over the ring of integers.

Proof: Write $s = \frac{x + \sqrt{x^2 + 4y}}{2}$ and $t = \frac{x - \sqrt{x^2 + 4y}}{2}$. By (1) and Theorem 4,

$$F_n(x, y) = \ell_n(s, t, 0) = \prod_{d|n} C_d(s, t, 0).$$

To see that the C_d's are irreducible as polynomials in x and y, suppose

$$C_d(s, t, 0) = p(x, y)q(x, y).$$

Then, since $x = s + t$ and $y = -st$, we have $C_d(s, t, 0)$ written as a product of two polynomials each in s and t. By Lemma 5, one of these polynomials is a constant polynomial, namely 1, since C_d is monic. Thus, either $p(x, y) = 1$ or $q(x, y) = 1$, as desired.

Theorem 7: For $k \geq 1$, let $L_k(x, y)$ be the kth generalized (unmodified) Lucas polynomial. For $n \geq 3$, the nth generalized Fibonacci cyclotomic polynomial is given by

$$\mathcal{F}_n(x, y) = \sum_{j=0}^{\phi(n)/2} \delta_j y^{\frac{\phi(n)}{2} - j} L_{2j}(x, y),$$

where $\delta_{\phi(n)/2} = 1$ and the numers $\delta_0, \delta_1, \delta_2, \ldots, \delta_{\frac{\phi(n)}{2}-1}$ are integers.

Proof: Suppose $n \geq 3$. With s and t as in the proof of Theorem 6,

$$\mathcal{F}_n(x, y) = C_n(s, t, 0) = t^{\phi(n)} C_n(s/t, 1, 0),$$

where

$$C_n(u, 1, 0) = u^{\phi(n)} + a_{\phi(n)-1} u^{\phi(n)-1} + \cdots + a_1 u + 1$$

is the nth cyclotomic polynomial. Thus, $C_n(s, t, 0)$ has the form

$$s^{\phi(n)} + a_{\phi(n)-1} s^{\phi(n)-1} t + \cdots + a_1 s t^{\phi(n)-1} + t^{\phi(n)}.$$

Since $C_n(s, t, 0)$ is symmetric in s and t, this polynomial is expressible as

$$s^{\phi(n)} + t^{\phi(n)} + a_{\phi(n)-1} st\left(s^{\phi(n)-2} + t^{\phi(n)-2}\right) + \cdots + a_{\frac{\phi(n)}{2}} (st)^{\frac{\phi(n)}{2}}.$$

Recalling $st = -y$ and the Binet formula $L_k(x, y) = s^k + t^k$ [in particular, $L_0(x, y) = 2$], we conclude that

$$\mathcal{F}_n(x, y) = L_{\phi(n)} - a_{\phi(n)-1} y L_{\phi(n)-2} + \cdots + \frac{(-1)^{\frac{\phi(n)}{2}}}{2} a_{\frac{\phi(n)}{2}} y^{\frac{\phi(n)}{2}} L_0,$$

as desired.

Corollary 7: Only for the purpose of facilitating the statement of this corolary, suppose $L_0(x, y) = 1$ (instead of 2). Then for $n \geq 1$, the nth Fibonacci cyclotomic polynomial $\mathcal{F}_n(x)$ is an integral linear combination of Lucas polynomials $L_n(x)$.

Proof: The proposition is easily verified for $n = 0, 1, 2$. For $n \geq 3$, put $y = 1$ in Theorem 7.

To illustrate Corollary 7, we write out, in Table 2, several Fibonacci cyclotomic polynomials $\mathcal{F}_n = \mathcal{F}_n(x, 1)$ in terms of the Lucas polynomials $L_n = L_n(x, 1)$. Recall that the $\mathcal{F}_n$'s are the irreducible divisors of the Fibonacci polynomials, in accord with the identity

$$F_n = \prod_{d|n} \mathcal{F}_d.$$

TABLE 2

Fibonacci Cyclotomic Polynomials

degree 0:	$\mathcal{F}_1 = 1$
degree 1:	$\mathcal{F}_2 = x = L_1$
degree 2:	$\mathcal{F}_3 = x^2 + 1 = L_2 - 1$
	$\mathcal{F}_4 = x^2 + 2 = L_2$
	$\mathcal{F}_6 = x^2 + 3 = L_2 + 1$
degree 4:	$\mathcal{F}_5 = x^4 + 3x^2 + 1 = L_4 - L_2 + 1$
	$\mathcal{F}_8 = x^4 + 4x^2 + 2 = L_4$
	$\mathcal{F}_{10} = x^4 + 5x^2 + 5 = L_4 + L_2 + 1$
	$\mathcal{F}_{12} = x^4 + 4x^2 + 1 = L_4 - 1$

TABLE 2 (continued)

degree 6:
$$\mathcal{F}_7 = x^6 + 5x^4 + 6x^2 + 1 = L_6 - L_4 + L_2 - 1$$
$$\mathcal{F}_9 = x^6 + 6x^4 + 9x^2 + 1 = L_6 - 1$$
$$\mathcal{F}_{14} = x^6 + 7x^4 + 14x^2 + 7 = L_6 + L_4 + L_2 + 1$$
$$\mathcal{F}_{18} = x^6 + 6x^4 + 9x^2 + 4 = L_6 + 1$$

degree 8:
$$\mathcal{F}_{15} = x^8 + 9x^6 + 26x^4 + 24x^2 + 1 = L_8 + L_6 - L_2 - 1$$
$$\mathcal{F}_{16} = x^8 + 8x^6 + 20x^4 + 16x^2 + 2 = L_8$$
$$\mathcal{F}_{20} = x^8 + 8x^6 + 19x^4 + 12x^2 + 1 = L_8 - L_4 + 1$$
$$\mathcal{F}_{24} = x^8 + 8x^6 + 20x^4 + 16x^2 + 1 = L_8 - 1$$
$$\mathcal{F}_{30} = x^8 + 7x^6 + 14x^4 + 8x^2 + 1 = L_8 - L_6 + L_2 - 1$$

degree >8:
$$\mathcal{F}_{11} = L_{10} - L_8 + L_6 - L_4 + L_2 - 1$$
$$\mathcal{F}_{32} = L_{16}$$
$$\mathcal{F}_{33} = L_{20} + L_{18} - L_{14} - L_{12} + L_8 + L_6 - L_2 - 1$$
$$\mathcal{F}_{36} = L_{12} - 1$$
$$\mathcal{F}_{40} = L_{16} - L_8 + 1$$
$$\mathcal{F}_{42} = L_{12} - L_{10} + L_6 - L_4 + 1$$
$$\mathcal{F}_{45} = L_{24} + L_{18} - L_6 - 1$$
$$\mathcal{F}_{48} = L_{16} - 1$$
$$\mathcal{F}_{50} = L_{20} + L_{10} + 1$$
$$\mathcal{F}_{105} = L_{48} - L_{46} + L_{44} + L_{38} - L_{36} + 2L_{34} - L_{32} + L_{30} + L_{24} - L_{22} + L_{20} - L_{18} + L_{16} - L_{14} - L_8 - L_4 - 1$$

Note in particular the coefficient of L_{34} in the polynomial $\mathcal{F}_{105}$.

Two reminders (e.g., [9]) about the cyclotomic polynomials $C_n(u, 1, 0) = \Phi_n(u)$ which are helpful in computing $\mathcal{F}_n$'s are the following:

(i) If p is a prime and $p \nmid n$, then $\Phi_{np}(u) = \Phi_n(u^p)/\Phi_n(u)$;

(ii) If p is a prime and $p \mid n$, then $\Phi_{np}(u) = \Phi_n(u^p)$.

As an example, we compute $\mathcal{F}_{45}$ as follows:

$$\Phi_{45}(u) = \Phi_{15}(u^3) = \Phi_3(u^{15})/\Phi_3(u^3) = \frac{u^{30} + u^{15} + 1}{u^6 + u^3 + 1}$$
$$= u^{24} - u^{21} + u^{15} - u^{12} + u^9 - u^3 + 1,$$

so that

$$\begin{aligned}\mathcal{F}_{45}(x, y) &= C_{45}(s, t, 0)\\ &= s^{24} - s^{21}t^3 + s^{15}t^9 - s^{12}t^{12} + s^9t^{15} - s^3t^{21} + t^{24}\\ &= s^{24} + t^{24} - (st)^3(s^{18} + t^{18}) + (st)^9(s^6 + t^6) - (st)^{12}\end{aligned}$$

$$= L_{24} + y^3 L_{18} - y^9 L_6 - y^{12} \quad \text{(Theorem 7)},$$
$$\mathcal{F}_{45}(x,\ 1) = L_{24} + L_{18} - L_6 - 1 \quad \text{(Corollary 7)}.$$

Since for highly composite values of n the cyclotomic polynomials tend to be complicated ([1], [3], [4], [11], [12]), the same is true for the corresponding Fibonacci cyclotomic polynomials.

In Theorem 12 of [6], Hoggatt and Long find an upper bound for the number $N(m)$ of polynomials of degree $2m$ that divide some Fibonacci polynomial. If we restrict $N(m)$ to irreducible polynomials, then $N(m)$ is the number of solutions n to the equation $\phi(n) = 2m$. For example, $N(720) = 72$. That is, there are 72 distinct Fibonacci cyclotomic polynomials $\mathcal{F}_n$ having degree 1440. See [10].

Still restricting $N(m)$ to irreducible polynomials, we ask if $N(m) = 0$ for any m. The answer is yes. C. L. Klee proved in [8] that $\phi(n) = 2m$ has no solution n if m has no divisor $d > 1$ for which $2d + 1$ is a prime. For example, no $\mathcal{F}_n$ has degree 14.

4. THE CASE $y = 0$: LUCAS CYCLOTOMIC POLYNOMIALS

Our main objective in this section is to determine the irreducible factors of the generalized modified Lucas polynomials $\ell_n(x,\ 0,\ z)$. First, however, we wish to justify the names *Lucas cyclotomic polynomials* and *generalized Lucas cyclotomic polynomials* for the sequences

$$C_n(x,\ 0,\ 1) \quad \text{and} \quad C_n(x,\ 0,\ z),$$

since these sequences are determined by (3) from the generalized *modified* Lucas sequence $\ell_n(x,\ 0,\ z)$ and not the generalized Lucas sequence $L_n(x,\ z)$. The justification is this: that, by Theorem 1, the quotient (3) defines polynomials analogous to cyclotomic polynomials in the former case, but does not generally define polynomials at all if the L_n's are substituted for the ℓ_n's. (Nevertheless, the irreducible factors of the L_n's will be easily determined otherwise in Section 5.)

In Section 1, the (not generalized) Lucas analogue of the Fibonacci cyclotomic polynomials were named Lucas cyclotomic polynomials and denoted by $\mathcal{L}_n(x)$. Here however, we shall deal with the natural generalization, the generalized Lucas cyclotomic polynomials, denoted $\mathcal{L}_n(x,\ z)$ and defined by

$$\mathcal{L}_n(x,\ z) = C_n(x,\ 0,\ z).$$

By Theorem 3 and the identity $\sinh iu = i \sin u$, the roots of $\mathcal{L}_n(x,\ z)$ are

$$2i\sqrt{z}\ \sin 2k\pi/n,\ (k,\ n) = 1,\ 1 \le k \le n - 1.$$

The roots of $F_n(x,\ z)$ are $2i\sqrt{z}\ \cos k\pi/n$ for $1 \le k \le n - 1$, as proved in [5] and [6], and consequently, the roots of $\mathcal{F}_n(x,\ z)$ are

$$2i\sqrt{z}\ \cos k\pi/n,\ (k,\ n) = 1,\ 1 \le k \le n - 1.$$

In order to reconcile roots of the $\mathcal{L}_n(x,\ z)$'s with those of the $\mathcal{F}_n(x,\ z)$'s let

$$Q_n = \{k : (k,\ n) = 1 \quad \text{and} \quad 1 \le k \le n - 1\}.$$

for $k \in Q_n$, we have

$$\sin 2k\pi/n = \cos(n - 4k)\pi/2n.$$

As k ranges through the set Q_n, it is natural to expect the numbers $n - 4k$ to range through residue sets modulo various divisors or multiples of n. Such expectations are fulfilled in the next theorem.

Theorem 8: Except for $\mathcal{L}_1(x, z) = 1$ and $\mathcal{L}_4(x, z) = x^2 + 4z$, the nth generalized Lucas cyclotomic polynomial $\mathcal{L}_n(x, z)$ can be expressed in terms of the generalized Fibonacci cyclotomic polynomials as follows:

$$\mathcal{L}_n(x, z) = \begin{cases} \mathcal{F}_{2n}(x, z) & \text{for odd } n,\ n \neq 1, \\ \mathcal{F}_n(x, z) & \text{for } n = 2q,\ q \text{ odd}, \\ \mathcal{F}_q^2(x, z) & \text{for } n = 4q,\ q \text{ odd},\ q \neq 1, \\ \mathcal{F}_{2^t q}^2(x, z) & \text{for } n = 2^{t+1}q,\ q \text{ odd},\ t \geq 2. \end{cases}$$

Proof:

Case 1. Suppose n is odd and $n \neq 1$. Then

$$\cos\frac{(n - 4k)\pi}{2n} = \begin{cases} \cos\dfrac{|n - 4k|\pi}{2n} & \text{for } 4k < 3n, \\ \cos\dfrac{(5n - 4k)\pi}{2n} & \text{for } 4k > 3n. \end{cases}$$

Let

$$A = \{|n - 4k| : k \in Q_n \text{ and } 4k < 3n\},$$
$$B = \{5n - 4k : k \in Q_n \text{ and } 4k > 3n\},$$

and

$$Q = A \cup B.$$

It suffices to show that $Q = Q_{2n}$ and that each element of Q_{2n} appears only once in forming the set Q. This will be shown in four steps:

(i) $A \cap B$ is empty;
(ii) Q consists of $\phi(2n)$ elements;
(iii) If $j \in Q$, then $1 \leq j \leq 2n - 1$;
(iv) If $j \in Q$, then $(j, 2n) = 1$.

To verify (i), suppose $n - 4k_1 = 5n - 4k_2$ where $4k_1 < 3n$ and $4k_2 > 3n$. Then $k_2 - k_1 = n$, contrary to the inequalities

$$1 \leq k_1 \leq n - 1 \quad \text{and} \quad 1 \leq k_2 \leq n - 1.$$

If $|n - 4k_1| = 4k_1 - n = 5n - 4k_2$, then $2(k_1 + k_2) = 3n$, contrary to our assumption that n is odd.

For (ii), we know from (i) that distinct k's in Q_n provide distinct elements in Q. Furthermore, every element k in Q_n does yield an element of A or B, since $4k = 3n$ is impossible for odd n. Thus, Q consists of the same number of elements as Q_n, which is $\phi(n)$. Since n is odd, we have $\phi(n) = \phi(2n)$.

To verify (iii), first suppose $4k < 3n$. If $n - 4k \geq 0$, then $1 \leq n - 4k$ since n is an odd positive integer and, clearly, $n - 4k \leq 2n - 1$; if $n - 4k < 0$, then, similarly, $1 \leq 4k - n$, and $4k - n \leq 2n - 1$ since $4k < 3n$. Now suppose $4k > 3n$. Then $5n - 4k \leq 2n - 1$, and also $1 \leq 5n - 4k$, since $k < n$.

For (iv), if $d|\,|(n-4k)|$ and $d|2n$, then d must be odd since $n-4k$ is odd. Consequently, $d|n$. But then $d|4k$, so that $d|k$. Since $(k, n) = 1$, we conclude that $(n-4k, 2n) = 1$. The same clearly holds for $4k-n$ and $5n-4k$.

Case 2. Suppose $n = 2q$, q odd. Then

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|q-2k|\pi}{n} & \text{for } 2k < 3q, \\[2ex] \cos\dfrac{(5q-2k)\pi}{n} & \text{for } 2k > 3q. \end{cases}$$

Here, the numbers $|q-2k|$ and $5q-2k$, as stipulated, range through the set Q_n as k ranges through the set Q_n. The proof is so similar to that in Case 1 that we omit it here.

Case 3. Suppose $n = 4q$, q odd, $q \neq 1$. Let

$$A = \{k \in Q_n : k < q\},\ B = \{k \in Q_n : q < k < 2q\},$$
$$C = \{k \in Q_n : 2q < k < 3q\},\ D = \{k \in Q_n : 3q < k\}.$$

Each k in Q_n in odd, so that $(q-k)/2$ is an integer, and

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|(q-k)/2|\pi}{q} & \text{for } k \in A \cup B, \\[2ex] \cos\dfrac{[(5q-k)/2]\pi}{q} & \text{for } k \in C \cup D. \end{cases}$$

We first claim that as k ranges through the set $A \cup C$, the numbers $|(q-k)/2|$ and $(5q-k)/2$, as stipulated, range through the set Q_q. This claim is verified as in the four steps in Case 1. Starting with

$$A^* = \{|(q-k)/2| : k \in A\} \quad\text{and}\quad C^* = \{(5q-k)/2 : k \in C\},$$

only step (ii) calls for anything new: To see that $A^* \cup C^*$ consists of $\phi(q)$ elements [granted from step (i) that distinct k's lead to distinct elements in $A \cup B \cup C \cup D$], we note that the number of k's in Q_n is

$$\phi(4q) = \phi(4)\phi(q) = 2\phi(q),$$

and precisely half of these lie in $A^* \cup C^*$ since, as is easily checked, the sets A, B, C, D are in one-to-one correspondence with one another:

$$\begin{aligned} A \to B &: k \to 2q - k, \\ A \to C &: k \to 2q + k, \\ C \to D &: k \to 6q - k. \end{aligned}$$

Thus, the roots of $\mathcal{L}_n(x, z)$ found for $k \in A \cup C$ are the roots of $\mathcal{F}_q(x, z)$. That the same is true for $k \in B \cup D$ will now be proved. Since

$$B = \{2q - k : k \in A\},$$

we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in B\right\} = \left\{\cos\frac{|(q-k)/2|\pi}{q} : k \in A\right\}.$$

Since $D = \{6q - k : k \in C\}$, we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in D\right\} = \left\{\cos\frac{[(5q-k)/2]\pi}{q} : k \in C\right\}.$$

Thus the roots of $\mathcal{L}_n(x, z)$ for $k \in B \cup D$ are the roots of $\mathcal{F}_q(x, z)$. We conclude that $\mathcal{L}_n(x, z) = \mathcal{F}_q^2(x, z)$.

Case 4. Suppose $n = 2^{t+1}q$, q odd, $t \geq 2$. Define sets A, B, C, D as in Case 3, and have the following one-to-one correspondences:

$$\begin{aligned} A \to B &: k \to 2^t q - k, \\ A \to C &: k \to 2^t q + k, \\ C \to D &: k \to 3 \cdot 2^t q - k. \end{aligned}$$

Now

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|2^{t-1}q - k|\pi}{2^t q} & \text{for } k \in A \cup B, \\[2ex] \cos\dfrac{(5 \cdot 2^{t-1}q - k)\pi}{2^t q} & \text{for } k \in C \cup D. \end{cases}$$

We claim that as k ranges through the set $A \cup C$, the numbers $|2^{t-1}q - k|$ and $(5 \cdot 2^{t-1}q - k)$, as stipulated, range through the set $Q_{2^t q}$. The four steps in Case 3 easily verify this claim. We omit the verification, except to note that for step (ii) we have $\phi(2^{t+1}q) = 2\phi(2^t q)$, so that $\phi(2^t q)$ roots are found for $k \in A \cup C$.

As in Case 3, we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in B \cup D\right\} = \left\{\cos\frac{(n-4k)\pi}{2n} : k \in A \cup C\right\}.$$

Therefore, $\mathcal{L}_n(x, z) = \mathcal{F}_{2^t q}^2$, and Theorem 8 is proved.

Theorem 8 and Theorem 4 enable us to factor the polynomials $\ell_n(x, 0, z)$ completely in terms of irreducible factors. For example,

$$\begin{aligned} \ell_{60}(x, 0, z) &= \prod_{d|60} C_d(x, 0, z) \\ &= \prod_{d|60} \mathcal{L}_d(x, z) \\ &= \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4\mathcal{L}_5\mathcal{L}_6\mathcal{L}_{10}\mathcal{L}_{12}\mathcal{L}_{15}\mathcal{L}_{20}\mathcal{L}_{30}\mathcal{L}_{60} \\ &= x(x^2 + 4z)(\mathcal{F}_3\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{15}\mathcal{F}_{30})^2. \end{aligned}$$

Recalling that $F_{30} = \mathcal{F}_2\mathcal{F}_3\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{15}\mathcal{F}_{30}$, that $x\ell_{60}(x, 0, z) = L_{60} - 2z^{30}$, and that $x^2 + 4z$ is the discriminant $D(x, z)$ of $t^2 - xt - z$, we rewrite L_{60} as follows:

$$L_{60}(x, z) = D(x, z)F_{30}^2(x, z) + 2z^{30}.$$

Putting $x = z = 1$, we find an identity $L_{60} = 5F_{30}^2 + 2$ involving the thirtieth Fibonacci number and the sixtieth Lucas number. These considerations lead to the following theorems and corollary.

Theorem 9a: Suppose $m = 2^t q$, q odd, $t \geq 2$. Then

$$L_{2m}(x, z) = (x^2 + 4z)F_m^2(x, z) + 2z^m. \tag{4}$$

Proof:
$$\begin{aligned} \ell_{2m} &= \mathcal{L}_1\mathcal{L}_2\mathcal{L}_4 \ \ldots\ \mathcal{L}_{2^{t+1}}\mathcal{L}_q\mathcal{L}_{2q}\mathcal{L}_{4q}\ \ldots\ \mathcal{L}_{2^{t+1}q} \\ &= x(x^2 + 4z)\mathcal{F}_4^2\mathcal{F}_8^2\ \ldots\ \mathcal{F}_{2^t}^2\mathcal{F}_{2q}^2\mathcal{F}_q^2\mathcal{F}_{4q}^2\mathcal{F}_{8q}^2\ \ldots\ \mathcal{F}_{2^tq}^2 \\ &= x(x^2 + 4z)F_m^2/x^2, \end{aligned}$$

and (4) follows immediately.

Theorem 9b: If m is odd, then

$$L_{2m}(x, z) - 2z^m = L_m^2(x, z). \tag{5}$$

Proof: The proof of this known identity is so similar to that of Theorem 9a that we omit it here.

Corollary 9: For $k > 0$, let F_k and L_k be the kth Fibonacci and Lucas numbers. If $m = 2^t q$, q odd, $t \geq 2$, then

$$L_{2m} = 5F_m^2 + 2.$$

If m is odd, then

$$L_{2m} = L_m^2 + 2.$$

Proof: Put $x = z = 1$ in (4) and (5).

5. THE IRREDUCIBLE FACTORS OF THE LUCAS POLYNOMIALS

Hoggatt and Bicknell prove in [5] that for $n \geq 1$ the roots of the nth Lucas polynomial $L_n(x, 1)$ are

$$2i \cos \frac{(2k + 1)\pi}{2n},\ k = 0, 1, \ldots, n - 1.$$

The methods of Section 4 could be used to compare these roots with those of the Fibonacci cyclotomic polynomials. However, we choose a different way, which depends on the well-known identity $F_{2n} = L_nF_n$.

Theorem 10: For $n \geq 1$, write $n = 2^t q$, where $t \geq 0$ and q is odd. The nth generalized Lucas polynomial $L_n(x, z)$ is a product of (irreducible) Fibonacci cyclotomic polynomials:

$$L_n(x, z) = \prod_{d|q} \mathcal{F}_{2^{t+1}d}\ (x, z).$$

Proof:

$$L_n = \frac{F_{2n}}{F_n} = \frac{\prod_{d|2n} \mathcal{F}_d}{\prod_{d|n} \mathcal{F}_d} = \prod_{\substack{d|2n \\ d\nmid n}} \mathcal{F}_d.$$

Now

$$\{d : d|2n \text{ and } d\nmid n\} = \{2^{\ +1}d : d|n \text{ and } d \text{ is odd}\},$$

so that the conditions $d|2n$, $d\nmid n$ are replaceable by the condition $2^{t+1}d|2n$, i.e., $d|q$.

Example:

$$L_{60} = \frac{\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4\mathcal{F}_5\mathcal{F}_6\mathcal{F}_8\mathcal{F}_{10}\mathcal{F}_{12}\mathcal{F}_{15}\mathcal{F}_{20}\mathcal{F}_{24}\mathcal{F}_{30}\mathcal{F}_{40}\mathcal{F}_{60}\mathcal{F}_{120}}{\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{12}\mathcal{F}_{15}\mathcal{F}_{20}\mathcal{F}_{30}\mathcal{F}_{60}}$$

$$= \mathcal{F}_8\mathcal{F}_{24}\mathcal{F}_{40}\mathcal{F}_{120}.$$

Corollary 10: For even $n \geq 2$, $L_n(x, z)$ is irreducible if and only if $n = 2^k$ for some $k \geq 1$.

Proof: Suppose $n = 2^k$ for some $k \geq 1$. Then by Theorem 10, we have $L_n = \mathcal{F}_{2n}$, which is irreducible by Theorem 6. If n is even but not a power of 2, then by Theorem 10, $\mathcal{F}_{2n}$ is a proper divisor of $L_n(x, z)$.

In [2], Bergum and Hoggatt prove Corollary 10 using Eisenstein's Criterion.

