The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 19 NUMBER 1

FEBRUARY 1981

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980),
Br. Alfred Brousseau, and I.D. Ruggles

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
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OF INTEGERS WITH SPECIAL PROPERTIES*

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NEWTON'S METHOD AND RATIOS OF FIBONACCI NUMBERS

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ABSTRACT

The sequence $\{F_{n+1}/F_n\}$ of ratios of consecutive Fibonacci numbers converges to the golden mean $\varphi = \frac{1}{2}(1 + \sqrt{5})$, the positive root of $x^2 - x - 1 = 0$. Newton's method for the equation $x^2 - x - 1 = 0$ with initial approximation 1 produces the subsequence $\{F_{2^n+1}/F_{2^n}\}$ of Fibonacci ratios. The secant method for this equation with initial approximations 1 and 2 produces the subsequence $\{F_{F_n+1}/F_{F_n}\}$. These results generalize to quadratic equations with roots of unequal magnitudes.

It is well known that the ratios of successive Fibonacci numbers converge to the golden mean. We recall that the Fibonacci numbers $\{F_n\}$ are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.* The golden mean, $\varphi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$, is the positive solution of the equation $x^2 - x - 1 = 0$.

The ratios $\{F_{n+1}/F_n\}$ of consecutive Fibonacci numbers are a sequence of rational numbers converging to φ *linearly*; that is, the number of digits of F_{n+1}/F_n which agree with φ is approximately a linear function of n. In fact, there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^n < |F_{n+1}/F_n - \varphi| < \beta\varepsilon^n$.

We can obtain sequences of rational numbers converging more rapidly to φ by using procedures of numerical analysis for approximating solutions of the equation $x^2 - x - 1 = 0$. Two common methods for solving an equation $f(x) = 0$ numerically are Newton's method and the secant method (*regula falsi*) [1, 3]. Each method generates a sequence $\{x_n\}$ converging to a solution of $f(x) = 0$. For Newton's method,

$$x_n = \mathrm{NEWTON}(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{1}$$

The secant method is obtained from Newton's method by replacing $f'(x_{n-1})$ by a difference quotient:

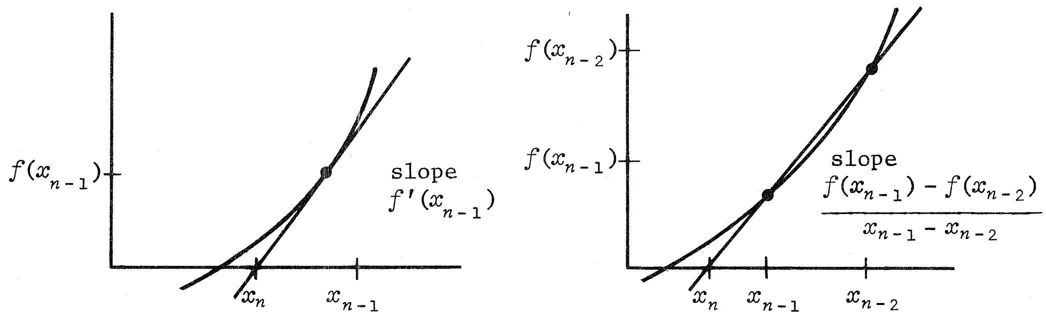
$$\begin{aligned} x_n = \mathrm{SECANT}(x_{n-1}, x_{n-2}) &= x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})} \\ &= \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}. \end{aligned} \tag{2}$$

[The first expression for $\mathrm{SECANT}(x_{n-1}, x_{n-2})$ is more useful for numerical calculations, while the second expression reveals the symmetric roles of x_{n-1} and x_{n-2}.] The familiar geometric interpretations of Newton's method and the secant method are given in Figure 1.

Newton's method requires an initial approximation x_0; the secant method requires two approximations x_0 and x_1. If the initial values are sufficiently close to a solution ξ of $f(x) = 0$, then the sequences $\{x_n\}$ defined by either method converge to ξ. Suppose that $f'(\xi) \neq 0$; that is, ξ is a simple zero of f. Then, the convergence of Newton's method is quadratic [1]: the number of correct digits of x_n is about twice that of x_{n-1}, since $|x_n - \xi| \approx \alpha|x_{n-1} - \xi|^2$

**For future reference, we note the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.*

for some $\alpha > 0$. Similarly, the order of convergence of the secant method is $\varphi \approx 1.618$. since $|x_n - \xi| \approx \alpha |x_{n-1} - \xi|^{\varphi}$ for some $\alpha > 0$ [3].



(a) Newton's method *(b) Secant method*

Fig. 1. Geometric interpretations of Newton's method and secant method

Both these methods applied to the equation $x^2 - x - 1 = 0$ yield sequences converging to φ more rapidly than $\{F_{n+1}/F_n\}$. For this equation, we calculate easily that

$$\text{(3)} \qquad \text{NEWTON}(x_{n-1}) = \frac{x_{n-1}^2 + 1}{2x_{n-1} - 1} \quad \text{and} \quad \text{SECANT}(x_{n-1}, x_{n-2}) = \frac{x_{n-1}x_{n-2} + 1}{x_{n-1} + x_{n-2} - 1}.$$

For initial approximations to φ, it is natural to choose Fibonacci ratios. For example, with $x_0 = 1$, Newton's method produces the sequence,

$$1,\ 2/1,\ 5/3,\ 34/21,\ 1597/987,\ \ldots,$$

which we recognize (see note on page 1) as a subsequence of Fibonacci ratios. From a few more sample calculations [e.g.,

$$\text{NEWTON}(3/2) = 13/8 \quad \text{or} \quad \text{NEWTON}(8/5) = 89/55]$$

we infer the identity:

$$\text{(4)} \qquad \text{NEWTON}(F_{n+1}/F_n) = F_{2n+1}/F_{2n}.$$

The sequence $\{x_n\}$ generated by Newton's method with $x_0 = 1$ is defined by $x_n = F_{2^n+1}/F_{2^n}$. Now it is obvious that the convergence of $\{x_n\}$ is quadratic, since there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{2^n} < |x_n - \varphi| < \beta\varepsilon^{2^n}$.

We can similarly apply the secant method with Fibonacci ratios as initial approximations. From examples such as

$$\text{SECANT}(1, 2) = 3/2,\ \text{SECANT}(2, 3/2) = 8/5,\ \text{and SECANT}(3/2, 8/5) = 34/21,$$

we infer the general rule:

$$\text{(5)} \qquad \text{SECANT}(F_{m+1}/F_m, F_{n+1}/F_n) = F_{m+n+1}/F_{m+n}.$$

In particular, if $x_1 = 1$ and $x_2 = 2$, then the sequence $\{x_n\}$ generated by the secant method is given by $x_n = F_{F_n+1}/F_{F_n}$. Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{\varphi^n} < |x_n - \varphi| < \beta\varepsilon^{\varphi^n}$, which dramatically illustrates that the order of convergence of the secant method is φ.

Equations (4) and (5) are interesting because they imply that the sequences of rational approximations to φ produced by Newton's method and by the secant method are simple subsequences of Fibonacci ratios.

We now verify (4) and (5). In fact, these identities are valid in general for any sequence $\{u_n\}$ defined by a second-order linear difference equation with $u_0 = 0$ and $u_1 = 1$, provided the sequence $\{u_{n+1}/u_n\}$ is convergent.

Lemma: Let $\{u_n\}$ be defined by $au_n + bu_{n-1} + cu_{n-2} = 0$ with $u_0 = 0$ and $u_1 = 1$. Then $au_{m+1}u_{n+1} - cu_m u_n = au_{m+n+1}$ for all $m, n \geq 0$.

Proof: By induction on n. For $n = 0$, the lemma holds for all m since

$$au_{m+1}u_1 - cu_m u_0 = au_{m+1}.$$

Now assume that for $n - 1$ the lemma is true for all m. Then

$$\begin{aligned} au_{m+1}u_{n+1} - cu_m u &= (-bu_n - cu_{n-1})u_{m+1} + (au_{m+2} + bu_{m+1})u_n \\ &= au_{m+2}u_n - cu_{m+1}u_{n-1} \\ &= au_{(m+1)+(n-1)+1} \\ &= au_{m+n+1}. \quad \square \end{aligned}$$

The lemma generalizes the Fibonacci identity [2]:

$$F_{m+1}F_{n+1} + F_m F_n = F_{m+n+1}.$$

Suppose that $ax^2 + bx + c$ has distinct zeros λ_1 and λ_2. Any sequence $\{u_n\}$ satisfying the recurrence $au_n + bu_{n-1} + cu_{n-2} = 0$ is of the form

$$u_n = k_1\lambda_1^n + k_2\lambda_2^n,$$

where k_1 and k_2 are constants determined by the initial values u_0 and u_1. If $|\lambda_1| > |\lambda_2|$ and $k_1 \neq 0$, then u_n is asymptotic to $k_1\lambda_1^n$, and so $\{u_{n+1}/u_n\}$ converges linearly to λ_1. We now show that if $u_0 = 0$ and $u_1 = 1$, then Newton's method and the secant method, starting with ratios from $\{u_{n+1}/u_n\}$, generate subsequences of $\{u_{n+1}/u_n\}$.

Theorem: Let $\{u_n\}$ be defined by $au_n + bu_{n-1} + cu_{n-2} = 0$ with $u_0 = 0$ and $u_1 = 1$. If the characteristic polynomial $f(x) = ax^2 + bx + c$ has zeros λ_1 and λ_2 with $|\lambda_1| > |\lambda_2|$, then:

(i) $u_n \neq 0$ for all $n > 0$;

(ii) $\lim_{n\to\infty} u_{n+1}/u_n = \lambda_1$;

(iii) $\text{NEWTON}(u_{n+1}/u_n) = u_{2n+1}/u_{2n}$;

(iv) $\text{SECANT}(u_{m+1}/u_m,\ u_{n+1}/u_n) = u_{m+n+1}/u_{m+n}$.

Proof:

(i) It is easily verified that $u_n = k(\lambda_1^n - \lambda_2^n)$, where $k = \pm a/\sqrt{b^2 - 4ac}$. (The sign of k depends on the signs of a and b.) Since $|\lambda_1| > |\lambda_2|$, if $n > 0$, then $|\lambda_1^n| > |\lambda_2^n|$ and, therefore, $u_n \neq 0$.

(ii) We note, as an aside, that the sequence $\{u_{n+1}/u_n\}$ satisfies the first-order recurrence $x_n = -(bx_{n-1} + c)/ax_{n-1}$. To verify (ii):

$$\frac{u_{n+1}}{u_n} = \frac{k(\lambda_1^{n+1} - \lambda_2^{n+1})}{k(\lambda_1^n - \lambda_2^n)} = \lambda_1\frac{1 - (\lambda_1/\lambda_2)^{n+1}}{1 - (\lambda_1/\lambda_2)^n} \to \lambda_1 \text{ as } n \to \infty, \text{ since } |\lambda_1/\lambda_2| < 1.$$

(iii) For the equation $ax^2 + bx + c = 0$, Newton's method and the secant method are given by

$$(6)\quad \text{NEWTON}(x_{n-1}) = \frac{ax_{n-1}^2 - c}{2ax_{n-1} + b} \quad\text{and}\quad \text{SECANT}(x_{n-1}, x_{n-2}) = \frac{ax_{n-1}x_{n-2} - c}{a(x_{n-1} + x_{n-2}) + b}.$$

Therefore, $\text{NEWTON}(x_{n-1}) = \text{SECANT}(x_{n-1}, x_{n-1})$, and so (iii) follows from (iv). Note that this identity holds for any polynomial equation $f(x) = 0$.

(iv) By (6),

$$\begin{aligned}\text{SECANT}(u_{m+1}/u_m,\ u_{n+1}/u_n) &= \frac{a(u_{m+1}/u_m)(u_{n+1}/u_n) - c}{a(u_{m+1}/u_m + u_{n+1}/u_n) + b}\\ &= \frac{au_{m+1}u_{n+1} - cu_mu_n}{au_{m+1}u_n + au_mu_{n+1} + bu_mu_n}\\ &= \frac{au_{m+1}u_{n+1} - cu_mu_n}{au_{m+1}u_n - cu_mu_{n-1}}\\ &= au_{m+n+1}/au_{m+n} \quad \text{(by the lemma)}\\ &= u_{m+n+1}/u_{m+n}. \ \square\end{aligned}$$

Remarks:

1. The theorem does not generalize to polynomials of degree higher than 2.

2. Not only do the ratios of the consecutive Fibonacci numbers converge to φ, they are the "best" rational approximation to φ; i.e., if $n > 1$, $0 < F \leq F_n$ and $P/F \neq F_{n+1}/F_n$, then $|F_{n+1}/F_n - \varphi| < |P/F - \varphi|$ by [4]. Since Newton's method and the secant method produce subsequences of Fibonacci ratios, they also produce the best rational approximation to φ.

ACKNOWLEDGMENTS

The above results were discovered in response to a question posed by Ernst Specker. This research was supported by the National Science Foundation under Grant GK-43121 at Stanford University and by the Forschungsinstitut für Mathematik, Eidgenössische Technische Hochschule, Zürich.

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A CHARACTERIZATION OF THE FUNDAMENTAL SOLUTIONS TO PELL'S EQUATION $u^2 - Dv^2 = C$

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Due to a confusion originating with Euler, the diophantine equation

(1) $$u^2 - Dv^2 = C,$$

where D is a positive integer that is not a perfect square and C is a nonzero integer, is usually called *Pell's equation*. In a previous article [1, Theorem 2], the following theorem was proved.

Theorem 1: Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to $x^2 - Dy^2 = 1$. If $k =$

$(y_1)/(x_1 - 1)$ and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = -N$, where $N > 0$, then $v_0 = |v_0| \geq k|u_0|$. If $k = (Dy_1)/(x_1 - 1)$ and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = N$, where $N > 1$, then $u_0 = |u_0| \geq k|v_0|$.

In Theorem 4, we shall prove the converse of this result. In the sequel, the definition of a fundamental solution to Eq. (1) given in [1] will be used. This definition differs from the one in [2, p. 205] only when $v_0 < 0$. In this case, if the fundamental solution given in [1] is denoted by $u_0 + v_0\sqrt{D}$, then the one given by the definition in [2] would be $-(u_0 + v_0\sqrt{D})$. We shall need to recall Remark A of [1] and to add to the three statements of this remark the statement:

(iv) If $C \leq 1$ and $-u_0 + v_0\sqrt{D}$ is in K then $u_0 \geq 0$. If $C \geq 1$ and $u_0 - v_0\sqrt{D}$ is in K then $v_0 \geq 0$.

Also, we shall need the following result (see [1, Theorem 5]).

Theorem 2: If $u + v\sqrt{D}$ is a solution in nonnegative integers to the diophantine equation $u^2 - Dv^2 = C$, where $C \neq 1$, then there exists a nonnegative integer n such that $u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n$ where $u_0 + v_0\sqrt{D}$ is the fundamental solution to the class of solutions of $u^2 - Dv^2 = C$ to which $u + v\sqrt{D}$ belongs and $x_1 + y_1\sqrt{D}$ is the fundamental solution to $x^2 - Dy^2 = 1$.

We now need to prove a lemma and a simple consequence of this lemma.

Lemma 3: Let $u_0 + v_0\sqrt{D}$ be a fundamental solution to a class of solutions to $u^2 - Dv^2 = C$. If, for $n \geq 1$, we let $u_n + v_n\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n$, then $u_n > 0$ and $v_n > 0$ for $n \geq 1$.

Proof: Since

$$u_1 + v_1\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D}) = (u_0x_1 + Dv_0y_1) + (u_0y_1 + v_0x_1)\sqrt{D},$$

we have that $u_1 = u_0x_1 + Dv_0y_1$ and $v_1 = u_0y_1 + v_0x_1$.

We now begin an induction proof of Lemma 3. First, suppose $u_0^2 - Dv_0^2 = C$, where $C < 0$. This implies, by Remark A [1], $v_0 > 0$. Hence $u_0 \geq 0$ implies $u_1 > u_0x_1 \geq u_0 \geq 0$ and $v_1 > v_0 > 0$. Thus suppose $u_0 < 0$. By Theorem 1,

$$v_0 \geq \frac{-u_0y_1}{x_1 - 1} = \frac{-u_0(x_1 + 1)}{Dy_1}.$$

Whence, $u_1 = u_0x_1 + Dv_0y_1 \geq -u_0 > 0$ and $v_1 = u_0y_1 + v_0x_1 \geq v_0 > 0$. Therefore, for $C < 0$, $u_1 > 0$ and $v_1 > 0$.

Next, suppose $u_0^2 - Dv_0^2 = C$, where $C > 0$. This implies $u_0 > 0$. Thus $v_0 \geq 0$ implies $u_1 > u_0 > 0$ and $v_1 > v_0 \geq 0$. Thus suppose $v_0 < 0$. Hence $C > 1$, so by Theorem 1,

$$u_0 \geq \frac{-Dv_0y_1}{x_1 - 1} = \frac{-v_0(x_1 + 1)}{y_1}.$$

Whence, $u_1 \geq u_0 > 0$ and $v_1 \geq -v_0 > 0$. This completes the proof of Lemma 3 for $n = 1$.

Since

$$(u_{n+1} + v_{n+1}\sqrt{D}) = (u_n + v_n\sqrt{D})(x_1 + y_1\sqrt{D}) = (u_nx_1 + Dv_ny_1) + (x_1v_n + y_1u_n)\sqrt{D}, \tag{2}$$

the assumption $u_n > 0$ and $v_n > 0$ implies $u_{n+1} > 0$ and $v_{n+1} > 0$.

Corollary: With u_0, v_0, u_n, and v_n defined as in Lemma 3, we have $u_{n+1} > u_n$ and $v_{n+1} > v_n$ for $n \geq 0$.

Proof: In the proof of Lemma 3, it was shown that $v_1 \geq v_0$ and that, in addition, for $u_0 \geq 0$ or $C > 0$ we actually have $v_1 > v_0$. For the case $u_0 < 0$ and $C < 0$, it follows from the proof of Lemma 3 that $v_1 = v_0$ implies $u_1 = -u_0$. So

$-u_0 + v_0\sqrt{D} = u_1 + v_1\sqrt{D}$ belongs to the same class of solutions to $u^2 - Dv^2 = C$ as $u_0 + v_0\sqrt{D}$. Since we are assuming $u_0 < 0$, this contradicts (iv) of Remark A [1]. Hence, even in this case, $v_1 > v_0$. In a similar manner, it is seen that we always have $u_1 > u_0$. Since $u_n > 0$ and $v_n > 0$ for $n \geq 1$, (2) implies that $u_{n+1} > u_n$ and $v_{n+1} > v_n$ for $n \geq 1$.

Theorem 4: If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = -N$, where $N \geq 1$, and if $v \geq ku$, where $k = (y_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = -N$. If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = N$, where $N > 1$, and if $u \geq kv$, where $k = (Dy_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = N$.

Proof: By Theorem 2, $u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n = u_n + v_n\sqrt{D}$, where n is a nonnegative integer and $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = \pm N$. We shall prove $u + v\sqrt{D} = u_0 + v_0\sqrt{D}$. So assume $n \geq 1$. Then we have

$$\begin{aligned} u_n + v_n\sqrt{D} &= (u_{n-1} + v_{n-1}\sqrt{D})(x_1 + y_1\sqrt{D}) \\ &= (x_1u_{n-1} + Dy_1v_{n-1}) + (x_1v_{n-1} + y_1u_{n-1})\sqrt{D}. \end{aligned}$$

Thus $u_{n-1} = x_1u_n - Dy_1v_n$ and $v_{n-1} = -y_1u_n + x_1v_n$.

First, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = -N$. We know that

$$v = v_n \geq ku_n = \frac{y_1u_n}{x_1 - 1}.$$

Hence

$$v_{n-1} = -y_1u_n + x_1v_n = (x_1 - 1)v_n - y_1u_n + v_n \geq v_n.$$

But by the corollary to Lemma 3, $v_{n-1} < v_n$ for $n \geq 1$. Thus $n = 0$ and the proof is complete for the case $u^2 - Dv^2 = -N$.

Now, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = N$. We know that

$$u_n \geq kv_n = \frac{Dy_1v_n}{x_1 - 1}.$$

(Please turn to page 92)

STRUCTURAL ISSUES FOR HYPERPERFECT NUMBERS

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ABSTRACT

An integer m is said to be n-hyperperfect if $m = 1 + n[\sigma(m) - m - 1]$. These numbers are a natural extension of the perfect numbers, and as such share remarkably similar properties. In this paper we investigate sufficient forms for hyperperfect numbers.

1. INTRODUCTION

Integers having "some type of perfection" have received considerable attention in the past few years. The most well-known cases are: perfect numbers ([1], [12], [13], [14], [15]); multiperfect numbers ([1]); quasiperfect numbers ([2]); almost perfect numbers ([3], [4], [5]); semiperfect numbers ([16], [17]);

and unitary perfect numbers ([11]). The related issue of amicable, unitary amicable, quasiamicable, and sociable numbers ([8], [10], [11], [9], [6], [7]) has also been investigated extensively.

The intent of these variations of the classical definition appears to have been the desire to obtain a set of numbers, of nontrivial cardinality, whose elements have properties resembling those of the perfect case. However, none of the existing definitions generates a rich theory and a solution set having structural character emulating the perfect numbers; either such sets are empty, or their euclidean distance from zero is greater than some very large number, or no particularly unique prime decomposition form for the set elements can be shown to exist.

This is in contrast with the abundance (cardinally speaking) and the crystalized form of the n-hyperperfect numbers (n-HP) first introduced in [18]. These numbers are a natural extension of the perfect case, and, as such, share remarkably similar properties, as described below.

In this paper we investigate sufficient forms for the hyperperfect numbers. The necessity of these forms, though highly corroborated by empirical evidence, remains to be established for many cases.

2. BASIC THEORY

Definition 1:

a. m is n-HP iff $m = 1 + n[\sigma(m) - m - 1]$, m and n positive integers.

b. $M_n = \{m|m \text{ is } n\text{-HP}\}$.

c. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} \ldots p_j^{\alpha_j} p_{j+1}^k$ be n-HP and be in canonical form

$$(p_1 < p_2 < \cdots < p_j < p_{j+1}).$$

Then $\rho(m) = \{p_1, p_2, \ldots, p_{j-1}, p_j\}$ are the roots of m [if $m = p_1$, $\rho(m) = \emptyset$].

d. $d_1(m) = |\rho(m)| = j$, $d_2(m) = k$.

e. ${}_nM_{h,L} = \{m|m \text{ is } n\text{-HP}, d_1(m) = h, d_2(m) = L\}$.

