In Memoriam

Verner E. Hoggatt, Jr.
June 26, 1921–August 11, 1980

Founding Editor, *The Fibonacci Quarterly*

With the end of summer came the end of the life of our editor, whom many of us loved and respected deeply. His passing has left a void that may never be filled for those who knew him either as a man or as a researcher.

Dr. Hoggatt's life was marked by dedication to the study of the properties of the Fibonacci sequence. His production and creativity were astounding. He wrote more than 150 research papers, which have appeared in numerous prestigious mathematics journals. He wrote a book on Fibonacci numbers, and was the editor of three other books. In addition, he was commissioned to write an article on Fibonacci numbers for the Encyclopaedia Britannica.

He guided 38 master's theses and one master's project, co-authored papers with some of the outstanding mathematicians of the world, was an enthusiastic conference speaker for mathematical societies several times each year, and the organizer of approximately 25 Fibonacci research conferences in the San Francisco Bay Area.

Dr. Hoggatt, Brother Alfred Brousseau, and I. Dale Ruggles founded The Fibonacci Association, and in 1963 he and Brother Alfred started *The Fibonacci Quarterly*, of which Dr. Hoggatt had been the general editor for 18 years.

Dr. Hoggatt received his M.A. under Dr. Howard Eves, and his Ph.D. under Dr. Charles C. Clark at Oregon State University. For the past 27 years, he had taught in the Mathematics Department at San Jose State University.

Even with as many as fifteen research projects going on at the same time, teaching full time, editing the *Quarterly*, and helping his wife, Herta, raise their family, Dr. Hoggatt always found the time to encourage his students and discuss their problems. He always had a ready smile and a quick witty pun.

We loved him and shall miss him.

Marjorie Bicknell-Johnson

GENERALIZED FIBONACCI NUMBERS

ANNE SILVA
University of Santa Clara, Santa Clara, CA 95053
VERNER E. HOGGATT, JR.
San Jose State University, San Jose, CA 95192

The sequence of generalized Fibonacci numbers is composed of terms derived from Pascal's triangle. The nth term of the sequence, u_n, is equal to the sum from $i = 0$ to $i = \dfrac{n+p-1}{p+1}$ of the terms $\dbinom{n-(i-1)p-1}{i}$, which represent binomial coefficients.

In the left-justified form of Pascal's triangle, u_n equals the sum of the $\dbinom{n+p-1}{0}$ term and the terms taken successively p units up and 1 unit over. For $p = 0$, this generates the powers of 2. For $p = 1$, the resulting sequence is the Fibonacci numbers.

The sequence for $p = k$, any given constant, begins as follows:

$$\begin{array}{cccccccc} u_0 & u_1 & u_2 & \ldots\ u_k & u_{k+1} & u_{k+2} & u_{k+3} & \ldots \\ 1 & 1 & 2 & \ldots\ k & k+1 & k+2 & k+4 & \ldots \end{array}$$

The rest of the sequence can be generated using the recursion formula $u_n = u_{n-1} + u_{n-k-1}$.

There are four important properties related to representations of integers which apply to the generalized Fibonacci sequence.

1. Completeness—Every positive integer N can be expressed as a sum of distinct u_n terms:

$$N = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0, 1\}.$$

2. Zeckendorf Form—Every positive integer N has a unique representation using a minimal number of u_n terms $\alpha_i \alpha_{i+j} = 0$ for $1 \le j \le k$.

3. Second Canonical Form—In this form, any positive integer M which contains $u_1 = 1$ in its representation has this u_1 replaced by $u_0 = 1$. This form is also unique for each positive integer.

4. Lexicographic Ordering—Both the Zeckendorf and Second Canonical forms of representations are lexicographic orderings meaning that, when comparing two numbers M and N, $M > N$ iff M has the larger coefficient for u_i, where u_i is the first summand for which the representations of M and N differ, going from highest to lowest.

The set of positive integers can be partitioned into $k + 1$ sets, using representations in terms of generalized Fibonacci numbers. Since the sequence of generalized Fibonacci numbers is complete with respect to the positive integers, each positive integer N is the sum of distinct u_n terms. The partitions are made according to the subscript on the smallest u_n term used in the Zeckendorf representation of an integer. If the subscript is congruent to i modulo $(k+1)$, then that integer is an element of the set A_i. Every integer is an element of one and only one set A_i for $1 \le i \le k+1$. The notation $A_i(n)$ denotes the nth element of the set A_i, when the elements are listed in natural order.

$A_i(n) = R + u_{m(k+1)+i}$, where R denotes the representation of N minus the smallest summand.

$u_{m(k+1)+i}$ can be rewritten using the recursion formula:

$$u_n = u_{n-1} + u_{n-k-1}.$$

$$\begin{aligned} u_{m(k+1)+i} &= u_{m(k+1)+i-1} + u_{m(k+1)+i-(k+1)} = u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i} \\ &= u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i-1} + u_{(m-2)(k+1)+i} \end{aligned}$$

$$= u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i-1} + \cdots + u_i.$$

Thus, $A_i(n) = u_{m(k+1)+i} + R$ can be rewritten

$$A_i(n) = u_i + R', \ 1 \le i \le k + 1.$$

R' is the rest of the representation of $A_i(n)$, so

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \alpha_{i+2}u_{i+2} + \alpha_{i+3}u_{i+3} + \cdots + \alpha_m u_m.$$

There are two mappings which are useful in discovering properties of the partitioned integers. The first is f, which advances by 1 the subscripts on the summands of N when N is written in Second Canonical form. The second mapping is f^*, which performs the same function as f on N when N is written in Zeckendorf form.

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \alpha_{i+2}u_{i+2} + \cdots + \alpha_n u_n.$$

$$A_i(n) \xrightarrow{f} u_{i+1} + \alpha_{i+1}u_{i+2} + \alpha_{i+2}u_{i+3} + \cdots + \alpha_n u_{n+1} = A_{i+1}(n).$$

By lexicographic ordering, $A_i(n)$ is mapped by f onto the nth element of A_{i+1}, $2 \le i \le k$.

$$A_1(n) = u_0 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m \text{ in Second Canonical form.}$$

$$A_1(n) \xrightarrow{f} u_1 + \alpha_2 u_3 + \alpha_3 u_4 + \cdots + \alpha_m u_{m+1},$$

which is an element of A_1.

$$A_{k+1}(n) = u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m.$$

$$A_{k+1}(n) \xrightarrow{f} u_{k+2} + \alpha_{k+2}u_{k+3} + \cdots + \alpha_m u_{m+1}.$$

By the recursion formula, $u_n = u_{n-1} + u_{n-k-1}$, $u_{k+2} = u_{k+1} + u_1$.

$$A_{k+1}(n) \xrightarrow{f} u_1 + u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_{m+1},$$

which is an element of A_1.

Since $A_1, A_2, \ldots, A_{k+1}$ cover the positive integers, we have that every integer n is mapped by f onto the set $(A_1 \cup A_3 \cup A_4 \cup \ldots \cup A_{k+1})$. By lexicographic ordering, n is mapped onto the nth element of this set. Call this set H_1. Then $n \xrightarrow{f} H_1(n)$.

An element of the set H_1 is mapped forward onto

$$(A_1 \cup A_4 \cup A_5 \cup \ldots \cup A_{k+1}),$$

since each set except A_1 and A_{k+1} map forward one set and these two map onto A_1. Call this second set H_2: $\quad n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n)$

In general,

$$H_i(n) = (A_1 \cup A_{i+2} \cup A_{i+3} \cup \ldots \cup A_{k+1})(n) \text{ and } H_i(n) \xrightarrow{f} H_{i+1}(n)$$

for $1 \le i \le k - 2$. For $i = k - 2$,

$$H_{k-2}(n) \xrightarrow{f} H_{k-1}(n) = (A_1 \cup A_{k+1})(n). \qquad H_{k-1}(n) \xrightarrow{f} A_1(n).$$

$$n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n) \xrightarrow{f} H_3(n) \xrightarrow{f} \ldots \xrightarrow{f} H_{k-1}(n) \xrightarrow{f} A_1(n).$$

Now using the f^* mapping,

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^*} u_2 + \alpha_2 u_3 + \cdots + \alpha_m u_{m+1} = A_2(n).$$

In general,

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \cdots + \alpha_m u_m \xrightarrow{f^*} A_{i+1}(n)$$

$$A_{i+1}(n) = u_{i+1} + \alpha_{i+1}u_{i+2} + \cdots + \alpha_m u_{m+1}, \text{ for } 1 \le i \le k.$$

f^* and f are the same mappings except when applied to elements of A_1, the only elements whose Second Canonical and Zeckendorf forms are different.

(1) $n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n) \xrightarrow{f} \cdots \xrightarrow{f} H_{k-1}(n) \xrightarrow{f} A_1(n) \xrightarrow{f^*} A_2(n) \xrightarrow{f^*} \cdots \xrightarrow{f^*} A_{k+1}(n)$.

The mappings can be used to identify a further relationship between the H_i and A_i sets. By (1) above, n is mapped by k successive applications of f onto $A_1(n)$. Denote this

$$n \xrightarrow{f^{(k)}} A_1(n).$$

$$n = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^{(k)}} u_1 + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m} = A_1(n).$$

$$A_1(n) \xrightarrow{f^*} u_2 + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = A_2(n).$$

$$n + A_1(n) = (u_1 + u_1) + \alpha_2(u_2 + u_{k+2}) + \cdots + \alpha_m(u_m + u_{k+m}).$$

Using the recursion formula,

$$u_n = u_{n-1} + u_{n-k-1},\ A_1(n) + n = u_2 + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = A_2(n).$$

By similar proofs, any element plus its image k steps forward in the scheme described in (1) equals the element one step further in scheme (1).

(2) $$A_1(n) + n = A_2(n),$$
$$A_i(n) + H_{i-1}(n) = A_{i+1}(n) \quad \text{for} \quad 2 \le i \le k.$$

Here is an example of the representations, partitions, and mappings for $k = 3$.

The sequence of u_n's for $k = 3$ begins as follows:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7 ...
1	1	2	3	4	5	7	10 ...

$$1 = u_0 \xrightarrow{f} u_1 = 1 = H_1(1)$$
$$2 = u_2 \xrightarrow{f} u_3 = 3 = H_1(2)$$
$$3 = u_3 \xrightarrow{f} u_4 = 4 = H_1(3)$$
$$4 = u_4 \xrightarrow{f} u_5 = 5 = H_1(4)$$
$$5 = u_0 + u_4 \xrightarrow{f} u_1 + u_5 = 1 + 5 = 6 = H_1(5)$$
$$6 = u_0 + u_5 \xrightarrow{f} u_1 + u_6 = 1 + 7 = 8 = H_1(6)$$
$$7 = u_2 + u_5 \xrightarrow{f} u_3 + u_6 = 3 + 7 = 10 = H_1(7)$$
$$8 = u_0 + u_6 \xrightarrow{f} u_1 + u_7 = 1 + 10 = 11 = H_1(8)$$
$$9 = u_2 + u_6 \xrightarrow{f} u_3 + u_7 = 3 + 10 = 13 = H_1(9)$$
$$10 = u_3 + u_6 \xrightarrow{f} u_4 + u_7 = 4 + 10 = 14 = H_1(10)$$

The other mappings described in scheme (1) are derived in a similar manner. The array for $k = 3$ from $1 \le n \le 10$ follows:

n	$H_1(n)$	$H_2(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$
1	1	1	1	2	3	4
2	3	4	5	7	10	14
3	4	5	6	9	13	18
4	5	6	8	12	17	23
5	6	8	11	16	22	30
6	8	11	15	21	29	40
7	10	14	19	26	36	50

n	$H_1(n)$	$H_2(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$
8	11	15	20	28	39	54
9	13	18	24	33	46	64
10	14	19	25	35	49	68

Examining this array, it is soon apparent that the differences between successive elements in a given set depend on which set the subscript belongs to. Thus, it is necessary to add another layer of subscripts to discuss these differences. We want to find a general description for

$$A_j(A_i(n) + 1) - A_j(A_i(n)).$$

Denote this difference, $\Delta A_j(A_i(n))$, as $\Delta g(j, i)$.

The simplest case to start with is $\Delta g(1, 2)$. The first step is to notice that by applying lexicographic ordering to mapping scheme (1), we can see that that number of integers $N \leq H_1(n)$ must equal the number of elements of A_1 that are $\leq A_2(n)$. The same idea applies to any two pairs of numbers an equal number of sets apart in the mapping scheme.

Since the number of integers $N \leq H_1(n) = H_1(n)$, we have that $\#A_1$ elements $\leq A_2(n) = H_1(n)$. Thus, the largest A_1 element $\leq A_2(n)$ is $A_1(H_1(n))$.

$$A_2(n) = u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m.$$

Since $u_2 = 2$,

$$A_2(n) - 1 = 1 + \alpha_3 u_3 + \cdots + \alpha_m u_m = u_1 + \alpha_3 u_3 + \cdots + \alpha_m u_m \in A_1.$$

Since we are dealing with integers, the closest two elements can be is 1 apart. Thus $A_2(n) - 1$ is the largest element less than A_2, and since we know it is an element of A_1, it must be $A_1(H_1(n))$.

$$A_1(H_1(n)) + 1 = A_2(n).$$

The set H_1 excludes A_2 elements, so $A_1(A_2(n))$ cannot equal any $A_1(H_1)$ element. $A_1(A_2(n)) + 1 \notin A_2$.

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n.$$

Since $u_1 = 1$,

$$A_1(n) + 1 = 2 + \alpha_2 u_2 + \cdots + \alpha_n u_n.$$

We know that $A_1(A_2(n)) + 1$ does not belong to A_2. Adding 1 to $A_1(A_2(n))$ must change the representation so that u_2 is not used. Since in the Zeckendorf form and the Second Canonical form we are dealing with you cannot have terms in the representation closer than k subscripts apart, $A_1(A_2(n)) + 1$ cannot be an element of $A_3, A_4, \ldots, A_{k+1}$. By process of elimination, $A_1(A_2(n)) + 1$ is an element of A_1. By lexicographic ordering, it must be the next element after the $A_2(n)$th element.

$$\Delta g(1, 2) = 1. \tag{3}$$

Next we want to find $\Delta g(1, i)$ for $3 \leq i \leq k$. We know from mapping scheme (1) that $n \xrightarrow{f} H_1(n)$. Therefore, $A_i(n) \xrightarrow{f} H_1(A_i(n))$. But we also know from the mapping scheme that $A_i(n) \xrightarrow{f} A_{i+1}(n)$ for $2 \leq i \leq k$, since f and f^* are the same mappings for these elements. Thus,

$$H_1(A_i(n)) = A_{i+1}(n), \quad 2 \leq i \leq k. \tag{4}$$

By lexicographic ordering and mapping scheme (1), $\#A_i$ elements $\leq A_{i+1}(n) = \#n$'s $\leq H_1(n) = H_1(n)$.

$$A_{i+1}(n) = u_{i+1} + \alpha_{i+2} u_{i+2} + \cdots + \alpha_m u_m.$$

$u_{i+1} = i + 1$ for $0 \leq i \leq k$, so $A_{i+1}(n) - 1 = i + \alpha_{i+1} + \cdots + \alpha_m u_m \in A_i$.

We know that there are $H_1(n)$ elements of $A_i \le A_{i+1}(n)$. Therefore, the largest $A_i \le A_{i+1}(n)$ is $A_i(H_1(n))$. We know that $A_{i+1}(n) - 1 \in A_i$, and that this is the closest any 2 integers can get. Therefore,

(5) $$A_i(H_1(n)) + 1 = A_{i+1}(n), \text{ for } 1 \le i \le k.$$

Equation (5) can be generalized further. By lexicographic ordering and mapping scheme (1), $\#A_i$ elements $\le A_{i+j}(n) = \#n$'s $\le H_j(n) = H_j(n)$, for $1 \le i \le k$; $1 \le j \le k-1$; $1 \le i+j \le k+1$. Thus the $H_j(n)$th element of A_i is the largest one $\le A_{i+j}(n)$.

$$A_{i+j}(n) = u_{i+j} + \alpha_{i+j+1}u_{i+j+1} + \cdots + \alpha_m u_m.$$

$u_{i+j} = i + j$ for $1 \le i+j \le k+1$, so

$$A_{i+j}(n) - j = i + \alpha_{i+j+1}u_{i+j+1} + \cdots + \alpha_m u_m.$$

$i = u_i$ for $1 \le i \le k+1$, so

$$A_{i+j}(n) - j \in A_i.$$

By mapping scheme (1), the closest any 2 elements of A_i and A_{i+j} can be is j units apart, so $A_{i+j}(n) - j$ is the largest A_i element $\le A_{i+j}(n)$. Thus,

(6) $$A_1(H\ (n)) + j = A_{i+j}(n) \quad \text{for } 1 \le i \le k;\ 1 \le j \le k-1;\ 1 \le i+j \le k+1.$$

$$A_1(H_1(A_{i-1}(n))) + 1 = A_1(A_i(n)) + 1, \text{ by (4)},$$

$$A_1(H_1(A_{i-1}(n))) + 1 = A_2(A_{i-1}(n)) \text{ by (5)}.$$

Thus (a) $$A_1(A_i(n)) + 1 = A_2(A_{i-1}(n)).$$

$$A_2(H_{i-3}(n)) + i - 3 = A_{i-1}(n) \text{ by (6)}.$$

$H_{i-3} = (A_1 \cup A_{i-1} \cup A_{i-1} \cup A_i \cup \cdots \cup A_{k+1})$ by definition of H_i (see p. 291). Thus $A_{i-1}(n) \in H_{i-3}$, and $A_2(A_{i-1}(n)) + i - 3 \in A_{i-1}$, say $A_{i-1}(t)$.

$$A_2(H_{i-2}(n)) + i - 2 = A_i(n) \text{ by (6)}.$$

$H_{i-2} = (A_1 \cup A_i \cup \cdots \cup A_{k+1})$ by definition of H_i. Thus $A_{i-1}(n) \notin H_{i-2}$, and $A_2(A_{i-1}(n)) + i - 2 \notin A_i$.

$$A_2(A_{i-1}(n)) + i - 2 = A_{i-1}(t) + 1 \notin A_i.$$

$$A_{i-1}(t) = u_{i-1} + \alpha_i u_i + \cdots + \alpha_m u_m.$$

Adding 1 to this particular A_{i-1} element must change the representation so that a u_i is not used. Since, in Zeckendorf and Second Canonical form, no two summands can have subscripts closer than k units apart, $A_{i-1}(t) + 1$ cannot use any summands from u_2, u_3, ... up to u_{k+1}. This means that $A_{i-1}(t) + 1 \notin A_2$, A_3, ... up to A_{k+1}. The only remaining set is A_1.

From (a) above, $A_1(A_i(n)) + 1 = A_2(A_{i-1}(n))$. $A_2(A_{i-1}(n)) + i - 2 \in A_1$, and this must be the next A_1 element after $A_1(A_i(n))$ by lexicographic ordering. Therefore

(7) $$\Delta g(1, i) = i - 1 \quad \text{for } 3 \le i \le k.$$

The next case we will examine is $\Delta g(1, 1)$. Since $n \xrightarrow{f} H_1(n)$, from mapping scheme (1),

$$A_1(n) \xrightarrow{f} H_1(A_1(n)).$$

From (1), we also know that

$$A_1(n) \xrightarrow{f^*} A_2(n).$$

The only difference between the two mappings is that f maps $u_0 = 1$ onto $u_1 = 1$, while f^* maps $u_1 = 1$ onto $u_2 = 2$. Therefore, we know that

$$H_1(A_1(n)) + 1 = A_2(n).$$

But we also know that $A_1(H_1(n)) + 1 = A_2(n)$ by (5). Thus

(8) $$A_1(H_1(n)) = H_1(A_1(n)).$$

By (4), $H_1(A_i(n)) = A_{i+1}(n)$ for $2 \leq i \leq k$. $H_1(A_{k+1}(n))$ is the only portion of the H_1 set not identified as a particular A_i. $H_1 = (A_1 \cup A_3 \cup A_4 \cup \ldots \cup A_{k+1})$. $A_3, A_4, \ldots, A_{k+1}$ are taken by $H_1(A_i(n))$ for $2 \leq i \leq k$, and $A_1(H_1(n))$ is taken by $H_1(A_1(n))$. Since the elements of each set A_i do not overlap, $H_1(A_{k+1}(n))$ must be an element of the only remaining portion of H_1: $A_1(A_2)$. By lexicographic ordering, it must be the nth.

(9) $$H_1(A_{k+1}(n)) = A_1(A_2(n)).$$

$$\begin{aligned} A_1(H_1(A_1(n))) + 1 &= A_2(A_1(n)) \text{ by (5)} \\ &= A_1(A_1(H_1(n))) + 1 \text{ by (8)}. \end{aligned}$$

$$\begin{aligned} A_1(H_1(A_{k+1}(n))) + 1 &= A_2(A_{k+1}(n)) \text{ by (5)} \\ &= A_1(A_1(A_2(n))) + 1 \text{ by (9)}. \end{aligned}$$

$(A_1 \cup A_{k+1}) = H_{k-1}$, so the first line of each of the above two equations defines $A_2(H_{k-1}(n))$.

$(H_1 \cup A_2)$ = all the integers; thus, the second line of each of the above two equations defines $A_1(A_1(n)) + 1$.

Thus $A_2(H_{k-1}(n)) = A_1(A_1(n)) + 1$.

$$A_2(H_{k-1}(n)) + k - 1 = A_{k+1}(n) \text{ by (6)}.$$

So

$$A_1(A_1(n)) + k = A_{k+1}(n).$$

$$A_{k+1}(n) = u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m.$$

$$A_{k+1}(n) + 1 = u_1 + u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m,$$

since 1 and $k+1$ are k units apart, making the combination of u_1 and u_{k+1} acceptable under Zeckendorf form.

$A_{k+1}(n) + 1 \in A_1$, since it has a u_1 in its representation. Thus

$$A_1(A_1(n)) + k + 1 \in A_1,$$

and this must be the next A_1 by lexicographic ordering.

(10) $$\Delta g(1, 1) = k + 1.$$

Finally, we examine $\Delta g(1, k+1)$. $A_1(H_{k-1}(n)) + (k-1) = A_k(n)$ by (6). $A_{k+1}(n) \in H_{k-1}$, so $A_1(A_{k+1}(n)) + k - 1 \in A_k$.

$A_1(A_1(n)) + k = A_2(H_{k-1}(n)) + k - 1 = A_{k+1}(n)$ from the preceding argument for $\Delta g(1, 1)$.

$A_1(A_{k+1}(n)) + k \notin A_{k+1}$, since A_1 and A_{k+1} are disjoint sets.

$A_1(A_{k+1}(n)) + k = A_k(t) + 1$. Since this $A_k(t) + 1$ is not an element of A_{k+1}, it can only be an A_1 element from the restrictions imposed by Zeckendorf form.

$A_1(A_{k+1}(n)) + k \in A_1$, and it must be the next A_1 by lexicographic ordering.

(11) $$\Delta g(i, k+1) = k.$$

Combining equations (3), (7), (10), and (11), we have that

$$\begin{aligned} \Delta g(1, 1) &= k + 1, \\ \Delta g(1, i) &= i - 1, \text{ for } 2 \leq i \leq k + 1. \end{aligned}$$

Since $u_i = 1$ for $1 \leq i \leq k + 1$, we can restate this as

$$\Delta g(1, 1) = u_{k+1},$$

$$\Delta g(1, i) = u_{i-1}.$$

Now we will use mathematical induction to prove what $\Delta g(j, i)$ is equal to.

(12) Induction Hypothesis: $\Delta g(j, 1) = u_{k+j}$,
$\Delta g(j, i) = u_{i+j-2}$, for $2 \leq i \leq k + 1$.

These differences apply for $1 \leq j \leq k + 1$.

Equations (3), (7), (10), and (11) prove that the induction hypothesis is true for $j = 1$, establishing an induction basis.

Assume

(13) $$\Delta g(m, i) = u_{i+m-2}, \text{ for } 2 \leq i \leq k + 1.$$

(a) $$A_{m+1}(A_i(n)) = A_m(H_1(A_i(n))) + 1 \text{ by (5).}$$

$$A_m(H_1(A_i(n))) + 1 = A_m(A_{i+1}(n)) + 1 \text{ by (4).}$$

$$A_m(A_{i+1}(n)) + u_{i+m-1} = A_m(A_{i+1}(n) + 1) \text{ by assumption (13), for } 1 \leq i \leq k.$$

(b) $$A_{i+1}(n) + 1 \in H_1 \text{ for } 1 \leq i \leq k.$$

Since $A_m(H_1(n)) + 1 = A_{m+1}(n)$ by (5), $A_m(A_{i+1}(n) + 1) + 1 \in A_{m+i}$.

$$A_m(H_1(A_i(n))) + 1 = A_{m+1}(A_i(n)) \text{ from (13a)}$$
$$= A_m(A_{i+1}(n)) + 1.$$

Thus

$$A_{m+1}(A_i(n)) = A_m(A_{i+1}(n)) + 1.$$

$$A_{m+1}(A_i(n)) + u_{i+m-1} = A_m(A_{i+1}(n)) + u_{i+m-1} + 1$$
$$= A_m(A_{i+1}(n) + 1) + 1 \text{ by (13).}$$

Thus $$A_m(A_{i+1}(n) + 1) + 1 \in A_{m+1} \text{ by 13b),}$$

and by (5), $$A_{m+1}(A_i(n)) + u_{i+m-1} \in A_{m+1}.$$

This must be the next A_{m+1}, so

$$\Delta g(m + 1, i) = u_i + (m + 1) - 2 = u_{i+m-1}.$$

Thus far, assuming $\Delta g(m, i) = u_{i+m-2}$ has implied that

$$\Delta g(m + 1, i) = u_{i+(m+1)-2}, \text{ for } 2 \leq i \leq k.$$

By mathematical induction, hypothesis (12) holds true for $2 \leq i \leq k$.

Assume

(14) $$\Delta g(m, k + 1) = u_{k+m-1} \text{ and } \Delta g(m, 1) = u_{k+m}.$$

$$A_{m+1}(A_{k+1}(n)) = A_m(H_1(A_{k+1}(n))) + 1 \text{ by (5)}$$
$$= A_m(A_1(A_2(n))) + 1 \text{ by (9).}$$

$$A_m(A_1(A_2(n))) + u_{k+m} = A_m(A_1(A_2(n)) + 1) \text{ by (14)}$$
$$= A_m(A_1(A_2(n) + 1)) \text{ by (3).}$$

$$A_1(A_2(n)) + 1 \in H_1, \text{ so } A_m(A_1(A_2(n) + 1)) + 1 \in A_{m+1} \text{ by (5).}$$

Putting the last few statements together,

$$A_{m+1}(A_{k+1}(n)) + u_{k+m} \in A_{m+1}.$$

This must be the next A_{m+1} element by lexicographic ordering.

$$\Delta g(m, k + 1) = u_{k+m-1} \text{ implies } \Delta g(m + 1, k + 1) = u_{k+m} = u_{k+(m+1)-1}.$$

Since $\Delta g(j, k + 1) = u_{k+j-1}$ was proved true for $j = 1$ in (11), and assuming this statement true for $j = m$ implies that it holds for $j = m + 1$, then, by mathematical induction $\Delta g(j, k + 1) = u_{k+j-1}$.

For the final case, we want to prove that $\Delta g(m, 1) = u_{k+m}$ implies that $\Delta g(m + 1, 1) = u_{k+m+1}$.

$$A_{m+1}(A_1(n)) = A_m(H_1(A_1(n))) + 1 \text{ by (5)} = A_m(A_1(H_1(n))) + 1 \text{ by (8)},$$

$$A_m(A_1(H_1(n))) + u_{k+m} = A_m(A_1(H_1(n)) + 1) \text{ by (14)} = A_m(A_2(n)) \text{ by (5)},$$

$$A_m(A_2(n)) + u_m = A_m(A_2(n) + 1) \text{ by (13)}.$$

Since $A_m(H_1(n)) + 1 = A_{m+1}(n)$ by (5), and $A_2(n) + 1 \in H_1$, then

$$A_m(A_2(n) + 1) + 1 \in A_{m+1}.$$

Combining the above statements,

$$A_{m+1}(A_1(n)) + u_{k+m} + u_m \in A_{m+1}.$$

This must be the next A_{m+1} element by lexicographic ordering.

$$\Delta g(m + 1, 1) = u_{k+m} + u_m.$$

By the recursion formula, $u_n = u_{n-1} + u_{n-k-1}$, $u_{k+m} + u_m = u_{k+m+1}$, so

$$\Delta g(m + 1, 1) = u_{k+m+1}.$$

By mathematical induction, hypothesis (12) has been proved true.

$$\Delta g(j, i) = u_{i+j-2}, \; 2 \le i \le k + 1,$$
$$\Delta g(j, 1) = u_{k+j},$$

for $1 \le j \le k + 1$.

Arrays for $k = 1$, 2, and 4 follow to help illustrate the difference formula $\Delta g(i, j)$. The array for $k = 3$ can be found on pages 291-292 above.

<u>$k = 1$</u>: The sequence of u_n's generated for $k = 1$ in Fibonacci numbers is:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8 ...
1	1	2	3	5	8	13	21	34 ...

n	$A_1(n)$	$A_2(n)$
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

<u>$k = 2$</u>:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7 ...
1	1	2	3	4	6	9	13 ...

n	$H_1(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$
1	1	1	2	3
2	3	4	6	9
3	4	5	8	12
4	5	7	11	16
5	7	10	15	22
6	9	13	19	28
7	10	14	21	31
8	12	17	25	37
9	13	18	27	40
10	14	20	30	44

$k = 4$:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9 ...
1	1	2	3	4	5	6	8	11	15 ...

n	$H_1(n)$	$H_2(n)$	$H_3(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$	$A_5(n)$
1	1	1	1	1	2	3	4	5
2	3	4	5	6	8	11	15	20
3	4	5	6	7	10	14	19	25
4	5	6	7	9	13	18	24	31
5	6	7	9	12	17	23	30	39
6	7	9	12	16	22	29	38	50
7	9	12	16	21	28	37	49	65
8	11	15	20	26	34	45	60	80
9	12	16	21	27	36	48	64	85
10	14	19	25	32	42	56	75	100

Another question suggested by these arrays is: How many elements of a set A_j are less than a given n? To find the answer, we need a function that increments only when it passes an A_j element. This function turns out to be the third difference of terms in successive A_j sets.

(15) $\#A_j\text{'s} < n = S(j, n) = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j}(n).$

Proof: First, we need to define the sets A_{k+2}, A_{k+3}, and A_{k+4} and show that their properties are consistent with those of $A_1, A_2, \ldots, A_{k+1}$.

$$A_{k+1}(n) \xrightarrow{f} H_1(A_{k+1}(n)) \text{ by mapping scheme (1).}$$

$$H_1(A_{k+1}(n)) = A_1(A_2(n)) \text{ by equation (9).}$$

$$A_{k+1}(n) \xrightarrow{f^*} A_1(A_2(n)) \xrightarrow{f^*} A_2(A_2(n)) \xrightarrow{f^*} A_3(A_2(n)), \text{ using (1)}$$

with the subscript $A_2(n)$ instead of n.

$A_1(n)$ is mapped onto $A_{k+1}(n)$ by k applications of f^*.

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^{*(k)}} u_{k+1} + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m} = A_{k+1}(n).$$

$$A_{k+1}(n) \xrightarrow{f^*} u_{k+2} + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = u_1 + u_{k+1} + \alpha_2 u_{k+3} + \cdots = A_1(A_2(n)).$$

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$$

$$A_{k+1}(n) = u_{k+1} + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m}$$

$$A_1(n) + A_{k+1}(n) = u_1 + u_{k+1} + \alpha_2(u_2 + u_{k+2}) + \cdots + \alpha_m(u_m + u_{k+m})$$

$$= u_1 + u_{k+1} + \alpha_2 u_{k+3} + \cdots + \alpha\, u_{k+m+1},$$

by the recursion formula.

$A_1(n) + A_{k+1}(n) = A_1(A_2(n))$. Relabel $A_1(A_2(n))$ as $A_{k+2}(n)$.

Since $A_2(n)$ and $A_{k+2}(n)$ are also k applications of f^* apart in the mapping scheme, $A_2(n) + A_{k+2}(n) = A_2(A_2(n))$ by the recursion formula, since

$$A_{k+2}(n) \xrightarrow{f^*} A_2(A_2(n)).$$

Relabel $A_2(A_2(n)) = A_{k+3}(n)$.

Similarly, $A_3(n) + A_{k+3}(n) = A_3(A_2(n)) = A_{k+4}(n)$.

$$\Delta g(k+2, i) = \Delta g(1, i) + \Delta g(k+1, i) = u_{i-1} + u_{k+i-1} = u_{k+i}, \quad \text{for } 2 \le i \le k+1.$$

$$\Delta g(k+2, 1) = \Delta g(1, 1) + \Delta g(k+1, 1) = u_{k+1} + u_{2k+1} = u_{2k+2}.$$

This result is consistent with formula (12) above.

$\Delta g(k+3, i) = \Delta g(2, i) + \Delta g(k+2, i) = u_i + u_{k+i} = u_{k+i+1}$ for $2 \le i \le k+1$.

$\Delta g(k + 3, 1) = \Delta g(2, 1) + \Delta g(k + 2, 1) = u_{k+2} + u_{2k+2} = u_{2k+3}$.
$\Delta g(k + 4, i) = \Delta g(3, i) + \Delta g(k + 3, i) = u_{i+1} + u_{k+i+1} = u_{k+i+2}$ for $2 \le i \le k + 1$.
$\Delta g(k + 4, 1) = \Delta g(3, 1) + \Delta g(k + 3, 1) = u_{k+3} + u_{2k+3} = u_{2k+4}$.
Thus, equation (12) can be extended to cover $1 \le j \le k + 4$:

$$\Delta g(j, i) = u_{i+j-2}, \quad 2 \le i \le k + 1,$$
$$\Delta g(j, 1) = u_{k+j}.$$

Now function (15) is defined for all values of j and i.
$S(j, n) = \#A_j\text{'s} < n = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j}(n)$ by (15).
To prove that $S(j, n)$ increments only when passing an A_j element, look at

$$S(j, n + 1) - S(j, n) = \Delta S(j, n).$$

$$\Delta S(j, n) = u_{k+3-j+i} - 3u_{k+2-j+i} + 3u_{k+1-j+i} - u_{k-j+i} \text{ for } 2 \le i \le k + 1.$$
$$\Delta S(j, n) = (u_{k+3-j+i} - u_{k+2-j+i}) - 2(u_{k+2-j+i} - u_{k+1-j+i}) + (u_{k+1-j+i} - u_{k-j+i}).$$

Using the recursion formula, $\Delta S(j, n)$ reduces to

$$u_{2-j+i} - 2u_{1-j+i} + u_{-j+i} = (u_{2+i-j} - u_{1+i-j}) - (u_{1+i-j} - u_{i-j}).$$

Looking at the series of u_n's, we find that the only time this function $= 1$ is when $i = j$, so $\Delta S(j, n) = (u_2 - u_1) - (u_1 - u_0) = 1 - 0 = 1$.

This happens because $u_i = i$ for $1 \le i \le k+1$, and because, by the recursion formula, $u_{-i} = 1$ for $-k < -i \le 0$. Any other successive difference of 3 consecutive u_i terms equals $1 - 1 = 0$ for $i > j$ or $0 - 0 = 0$ for $i < j$.

Thus, $S(j, n)$ increments 1 iff $n \in A_j$ for $2 \le i \le k + 1$. Since $i = 1$ has a distinct difference, that case has to be proved separately.

$$\begin{aligned}\Delta S(j, n) &= u_{2k+5-j} - 3u_{2k+4-j} + 3u_{2k+3-j} - u_{2k+2-j}\\ &= (u_{2k+5-j} - u_{2k+4-j}) - 2(u_{2k+4-j} + u_{2k+3-j}) + (u_{2k+3-j} - u_{2k+2-j})\\ &= u_{k+4-j} - 2u_{k+3-j} + u_{k+2-j}\\ &= (u_{k+4-j} - u_{k+3-j}) - (u_{k+3-j} - u_{k+2-j})\\ &= u_{3-j} - u_{2-j}\\ &= 0 \text{ except for } j = 1, \text{ when it equals } 1.\end{aligned}$$

Thus, $S(j, n)$ increments 1 for $i = 1$ only when j also $= 1$.

The function $S(j, n)$ has been proved to be accurate to within a constant by examining $\Delta S(j, n)$. If a constant were present at the end of the function, it would cancel out in the incrementation process. To find out the value of the constant, it is necessary to check $S(j, 1)$ for $1 \le j \le k + 1$.

$$A_1(1) = 1,\ A_2(1) = 2,\ \ldots,\ A_{k+1}(1) = k + 1,$$
$$A_1(A_2(1)) = A_{k+2}(1) = A_1(2) = k + 2,$$
$$A_2(A_2(1)) = A_{k+3}(1) = A_2(2) = k + 4,$$
$$A_3(A_2(1)) = A_{k+4}(1) = A_3(2) = k + 7.$$

These values were derived from the difference formula (12) above.

$$\begin{aligned}S(1, 1) &= A_{k+4}(1) - 3A_{k+3}(1) + 3A_{k+2}(1) - A_{k+1}(1)\\ &= (k + 7) - 3(k + 4) + 3(k + 2) - (k + 1) = 0,\end{aligned}$$

$$\begin{aligned}S(2,\ 1) &= A_{k+3}(1) - 3A_{k+2}(1) + 3A_{k+1}(1) - A_k(1)\\ &= (k+4) - 3(k+2) + 3(k+1) - k = 1,\\ S(3,\ 1) &= A_{k+2}(1) - 3A_{k+1}(1) + 3A_k(1) - A_{k-1}(1)\\ &= (k+2) - 3(k+1) + 3(k) - (k-1) = 0,\\ S(j,\ 1) &= A_{k+5-j} - 3(1)A_{k+4-j}(1) + 3A_{k+3-j}(1) = A_{k+2-j}(1)\\ &= (k+5-j) - (k+4-j) + (k+3-j) - (k+2-j)\\ &= 0 \text{ for } 4 \le j \le k+1.\end{aligned}$$

Finally,

$$\begin{aligned}S(1,\ n) &= A_{k+4}(n) - 3A_{k+3}(n) + 3A_{k+2}(n) - A_{k+1}(n),\\ S(2,\ n) &= A_{k+3}(n) - 3A_{k+2}(n) + 3A_{k+1}(n) - A_k(n) - 1,\\ S(j,\ n) &= A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j},\\ &\quad \text{for } 3 \le j \le k+1.\end{aligned}$$

REFERENCE

1. Verner E. Hoggatt, Jr. "A New Angle on Pascal's Triangle." *The Fibonacci Quarterly* 6 (1968):221-234.

*****

A NOTE ON TAKE-AWAY GAMES*

ROBERT J. EPP AND THOMAS S. FERGUSON
Department of Mathematics, UCLA, Los Angeles, CA 90024

1. SUMMARY

Schwenk [1] considers take-away games where the players alternately remove a positive number of counters from a single pile, the player removing the last counter being the winner. On his initial move, the player moving first can remove at most a given number m of counters. On each subsequent move, a player can remove at most $f(n)$ counters, where n is the number of counters removed by his opponent on the preceding move. In [1], Schwenk solves the case when $f(n)$ is nondecreasing and $f(n) \ge n$. This solution is extended to the case when $f(n)$ is nondecreasing and $f(1) \ge 1$.

2. THE WINNING REPRESENTATION

Let $f(n) \ge 1$ be a nondecreasing function defining a take-away game. If a player whose turn it is to move is confronted with a pile of $n \ge 1$ counters, let $L(n)$ be the minimal number of counters he must remove in order to assure a win. Let $L(0) = \infty$. Note that $L(n) \le n$ for $n \ge 1$ and that equality might hold. Note also that removing k counters from a pile of n is a winning strategy if and only if $f(k) < L(n-k)$.

Theorem 2.1: Suppose $f(k) < L(n-k)$; then $k = L(n)$ if and only if $L(k) = k$.

Proof: Suppose that $L(k) < k$. By removing $L(k)$ counters from a pile of counters, a player can then guarantee he will eventually remove the last of the first k counters, and that he will do this by removing $\ell < k$ counters. His opponent will than face a pile of $n - k$ counters and be able to remove at

*Research partially supported by NSF Grant MCS 72-04591.

most $f(\ell) \le f(k) < L(n-k)$ counters, implying the opponent cannot win. Thus, removing $L(k) < k$ counters is a winning strategy and k can be minimal winning, i.e., $L(n) = k$, only if $L(k) = k$.

Conversely, if $L(k) = k$ and a player removes fewer than k counters, his opponent can eventually remove the last of the first k counters. Since the opponent will do this by removing $\ell < k$ counters, and since

$$f(\ell) \le f(k) < L(n - k),$$

we see that the opponent can win. Thus, if $L(k) = k$, then k is minimal winning and $L(n) = k$.