We conclude this section by noting that the divisibility properties that are already established for the polynomials F_n, L_n, and ℓ_n in terms of the irreducible polynomials $\mathcal{F}_n$ now carry over to divisibility properties of Chebyshev polynomials of the first and second kinds.

It is well known that the nth Chebyshev polynomial of the first kind is

$$T_n(x) = \frac{1}{2}L_n(2x, -1), \; n = 0, 1, \ldots .$$

Accordingly, the factorization of $T_n(x)$ in terms of factors which are irreducible over the ring of integers is given by Theorem 10.

Let us define *modified Chebyshev polynomials of the first kind* by

$$t_n(x) = \begin{cases} \dfrac{1}{x}T_n(x) & \text{for odd } n, \\[2ex] \dfrac{1}{x}\left[T_n(x) - (-1)^{\frac{n}{2}}\right] & \text{for even } n > 0. \end{cases}$$

Then we have $t_n(x) = \frac{1}{2}\ell_n(2x, 0, -1)$, so that the divisibility properties of the t_n's are the same as those of the ℓ_n's. In particular, the irreducible factors are given by Theorem 8. Moreover, many of the results proved in [7] [e.g., concerning greatest common divisors, $(\ell_m, \ell_n) = \ell_{(m,n)}$] carry over to similar results for the modified Chebyshev polynomials.

It is well known that the nth Chebyshev polynomial of the second kind is

$$U_n(x) = F_{n+1}(2x, -1), \; n = 0, 1, \ldots .$$

Accordingly, the factorization of $U_n(x)$ in terms of irreducible factors is given by Theorem 6.

Finally, note that the roots of the Chebyshev and modified Chebyshev polynomials, and also the roots of their irreducible factors, are easily obtained from Theorem 1 and Theorem 3.

6. TRANSFORMED FIBONACCI AND LUCAS POLYNOMIALS

For any integers (or indeterminants) a, b, c, where $a \neq 0 \neq c$, let

$$U_n(x, z) = F_n(ax, bx^2 + cz^2),$$

$$V_n(x, z) = \frac{1}{2}L_n(ax, bx^2 + cz^2),$$

and $$W_n(x,\ z) = \ell_n(ax,\ 0,\ bx^2 + cz^2).$$

Then the quotients (3) are clearly polynomials for each of the sequences

$$U_n(x,\ z) \quad \text{and} \quad W_n(x,\ z),$$

since this is true for the sequences F_n and ℓ_n. Similarly, the divisibility properties of the V_n's follow from those of the L_n's, as given in [2] and Section 5.

One of the most attractive special cases is $(a,\ b,\ c) = (2,\ -1,\ 1)$. We tabulate the first few U_n's and V_n's in this case. Then we tabulate the first few W_n's and the first few *transformed Fibonacci cyclotomic polynomials*; i.e., the quotients (3) formed from the U_n's. These, we shall show, are irreducible except for a constant multiple; hence, they are the irreducible factors not only of the U_n's, but also of the V_n's and the W_n's. After the tables, we shall return to arbitrary a, b, c satisfying $a^2 + 4b = 0$ and find roots, Binet forms, etc.

TABLE 3

Transformed Generalized Fibonacci Polynomials $U_n = F_n(2x,\ z^2 - x^2)$

and Transformed Generalized Lucas Polynomials $V_n = \frac{1}{2}L_n(2x,\ z^2 - x^2)$

n	U_n	V_n
1	1	x
2	$2x$	$x^2 + z^2$
3	$3x^2 + z^2$	$x^3 + 3xz^2$
4	$4x^3 + 4xz^2$	$x^4 + 6x^2z^2 + z^4$
5	$5x^4 + 10x^2z^2 + z^4$	$x^5 + 10x^3z^2 + 5xz^4$
6	$6x^5 + 20x^3z^2 + 6xz^4$	$x^6 + 15x^4z^2 + 15x^2z^4 + z^6$
7	$7x^6 + 35x^4z^2 + 21x^2z^4 + z^6$	$x^7 + 20x^5z^2 + 35x^3z^4 + 7xz^6$

One immediately detects Pascal's triangle lurking within Table 3. We shall soon ascertain that $zU_n + V_n = (x + z)^n$ for $n \geq 1$.

TABLE 4

Transformed Generalized Modified Lucas Polynomials

$$W_n = \ell_n(2x,\ 0,\ z^2 - x^2)$$

$W_1 = 1$
$W_2 = 2x$
$W_3 = x^2 + 3z^2$
$W_4 = 8xz^2$
$W_5 = x^4 + 10x^2z^2 + 5z^4$
$W_6 = 2x^5 + 12x^3z^2 + 18xz^4$
$W_7 = x^6 + 21x^4z^2 + 35x^2z^4 + 7z^6$
$W_8 = 32x^5z^2 + 64x^3z^4 + 32xz^6$

TABLE 5

Transformed Generalized Fibonacci Cyclotomic Polynomials

$$\mathcal{U}_n = \mathcal{F}_n(2x,\ z^2 - x^2)$$

$\mathcal{U}_1 = 1$
$\mathcal{U}_2 = 2x$
$\mathcal{U}_3 = 3x^2 + z^2$
$\mathcal{U}_4 = 2x^2 + 2z^2$
$\mathcal{U}_5 = 5x^4 + 10x^2z^2 + z^4$
$\mathcal{U}_6 = x^2 + 3z^2$
$\mathcal{U}_8 = 2x^4 + 12x^2z^2 + 2z^4$
$\mathcal{U}_{10} = x^4 + 10x^2z^2 + 5z^4$
$\mathcal{U}_{12} = x^4 + 14x^2z^2 + z^4$

Lemma 11: Suppose n is an odd positive integer ≥ 3. Then

$$\prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = 2^{1-n}, \quad \prod_{k=0}^{\frac{n-3}{2}} \cos^2 \frac{(2k+1)\pi}{2n} = n2^{1-n}, \text{ and } \prod_{k=1}^{\frac{n-1}{2}} \sin^2 \frac{2k\pi}{n} = n2^{1-n}.$$

Suppose n is an even positive integer ≥ 4. Then

$$\prod_{k=1}^{\frac{n-2}{2}} \cos^2 \frac{k\pi}{n} = n2^{1-n} \quad \text{and} \quad \prod_{k=1}^{\frac{n-2}{2}} \sin^2 \frac{2k\pi}{n} = n^2 2^{-n}.$$

Suppose n is an even positive integer ≥ 2. Then

$$\prod_{k=1}^{\frac{n-2}{2}} \cos^2 \frac{(2k+1)\pi}{2n} = 2^{1-n}.$$

Proof: For odd $n \geq 3$, we have

$$\prod_{k=1}^{n-1} 2i \cos \frac{k\pi}{n} = F_n(0) = 1,$$

so that

$$2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = 1.$$

For even $n \geq 4$, let $G_n(x) = \frac{1}{x} F_n(x)$. Then $G_n(0) = n/2$, and

$$\prod_{k=1}^{n-1} \left(x - 2i \cos \frac{k\pi}{n}\right) = x \prod_{\substack{1 \leq k \leq n-1 \\ k \neq n/2}} \left(x - 2i \cos \frac{k\pi}{n}\right) = xG_n(x),$$

so that

$$\prod_{\substack{1 \leq k \leq n-1 \\ k \neq n/2}} 2i \cos \frac{k\pi}{n} = G_n(0) = n/2,$$

and

$$2^{n-2} \prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = n/2.$$

Proofs of the other four formulas follow from similar considerations of $L_n(0)$ and $\ell_n(0, 0, 1)$.

Theorem 11: Suppose $a^2 + 4b = 0$. Then, for $n \geq 3$, the roots of the polynomials $U_n(x, z)$, $V_n(x, z)$, and $W_n(x, z)$ are given by the following factorizations.

$$U_n(x, z) = \begin{cases} \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for odd } n \geq 3, \\ \displaystyle\frac{nax}{2} \prod_{k=1}^{\frac{n-2}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for even } n \geq 4. \end{cases}$$

$$V_n(x, z) = \begin{cases} \dfrac{nax}{2} \displaystyle\prod_{k=0}^{\frac{n-3}{2}} \left[cz^2 - bx^2 \tan^2 \frac{(2k+1)\pi}{2n}\right] & \text{for odd } n \geq 3, \\ \displaystyle\prod_{k=0}^{\frac{n-2}{2}} \left[cz^2 - bx^2 \tan^2 \frac{(2k+1)\pi}{2n}\right] & \text{for even } n \geq 2. \end{cases}$$

$$W_n(x, z) = \begin{cases} n \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \left[cz^2 - bx^2 \cot^2 \frac{2k\pi}{n}\right] & \text{for odd } n \geq 3, \\ \dfrac{n^2 ax}{4} \displaystyle\prod_{k=1}^{\frac{n-2}{2}} \left[cz^2 - bx^2 \cot^2 \frac{2k\pi}{n}\right] & \text{for even } n \geq 4. \end{cases}$$

Proof: $U_n(x, z) = F_n(ax, bx^2 + cz^2) = \displaystyle\prod_{k=1}^{n-1} \left(ax - 2i\sqrt{bx^2 + cz^2} \cos \frac{k\pi}{n}\right)$.

If n is odd and ≥ 3, then the $n - 1$ roots of $U_n(x, z)$ occur in conjugate pairs, so that

$$\begin{aligned} U_n(x, z) &= \prod_{k=1}^{\frac{n-1}{2}} \left[a^2x^2 + 4(bx^2 + cz^2)\cos^2 \frac{k\pi}{n}\right] \\ &= \prod_{k=1}^{\frac{n-1}{2}} \left(-4bx^2 \sin^2 \frac{k\pi}{n} + 4cz^2 \cos^2 \frac{k\pi}{n}\right) \\ &= \prod_{k=1}^{\frac{n-1}{2}} 4\left(\cos^2 \frac{k\pi}{n}\right)\left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) \\ &= \prod_{k=1}^{\frac{n-1}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) \end{aligned}$$

by Lemma 11.

If n is even and ≥ 4, then the $n - 2$ roots of $U_n(x, z)$ remaining after the root 0 is excluded occur in conjugate pairs, and we find as above that

$$U_n(x, z) = \frac{nax}{2} \prod_{k=1}^{\frac{n-2}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right).$$

With the help of Lemma 11, the remaining four factorizations are proved in the same way.

Lemma 12: Suppose $a^2 + 4b = 0$. For $n \geq 3$, the transformed generalized Fibonacci cyclotomic polynomial $\mathcal{U}_n(x, z) = \mathcal{F}_n(ax, bx^2 + cz^2)$ is given by

$$\mathcal{U}_n(x,\ z) = \begin{cases} \displaystyle\prod_{\substack{1 \le k \le (n-1)/2 \\ (k,n)=1}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for odd } n \ge 3, \\ \displaystyle\frac{nax}{2} \prod_{\substack{1 \le k \le (n-2)/2 \\ (k,n)=1}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for even } n \ge 4. \end{cases}$$

Proof: This is an obvious consequence of Theorem 11 and the fact that the roots of $\mathcal{F}_n(x,\ z)$ are

$$2i\sqrt{z} \cos \frac{k}{n},\ (k,\ n) = 1,\ 1 \le k \le n - 1.$$

Theorem 12: Suppose a, b, c are integers and $a^2 + 4b = 0$. Except for an integer multiple, for $n \ge 1$, the polynomial $\mathcal{U}_n(x,\ z)$ is irreducible over the ring of integers.

Proof: The proposition is clearly true for $n = 1$ and $n = 2$. Suppose, for $n \ge 3$, that $\mathcal{U}_n(x,\ z) = p(x,\ z)q(x,\ z)$. By Lemma 12 and the irreducibility (since $-b > 0$) of the factors

$$cz^2 - bx^2 \tan^2 \frac{k\pi}{n}$$

over the real number field, $p(x,\ z)$ has the form $P(x,\ z^2)$ and $q(x,\ z)$ has the form $Q(x,\ z^2)$. Thus, putting $r = ax$ and $s = bx^2 + cz^2$, we find

$$\mathcal{F}_n(r,\ s) = P\left(\frac{r}{a},\ \frac{a^2 s - br^2}{a^2 c}\right) Q\left(\frac{r}{a},\ \frac{a^2 s - br^2}{a^2 c}\right).$$

Since $\mathcal{F}_n(r,\ s)$ is irreducible, one of the polynomials P and Q must be constant. But then $p(x,\ z)$ or $q(x,\ z)$ is constant, as desired.

Theorem 13: Suppose $(a,\ b,\ c) = (2, -1,\ 1)$. The Binet formulas for the polynomials U_n, V_n, and W_n are as follows:

$$U_n(x,\ z) = \frac{(x + z)^n - (x - z)^n}{2z}$$

$$V_n(x,\ z) = \frac{(x + z)^n + (x - z)^n}{2}$$

$$W_n(x,\ z) = \begin{cases} \dfrac{1}{x} V_n(x,\ z) & \text{for odd } n, \\ \dfrac{(x + z)^n + (x - z)^n - 2(z^2 - x^2)^{n/2}}{2x} & \text{for even } n. \end{cases}$$

Proof: Let $t_1 = \dfrac{r + \sqrt{r^2 + 4s}}{2}$, $t_2 = \dfrac{r - \sqrt{r^2 + 4s}}{2}$, $t_3 = \sqrt{s}$, $t_4 = -\sqrt{s}$. Putting $r = 2x$ and $s = z^2 - x^2$, the desired formulas follow immediately from the Binet formulas

$$F_n(r,\ s) = \frac{t_1^n - t_2^n}{t_1 - t_2},$$

$$L_n(r,\ s) = t_1^n + t_2^n$$

$$\ell_n(r, 0, s) = \frac{t_1^n + t_2^n - t_3^n - t_4^n}{t_1 + t_2 - t_3 - t_4}.$$

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GEOMETRIC RECURRENCE RELATION

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1. INTRODUCTION

In a previous paper [1], we considered r, s sequences $\{U_k\}$ and obtained explicit formulations for the general term in powers of r and s. We noted 2 special sequences $\{G_k\}$ and $\{M_k\}$. These are sequences that specialize to the Fibonacci and Lucas sequences where $r = s = 1$.

In this paper, we propose to consider the relationship between r, s recurrence relations and geometric sequences. We give a necessary and sufficient condition on r and s for the recurrence relation to be geometric. We conclude the section by showing how to write any geometric sequence as an r, s recurrence relation.

In the final section, we briefly consider a special Fibonacci sequence. We give an explicit formulation for its general term. We are then able to note when it is a geometric sequence.

2. GEOMETRIC r, s SEQUENCES

In the previous paper [1] we considered the special r, s relations $\{G_k\}$ and $\{M_k\}$ which were characterized by the initial values $G_0 = 0$, $G_1 = 1$, $M_0 = 2$, and $M_1 = r$. We further specialize r and s so that the characteristic equation of the sequence has a multiple root λ. We then have $r = 2\lambda$ and $s = -\lambda^2$. It can be readily verified that the expression for the general terms are

$$G_k = k\lambda^{k-1} \quad \text{and} \quad M_k = 2\lambda^k.$$

Note that the M_k sequence is geometric with ratio of λ and first term of $M_0 = 2$. But the other sequence is not geometric. We shall develop the general conditions for which these two results are special cases.

Before going to the main theorem, we will make a few observations. Consider the general term of the r, s sequence $\{U_k\}$:

$$U_n = rU_{n-1} + sU_{n-2};\ U_0,\ U_1 \text{ arbitrary}.$$

If $s = 0$, this would be a geometric sequence starting with U_1. Further, if the initial values were such that $U_1 = rU_0$, the sequence would be geometric with U_0 as the first term.

If $r = 0$, we have two geometric sequences with ratio s. One of these is the even indexed U_k with U_0 as initial value. The other geometric sequence is the odd indexed U_k with U_1 as starting value.

We shall call these two cases the trivial cases. In other words, an r, s relation for which $rs = 0$ is trivially geometric.

There is a whole class of r, s sequences that are geometric only in this trivial case. These are the sequences, for which $U_0 = 0$, for in this case

$$U_2 = rU_1 + sU_0 = rU_1,$$
$$U_3 = rU_2 + sU_1 = (r^2 + s)U_1.$$

Now this is geometric only if $r^2 + s = r^2$. But this can only happen for $s = 0$. Included in this class is the $\{G_k\}$ sequence.

We shall assume in the rest of this section that U_0, r, and s are all nonzero. We are ready to state and prove our theorem.

Theorem 2.1: The r, s sequence $\{U_k\}$ is geometric if and only if

$$\frac{r + e}{2} = \frac{U_1}{U_0}, \text{ where } e = \pm\sqrt{r^2 + 4s}.$$

For convenience, we shall denote the ratio as m so that $r + e = 2m$ or $r = 2m - e$. We find that

$$s = \frac{e^2 - r^2}{4} = \frac{e^2 - (2m - e)^2}{4} = m(e - m).$$

We also need the result that

$$rm + s = 2m^2 - me + me - m^2 = m^2.$$

From the expression for U_2 and the assumption that $U_1 = mU_0$, we have

$$U_2 = rU_1 + sU_0 = r(mU_0) + sU_0 = (rm + s)U_0 = m^2U_0 = mU_1.$$

Assume that $U_k = mU_{k-1}$ for $k = 2, \ldots, i - 1$. For

$$U_i = rU_{i-1} + sU_{i-2} = r(mU_{i-2}) + sU_{i-2} = (rm + s)U_{i-2} = m^2U_{i-2} = mU_{i-1}.$$

Hence, the sequence is geometric with U_0 as first term and ratio of m.

Conversely, assume $\{U_k\}$ is geometric with ratio m so that $U_k = mU_{k-1}$ for all k. Since

$$U_k = rU_{k-1} + sU_{k-2} = (rm + s)U_{k-2},$$

and, by assumption,

$$U_k = mU_{k-1} = m(mU_{k-2}) = m^2U_{k-2},$$

it follows that $rm + s = m^2$. This means that m is a solution of the equation $x^2 - rx - s = 0$. The roots of this equation are $\frac{r \pm e}{2}$, so $m = \frac{r + e}{2}$. Further, $U_1 = mU_0$ so $\frac{U_1}{U_0} = m$. But these are the given equivalent conditions.

In the proof, it was not necessary that r and s be integers. The results are then valid for a more general recurrence relation. In the corollary that follows, we note how any geometric sequence can be expressed as an r, s relation.

Corollary 2.1: The geometric sequence $U_k = at^k$ can be represented as the r, s sequence with $U_0 = a$, $U_1 = at$, $r = 2t - \lambda$, $s = t\lambda - t^2$ for any λ.

By the choice of U_0 and U_1, we have $U_1 = tU_0$. Also,

$$e^2 = r^2 + 4s = 4t^2 - 4t\lambda + \lambda^2 + 4t\lambda - 4t^2 = \lambda^2,$$

so that

$$\frac{r + e}{2} = \frac{2t - \lambda + \lambda}{2} = t.$$

Hence, by the theorem, this r, s sequence is geometric.

3. A SPECIAL TRIBONACCI SEQUENCE

There is a special Tribonacci sequence that is geometric under some conditions. It can be verified that the sequence

$$T_n = rT_{n-1} + sT_{n-2} - rsT_{n-3};\ T_0,\ T_1,\ T_2 \text{ arbitrary}$$

has for a solution

$$T_{2k+2} = \sum_{j=0}^{k} r^{2k-2j}s^j(T_2 - sT_0) + s^{k+1}T_0;$$

$$T_{2k+3} = \sum_{j=0}^{k} r^{2k+1-2j}s^j(T_2 - sT_0) + s^{k+1}T_1.$$

The roots of the characteristic equation of the sequence are r, $\pm\sqrt{s}$. In case $T_2 - sT_0 = 0$, we see that the even-indexed terms form a geometric sequence with ratio s and initial value T_0. Note that the condition imposed has $T_2 = sT_0$. The odd-indexed terms also form a geometric sequence with ratio s and initial value T_1.

We have another important special case to be noted. If $T_0 = T_1 = 0$, we do not need to differentiate between even- and odd-indexed terms. We have for solution

$$T_m = \sum_{j=0}^{\left[\frac{m-2}{2}\right]} r^{m-2-2j}s^jT_2$$

if $T_2 = 1$, we have represented the restricted partitions of $m - 2$ as a sum of $(m - 2 - 2j)$ 1's and (j) 2's.

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REPRESENTATIONS FOR r, s RECURRENCE RELATIONS

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1. STATEMENT OF THE PROBLEM

Recently, Buschman [1], Horadam [2], and Waddill [3] considered properties of the recurrence relation

$$U_k = rU_{k-1} + sU_{k-2}$$

where r, s are nonnegative integers. Buschman and Horadam gave representations for U_k in powers of r and $e = (r^2 + 4s)^{1/2}$. In this paper we give them in powers of r and s. We write the K_n of Waddill as G_k. It is a generalization of the Fibonacci sequence. We also consider a sequence $\{M_k\}$ that is a generalization of the Lucas sequence.

For the $\{G_k\}$ and $\{M_k\}$ sequences, we obtain two representations for their general terms. From this, we move to a representation for the general term of the basic sequence. A computer program has been written that gives this term for specified values of the parameters.

In this paper we use some standard notation. We start by defining

$$e^2 = r^2 + 4s,$$

where e could be irrational. We also need to define

$$\alpha = (r + e)/2 \quad \text{and} \quad \beta = (r - e)/2.$$

In other words, α and β are solutions of the quadratic equation

$$x^2 - rx - s = 0.$$

We can easily show that $\alpha + \beta = r$, $\alpha - \beta = e$, and $\alpha\beta = -s$.

2. GENERALIZATIONS OF THE FIBONACCI AND LUCAS SEQUENCES

Using the α and β given in the first section, we can define two special r, s sequences. These are given by

$$G_k = \frac{\alpha^k - \beta^k}{e} (e \neq 0), \quad M_k = \alpha^k + \beta^k.$$

It is easy to verify that

$$G_0 = 0,\ G_1 = 1,\ G_2 = r,\ G_3 = r^2 + s,\ G_4 = r^3 + 2rs;$$
$$M_0 = 2,\ M_1 = r,\ M_2 = r^2 + 2s,\ M_3 = r^3 + 3rs,$$
$$M_4 = r^4 + 4r^2s + 2s^2;$$

and that they satisfy the basic r, s recurrence relation; i.e.,

$$G_2 = rG_1 + sG_0 \qquad M_2 = rM_1 + sM_0$$
$$G_3 = rG_2 + sG_1 \qquad M_3 = rM_2 + sM_1$$
$$G_4 = rG_3 + sG_2 \qquad M_4 = rM_3 + sM_2$$

In the next theorem, we prove that these two sequences are indeed r, s sequences.

Theorem 1: The sequences $\{G_k\}$ and $\{M_k\}$ are r, s sequences.

The proofs for both utilize mathematical induction. We have already indicated the validity of the theorem for $k = 2, 3$, and 4. We assume the terms satisfy the r,s relation for $k = 2, 3, \ldots, i - 1$. We form

$$\begin{aligned} rG_{i-1} + sG_{i-2} &= (\alpha + \beta)\frac{\alpha^{i-1} - \beta^{i-1}}{e} + (-\alpha\beta)\frac{\alpha^{i-2} - \beta^{i-2}}{e} \\ &= \frac{\alpha^i - \beta^i + \alpha^{i-1}\beta - \alpha\beta^{i-1} - \alpha^{i-1}\beta + \alpha\beta^{i-1}}{e} \\ &= \frac{\alpha^i - \beta^i}{e}. \end{aligned}$$

This is G_i by definition, so this sequence is an r, s sequence.

For the second part, we once more assume that the terms satisfy the r, s relation for $k = 2, \ldots, i - 1$. We form this time

$$\begin{aligned} rM_{i-1} + sM_{i-2} &= (\alpha + \beta)(\alpha^{i-1} + \beta^{i-1}) + (-\alpha\beta)(\alpha^{i-2} + \beta^{i-2}) \\ &= \alpha^i + \beta^i + \alpha^{i-1}\beta + \alpha\beta^{i-1} - \alpha^{i-1}\beta - \alpha\beta^{i-1} \\ &= \alpha^i + \beta^i. \end{aligned}$$

This is M by definition, so this too is an r, s sequence.

We obtain the Fibonacci and Lucas sequences from these two by letting $r = s = 1$. This can be readily verified.

In the next two theorems we give a more explicit formulation for G_k and M_k that can be easily programmed for a computer.