Note that for $n = 1$ the perfect numbers are recaptured. Clearly one has

$$M_n = \left[\bigcup_L {}_nM_{0,L}\right] \cup \left[{}_nM_{1,1}\right] \cup \left[\bigcup_{h=2}^{\infty} {}_nM_{h,1}\right] \cup \left[\bigcup_{h=1}^{\infty} \bigcup_{L=2}^{\infty} {}_nM_{h,L}\right].$$

Definition 2:

a. If $m \in \bigcup_L {}_nM_{0,L}$ we say that m is a *Sublinear HP*.

b. If $m \in {}_nM_{1,1}$ we say that m is a *Linear HP*.

c. If $m \in \bigcup_{h=2}^{\infty} {}_nM_{h,1}$ we say that m is a *Superlinear HP*.

d. If $m \in \bigcup_{h=1}^{\infty} \bigcup_{L=2}^{\infty} {}_nM_{h,L}$ we say that m is a *Nonlinear HP*.

It has already been shown [18] that

Proposition 1: There are no Sublinear n-HPs.

Table 1 below shows the n-HP numbers less than 1,500,000. In each case, m is a Linear HP. We thus give an exhaustive theory for Linear HPs. Superlinear and Nonlinear results will be presented elsewhere; however, it appears that the only n-HP are *Linear* n-HP. In fact, several nonlinear forms have been shown to be impossible.

Table 1. n-HP up to 1,500,000, $n \geq 2$

n	m	Prime Decomposition for m	n	m	Prime Decomposition for m
2	21	3×7	2	176,661	$3^5 \times 727$
6	301	7×43	31	214,273	$47^2 \times 97$
3	325	$5^2 \times 13$	168	250,321	193×1297
12	697	17×41	108	275,833	133×2441
18	1,333	31×43	66	296,341	67×4423
18	1,909	23×83	35	306,181	$53^2 \times 109$
12	2,041	13×157	252	389,593	317×1229
2	2,133	$3^3 \times 79$	18	486,877	79×6163
30	3,901	47×83	132	495,529	137×3617
11	10,693	$17^2 \times 37$	342	524,413	499×1087
6	16,513	$7^2 \times 337$	366	808,861	463×1747
2	19,521	$3^4 \times 241$	390	1,005,421	479×2099
60	24,601	73×337	168	1,005,649	173×5813
48	26,977	53×509	348	1,055,833	401×2633
19	51,301	$29^2 \times 61$	282	1,063,141	307×3463
132	96,361	173×557	498	1,232,053	691×1783
132	130,153	157×829	540	1,284,121	829×1549
10	159,841	$11^2 \times 1321$	546	1,403,221	787×1783
192	163,201	293×557	59	1,433,701	$89^2 \times 181$

3. LINEAR THEORY

The following basic theorem of Linear n-HP gives a sufficient form for a hyperperfect number.

Theorem 1: m is a Linear n-HP if and only if

$$p_2 = \frac{np_1^{\alpha_1+1} - (n-1)p_1 - 1}{p_1^{\alpha_1+1} - (n+1)p_1^{\alpha_1} + n}.$$

Proof: ($\rightarrow$) m is a Linear n-HP, if $m = p_1^{\alpha_1} p_2$; then

$$\sigma(m) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}(1 + p_2).$$

But m n-HP implies that $(n+1)m = (1-n) + n\sigma(m)$. Substituting for $\sigma(m)$ and solving for p_2, we obtain the desired result. Note that p_2 must be a prime.

$$(\leftarrow) \text{ if } m = p_1^{\alpha_1} \frac{np_1^{\alpha_1+1} - (n-1)p_1 - 1}{p_1^{\alpha_1+1} - (n+1)p_1^{\alpha_1} + n}$$

where the second term is prime, then

$$\sigma(m) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\left[1 + \frac{np_1^{\alpha_1+1} - (n - 1)p_1 - 1}{p_1^{\alpha_1\ 1} - (n + 1)p_1^{\alpha_1} + n}\right],$$

from which one sees that the condition for a *Linear n-HP* is satisfied. Q.E.D. We say that n is convolutionary if $n + 1$ is prime p_1.

Corollary 1: If n is convolutionary, a sufficient form for $m = p_1^{\alpha_1+1}p_2$ to be Linear n-HP is that for some α_1, $p_2 = (n + 1)^{\alpha_1} - n$ is a prime. In this case,

$$m = (n + 1)^{\alpha_1+1}[(n + 1)^{\alpha_1} - n].$$

Corollary 2: If $m = p_1p_2$ is a Linear n-HP, then

$$p_2 = \frac{np_1^2 - (n - 1)p_1 - 1}{p_1^2 - (n + 1)p_1 + n}.$$

We would expect these n-HPs to be the most abundant, since they have the simplest structure. This appears to be so, as indicated by Table 1.

Corollary 3: If $m = p_1p_2$ is a Linear n-HP with $p_1 = n + 1$, then $p_2 = n^2 + n + 1$, so that

$$m = (n + 1)(n^2 + n + 1).$$

In view of these corollaries, the following issues are of capital importance for cardinality considerations of Linear n-HP.

Definition 3:

a. We say that $(n + 1)^{\alpha} - n$, $\alpha = 1, 2, 3, \ldots,$ is a Legitimate Mersenne sequence rooted on n (n-LMS), if $n + 1$ is a prime.

b. Given an n-LMS, we say that $(n + 1)^{\alpha} - n$ is an nth-order Mersenne prime (n-MP), if $(n + 1)^{\alpha} - n$ is prime.

A 1-LMS is the well-known sequence $2^{\alpha} - 1$.

Question 1. Does there exist an n-MP for each n?

Question 2. Do there exist infinitely many n-MP for each n?

Question 3. Are there infinitely many primes of the form $n^2 + n + 1$, where $n + 1$ is prime?

Extensive computer searches (not documented here) seem to indicate that the answer to these questions is affirmative.

Theorem 2: If m is a Linear n-HP, then $n + 1 \leq p_1 \leq 2n - 1$ if $n > 1$ and $p_1 \leq 2$ if $n = 1$.

Proof: It can be shown that if m is n-HP and $j|m$, then $j > n$. Thus, for a Linear n-HP, $p_1 > n$; equivalently, $p_1 \geq n + 1$. Now, since

$$p_2 = p_1\frac{np_1^{\alpha_1} - (n - 1) - 1/p_1}{(p_1 - n - 1)p_1^{\alpha_1} + n},$$

we let

$$p_1 = n + 1 + \mu;$$

then,

$$p_2 = p_1\frac{np_1^{\alpha_1} - (n - 1) - 1/p_1}{\mu p_1^{\alpha_1} + n}.$$

(Note: $\mu = 0$ implies $p_1 = n + 1 < 2n$.) Since we want the second factor of this expression larger than 1, we must have $\mu < n$ or $n < p_1 \leq 2n$, from which we get $n + 1 \leq p_1 \leq 2n - 1$ if $n > 1$, for primality, and $1 < p_1 \leq 2$ if $n = 1$. Q.E.D.

Observe that the upper bound is necessary for a Linear n-HP, *but not for a general n-HP.*

Corollary 4: m is a Linear 1-HP iff it is of the form $m = 2^{t-1}(2^t - 1)$.

Proof: From Theorem 2, $p_1 = 2$. Now we can apply Corollary 1 to obtain the necessary part of this result. The sufficiency part follows from the definition.

4. BOUNDS FOR LINEAR n-HP

We now establish important bounds for Linear n-HP.

Proposition 2: Let m be Linear n-HP. Consider $p_2 = F(\alpha)$. The p_2 is monotonically increasing on α.

Proof: Omitted.

Proposition 3:

$$\lim_{\alpha \to \infty} p_2 = \begin{cases} \dfrac{np_1}{p_1 - n - 1} & p_1 \neq n + 1 \\ \infty & p_1 = n + 1 \end{cases}$$

This follows directly from Theorem 1 and Corollary 1. Using Proposition 2, we obtain

Proposition 4:

$$\frac{np_1^2 - (n-1)p_1 - 1}{p_1^2 - (n+1)p_1 + n} \leq p_2 \leq \frac{np_1}{p_1 - n - 1} \qquad p_1 \neq n + 1$$

$$n^2 + n + 1 \leq p_2 < \infty \qquad p_1 = n + 1$$

Using these propositions, we have essentially proved the following important theorem.

Theorem 3: Given n, $n+1 \leq p_1 \leq 2n - 1$, if n is not convolutionary, then there can be at most finitely many n-HP of the form $m = p_1^{\alpha_1} p_2$.

Table 2 and Table 3 show the allowable values for p_1, given n, along with the bounds for p_2. We can now obtain results similar to those of Corollary 4.

Corollary 5: If m is Linear 2-HP, then it can only be of the form

$$3^{t-1}(3^t - 2).$$

Corollary 6: a) If m is Linear 3-HP, then it must be of the form

$$5^{t-1}\frac{3 \cdot 5^t - 11}{5^{t-1} + 3}$$

b) There is exactly one Linear 3-HP (see Tables 2 and 3).

Corollary 7: a) There are no Linear 4-HP rooted on 7 (see Tables 2 and 3).
b) There are Linear 4-HP rooted on 5. For example,

$$m = 5^4(5^4 - 4) = 5^4(3121).$$

Corollary 8: There are no Linear 5-HP.

Corollary 9: There are no Linear 7-HP.

Table 2. Allowable Values of p_1 and Bounds on p_2

n	Allowable Roots			
2	3(7, ∞)			
3	5(8, 15)			
4	5(21, ∞)	7(9, 14)		
5	7(18, 35)			
6	7(43, ∞)	11(13, 17)		
7	11(19, 26)	13(15, 19)		
8	11(29, 44)	13(21, 26)		
9	11(50, 99)	13(29, 39)	17(19, 22)	
10	11(111, ∞)	13(43, 65)	17(24, 29)	19(21, 24)

Table 3. p_2 as a Function of p_1, n, and α

α	$p_1 = 5$		$p_1 = 7$	
↓	$n = 3$	$n = 4$	$n = 4$	$n = 5$
1	8	21	9.66	18
2	13	121	13.23	31.22
3	14.56	621	13.88	34.41
4	14.91	3121	13.98	34.91
5	14.98	15621	13.99	34.98
6	14.99	78121	13.999	34.99
7	14.999	390621	13.9999	34.999
8	14.999	1953121	13.99999	34.9999

α	$p_1 = 11$			$p_1 = 13$	
↓	$n = 7$	$n = 8$	$n = 9$	$n = 7$	$n = 8$
1	19.50	29.66	50	15.33	21
2	25	42.28	91.46	17.95	25.56
3	25.60	43.83	98.26	18.18	25.96
4	25.66	43.98	98.93	18.19	25.997
5	25.666	43.99	98.99	18.199	25.9997
6	25.6666	43.999	98.999	18.1999	25.99998
7	25.66666	43.9999	98.9999	18.19999	
8		43.99999	98.99999		

Further bounds are derived below. We have already given one such bound:

$$p_2 \leq \frac{np_1}{p_1 - n - 1} \quad \text{for } n \text{ nonconvolutionary}$$

$$p_1 = n + 2 \qquad p_2 \leq n^2 + 2n$$

$$p_1 = n + 3 \qquad p_2 \leq \frac{n(n + 3)}{2}$$

$$p_1 = n + 4 \qquad p_2 \leq \frac{n(n + 4)}{3}$$

Therefore,

Proposition 5: Let n be nonconvolutionary. If m is Linear n-HP, then

$$p_2 \leq n^2 + 2n.$$

More generally,

Proposition 6: Let $d_n = p - n$, where p is the first prime larger than n. Then

$$\frac{n(n + d_n)^2 - (n - 1)(n + d) - 1}{(n + d_n)^2 \leq (n + 1)(n + d_n) + n} \leq p_2 \leq \frac{n(n + d_n)}{d_n - 1},$$

which is valid for $d_n \geq 1$.

Proposition 7: Let m be Linear n-HP, n nonconvolutionary. Then $p_2 \geq 2n + 1$.

Proof: (From the previous general bound on p_2, we see that this statement is also true for convolutionary n.) The proof involves looking at the expression for p_2, given that $p_1 = n + i$, $2 \leq i \leq n - 1$. Suppose $p_1 = n + 2$. Since m is Linear n-HP, we have

$$p_2 \geq \frac{np_1^2 - (n - 1)p_1 - 1}{p_1^2 - (n + 1)p_1 + 1}.$$

But $p_1 = n + 2$, so that

$$p_2 \geq \frac{n(n + 2)^2 - (n - 1)(n + 2) - 1}{(n + 2)^2 - (n + 1)(n + 2) + n} = \frac{(n + 1)^3}{2(n + 1)} = \frac{(n + 1)^2}{2} = \frac{n^2 + 2n + 1}{2}.$$

However, $n^2 > 2n$ $(n > 2)$, so that for this case $p_2 \geq 2n$ or $p_2 \geq 2n + 1$. Similar arguments hold for $p = n + 3, n + 4, \ldots$. We show the case $p = 2n - 1$. We have

$$p_2 \geq \frac{n(2n - 1)^2 - (n - 1)(2n - 1) - 1}{(2n - 1)^2 - (n + 1)(2n - 1) + n} = \frac{2n^3 - 3n^2 + 2n - 1}{n^2 - 2n + 1}$$

$$= 2n + 1 + \frac{2n - 2}{n^2 - 2n + 1}.$$

(Note that $n \neq 1$.) Therefore, again, $p_2 \geq 2n + 1$. Q.E.D.

Proposition 8: If $m = p_1^{\alpha_1} p_2$ is a Linear n-HP, n nonconvolutionary, then

$$\alpha_1 \leq \frac{\log\left[\dfrac{n^2 p_1}{p_1 - n - 1} + (n - 1)p_1 + 1\right]}{\log p_1}$$

Proof: We have shown that p_2 tends monotonically to $e = (np_1)/(p_1 - n - 1)$ as $\alpha \to \infty$. Let e' be the greatest integer smaller than e. Setting

$$\frac{np_1^{\alpha_1+1} - (n - 1)p_1 - 1}{p_1^{\alpha_1}(p_1 - n - 1) + n} = e'$$

and solving for α_1, we obtain

$$\alpha_1 = \frac{\log\left[\dfrac{ne' + (n-1)p_1 + 1}{np_1 - e'(p_1 - n - 1)}\right]}{\log p_1}.$$

However,

$$\frac{ne' + (n-1)p_1 + 1}{np_1 - e'(p_1 - n - 1)} \leq n\,\frac{np_1}{p_1 - n - 1} + (n-1)p_1 + 1,$$

and, in fact, the equality holds in many cases. The result follows. Q.E.D.

The following statement summarizes the bounds for a linear n-HP:

1. $n + 1 \leq p_1 \leq 2n - 1$;
2. $\begin{cases} \text{if } p_1 = n + 1, \text{ then } n^2 + n + 1 \leq p_2 < \infty, \\ \text{if } p_1 > n + 1, \text{ then } 2n + 1 \leq p_2 \leq n^2 + 2n; \end{cases}$
3. if $p_1 > n + 1$, then $\alpha_1 < \dfrac{\log\left[\dfrac{n^2 p_1}{p_1 - n - 1} + (n-1)p_1 + 1\right]}{\log p_1}$.

Notwithstanding the fact that no Superlinear and Nonlinear n-HP have been observed, we can still derive sufficient forms for these numbers (if they exist). It may be shown that

Proposition 9: $m = p_1^{\alpha_1} p_2^{\alpha} \ldots p_{j-1}^{\alpha_{j-1}} p_j$ is a Superlinear n-HP if and only if

$$p_j = \frac{n\Pi(p_1^{\alpha_i + 1} - 1) + (1 - n)\Pi(p_i - 1)}{(n+1)\Pi(p_i - 1)\Pi p_i^{\alpha_i} - n\Pi(p^{\alpha_i + 1} - 1)}.$$

3. CONCLUSION

Theorem 1 and Proposition 9 guarantee that, if an integer has a specific prime decomposition, then it is n-HP. However, no n-HP with these forms was observed in the search up to 1,500,000. One reason for such an unavailability could be the fact that the search was limited. The last term required by these theorems is a fraction or even involves a radical; hence, to ask that this expression turn out to be an integer and, moreover, a prime, is a strong demand. Possibly, very rare combinations of primes could generate the required conditions. It has been shown that indeed some forms are impossible.

The other explanation is that there are only Linear n-HP, and thus Theorem 1 is *necessary* and *sufficient* for a number to be n-HP, just as in the regular perfect number case. Such a statement would have a critical impact on the generalized perfect number problem. In fact, in view of the corollaries presented above, there would be no n-HP for various values of n.

Computer time (PDP 11/70) for Table 1 was over ten hours.

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ON RECIPROCAL SERIES RELATED TO FIBONACCI NUMBERS WITH SUBSCRIPTS IN ARITHMETIC PROGRESSION

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1. INTRODUCTION

Recently, interest has been shown in summing infinite series of reciprocals of Fibonacci numbers [1], [2], and [3]. As V. E. Hoggatt, Jr., and Marjorie Bicknell state [2]: "It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci numbers such that the subscripts are terms of geometric progressions." It seems even more difficult if the subscripts are in arithmetic progression. To take a very simple example, to my knowledge the series

$$\sum_{1}^{\infty} \frac{1}{F_n} \tag{1.1}$$

has not been evaluated in closed form, although Brother U. Alfred has derived formulas connecting it with other highly convergent series [4].

In this note, we develop formulas for closely related series of the form

$$\sum_{0}^{\infty} \frac{1}{F_{an+b} + c} \tag{1.2}$$

for certain values of a, b, and c. Examples include the following:

$$\sum_0^\infty \frac{1}{F_{2n+1} + 1} = \sqrt{5}/2, \qquad \sum_0^\infty \frac{1}{F_{2n+1} + 2} = 3\sqrt{5}/8,$$

(1.3)

$$\sum_0^\infty \frac{1}{F_{2n+1} + 5} = 5\sqrt{5}/22, \qquad \sum_0^\infty \frac{1}{F_{2n+1} + 13} = 7\sqrt{5}/58.$$

In fact, much more than this is true. Each of these series may be further broken down into a remarkable set of symmetric series illustrated by the following examples:

$$\sum_0^\infty \frac{1}{F_{14n+1} + 13} = (\sqrt{5} + 2)/58, \qquad \sum_0^\infty \frac{1}{F_{14n+13} + 13} = (\sqrt{5} - 2)/58,$$

$$\sum_0^\infty \frac{1}{F_{14n+3} + 13} = (\sqrt{5} + 5/3)/58, \qquad \sum_0^\infty \frac{1}{F_{14n+11} + 13} = (\sqrt{5} - 5/3)/58,$$

(1.4)

$$\sum_0^\infty \frac{1}{F_{14n+5} + 13} = (\sqrt{5} + 1)/58, \qquad \sum_0^\infty \frac{1}{F_{14n+9} + 13} = (\sqrt{5} - 1)/58,$$

$$\sum_0^\infty \frac{1}{F_{14n+7} + 13} = \sqrt{5}/58.$$

It will be noted that the sum of the series in (1.4) agrees with that given in (1.3)—namely, $7\sqrt{5}/58$—since the rational terms cancel out in pairs. Also, the reader will have noticed the use of $c = 1$, 2, 5, and 13 in these examples. They are, of course, the Fibonacci numbers with odd subscripts. Unfortunately, the methods of this note do not apply to values of c which are Fibonacci numbers with even subscripts.

2. MAIN RESULTS

The main results of this note are summarized in three theorems: Theorem I provides a formulation of series of the form (1.3); Theorem II gives finer results where the sums are broken down into individual series similar to those in (1.4); Theorem III reveals even more detailed information in the form of explicit formulas for the partial sums of series in Theorem II.

In the following discussion, it will be assumed that K represents an odd integer and that t is an integer in the range $-(K-1)/2$ to $(K-1)/2$ inclusive.

Theorem I:

$$S(K) = \sum_0^\infty \frac{1}{F_{2n+1} + F_K} = K\sqrt{5}/2L_K.$$

Theorem II:

$$S(K,\ t) = \sum_0^\infty \frac{1}{F_{(2n+1)K+2t} + F_K} = (\sqrt{5} - 5F_t/L_t)/2L_K \qquad t \text{ even},$$

$$= (\sqrt{5} - L_t/F_t)/2L_K \qquad t \text{ odd}.$$

Theorem III:

$$S_N(K,\ t) = \sum_0^N \frac{1}{F_{(2n+1)K+2t} + F_K} = \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K \qquad N \text{ even},\ t \text{ even} \qquad \text{(a)}$$

(continued)

$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/2L_K \qquad N \text{ even, } t \text{ odd;} \tag{b}$$

$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K \qquad N \text{ odd, } t \text{ even;} \tag{c}$$

$$= \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/2L_K \qquad N \text{ odd, } t \text{ odd.} \tag{d}$$

3. ELEMENTARY RESULTS

We shall adopt the usual Fibonacci and Lucas number definitions:

$$F_{n+2} = F_{n+1} + F_n \text{ with } F_0 = 0 \text{ and } F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

We shall also employ the well-known Binet forms:

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Other elementary results which will be required include $\alpha\beta = -1$ and $F_{2a} = F_a L_a$.

4. PROOF OF MAIN RESULTS

To prove Theorems I, II, and III, it will be sufficient to prove Theorem III together with several short lemmas that establish the connection with Theorems I and II.

Lemma 1: $\displaystyle\lim_{n\to\infty} \frac{L_n}{F_n} = \lim_{n\to\infty} \frac{5F_n}{L_n} = \sqrt{5}.$

Proof: From the Binet forms, we have

$$\frac{L_n}{F_n} = \frac{\sqrt{5}(\alpha^n + \beta^n)}{(\alpha^n - \beta^n)} = \frac{\sqrt{5}(1 + (-1)^n\beta^{2n})}{(1 - (-1)^n\beta^{2n})} \to \sqrt{5} \text{ as } n \to \infty.$$

The second part follows immediately, since $5/\sqrt{5} = \sqrt{5}$.