The integers H such that $L(H) = H$ form an increasing, possibly finite sequence H_j satisfying the following theorems.

Theorem 2.2: If $N = \sum_{i=1}^{n} H_{j_k}$ and if $f(H_{j_i}) < H_{j_{i+1}}$ for $i \le n - 1$, then

$$L(N) = H_{j_i}.$$

Proof: The theorem is true by definition when $n = 1$. Suppose the theorem is true for n and

$$N = \sum_{i=1}^{n+1} H_{j_i}, \quad f(H_{j_i}) < H_{j_{i+1}}, \quad i \le n.$$

Then $f(H_{j_1}) < H_{j_2} = L(N - H_{j_1})$. But since $L(H_{j_1}) = H_{j_1}$, Theorem 2.1 gives

$$L(N) = H_{j_1},$$

completing the proof.

Theorem 2.3: Any positive integer N can be written uniquely as

$$N = \sum_{i=1}^{n} H_{j_i}, \quad f(H_{j_i}) < H_{j_{i+1}}, \quad i \le n - 1.$$

Proof: Let $H_{j_1} = L(N)$ and define

$$H_{j_i} = L\left(N - \sum_{k=1}^{i-1} H_{j_k}\right) \text{ until } \sum_{i=1}^{n} H_{j_i} = N.$$

Then

$$f(H_{j_i}) < L\left(N - \sum_{k=1}^{i-1} H_{j_k} - H_{j_i}\right) = L\left(N - \sum_{k=1}^{i} H_{j_k}\right) = H_{j_{i+1}} \quad \text{for } i \le n - 1.$$

Uniqueness follows easily from Theorem 2.2 and a simple induction.

The winning strategy for the game is now clear. Represent the number of counters N as

$$N = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \quad \text{for } i \le n - 1,$$

and remove H_{j_i} counters.

3. CALCULATION OF THE H_i's

To complete the picture, we have the following theorem on the calculation of the H_i's.

Theorem 3.1: $H_1 = 1$ and if $f(H_j) \ge H_j$, then $H_{j+1} = H_j + H_\ell$, where

$$H_\ell = \min_{i \le j}\{H_i \mid f(H_i) \ge H_j\}.$$

If $f(H_j) < H_j$, the sequence H_i is finite and H_j is the final term.

Proof: $H_1 = 1$ is obvious. If $f(H_j) \geq H_j$, define $H_j + H_\ell$ as in the statement of the theorem. We must show that $L(H_j + H_\ell) = H_j + H_\ell$, and to do this, we show that

$$f(k) \geq L(H_j + H_\ell - k) \text{ for } 1 \leq k < H_j + H_\ell.$$

First, if $H_\ell < k < H_j + H_\ell$, then $k - H_\ell < H_j$, and so

$$f(k) \geq f(k - H_\ell) \geq L(H_j - (k - H_\ell)) = L(H_j + H_\ell - k).$$

If $k = H_\ell$, then

$$f(k) = f(H_\ell) \geq H_j = L(H_j) = L(H_j + H_\ell - k).$$

If $1 \leq k < H_\ell$, then $f(k) \geq L(H_\ell - k)$. But

$$H_\ell - k = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1, \text{ and } L(H_\ell - k) = H_{j_1}.$$

As a result,

$$H_j + H_\ell - k = \sum_{i=1}^{n} H_{j_i} + H_j,$$

and since H_ℓ is the smallest H_i with $f(H_i) \geq H_j$, it follows that $f(H_{j_n}) < H_j$. Therefore, Theorem 2.2 gives $L(H_j + H_\ell - k) = H_{j_1} = L(H_\ell - k) \leq f(k)$.

We have just shown that $L(H_j + H_\ell) = H_j + H_\ell$. To show that $H_j + H_\ell$ is indeed the next term in the H_i sequence, we need only show that

$$L(H_j + k) < H_j + k \text{ for } 1 \leq k < H_\ell.$$

But such a k can be represented as

$$k = \sum_{i=1}^{n} H_{j_i} \text{ with } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1,$$

and since $H_{j_n} < H_\ell$, we have $f(H_{j_n}) < H_j$. Hence,

$$L(H_j + k) = L\left(\sum_{i=1}^{n} H_{j_i} + H_j\right) = H_{j_1} < H_j + k$$

by Theorem 2.2, and we have shown that $H_{j+1} = H_j + H_\ell$.

Suppose now that $f(H_j) < H_j$, any positive integer N can be written as $N = k + mH_j$ where k, $m \geq 0$ are integers and $0 < k \leq H_j$. But we can represent k as

$$k = \sum_{i=1}^{n} H_{j_i} \text{ where } f(H_{j_i}) < H_{j_{i+1}} \text{ for } i \leq n - 1,$$

and since $f(H_{j_n}) \leq f(H_j) < H_j$, the representation

$$N = \sum_{i=1}^{n} H_{j_i} + H_j + H_j + \cdots + H_j$$

and Theorem 2.2 tells us that $L(N) = H_{j_1} < H_j$. Thus, H_j is the largest H_i and the theorem is proved.

It may be noted that take-away games with the last player losing may be played with the same strategy but regarding the pile as having one less counter than is actually the case.

REFERENCE

1. A. J. Schwenk. "Take-Away Games." *The Fibonacci Quarterly* 8 (1970): 225-234.

*****

ASSOCIATED STIRLING NUMBERS

F. T. HOWARD
Wake Forest University, Winston-Salem, NC 27109

1. INTRODUCTION

For $r \geq 0$, define the integers $s_r(n, k)$ and $S_r(n, k)$ by means of

$$\text{(1.1)} \qquad \left(\log(1 - x)^{-1} - \sum_{i=1}^{r} x^i/i\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j\right)^k = k! \sum_{n=(r+1)k}^{\infty} s_r(n, k)x^n/n!,$$

$$\text{(1.2)} \qquad \left(e^x - \sum_{i=0}^{r} x^i/i!\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j!\right)^k = k! \sum_{n=(r+1)k}^{\infty} S_r(n, k)x^n/n!.$$

We will call $s_r(n, k)$ the r-associated Stirling number of the first kind, and $S_r(n, k)$ the r-associated Stirling number of the second kind. The terminology and notation are suggested by Comtet [6, pp. 221, 257]. When $r = 0$, we have $s_0(n, k) = (-1)^{n+k}s(n, k)$, where $s(n, k)$ is the Stirling number of the first kind, and $S_0(n, k) = S(n, k)$ is the Stirling number of the second kind. (In Comtet's notation this is true when $r = 1$.) If we define the polynomials $s_{r,n}(y)$ and $S_{r,n}(y)$ by means of

$$\text{(1.3)} \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j\right) = \sum_{n=0}^{\infty} s_{r,n}(y)x^n/n!,$$

$$\text{(1.4)} \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j!\right) = \sum_{n=0}^{\infty} S_{r,n}(y)x^n/n!,$$

it follows immediately that

$$\text{(1.5)} \qquad s_{r,n}(y) = \sum_{j=0}^{[n/r+1]} s_r(n, j)y^j,$$

and

$$\text{(1.6)} \qquad S_{r,n}(y) = \sum_{j=0}^{[n/r+1]} S_r(n, j)y^j.$$

Since the r-associated Stirling numbers of the second kind have appeared in two recent papers [7] and [9], it may be of interest to examine their combinatorial significance, their history, and their basic properties. We do this in §2, §3, and §4 for both the numbers of the first and second kind. Another purpose of this paper is to show how all the results of two recently published articles concerned with Stirling and Bell numbers, [7] and [16] can be generalized by the use of (1.2), (1.4), and (1.6). This is done in §5 and §6. To the writer's knowledge, the r-associated Stirling numbers of the first kind have not been studied before. Since most of their properties and formulas are analogous to those of the numbers of the second kind, it seems appropriate to include them in this paper.

2. COMBINATORIAL SIGNIFICANCE

Let $a_1, a_2, a_3, \ldots$ be any strictly increasing sequence of positive integers. It follows from [12, Ch. 4] that the numbers $t(n, k)$ and $T(n, k)$ defined by means of

$$\left(\sum_{j=1}^{\infty} x^{a_j}/a_j\right)^k = k!\sum_{n=0}^{\infty} t(n, k)x^n/n!, \tag{2.1}$$

and

$$\left(\sum_{j=1}^{\infty} x^{a_j}/(a_j)!\right)^k = k!\sum_{n=0}^{\infty} T(n, k)x^n/n! \tag{2.2}$$

have the following combinatorial interpretation: $t(n,k)$ is the number of permutations of $1, 2, \ldots, n$ having exactly k cycles such that the number of elements in each cycle is equal to one of the a_i; $T(n, k)$ is the number of set partitions of $1, 2, \ldots, n$ consisting of exactly k blocks (subsets) such that the number of elements in each block is equal to one of the a_i. Furthermore, if we define $t_n(y)$ and $T_n(y)$ by means of

$$\exp\left(y\sum_{k=1}^{\infty} x^{a_k}/a_k\right) = \sum_{n=0}^{\infty} t_n(y)x^n/n!, \tag{2.3}$$

and

$$\exp\left(y\sum_{k=1}^{\infty} x^{a_k}/(a_k)!\right) = \sum_{n=0}^{\infty} T_n(y)x^n/n!, \tag{2.4}$$

it follows that

$$t_n(y) = \sum_{j=0}^{n} t(n, j)y^j, \tag{2.5}$$

and

$$T_n(y) = \sum_{j=0}^{n} T(n, j)y^j. \tag{2.6}$$

Thus $t_n(1)$ is the number of permutations of $1, 2, \ldots, n$ such that the number of elements in each cycle is equal to one of the a_i, and $T_n(1)$ is the number of set partitions of $1, 2, \ldots, n$ such that the number of elements in each block is equal to one of the a_i.

As Riordan [12, p. 74] points out, the presence or absence of cycles (or blocks) of various lengths can easily be included in the generating functions (2.1) and (2.2), though the mathematics required to obtain numerical results may be very elaborate. There are many examples in the problems of [12, pp. 80-89]. Other interesting examples can be found in [1] and [3].

It is clear, then, that the r-associated Stirling numbers have the following interpretations:

The number $s_r(n, k)$ is equal to the number of permutations of $1, 2, \ldots, n$ having exactly k cycles such that each cycle has at least $r+1$ elements. It is understood that in any cycle the smallest element is written first. The number $s_{r,n}(1)$ is equal to the number of permutations of $1, 2, \ldots, n$ such that each cycle has at least $r+1$ elements. If we give a permutation with exactly j cycles a "weight" of y^j, then $s_{r,n}(y)$ is the sum of the weights of all the permutations of $1, 2, \ldots, n$ such that each cycle has at least $r+1$ elements.

The number $S_r(n, k)$ is equal to the number of set partitions of $1, 2, \ldots, n$ consisting of exactly k blocks such that each block contains at least $r+1$

The number $S_{r,n}(1)$ is equal to the number of set partitions of 1, 2, ..., n such that each block has at least $r + 1$ elements. If we give a set partition with exactly j blocks a weight of y^j, then $S_{r,n}(y)$ is the sum of the weights of all the set partitions of 1, 2, ..., n such that each block has at least $r + 1$ elements.

3. HISTORY OF THE r-ASSOCIATED STIRLING NUMBERS

The Stirling numbers of the first kind, $s(n, k)$, and of the second kind, $S(n, k)$, were evidently first introduced in 1730 by James Stirling [13, pp. 8, 11]. They are usually defined in the following way:

$$(x)_n = x(x - 1) \ldots (x - n + 1) = \sum_{j=0}^{n} s(n, j)x^j; \tag{3.1}$$

$$x^n = \sum_{j=0}^{n} S(n, j)(x)_j. \tag{3.2}$$

It is not the purpose of this paper to review the history or well-known properties of the Stirling numbers; there are many good references, including [6, Ch. 5], [10, Ch. 4], and [12, pp. 32-38 and Ch. 4]. We are using the notation of Riordan [12] for the Stirling numbers of the first and second kind.

The numbers $s_1(n, k)$ and $S_1(n, k)$ were introduced in 1933-34 by Jordan [11] and Ward [17]. Using different notations, these authors defined $s_1(n, j)$ and $S_1(n, j)$ by means of

$$s(n, n - k) = (-1)^k \sum_{j=0}^{k} s_1(2k - j, k - j)\binom{n}{2k - j}, \tag{3.3}$$

$$S(n, n - k) = \sum_{j=0}^{k} S_1(2k - j, k - j)\binom{n}{2k - j}. \tag{3.4}$$

The purpose of these definitions was to prove that $s(n, n - k)$ and $S(n, n - k)$ are both polynomials in n of degree $2k$, and also to show how $s(n, n - k)$ and $S(n, n - k)$ can be written as linear combinations of binomial coefficients. Formulas (3.3) and (3.4) can also be useful in determining $s(n, n - k)$ and $S(n, n - k)$ when n is large and k is small. The generating functions (1.1) and (1.2) were not given in [11] or [17]. This approach to $s_1(n, k)$ and $S_1(n, k)$ is also discussed in [11, Ch. 4]. In [12, Ch. 4], the generating functions are given, and the combinatorial interpretations are thoroughly discussed. It is also shown that

$$(-1)^{n+k} s(n, k) = \sum_{j=0}^{k} \binom{n}{j} s_1(n - j, k - j). \tag{3.5}$$

$$S(n, k) = \sum_{j=0}^{k} \binom{n}{j} S_1(n - j, k - n), \tag{3.6}$$

$$s_1(n + 1, k) = ns_1(n, k) + ns_1(n - 1, k - 1), \tag{3.7}$$

$$S_1(n + 1, k) = kS_1(n, k) + nS_1(n - 1, k - 1). \tag{3.8}$$

Applications for $s_1(n, k)$ and $S_1(n, k)$ have been found; see [1], [4], and [5], for example. A good discussion of these numbers can also be found in [2].

The r-associated Stirling number of the second kind, for arbitrary r, was apparently first defined and used by Tate and Goen [15] in 1958. They made the following definition:

$$G_r(n,\ k) = (-1)^k n! \sum \frac{(-1)^{k_1}(k_1)^A}{k_1!k_2!\ \ldots\ k_{r+2}!A!Q}, \tag{3.9}$$

where

$$A = A(k_1,\ \ldots,\ k_{r+2}) = n - \sum_{i=0}^{r} ik_{i+2},$$

$$Q = Q(k_1,\ \ldots,\ k_{r+2}) = \prod_{i=0}^{r} (i!)^{k_{i+2}},$$

and the sum is over all $k_1,\ k_2,\ \ldots,\ k_{r+2}$ such that $k_1 + k_2 + \cdots + k_{r+2} = k$, and $0 \le k_i \le k$. For $r = 0$, (3.9) reduces to the familiar formula for $S(n,\ k)$:

$$G_0(n,\ k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n = S(n,\ k). \tag{3.10}$$

Now by induction we can show that $G_r(n,\ k) = S_r(n,\ k)$. It is true for $r = 0$; assume it is true for a fixed r. Then, by (1.2), we have

$$\begin{aligned}\sum_{n=(r+2)k}^{\infty} k!S_{r+1}(n,\ k)x^n/n! &= \left(\sum_{j=r+1}^{\infty} x^j/j! - x^{r+1}/(r+1)!\right)^k \\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}[(r+1)!]^{i-k}x^{(r+1)(k-i)}\left(\sum_{j=r+1}^{\infty} x^j/j!\right)^i \\ &= \sum_{i=0}^{k}\sum_{m=0}^{\infty}\binom{k}{i}(-1)^{k-i}[(r+1)!]^{i-k}i!(m!)^{-1}G_r(m,\ i)x^{m+(r+1)(k-i)}.\end{aligned}$$

By using (3.9) to rewrite $G_r(m,\ i)$ and then comparing coefficients of x^n, we have $G_{r+1}(n,\ k) = S_{r+1}(n,\ k)$. For example, we have

$$S_1(n,\ k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(n)_m(k-j)^{n-m}. \tag{3.11}$$

A formula equivalent to (3.11) was also proved by Carlitz [2].

The r-associated Stirling numbers of the second kind have appeared in problems in [6, pp. 221-222] and [12, p. 102]. Recently, Enneking and Ahuja [7] have used these numbers to extend earlier results of Uppuluri and Carpenter [16] concerning the Bell numbers. In another recent paper the writer [9] has shown the relationship of $S_r(n,\ k)$ to the numbers $A_{r,n}$ defined by

$$(x^r/r!)\left(\sum_{j=r}^{\infty} x^j/j!\right)^{-1} = \sum_{n=0}^{\infty} A_{r,n}x^n/n!. \tag{3.12}$$

The relationship is

$$A_{r,n} = \sum_{j=1}^{n}(-r!)\ j!n!S_r(n+rj,\ j)/(n+rj)!. \tag{3.13}$$

The number $A_{1,n}$ is the nth Bernoulli number.

Evidently, the r-associated Stirling numbers of the first kind have not been studied, though they do appear in a problem in [6, pp. 256-257].

4. BASIC FORMULAS

In [7] and [9] formulas for $S_r(n,\ k)$ and the polynomials defined by (1.4) and (1.6) were derived. The notation for $S_r(n,\ k)$ is $d_r(n,\ k)$ in [7] and

$b(r;\, n,\, k)$ in [9]. In this section we are concerned mainly with the analogous formulas for the r-associated numbers of the first kind. The following formulas have been proved:

$$S_r(n + 1,\, k) = kS_r(n,\, k) = \binom{n}{r} S_r(n - r,\, k - 1), \quad \text{with } S_r(0,\, 0) = 1, \tag{4.1}$$

$$S_r(n,\, k) = \sum \frac{n!}{k!u_1!u_2! \ \ldots \ u_k!}, \tag{4.2}$$

the sum over all compositions (ordered partitions) $u_1 + u_2 + \cdots + u_k = n$, each $u_i \geq r + 1$,

$$S_{r,n}(y) = \sum_{i=0}^{n} \frac{n!(r!)^{-i}(-y)^i}{i!(n - ri)!} S_{r-1,\, n-ri}(y), \tag{4.3}$$

$$S_{r-1,n}(y) = \sum_{i=0}^{n} \frac{n!(r!)^{-i}y^i}{i!(n - ri)!} S_{r,\, n-ri}(y), \tag{4.4}$$

$$S_{r-1}(n,\, k) = \sum_{j=0}^{k} \frac{n!(r!)^{-j}}{j!(n - rj)!} S_r(n - jr,\, k - j), \tag{4.5}$$

$$S_r(n,\, k) = \sum_{j=0}^{k} \frac{(-1)^j n!(r!)^{-j}}{j!(n - jr)!} S_{r-1}(n - jr,\, k - j), \tag{4.6}$$

$$S_{r,\, n+1}(y) = y \sum_{i=0}^{n-r} \binom{n}{i} S_{r,\, i}(y). \tag{4.7}$$

It should be noted that there are misprints in formulas (5.14) and (5.16) of [9], which correspond to (4.7) and (4.4), respectively, in this paper. Also, in the table following (5.11) in [9], the value of $g(6,\, 2)$ is 10, not 0. We also note that the Tate-Goen formula (3.9) can be proved inductively by means of (4.6).

We now look at the analogous formulas for $s_r(n,\, k)$. First, we have the recurrence

$$s_r(n + 1,\, k) = ns_r(n,\, k) + (n)_r s_r(n - r,\, k - 1), \tag{4.8}$$

where $(n)_r = n(n - 1) \ldots (n - r + 1)$ and $s_r(0,\, 0) = 1$, $s_r(n,\, 0) = 0$ if $n \neq 0$. We shall use a combinatorial argument to prove (4.8). In the permutations of $n + 1$ elements which have k cycles, each cycle containing at least $r + 1$ elements, enumerated by $s_r(n + 1,\, k)$, element $n + 1$ is in some $r + 1$ cycle or it is not. If it is not, it is inserted into one of the k cycles of n elements enumerated by $s_r(n,\, k)$, and this can be done in n ways. If it is, there are $\binom{n}{r}$ ways to choose the other r elements of the $r + 1$ cycle, and since the smallest element must be first, there are $r!$ ways the elements can be arranged in the cycle. Note that $r!\binom{n}{r} = (n)_r$. There are then $n - r$ elements left to be arranged in $k - 1$ cycles.

By comparing coefficients of x^n on both sides of (1.1), we have

$$s_r(n,\, k) = \frac{n!}{k!u_1u_2 \ \ldots \ u_k}, \tag{4.9}$$

where the sum is over all compositions $u_1 + u_2 + \cdots + u_k = n$, each $u_i \geq r + 1$. This generalizes the formula for $s(n,\, k)$ given in [10, p. 146, formula (5)].

Formulas analogous to (4.3)-(4.7) can be derived. From (1.3), we have

$$\sum_{n=0}^{\infty} s_{r,n}(y)x^n/n! = \exp\left(y\sum_{j=r}^{\infty} x^j/j\right)\exp(-yx^r/r)$$

$$= \sum_{n=0}^{\infty} s_{r-1,n}(y)x^n/n! \sum_{j=0}^{\infty} (-y)^j x^{rj} r^{-j}/j!.$$

Comparing coefficients of x^n, we have

$$s_{r,n}(y) = \sum_{j=0}^{[n/r]} \binom{n}{rj} f_r(rj)(-y)^j s_{r-1,n-rj}(y), \tag{4.10}$$

when $f_r(0) = 1$ and for $j > 0$,

$$f_r(rj) = (rj)!/r(2r)(3r)\ \ldots\ (jr), \tag{4.11}$$

that is, $f_r(rj)$ is the same as $(rj)!$ with every rth term divided out. With a similar argument, we have

$$s_{r-1,n}(y) = \sum_{j=0}^{[n/r]} \binom{n}{rj} f_r(rj) y^j s_{r,n-rj}(y). \tag{4.12}$$

It follows from (1.5), (4.10), and (4.12) that

$$s_r(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n}{rj} f_r(rj) s_{r-1}(n - rj, k - j), \tag{4.13}$$

and

$$s_{r-1}(n, k) = \sum_{j=0}^{k} \binom{n}{rj} f_r(rj) s_r(n - rj, k - j). \tag{4.14}$$

Equation (4.14) generalizes (3.5) and shows how to write $s_{r-1}(n, k)$ as a linear combination of binomial coefficients. In fact it is not difficult to see from (4.14) and (4.5) that, for $k > 0$ and fixed r,

$$r^m s_{r-1}(rm + k, m) = (rm + k)(rm + k - 1)\ \ldots\ mR_k(m), \tag{4.15}$$

and

$$(r!)^m S_{r-1}(rm + k, m) = (rm + k)(rm + k - 1)\ \ldots\ mQ_k(m), \tag{4.16}$$

where $R_k(m)$ and $Q_k(m)$ are polynomials in m of degree $k - 1$. By differentiating (1.3) with respect to x and comparing coefficients of x^n, we have

$$s_{r,n+1}(y) = y\sum_{i=0}^{n-r} (n)_i s_{r,i}(y). \tag{4.17}$$

If we define the numbers $d_{r,n}$ by means of

$$(x^r/r)\left(\sum_{j=r}^{\infty} x^j/j\right)^{-1} = \sum_{n=0}^{\infty} d_{r,n}x^n, \tag{4.18}$$

then it follows from [9, formulas 4.11 and 4.12] that

$$d_{r,n} = \sum_{j=1}^{n} (-1)^j [f_r(rj)(n + rj)_n]^{-1} s_r(n + rj, j), \tag{4.19}$$

$$d_{r,n} = \sum_{j=1}^{n} (-1)^j \binom{n+1}{j+1} [f_r(rj)(n + rj)_n]^{-1} s_{r-1}(n + rj, j). \tag{4.20}$$

When $r = 1$ in (4.16), we have

$$-x[\ln(1 - x)]^{-1} = \sum_{n=0}^{\infty} d_{1,n}x^n, \tag{4.21}$$

so that $d_{1,n} = (-1)^n b_n$, where b_n is the Bernoulli number of the second kind [10, pp. 265-287]. Thus, by (4.17) and (4.18), we have

$$b_n = \sum_{j=1}^{n} (-1)^{n+j}[(n + j)_n]^{-1}s_1(n + j, j), \tag{4.22}$$

$$b_n = \sum_{j=1}^{n} (-1)^j \binom{n + 1}{j + 1}[(n + j)_n]^{-1}s(n + j, j). \tag{4.23}$$

It can also be proved [see 10, p. 267] that

$$n!b_n = \sum_{k=0}^{n} s(n, k)/(k + 1). \tag{4.24}$$

We can compare formulas (4.22), (4.23), and (4.24) to similar formulas involving the ordinary Bernoulli numbers and the Stirling numbers of the second kind [10, pp. 182, 219, and 599].

5. GENERALIZATION OF THE PAPER BY UPPULURI AND CARPENTER

In [16] Uppuluri and Carpenter defined a sequence C_0, C_1, C_2 ... by means of

$$\exp(1 - e^x) = \sum_{j=0}^{\infty} C_j x^j/j!, \tag{5.1}$$

and they derived some formulas involving the C_j and Bell numbers B_1, B_2, ..., defined by

$$B_n = \sum_{j=1}^{n} S(n, j). \tag{5.2}$$

In this section, we show how all the results of [16] can be extended by using (1.4) and (1.6). In Propositions 5.1-5.10, which correspond to Propositions 1-10 in [16], we use the notation

$$S_{0,n}(y) = S_n(y), \tag{5.3}$$

so clearly $B_n = S_n(1)$ and $C_n = S_n(-1)$. We omit any proof which is obvious or which is analogous to the corresponding proof in [16].

Proposition 5.1: $S_k(y) = e^{-y}\sum_{m=0}^{\infty} y^m m^k/m!$, $k = 0, 1, 2, \ldots$.

Proposition 5.2: Equation (1.6) of this paper.

Proposition 5.3: Equation (4.7) of this paper.

Proposition 5.4: $\Delta^n S_1(y) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} S_{j+1}(y) = yS_n(y)$.

Using Proposition 5.4 and $S_1(y) = y$, we can compute $S_2(y)$, ..., $S_n(y)$ for small values of n. For example, $\Delta S_1(y) = yS_1(y) = y^2$, so

$$S_2(y) = S_1(y) + \Delta S_1(y) = y + y^2,$$

and

$$\begin{aligned} S_3(y) &= S_2(y) + \Delta S_2(y) = S_2(y) + \Delta^2 S_1(y) + \Delta S_1(y) \\ &= (y + y^2) + (y^2 + y^3) + y^2 = y + 3y^2 + y^3. \end{aligned}$$

Proposition 5.5: $\sum_{k=0}^{n} \binom{n}{k} S_k(y) S_{n-k}(-y) = 0$, $n = 1, 2, \ldots$, and $S_0(y) = 1$.

Proposition 5.6: $\sum_{j=0}^{n} \binom{n}{j} S_j(-y) S_{n+1-j}(y) = y$, $n = 0, 1, 2, \ldots$.

Proposition 5.7: Same as Proposition 5.6.

Proposition 5.8: Let $a_i = S_i(y)/i!$. Then

(a)
$$S_n(-y) = (-1)^n n! \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & & \cdots & & a_1 \end{vmatrix}$$
$$= (-1)^n n! g_n,$$

(b)
$$(-1)^n S_n(-y) = n! \sum_{k=0}^{n-1} (-1)^k g_{n-k-1} S_{k+1}(y)/(k+1)!.$$

Proposition 5.9:
$$S_{n+1}(-y) = (-1)^{n+1} \begin{vmatrix} y & 1 & 0 & 0 & \cdots & 0 \\ y & y & 1 & 0 & \cdots & 0 \\ y & 2y & y & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \binom{n}{0} & \binom{n}{1}y & & \cdots & & \binom{n}{n}y \end{vmatrix}$$

In Proposition 5.9, the element in the ith row, jth column, for $j \le i$, is $\binom{i-1}{j-i} y$.

Proposition 5.10:
$$S_{n+1}(-y) = (-1)^{n+1} \begin{vmatrix} y & 1 & 0 & 0 & \cdots & 0 \\ y & y & 2 & 0 & \cdots & 0 \\ y/2 & y & y & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y/n! & y/(n-1)! & & & \cdots & y/0! \end{vmatrix}$$

In Proposition 5.10, the element in the ith row, jth column, for $j \le i$, is $y/(i - j)!$.

The proof of Proposition 10 in [16] is not given. A reference is given to a formula of Ginsburg [8] for the Bell numbers, but unfortunately Ginsburg's proof is obscure. Proposition 5.10 is easily proved, however, by multiplying the k + 1st row of the determinant in Proposition 5.9 by $1/k!$ and the k + 1st column by $k!$ ($k = 1, 2, \ldots, n$).

The motivation given in [16] for studying the numbers C_j defined by (5.1) is the following: Define $B_n^{(k)}$ by

(5.3) $$B_n^{(k)} = \sum_{j=1}^{n} j^k S(n, j).$$

Then

$$B_n^{(0)} = B_n,$$
$$B_n^{(1)} = B_{n+1} - B_n,$$
$$B_n^{(2)} = B_{n+2} - 2B_{n+1},$$

and these equations lead to a search for a general expression for $B_n^{(k)}$ in terms of the Bell numbers $B_n, B_{n+1}, \ldots, B_{n+k}$. It is stated, though not actually proved, that

(5.4) $$B_n^{(k)} = \sum_{j=0}^{k} \binom{k}{j} C_j B_{n+k-j}.$$

We now generalize this result by defining $S_n^{(k)}(y)$ by

(5.5) $$S_n^{(k)}(y) = \sum_{j=1}^{n} j^k S(n, j) y^j,$$

and showing that

(5.6) $$S_n^{(k)}(y) = \sum_{j=0}^{k} \binom{k}{j} S_j(-y) S_{n+k-j}(y).$$

For example,

$$S_n^{(1)}(y) = S_{n+1}(y) - yS_n(y),$$
$$S_n^{(2)}(y) = S_{n+2}(y) - 2yS_{n+1}(y) + (y^2 - y)S_n(y).$$

To prove (5.6), we start with (1.4) with $r = 0$. Differentiating n times with respect to x, we have

(5.7) $$D^{(n)} \exp y(e^x - 1) = \sum_{j=0}^{\infty} S_{n+j}(y) x^j / j!.$$

Now consider the numbers $q_n^{(m)}(y)$ defined by

(5.8) $$(\exp y(1 - e^x)) D^{(n)} \exp y(e^x - 1) = \sum_{m=0}^{\infty} q_n^{(m)}(y) x^m / m!.$$

It follows from (1.4), (5.7), and (5.8) that

$$q_n^{(k)}(y) = \sum_{j=0}^{k} S_j(-y) S_{n+k-j}(y) \binom{k}{j}.$$

Now we show by induction that

(5.9) $$q_n^{(k)}(y) = S_n^{(k)}(y).$$

For $n = 1$, we have, from (5.8),

$$ye^x = \sum_{m=0}^{\infty} q_1^{(m)}(y) x^m / m!,$$

so

$$q^{(k)}(y) = y = S_1^{(k)}(y).$$

Assume (5.9) holds for a fixed n, and also assume

$$D^{(n)} \exp y(e^x - 1) = (\exp y(e^x - 1)) \sum_{i=1}^{n} e^{xi} S(n, i) y^i.$$

Then we have

$$(5.10)\quad D^{(n+1)}\exp y(e^x - 1) = (\exp y(e^x - 1))\sum_{i=1}^{n} e^{xi}(iS(n, i) + S(n, i - 1))y^i$$
$$= (\exp y(e^x - 1))\sum_{i=1}^{n+1} e^{xi}S(n + 1, i)y^i.$$

Multiplying both sides of (5.10) by $\exp y(1 - e^x)$ and comparing coefficients of x, we see that $q_{n+1}^{(k)}(y) = S_{n+1}^{(k)}(y)$.

6. GENERALIZATION OF THE PAPER BY ENNEKING AND AHUJA

In [7] Enneking and Ahuja defined a generalized Bell number by

$$(6.1)\quad B_r(n) = \sum_{j=0}^{n} S_r(n, j),$$

and they were able to generalize some of the formulas in [16]. Note that

$$B_r(n) = S_{r,n}(1).$$

By considering $S_{r,n}(y)$, we can extend each of the twelve properties in [7]; Properties 6.1-6.12 in this paper correspond to Properties 1-12 in [7]. We omit any proof which is obvious or is analogous to the corresponding proof in [7].

Property 6.1: Equation (1.4) of this paper.

Property 6.2: Equation (4.7) of this paper.

Property 6.3: Equation (4.3) of this paper.

Property 6.4: $S_{1,n}(y) = e^{-y}\sum_{m=0}^{\infty} y^m(m - y)^n/m!.$

Proof: We have, from (1.4),

$$\sum_{n=0}^{\infty} S_{1,n}(y)x^n/n! = e^{-y}e^{-xy}\exp(ye^x) = e^{-y}\left(\sum_{i=0}^{\infty}(-y)^i x^i/i!\right)\left(\sum_{m=0}^{\infty} y^m e^{xm}/m!\right)$$
$$= e^{-y}\left(\sum_{m=0}^{\infty} y^m/m! \sum_{j=0}^{\infty}(xm)^j/j!\right)\left(\sum_{i=0}^{\infty}(-y)^i x^i/i!\right)$$
$$= e^{-y}\sum_{m=0}^{\infty} y^m/m! \sum_{n=0}^{\infty}(m - y)^n x^n/n!,$$

and Property 6.4 is proved when we compare coefficients of x^n.

For Property 6.5, we need the following definition of $H_r(x)$:

$$(6.2)\quad \exp y(e^x - 1 - x - \cdots - x^r/r!) = (\exp y(e^x - 1))H_r(x),$$

where

$$(6.3)\quad H_r(x) = \sum_{i=0}^{\infty} h_{r,i}(y)x^i/i!,\ r \geq 1.$$

Throughout the remainder of this paper we will also continue to use the notation of (5.3).

Property 6.5: $S_{r,n}(y) = \sum_{i=0}^{n} \binom{n}{i} h_{r,i}(y) S_{n-i}(y)$, where

$$h_{r,n+1}(y) = -y \sum_{j=0}^{r-1} \binom{n}{j} h_{r,n-j}(y), \; h_{0,n}(y) = 0 \text{ for } r \geq 0, \; h_{r,0}(y) = 1.$$

To generalize (5.5), we make the following definition:

$$S_{r,n}^{(k)}(y) = \sum_{m=1}^{n} m^k S_r(n, m) y^m. \tag{6.4}$$

Property 6.6: $S_{r,n}^{(k+1)}(y) = S_{r,n+1}^{(k)}(y) - y\binom{n}{r} \sum_{j=0}^{k} \binom{k}{j} S_{r,n-r}^{(j)}(y)$.

Property 6.7: $S_n^{(k+1)}(y) = S_{n+1}^{(k)}(y) - y \sum_{j=0}^{k} \binom{k}{j} S_n^{(j)}(y)$.

Now we want to generalize (5.6); that is, we want to express $S_{r,n}^{(k)}(y)$ in terms of the $S_{r,n}(y)$. For example,

$$S_{r,n}^{(1)}(y) = S_{r,n+1}(y) - \binom{n}{r} y S_{r,n-r}(y),$$
$$S_{r,n}^{(2)}(y) = S_{r,n+2}(y) - y\left[\binom{n+1}{r} + \binom{n}{r}\right] S_{r,n+1-r}(y) + y^2 \binom{n-r}{r}\binom{n}{r} S_{r,n-2r}(y) - y\binom{n}{r} S_{r,n-r}(y).$$

Property 6.8: $S_{r,n}^{(k)}(y) = \sum_{i=0}^{k} \sum_{j=0}^{i} a_{ij}(n, k, r) S_{r,n+k-i-jr}(y)$, where

$$a_{0,0}(n, k, r) = 1, \; a_{ij}(n, k, r) = 0 \text{ if } j = 0 \text{ and } i > 0,$$

and

$$a_{ij}(n, k+1, r) = a_{ij}(n+1, k, r) - y\binom{n}{r} \sum_{m=k-i+j}^{k} a_{i+m-k-1,j-1}(n-r, m, r).$$

When $r = 0$, we have

$$\sum_{j=0}^{i} a_{ij}(n, k, 0) = \binom{k}{i} S_i(-y), \tag{6.5}$$

independent of n for $i = 1, 2, \ldots, k$. Letting $r = 0$ in Property 6.8, letting $i = k + 1$, and summing on j, we have

$$S_{k+1}(-y) = -y \sum_{m=0}^{k} \binom{k}{m} S_m(-y),$$

which agrees with Proposition 5.3.

Now let

$$W_{r,n}^{(k)}(y) = \sum_{j=0}^{n} (j)_k S_r(n, j) y^j. \tag{6.6}$$

We shall use the notation $W_{0,n}^{(k)}(y) = W_n^{(k)}(y)$.

Property 6.9: $W_{r,n}^{(k+1)}(y) = W_{r,n+1}^{(k)}(y) - kW_{r,n}^{(k)}(y) - \binom{n}{r} y\left(kW_{r,n-r}^{(k+1)}(y) + W_{r,n-r}^{(k)}(y)\right)$.

Property 6.10: $W_n^{(k+1)}(y) = W_{n+1}^{(k)}(y) - (k+y)W_n^{(k)}(y) - ykW_n^{(k-1)}(y)$.

Property 6.11: $W_n^{(k)}(y) = \sum_{i=0}^{k} w(k, i, y)S_{n+k-i}(y)$,

where the $w(k, i, y)$ satisfy $w(k, 0, y) = 1$ and

$$w(k+1, i, y) = w(k, i, y) - (k+y)w(k, i-1, y) - ykw(k-1, i-2, y).$$

For example,

$$W_n^{(1)}(y) = S_{n+1}(y) - yS_n(y),$$
$$W_n^{(2)}(y) = S_{n+2}(y) - (2y+1)S_{n+1}(y) + y^2S_n(y).$$

It is noted in [7], without proof, that for $y = 1$ the $w(k, i, y)$ are the coefficients of a special case of the Poisson-Charlier polynomials $P_n(x)$ [14, p. 34]. These polynomials can be defined by

$$P_k(x) = \sum_{i=0}^{k} p(k, i, u)x^{k-i}, \tag{6.7}$$

$$p(k, i, u) = \sum_{j=0}^{i} (-1)^j \binom{k}{j} u^{j-k} s(k-j, k-i). \tag{6.8}$$

(This definition is slightly different from the one given by Szegö [14].) We now show that when $u = y$,

$$w(k, i, y) = y^k p(k, i, y). \tag{6.9}$$

We prove (6.9) by showing that $y^k p(k, i, y)$ satisfies the same recurrence as $w(k, i, y)$. For convenience, in the proof we use the notation $p(k, i) = y^k p(k, i, y)$. Then we have $p(k, 0) = 1$ and

$$\begin{aligned} p(k+1, i) &= \sum_{j=0}^{i} (-1)^j \binom{k+1}{j} s(k+1-j, k+1-i)y^j \\ &= \sum_{j=0}^{i} (-1)^j \left[\binom{k}{j} + \binom{k}{j-1}\right][s(k-j, k-i) - (k-j)s(k-j, k+1-i)]y^j \\ &= p(k, i) - \sum_{j=0}^{i-1} (-1)^j \binom{k}{j} s(k-j, k+1-i)y^{j+1} \\ &\quad - k\sum_{j=0}^{i} (-1)^j \binom{k-1}{j} s(k-j, k+1-i)y^j. \end{aligned}$$

Replacing $\binom{k-1}{j}$ by $\binom{k}{j} - \binom{k-1}{j-1}$, we have

$$p(k+1, i) = p(k, i) - yp(k, i-1) - kp(k, i-1) - ykp(k-1, i-2).$$

This completes the proof of (6.9).