Theorem 2: For the sequence $\{G_k\}$,

$$G_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k - 1 - j}{j} r^{k-1-2j} s^j, \; k > 0; \; G_0 = 0.$$

We shall prove this by induction. We first note that this formulation for $k = 1, 2, 3, 4$ gives the same results as the previous one.

$$\begin{aligned} G_1 &= \binom{0}{0} r^0 s^0 = 1 \\ G_2 &= \binom{1}{0} r = r \\ G_3 &= \binom{2}{0} r^2 + \binom{1}{1} s = r^2 + s \\ G_4 &= \binom{3}{0} r^3 + \binom{2}{1} rs = r^3 + 2rs \end{aligned}$$

We assume that the result is valid for $k = 1, \ldots, i - 1$. We now show that $rG_{i-1} + sG_{i-2}$ does give the expression for G_i. Consider then

$$rG_{i-1} + sG_{i-2} = r\sum_{j=0}^{\left[\frac{i-2}{2}\right]}\binom{i-2-j}{j}r^{i-2-2j}s^j + s\sum_{j=0}^{\left[\frac{i-3}{2}\right]}\binom{i-3-j}{j}r^{i-3-2j}s^j$$

$$= \sum_{j=0}^{\left[\frac{i-2}{2}\right]}\binom{i-2-j}{j}r^{i-1-2j}s^j + \sum_{j=0}^{\left[\frac{i-3}{2}\right]}\binom{i-3-j}{j}r^{i-3-2j}s^{j+1}.$$

We now introduce a standard change that we use in several proofs. We first remove the first term of the first summation; then we shift the index of the second summation by replacing j by $j - 1$. This gives the same exponents for r and s in both summations. We then have

$$r^{i-1} + \sum_{j=1}^{\left[\frac{i-2}{2}\right]}\binom{i-2-j}{j}r^{i-1-2j}s^j + \sum_{j=1}^{\left[\frac{i-1}{2}\right]}\binom{i-2-j}{j-1}r^{i-1-2j}s^j.$$

If i is even, the upper limits of both summations are equal, so we can combine them into the single summation:

$$r^{i-1} + \sum_{j=1}^{\left[\frac{i-1}{2}\right]}\left[\binom{i-2-j}{j} + \binom{i-2-j}{j-1}\right]r^{i-1-2j}s^j$$

$$= r^{i-1} + \sum_{j=1}^{\left[\frac{i-1}{2}\right]}\binom{i-1-j}{j}r^{i-1-2j}s^j.$$

We see that the summand is r^{i-1} for $j = 0$. We include that term in the summation and obtain the desired expression for G_i.

If i is odd, then the upper limit on the second summation is one larger than that on the first. We break off the last term on the second summation and combine the two summands. This gives

$$r^{i-1} + \sum^{\left[\frac{i-3}{2}\right]}\left[\binom{i-2-j}{j} + \binom{i-2-j}{i-1}\right]r^{i-1-2j}s^j + s^{(i-1)/2}$$

$$= r^{i-1} + \sum^{\left[\frac{i-3}{2}\right]}\binom{i-1-j}{j}r^{i-1-2j}s^j + s^{(i-1)/2}.$$

We see that the summand gives r^{i-1} for $i = 0$ and $s^{(i-1)/2}$ for $i = \left[\frac{i-1}{2}\right]$. We combine these terms into the summation and we have the expression for G_i.

Hence, in any case, we do obtain the desired formula for G_i, so it must be valid for all terms of the sequence.

In passing, we might note that for the Fibonacci sequence we have

$$F_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-j}{j},\ k > 0;\ F_0 = 0.$$

In the next theorem for the $\{M_k\}$, we need the following property of binomial coefficients:

$$\frac{i-1}{i-1-j}\binom{i-1-j}{j} + \frac{i-2}{i-1-j}\binom{i-1-j}{j-1} = \frac{i}{i-j}\binom{i-j}{j}.$$

This can be readily verified using factorials.

Theorem 3: For the sequence $\{M_k\}$,

$$M_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k}{k-j}\binom{k-j}{j} r^{k-2j}s^j, \; k > 0; \; M_0 = 2.$$

The proof is by induction, so we first note that it is valid for $k = 1, 2, 3$.

$$M_1 = \sum_{j=0}^{0} \frac{1}{1-j}\binom{1-j}{j} r^{1-2j}s^j = \frac{1}{1}\binom{1}{0} r^1 s^0 = r;$$

$$M_2 = \sum_{j=0}^{1} \frac{2}{2-j}\binom{2-j}{j} r^{2-2j}s^j = \frac{2}{2}\binom{2}{0} r^2 + \frac{2}{1}\binom{1}{1} s = r^2 + 2s;$$

$$M_3 = \sum_{j=0}^{1} \frac{3}{3-j}\binom{3-j}{j} r^{3-2j}s^j = \frac{3}{3}\binom{3}{0} r^3 + \frac{3}{2}\binom{2}{1} r^2 = r^3 + 3r^2.$$

We assume that the formula is valid for $k = 2, 3, \ldots, i-1$ and show it is valid for M. The proof is similar to that of Theorem 2 except that we have an extra term for the case i is even.

We start with the basic

$$rM_{i-1} + sM_{i-2} = \sum_{j=0}^{\left[\frac{i-1}{2}\right]} \frac{i-1}{i-1-j}\binom{i-1-j}{j} r^{i-2j}s^j$$
$$+ \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \frac{i-2}{i-2-j}\binom{i-2-j}{j-1} r^{i-2-2j}s^{j+1}.$$

Once more we break off the first term in the first summation and shift the second summation index to give

$$r^i + \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \frac{i-1}{i-1-j}\binom{i-1-j}{j} r^{i-2j}s^j + \sum_{j=1}^{\left[\frac{i}{2}\right]} \frac{i-2}{i-1-j}\binom{i-1-j}{j-1} r^{i-2j}s^j.$$

If i is odd, the two summations have the same upper limit; thus, we can combine them using the property of binomial coefficients given before the theorem. This gives, for the summation,

$$r^i + \sum_{j=1}^{\left[\frac{i}{2}\right]} \frac{i}{i-j}\binom{i-j}{j} r^{i-2j}s^j.$$

Finally, note that the summand is r^i for $j = 0$. We combine into a single sum that is the formula for M_i.

In case i is even, the second summation has an extra term of $2s^{i/2}$. If we separate it from the summation, we can combine the two summations to get

$$r^i + \sum_{j=1}^{\left[\frac{i-2}{2}\right]} \frac{i}{i-j}\binom{i-j}{j} r^{i-2j}s^j + 2s^{i/2}.$$

The summand is r^i for $j = 0$ and $2s^{i/2}$ for $j = i/2$, so we can combine these and obtain the expression for M_i. Hence, in either case, the formula is valid for all integers k.

This theorem gives, for the general term of the Lucas sequence,

$$L_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k}{k-j}\binom{k-j}{j}, \ k > 0; \ L_0 = 2.$$

3. THE FORMULATION FOR U_k

In this section, we first prove a basic result for $\{U_k\}$. It is comparable to the result in Waddill's paper for $K_n = G_n$.

Theorem 4: The general term of $\{U_k\}$ can be expressed as

$$U_k = U_{t+j} = G_j U_{t+1} + G_{j-1} s U_t.$$

Once more the proof is by induction. For $j = 2$, we have

$$U_{t+2} = G_2 U_{t+1} + G_1 s U_t = rU_{t+1} + sU_t,$$

which is true for all t. Assume that the expression is true for $j = 2, \ldots, i - 1$. Then, since U_{t+i} is an r, s sequence,

$$\begin{aligned} U_{t+i} &= rU_{t+i-1} + sU_{t+i-2} = r(G_{i-1}U_{t+1} + G_{i-2}sU_t) + s(G_{i-2}U_{t+1} + G_{i-3}sU_t) \\ &= (rG_{i-1} + sG_{i-2})U_{t+1} + (rG_{i-2} + sG_{i-3})sU_t = G_i U_{t+1} + G_{i-1}U_t. \end{aligned}$$

Hence, the result is true for $j = i$ and so is true for all integers.

We can now give a formulation for U_k in terms of its initial values U_0 and U_1. This is given in the next theorem.

Theorem 5: The general term of the r, s sequence $\{U_k\}$ is given by

$$U_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j.$$

In Theorem 4, we take $t = 0$, so $j = k$, and we have

$$U_k = G_k U_1 + G_{k-1} s U_0.$$

Substituting the result of Theorem 2 for G_k, G_{k-1},

$$U_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j} r^{k-1-2j}s^j U_1 + \sum_{j=0}^{\left[\frac{k-2}{2}\right]} \binom{k-2-j}{j} r^{k-2-2j}s^j (sU_0).$$

Once more we break off the first term of the first summation and shift the index of the second summation to give

$$r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j} r^{k-1-2j}s^j U_1 + \sum_{j=1}^{\left[\frac{1}{2}\right]} \binom{k-1-j}{j-1} r^{k-2j}s^j U_0.$$

Again, we consider the two cases where k is odd or even. For k odd, the two upper indices are equal, so we can combine the two summations to obtain

$$r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k-1-j}{j} U_1 + \binom{k-1-j}{j-1} rU_0 \Bigg] r^{k-1-2j}s^j.$$

It can be verified that the summand can be written so that we have

$$\begin{aligned} U_k &= r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j \\ &= \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j \end{aligned}$$

For k even, we break off the last term in the second summation and have

$$\begin{aligned} &r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-2}{2}\right]} \binom{k-1-j}{j} U_1 + \binom{k-1-j}{j-1} rU_0 \Bigg] r^{k-1-2j}s^j + s^{k/2}U_0 \\ &= r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-2}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j + s^{k/2}U_0 . \end{aligned}$$

we note that the summand gives $r^{k-1}U_1$ for $j = 0$ and $s^{k/2}U_0$ for $j = k/2$. Thus we can write, for the general k,

$$U_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j.$$

It can be verified that by letting $U_1 = M_1 = r$ and $U_0 = M_0 = 2$, we obtain the expression for M_k given in Theorem 3.

We can obtain an expression for $\{U_k\}$ in terms of $\{M_k\}$. This is shown in the next theorem.

Theorem 6: The $\{U_k\}$ is given by

$$U_k = U_{t+j} = \frac{M_1M_j + sM_0M_{j-1}}{M_1^2 + sM_0^2} U_{t+j} + \frac{M_1M_{j-1} + sM_0M_{j-2}}{M_1^2 + sM_0^2} U_t .$$

We can obtain this result from Theorem 4 by determining G_j and G_{j-1} in terms of $\{M_k\}$. For this, we start with

$$M_{j-1} = G_{j-1}M_1 + G_{j-2}sM_0 = rG_{j-1} + 2sG_{j-2}.$$

Since $G_j = rG_{j-1} + sG_{j-2}$, it follows that $2sG_{j-2} = 2G_j - 2rG_{j-1}$. We substitute this into the expression for M_{j-1}, and also write the expression for M_j to give the two equations:

$$M_{j-1} = 2G_j - rG_{j-1};$$
$$M_j = rG_j + 2sG_{j-1}.$$

The solutions for G_i and G_{i-1} are

$$G_j = \frac{rM_j + 2sM_{j-1}}{r^2 + 4s} = \frac{M_1M_j + sM_0M_{j-1}}{M_1^2 + sM_0^2}$$

and

$$G_{j-1} = \frac{2M_j - rM_{j-1}}{r^2 + 4s} = \frac{2(rM_{j-1} + sM_{j-2}) - rM_{j-1}}{r^2 + 4s} = \frac{M_1M_{j-1} + sM_0M_{j-2}}{M_1^2 + sM_0^2}.$$

Substituting the results in the expression for U_k of Theorem 4 gives the required expression for this theorem.

The formulation for U_k given in Theorem 5 has been programmed by Robert C. Fitzgerald. He is a senior in Computer Science. We can generate the U_k for specified values of r, s, U_1 and U_0.

Special cases of this result for $e = 0$ and other particular values of r and s will be considered in a future paper.

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THORO'S CONJECTURE AND ALLIED DIVISIBILITY PROPERTY OF LUCAS NUMBERS

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In [3], Thoro made a conjecture that for any prime $p \equiv 3 \pmod 4$, the congruence $F_{2n+1} \equiv 0 \pmod p$ is not solvable where F_{2n+1} is an arbitrary Fibonacci number of odd index. The conjecture has already been proved. In what follows, we give a different proof of this and discuss another problem that arose during this investigation.

Proof: If possible, let the above congruence be true: since $F_{2n+1} = F_n^2 + F_{n+1}^2$ (see [1], p. 56), we get

(1) $$F_n^2 + F_{n+1}^2 \equiv 0 \pmod p$$

Under this hypothesis, it follows that p divides neither F_n nor F_{n+1}. This is justified because if, on the contrary, p divides F_n, then (1) would enable us to conclude that p divides F_{n+1}, forcing us to the invalid result that p divides (F_n, F_{n+1}) or p divides 1. Hence,

$$F_n^2 \equiv -F_{n+1}^2 \pmod{p}.$$

Using Legendre symbol, it means that

$$\left(\frac{-F_{n+1}^2}{p}\right) = 1 \quad \text{or} \quad \left(\frac{-1}{p}\right) = 1.$$

This is not valid, since the prime p is $\equiv 3 \pmod 4$. The required conclusion is now immediate.

Further analysis in regard to divisibility property possessed by Lucas numbers yielded the following theorem.

Theorem: If L_{2n} is an arbitrary Lucas number of even index, then there always exists a prime $p \equiv 3 \pmod 4$ which satisfies the congruence $L_{2n} \equiv 0 \pmod p$.

Proof: Using the result $F_{2n+1} \equiv 1, 2, 5 \pmod 8$ of [3] and the fact that $L_{2n} = F_{2n-1} + F_{2n+1}$ (see [1], p. 56), we obtain $L_{2n} \equiv 2, 3, 4, 6, 7 \pmod 8$. This means that $L_{2n} \not\equiv 1 \pmod 4$. Since the case of L_{2n} being even arises only when $3|n$, we conclude that $L_{6n\pm2} \equiv 3 \pmod 4$. This means that $L_{6n\pm2}$ always contains at least one prime factor p with $p \equiv 3 \pmod 4$. In fact, in this case, either this Lucas number is prime of this type or it will contain an odd number of prime factors of this type. For discussion of the case L_{6k}, we first observe that all the members of the family L_{6k} can be obtained from $L_{2^m(6n+3)}$ by choosing suitable values of m and n, where $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$. Now, using the fact that

$$L_t | L_s \text{ iff } s = (2k - 1)t$$

(see [1], p. 40), we get

$$L_{2^m} | L_{2^m(6n+3)}.$$

Since $(2^m, 3) = 1$, by previous discussion, there always exists a prime $p \equiv 3 \pmod 4$ such that $p|L_{2^m}$, which implies that $p|L_{2^m(6n+3)}$ and the proof is complete. It is easy to verify that $3|L_6$, $7|L_{12}$, $3|L_{18}$, $47|L_{24}$ and so on. For a strong result, namely $2 \cdot 3^k | L_{2\cdot3^k}$, refer to [2].

Corollary: L_{6n} contains an even number of prime factors p where $p \equiv 3 \pmod 4$.

Proof: From the well-known identities (see [1], p. 56), we have

$$L_{2n} = F_{n-1}^2 + 2F_n^2 + F_{n+1}^2,$$

which yields

$$L_{6n} = F_{3n-1}^2 + 2F_{3n}^2 + F_{3n+1}^2.$$

Since F_{3n} is even whereas F_{3n-1} and F_{3n+1} are odd, we have $F_{3n-1}^2 \equiv 1 \pmod 8$, $F_{3n+1}^2 \equiv 1 \pmod 8$, and $2F_{3n}^2 \equiv 0 \pmod 8$. Therefore, $L_{6n} \equiv 2 \pmod 8$ or $L_{6n} = 2(4\alpha + 1)$ for a suitable α.

From the above theorem, we have the existence of at least one prime $p \equiv 3 \pmod 4$ such that $p|L_{6n}$. We conclude that L_{6n} must have an even number of such factors for justifying the odd factor $(4\alpha + 1)$ stated above.

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A CLASS OF SOLUTIONS OF THE EQUATION $\sigma(n) = 2n + t$

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INTRODUCTION

Let the nondeficient natural number n satisfy

(1) $$f(n) = t,$$

where $f(n) = \sigma(n) - 2n$, and t is a given nonnegative integer. Clearly, (1) is equivalent to

(1*) $$\sigma(n) = 2n + t.$$

Definition 1: m is acceptable with respect to n if m is a nondeficient proper divisor of n.

Definition 2: n is primitive if no number is acceptable with respect to n; otherwise, n is nonprimitive.

Remark 1: Primitive nondeficient numbers were defined by L. E. Dickson [3], p. 413.

If $t = 0$ in (1), then n is called perfect. It is known that when n is perfect:

(a) if n is even, then $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime (Euclid-Euler);
(b) if n is odd, then n has at least 8 distinct prime factors [4] and exceeds 10^{50} [5];
(c) n is primitive.

If $t = 1$ in (1), then n is called quasiperfect [2]. It is known that if n is quasiperfect, then:

(a) n is odd and primitive [2];
(b) n has at least 6 distinct prime factors and exceeds 10^{30} [6].

On the other hand, for $t = 3$, by inspection we obtain the nonprimitive solution $n = 18$. This suggests that nonprimitive solutions of (1), when they exist, are more easily obtained than primitive ones.

In this article, we shall determine the set of all nonprimitive solutions of (1) for each t such that $2 \le t \le 100$. Theorem 1 states that Table 5 contains all such solutions for the given range of values of t.

* * *

Definition 3: For given nonnegative t, let $S(t)$ denote the set of all nonprimitive solutions of (1).

Pomerance [7] showed that $S(t)$ is finite unless there exists k such that $t = \sigma(k) = 2k$.

Remark 2: In this case, a subset of $S(t)$ consists of all numbers kq where q is prime and $(k, q) = 1$. If also k is even, so that $t = 2^p(2^p - 1)$ and $2^p - 1$ is prime, then it is easily verified that $2^{2p-1}(2^p - 1)$ and $2^{p-1}(2^p - 1)^3$ also belong to $S(t)$.

Lemma 1: If m is acceptable with respect to n, then $f(m) < f(n)$.

Proof: By [7], Lemma 5, we have $\sigma(m)/m < \sigma(n)/n$. Therefore,

$$(\sigma(m) - 2m)/m < (\sigma(n) - 2n)/n,$$

i.e., $f(m)/m < f(n)/n$. Now,

$$f(m) \geq 0 \Rightarrow f(m)/n < f(m)/m \Rightarrow f(m)/n < f(n)/n \Rightarrow f(m) < f(n).$$

Definition 4: m is maximal with respect to n if m is the largest number that is acceptable with respect to n.

Lemma 2: If n is nonprimitive and m is maximal with respect to n, then there exists a prime, p, such that $n = mp$.

Proof: Let p be a prime which divides n/m, i.e., mp divides n. Now $mp > m$, so that, by hypothesis and Lemma 1, we have

$$f(mp) > f(m) \geq 0.$$

Since m is maximal with respect to n, mp is not a proper divisor of n. Thus, $mp = n$.

Corollary 2.1: m is maximal with respect to n if and only if $m = n/p$, where p is the least prime such that n/p is an integer which is acceptable with respect to n.

Proof: The proof follows directly from Lemma 2.

Corollary 2.2: If $n/2$ is a nondeficient integer, then $n/2$ is maximal with respect to n.

Proof: The proof follows directly from Corollary 2.1.

In order to construct Table 5, we first determine all nonprimitive n such that $f(n) \leq 100$. Assume, furthermore, that $n = mp$ where p is prime and m is maximal with respect to n. The need for the latter condition will be justified below.

Case 1. Suppose $(m, p) = 1$. Then

$$f(n) = f(mp) = \sigma(mp) - 2mp = (p + 1)\sigma(m) - 2mp = pf(m) + \sigma(m).$$

Thus, $2m \leq \sigma(m) \leq f(n) \leq 100$, so that $m \leq 50$. Now,

$$f(m) \geq 0 \Rightarrow m \in \{6, 12, 18, 20, 24, 28, 30, 36, 40, 42, 48\}.$$

Suppose that $m = 2^a 3^b c > 6$, where a, b, and c are natural numbers and $(6, c) = 1$. Then $n = 2^a 3^b cp$, with $(6c, p) = 1$. If $c = 1$, then $a > 1$ or $b > 1$. If $a > 1$, then $2^{a-1}3^b p$ is acceptable with respect to n, so that $2^{a-1}3^b p < 2^a 3^b$, which implies $p < 2$, an impossibility. Similarly, $b > 1$ implies $p < 3$. If $c > 1$, then $2^a 3^b p$ is acceptable with respect to n, so that $2^a 3^b p < 2^a 3^b c$, and $p < c$. Now,

$$(6c,\ p) = 1 \Rightarrow p \geq 5 \Rightarrow c \geq 6.$$

But $6c \leq m \leq 50 \Rightarrow c \leq 8$. Thus,

$$(6,\ c) = 1 \Rightarrow c = 7 \Rightarrow m = 42 \Rightarrow f(n) = 12p + 96 \geq 156,$$

contradicting the hypothesis. Likewise, $m = 40 \Rightarrow f(n) = 10p + 90 \geq 120$. If $m = 6$ and $p \geq 5$, then $f(6p) = 12$. By Corollary 2.1, it is easily verified that 6 is maximal with respect to $6p$. If $m = 28$ and $(14, p) = 1$, then $f(28p) = 56$. If $p < 11$, then $14p$ is maximal with respect to $28p$; if $p \geq 11$, then 28 is maximal with respect to $28p$. If $m = 20$ and $(10, p) = 1$, then $f(20p) = 42 + 2p$. As above, 20 is maximal with respect to $20p$ if and only if $p \geq 11$. Also,

$$f(n) = f(20p) \leq 100 \Rightarrow p \leq 29.$$

For each $m \in \{6, 28, 20\}$, and for each prime p such that m is maximal with respect to $n = mp$, and $f(n) \leq 100$, we list m, p, n, and $f(n)$ in Table 1.

TABLE 1

m	p	n	$f(n)$
6	≥ 5	$6p$	12
28	≥ 11	$28p$	56
20	11	220	64
20	13	260	68
20	17	340	76
20	19	380	80
20	23	460	88
20	29	580	100

Case 2. Suppose p divides m. Let $m = p^k r$, $n = p^{k+1}r$, where $(p, r) = 1$. Now,

$$\begin{aligned} f(m) &= \sigma(m) - 2m = \sigma(p^k r) - 2p^k r = \sigma(p^k)\sigma(r) - 2p^k r \\ &= (p^k + \sigma(p^{k-1}))\sigma(r) - 2p^k r = p^k(\sigma(r) - 2r) + \sigma(p^{k-1})\sigma(r). \end{aligned}$$

Similarly,

$$f(n) = p^{k+1}(\sigma(r) - 2r) + \sigma(p^k)\sigma(r).$$

Therefore,

$$f(n) - f(m) = (p^{k+1} - p^k)(\sigma(r) - 2r) + p^k\sigma(r) = p^k(p\sigma(r) - (p - 1)2r).$$

Now,

$$f(n) = t \Rightarrow 0 \leq f(n) - f(m) = d \leq t.$$

Therefore, the solutions of (1) may be found among the solutions of

(2) $$f(n) - f(m) = d, \text{ where } d \leq 100.$$

Let $h(p, k, r) = p^k(p\sigma(r) - (p - 1)2r)$. Then (2) is equivalent to

(3) $$h(p, k, r) = d,$$

with the restriction that

(4) $$f(p^k r) \geq 0.$$

Furthermore, (4) implies

(5) $$r \geq 2,$$

since $f(p^k) < 0$ for all primes p and all exponents k. Henceforth we consider (3).

Definition 5: Let $g(r) = \sigma(r) - r$, where r is a natural number.

Lemma 3: If

(6) $$h(2,\ k,\ r) = d, \text{ where } r \text{ is odd,}$$

then $d \equiv 0 \pmod 4$. All solutions of (6) for $d \leq 100$ are given in Table 2.

TABLE 2

d	k	$g(r)$	r	s	n	$f(n)$	d	k	$g(r)$	r	s	n	$f(n)$
4	1	1	3	+	12	4	64	1	16	*	*	*	*
8	1	2	*	*	*	*	64	2	8	49	+	392	71
8	2	1	3	+	24	12	64	3	4	9	+	144	115
8	2	1	5	+	40	10	64	4	2	*	*	*	*
8	2	1	7	+	56	8	64	5	1	3	+	192	124
12	1	3	*	*	*	*	64	5	1	5	+	320	122
16	1	4	9	+	36	19	64	5	1	7	+	448	120
16	2	2	*	*	*	*	64	5	1	11	+	704	116
16	3	1	3	+	48	28	64	5	1	13	+	832	114
16	3	1	5	+	80	26	64	5	1	17	+	1088	110
16	3	1	7	+	112	24	64	5	1	19	+	1216	108
16	3	1	11	+	176	20	64	5	1	23	+	1472	104
16	3	1	13	+	208	18	64	5	1	29	+	1856	98
20	1	5	*	*	*	*	64	5	1	31	+	1984	96
24	1	6	25	–	*	*	64	5	1	37	+	2368	90
24	2	3	*	*	*	*	64	5	1	41	+	2824	86
28	1	7	*	*	*	*	64	5	1	43	+	2952	84
32	1	8	49	–	*	*	64	5	1	47	+	3008	80
32	2	4	9	+	72	51	64	5	1	53	+	3392	74
32	3	2	*	*	*	*	64	5	1	59	+	3776	68
32	4	1	3	+	96	60	64	5	1	61	+	3904	66
32	4	1	5	+	160	58	68	1	17	39	+	156	80
32	4	1	7	+	224	56	68	1	17	55	–	*	*
32	4	1	11	+	352	52	72	1	18	289	–	*	*
32	4	1	13	+	416	50	72	2	9	15	+	120	120
32	4	1	17	+	544	46	76	1	19	65	–	*	*
32	4	1	19	+	608	44	76	1	19	77	–	*	*
32	4	1	23	+	736	40	80	1	20	361	–	*	*
32	4	1	29	+	928	34	80	2	10	*	*	*	*
32	4	1	31	+	992	32	80	3	5	*	*	*	*
36	1	9	15	+	60	48	84	1	21	51	+	204	96
40	1	10	*	*	*	*	84	1	21	91	–	*	*
40	2	5	*	*	*	*	88	1	22	*	*	*	*
44	1	11	21	+	84	56	88	2	11	21	+	168	144
48	1	12	121	–	*	*	92	1	23	57	+	228	104
48	2	6	25	+	200	65	92	1	23	85	–	*	*
48	3	3	*	*	*	*	96	1	24	529	–	*	*
52	1	13	27	+	108	64	96	2	12	121	–	*	*
52	1	13	35	+	140	56	96	3	6	25	+	400	161
56	1	14	169	–	*	*	96	4	3	*	*	*	*
56	2	7	*	*	*	*	100	1	25	95	–	*	*
60	1	15	33	+	132	72	100	1	25	119	–	*	*
							100	1	25	143	–	*	*

Proof:

$$h(2, k, r) = 2^k(2\sigma(r) - 2r) = 2^{k+1}g(r) = d;\ k \geq 1 \Rightarrow d \equiv 0 \pmod 4.$$

To solve (6) for $d \leq 100$, we proceed as follows. For each d such that $d \equiv 0 \pmod 4$, and for each k such that $d \equiv 0 \pmod{2^{k+1}}$, we compute $g(r) = 2^{-(k+1)}d$. Next, we list the corresponding odd values of r, if any, using [1], Table 6.1. If no such r exists, then there is no solution of (6) corresponding to the chosen values of d and k. In this case, the r column and all columns to its right contain asterisks. For each possible r, we compute $f(2^k r)$ and list its sign, s, considering 0 to be positive. If $f(2^k r) < 0$, then there is no solution, and the last two columns contain asterisks. If $f(2^k r) \geq 0$, then we have obtained a solution of (6), and $n = 2^{k+1}r$ corresponds to a solution of (2). In this case, we list n and $f(n)$. If $g(r) = 1$, then r is prime and (4) implies $r \leq 2^k - 1$. In this case, we list only such r.