Lemma 2: $\displaystyle\frac{L_t}{F_t} = -\frac{L_{-t}}{F_{-t}}.$

Proof: Again using the Binet forms, we have

$$-\frac{L_{-t}}{F_{-t}} = \frac{-\sqrt{5}(\alpha^{-t} + \beta^{-t})}{(\alpha^{-t} - \beta^{-t})} = \frac{-\sqrt{5}((-1)^t\beta^t + (-1)^t\alpha^t)}{((-1)^t\beta^t - (-1)^t\alpha^t)}$$

$$= \frac{-\sqrt{5}(\beta^t + \alpha^t)}{(\beta^t - \alpha^t)} = \frac{\sqrt{5}(\alpha^t + \beta^t)}{(\alpha^t - \beta^t)} = \frac{L_t}{F_t}. \qquad \text{Q.E.D.}$$

Theorem II may therefore be deduced from Theorem III and Lemma 1 and taking the limits as N approaches infinity. Summation of the results of Theorem II over the K values of t ranging from $-(K - 1)/2$ to $(K - 1)/2$ inclusive implies the truth of Theorem I, since the rational terms cancel out in pairs (as guaranteed by Lemma 2).

Before proceeding to the proof of Theorem III, we will need the results of the following four lemmas.

Lemma 3: $F_{a+2b} + F_a = F_{a+b} \cdot L_b$ for b even.

Proof: Since b is even, $(\alpha\beta)^b = +1$.

$$\begin{aligned}\text{RHS} &= (\alpha^{a+b} - \beta^{a+b})(\alpha^b + \beta^b)/\sqrt{5}\\ &= (\alpha^{a+2b} + \alpha^{a+b} \cdot \beta^b - \beta^{a+b} \cdot \alpha^b - \beta^{a+2b})/\sqrt{5}\\ &= (\alpha^{a+2b} - \beta^{a+2b} + (\alpha\beta)^b(\alpha^a - \beta^a))/\sqrt{5}\\ &= F_{a+2b} + F_a = \text{LHS}.\end{aligned}$$

Lemma 4: $F_{2a+b} + F_b = F_a \cdot L_{a+b}$ for a odd.

Proof: Since a is odd, $(\alpha\beta)^a = -1$.

$$\begin{aligned}\text{RHS} &= (\alpha^a - \beta^a)(\alpha^{a+b} + \beta^{a+b})/\sqrt{5}\\ &= (\alpha^{2a+b} + \alpha^a \cdot \beta^{a+b} - \beta^a \cdot \alpha^{a+b} - \beta^{2a+b})/\sqrt{5}\\ &= (\alpha^{2a+b} - \beta^{2a+b} - (\alpha\beta)^a(\alpha^b - \beta^b))/\sqrt{5}\\ &= F_{2a+b} + F_b = \text{LHS}.\end{aligned}$$

Lemma 5: $L_a \cdot L_b - 5F_a \cdot F_b = 2L_{a-b}$ for b even.

Proof: Since b is even, $(\alpha\beta)^b = +1$.

$$\begin{aligned}\text{LHS} &= (\alpha^a + \beta^a)(\alpha^b + \beta^b) - (\alpha^a - \beta^a)(\alpha^b - \beta^b)\\ &= \alpha^{a+b} + \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b + \beta^{a+b} - \alpha^{a+b} + \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b - \beta^{a+b}\\ &= (\alpha\beta)^b(\alpha^{a-b} + \beta^{a-b} + \alpha^{a-b} + \beta^{a-b})\\ &= 2(\alpha^{a-b} + \beta^{a-b}) = 2L_{a-b} = \text{RHS}.\end{aligned}$$

Lemma 6: $L_a \cdot F_b - F_a \cdot L_b = 2F_{a-b}$ for b odd.

Proof: Since b is odd, $(\alpha\beta)^b = -1$.

$$\begin{aligned}\text{LHS} &= ((\alpha^a + \beta^a)(\alpha^b - \beta^b) - (\alpha^a - \beta^a)(\alpha^b + \beta^b))/\sqrt{5}\\ &= (\alpha^{a+b} - \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b - \beta^{a+b} - \alpha^{a+b} - \alpha^a \cdot \beta^b + \beta^a \cdot \alpha^b + \beta^{a+b})/\sqrt{5}\\ &= -(\alpha\beta)^b(\alpha^{a-b} - \beta^{a-b} + \alpha^{a-b} - \beta^{a-b})/\sqrt{5}\\ &= 2(\alpha^{a-b} - \beta^{a-b})/\sqrt{5} = 2F_{a-b} = \text{RHS}.\end{aligned}$$

We shall prove part (a) of Theorem III in full and leave the details of parts (b), (c), and (d) to the reader, since they follow exactly the same pattern. In the discussion that follows, we will assume both N and t to be even. We shall proceed by induction on N.

$N = 0$: We must prove that

$$\frac{1}{F_{K+2t} + F_K} = \left(\frac{L_{K+t}}{F_{K+t}} - \frac{5F_t}{L_t}\right)/2L_K.$$

Using Lemma 3 with $a = K$ and $b = t$ gives $F_{K+2t} + F_K = F_{K+t} \cdot L_t$. Hence

$$\text{LHS} = \frac{1}{F_{K+t} \cdot L_t} \quad \text{and} \quad \text{RHS} = \frac{L_{K+t} \cdot L_t - 5F_{K+t} \cdot F_t}{2F_{K+t} \cdot L_t \cdot L_K}.$$

Using Lemma 5 with $a = K + t$ and $b = t$ gives $L_{K+t} \cdot L_t - 5F_{K+t} \cdot F_t = 2L_K$. Hence

$$\text{RHS} = \frac{2L_K}{2F_{K+t} \cdot L_t \cdot L_K} = \frac{1}{F_{K+t} \cdot L_t} = \text{LHS}.$$

Assuming that Theorem III (a) is true for $N = M$ (where M is even), we must prove it true for $N = M + 2$. Hence, the sum of the two extra terms on the LHS corresponding to $N = M + 1$ and $N = M + 2$ must equal the difference in the RHS formulas for $N = M + 2$ and $N = M$. Therefore, we must prove that

$$\frac{1}{F_{(2M+3)K+2t} + F_K} + \frac{1}{F_{(2M+5)K+2t} + F_K} = \left(\frac{L_{(M+3)K+t}}{F_{(M+3)K+t}} - \frac{5F_t}{L_t}\right)/2L_K - \left(\frac{L_{(M+1)K+t}}{F_{(M+1)K+t}} - \frac{5F_t}{L_t}\right)/2L_K.$$

To simplify the following algebra, we introduce the odd integer P, where

$$P = (M + 1)K + t.$$

This means that we must now prove that

$$\frac{1}{F_{2P+K} + F_K} + \frac{1}{F_{2P+3K} + F_K} = \left(\frac{L_{P+2K}}{F_{P+2K}} - \frac{L_P}{F_P}\right)/2L_K$$

Using Lemma 4 with $a = P$ and $b = K$ gives

$$F_{2P+K} + F_K = F_P \cdot L_{P+K}.$$

Using Lemma 3 with $a = K$ and $b = P + K$ gives

$$F_{2P+3K} + F_K = F_{P+2K} \cdot L_{P+K};$$

$$\text{LHS} = \frac{1}{F_P \cdot L_{P+K}} + \frac{1}{F_{P+2K} \cdot L_{P+K}} = \frac{F_{P+2K} + F_P}{F_P \cdot L_{P+K} \cdot F_{P+2K}}$$

Using Lemma 4 with $a = K$ and $b = P$ gives

$$F_{P+2K} + F_P = F_K \cdot L_{P+K};$$

$$\text{LHS} = \frac{F_K \cdot L_{P+K}}{F_P \cdot L_{P+K} \cdot F_{P+2K}} = \frac{F_K}{F_P \cdot F_{P+2K}};$$

$$\text{RHS} = \frac{L_{P+2K} \cdot F_P - F_{P+2K} \cdot L_P}{2F_{P+2K} \cdot F_P \cdot L_K}.$$

Using Lemma 6 with $a = P + 2K$ and $b = P$ gives

$$L_{P+2K} \cdot F_P - F_{P+2K} \cdot L_P = 2F_{2K} = 2F_K \cdot L_K;$$

$$\text{RHS} = \frac{2F_K \cdot L_K}{2F_{P+2K} \cdot F_P \cdot L_K} = \frac{F_K}{F_{P+2K} \cdot F_P} = \text{LHS}.$$

5. EXTENSION TO LUCAS NUMBERS

Similar results may be obtained by substituting Lucas numbers for the Fibonacci numbers in (1.2). In this case, however, even subscripts are required. Examples equivalent to those in (1.3) include the following:

(5.1)
$$\sum_0^\infty \frac{1}{L_{2n} + 3} = (2\sqrt{5} + 1)/10 \qquad \sum_0^\infty \frac{1}{L_{2n} + 7} = (4\sqrt{5} + 5/3)/30$$
$$\sum_0^\infty \frac{1}{L_{2n} + 18} = (6\sqrt{5} + 2)/80 \qquad \sum_0^\infty \frac{1}{L_{2n} + 47} = (8\sqrt{5} + 15/7)/210$$

These series may also be broken down into subseries similar to those in (1.4). For example:

$$\sum_0^\infty \frac{1}{L_{12n} + 18} = (\sqrt{5} + 2)/80$$

(5.2)
$$\sum_0^\infty \frac{1}{L_{12n+2} + 18} = (\sqrt{5} + 5/3)/80 \qquad \sum_0^\infty \frac{1}{L_{12n+10} + 18} = (\sqrt{5} - 5/3)/80$$

$$\sum_0^\infty \frac{1}{L_{12n+4} + 18} = (\sqrt{5} + 1)/80 \qquad \sum_0^\infty \frac{1}{L_{12n+8} + 18} = (\sqrt{5} - 1)/80$$

$$\sum_0^\infty \frac{1}{L_{12n+6} + 18} = \sqrt{5}/80$$

Notice that, in this case, the rational terms occur in pairs except for the first series. This explains the presence of the residual rational terms in (5.1) above.

The following three theorems (IV-V) summarize the above results. They are given without proof, since the methods required exactly parallel those of Section 4. In these theorems, We assume that K is an even integer and that t is an integer in the range $-K/2$ to $K/2 - 1$ inclusive.

Theorem IV:

$$T(K) = \sum_0^\infty \frac{1}{L_{2n} + L_K} = K\sqrt{5}/10F_K + 1/2L_{K/2}^2 \qquad K/2 \text{ even,}$$
$$= K\sqrt{5}/10F_K + 1/10F_{K/2}^2 \qquad K/2 \text{ odd.}$$

Theorem V:

$$T(K, t) = \sum_0^\infty \frac{1}{L_{(2n+1)K+2t} + L_K} = (\sqrt{5} - 5F_t/L_t)/10F_K \qquad t \text{ even,}$$
$$= (\sqrt{5} - L_t/F_t)/10F_K \qquad t \text{ odd.}$$

Theorem VI:

$$T_N(K, t) = \sum_0^N \frac{1}{L_{(2n+1)K+2t} + L_K}$$
$$= \left(\frac{5F_{(N+1)K+t}}{L_{(N+1)K+t}} - \frac{5F_t}{L_t}\right)/10F_K \qquad t \text{ even,}$$
$$= \left(\frac{L_{(N+1)K+t}}{F_{(N+1)K+t}} - \frac{L_t}{F_t}\right)/10F_K \qquad t \text{ odd.}$$

6. A TANTALIZING PROBLEM

If we let $K = 0$ in Theorem V or VI, we find that they give divergent series. However, if we formally substitute $K = 0$ into Theorem IV (without, as yet, any mathematical justification), we find that the LHS is finite, namely:

(6.1)
$$\sum_0^\infty \frac{1}{L_{2n} + 2} = .64452\ 17830\ 67274\ 44209\ 92731\ 19038$$

(to 30 decimal places). The RHS, however, contains the indeterminate form K/F_K.

If we take the liberty of defining a Fibonacci function such as

$$\begin{aligned} f(x) &= (\alpha^x - (-1)^x\alpha^{-x})/\sqrt{5} \\ &= (\alpha^x - (\cos \pi x + i \sin \pi x)\alpha^{-x})/\sqrt{5} \\ &= ((\alpha^x - \cos \pi x \cdot \alpha^{-x}) - i \sin \pi x \cdot \alpha^{-x})/\sqrt{5} \end{aligned}$$

and differentiate with respect to x, the real part becomes:

$$\mathrm{Re}[f'(x)] = (\ln \alpha \cdot \alpha^x + \pi \sin \pi x \cdot \alpha^{-x} + \cos \pi x \cdot \ln \alpha \cdot \alpha^{-x})/\sqrt{5}$$

and

$$\mathrm{Re}[f'(0)] = (\ln \alpha \cdot 1 + \pi \cdot 0 \cdot 1 + 1 \cdot \ln \alpha \cdot 1)/\sqrt{5} = 2 \ln \alpha/\sqrt{5}.$$

Substituting this value into the RHS of Theorem IV gives:

$$1/(4 \ln \alpha) + 1/8 = .64452\ 17303\ 08756\ 88440\ 03306\ 51529 \tag{6.2}$$

(to 30 decimal places). The difference between the values in (6.1) and (6.2) is obvious, but can any reader resolve this most tantalizing problem?

7. CONCLUSIONS

In this note, we have established explicit formulas for a number of series of the form

$$\sum_0^\infty \frac{1}{F_{an+b} + c} \quad \text{and} \quad \sum_0^\infty \frac{1}{L_{an+b} + c} \tag{7.1}$$

for certain values of a, b, and c positive. Similar results apply for c negative, but because of the possibility of a zero denominator, the series must begin with the term in which $an + b > K$. This leads to less elegant formulas, such as the following:

$$\sum_0^\infty \frac{1}{F_{6n+5} - 2} = (5 - \sqrt{5})/8$$

$$\sum_0^\infty \frac{1}{F_{6n+7} - 2} = (3 - \sqrt{5})/8 \qquad \sum_0^\infty \frac{1}{F_{6n+9} - 2} = (5/2 - \sqrt{5})/8. \tag{7.2}$$

Summing these three series gives

$$\sum_0^\infty \frac{1}{F_{2n+5} - 2} = (21/2 - 3\sqrt{5})/8, \tag{7.3}$$

where the symmetric form of (1.4) appears to have been lost. Similar results may be obtained using the Lucas numbers in (7.1). We leave the reader to investigate these formulas and to determine the true value of the series:

$$\sum_0^\infty \frac{1}{L_{2n} + 2}. \tag{7.4}$$

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ON THE EQUATION $\sigma(m)\sigma(n) = (m+n)^2$

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1. A pair of positive integers m and n are called amicable if

$$\sigma(m)\sigma(n) = (m+n)^2 \quad \text{and} \quad \sigma(m) = \sigma(n).$$

Although over a thousand pairs of amicable numbers are known, no pairs of relatively prime amicable numbers are known. Some necessary conditions for existence of such numbers are given in [1], [2], and [3].

In this paper, we show that some of the conditions are also necessary for the existence of m and n satisfying

$$\sigma(m)\sigma(n) = (m+n)^2, \tag{1}$$

and

$$(m, n) = 1. \tag{2}$$

In particular we prove

Theorem: If m and n satisfy (1) and (2), mn is divisible by at least twenty-two distinct primes.

Corollary (Hagis [3]): The product of relatively prime amicable numbers are divisible by twenty-two distinct primes.

2. Throughout this paper, let m and n be positive integers satisfying (1) and (2), and let

$$mn = \prod_{i=1}^{r} p_i^{a_i}$$

where $p_1 < \cdots < p_r$ are primes and the a_i's are positive integers. Since σ is multiplicative,

$$\prod_{i=1}^{r} \sigma(p_i^{a_i}) = \sigma(mn) = (m+n)^2.$$

If k and a are positive integers, p is a prime and if $p^a | k$ and $p^{a+1} \nmid k$, then we write $p^a || k$. $\omega(k)$ denotes the number of distinct prime factors of k.

Lemma 1: $\sigma(mn)/mn > 4$.

Proof: By (1) and (2)

$$\frac{\sigma(mn)}{mn} = \frac{(m+n)^2}{mn} = 4 + \frac{(m-n)^2}{mn} > 4. \quad \text{Q.E.D.}$$

Lemma 2: If q is a prime, $q | mn$ and if $p^a || mn$, $q \nmid \sigma(p^a)$.

Proof: Suppose q is a prime, $q | mn$, $p^a || mn$, and $q | \sigma(p^a)$. Since

$$\sigma(p^a) \,|\, (m+n)^2,$$

$q | m + n$. Then $q | m$ and $q | n$, contradicting (2). Q.E.D.

Lemma 3: If $\omega(mn) \le 21$, $2|mn$.

Proof: Suppose

$$mn = \prod_{i=1}^{r} p_i^{a_i}, \quad r \le 21, \text{ and } 3 \le p_1.$$

If q_i is the ith prime, we have, by Lemma 1,

$$4 < \frac{\sigma(mn)}{mn} = \prod_{i=1}^{r} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \le \prod_{i=2}^{r+1} \frac{q_i}{q_i - 1}.$$

Since

$$\prod_{i=2}^{r+1} q_i/(q_i - 1) < 4 \text{ if } r \le 20, \; r = 21.$$

Then

(3) $\quad 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 | mn$

and

(4) $\quad p_{21} \le 113$

because

$$\prod_{i=2}^{17} \frac{q_i}{q_i - 1} \prod_{i=19}^{22} \frac{q_i}{q_i - 1} < 4 \quad \text{and} \quad \prod_{i=2}^{21} \frac{q_i}{q_i - 1} \frac{127}{126} < 4.$$

Suppose $p^d || mn$ and $p \ne 3, 7, 31$. Then $p \le 113$ and $p \equiv -1(q)$ for some prime $3 \le q \le 37$. If d is odd, then $1 + p|\sigma(p^d)$, and we have $q|\sigma(p^d)$ and $q|mn$, contradicting Lemma 2. Hence, d is even, and $mn = 3^a 7^b 31^c e^2$, where

$$(e, 2 \cdot 3 \cdot 7 \cdot 31) = 1.$$

Since

$$\prod_{i=1}^{21} \sigma(p_i^{a_i}) = (m+n)^2$$

is even and $\sigma(p_i^{a_i})$ is odd if a_i is even, at least one of a, b, or c is odd.

Suppose at least two of them are odd. Then

$$32|\sigma(3^a)\sigma(7^b)\sigma(31^c)\sigma(e^2) = (m+n)^2,$$

or $8|m + n$. Hence, $m \equiv -n(8)$, or $mn = -n^2 \equiv -1(8)$. If a is even, then b and c are odd and $mn = 3^a 7^b 31^c e^2 \equiv 1(8)$, while, if a is odd, then $mn \equiv \pm 3(8)$, a contradiction in both cases. Hence, only one of a, b, or c is odd.

Suppose a is odd and b and c are even. Then

$$m = 3^a f^2 \equiv 3(8)$$

and

$$n = g^2 \equiv 1(8) \quad [\text{or } m \equiv 1(8) \text{ and } n = 3(8)].$$

Hence, $m = 3 + 8h$ and $n = 1 + 8i$, for some h and i, and we have

$$\sigma(3^a)\sigma(f^2g^2) = (m+n)^2 = (3 + 8h + 1 + 8i)^2 \equiv 0(16),$$

or $16|\sigma(3^a)$. Since

$$\sigma(3^a) = (1+3)(1 + 3^2 + 3^4 + \cdots + 3^{a-1}),$$

$4|1 + 3^2 + 3^4 + \cdots + 3^{a-1}$, or $a \equiv 7(8)$. Then

$$\sigma(3^a) = (1+3)(1+3^2)(1+3^4)(1 + 3^8 + \cdots + 3^{a-7}),$$

or $5|\sigma(3^a)$ contradicting Lemma 2. Suppose b is odd and a and c are even. Then $m \equiv 7(8)$ and $n \equiv 1(8)$ [or $m \equiv 1(8)$ and $n = 7(8)$], $(m+n)^2 \equiv 0(64)$, $64|\sigma(7^b)$, $b \equiv 7(8)$, $1 + 7^2|\sigma(7^b)$, or $5|\sigma(7^b)$, a contradiction. Suppose c is odd and a and

b are even. Then $64|\sigma(31^e)$, $e \equiv 3(4)$, $1 + 31^2|\sigma(31^e)$, or $13|\sigma(31^e)$, a contradiction. Since we get a contradiction in every case,

$$2|mn \text{ if } \omega(mn) \le 21. \quad \text{Q.E.D.}$$

Lemma 4: $4\nmid mn$.

Proof: Suppose

$$mn = 2^a \prod_{i=2}^{r} p_i^{a_i}, \text{ with } a \ge 1.$$

Since

$$\sigma(mn) = (2^{a+1} - 1)\prod_{i=2}^{r} \sigma(p_i^{a_i}) = (m+n)^2$$

is odd, a_i is even, and we have

$$m = 2^a b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2^a b^2].$$

Suppose a is even. Then $m = d^2$, and $2^{a+1} - 1$ has a prime factor $q \equiv 3(4)$. Since $q|m + n$, $c^2 \equiv -d^2(q)$. Since $(q, dc) = 1$ by Lemma 2, $(-d^2/q) = 1$, where (e/q) is the Legendre symbol. However,

$$(-d^2/q) = (-1/q) = (-1)^{\frac{q-1}{2}} = -1,$$

a contradiction. Hence, a is odd.

Suppose $a \ge 3$ is odd. Then

$$m = 2b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2b^2],$$

and $2^{a+1} - 1$ has a prime factor $q \equiv 5$ or $7(8)$. Since $q|m + n$ and $(q, 2bc) = 1$, $c^2 \equiv -2b^2(q)$, or $(-2b^2/q) = 1$. However,

$$(-2b^2/q) = (-2/q) = (-1/q)(2/q) = (-1)^{\frac{q-1}{2}}(-1)^{\frac{q^2-1}{8}} = -1,$$

a contradiction, Hence, $a = 1$. Q.E.D.

Lemma 5: If $2|mn$, $\omega(mn) \ge 22$.

Proof: Suppose

$$mn = 2\prod_{i=2}^{r} p_i^{a_i} \quad \text{and} \quad r \le 21.$$

Since

$$3\prod_{i=2}^{r} \sigma(p_i^{a_i}) = (m+n)^2$$

is odd, $3\nmid mn$, by Lemma 2, and so $5 \le p_i$, a_i is even, and $3|\sigma(p_j^{a_j})$ for some j. Then, as in Lemma 3, we have $r = 21$, (3) and (4). We can also show that $p_{20} \le 83$. Suppose $p^a||mn$, $p \ge 5$, q is a prime, and $q|\sigma(p^a)$. Then, by Lemma 2, $q\nmid mn$, and by (3), $q > 61$; moreover, since $q|m + n$,

$$m = 2b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2b^2],$$

we have $c^2 \equiv -2b^2(q)$, or $(-2b^2/q) = 1$, and so $q \equiv 1$ or $3(8)$. Hence, if

$$(5) \qquad 5 \le q \le 61, \text{ or } q \equiv 5 \text{ or } 7(8),$$

$q\nmid\sigma(p^a)$.