Now we want to express $W_{r,n}^{(k)}(y)$ in terms of the $S_{r,j}(y)$. For example

$$W_{r,n}^{(1)}(y) = S_{r,n+1}(y) - \binom{n}{r} yS_{r,n-r}(y),$$
$$W_{r,n}^{(2)}(y) = S_{r,n+2}(y) - S_{r,n+1}(y) - y\left[\binom{n+1}{r} + \binom{n}{r}\right] S_{r,n-r+1}(y) + \binom{n}{r}\binom{n-r}{r} y^2 S_{r,n-2r}(y).$$

Property 6.12: $W_{r,n}^{(k)}(y) = \sum_{i=0}^{k} \sum_{j=0}^{i} b_{ij}(n, k, r) S_{r,\, n+k-i-jr}(y),$

where the $b_{ij}(n, k, r)$ satisfy $b_{0,0}(n, k, r) = 1$, $b_{kj}(n, k, r) = 0$ for $j = 0, \ldots, k-1$, and

$$b_{ij}(n, k+1, r) = b_{ij}(n+1, k, r) - kb_{i-1,\,j}(n, k, r) - \binom{n}{r} y [b_{i-1,j-1}(n-r, k, r) + kb_{i-2,j-1}(n-r, k-1, r)].$$

REFERENCES

1. L. Carlitz. "A Special Class of Triangular Arrays." *Seminario Matematico de Barcelona* XXVII (1976):3-38.
2. L. Carlitz. "Note on the Numbers of Jordan and Ward." *Duke Math. Journal* 38 (1971):783-790.
3. L. Carlitz. "Set Partitions." *The Fibonacci Quarterly* 14 (1976):327-342.
4. L. Carlitz & M. S. Klamkin. "Stirling Operators." *Seminario Matematico de Barcelona* XXV (1974):3-30.
5. L. Carlitz. "The Coefficients in an Asymptotic Expansion and Certain Related Numbers." *Duke Math. Journal* 35 (1968):83-90.
6. L. Comtet. *Advanced Combinatorics*. Dordrecht: Reidel, 1974.
7. E. A. Enneking & J. C. Ahuja. "Generalized Bell Numbers." *The Fibonacci Quarterly* 14 (1976):67-73.
8. J. Ginsburg. "Iterated Exponentials." *Scripta Mathematica* 11 (1945):340-353.
9. F. T. Howard. "Numbers Generated by the Reciprocal of $e^x - x - 1$." *Mathematics of Computation* 31 (1977):581-598.
10. C. Jordan. *Calculus of Finite Differences*. New York: Chelsea, 1950.
11. C. Jordan. "On Stirling Numbers." *The Tôhoku Math. Journal* 37 (1933):254-278.
12. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1958.
13. James Stirling. *Methodus Differentialis Sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*. London, 1930.
14. G. Szegö. *Orthogonal Polynomials*. New York: The American Mathematical Society, 1959.
15. R. F. Tate & R. L. Goen. "Minimum Variance Unbiased Estimation for the Truncated Poisson Distribution." *Annals of Math. Statistics* 29(1958):755-765.
16. V. Uppuluri & J. Carpenter. "Numbers Generated by the Function $\exp(1 - e^x)$." *The Fibonacci Quarterly* 7 (1969):437-447.
17. Morgan Ward. "The Representation of Stirling's Numbers and Stirling's Polynomials as Sums of Factorials." *American Journal of Math.* 56 (1934):87-95.

*****

THE DIVISIBILITY PROPERTIES OF PRIMARY LUCAS RECURRENCES WITH RESPECT TO PRIMES

LAWRENCE SOMER
U.S. Dept. of Agriculture, FSQS, Washington, D.C. 20250

1. INTRODUCTION

In this paper we will extend the results of D. D. Wall [12], John Vinson [11], D. W. Robinson [9], and John H. Halton [3] concerning the divisibility properties of the Fibonacci sequence to the general Lucas sequence

$$(r_1^n - r_2^n)/(r_1 - r_2).$$

In particular, we will improve their theorems for the Fibonacci sequence. Their results are inconclusive for those primes for which

$$(5/p) = (-1/p) = 1,$$

where (x/p) is the Legendre symbol for the quadratic character of x with respect to the prime p. We will obtain sharper results in these cases.

Let

(1) $$u_{n+2} = au_{n+1} + bu_n,$$

where u_0, u_1, a, and b are integers, be an integral second-order linear recurrence. The integers a and b will be called the parameters of the recurrence. If $u_0 = 0$ and $u_1 = 1$, such a recurrence will be called a primary recurrence (PR) and will be denoted by $u(a, b)$. Associated with PR $u(a, b)$ is its characteristic polynomial

$$x^2 - ax - b = 0$$

with roots r_1 and r_2 where $r_1 + r_2 = a$ and $r_1 r_2 = -b$. Let

$$D = a^2 + 4b = (r_1 - r_2)^2$$

be the discriminant of the characteristic polynomial. If $D \neq 0$, then, by the Binet formula

(2) $$u_n = (r_1^n - r_2^n)/(r_1 - r_2).$$

One other type of sequence will be of interest: the Lucas sequence $v(a, b)$ in which

(3) $$v_{n+2} = av_{n+1} + bv_n,\ v_0 = 2,\ v_1 = a.$$

As is well known, the Lucas sequence is given by the Binet formula

(4) $$v_n = r_1^n + r_2^n.$$

To continue, we need the following definitions which are modeled after the notation of Halton [3]. The letter p will always denote a rational prime.

Definition 1: $\nu(a, b, p)$ is the numeric of the PR $u(a, b)$ modulo p. It is the number of nonrepeating terms modulo p.

Definition 2: $\mu(a, b, p)$ is the period of the PR $u(a, b)$ modulo p. It is the least positive integer k such that

$$u_{n+k} \equiv u_n \pmod{p}$$

is true for all $n \geq \nu(a, b, p)$.

Clearly, if $\nu(a, b, p) = 0$,

$$u_{\mu(a,b,p)} \equiv 0 \quad \text{and} \quad u_{\mu(a,b,p)+1} \equiv 1 \pmod{p}.$$

Definition 3: $\alpha(a, b, p)$ is the restricted period of the PR $u(a, b)$ modulo p. It is the least positive integer k such that

$$u_{n+k} \equiv su_n \pmod{p}$$

for all $n \geq \nu(a, b, p)$ and some nonzero residue s. Then $s = s(a, b, p)$ is called the multiplier of the PR $u(a,b)$. If $u_k \equiv 0 \pmod{p}$ for $k \geq \nu(a, b, p)$, we say that $s(a, b, p) = 0$ by convention.

Definition 4: $\beta(a, b, p)$ is called the exponent of the multiplier $s(a, b, p)$ modulo p. It is clearly equal to

$$\mu(a, b, p)/\alpha(a, b, p).$$

Definition 5: In the PR $u(a, b)$ the rank of apparition of p is the least positive integer, if it exists, such that $u_k \equiv 0 \pmod{p}$.

We will restrict our attention chiefly to the PR's $u(a, b)$, because, as we shall see, if $b \neq 0$, then for these sequences the rank of apparition of p exists. By [10], primary recurrences are essentially the only recurrences having this property.

2. PRELIMINARY RESULTS

The following well-known properties of Lucas sequences will be necessary for our future proofs. Proofs of these results can be found in the papers of Lucas [8] or Carmichael [2].

(5) In the PR $u(a, b)$ suppose that $b \not\equiv 0 \pmod{p}$ and that $p \neq 2$. Then

$$u_{p-(D/p)} \equiv 0 \pmod{p}.$$

(6) $$u_{m+n} = bu_m u_{n-1} + u_n u_{m+1}.$$

(7) $$u_n^2 - u_{n-1}u_{n+1} = (-b)^{n-1}, \; n \geq 1.$$

(8) $$v_n^2 - Du_n^2 = 4(-b)^n.$$

(9) $$u_{2n} = u_n v_n.$$

(10) If $p \nmid bD$, then p is a divisor of the Lucas sequence $v(a, b)$ if and only if $\alpha(a, b, p) \equiv 0 \pmod{2}$ for the PR $u(a, b)$. Then the rank of apparition of p in $v(a, b)$ is $(1/2)\alpha(a, b, p)$.

The following two lemmas will determine the possible numerics $\nu(a, b, p)$ for the PR $u(a, b)$ modulo p.

Lemma 1: In the PR $u(a, b)$ if $b \not\equiv 0 \pmod{p}$, then $\nu(a, b, p) = 0$ and $\alpha(a, b, p)$ is also the rank of apparition of p. Also, if $u_k \equiv 0 \pmod{p}$, then

$$\alpha(a, b, p) \mid k.$$

Further

$$\alpha(a, b, p) \mid p - (D/p).$$

Proof: Since there are only p^2 possible pairs of consecutive terms $(u_n, u_{n+1}) \pmod{p}$, some pair must repeat. Suppose that the pair (u_k, u_{k+1}) is the first such pair to repeat modulo p and that $k \neq 0$. Let $m = \mu(a, b, p)$. Then,

$$u_{k+m} \equiv u_k \quad \text{and} \quad u_{k+1+m} \equiv u_{k+1} \pmod{p}.$$

However, by the recurrence relation (1),

$$bu_{k-1} \equiv u_{k+1} - au_k.$$

Since $b \not\equiv 0 \pmod{p}$,

$$u_{k-1} \equiv (u_{k+1} - au_k)/b \pmod{p}.$$

Hence, the pair (u_{k-1}, u_k) repeats modulo p which is a contradiction if $k \neq 0$. Thus, the pair $(u_0, u_1) = (0, 1)$ repeats modulo p. Hence, the numeric is 0 modulo p and the PR $u(a, b)$ is purely periodic modulo p.

Now, let $n = \alpha(a, b, p)$. As in the above argument, (u_0, u_1) is the first pair (u_k, u_{k+1}) such that

$$u_{k+n} \equiv su_k \quad \text{and} \quad u_{k+1+n} \equiv su_{k+1} \pmod{p}$$

for some residue $s \pmod{p}$. The assertion that $\alpha(a, b, p) | k$ now follows from the fact that the PR $u(a, b)$ is purely periodic modulo p. The rest of the lemma follows from (5).

Lemma 2: In the PR $u(a, b)$, assume that $b \equiv 0 \pmod{p}$.

(i) If $a \not\equiv 0 \pmod{p}$, then $\nu(a, b, p) = 1$ and $u_n \equiv a^{n-1} \pmod{p}$, $n \geq 1$.

(ii) If $a \equiv 0 \pmod{p}$, then $\nu(a, b, p) = 2$ and $u_n \equiv 0 \pmod{p}$, $n \geq 2$.

Proof: This follows by simple verification.

3. RESULTS FOR SPECIAL CASES

For certain special classes of PR's, we can easily determine $\mu(a, b, p)$, $\alpha(a, b, p)$, and $s(a, b, p)$. Of course, if $\mu(a, b, p)$ and $\alpha(a, b, p)$ are known exactly, $\beta(a, b, p)$ is immediately determined. Theorems 1-4 will discuss these cases. The proofs follow by induction and direct verification.

Theorem 1: In the PR $u(a, b)$, suppose that $b = 0$.

(i) If $a \not\equiv 0 \pmod{p}$, then $u_n = a^{n-1}$, $n \geq 1$.

Further,

$$\nu(a, b, p) = 1, \ \alpha(a, b, p) = 1, \ \mu(a, b, p) = \operatorname{ord}_p(a), \text{ and } s(a, b, p) = a$$

for all primes p, where $\operatorname{ord}_p(x)$ denotes the exponent of x modulo p.

(ii) If $a \equiv 0 \pmod{p}$, then $u_n = 0$, $n \geq 2$,

$$\nu(a, b, p) = 2, \ \alpha(a, b, p) = 1, \ \mu(a, b, p) = 1, \text{ and } s(a, b, p) = 0.$$

Theorem 2: In the PR $u(a, b)$ let $a = 0$ and $b \not\equiv 0 \pmod{p}$. Then

$$u_{2n} = 0 \quad \text{and} \quad u_{2n+1} = b, \ n \geq 0.$$

Further,

$$\nu(a, b, p) = 0, \ \alpha(a, b, p) = 2, \ \mu(a, b, p) = 2 \operatorname{ord}_p(b), \text{ and } s(a, b, p) = b.$$

Theorem 3: In the PR $u(a, b)$ suppose that $D = 0$, $a \not\equiv 0 \pmod{p}$, and $b \not\equiv 0 \pmod{p}$. Then

$$u_n = n(a/2)^{n-1}, \ n \geq 0.$$

Further

$$\alpha(a, b, p) = p, \ \mu(a, b, p) = p \operatorname{ord}_p(a/2), \text{ and } s(a, b, p) = a/2.$$

Theorem 4: In the PR $u(a, b)$ suppose that r_1/r_2 is a root of unity. Let k be the order of the root of unity. Let ζ_k be a primitive kth root of unity.

(i) If $k = 1$, then $a = 2N$, $b = -N$, $D = 0$, $r_1 = N$, $r_2 = N$, and $r_1/r_2 = 1$. Theorem 3 characterizes the terms of this sequence.

(ii) If $k = 2$, then $a = 0$, $b = N$, $D = 4N$, $r_1 = \sqrt{N}$, $r_2 = -\sqrt{N}$, and $r_1/r_2 = -1$. Theorem 2 characterizes the terms of this sequence.

(iii) If $k = 3$, $a = N$, $b = -N^2$, $D = -3N^2$, $r_1 = -\zeta_3 N$, $r_2 = -\zeta_3^2 N$, and $r_1/r_2 = \zeta_3^{-1}$.

(iv) If $k = 4$, $a = 2N$, $b = -2N^2$, $D = -4N^2$, $r_1 = (1 + i)N$, $r_2 = (1 - i)N$, and $r_1/r_2 = i$ where $i = \sqrt{-1}$.

(v) If $k = 6$, $a = 3N$, $b = -3N^2$, $D = -3N^2$, $r_1 = -i\zeta_3\sqrt{3}$, $r_2 = i\zeta_3^2(\sqrt{3})N$, and $r_1/r_2 = \zeta_6$.

Moreover, if $k \geq 2$, then

$$\alpha(a, b, p) = k, \ \mu(a, b, p) = k \operatorname{ord}_p(s),$$

and

$$s(a, b, p) = s \equiv \operatorname{sgn}(a^k)(-(-b)^{k/2}) \pmod{p},$$

where $\operatorname{sgn}(x)$ denotes the sign of x. Furthermore, if $n = qk + r$, $0 \geq r \geq k$, and $k \geq 3$, then

$$u_n = s^q u_r = (-1)^q N^{qk} u_r.$$

In Theorem 4, note that $k = 1, 2, 3, 4$, or 6 are the only possibilities for k since these are the only orders of roots of unity that satisfy a quadratic polynomial over the rationals.

Just as we treated the divisibility properties of certain special recurrences with respect to a general prime, we now consider the special case of the prime 2 in the following theorem. We have already handled the cases where $b \equiv 0$ or $a \equiv 0 \pmod{2}$ in Theorems 1 and 2.

Theorem 5: Consider the PR $u(a, b)$. Suppose that $2 \nmid ab$. Then $\nu(a, b, 2) = 0$, $\mu(a, b, 2) = 3$, $\alpha(a, b, 2) = 3$, and $s(a, b, 2) = 1$. The reduced recurrence modulo 2 is then

$$(0, 1, 1, 0, 1, 1, \ldots) \pmod{2}.$$

4. GENERAL RESULTS

From this point on, p will always denote an odd prime unless otherwise specified. Theorem 6 gives criteria for determining $\mu(a, b, p)$, $\alpha(a, b, p)$, and $s(a, b, p)$ for the general PR $u(a, b)$. For the rest of the paper, D' will denote the square-free part of the discriminant D, and K will denote the algebraic number field $Q(\sqrt{D'})$, where Q as usual stands for the rationals.

Theorem 6: In the PR $u(a, b)$, suppose that $p | bD$. Let P be a prime ideal in K dividing p. If $(D/p) = 1$, we will identify P with p.

(i) $\mu(a, b, p)$ is the least common multiple of the exponents of r_1 and r_2 modulo P.

(ii) $\alpha(a, b, p)$ is the exponent of r_1/r_2 modulo P. If $(D/p) = -1$, then $\alpha(a, b, p)$ is also the least positive integer n such that r_1 is congruent to a rational integer modulo P.

(iii) If $k = \alpha(a, b, p)$, then $s(a, b, p) \equiv r_1^k \pmod{P}$.

Proof: Let R denote the integers of K. Since $b \not\equiv 0 \pmod{p}$, neither r_1 nor $r_2 \equiv 0 \pmod{p}$. Since R/P is a field of p or p^2 elements, r_1/r_2 is well-defined modulo P. Further, since $D = (r_1 - r_2)^2 \not\equiv 0 \pmod{P}$, $u_n = (r_1^n - r_2^n)/(r_1 - r_2)$ is also well-defined modulo P.

(i) Let $n = \mu(a, b, p)$. Then

$$u_n = (r_1^n - r_2^n)/(r_1 - r_2) \equiv 0 \pmod{p} \equiv 0 \pmod{P}$$

and

$$u_{n+1} \equiv 1 \pmod{p} \equiv 1 \pmod{P}.$$

Thus, $r_1^n \equiv r_2^n \pmod{P}$. Hence,

$$u_{n+1} = (r_1^{n+1} - r_2^{n+1})/(r_1 - r_2) \equiv (r_1^n(r_1) - r_1^n(r_2))/(r_1 - r_2) \equiv r_1^n \equiv 1 \pmod{P}$$

Thus, $r_1^n \equiv r_2^n \equiv 1 \pmod{P}$. Conversely, if $r_1^k \equiv r_2^k \equiv 1 \pmod{P}$ for some positive integer k, then it follows that $u_k \equiv 0$ and $u_{k+1} \equiv 1 \pmod{p}$. Assertion (i) now follows.

(ii) Now let $n = \alpha(a, b, p)$. Then $u_n = (r_1^n - r_2^n)/(r_1 - r_2) \equiv 0 \pmod{P}$. This occurs only if $r_1^n \equiv r_2^n \pmod{P}$. Dividing through by r_2^n, we obtain

$$(r_1/r_2)^n \equiv 1 \pmod{P}.$$

Hence, $\alpha(a, b, p)$ is the exponent of r_1/r_2 modulo P.

Further, if $(D/p) = -1$, then

$$\sigma(r_1) = r_1^p \equiv r_2 \pmod{P} \quad\text{and}\quad \sigma(r_1^n) = (r_1^p)^n \equiv r_2^n \pmod{P},$$

where σ is the Frobenius automorphism of R/P. This follows, since r_1 and r_2 are both roots of the irreducible polynomial modulo P, $x^2 - ax - b$. Thus, if $r_1^n \equiv r_2^n \pmod{P}$, we obtain

$$(r_1^n)^p \equiv r_2^n \equiv r_1^n \pmod{P}.$$

Let Z_p denote the finite field of p elements. Now,

$$R/P = Z_p[\sqrt{D'}].$$

In $Z_p[\sqrt{D'}]$, the only solutions of the equation $x^p - x = 0$ are those in Z_p by Fermat's theorem. Assertion (ii) now follows.

(iii) Let $k = \alpha(a, b, p)$. Then

$$u_{k+1} \equiv s(a, b, p) \pmod{p} \equiv s(a, b, p) \pmod{P}.$$

By the proof of (ii), $r_1^k \equiv r_2^k \pmod{P}$. Thus,

$$\begin{aligned} u_{k+1} = (r_1^{k+1} - r_2^{k+1})/(r_1 - r_2) &\equiv (r_1^k(r_1) - r_1^k(r_2))/(r_1 - r_2) \\ &\equiv r_1^k \equiv s(a, b, p) \pmod{P}. \end{aligned}$$

The proof is now complete.

Theorem 6, while definitive, is impractical for actually computing

$$\mu(a, b, p),\ \alpha(a, b, p),\ \text{and } s(a, b, p).$$

We will develop more practical methods of determining these numbers, although our results will not be as complete. The most easily applied of our methods will use the quadratic character modulo p and pertain to certain special classes of PR's. For sharper results, we will also utilize the less convenient 2^n-*ic* characters modulo p.

A good theory of the divisibility properties of the PR $u(a, b)$ with respect to p should give limitations for the restricted period modulo p. Given the restricted period, one should then be able to determine exactly the exponent of the multiplier modulo p and, consequently, the period modulo p. Further, we should be able to specify the multiplier modulo p. This will be our program from here on. As a first step toward fulfilling this project, we now present Theorems 7 and 8. Theorem 7 is due to Wyler [14] and, in most cases, determines $\mu(a, b, p)$ when $\alpha(a, b, p)$ and $\mathrm{ord}_p(-b)$ are known. Theorem 8 is the author's application of Wyler's Theorem 7.

Theorem 7: Consider the PR $u(a, b)$. Suppose $b \not\equiv 0 \pmod{p}$. Let $h = \mathrm{ord}_p(-b)$. Suppose $h = 2^c h'$, where h' is an odd integer. Let $k = \alpha(a, b, p) = 2^d k'$, where k' is an odd integer. Let H be the least common multiple of h and k.

(i) $\mu(a, b, p) = H$ or $2H$; $\beta(a, b, p) = H/k$ or $2H/k$.

(ii) If $c \neq d$, then $\mu(a, b, p) = 2H$. If $c = d > 0$, then $\mu(a, b, p) = H$.

This theorem is complete in the sense that if $c = d = 0$, then $\mu(a, b, p)$ may be either H or $2H$. For example, look at the PR $u(3, -1)$. For all primes p, $h = \mathrm{ord}_p(1) = 1 = 2^0(1)$.

If $p = 13$, then $k = \alpha(3, -1, 13) = 7 = 2^0(7)$. Further, $H = [1, 7] = 7$. By inspection, $\mu(3, -1, 13) = 14 = 2H$.

If $p = 29$, then $k = \alpha(3, -1, 29) = 7$. As before, $H = 7$. But now we have $\mu(3, -1, 29) = 7 = H$.

Theorem 8: Let p be an odd prime. Consider the PR $u(a, b)$, where $b \not\equiv 0 \pmod{p}$. Let $h = \mathrm{ord}_p(-b)$. Suppose $h = 2^c h'$, where h' is an odd integer. Let $k = \alpha(a, b, p) = 2^d k'$, where k' is an odd integer. Let $H = [h, k]$, where $[x, y]$ is the least common multiple of x and y. Let $s = s(a, b, p)$.

(i) $s^2 \equiv (-b)^k \pmod{p}$.
(ii) If $c = d = 0$ and $\mu(a, b, p) = H$, then $s \equiv (-b)^{(k+h)/2} \pmod{p}$.
(iii) If $c = d = 0$ and $\mu(a, b, p) = 2H$, then $s \equiv -(-b)^{(k+h)/2} \pmod{p}$.
(iv) If $c = d > 0$, then $s \equiv -(-b)^{k/2} \pmod{p}$.
(v) If $d > c$, then $s \equiv -(-b)^{k/2} \pmod{p}$.
(vi) If $c > d$, then $s \equiv \pm r$, where $r^2 \equiv (-b)^k \pmod{p}$ and $0 \leq r \leq (p-1)/2$.
Further, both possibilities do in fact occur.

Proof:
(i) This follows immediately from (7), letting $n = k$.
(ii) Let $c = d = 0$ and assume that $\mu(a, b, p) = H$. Then,

$$\text{ord}_p(s) = \beta(a, b, p) = H/k = [h, k]/k.$$

Further, by (i),

$$s^2 \equiv (-b)^k \pmod{p}.$$

Thus,

$$s \equiv (-b)^{(k+h)/2} \quad \text{or} \quad s \equiv -(-b)^{(k+h)/2} \pmod{p}.$$

In general, it is easy to see that if r is a positive integer,

$$\text{ord}_p(-b)^r = [h, r]/r.$$

Therefore,

$$\text{ord}_p((-b)^{(k+h)/2}) = [h, (k+h)/2]/((k+h)/2).$$

Suppose $g = (h, k)$. Let $h = gm$ and $k = gn$, where $(m, n) = 1$. Then,

$$\begin{aligned}[h, (k+h)/2]/((k+h)/2) &= [gm, g(m+n)/2]/(g(m+n)/2)\\ &= g[m, (m+n)/2]/(g(m+n)/2).\end{aligned}$$

Clearly, $(m, m+n) = 1$ and, a fortiori, $(m, (m+n)/2) = 1$. Hence,

$$g[m, (m+n)/2]/(g(m+n)/2) = (gm(m+n)/2)/(g(m+n)/2) = m.$$

But,

$$[h, k]/k = [gm, gn]/(gn) = gmn/(gn) = m.$$

Thus,

$$\text{ord}_p((-b)^{(k+h)/2}) = \text{ord}_p(s) = m.$$

However, since m is odd,

$$\text{ord}_p(-(-b)^{(k+h)/2}) = 2m.$$

Thus, $s \equiv (-b)^{(k+h)/2} \pmod{p}$.

(iii)-(v) The proofs of these assertions are similar to that of (ii). In calculating $\text{ord}_p(s)$ for (iv) and (v), we make use of Wyler's Theorem 7.

(vi) To see that both possibilities actually occur, consider $s(1, 1, 13)$ and $s(1, 1, 17)$.

Now, $\alpha(1, 1, 13) = 7$ and $\text{ord}_{13}(-1) = 2$, so $c > d$. By inspection, we see that

$$s(1, 1, 13) \equiv 8 > (13-1)/2 = 6 \pmod{13}.$$

Also, $\alpha(1, 1, 17) = 9$ and $\text{ord}_{17}(-1) = 2$. Hence, $c > d$. However, we now find that

$$s(1, 1, 17) \equiv 4 \leq (17-1)/2 = 8 \pmod{17},$$

and we are done.

Unfortunately, Theorems 7 and 8 depend on knowing the highest power of 2 dividing $\alpha(a, b, p)$ and $\text{ord}_p(-b)$ to determine $\beta(a, b, p)$ and $\mu(a, b, p)$. Our project will be to find classes of PR's (excluding the special cases already treated) in which for almost all primes p the exponent of the multiplier modulo p, $\beta(a, b, p)$, can be determined by knowing the residue class modulo m to which $\alpha(a, b, p)$ belongs for some fixed positive integer m. In addition, we would like a set of conditions, preferably involving the quadratic character modulo p, for determining $\alpha(a, b, p)$ modulo m without explicitly computing $\alpha(a, b, p)$.

By Theorem 7, these conditions can be satisfied if either

(i) $\text{ord}_p(-b)|m$ for a fixed positive integer m and for almost all primes p, or

(ii) $2H/\alpha(a, b, p)|m$ for a fixed positive integer m and for almost all primes p.

Now, condition (i) can be satisfied for almost all p iff $b = \pm 1$. Thus, we will consider the PR's $u(a, 1)$ and $u(a, -1)$. If $b = 1$, then $\text{ord}_p(-b) = 2$ for all odd primes p and, by Theorem 7, $H = \alpha(a, 1, p)$ or $H = 2\alpha(a, 1, p)$. Hence, $\beta(a, 1, p)|4$ and $\beta(a, 1, p)$ is largely determined if $\alpha(a, 1, p)$ is known modulo 4. Similarly, if $b = -1$, then $\beta(a, -1, p)$ is largely determined if $\alpha(a, -1, p)$ is known modulo 2.

By Theorems 6 and 7, $H = [\text{ord}_p(r_1/r_2), \text{ord}_p(-b)]$. Hence, condition (ii) can be satisfied if

$$r_1/r_2 = \pm b. \tag{11}$$

Since $r_1 r_2 = -b$, equation (11) is equivalent to requiring that

$$r_1/r_2 = \pm r_1 r_2. \tag{12}$$

Solving, we see that $r_2^2 = 1$ or $r_2^2 = -1$. But, if $r_2^2 = -1$, then $r_2 = \pm i$ and $r_1 = \mp i$. However, this case is already treated by Theorem 4(ii). If $r_2^2 = 1$, then $r_2 = \pm 1$. If $r_2 = 1$, then by Theorem 6 we see that $\beta(a, b, p) = 1$ always no matter what $\alpha(a, b, p)$ is. If $r_2 = -1$, then Theorem 6 and a little analysis shows that $\beta(a, b, p)|2$ and depends upon the residue class of $\alpha(a, b, p)$ modulo 2. Note that if $r_2 = 1$, then

$$r_1 = -b/r_2 = -b \quad \text{and} \quad a = r_1 + r_2 = -b + 1. \tag{13}$$

If $r_2 = -1$, then

$$r_1 = b \quad \text{and} \quad a = b - 1. \tag{14}$$

Hence, we will also investigate the divisibility properties of the PR's

$$u(-b + 1, b) \quad \text{and} \quad u(b - 1, b).$$

From our preceding discussion, it will be very helpful if we can find conditions to determine $\alpha(a, b, p)$ modulo 4. The following two lemmas and two theorems determine the residue class of $\alpha(a, b, p)$ modulo 4 for a general PR $u(a, b)$.

Lemma 3: Let p be an odd prime. Consider the PR $u(a, b)$. Suppose that $p \nmid bD$.

(i) If $\alpha(a, b, p) \equiv 1 \pmod 2$, then $(-b/p) = 1$.

(ii) If $\alpha(a, b, p) \equiv 2 \pmod 4$, then $(bD/p) = 1$.

(iii) If $\alpha(a, b, p) \equiv 0 \pmod 4$, then $(bD/p) = (-b/p)$.

Proof: Firstly, note that by (8),

$$v_n^2 - Du_n^2 = 4(-b)^n. \tag{15}$$

(i) Let $k = \alpha(a, b, p) \equiv 1 \pmod 2$. By (15),

$$v_k^2 \equiv 4(-b)^k \pmod p.$$

Since $k \equiv 1 \pmod 2$, this is possible only if $(-b/p) = 1$.

(ii) Let $2k = \alpha(a, b, p)$. Then $k \equiv 1 \pmod 2$. By (10), $v_k \equiv 0 \pmod p$. Then by (15),

$$-Du_k^2 \equiv 4(-b)^k \pmod p.$$

If $(-b/p) = 1$, then clearly, $(-D/p) = 1$. If $(-b/p) = -1$, then $(-D/p) = -1$, since $k \equiv 1 \pmod 2$. In both cases, $(bD/p) = 1$.

(iii) Let $2k = \alpha(a, b, p)$. Then $k \equiv 0 \pmod 2$. By (10), $v_k \equiv 0 \pmod p$. Then by (15),

$$-Du_k^2 \equiv 4(-b)^k \pmod p.$$

Since $k \equiv 0 \pmod 2$, $(-D/p) = 1$ in all cases. It follows that $(bD/p) = (-b/p)$.

Theorem 9: Let p be an odd prime. Consider the PR $u(a, b)$. Suppose $p \nmid bD$.

(i) If $(-b/p) = 1$ and $(bD/p) = -1$, then $\alpha(a, b, p) \equiv 1 \pmod 2$.

(ii) If $(-b/p) = -1$ and $(bD/p) = 1$, then $\alpha(a, b, p) \equiv 2 \pmod 4$.

(iii) If $(-b/p) = (bD/p) = -1$, then $\alpha(a, b, p) \equiv 0 \pmod 4$.

Proof: This follows immediately from Lemma 3.

As we can see from Theorem 9, the only doubtful case occurs when

$$(-b/p) = (bD/p) = 1.$$

Lemma 4 and Theorem 10 give a new criterion for determining the restricted period in some instances when $(-b/p) = (bD/p) = 1$.

Lemma 4: Let p be an odd prime. Consider the PR $u(a, b)$. Suppose $p \nmid bD$ and $\alpha(a, b, p) \equiv 1 \pmod 2$. Then $(-b/p) = 1$. Let $r^2 \equiv -b$, where $0 \leq r \leq (p-1)/2$. Then

$$(-2b + ar/p) = 1 \quad \text{or} \quad (-2b - ar/p) = 1, \tag{16}$$

where $(-2b + ar/p)$ denotes the Legendre symbol.

Proof: By Lemma 3(i), we know that $(-b/p) = 1$. Let $k = \alpha(a, b, p)$. By (6),

$$u_k = bu^2_{(k-1)/2} + u^2_{(k+1)/2} \equiv 0 \pmod p.$$

Hence,

$$u^2_{(k+1)/2} \equiv -bu^2_{(k-1)/2} \pmod p.$$

Thus,

$$u_{(k+1)/2} \equiv \pm ru_{(k-1)/2} \pmod p.$$

Suppose that $u_{(k+1)/2} \equiv ru_{(k-1)/2} \pmod p$. Then

$$\begin{aligned} u_{(k+3)/2} \equiv au_{(k+1)/2} + bu_{(k-1)/2} &\equiv aru_{(k-1)/2} + bu_{(k-1)/2} \\ &\equiv (ar + b)u_{(k-1)/2} \pmod p. \end{aligned}$$

Now, by (7),

$$\begin{aligned} u^2_{(k+1)/2} - u_{(k-1)/2}u_{(k+3)/2} &\equiv -bu^2_{(k-1)/2} - (ar + b)u^2_{(k-1)/2} \\ &\equiv (-ar - 2b)u^2_{(k-1)/2} \equiv (-b)^{(k-1)/2} \\ &\equiv r^{k-1} \pmod p. \end{aligned}$$

Since $k - 1$ is even, this implies that $(-2b - ar/p) = 1$.

Now suppose that $u_{(k+1)/2} \equiv -ru_{(k-1)/2} \pmod p$. Continuing as before, we obtain

$$(-2b + ar)u^2_{(k-1)/2} \equiv r^{k-1} \pmod p.$$

This similarly implies that $(-2b + ar/p) = 1$ and we are done.

In our statement of Lemma 4, note that

$$(-2b + ar)(-2b - ar) = bD.$$

Theorem 10: Consider the PR $u(a, b)$. Let p be an odd prime. Suppose $p \nmid bD$ and $(-b/p) = 1$. Let r be as in Lemma 4.

(i) If $(-b/p) = (bD/p) = 1$ and $(-2b + ar/p) = (-2b - ar/p) = -1$, then, $\alpha(a, b, p) \equiv 0$ or $2 \pmod 4$.

(ii) If $(-b/p) = (bD/p) = (-2b + ar/p) = (-2b - ar/p) = 1$, then $\alpha(a, b, p)$ can be congruent to 0, 1, 2, or 3 (mod 4).

Proof: This follows immediately from Lemma 4.

The following examples in Table 1 from the Fibonacci sequence show the completeness of Theorem 10. For the Fibonacci sequence,

$$a = b = 1,\ D = 5,\ bD = 5,\ -2b + ar = -2 + i, \text{ and } -2b - ar = -2 - i.$$

TABLE 1

Examples from the Fibonacci Sequence in Which $(-b/p) = bD/p) = 1$ and $\alpha(a, b, p)$ Takes on All Possible Values Modulo 4

p	$(-b/p)$	(bD/p)	$(-2b + ar/p)$	$(-2b - ar/p)$	$\alpha(1, 1, p)$ (mod 4)
29	1	1	-1	-1	2
41	1	1	-1	-1	0
61	1	1	1	1	3
421	1	1	1	1	1
809	1	1	1	1	2
1601	1	1	1	1	0

By Theorems 9 and 10, we are so far unable to determine whether the restricted period modulo p is even or odd only when

$$(-b/p) = (bD/p) = (-2b + ar/p) = (-2b - ar/p) = 1.$$

The next theorem will settle this case. We will use the notation $[x/p]_n$ to denote the 2^n - *ic* character of x modulo p.

Theorem 11: Let p be an odd prime and suppose that $p - (D/p) = 2^k q$, where q is an odd integer. Consider the PR $u(a, b)$ and suppose that $p \nmid bD$. Let P be a prime ideal in $K = Q(\sqrt{D})$. Then $\alpha(a, b, p) \equiv 1 \pmod 2$ if and only if

$$r_1^{2q} \equiv (-b)^q \pmod P.$$

If $(D/p) = 1$, then $\alpha(a, b, p) \equiv 1 \pmod 2$ if and only if

$$[r_1/p]_{k-1} \equiv (-b)^q \pmod p.$$

Proof: This is proved by Morgan Ward [13] for the Fibonacci sequence in which case $b = 1$. Our proof will be an immediate generalization of Ward's.

First we note that $u_k \equiv 0 \pmod p$ if and only if

$$r_1^{2k} \equiv (-b)^k \pmod P.$$

This follows from the fact that

$$u_k = r_1^k(r_1^k - r_2^k)/(r_1^k(r_1 - r_2)) = (r_1^{2k} - (r_1r_2)^k)/(r_1^k(r_1 - r_2))$$
$$= (r_1^{2k} - (-b)^k)/(r_1^k(r_1 - r_2)).$$

The result now follows easily.

Assume that $\alpha(a, b, p) \equiv 1 \pmod 2$. Then, $u_{p-(D/p)} \equiv 0 \pmod p$ by (5). Further, by (6) it follows that $u_m | u_n$ if $m | n$. Thus, $u_q \equiv 0 \pmod p$ since any odd divisor of $p - (D/p)$ must divide q. Thus, by our result earlier in this proof,

$$r_1^{2q} \equiv (-b)^q \pmod P.$$

Conversely, if $r_1^{2q} \equiv 0 \pmod P$, then $u_q \equiv 0 \pmod p$ by the same result. It thus follows that $\alpha(a, b, p) \equiv 1 \pmod 2$. The last remark in the theorem follows from the definition of $[r_1/p]_{k-1}$.

We will generalize the previous theorem in Theorem 12, which will determine when $\alpha(a, b, p) \equiv 2^m \pmod{2^{m+1}}$. First, we will have to prove the following lemma.

Lemma 5: Consider the PR $u(a, b)$. Let p be an odd prime. Suppose that $p \nmid bD$. Let $k = p - (D/p)$. Then

$$p \mid u_{k/2} \text{ iff } (-b/p) = 1.$$

Proof: This was first proved by D. H. Lehmer [4]. Backstrom [1] also gives a proof.

Theorem 12: Consider the PR $u(a, b)$. Let p be an odd prime and suppose that $p - (D/p) = 2^k q$, where q is an odd integer. Suppose $p \nmid bD$. Let P be a prime ideal in K dividing p.

(i) If $(-b/p) = -1$, then $\alpha(a, b, p) \equiv 2^k \pmod{2^{k+1}}$.

(ii) If $(-b/p) = 1$, then $\alpha(a, b, p) \equiv 2^m \pmod{2^{m+1}}$, where $0 < m < k$, if and only if

$$r_1^{2^{m+1}q} \equiv (-b)^{2^m q} \pmod{P}.$$

but

$$r_1^{2^m q} \not\equiv (-b)^{2^{m-1}q} \pmod{P}.$$

(iii) If $(-b/p) = (D/p) = 1$, then $\alpha(a, b, p) \equiv 2^m \pmod{2^{m+1}}$, where $0 < m < k$, if and only if

$$[r_1/p]_{k-m-1} \equiv (-b)^{2^m q} \pmod{p},$$

but

$$[r_1/p]_{k-m} \not\equiv (-b)^{2^{m-1}q} \pmod{p}.$$

Proof:

(i) This follows from Lemma 5, which implies that

$$\alpha(a, b, p) \nmid (p - (D/p))/2.$$

(ii) First, $m < k$, since by Lemma 5,

$$\alpha(a, b, p) \mid (p - (D/p))/2.$$

Further, $\alpha(a, b, p) \equiv 2^m \pmod{2^{m+1}}$ if and only if $p \mid u_{2^m q}$, but $p \nmid u_{2^{m-1}q}$. Now apply the arguments of the preceding theorem, Theorem 11.

(iii) This follows from the definition of the 2^n-*ic* character modulo p and part (ii).

Note, however, that the criteria of Theorems 11 and 12 are not really simpler than direct verification that p is a divisor of some specified term of $\{u_n\}$. For example, in Theorem 11, we can show that $\alpha(a, b, p) \equiv 1 \pmod 2$, if we can show that $p \mid u_q$, where q is the largest odd integer dividing $p - (D/p)$. This is equivalent to the criterion of Theorem 11. In the next section, we will assume that $b = \pm 1$. In this case, the criteria of Theorems 11 and 12 will be easier to apply.

5. THE SPECIAL CASE $b = \pm 1$

In this section we will obtain more complete results than those of Theorems 7 and 8 for those particular PR's for which $b = \pm 1$. We will first treat the case in which $b = 1$ in the following theorems.

Theorem 13: Consider the PR $u(a, 1)$. Let p be an odd prime. Suppose that $(D/p) \neq 0$. If $(-1/p) = 1$, let $i \equiv \sqrt{-1}$, where $0 \leq i \leq (p - 1)/2$.

(i) $\beta(a, 1, p) = 1, 2,$ or 4; $s(a, 1, p) \equiv 1, -1,$ or $\pm i \pmod{p}$.

(ii) $\beta(a, 1, p) = 1$ iff $\alpha(a, 1, p) \equiv 2 \pmod 4$ and $\mu(a, 1, p) \equiv 2 \pmod 4$.

(iii) $\beta(a, 1, p) = 2$ iff $\alpha(a, 1, p) \equiv 0 \pmod 4$ and $\mu(a, 1, p) \equiv 0 \pmod 8$.

(iv) $\beta(a, 1, p) = 4$ iff $\alpha(a, 1, p) \equiv 1 \pmod 2$ and $\mu(a, 1, p) \equiv 4 \pmod 8$.

(v) If $(-1/p) = -1$ and $(a^2 + 4/p) = 1$, then $\alpha(a, 1, p) \equiv 2 \pmod 4$, $\beta(a, 1, p) = 1$, and $\mu(a, 1, p) \equiv 2 \pmod 4$.

(vi) If $(-1/p) = -1$ and $(a^2 + 4/p) = -1$, then $\alpha(a, 1, p) \equiv 0 \pmod 4$, $\beta(a, 1, p) = 2$, and $\mu(a, 1, p) \equiv 0 \pmod 8$.

(vii) If $(-1/p) = 1$ and $(a^2 + 4/p) = -1$, then $\alpha(a, 1, p) \equiv 1 \pmod 2$, $\beta(a, 1, p) = 4$, and $\mu(a, 1, p) \equiv 4 \pmod 8$.