Lemma 4: If

(7) $$p\sigma(r) - (p - 1)2r = v,$$

where p is an odd prime, $(p, r) = 1$, and (4) holds, then we must have:

(8) $$\sigma(r) = pv + (p - 1)2u;$$

(9) $$r = (p + 1)v/2 + pu;$$

(10) $$(p, v) = 1;$$

(11) $$r \leq v\sigma(p^k)/2.$$

Proof: Solving (7) for $\sigma(r)$ and $2r$ in terms of p and v, one has

(8*) $$\sigma(r) = pv + (p + 1)w;$$

(9*) $$2r = (p + 1)v + pw.$$

(9*) $\Rightarrow w$ is even. Setting $w = 2u$, one obtains (8) and (9). (10) follows directly from the hypothesis. (11) is derived from (4) as follows:

$$f(p^k r) \geq 0 \Rightarrow \sigma(p^k)\sigma(r) \geq 2p^k r \Rightarrow p\sigma(p^k)\sigma(r) \geq 2p^{k+1}r;$$
$$(7) \Rightarrow p\sigma(p^k)\sigma(r) - (p - 1)\sigma(p^k)2r = v\sigma(p^k).$$

Therefore,

$$2p^{k+1}r - (p^{k+1} - 1)2r \leq v\sigma(p^k) \Rightarrow 2r \leq v\sigma(p^k) \Rightarrow r \leq v\sigma(p^k)/2.$$

Corollary 4.1: If

(12) $$h(p, k, r) = p^j s,$$

where p is an odd prime, $s \geq 1$, and $(p, s) = 1$, then $k = j$.

Proof: By hypothesis, (7) holds with $v = p^{j-k}s$. Now (10) implies $j - k = 0$, i.e., $k = j$.

Lemma 5: If

(13) $$h(p, k, r) = q,$$

where q is an odd prime, then $k = 1$, and for some integer, a, we have

$$p = q = 2^a - 1,\ r = 2^{a-1}.$$

Proof: p divides $q \Rightarrow p = q$. Hypothesis and Corollary 4.1 $\Rightarrow k = 1$. Thus, (13) reduces to (7) with $v = 1$. From (11), we have $r \leq (p + 1)/2$, so that

$u \le 0$ in (9). But (5) and (9) $\Rightarrow u \ge (3 - p)/2p$. Therefore, $u = 0$, i.e., $\sigma(r) = p$, $r = (p + 1)/2$. $\sigma(r) = p \Rightarrow r = s^{a-1}$ for some prime, s, and some integer $a \ge 2$. Now,

$$s^{a-1} + s^{a-2} + \cdots + s + 1 = \sigma(s^{a-1}) = \sigma(r) = p = 2r - 1 = 2s^{a-1} - 1.$$

Therefore, 2 divides s, i.e., $s = 2$. Thus, $r = 2^{a-1}$, $p = 2^a - 1$.

Lemma 6: For any j, the unique solution of

$$h(p,\ k,\ r) = 3^j \tag{14}$$

is: $p = 3$, $k = j$, $r = 2$.

Proof: Clearly, $p = 3$, $k = j$, and (14) reduces to $3\sigma(r) - 4r = 1$. (8) and (9) $\Rightarrow r = 2 + 3u$, $\sigma(r) = 3 + 4u \Rightarrow \sigma(r)$ is odd $\Rightarrow r = 2^a b^2$ with $a \ge 0$ and b odd. Furthermore, $(3, r) = 1 \Rightarrow (6, b) = 1$. $r \equiv 2 \pmod 3 \Rightarrow 2^a b^2 \equiv 2 \pmod 3 \Rightarrow 2^a \equiv 2 \pmod 3 \Rightarrow a \ge 1 \Rightarrow r$ is even $\Rightarrow \sigma(r)/r \ge 3/2 \Rightarrow 2\sigma(r) \ge 3 \Rightarrow 6 + 8u \ge 6 + 9u \Rightarrow u \le 0 \Rightarrow r \le 2$. By (5), $r = 2$.

Lemma 7: For no j does

$$h(p,\ k,\ r) = 5^j \tag{15}$$

have a solution.

Proof: If a solution exists, then $p = 5$, $k = j$, and (15) reduces to $5\sigma(r) - 8r = 1$, so that $r = 3 + 5u$, $\sigma(r) = 5 + 8u$, and $r = 2^a b^2$ with $a \ge 0$ and $(10, b) = 1$. Now $r \equiv 3 \pmod 5 \Rightarrow 2^a b^2 \equiv 3 \pmod 5 \Rightarrow 2^a \equiv 2$ or $3 \pmod 5 \Rightarrow a = 2c + 1$. But $\sigma(2^{2c+1}) \equiv 0 \pmod 3$. Thus,

$$\sigma(r) \equiv 0 \pmod 3 \Rightarrow u \equiv 2 \pmod 3 \Rightarrow r \equiv 1 \pmod 3$$
$$\Rightarrow 2^{2c+1} b^2 \equiv 1 \pmod 3 \Rightarrow b^2 \equiv 2 \pmod 3,$$

an impossibility.

Lemma 8: If

$$h(p,\ k,\ r) = q^j, \tag{16}$$

where q is an odd prime, $j \ge 2$, and $q^j \le 100$, then $k = j$ and either

(i) $p = 3$, $r = 2$, $2 \le j \le 4$; or

(ii) $p = 7$, $r = 4$, $j = 2$.

Proof:

$$q^2 \le q^j \le 100 \Rightarrow q \le 10 \Rightarrow q \in \{3,\ 5,\ 7\}.$$

If $q = 3$, then $3^j \le 100 \Rightarrow j \le 4$, and the solutions of (16) are given by Lemma 6. Lemma 7 $\Rightarrow q \ne 5$. If $q = 7$, then $7^j \le 100 \Rightarrow j = 2$, and (16) reduces to $7\sigma(r) - 12r = 1$. Therefore, by Lemma 4, we have

$$\sigma(r) = 7 + 12u,\ r = 4 + 7u,\ r \le 28.$$

By inspection, we must have $r = 4$.

Combining the results of Lemmas 5 and 8, we list all solutions of

$$h(p,\ k,\ r) = q^j, \tag{17}$$

with q an odd prime and $q^j \le 100$, in Table 3. For each q^j, we list p, k, r, as well as the m, n of the corresponding solution of (2), and $f(n)$. It is easily verified that in each case m is maximal with respect to n.

TABLE 3

q	p	k	r	m	n	$f(n)$
3	3	1	2	6	18	3
7	7	1	4	28	196	7
9	3	2	2	18	54	12
27	3	3	2	54	162	39
31	31	1	16	496	15736	31
49	7	2	4	196	1372	56
81	3	4	2	162	486	120

Lemma 9: For no odd prime q does

(18) $$h(p, k, r) = 2q$$

have a solution.

Proof: If a solution exists, then by hypothesis, Lemma 3, and Corollary 4.1, we have $p \neq 2$, $p = q$, and $k = 1$. Thus, (18) reduces to (7) with $v = 2$, and we have $\sigma(r) = 2p + (p - 1)2u$, $r = p + 1 + pu$, $r \leq p + 1$. Now, (5) $\Rightarrow u = 0$, $r = p + 1$, $\sigma(r) = 2p$. Let $r = 2^a b$ with $a \geq 1$ and b odd. Then,

$$\sigma(r) = \sigma(2^a)\sigma(b) = 2p, \text{ so that } \sigma(b) = 2,$$

an impossibility.

Definition 6: If $0 \leq a \leq 3$, let

$$C_a = \{r : 2 \leq r \leq 100, \text{ and } \sigma(r) \equiv a \pmod 4\}.$$

By inspection, we have

$C_0 = \{3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38, 39, 42, 43, 44, 46, 47, 48, 51, 54, 55, 56, 57, 59, 60, 62, 63, 65, 66, 67, 69, 70, 71, 75, 76, 77, 78, 79, 83, 84, 85, 86, 87, 88, 91, 92, 93, 94, 95, 96, 99\}$;

$C_1 = \{9, 49, 50, 81, 100\}$;

$C_2 = \{5, 10, 13, 17, 20, 26, 29, 34, 37, 40, 41, 45, 52, 53, 58, 61, 68, 73, 74, 80, 82, 89, 90, 97\}$;

$C_3 = \{2, 4, 8, 16, 18, 25, 32, 36, 64, 72, 98\}$.

Lemma 10: In (3), if $r = q^b$, where q is prime, then $q = 2$ and $r \in C_3$.

Proof: (4) implies

$$(p/(p - 1))(q/(q - 1)) > \sigma(p^k r)/p^k r \geq 2 \Rightarrow q < 2(p - 1)/(p - 2).$$

If $p = 3$, then $q < 4 \Rightarrow q = 2$, since $(p, r) = 1$. If $p \geq 5$, then $q < 8/3 \Rightarrow q = 2$. $\sigma(2^b) = 2^{b+1} - 1 \equiv 3 \pmod 4) \Rightarrow r \in C_3$.

Lemma 11: All solutions of (3) such that p is odd, $d \leq 100$, $d \neq q^j$, where q is an odd prime, are given in Table 4.

Proof: To obtain the desired solutions of (3), we proceed as follows: for each $d \neq 2q$, $\neq q^j$, for each odd prime p such that $p^k v = d$, $(p, v) = 1$, we list p, k, v. If r exists such that (7) holds, we must have:

(i) $r \leq \underline{r} = [v\sigma(p^k)/2]$;

(ii) $r \equiv v(p + 1)/2 \pmod p$;

(iii) $r \in C_a$, where $pv \equiv a \pmod 4$;

(iv) r is not a power of a prime unless $r = 2^b$ and $a = 3$.

For convenience, we list $\underline{r}$, r_p [the least positive residue (mod p) of $v(p+1)/2$], and a. If no r exists satisfying the above conditions, then (3) has no solution corresponding to that particular choice of p, d. In this case, the r column and all remaining columns contain asterisks. For each r which does satisfy the conditions, we compute and list $w = p\sigma(r) - (p - 1)2r$. If $w \neq v$, then we have no solution, and the remaining columns contain asterisks. If $w = v$, we have a solution. We list the values m and n of the corresponding solution of (2). Finally, we test m for maximality with respect to n using Corollaries 2.1 and 2.2. If the test is positive, the max column says yes and the final column lists $f(n)$; otherwise, the max column says no and the final column contains an asterisk.

TABLE 4

d	p	k	v	$\underline{r}$	r_p	a	r	w	m	n	max	$f(n)$
12	3	1	4	8	2	0	*	*	*	*	*	*
15	3	1	5	10	1	3	4	5	12	36	no	*
15	5	1	3	9	4	3	4	3	20	100	yes	17
18	3	2	2	13	1	2	10	14	*	*	*	*
20	5	1	4	12	2	0	12	44	*	*	*	*
21	3	1	7	14	2	1	9	3	*	*	*	*
21	7	1	3	12	5	1	9	−17	*	*	*	*
24	3	1	8	16	1	0	*	*	*	*	*	*
28	7	1	4	16	2	0	*	*	*	*	*	*
30	3	1	10	20	2	2	20	46	*	*	*	*
30	5	1	6	18	3	2	18	51	*	*	*	*
33	3	1	11	22	1	1	*	*	*	*	*	*
33	11	1	3	18	7	1	*	*	*	*	*	*
35	5	1	7	21	1	3	16	27	*	*	*	*
35	7	1	5	20	6	3	*	*	*	*	*	*
36	3	2	4	26	2	0	14	129	*	*	*	*
39	3	1	13	26	2	3	8	13	24	72	no	*
39	13	1	3	21	8	3	8	3	104	1352	yes	41
40	5	1	8	24	4	0	14	8	70	350	yes	44
42	3	1	14	28	1	2	10	14	30	90	yes	54
42	7	1	6	24	3	2	10	6	70	490	yes	46
44	11	1	4	24	2	0	24	180	*	*	*	*
45	3	2	5	32	1	3	4	5	36	108	no	*
45	3	2	5	32	1	3	16	29	*	*	*	*
45	3	2	5	32	1	3	25	−7	*	*	*	*
45	5	1	9	27	2	1	*	*	*	*	*	*
48	3	1	16	32	2	0	14	16	42	126	yes	60
50	5	2	2	31	1	2	26	2	650	3250	yes	52
51	3	1	17	34	1	3	4	5	*	*	*	*
51	3	1	17	34	1	3	16	29	*	*	*	*
51	3	1	17	34	1	3	25	−7	*	*	*	*
51	17	1	3	27	10	3	*	*	*	*	*	*
52	13	1	4	28	2	0	15	−48	*	*	*	*
52	13	1	4	28	2	0	28	56	*	*	*	*
54	3	3	2	40	1	2	10	14	*	*	*	*

d	p	k	v	$\underline{r}$	r	a	r	w	m	n	max	$f(n)$
54	3	3	2	40	1	2	34	26	*	*	*	*
54	3	3	2	40	1	2	40	110	*	*	*	*
55	5	1	11	33	3	1	8	11	40	200	no	*
55	5	1	11	33	3	1	18	51	*	*	*	*
55	11	1	5	30	8	1	8	5	88	968	yes	59
56	7	1	8	32	4	0	*	*	*	*	*	*
57	3	1	19	38	2	1	*	*	*	*	*	*
57	19	1	3	30	11	1	*	*	*	*	*	*
60	3	1	20	40	1	0	22	20	66	198	yes	72
60	5	1	12	36	1	0	6	12	30	150	yes	72
63	3	2	7	45	2	1	*	*	*	*	*	*
63	7	1	9	36	1	3	8	9	56	392	no	*
65	5	1	13	39	4	1	*	*	*	*	*	*
65	13	1	5	35	9	1	9	−47	*	*	*	*
66	3	1	22	44	2	2	20	46	*	*	*	*
66	3	1	22	44	2	2	26	22	78	234	yes	78
66	11	1	6	36	3	2	*	*	*	*	*	*
68	17	1	4	36	2	0	*	*	*	*	*	*
69	3	1	23	46	1	1	*	*	*	*	*	*
69	23	1	3	36	13	1	*	*	*	*	*	*
70	5	1	14	42	2	2	*	*	*	*	*	*
70	7	1	10	40	5	2	26	−18	*	*	*	*
70	7	1	10	40	5	2	40	150	*	*	*	*
72	3	2	8	52	1	0	22	20	*	*	*	*
72	3	2	8	52	1	0	28	56	*	*	*	*
72	3	2	8	52	1	0	46	32	*	*	*	*
75	3	1	25	50	2	3	8	13	*	*	*	*
75	3	1	25	50	2	3	32	61	*	*	*	*
75	5	2	3	46	4	3	4	3	100	500	yes	92
76	19	1	4	40	2	0	21	−148	*	*	*	*
77	7	1	11	44	2	1	9	−17	*	*	*	*
77	11	1	7	42	9	1	9	−37	*	*	*	*
78	3	1	26	52	1	2	10	14	*	*	*	*
78	3	1	26	52	1	2	34	26	102	306	yes	90
78	3	1	26	52	1	2	40	110	*	*	*	*
78	3	1	26	52	1	2	52	86	*	*	*	*
78	13	1	6	42	3	2	*	*	*	*	*	*
80	5	1	16	48	3	0	28	56	*	*	*	*
80	5	1	16	48	3	0	33	−24	*	*	*	*
80	5	1	16	48	3	0	38	−4	*	*	*	*
80	5	1	16	48	3	0	48	296	*	*	*	*
84	3	1	28	56	2	0	38	28	114	342	yes	96
84	7	1	12	48	6	0	6	12	42	294	yes	96
84	7	1	12	48	6	0	27	−44	*	*	*	*
85	5	1	17	51	1	1	*	*	*	*	*	*
85	17	1	5	45	11	1	*	*	*	*	*	*
87	3	1	29	58	1	3	4	5	*	*	*	*
87	3	1	29	58	1	3	16	29	48	144	no	*
87	3	1	29	58	1	3	25	−7	*	*	*	*
87	29	1	3	45	16	3	16	3	474	13456	yes	89
88	11	1	8	48	4	0	15	−36	*	*	*	*

d	p	k	v	$\underline{r}$	r	a	r	w	m	n	max	$f(n)$
88	11	1	8	48	4	0	48	404	*	*	*	*
90	3	2	10	65	2	2	20	46	*	*	*	*
90	3	2	10	65	2	2	26	22	*	*	*	*
90	5	1	18	54	4	2	*	*	*	*	*	*
91	7	1	13	52	3	3	*	*	*	*	*	*
91	13	1	7	49	10	3	36	319	*	*	*	*
92	23	1	4	48	2	0	48	80	*	*	*	*
93	3	1	31	62	2	1	50	79	*	*	*	*
93	31	1	3	48	17	1	*	*	*	*	*	*
95	5	1	19	57	2	3	32	59	*	*	*	*
95	19	1	5	50	12	3	*	*	*	*	*	*
96	3	1	32	64	1	0	22	20	*	*	*	*
96	3	1	32	64	1	0	28	56	*	*	*	*
96	3	1	32	64	1	0	46	32	138	414	yes	108
96	3	1	32	64	1	0	55	-4	*	*	*	*
98	7	2	2	57	1	2	*	*	*	*	*	*
99	3	2	11	71	2	1	50	79	*	*	*	*
99	11	1	9	54	10	3	32	53	*	*	*	*
100	5	2	4	62	2	0	12	44	*	*	*	*
100	5	2	4	62	2	0	22	4	550	2750	yes	116
100	5	2	4	62	2	0	27	-16	*	*	*	*
100	5	2	4	62	2	0	42	144	*	*	*	*
100	5	2	4	62	2	0	57	-76	*	*	*	*
100	5	2	4	62	2	0	62	-6	*	*	*	*

Combining the results of Tables 1, 2, 3, and 4, we form Table 5. For each t such that $2 \leq t \leq 100$ and $S(t)$ is nonempty, we list the members of $S(t)$. If $S(t)$ is empty, then t does not appear as an entry. The requirement that the solutions listed in Tables 1, 2, 3, and 4 satisfy a maximality condition assures that distinct entries from these tables yield distinct corresponding entries in Table 5. Therefore, we have proved:

Theorem 1: All solutions of (1) such that n is nonprimitive and $2 \leq t \leq 100$ are given in Table 5.

TABLE 5

t	$S(t)$	t	$S(t)$	t	$S(t)$	t	$S(t)$
3	18	28	48	52	352, 3250	74	3392
4	12	31	15736	54	90	76	340
7	196	32	992	56	224, 1372, $28p$**	78	234
8	56	34	928	58	160	80	156, 380, 3008
10	40	39	162	59	968	84	2952
12	24, 54, $6p$*	40	736	60	96, 126	86	2824
17	100	41	1352	64	108, 220	88	460
18	208	44	350, 608	65	200	89	13456
19	36	46	490, 544	66	3904	90	306, 2368
20	176	48	60	68	260, 3776	92	500
24	112	50	416	71	392	96	204, 294, 342, 1984
26	80	51	72	72	132, 150, 198	98	1856
						100	580

*p prime, $(6, p) = 1$ **p prime, $(14, p) = 1$

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WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—I

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$(x)_n \equiv x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k)x(x-1)\cdots(x-k+1), \tag{1.2}$$

respectively.

It is well known that $S_1(n, k)$ is the number of permutations of

$$Z_n = \{1, 2, \ldots, n\}$$

with k cycles and that $S(n, k)$ is the number of partitions of the set Z_n into k blocks [1, Ch. 5], [2, Ch. 4]. These combinatorial interpretations suggest the following extensions.

Let n, k be positive integers, $n \geq k$, and let $k_1, k_2, \ldots, k$ be nonnegative integers such that

$$\begin{cases} k = k_1 + k_2 + \cdots + k_n \\ n = k_1 + 2k_2 + \cdots + nk_n. \end{cases} \tag{1.3}$$

We define $\overline{S}(n, k, \lambda)$, $\overline{S}_1(n, k, \lambda)$, where λ is a parameter, in the following way.

$$\overline{S}(n, k, \lambda) = \sum\sum(k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n), \tag{1.4}$$

where the inner summation is over all partitions of Z_n into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n; the outer summation is over all $k_1, k_2, \ldots, k_n$ satisfying (1.3).

(1.5) $$\overline{S}_1(n,\ k,\ \lambda) = \sum\sum\left\{k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)}{(n-1)!}\right\},$$

where the inner summation is over all permutations of Z_n with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n; the outer summation is over all $k_1,\ k_2,\ \ldots,\ k_n$ satisfying (1.3).

We now put

(1.6) $$\begin{cases} S(n,\ k,\ \lambda) = \dfrac{1}{k}\overline{S}\,(n,\ k,\ \lambda) \\ S_1(n,\ k,\ \lambda) = \dfrac{1}{n}\overline{S}_1\,(n,\ k,\ \lambda). \end{cases}$$

It is evident from (1.4) and (1.5) that

(1.7) $$S(n,\ k,\ 1) = S(n,\ k),\ S_1(n,\ k,\ 1) = S_1(n,\ k).$$

Indeed we shall show that if λ is an integer, then $S(n,\ k,\ \lambda)$ and $S_1(n,\ k,\ \lambda)$ are also integers. More precisely, we show that, for arbitrary λ,

(1.8) $$\overline{S}(n,\ k,\ \lambda) = \sum_{j=1}^{n-k+1}(k)_j S(n,\ j+k-1)\binom{\lambda}{j},$$

(1.9) $$\overline{S}_1(n,\ k,\ \lambda) = \sum_{j=1}^{n-k+1}\binom{n}{j}(\lambda)_j S_1(n-j,\ k-1).$$

We obtain recurrences and generating functions for both $S(n,\ k,\ \lambda)$ and $S_1(n,\ k,\ \lambda)$. Simpler results hold for the functions

(1.10) $$\begin{cases} R(n,\ k,\ \lambda) = \overline{S}(n,\ k+1,\ \lambda) + S(n,\ k) \\ R_1(n,\ k,\ \lambda) = \overline{S}_1(n,\ k+1,\ \lambda) + S_1(n,\ k). \end{cases}$$

For example, we have the recurrences

(1.11) $$\begin{cases} R(n+1,\ k,\ \lambda) = R(n,\ k-1,\ \lambda) + (k+\lambda)R(n,\ k,\ \lambda) \\ R_1(n+1,\ k,\ \lambda) = R_1(n,\ k-1,\ \lambda) + (n+\lambda)R_1(n,\ k,\ \lambda) \end{cases}$$

and the orthogonality relations

(1.12) $$\sum_{j=0}^{n} R(n,\ j,\ \lambda)\cdot(-1)^{j-k}R_1(j,\ k,\ \lambda)$$
$$= \sum_{j=0}^{n}(-1)^{n-j}R_1(n,\ j,\ \lambda)R(j,\ k,\ \lambda) = \begin{cases} 1 & (n=k) \\ 0 & (n \neq k). \end{cases}$$

For $\lambda = 0$ and $\lambda = 1$, (1.11) and (1.12) reduce to familiar formulas for $S(n,\ k)$ and $S_1(n,\ k)$.

The definitions (1.4) and (1.5) furnish combinatorial interpretations of $\overline{S}(n,\ k,\ \lambda)$ and $\overline{S}_1(n,\ k,\ \lambda)$ when λ is arbitrary. For λ a nonnegative integer, the recurrences (1.11) suggest combinatorial interpretations for $R(n,\ k,\ \lambda)$ and $R_1(n,\ k,\ \lambda)$ that generalize the interpretation of $S(n,\ k)$ and $S_1(n,\ k)$ described above. For the statement of the generalized interpretations, see Section 7 below.