In [3] Hagis showed that if $3|\sigma(p^a)$ then $\sigma(p^a)$ is divisible by a prime q satisfying (5), except when $p = 31$, 73, 97, or 103, in which case $\sigma(p^a)$ is divisible by $s = 331$, 1801, 3169, or 3571, respectively. Since

$$3\prod_{i=2}^{r}\sigma(p_i^{a_i}) = (m+n)^2, \quad s^2 \Big| \prod_{i=2}^{r}\sigma(p_i^{a_i}).$$

However, Hagis also showed that, if $t^b||mn$, $t \neq p$ and $s|\sigma(t^b)$, then $\sigma(t^b)$ is divisible by a prime q satisfying (5). Hence, $s^2|\sigma(p^a)$. Then $\sigma(p^a)$ is divisible by a prime q = 5564773, 13925333, 570421, or 985597, respectively, satisfying (5), a contradiction. Hence, $r > 21$. Q.E.D.

Lemmas 3 and 5 prove our Theorem.

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A SPECIAL mTH-ORDER RECURRENCE RELATION

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1. INTRODUCTION

In this paper, we consider mth-order recurrence relations whose characteristic equation has only one distinct root. We express the solution for the relation in powers of the single root. The proof for the solution depends upon a special property of factorial polynomials that is given in the first lemma. We conclude the paper by noting the simple form of the result for $m \geq 2$, 3.

2. A SPECIAL mTH-ORDER RECURRENCE RELATION

In this section, we shall consider an mth-order recurrence relation whose characteristic equation has only one distinct root λ. It is of the form

$$T = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{n-j}$$

with initial values $T_0, \ldots, T_{m-1}$.

Before we can prove the solution for this relation, we must establish two lemmas. The first lemma gives a useful property of the factorial polynomials. With the second lemma, we obtain an evaluation for more general polynomials. These are actually elements in the vector space $\mathbf{V}_m$ of all polynomials in j of degree less than m. This vector space has a basis that consists of the constant ${}_jP_0 = 1$ and the monic factorial polynomials in j:

$${}_jP_w = \frac{j!}{(j-w)!} = (j-0)(j-1)\ldots(j-(w-1)); \; w = 1, \ldots, m-1.$$

We will make use of the fact that the zeros of these polynomials are the integers 0, ..., $w - 1$. We are now ready to state and prove the first lemma.

Lemma 2.1: For any integers m, w where $0 \leq w < m$,

$$\sum_{j=0}^{m} (-1)^j\binom{m}{m-j}{}_jP_w = 0.$$

We first of all observe that for $w = 0$ the factorial polynomials are just the constant 1. For the summation, we then have:

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} = (1-1)^m = 0.$$

If $0 < w$ the polynomial ${}_jP_w = 0$ for $j = 0, \ldots, w - 1$. Hence in the given summation, we can omit the zero terms and start the summation at $j = w$. This gives

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = \sum_{j=w}^{m} (-1)^j \binom{m}{m-j} {}_jP_w.$$

We change our summation variable to t by letting $j = w + t$. Then, we have for the summation:

$$\sum_{t=0}^{m-w} (-1)^{w+t} \binom{m}{m-w-t} {}_{w+t}P_w = (-1)^w \sum_{t=0}^{m-w} (-1)^t \frac{m!}{(m-w-t)!\,(w+t)!} \frac{(w+t)!}{t!}.$$

When we multiply the numerator and denominator by $(m - w)!$, we have the form:

$$(-1)^w \frac{m!}{(m-w)!} \sum_{t=0}^{m-w} (-1)^t \binom{m-w}{m-w-t} = (-1)^w \frac{m!}{(m-w)!} (1-1)^{m-w} = 0,$$

which is the result we set out to prove.

In the next lemma, we use the above result to prove a property of general polynomials $f(j) \in \mathbf{V}_m$.

Lemma 2.2: For any polynomial $f(j) \in \mathbf{V}_m$,

$$\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j) = f(0).$$

We shall first prove that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} f(j) = 0,$$

which is the comparable result for $f(j)$ to that of ${}_jP_w$ in Lemma 2.1.

Since $f(j) \in \mathbf{V}_m$, there exist constants c_w such that

$$f(j) = \sum_{w=0}^{m-1} {}_jP_w c_w.$$

Using this expression for $f(j)$, we have

$$\begin{aligned}\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} f(j) &= \sum_{j=0}^{m} (-1)^j \binom{m}{m-j} \sum_{w=0}^{m-1} {}_jP_w c_w \\ &= \sum_{w=0}^{m-1} c_w \sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = \sum_{w=0}^{m-1} c_w(0) = 0\end{aligned}$$

by Lemma 2.1.

We now break off the first term in the summation to obtain

$$\binom{m}{m} f(0) - \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j) = 0,$$

so that

$$f(0) = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j),$$

which is the desired conclusion.

We shall apply this result to polynomials in $\mathbf{V}_m$ of the form

$$\binom{m+i-j}{m-u} \binom{u-1+i-j}{u-1} = \frac{(m+i-j) \cdots (i+1-j)}{(m-u)!\,(u-1)!\,(u+i-j)}.$$

The zeros of these polynomials are the integers from $i + 1$ to $m + i$ with $u + i$ omitted. We are now ready to prove our major result.

Theorem 2.3: The mth-order recurrence relation

(1) $$T_n = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{n-j},$$

$T_0, \ldots, T_{m-1}$ arbitrary, has for its solution:

(2) $$T_{m+k} = \sum_{u=1}^{m} (-1)^{u-1}\binom{m+k}{m-u}\binom{u-1+k}{u-1}\lambda^{u+k}T_{m-u}.$$

Before going to the proof by induction, we need to show that (2) is valid for $-m \leq k < 0$. In other words, it reduces to the arbitrary value. To show this we first write (2) as a polynomial in k:

$$T_{m+k} = \sum_{u=1}^{m} (-1)^{u-1}\frac{(m+k)\cdots(1+k)}{(m-u)!(u-1)!(u+k)}\lambda^{u+k}T_{m-u}.$$

The integer k is negative, so we let $k = -s$. The polynomial in s now becomes

$$\frac{(m-s)\cdots(1-s)}{(m-u)!(u-1)!(u-s)}$$

which has for zeros the integers from 1 to m with u omitted. This means that in the summation for a fixed $k = -s$, all terms are zero except when $u = s$. The summand reduces to

$$\frac{(-1)^{s-1}(m-s)\cdots 1(-1)\cdots(1-s)}{(m-s)!(s-1)!}\lambda^0 T_{m-s} = \frac{(-1)^{s-1}(m-s)!(-1)^{s-1}(s-1)!}{(m-s)!(s-1)!}\lambda^0 T_{m-s}$$
$$= T_{m-s} = T_{m+k}.$$

This is the result we said is true.

To prove the theorem by induction on k, we first show that it is valid for $k = 0$. For this, we take $n = m$ in (1), so we have the relation

$$T_m = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{m-j}.$$

For $k = 0$ in (2), we have the solution

$$T_m = \sum_{u=1}^{m} (-1)^{u-1}\binom{m}{m-u}\binom{u-1}{u-1}\lambda^u T_{m-u}.$$

These two results are equal for $u = j$.

We assume that the solution (2) is valid for $k = 0, \ldots, i-1$, and we shall show it is true for $k = i$ and, hence, for all k.

We have for (1) when $n = m + i$,

$$T_{m+i} = \sum_{j-1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{m+i-j}$$

Substituting the solution (2) for T_{m+i-j}, we have

$$T_{m+i} = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j \left(\sum_{u=1}^{m} (-1)^{u-1}\binom{m+i-j}{m-u}\binom{u-1+i-j}{u-1}\lambda^{u+i-j}T_{m-u}\right)$$
$$= \sum_{u=1}^{m} (-1)^{u-1}\lambda^{u+i}T_{m-u}\sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\binom{m+i-j}{m-u}\binom{u-1+i-j}{u-1}.$$

The inside summation involves a polynomial $f(j) \in \mathbf{V}_m$ so we can apply Lemma 2.2. Evaluating $f(0)$ gives, for the summation:

$$T_{m+i} = \sum_{u=1}^{m} (-1)^{u-1}\lambda^{u+1}T_{m-u}\binom{m+i}{m-u}\binom{u-1+i}{u-1}$$

Interchanging the order of factors in the summand gives (2) for $k = i$.

3. SPECIAL CASES

It may be helpful to consider the form of the problem for $m = 2, 3$. For $m = 2$, the relation is

$$T_n = 2\lambda T_{n-1} - \lambda^2 T_{n-2}$$

and the solution is

$$T_{2+k} = (2+k)\lambda^{1+k}T_1 - (1+k)\lambda^{2+k}T_0.$$

For $m = 3$, the relation is

$$T_n = 3\lambda T_{n-1} - 3\lambda^2 T_{n-2} + \lambda^3 T_{n-3},$$

and the solution is

$$T_{3+k} = \binom{3+k}{2}\lambda^{1+k}T_2 - \binom{3+k}{1}\binom{1+k}{1}\lambda^{2+k}T_1 + \binom{2+k}{2}\lambda^{3+k}T_0$$

$$= \frac{(3+k)(2+k)}{2}\lambda^{1+k}T_2 - \frac{(3+k)(1+k)}{1}\lambda^{2+k}T_1 + \frac{(2+k)(1+k)}{2}\lambda^{3+k}T_0$$

For other small values of m, the solutions can be written out quite readily. The form of the solution suggests a couple of other ways to write it. For instance

$$T_{2+k} = (2+k)(1+k)\left[\frac{\lambda^{1+k}}{1+k}T_1 - \frac{\lambda^{2+k}}{2+k}T_0\right] = 2\binom{2+k}{2}\left[\frac{\lambda^{1+k}}{1+k}T_1 - \frac{\lambda^{2+k}}{2+k}T_0\right]$$

and

$$T_{3+k} = \frac{(3+k)(2+k)(1+k)}{2}\left[\frac{\lambda^{1+k}}{1+k}T_2 - \frac{2\lambda^{2+k}}{2+k}T_1 + \frac{\lambda^{3+k}}{3+k}T_0\right]$$

$$= 3\binom{3+k}{3}\left[\binom{2}{2}\frac{\lambda^{1+k}}{1+k}T_2 - \binom{2}{1}\frac{\lambda^{2+k}}{2+k}T_1 + \binom{2}{0}\frac{\lambda^{3+k}}{3+k}T_0\right].$$

These two forms, when applied to the general case, give a solution of the form

$$T_{m+k} = \frac{(m+k)\ \cdots\ (1+k)}{(m-1)!}\sum_{u=1}^{m}(-1)^{u-1}\binom{m-1}{m-u}\frac{\lambda^{u+k}}{u+k}T_{m-u}$$

$$= m\binom{m+k}{m}\sum_{u=1}^{m}(-1)^{u-1}\binom{m-1}{m-u}\frac{\lambda^{u+k}}{u+k}T_{m-u}.$$

These two forms may be more suitable than the first form of the solution. Other forms could also be obtained.

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FIBONACCI NUMBERS IN TREE COUNTS FOR MAXIMAL OUTERPLANE AND RELATED GRAPHS

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ABSTRACT

Let G denote a plane multigraph that is obtained from a maximal outerplane graph by adding a collection of multiedges. We associate with each such G an M-tree (a tree in which some vertices are designated as type M), and we observe that many such graphs can be associated with the same M-tree. Formulas for counting spanning trees are given and are used to generate some Fibonacci identities. The path P_n is shown to be the tree on n vertices whose associated graph has the maximum number of spanning trees, and a class of trees on n vertices whose associated graph yields the minimum is conjectured.

1. INTRODUCTION

The occurrence of Fibonacci numbers in spanning tree counts has been noted by many authors ([5], [6], [7], [8], [9], and [10]). In particular, the labeled *fan* on $n + 2$ vertices has F_{2n+2} spanning trees, where F_n denotes the nth Fibonacci number. The fan is actually a special case of the class of maximal outerplane graphs. In [11] it is shown that any labeled maximal outerplane graph on $n+2$ vertices with exactly two vertices of degree 2 also has F_{2n+2} spanning trees. In [1] the unifying concept of the "associated tree of a maximal outerplane graph" is presented; it is shown that Fibonacci numbers occur naturally in the count of spanning trees of these graphs and depend upon the structure of the associated trees.

The purpose of this paper is to extend the idea of the associated tree to maximal outerplane graphs with multiple edges, to give formulas for counting spanning trees, and to generate Fibonacci identities. In the final section, bounds are given on the number of spanning trees of any maximal outerplane graph on $n + 2$ vertices.

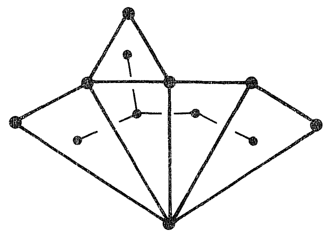
The associated tree T of a maximal outerplane graph G is simply the "inner dual" of G; that is, T is the graph formed by constructing the usual dual G^* and deleting the vertex in the infinite region of G. In [2] it is shown that all labeled maximal outerplane graphs that have the same associated tree have the same number of labeled spanning trees. When multiple edges are allowed in the maximal outerplane graph, the above construction can be carried out, but vertices of degree 1 or 2 in T can result from either vertices of degree 2 or 3 in G^*.

To avoid this ambiguity, we shall adopt the following convention in constructing T: place a vertex of "type R" in any interior region of G bounded by

*This work was supported in part by an Institutional Research Grant from the University of Wisconsin-La Crosse.

†This work was supported in part by the U.S. Energy Research and Development Administration (ERDA) under Contract No. AT(29-1)-789.

three edges and a vertex of "type M" in any interior region by a pair of multiedges. We shall call a tree that contains any vertices of type M an *M-tree*. Figure 1 gives examples of some graphs with their associated trees and M-trees in which circles denote vertices of type M.



(a) (b) (c)

Fig. 1. Some graphs with associated trees and M-trees

If T is a tree (respectively, an M-tree), $G(T)$ will denote a maximal outerplane graph (respectively, a multigraph) associated with T. Note that there may be many nonisomorphic graphs (or multigraphs) associated with T, but that each has the same number of labeled spanning trees (STs). Consequently, we can let ST $G(T)$ denote this number. We emphasize that we are considering the edges of $G(T)$ to be labeled. For example, for the graph G_1 in Figure 2, there are 12 labeled spanning trees: four containing e_1, four containing e_2, and four that do not contain e_1 or e_2.

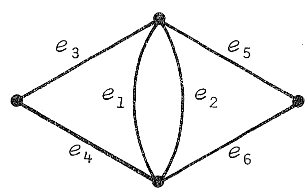
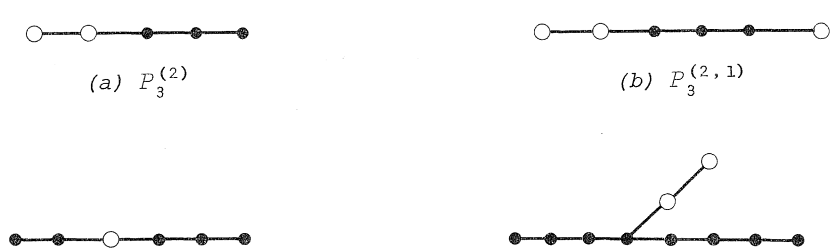


Fig. 2. A labeled multigraph G_1

It will be convenient to have a notation for some useful M-trees. As usual, P_n will denote the path with n vertices. Let $P_n^{(i)}$ denote the M-tree obtained from P_n by adjoining a path of i vertices of type M at an endpoint of P_n. A path P_n to which is attached a path of i type M vertices at one end and a path of j such vertices at the other end will be denoted $P_n^{(i,j)}$. The M-tree consisting of $P_n^{(1)}$ with P_m adjoined at the type M vertex will be denoted $P_{n,m}^{(0)}$. The M-tree constructed by adjoining a path of i vertices of type M at the $(n + 1)$st vertex of P_{n+m+1} will be denoted $P_{n,m}^{(i)}$. Figure 3 shows $P_3^{(2)}$, $P_3^{(2,1)}$, $P_{2,3}^{(0)}$, and $P_{3,4}^{(2)}$.



(a) $P_3^{(2)}$ (b) $P_3^{(2,1)}$

(c) $P_{2,3}^{(0)}$ (d) $P_{3,4}^{(2)}$

Fig. 3. Some M-trees

2. TREE COUNTS AND FIBONACCI IDENTITIES

As mentioned in the previous section, it is known that

$$\text{ST } G(P_n) = F_{2n+2}. \tag{2.1}$$

It is also known that Fibonacci numbers give the count of spanning trees of maximal outerplane graphs associated with the M-trees $P_n^{(1)}$ and $P_n^{(1,1)}$; namely,

$$\text{ST } G(P_n^{(1)}) = F_{2n+3} \tag{2.2}$$

and

$$\text{ST } G(P_n^{(1,1)}) = F_{2n+4}. \tag{2.3}$$

Counting the spanning trees of a graph G is often done with a basic reduction formula which sums those that contain a given edge e of G and those that do not (as in [3, p. 33]):

$$\text{ST } G = \text{ST } G\cdot e + \text{ST}(G - \{e\}), \tag{2.4}$$

where $G\cdot e$ denotes the graph obtained by identifying the end vertices of e, and removing the self-loops. This reduction performed on $G(P_n)$ at an edge containing a vertex of degree 2 demonstrates the basic Fibonacci equation

$$F_n = F_{n-1} + F_{n-2}. \tag{2.5}$$

Repeated applications of the reduction (2.4) will give recurrences leading to several well-known identities. For example,

$$\begin{aligned}\text{ST } G(P_n) &= \text{ST } G(P_{n-1}) + \text{ST } G(P_{n-1}^{(1)}) = \text{ST } G(P_{n-2}) + \text{ST } G(P_{n-2}^{(1)}) + \text{ST } G(P_{n-1}^{(1)})\\ &= \text{ST } G(P_1) + \text{ST } G(P_1^{(1)}) + \text{ST } G(P_2^{(1)}) + \cdots + \text{ST } G(P_{n-1}^{(1)}),\end{aligned}$$

which, from (2.2) gives

$$F_{2n+2} = F_1 + F_3 + \cdots + F_{2n+1} \tag{2.6}$$

since $\text{ST } G(P_1) = 3 = F_1 + F_3$.

The corresponding identity for the odd Fibonacci numbers

$$F_{2n+3} = 1 + F_2 + F_4 + \cdots + F_{2n+2} \tag{2.7}$$

follows from a similar recurrence developed by applying (2.4) to a multiple edge of $G(P_n^{(1)})$.

As another example, consider the recurrence obtained by starting again with $G(P_n)$ and alternating use of (2.4) on the resulting graphs of $G(P_k)$ and $G(P_k^{(1)})$:

$$\begin{aligned}\text{ST } G(P_n) &= \text{ST } G(P_{n-1}) + \text{ST } G(P_{n-1}^{(1)}) = \text{ST } G(P_{n-1}) + \text{ST } G(P_{n-2}^{(1)}) + \text{ST } G(P_{n-1})\\ &= \text{ST } G(P_{n-1}) + \text{ST } G(P_{n-2}^{(1)}) + \text{ST } G(P_{n-2}) + \text{ST } G(P_{n-2}^{(1)})\\ &= \text{ST } G(P_{n-1}) + \text{ST } G(P_{n-2}^{(1)}) + \text{ST } G(P_{n-2}) + \text{ST } G(P_{n-3}^{(1)}) + \cdots\\ &\qquad + \text{ST } G(P_1^{(1)}) + \text{ST } G(P_1) + \text{ST } G(P_1^{(1)}).\end{aligned}$$

Since $\text{ST } G(P_1^{(1)}) = F_5 = F_3 + F_2 + F_1 + 1$, this reduction yields the identity

$$F_{2n+2} = F_{2n} + F_{2n-1} + \cdots + F_2 + F_1 + 1.$$

The parallel identity

$$F_{2n+3} = F_{2n+1} + F_{2n} + \cdots + F_2 + F_1 + 1$$

can be obtained by beginning with $G(P_n^{(1)})$, and we then have the general identity

$$F_n = F_{n-2} + F_{n-3} + \cdots + F_1 + 1. \tag{2.8}$$

Other spanning tree counts that we shall find useful later can be obtained readily from (2.1), (2.2), and the reduction formula (2.4) applied at the appropriate edge:

$$\text{ST } G(P_{h,k}^{(0)}) = F_{2h+2k+2} + F_{2h+1}F_{2k+1} \tag{2.9}$$

and
$$\text{ST } G(P_{h,k}^{(j)}) = F_{2h+2k+4} + j(F_{2h+2k+2} + F_{2h+1}F_{2k+1}). \tag{2.10}$$
Each of these counts has been discovered by other means [5].

3. FURTHER TREE COUNTING FORMULAS AND FIBONACCI IDENTITIES

A second reduction formula for counting spanning trees is useful for any 2-connected graph G with cut-set $\{u, v\}$. Let $G = H \cup K$, where $H \cap K = \{u, v\}$ and each of H and K has at least one vertex other than u or v. [Any edge (u, v) may be arbitrarily assigned either to H or to K.] Then
$$\text{ST } G = [\text{ST } H][\text{ST } K\cdot(u, v)] + [\text{ST } K][\text{ST } H\cdot(u, v)], \tag{3.1}$$
where $K\cdot(u, v)$ means graph G with vertices u and v identified. To see this, we observe that a spanning tree of G contains exactly one path between u and v. Since G is 2-connected, we may first count all the ways that this path lies entirely in H and add the ways that it lies entirely in K.