(viii) If $(-1/p) = (a^2 + 4/p) = 1$ and $(-2 + ai/p) = (-2 - ai/p) = -1$, then $\alpha(a, 1, p) \equiv 0$ or $2 \pmod 4$ and $\beta(a, 1, p) = 1$ or 2.

(ix) If $(-1/p) = (a^2 + 4/p) = 1$ and $p \equiv 5 \pmod 8$, then $\alpha(a, 1, p) \not\equiv 0 \pmod 4$ and $\beta(a, 1, p) \neq 2$.

Proof:

(i) Apply Theorem 7. Since $-b = -1$, $\mathrm{ord}_p(-b) = 2$; hence, $H = \alpha(a, 1, p)$ or $H = 2\alpha(a, 1, p)$. Since $\beta(a, 1, p) = H/\alpha(a, 1, p)$ or $\beta(a, 1, p) = 2H/\alpha(a, 1, p)$, $\beta(a, 1, p) = 1$, 2, or 4.

(ii)-(iv) These follow from Theorem 7.

(v)-(vii) These follow from Theorem 9.

(viii) This follows from Theorem 10.

(ix) Suppose $p \equiv 5 \pmod 8$. Then I claim that $\alpha(a, 1, p) \not\equiv 0 \pmod 4$, and, consequently, $\beta(a, 1, p) \neq 2$. Let $k = \alpha(a, 1, p)$, then by part (iii) of this theorem,

$$2k = \mu(a, 1, p) \equiv 0 \pmod 8.$$

Since $(a^2 + 4/p) = (D/p) = 1$, $2k \mid p - 1$ by Theorem 6(i). But then $p \equiv 1 \pmod 8$, which contradicts the fact that $p \equiv 5 \pmod 8$.

Theorem 14: Consider the PR $u(a, 1)$. Let p be an odd prime such that $(-1/p) = (D/p) = 1$. Let $p - 1 = 2^k q$, where q is an odd integer. Let $\varepsilon = (a_0 + c_0\sqrt{D'})/2$ be the fundamental unit in $K = Q(\sqrt{D'})$, where D' is the square-free part of D. Let $\overline{\varepsilon} = -1/\varepsilon$. Consider further the PR $u(a_0, 1)$.

(i) $N(\varepsilon) = -1$, $r_1 = \varepsilon^m$, and $r_2 = -\varepsilon^{-m} = (\overline{\varepsilon})^m$, where $m \equiv 1 \pmod 2$ and r_1 and r_2 correspond to the PR $u(a, 1)$.

(ii) $\alpha(a, 1, p) \mid \alpha(a_0, 1, p)$.

(iii) Either $\alpha(a, 1, p) \equiv \alpha(a_0, 1, p) \equiv 1 \pmod 2$ or $\alpha(a, 1, p) \equiv \alpha(a_0, 1, p) \pmod 4$.

(iv) If $[\varepsilon/p]_{k-1} = -1$, then $\alpha(a, 1, p) \equiv 1 \pmod 2$, $\beta(a, 1, p) = 4$, and $\mu(a, 1, p) \equiv 4 \pmod 8$.

(v) If $[\varepsilon/p]_{k-1} = 1$, then $\alpha(a, 1, p) \equiv 2 \pmod 4$, $\beta(a, 1, p) = 1$, and $\mu(a, 1, p) \equiv 2 \pmod 4$.

(vi) If $[\varepsilon/p]_{k-2} \neq 1$, then $\alpha(a, 1, p) \equiv 0 \pmod 4$, $\beta(a, 1, p) = 2$, and $\mu(a, 1, p) \equiv 0 \pmod 8$.

Proof:

(i) Since $N(r_1) = r_1 r_2 = -1$, it follows that $N(\varepsilon) = -1$, $r_1 = \varepsilon^m$, and $r_2 = -\varepsilon^{-m} = (\overline{\varepsilon})^m$, where $m \equiv 1 \pmod 2$.

(ii) First, we will see that ε and $\overline{\varepsilon}$ are roots of the characteristic polynomial

$$x^2 - a_0 x - 1 = 0$$

associated with the PR $u(a_0, 1)$. Let

$$r_1' = (a_0 + \sqrt{a_0^2 + 4})/2 \quad \text{and} \quad r_2' = (a_0 - \sqrt{a_0^2 + 4})/2$$

be the roots of the characteristic polynomial. By definition of the fundamental unit ε, it is easily seen that

$$a_0^2 - D'c_0^2 = -4.$$

Hence, $\sqrt{a_0^2 + 4} = c_0\sqrt{D'}$. Thus,

$$\varepsilon = (a_0 + c_0\sqrt{D'})/2 = r_1' \quad \text{and} \quad \overline{\varepsilon} = (a_0 - c_0\sqrt{D'})/2 = r_2'.$$

Now, by Theorem 6(ii), $\alpha(a_0, 1, p)$ is the exponent of $\varepsilon/\bar{\varepsilon} = -\varepsilon^2$ modulo p. Similarly, $\alpha(a, 1, p)$ is the exponent of $r_1/r_2 = (-\varepsilon^2)^m$ modulo p. It is now easy to see that

$$\alpha(a, 1, p) = \alpha(a_0, 1, p)/(m, \alpha(a_0, 1, p)). \tag{17}$$

Clearly, $\alpha(a, 1, p) | \alpha(a_0, 1, p)$.

(iii) Since m is odd, it is easy to see from (17) that (iii) holds.

(iv) By definition,

$$[\varepsilon/p]_{k-1} = \varepsilon^{(p-1)/2^{k-1}} = \varepsilon^{2q} \equiv -1 \equiv (-1)^q \pmod{p}.$$

By Theorem 11, it now follows that $\alpha(a_0, 1, p) \equiv 1 \pmod 2$. By part (iii),

$$\alpha(a, 1, p) \equiv \alpha(a_0, 1, p) \equiv 1 \pmod 2.$$

The result now follows by Theorem 13(iv).

(v) and (vi) The proofs of these parts are similar to that of part (iv).

The advantage of Theorem 14 is that it gives results for the infinite number of PR's $u(a, 1)$, for which the discriminants D all have the same square-free part D', by analyzing only one PR $u(a_0, 1)$. When the 2^n - *ic* characters modulo p in Theorem 14 are merely the quadratic characters, computations are considerably easier. Further, when D' is a prime, we can make use of several identities to calculate the quadratic characters. The following theorem discusses this in more detail.

Theorem 15: Consider the PR $u(a, 1)$. Suppose that D', the square-free part of D, is an odd prime. Let p be an odd prime. Suppose that

$$(-1/p) = (-1/D') = (p/D') = (D'/p) = 1.$$

Let $\varepsilon_1 = (a_1 + c_1\sqrt{D'})/2$ be the fundamental unit in $K = Q(\sqrt{D'})$.
Let $\varepsilon_2 = (a_2 + c_2\sqrt{p})/2$ be the fundamental unit in $Q(\sqrt{p})$.
Let $D' = m_1^2 + 4n_1^2$ and $p = m_2^2 + 4n_2^2$.
Let $\delta_1 = (m_1 + \sqrt{D'})/2$ and $\delta_2 = (m_2 + \sqrt{p})/2$.
Let $i = \sqrt{-1}$.

(i) $$\begin{aligned}(\varepsilon_1/p) &= (\delta_1/p) = (m_1 + 2n_1 i/p) = (a_1 + 2i/p) = (m_1n_2 - m_2n_1/p)\\ &= (\varepsilon_2/D') = (\delta_2/D') = (m_2 + 2n_2 i/D') = (a_2 + 2i/D')\\ &= (m_1n_2 - m_2n_1/D').\end{aligned}$$

(ii) If $(\varepsilon_1/p) = 1$ and $p \equiv 5 \pmod 8$, then

$$\alpha(a, 1, p) \equiv 2 \pmod 4,\ \beta(a, 1, p) = 1, \text{ and } \mu(a, 1, p) \equiv 2 \pmod 4.$$

(iii) If $(\varepsilon_1/p) = -1$ and $p \equiv 5 \pmod 8$, then

$$\alpha(a, 1, p) \equiv 1 \pmod 2,\ \beta(a, 1, p) = 4, \text{ and } \mu(a, 1, p) \equiv 4 \pmod 8.$$

(iv) If $(\varepsilon_1/p) = -1$ and $p \equiv 1 \pmod 8$, then

$$\alpha(a, 1, p) \equiv 0 \pmod 4,\ \beta(a, 1, p) = 2, \text{ and } \mu(a, 1, p) \equiv 0 \pmod 8.$$

(v) If $(\varepsilon_1/p) = 1$ and $p \equiv 9 \pmod{16}$, then

$$\alpha(a, 1, p) \not\equiv 0 \pmod 4,\ \beta(a, 1, p) \neq 2, \text{ and } \mu(a, 1, p) \not\equiv 0 \pmod 8.$$

Proof:

(i) This is proved by Emma Lehmer in [6].
(ii) This follows from Theorem 14(v).
(iii) This follows from Theorem 14(iv).
(iv) and (v) These follow from Theorem 14(iv)-(vi).

In the case of the Fibonacci sequence, $a = b = 1$ and $D = D' = 5$, which is a prime. Further, the fundamental unit of $Q(\sqrt{5})$ is $\varepsilon_5 = (1 + \sqrt{5})/2$, and 5 can be partitioned as

$$5 = 1^2 + 4(1)^2.$$

With these facts, we can easily apply the criteria of Theorem 15 to the Fibonacci sequence. Wherever possible, we prefer to use the criteria of Theorems 13 and 15, since these involve only quadratic characters rather than the higher-order 2^{ic} characters used in Theorem 14. Theorems 13 and 15 suffice to determine $\alpha(1, 1, p) \pmod 4$ and, consequently, $\beta(1, 1, p)$ for all odd primes $p <$ 1,000 except p = 89, 401, 521, 761, 769, and 809. Further, we know from Theorem 15(v) that none of $\beta(1, 1, 89)$, $\beta(1, 1, 521)$, $\beta(1, 1, 761)$, or $\beta(1, 1, 809)$ are equal to 2.

There are additional rules to determine (ε_5/p) in addition to those of Theorem 15. These are given by Emma Lehmer [5], [6], and [7]. Suppose that $p \equiv 1 \pmod 4$ and $(5/p) = 1$. Then the prime p can be represented as

$$p = m^2 + n^2, \tag{18}$$

where $m \equiv 1 \pmod 4$ and $5|m$ or $5|n$. Another quadratic partition of p is

$$p = c^2 + 5d^2. \tag{19}$$

Further, if we express the fundamental unit of $Q(\sqrt{p})$ as $(f + g\sqrt{p})/2$, then either $5|f$ or $5|g$. We then have the following criteria for determining (ε_5/p):

$$(\varepsilon_5/p) = 1 \text{ iff } p \equiv 1 \pmod{20} \text{ and } n \equiv 0 \pmod 5, \text{ or } \atop p \equiv 9 \pmod{20} \text{ and } m \equiv 0 \pmod 5. \tag{20}$$

$$(\varepsilon_5/p) = (-1)^d. \tag{21}$$

$$(\varepsilon_5/p) = 1 \text{ iff } f \equiv 0 \pmod 5. \tag{22}$$

Now, suppose that p and q are both odd primes and that $(-1/p) = (-1/q) = (p/q) = (q/p) = 1$. Let ε_q be the fundamental unit of $Q(\ p)$. Emma Lehmer [7] has given an analogous rule to that of equation (21) to determine (ε_q/p) in terms of the representability of p or $2p$ by the form

$$c^2 + qd^2$$

in the cases q = 13, 17, 37, 41, 73, 97, 113, 137, 193, 313, 337, 457, and 577. These results are applicable to Theorem 15 when $D' = q$.

We now treat the PR's for which $b = -1$ and $|a| \geq 3$. The PR's $u(a, -1)$ for which $|a| \leq 2$ are treated in Theorem 4.

Theorem 16: Consider the PR $u(a, -1)$. Let p be an odd prime. Suppose $p \nmid D$.

(i) $\beta(a, -1, p) = 1$ or 2; $s(a, -1, p) \equiv 1$ or $-1 \pmod p$.

(ii) If $\alpha(a, -1, p) \equiv 0 \pmod 2$, then $\beta(a, -1, p) = 2$ and $\mu(a, -1, p) \equiv 0 \pmod 4$.

(iii) If $\alpha(a, -1, p) \equiv 1 \pmod 2$, then $\beta(a, -1, p)$ may be 1 or 2, and $\mu(a, -1, p)$ may be congruent to $1 \pmod 2$ or $2 \pmod 4$.

(iv) If $(2 - a/p) = (2 + a/p) = -1$, then

$$\alpha(a, -1, p) \equiv 0 \pmod 2,\ \beta(a, -1, p) = 2, \text{ and } \mu(a, -1, p) \equiv 0 \pmod 4.$$

(v) If $(2 - a/p) = 1$ and $(2 + a/p) = -1$, then

$$\alpha(a, -1, p) \equiv 1 \pmod 2,\ \beta(a, -1, p) = 2, \text{ and } \mu(a, -1, p) \equiv 2 \pmod 4.$$

(vi) If $(2 - a/p) = -1$ and $(2 + a/p) = 1$, then

$$\alpha(a, -1, p) \equiv 1 \pmod 2,\ \beta(a, -1, p) = 1, \text{ and } \mu(a, -1, p) \equiv 1 \pmod 2.$$

Proof:

(i) By Theorem 7,

$$\beta(a, -1, p) = H/\alpha(a, -1, p) \quad \text{or} \quad \beta(a, -1, p) = 2H/\alpha(a, -1, p).$$

Since $-b = 1$, $\text{ord}_p(-b) = 1$, and $H = \alpha(a, -1, p)$. Thus, $\beta(a, -1, p) = 1$ or 2.

(ii) and (iii) These follow from Theorem 7 and the comment following Theorem 7.

(iv) This follows from part (ii) and Theorem 10(i).

(v) and (vi) First notice that in both cases,

$$(4 - a^2/p) = -1 = (bD/p).$$

Thus, by Theorem 9(i), $\alpha(a, -1, p) \equiv 1 \pmod 2$. Now, let $k = \alpha(a, -1, p) \equiv 1 \pmod 2$. Then, by (6),

$$u_k = -u^2_{(k-1)/2} + u^2_{(k+1)/2} \equiv 0 \pmod p. \tag{23}$$

Hence,

$$u_{(k+1)/2} \equiv \pm u_{(k-1)/2} \pmod p.$$

First, suppose that $u_{(k+1)/2} \equiv u_{(k-1)/2} \pmod p$. Then,

$$u_{(k+3)/2} = -u_{(k-1)/2} + au_{(k+1)/2} \equiv (a - 1)u_{(k+1)/2} \pmod p.$$

Then, by (7),

$$\begin{aligned} u^2_{(k+1)/2} - u_{(k+3)/2} \cdot u_{(k-1)/2} &\equiv u^2_{(k+1)/2} - (a - 1)u^2_{(k+1)/2} \\ &\equiv (2 - a)u^2_{(k+1)/2} \equiv 1^{(k-1)/2} \equiv 1 \pmod p. \end{aligned}$$

Thus, $u^2_{(k+1)/2} \equiv 1/(2 - a) \pmod p$, and $(2 - a/p) = 1$. Now, by (6),

$$\begin{aligned} u_{k+1} &= -u_{(k+1)/2} \cdot u_{(k-1)/2} + u_{(k+1)/2} \cdot u_{(k+3)/2} \\ &\equiv -u^2_{(k+1)/2} + (a - 1)u^2_{(k+1)/2} \equiv (a - 2)u^2_{(k+1)/2} \\ &\equiv (a - 2)/(2 - a) \equiv -1 \pmod p. \end{aligned}$$

Thus, if $\alpha(a, -1, p) \equiv 1 \pmod 2$ and $u_{(k+1)/2} \equiv u_{(k-1)/2} \pmod p$, then,

$$(2 - a/p) = 1 \quad \text{and} \quad \beta(a, -1, p) = 2.$$

Now, suppose that $u_{(k+1)/2} \equiv -u_{(k-1)/2} \pmod p$. Then,

$$u_{(k+3)/2} = -u_{(k-1)/2} + au_{(k+1)/2} \equiv (a + 1)u_{(k+1)/2} \pmod p.$$

Further,

$$u^2_{(k+1)/2} - u_{(k-1)/2} \cdot u_{(k+3)/2} \equiv (a + 2)u^2_{(k+1)/2} \equiv 1^{(k-1)/2} \equiv 1 \pmod p.$$

Then, $u^2_{(k+1)/2} \equiv 1/(2 + a) \pmod p$, and $(2 + a/p) = 1$. Now,

$$\begin{aligned} u_{k+1} &= -u_{(k+1)/2} \cdot u_{(k-1)/2} + u_{(k+1)/2} \cdot u_{(k+3)/2} \equiv (a + 2)u^2_{(k+1)/2} \\ &\equiv (a + 2)/(a + 2) \equiv 1 \pmod p. \end{aligned}$$

Hence, if $(a, -1, p) \equiv 1 \pmod 2$ and $u_{(k+1)/2} \equiv -u_{(k-1)/2} \pmod p$, then,

$$(2 + a/p) = 1 \quad \text{and} \quad \beta(a, -1, p) = 1.$$

Parts (v) and (vi) now follow immediately.

Theorem 17: Consider the PR $u(a, -1)$, where $|a| \geq 3$. Let p be an odd prime such that $(4 - a^2/p) = (2 - a/p) = (2 + a/p) = 1$. Let $\varepsilon = (a_0 + c_0\sqrt{D'})/2$ be the fundamental unit of $Q(\sqrt{D'})$. Suppose $N(\varepsilon) = -1$. Consider the PR $u(a_0, 1)$. Suppose $\alpha(a_0, 1, p) = 2^k q$, where $q \equiv 1 \pmod 2$.

(i) $r_1 = (a + \sqrt{D})/2 = \varepsilon^m$, where $m = 2^c d$, $c \geq 1$, and $d \equiv 1 \pmod 2$.

(ii) $\alpha(a, -1, p) | \alpha(a_0, 1, p)$.

(iii) If $k = c$, then $(a, -1, p) \equiv 1 \pmod 2$ and

$$s(a, -1, p) \equiv s(a_0, 1, p) \pmod{p}.$$

Further,

$$\beta(a, -1, p) = 1 \text{ if } \alpha(a_0, 1, p) \equiv 2 \pmod{4}.$$

Moreover,

$$\beta(a, -1, p) = 2 \text{ if } \alpha(a_0, 1, p) \equiv 0 \pmod{4}.$$

(iv) If $k > c$, then $\alpha(a, -1, p) \equiv 0 \pmod{2}$ and $\beta(a, -1, p) = 2$.

(v) If $k < c$, then $\alpha(a, -1, p) \equiv 1 \pmod{2}$. If $k = 0$ and $c = 1$, then $\beta(a, -1, p) = 2$. If $c \neq 1$ and $k < c$, then $\beta(a, -1, p) = 1$.

Proof:

(i) Since $N(\varepsilon) = -1$, where ε is the fundamental unit, and

$$N(r_1) = r_1 r_2 = -b = 1,$$

it follows that $r_1 = \varepsilon^m$ where m is even.

(ii) Just as in the proof of Theorem 14(ii), we see that ε and $\bar{\varepsilon}$ are the roots of the characteristic polynomial of the PR $u(a_0, 1)$. Again, just as in equation (17) of the proof of Theorem 14(ii), it follows that

$$\alpha(a, -1, p) = \alpha(a_0, 1, p)/(m, \alpha(a_0, 1, p)). \tag{24}$$

Clearly, $\alpha(a, -1, p) \mid \alpha(a_0, 1, p)$.

(iii) Since m and $\alpha(a_0, 1, p)$ are both even and divisible by the same power of 2, it follows from equation (24) that $\alpha(a, -1, p) \equiv 1 \pmod{2}$. Since $\alpha(a_0, 1, p) \equiv 0 \pmod{2}$, it follows from Theorem 13 that $s(a_0, 1, p) \equiv \pm 1 \pmod{p}$. Now, by Theorem 6(iii),

$$s(a_0, 1, p) \equiv \varepsilon^{\alpha(a_0, 1, p)} \equiv \pm 1 \pmod{p}. \tag{25}$$

Also, by Theorem 6(iii),

$$s(a, -1, p) \equiv (r_1)^{\alpha(a, -1, p)} \equiv (\varepsilon^m)^{\alpha(a_0, 1, p)/(m, \alpha(a_0, 1, p))} \pmod{p}. \tag{26}$$

The last congruence follows by equation (24) in the proof of part (ii). However, since the same power of 2 divides both m and $\alpha(a_0, 1, p)$, it follows that

$$m/(m, \alpha(a_0, 1, p)) = r,$$

where $r \equiv 1 \pmod{2}$. Hence,

$$s(a, -1, p) \equiv [\varepsilon^{\alpha(a_0, 1, p)}]^r \equiv [s(a_0, 1, p)]^r \equiv (\pm 1)^r$$
$$\equiv \pm 1 \equiv s(a_0, 1, p) \pmod{p}.$$

Since $s(a, -1, p) \equiv s(a_0, 1, p)$, $\beta(a, -1, p) = \beta(a_0, 1, p)$. If $\alpha(a_0, 1, p) \equiv 2 \pmod{4}$, then $\beta(a_0, 1, p) = 1$ by Theorem 13(ii). Consequently, $\beta(a, -1, p) = 1$. If $\alpha(a_0, 1, p) \equiv 0 \pmod{4}$, then $\beta(a_0, 1, p) = 2 = \beta(a, -1, p)$ by Theorem 13(iii).

(iv) If $k > c$, it follows from equation (24) that $\alpha(a, -1, p) \equiv 0 \pmod{2}$. The result now follows from Theorem 16(ii).

(v) If $k < c$, it follows from equation (24) that $\alpha(a, -1, p) \equiv 1 \pmod{2}$. By (25) and (26),

$$s(a, -1, p) \equiv [\varepsilon^{\alpha(a_0, 1, p)}]^{m/(m, \alpha(a_0, 1, p))}. \tag{27}$$

If $k = 0$ and $c = 1$, then $\varepsilon^{\alpha(a_0, 1, p)} \equiv \pm\sqrt{-1} \pmod{p}$ and $\alpha(a_0, 1, p) = 4$ by Theorem 13(iv). Further,

$$m/(m, \alpha(a_0, 1, p)) \equiv 2 \pmod{4},$$

since $k = 0$ and $c = 1$. Thus, by (27),

$$s(a, -1, p) \equiv (\pm\sqrt{-1})^2 \equiv -1 \pmod{p},$$

and hence $\beta(a, -1, p) = 2$.

Now, suppose $c \neq 1$ and $k < c$. If $k = 0$, then $c \geq 2$ and

$$4 \mid m/(m, \alpha(a_0, 1, p)).$$

Then, again, $\varepsilon^{\alpha(a_0, 1, p)} \equiv \pm\sqrt{-1} \pmod p$, and by (27),

$$s(a, -1, p) \equiv [\varepsilon^{\alpha(a_0, 1, p)}]^{m/(m, \alpha(a_0, 1, p))} \equiv (\pm\sqrt{-1})^4 \equiv 1 \pmod p.$$

Thus, $\beta(a, -1, p) = 1$. If $k \neq 0$ and $k < c$, then,

$$2 \mid m/(m, \alpha(a_0, 1, p)).$$

Further, by Theorem 13 and Theorem 6(iii),

$$\varepsilon^{\alpha(a_0, 1, p)} \equiv \pm 1 \pmod p.$$

Thus, by (27),

$$s(a, -1, p) \equiv [\varepsilon^{\alpha(a_0, 1, p)}]^{m/(m, \alpha(a_0, 1, p))} \equiv (\pm 1)^2 \equiv 1 \pmod p.$$

Therefore, $\beta(a_0, 1, p) = 1$, and we are done.

Note that in Theorem 17 we obtain results for the infinite number of PR's $u(a, -1)$ which have the same square-free part of the discriminant D' by considering only one PR $u(a_0, 1)$. Since $b = 1$ for this PR, we are able to make use of Theorems 13-15. Further, note that in Theorem 17 we are able to calculate the exponent k for which $\alpha(a_0, 1, p) \equiv 2^k \pmod{2^{k+1}}$ by Theorem 12. In Theorem 18, we will consider the remaining case where $N(\varepsilon) = 1$.

Theorem 18: Consider the PR $u(a, -1)$. Let p be an odd prime such that

$$(4 - a^2/p) = (2 - a/p) = (2 + a/p) = 1.$$

Let $\varepsilon = (a_0 + c_0\sqrt{D'})/2$ be the fundamental of $Q(\sqrt{D'})$. Suppose that $N(\varepsilon) = 1$. Consider the PR $u(a_0, -1)$. Suppose that $\alpha(a_0, -1, p) = 2^k q$, where $q \equiv 1 \pmod 2$.

(i) $r_1 = (a + \sqrt{D})/2 = \varepsilon^m$, where $m = 2^c d$, $c \geq 0$, and $d \equiv 1 \pmod 2$.

(ii) $\alpha(a, -1, p) \mid \alpha(a_0, -1, p)$.

(iii) If $k = c$ and $k \geq 1$, then $\alpha(a, -1, p) \equiv 1 \pmod 2$ and $\beta(a, -1, p) = 2$.

(iv) If $k = c = 0$, then $\alpha(a, -1, p) \equiv 1 \pmod 2$. If

$$s(a_0, -1, p) = \varepsilon^{2^k q} \equiv 1 \pmod p,$$

then $\beta(a, -1, p) = 1$; otherwise, $\beta(a, -1, p) = 2$.

(v) If $k > c$, then $\alpha(a, -1, p) \equiv 0 \pmod 2$ and $\beta(a, -1, p) = 2$.

(vi) If $k < c$, then $\alpha(a, -1, p) \equiv 1 \pmod 2$ and $\beta(a, -1, p) = 1$.

Proof:

(i) This follows since $N(r_1) = r_1 r_2 = 1$ and ε is the fundamental unit of $Q(\sqrt{D'})$.

(ii) It is easy to see that ε and $\bar{\varepsilon}$ are the roots of the characteristic polynomial

$$x^2 - a_0 x + 1 = 0$$

of the PR $u(a_0, -1)$. The rest of the proof follows as in the proofs of Theorem 14(ii) and Theorem 17(ii).

(iii) Just as in the proof of Theorem 17(ii), it follows that

$$\alpha(a, -1, p) = \alpha(a_0, -1, p)/(m, \alpha(a_0, -1, p)). \tag{28}$$

Since $k = c$, it follows that $\alpha(a, -1, p) \equiv 1 \pmod 2$. Since $\alpha(a_0, -1, p) \equiv 0 \pmod 2$, it follows from Theorem 13(ii) that $\beta(a_0, -1, p) = 2$ and $s(a_0, -1, p) \equiv -1 \pmod p$. By (25) and (26), it follows that

(29) $$s(a,\ -1,\ p) \equiv s(a_0,\ -1,\ p)^{m/(m,\,\alpha(a_0,\,1,\,p))}$$
$$\equiv -1^{m/(m,\,\alpha(a_0,\,1,\,p))} \equiv -1 \pmod{p},$$

since $k = c$. Thus, $\beta(a,\ -1,\ p) = 2$.

(iv) It follows just as in the proof of part (iii) that $\alpha(a,\ -1,\ p) \equiv 1 \pmod{2}$. By (29),

$$s(a,\ -1,\ p) \equiv s(a_0,\ -1,\ p)^{m/(m,\,\alpha(a_0,\,1,\,p))}.$$

Since $k = c$ and $s(a_0,\ -1,\ p) \equiv \pm 1 \pmod{p}$ by Theorem 16, it follows that

$$s(a_0,\ -1,\ p) \equiv s(a_0,\ -1,\ p) \pmod{p}.$$

The rest follows from Theorem 6(iii).

(v) If $k > c$, it follows from (28) that $\alpha(a,\ -1,\ p) \equiv 0 \pmod{2}$. It now follows from Theorem 16(ii) that $\beta(a,\ -1,\ p) = 2$.

(vi) If $k < c$, it follows from (28) that $\alpha(a,\ -1,\ p) \equiv 1 \pmod{2}$. By (29),

$$s(a,\ -1,\ p) \equiv s(a_0,\ -1,\ p)^{m/(m,\,\alpha(a_0,\,1,\,p))}.$$

Since $k < c$, $m/(m,\ \alpha(a_0,\ -1,\ p)) \equiv 0 \pmod{2}$. Since $s(a_0,\ -1,\ p) \equiv \pm 1 \pmod{p}$, it now follows that

$$s(a,\ -1,\ p) \equiv (\pm 1)^2 \equiv 1 \pmod{p}.$$

Thus, $\beta(a,\ -1,\ p) = 1$.

In Theorem 18, we are again able to calculate the exponent k for which $\alpha(a_0,\ -1,\ p) \equiv 2^k \pmod{2^{k+1}}$ by Theorem 12. Theorem 18 just reduces the problem of finding the restricted period modulo p of a PR $u(a,\ -1)$ for which $b = -1$ to that of considering another PR $u(a_0, -1)$ for which also $b = -1$. However, since $r_1 = \varepsilon^m$, $|a_0| \le |a|$, and it is easier to work with the PR $u(a_0,\ -1)$ instead of the PR $u(a,\ -1)$.

6. THE SPECIAL CASE $r_2 = \pm 1$

In this section, we will conclude our paper by considering those PR's for which one of the characteristic roots is ± 1. Theorems 19 and 20 will treat these cases.

Theorem 19: Consider the PR $u(-b + 1,\ b)$, where $b \ne 0$ and $b \ne 1$. Then $r_1 = -b$, $r_2 = 1$, and $D = (b + 1)^2$. Let p be an odd prime such that $b \not\equiv 0$ and $b \not\equiv -1 \pmod{p}$. If $(-b/p) = 1$, let $r^2 \equiv -b \pmod{p}$, where $0 \le r \le (p - 1)/2$.

(i) $\alpha(-b + 1,\ b,\ p) = \mathrm{ord}_p(-b)$.

(ii) $\beta(-b + 1,\ b,\ p) = 1$ always; $s(-b + 1,\ b,\ p) \equiv 1 \pmod{p}$ always.

(iii) If $(-b/p) = -1$ and $p \equiv 3 \pmod{4}$, then

$$\alpha(-b + 1,\ b,\ p) = \mu(-b + 1,\ b,\ p) \equiv 2 \pmod{4}.$$

(iv) If $(-b/p) = -1$ and $p \equiv 1 \pmod{4}$, then

$$\alpha(-b + 1,\ b,\ p) = \mu(-b + 1,\ b,\ p) \equiv 0 \pmod{4}.$$

(v) If $(-b/p) = 1$ and $p \equiv 3 \pmod{4}$, then

$$\alpha(-b + 1,\ b,\ p) = \mu(-b + 1,\ b,\ p) \equiv 1 \pmod{2}.$$

(vi) If $(-b/p) = 1$, $p \equiv 1 \pmod{4}$, and

$$(-2b + (1 - b)r/p) = (-2b - (1 - b)r/p) = -1,$$

then $\alpha(-b + 1,\ b,\ p)$ is congruent to 0 or 2 modulo 4.

(vii) Suppose that $p - 1 = 2^k q$, where $q \equiv 1 \pmod{2}$. If $(-b/p) = -1$, then $\alpha(-b + 1,\ b,\ p) \equiv 2^k \pmod{2^{k+1}}$. If $(-b/p) = 1$, then $\alpha(-b + 1,\ b,\ p) \equiv 2^m \pmod{2^{m+1}}$, where $0 < m < k$ iff

$$[-b/p]_{k-m} \equiv 1 \pmod{p}, \text{ but } [-b/p]_{k-m+1} \equiv -1 \pmod{p}.$$

Further,

$$\alpha(-b+1, b, p) \equiv 1 \pmod{2} \text{ iff } [-b/p]_k \equiv 1 \pmod{p}.$$

Proof:

(i) and (ii) Since $a = -b + 1$, it easily follows that $r_1 = -b$ and $r_2 = 1$. By Theorem 6(ii), it follows that

$$\mu(-b+1, b, p) = \mathrm{ord}_p(r_1/r_2) = \mathrm{ord}_p(-b).$$

Further, by Theorem 6(i),

$$(-b+1, b, p) = [\mathrm{ord}_p(-b), \mathrm{ord}_p(1)] = \mathrm{ord}_p(-b).$$

The results now follow.

(iii)-(vi) These follow from Theorems 9 and 10.

(vii) This follows from Theorem 12 and Theorem 11.

Theorem 20: Consider the PR $u(b-1, b)$, where $b \neq 0$ and $b \neq -1$. Then $r_1 = b$, $r_2 = -1$, and $D = (b+1)^2$. Let p be an odd prime such that $b \not\equiv 0$ and $b \not\equiv -1 \pmod{p}$. Suppose $p = 2^k q$, where $k \equiv 1 \pmod{2}$. If $(-b/p) = 1$, let $r^2 \equiv -b \pmod{p}$, where $0 \leq r \leq (p-1)/2$.

(i) $\alpha(b-1, b, p) = \mathrm{ord}_p(-b)$.

(ii) $\beta(b-1, b, p) = 1$ or 2; $s(b-1, b, p) \equiv \pm 1 \pmod{p}$.

(iii) If $\alpha(b-1, b, p) \equiv 1 \pmod{2}$, then $\beta(b-1, b, p) = 2$.
If $\alpha(b-1, b, p) \equiv 0 \pmod{2}$, then $\beta(b-1, b, p) = 1$.

(iv) If $(-b/p) = -1$ and $p \equiv 3 \pmod{4}$, then

$$\alpha(b-1, b, p) = \mu(b-1, b, p) \equiv 2 \pmod{4}.$$

(v) If $(-b/p) = -1$ and $p \equiv 1 \pmod{4}$, then

$$\alpha(b-1, b, p) = \mu(b-1, b, p) \equiv 0 \pmod{4}.$$

(vi) If $(-b/p) = 1$ and $p \equiv 3 \pmod{4}$, then

$$\alpha(b-1, b, p) \equiv 1 \pmod{2} \text{ and } \mu(b-1, b, p) \equiv 2 \pmod{4}.$$

Hence, if $p \equiv 3 \pmod{4}$, then $\mu(b-1, b, p) \equiv 2 \pmod{4}$.

(vii) If $(-b/p) = 1$, $p \equiv 1 \pmod{4}$, and

$$(-2b + (b-1)r/p) = (-2b - (b-1)r/p) = -1,$$

then $\alpha(b-1, b, p)$ is congruent to 0 or 2 (mod 4).

(viii) If $(-b/p) = -1$, then $\alpha(b-1, b, p) \equiv 2^k \pmod{2^{k+1}}$.
If $(-b/p) = 1$, then $\alpha(b-1, b, p) \equiv 2^m \pmod{2^{m+1}}$, where $0 < m < k$ iff

$$[-b/p]_{k-m} \equiv 1 \pmod{p}, \text{ but } [-b/p]_{k-m+1} \equiv -1 \pmod{p}.$$

Further, $\alpha(b-1, b, p) \equiv 1 \pmod{2}$ iff $[-b/p]_k \equiv 1 \pmod{p}$.

Proof:

(i)-(iii) If $a = b - 1$, it follows that $r_1 = b$ and $r_2 = -1$. Now, by Theorem 6(i),

$$\mu(b-1, b, p) = [\mathrm{ord}_p(b), \mathrm{ord}_p(-1)].$$

If $\mathrm{ord}_p(b) \equiv 0 \pmod{4}$, then $\mathrm{ord}_p(b) = \mathrm{ord}_p(-b) = \mu(b-1, b, p)$.
If $\mathrm{ord}_p(b) \equiv 2 \pmod{4}$, then $\mathrm{ord}_p(-b) \equiv 1 \pmod{2}$.

Thus,

$$\mathrm{ord}_p(b) = \mu(b-1, b, p) = 2 \cdot \mathrm{ord}_p(-b).$$

If $\mathrm{ord}_p(b) \equiv 1 \pmod{2}$, then $\mathrm{ord}_p(-b) \equiv 2 \pmod{4}$.

Hence,

$$\mathrm{ord}_p(-b) = 2 \cdot \mathrm{ord}_p(b) = \mu(b-1, b, p).$$

Now, by Theorem 6(ii),

$$\alpha(b-1, b, p) = \mathrm{ord}_p(r_1/r_2) = \mathrm{ord}_p(-b).$$

Thus, by our above argument, if $\alpha(b-1, b, p) \equiv 0 \pmod 2$, then

$$\alpha(b-1, b, p) = \mu(b-1, b, p), \text{ and } \beta(b-1, b, p) = 1.$$

If $\alpha(b-1, b, p) \equiv 1 \pmod 2$, then

$$\mu(b-1, b, p) = 2\alpha(b-1, b, p), \text{ and } \beta(b-1, b, p) = 2.$$

The results of parts (i)-(iii) now follows.

(iv)-(vii) These follow from Theorems 9 and 10.

(viii) This follows from Theorems 11 and 12.

REFERENCES

1. Robert P. Backstrom. "On the Determination of the Zeros of the Fibonacci Sequence." *The Fibonacci Quarterly* 4, No. 4 (1966):313-322.
2. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n + \beta^n$." *Annals of Mathematics*, 2nd Ser. 15 (1913):30-70.
3. John H. Halton. "On the Divisibility Properties of Fibonacci Numbers." *The Fibonacci Quarterly* 4, No. 3 (1966):217-240.
4. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Annals of Mathematics*, 2nd Ser. 31 (1930):419-448.
5. Emma Lehmer. "On the Quadratic Character of the Fibonacci Root." *The Fibonacci Quarterly* 4, No. 2 (1966):135-138.
6. Emma Lehmer. "On the Quadratic Character of Some Quadratic Surds." *Journal für die Reine und Angewandte Mathematik* 250 (1971):42-48.
7. Emma Lehmer. "On Some Special Quartic Reciprocity Laws." *Acta Arithmetica* 21 (1972):367-377.
8. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* 1 (1878):184-240, 289-321.
9. D. W. Robinson. "The Fibonacci Matrix Modulo m." *The Fibonacci Quarterly* 1, No. 1 (1963):29-36.
10. Lawrence Somer. "Which Second-Order Recurrences Have Almost All Primes as Divisors?" *The Fibonacci Quarterly* 17, No. 2 (1979):111-116.
11. John Vinson. "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence." *The Fibonacci Quarterly* 1, No. 1 (1963):37-45.
12. D. D. Wall. "Fibonacci Series Modulo m." *American Mathematical Monthly* 67 (1960):525-532.
13. Morgan Ward. "The Prime Divisors of Fibonacci Numbers." *Pacific Journal of Mathematics* 11 (1961):379-386.
14. O. Wyler. "On Second-Order Recurrences." *American Mathematical Monthly* 72 (1965):500-506.

*****

MIXING PROPERTIES OF MIXED CHEBYSHEV POLYNOMIALS

CLARK KIMBERLING
University of Evansville, Evansville, Indiana 47702

The *Chebyshev polynomials of the first kind*, defined recursively by

$$t_0(x) = 1,\ t_1(x) = x,\ t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x) \text{ for } n = 2, 3, \ldots,$$

or equivalently, by

$$t_n(x) = \cos(n \cos^{-1} x) \text{ for } n = 0, 1, \ldots,$$

commute with one another under composition; that is

$$t_m(t_n(x)) = t_n(t_m(x)).$$

In [1], Adler and Rivlin use this well-known fact to prove that in an appropriate measure-theoretic setting the mappings $t_1, t_2, \ldots$ are measure-preserving and the sequence $\{t_1, t_2, \ldots\}$ is strongly mixing. In another setting, Johnson and Sklar [2] obtain related results. The purpose of the present note is to establish results analogous to those in [1] for sequences involving not only t_n's but also the *Chebyshev polynomials of the second kind*; these are defined recursively by

$$u_0(x) = 1,\ u_1(x) = 2x,\ u_n(x) = 2xu_{n-1}(x) = u_{n-2}(x) \text{ for } n = 2, 3, \ldots,$$

or equivalently, by

$$u_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sqrt{1 - x^2}} \text{ for } n = 0, 1, \ldots .$$

Concerning compositions of Chebyshev polynomials of both kinds, we have the following lemma from [3], where a trigonometric proof may be found.