2. THE FUNCTION $\overline{S}(n, k, \lambda)$

Let n, k be positive integers, $n \geq k$, and $k_1, k_2, \ldots, k_n$ nonnegative such that

$$\begin{cases} k = k_1 + k_2 + \cdots + k_n \\ n = k_1 + 2k_2 + \cdots + nk_n. \end{cases} \tag{2.1}$$

Put

$$S(n; k_1, k_2, \ldots, k_n; \lambda) = \sum(k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n), \tag{2.2}$$

where the summation is over all partitions of $Z_n = 1, 2, \ldots, n$ into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n. Then we have (compare [2, p. 75]):

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \ldots} S(n; k_1, k_2, \ldots; \lambda) \frac{y_1^{k_1} y_2^{k_2} \cdots}{k_1! k_2! \cdots}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \ldots} (k_1\lambda + k_2\lambda^2 + \cdots) \frac{n!}{1!^{k_1} 2!^{k_2} \cdots} \frac{y_1^{k_1} y_2^{k_2} \cdots}{k_1! k_2! \cdots}$$

$$= \left(\frac{y_1 \lambda x}{1!} + \frac{y_2 \lambda^2 x^2}{2!} + \cdots\right) \exp\left\{\frac{y_1 x}{1!} + \frac{y_2 x^2}{2!} + \cdots\right\}.$$

For $y_1 = y_2 = \cdots = y$, the extreme right member becomes

$$y(e^{\lambda x} - 1) \exp\{y(e^x - 1)\}.$$

Hence, we get the generating function

$$\sum_{n,k} \overline{S}(n, k, \lambda) \frac{x^n}{n!} y^k = y(e^{\lambda x} - 1) \exp\{y(e^x - 1)\}. \tag{2.3}$$

Recall that

$$\sum_{n,k} S(n, k) \frac{x^n}{n!} y^k = \exp\{y(e^x - 1)\}. \tag{2.4}$$

Thus, the right-hand side of (2.3) is equal to

$$y \sum_{m=1}^{\infty} \frac{\lambda^m x^m}{m!} \sum_{n,k} S(n, k) \frac{x^n}{n!} y^k$$

and therefore,

$$\overline{S}(n, k, \lambda) = \sum_{m=1}^{n-k+1} \binom{n}{m} \lambda^m S(n - m, k - 1). \tag{2.5}$$

Note that, for $\lambda = 1$, (2.3) reduces to

$$\sum_{n,k} \overline{S}(n, k, 1) \frac{x^n}{n!} y^k = y(e \;\; - 1) \exp\{y(e^x - 1)\} = y \frac{\partial}{\partial y} \exp\{y(e^x - 1)\}$$

$$= \sum_{n,k} kS(n, k) \frac{x^n}{n!} y^k, \quad \text{by (2.4)}.$$

Thus, we again get

$$\bar{S}(n, k, 1) = kS(n, k).$$

By (1.2),

$$\lambda^m = \sum_{j=0}^{m} S(m, j)j!\binom{\lambda}{j}.$$

Thus, (2.5) becomes

$$\begin{aligned}\bar{S}(n, k, \lambda) &= \sum_{m=1}^{n-k+1} \binom{n}{m} S(n-m, k-1) \sum_{j=1}^{m} S(m, j)j!\binom{\lambda}{j} \\ &= \sum_{j=1}^{n-k+1} j!\binom{\lambda}{j} \sum_{m=j}^{n} \binom{n}{m} S(m, j)S(n-m, k-1).\end{aligned}$$

The inner sum is equal to

$$\binom{j+k-1}{j} S(n, j+k-1),$$

so that

$$\begin{aligned}\bar{S}(n, k, \lambda) &= \sum_{j=1}^{n-k+1} j!\binom{\lambda}{j}\binom{j+k-1}{j} S(n, j+k-1) \\ &= \sum_{j=1}^{n-k+1} (k)_j S(n, j+k-1)\binom{\lambda}{j}.\end{aligned} \tag{2.6}$$

Hence,

$$S(n, k, \lambda) = \frac{1}{k}\bar{S}(n, k, \lambda) = \sum_{j=1}^{n-k+1} (k+1)_{j-1} S(n, j+k-1)\binom{\lambda}{j}. \tag{2.7}$$

Thus, for λ an integer, $S(n, k, \lambda)$ is an integer. For example, we have

$$\begin{aligned}S(n, k, 1) &= S(n, k) \\ S(n, k, 2) &= 2S(n, k) + (k+1)S(n, k+1) \\ S(n, k, 3) &= 3S(n, k) + 3(k+1)S(n, k+2).\end{aligned}$$

It follows readily from (2.7) that

$$\begin{aligned}&\sum_{t=0}^{m} (-1)^t \binom{m}{t} S(n, k, \lambda - t) \\ &= \sum_{j=m}^{n-k+1} (k+1)_{j-1} S(n, j+k-1)\binom{\lambda - m}{j - m}, \quad (m \geq 1).\end{aligned} \tag{2.8}$$

This result holds for all λ. However, if λ is a positive integer, then

$$\sum_{t=0}^{\lambda} (-1)^t \binom{\lambda}{t} S(n, k, \lambda - t) = (k+1)_{\lambda-1} S(n, \lambda + k - 1), \tag{2.9}$$

and

$$\sum_{t=0}^{\lambda+1} (-1)^t \binom{\lambda+1}{t} S(n, k, \lambda - t) \tag{2.10}$$
$$= \sum_{j=\lambda+1}^{n-k+1} (-1)^{j-\lambda-1} (k+1)_{j-1} S(n, j+k-1).$$

3. THE FUNCTION $R(n, k, \lambda)$

It is convenient to define

$$R(n, k, \lambda) = \overline{S}(n, k+1, \lambda) + S(n, k). \tag{3.1}$$

Thus, (2.5) implies

$$R(n, k, \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} \lambda^m S(n-m, k), \tag{3.2}$$

while (2.7) gives

$$R(n, k, \lambda) = \sum_{j=0}^{n-k} (k+1)_j S(n, j+k) \binom{\lambda}{j}. \tag{3.3}$$

Multiplying (3.2) by $k!\binom{y}{k}$ and summing over k, we get

$$\sum_{k=0}^{n} k!\binom{y}{k} R(n, k, \lambda) = \sum_{m=0}^{n} \binom{n}{m} \lambda^m \sum_{k=0}^{n-m} S(n-m, k) y(y-1) \cdots (y-k+1)$$
$$= \sum_{m=0}^{n} \binom{n}{m} \lambda^m y^{n-m}.$$

Hence,

$$\sum_{k=0}^{n} k!\binom{y}{k} R(n, k, \lambda) = (y+\lambda)^n. \tag{3.4}$$

It follows from (3.4) that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} k!\binom{y}{k} R(n, k, \lambda) = e^{x(y+\lambda)}. \tag{3.5}$$

To obtain a recurrence for $R(n, k, \lambda)$, take

$$\sum_{k=0}^{n} k!\binom{y}{k} (R(n+1, k, \lambda) - \lambda R(n, k, \lambda)) = (y+\lambda)^{n+1} - \lambda(y+\lambda)^n$$
$$= y(y+\lambda)^n.$$

Since

$$k!\binom{y}{k} y = (k+1)!\binom{y}{k+1} + k.k!\binom{y}{k},$$

it is clear that (3.4) gives

$$R(n+1, k, \lambda) - \lambda R(n, k, \lambda) = kR(n, k, \lambda) + R(n, k-1, \lambda),$$

that is

$$R(n+1, k, \lambda) = (\lambda + k)R(n, k, \lambda) + R(n, k-1, \lambda). \tag{3.6}$$

An equivalent result is

$$\overline{S}(n+1, k+1, \lambda) = (\lambda + k)\overline{S}(n, k+1, \lambda) + \overline{S}(n, k, \lambda) + S(n, k). \tag{3.7}$$

To get an explicit formula for $R(n, k, \lambda)$ we recall that

$$S(n, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Thus, by (3.2),

$$R(n, k, \lambda) = \frac{1}{k!}\sum_{m=0}^{n-k}\binom{n}{m}\lambda^m \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{n-m}.$$

For $n - k < m \le n$, the inner sum vanishes, so that

$$\begin{aligned} R(n, k, \lambda) &= \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}\lambda^m \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{n-m} \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{m=0}^{n}\binom{n}{m}\lambda^m j^{n-m}. \end{aligned}$$

Thus,

$$R(n, k, \lambda) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda + j)^n = \frac{1}{k!}\Delta^k \lambda^n. \tag{3.8}$$

It follows from (3.8) that

$$\sum_{n=k}^{\infty} R(n, k, \lambda)\frac{z^n}{n!} = \frac{1}{k!}e^{\lambda z}(e^z - 1)^k \tag{3.9}$$

in agreement with previous results. Also, since

$$\begin{aligned} \frac{1}{k!}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda + j)^n &= \frac{1}{k!}\sum_{j=0}^{k}\frac{(-1)^{k-j}\binom{k}{j}}{1 - (\lambda + j)z} \\ &= \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z)\ \ldots\ (1 - (\lambda + k)z)}, \end{aligned}$$

we have

$$\sum_{n=0}^{\infty} R(n, k, \lambda)z^n = \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z)\ \ldots\ (1 - (\lambda + k)z)}. \tag{3.10}$$

We also note that (3.9) implies the "addition theorem":

$$R(n, j + k, \lambda + \mu) = \binom{j + k}{j}^{-1}\sum_{m=0}^{n}\binom{n}{m}R(m, j, \lambda)R(n - m, k, \mu). \tag{3.11}$$

By the recurrence (3.6) together with $R(0, 0, \lambda) = 1$, or by means of (3.8), we have

$$R(n, 0, \lambda) = \lambda^n, \quad R(n, n, \lambda) = 1. \tag{3.12}$$

Moreover, if we put

$$x^n = \sum_{k=0}^{n} \overline{R}(n, k, \lambda)(x - \lambda)(x - \lambda - 1) \cdots (x - \lambda - k + 1),$$

then

$$\overline{R}(n + 1, k, \lambda) = (\lambda + k)\overline{R}(n, k, \lambda) + \overline{R}(n, k - 1, \lambda),$$

so that $\overline{R}(n, k, \lambda) = R(n, k, \lambda)$. Thus, we have

$$y^n = \sum_{k=0}^{n} R(n, k, \lambda)(y - \lambda)(y - \lambda - 1) \cdots (y - \lambda - k + 1), \tag{3.13}$$

or, replacing y by $-y$,

$$y^n = \sum_{k=0}^{n} (-1)^{n-k} R(n, k, \lambda)(y + \lambda)_k. \tag{3.14}$$

This, of course, is equivalent to (3.4).

It is clear from (3.8) or (3.13) that

$$R(n, k, 0) = S(n, k). \tag{3.15}$$

For $\lambda = 1$, since $\overline{S}(n, k, 1) = kS(n, k)$, then by (3.1)

$$R(n, k, 1) = (k + 1)S(n, k + 1) + S(n, k),$$

so that

$$R(n, k, 1) = S(n + 1, k + 1). \tag{3.16}$$

The function

$$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda) \tag{3.17}$$

evidently reduces, for $\lambda = 0$, to the Bell number [1, p. 210]

$$B(n) = \sum_{k=0}^{n} S(n, k).$$

A few formulas may be noted. It follows from (3.2) that

$$B(n, \lambda) = \sum_{m=0}^{n} \binom{n}{m} \lambda^m B(n - m). \tag{3.18}$$

Also, by (3.9), we have

$$\sum_{n=0}^{\infty} B(n, \lambda)\frac{z^n}{n!} = e^{\lambda z} \exp(e^z - 1), \tag{3.19}$$

which, indeed, is implied by (3.18).

Differentiation of (3.19) gives

$$\sum_{n=0}^{\infty} B(n+1, \lambda)\frac{z^n}{n!} = \lambda e^{\lambda z} \exp(e^z - 1) + e^{(\lambda+1)z} \exp(e^z - 1).$$

Hence,

$$B(n+1, \lambda) = \lambda B(n, \lambda) + B(n, \lambda + 1) \tag{3.20}$$

$$= B(n, \lambda) + \sum_{m=0}^{n} \binom{n}{m} B(m, \lambda).$$

Iteration of the first half of (3.20) gives

$$B(n+m, \lambda) = \sum_{j=0}^{m} \frac{1}{j!}\Delta^j \lambda^m \cdot B(n, \lambda + j), \tag{3.21}$$

as can be proved by induction on m. Incidentally, by (3.8), (3.21) can be written in the form

$$B(n+m, \lambda) = \sum_{j=0}^{m} R(m, j, \lambda) B(n, \lambda + j). \tag{3.22}$$

To anticipate the first result in Section 6, the inverse of (3.22) is

$$B(n, \lambda + m) = \sum_{j=0}^{m} (-1)^{m-j} R_1(m, j, \lambda) B(n + j, \lambda), \tag{3.23}$$

where $R_1(m, j, \lambda)$ is defined by (5.1).

*
* *

Returning to (3.9), note that

$$\sum_{n=k}^{\infty} R(n, k, \lambda + 1)\frac{z^n}{n!} = \frac{1}{k!}e^{(\lambda+1)z}(e^z - 1)^k$$

$$= \frac{1}{k!}e^{\lambda z}(e^z - 1)^{k+1} + \frac{1}{k!}e^{\lambda z}(e^z - 1)^k,$$

which implies

$$R(n, k, \lambda + 1) = (k+1)R(n, k+1, \lambda) + R(n, k, \lambda). \tag{3.24}$$

More generally, since

$$e^{mz} = ((e^z - 1) + 1)^m = \sum_{j=0}^{m} \binom{m}{j} (e^z - 1)^j,$$

we get

$$R(n, k, \lambda + m) = \sum_{j=0}^{m} \binom{m}{j} (k+1)_j R(n, k + j, \lambda). \tag{3.25}$$

We may also write (3.24) in the form

$$\Delta_\lambda R(n,\ k,\ \lambda) = (k + 1)R(n,\ k + 1,\ \lambda), \tag{3.26}$$

where Δ_λ is the finite difference operator. Iteration of (3.26) gives

$$\Delta_\lambda^m R(n,\ k,\ \lambda) = (k + 1)_m R(n,\ k + m,\ \lambda). \tag{3.27}$$

4. THE FUNCTION $\overline{S}_1(n,\ k,\ \lambda)$

Corresponding to (2.2), we define

$$S_1(n;\ k_1,\ k_2,\ \ldots,\ k_n;\ \lambda) = k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)_n}{(n-1)!}, \tag{4.1}$$

where the inner summation is over all permutations of Z_n,

$$n = k_1 + 2k_2 + \cdots + nk_n,$$

with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n. Then (compare [2, p. 68]), we have

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{k_1,k_2,\ldots} S_1(n;\ k_1,k_2,\ldots,k_n;\lambda)\frac{y_1^{k_1}y_2^{k_2}\ \cdots}{k_1!k_2!\ \cdots}$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{k_1,k_2,\ldots} k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)_n}{(n-1)!}\left\{\frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}}\right\}\frac{y_1^{k_1}y_2^{k_2}\ \cdots}{k_1!k_2!\ \cdots}$$

$$= \left\{\frac{(\lambda)_1}{1!}y_1x + \frac{(\lambda)_2}{2!}y_2x^2 + \frac{(\lambda)_3}{3!}y_3x^3 + \cdots\right\}\exp\left\{y_1x + \frac{1}{2}y_2x^2 + \frac{1}{3}y_3x^3 + \cdots\right\}.$$

For $y_1 = y_2 = \cdots y$, the extreme right member becomes

$$y((1-x)^{-\lambda} - 1)(1-x)^{-y}.$$

Hence, we get

$$\sum_{n,k}\overline{S}_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k = y((1-x)^{-\lambda} - 1)(1-x)^{-y}, \tag{4.2}$$

where

$$\overline{S}_1(n,\ k,\ \lambda) = \sum S_1(n;\ k_1,\ k_2,\ \ldots,\ k_n;\lambda), \tag{4.3}$$

and the summation on the right is over all nonnegative $k_1,\ k_2,\ \ldots,\ k_n$ satisfying $n = k_1 + 2k_2 + \cdots + nk_n$.

Since (see [2, p. 71]),

$$\sum_{n,k} S_1(n,\ k)\frac{x^n}{n!}y^k = (1-x)^{-y}, \tag{4.4}$$

it follows from (4.2) that

$$\sum_{n,k}\overline{S}_1(n,\ k+1,\ \lambda)\frac{x^n}{n!}y^k = \sum_{n,k} S_1(n,\ m)\frac{x^n}{n!}((\lambda + y)^m - y^m)$$

$$= \sum_{n,m} S_1(n,\ m)\frac{x^n}{n!}\sum_{k=0}^{m-1}\binom{m}{k}\lambda^{m-k}y^k = \sum_{n,k}\frac{x^n}{n!}y^k\sum_{m=k+1}^{n}\binom{m}{k}\lambda^{m-k}S_1(n,\ m).$$

Therefore,

$$\overline{S}_1(n,\ k+1,\ \lambda) = \sum_{j=1}^{n-k} \binom{j+k}{j} \lambda^j S_1(n,\ j+k). \tag{4.5}$$

In the next place, it also follows from (4.2) that

$$\sum_{n,k} \overline{S}_1(n,\ k+1,\ \lambda)\frac{x^n}{n!}y^k = ((1-x)^{-\lambda} - 1)(1-x)^{-y}$$
$$= \sum_{m=1}^{\infty} (\lambda)_m \frac{x^m}{m!} \sum_{n,k} S_1(n,\ k)\frac{x^n}{n!}y^k.$$

Equating coefficients, we get

$$\overline{S}_1(n,\ k+1,\ \lambda) = \sum_{m=1}^{n-k} \binom{n}{m} (\lambda)_m S_1(n-m,\ k)$$
$$= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!} n(n-1)\ \cdots\ (n-m+1)S_1(n-m,\ k).$$

Thus,

$$S_1(n,\ k+1,\ \lambda) = \frac{1}{n}\overline{S}_1(n,\ k+1,\ \lambda) \tag{4.7}$$
$$= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!}(n-1)\ \cdots\ (n-m+1)S_1(n-m,\ k).$$

It follows at once from (4.7) that, for λ integral, $S_1(n,\ k+1,\ \lambda)$ is also integral.

It is evident from (4.1) and (4.3) that

$$\overline{S}_1(n,\ k,\ 1) = nS_1(n,\ k). \tag{4.8}$$

Thus, for example, (4.5) and (4.6) yield

$$\sum_{j=1}^{n-k} \binom{j+k}{j} S_1(n,\ j+k) = nS_1(n,\ k+1), \tag{4.9}$$

and

$$\sum_{m=1}^{n-k} n(n-1)\ \cdots\ (n-m+1)S_1(n-m,\ k) = nS_1(n,\ k+1), \tag{4.10}$$

respectively.

5. THE FUNCTION $R_1(n,\ k,\ \lambda)$

We define the function $R_1(n,\ k,\ \lambda)$ by means of

$$R_1(n,\ k,\ \lambda) = \overline{S}_1(n,\ k+1,\ \lambda) + S_1(n,\ k). \tag{5.1}$$

Then, by (4.5),

$$R_1(n,\ k,\ \lambda) = \sum_{j=0}^{n-k} \binom{j+k}{j} \lambda^j S_1(n,\ j+k), \tag{5.2}$$

and by (4.6),

$$R_1(n,\ k,\ \lambda) = \sum_{m=0}^{n-k} \binom{n}{m} (\lambda)_m S_1(n - m,\ k) \tag{5.3}$$
$$= \sum_{m=0}^{n-k} \frac{(\lambda)_m}{m!} n(n - 1) \cdots (n - m + 1) S_1(n - m,\ k).$$

It is also evident from (4.2) and (4.4) that

$$\sum_{n,k} R_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k = (1 - x)^{-\lambda - y}. \tag{5.4}$$

Differentiation of (5.4) with respect to x gives

$$\sum_{n,k} R_1(n + 1,\ k,\ \lambda)\frac{x^n}{n!}y^k = (\lambda + y)(1 - x)^{-\lambda - y - 1},$$

so that

$$(1 - x)\sum_{n,k} R_1(n+1,\ k,\ \lambda)\frac{x^n}{n!}y^k = (\lambda + y)\sum_{n,k} R_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k.$$

Equating coefficients, we get

$$R_1(n + 1,\ k,\ \lambda) = nR_1(n,\ k,\ \lambda) = \lambda R_1(n,\ k,\ \lambda) + R_1(n,\ k = 1,\ \lambda),$$

that is,

$$R_1(n + 1,\ k,\ \lambda) = (\lambda + n)R_1(n,\ k,\ \lambda) + R_1(n,\ k - 1,\ \lambda). \tag{5.5}$$

It follows at once from (5.5) and $R_1(0,\ 0,\ \lambda) = 1$ that

$$R_1(n,\ 0,\ \lambda) = (\lambda)_n,\quad R_1(n,\ n\lambda) = 1. \tag{5.6}$$

Also, taking $y = 1$ in (5.4), we get

$$\sum_{k=0}^{n} R_1(n,\ k,\ \lambda) = (\lambda + 1)_n. \tag{5.7}$$

More generally, we have

$$\sum_{k=0}^{n} R_1(n,\ k,\ \lambda)y^k = (\lambda + y)_n. \tag{5.8}$$

Clearly, (5.5) is implied by (5.8).

It is clear from (5.4) that

$$R_1(n,\ k,\ 0) = S_1(n,\ k). \tag{5.9}$$

For $\lambda = 1$, we have, by (4.8) and (5.1),

$$R_1(n,\ k,\ 1) = S_1(n + 1,\ k + 1). \tag{5.10}$$

These formulas may be compared with (3.15) and (3.16).

In view of (5.10), (5.2) and (5.3) reduce to

$$S_1(n + 1,\ k + 1) = \sum_{j=0}^{n-k} \binom{j + k}{j} S_1(n,\ j + k), \tag{5.11}$$

and

$$(5.12)\qquad S_1(n+1,\ k+1) = \sum_{m=0}^{n-k} n(n-1)\cdots(n-m+1)S_1(n-m,\ k).$$

It is not difficult to give direct proofs of (5.11) and (5.12).

Returning to (5.4), note that

$$(1-x)\sum_{n,k} R_1(n,\ k,\ \lambda+1)\frac{x^n}{n!}y^k = (1-x)^{-\lambda-y}.$$

This gives

$$(5.13)\qquad R_1(n,\ k,\ \lambda) = R_1(n,\ k,\ \lambda+1) - nR_1(n-1,\ k,\ \lambda+1),$$

and generally,

$$(5.14)\qquad R_1(n,\ k,\ \lambda) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}n(n-1)\cdots(n-j+1)R_1(n-j,\ k,\ \lambda+m).$$

The inverse of (5.14) is

$$(5.15)\qquad R_1(n,\ k,\ \lambda+m) = \sum_{j=0}^{n}\binom{n}{j}(m)_j R_1(n-j,\ k,\ \lambda).$$

We may write (5.13) in the form

$$(5.16)\qquad \Delta_\lambda R_1(n,\ k,\ \lambda) = nR_1(n-1,\ k,\ \lambda+1).$$

Iteration gives

$$(5.17)\qquad \Delta_\lambda^m R_1(n,\ k,\ \lambda) = n(n-1)\cdots(n-m+1)R_1(n-m,\ k,\ \lambda+m).$$

6. ORTHOGONALITY RELATIONS

Comparing (5.8) with (3.14), we have immediately the orthogonality relations

$$(6.1)\qquad \sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \lambda)$$
$$= \sum_{k=0}^{n}R_1(n,\ k,\ \lambda)\cdot(-1)^{k-j}R(k,\ j,\ \lambda) = \delta_{n,j},$$

the Kronecker delta.

It is of some interest to give a proof of (6.1) making use of (3.2) and (5.2). We have

$$\sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \lambda)$$
$$= \sum_{k=0}^{n}(-1)^{n-k}\sum_{m=0}^{n-k}\binom{n}{m}\lambda^m S(n-m,\ k)\sum_{t=0}^{k-j}\binom{j+t}{t}\lambda^t S_1(k,\ k+t)$$
$$= \sum_{m=0}^{n}\sum_{t=0}^{n-j}(-1)^m\binom{n}{m}\binom{j+t}{t}\lambda^{m+t}\sum_{k=0}^{n-m}(-1)^{n-m-k}S(n-m,\ k)S_1(k,\ j+t).$$

The inner sum is equal to 1 if $n - m = j + t$, and vanishes otherwise. Thus, we have

$$\lambda^{n-j}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\binom{n-m}{j} = \lambda^{n-j}\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\binom{m}{j} = \delta_{n,j},$$

so that

$$\sum_{k=0}^{n}(-1)^{n-k}R(n, k, \lambda)R_1(k, j, \lambda) = \delta_{n,j}. \tag{6.2}$$

As for the second half of (6.1), we have

$$\begin{aligned}
&\sum_{k=0}^{n} R_1(n, k, \lambda)\cdot(-1)^{k-j}R(k, j, \lambda)\\
&= \sum_{k=0}^{n}\sum_{t=0}^{n-k}\binom{t+k}{t}\lambda^t S_1(n, t+k)\cdot(-1)^{k-j}\sum_{m=0}^{k-j}\binom{k}{m}\lambda^m S(k-m, j)\\
&= \sum_{k=0}^{n}\sum_{t=k}^{n}\binom{t}{k}\lambda^{t-k}S_1(n, t)\cdot(-1)^{k-j}\sum_{m=j}^{k}\binom{k}{m}\lambda^{k-m}S(m, j)\\
&= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\lambda^{t-m}S_1(n, t)S(m, j)\sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}\binom{k}{m}\\
&= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\lambda^{t-m}S_1(n, t)S(m, j)\delta_{t,m}\\
&= \sum_{t=j}^{n}(-1)^{t-j}S_1(n, t)S(t, j) = \delta_{n,j}.
\end{aligned}$$

This, together with (6.2), completes the proof of (6.1).