If this formula is applied to the maximal outerplane graph whose associated tree is the path P_{h+k}, we have
$$\text{ST } G(P_{h+k}) = [\text{ST } G(P_h)][\text{ST } G(P_{k-1}^{(1)})] + [\text{ST } G(P_{k-1})][\text{ST } G(P_{h-1}^{(1)})],$$
or using (2.1) and (2.2), we obtain
$$F_{2h+2k+2} = F_{2h+2}F_{2k+1} + F_{2k}F_{2h+1}, \tag{3.2}$$
which appears in [8]. A similar application on $G(P_{h+k}^{(1)})$ and use of (2.1), (2.1), and (2.3) gives
$$F_{2h+2k+3} = F_{2h+2}F_{2k+2} + F_{2h+1}F_{2k+1}. \tag{3.3}$$
Combining (3.2) and (3.3) will produce the general identity
$$F_n = F_{j+1}F_{n-j} + F_jF_{n-j-1}, \quad 1 \le j \le n-1. \tag{3.4}$$

The next identity will be obtained by counting the spanning trees of a graph $G(T)$ in two ways. Let S be the tree formed by joining paths of lengths j, h, and k to a vertex w (see Fig. 4). The first computation of ST $G(S)$ is obtained by applying (3.1) to $S = H \cup K$, where $H = G(P_h \cup \{w\} \cup P_k)$, and using (2.9) to get
$$\text{ST } G(S) = F_{2h+2k+4}F_{2j+1} + F_{2j}[F_{2h+2k+2} + F_{2h+1}F_{2k+1}]. \tag{3.5}$$

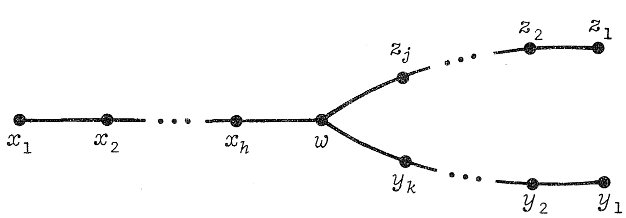


Fig. 4. The tree S

The second count is obtained by applying the reduction formula (2.4) successively to the exterior edges of $G(S)$ associated with vertices $z_j, z_{j-1}, \ldots, z_1$ and using (2.10) to get
$$\begin{aligned}\text{ST } G(S) = F_{2h+2k+4}F_{2j} &+ [F_{2h+2k+4} + 1(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_{2j-2}\\ &+ [F_{2h+2k+4} + 2(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_{2j-4} + \cdots\\ &+ [F_{2h+2k+4} + (j-1)(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_2\\ &+ [F_{2h+2k+4} + j(F_{2h+2k+2} + F_{2h+1}F_{2k+1}] \cdot 1.\end{aligned}$$

Collecting terms and using identity (2.7) we have

$$\mathrm{ST}\ G(S) = F_{2h+2k+4}F_{2j+1} + \left[\sum_{r=1}^{j-1} rF_{2j-2r} + j\right]\left[F_{2h+2k+2} + F_{2h+1}F_{2k+1}\right]. \tag{3.6}$$

Equating (3.5) and (3.6) produces the identity

$$F_{2j} - j = \sum_{r=1}^{j-1} rF_{2j-2r}. \tag{3.7}$$

A corresponding formula for odd Fibonacci numbers can be obtained by starting with the multi-tree consisting of tree S with a vertex of type M attached at Z_1:

$$F_{2j+1} - 1 = \sum_{r=1}^{j-1} rF_{2j-2r+1}. \tag{3.8}$$

The identity (3.8) appears in [9], but we think that (3.7) may be new.

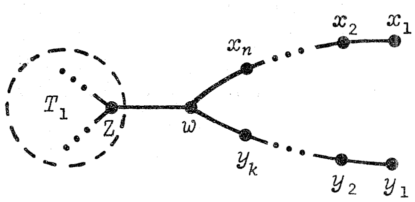
4. BOUNDS ON ST $G(T_n)$

The reduction formula (3.1) can also be used to derive a formula for "moving a branch path" in an associated tree T. Let w be a vertex of degree 3 in T with subtrees T_1, P_h, and P_k attached as in Figure 5(a). Let T' be the tree with path P_k "moved" as in Figure 5(b). Then the following formula holds

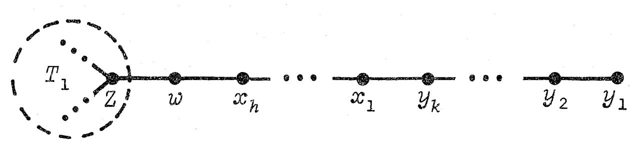
$$\mathrm{ST}\ G(T') = \mathrm{ST}\ G(T) + \mathrm{ST}[G(T_1) - (u, v)]F_{2h}F_{2k}, \tag{4.1}$$

where (u, v) is the edge in $G(T)$ separating vertices z and w. To see this, apply the reduction (3.1) to $G(T)$ and $G(T')$ with subgraphs

$$H = G(P_h \cup \{w\} \cup P_k) \quad \text{and} \quad K = G(T_1) = (u, v).$$



(a) Tree T



(b) Tree T'

Fig. 5

For the graph $G(T)$ we have

$$\begin{aligned}\mathrm{ST}\ G(T) &= F_{2h+2k+4}\mathrm{ST}[G(T_1) - (u, v)] \cdot (u, v)\\ &\quad + \mathrm{ST}[G(T_1) - (u, v)][F_{2h+2k+2} + F_{2h+1}F_{2k+1}],\end{aligned}$$

where we have used (2.9). For the graph $G(T')$ we have

$$\begin{aligned}\mathrm{ST}\ G(T') &= F_{2h+2k+4}\mathrm{ST}[G(T_1) - (u, v)] \cdot (u, v)\\ &\quad + \mathrm{ST}[G(T_1) - (u, v)]F_{2h+2k+3}.\end{aligned}$$

Subtracting these equations gives us

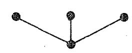
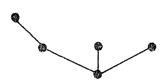
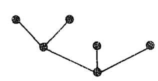
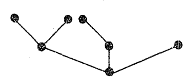
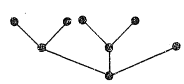
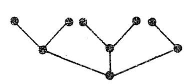
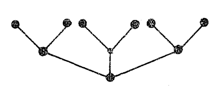
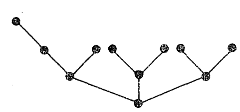
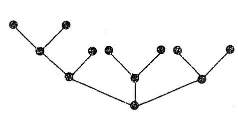
$$\begin{aligned}\mathrm{ST}\ G(T') - \mathrm{ST}\ G(T) &= \mathrm{ST}[G(T_1) - (u, v)][F_{2h+2k+1} - F_{2h+1}F_{2k+1}]\\ &= \mathrm{ST}[G(T_1) - (u, v)]F_{2h}F_{2k}\end{aligned}$$

by (3.4).

As a corollary to (4.1) we have an upper bound on ST $G(T)$: if T_n is a tree with maximum degree of a vertex equal to three, then ST $G(T_n)$ < ST $G(P_n)$.

To form a class of trees U_n whose associated maximal outerplane graph has the minimum number of spanning trees, it seems reasonable that U_n should be as "far" from P_n as possible. We conjecture that the trees U_n have the form given in Table 1.

Table 1

n	U_n	ST $G(U_n)$	ST $G(P_n) = F_{2n+2}$
1		3	3
2		8	8
3		21	21
4		54	55
5		141	144
6		360	377
7		939	987
8		2,394	2,584
9		6,237	6,765
10		15,876	17,711
11		41,391	46,368
12		105,462	121,393

The construction of the U_n can be described as follows. Let the vertex of U_1 be labeled v_1. To form U_{n+1} from U_n for $n \geq 2$, join a vertex v_{n+1} of degree 1 to U_n so that v_{n+1} is adjacent to v_i, where i is the smallest possible index subject to the requirement that all vertices of U_{n+1} have degree 3 or less.

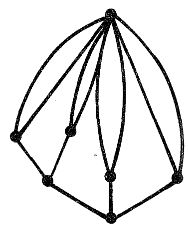
The values for ST $G(U_n)$ in Table 1 were computed using the fact that if U_n^* is the dual of $G(U_n)$, then ST $G(U_n)$ = ST U_n^*. By standard tree counting methods

$$\text{ST } U_n^* = \det(A_f A_f^t) = |a_{ij}|,$$

where A_f is the reduced incidence matrix of U_n^*. By labeling the vertices and edges of U_n^* consecutively from left to right and bottom to top, all the a_{ij} are zero except that

$$a_{ii} = 3,\ a_{12} = a_{21} = 1$$
$$a_{i,\,2i+1} = a_{2i+1,\,i} = 1,\ a_{i,\,2i+2} = a_{2i+2,\,i} = 1.$$

For example, with $n = 6$,

U_6^* is  and ST $U_6^* = \begin{vmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 \end{vmatrix}.$

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A NEW ANGLE ON THE GEOMETRY OF THE FIBONACCI NUMBERS

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The "angle" we have in mind is a *gnomon*, a planar region that has the general shape of a carpenter's square. At the time of Pythagoras, a carpenter's square was in fact called a gnomon. The term came from Babylonia, where it originally referred to the vertically placed bar that cast the shadow on a sundial. The ancient Greeks also inherited a large body of algebra from the Babylonians, which they proceeded to recast into geometric terms. The gnomon became a recurrent figure in the Greek geometric algebra.

There are several reasons why Babylonian algebra was not adopted as it was, principally the discovery of irrationals: an irrational was acceptable to the Greeks as a length but not as a number. A secondary reason but, nevertheless, one of significance, was the Greek "delight in the tangible and visible" [2].

In this note we shall attempt to make the numbers $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ... in the Fibonacci sequence "tangible and visible" by representing each F_m with a gnomon. These figures will enable us geometrically to derive or interpret many of the standard identities for the Fibonacci numbers. The ideas work equally well for the Lucas numbers and other generalized Fibonacci number sequences.

The gnomons we shall associate with the Fibonacci numbers are depicted in Figure 1. The angular shape that represents the mth Fibonacci number will be called the F_m-*gnomon*. In particular, "observe" the F_0 = 0-gnomon!

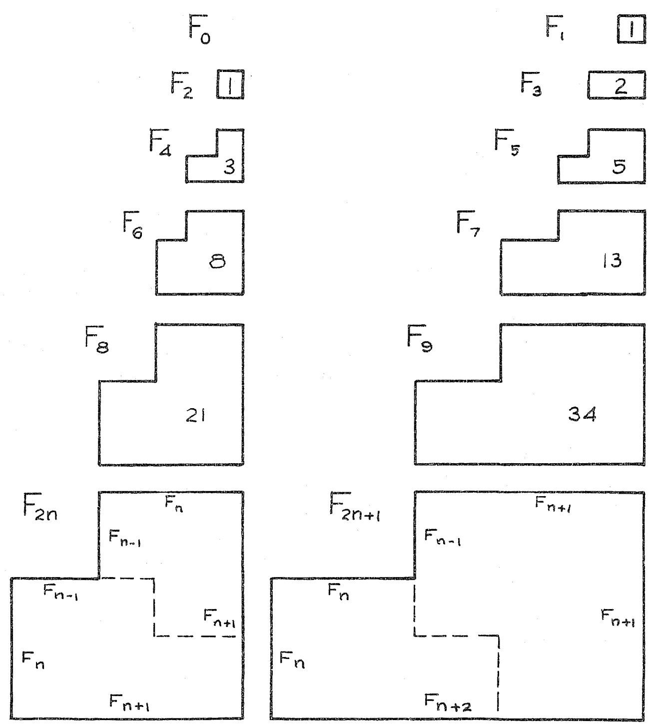


Fig. 1

The dashed lines in the lowermost gnomons indicate how the F_m- and F_{m+1}-gnomons can be combined to form the F_{m+2}-gnomon. This geometrically illustrates the basic recursion relation

$$F_{m+2} = F_{m+1} + F_m, \; m \geq 0. \tag{1}$$

The left-hand column of Figure 1 shows rather strikingly that the evenly indexed Fibonacci numbers are differences of squares of Fibonacci numbers. Indeed

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \; n \geq 1. \tag{2}$$

Equally obvious from the right-hand column is the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \; n \geq 0. \tag{3}$$

Several other identities can be read off easily:

$$F_{2n+1} = F_{n-1}F_{n+1} + F_n F_{n+2}, \; n \geq 1; \tag{4}$$

$$F_{2n+1} = F_{n+2}F_{n+1} - F_n F_{n-1}, \; n \geq 1; \tag{5}$$

$$F_{2n} = F_{n-1}F_n + F_n F_{n+1}, \; n \geq 1. \tag{6}$$

Since $L_n = F_{n+1} + F_{n-1}$ is the nth Lucas number, it follows from (6) that

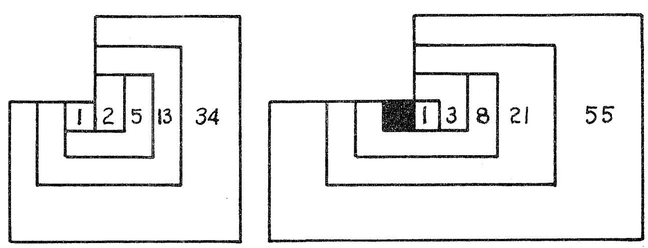
$$F_{2n} = L_n F_n, \; n \geq 1. \tag{7}$$

The gnomons in the left-hand column of Figure 1 can be superimposed in the manner shown in Figure 2(a). This shows how the F_{2n}-gnomon can be decomposed into "triple" gnomons of area $F_{2j} - F_{2j-2}$, $j = 1, \ldots, n$. From identity (1), we already know $F_{2j-1} = F_{2j} - F_{2j-2}$, and so

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}, \; n \geq 1. \tag{8}$$

In a similar manner (note the shaded unit square "hole") we see from Figure 2(b) that

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1, \; n \geq 1. \tag{9}$$



(a) *(b)*

Fig.2

We have noted that the F_{m+2}-gnomon can be dissected into an F_{m+1}- and F_m-gnomon. The larger of these can, in turn, be dissected into an F_m- and F_{m-1}-gnomon, and the larger of these can then be dissected into an F_{m-1}- and F_{m-2}-gnomon. Continuing this process dissects the original F_{m+2}-gnomon into a spiral that consists of the F_j-gnomons, $j = 1, \ldots, m$, together with an additional unit square (shown black), as illustrated in Figure 3. The separation of the gnomons into quadrants is rather unexpected.

From Figure 3, we conclude that

$$F_1 + F_2 + \cdots + F_m = F_{m+2} - 1, \; m \geq 0. \tag{10}$$

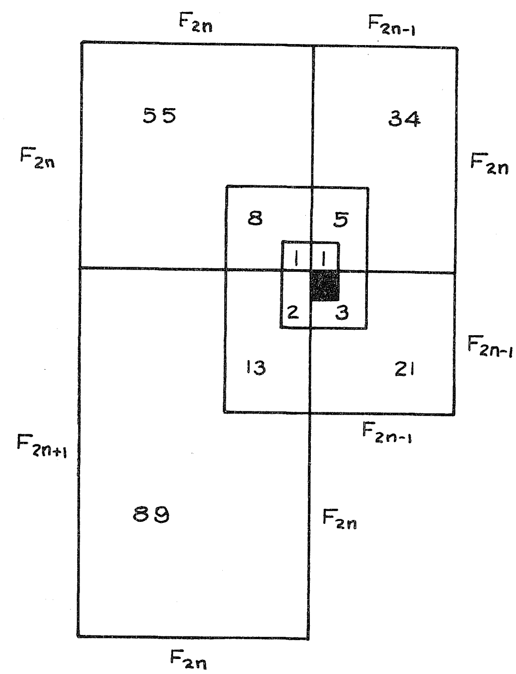


Fig. 3

Geometrically, we see that the first m Fibonacci gnomons can be combined with an additional unit square to form the F_{m+2}-gnomon. It is interesting to check this out successively for the special cases $m = 0, 1, 2, \ldots$.

The spiral pattern gives rise to additional identities. For example, by adding the areas of the gnomons in the first quadrant, we find

$$F_1 + F_5 + \cdots + F_{4n-3} = F_{2n-1}F_{2n}, \; n \geq 1. \tag{11}$$

The same procedure for the other three quadrants yields:

$$F_2 + F_6 + \cdots + F_{4n-2} = F_{2n}^2, \; n \geq 1; \tag{12}$$

$$F_3 + F_7 + \cdots + F_{4n-1} = F_{2n}F_{2n+1}, \; n \geq 1; \tag{13}$$

$$F_4 + F_8 + \cdots + F_{4n} = F_{2n+1}^2 - 1, \; n \geq 1. \tag{14}$$

The gnomons in the first quadrant are each a sum of two squares. (Some additional horizontal segments can be imagined in Figure 3.) We see that

$$F_1^2 + F_2^2 + \cdots + F_{2n-1}^2 = F_{2n-1}F_{2n}, \; n \geq 1. \tag{15}$$

Similarly, the third quadrant demonstrates

$$F_1^2 + F_2^2 + \cdots + F_{2n}^2 = F_{2n}F_{2n+1}, \; n \geq 1. \tag{16}$$

Of course, identities (15) and (16) are more commonly written simultaneously in the form

$$F_1^2 + F_2^2 + \cdots + F_m^2 = F_mF_{m+1}, \; m \geq 1. \tag{17}$$

Next, consider the F_{2n-1} by F_{2n+1} rectangle that the spiral covers in the right half-plane. Evidently, the area of this rectangle is one unit more than the area of the F_{2n} by F_{2n} square covered by the spiral in the third quadrant. Thus

$$F_{2n-1}F_{2n+1} = F_{2n}^2 + 1, \; n \geq 1. \tag{18}$$

An analogous consideration of the F_{2n} by F_{2n+2} rectangle covered by the spiral in the left half-plane shows

$$F_{2n}F_{2n+2} = F_{2n+1}^2 - 1, \; n \geq 1. \tag{19}$$

The black square at the center of the spiral plays an interesting role in the geometric derivation of these relations.

The geometric approach used above can be extended easily to deal with generalized Fibonacci sequences $T_1 = p$, $T_2 = q$, $T_3 = p+q$, $T_4 = p+2q$, ..., where p and q are positive integers. The T_m-gnomons can be taken as shown in Figure 4 (however, it should be mentioned that other gnomon shapes can be adopted, and will do just as well).

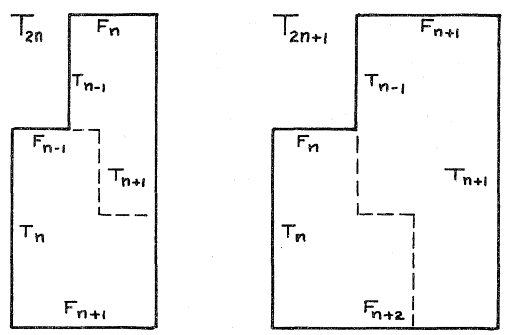


Fig. 4

From Figure 4, it is clear that

$$T_{m+2} = T_{m+1} + T_m, \; m \geq 1; \tag{20}$$

$$T_{2n} = T_{n-1}F_n + T_nF_{n+1}, \; n \geq 1; \tag{21}$$

$$T_{2n+1} = T_nF_n + T_{n+1}F_{n+1}, \; n \geq 1. \tag{22}$$

As before, a spiral pattern can be obtained readily. Figure 5 shows the spiral that corresponds to the Lucas sequence $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, ..., where $p = 1$ and $q = 3$.

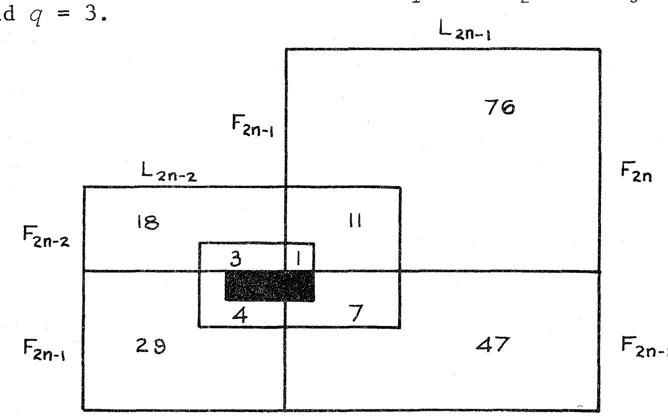


Fig. 5

It is clear from Figure 5 that

$$L_1 + L_2 + \cdots + L_m = L_{m+2} - 3,\ m \geq 1. \tag{23}$$

For the generalized sequence, one would find

$$T_1 + T_2 + \cdots + T_m = T_{m+2} - q,\ m \geq 1. \tag{24}$$

Beginning with a $q \times 1$ (black) rectangle, one can use identity (24) successively for $m = 1, 2, \ldots$ to generate T_m-gnomons. A variety of identities for generalized Fibonacci numbers can be observed and discovered by mimicking the procedures followed earlier.

It seems appropriate to conclude with a remark of Brother Alfred Brousseau: "It appears that there is a considerable wealth of enrichment and discovery material in the general area of Fibonacci numbers as related to geometry" [1]. Additional geometry of Fibonacci numbers can be found in Bro. Alfred's article.

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FIBONACCI AND LUCAS CUBES

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1. INTRODUCTION

The Fibonacci numbers are defined by the well-known recursion formulas

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

and the Lucas numbers by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}.$$

J. H. E. Cohn [2] determined the Fibonacci and Lucas numbers that are perfect squares. R. Finkelstein and H. London [3] gave a rather complicated determination of the cubes in the Fibonacci and Lucas sequences. Diophantine equations whose solutions must be Fibonacci and Lucas cubes occur in C. L. Siegel's proof [7] of H. M. Stark's result that there are exactly nine complex quadratic fields of class number one. This paper presents a simple determination of all Fibonacci numbers F_n of the form $2^a 3^b X^3$ and all Lucas numbers L_n of the form $2^a X^3$.

2. PRELIMINARY REDUCTIONS

From the recursion formulas defining the Fibonacci and Lucas numbers, it is easily verified by induction that the sequence of residues of F_n and L_n (mod p) are periodic, and in particular that

$$2 | F_n \text{ iff } 3 | n \tag{1}$$

$$2 | L_n \text{ iff } 3 | n \tag{2}$$

$$3 | F_n \text{ iff } 4 | n \tag{3}$$

$$3 \mid L_n \text{ iff } n \equiv 2 \pmod 4 \tag{4}$$

$$5 \nmid L_n \tag{5}$$

$$7 \mid L_n \text{ iff } n \equiv 4 \pmod 8 \tag{6}$$

If $\varepsilon_0 = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varepsilon}_0 = \dfrac{1-\sqrt{5}}{2}$, it is also easily verified by induction that:

$$\varepsilon_0 = \frac{L_n + F_n\sqrt{5}}{2},\quad F_n = \frac{1}{\sqrt{5}}(\varepsilon_0^n - \overline{\varepsilon}_0^n),\quad L_n = \varepsilon_0^n + \overline{\varepsilon}_0^n.$$

From these formulas, the following identities are easily derived:

$$5F_n^2 - L_n^2 = 4(-1)^{n+1} \tag{7}$$

$$F_{2n} = F_n L_n \tag{8}$$

$$4F_{3n} = F_n(5F_n^2 + 3L_n^2) \tag{9}$$

$$4L_{3n} = L_n(15F_n^2 + L_n^2) \tag{10}$$

Further, from (1), (2), and (7), we find that

$$(F_n, L_n) = \begin{cases} 2 & \text{if } 3 \mid n \\ 1 & \text{otherwise} \end{cases} \tag{11}$$

Finally, since $F_n = (-1)^n F_{-n}$ and $L_n = (-1)^n L_{-n}$, it suffices to consider the case $n > 0$ in what follows.