Lemma 1: Let $\{t_0, t_1, \ldots\}$ and $\{u_0, u_1, \ldots\}$ be the sequences of Chebyshev polynomials of the first and second kinds, respectively. Put $\overline{u}_{-1}(x) \equiv 0$ and define

$$\overline{u}_n(x) = u_n(x)\sqrt{1 - x^2} \text{ for } n = 0, 1, \ldots .$$

Then for nonnegative m and n,

$$t_m(t_n) = t_{mn}, \tag{1}$$

$$\overline{u}_m(t_n) = \overline{u}_{mn+n-1}, \tag{2}$$

$$t_m(\overline{u}_n) = \begin{cases} (-1)^{\frac{m}{2}} t_{mn+n} & \text{for even } m \\ (-1)^{\frac{m-1}{2}} \overline{u}_{mn+m-1} & \text{for odd } m, \end{cases} \tag{3}$$

$$\overline{u}_m(\overline{u}_n) = \begin{cases} (-1)^{\frac{m}{2}} t_{(m+1)(n+1)} & \text{for even } m \\ (-1)^{\frac{m-1}{2}} \overline{u}_{mn+m+n} & \text{for odd } m. \end{cases} \tag{4}$$

We introduce some notation:

I = the closed interval $[-1, 1]$
I' = the closed interval $[0, \pi]$
$\mathcal{B}$ = the family of Borel subsets of I
$\mathcal{B}'$ = the family of Borel subsets of I'
λ = Legesgue measure on $\mathcal{B}$
λ' = Lebesgue measure on $\mathcal{B}'$

Let μ be the measure defined on $\mathcal{B}$ by the Lebesgue integral

$$\mu(B) = \frac{2}{\pi}\int_B \frac{dx}{\sqrt{1 - x^2}},\ B \in \mathcal{B}.$$

Rivlin [4] proves that each t_n for $n \geq 1$ preserves the measure μ; that is, the inverse mapping t_n^{-1}, which is an n-valued mapping (except at ± 1) from I' onto I, satisfies

$$\mu(t_n^{-1}(B)) = \mu(B),\ B \in \mathcal{B}.$$

Using the same method of proof, we establish the following lemma.

Lemma 2a: Let $\overline{u}_n = u_n(x)\sqrt{1 - x^2}$ for $n = 0, 1, \ldots$. For odd n, the mapping $\overline{u}_n$ preserves the measure μ on $\mathcal{B}$.

Proof: Let ϕ be the one-to-one measurable mapping of I onto I' defined by

$$\phi(x) = \theta = \cos^{-1} x,$$

and put $v_n = \phi(\overline{u}_n(\phi^{-1}))$. Then, for odd n and

$$\frac{(2k+1)\pi}{2(n+1)} \le \theta \le \frac{(2k+3)\pi}{2(n+1)},\ k = 0, 1, \ldots, n-1,$$

we find

$$v_n(\theta) = \begin{cases} -(n+1)\theta + \frac{\pi}{2}, & 0 \le \theta \le \frac{\pi}{2(n+1)} \\ (n+1)\theta - \frac{2k+1}{2}\pi, & \text{even } k \\ -(n+1)\theta + \frac{2k+3}{2}\pi, & \text{odd } k \\ -(n+1)\theta + \frac{2n+3}{2}\pi, & \frac{(2n+1)\pi}{2(n+1)} \le \theta \le \pi. \end{cases}$$

An open subinterval of $[0, \pi/2]$ or $[\pi/2, \pi]$ having length ℓ is the image under v_n of $n+1$ subintervals of I' (on the horizontal axis in Figure 1) in case n is odd, where each of these subintervals has length $\ell/(n+1)$. It follows that the mapping v_n preserves the measure λ'. Now, if $-1 \le a < b < 1$, then

$$\int_a^b \frac{dx}{\sqrt{1-x^2}} = \int_{\phi(b)}^{\phi(a)} d\theta,$$

so that $\mu(B) = \frac{2}{\pi}\lambda'(\phi(B))$ for $B \in \mathcal{B}$. Consequently (omitting parentheses),

$$\mu(\overline{u}_n^{-1}(B)) = \frac{2}{\pi}\lambda'(\phi\overline{u}_n^{-1}B) = \frac{2}{\pi}\lambda'(\phi\overline{u}_n^{-1}\phi^{-1}\phi B) = \frac{2}{\pi}\lambda'(v_n^{-1}\phi B) = \frac{2}{\pi}\lambda'(\phi B) = \mu(B).$$

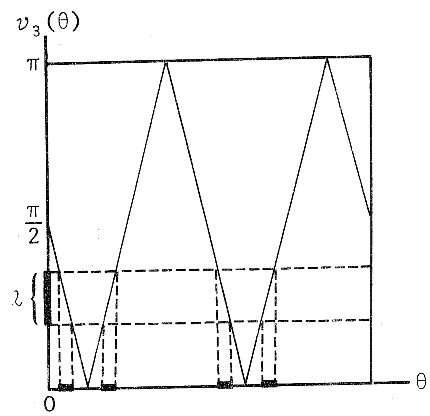

Fig. 1. v_3 preserves λ' on $[0, \pi]$.

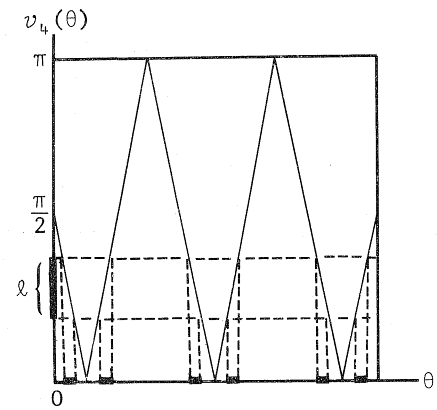

Fig. 2. v_4 preserves λ' on $\left[0, \frac{4\pi}{5}\right]$.

For even n, the result is not so simple, since in this case v_n fails to preserve λ' on all of I'. However, one may prove the following lemma with an argument similar to that just given.

Lemma 2b: Let $\overline{u}_n(x) = u_n(x)\sqrt{1-x^2}$ for $n = 0, 1, \ldots$. For even n, the mapping $\overline{u}_n$ preserves the restriction of the measure μ to the family of Borel sets of the closed interval $\left[\cos^{-1}\frac{n\pi}{n+1}, 1\right]$. (See Figure 2.)

Turning now to orthogonality of Chebyshev polynomials of both kinds, let $L^2(I, \mathcal{B}, \mu)$ denote the set of square μ-integrable functions f which are μ-measurable on $\mathcal{B}$:

$$\int_{-1}^{1} f^2(x)d\mu(x) < \infty.$$

For f and g in $L^2(I, \mathcal{B}, \mu)$, let $\langle f, g\rangle$ denote the inner product

$$\frac{2}{\pi}\int_{-1}^{1} f(x)g(x)d\mu(x),$$

and let $\|f\|$ denote the norm $\langle f, f\rangle^{1/2}$.

Lemma 3: Let $\{t_0, t_1, \ldots\}$ and $\{u_0, u_1, \ldots\}$ be the sequences of Chebyshev polynomials of the first and second kinds, respectively. Put

$$\overline{u}_n(x) = u_n(x)\sqrt{1 - x^2} \text{ for } n = 0, 1, \ldots .$$

Then for nonnegative m and n,

$$\langle t_m, t_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \neq 0 \\ 2 & m = n = 0 \end{cases} \tag{5}$$

$$\langle \overline{u}_m, \overline{u}_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{6}$$

$$\langle \overline{u}_m, t_n\rangle = \begin{cases} 0 & m + n \text{ odd} \\ \dfrac{4(m+1)}{\pi[(m+1)^2 - n^2]} & m + n \text{ even} \end{cases} \tag{7}$$

Proof: Equations (5) and (6) are well known. Proof of (7) follows from

$$\int_0^{\pi} \sin(m+1)\theta \cos n\theta \, d\theta = \frac{1}{2}\int_0^{\pi} [\sin(m+1-n)\theta + \sin(m+1+n)\theta]d\theta,$$

where $\cos\theta = x$.

Lemma 3 shows that the sequences

$$\left\{\frac{1}{\sqrt{2}}t_0, t_1, t_2, \ldots\right\} \text{ and } \{\overline{u}_0, \overline{u}_1, \overline{u}_2, \ldots\}$$

are orthonormal over I, a well-known fact. It is well known, a fortiori, that these are complete orthonormal sets in the space $L^2(I, \mathcal{B}, \mu)$; i.e., for each f in $L^2(I, \mathcal{B}, \mu)$ and $\varepsilon > 0$, there exists a finite linear combination

$$s_n(x) = \sum_{k=0}^{n} a_k t_k(x)$$

such that $\| f - s_n \| < \varepsilon$ [and similarly for the $\overline{u}_k(x)$'s].

Now let $\{F_n\} = \{F_0, F_1, F_2, \ldots\}$ denote the sequence

$$\frac{1}{\sqrt{2}}t_0, \overline{u}_1, t_2, \overline{u}_3, \ldots$$

and let $\{G_n\} = \{G_0, G_1, G_2, \ldots\}$ denote the sequence

$$\{\overline{u}_0, t_1, \overline{u}_2, t_3, \ldots\}.$$

These are orthonormal sequences by Lemma 3. For f in $L^2(I, \mathcal{B}, \mu)$, we define the *F-Chebyshev series* for f to be the series

$$\sum_{k=0}^{\infty} f_k F_k(x),$$

where the coefficients $f_0, f_1, \ldots$ are given by $f_k = \langle f, F_k \rangle$. Similarly, the *G-Chebyshev series* for given g in $L^2(I, \mathcal{B}, \mu)$ is defined by

$$\sum_{k=0}^{\infty} g_k G_k(x),$$

where $g_k = \langle g, G_k \rangle$ for $k = 0, 1, \ldots$.

Lemma 4: If n is an odd positive integer and $\varepsilon > 0$, then there exists a sum of the form

$$s_m(x) = \sum_{k=0}^{m} a_{2k+1}\overline{u}_{2k+1}(x)$$

such that $\| t_n - s_m \| < \varepsilon$. If n is an even nonnegative integer and $\varepsilon > 0$, then there exists a sum of the form

$$s_m(x) = \sum_{k=0}^{m} a_{2k} t_{2k}$$

such that $\| \overline{u}_n - s_m \| < \varepsilon$.

Proof: Suppose that n is an odd positive integer. It suffices, by the Riesz-Fischer Theorem (see [5], p. 127) to show that the sequence $\tau_{2k+1} = \langle t_n, \overline{u}_{2k+1} \rangle$ satisfies

$$\sum_{k=0}^{\infty} \tau_{2k+1}^2 < \infty.$$

This is clearly the case, since, by (7),

$$\tau_{2k+1} = \frac{8}{\pi} \frac{k+1}{[(2k+2)^2 - n^2]}.$$

Similarly, for even nonnegative n and $\tau_{2k} = \langle \overline{u}_n, t_{2k} \rangle$, we have

$$\tau_{2k} = \frac{4}{\pi} \frac{n+1}{(n+1)^2 - 4k^2}.$$

Theorem 1: The orthonormal sequences $\{F_n\}$ and $\{G_n\}$ for $n = 0, 1, \ldots$ are complete in $L^2(I, \mathcal{B}, \mu)$.

Proof: We deal first with $\{F_n\}$. Suppose $f \in L^2(I, \mathcal{B}, \mu)$ and $\varepsilon > 0$. Since

$$\left\{\frac{1}{\sqrt{2}} t_0, t_1, t_2, \ldots\right\}$$

is a complete orthonormal sequence in $L^2(I, \mathcal{B}, \mu)$, we choose odd m and numbers $a_0, a_1, \ldots, a_m$ satisfying

$$\left\| f - \sum_{k=0}^{m} a_k t_k \right\| < \varepsilon/2.$$

By Lemma 4, there exist sums $s_k = c_{k1}\overline{u}_1 + c_{k3}\overline{u}_3 + \cdots + c_{kq_k}\overline{u}_{q_k}$ such that

$$\| a_k t_k - a_k s_k \| < \varepsilon/m \text{ for } k = 1, 3, 5, \ldots, m.$$

Let $Q = \max\{q_k : k = 1, 3, 5, \ldots, m\}$ and put

$$q = \begin{cases} Q & \text{if } Q \text{ is odd} \\ Q + 1 & \text{if } Q \text{ is even.} \end{cases}$$

Put $c_{kp} = 0$ for $q_k < p \leq q$, $k = 1, 3, 5, \ldots, m$. Next, let

$$b_j = \begin{cases} a_1c_{1j} + a_3c_{3j} + \cdots + a_mc_{mj} & \text{for } j = 1, 3, 5, \ldots, q \\ a_j & \text{for even } j < m \\ 0 & \text{for even } j > m. \end{cases}$$

Then,

$$\begin{aligned} \| f - (b_0t_0 + b_1\overline{u}_1 + \cdots + b_q\overline{u}_q) \| \leq & \| f - b_0t_0 - a_1t_1 - b_2t_2 - a_3t_3 - \cdots - a_mt_m \| \\ & + \| a_1t_1 - a_1(c_{11}\overline{u}_1 + \cdots + c_{1q}\overline{u}_q) \| \\ & + \| a_3t_3 - a_3(c_{31}\overline{u}_1 + \cdots + c_{3q}\overline{u}_q) \| + \cdots \\ & + \| a_mt_m - a_m(c_{m1}\overline{u}_1 + \cdots + c_{mq}\overline{u}_q) \| < \varepsilon. \end{aligned}$$

This proves completeness of the sequence $\{F_n\}$. The proof for $\{G_n\}$ is quite similar.

We wish to use all the foregoing results to prove that the sequences of mappings $\{F_n^{-1}\}$, $\{G_n^{-1}\}$, and $\{\overline{u}_n^{-1}\}$, when applied to any B in $\mathcal{B}$, *increasingly homogenize* or *mix* B throughout I. This vague description is made precise for a μ-preserving sequence of mappings $\{\tau_n\}$ by the notion that $\{\tau_n\}$ is a *strongly mixing sequence with respect to* μ if

$$\lim_{n\to\infty} \mu[(\tau_n^{-1}A) \cap B] = \frac{\mu(A)\mu(B)}{\mu(I)} \tag{8}$$

for all A and B in $\mathcal{B}$.

<u>*Theorem 2*</u>: The sequence of mappings $\{F_1, F_2, \ldots\}$ is strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

<u>*Proof*</u>: To establish (8), it suffices to prove

$$\lim_{n\to\infty} \langle f(F_n), g\rangle = \frac{1}{2}\langle f, 1\rangle\langle g, 1\rangle \tag{9}$$

for all f and g in $L^2(I, \mathcal{B}, \mu)$, since (9) is merely a restatement of (8) in case f is the characteristic function of A and g is the characteristic function of B. [That is, $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$; similarly for g and B.] First, assume f and g are terms of the sequence $\{F_0, F_1, \ldots\}$. Then for some $j \geq 0$ and $k \geq 0$, with $n \geq 1$, Lemmas 1 and 3 show that

$$\begin{aligned} \langle f(F_n), g\rangle &= \langle F_j(F_n), F_k\rangle \\ &= \begin{cases} \langle t_{jn}, F_k\rangle & j \text{ even, } n \text{ even, } j \neq 0 \\ \langle t_0/\sqrt{2}, F_k\rangle & j = 0 \\ (-1)^{j/2}\langle t_{jn+j}, F_k\rangle & j \text{ even, } n \text{ odd, } j \neq 0 \\ \langle \overline{u}_{jn+n-1}, F_k\rangle & j \text{ odd, } n \text{ even} \\ (-1)^{\frac{j-1}{2}}\langle \overline{u}_{jn+j+n}, F_k\rangle & j \text{ odd, } n \text{ odd} \end{cases} \\ &= \begin{cases} 1 & 0 \neq k = jn, \quad j \text{ even, } n \text{ even} \\ \sqrt{2} & 0 = j = k \\ (-1)^{j/2} & k = (j+1)n, \quad j \text{ even, } n \text{ odd} \\ (-1)^{\frac{j-1}{2}} & k = (j+1)n + j, \ j \text{ odd, } n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$\lim_{n\to\infty}\langle f(F_n),\ g\rangle = 0 \text{ for } j > 0,$$

and in this case (9) clearly holds. If $j = 0$, then (9) is satisfied by

$$\langle f(F_n),\ g\rangle = 1 \text{ for all } n \geq 1.$$

We have shown so far that (9) holds if f and g are both terms of the sequence $\{F_0, F_1, \ldots\}$. We continue now as in Rivlin [4, p. 171]: Suppose f and g are *any* functions in $L^2(I, \mathcal{B}, \mu)$ and let $\varepsilon > 0$. By Theorem 1, there exist finite linear combinations u and v of the mappings F_n such that

$$\|f - u\| < \varepsilon^2 \quad \text{and} \quad \|g - v\| < \varepsilon^2 . \tag{10}$$

We write

$$\begin{aligned} C &= \langle f(F_n),\ g\rangle - \tfrac{1}{2}\langle f,\ 1\rangle\langle g,\ 1\rangle \\ &= [\langle f(F_n) - u(F_n),\ g - v\rangle + \langle v,\ f(F_n) - u(F_n)\rangle + \langle u(F_n),\ g - v\rangle] + \\ &\quad \left[\langle u(F_n),\ v\rangle - \tfrac{1}{2}\langle u,\ 1\rangle\langle v,\ 1\rangle\right] + \left[\tfrac{1}{2}\langle u,\ 1\rangle\langle v,\ 1\rangle - \tfrac{1}{2}\langle f,\ 1\rangle\langle g,\ 1\rangle\right] \\ &= [J] + [K] + [L]. \end{aligned}$$

Since F_n is measure perserving,

$$\|f(F_n) - u(F_n)\| = \|f - u\| \text{ and } \|u(F_n)\| = \|u\| .$$

(See, for example, [4, p. 169].) Thus, the Schwarz inequality with (10) shows that $|J| < j\varepsilon$ for some constant $j > 0$. For large enough n, $|K| < \varepsilon$ since the theorem is already proved for u and v. Now

$$L = \tfrac{1}{2}[\langle f - u,\ 1\rangle\langle g - v,\ 1\rangle - \langle g,\ 1\rangle\langle f - u,\ 1\rangle - \langle f,\ 1\rangle\langle g - v,\ 1\rangle],$$

so that $|L| < \ell\varepsilon$ for some constant $\ell > 0$, again by the Schwarz inequality and (10). Thus $|C| < (1 + j + \ell)\varepsilon$ for large enough n, and this proves the theorem.

Is the sequence $\{G_1, G_2, \ldots\}$ strongly mixing, too? This question is presumptuous, since "strongly mixing" has been defined only for measure-preserving (on I) mappings. However, while no single G_n is measure-preserving on all of I, Lemma 2b shows G_n to be measure-preserving on

$$\left[\cos^{-1}\frac{n\pi}{n+1},\ 1\right],$$

and since "strongly mixing" involves $\lim\limits_{n\to\infty}$, we are led to the following definition:

> A sequence of mappings $\{\tau_n\}$, not necessarily measure-preserving on I, is *limit-strongly mixing* if (8) holds for all f and g in $L^2(I, \mathcal{B}, \mu)$.

One may now prove the following two theorems, using Lemma 2b and a modification of the proof of Theorem 2.

Theorem 3: The sequence $\{G_1, G_2, \ldots\}$ is limit-strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

Theroem 4: The sequence $\{\overline{u}_1, \overline{u}_2, \ldots\}$ is limit-strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

Finally, we note that the *mapping* F_n, for $n \geq 1$, is *strongly mixing* and, therefore, *ergodic* in the sense given in [4, p. 169]. In the limiting sense of Theorems 3 and 4 above, the same properties hold for the mappings G_n and $\overline{u}_n$ for $n \geq 1$.

REFERENCES

1. R. L. Adler & T. J. Rivlin. "Ergodic and Mixing Properties of Chebyshev Polynomials." *Proc. Amer. Math. Soc.* 15 (1964):794-796.
2. P. Johnson & A. Sklar. "Recurrence and Dispersion under Iteration of Čebyšev Polynomials." To appear.
3. C. H. Kimberling. "Four Composition Identities for Chebyshev Polynomials." This issue, pp. 353-369.
4. T. J. Rivlin. *The Chebyshev Polynomials*. New York: Wiley, 1974.
5. A. Zygmund. *Trigonometric Series*. I. Cambridge: Cambridge Univ. Press, 1969.

*****

ON THE CONVERGENCE OF ITERATED EXPONENTIATION—I

MICHAEL CREUTZ and R. M. STERNHEIMER*
Brookhaven National Laboratory, Upton, NY 11973

We have investigated the properties of the function $f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ with an infinite number of x's in the region $0 < x < e^{1/e}$. We have also defined a class of functions $F_n(x)$ which are a generalization of $f(x)$, and which exhibit the property of "dual convergence," i.e., convergence to different values of $F_n(x)$ as $n \to \infty$, depending upon whether n is even or odd.

An elementary exercise is to find a positive x satisfying

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2 \tag{1}$$

when an infinite number of exponentiations is understood [1], [2]. The standard solution is to note that the exponent of the first x must be 2, and thus $x = \sqrt{2}$. Indeed, the sequence f_n defined by

$$f_0 = 1, \qquad f_{n+1} = 2^{f_n/2} \tag{2}$$

does converge to 2 as n goes to infinity. Now consider the problem

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{1}{3}. \tag{3}$$

By analogy, one might assume that

$$x = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

is the solution; however, this is too naive because the sequence f_n defined by

$$f_0 = 1, \qquad f_{n+1} = \left(\frac{1}{27}\right)^{f_n} \tag{4}$$

does not converge.

The purpose of this article is to discuss some criteria for convergence of sequences of the form

---

*This article was authored under contract EY-76-C-02-0016 with the U.S. Department of Energy.

(5) $$f_n = g_1^{g_2^{\cdot^{\cdot^{\cdot^{g_n}}}}}$$

where g_i is some given sequence of positive numbers. Applying these criteria to the case where $g_i = x$ for all i, we will show convergence of the resulting sequence for x in the range

$$\left[\left(\frac{1}{e}\right)^e,\ e^{\left(\frac{1}{e}\right)}\right]$$

where e is the base of the natural logarithm. For x larger than $e^{\frac{1}{e}}$, the sequence f_n diverges to infinity, while for x in the range $(0,\ e^{-e})$ the even and odd sequences f_{2n} and f_{2n+1} both converge, but to different values. This property of "dual convergence" occurs for many starting sequences g_j, some of which we will discuss briefly.

Before proceeding, we should comment on the order in which the exponentiations of equation (5) are to be carried out. Rather than insert cumbersome parentheses, we will understand throughout this paper that this expression is to be evaluated "from the top down." More precisely g_{n-1} is taken to the g_nth power, g_{n-2} is taken to the resulting power, and so on. The only other simple specification of the ordering of the exponentiations is "from the bottom up," but this merely reduces to g_1 raised to the product of the remaining g's.

It is convenient at this point to introduce a shorthand notation for expression of the form in equation (5). We thus write for $m \geq n$,

(6) $$\mathop{\Xi}_{j=n}^{m} g_j = g_n^{g_{n+1}^{\cdot^{\cdot^{\cdot^{g_m}}}}}$$

A simple recursive definition of this quantity is

(7) $$\mathop{\Xi}_{j=n}^{m} g_j = \begin{cases} g_n, & n = m \\ \exp\left\{\left(\mathop{\Xi}_{j=n+1}^{m} g_j\right)\cdot \log g_n\right\}, & m > n \end{cases}$$

We now prove two theorems on the convergence of these sequences.

Theorem 1: If there exists a positive integer i such that for all $j \geq i$ we have $1 \leq g_j \leq e^{\frac{1}{e}}$, then the sequence $\mathop{\Xi}_{j=1}^{n} g_j$ converges as $n \to \infty$.

Proof: When $n > i$, we have

(8) $$\mathop{\Xi}_{j=1}^{n} g_j = g_1^{\cdot^{\cdot^{\cdot^{g_{i-1}^{\left(\mathop{\Xi}_{j=i}^{n} g_j\right)}}}}}$$

consequently, we need only prove the theorem when all g_i lie in the range $\left[1,\ e^{\frac{1}{e}}\right]$. In this case, $\mathop{\Xi}_{j=1}^{n} g_j$ is easily shown to be a monotonic increasing function of any g_i. This, in turn, implies $\mathop{\Xi}_{j=1}^{n} g_j > \mathop{\Xi}_{j=1}^{n-1} g_j$; i.e., we have an increasing sequence. However, the sequence is also bounded because

(9) $$\mathop{\Xi}_{j=1}^{n} g_j \le \mathop{\Xi}_{1}^{n}\left(e^{\frac{1}{e}}\right) < \left(e^{\frac{1}{e}}\right)^{\cdot^{\cdot^{\cdot^{\left(e^{\frac{1}{e}}\right)^e}}}} = e.$$

Now, by an elementary theorem [3], any bounded and monotonic sequence is convergent.

Theorem 2: If there exists a positive integer i such that for all $j > i$ we have $0 < g_j \le 1$, then the even and odd sequences

$$\mathop{\Xi}_{j=1}^{2n} g_j \quad \text{and} \quad \mathop{\Xi}_{j=1}^{2n+1} g_j$$

are both convergent as $n \to \infty$.

Proof: Again, we need only prove the theorem when all g_i are in the range $[0, 1]$. Also, we need only consider the even sequence because the odd sequence is merely g_1 raised to an even sequence. Now for x and y in the range $[0, 1]$ the quantity x^y is a monotonic decreasing function of y. Using this inductively on f_{2n}, we find f_{2n} is a monotonic decreasing function of g_{2n}. If we now replace g_{2n} with

$$g_{2n}^{\,g_{2n+1}^{\,g_{2n+2}}} > g_2$$

we can conclude that

(10) $$f_{2n+2} < f_{2n}.$$

However, f_{2n} is always bounded below by zero. Thus, we again have a monotonic bounded sequence which must converge.

With the help of these theorems we now return to the case $g_i = x$ independent of i. We state the result as a theorem.

Theorem 3: For positive x,

$$\mathop{\Xi}_{1}^{n} x \text{ converges as } n \to \infty \text{ iff } x \text{ lies in the interval } \left[\left(\frac{1}{e}\right)^e,\ e^{\frac{1}{e}}\right].$$

Proof: For x in the interval $\left[1,\ e^{\frac{1}{e}}\right]$, Theorem 1 immediately implies convergence. For x larger than $e^{\frac{1}{e}}$ the sequence cannot converge because, if it did, it would converge to a solution f of the equation (see [4])

(11) $$x^f - f = 0.$$

Whenever $x > e^{\frac{1}{e}}$, the lefthand side of this equation is strictly positive for all real f and the equation has no solution. The curves of x versus f as obtained (see [2]) from equation (11) for $f < e$ are shown in Figures 1 and 2, which pertain to $x > 1$ and $x < 1$, respectively.

When $x < 1$ Theorem 2 applies and we have convergence of the even and odd sequences. Both these sequences must converge to solutions f of the equation

(12) $$x^{x^f} - f = 0.$$

We will now show that, for $\left(\frac{1}{e}\right)^e \le x < 1$, this equation has only one solution and therefore the even and odd sequences converge to the same number. Take the derivative of the lefthand side of equation (12) with respect to f,

(13) $$\frac{d}{df}(x^{x^f} - f) = \log^2 x \cdot x^f \cdot x^{x^f} - 1.$$

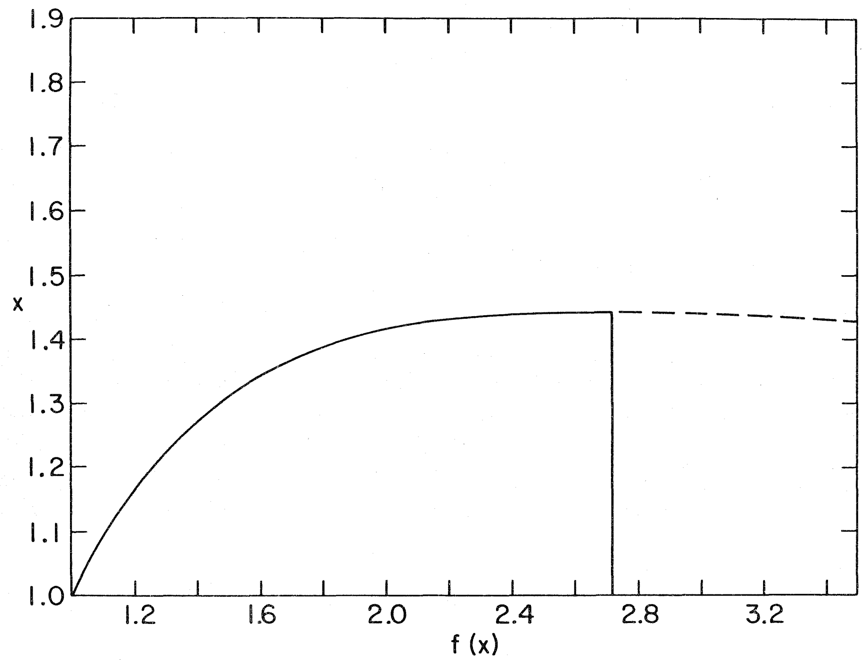

Fig. 1. The variable x as a function of $f(x)$, with $f(x)$ defined by (11), for values of $f(x)$ in the region $1 < f(x) < e$. The dashed part of the curve to the right of $f(x) = e$ is not meaningful.

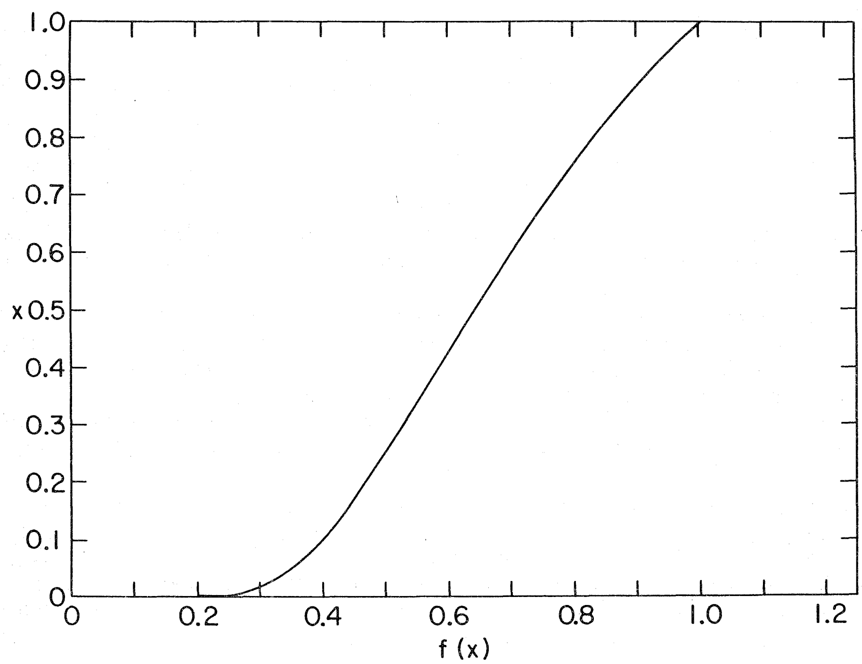

Fig. 2. The variable x as a function of $f(x)$, with $f(x)$ defined by (11) for values of $f(x)$ in the region $0 < f(x) < 1$.

Keeping $x < 1$ and maximizing the righthand side of (13) over f we obtain

$$\frac{d}{df}(x^{x^f} - f) \leq -\frac{1}{e} \log x - 1. \tag{14}$$

If the righthand side of this inequality is negative, i.e., when

$$1 > x \geq \left(\frac{1}{e}\right)^e, \tag{15}$$

then the quantity $x^{x^f} - f$ is a monotonic decreasing function of f and can only vanish at one point. This value of f is the number to which both the even and odd sequences must converge.

Finally we show by contradiction that $\mathop{\Xi}\limits_1^n x$ cannot converge for $x < \left(\frac{1}{e}\right)^e$. Assume it does converge to some number f which must satisfy (11). Define the sequence ε_n by

$$\varepsilon_n = f_n - f. \tag{16}$$

In the proof of Theorem 2 we showed the even and odd sequences are both monotonic, and thus ε_n cannot vanish for finite n. The relation between ε_{n+1} and ε_n is

$$\varepsilon_{n+1} = x^{f+\varepsilon_n} - f. \tag{17}$$

Expanding in powers of ε_n and using equation (11) gives

$$\varepsilon_{n+1} = \varepsilon_n \log f + \mathcal{O}(\varepsilon_n^2). \tag{18}$$

Consequently the sequence cannot converge if $|\log f| > 1$ which corresponds to $x < \left(\frac{1}{e}\right)^e$. This completes the proof of Theorem 3.

We now return to the case of general g_j in equation (5). The above discussion of $g_j = x$ shows that under the conditions of Theorem 2, the limits of the even and odd sequences are not in general equal. The special role played by $\left(\frac{1}{e}\right)^e$ impels us to conjecture that the simple convergence of Theorem 1 may be extended for g_j in the range $\left[\left(\frac{1}{e}\right)^e, e^{\frac{1}{e}}\right]$, but we have no proof of this. Note that neither Theorem 1 nor 2 needs any assumption of the existence of a limit for g_j; this suggests it might be amusing to study g_j alternately inside and outside the above region.

In an informal report [5], we have studied several sequences where g_j goes to zero as j goes to infinity. In general upon iterated exponentiation these give rise to dual convergent sequences in the sense of Theorem 2, the even and odd sequences both converging to different numbers. As a particular example take $g_j = \frac{x}{j^2}$, and consider $\mathop{\Xi}\limits_{j=1}^n g_j$ as a function of x. In Figure 3, we have plotted this function versus x for $n = 10$ and 11. Increasing n further makes no visually discernible difference between the curves; even n essentially reproduce the $n = 10$ curve and odd n the $n = 11$ curve. Note the crossing points at $x = 1$ and 4 where one of the g_j is one and therefore the sequence converges after a finite number of steps.

In [5] we have also considered the sequence resulting from $g_j = jx$. Here g_j goes to infinity as j does; nonetheless, the resulting $\mathop{\Xi}\limits_{j=1}^n g_j$ converges as long as x is less than one. The amusing function resulting is piecewise continuous with discontinuities at $x = \frac{1}{k}$ where k is any positive integer. Three different values for $\mathop{\Xi}\limits_{j=1}^{n}(xj)$ are obtained by taking $x = \frac{1}{k}$ and $x = \frac{1}{k} \pm \varepsilon$ in the limit of vanishing ε.

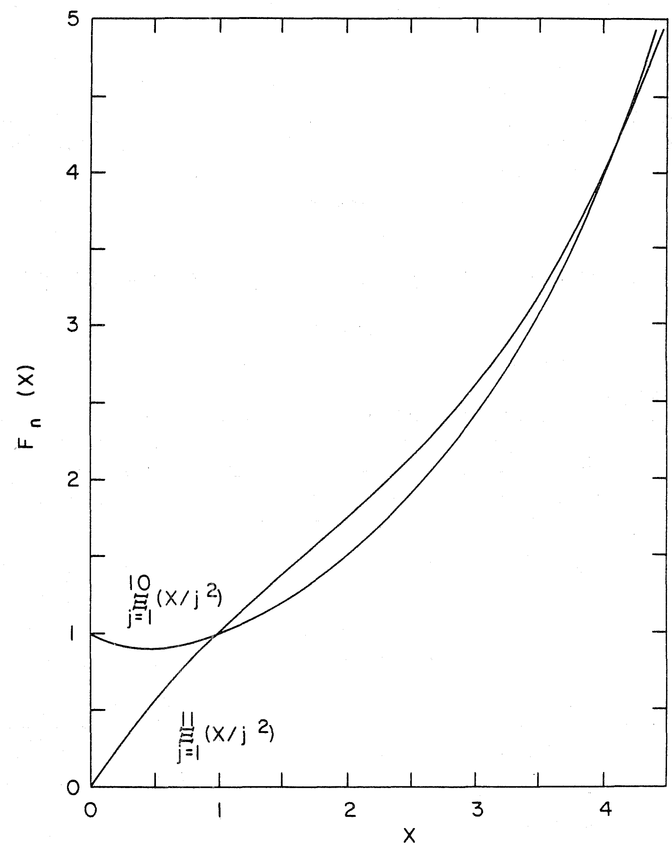

Fig. 3. The function $F_n(x) = \mathop{\Xi}\limits_{j=1}^{n}(x/j^2)$ *for* $n = 10$ *and* $n = 11$, *showing the dual convergence of* $F_n(x)$. *We note the "crossing points" at* $x = 1$ *and* $x = 4$, *where the two functions are equal.*

ACKNOWLEDGMENTS

One of us (R. M. S.) wishes to thank Dr. J. F. Herbst, Mr. B. A. Martin, and Dr. M. C. Takats for helpful discussions. He is particularly indebted to the late Dr. Hartland Snyder for a stimulating discussion concerning the function $f(x)$ in 1960.

Note: After completing this paper, we became aware of a similar calculation by Perry B. Wilson, in which some of the present results have been obtained (Stanford Linear Accelarator Report PEP-232, February 1977). We wish to thank Dr. S. Krinsky for calling our attention to this report.

REFERENCES

1. M. Gardner. *Scientific American* 228 (1973):105. This article is based in part on unpublished work of Dr. A. V. Grosse.
2. R. M. Sternheimer. "On a Set of Non-Associative Functions of a Single Positive Real Variable." Brookhaven Informal Report PD-128; BNL-23081 (June 1977).
3. T. M. Apostol. *Calculus*. Vol. I, p. 417. New York: Blaisdell, 1961.
4. Hartland S. Snyder. Private communication to R. M. S., 1960.
5. M. Creutz & R. M. Sternheimer. "On a Class of Non-Associative Functions of a Single Positive Real Variable." Brookhaven Informal Report PD-130; BNL-23308 (September 1977).

*****

THE NUMBER OF PERMUTATIONS WITH A GIVEN NUMBER OF SEQUENCES

L. CARLITZ

Duke University, Durham, N.C. 27706

1. Let $P(n, s)$ denote the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ with s ascending or descending sequences. For example, the permutation 24315 has the ascending sequences 24, 15 and the descending sequence 431; the permutation 613254 has ascending sequences 13, 25 and descending sequences 61, 32, 54. André proved that $P(n, s)$ satisfies the recurrence

$$(1.1)\qquad P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2), \quad (n \geq 1),$$

where $P(0, s) = P(1, s) = \delta_{0,s}$; for proof see Netto [3, pp. 105-112].

Using (1.1), the writer [1] obtained the generating function

$$(1.2)\qquad \sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{\infty}P(n+1, s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2} + \sin z}{x - \cos z}\right)^2.$$

However, an explicit formula for $P(n, s)$ was not found.

In the present note, we obtain an explicit result, namely

$$(1.3)\qquad \begin{cases} P(2n-1, 2n-s-2) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+2}(2j-1)!\overline{K}_{n,j}M_{n,j,s} \\ P(2n, 2n-s-1) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+1}(2j)!\overline{K}_{n,j}M_{n,j,s}, \end{cases}$$

where

$$\overline{K}_{n,j} = \frac{1}{(2j)!}\sum_{t=0}^{2j}(-1)^t\binom{2j}{t}(j-t)^{2n}$$

and

$$M_{n,j,s} = \sum_{t=0}^{n-j}(-1)^t\binom{n-j}{t}\binom{n-2}{s-t}.$$

2. Put $y = \csc^2 x$. Then it is easily verified that $(D \equiv d/dx)$

$$\begin{aligned} Dy &= -2 \csc^2 x \cot x \\ D^2 y &= -4 \csc^2 x + 6 \csc^4 x \\ D^3 y &= 8 \csc^2 x \cot x - 24 \csc^4 x \cot x \\ D^4 y &= 16 \csc^2 x - 120 \csc^4 x + 120 \csc^6 x. \end{aligned}$$

Generally, we can put

$$D^{2n-2} y = \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \csc^{2j} x \quad (n \geq 1). \tag{2.1}$$

Differentiation of (2.1) gives

$$D^{2n-1} y = \sum_{j=1}^{n} (-1)^{n-j+1} \cdot 2j a_{n,j} \csc^{2j} x \cot x$$

$$\begin{aligned} D^{2n} y &= \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \{4j^2 \csc^{2j} x \cot^2 x + 2j \csc^{2j+2} x\} \\ &= \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \{2j(2j+1) \csc^{2j+2} x - 4j^2 \csc^{2j} x\}. \end{aligned}$$

Comparing this with

$$D^{2n} y = \sum_{j=1}^{n+1} (-1)^{n-j+1} a_{n+1,j} \csc^{2j} x,$$

we get the recurrence

$$a_{n+1,j} = (2j-1)(2j-2) a_{n,j-1} + 4j^2 a_{n,j} \quad (n \geq 1). \tag{2.2}$$

It follows easily from (2.2) that $a_{n,j}$ is divisible by $(2j-1)!$. Thus, if we put

$$a_{n,j} = (2j-1)! b_{n,j}, \tag{2.3}$$

(2.2) becomes

$$b_{n+1,j} = b_{n,j-1} + 4j^2 b_{n,j} \quad (n \geq 1). \tag{2.4}$$

Now put

$$b_{n,j} = 2^{2n-2j} \overline{K}_{n,j}, \tag{2.5}$$

so that (2.4) reduces to

$$\overline{K}_{n+1,j} = \overline{K}_{n,j-1} + j^2 \overline{K}_{n,j} \quad (n \geq 1). \tag{2.6}$$

The $\overline{K}_{n,j}$ are evidently positive integers. Table 1 was obtained by means of (2.6).