The proof of (6.2) above suggests a more general result. As in the above proof, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{n-k}R(n, k, \lambda)R_1(k, j, \mu) &= \sum_{m=0}^{n}\sum_{t=0}^{n-j}(-1)^m\binom{n}{m}\binom{j+t}{j}\lambda^m\mu^t\delta_{n-m,j+t}\\
&= \sum_{m=0}^{n}(-1)^m\binom{n}{m}\binom{n-m}{j}\lambda^m\mu^{n-m-j}\\
&= \sum_{m=j}^{n}(-1)^{n-m}\binom{n}{m}\binom{m}{j}\lambda^{n-m}\mu^{m-j}\\
&= \binom{n}{j}\sum_{m=1}^{n}(-1)^{n-m}\binom{n-j}{m-j}\lambda^{n-m}\mu^{m-j}\\
&= (-1)^{n-j}\binom{n}{j}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\lambda^{n-j-m}\mu^m,
\end{aligned}$$

and therefore,

$$\sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \mu) = \binom{n}{j}(\mu - \lambda)^{n-j}. \tag{6.3}$$

For $\mu = \lambda$, (6.3) reduces to (6.2).

In the next place

$$\begin{aligned}
&\sum_{k=0}^{n} R_1(n,\ k,\ \mu)\cdot(-1)^{k-j}R(k,\ j,\ \lambda)\\
&= \sum_{k=0}^{n}\sum_{t=k}^{n}\binom{t}{k}\mu^{t-k}S_1(n,\ t)\cdot(-1)^{k-j}\sum_{m=j}^{k}\binom{k}{m}\lambda^{k-m}S(m,\ j)\\
&= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\binom{t}{m}S_1(n,\ t)S(m,\ j)\sum_{k=m}^{t}(-1)^{t-k}\binom{t-m}{k-m}\mu^{t-k}\lambda^{k-m}\\
&= \sum_{t=0}^{n}\sum_{m=j}^{t}(-1)^{t-j}\binom{t}{m}S_1(n,\ t)S(m,\ j)(\lambda - \mu)^{t-m}.
\end{aligned}$$

Let $U(n,\ j)$ denote this sum. Then,

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j U(n,\ j)j!\binom{x}{j} &= \sum_{t=0}\sum_{m=0}(-1)^t\binom{t}{m}S_1(n,\ t)(\lambda - \mu)^{t-m}\sum_{j=0}^{m}S(m,\ j)j!\binom{x}{j}\\
&= \sum_{t=0}^{n}\sum_{m=0}^{t}(-1)^t\binom{t}{m}S_1(n,\ t)(\lambda - \mu)^{t-m}x^m\\
&= \sum_{t=0}^{n}(-1)^t S_1(n,\ t)(x + \lambda - \mu)^t\\
&= (-1)^n(x + \lambda - \mu)(x+\lambda-\mu-1)\cdots(x+\lambda-\mu-n+1).
\end{aligned}$$

Replacing x by $-x$, this becomes

$$\sum_{j=0}^{n}U(n,\ j)(x)_j = (x - \lambda + \mu)_n. \tag{6.4}$$

Since

$$(x + y)_n = \sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j},$$

it follows from (6.4) that

$$U(n,\ j) = \binom{n}{j}(\mu - \lambda)_{n-j}.$$

Therefore, we have

$$\sum_{k=0}^{n}R_1(n,\ k,\ \mu)\cdot(-1)^{k-j}R(k,\ j,\ \lambda) = \binom{n}{j}(\mu - \lambda)_{n-j}. \tag{6.5}$$

This result may be compared with (6.3). If we define matrices

$$M = [(-1)^{n-k}R(n,\ k,\ \lambda)] \qquad (n,\ k = 0,\ 1,\ 2,\ \ldots),$$

and

$$M_1 = [R_1(n,\ k,\ \mu)] \qquad (n,\ k = 0,\ 1,\ 2,\ \ldots),$$

then (6.3) and (6.5) become

$$(6.3)' \qquad MM_1 = \left[\binom{n}{k}(\lambda - \mu)^{n-k}\right],$$

and

$$(6.5)' \qquad M_1M = \left[\binom{n}{k}(\mu - \lambda)_{n-k}\right],$$

respectively.

7. COMBINATORIAL INTERPRETATION OF $R(n,\ k,\ \lambda)$ AND $R_1(n,\ k,\ \lambda)$

Let λ be a nonnegative integer and let $B_1,\ B_2,\ \ldots,\ B_\lambda$ denote λ open boxes. Let $P(n,\ k,\ \lambda)$ denote the number of partitions of $Z_n = \{1,\ 2,\ \ldots,\ n\}$ into k blocks with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes. For brevity, we shall call these "λ-partitions." Clearly,

$$(7.1) \qquad P(n,\ k,\ 0) = S(n,\ k).$$

To evaluate $P(n,\ 0,\ \lambda)$, we place x_1 elements of Z_n in B_1, x_2 in B_2, ..., x_λ in B_λ. Thus,

$$P(n,0,\lambda) = \sum_{x_1+x_2+\cdots+x_\lambda} \frac{n!}{x_1!x_2!\ \ldots\ x_\lambda!}.$$

Hence,

$$(7.2) \qquad P(n,\ 0,\ \lambda) = \lambda^n.$$

Also, clearly,

$$(7.3) \qquad P(0,\ k,\ \lambda) = \delta_{0,k}.$$

To get a recurrence for $P(n,k,\lambda)$, we consider the effect of adding the element $n+1$ to a λ-partition of Z_n into k blocks. The added element may be placed in any of the blocks or any of the boxes without changing the value of k. On the other hand, if it constitutes an additional block, then of course the number of blocks becomes $k+1$. Thus, we have

$$(7.4) \qquad P(n+1,\ k,\ \lambda) = (\lambda + k)P(n,\ k,\ \lambda) + P(n,\ k-1,\ \lambda).$$

Since

$$P(0,\ k,\ \lambda) = R(0,\ k,\ \lambda) = \delta_{0,k},$$

comparison of (7.4) with (3.6) gives

$$(7.5) \qquad P(n,\ k,\ \lambda) = R(n,\ k,\ \lambda).$$

Hence, $R(n,\ k,\ \lambda)$ *is equal to the number of* λ*-partitions of* Z_n *into* k *blocks.*

Turning next to $R\ (n,\ k,\ \lambda)$, again let $B_1,\ B_2,\ \ldots,\ B_\lambda$ denote λ open boxes. Let $P_1(n,\ k,\ \lambda)$ denote the number of permutations of Z_n with k cycles with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes and then permuted in all possible ways in each box. For brevity, we call these "λ-permutations."

Clearly,

$$P_1(n, k, 0) = S_1(n, k). \tag{7.6}$$

To evaluate $P(n, 0, \lambda)$, note that $P(1, 0, \lambda) = \lambda$ and

$$P(n + 1, 0, \lambda) = (\lambda + n)P(n, 0, \lambda),$$

since the element $n + 1$ may occupy any one of the $n + \lambda$ positions. Thus,

$$P_1(n, 0, \lambda) = (\lambda)_n. \tag{7.7}$$

Also clearly,

$$P_1(0, k, \lambda) = \delta_{0,k}. \tag{7.8}$$

A recurrence for $P_1(n, k, \lambda)$ is obtained using the method of proof of (7.4); however, there are now $\lambda + n$ possible positions for the element $n + 1$. Thus, we get

$$P_1(n + 1, k, \lambda) = (\lambda + n)P_1(n, k, \lambda) + P_1(n, k - 1, \lambda). \tag{7.9}$$

Comparison of (7.9) with (5.5) gives

$$P_1(n, k, \lambda) = R_1(n, k, \lambda). \tag{7.10}$$

Hence, $R_1(n, k, \lambda)$ *is equal to the number of* λ*-permutations of* Z_n *with* k *cycles.*

We remark that (7.5) can also be proved using (3.2) and that (7.10) can be proved using (5.3).

Finally, we note that the generalized Bell number defined by (3.17),

$$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda),$$

is equal to the total number of λ*-partitions of* Z_n.

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A PROPERTY OF QUASI-ORTHOGONAL POLYNOMIALS

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We define the simple set of polynomials $\phi_n(x)$ to be quasi-orthogonal if

$$(\phi_n, \phi_k) = \int_a^b w(x)\phi_n(x)\phi_k(x)dx = \begin{cases} A_n & \text{if } k = n - 1 \\ B_n & \text{if } k = n \\ C_n & \text{if } k = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

We shall require A_n and C_n to be nonvanishing. It is to be noted that the $\phi_n(x)$ may or may not be orthogonal over some other combination of range $[a, b]$ and weighting function $w(x)$. Consider, for example, if the range is $[-1, 1]$, $w(x) = 1 + x$, and $\phi_n(x) = P_n(x)$, the Legendre Polynomial,

$$\int_{-1}^1 (1 + x)P_n(x)P_m(x)dx = \begin{cases} \dfrac{2n}{(2n - 1)(2n + 1)} & \text{if } m = n - 1 \\ \dfrac{2}{2n + 1} & \text{if } m = n \\ \dfrac{2(n + 1)}{(2n + 1)(2n + 3)} & \text{if } m = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, P_n is quasi-orthogonal, but, of course, if $w(x) = 1$, P_n is also orthogonal.

However, the simple set

$$\psi_n = (2n + 1)P_n + P_{n-1}$$

is quasi-orthogonal, but it is not orthogonal with respect to any range and weighting function. This is easily illustrated by noting that:

$$x\psi_3 = \frac{4}{9}\psi_4 + \frac{1}{45}\psi_3 + \frac{403}{15(45)}\psi_2 - \frac{133}{(45)^2}\psi_1 - \frac{281}{90(45)}\psi_0.$$

Since the ψ_n do not satisfy a three-term recursion formula, they, by the converse of Favard's Theorem, are not an orthogonal set, no matter what $w(x)$ or $[a, b]$ is selected. Favard's Theorem and converse are as follows.

Theorem: If the $\psi_n(x)$ are a set of simple polynomials which satisfy a three-term recursion formula, $x\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}$, then the ψ_n are orthogonal with respect to some weighting function $w(x)$ and some range $[a, b]$ if the integration be considered in the Stieltjes sense.

Converse: If the ψ_n are a simple set of polynomials orthogonal with respect to a weighting function $w(x)$ and some range $[a, b]$, then the ψ_n satisfy the three-term recursion formula:

$$x\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}.$$

For quasi-orthogonal polynomials, the following property will be satisfied:

Theorem: If $R_n(x)$ is a set of simple quasi-orthogonal polynomials over $[a, b]$ with respect to $w(x)$, then the necessary and sufficient condition that $R_n(x)$ also be orthogonal over some range $[c, d]$ with respect to some weighting function $w_1(x)$ is given by the expression:

$$xR_{n-1} = \sum_{k=0}^{n} c_k R_k, \; n \geq 2,$$

$C_0 \neq 0$ if $n = 2$ and $C_0 = 0$ if $n \geq 3$.

Proof: The quasi-orthogonal character of R_n leads at once to the set of equations:

$$\begin{aligned}
(xR_{n-1}, R_{n+1}) &= C_n(R_n, R_{n+1}) \\
(xR_{n-1}, R_n) &= C_{n-1}(R_{n-1}, R_n) + C_n(R_n, R_n) \\
(xR_{n-1}, R_{n-1}) &= C_{n-2}(R_{n-2}, R_{n-1}) + C_{n-1}(R_{n-1}, R_{n-1}) + C\;(R\;, R_{n-1}) \\
(xR_{n-1}, R_{n-2}) &= C_{n-3}(R_{n-3}, R_{n-2}) + C_{n-2}(R_{n-2}, R_{n-2}) + C_{n-1}(R_{n-1}, R_{n-2}) \\
(xR_{n-1}, R_{n-3}) &= C_{n-4}(R_{n-4}, R_{n-3}) + C_{n-3}(R_{n-3}, R_{n-3}) + C_{n-2}(R_{n-2}, R_{n-3}) \\
0 &= C_{n-5}(R_{n-5}, R_{n-4}) + C_{n-4}(R_{n-4}, R_{n-4}) + C_{n-3}(R_{n-3}, R_{n-4}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= C_0(R_0, R_1) + C_1(R_1, R_1) + C_2(R_2, R_1) \\
0 &= C_0(R_0, R_0) + C_1(R_1, R_0).
\end{aligned}$$

The zero terms on the right occur, since

$$(xR_{n-1}, R_{n-4}) = (R_{n-1}, xR_{n-4}) = \left(R_{n-1}, \sum_{k=0}^{n-2} b_k R_k\right)$$

and

$$(R_{n-1}, R_k) = 0 \text{ for } k \leq n - 3.$$

We begin at the bottom of the chain and observe that if $C_0 = 0$, C_1 is also 0. Then the penultimate equation yields $C_2 = 0$. Continuing,

$$C_0 = C_1 = \cdots = C_{n-3} = 0.$$

To show $C_{n-2} \neq 0$, note that when $C_{n-2} = 0$, the fifth equation of the chain requires:

$$0 = (xR_{n-1}, R_{n-3}) = (R_{n-1}, xR_{n-3}) = \left(R_{n-1}, \sum_{k=0}^{n-2} a_k R_k\right) = a_{n-2}(R_{n-1}, R_{n-2}).$$

Now, $a_{k-2} \neq 0$, since from the equation

$$xR_{n-3} = \sum_{k=0}^{n-2} a_k R_k$$

we see that $a_{n-2} = \dfrac{h_{n-3}}{h_{n-2}}$, where h_{n-3} is the coefficient of x^{n-3} and h_{n-2} is the coefficient of x^{n-2} in R_{n-2}. Since the R_n are a simple set of polynomials, these cannot vanish. Therefore,

$$C_{n-2} \neq 0.$$

So, $C_0 = 0$ implies

$$xR_{n-1} = C_nR_n + C_{n-1}R_{n-1} + C_{n-2}R_{n-2}.$$

Hence, by Favard's Theorem, these R_n must be an orthogonal set with respect to some weighting function $w_1(x)$ and some range $[c, d]$ if the integral be considered a Stieltjes integral.

If $C_0 \neq 0$, the R_n do not satisfy a three-term recursion formula (unless $n = 2$) and by applying the contrapositive of the converse, we see that the R_n cannot be an orthogonal set with respect to any weighting function and range.

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ON SOME SYSTEMS OF DIOPHANTINE EQUATIONS INCLUDING THE ALGEBRAIC SUM OF TRIANGULAR NUMBERS

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The natural number of the form

$$t_n = \binom{n+1}{2} = \frac{1}{2}n(n+1),$$

where n is a natural number, is referred to as the nth triangular number. The aim of this work is to give solutions of some equations and systems of equations in triangular numbers.

1. THE EQUATION $t_{t_x} + t_{t_y} = t_{t_z}$

It is well known that the equation

$$t_x + t_y = t_z \tag{1}$$

has infinitely many solutions in triangular numbers t_x, t_y, and t_z. For example, it follows immediately from the formula:

$$t_{(2n+1)k} + t_{4t_nk+n} = t_{(4t_n+1)k+n}. \tag{2}$$

We can ask whether there exists a solution of the equation:

$$t_{t_x} + t_{t_y} = t_{t_z}. \tag{3}$$

The answer to this question is positive, because there exist two solutions:

$$t_{t_{59}} + t_{t_{77}} = t_{t_{83}} \quad \text{and} \quad t_{t_{104}} + t_{t_{213}} = t_{t_{216}}.$$

The problem of finding all the solutions to equation (1) above will be solved in a subsequent paper.

2. TRIPLES OF TRIANGULAR NUMBERS, THE SUMS OR DIFFERENCES OF ANY TWO OF WHICH ARE ALSO TRIANGULAR NUMBERS

The system of three equations:

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + 2t_z &= t_q, \end{aligned} \tag{4}$$

has infinitely many solutions in triangular numbers t_x, t_y, t_z, t_u, t_v, and t_q. This theorem can be proved by insertion of the following formulas into equations (4):

$$\begin{aligned} &x = n, \quad y = \frac{1}{2}(t_n - 3), \quad z = t_n - 1, \\ &u = \frac{1}{2}(t_n + 1), \quad v = t_n, \quad q = \frac{3}{2}(t_n - 1), \end{aligned} \tag{5}$$

where n is a natural number of the form $4k + 1$ or $4k + 2$ for natural k.

In particular, putting $n = 14$, we have:

$$\begin{aligned} &x = 14, \quad y = 51, \quad z = 104, \\ &u = 53, \quad v = 105, \quad q = 156. \end{aligned}$$

Since $t_q - t_z = t_{156} - t_{104} = t_{116} = t_w$, we obtain a solution of the system of equations:

$$\begin{aligned} t_x + t_y &= t_u, & &\text{in the numbers:} & t_{14} + t_{51} &= t_{53}, \\ t_x + t_z &= t_v, & & & t_{14} + t_{104} &= t_{105}, \\ t_y + t_z &= t_w, & & & t_{51} + t_{104} &= t_{116}. \end{aligned} \tag{6}$$

We see that there exists a triple of triangular numbers whose sums in pairs are also triangular numbers. The problem of whether there exist three different triangular numbers, the sum of any two of which is a triangular number was formulated by W. Sierpiński [1].

Theorem: Suppose that $x > y > z$; then each of the systems of equations:

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + t_z &= t_w; \end{aligned} \tag{7.1}$$

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned} \tag{7.2}$$

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned} \tag{7.3}$$

$$\begin{aligned} t_x - t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ z \neq u; \end{aligned} \tag{7.4}$$

$$\begin{aligned} t_x - t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v,\ z \neq u; \end{aligned} \tag{7.5}$$

$$\begin{aligned} t_x - t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned} \tag{7.6}$$

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y - t_z &= t_w; \end{aligned} \tag{7.7}$$

$$\begin{aligned} t_x - t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v,\ z \neq u; \end{aligned} \tag{7.8}$$

has infinitely many solutions in triangular numbers t_x, t_y, t_z, t_u, t_v, and t_w.

Proof: We prove even more. Each of the following systems of equations has infinitely many solutions in natural numbers x and y.

$$\begin{aligned} t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{16x+2} + t_{9x+2} &= t_y, \\ t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned} \tag{8.1}$$

$$\begin{aligned} t_{16x+2} + t_{13x+2} &= t_y, \\ t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{13x+2} - t_{12x+2} &= t_{5x}; \end{aligned} \tag{8.2}$$

$$\begin{aligned} t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{16x+2} - t_{9x+2} &= t_y, \\ t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned} \tag{8.3}$$

$$\begin{aligned} t_{15x+3} - t_{12x+2} &= t_{9x+2}, & \qquad t_{13x+2} - t_{12x+2} &= t_{5x}, \\ t_{15x+3} + t_{5x} &= t_y, & \text{or} \qquad t_{13x+2} + t_{9x+2} &= t_y, \\ t_{12x+2} + t_{5x} &= t_{13x+2}, & \qquad t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned} \tag{8.4}$$

$$(8.5)\qquad \begin{aligned} t_{15x+3} - t_{12x+2} &= t_{9x+2},\\ t_{15x+3} - t_{5x} &= t_y,\\ t_{12x+2} + t_{5x} &= t_{13x+2}; \end{aligned}$$

$$(8.6)\qquad \begin{aligned} t_{16x+2} - t_{13x+2} &= t_y,\\ t_{16x+2} + t_{12x+2} &= t_{20x+3},\\ t_{13x+2} - t_{12x+2} &= t_{5x}; \end{aligned}$$

$$(8.7)\qquad \begin{aligned} t_{52x+2} + t_{39x+2} &= t_{65x+3},\\ t_{52x+2} - t_{36x+2} &= t_y,\\ t_{39x+2} - t_{36x+2} &= t_{15x}; \end{aligned}$$

$$(8.8)\qquad \begin{aligned} t_{15x+3} - t_{13x+2} &= t_y,\\ t_{15x+3} - t_{12x+2} &= t_{9x+2},\\ t_{13x+2} - t_{12x+2} &= t_{5x}. \end{aligned}$$

The systems of equations (8.1)-(8.8) are, respectively, equivalent to the following equations, for which there exist initial solutions given below:

$$(9.1)\qquad 337x^2 + 125x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.2)\qquad 425x^2 + 145x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.3)\qquad 175x^2 + 35x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.4)\qquad 250x^2 + 110x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.5)\qquad 200x^2 + 100x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.6)\qquad 87x^2 + 15x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.7)\qquad 1408x^2 + 80x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.8)\qquad 56x^2 + 40x - y^2 - y = -6,\quad x_0 = 0,\quad y_0 = 2.$$

From the theory of Pell's equation (also referred to as Fermat's equation), it follows that if, simultaneously, k and m are natural numbers, 1, n, and q are integers, then the product $k \cdot m$ is not a square, and if there exists an initial solution of the equation,

$$(10)\qquad kx^2 + 1x - my^2 - ny = q,$$

in integers x_0 and y_0, where $\left(x_0 + \frac{1}{2k}\right)^2 + \left(y_0 + \frac{n}{2m}\right)^2 \neq 0$, then equation (10) has infinitely many solutions in natural numbers x and y. Applying this to equations (9.1)-(9.8) we prove that all the systems of equations (8.1)-(8.8) have infinitely many solutions in natural numbers x and y. This theorem is thus proved.

Some years ago, A. Schinzel found the following proof for the statement that there exist infinitely many triples of different triangular numbers for which the sum of any two is a triangular number [private communication from A. Schinzel].

Schinzel's Proof (unpublished): It is well known that the equation

$$x^2 - 424y^2 = 1$$

has infinitely many solutions, where $x \equiv 1 \pmod{106}$ [in every solution, we have $\pm x \equiv 1 \pmod{106}$]. Putting

$$k = 5y - \frac{25}{106}(x - 1) - 1,$$
$$1 = \frac{5}{2}(x - 1) - 50y + 2,$$

we find

$$t_{5k+4} + t_{9k+6} = t_1,$$
$$t_{5k+4} + t_{12k+9} = t_{13k+10},$$
$$t_{9k+6} + t_{12k+9} = t_{15k+11}.$$

3. SYSTEMS OF EQUATIONS INCLUDING THE ALGEBRAIC SUM AND THE PRODUCT OF TRIANGULAR NUMBERS

W. Sierpiński [1] has asked whether there exists a pair of triangular numbers such that the sum and the product of these numbers are triangular numbers. We have found some such systems of equations for which there exist one or two solutions in triangular numbers, e.g.:

(11) 1. $t_x - t_y = t_u, \quad t_x + t_y = t_v, \quad t_x t_y = t_w,$

$$t_{18} - t_{14} = t_{11}, \quad t_{18} + t_{14} = t_{23}, \quad t_{18}t_{14} = t_{189}.$$

(This solution was found by K. Szymiczek [2].)

(12) 2. $t_x + t_y = t_u, \quad t_x t_y = t_v, \quad (t_x + 1)t_y = t_w,$

$$t_9 + t_{13} = t_{16}, \quad t_9 t_{13} = t_{90}, \quad (t_9 + 1)t_{13} = t_{91}.$$

(13) 3. $t_x - t_y = t_u, \quad t_x t_y = t_v, \quad t_x/t_y - 1 = t_w,$

$$t_{21} - t_6 = t_{20}, \quad t_{21}t_6 = t_{98}, \quad t_{21}/t_6 - 1 = t_4.$$

(14) 4. $t_x - t_y = t_q, \quad t_x + t_z = t_u,$

$$t_x t_y = t_v, \quad t_x t_z = t_w,$$

$$t_{21} - t_6 = t_{20}, \quad t_{21} + t_{35} = t_{41},$$
$$t_{21}t_6 = t_{98}, \quad t_{21}t_{35} = t_{539},$$

and

$$t_{63} - t_{38} = t_{50}, \quad t_{63} + t_{219} = t_{228},$$
$$t_{63}t_{38} = t_{1728}, \quad t_{63}t_{219} = t_{9855}.$$

(15) 5. $t_x + t_z = t_q, \quad t_y - t_z = t_u,$

$$t_x t_z = t_v, \quad t_y t_z = t_w,$$

(continued)

5. *continued*

$$t_{29} + t_{69} = t_{75}, \quad t_{168} - t_{69} = t_{153},$$
$$t_{29}t_{69} = t_{1449}, \quad t_{168}t_{69} = t_{8280}.$$

6. For the system of equations,

$$t_x + t_y = t_u, \qquad t_xt_y = t_v, \tag{16}$$

there exists also the solution:

$$t_{505} + t_{531} = t_{733}, \quad t_{505}t_{531} = t_{189980}.$$

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ON EULER'S SOLUTION TO A PROBLEM OF DIOPHANTUS—II

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1. INTRODUCTION

In an earlier paper [1] we considered solutions to a system of equations:

$$x_ix_j + 1 = y_{ij}^2; \quad 1 \le i < j \le n.$$

In this note we look at the generalized problems:

$$x_ix_j + a = y_{ij}^2, \quad a \ne 0. \tag{1.1}$$

In Section 2 we apply the results of [1] to the solutions of (1.1). In Section 3 we consider the following problem: Find $n \times 2$ matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

so that $a_ib_j \pm a_jb_i = \pm 1$ for all $1 \le i < j \le n$. In Section 4 we apply the results of Section 3 to get two-parameter families of solutions of (1.1), linear in a, for $n = 4$.