The identity (7) is the basis of a reduction of the determination of Fibonacci or Lucas cubes (or, more generally, Fibonacci and Lucas Pth powers) to solving particular Diophantine equations. It turns out that this identity actually characterizes Fibonacci and Lucas numbers, in the sense that (L_{2n}, F_{2n}) for $n > 0$ is the complete set of positive solutions to the Diophantine equation $X^2 - 5Y^2 = 4$, and (L_{2n+1}, F_{2n+1}) for $n \geq 0$ is the complete set of positive solutions to the Diophantine equation $X^2 - 5Y^2 = -4$. From these facts, it follows that the positive Fibonacci cubes are exactly those positive Y^3 for which $X^2 - 5Y^6 = \pm 4$ is solvable in integers, and the positive Lucas cubes are those positive X^3 for which $X^6 - 5Y^2 = \pm 4$ is solvable in integers. For our purposes, it suffices to know only that (7) holds, so that the Fibonacci and Lucas cubes are a *subset* of the solutions of these Diophantine equations.

We now show that the addition formulas (8)-(10) can be used to relate Fibonacci numbers of the form $2^a 3^b X^3$ to those of the form X^3, and Lucas numbers of the form $2^a X^3$ to those of the form X^3.

Lemma 1: (i) If F_{2n} is of the form $2^a 3^b X^3$, so is F_n.
(ii) If F_{3n} is of the form $2^a 3^b X^3$, so is F_n.
(iii) If L_{3n} is of the form $2^a X^3$, so is L_n.

Proof: (i) follows from (8) and (11). (ii) follows from (9) and (11), where we note that $(F_n, 3L_n^2) \mid 12$. Finally, (iii) follows from (10), (11), and (5), noting that $(L_n, 15F_n^2) \mid 12$.

Lemma 2: (i) If $F_n = 2^a 3^b X^3$ and $n = 2^c 3^d k$ with $(6, k) = 1$, then $F_k = Z^3$.
(ii) If $L_n = 2^a X^3$ and $n = 3^d k$ with $(3, k) = 1$, then $L_k = Z^3$.

Proof: For (i), note that F_k is of the form $2^a 3^b X^3$ by repeated application of Lemma 1, while $(F_k, 6) = 1$ by (1) and (3), so $F_k = Z^3$. (ii) has a similar proof using (2).

Remark: The preceding two lemmas are both valid in the more general case where "cube" is replaced by "Pth power" throughout, using the same proofs.

3. MAIN RESULTS

Theorem 1: The only F_n with $(n, 6) = 1$ that are cubes are $F_1 = 1$ and $F_{-1} = -1$.

Proof: Let $F_n = Z^3$ and note that $(n, 6) = 1$ and (1) and (7) yield

$$5Z^6 - 4 = L_n^2 \quad \text{and} \quad (2, Z) = 1. \tag{12}$$

Setting $X = 5Z^2$ and $Y = 5L_n$ yields

$$X^3 - 100 = Y^2 \tag{13}$$

and (2) and (4) require $(Y, 6) = 1$. We examine (13) over the ring of integers of $Q(\sqrt[3]{10})$. It has been shown (see [6] and [8]) that this ring has unique factorization, that its members are exactly those $(1/3)(A + B\sqrt[3]{10} + C\sqrt[3]{100})$ where A, B, and C are integers with $A \equiv B \equiv C \pmod 3$, and that the units in this ring are of the form $\pm\varepsilon^K$ where $\varepsilon = (1/3)(23 + 11\sqrt[3]{10} + 5\sqrt[3]{100})$. Equation (13) factors as

$$(X - \sqrt[3]{100})(X^2 + \sqrt[3]{100}X + 10\sqrt[3]{10}) = Y^2. \tag{14}$$

Write

$$X - \sqrt[3]{100} = \eta\alpha^2, \tag{15}$$

where η in square free and divides both $X - \sqrt[3]{100}$ and $X^2 + \sqrt[3]{100} + 10\sqrt[3]{10}$. Then

$$\eta \mid (X^2 + \sqrt[3]{100}X + 10\sqrt[3]{10}) - (X + 2\sqrt[3]{100})(X - \sqrt[3]{100}) = 30\sqrt[3]{10}.$$

Since $(Y, 3) = 1$, $(\eta, 3) = 1$, and $\eta \mid 10\sqrt{10}$. Now $(\sqrt{10})^3 = 2 \cdot 5$ and $(2, 5) = 1$, so by unique factorization we can find Δ and Φ such that $\sqrt[3]{10} = \Delta\Phi$, $5 = \Delta^3\varepsilon^K$, and $2 = \Phi^3\varepsilon^{-K}$. Then $\eta \mid 10\sqrt[3]{10} = \Delta^4\Phi^4$. Now $Y = 5L_n$ and $(2, L_n) = 1$ by (2), so (14) shows that $(\Phi, X - \sqrt[3]{100}) = 1$. Hence $\eta \mid \Delta^4$. But $5 \mid X$, so $\Delta^3 \mid X$, and hence $\Delta^2 \| X - \sqrt[3]{100}$. Since η is square free, η must be a unit. By absorbing squares of units into α, we need only consider $\eta = \pm 1$ and $\eta = \pm\varepsilon$ in (15).

Case 1: $X - \sqrt[3]{100} = \pm\alpha^2$. Let $\alpha = (1/3)(A + B\sqrt[3]{10} + C\sqrt[3]{100})$. Since representation of integers in this form is unique.

$$X = \pm\frac{1}{9}(A^2 + 20BC) \tag{16}$$

$$0 = \pm\frac{1}{9}(2AB + 10C^2) \tag{17}$$

$$-1 = \pm\frac{1}{9}(B^2 + 2AC) \tag{18}$$

Equation (17) shows $B \mid 5C^2$. Squaring (18) and multiplying both sides by $3^4 \cdot 5$, we see that B divides each term on the right side so $B \mid 3^4 \cdot 5$. For each of the twenty values of B satisfying $B \mid 3^4 \cdot 5$, we can solve (17) and (18) for A and C, and verify the only integer solutions (A, B, C) are $(-5, 1, 1)$ and $(5, -1, -1)$ when $\eta = 1$, and $(0, \pm 3, 0)$ when $\eta = -1$. Evaluating X by (16) we find that the first two solutions yield $Z = \pm 1$ in (12), and thus $F_1 = 1$ and $F_{-1} = -1$, while the third solution is extraneous to (13).

Case 2: $X - \sqrt[3]{100} = \pm\varepsilon\alpha^2$. Proceeding as in Case 1, we obtain

$$X = \pm\frac{1}{27}(23A^2 + 110B^2 + 500C^2 + 100AB + 220AC + 460BC) \tag{19}$$

$$0 = \pm\frac{1}{27}(11A^2 + 50B^2 + 230C^2 + 46AB + 100AC + 230BC) \tag{20}$$

$$-1 = \pm\frac{1}{27}(5A^2 + 23B^2 + 110C^2 + 22AB + 46AC + 100BC) \tag{21}$$

From (20) $2 \mid A$ so that $2 \mid X$ in (19), and such solutions are extraneous to (12).

Remark: It can be shown (see [1] and [3]) that the complete set of solutions (X, Y) to (13) is $(5, \pm 5)$, $(10, \pm 30)$, and $(34, \pm 198)$.

Theorem 2: The set of Fibonacci numbers F_n with $n > 0$ of the form $2^a 3^b X^3$ is $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_6 = 8$, and $F_{12} = 144$.

Proof: Let $F_n = 2^a 3^b X^3$ with $n = 2^c 3^d k$ and $(k, 6) = 1$. By Lemma 2, $F_k = Z^3$ and by Theorem 1, $k = 1$. If $c \geq 3$, repeated application of Lemma 1(ii) would show $F_8 = 21$ is of the form $2^a 3^b X^3$, which is false. If $d \geq 2$, repeated application of Lemma 1(i) would show $F_9 = 34$ is of the form $2^a 3^b X^3$, which is false. The values $0 \leq c \leq 2$ and $0 \leq d \leq 1$ give the stated solutions.

Theorem 3: The equation $L_{2n} = X^3$ has no solutions.

Proof: Suppose $L_{2n} = X^3$. Then (7) yields

$$5F_{2n}^2 + 4 = X^6.$$

All solutions to this equation (mod 7) require $7|X$. Then (6) shows $4|2n$ hence $3|F_{2n}$ by (2), so $X^6 \equiv 4 \pmod 9$, which is impossible.

Theorem 4: The equation $L_n = X^3$ with $(n, 6) = 1$ has only the solutions $L_1 = 1$ and $L_{-1} = -1$.

Proof: Suppose $L_n = X^3$ with $(n, 6) = 1$. Then (2) and (7) yield

$$5F_n^2 - 4 = X^6 \quad \text{and} \quad (6, X) = 1. \tag{22}$$

We examine (22) over the ring of integers of $Q(\sqrt{5})$. It is known that this ring has unique factorization, that these integers are of the general form

$$\frac{1}{2}(A + B\sqrt{5})$$

with $A \equiv B \pmod 2$, and that the units are of the form $\pm\varepsilon_0^K$, where

$$\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5}).$$

Now (22) gives

$$(\sqrt{5}F_n + 2)(\sqrt{5}F_n - 2) = Z^3,$$

where $Z = X^2$. Then

$$\sqrt{5}F_n + 2 = \eta\alpha^3,$$

where η divides both $\sqrt{5}F_n + 2$ and $\sqrt{5}F_n - 2$. Then we have $\eta|4$. But $(2, Z) = 1$, so $(2, \sqrt{5}F_n + 2) = 1$ and η is a unit. By absorbing cubes of units, we need to consider only $\eta = 1$, ε_0, and ε_0^{-1}.

Case 1: $2 + F_n\sqrt{5} = \alpha^3$. Let $\alpha = (1/2)(A + B\sqrt{5})$. Substituting this yields the equations

$$2 = \frac{1}{8}A(A^2 + 15B^2)$$
$$F_n = \frac{1}{8}B(3A^2 + 5B^2). \tag{23}$$

Then (23) shows that $A|16$ and $|B| \leq 1$, from which $A = 1$ and $B = \pm 1$ are the only solutions, yielding $F_n = \pm 1$ and, finally, $L_1 = 1$ and $L_{-1} = -1$.

Case 2: $2 + F_n\sqrt{5} = \varepsilon_0\alpha^3$. Let $\alpha = (1/2)(A + B\sqrt{5})$ with $A \equiv B \pmod 2$, which yields

$$2 = \frac{1}{16}(A^3 + 15A^2B + 15AB^2 + 25B^3)$$

and

$$F_n = \frac{1}{16}(A^3 + 3A^2B + 15AB^2 + 5B^3).$$

Then

$$4(2 - F_n) = B(3A^2 + 5B^2) \equiv 4 \pmod 8,$$

because $2 \nmid F_n$ since $(n, 6) = 1$. This congruence has no solutions with $A \equiv B \pmod 2$.

Case 3: $2 + F_n\sqrt{5} = \varepsilon_0^{-1}\alpha^3$. Noting $\varepsilon_0^{-1} = (1/2)(1 - \sqrt{5})$, we argue as in Case 2, using instead

$$4(2 + F_n) = -B(3A^2 + 5B^2) \equiv 4 \pmod 8,$$

which has no solutions with $A \equiv B \pmod 2$.

Theorem 5: The set of Lucas numbers L_n with $n > 0$ of the form 2^aX^3 are $L_1 = 1$ and $L_3 = 4$.

Proof: Let $L_n = 2^aX^3$ with $n = 3^ck$ and $(k, 3) = 1$. By Lemma 2, $L_k = X^3$ so by Theorems 3 and 4, $k = 1$. If $c \geq 2$, then Lemma 2(ii) would show $L_9 = 76$ was of the form 2^aX^3, which is false.

Remark: The set of Lucas numbers of the form $2^a3^bX^3$ leads to consideration of the equation $X^3 = Y^2 + 18$. The only solutions to this equation are $(3, \pm 3)$, but the available proofs (see [1] and [3]) are complicated. General methods for solving the equation $X^3 = Y^2 + K$ for fixed K are given in [1], [4], and [5].

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THE NUMBER OF STATES IN A CLASS OF SERIAL QUEUEING SYSTEMS

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ABSTRACT

It is shown that the number of states in a class of serial production or service systems with N servers is the $(2N - 1)$st Fibonacci number. This has proved useful in designing efficient systems.

This research was supported by National Research Council Grant No. A4142.

In studying queueing systems in series, it is useful to know precisely the number of different states that might occur. In particular, in [1], this number is crucial in determining approximate solutions to the allocation of a fixed resource to the individual servers or for scheduling servers with variable serving times. For a particular class of these problems, this number possesses an interesting property.

The system can be described in general as follows:

> N (single-server) service facilities (usually corresponding to N work stations of a production line) are arranged in series. Customers completing service at station i proceed to station $i+1$ and commence service there if it is free, or join a queue if the server is busy. The limitation on space restricts the number who can wait before station i to be W_i. If service is completed at station i and the waiting space before station $i+1$ is full, then the customer completing service cannot advance and station i becomes "blocked." Any station that is idle is said to be "starved." Station 1 cannot be starved, as a customer is always ready for processing (raw materials) and station N can never be blocked. Customers are not permitted to renege (see Figure 1).

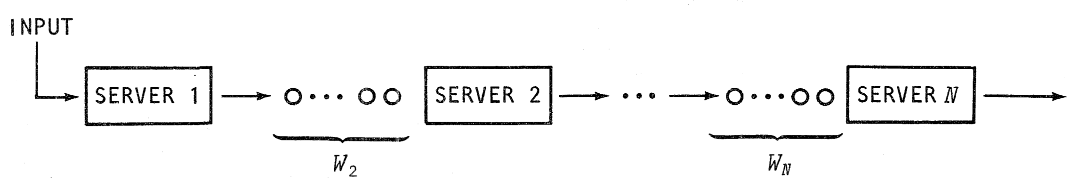

Fig. 1

The design problem is to consider how to divide the work among the N stations (or, equivalently, to determine the order of service) to maximize, among other objectives, the rate at which customers leave the system. The problem is complicated by having operation times that are not deterministic and are given only by a random variable. This optimization involves inverting a stochastic matrix whose dimension is the number of states in the system. Our problem here is to determine the number of possible states.

Without loss of generality, we can assume that $W_i = 0$, $i = 2, 3, \ldots, N$, that is, there is no waiting space before each server. This is done by assuming each waiting space is another service station with 0 service time. Hence, each station can be busy (state 1), all but station 1 can be starved (state 0), and all but station N can be blocked (state b). An N-tuple of 1's, 0's, and b's represents a state of the system. Obviously, not all combinations are allowed, for instance, a "b" must be followed by a "b" or a "1."

Theorem: Let S_N be the number of states when N servers are in series. Then $S_N = F_{2N-1}$.

Proof: When $N = 2$, the only possible states are $(1, 1), (1, 0), (b, 1)$ and $S_2 = F_3 = 3$. Assume that $S_k = F_{2k-1}$. All possible states, when $N = k+1$, can be generated from the S_k states as follows: catenate a "1" to the right of each of the S_k states [corresponding to the $(k+1)$st server being busy]; catenate a "0" to the right of each of the S_k states; and, for each state with a "1" in the kth position, change this to a "b" and add a "1" in the $(k+1)$st position. The states with the "1" in the kth position had been similarly generated from the S_{k-1} states. This leads to the recursive relationship

$$S_{k+1} = S_k + S_k + (S_{k-1} + \cdots + S_1 + 1) = 2F_{2k-1} + \sum_{j=1}^{k-1} F_{2j-1} + F_0 = F_{2k+1}.$$

This result has been most useful in developing numerical procedures for calculating or approximating the probabilities that a server is busy, which is used in finding efficient designs for this class of production systems.

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THE DETERMINATION OF ALL DECADIC KAPREKAR CONSTANTS

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0. INTRODUCTION

Choose a to be any r-digit integer expressed in base 10 with not all digits equal. Let a' be the integer formed by arranging these digits in descending order, and let a'' be the integer formed by arranging these digits in ascending order. Define $T(a) = a' - a''$. When $r = 3$, repeated applications of T to any starting value a will always lead to 495, which is self-producing under T, that is, $T(495) = 495$. Any r-digit integer exhibiting the properties that 495 exhibits in the 3-digit case will be called a "Kaprekar constant." It is well known (see [2]) that 6174 is such a Kaprekar constant in the 4-digit case.

In this paper we concern ourselves only with self-producing integers. After developing some general results which hold for any base g, we then characterize all decadic self-producing integers. From this it follows that the only r-digit Kaprekar constants are those given above for $r = 3$ and 4.

1. THE DIGITS OF $T(a)$

Let $r = 2n + \delta$, where

$$\delta = \begin{cases} 1 & r \text{ odd} \\ 0 & r \text{ even.} \end{cases}$$

Let a be an r-digit g-adic integer of the form

$$a = \alpha_{r-1}g^{r-1} + \alpha_{r-2}g^{r-2} + \cdots + \alpha_1 g + \alpha_0 \tag{1.1}$$

with

$$g > \alpha_{r-1} \geq \alpha_{r-2} \geq \cdots \geq \alpha_1 \geq \alpha_0, \ \alpha_{r-1} > \alpha_0.$$

Let a' be the corresponding reflected integer

$$a' = \alpha_0 g^{r-1} + \alpha_1 g^{r-2} + \cdots + \alpha_{r-2}g + \alpha_{r-1}. \tag{1.2}$$

The operation $T(a) = a - a'$ will give rise to a new r-digit integer (permitting leading zeros) whose digits can be arranged in descending and ascending order as in (1.1) and (1.2). Define

$$d_{n-i+1} = \alpha_{r-i} - \alpha_{i-1}, \ i = 1, 2, \ldots, n. \tag{1.3}$$

Thus associated with the integer a given in (1.1) is the n-tuple of differences $D = (d_n, d_{n-1}, \ldots, d_1)$ with $g > d_n \geq d_{n-1} \geq \cdots \geq d_1$. Note that $T(a)$ depends entirely upon the values of these differences. The digits of $T(a)$ are given by the following, viz.,

$$\delta = 0 \text{ and } d_1 \neq 0 \tag{1.4a}$$

$$d_n \quad d_{n-1} \ldots d_2 \quad d_1 - 1 \quad g - d_1 - 1 \quad g - d_2 - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 0 \text{ and } d_1 = d_2 = \cdots = d_{j-1} = 0,\ d_j \neq 0,\ 1 < j \leq n \tag{1.4b}$$

$$d_n \quad d_{n-1} \ldots d_{j+1} \quad d_j - 1 \quad \overbrace{g - 1 \ldots g - 1}^{2(j-1) \text{ terms}} \quad g - d_j - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 1 \text{ and } d_1 \neq 0 \tag{1.4c}$$

$$d_n \quad d_{n-1} \ldots d_2 \quad d_1 - 1 \quad g - 1 \quad g - d_1 - 1 \quad g - d_2 - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 1 \text{ and } d_1 = d_2 = \cdots = d_{j-1} = 0,\ d_j \neq 0,\ 1 < j \leq n \tag{1.4d}$$

$$d_n \quad d_{n-1} \ldots d_{j+1} \quad d_j - 1 \quad \overbrace{g - 1 \ldots g - 1}^{2(j-1)+1 \text{ terms}} \quad g - d_j - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

Differences $D' = (d'_n, d'_{n-1}, \ldots, d'_1)$ can now be assigned to the integers $T(a)$ as in (1.3). We say that $(d_n, d_{n-1}, \ldots, d_1)$ is mapped to $(d'_n, d'_{n-1}, \ldots, d'_1)$ under T.

2. PROPERTIES OF ONE-CYCLES

We shall focus attention on the determination of all a such that $T(a) = a$. Such integers are said to generate a one-cycle a. This is equivalent to finding all n-tuples $(d_n, d_{n-1}, \ldots, d_1)$ that are mapped to themselves under T.

Theorem 2.1: Suppose $(d_n, d_{n-1}, \ldots, d_1)$ represents a one-cycle with $d_j \neq 0$, $j \geq 1$, and $d_k = 0$ for $k < j$. Further suppose that $d_n \neq d_j$. Then

(i) $d_n + d_j = g$ if $\delta = 1$ or if $\delta = 0$ and $j > 1$,

or

(ii) $\begin{cases} d_n + 2d_1 = g \\ \quad\text{or} \\ d_n = g - 1,\ d_1 = 1 \end{cases}$ if $\delta = 0$ and $j = 1$

Proof: (i) Since either $j > 1$ or $\delta = 1$, (1.4a) does not apply. Thus the largest digit in $T(a)$ is $g - 1$. The smallest digit could be one of three:

$$\begin{cases} d_j - 1 & \text{if } d_j + d_n - 1 < g \\ g - d_n & \text{if } d_j + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ g - d_n - 1 & \text{if } d_j + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

Therefore,

$$d'_n = \begin{cases} g - d_j & \text{if } d_j + d_n - 1 < g \\ d_n - 1 & \text{if } d_j + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ d_n & \text{if } d_j + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

Since $d_n = d'_n$, if $d_j + d_n - 1 < g$, then $d_n + d_j = g$. If $d_j + d_n - 1 \geq g$, then since $d'_n = d_n \neq d_n - 1$, it must be that $d_n = d_{n-1}$. This condition restricts the second largest digit to be either d_n or $g - 1$, and the second smallest to be $g - d_n$ if $d_n \neq d_{n-2}$ or $g - d_n - 1$ if $d_n = d_{n-2}$. Since $d'_{n-1} = d_{n-1} = d_n \neq g$, we must have $d_n = d_{n-2}$. Continuing in this fashion, one finds that $d_n = d_j$, which contradicts the hypothesis. Thus $d_n + d_j = g$.