The numbers $\overline{K}_{n,j}$ are called the divided central differences of zero [2], [5]. They are related to the $K_{n,j}$ of [2] by

$$\overline{K}_{n,j} = K_{n+1,j}. \tag{2.7}$$

In the notation of divided central differences, we have

$$\overline{K}_{rs} = \delta^{2s} 0^{2r} / (2s)!, \tag{2.8}$$

where

Table 1

$n \backslash j$	1	2	3	4	5
1	1				
2	1	1			
3	1	5	1		
4	1	21	30	1	
5	1	85	501	46	1

$$\delta f(x) = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right).$$

Thus,

$$\overline{K}_{rs} = \frac{1}{(2s)!}\sum_{t=0}^{2s}(-1)^t\binom{2s}{t}(s - t)^{2r}, \tag{2.9}$$

which is equivalent to

$$\sum_{r=1}^{\infty} \overline{K}_{r,s}\, \frac{x^{2r}}{(2r)!} = \frac{1}{(2s)!}(e^{(1/2)x} - e^{-(1/2)x})^s \quad (s \geq 1). \tag{2.10}$$

Substituting from (2.3) and (2.5) in (2.1), we get

$$D^{2n-2}\csc^2 x = \sum_{j=1}^{n} (-1)^{n-j}2^{2n-2j}(2j - 1)!\overline{K}_{n,j}\csc^{2j} x \quad (n \geq 1). \tag{2.11}$$

Differentiation gives

$$D^{2n-1}\csc^2 x = -\sum_{j=1}^{n}(-1)^{n-j}2^{2n-2j}(2j)!\overline{K}_{n,j}\csc^{2j}\phi \cot \phi \quad (n \geq 1). \tag{2.12}$$

3. Returning to the generating function (1.2), we take $x = \cos 2\phi$ and replace z by $2z$. Thus, the lefthand side becomes

$$\sum_{n=0}^{\infty}(\sin 2\phi)^{-n}\frac{2^n z^n}{n!}\sum_{s=0}^{n} P(n + 1, s)\cos^{n-s}2\phi.$$

The right-hand side is equal to

$$\frac{1 - \cos 2\phi}{1 + \cos 2\phi}\left(\frac{\sin 2\phi + \sin 2z}{\cos 2\phi - \cos 2z}\right)^2 = \frac{\sin^2\phi}{\cos^2\phi}\left(\frac{\cos(z - \phi)}{\sin(z - \phi)}\right)^2.$$

Hence, we have

$$\sum_{n=0}^{\infty}(\sin 2\phi)^{-n}\frac{2^n z^n}{n!}\sum_{s=0}^{n}P(n + 1, s)\cos^{n-s}2\phi = \tan^2\phi \cos^2(z - \phi).$$

Replacing ϕ by $-\phi$, this becomes

$$\sum_{n=0}^{\infty}(-1)^n(\sin 2\phi)^{-n}\frac{2^n z^n}{n!}\sum_{=0}^{n}P(n+1,\ s)\cos^{n-s}2\phi \tag{3.1}$$
$$= \tan^2\phi\ \csc^2(z+\phi) - \tan^2\phi.$$

By Taylor's theorem,

$$\csc^2(z+\phi) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{d^n}{d\phi^n}\csc^2\phi.$$

Hence, (3.1) yields

$$(-1)^n 2^n(\sin 2\phi)^{-n}\sum_{s=0}^{n}P(n+1,\ s)\cos^{n-s}2\phi = \tan^2\phi\ \frac{d^n}{d\phi^n}\csc^2\phi,$$

so that

$$\sum_{s=0}^{n}P(n+1,\ s)\cos^{n-s}2\phi = (-1)^n\sin^{n+2}\phi\ \cos^{n-2}\phi\frac{d^n}{d\phi^n}\csc^2\phi \quad (n \geq 1). \tag{3.2}$$

Replacing n by $2n - 2$ and making use of (2.11), we get

$$\sum_{s=0}^{2n-2}P(2n-1,\ s)\cos^{2n-s-2}2\phi \tag{3.3}$$
$$= \sin^{2n}\phi\ \cos^{2n-4}\phi\sum_{j=1}^{n}(-1)^{n-j}2^{2n-2j}(2j-1)!\overline{K}_{n,j}\csc^{2j}\phi \quad (n \geq 1).$$

Similarly, by (2.12),

$$\sum_{s=0}^{2n-1}P(2n,\ s)\cos^{2n-s-1}2\phi \tag{3.4}$$
$$= \sin^{2n}\phi\ \cos^{2n-4}\phi\sum_{j=1}^{n}(-1)^{j}2^{2n-2j}(2j)!\overline{K}_{n,j}\csc^{2j}\phi \quad (n \geq 1).$$

We have, for $1 \leq j \leq n$,

$$2^{2n-2j}\ \sin^{2n-2j}\phi\ \cos^{2n-4}\phi = 2^{-j+2}(1-\cos 2\phi)^{n-j}(1+\cos 2\phi)^{n-2}$$
$$= 2^{-j+2}\sum_{r=0}^{n-j}\sum_{t=0}^{n-2}(-1)^r\binom{n-j}{r}\binom{n-2}{t}\cos^{r+t}2\phi.$$

For $r + t = 2n - s - 2$, comparison with (3.3) gives

$$P(2n-1,\ s) = \sum_{j=1}^{n}(-1)^{n-j}2^{-j+2}(2j-1)!\overline{K}_{n,j}\cdot\sum_{r=0}^{n-j}(-1)^r\binom{n-j}{r}\binom{n-2}{2n-r-s-2}.$$

Replacing s by $2n - s - 2$, we have

$$P(2n-1,\ 2n-s-2) = \sum_{j=1}^{n}(-1)^{n-j}2^{-j+1}(2j-1)!\overline{K}_{n,j} \tag{3.5}$$
$$\cdot\sum_{r=0}^{n-j}(-1)^r\binom{n-j}{r}\binom{n-2}{s-r}.$$

The corresponding result for $P(2n, 2n - s - 1)$ is

$$P(2n, 2n - s - 1) = \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j)!\overline{K}_{n,j} \cdot \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r}\binom{n-2}{s-r}. \tag{3.6}$$

This completes the proof of the following theorem.

Theorem: Let $n > 1$. The number of permutations of Z_n with a given number of sequences is determined by

$$\begin{aligned} P(2n - 1, 2n - s - 2) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+2} (2j - 1)!\overline{K}_{n,j} M_{n,j,s} \\ P(2n, 2n - s - 1) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j)!\overline{K}_{n,j} M_{n,j,s}, \end{aligned} \tag{3.7}$$

where

$$\overline{K}_{n,j} = \frac{1}{(2j)!}\sum_{t=0}^{2j} (-1)^t \binom{2j}{t}(j - t)^{2n} \tag{3.8}$$

and

$$M_{n,j,s} = \sum_{t=0}^{n-j} (-1)^t \binom{n-j}{t}\binom{n-2}{s-t}. \tag{3.9}$$

4. It follows from the definition that, for $n > 1$, $P(n, 1) = 2$. In the first of (3.7), take $s = 2n - 3$. Then, by (3.9),

$$M_{n,j,2n-3} = \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r}\binom{n-2}{2n-r-3},$$

so that $2n - r - 3 \le n - 2$, $n - 1 \le r$ and $j = 0$ or 1. Since $\overline{K}_{n,0} = 0$, $\overline{K}_{n,1} = 1$, $M_{n,1,2n-3} = (-1)^{n-1}$, we get

$$P(2n - 1, 1) = (-1)^{n-1} 2 \cdot (-1)^{n-1} = 2.$$

Similarly, by the second of (3.7), $P(2n, 1) = 2$.

A permutation of Z_n with $n - 1$ ascents and descents is either an up-down or a down-up permutation. Since the number of up-down permutations is equal to the number of down-up permutations, we have

$$P(n, n - 1) = 2A(n) \quad (n \ge 2), \tag{4.1}$$

where $A(n)$ is the number of up-down permutations of Z_n. Hence, in applying (3.7) to this case it is only necessary to take $s = 0$. By equation (3.9), we have $M_{n,j,0} = 0$. Thus (3.7) implies

$$\begin{aligned} A(2n - 1) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j - 1)!\overline{K}_{n,j} \\ A(2n) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j} (2j)!\overline{K}_{n,j}. \end{aligned} \tag{4.2}$$

André [3] proved that

$$\tag{4.3} \sum_{n=1}^{\infty} A(2n-1)\frac{x^{2n-1}}{(2n-1)!} = \tan x \qquad \sum_{n=0}^{\infty} A(2n)\frac{x^{2n}}{(2n)!} = \sec x.$$

On the other hand, in the notation of Nörlund [4, Ch. 2],

$$\tan x = \sum_{n=1}^{\infty} (-1)^n C_{2n-1}\frac{x^{2n-1}}{(2n-1)!}$$

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n}\frac{x^{2n}}{(2n)!},$$

where

$$C_{n-1} = 2^n(1-2^n)\frac{B_n}{2!},$$

and B_n, C_n are the Bernouli and Euler numbers, respectively. Thus, by (4.3),

$$\tag{4.4} A_{2n-1} = (-1)^n C_{2n-1} = (-1)^n 2^{2n}(1-2^{2n})\frac{B_{2n}}{2n} \qquad A(2n) = (-1)^n E_{2n}.$$

Therefore, by (4.2) and (4.4),

$$\tag{4.5} 2^{2n}(1-2^{2n})\frac{B_{2n}}{2n} = \sum_{j=1}^{n} (-1)^j 2^{-j+1}(2j-1)!\overline{K}_{n,j}$$

and

$$\tag{4.6} E_{2n} = \sum_{j=1}^{n} (-1)^j 2^{-j}(2j)!\overline{K}_{n,j}.$$

The representation (4.5) may be compared with the following formula in [2]:

$$\tag{4.7} (2r+1)B_{2r} = \sum_{s=1}^{r+1} (-1)^{s-1}((s-1)!)^2 s^{-1} K_{r+1,s}.$$

We remark that it is proved in [1] that

$$\tag{4.8} P(n, n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \quad (1 \le s \le n),$$

where the $f_{sj}(n)$ are polynomials in n that satisfy $f_{s1}(n) = 1$ and

$$sf_{s+1,j}(n) = f_{s,j}(n+1) - (n-s+1)f_{s-1,j-2}(n) - 2f_{s,j-1}(n).$$

Thus, it would be of interest to evaluate the $f_{sj}(n)$.

REFERENCES

1. L. Carlitz. "Enumeration of Permutations by Sequences." *The Fibonacci Quarterly* 16 (1978):259-268.
2. L. Carlitz & John Riordan. "The Divided Central Differences of Zero." *Canadian Journal of Mathematics* 15 (1963):94-100.
3. E. Netto. *Lehrbuch der Combinatorik*. Leipzig: Teubner, 1927.
4. N. E. Nörlund. *Vorlesungen über Differenzenrechnung*. Berlin: Springer, 1924.
5. J. F. Steffensen. *Interpolation*. Baltimore: Williams and Wilkens, 1927.

*****

FOUR COMPOSITION IDENTITIES FOR CHEBYSHEV POLYNOMIALS

CLARK KIMBERLING*
University of Evansville, Evansville, IN 47702

1. INTRODUCTION

Let $\{t_n(x)\}_{n=0}$ be the sequence of Chebyshev polynomials defined by

$$t_0(x) = 1,\ t_1(x) = x,\ t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x) \text{ for } n \geq 2.$$

These are often called *Chebyshev polynomials of the first kind* to distinguish them from *Chebyshev polynomials of the second kind*, which are defined by

$$u_0(x) = 1,\ u_1(x) = 2x,\ u_n(x) = 2xu_{n-1}(x) - u_{n-2}(x) \text{ for } n \geq 2.$$

It is well known that any two Chebyshev polynomials of the first kind commute under composition. Explicitly,

$$t_m(t_n(x)) = t_n(t_m(x)) = t_{mn}(x) \text{ for nonnegative } m \text{ and } n.$$

Similar identities involving Chebyshev polynomials of the second kind are not well known. This paper offers three such identities, one for each of the expressions $\overline{u}_m(\overline{u}_n(x))$, $t_m(\overline{u}_n(x))$, and $\overline{u}_m(t_n(x))$, where $\overline{u}_m(x) = u_m(x)\sqrt{1 - x^2}$.

Literature relating to the identity $t_m(t_n) = t_n(t_m)$ shows that this commutativity, also called *permutability*, is, among polynomials with coefficients in a field of characteristic 0, a distinctive property of Chebyshev polynomials of the first kind. For example, Bertram [1] shows that if p is a polynomial of degree $m > 1$ which is permutable with some t_n for $n \geq 2$, then $p = \pm t_m$. Another theorem (e.g., Kuczma [5, pp. 215-218] and Rivlin [6, pp. 160-164]) characterizes the sequence $\{t_n\}$ as the only nontrivial *semipermutable chain* (up to equivalence, as described below). Sections 3 and 4 of this paper deal with analogous results for the functions $\overline{u}_n$.

We deal with the Chebyshev polynomials in slightly altered form. Assume throughout that all numbers, including coefficients of all polynomials, lie in a field of characteristic 0. With this in mind, the nonmonic polynomials t_n and u_n are altered as follows: define

$$T_0(x, y) = 2,\ T_1(x, y) = x,\ T_n(x, y) = xT_{n-1}(x, y) - yT_{n-2}(x, y) \text{ for } n \geq 2;$$

$$U_0(x, y) = 0,\ U_1(x, y) = 1,\ U_n(x, y) = xU_{n-1}(x, y) - yU_{n-2}(x, y) \text{ for } n \geq 2.$$

In the sequel, the polynomials T_n are regarded as Chebyshev polynomials of the first kind, and the polynomials U_n are regarded as Chebyshev polynomials of the second kind. The connections with the polynomials t_n and u_n are simply

$$T_n(x, 1) = 2t_n(x/2) \text{ for } n \geq 0 \quad \text{and} \quad U_n(x, 1) = u_{n-1}(x/2) \text{ for } n \geq 1.$$

All the results obtained below for $\{T_n\}$ and $\{U_n\}$ carry over, as in Corollary 1, to $\{t_n\}$ and $\{u_n\}$. We also wish to carry over some results to certain polynomials of number-theoretic interest, namely the *generalized Lucas polynomials* $L_n(x, y)$ and *generalized Fibonacci polynomials* $F_n(x, y)$, discussed in [4] and elsewhere. For these, we have

$$T_n(x, y) = L_n(x, -y) \quad \text{and} \quad U_n(x, y) = F_n(x, -y).$$

2. THE FOUR IDENTITIES

Consistent with the modification $\overline{u}_n(x)$ of $u_n(x)$ already mentioned, we introduce a modification of $U_n(x, y)$:

$$\overline{U}_n(x, y) = U_n(x, y)\sqrt{4y - x^2} \text{ for } n \geq 0.$$

*Supported by a University of Evansville Alumni Research Fellowship.

Although $\overline{U}_n$ is not a polynomial for $n \geq 1$, it is convenient to say that $\overline{U}_n(x, y)$ has degree n in x. [The polynomial $U_n(x, y)$ has degree $n - 1$ in x.] Generally, a function $P(x, y)\sqrt{S(x, y)}$, where $P(x, y)$ and $S(x, y)$ are polynomials of degrees n and $2k$, respectively, in x, is regarded as a *function of degree* $n + k$ *in* x.

Definition: Suppose $P(x, y)$ and $Q(x, y)$ are functions of degrees m and n, respectively, in x. The composite function $P \circ Q$ is defined by

$$P \circ Q(x, y) = P[Q(x, y), y^n].$$

Theorem 1: Suppose m and n are nonnegative integers. Then

$$T_m \circ T_n(x, y) = T_{mn}(x, y) \tag{1}$$

$$\overline{U}_m \circ T_n(x, y) = \overline{U}_{mn}(x, y) \tag{2}$$

$$T_m \circ \overline{U}_n(x, y) = \begin{cases} (-1)^{m/2} T_{mn}(x, y) & \text{for even } m \\ (-1)^{(m-1)/2} \overline{U}_{mn}(x, y) & \text{for odd } m \end{cases} \tag{3}$$

$$\overline{U}_m \circ \overline{U}_n(x, y) = \begin{cases} (-1)^{(m-2)/2} \overline{U}_{mn}(x, y) & \text{for even } m \\ (-1)^{(m-1)/2} T_{mn}(x, y) & \text{for odd } m. \end{cases} \tag{4}$$

Proof: It is easy to establish (as in [4]) that

$$T_m(x, y) = 2y^{m/2}\cos(m \cos^{-1} x/2\sqrt{y})$$

and

$$U_m(x, y) = (4y - x^2)^{-1/2}\, 2y^{m/2}\sin(m \cos^{-1} x/2\sqrt{y}),$$

so that

$$\overline{U}_m(x, y) = 2y^{m/2}\sin(m \cos^{-1} x/2\sqrt{y}).$$

Then

$$T_m \circ T_n(x, y) = 2y^{mn/2}\cos\left[m \cos^{-1} \frac{2y^{n/2}\cos(n \cos^{-1} x/2\sqrt{y})}{2y^{n/2}}\right] = T_{mn}(x, y).$$

Similarly,

$$\overline{U}_m \circ T_n(x, y) = 2y^{mn/2}\sin\left[m \cos^{-1} \frac{2y^{n/2}\cos(n \cos^{-1} x/2\sqrt{y})}{2y^{n/2}}\right] = \overline{U}_{mn}(x, y).$$

Next,

$$\begin{aligned} T_m \circ \overline{U}_n(x, y) &= 2y^{mn/2}\cos[m \cos^{-1} \sin(n \cos^{-1} x/2\sqrt{y})] \\ &= 2y^{mn/2}\cos[m(\pi/2 - n \cos^{-1} x/2\sqrt{y})] \\ &= 2y^{mn/2}[\cos m\pi/2 \cos(mn \cos^{-1} x/2\sqrt{y}) \\ &\quad + \sin m\pi/2 \sin(mn \cos^{-1} x/2\sqrt{y})], \end{aligned}$$

and from this, (3) clearly follows. Finally,

$$\begin{aligned} \overline{U}_m \circ \overline{U}_n(x, y) &= 2y^{mn/2}\sin[m \cos^{-1} \sin(n \cos^{-1} x/2\sqrt{y})] \\ &= 2y^{mn/2}\sin[m(\pi/2 - n \cos^{-1} x/2\sqrt{y})] \\ &= 2y^{mn/2}[\sin m\pi/2 \cos(mn \cos^{-1} x/2\sqrt{y}) \\ &\quad - \cos m\pi/2 \sin(mn \cos^{-1} x/2\sqrt{y})], \end{aligned}$$

and this proves (4).

Corollary 1: Let $\{t_n\}_{n=0}$ and $\{u_n\}_{n=0}$ be the sequences of (unaltered) Chebyshev polynomials of the first and second kinds, respectively. Put $\overline{u}_{-1}(x) \equiv 0$ and $\overline{u}_n(x) = u_n(x)\sqrt{1 - x^2}$ for $n \geq 0$. Then for nonnegative m and n, (1′) $$t_m(t_n(x)) = t_{mn}(x)$$

(2′) $$\overline{u}_m(t_n(x)) = \overline{u}_{mn+n-1}(x)$$

(3′) $$t_m(\overline{u}_n(x)) = \begin{cases} (-1)^{m/2}t_{mn+m}(x) & \text{for even } m \\ (-1)^{(m-1)/2}\overline{u}_{mn+m-1}(x) & \text{for odd } m \end{cases}$$

(4′) $$\overline{u}_m(\overline{u}_n(x)) = \begin{cases} (-1)^{m/2}t_{(m+1)(n+1)}(x) & \text{for even } m \\ (-1)^{(m-1)/2}\overline{u}_{mn+m+n}(x) & \text{for odd } m. \end{cases}$$

Proof: These identities come directly from Theorem 1 via the transformations

$$t_n(x) = \frac{1}{2}T_n(2x, 1) \quad \text{and} \quad \overline{u}_n(x) = \frac{1}{2}\overline{U}_{n+1}(2x, 1) \text{ for } n \geq 0.$$

We turn now to the problem of expressing (1)-(4) in terms of generalized Lucas and Fibonacci polynomials. Corresponding to the functions $\overline{U}_n(x, y)$ we define

$$\overline{F}_n(x, y) = F_n(x, y)\sqrt{x^2 + 4y} \text{ for } n \geq 0,$$

noting that this equals $i\overline{U}_n(x, -y)$. Two lemmas are helpful.

Lemma 2a: For $0 \leq m \leq n$,

(5) $$L_m(x, y)L_n(x, y) - \overline{F}_m(x, y)\overline{F}_n(x, y) = 2(-y)^m L_{n-m}(x, y).$$

Proof: It is well known and easily shown by induction that

$$L_n(x, y) = \alpha^n + \beta^n \quad \text{and} \quad \overline{F}_n(x, y) = \alpha^n - \beta^n,$$

where $\alpha + \beta = x$ and $\alpha\beta = -y$. The desired identity now follows immediately.

Lemma 2b: For $m \geq 0$,

$$L_m(ix, -y) = i^m L_m(x, y) \quad \text{and} \quad \overline{F}_m(ix, -y) = i^m\overline{F}_m(x, y).$$

Proof: This is easily seen by induction, using the recurrence relation

$$H_m(x, y) = xH_{m-1}(x, y) + yH_{m-2}(x, y)$$

satisfied by both $\{L_m\}$ and $\{\overline{F}_m\}$ for $m \geq 2$.

From (1) and the relation $T_n(x, -y) = L_n(x, y)$ comes

$$T_m[L_n(x, y), (-1)^n y^n] = L_{mn}(x, y),$$

so that

(1a) $$L_m \circ L_n(x, y) = L_{mn}(x, y) \text{ for odd } n.$$

But, for even n,

(6) $$L_n^m - a_{m-2}L_n^{m-2}y^n + a_{m-4}L_n^{m-4}y^{2n} - \cdots + (-1)^{\left[\frac{m}{2}\right]}a_\ell L_n^\ell y^{\left[\frac{m}{2}\right]n} = L_{mn}(x, y),$$

where the a_k's are coefficients in the polynomial

$$T_m(x, y) = x^m - a_{m-2}x^{m-2}y + a_{m-4}x^{m-4}y - \cdots + (-1)^{\left[\frac{m}{2}\right]}a_\ell x^\ell y^{\left[\frac{m}{2}\right]};$$

here, $\ell = 0$ if m is even and $\ell = 1$ if m is odd (see Lemma 2e). Adding

$$2a_{m-2}L_n^{m-2}y^n + 2a_{m-6}L_n^{m-6}y^{3n} + \cdots$$

to both sides of (6) gives

(1b) $$L_m \circ L_n(x, y) = L_{mn}(x, y) + 2(a_{m-2}L_n^{m-2}y + a_{m-6}L_n^{m-6}y^{3n} + \cdots + a_s L_n^s y^{tn})$$

for even n, where

$$s = \begin{cases} 0 \text{ if } m \equiv 2 \bmod 4 \\ 1 \text{ if } m \equiv 3 \bmod 4 \\ 2 \text{ if } m \equiv 0 \bmod 4 \\ 3 \text{ if } m \equiv 1 \bmod 4 \end{cases} \quad \text{and} \quad t = 1\left[\frac{m-2}{4}\right] + 1.$$

Now from (2) and the relation $\overline{U}_n(x, -y) = -i\overline{F}_n(x, y)$ comes

$$i\overline{U}_m[L_n(x, y), (-1)^n y^n] = \overline{F}_{mn}(x, y),$$

so that

(2a) $$\overline{F}_m \circ L_n(x, y) = \overline{F}_{mn}(x, y) \text{ for odd } n.$$

But for even n,

$$\begin{aligned} \overline{F}_{mn}(x, y) &= i\overline{U}_m[L_n(x, y), y^n] \\ &= \overline{F}_m[L_n(x, y), -y^n] \\ &= \sqrt{L_n^2 - 4y^n}F_m[L_n(x, y), -y^n] \\ &= \overline{F}_n(x, y)F_m[L_n(x, y), -y^n], \end{aligned}$$

by Lemma 2a. Thus,

(7) $$\overline{F}_{mn}(x, y) = \overline{F}_n(x, y)\Big\{L_n^{m-1} - b_{m-3}L_n^{m-3}y + b_{m-5}L_n^{m-5}y^{2n} - \cdots + (-1)^{\left[\frac{m-1}{2}\right]}b_\ell L_n^\ell y^{\left[\frac{m-1}{2}\right]n}\Big\},$$

where the b_k's are the coefficients of the polynomial

$$F_m(x, y) = x^{m-1} + b_{m-3}x^{m-3}y + b_{m-5}x^{m-5}y^2 + \cdots + b_\ell x^\ell y^{\left[\frac{m-1}{2}\right]};$$

here, $\ell = 0$ if m is even and $\ell = 1$ if m is odd (see Lemma 2e). Adding

$$2\overline{F}_n(x, y)(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots)$$

to both sides of (7) gives

$$\overline{F}_n(x, y)F_m \circ L_n(x, y) = \overline{F}_{mn}(x, y) + 2\overline{F}_n(x, y)(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots).$$

For $n > 0$, we divide both sides by $\overline{F}_n(x, y)$ and have

(2b) $$F_m \circ L_n(x, y) = \frac{F_{mn}(x, y)}{F_n(x, y)} + 2(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots + b_s L_n^s y^{tn}) \text{ for even } n > 0,$$

where

$$s = \begin{cases} 0 \text{ if } m \equiv 3 \bmod 4 \\ 1 \text{ if } m \equiv 0 \bmod 4 \\ 2 \text{ if } m \equiv 1 \bmod 4 \\ 3 \text{ if } m \equiv 2 \bmod 4 \end{cases} \quad \text{and} \quad t = 2\left[\frac{m-3}{4}\right] + 1.$$

Identity (3) leads to

(8) $$T_m[-i\overline{F}_n(x, y), (-1)^n y^n] = \begin{cases} (-1)^{\frac{m}{2}}L_{mn}(x, y) & \text{for even } m \\ (-1)^{\frac{m+1}{2}}i\overline{F}_{mn}(x, y) & \text{for odd } m. \end{cases}$$

For even $n \geq 0$, we apply Lemma 2b to find, without difficulty, that

(3a) $$L_m \circ \overline{F}_n = \begin{cases} L_{mn} \text{ for even } n \text{ and even } m \\ \overline{F}_{mn} \text{ for even } n \text{ and odd } m. \end{cases}$$

For odd n, suppose first that m is odd also. Then (8) with Lemma 2a gives

$$L_m[\overline{F}_n(x, y), -y^n] = \overline{F}_{mn}(x, y).$$

As in the derivation of (1b), we add

$$2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots)$$

to both sides. This gives

(3b) $$L_m \circ \overline{F}_n(x, y) = \overline{F}_{mn}(x, y) + 2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots + a_s\overline{F}_n^s y^{tn}) \text{ for odd } n \text{ and odd } m,$$

where the a_k's, s, and t are the same as for (1b).

Continuing with odd n, suppose now that m is even. Using (8) and Lemma 2a, we find

(3c) $$L_m \circ F_n(x, y) = L_{mn}(x, y) + 2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots + a_s\overline{F}_n^s y^{tn}) \text{ for odd } n \text{ and even } m,$$

where the a_k's, s, and t are the same as for (1b).

Identity (4) leads to

(9) $$\overline{U}_m[-i\overline{F}_n(x, y), (-1)^n y^n] = \begin{cases} (-1)^{\frac{m}{2}} i\overline{F}_{mn}(x, y) & \text{for even } m \\ (-1)^{\frac{m-1}{2}} L_{mn}(x, y) & \text{for odd } m. \end{cases}$$

whence,

(4a) $$\overline{F}_m \circ \overline{F}_n = \begin{cases} \overline{F}_{mn} & \text{for even } n \text{ and even } m \\ L_{mn} & \text{for even } n \text{ and odd } m. \end{cases}$$

For odd n, suppose first that m is odd also. Then (9) and Lemmas 2a and 2b apply, and we find

$$L_{mn}(x, y) = \overline{F}_m[\overline{F}_n(x, y), -y^n] = \sqrt{\overline{F}_n^2 - 4y^n}F_m[\overline{F}_n(x, y), -y^n]$$
$$= L_n(x, y)(\overline{F}_n^{m-1} - b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-5}\overline{F}_n^{m-5}y^{2n} - \cdots).$$

At this point, we add $2L_n(x, y)(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots)$ to both sides and then divide both sides by $L_n(x, y)$, getting

(4b) $$F_m \circ \overline{F}_n(x, y) = \frac{L_{mn}(x, y)}{L_n(x, y)} + 2(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots + b_s\overline{F}^s y^{tn}) \text{ for odd } n \text{ and odd } m,$$

where the b_k's, s, and t are the same as for (2b).

Continuing with odd n, suppose now that m is even. With the method which is now familiar, we find

(4c) $$F_m \circ F_n(x, y) = \frac{F_{mn}(x, y)}{F_n(x, y)} + 2(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots + b_s\overline{F}_n^s y^{tn}) \text{ for odd } m \text{ and even } m,$$

where the b_k's, s, and t are the same as for (2b).

Table 1. Examples of Composites Involving Generalized Lucas and Fibonacci Polynomials

From (1b) and (2b), for even $n > 0$:

$L_2 \circ L_n = L_{2n} + 4y^n$	$F_2 \circ L_n = F_{2n}/F_n$
$L_3 \circ L_n = L_{3n} + 6L_n y^n$	$F_3 \circ L_n = F_{3n}/F_n + 2y^n$
$L_4 \circ L_n = L_{4n} + 8L_n^2 y^n$	$F_4 \circ L_n = F_{4n}/F_n + 4L_n y^n$
$L_5 \circ L_n = L_{5n} + 10L_n^3 y^n$	$F_5 \circ L_n = F_{5n}/F_n + 6L_n^2 y^n$
$L_6 \circ L_n = L_{6n} + 12L_n^4 y^n + 4y^{3n}$	$F_6 \circ L_n = F_{6n}/F_n + 8L_n^3 y^n$
$L_7 \circ L_n = L_{7n} + 14L_n^5 y^n + 14L_n y^{3n}$	$F_7 \circ L_n = F_{7n}/F_n + 10L_n^4 y^n + 2y^{3n}$

Table 1—continued

From (3b) and (3c), for odd $n \geq 1$:

$L_3 \circ \overline{F}_n = \overline{F}_{3n} + 6\overline{F}_n y^n$	$L_2 \circ \overline{F}_n = L_{2n} + 4y^n$
$L_5 \circ \overline{F}_n = \overline{F}_{5n} + 10\overline{F}_n^3 y^n$	$L_4 \circ \overline{F}_n = L_{4n} + 8\overline{F}_n^2 y^n$
$L_7 \circ \overline{F}_n = \overline{F}_{7n} + 14\overline{F}_n^5 y^n + 14\overline{F}_n^3 y^n$	$L_6 \circ \overline{F}_n = L_{6n} + 12\overline{F}_n^4 y^n + 4y^{3n}$
$L_9 \circ \overline{F}_n = \overline{F}_{9n} + 18\overline{F}_n^7 y^n + 60\overline{F}_n^3 y^n$	$L_8 \circ \overline{F}_n = L_{8n} + 16\overline{F}_n^6 y^n + 32\overline{F}_n^2 y^{3n}$

From (4b) and (4c), for odd $n \geq 1$:

$F_1 \circ \overline{F}_n = 1$	$F_2 \circ \overline{F}_n = \overline{F}_{2n}/L_n$
$F_3 \circ \overline{F}_n = L_{3n}/L_n + 2y^n$	$F_4 \circ \overline{F}_n = \overline{F}_{4n}/L_n + 4\overline{F}_n y^n$
$F_5 \circ \overline{F}_n = L_{5n}/L_n + 6\overline{F}_n^2 y^n$	$F_6 \circ \overline{F}_n = \overline{F}_{6n}/L_n + 8\overline{F}_n^3 y^n$
$F_7 \circ \overline{F}_n = L_{7n}/L_n + 10\overline{F}_n^4 y^n + 2y^{3n}$	$F_8 \circ \overline{F}_n = \overline{F}_{8n}/L_n + 12\overline{F}_n^5 y^n + 8\overline{F}_n y^{3n}$
$F_9 \circ \overline{F}_n = L_{9n}/L_n + 14\overline{F}_n^6 y^n + 20\overline{F}_n^2 y^{3n}$	$F_{10} \circ \overline{F}_n = \overline{F}_{10n}/L_n + 16\overline{F}_n^7 y^n + 40\overline{F}_n y^{3n}$

For $m \geq 0$, define

$$V_m(x, y) = \binom{m}{0}x^m + \binom{m}{1}x^{m-2}y + \cdots + \binom{m}{[m/2]}x^{\ell}y^{\left[\frac{m}{2}\right]n}$$

and

$$W_m(x, y) = V_m(x, -y),$$

where $\ell = 0$ for even m and $\ell = 1$ for odd m.

Lemma 2c: Suppose m and n are nonnegative integers. Then

$$L_n^m(x, y) = \begin{cases} V_m \circ L_1^n(x, y) \text{ for even } n \\ W_m \circ L_1^n(x, y) \text{ for odd } n, \end{cases}$$

$$\overline{F}_n^m(x, y) = \begin{cases} W_m \circ L_1^n(x, y) \text{ for even } m \text{ and even } n \\ V_m \circ L_1^n(x, y) \text{ for even } m \text{ and odd } n, \end{cases}$$

$$\overline{F}_n^m(x, y) = \begin{cases} W_m \circ \overline{F}_1^n(x, y) \text{ for odd } m \text{ and even } n \\ V_m \circ \overline{F}_1^n(x, y) \text{ for odd } m \text{ and odd } n; \end{cases}$$

in these formulas, after expansions on the right sides, each symbol of the form L_1^j (or $\overline{F}_1^j$) is to be changes to L_j (or $\overline{F}_j$). (This "symbolic substitution" is discussed in Hoggatt and Lind [3].)

Proof: These are direct results of writing

$$L_n(x, y) = \alpha^n + \beta^n \quad \text{and} \quad \overline{F}_n(x, y) = \alpha^n - \beta^n$$

and applying the binomial formula, where $\alpha + \beta = x$ and $\alpha\beta = -y$.

Lemma 2d: Suppose m and n are nonnegative integers. Then

$$L_{mn}(x, y) = \begin{cases} T_m \circ L_n(x, y) \text{ for even } n \\ L_m \circ L_n(x, y) \text{ for odd } n \end{cases}$$

and

$$\frac{F_{mn}(x, y)}{F_n(x, y)} = \begin{cases} U_m \circ L_n(x, y) \text{ for even } n > 0 \\ F_m \circ L_n(x, y) \text{ for odd } n. \end{cases}$$

Proof: Near (1a) and (2a) these two are already proved. (They are restated here for later convenience and as inverse formulas for the formulas in Lemma 2c. Tables of coefficients for these two formulas are found in Brousseau [2, pp. 145-150].)

Lemma 2e: For $m \geq 0$,

$$L_m(x, y) = \sum_{j=0}^{p} \frac{m}{m-2j}\binom{m-j-1}{j}x^{m-2j}y^j \text{ with } p = \begin{cases} m/2 & \text{for even } m \\ (m-1)/2 & \text{for odd } m \end{cases}$$

where the summand on the right equals $2y^p$, by definition, in case $j = p = m/2$. Also

$$F_m(x, y) = \sum_{j=0}^{q} \binom{m-j-1}{j}x^{m-2j-1}y^j \text{ with } q = \begin{cases} (m-2)/2 & \text{for even } m \\ (m-1)/2 & \text{for odd } m. \end{cases}$$

Proof: These well-known formulas are easily proved by induction.

The composite functions in Table 1 can also be expressed as linear combinations of terms of the form $L_{jn}y^k$ or $\overline{F}_{jn}y^k$. To obtain such expressions, one may use Table 1 with substitutions from Lemma 2c, or one may use Binet forms (e.g., $\overline{F}_n = \alpha^n - \beta^n$) and binomial expansions. These methods give the following results.

For even n, the coefficients c_{m-2j} in the expression

$$L_m \circ L_n = c_m L_{mn} + c_{m-2}L_{(m-2)n}y^n + \cdots + c_{m-2p}L_{(m-2p)n}y^{pn},$$

where p is as in Lemma 2e and for temporary convenience $L_0 \equiv 1$ (instead of 2):

Table 2

	c_m	c_{m-2}	c_{m-4}	c_{m-6}	c_{m-8}	c_{m-10}
$m = 2$	1	4				
3	1	6				
4	1	8	16			
5	1	10	30			
6	1	12	48	76		
7	1	14	70	154		
8	1	16	96	272	384	
9	1	18	126	438	810	
10	1	20	160	660	1520	2004

Formula: $c_{m-2j} =$

$$\sum_{k=0}^{j} \frac{m}{m-2k}\binom{m-k-1}{k}\binom{m-2k}{j-k}$$

for $0 \leq j \leq p$, where the summand on the right = 2, by definition, in case

$$k = m/2$$

(which occurs in c_{m-2p} for even m).

For even n, the coefficients c_{m-2j-1} in the expression

$$F_m \circ L_n = c_{m-1}L_{(m-1)n} + c_{m-3}L_{(m-3)n}y^n + \cdots + c_{m-2q-1}L_{(m-2q-1)n}y^{qn},$$

where q is as in Lemma 2e and for temporary convenience $L_0 \equiv 1$ (instead of 2):

Table 3

	c_{m-1}	c_{m-3}	c_{m-5}	c_{m-7}	c_{m-9}
$m = 2$	1				
3	1	3			
4	1	5			
5	1	7	13		
6	1	9	25		
7	1	11	41	63	
8	1	13	61	129	
9	1	15	85	231	321
10	1	17	113	377	681

<u>Formula</u>: $c_{m-2j-1} = \sum_{k=0}^{j} \binom{m-k-1}{k}\binom{m-2k-1}{j-k}$ for $0 \le j \le q$.

For odd $n \ge 1$, the coefficients c_{m-2j} in the expression

$$L_m \circ \overline{F}_n = \begin{cases} c_m L_{mn} + c_{m-2}L_{(m-2)n}y^n + \cdots + c_{m-2p}L_{(m-2p)n}y^{pn} & \text{for even } m \ge 0 \\ c_m \overline{F}_{mn} + c_{m-2}\overline{F}_{(m-2)n}y^n + \cdots + c_{m-2p}\overline{F}_{(m-2p)n}y^{pn} & \text{for odd } m \ge 1 \end{cases}$$

are precisely the same as in Table 2. Similarly, for odd $n \ge 1$, the coefficients c_{m-2j-1} in the expression

$$F_m \circ \overline{F}_n = \begin{cases} c_{m-1}\overline{F}_{(m-1)n} + c_{m-3}\overline{F}_{(m-3)n}y^n + \cdots + c_{m-2q-1}\overline{F}_{(m-2q-1)n}y^{qn} & \text{for even } m \ge 2 \\ c_{m-1}L_{(m-1)n} + c_{m-3}L_{(m-3)n}y^n + \cdots + c_{m-2q-1}L_{(m-2q-1)n}y^{qn} & \text{for odd } m \ge 1. \end{cases}$$

are precisely the same as in Table 3.

Now let us recall (1a), (2a), (3a), and (4a): For odd $n \ge 1$,

$$\overline{F}_m \circ L_n = \overline{F}_{mn} \quad \text{and} \quad L_m \circ L_n = L_{mn};$$

for even $n \ge 0$,

$$L_m \circ \overline{F}_n = \begin{cases} L_{mn} & \text{for even } m \ge 0 \\ \overline{F}_{mn} & \text{for odd } m \ge 1 \end{cases} \quad \text{and} \quad \overline{F}_m \circ \overline{F}_n = \begin{cases} \overline{F}_{mn} & \text{for even } m \ge 0 \\ L_{mn} & \text{for odd } m \ge 1. \end{cases}$$

These four identities lead to identities for products of composites. For example, suppose s and σ are odd positive integers and t and τ are even nonnegative integers. Then

$$\overline{F}_s \circ \overline{F}_t = L_{st} \quad \text{and} \quad \overline{F}_\sigma \circ \overline{F}_\tau = L_{\sigma\tau}.$$

By identity (5) in [4], $L_{st}L_{\sigma\tau} = L_{st+\sigma\tau} + L_{st-\sigma\tau}$. Therefore,

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = L_{st+\sigma\tau} + L_{st-\sigma\tau}.$$

Ten identities are obtainable in this way. To facilitate listing them, we make certain abbreviations. The identity just derived appears below in (10) as

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = L_\sharp + L_\flat, \text{ oeoe},$$

where the designation "oeoe" means "for odd s, even t, odd σ, even τ."

Table 4. Product-Composition Identities

<u>Notation</u>: s, t, σ, τ are nonnegative integers and $st \ge \sigma\tau$.
Also, $\sharp = st + \sigma\tau$ and $\flat = st - \sigma\tau$ as in the example above.