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2. SOLUTIONS

Solutions of

$$x_i x_3 + a = y_{i3}^2; \quad i = 1, 2,$$

where

$$x_3, y_{i3} \in R = k[x_1, x_2, \sqrt{x_1 x_2 + a}]$$

and k is a field of characteristic $\neq 2$; x_1, x_2 algebraically independent over k.

We saw in [1] that for $a = 1$ the general solution could be represented by

$$\sqrt{x_1} y_{23} + \sqrt{x_2} y_{13} = \pm(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} + \sqrt{x_1 x_2})^n; \; n = 0, \pm 1, \pm 2, \ldots . \tag{2.1}$$

where $y_{12} = \sqrt{x_1 x_2 + a}$. We arrived at (2.1) by solving the Pell's equation,

$$x_1 y_{23}^2 - x_2 y_{13}^2 = x_1 - x_2, \tag{2.2}$$

which arises from the elimination of x_3 between the two equations (1.1). For general a, equation (2.2) becomes

$$x_1 y_{23}^2 - x_2 y_{13}^2 = a(x_1 - x_2). \tag{2.3}$$

If a is a square in b, say $a = b^2$, then the solution of (2.3) is entirely analogous to (2.1).

Theorem (2.4): If $a = b^2$, then the general solution of (2.3) in R is given by

$$\sqrt{x_1} y_{23} + \sqrt{x_2} y_{13} = \pm b(\sqrt{x_1} \pm \sqrt{x_2})\left(\frac{y_{12} + \sqrt{x_1 x_2}}{b}\right)^n; \; n = 0, \pm 1, \pm 2, \ldots .$$

Proof: We just take the general solution (2.1) for the case $a = 1$ and rename x_i by x_i/b and y_{ij} by y_{ij}/b to get the solution for $a = b^2$.

In case a is not a square in k, we can use Theorem 2.4 to give the general solution in the extended ring $R^* = k^*[x_1, x_2, y_{12}]$ where $k^* = k(\sqrt{a})$. The solutions in R are therefore given by the following.

Theorem (2.5): If a is not a square in k, then the general solution of (2.3) in R is given by

$$\sqrt{x_1} y_{23} + \sqrt{x_2} y_{13} = \pm(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} \pm \sqrt{x_1 x_2})^{2n+1} a^{-n}; \; n = 0, 1, 2, \ldots .$$

For example, if $k = 0$ and a is an integer, then either $a = \pm 1$ or the only solution with integral coefficients is

$$x_3 = x_1 + x_2 + 2y_{12}, \; y_{i3} = x_i + y_{12}. \tag{2.6}$$

Following [1], we see that in case $a = b^2$ we can find

$$x_4, y_{i4} \in R_1 = k[x_1, x_2, x_3, y_{12}, y_{13}, y_{23}]$$

so that $x_i x_4 + a = y_{i4}^2$. Namely,

$$x_4 = x_1 + x_2 + x_3 + 2\frac{x_1 x_2 x_3}{a} + 2\frac{y_{12} y_{13} y_{23}}{a} \tag{2.7}$$

$$y_{i4} = \frac{1}{b}(x_i y_{jk} + y_{ij} y_{ik}); \; \{i, j, k\} = \{1, 2, 3\}.$$

If a is not a square, then there is no x_4 element in R_1 so that $x_i x_4 + a$ are squares in R_1 for $i = 1, 2, 3$.

The construction in [1] for an $x_5 \in K = k(x_1, x_2, x_3, y_{12}, y_{13}, y_{23})$ so that $x_i x_5 + a = y_{i5}^2$; $i = 1, 2, 3, 4$ can be extended in case $a = b^2$ but not if a is not a square in k.

3. ON REAL $n \times 2$ MATRICES SATISFYING $a_i b_j \pm a_j b_i = \pm 1$

If we first consider the case where all the 2×2 determinants are ± 1, then it is clear that we must have $n \leq 3$, since for $n = 4$ the 6 determinants A_{ij} satisfy the identity

$$A_{12}A_{34} + A_{31}A_{24} + A_{23}A_{14} = 0$$

which makes it impossible that all A_{ij} are odd integers. Of course, there are many solutions for $n = 3$, for example

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

There is no restriction on the size of the matrix if we require only that the permanents of the 2×2 submatrices are ± 1. In fact, given any a, b so that $2ab = \pm 1$, then the matrix

$$a_1 = a_2 + \cdots + a_n = a; \quad b_1 = \cdots = b_n = b$$

obviously has all permanents ± 1.

If we call a matrix *admissible* when it satisfies $a_i b_j \pm a_j b_i = \pm 1$ for all $1 \leq i < j \leq n$, then admissibility is preserved under the following operations.

(i) Change of sign of any element.
(ii) Interchange of the two rows and permutations of columns.
(iii) Multiplication of one row by any nonzero constant and division of the other row by the same constant.

We therefore normalize to consider only matrices with nonnegative entries and without repeated columns. We call such matrices *permissible*.

Lemma (3.1): A permissible matrix with an entry 0 has no more than three columns.

Proof: We normalize the matrix so that $a_1 = 1$, $b_1 = 0$. Then

$$b_2 = \cdots = b_n = 1.$$

Thus, if we order the columns by $a_2 \leq a_3 \leq \cdots \leq a_n$, we get $a_j \pm a_i = 1$ for $2 \leq i < j \leq n$. If $n > 3$, this leaves only the possibilities

$$a_3 = 1 - a_2, \quad a_4 = 1 + a_2.$$

But then, $a_4 + a_3 = 2$ and $a_4 - a_3 = 2a_2 = 1$ leads to $a_2 = a_3 = 1/2$. Thus, $n \leq 3$.

We then assume that all entries are positive, and normalize to the form

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ b & b_2 & \cdots & b_n \end{pmatrix} \text{ with } 1 \leq a_2 \leq \cdots \leq a_n.$$

Then $b_i = 1 + ba_i$ or $|1 - ba_i|$.

Case 1. $b_2 = 1 + ba_2$. From the equations

$$a_2|1 \pm ba_i| \pm (1 + ba_2)a_i = \pm 1,$$

we get three possibilities:

$$a_2(1 + ba_i) - a_i(1 + ba_2) = -1, \qquad a_i = a_2 + 1$$

or

$$a_2(1 - ba_i) - a_i(1 + ba_2) = -1, \qquad a_i = \frac{a_2 + 1}{1 + 2ba_2}$$

or

$$a_2(ba_i - 1) + a_i(1 + ba_2) = 1, \qquad a_i = \frac{a_2 + 1}{1 + 2ba_2}.$$

Thus, $n \leq 4$, and for $n = 4$ we have

$$a_3 = \frac{a_2 + 1}{1 + 2ba_2}, \qquad b_3 = \frac{1 - b + ba_2}{1 + 2ba_2};$$

$$a_4 = a_2 + 1, \qquad b_4 = 1 + b + ba_2.$$

The equation $a_3b_4 \pm a_4b_3 = \pm 1$ becomes

$$(a_2 + 1)[(1 + b + ba_2) \pm (1 - b + ba_2)] = 1 + 2ba_2,$$

and hence,

$$2(a_2 + 1)(1 + ba_2) = 1 + 2ba_2,$$

which is impossible, or

$$2b(a_2 + 1) = 1 + 2ba_2, \quad b = 1/2.$$

But then $a_3 = 1$, $b_3 = 1/2$ which is not permissible. Thus $n \leq 3$ in this case.

Case 2. $b_2 = 1 - ba_2$. We get the possibilities:

$$(3.1)\qquad \begin{aligned} &a_2(1 + ba_i) - (1 - ba_2)a_i = \pm 1, && a_i = \frac{a_2 \pm 1}{1 - 2ba_2} \\ &a_2(1 - ba_i) + (1 - ba_2)a_i = 1, && a_i = \frac{a_2 - 1}{2ba_2 - 1} \\ &a_2(1 - ba_i) - (1 - ba_2)a_i = -1, && a_i = a_2 + 1 \\ &a_2(ba_i - 1) + (1 - ba_2)a_i = 1, && a_i = a_2 + 1 \\ &a_2(ba_i - 1) - (1 - ba_2)a_i = \pm 1, && a_i = \frac{a_2 \pm 1}{2ba_2 - 1} \end{aligned}$$

So the possible coices of a_i, $i = 3, 4, \ldots$, depend on the magnitude of ba_2.

(i) For $ba_2 < 1/2$, we get the possibilities:

$$(3.2)\qquad \begin{aligned} &a_i = \frac{a_2 - 1}{1 - 2ba_2}, && b_i = \frac{1 - b - ba_2}{1 - 2ba_2}; \\ &a_i = \frac{a_2 + 1}{1 - 2ba_2} && b_i = \frac{1 + b - ba_2}{1 - 2ba_2}; \\ &a_i = a_2 + 1, && b_i = 1 - b - ba_2. \end{aligned}$$

(ii) For $1/2 = ba_2$, we get only one possibility

$$a_i = a_2 + 1, \quad b_i = 1 - b - ba_2.$$

(iii) For $1/2 < ba_2 < 1$, we get the possibilities:

$$a_i = \frac{a_2 - 1}{2ba_2 - 1}, \quad b_i = \frac{|1 - b - ba_2|}{2ba_2 - 1}$$

$$(3.2)' \qquad a_i = \frac{a_2 + 1}{2ba_2 - 1}, \quad b_i = \frac{1 + b - ba_2}{2ba_2 - 1}$$

$$a_i = a_2 + 1, \quad b_i = |1 - b - ba_2|.$$

The first and third lines in (3.2) lead to

$$(1 - b - ba_2)[(a_2 + 1) \pm (a_2 - 1)] = 1 - 2ba_2;$$

that is, either

$$2a_2(1 - b - ba_2) = 1 - 2ba_2 \quad \text{or} \quad 2a_2(1 - ba_2) = 1,$$

which is impossible, since $a_2 > 1$ and $1 - ba_2 > 1/2$; or

$$2(1 - b - ba_2) = 1 - 2ba_2 \quad \text{or} \quad b = 1/2,$$

which violates the condition $ba_2 < 1/2$.

The second and third lines in (3.2) lead to

$$(a_2 + 1)[1 + b - ba_2 \pm (1 - b - ba_2)] = 1 - 2ba_2;$$

that is, either

$$2(a_2 + 1)(1 - ba_2) = 1 - 2ba_2 \quad \text{or} \quad a_i = \frac{a_2 + 1}{1 - 2ba_2} = \frac{1}{2(1 - ba_2)} < 1,$$

contrary to hypothesis, or

$$(3.3) \qquad 2b(a_2 + 1) = 1 - 2ba_2$$

$$b = \frac{1}{2(2a_2 + 1)}$$

which yields the 4×2 matrix

$$(3.4) \qquad \begin{pmatrix} 1 & a & a + 1 & 2a + 1 \\ \frac{1}{4a + 2} & \frac{3a + 2}{4a + 2} & \frac{3a + 1}{4a + 2} & \frac{3}{2} \end{pmatrix}$$

where the parameter, a, is chosen ≥ 1.

The first and second lines of (3.2) lead to

$$(a_2 + 1)(1 - b - ba_2) \pm (a_2 - 1)(1 + b - ba_2) = \pm(1 - 2ba_2)^2$$

which gives

$$(2b + 1)(2ba_2^2 - 2a_2 + 1) = 0 \quad \text{or} \quad 2(1 - 2ba_2) = (1 - 2ba_2)^2.$$

The first violates $2ba_2 < 1$, and the second violates $2ba_2 > 0$. Thus, (3.4) is the only matrix with $n > 3$ for Case 2(i).

The second and third lines of (3.2)′ lead to

$$(a_2 + 1)[1 + b - ba_2 \pm (1 - b - ba_2)] = 2ba_2 - 1.$$

Thus, either

$$2(a_2 + 1)(1 - ba_2) = 2ba_2 - 1, \quad b = \frac{2a_2 + 3}{2a_2(a_2 + 2)},$$

or

$$2b(a_2 + 1) = 2ba_2 - 1,$$

which is impossible.

The first case leads to the matrix

$$\begin{pmatrix} 1 & a & a+1 & a+2 \\ \dfrac{2a+3}{2a(a+2)} & \dfrac{1}{2a(a+2)} & \dfrac{a+3}{2a(a+2)} & \dfrac{3}{2a} \end{pmatrix} \tag{3.4'}$$

This is the same as the matrix (3.4) in case $0 < a \leq 1$, after we renormalize by replacing a by $1/a$, multiplying the first row by a and the second row by $1/a$ and interchanging the first two columns.

The first and third lines of (3.2) lead to

$$|1 - b - ba_2|[(a_2 + 1) \pm (a_2 - 1)] = 2ba_2 - 1,$$

both of which lead to

$$a_i = \frac{|1 - b - ba_2|}{2ba_2 - 1} \leq \frac{1}{2} < 1,$$

contrary to hypothesis.

To consider the first and third lines we first note that the conditions $1 - b - ba_2 < 0$, that is,

$$b > 1/(1 + a_2)$$

and

$$a_i = (a_2 - 1)/(2ba_2 - 1) \geq a_2 \geq 1$$

and incompatible. Thus, we get

$$(a_2 + 1)(1 - b - ba_2) \pm (a_2 - 1)(1 + b - ba_2) = (2ba_2 - 1)^2,$$

which leads either to

$$2a_2(1 - ba_2) - 2b = (2ba_2 - 1)^2,$$

and hence,

$$2(1 - b - ba_2) \leq (2ba_2 - 1)^2, \quad a_i \leq \frac{1}{2};$$

or to $ab_2 = \frac{1}{2}$. Both cases are excluded.

Thus (3.4) is the only normalized 4×2 matrix in Case 2.

Case 3. $b_2 = ba_2 - 1$. In this case, $b_i = ba_i - 1$ for all i and the possibilities reduce to:

$$\begin{aligned} a_2(ba_i - 1) - a_i(ba_2 - 1) &= 1, \quad a_i = a_2 + 1, \\ a_2(ba_i - 1) + a_i(ba_2 - 1) &= 1, \quad a_i = \frac{a_2 + 1}{2ba_2 - 1}. \end{aligned} \tag{3.5}$$

The two lines of (3.5) lead to

$$(a_2 + 1)[(ba_2 + b - 1) \pm (-ba_2 + b + 1)] = 2ba_2 - 1.$$

The resulting equations are $2b(a_2 + 1) = 2ba_2 - 1$, which is impossible, and

$$b = \frac{2a_2 + 1}{2a_2^2}$$

which makes

$$a_3 = \frac{a_2 + 1}{2ba_2 - 1} = a_2.$$

To sum up.

Theorem (3.6): There are no 5 x 2 permissible real matrices, and there is a one-parameter family of normalized permissible 4 x 2 matrices, given by (3.4).

We have limited the discussion to real matrices in order to reduce the number of cases. However, the family of permissible matrices (3.4) is valid for all fields of characteristic $\neq 2$ or 3, as long as we exclude the values $a = 0, -1/3, -1/2, -2/3$, and -1.

4. PARAMETRIC SOLUTIONS OF (1.1) WITH THE USE OF ADMISSIBLE MATRICES

Theorem (4.1): Given an admissible matrix $\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}$ then for any a, the elements

$$x_i = a_i^2 a - b_i^2; \quad i = 1, 2, \dots, n,$$

satisfy (1.1) with $y_{ij} = a_i a_j a \pm b_i b_j$.

Proof: For $1 \leq i < j \leq n$, we have

$$\begin{aligned} x_i x_j + a &= (a_i^2 a - b_i^2)(a_j^2 a - b_j^2) + a \\ &= a_i^2 a_j^2 a^2 + (1 - a_i^2 b_j^2 - a_j^2 b_i^2)a^2 + b_i^2 b_j^2. \end{aligned} \tag{4.2}$$

Now, since $a_i b_j \pm a_j b_i = \pm 1$, we have

$$1 - a_i^2 b_j^2 - a_j^2 b_i^2 = \pm 2a_i a_j b_i b_j.$$

Substituting in (4.2), we get

$$x_i x_j + a = a_i^2 a_j^2 a^2 \pm 2a_i a_j b_i b_j a + b_i^2 b_j^2 = (a_i a_j a \pm b_i b_j)^2.$$

In view of (3.4), we get a two-parameter family of 4 x 2 admissible matrices,

$$\begin{pmatrix} s & st & s(t + 1) & s(2t + 1) \\ \frac{1}{2s(2t + 1)} & \frac{3 \quad + 2}{2s(2t + 1)} & \frac{3 \quad + 1}{2s(2t + 1)} & \frac{3}{2s} \end{pmatrix}$$

which yield a corresponding three-parameter solution,

$$x_i = x_i(s, t, a), \quad y_{ij} = y_{ij}(s, t, a),$$

of (1.1), which is linear in a. In general, x_3 and x_4 are algebraic, but not rational, functions of x_1 and x_2.

REFERENCE

1. Joseph Arkin, V. E. Hoggatt, Jr., & E. G. Straus. "On Euler's Solution to a Problem of Diophantus." *The Fibonacci Quarterly* 17 (1979):333-339.

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FIBONACCI NOTES

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6. ZERO-ONE SEQUENCE ONCE MORE

1. Let $f(m, n, r, s)$ denote the number of zero-one sequences of length $m+n$:

$$\sigma = (a_1, a_2, \ldots, a_{m+n}) \qquad (a \ = 0 \text{ or } 1) \tag{1.1}$$

with m zeros, n ones, r occurrences of (00), and s occurrences of (11). It is proved in [1] that

$$f(m, n, r, s) = \begin{cases} 2\dbinom{m-1}{r}\dbinom{n-1}{s} & (m - r = n - s) \\ \dbinom{m-1}{r}\dbinom{n-1}{s} & (m - r = n - s \pm 1) \\ 0 & (\text{otherwise}). \end{cases} \tag{1.2}$$

The proof in [1] makes use of generating functions; we shall now give a combinatorial proof of (1.2).

Arrange the m zeros and n ones in the following way. We first place m_0 zeros on the extreme left, then n_1 ones, m_1 zeros, n_2 ones, n_2 zeros, ..., n_k ones, m_k zeros, where k is some nonnegative integer,

$$\begin{aligned} &m = m_0 + m_1 + \cdots + m_k,\ n = n_1 + \cdots + n_k, \\ &m_0 \geq 0,\ m \ \geq 0,\ m \ \geq 1 \quad (1 \leq i < k) \\ &n_1 \geq 0 \quad (1 \leq i \leq k) \end{aligned} \tag{1.3}$$

and

$$\begin{cases} r = \displaystyle\sum_{i=0}^{k} (m_i - 1) + \delta + \delta' = m - k - 1 + \delta + \delta' \\ s = \displaystyle\sum_{i=1}^{k} (n_i - 1) = n - k, \end{cases} \tag{1.4}$$

where

$$\delta = \begin{cases} 1 & (m_0 = 0) \\ 0 & (m_0 > 0), \end{cases} \qquad \delta' = \begin{cases} 1 & (m_k = 0) \\ 0 & (m_k > 0). \end{cases} \tag{1.5}$$

It follows from (1.3) and (1.4) that

$$r - s = m - n + \delta + \delta' - 1. \tag{1.6}$$

It is now convenient to consider four cases:

$$\text{(i)} \quad m_0 = m_k = 0; \qquad \text{(ii)} \quad m_0 = 0,\ m_k > 0;$$

$$\text{(iii)} \quad m_0 > 0,\ m_k = 0; \quad \text{(iv)} \quad m_0 > 0,\ m_k > 0.$$

The number of solutions of

$$a = x_1 + \cdots + x_k, \; x_i > 0 \quad (i = 1, \ldots, k)$$

is equal to $\binom{a-1}{k-1}$.

Thus, the number of solutions

$$(m_0, m_1, \ldots, m_k; n_1, \ldots, n_k)$$

of (1.3) is equal to:

$$\text{(i)} \qquad \binom{m-1}{k-2}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s - 1)$$

$$\text{(ii)} \qquad \binom{m-1}{k-1}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s)$$

$$\text{(iii)} \qquad \binom{m-1}{k-1}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s)$$

$$\text{(iv)} \qquad \binom{m-1}{k}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s + 1).$$

The first part of (1.2) is implied by (ii) together with (iii), the second part by (i) and (iv). The last part of (1.2) is equivalent to the statement that k cannot exist satisfying both parts of (1.4).

This evidently completes the proof of (1.2).

2. The above proof is applicable to a much more general problem. Let

$$(2.1) \qquad \boldsymbol{r} = (r_1, r_2, r_3, \ldots), \quad \boldsymbol{s} = (s_1, s_2, s_3, \ldots)$$

be two sequences of nonnegative integers. We again consider zero-one sequences of length $m+n$ with m zeros and n ones. Let $f(\boldsymbol{r}, \boldsymbol{s})$ denote the number of such sequences, where $r_1 = m$, $s_1 = n$, with r_i blocks of zeros of length i and s_i blocks of ones of length i for $i = 2, 3, 4, \ldots$. Thus, r_1 can be thought of as the number of blocks of zeros of length one and s_1 the number of blocks of length one.

As in §1, we envisage an arbitrary sequence σ as broken into a block of zeros (possibly vacuous), a block of ones, a block of zeros, and so on. However, we shall now enumerate the blocks by their cardinality. If k denotes the number of blocks of ones, then the number of blocks of zeros is either $k - 1$, k, or $k + 1$. Hence, we have the following relations,

$$(2.2) \qquad \begin{cases} r_1 = k_1' + 2k_2' + 3k_3' + \cdots \\ r_2 = k_2' + 2k_3' + 3k_4' + \cdots \\ r_3 = k_3' + 2k_4' + 3k_5' + \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{cases}$$

and

$$(2.3)\qquad \begin{cases} s_1 = k_1 + 2k_2 + 3k_3 + \cdots \\ s_2 = k_2 + 2k_3 + 3k_4 + \cdots \\ s_3 = k_3 + 2k_4 + 3k_5 + \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{cases}$$

together with

$$(2.4)\qquad \begin{cases} k' = k_1' + k_2' + k_3' + \cdots \\ k = k_1 + k_2 + k_3 + \cdots, \end{cases}$$

where $k' = k - 1$, k, or $k + 1$.

The k_i' denote the multiplicity of blocks of zeros of length i, and the k_i denote the multiplicity of blocks of ones of length i. Thus, the first of (2.2) enumerates the number of blocks of zeros of length one, that is, the total number of zeros. The second of (2.2) enumerates the number of blocks of zeros of length two, and so on. Similar remarks apply to (2.3) for the blocks of ones.

It is easily verified that (2.2) is equivalent to the system of equations

$$(2.5)\qquad \begin{cases} k_1' = r_1 - 2r_2 + r_3 \\ k_2' = r_2 - 2r_3 + r_4 \\ k_3' = r_3 - 2r_4 + r_5 \\ \cdots\cdots\cdots\cdots \end{cases}$$

while (2.3) is equivalent to

$$(2.6)\qquad \begin{cases} k_1 = s_1 - 2s_2 + s_3 \\ k_2 = s_2 - 2s_3 + s_4 \\ k_3 = s_3 - 2s_4 + s_5 \\ \cdots\cdots\cdots\cdots \end{cases}$$

Thus, the r_i and s_i must satisfy the following conditions, but are otherwise unrestricted.

$$(2.7)\qquad \begin{cases} r_i - 2r_{i+1} + r_{i+2} \geq 0 \\ s_i - 2s_{i+1} + s_{i+2} \geq 0 \end{cases} \qquad (i = 1, 2, 3, \ldots).$$

It follows from (2.5), (2.6), and (2.4) that

$$(2.8)\qquad \begin{cases} k' = r_1 - r_2 \\ k = s_1 - s_2. \end{cases}$$

Clearly,

$$f(\mathbf{r},\ \mathbf{s}) = \frac{k'!}{k_1'!k_2'!k_3'!' \ldots} \cdot \frac{k!}{k_1!k_2!k_3!\ \ldots} \tag{2.9}$$

In terms of r_i and s_i, this becomes

$$f(\mathbf{r},\ \mathbf{s}) = \frac{(r_1 - r_2)!}{(r_1 - 2r_2 + r_3)!(r_2 - 2r_3 + r_4)!\ \ldots} \cdot \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)!\ \ldots} \tag{2.10}$$

3. For applications, it is convenient to use generating functions. By the multinomial theorem, we have

$$\sum_{k_1+k_2+k_3+\cdots=k} \frac{k!}{k_1!k_2!k_3!\ \ldots} x_1^{k_1}x_2^{k_2}x_3^{k_3} \ldots = (x_1 + x_2 + x_3 + \cdots)^k, \tag{3.1}$$

where it is assumed that the series $x_1 + x_2 + x_3 + \cdots$ is absolutely convergent. By (2.6), the left-hand side of (3.1) is equal to

$$\sum_{\substack{\mathbf{s}\\ s_1-s_2=k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_1 - 2s_2 + s_3)!\ \ldots} x_1^{s_1-2s_2+s_3} x_2^{s_2-2s_3+s_4} \ldots$$

$$= \sum_{\substack{\mathbf{s}\\ s_1-s_2=k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)!\ \ldots} x_1^{s_1}(x_1^{-2}x_2)^{s_2}(x_1x_2^{-2}x_3)^{s_3} \ldots .$$

Hence, if we take

$$\left\{\begin{array}{l} x_1 = y_1 \\ x_2 = y_1^2 y_2 \\ x_3 = y_1^3 y_2^2 y_3 \\ x_4 = y_1^4 y_2^3 y_3^2 y_4 \\ \ldots\ldots\ldots \end{array}\right.$$

(3.1) becomes

$$(y_1 + y_1^2y_2 + y_1^3y_2^2y_3 + \cdots)^k \tag{3.2}$$

$$= \sum_{\substack{\mathbf{s}\\ s_1-s_2=k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)!\ \ldots} y_1^{s_1}y_2^{s_2}y_3^{s_3} \ldots .$$

As a first application of (3.2), we take $y_3 = y_4 = y_5 = \cdots = 1$. Then, the left-hand side of (4.2) reduces to

$$\begin{aligned}(y_1 + y_1^2y_2 + y_1^3y_2^2 + \cdots)^k &= y_1^k(1 - y_1y_2)^{-k} \\ &= \sum_{s=0}^{\infty}\binom{k+s-1}{s} y_1^{s+k}y_2^s \\ &= \sum_{s_1 - s_2 = k}\binom{s_1 - 1}{s_2} y_1^{s_1}y_2^{s_2},\end{aligned}$$

in agreement with (1.2).