(ii) Suppose first that $d_n > g - d_1 - 1$, then d_n is the largest digit in (1.4a). Then

$$d_n' = \begin{cases} d_n - d_1 + 1 & \text{if } d_1 + d_n - 1 < g \\ 2d_n - g & \text{if } d_1 + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ 2d_n - g + 1 & \text{if } d_1 + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

If $d_1 + d_n - 1 < g$ and $g < d_1 + d_n + 1$, then $g = d_1 + d_n$. Since $d_n' = d_n$, one must have $d_1 = 1$ and $d_n = g - 1$. If $d_1 + d_n - 1 \geq g$, then $d_n = d_{n-1}$ as shown in (i). Hence $d_n = d_{n-1} = \cdots = d_1 = g - 1$. This cannot occur in a one-cycle unless $g = 2$, in which case $d_n = g - 1 = 1 = d_1$. Thus, if $d_n > g - d_1 - 1$, $d_n = g - 1$ and $d_1 = 1$.

Now suppose that $d_n \leq g - d_1 - 1$. Then the largest digit in (1.4a) is $g - d_1 - 1$ and the smallest is $d_1 - 1$. Hence

$$d_n = d_n' = (g - d_1 - 1) - (d_1 - 1) = g - 2d_1$$

and

$$d_n + 2d_1 = g.$$

Theorem 2.2: If $D = (d_n, d_{n-1}, \ldots, d_1)$ represents a one-cycle with $d_n = \cdots = d_j \neq 0$, $j \geq 1$, and $d_k = 0$ for $k < j$, then $d_n = \cdots = d_j = g/2$. Further,

(i) if $g \neq 2$, then $r \equiv 0 \pmod 3$ and $g \equiv 0 \pmod 2$. In particular

$$D = \begin{cases} \Big(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{r/3 \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{r/6 \text{ terms}}\Big) & \text{when } r \equiv 0 \pmod 2 \\ \Big(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{r/3 \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{(r-3)/6 \text{ terms}}\Big) & \text{when } r \equiv 1 \pmod 2 \end{cases}$$

(ii) if $g = 2$, then every n-tuple D is a one-cycle.

Proof: (i) If $g > 2$, then $j > 1$ from (1.4). From (1.4b) and (1.4d), any n-tuple $(k, k, \ldots, k, 0, 0, \ldots, 0)$ will give rise to a successor with digits

$$\overbrace{k\ k \ldots k}^{(n-j) \text{ terms}}\ \ k-1\ \ \overbrace{g-1 \ldots g-1}^{2(j-1)+\delta \text{ terms}}\ \ \overbrace{g-k-1 \ldots g-k-1}^{(n-j) \text{ terms}}\ \ g-k.$$

Clearly the largest digit is $g - 1$. The smallest is either $k - 1$, forcing $k = g/2$, or $g - k - 1$, forcing $k - (g - k) = k$, which is impossible. Hence

$$d_n = d_{n-1} = \cdots = d_j = \frac{g}{2}.$$

Consider

$$D = \Big(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{a \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{(n-a) \text{ terms}}\Big), \quad a = n - j + 1.$$

The digits of the successor of D are

$$\overbrace{\tfrac{g}{2}\ \tfrac{g}{2} \ldots \tfrac{g}{2}}^{(a-1) \text{ terms}}\ \ \tfrac{g}{2} - 1\ \ \overbrace{g-1 \ldots g-1}^{2(n-a)+\delta \text{ terms}}\ \ \overbrace{\tfrac{g}{2}-1 \ldots \tfrac{g}{2}-1}^{(a-1) \text{ terms}}\ \ \tfrac{g}{2}. \tag{2.1}$$

Ordering the digits of (2.1) in descending order, one obtains

$$\overbrace{g-1 \ldots g-1}^{2(n-a)+\delta \text{ terms}}\ \ \overbrace{\tfrac{g}{2} \ldots \tfrac{g}{2}}^{a \text{ terms}}\ \ \overbrace{\tfrac{g}{2}-1 \ldots \tfrac{g}{2}-1}^{a \text{ terms}}. \tag{2.2}$$

Differences equal to $g/2$ will be generated by the pairs $(g - 1, g/2 - 1)$, and differences will be generated by the pairs $(g/2, g/2)$. Hence, if D is a one-cycle, then $2(n - a) + \delta = a$, that is, $r = 2n + \delta = 3a$. In addition,

$$n - a = \begin{cases} \dfrac{r}{6} & \text{if } r \equiv 0 \pmod 2 \\[2ex] \dfrac{r - 3}{6} & \text{if } r \equiv 1 \pmod 2). \end{cases}$$

(ii) If $g = 2$, then the digits of the successor of D ordered in descending order, from (2.2), are

$$\overbrace{1\ \ 1\ \ldots\ 1}^{2(n-a)+\delta \text{ terms}} \quad \overbrace{1\ \ldots\ 1}^{a \text{ terms}} \quad \overbrace{0\ \ldots\ 0}^{a \text{ terms}}. \tag{2.3}$$

Clearly the first a succeeding differences in (2.3) are equal to 1 and the remaining $(n - a)$ differences are equal to 0. Therefore, a is a one-cycle for all $1 \leq a \leq n$.

Definition 2.1: For $i = 0, 1, \ldots, g - 1$, let ℓ_i be the number of entries in $(d_n, d_{n-1}, \ldots, d_1)$ that equal i, and let c_i be the number of digits of $T(a)$ that equal i.

For example, if $g = 10$, $\delta = 0$, and $D = (9, 9, 7, 7, 3, 1, 0, 0)$, then

$\ell_9 = 2$, $\ell_8 = 0$, $\ell_7 = 2$, $\ell_6 = \ell_5 = \ell_4 = 0$, $\ell_3 = 1$, $\ell_2 = 0$, $\ell_1 = 1$, and $\ell_0 = 2$

From (1.4), the digits of D' are

$$9\ \ 9\ \ 7\ \ 7\ \ 3\ \ 0\ \ 9\ \ 9\ \ 9\ \ 9\ \ 8\ \ 6\ \ 2\ \ 2\ \ 0\ \ 1$$

giving rise to the digit counters

$c_9 = 6$, $c_8 = 1$, $c_7 = 2$, $c_6 = 1$, $c_5 = c_4 = 0$, $c_3 = 1$, $c_2 = 2$, $c_1 = 1$, and $c_0 = 2$

Using the results of Section 1, we now obtain the following corollary.

Corollary 2.1: If $d_n + d_j = g$, where d_j is the smallest nonzero entry in

$$D = (d_n, d_{n-1}, \ldots, d_1),$$

then

$$\begin{aligned} c_{g-1} &= \ell_{g-1} + 2\ell_0 + \delta \\ c_i &= \ell_i + \ell_{g-i-1} \qquad i = 1, 2, \ldots, g - 2 \\ c_0 &= \ell_{g-1} \end{aligned}$$

Proof: This result follows directly from (1.4).

3. *THE DETERMINATION OF ALL DECADIC ONE-CYCLES*

If one fixes $g = 10$, then each one-cycle $D = (d_n, d_{n-1}, \ldots, d_2, d_1)$ falls into one of four classes. These classes can be described using the difference counters ℓ_i, $i = 0, 1, 2, \ldots, g - 1$ introduced in Definition 2.1. The following conditions on the difference counters must hold for $D = (d_n, d_{n-1}, \ldots, d_1)$ to be in a given class.

Class A: $\ell_8 = \ell_6 = \ell_4 = \ell_2 = 2\ell_0 + \delta$
$\ell_7 = \ell_5 = \ell_1$
$\ell_9 = 0$ iff $\ell_1 = 0$
ℓ_0, ℓ_1, or δ is nonzero

Class B: $\ell_9 = \ell_1 = 0$ $\qquad$ $\ell_4 = \ell_2 = \ell_0 + \dfrac{\ell_8}{2}$
$\ell_7 = 2\ell_0 - \ell_8$ $\qquad$ $\delta = 0$
$\ell_6 = \ell_8 \neq 0$ $\qquad$ $\ell_8 \equiv 0 \pmod 2$

and $$\ell_5 = \ell_3 = \ell_0 - \frac{\ell_8}{2} \qquad \ell_0 \geq \frac{\ell_8}{2}$$

<u>*Class C*</u>: $\ell_6 = \ell_2 = 1$
$\ell_i = 0, \; i \neq 2, 3, 6$
$\delta = 0$

<u>*Class D*</u>: $\ell_5 = 2\ell_0 + \delta$
$\ell_i = 0, \; i \neq 0, 5$
ℓ_0 or δ is nonzero

<u>*Theorem 3.1*</u>: Let $(d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n + d_j = 10$ and $\ell_0 = j - 1$. Suppose that $d_j \neq 5$ and either $j \neq 1$ or $\delta \neq 0$. Then $(d_n, d_{n-1}, \ldots, d_1)$ is in either Class A or Class B.

<u>*Proof*</u>: We wish to determine the difference counters ℓ_i, $i = 0, 1, 2, \ldots, g - 1$. To do this, we shall explore the various ways these differences can be computed from the digits in a self-producing integer. From Corollary 2.1,

$$c_9 = \ell_9 + 2\ell_0 + \delta$$
$$c_i = \ell_i + \ell_{9-i} \qquad i = 1, 2, \ldots, 8$$
$$c_0 = \ell_9$$

Certainly, $\ell_9 = \min(c_9, c_0) = c_0$, since a difference of 9 can only be obtained from the digits 9 and 0. Hence

$$\ell_8 = \min(2\ell_0 + \delta, c_1) = \min(2\ell_0 + \delta, \ell_1 + \ell_8) = \begin{cases} 2\ell_0 + \delta & \ell_1 \neq 0 \\ \ell_8 & \ell_1 = 0 \end{cases} \tag{3.1}$$

Thus the value of ℓ_8 depends on whether ℓ_1 is zero or nonzero. If $\ell_1 \neq 0$, then there are fewer 9's than 1's remaining and hence there will be as many differences of 8 as there are 9's remaining. If $\ell_1 = 0$, then there are fewer 1's in the self-producing integer than remaining 9's, and there will be as many differences of 8 as there are 1's. This technique of evaluating the difference counters is used throughout this section.

Suppose first that $\ell_1 \neq 0$. Note that if $\ell_1 \neq 0$, $d_j = 1$, and hence $d_n = 9$. Then we have

$$\begin{aligned} \ell_9 &= \ell_9 \neq 0 \\ \ell_8 &= 2\ell_0 + \delta \\ \ell_7 &= \ell_1 \end{aligned} \tag{3.2}$$

and $$\ell_6 = \min(2\ell_0 + \delta, \ell_2 + \ell_7).$$

Now if $\ell_2 + \ell_7 < 2\ell_0 + \delta$, then one finds either

$$\begin{aligned} \ell_6 &= \ell_2 + \ell_7 \\ \ell_5 &= \ell_8 - (\ell_2 + \ell_7) \\ \ell_4 &= \ell_7 + \ell_2 \\ \ell_3 &= \ell_3 + \ell_6 - \ell_8 \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \ell_6 &= \ell_2 + \ell_7 \\ \ell_5 &= \ell_3 + \ell_6 \\ \ell_4 &= \ell_8 - (\ell_2 + \ell_7 + \ell_3 + \ell_6) \\ \ell_3 &= \ell_7 + \ell_2 \\ \ell_2 &= \ell_3 \end{aligned} \tag{3.4}$$

and $$\ell_1 = \min(\ell_2 + \ell_7, \ell_8 - \ell_2 - \ell_7)$$

Equations (3.3) imply that $\ell_6 = \ell_8$ or $\ell_2 + \ell_7 = 2\ell_0 + \delta$, which is a contradiction. Equations (3.4) imply that $\ell_1 = 0$, again a contradiction. Thus we must have $2\ell_0 + \delta \leq \ell_2 + \ell_7$. Continuing in like fashion,

$$\begin{aligned}
\ell_6 &= 2\ell_0 + \delta \\
\ell_5 &= \ell_2 + \ell_7 - (2\ell_0 + \delta) \\
\ell_4 &= 2\ell_0 + \delta \\
\ell_3 &= \ell_3 \\
\ell_2 &= 2\ell_0 + \delta \\
\ell_1 &= \ell_5 \\
\ell_0 &= \frac{\ell_4 - \delta}{2}
\end{aligned} \tag{3.5}$$

Equations (3.5) together with equations (3.2) determine the relations given in Class A with ℓ_1 and ℓ_9 nonzero.

Suppose now that $\ell_1 = 0$. From (3.1),

$$\begin{aligned}
\ell_8 &= \ell_8 \\
\ell_7 &= \min(2\ell_0 + \delta - \ell_8,\ \ell_2 + \ell_7),\text{ or} \\
\ell_7 &= \begin{cases} 2\ell_0 + \delta - \ell_8 & \ell_2 \neq 0 \\ \ell_7 & \ell_2 = 0 \end{cases}
\end{aligned} \tag{3.6}$$

We first consider the case where $\ell_2 \neq 0$. From (3.1) and (3.6) it is clear that

$$\begin{aligned}
\ell_9 &= 0 \\
\ell_8 &= \ell_8 \\
\ell_7 &= 2\ell_0 + \delta - \ell_8 \\
\ell_6 &= \min(\ell_8,\ \ell_2)
\end{aligned} \tag{3.7}$$

If $\ell_2 < \ell_8$, then

$$\begin{aligned}
\ell_6 &= \ell_2 \\
\ell_5 &= \ell_8 - \ell_2 \\
\ell_4 &= \ell_2 + \ell_7 \\
\ell_3 &= \ell_3 + \ell_6 - \ell_8 - \ell_7 \\
\ell_2 &= \ell_3 = 2\ell_0 + \delta
\end{aligned} \tag{3.8}$$

or

$$\begin{aligned}
\ell_6 &= \ell_2 \\
\ell_5 &= \ell_8 - \ell_2 \\
\ell_4 &= \ell_2 + \ell_3 + \ell_6 - \ell_8 \\
\ell_3 &= 2\ell_0 + \delta - \ell_3 - \ell_6 \\
\ell_2 &= \ell_6 + \ell_3 = \ell_2 + \ell_3
\end{aligned} \tag{3.9}$$

or

$$\begin{aligned}
\ell_6 &= \ell_2 \\
\ell_5 &= \ell_3 + \ell_6 \\
\ell_4 &= \ell_8 - \ell_2 - \ell_3 - \ell_6 \\
\ell_3 &= \ell_7 + \ell_2 \\
\ell_2 &= \ell_3 + \ell_6
\end{aligned} \tag{3.10}$$

In (3.8), $\ell_5 = -\ell_7 = 0$, so $\ell_2 = \ell_8$. In (3.9), $\ell_3 = 0$, which implies $\ell_2 = \ell_8$. In (3.10), $\ell_2 = 0$, so all three circumstances lead to a contradiction. Hence, it must be that $\ell_8 \leq \ell_2$, and, therefore, in (3.7) one finds $\ell_6 = \ell_8$. In this case, there are two possible values for ℓ_4, viz., $\ell_4 = \ell_6 + \min(\ell_7, \ell_3)$.

If $\ell_4 = \ell_6 + \ell_7$, then

$$\begin{aligned}
\ell_6 &= \ell_8 & \qquad \ell_2 &= \ell_6 = \ell_6 + \ell_5 \\
\ell_5 &= \ell_2 - \ell_8 & \ell_1 &= \ell_5 \\
\ell_4 &= \ell_6 + \ell_7 & \ell_0 &= \frac{\ell_6 - \delta}{2} \\
\ell_3 &= \ell_3 - \ell_7 & &
\end{aligned} \tag{3.11}$$

This implies that $\ell_5 = \ell_7 = 0$, $\ell_8 = \ell_6 = \ell_4 = \ell_2 = 2\ell_0 + \delta$, so (3.11) falls into Class A with $\ell_1 = 0$. Otherwise,

$$\begin{aligned} \ell_6 &= \ell_8 & \ell_2 &= \ell_3 + \ell_6 \\ \ell_5 &= \ell_2 - \ell_8 & \ell_1 &= 0 \\ \ell_4 &= \ell_6 + \ell_3 & & \\ \ell_3 &= \ell_7 - \ell_3 & \ell_0 &= \frac{\ell_4 + \ell_5 - \delta}{2} \end{aligned} \tag{3.12}$$

Equations (3.12) fall into Class B.

It can easily be checked that there exist no one-cycles with $d_n = 7$ and $d_j = 3$ or $d_n = 6$ and $d_j = 4$. This completes the proof of the theorem.

Theorem 3.2: Let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n = 9$, $d_1 = 1$ and $\delta = 0$. Then

$$\ell_7 = \ell_5 = \ell_1 \neq 0$$
$$\ell_8 = \ell_6 = \ell_4 = \ell_2 = \ell_0 = 0,$$

and this one-cycle falls into Class A.

Proof: This results immediately from Corollary 2.1, since $\ell_0 = \delta = 0$.

Theorem 3.3: Let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n + 2d_1 = 10$ and $\delta = 0$. Then

$$\ell_6 = \ell_2 = 1$$
$$\ell_i = 0,\ i \neq 2,\ 3,\ 6;$$

hence, this one-cycle will fall into Class C.

Proof: If $d_1 = 1$, one obtains the following system of inconsistent equations:

$$\begin{aligned} \ell_8 &= 1 \\ \ell_7 &= \ell_1 - 1 \\ \ell_6 &= 1 \\ \ell_5 &= \ell_7 + \ell_2 \\ \ell_4 &= 0 \\ \ell_3 &= \ell_3 + \ell_6 = \ell_3 + 1 \end{aligned}$$

If $d_1 = 2$, then

$$\begin{aligned} \ell_6 &= 1 \\ \ell_5 &= \ell_2 - 1 \\ \ell_4 &= 0 \\ \ell_3 &= \ell_3 \\ \ell_2 &= 1 \end{aligned}$$

which falls into Class C. It can easily be checked that $d_1 = 3$ implies that $\ell_3 = \ell_3 - 1$, so the proof is complete.

Since Class D consists of all the remaining one-cycles, namely, those with $d_j = 5$ from Theorem 3.1, this completes the classification of all dedadic one-cycles.

4. THE DETERMINATION OF KAPREKAR CONSTANTS

An r-digit Kaprekar constant is an r-digit, self-producing integer such that repeated iterations of T applied to *any* starting value a will always lead to this integer. Utilizing the results of Section 3, one can now show that only for $r = 3$ and $r = 4$ does such an integer exist.

Lemma 4.1: For $r = 2n$ with $n \geq 3$, there exist at least two distinct one-cycles.

Proof: If $r = 6$, then one finds the one cycles

$$D_1 = (6,\ 3,\ 2) \quad \text{and} \quad D_2 = (5,\ 5,\ 0).$$

If $r = 2n$, $n \geq 4$, then two distinct one-cycles are

D_1: $\ell_6 = \ell_2 = 1;\ \ell_3 = n - 2;\ \ell_i = 0,\ i \neq 2, 3, 6$

D_2: $\ell_7 = \ell_5 = \ell_1 = 1;\ \ell_9 = n - 3;\ \ell_i = 0,\ i \neq 1, 5, 7, 9.$

Lemma 4.2: For $r = 2n + 1$ with $n \geq 7$, there exist at least two distinct one-cycles.

Proof: If $n = 7$, then one finds the one-cycles:

$$D_1 = (8, 6, 4, 3, 3, 3, 2) \text{ and } D_2 = (5, 5, 5, 5, 5, 0, 0).$$

If $r = 2n + 1$, $n \geq 8$, then two distinct one-cycles are:

$$D_1: \ell_8 = \ell_7 = \ell_6 = \ell_5 = \ell_4 = \ell_2 = \ell_1 = 1;\ \ell_9 = n - 7;\ \ell_3 = \ell_0 = 0$$

$$D_2: \ell_8 = \ell_6 = \ell_4 = \ell_2 = 1;\ \ell_3 = n - 4;\ \ell_9 = \ell_7 = \ell_5 = \ell_1 = \ell_0 = 0.$$

Lemma 4.3: If $r = 2, 5, 7, 9, 11$, or 13, then there does not exist a Kaprekar constant.

Proof: When $r = 2, 5$, and 7 there are no one-cycles. When $r = 9$ there are two distinct one-cycles:

$$D_1 = (5, 5, 5, 0) \text{ and } D_2 = (8, 6, 4, 2).$$

If $r = 11$ the only one-cycle is $D_1 = (8, 6, 4, 3, 2)$, but there is also a cycle of length four, viz.,

$$(8, 8, 4, 3, 2) \to (8, 6, 5, 4, 2) \to (8, 6, 4, 2, 1) \to (9, 6, 6, 4, 2).$$

If $r = 13$ the only one-cycle is $D_1 = (8, 6, 4, 3, 3, 2)$, but there is also a cycle of length two, viz.,

$$(8, 7, 3, 3, 2, 1) \to (9, 6, 6, 5, 4, 3).$$

Theorem 4.1: The only decadic Kaprekar constants are 495 and 6174.

Proof: This follows from Lemmas 4.1-4.3.

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THE DECIMAL EXPANSION OF 1/89 AND RELATED RESULTS

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One of the more bizarre and unexpected results concerning the Fibonacci sequence is the fact that

$$\begin{aligned}\frac{1}{89} = .0112358&\\ 13&\\ 21&\\ 34&\\ 55&\\ 89&\\ 144&\\ 233&\\ \ldots&\\ = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^i},&\end{aligned} \tag{1}$$

where F_i denotes the ith Fibonacci number. The result follows immediately from Binet's formula, as do the equations

$$\frac{19}{89} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{10^i} \tag{2}$$

$$\frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i} \tag{3}$$

and

$$-\frac{21}{109} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{(-10)^i}. \tag{4}$$

where L_i denotes the ith Lucas numbers. It is interesting that all these results can be obtained from the following unusual identity, which is easily proved by mathematical induction.

Theorem 1: Let a, b, c, d, and B be integers. Let $\{\mu_n\}$ be the sequence defined by the recurrence $\mu_0 = c$, $\mu_1 = d$, $\mu_{n+2} = a\mu_{n+1} + b\mu_n$ for all $n \geq 2$. Let m and N be integers defined by the equations

$$B^2 = m + Ba + b \quad \text{and} \quad N = cm + dB + bc.$$

Then

$$B^nN = m\sum_{i=1}^{n+1} B^{n+1-i}\mu_{i-1} + B\mu_{n+1} + b\mu_n \tag{5}$$

for all $n \geq 0$. Also, $N \equiv 0 \pmod{B}$.