(10)

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp + L_\flat, & \text{oeoe} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{oeee} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeoe} \\ L_\sharp - L_\flat, & \text{eeee} \end{cases}$$

(11)

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp + \overline{F}_\flat, & \text{oeoo} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{oeeo} \\ L_\sharp + L_\flat, & \text{eeoo} \\ L_\sharp - L_\flat, & \text{eeeo} \end{cases}$$

Table 4.—continued

(12)
$$(\overline{F}_s \circ \overline{F}_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oeoe} \\ L_\sharp + L_\flat, & \text{oeee} \\ L_\sharp - L_\flat, & \text{eeoe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeee} \end{cases}$$

(13)
$$(\overline{F}_s \circ \overline{F}_t)(L_\sigma \circ L_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oeoo} \\ L_\sharp + L_\flat, & \text{oeeo} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{eeoo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeeo} \end{cases}$$

(14)
$$(L_s \circ \overline{F}_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oeoe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{oeee} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{eeoe} \\ L_\sharp + L_\flat, & \text{eeee} \end{cases}$$

(15)
$$(L_s \circ \overline{F}_t)(L_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oeoo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{oeeo} \\ L_\sharp - L_\flat, & \text{eeoo} \\ L_\sharp + L_\flat, & \text{eeeo} \end{cases}$$

(16)
$$(\overline{F}_s \circ L_t)(\overline{F}_\sigma \circ L_\tau) = \begin{cases} L_\sharp + L_\flat, & \text{oooo and eooo} \\ L_\sharp - L_\flat, & \text{ooeo and eoeo} \end{cases}$$

(17)
$$(\overline{F}_s \circ L_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oooe and eooe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{ooee and eoee} \end{cases}$$

(18)
$$(\overline{F}_s \circ L_t)(L_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oooo and eooo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{ooeo and eoeo} \end{cases}$$

(19)
$$(L_s \circ L_t)(L_\sigma \circ L_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oooo and eooo} \\ L_\sharp + L_\flat, & \text{ooeo and eoeo} \end{cases}$$

3. FUNCTIONS COMMUTING WITH $\overline{U}$ (x)

Bertram [1] proves that, except for a possible factor -1, the only nonconstant polynomials that are permutable (i.e., commute) with nonlinear Chebyshev polynomials (of the first kind) are Chebyshev polynomials (of the first kind). Here we obtain analogous results for Chebyshev polynomials of the second kind. The same arguments give further analogous results for composites involving one Chebyshev polynomial of each kind.

There is no real loss in disregarding the symbol y in $T_n(x, y)$, $U_n(x, y)$, and $\overline{U}_n(x, y)$ in this section. Accordingly, we write $T_n(x)$ for $T_n(x, 1)$, $U_n(x)$ for $U_n(x, 1)$, and $\overline{U}_n(x)$ for $\overline{U}_n(x, 1)$. Following the notation and arguments in Bertram, if P and Q are functions, the substitution of $Q(x)$ for x in $P(x)$ is denoted either by $P(Q(x))$ or $P(Q)$. Ordinary multiplication of functions is given by juxtaposition, as in $\sqrt{4 - x^2}U_n(x)$, or by brackets, as in $A[\overline{P}']^j$ and $(4 - x^2)[U_n'(x)]$, in order to avoid confusion with the composition (i.e., substitution) operation.

Proofs in this section are abbreviated or omitted, but the interested reader with [1] at hand should have no trouble writing out the proofs in full. One must of course bear in mind the transformations already given between T_n, $\overline{U}_n$, and t_n, $\overline{u}_n$.

Lemma 3a: Suppose $\overline{P}(x)$ satisfies the following differential equation for some positive integer n:

$$(4 - x^2)[\overline{P}'(x)]^2 = n^2[4 - \overline{P}^2(x)]. \tag{20}$$

If $\overline{P}(x)$ is of the form $\sqrt{4 - x^2}P(x)$, where $P(x)$ is a polynomial, then

$$P(x) = \pm U_n(x). \quad [\text{That is, } \overline{P}(x) = \pm\overline{U}_n(x).]$$

Lemma 3b: Suppose $A(x)$, a polynomial of degree $j \geq 0$, and $\overline{Q}(x) = \sqrt{4 - x^2}Q(x)$, where $Q(x)$ is a polynomial of degree $n - 1 \geq 1$, satisfy the differential equation

$$\{A(x)[\overline{Q}'(x)]^j\}^2 = [n^j A(\overline{Q}(x))]^2. \tag{21}$$

If $\overline{P}(x) = \sqrt{4 - x^2}P(x)$, where $P(x)$ is a polynomial of degree $m - 1 \geq 0$, is permutable with $\overline{Q}(x)$, then $\overline{P}(x)$ satisfies the same differential equation with n replaced by m.

Proof: Let

$$G = \{A[\overline{P}']^j\}^2 - [m^j A(\overline{P})]^2,$$

and suppose $G \not\equiv 0$. The highest degree term of both $\{A[\overline{P}']^j\}^2$ and $[m^j A(\overline{P})]^2$ is

$$(-1)^j m^{2j} a_j^2 p_{m-1}^{2j} x^{2mj},$$

so that the degree d of G is strictly less than $2jm$. We next prove that $G \not\equiv 0$ also implies $d = 2jm$. Using (21), the commutativity, and the chain rule,

$$\begin{aligned} n^{2j} G(\overline{Q}) &= n^{2j} \{A(\overline{Q})[\overline{P}'(\overline{Q})]^j\}^2 - m^{2j} n^{2j} [A(\overline{P}(\overline{Q}))]^2 \\ &= A^2[\overline{Q}']^{2j} [\overline{P}'(\overline{Q})]^{2j} - m^{2j} [A(\overline{P})]^2 [\overline{Q}'(\overline{P})]^{2j} \\ &= A^2[\overline{P}']^{2j} [\overline{Q}'(\overline{P})]^{2j} - m^{2j} [A(\overline{P})]^2 [\overline{Q}'(\overline{P})]^{2j} \\ &= [\overline{Q}'(\overline{P})]^{2j} \{A^2[\overline{P}']^{2j} - m^{2j} [A(\overline{P})]^2\} = [\overline{Q}'(\overline{P})]^{2j} G. \end{aligned}$$

Equating degrees gives $nd = d + 2j(n-1)m$, so that $d = 2jm$ since $n \neq 1$. This contradiction shows that $G \equiv 0$, as desired.

Theorem 3: Let $\{U_n\}_{n=0}$ be the sequence of (altered) Chebyshev polynomials of the second kind. Suppose P is a polynomial of degree $m - 1 \geq 0$ such that the functions

$$\overline{U}_n(x) = \sqrt{4 - x^2}\, U_n(x) \quad \text{and} \quad \overline{P}(x) = \sqrt{4 - x^2}\, P(x)$$

are permutable for some positive integer n. Then $P = U_m$ if n is odd, and $P = \pm U_m$ if n is even.

Proof: First suppose $n = 1$. If $m = 1$ also, then the desired result is easily obtained. If $m > 1$, then the method of proof of Theorem 6 below shows that $P = \pm U_m$. Now suppose $n > 1$. By Lemma 3a, $\pm U_n$ are the only polynomials Y of degree $n - 1 \geq 1$ which satisfy the differential equation

$$A^2[\overline{Y}']^4 = n^4[A(\overline{Y})]^2,$$

where $\overline{Y}(x) = \sqrt{4 - x^2}\, Y(x)$ and $A(x) = 4 - x^2$. But the hypothesis that $\overline{U}_n(\overline{P}) = \overline{P}(\overline{U}_n)$ for $n \geq 1$, together with Lemma 3b implies that $\overline{P}$ satisfies this differential equation with n replaced by m. Thus, taking square roots,

$$(4 - x^2)[\overline{P}'(x)]^2 = n^2[4 - \overline{P}^2(x)] \quad \text{or} \quad -n^2[4 - \overline{P}^2(x)].$$

The latter leads to $m^2 + n^2 = 0$, which is impossible. Therefore, Lemma 3a applies, and $P = \pm U_m$. If n is odd, then U_n is an even function, and $P = U_m$; if n is even, then U_n is an odd function, and $P = \pm U_m$.

Identities (2) and (3) show that $\overline{U}_m$ and T_n sometimes commute. Theorems 4 and 5 below tell precisely when this happens and also answer the following questions: What polynomials Q commute with a given $\overline{U}_m$? What functions of the form $\sqrt{4 - x^2}\, P(x)$ commute with a given T_n for $n \geq 2$? The proofs, which are omitted, follow closely the arguments already used in this section.

Theorem 4: Suppose $Q(x)$ is a polynomial of degree $m \geq 2$ and $Q(x)$ commutes with $\overline{U}_n(x)$ for some $n \geq 1$. Then $m \equiv 1 \bmod 4$ and $Q(x) = T_m(x)$. Moreover, if

$$Q(\overline{U}_n(x)) \equiv -\overline{U}_n(Q(x)) \text{ for some } n \geq 1,$$

then $m \equiv 3 \bmod 4$ and $P(x) = T_m(x)$.

Theorem 5: Suppose $P(x)$ is a polynomial of degree $m - 1 \geq 0$ and

$$\overline{P}(x) = \sqrt{4 - x^2}\, P(x).$$

If $\overline{P}(x)$ commutes with $T_n(x)$ for some $n \geq 2$, then $m \equiv 1 \bmod 4$ and $P(x) = U_m(x)$. Moreover, if $\overline{P}(T_n(x)) \equiv -T_n(\overline{P}(x))$ for some $n \geq 2$, then $m \equiv 3 \bmod 4$ and $P(x) = U_m(x)$.

4. SEMIPERMUTABLE CHAINS

Two functions $f(x)$ and $g(x)$ are defined in Kuczma [5, p. 215] to be *semi-permutable* if there exists a function

$$\Phi(x) = \frac{Kx + L}{Mx + N}$$

such that

$$f(g(x)) = \Phi[g(f(x))]. \tag{22}$$

Two functions $f(x)$ and $v(x)$ are *equivalent* if there exists a function

$$\phi(x) = rx + s, \text{ where } r \neq 0, \tag{23}$$

such that

$$\phi^{-1}[f(\phi(x))] = v(x).$$

Lemma 6a: Suppose $\phi(x)$ and $\Phi(x)$ are as just described and that (22) holds. Then the functions

$$F(x) = \phi^{-1}[f(\phi(x))] \quad \text{and} \quad G(x) = \phi^{-1}[g(\phi(x))]$$

are semipermutable.

Proof: For $\Psi(x) = \dfrac{Ax + B}{Cx + D}$, where $A = K - sM$, $B = L - sN$, $C = rM$, and $D = rN$, we have

$$\begin{aligned} F(G(x)) &= \phi^{-1} \circ f \circ g \circ \phi(x) = \phi^{-1} \circ \Phi \circ g \circ f \circ \phi(x) \\ &= \Psi \circ \phi^{-1} \circ g \circ f \circ \phi(x) = \Psi[G(F(x))], \end{aligned}$$

where the symbol $\circ$ indicates composition.

Suppose Γ is a sequence of positive integers and

$$P = \{p_n(x)\} \quad \text{and} \quad D = \{d_n(x)\}$$

are sequences of functions indexed by Γ. We define P to be an *SP chain under* D if every pair of functions in the set

$$\{p_n(x)d_n(x) : n \in \Gamma\}$$

are semipermutable. This definition generalizes that for SP chains given in [5], which is obtainable from the present definition in the case $d_n(x) \equiv 1$ for all positive integers n.

If $P = \{p_n(x)\}_{n \in \Gamma}$ is an SP chain under $D = \{d_n(x)\}_{n \in \Gamma}$ and $Q = \{q_n(x)\}_{n \in \Gamma}$ is an SP chain under $E = \{e_n(x)\}_{n \in \Gamma}$, then P and Q are *equivalent* if there exists $\phi(x)$ as in (23) such that

$$\phi^{-1}[p_n(\phi(x))d_n(\phi(x))] = q_n(x)e_n(x) \text{ for all } n \text{ in } \Gamma.$$

Corollary to Lemma 6a: Suppose $\{u_n(x)\}$ is an SP chain under $\{d_n(x)\}$ and $\phi(x) = rx + s$, where $r \neq 0$. Write

$$\phi^{-1}[p_n(\phi(x))d_n(\phi(x))] \text{ as } q_n(x)e_n(x).$$

[This is always possible, since we may choose $e_n(x) \equiv 1$ for all n in Γ.] Then $\{q_n(x)\}$ is an SP chain under $\{e_n(x)\}$.

If Γ is the sequence of odd positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is an *even SP chain under* D. Similarly, if Γ is the sequence of even positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is an *odd SP chain under* D. In particular, we define a *Chebyshev even chain* by

$$|p_n(x)| = U_n(x) \quad \text{and} \quad d_n(x) = \sqrt{4 - x^2} \text{ for } n = 1, 3, 5, \ldots;$$

and a *Chebyshev odd chain* by the same symbols, for $n = 2, 4, 6, \ldots$.

Finally, if Γ is the sequence of all the positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is a *complete SP chain under D*.

Lemma 6b: Suppose α, a, and e are nonzero, $\beta^2 \neq 4\alpha\gamma$, $F(x) = e\sqrt{\alpha x^2 + \beta x + \gamma}$, and $G(x) = \sqrt{\alpha x^2 + \beta x + \gamma}(ax^2 + bx + c)$. If $F(x)$ and $G(x)$ are semipermutable, then $F(x)$ and $G(x)$ are equivalent [with the same ϕ in (23)], respectively, to the functions

$$\overline{U}_1(x) = \sqrt{4 - x^2} \quad \text{and} \quad a_3\overline{U}_3(x) = a_3(x^2 - 1)\sqrt{4 - x^2}, \text{ where } a_3^2 = 1.$$

Proof:

$$(24) \qquad [F(G(x))]^2 = e^2[\alpha^2a^2x^6 + (2\alpha^2ab + \alpha\beta a^2)x^5 + (\alpha^2b^2 + 2\alpha^2ac + 2\alpha\beta ab + \alpha\gamma a^2)x^4 + (2\alpha^2bc + \alpha\beta b^2 + 2\alpha\beta ac + 2\alpha\gamma ab)x^3 + (\alpha^2c^2 + 2\alpha\beta bc + \alpha\gamma b^2 + 2\alpha\gamma ac)x^2 + (\alpha\beta c^2 + 2\alpha\beta bc)x + (\alpha\gamma c^2 + \gamma) + \beta(ax^2 + bx + c)\sqrt{\alpha x^2 + \beta x + \gamma}],$$

and

$$(25) \qquad [KG(F(x)) + L]^2 = K^2[\alpha^4a^2e^6x^6 + 3\alpha^3\beta a^2e^6x^5 + \alpha^2ae^5(\beta a + 2\alpha b)x^4\sqrt{\alpha x^2 + \beta x + \gamma} + \cdots + 2KL(\quad) + L^2,$$

where the expression indicated parenthetically after $2KL$ contains no nonzero constant multiple of $x^4\sqrt{\alpha x^2 + \beta x + \gamma}$.

In (22), suppose $M \neq 0$. Then, squaring both sides of (22) and writing

$$[MG(F(x)) + N]^2[F(G(x))]^2 = [KG(F(x)) + L]^2,$$

the left side contains for its highest degree term a multiple of x^{12}, whereas the highest degree term on the right side is $K^2\alpha^4a^2e^6x^6$. Therefore, $M = 0$, and there is no loss in assuming that $\Phi(x)$ is simply $Kx + L$.

Equating coefficients of x^6 and x^5 in (24) and (25) gives $K^2\alpha^2e^4 = 1$ and $\alpha b = \beta a$. The assumption $\beta^2 \neq 4\alpha\gamma$ keeps $\sqrt{\alpha x^2} + \beta x + \gamma$ from being a polynomial, and this implies that the coefficient $(\alpha^2\beta a^2 + 2\alpha^3ab)e^5$ in (25) equals 0; together with $\alpha b = \beta a$ and $\alpha \neq 0$, this means $\beta = b = 0$. Thus,

$$(26) \qquad [F(G(x))]^2 = e^2[\alpha^2a^2x^6 + (2\alpha^2ac + \alpha\gamma a^2)x^4 + (\alpha^2c^2 + 2\alpha\gamma ac)x^2 + \alpha\gamma c^2 + \gamma]$$

and

$$(27) \quad [KG(F(x)) + L]^2 = K^2[\alpha^2e^2x^2 + \gamma(\alpha e^2 + 1)][\alpha^2a^2e^4x^4 + 2\alpha ae^2(\gamma ae^2 + c)x^2 + (\gamma ae^2 + c)^2] + 2KL\sqrt{\alpha^2e^2x^2 + \gamma(\alpha e^2 + 1)}(\alpha ae^2x^2 + \gamma ae^2 + c) + L^2.$$

Again comparing coefficients, we see that either $L = 0$ or $\sqrt{\alpha^2e^2x^2 + \gamma(\alpha e^2 + 1)}$ is a polynomial. The latter implies $\alpha e^2 = -1$, which, by comparison of odd powers of x, leads to $L = 0$.

Multiplying out the right side of (27) and again comparing coefficients with (26), we find

$$(28) \qquad \gamma a(2\alpha e^2 + 1) + 2\alpha c(1 - \alpha e^2) = 0,$$

$$(29) \qquad \alpha^2ce^4(\alpha c + 2\gamma a) - (\gamma ae^2 + c)(3\alpha\gamma ae^2 + \alpha c + 2\gamma a) = 0,$$

$$(30) \qquad c^2(1 - \alpha e^2 + \alpha^3e^6) - \alpha^2e^6 + \gamma ae^2(\alpha e^2 + 1)(\gamma ae^2 + 2c) = 0.$$

Evaluating (26) and (27) at $x^2 = -\gamma/\alpha$ and equating them gives $e^2 = K^2c^2$, so that $c^2 = \alpha^2e^2$. We now rewrite (28), (29), and (30) with $q = \gamma a$ and $\alpha e = \delta c$, where $|\delta| = 1$:

(31) $\quad 2c^3e - 2\delta c^2 - 2\delta ce^2q - qe = 0,$

(32) $\quad \delta c^5e^2 + (2qe^3 - \delta)c^3 - 4q\delta c^2e^2 - (3\delta qe^3 + 2)qce - 2q^2e^3 = 0,$

(33) $\quad c^3e(c^2 - 1) + \delta c^2(1 - e^4) + qe^2(ce + \delta)(qe^2 + 2c) = 0.$

If $2\delta ce + 1 = 0$, no q satisfies both (32) and (33). Therefore, $2\delta ce + 1 \neq 0$, and in this case we find

$$q = \frac{2c^2(ce - \delta)}{e(2\delta ce + 1)}$$

from (31) and substitute into (32) to obtain $c^2e^2 = 1$. For $\delta = 1$, we find from $c^2e^2 = 1$ that $ce = -1$, since if $ce = 1$ then $q = 0$, contrary to $\gamma \neq 0 \neq a$. Simplifying the expression for q gives $\gamma ae = 4c^2$. Also, from $\alpha e = \delta c$ comes $\alpha e^2 = -1$. Similarly for $\delta = -1$, we determine $ce = 1$, $\gamma ae = -4c^2$, and $\alpha e^2 = -1$.

Now for $\phi(x) = e\sqrt{\gamma}x/2$, it is easy to verify that

$$\phi^{-1}[F(\phi(x))] = \sqrt{4 - x^2}$$

and, using the fact $\gamma ae^3 = 4\delta$, that

$$\phi^{-1}[G(\phi(x))] = e^{-2}(x^2 - 1)\sqrt{4 - x^2}.$$

Finally, it is easy to check directly that these two functions are semipermutable if and only if $e^2 = \pm 1$, and this completes the proof.

Theorem 6: Every even SP chain under a constant sequence of the form

$$d_n(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$$

is equivalent to a Chebyshev even chain $\{a_nU_n(x)\}$, $a_n^2 = 1$, $n = 1, 3, 5, \ldots$.

Proof: Suppose $\{y_1, y_3, y_5, \ldots\}$ is an even SP chain under $d(x) = d_n(x)$ as above. Let $\overline{y}_n(x) = y_n(x)d(x)$. By Lemma 6b, we may assume that $d(x) = \sqrt{4 - x^2}$. Since every even polynomial $y_n(x)$ of degree $n - 1$ is a linear combination of even $U_i(x)$'s up to degree $n - 1$, we write

$$\overline{y}_n(x) = a_n\overline{U}_n(x) + \sum_{i=1}^{m} b_i\overline{U}_i(x), \quad n > m \geq 1,$$

where $b_i = 0$ for even i. Suppose $b_m \neq 0$. Then

(34) $$[\overline{y}_1(\overline{y}_n(x))]^2 = (4 - \overline{y}_n^2(x)) = \left\{-a_n^2\overline{U}_n^2(x) - 2a_n\overline{U}_n(x)\sum_{i=1}^{m} b_i\overline{U}_i(x) - \left[\sum_{i=1}^{m} b_i\overline{U}_i(x)\right]^2 + 4\right\}$$

and

(35) $$[K\overline{y}_n(\overline{y}_1(x)) + L]^2 = K^2\left[a_n\overline{U}_n(\overline{U}_1(x)) + \sum_{i=1}^{m} b_i\overline{U}_i(\overline{U}_1(x))\right]^2 + 2KL\left[a_n\overline{U}_n(\overline{U}_1(x)) + \sum_{i=1}^{m} b_i\overline{U}_i(\overline{U}_1(x))\right] + L^2.$$

The highest degree term on the right side of (34) is $a_n^2x^{2n}$, while that on the right side of (35) is $(-1)^{n-1}K^2a_n^2x^{2n}$. Thus, $K^2 = 1$, so subtracting (35) from (34) and using Lemma 2a [rewritten as $T_m(x)T_n(x) + \overline{U}_m(x)\overline{U}_n(x) = 2T_{n-m}(x)$ for $0 \leq m \leq n$],

$$0 = [\overline{y}_1(\overline{y}_n(x))]^2 - [K\overline{y}_n(\overline{y}_1(x)) + L]^2 = -a_n^2\overline{U}_n^2(x) - 2a_n\sum_{i=1}^m b_i\overline{U}_n(x)\overline{U}_i(x)$$
$$- \left[\sum_{i=1}^m b_i\overline{U}_i(x)\right]^2 + 4 - \left\{a_n^2T_n^2(x) + 2a_n\sum_{i=1}^m b_iT_n(x)T_i(x) + \left[\sum_{i=1}^m b_iT_i(x)\right]^2\right\}$$
$$- 2KL\left[a_nT_n(x) + \sum_{i=1}^m b_iT_i(x)\right] - L^2$$
$$= -a_n^2[\overline{U}_n^2(x) + T_n^2(x)] - 2a_n\sum_{i=1}^m b_i[\overline{U}_n(x)\overline{U}_i(x) + T_n(x)T_i(x)]$$
$$-\left\{\left[\sum_{i=1}^m b_i\overline{U}_i(x)\right]^2 + \left[\sum_{i=1}^m b_iT_i(x)\right]^2\right\} - 2KL\left[a_nT_n(x) + \sum_{i=1}^m b_iT_i(x)\right]$$
$$- L^2 + 4.$$

Thus,

$$0 = -4a_n^2 - 4a_n\sum_{i=1}^m b_iT_{n-i}(x) - \left[2\sum_{i=1}^m b_i^2 + 4\sum_{1\le i<j\le m} b_ib_jT_{j-i}(x)\right]$$
$$- 2KL\left[a_nT_n(x) + \sum_{i=1}^m b_iT_i(x)\right] - L^2 + 4. \tag{36}$$

If $L \neq 0$, the right side of (36) is a polynomial of degree n. Therefore, $L = 0$. If $b_m \neq 0$, the right side of (36) is a polynomial of degree $n - 1$, again a contradiction. Therefore, $m = 0$, so that

$$\overline{y}_n(x) = a_n\overline{U}_n(x) \text{ for } n > 1,$$

and (36) shows that $a_n^2 = 1$ for $n > 1$.

Lemma 7a: Suppose α, a, and e are nonzero, $\beta^2 \neq 4\alpha\gamma$,

$F(x) = (ex + f)\sqrt{\alpha x^2 + \beta x + \gamma}$ and $G(x) = (ax^3 + bx^2 + cx + d)\sqrt{\alpha x^2 + \beta x + \gamma}$.

If $F(x)$ and $G(x)$ are semipermutable, then $F(x)$ and $G(x)$ are equivalent [with the same ϕ in (23)], respectively, to the functions

$$\overline{U}_2(x) = x\sqrt{4 - x^2} \text{ and } a_4\overline{U}_4(x) = a_4(x^3 - 2x)\sqrt{4 - x^2}, \text{ where } a_4^2 = 1.$$

Proof: Write $A = \sqrt{\alpha x^2 + \beta x + \gamma}$ and $B = ax^3 + bx^2 + cx + d$, so that

$$F(x) = (ex + f)A \quad \text{and} \quad G(x) = BA.$$

Direct computations show

$$[F(G(x))]^2 = \alpha e^2G^4(x) + (\alpha f^2 + 2\beta ef + \gamma e^2)G^2(x)$$
$$+ [e(2\alpha f + \beta e)G^2(x) + f(\beta f + 2\gamma e)]BA \tag{37}$$

and

$$[KG(F(x)) + L]^2 = K^2[Q_8F^8(x) + Q_7F^7(x) + \cdots + Q_1F(x) + Q_0]$$
$$+ 2KLG(F(x)) + L^2, \tag{38}$$

where

$Q_8 = \alpha a^2$, $Q_7 = a(2\alpha b + \beta a)$,
$Q_6 = 2\alpha ac + \alpha b^2 + 2\beta ab + \gamma a^2$, $Q_5 = 2\alpha ad + 2\alpha bc + 2\beta ac + \beta b^2 + 2\gamma ab$,
$Q_4 = 2\alpha bd + \alpha c^2 + 2\beta ad + 2\beta bc + 2\gamma ac + \gamma b^2$,
$Q_3 = 2\alpha cd + 2\beta bd + \beta c^2 + 2\gamma ad + 2\gamma bc$, $Q_2 = \alpha d^2 + 2\beta cd + 2\gamma bd + \gamma c^2$,
$Q_1 = d(\beta d + 2\gamma c)$, $Q_0 = \gamma d^2$.

Comparing coefficients of x^{16} in (37) and (38) gives $a^2 = K^2\alpha^2 e^6$. In (38) only the expression $K^2 a(2\alpha b + \beta a)F^7(x)$ comtains a nonzero multiple of $x^{13}A$, and (37) contains no such term. Specifically, (38) contains the term

$$K^2\alpha^3 a e^7(2\alpha b + \beta a)x^{13}A.$$

The condition $\beta^2 \neq 4\alpha\gamma$ keeps A from being a polynomial, and since $K^2\alpha^3 a e^7 \neq 0$, comparison with terms in (37) gives

$$\beta a = -2\alpha b. \tag{39}$$

In (37) only the expression $e(2\alpha f + \beta e)G^2(x)BA$ contains a nonzero multiple of $x^{11}A$, and (38) contains no such term. Writing this expression as

$$e(2\alpha f + \beta e)(\alpha a^2 x^8 + \cdots)(ax^3 + \cdots)A,$$

we find by comparison with (37) that

$$\beta e = -2\alpha f. \tag{40}$$

Since A is not a polynomial, the expression

$$\sqrt{\alpha A^2(ex + f)^2 + \gamma + \beta A}[bF^2(x) + d + (aF^2(x) + c)(ex + f)A] \tag{41}$$

for $G(F(x))$ in (38) cannot be of the form $R(x) + Q(x)A$ for any polynomials $R(x)$ and $Q(x)$ unless perhaps $\beta = 0$. Thus, for $\beta \neq 0$, the expression (41) is linearly independent of the other terms in (38) and all those in (37), so that $L = 0$. On the other hand, if $\beta = 0$, then $b = f = 0$ by (39) and (40). Then (37) shows $[F(G(x))]$ to be a polynomial, and (41) reduces to

$$\sqrt{\alpha A^2 e^2 x^2 + \gamma}[d + (ae^2 A^2 x^2 + c)ex\sqrt{\alpha x^2 + \gamma}].$$

For this to be a polynomial requires $\gamma = 0$, contrary to $\beta^2 \neq 4\alpha\gamma$. Consequently, for $\beta = 0$, we still have $L = 0$.

Equation (40) shows that no multiple of $x^p A$ occurs in $[F(G)]^2$ for any $p > 3$. Since only $Q_5 F^5(x)$ in (38) contains such a multiple for $p = 9$, we have $Q_5 = 0$. Because of this and the fact that $Q_3 F^3(x)$ alone in (38) contains a multiple of $x^5 A$, we have $Q_3 = 0$. This leaves (38) with no multiple of $x^3 A$, so that the coefficient of $x^3 A$ in (37), namely $f(\beta f + 2\gamma e)$, must equal 0. If $f \neq 0$, then eliminating e from $\beta f + 2\gamma e = 0$ and $\beta e + 2\alpha f = 0$ gives $\beta^2 = 4\alpha\gamma$, which is forbidden. Therefore, $f = 0$. By (40) and (39), $\beta = b = 0$ also.

For x_0 a root of $\alpha x^2 + \beta x + \gamma$,

$$F[G(x_0)] = F(0) = \sqrt{\gamma} f = 0 \quad \text{and} \quad G[F(x_0)] = G(0) = \sqrt{\gamma} d;$$

since $L = 0$, we have $\sqrt{\gamma} d = 0$. The condition $\beta^2 \neq 4\alpha\gamma$ implies $\gamma \neq 0$. We summarize our findings:

$$\beta = 0,\ b = 0,\ f = 0,\ d = 0,\ L = 0,\ Q_5 = 0,\ Q_3 = 0,\ Q_1 = 0. \tag{42}$$

These enable us to simplify (37) and (38) as follows:

$$\begin{aligned}[F(G(x))]^2 = {} & \alpha^3 a^4 e^2 x^{16} + 2\alpha^2 a^3 e^2(2\alpha c + \gamma a)x^{14} \\ & + \alpha a^2 e^2(6\alpha^2 c^2 + 8\alpha\gamma ac + \gamma^2 a^2)x^{12} \\ & + 2\alpha ace^2(5\alpha\gamma ac + 2\gamma^2 a^2 + 2\alpha^2 c^2 + \alpha\gamma ac)x^{10} \\ & + \alpha e^2(6\gamma^2 a^2 c^2 + 8\alpha\gamma ac^3 + \alpha^2 c^4 + \gamma a^2)x^8 \\ & + \gamma e^2(4\alpha\gamma ac^3 + 2\alpha^2 c^2 + 2\alpha ac + \gamma a^2)x^6 \\ & + \gamma ce^2(\alpha\gamma c^3 + 2\gamma a + \alpha c)x^4 + \gamma^2 c^2 e^2 x^2;\end{aligned} \tag{43}$$

$$\begin{aligned}K^2[G(F(x))]^2 = {} & K^2\{\alpha^5 a^2 e^8 x^{16} + 4\alpha^4 a^2\gamma e^8 x^{14} + \alpha^3 e^6(6\gamma^2 a^2 e^2 \\ & + 2\alpha ac + \gamma a^2)x^{12} + \alpha^2\gamma e^6(4\gamma^2 a^2 e^2 + 6\alpha ac + 3\gamma a^2)x^{10} \\ & + \alpha e^4(\gamma^4 a^2 e^4 + 6\alpha\gamma^2 ace^2 + 3\gamma^3 a^2 e^2 + \alpha^2 c^2 + 2\alpha\gamma ac)x^8 \\ & + \gamma e^4(2\alpha\gamma^2 ace^2 + \gamma^3 a^2 e^2 + 2\alpha^2 c^2 + 4\alpha\gamma ac)x^6 \\ & + \gamma ce^2(\alpha\gamma ce^2 + 2\gamma^2 ae^2 + \alpha c)x^4 + \gamma^2 c^2 e^2 x^2\}.\end{aligned} \tag{44}$$

Comparing coefficients of x^{16}, x^{14}, ..., x^2, in order, gives

(45) $$a^2 = \alpha^2 e^6 \text{ [because of (52) below]}$$

(46) $$2\alpha c = \gamma a$$

(47) $$13a^2c^2 = e^4(3\gamma^2 a^2 e^2 + 2\alpha a c)$$

(48) $$11\alpha a c = e^4(\gamma^3 a^2 e^2 + 6\alpha^2 c^2)$$

(49) $$41\alpha^2 c^4 + 2\alpha a c = \gamma^4 a^2 e^6 + 24\alpha^2 \gamma c^2 e^4 + 5\alpha^2 c^2 e^2$$

(50) $$5\alpha c^2 + 2a = 4\alpha\gamma c e^4 + 5\alpha c e^2$$

(51) $$\alpha c^3 + 2a = 5\alpha c e^2$$

(52) $$K^2 = 1.$$

Subtracting (51) from (50) gives

(53) $$c^2 = \gamma e^4.$$

Eliminating α from (46) and (47) gives

(54) $$13c^2 = \gamma e^4(3\gamma e^2 + 1).$$

Eliminating c^2 from (53) and (54) gives

(55) $$\gamma e^2 = 4.$$

With (45), (53), and (55) in mind, we now discern four possibilities for given a and e:

(56) $$\alpha = -ae^3 \quad \text{and} \quad c = -2e$$

(57) $$\alpha = ae^3 \quad \text{and} \quad c = -2e$$

(58) $$\alpha = -ae^3 \quad \text{and} \quad c = 2e$$

(59) $$\alpha = ae^3 \quad \text{and} \quad c = 2e.$$

For (56), we have

$$F(x) = x\sqrt{4 - ae^5x^2} \quad \text{and} \quad G(x) = e^{-1}(ax^3 + cx)\sqrt{4 - ae^5x^2}.$$

For $\phi(x) = x/\sqrt{ae^5}$ we find that $\phi^{-1}[F(\phi(x))] = x\sqrt{4 - x^2}$ and, using the assumption $c = -2e$, that

$$\phi^{-1}[G(\phi(x))] = (e^{-6}x^3 - 2x)\sqrt{4 - x^2}.$$

It is easily checked directly that these two functions are semipermutable iff $e^6 = 1$.

Direct checking for semipermutability further shows that (57) gives F and G respectively equivalent to $\overline{U}_2$ and $\overline{U}_4$, while (58) and (59) give functions respectively equivalent to $\overline{U}_2$ and $-\overline{U}_4$ as desired.

Theorem 7: Every odd SP chain under a constant sequence of the form

$$d_n(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$$

is equivalent to a Chebyshev odd chain $\{a_n U_n(x)\}$, $a^2 = 1$, $n = 2, 4, 6, \ldots$.

Proof: Suppose $\{y_2, y_4, \ldots\}$ is an odd SP chain under $d(x) = d_n(x)$ as above. Let $\overline{y}_n(x) = y_n(x)d(x)$. By Lemma 7a, we may assume that $d(x) = \sqrt{4 - x^2}$. Since every odd polynomial $y_n(x)$ of degree $n - 1$ is a linear combination of odd $U_i(x)$'s up to degree $n - 1$, we write

$$\overline{y}_n(x) = a_n\overline{U}_n(x) + \sum_{i=1}^{m} b_i\overline{U}_i(x), \; n > m \geq 1,$$

where $b_i = 0$ for odd i. The rest of the proof follows that of Theorem 6 exactly.

Theorem 8: Suppose $d(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$ where $\alpha \neq 0$ and $\beta^2 \neq 4\alpha\gamma$. There exists no complete SP chain under D.

Proof: Referring to the definitions given just before Lemma 6b, if such a chain $\{p_1(x), p_2(x), \ldots\}$ exists, then the chain $\{p_1(x), p_3(x), \ldots\}$ is an even SP chain. The proof of Lemma 6b shows that we may assume $\Phi(x) = Kx + L$ in (22) and $\alpha = -1$ and $\beta = 0$. Thus, we write

$$\overline{p}_1(x) = a\sqrt{-x^2 + \gamma} \quad \text{and} \quad \overline{p}_2(x) = (bx + c)\sqrt{-x^2 + \gamma}$$

where a, b, and γ are nonzero. Writing out the assumption

$$[\overline{p}_1(\overline{p}_2(x))]^2 = [K\overline{p}_2(\overline{p}_1(x)) + L]^2,$$

we find the term $2K^2a^3bcx^2\sqrt{-x^2 + \gamma}$ on the right side and all other terms in this equation linearly independent of this term. Thus $c = 0$, so that

$$[\overline{p}_1(\overline{p}_2(x))]^2 = a^2b^2x^4 - \gamma a^2b^2x^2 + \gamma a^2.$$

It is easily checked that $L = 0$, so that

$$\begin{aligned}[K\overline{p}_2(\overline{p}_1(x)) + L]^2 &= -a^4b^2K^2x^4 + \gamma a^2b^2K^2(2a^2 - 1)x^2 \\ &\quad + K^2\gamma^2a^2b^2(1 - a^2).\end{aligned}$$

Comparison of coefficients of x^4 gives $a^2K^2 = -1$, which along with comparison of coefficients of x^2 implies $K^2 = -1$. But this leads to a contradiction, since comparison of constant terms gives $1 = \gamma b^2(K^2 + 1)$.

REFERENCES

1. E. A. Bertram. "Polynomials which Commute with a Tchebycheff Polynomial." *American Math. Monthly* 78 (1971):650-653.
2. Brother Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables*. San Jose, Calif.: The Fibonacci Association, 1972.
3. V. E. Hoggatt, Jr., & D. A. Lind. "Symbolic Substitution into Fibonacci Polynomials." *The Fibonacci Quarterly* 6, No. 5 (1968):55-74.
4. C. H. Kimberling. "Greatest Common Divisors of Sums and Differences of Fibonacci, Lucas, and Chebyshev Polynomials." *The Fibonacci Quarterly*, 17 No. 1 (1979):18-22.
5. M. Kuczma. *Functional Equations in a Single Variable*. Monografie Matematyczne, Tom 46. Warszawa: Akademia Nauk, 1968.
6. T. J. Rivlin. *The Chebyshev Polynomials*. New York: Wiley, 1974.

*****

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, NM 87131

Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1 \text{ and } L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-436 *Proposed by Sahib Singh, Clarion State College, Clarion, PA.*

Find an appropriate expression for the nth term of the following sequence and also find the sum of the first n terms:

$$4,\ 2,\ 10,\ 20,\ 58,\ 146,\ 388,\ 1010,\ \ldots\ .$$

B-437 *Proposed by G. Iommi Amunategui, Universidad Católica de Valparaíso, Valparaíso, Chile.*

Let $[m, n] = mn(m + n)/2$ for positive integers m and n. Show that:

(a) $[m + 1, n][m, n + 2][m + 2, n + 1] = [m, n + 1][m + 2, n][m + 1, n + 2]$.

(b) $\displaystyle\sum_{k=1}^{m} [m + 1 - k, k] = m(m + 1)^2(m + 2)/12$.

(We note that part (a) is the Hoggatt-Hansell "Star of David" property for the $[m, n]$.)

B-438 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Let n and w be integers with w odd. Prove or disprove the proposed identity

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} + F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

B-439 *Proposed by A. P. Hillman, University of New Mexico, Albuquerque, NM.*

Can the proposed identity of B-438 be proved by mere verification for a finite set of ordered pairs (n, w)? If so, how few pairs suffice?

B-440 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA.*

(a) Let $n = x^2 + y^2$, with x and y integers not both zero. Prove that there is a nonnegative integer k such that $n \equiv 2^k \pmod{2^{k+2}}$.

(b) If $n \equiv 2^k \pmod{2^{k+2}}$, must n be a sum of two squares?

B-441 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA.*

A base-b palindrome is a positive integer whose base-b representation reads the same forward and backward. Prove that the sum of the reciprocals of all base-b palindromes converges for any given integer $b \geq 2$.

SOLUTIONS

GCD Not LCM

B-412 *Proposed by Phil Mana, Albuquerque, NM.*

Find the least common multiple of the integers in the infinite set

$$\{2^9 - 2,\ 3^9 - 3,\ 4^9 - 4,\ \ldots,\ n^9 - n,\ \ldots\}.$$

Solution by Sahib Singh, Clarion College, Clarion, PA.

The least common multiple is infinite because every positive integer n is to be its factor. If we want the greatest common divisor of the members of the set, we note that

$$n^9 - n = n(n - 1)(n + 1)(n^2 + 1)(n^4 + 1) = (n^5 - n)(n^4 + 1).$$

Since $n(n - 1)(n + 1) \equiv 0 \pmod 6$ and $n^5 - n \equiv 0 \pmod 5$, we conclude that

$$n^9 - n \equiv 0 \pmod{30} \text{ for } n = 2, 3, \ldots .$$

By examining the first two terms of the set, we see that the greatest common divisor is 30.

Also solved by Paul. S. Bruckman, Lawrence Somer, and the proposer.

Counting Equilateral Triangles

B-413 *Proposed by Herta T. Freitag, Roanoke, VA.*

For every positive integer n, let U_n consist of the points $j + ke^{2\pi i/3}$ in the Argand plane with

$$j \in \{0, 1, 2, \ldots, n\} \quad \text{and} \quad k \in \{0, 1, \ldots, j\}.$$

Let $T(n)$ be the number of equilateral triangles whose vertices are subsets of U_n. For example, $T(1) = 1$, $T(2) = 5$, and $T(3) = 13$.