If we take $y_3 = y_4 = \cdots = 0$, we get

$$\begin{aligned}(y_1 + y_1^2y_2)^k &= y_1^k \sum_{s=0}^{\infty}\binom{k}{s} y_2^s \\ &= \sum_{s_1 - s_2 = k}\binom{s_1 - s_2}{s_2} y_1^{s_1}y_2^{s_2}.\end{aligned}$$

Thus, in this case, we have

$$f(\mathbf{r}, \mathbf{s}) = \binom{r_1 - r_2}{r_2}\binom{s_1 - s_2}{s_2}, \tag{3.3}$$

where $r_1 - r_2 = k'$, $s_1 - s_2 = k$, while

$$r_3 = r_4 = \cdots = 0, \quad s_3 = s_4 = \cdots = 0.$$

That is, (3.3) furnishes the enumerant when all blocks are of length one or two.

4. In (3.2), we now take

$$y_4 = y_5 = y_6 = \cdots = 1. \tag{4.1}$$

Then, the left-hand side of (3.2) becomes

$$\begin{aligned}&(y_1 + y_1^2y_2 + y_1^3y_2^2y_3 + y_1^4y_2^3y_3^2 + \cdots)^k \\ &= y_1^k\left\{1 + y_1y_2(1 + y_1y_2y_3 + y_1^2y_2^2y_3^2 + \cdots)\right\}^k \\ &= y_1^k\left\{1 + \frac{y_1y_2}{1 - y_1y_2y_3}\right\}^k \\ &= y_1^k \sum_{t=0}^{k} \begin{matrix}k\\t\end{matrix}\ (y_1y_2) \sum_{s=0}^{\infty}\binom{t+s-1}{s}(y_1y_2y_3) \\ &= \sum_{\substack{s_1, s_2, s_3 \\ s_1 - s_2 = k}}\binom{s_1 - s_2}{s_2 - s_3}\binom{s_2 - 1}{s_3} y_1^{s_1}y_2^{s_2}y_3^{s_3}.\end{aligned}$$

Hence, we have

$$f(\mathbf{r}, \mathbf{s}) = \binom{r_1 - r_2}{r_2 - r_3}\binom{r_2 - 1}{r_3}\binom{s_1 - s_2}{s_2 - s_3}\binom{s_2 - 1}{s_3}, \tag{4.2}$$

where $r_1 - r_2 = k'$, $s_1 - s_2 = k$.

Thus (4.2) furnishes the enumerant by blocks of length 1, 2, and 3.

If, instead of (4.1), we take

$$y_4 = y_5 = y_6 = \cdots = 0, \tag{4.3}$$

we have

$$(y_1 + y_1^2y_2 + y_1^3y_2^2y_3)^k = \sum_{t_1+t_2+t_3=k} \frac{k!}{t_1!t_2!t_3!} y_1^{t_1+2t_2+t_3} y_2^{t_2+2t_3} y_3^{t_3}$$

$$= \sum_{\substack{s_1, s_2, s_3 \\ s_1 - s_2 = k}} \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3)!s_3!} y_1^{s_1} y_2^{s_2} y_3^{s_3}$$

so that

$$f(\mathbf{r}, \mathbf{s}) = \frac{(r_1 - r_2)!}{(r_1 - 2r_2 + r_3)!(r_2 - 2r_3)!r_3!} \cdot \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3)!s_3!}, \tag{4.4}$$

the enumerant when *all* blocks are of length 1, 2, or 3.

5. The general cases corresponding to (4.2) and (4.4) are now readily obtained. Let p be a fixed positive integer, and take

$$y_{p+1} = y_{p+2} = \cdots = 1. \tag{5.1}$$

Then we have

$$\left\{y_1 + y_1^2y_2 + \cdots + y_1^{p-2}y_2^{p-3} \cdots y_{p-2} + \frac{y_1^{p-1}y_2^{p-2} \cdots y_{p-1}}{1 - y_1y_2 \cdots y_p}\right\}^k \tag{5.2}$$

$$= \sum_{t_1+\cdots+t_{p-1}=k} (t_1, t_2, \ldots, t_{p-1}) y_1^{t_1'} y_2^{t_2'}$$

$$\cdots y_{p-1}^{t_{p-1}'} \sum_{s=0}^{\infty} \binom{t_{p-1} + s - 1}{s} (y_1y_2 \cdots y_p)^s,$$

where

$$(t_1, t_2, \ldots, t_{p-1}) = \frac{(t_1 + t_2 + \cdots + t_{p-1})!}{t_1!t_2! \ldots t_{p-1}!}$$

and

$$\begin{cases} t_1' = t_1 + 2t_2 + \cdots + (p-1)t_{p-1} \\ t_2' = t_2 + 2t_3 + \cdots + (p-2)t_{p-2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ t_{p-2}' = t_{p-2} + 2t_{p-1} \\ t_{p-1}' = t_{p-1}. \end{cases}$$

Put

$$t_i' + s = s_i \quad (1 \le i < p), \quad s = s_p.$$

It follows that

$$(5.3) \qquad \begin{cases} t_{p-1} = s_{p-1} - s_p \\ t_{p-2} = s_{p-2} - 2s_{p-1} + s_p \\ t_{p-3} = s_{p-3} - 2s_{p-2} + s_{p-1} \\ \cdots\cdots\cdots\cdots\cdots \\ t_1 = s_1 - 2s_2 + s_3. \end{cases}$$

Hence, the coefficient of $y_1^{s_1} y_2^{s_2} \ldots y_p^{s_p}$ in (5.2) is equal to

$$(5.4) \qquad (t_1, t_2, \ldots, t_{p-1})\binom{s_{p-1} - 1}{s},$$

where $t_1, t_2, \ldots, t_{p-1}$ are given by (5.3).

The enumerant $f(\mathbf{r}, \mathbf{s})$ is therefore equal to (5.4) times the corresponding factor containing the r_i.

Corresponding to

$$(5.5) \qquad y_{p+1} = y_{p-2} = \cdots = 0,$$

we have

$$(5.6) \qquad \begin{aligned} &(y_1 + y_1^2 y_2 + \cdots + y_1^p y_2^{p-1} \cdots y_p)^k \\ &= \sum_{t_1 + \cdots + t_p = k} (t_1, t_2, \ldots, t_p) y_1^{s_1} y_2^{s_2} \ldots y_p^{s_p}, \end{aligned}$$

where now

$$\begin{cases} t_1 + 2t_2 + 3t_3 + \cdots + pt_p = s_1 \\ t_2 + 2t_3 + 3t_4 + \cdots + (p-1)t_p = s_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t_{p-1} + 2t_p = s_{p-1} \\ t_p = s_p. \end{cases}$$

This gives

$$(5.7)\qquad \begin{cases} t_p = s_p \\ t_{p-1} = s_{p-1} - 2s_p \\ t_{p-2} = s_{p-2} - 2s_{p-1} + s_p \\ t_{p-3} = s_{p-3} - 2s_{p-2} + s_{p-1} \\ \cdots\cdots\cdots\cdots\cdots \\ t_1 = s_1 - 2s_2 + s_3. \end{cases}$$

Hence, the coefficient of $y_1^{s_1}y_2^{s_2} \ldots y_p^{s_p}$ is the multinomial coefficient $(t_1, t_2, \ldots, t_p)$, with the t_i determined by (5.7). The enumerant $f(\boldsymbol{r}, \boldsymbol{s})$ is the product of this coefficient times the corresponding factor containing the r_i.

6. Some curious combinatorial identities are implied by the above results. To illustrate with a simple case, we return to §3. It follows from (3.1) that, for $s_1 > s_2$, we have

$$(6.1)\qquad \sum (t_1, t_2, t_3, \ldots) = \binom{s_1 - 1}{s_2},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots),$$

and the summation is over all $s_3, s_4, s_5, \ldots$.

Similarly, from the proof of (4.2), we have, for

$$s_1 - 2s_2 + s_3 \geq 0, \quad s_2 > s_3,$$

$$(6.2)\qquad \sum (t_1, t_2, t_3, \ldots) = \binom{s_1 - s_2}{s_2 - s_3}\binom{s_2 - 1}{s_3},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots),$$

and the summation is over all $s_4, s_5, s_6, \ldots$.

The general case implied by (5.2) and (5.4) is readily stated. We have

$$(6.3)\qquad \sum (t_1, t_2, t_3, \ldots) = (\overline{t}_1, \overline{t}_2, \ldots, \overline{t}_{p-1})\binom{s_{p-1} - 1}{s_p},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots)$$

$$\overline{t}_i = t_i \quad (i = 1, \ldots, p - 2), \quad \overline{t}_{p-1} = s_{p-1} - s_p,$$

and the summation on the left of (6.3) is over all $s_{p+1}, s_{p+2}, s_{p+3}, \ldots$.

There are various other possibilities; for example, taking $y = 1$ in (3.2). However, we leave this for another occasion.

REFERENCE

1. L. Carlitz. "Fibonacci Notes: 5. Zero-One Sequences Again." *The Fibonacci Quarterly* 15 (1977):49-56.

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-424 *Proposed by Richard M. Grassl, University of New Mexico*

Of the $\binom{52}{5}$ possible 5-card poker hands, how many form a:

(i) full house?
(ii) flush?
(iii) straight?

B-425 *Proposed by Richard M. Grassl, University of New Mexico*

Let k and n be positive integers with $k < n$ and let S consist of all k-tuples $X = (x_1, x_2, \ldots, x_k)$ with each x_j an integer and

$$1 \leq x_1 < x_2 < \cdots < x_k \leq n.$$

For $j = 1, 2, \ldots, k$, find the average value $\overline{x}_j$ of x_j over all X in S.

B-426 *Proposed by Herta T. Freitag, Roanoke, VA*

Is $(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2$ a perfect square for all positive integers n, i.e., are there integers c_n such that $(F_nF_{n+3}, 2F_{n+1}F_{n+2}, c_n)$ is always a Pythagorean triple?

B-427 *Proposed by Phil Mana, Albuquerque, NM*

Establish a closed form for $\displaystyle\sum_{k=1}^{n} k\binom{k}{2}\binom{n-k}{3}$.

B-428 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For odd positive integers w, establish a closed form for

$$\sum_{k=0}^{2s+1} \binom{2s+1}{k} F^2_{n+kw}.$$

B-429 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Is the function

$$F^4_{n+10r} + F^4_n - (L_{8r} + L_{4r} - 1)(F^4_{n+8r} + F^4_{n+2r}) + (L_{12r} - L_{8r} + 2)(F^4_{n+6r} + F^4_{n+4r})$$

independent of n? Here n and r are integers.

SOLUTIONS

Multiples of Some Triangular Numbers

B-400 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the nth triangular number $n(n+1)/2$. For which positive integers n is $T_1^2 + T_2^2 + T_3^2 + \cdots + T_n^2$ an integral multiple of T_n?

Solution by C. C. Thompson, Roanoke, VA

Let $S = \sum_{k=1}^{n} T_n^2$, where n is a positive integer; then S is an integral multiple of T_n iff $n \equiv 1, 7, 13 \pmod{15}$. To see this, use the formulas for sums of powers of the first n positive integers (or the method of differences) and a bit of manipulative algebra to get

$$S = T_n \cdot (3n^3 + 12n^2 + 13n + 2)/30.$$

From this, the sum S is an integral multiple of T_n iff

$$f(n) = 3n^3 + 12n^2 + 13n + 2 \equiv 0 \pmod{2 \cdot 3 \cdot 5}.$$

Now $f(n) \equiv n^3 + n \equiv n(n+1)^2 \equiv 0 \pmod 2$ is satisfied by any positive integer; $f(n) \equiv n + 2 \equiv 0 \pmod 3$ has $n \equiv 1 \pmod 3$ as its only solution; $f(n) \equiv (3n+2)(n^2+1) \equiv 0 \pmod 5$ has $n \equiv 1, 2, 3 \pmod 5$ as solutions. From this, $f(n) \equiv 0 \pmod{30}$ has the solutions $n \equiv 1, 7, 13 \pmod{15}$.

Also solved by Paul S. Bruckman, Edilio A. Escalona Fernández, Bob Prielipp, Sahib Singh, M. Wachtel (Switzerland), Jonathan Weitsman, Gregory Wulczyn, and the proposer.

Change of Pace for F.Q.

B-401 *Proposed by Gary L. Mullen, Pennsylvania State University*

Show that $\lim\limits_{n\to\infty}[(n!)^{2n}/(n^2)!] = 0$.

Solution by Edilio A. Escalona Fernández, Caracas, Venezuela

Let's call $R_n = (n!)^{2n}/(n^2)!$, and $T_n = Ln(R_n)$. Then,

$$T_n = 2nLn(n!) - Ln((n^2)!),$$

so that by applying the formula $Ln(n!) = nLn(n) - n + O(Ln(n))$, we have

$$T_n = -n^2 + 2nO(Ln(n)) + O(Ln(n)) = -n^2 + O(nLn(n)),$$

and this means that $T_n \to -\infty$ as $n \to \infty$; hence, by continuity of $\exp(x)$:

$$\exp(T_n) = R_n \to 0 \text{ as } n \to \infty.$$

Also solved by Paul S. Bruckman, M. Wachtel (Switzerland), Jonathan Weitsman, Gregory Wulczyn, and the proposer.

Pythagorean Triple

B-402 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Show that $(L_nL_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$ is a Pythagorean triple.

Solution by Sahib Singh, Clarion College, Clarion, PA

Let $A = L_{n+2}$, $B = L_{n+1}$, then

$$A^2 - B^2 = (L_{n+2} - L_{n+1})(L_{n+2} + L_{n+1}) = L_nL_{n+3}.$$
$$A^2 + B^2 = L^2_{n+2} + L^2_{n+1} = 5(F^2_{n+2} + F^2_{n+1}) = 5F_{2n+3}.$$

Thus, the given triple is $A^2 - B^2$, $2AB$, $A^2 + B^2$, which is Pythagorean.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, John W. Milsom, Bob Prielipp, and the proposer.

Lucas Congruence

B-403 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let $m = 5^n$. Show that $L_{2m} \equiv -2 \pmod{5m^2}$.

Solution by Graham Lord, Université Laval, Québec;
Bob Prielipp, University of Wisconsin-Oshkosh; and
Sahib Singh, Clarion College, Clarion, Pa (independently)

It is known that $m|F_m$. [See B-248, vol. 11 (1973):553.] Hence,

$$(5m^2)\,|\,(5F^2_m).$$

Since m is odd, we also have $L_{2m} = 5F^2_m - 2$, and it follows that

$$L_{2m} \equiv -2 \pmod{5m^2}.$$

Also solved by Paul S. Bruckman, Lawrence Somer, and the proposer.

Golden Approximations

B-404 *Proposed by Phil Mana, Albuquerque, NM*

Let x be a positive irrational number. Let a, b, c, and d be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with r rational, implies that the denominator of r exceeds b, we call a/b a good lower approximation (GLA) for x. If $x < r < c/d$, with r rational, implies that the denominator of r exceeds d, c/d is a good upper approximation (GUA) for x. Find all the GLAs and all the GUAs for $(1 + \sqrt{5})/2$.

Solution by Paul S. Bruckman, Concord, CA

Let

(1) $$x_n = F_{2n}/F_{2n-1},\ y_n = F_{2n+1}/F_{2n},\ n = 1, 2, 3, \ldots;$$

let

(2) $$X = (x_n)^\infty_{n=1},\ Y = (y_n)^\infty_{n=1}.$$

It is well known that X and Y provide the convergents for the continued fraction of α, and moreover:

(3) $$1 = x_1 < x_2 < \cdots < x_n < \cdots \alpha \cdots < y_n < \cdots < y_2 < y_1 = 2.$$

Let L and U denote the set of GLAs and GUAs, respectively, for α. We will prove that

(4) $$L = X,\ U = Y.$$

We will use the following result, readily proved by applying the Binét definitions:

(5) $$F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1} = 1.$$

Proof of (4): Given any positive integer n, and any rational $r = u/v$, such that $x_n < r \le x_{n+1}$, then, $x_{n+1} - x_n \ge r - x_n > 0$, i.e.,

$$\frac{F_{2n+2}}{F_{2n+1}} - \frac{F_{2n}}{F_{2n-1}} \ge \frac{u}{v} - \frac{F_{2n}}{F_{2n-1}} > 0$$

$$\Rightarrow v(F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1}) \ge F_{2n+1}(uF_{2n-1} - vF_{2n}) > 0.$$

But, since $u/v > F_{2n}/F_{2n-1}$, thus $uF_{2n-1} - vF_{2n} \ge 1$; using (5), this implies

(6) $$v \ge F_{2n+1}.$$

Since $F_{2n-1} < F_{2n+1}$, thus $v > F_{2n-1}$, which implies that $x_n \in L$. Hence,

(7) $$X \subseteq L.$$

Conversely, suppose $r = u/v \in L$. Then, for some n, $x_n < r \le x_{n+1}$, which again implies (6), as above. Assume that $r < x_{n+1}$. Then, by definition of L, $v < F_{2n+1}$, which contradicts (6). It follows that $r = x_{n+1} \Rightarrow r \in X$. Hence,

(8) $$L \subseteq X.$$

Combining (7) and (8) implies $L = X$. Proceeding in a totally analogous manner, we may likewise prove that $U = Y$.

Also solved by Sahib Singh, Gregory Wulczyn, and the proposer.

Good Rational Approximations

B-405 *Proposed by Phil Mana, Albuquerque, NM*

Prove that for every positive irrational x, the GLAs and GUAs for x (as defined in B-404) can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_nq_{n+1} = \pm 1 \text{ for all } n.$$

Solution by the proposer.

Let $p = [x]$, the greatest integer in x. Clearly p is a GLA and $p + 1$ is a GUA. So we let $p_1 = p$, $q_1 = 1 = q_2$, and $p_2 = p + 1$. Then we assume inductively that p_n and q_n have been defined for $n = 1, 2, \ldots, k$. Let s be the largest such n for which p_n/q_n is a GLA and t be the largest such n for which p_n/q_n is a GUA; then define $p_{n+1} = p_s + p_t$ and $q_{n+1} = q_s + q_t$. This defines p_n and q_n for all positive integers n and we let $r = p_n/q_n$. It follows from the theory of Farey sequences [see Ivan Niven & Herbert S. Zuckerman, *An Introduction to the Theory of Numbers* (New York: Wiley, 1960), pp. 128-133) that the r_n give us all the GLAs and GUAs and that $p_{n+1}q_n - p_nq_{n+1} = \pm 1$.

Also solved by Paul S. Bruckman and Sahib Singh.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-313 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

A. Show that the Fibonacci numbers partition the Fibonacci numbers.
B. Show that the Lucas numbers partition the Fibonacci numbers.
(See "Additive Partitions I," FQJ, April 1977, p. 166.)

H-314 *Proposed by P. Bruckman, Concord, CA*

Given $x_0 \in (-1, 0)$, define the sequence $S = (x_n)_{n=0}^{\infty}$ as follows:

(1) $$x_{n+1} = 1 + (-1)^n\sqrt{1 + x_n},\ n = 0, 1, 2, \ldots .$$

Find the limit point(s) of S, if any exist.

H-315 *Proposed by D. P. Laurie, National Research Institute for Mathematical Sciences, Pretoria, South Africa*

Let the polynomial P be given by

$$P(z) = z_n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

and let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. The following iteration scheme for factorizing P has been suggested by Kerner [1]:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\sum_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)};\quad i = 1, 2, \ldots, n.$$

Prove that if $\sum_{i=1}^{n} z_i = -a_{n-1}$, then also $\sum_{i=1}^{n} \hat{z}_i = -a_{n-1}$.

REFERENCE

1. I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen." *Numer. Math.* 8 (1966):290-294.

H-316 *Proposed by B. R. Myers, University of British Columbia, Vancouver, Canada*

The enumerator of compositions with exactly k parts is $(x + x^2 + \cdots)^k$, so that

$$(1) \qquad [W(x)]^k = (w_1x + w_2x^2 + \cdots)^k$$

is then the enumerator of weighted k-part compositions. After Hoggatt & Lind ("Compositions and Fibonacci Numbers," *The Fibonacci Quarterly* 7 (1969):253-266), the number of weighted compositions of n can be expressed in the form,

$$(2) \qquad C_n(w) = \sum_{\Upsilon(n)} w_{a_1} \ldots w_{a_k} \qquad (n > 0),$$

where $w = \{w_1, w_2, \ldots\}$ and where the sum is over all compositions $a_1 + \cdots + a_k$ of n (k variable). In particular (*ibid.*),

$$(3) \qquad \sum_{\Upsilon(n)} a_1 \ldots a_k = F_{2n}(1, 1),$$

where $F_k(p, q)$ is the kth number in the Fibonacci sequence

$$(4) \qquad \begin{aligned} F_1(p, q) &= p \quad (\geq 0) \\ F_2(p, q) &= q \quad (\geq p) \\ F_{n+2}(p, q) &= F_{n+1}(p, q) + F_n(p, q) \quad (n \geq 1). \end{aligned}$$

Show that

$$(5) \qquad \sum_{\Upsilon(n)} (a_1 \pm 1)a_1 \ldots a_k = 2[F_{2n\pm1}(1, 1) - 1]$$

and, hence, that

$$(6) \qquad \sum_{\Upsilon(n)} (a_1 - 1)a_1 \ldots a_k + \sum_{\Upsilon(n)} a_1 \ldots a_k = F_{2n}(1, 1 + 2m) - 2m \quad (m \geq 0).$$

SOLUTIONS

Umbral - a

H-285 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA*
(A generalization of R. G. Buschman's H-18)
(Vol. 16, No. 2, April 1978)

Show that

$$(a) \qquad \sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{nr-rk} = 2^n F_{rn} \text{ or } (F^r + L^r)^n \doteq (2F^r)^n.$$

Solution by L. Carlitz, Duke University, Durham, NC

Much more can be proved readily. Let $C(n, k)$, $0 \leq k \leq n$, be numbers that satisfy the symmetry condition:

$$C(n, k) = C(n, n - k) \qquad (0 \leq k \leq n).$$

Let a, b be arbitrary, and define

$$F_n = \frac{a^n - b^n}{a - b}, \quad L_n = a^n + b^n.$$

Then

$$\sum_{k=0}^{n} C(n, k) F_{rk} L_{rn-rk} = \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rk} - b^{rk})(a^{rn-rk} + b^{rn-rk})$$

$$= \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rn} - b^{rn}) + \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rk}b^{rn-rk} - a^{rn-rk}b^{rk}).$$

Since

$$\sum_{k=0}^{n} C(n,\ k)a^{rn-rk}b^{rk} = \sum_{k=0}^{n} C(n,\ n-k)a^{rk}b^{rn-rk} = \sum_{k=0}^{n} C(n,\ k)a^{rk}b^{rn-rk},$$

it follows that

$$\sum_{k=0}^{n} C(n,\ k)(a^{rk}b^{rn-rk} - a^{rn-rk}b^{rk}) = 0.$$

Therefore

$$(*) \qquad \sum_{k=0}^{n} C(n,\ k)F_{rk}L_{rn-rk} = F_{rn}\sum_{k=0}^{n} C(n,\ k).$$

For example, if $C(n,\ k) = \binom{n}{k}$, we get

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk}L_{rn-rk} = 2\ F_{rn},$$

while, if $C(n,\ k) = \binom{n}{k}^2$, we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{rk}L_{rn-rk} = \binom{2n}{n}F_{rn}.$$

To take a less obvious example, let $A_{n,k}$ denote the Eulerian number defined by

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}};\ A_n(x) = \sum_{k=1}^{n} A_{n,k}x^k \quad (n \ge 1).$$

It is well known that

$$A_{n,k} = A_{n,n-k} \quad (1 \le k \le n)$$

and

$$\sum_{k=1}^{n} A_{n,k} = n!$$

Take

$$C(n,\ k) = A_{n+1,\,k+1}\ (0 \le k \le n),$$

so that

$$C(n,\ k) = C(n,\ n-k) \qquad (0 \le k \le n).$$

It follows that

$$\sum_{k=0}^{n} A_{n+1,\,k+1}F_{rk}L_{rn-rk} = (n+1)!F_{rn}.$$

Also solved by P. Bruckman, J. Vogel, and the proposer.

<u>*LATE ACKNOWLEDGMENTS:*</u>

H-281, also solved by J. Shallit.
H-283, also solved by A. Shannon, A. Philippou, and P. Yff.

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