(a) Obtain a formula for $T(n)$.
(b) Find all n for which $T(n)$ is an integral multiple of $2n + 1$.

Solution by W. O. J. Moser, McGill University, Montreal, P.Q., Canada.

For the problem as given, $T(3)$ is 15 and not 13 as stated in the problem. The difference may be accounted for by the triangles $\{[2, 2], [1, 0], [3, 1]\}$ and $\{[1, 1], [2, 0], [3, 2]\}$, where $[j, k]$ denotes $j + ke^{2\pi i/3}$. The proposer probably meant to count only the triangles with a side parallel to the real axis. The intended problem is the same as Problem 889, *Math. Mag.* 47 (1974), solution *ibid.* 47 (1974):289-91, where other references are given.

Using the following well-known result one can count various sets of vertices forming equilateral triangles in U_n:

<u>*Lemma*</u>: Let m and r be integers, $m \geq 0$, $r \geq 1$. The number of ordered r-tuples $(a_1, \ldots, a_r)$ of nonnegative integers a_i satisfying $a_1 + \cdots + a_r = m$ is

$$\binom{m + r - 1}{r - 1}.$$

Triples of the form

$$\{[j, k], [j - i, k], [j, k + i]\},\ \{[j, k], [j, k - i], [j + i, k]\},$$
$$\{[j + i, k], [j, k + i], [j - i, k - i]\},$$
$$\{[j - i, k], [j, k - i], [j + i, k + i]\}$$

all form equilateral triangles.

Let $A_s(n)$ for $s = 1, 2, 3, 4$ denote the numbers of triples in U of these forms in the order listed. Geometrically, one sees easily that $A_3(n) = A_4(n)$.

Since $[j, k] \in U_n$ if and only if j and k are integers with $0 \le k \le j \le n$, $A_1(n)$ is the number of ordered triples (i, j, k) of nonnegative integers satisfying $1 \le i$ and $k + i \le j \le n$. Letting $x = i - 1$, $y = k$, $z = j - i - k$, and $w = n - j$, we see that $A_1(n)$ is the set of ordered quadruples (x, y, z, w) of nonnegative integers with $x + y + z + w = n - 1$; hence, $A_1(n) = \binom{n+2}{3}$ by the lemma. Other types of triangles may be enumerated similarly.

The answer for the intended problem is

$$T(n) = [n(2n + 1)(n + 2) - \theta_n]/8,$$

with $\theta_n = 0$ for n even and $\theta_n = 1$ for n odd. Hence, $(2n + 1)|T(n)$ iff n is even.

Also solved by Paul S. Bruckman and the proposer.

B-414 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let

$$S_n = L_{n+5} + \binom{n}{2} L_{n+2} - \sum_{i=2}^{n} \binom{i}{2} L_i - 11.$$

Determine all n in $\{2, 3, 4, \ldots\}$ for which S_n is (a) prime; (b) odd.

Solution by Paul S. Bruckman, Concord, CA.

Note that

$$\begin{aligned}\Delta S_n &= S_{n+1} - S_n = L_{n+4} + \binom{n+1}{2} L_{n+3} - \binom{n}{2} L_{n+2} - \binom{n+1}{2} L_{n+1}\\ &= L_{n+4} + \left\{\binom{n+1}{2} - \binom{n}{2}\right\} L_{n+2} = L_{n+4} + nL_{n+3}\\ &= (n+1)L_{n+4} - nL_{n+3}.\end{aligned}$$

Hence, $S_n = nL_{n+3} + c$, for some constant c. Now

$$S_2 = L_7 + L_4 - L_2 - 11 = 29 + 7 - 3 - 11 = 22;$$

but also,

$$S_2 = 2L_5 + c = 2 \cdot 11 + c = 22 + c.$$

Hence, $c = 0$. Therefore,

$$S_n = nL_{n+3},\ n = 2, 3, 4, \ldots . \tag{1}$$

Clearly, since n and L_{n+3} are each integers greater than 1 (for $n \ge 2$), S_n is *never* prime. In order for S_n to be odd, both n and L_{n+3} must be odd. Now L_n is even iff $3|n$, as is readily seen by inspection of the first few values (mod 2) of the Lucas sequence. Hence, L_{n+3} is odd iff $3 \nmid n$. It follows that S_n is odd iff $n \equiv \pm 1 \pmod 6$.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

PROPOSALS TABLED

No solutions to problem B-415 were received. The problem was restated by the Elementary Problems Editor in a form not equivalent to the original problem.

No solutions to problem B-416 were received.

Not a Bracket Function

B-417 *Proposed by R. M. Grassl and P. L. Mana, University of New Mexico, Albuquerque, NM*

Here let $[x]$ be the greatest integer in x. Also, let $f(n)$ be defined by

$$f(0) = 1 = f(1),\ f(2) = 2,\ f(3) = 3,$$

and

$$f(n) = f(n-4) + [1 + (n/2) + (n^2/12)] \text{ for } n \in \{4, 5, 6, \ldots\}.$$

Do there exist rational numbers a, b, c, and d such that

$$f(n) = [a + bn + cn^2 + dn^3]?$$

Solution by Paul S. Bruckman, Concord, CA.

We first prove the following:

(1) $$f(12n) = 12n^3 + 15n^2 + 6n + 1,\ n = 0, 1, 2, \ldots.$$

Let S denote the set of all nonnegative integers n for which (1) is true. Since $f(0) = 1$, it is clear that $0 \in S$. Now $f(12n + 12) - f(12n)$

$$= \sum_{k=0}^{2}\Big(f(12n + 4k + 4) - f(12n + 4k)\Big) = \sum_{k=0}^{2}\left(1 + 6n + 2k + 2 + \left[\frac{16}{12}(3n+k+1)^2\right]\right)$$

$$= \sum_{k=0}^{2}\left(3 + 6n + 2k + \left[\frac{4}{3}\left\{9n^2 + 6n(k+1) + (k+1)^2\right\}\right]\right)$$

$$= \sum_{k=0}^{2}\Big\{\Big(3 + 6n + 2k + 12n^2 + 8n(k+1)\Big)\Big\} + [4/3] + [16/3] + 12$$

$$= 1 + 5 + 12 + \sum_{k=0}^{2}\left\{3 + 14n + 12n^2 + (8n+2)k\right\}$$

$$= 3(12n^2 + 14n + 3) + 3(8n + 2) + 18, \text{ or}$$

(2) $$f\Big(12(n+1)\Big) - f(12n) = 36n^2 + 66n + 33.$$

Suppose $n \in S$. Then

$$\begin{aligned} f\Big(12(n+1)\Big) &= 12n^3 + 15n^2 + 6n + 1 + 36n^2 + 66n + 33 \\ &= 12n^3 + 51n^2 + 72n + 34 \\ &= 12(n+1)^3 + 15(n+1)^2 + 6(n+1) + 1. \end{aligned}$$

Hence, $n \in S \Rightarrow (n+1) \in S$. By induction, (1) is proved.

Now, suppose that for *all* $n \geq 0$,

(3) $$f(n) = [a + bn + cn^2 + dn^3]$$

for some rational a, b, c, and d independent of n. Then

$$f(n) = a + bn + cn^2 + dn^3 + e_n,$$

where $e_n = 0(n)$ as $n \to \infty$. In particular, substituting $12n$ for n:

(4) $$f(12n) = a + 12bn + 144cn^2 + 1728dn^3 + e_{12n}.$$

By comparison of (1) and (4), it follows that $12b = 6$, $144c = 15$, $1728d = 12$, i.e.,

(5) $$b = 1/2 = 72/144,\ c = 15/144,\ d = 1/144.$$

Hence,

(6) $$f(n) = \left[\frac{n^3 + 15n^2 + 72n}{144} + a\right], \; n = 0, 1, 2, \ldots .$$

Note that

$$f(5) = f(1) + [1 + 5/2 + 25/12] = 1 + 1 + [55/12] = 6,$$

and

$$f(9) = f(5) + [1 + 9/2 + 81/12] = 6 + 1 + [45/4] = 18.$$

Setting $n = 0$ in (6) yields:

$$f(0) = 1 = [a];$$

however, setting $n = 9$ in (6) yields:

$$f(9) = 18 = [18 + a],$$

which implies $[a] = 0$. This contradiction establishes that the supposition in (3) is false.

Also solved by the proposers.

*****

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months of publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-320 *Proposed by Paul S. Bruckman, Concord, CA.*

Let

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \text{ Re}(s) > 1, \text{ the Riemann Zeta function.}$$

Also, let

$$H_n = \sum_{k=1}^{n} k^{-1}, \; n = 1, 2, 3, \ldots, \text{ the harmonic sequence.}$$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

H-321 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identity

$$\begin{aligned} F_{n+14r}^6 + F_n^6 &- (L_{12r} + L_{8r} + L_{4r} - 1)(F_{n+12r}^6 + F_{n+2r}^6) \\ &+ (L_{20r} + L_{16r} + L_{4r} + 3)(F_{n+10r}^6 + F_{n+4r}^6) \\ &- (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1)(F_{n+8r}^6 + F_{n+6r}^6) \\ &= 40(-1)^n \prod_{i=1}^{3} F_{2ri}^2. \end{aligned}$$

SOLUTIONS

Δ Dawn

H-294 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA. (Vol. 16, No. 6, December 1978)*

Evaluate

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & -F_{12r+6} & F_{20r+10} & -F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{30r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & -F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{30r+15} & F_{50r+25} & F_{70r+35} & F_{90r+45} \end{vmatrix}$$

Solution by the proposer.

After simplification,

$$\Delta = F_{2r+1}F_{6r+3}F_{10r+5}F_{14r+7}F_{18r+9}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ L_{2r+1} & -L_{6r+3} & L_{10r+5} & -L_{14r+7} & L_{18r+9} \\ L_{4r+2} & L_{12r+6} & L_{20r+10} & L_{28r+14} & L_{36r+18} \\ L_{6r+3} & -L_{18r+9} & L_{30r+15} & -L_{42r+21} & L_{54r+27} \\ L_{8r+4} & L_{24r+12} & L_{40r+20} & L_{56r+28} & L_{72r+36} \end{vmatrix}$$

$$= F_{2r+1}F_{6r+3}F_{10r+5}F_{14r+7}F_{18r+9}(L_{6r+3} + L_{2r+1})(L_{10r+5} - L_{2r+1})$$
$$\cdot\ (L_{14r+7} + L_{2r+1})(L_{18r+9} - L_{2r+1})(L_{10r+5} + L_{6r+3})(L_{14r+7} - L_{6r+3})$$
$$\cdot\ (L_{18r+9} + L_{6r+3})(L_{14r+7} + L_{10r+5})(L_{18r+9} - L_{10r+5})(L_{18r+9} + L_{14r+7})$$
$$= 5^{10}F^5_{2r+1}F^4_{4r+2}F^4_{6r+3}F^3_{8r+4}F^3_{10r+5}F^2_{12r+6}F^2_{14r+7}F_{16r+8}F_{18r+9}$$
$$= 5^{10}F^5_wF^4_{2w}F^4_{3w}F^3_{4w}F^3_{5w}F^2_{6w}F^2_{7w}F_{8w}F_{9w},\ \text{where } w = 2r+1.$$

More Identities

H-295 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*
(Vol. 17, No. 1, February 1979)

Establish the identities

(a) $F_kF^2_{k+6r+3} - F_{k+8r+4}F^2_{k+2r+1} = (-1)^{k+1}F^3_{2r+1}L_{2r+1}L_{k+4r+2}$

and

(b) $F_kF^2_{k+6r} - F_{k+8r}F^2_{k+2r} = (-1)^{k+1}F^3_{2r}L_{2r}L_{k+4r}$.

Solution by the proposer.

(a) $F_kF^2_{k+6r+3} - F_{k+8r+4}F^2_{k+2r+1}$

$$= \frac{1}{5\sqrt{5}}\Big\{(\alpha^k - \beta^k)[\alpha^{2k+12r+6} + \beta^{2k+12r+6} + 2(-1)^k]$$
$$- (\alpha^{k+8r+4} - \beta^{k+8r+4})[\alpha^{2k+4r+2} + \beta^{2k+4r+2} + 2(-1)^k]\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}(\alpha^{16r+8} - 2\alpha^{4r+2} + 2\alpha^{12r+6} - 1)$$
$$- \beta^{k-4r-2}(\beta^{16r+8} + 2\beta^{12r+6} - 2\beta^{4r+2} - 1)\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}(\alpha^{4r+2} - 1)(\alpha^{4r+2} + 1)^3 - \beta^{k-4r-2}(\beta^{4r+2} - 1)(\beta^{4r+2} + 1)^3\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}\alpha^{2r+1}(\alpha^{2r+1} + \beta^{2r+1})(\alpha^{2r+1})^3(\alpha^{2r+1} - \beta^{2r+1})^3$$
$$+ \beta^{k+4r+2}(\alpha^{2r+1} + \beta^{2r+1})(\alpha^{2r+1} - \beta^{2r+1})^3\Big\}$$
$$= (-1)^{k+1}F^3_{2r+1}L_{2r+1}L_{k+4r+2}.$$

(b) $F_kF^2_{k+6r} - F_{k+8r}F^2_{k+2r}$

$$= \frac{1}{5\sqrt{5}}\Big\{(\alpha^k - \beta^k)[\alpha^{2k+12r} + \beta^{2k+12r} + 2(-1)^{k+1}]$$
$$- (\alpha^{k+8r} - \beta^{k+8r})[\alpha^{2k+4r} + \beta^{2k+2r} + 2(-1)^{k+1}]\Big\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k+4r}(\alpha^{16r} - 2\alpha^{12r} + 2\alpha^{4r} - 1) - \beta^{k-4r}(\beta^{16r} - 2\beta^{12r} + 2\beta^{4r} - 1)\right\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k-4r}(\alpha^{4r} - 1)^3(\alpha^{4r} + 1) - \beta^{k-4r}(\beta^{4r} - 1)^3(\beta^{4r} + 1)\right\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k+4r}(\alpha^{2r} - \beta^{2r})^3(\alpha^{2r} + \beta^{2r}) + \beta^{k+4r}(\alpha^{2r} - \beta^{2r})^3(\alpha^{2r} + \beta^{2r})\right\}$$

$$= (-1)^{k+1}F_{2r}^3L_{2r}L_{k+4r}.$$

Also solved by P. Bruckman.

Bracket Your Answer

H-296 *Proposed by C. Kimberling, University of Evansville, Evansville, IN. (Vol. 17, No. 1, February 1979)*

Suppose x and y are positive real numbers. Find the least positive integer n for which

$$\left[\frac{x}{n+y}\right] = \left[\frac{x}{n}\right]$$

where $[z]$ denotes the greatest integer less than or equal to z.

Partial solution by the proposer.

Solution for the Special Case $y = 1$:

$$\text{Let } m = [\sqrt{x}] \text{ and } A = \begin{cases} \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{if } m^2 \le x \le m^2 + m - 1 \\ 1 + \sqrt{(m+1)^2 - x} & \text{if } m^2 + m \le x \le m^2 + 2m \end{cases}$$

Then the least positive integer n satisfying $\left[\frac{x}{n+1}\right] = \left[\frac{x}{n}\right]$ is given by

$$n = \begin{cases} m - 1 + A & \text{if } A \text{ is an integer} \\ m + [A] & \text{otherwise.} \end{cases}$$

Proof: First suppose $m^2 \le x \le m^2 + m - 1$, where $m = [\sqrt{x}]$. Let $L = x - m^2$. Then writing $k = n - m$ gives

$$\frac{x}{n} = \frac{x}{m+k} = \frac{m^2 + L}{m+k} = m - k + \frac{k^2 + L}{m+k}.$$

Similarly

$$\frac{x}{n+1} = m - k - 1 + \frac{(k+1)^2 + L}{m+k+1}.$$

The least k satisfying $\frac{(k+1)^2 + L}{m+k+1} \ge 1$ is easily found to satisfy

$$k \ge -\frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x}.$$

Thus, for

$$k = \begin{cases} -\frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{if this is an integer} \\ \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{otherwise} \end{cases}$$

we find that $k < \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} = \frac{1}{2} + \sqrt{m - L + \frac{1}{4}}$, so that

$$\left(k - \frac{1}{2}\right)^2 < n - L + \frac{1}{4} \quad \text{and} \quad \frac{k^2 + L}{m + k} > 1.$$

Consequently, $\left[\frac{x}{n}\right] = m - k$. Furthermore, if

$$\frac{(k + 1)^2 + L}{m + k + 1} \geq 2$$

then $\frac{x}{1 + n} \geq m - k + 1$, contrary to $\frac{x}{n + 1} < \frac{x}{n} < m - k$. This shows that

$$\frac{(k + 1)^2 + L}{m + k + 1} < 2$$

so that $\left[\frac{x}{1 + n}\right] = m - k$.

If $n' < n$, then for $n = m + k'$ we have $k' < k$; by definition of k this implies

$$\frac{(k' + 1)^2 + L}{m + k' + 1} < 1$$

so that

$$\frac{x}{1 + n'} = m - k' - 1 + \frac{(k' + 1)^2 + L}{m + k' + 1} \quad \text{and} \quad \left[\frac{x}{1 + n'}\right] \leq m - k' - 1.$$

On the other hand, $\frac{x}{n'} = m - k' + \frac{k'^2 + L}{m + k'}$, so that $\left[\frac{x}{n'}\right] \geq m - k'$. This shows that n is indeed the *least* positive integer for which $\left[\frac{x}{n + 1}\right] = \left[\frac{x}{n}\right]$.

Now suppose $m^2 + m \leq x \leq m^2 + 2m$, where again $m = [\sqrt{x}]$. Let

$$L = x - m^2 - m \quad \text{and} \quad k = n - m.$$

An argument analogous to that above shows that the least k for which

$$\left[\frac{x}{m + k}\right] = \left[\frac{x}{m + k + 1}\right]$$

is given by

$$k = \begin{cases} \sqrt{(m + 1)^2 - x} & \text{if this is an integer} \\ 1 + \left[\sqrt{(m + 1)^2 - x}\right] & \text{otherwise.} \end{cases}$$

The solution stated above now follows from $k = n - m$.

Note: It appears likely that for any y, at least for any positive *integer* y, the solution can be written in the form $[x/j] + 1$, where j is an integer.

Also solved by C. B. A. Peck.

The Limit

H-297 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA. (Vol. 17, No. 1, February 1979)*

Let $P_0 = P_1 = 1$, $P_n(\lambda) = P_{n-1}(\lambda) - \lambda P_{n-2}(\lambda)$. Show

$$\lim_{n \to \infty} P_{n-1}(\lambda)/P_n(\lambda) = (1 - \sqrt{1 - 4\lambda})/2\lambda = \sum_{n=0}^{\infty} C_{n+1}x^n,$$

where C_n is the nth Catalan number. Note that the coefficients of $P_n(\lambda)$ lie along the rising diagonals of Pascal's triangle with alternating signs.

Solution by Paul S. Bruckman, Concord, CA.

The characteristic polynomial of the P_n's is $x^2 - x + \lambda = (x - u)(x - v)$, where

$$u = u(\lambda) = \frac{1}{2}(1 + \sqrt{1 - 4\lambda}), \quad v = v(\lambda) = \frac{1}{2}(1 - \sqrt{1 - 4\lambda}).$$

It follows readily from the initial conditions that

$$P_n(\lambda) = (u^{n+1} - v^{n+1})/(u - v), \; n = 0, 1, 2, \ldots .$$

Although it is not stated in the problem, we assume that $|\lambda| < 1/4$, to avoid possible problems of convergence. Being acquainted to some degree with the proposer of the problem, it is nearly safe to say that he did not intend the problem to involve a rigorous treatment with λ ranging over all admissible values, but rather a formal result valid for "nice" values of λ. Moreover, we assume λ is real.

Let $r_n = P_{n-1}(\lambda)/P_n(\lambda)$, and $f(\lambda) = u(\lambda)/v(\lambda)$. Since $uv = \lambda$,

$$f(\lambda) = u^2/\lambda = (u - \lambda)/\lambda = u(\lambda)/\lambda - 1.$$

Also

$$r_n(\lambda) = (u^n - v^n)/(u^{n+1} - v^{n+1}) = \frac{(u/v)^n - 1}{v\{(u/v)^{n+1} - 1\}} = \frac{f^n - 1}{v(f^{n+1} - 1)}.$$

If we consider the graph of f, we see that the graph has asymptotes at $\lambda = 0$ and at $f = -1$; however, the latter asymptote is approached only as $\lambda \to -\infty$, and we exclude this possibility, by hypothesis. If $\lambda > 0$, clearly $u > v$. If $\lambda < 0$, then $u > 1$, $v < 0$, and $f < -1$. It follows that, if $|\lambda| < 1/4$, then $|f| > 1$.

Hence, $r = \lim\limits_{n\to\infty} r_n$ exists and

$$r = 1/vF = 1/u = v/\lambda = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda}.$$

Now, by the binomial theorem,

$$(1 - 4\lambda)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n}(-4\lambda)^n \quad (\text{provided } |\lambda| < 1/4) = \sum_{n=0}^{\infty} \binom{2n}{n}\lambda^n.$$

Integrating with respect to λ, we see that

$$(-1/2)(1 - 4\lambda)^{1/2} = c + \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1}\lambda^{n+1}, \text{ for some constant } c.$$

Setting $\lambda = 0$, we find that $c = -1/2$. Also, observe that $C_{n+1} = \dfrac{\binom{2n}{n}}{n+1}$, the $(n + 1)$th Catalan number. Therefore,

$$r = r(\lambda) = \sum_{n=0}^{\infty} C_{n+1}\lambda^n. \quad \text{Q.E.D.}$$

Also solved by the proposer.

The Big Six

H-298 *Proposed by L. Kuipers, Mollens, Valais, Switzerland.*
(Vol. 17, No. 1, February 1979)

Prove:

(i) $F_{n+1}^6 - 3F_{n+1}^5F_n + 5F_{n+1}^3F_n^3 - 3F_{n+1}F_n^5 - F_n^6 = (-1)^n, \; n = 0, 1, \ldots;$

(ii) $F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6 + F_n^6$
$= (-1)^n 80, \; n = 0, 1, \ldots;$

(iii) $F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6$
$\equiv -40 + \frac{1}{2}(1 + (-1)^n)80 \pmod{144}.$

Solution by L. Carlitz, Duke University, Durham, NC.

(i) Let $P(x, y) = x^6 - 3x^5y + 5x^3y^3 - 3xy^5 - y^6$. It is easily verified that $P(y + z, y) = -P(y, z)$.

In this identity, take $y = F_n$, $z = F_{n-1}$. This gives

$$P(F_{n+1}, F_n) = -P(F_n, F_{n-1}) \quad (n = 1, 2, 3, \ldots).$$

Thus,

$$P(F_{n+1}, F_n) = (-1)^n P(F_1, F_0) = (-1)^n P(1, 0).$$

Since $P(1, 0) = 1$, we get

$$P(F_{n+1}, F_n) = (-1)^n \quad (n = 0, 1, 2, \ldots),$$

as asserted.

(ii) Put

$$f_k(x) = \sum_{n=0}^{\infty} F_{n+1}^k x^n \quad (k = 0, 1, 2, \ldots).$$

It has been proved (L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math. Journal* 29 (1962):521-537; see also Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. Journal* 29 (1962):5-12) that

$$(*) \qquad f_k(x) = \frac{U_k(x)}{D_k(x)},$$

where

$$D_k(x) = \sum_{r=0}^{k+1} (-1)^{(1/2)r(r+1)} \frac{F_{k+1}F_k \cdots F_{k-r+2}}{F_1F_2 \cdots F_r} x^r$$

and

$$U_k(x) = \sum_{j=0}^{k-1} U_{kj}x^j \quad (k = 1, 2, 3, \ldots)$$

can be computed recursively by means of

$$U_{k+1,j} = F_{j+1}U_{k,j} + (-1)^j F_{k-j+1}U_{k,j-1}.$$

For $k = 6$, we find that

$$D_6(x) = 1 - 13x - 104x^2 + 260x^3 + 260x^4 - 104x^5 - 13x^6 + x^7$$

and

$$U_6(x) = 1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5.$$

Moreover, it can be verified that

$$D_6(x) = (1 + x)(1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6)$$

and

$$U_6(x) = (1 + x)(1 - 13x - 40x^2 + 93x^3 - 81x^4) + 80x^5.$$

Thus, taking $k = 6$ in (*), we have

$$(1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6)\sum_{n=0}^{\infty} F_{n+1}^6 x^n = \frac{U_6(x)}{1 + x}$$

$$= 1 - 13x - 40x^2 + 93x^3 - 81x^4 + \frac{80x^5}{1 + x}$$

$$= 1 - 13x - 40x^2 + 93x^3 - 81x^4 + 80\sum_{n=0}^{\infty}(-1)^n x^{n+5}.$$

Comparing coefficients of x^{n+5}, we get

$$F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6 + F_n^6$$
$$= (-1)^n 80 \quad (n = 0, 1, 2, \ldots).$$

(iii) We have

$$\begin{aligned}&1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6\\ &\equiv (1 - x)(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5) \pmod{144},\end{aligned}$$

so that

$$D_6(x) \equiv (1 - x^2)(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5) \pmod{144}.$$

It follows that

$$\begin{aligned}(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5)\sum_{n=0}^{\infty} F_{n+1}^6 x^n &\equiv \frac{U_6(x)}{1 - x^2}\\ &\equiv \frac{1 - 11x - 64x^2 - 11x^3 + x^4}{1 + x}\\ &\equiv 1 - 12x - 52x^2 + 41x^3 - \frac{40x^4}{1+x} \pmod{144}.\end{aligned}$$

Comparing coefficients of x^{n+5}, we get

$$\begin{aligned}&F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6\\ &\equiv (-1)^n 40 \pmod{144} \quad (n = 0, 1, 2, \ldots).\end{aligned}$$

Also solved by P. Bruckman, G. Wulczyn, D. Zeitlin, and the proposer.

*****

VOLUME INDEX

AHUJA, MANGHO. Problems Solved: B-394, B-395, 18(1):85-87.

AINSWORTH, O. R. "A Property of Quasi-Orthogonal Polynomials" (co-author, Joseph E. Morris, Jr.), 18(2):163-165.

AMUNATEGUI, G. IOMNI. Problem Proposed: B-437, 18(4):370.

ARKIN, JOSEPH. "On Euler's Solution to a Problem of Diophantus—II" (co-authors, Verner E. Hoggatt, Jr., & E. G. Straus), 18(2):170-176.

BABU, A. G. F. "A Note on the Multiplication of Two 3 x 3 Fibonacci-Rowed Matrices" (co-author, Wei Shen Hsia), 18(1):43.

BANERJEE, MAHADEV. "Summation of the Series $y^n + (y + 1)^n + \cdots + x^n$" (co-author, W. G. Waller), 18(1):35.

BAUDERT, F. R. "The Apollonius Problem," 18(1):33-34.

BERGUM, GERALD E. "Some Extensions of Wythoff Pair Sequences" (co-author, V. E. Hoggatt, Jr.), 18(1):28-33.

BERZSENYI, GEORGE. Problem Solved: H-291, 18(3):286.

BICKNELL-JOHNSON, MARJORIE. "Multisection of the Fibonacci Convolution Array and Generalized Lucas Sequence" (co-author, V. E. Hoggatt, Jr.), 18(1):51-58.

BRADY, WRAY G. Problems Solved: B-406, 18(3):274; H-291, 18(3):286.

BRIDGER, CLYDE A. Problem Solved: B-395, 18(1):87.

BRUCKMAN, PAUL S. Problems Proposed: H-311, 18(1):90; H-285, H-314, 18(2):191-192; H-320, 18(4):375. Problems Solved: B-394, B-395, B-396, B-397, B-398, B-399, 18(1):85-89; B-400, B-401, B-402, B-403, B-404, B-405, 18(2):188-190; B-406, B-407, B-410, B-411, 18(3):274-277; B-412, B-413, B-414, B-417, 18(4): 371-374; H-281, H-282, H-283, 18(1):91-96; H-285, H-286, H-288, H-289, H-290, H-291, H-293, 18(3):281-288; H-295, H-297, H-298, 18(4):376-382.

BUDGOR, AARON B. "Star Polygons, Pascal's Triangle, and Fibonacci Numbers," 18 (3):229-231.

BYRD, PAUL F. Problem Solved: H-293, 18(3):287-288.

CARLITZ, LEONARD. "Some Restricted Multiple Sums," 18(1):58-65; "Some Remarks on the Bell Numbers," 18(1):66-73; "Weighted Stirling Numbers of the First and Second Kind—I," 18(2):147-162; "Weighted Stirling Numbers of the First and Second Kind—II," 18(3):242-257; "The Number of Permutations with a Given Number of Sequences," 18(4):347-352. Problem Proposed: H-312, 18(1):90. Problems Solved: B-394, 18(1):85-86; H-282, H-283, 18(1):93-96; H-285, 18(2):191-192; H-289, H-293, 18(3):283-288; H-298, 18(4): 380-381.

CATER, F. S. Problem Solved: H-292, 18(3):286-287.

CHAINBUS, B. Problem Solved: H-291, 18(3):286.

CREUTZ, MICHAEL. "On the Convergence of Iterated Exponentiation—I" (co-author, R. M. Sternheimer), 18(4):341-347.

DAILY, J. Problem Solved: H-292, 18(3):286-287.

DRESEL, L. A. G. Letter to the Editor, 18(1):34.

EPP, ROBERT J. "A Note on Take-Away Games" (co-author, Thomas S. Ferguson), 18 (4):300-303.

EWELL, JOHN A. "Recurrences for Two Restricted Partition Functions," 18(1):1-2.

FERGUSON, THOMAS S. "A Note on Take-Away Games" (co-author, Robert J. Epp), 18 (4):300-303.

FRAME, J. S. "Factors of the Binomial Circulant Determinant," 18(1):9-23.

FREITAG, HERTA. Problems Proposed: B-418, 18(1):84; B-426, 18(2):187; B-434, 18(3):274. Problems Solved: B-394, B-395, B-398, 18(1):85-88; B-400, B-402, 18(2):187-188; B-406, 18(3):274; B-413, B-414, 18(4):371-372.

FULLER, LEONARD E. "Geometric Recurrence Relation," 18(2):126-129; "Representations for r, s Recurrence Relations," 18(2):129-135.

GIULI, R. Problems Solved: B-407, 18(3):274; H-281, H-282, 18(1):91-94.

GRASSL, RICHARD M. Problems Proposed: B-424, B-425, 18(2):187. Problem Solved: B-417, 18(4):373-374 (co-solver, P. L. Mana).

HILLMAN, A. P. (Ed.). Elementary Problems and Solutions, 18(1):84-89; 18(2):187-189; 18(3):273-379; 18(4):370-374.

HILLMAN, A. P. "Recursive, Spectral, and Self-Generating Sequences" (co-author, V. E. Hoggatt, Jr.), 18(2):97-103. Problem Proposed: B-439, 18(4):370.

HINDIN, HARVEY J. "A Theorem Concerning Heptagonal Numbers," 18(3):258-261.

HOGGATT, VERNER E., JR. "Some Extensions of Wythoff Pair Sequences" (co-author, Gerald E. Bergum), 18(1):28-33; "Roots of $(H - L)/15$ Recurrence Equations in Generalized Pascal Triangles" (co-author, Claudia Smith), 18(1):36-42; "Multisection of the Fibonacci Convolution Array and Generalized Lucas Sequence (co-author, M. Bicknell-Johnson), 18(1):51-58; "Another Proof that $\phi(F_n) \equiv 0$ Mod 4 for all $n > 4$" (co-author, Hugh Edgar), 18(1):80-82; "Recursive, Spectral, and Self-Generating Sequences" (co-author, A. P. Hillman), 18(2):97-103; "On Euler's Solution to a Problem of Diophantus—II" (co-authors, J. Arkin & E. G. Straus), 18(2):170-176; "Additive Partitions of the Positive Integers," 18(3):220-226; "Generalized Fibonacci Numbers" (co-author, Anne Silva), 18(4):290-300. Problems Proposed: B-421, B-422, 18(1):84-85; B-431, B-432, 18(3):273; H-313, 18(2):191; H-319, 18(3):280. Problems Solved: B-395, 18(1):86-87; H-297, 18(4):378-379.

HORADAM, A. F. "Extensions of a Paper on Diagonal Functions," 18(1):3-8.

HOWARD, F. T. "Associated Stirling Numbers," 18(4):303-315.

HSIA, WEI SHEN. "A Note on the Multiplication of Two 3 x 3 Fibonacci-Rowed Matrices" (co-author, A. G. L. Babu), 18(1):43.

KAHAN, STEVEN. "Mutually Counting Sequences," 18(1):47-50.

KIMBERLING, CLARK. "Generalized Cyclotomic Polynomials, Fibonacci Cyclotomic Polynomials, and Lucas Cyclotomic Polynomials," 18(2):108-126; "Generating Functions of Linear Divisibility Sequences," 18(3):193-208; "Mixing Properties of Mixed Chebyshev Polynomials," 18(4):334-341; "Four Composition Identities for Chebyshev Polynomials," 18(4):353-369. Problem Solved: H-296, 18(4):377.

KLAUSER, H. Problems Proposed: B-430, B-435, 18(3):273-274 (co-proposers, M. Wachtel & E. Schmutz). Problem Solved: H-291, 18(3):286.

KREWARAS, G. "The Number of More or Less 'Regular' Permutations," 18(3):226-229.

KUIPERS, L. Problem Solved: H-298, 18(4):380-381.

LAURIE, D. P. Problem Proposed: H-315, 18(2):190.

LEWIN, MORDECHAI. "Some Combinatorial Identities," 18(3):214-220.

LORD, GRAHAM. Problems Solved: B-394, B-395, 18(1):85-87; B-402, B-403, 18(2):189; H-285, 18(3):281.

MANA, PHIL. Problem Proposed: B-427, 18(2):187. Problems Solved: B-294, 18(1):85-86; B-404, B-405, 18(2):189-190; B-412, 18(4):371; B-417, 18(4):373-374 (co-solver, R. M. Grassl).

MILSOM, JOHN W. Problem Solved: B-406, 18(3):274.

MORRIS, JOSEPH E., JR. "A Property of Quasi-Orthogonal Polynomials" (co-author, O. R. Ainsworth), 18(2):163-165.

MOSER, W. O. J. Problems Solved: B-407, 18(3):274; B-413, 18(4):371-372; H-283, 18(1):94-96.

MULLEN, GARY. "Local Permutation Polynomials over Z_p," 18(2):104-108; "Local Permutation Polynomials in Three Variables over Z_p," 18(3):208-214. Problem Solved: B-401, 18(2):188.

MYERS, B. R. Problem Proposed: H-316, 18(2):190-191.

PECK, C. B. A. Problems Solved: B-394, B-395, B-396, 18(1):85-87; H-296, 18(4):377-378.

PETERS, J. F. Problem Proposed: B-433, 18(3):273 (co-proposer, R. Pletcher).

PHILIPPOU, A. Problem Solved: H-283, 18(1):94-96.

PLETCHER, R. Problem Proposed: B-433, 18(3):273 (co-proposer, J. F. Peters).

POLLIN, JACK M. "On the Matrix Approach to Fibonacci Numbers and the Fibonacci Pseudoprimes" (co-author, I. J. Schoenberg), 18(3):261-268.

PRIELIPP, BOB. Problems Solved: B-395, B-398, 18(1):87-88; B-400, B-402, B-403, 18(2):188-189; B-406, B-409, 18(3):274-275; B-414, 18(4):372; H-281, H-183, 18(1):94-96; H-291, 18(3):286.

PROPP, JAMES. Problem Proposed: H-318, 18(3):280.

READ, RONALD C. "A Note on Tiling Rectangles with Dominoes," 18(1):24-27.

RICE, BART. Problem Solved: B-411, 18(3):277.

ROBBINS, NEVILLE. "A Class of Solutions of the Equation $\sigma(n) = 2n + t$," 18(2):137-147.

ROBINSON, E. D. Problems Solved: B-395, 18(1):87; H-282, 18(1):93-94.
RUSSELL, DAVID L. Letter to the Editor, 18(1):82-83.
RUSSELL, P. Problems Solved: H-281, H-283, 18(1):91-96.
SCHMUTZ, E. Problems Proposed: B-430, B-435, 18(3):273-274 (co-proposers, M. Wachtel & H. Klauser). Problem Solved: B-406, 18(3):274.
SCHOENBERG, I. J. "On the Matrix Approach to Fibonacci Numbers and the Fibonacci Pseudoprimes" (co-author, Jack M. Pollin), 18(3):261-268.
SHALLIT, JEFFREY. Problems Proposed: B-423, 18(1):85; B-440, B-441, 18(4):370. Problem Solved: H-281, 18(1):91-93.
SHANNON, A. G. "Some Lacunary Recurrence Relations," 18(1):73-79. Problems Solved: B-406, B-407, 18(3):274; H-283, 18(1):95-96; H-293, 18(3):287-288.
SHIELDS, CHARLES B. Problems Solved: B-394, B-395, 18(1):85-87.
SILVA, ANNE. "Generalized Fibonacci Numbers" (co-author, Verner E. Hoggatt, Jr.), 18(4):290-300.
SINGH, SAHIB. "Thoro's Conjecture and Allied Divisibility Property of Lucas Numbers," 18(2):135-137. Problem Proposed: B-436, 18(4):370. Problems Solved: B-394, B-395, B-396, B-398, 18(1):85-88; B-400, B-402, B-403, B-404, B-405, 18(2):198-190; B-406, B-407, B-409, 18(3):274-275; B-412, B-414, 18(4):371-372; H-291, 18(3):286.
SMITH, CLAUDIA. "Roots of $(H - L)/15$ Recurrence Equations in Generalized Pascal Triangles" (co-author, Verner E. Hoggatt, Jr.), 18(1):36-42.
SOMER, LAWRENCE. "The Divisibility Properties of Primary Lucas Recurrences with Respect to Primes," 18(4):316-334. Problem Proposed: H-317, 18(3):280. Problems Solved: B-395, B-396, 18(1):87; B-403, 18(2):189; B-406, 18(3):274; B-412, 18(4):371; H-285, H-291, 18(3):281, 286.
STANLEY, T. E. "Powers of the Period Function for the Sequence of Fibonacci Numbers," 18(1):44-45; "Some Remarks on the Periodicity of the Sequence of Fibonacci Numbers—II, 18(1):45-47.
STARKE, E. Problem Solved: H-291, 18(3):286.
STERNHEIMER, R. M. "On the Convergence of Iterated Exponentiation—I" (co-author, Michael Creutz), 18(4):341-347.
STRAUS, E. G. "On Euler's Solution to a Problem of Diophantus—II" (co-authors, J. Arkin & V. E. Hoggatt, Jr.), 18(2):170-176.
THOMPSON, C. C. Problem Solved: B-400, 18(2):188.
VOGEL, J. Problem Solved: H-285, 18(2):191-192.
WACHTEL, M. Problems Proposed: B-430, B-435, 18(3):273-274 (co-proposers, H. Klauser & E. Schmutz). Problems Solved: B-400, B-401, 18(2):188; B-406, B-410, 18(3):274-276.
WALLER, W. G. "Summation of the Series $y^n + (y + 1)^n + \cdots + x^n$" (co-author, Mahadev Banerjee), 18(1):35.
WALTHER, G. "Free Group and Fibonacci Sequence," 18(3):268-272.
WEITSMAN, JONATHAN. Problems Solved: B-399, 18(1):89; B-400, B-401, 18(2):188.
WHITNEY, RAYMOND E. (Ed.). Advanced Problems and Solutions, 18(1):90-96; 18(2): 191-192; 18(3):280-288; 18(4):375-381.
WIECKOWSKI, ANDRZEJ. "On Some Systems of Diophantine Equations Including the Algebraic Sum of Tirangular Numbers," 18(2):165-170.
WILSON, JOHN W. Problems Solved: B-394, 18(1):85-86; B-402, 18(2):189.
WULCZYN, G. Problems Proposed: B-419, B-420, 18(1):84; B-428, B-429, 18(2):187-188; B-438, 18(4):370; H-294, H-295 (corrected), 18(3):280-281; H-321, 18(4): 375. Problems Solved: B-396, B-397, B-398, 18(1):87-88; B-400, B-401, B-402, B-403, B-404, 18(2):188-190; B-406, B-409, 18(3):274-275; H-285, 18(2):191-192; H-288, H-290, H-291, 18(3):282-286; H-294, H-295, H-298, 18(4):375-382,
YAMADA, MASAJI. "A Convergence Proof about an Integral Sequence," 18(3):231-242.
YFF, P. Problem Solved: H-283, 18(1):95-96.
ZEITLIN, D. Problem Solved: H-298, 18(4):380-382.