Proof: The result is clearly true for $n = 0$, since it then reduces to the equation

$$N = cm + dB + bc$$

of the hypotheses. Assume that

$$B^kN = m\sum_{i=1}^{k+1} B^{k+1-i}\mu_{i-1} + B\mu_{k+1} + b\mu_k.$$

Then

$$B^{k+1}N = m\sum_{i=1}^{k+1} B^{k+2-i}\mu_{i-1} + B^2\mu_{k+1} + Bb\mu_k$$

$$= m\sum_{i=1}^{k+1} B^{k+2-i}\mu_{i-1} + (m + Ba + b)\mu_{k+1} + Bb\mu_k$$

$$= m\sum_{i=1}^{k+2} B^{k+2-i}\mu_{i-1} + B(a\mu_{k+1} + b\mu_k) + b\mu_{k+1}$$

$$= m\sum_{i=1}^{k+2} B^{k+2-i}\mu_{i-1} + B\mu_{k+2} + b\mu_{k+1}.$$

This completes the induction. Finally, to see that $N \equiv 0 \pmod{B}$, we have only to note that

$$N = cm + dB + bc = c(B^2 - Ba - b) + dB + bc = cB^2 - caB + dB \equiv 0 \pmod{B}.$$

Now, it is well known that the terms of the sequence defined in Theorem 1 are given by

$$\mu_n = \left(\frac{c}{2} + \frac{2d - c}{\sqrt{a^2 + 4b}}\right)\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{c}{2} - \frac{2d - c}{\sqrt{a^2 + 4b}}\right)\left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n. \tag{6}$$

Thus it follows from (5) that

$$\frac{N}{Bm} = \sum_{i=1}^{n+1} \frac{\mu_{i-1}}{B^i} + \frac{B\mu_{n+1} + b\mu_n}{mB^{n+1}} = \sum_{i=1}^{\infty} \frac{\mu_{i-1}}{B^i}, \tag{7}$$

provided that the remainder term tends to 0 as n tends to infinity, and a sufficient condition for this is that

$$\left|\frac{a + \sqrt{a^2 + 4b}}{2B}\right| < 1 \quad \text{and} \quad \left|\frac{a - \sqrt{a^2 + 4b}}{2B}\right| < 1.$$

Thus we have proved the following theorem.

Theorem 2: If a, b, c, d, m, N, and B are integers, with m and N as defined above and if

$$\left|\frac{a + \sqrt{a^2 + 4b}}{2B}\right| < 1 \quad \text{and} \quad \left|\frac{a - \sqrt{a^2 + 4b}}{2B}\right| < 1,$$

then

$$\frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{\mu_{i-1}}{B^i}. \tag{8}$$

Of course, equations (1)-(4) all follow from (8) by particular choices of a, b, c, and d. To obtain (2), for example, we set $c = 2$, $a = b = d = 1$, and $B = 10$. It then follows that

$$m = B^2 - Ba - b = 100 - 10 - 1 = 89$$

$$N = cm + dB + bc = 178 + 10 + 2 = 190$$

and

$$\frac{19}{89} = \frac{190}{10 \cdot 89} = \frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{10^i} \quad \text{as claimed.}$$

To obtain (3), we set $c = 0$, $a = b = d = 1$, and $B = -10$. Then

$$m = B^2 - Ba - b = 100 + 10 - 1 = 109,$$

$$N = cm + dB + bc = -10,$$

and

$$\frac{N}{Bm} = \frac{-10}{-10 \cdot 109} = \frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i} \quad \text{as indicated.}$$

Finally, we note that interesting results can be obtained by setting B equal to a power of 10. For example, if $B = 10^h$ for some integer h, $c = 0$, and $a = b = d = 1$,

$$m = 10^{2h} - 10^h - 1, \; N = 10^h,$$

and (8) reduces to

$$\frac{1}{10^{2h} - 10^h - 1} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{hi}}. \tag{9}$$

For successive values of h this gives

$$\frac{1}{89} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^i} \tag{10}$$

as we already know,

$$\begin{aligned}\frac{1}{9899} &= \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{2i}} \\ &= .000101020305081321\ldots,\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{1}{998999} &= \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{3i}} \\ &= .000001001002003005008013\ldots,\end{aligned} \tag{12}$$

and so on. In case $B = (-10)^h$ for successive values of h, $c = 0$, and $a = b = d = 1$, we obtain

$$\frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i}, \tag{13}$$

$$\frac{1}{10099} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-100)^i}, \tag{14}$$

$$\frac{1}{1000999} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-1000)^i}, \tag{15}$$

and so on. Other fractions corresponding to (2) and (3) above are

$$\frac{19}{89}, \frac{199}{9899}, \frac{1999}{998999}, \ldots$$

and

$$-\frac{21}{109}, -\frac{201}{10099}, -\frac{2001}{1000999}, \ldots .$$

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A ROOT PROPERTY OF A PSI-TYPE EQUATION

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1. INTRODUCTION

By counting the number of roots between the asymptotes of the graph of

(1) $$y = f(x) = 1/x + 1/(x + 1) + \cdots + 1/(x + k - 1) - 1/(x + k) - \cdots - 1/(x + 2k)$$

we find that $f(x)$ possesses zeros which are all negative except for one, say r, and this positive r has the interesting property that

$$[r] = k^2,$$

where the brackets denote the greatest integer function.

2. THE POSITIVE ROOT

The existence of r is obtained by direct calculation.

Theorem 1: $f(x) = 0$ possesses a positive root r, and $[r] = k^2$.

Proof:

(2) $$f(x) = \sum_{j=0}^{k-1} \frac{1}{x+j} - \sum_{j=0}^{k} \frac{1}{x+k+j} = \sum_{j=0}^{k-1} \frac{1}{(x+j)(x+k+j)} - \frac{1}{x+2k}.$$

Similarly, we remove the first term from the second summation and combine the series parts to get

(3) $$f(x) = \sum_{j=0}^{k-1} \frac{k+1}{(x+j)(x+k+1+j)} - \frac{1}{x+k}.$$

Now, if we multiply equation (2) by $x + 2k$, and equation (3) by $-(x + k)$ and add the two resulting equations, we get, after replacing x by $k^2 + h$, the result

(4) $$kf(k^2 + h) = \sum_{j=0}^{k-1} \frac{1}{k^2+h+j} \cdot \frac{(1-h)k^2 - (h+j)k - h(h+j)}{(k^2+k+h+j)(k^2+k+h+1+j)}.$$

We now see at once that $f(k^2) > 0$ and $f(k^2 + 1) < 0$, since k is positive, and Theorem 1 is proved.

3. THE NUMBER OF ROOTS

The function $f(x)$ given in (1) is defined for $k = 1, 2, 3, \ldots$.

Theorem 2: $f(x) = 0$ possesses exactly $2k - 1$ negative roots and exactly one positive root.

Proof: As $x \to 0^-$, $f(x) \to -\infty$, and as $x \to -1^+$, $f(x) \to +\infty$; therefore, $f(x) = 0$ for some x in $-1 < x < 0$. Similarly for the other asymptotes, and we get

(5) $$-j + 1 < x < -j + 2, \; j = 2, 3, 4, \ldots, k,$$

implies the existence of a root in each such interval.

The branch of the curve between $-k$ and $-k + 1$ is skipped for the moment. Continuing, we find as above that (5) implies roots for

$$j = k + 2, k + 3, \ldots, 2k + 1.$$

Thus, $f(x)$ possesses at least $2k - 1$ negative roots.

Now we combine the fractions in the expression for $f(x)$ to get

$$f(x) = P(x)/[x(x + 1) \ldots (x + 2k)] \tag{6}$$

and observe that these negative roots are also zeros of $P(x)$, since the factors in the denominator of (6) cannot be zero at these values of x. But the degree of $P(x)$ is $2k$. Therefore, $P(x)$ possesses one more zero, and this is then the r obtained in Section 2. Q.E.D.

Remark: The branch of the curve, skipped in the above argument, then does not cut the x-axis at all.

4. THE PSI FUNCTION

The psi function, denoted by $\Psi(x)$, is defined by some authors [2, p. 241] by means of

$$\Delta^{-1}\left(\frac{1}{x}\right) = \Psi(x) + C, \tag{7}$$

where C is an arbitrary periodic function. This is the analog for defining $\ln(x)$ in the elementary calculus by means of

$$\int \frac{1}{x}\,dx = \ln(x) + c.$$

We employ (7) to obtain

$$f(x) = 2\Psi(x + k) - \Psi(x) - \Psi(x + 2k + 1).$$

This provides us with an iteration method for the calculation of r, starting with $r_1 = k^2$.

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RECOGNITION ALGORITHMS FOR FIBONACCI NUMBERS

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A FORTRAN, BASIC, or ALGOL program to generate Fibonacci numbers is not unfamiliar to many mathematicians. A Turing machine or a Markov algorithm to recognize Fibonacci numbers is, however, considerably more abstruse.

A Turing machine, an abstract mathematical system which can simulate many of the operations of computers, is named after A. M. Turing who first described such a machine in [2]. It consists of three main parts: (1) a finite set of states or modes; (2) a tape of infinite length with tape reader; (3) a set of instructions or rules. The tape reader can read only one character at a time, and, given the machine state and tape symbol, each instruction gives us information consisting of three parts; (1) the character to be written on the tape, (2) the direction in which the tape reader is to move; (3) the new state the machine is to be in.

A Turing machine can be described by either a diagram or a table. An example of a Turing machine that adds two numbers is shown in Fig. 1. The figure shows both the table form and the diagram form of this Turing machine.

Character \ State	*A*	*B*
1	1 *R A*	¢ Halt
b	1 *R A*	
¢	¢ *L B*	

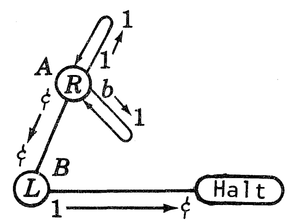

Fig. 1

Let us now consider the tape shown in (1). This tape shows a

(1) 1 1 *b* 1 1 1 ¢

two represented by two ones, a blank space represented by *b*, a three shown by three ones, and the ¢ which will mean the end of the information. The Turing machine shown in Fig. 1, when started in State A at the leftmost character of the tape in (1) will produce the following tape which shows a five, the sum of two and three.

(2) 1 1 1 1 1 ¢ ¢

The above table is read in the following way. The first row represents the states that the machine can be in, and the first column shows the characters that the machine can read. Let us, for example, look at the entry under State *A* and Character *b*. That entry, 1 *R A*, like every entry, save one, consists of three parts. The first part of the entry, 1, means change the character that is being read, *b* in this case, to a 1; the *R*, the second part of of the entry, means move one space to the right on the tape; and the *A*, the third part of the entry, says that the machine is to be in State *A* before reading the next character. Thus, if the machine is in State *A* and sees ¢, the table says that it changes the ¢ to ¢, moves left one place, and goes into State *B*.

The above diagram, which is equivalent to the table, can be most easily explained by considering only a portion of it. The states of the

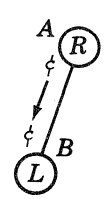

Fig. 2

machine are shown on the outside of the circles; the direction of the move is shown inside the next circle; and the character change is shown along the line connecting the circles. Thus, Fig. 2 says that a machine in State A and seeing ¢, changes ¢ to ¢, moves left one place, and goes into State B.

The Turing machine diagram which appears in Fig. 3 exhibits a machine which will halt only when presented with a string of consecutive ones, whose length is a Fibonacci number. If the total number of consecutive ones is not a Fibonacci number, the machine will loop endlessly. A basic assumption is that the string of ones is bounded on each side by at least one zero.

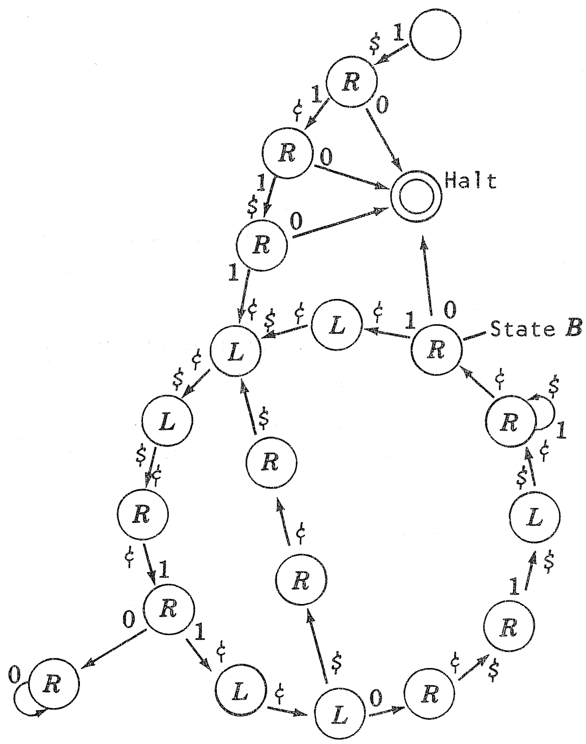

Fig. 3

The machine depicted examines the string of ones, starting at the left end, and repeatedly builds larger and larger Fibonacci numbers within this string. It keeps track of its place, and of previously constructed Fibonacci numbers, by slowly changing the ones to a series of dollar signs and cent signs as it moves through the string of ones. Each time the machine reaches the states labeled B in Fig. 3, the segment of the tape which has been examined has been changed to a string of dollar signs with the exception of a cent sign in the F_n place (which is the place immediately to the left of the tape digit being read while in State B), and a second cent sign in the F_{n-1} place.

After the machine finishes constructing a Fibonacci number within the string of ones, that is, each time it reaches State B, it checks to see if the next digit on the tape is zero or not. If so, the number of ones in the original string is a Fibonacci number and the machine halts. If, however, the next digit is a one, the machine attempts to build the next larger Fibonacci number within the string of ones (and, at this point, dollar and cent signs). If it encounters a zero on the tape before completing the construction of this next Fibonacci number, the machine goes into an endless loop. Thus, it halts only when the original number of consecutive ones is a Fibonacci number.

As an example, suppose the initial input string was ...001111111100... . By the time the Turing machine reaches State B, the string would be changed to ...00$$¢$¢11100... with the first cent sign replacing the third "1" (3 being a Fibonacci number) and the second cent sign replacing the fifth "1" (5 being the next Fibonacci number), and the "tape reader" would be "reading" the first remaining "1" in the string, as indicated. The next time around the major loop the string would be changed to ...00$$$$¢$$¢00... (the first cent sign replacing the fifth "1," and the second cent sign replacing the eighth "1" in the original string). Since the tape reader now reads a zero, the Turing machine moves to the Halt state and stops.

A Markov algorithm provides an alternate but equivalent approach to having a recognition algorithm for Fibonacci numbers. A Markov algorithm, like the Turing machine, operates on a string of elements over a given alphabet and consists of a sequence of rules which specify operations on the given string. Each rule ends with a number indicating the number of the next rule to be executed. If that rule is inapplicable, then the next rule in order is taken. The algorithm starts with rule number zero and each rule is applied to the leftmost occurrence of the element in the string. A rule ending with a period indicates a terminating rule, after which the algorithm is completed.

The Markov algorithm given below operates in a manner similar to the Turing machine given above. Both the Markov algorithm and the Turing machine generate Fibonacci numbers inside the given string of 1's and check to see if the constructed string and the given string are equal.

MARKOV ALGORITHM TO RECOGNIZE FIBONACCI NUMBERS

0:	$1 \rightarrow \alpha$, 1	first 1 converted to α
1:	$\alpha 1 \rightarrow \beta\alpha$, 3	first α changed to β, next available 1 to α
2:	$\Lambda \rightarrow \Lambda$.	nothing changed and Markov algorithm stops
3:	$\alpha \rightarrow \delta$, 3	repeated step, α's to deltas
4:	$\Lambda \rightarrow \gamma$, 5	gamma inserted at beginning of string
5:	$\gamma\beta \rightarrow \beta\gamma$, 9	gamma shifted right one through β's
6:	$\gamma \rightarrow \Lambda$, 7	delete gamma
7:	$\delta \rightarrow \beta$, 7	repeated step, deltas to β's
8:	$\Lambda \rightarrow \Lambda$, 3	dummy step—if rule 7 is nonapplicable, do nothing and skip to rule 3
9:	$1 \rightarrow \alpha$, 5	change next available 1 to an α
10:	$\gamma \rightarrow \Lambda$, 11	delete gamma
11:	$\alpha \rightarrow 1$, 13	change first α back to a 1
12:	$\Lambda \rightarrow \Lambda$.	nothing changed and Markov algorithm stops
13:	$\alpha \rightarrow 1$, 13	repeated step, α's to 1's
14:	$1 \rightarrow 1$, 14	does nothing, endless loop which occurs if original string is NOT a Fibonacci number

Lambda is the null symbol. Thus, rules 2 and 12 say "do nothing and stop." Rule 4 says to insert a gamma at the beginning of the string, and rule 6 says to delete the first gamma.

This Markov algorithm works as follows: it converts a given string of 1's into a string of β's and α's that represent F_i and F_{i+1} within the string of 1's. At the end of a loop, the α's are changed to deltas and more 1's are changed into α's to correspond to the number of β's which begin the string. The deltas are then changed to β's. Thus, after one loop, the number of α's has changed from F_i to F_{i+1}, and the number of β's has changed from F_{i+1} to

$$F_i + F_{i+1} = F_{i+2}.$$

If there are no more 1's to be changed at the end of a loop, the Markov algorithm stops at rule 12, indicating that the original string of 1's was a Fibonacci number. If, however, the string was not a Fibonacci number, the Markov algorithm jumps out of the loop in midstream of changing 1's to α's and goes into an endless loop at rule 14 after changing the α's back to 1's.

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ON SOME CONJECTURES OF GOULD ON THE PARITIES OF THE BINOMIAL COEFFICIENTS

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In studying the parities of the binomial coefficients, Gould [1] noted several interesting relationships about the signs of the sequence of numbers

$$(-1)^{\binom{n}{0}},\ (-1)^{\binom{n}{1}},\ \ldots,\ (-1)^{\binom{n}{n}}.$$

Further interesting relationships may be discovered by converting each such sequence to a binary number, $f(2, n)$, by

$$f(x, n) = \sum_{k=0}^{n} x^k \frac{1 - (-1)^{\binom{n}{k}}}{2} \tag{1}$$

and then comparing the numbers of the sequence $f(2, 0)$, $f(2, 1)$, $f(2, 2)$, ... The following conjectures were then proposed by Gould.

<u>*Conjecture 1*</u>: $f(2, 2^m - 1) = 2^{2^m} - 1$.

<u>*Conjecture 2*</u>: $f(2, 2\ ) = 2^{2^m} + 1$.

<u>*Conjecture 3*</u>: $f(x, 2n + 1) = (x + 1)f(x, 2n)$.

We will prove these conjectures and present some related results.

The following lemma provides a convenient recursive scheme for generating the sequence of numbers $f(x, 0)$, $f(x, 1)$, ... We use the notation $(.)_x$ to denote the representation of a number to the base x.

<u>*Lemma 1*</u>: The sequence $f(x, n)$ may be defined by $f(x, 0) = 1$, and if

$$f(x, n - 1) = (a_{n-1}, \ldots, a_0)_x$$

for $n > 0$, then

$$f(x,\ n) = x^n + 1 + \sum_{k=1}^{n-1} x^k |a_k - a_{k-1}|. \tag{2}$$

Proof: It follows directly from (1) that

$$f(x,\ n) = x^n + 1 + \sum_{k=1}^{n-1} x^k \frac{1 - (-1)^{\binom{n}{k}}}{2}.$$

By the well-known recursion for binomial coefficients,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

so that

$$(-1)^{\binom{n}{k}} = \begin{cases} +1 & \text{if } (-1)^{\binom{n-1}{k}} = (-1)^{\binom{n-1}{k-1}} \\ -1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{1 - (-1)^{\binom{n}{k}}}{2} = |a_k - a_{k-1}| \quad \text{for } n - 1 \geq k \geq 1.$$

Theorem 1: $f(x,\ 2^m - 1) = \sum_{k=0}^{2^m - 1} x^k.$

Proof: The theorem is clearly satisfied for $m = 1$. Assume that

$$f(x,\ 2^m - 1) = \sum_{k=0}^{2^m-1} x^k = (a_{2^m-1},\ \ldots,\ a_0)_x,$$

where $a_k = 1$ for $2^m - 1 \geq k \geq 0$. By Lemma 1,

$$f(x,\ 2^m) = x^{2^m} f(x,\ 0) + f(x,\ 0).$$

We may apply (2) to both parts of $f(x,\ 2^m)$ independently for $2^m - 1$ times, and then add the results to obtain

$$f(x,\ 2^m + 2^m - 1) = x^{2^m} f(x,\ 2^m - 1) + f(x,\ 2^m - 1).$$

By the induction hypothesis,

$$f(x,\ 2^{m+1} - 1) = x^{2^m} \sum_{k=0}^{2^m-1} x^k + \sum_{k=0}^{2^m-1} x^k = \sum_{k=2^m}^{2^{(m+1)}-1} x^k + \sum_{k=0}^{2^m-1} x^k = \sum_{k=0}^{2^{(m+1)}-1} x^k.$$

Corollary 1 (Conjecture 1): $f(2,\ 2^m - 1) = 2^{2^m} - 1.$

Corollary 2: $f(x,\ 2^m) = x^{2^m} + 1.$

Proof: Apply (2) to the result of Theorem 1.

Corollary 3 (Conjecture 2): $f(2,\ 2^m) = 2^{2^m} + 1.$

Let $L(n)$ denote $2^{\lfloor \log_2 n \rfloor}$, where $\lfloor y \rfloor$ denotes the integer part of y. Examining each number $f(x,\ n)$ as a number to the base x, the following striking symmetry may be noticed: the sequence of the least significant $L(n)$ digits of $f(x,\ n)$, is equal to the sequence of the next most significant $L(n)$ digits of $f(x,\ n)$, which is also equal to the sequence of the least most significant $L(n)$ digits of $f(x,\ n - L(n))$. The following lemma, which is based on this symmetry provides another recursive scheme for generating the sequence $f(x,\ 0), f(x,\ 1),\ \ldots$.

Lemma 2: For $n > 0$, $f(x,\ n) \bmod (x^{L(n)}) = \left\lfloor \dfrac{f(x,\ n)}{x^{L(n)}} \right\rfloor = f(x,\ n - L(n)).$

Proof: We distinguish between the two cases of whether or not there exists an integer m such that $n = 2^m$. If $n = 2^m$ for some integer m, then from Corollary 2 it follows that $f(x, n) = x^n + 1$ and

$$f(x, n) \bmod (x^n) = 1 = \left\lfloor \frac{f(x, n)}{x^n} \right\rfloor.$$

Furthermore, since $L(n) = n$, it follows that $f(x, n - L(n)) = f(x, 0) = 1$, and the lemma is established for this case.

For the case $n \neq L(n)$, it follows from Corollary 2 that

$$f(x, L(n)) = x^{L(n)} f(x, 0) + f(x, 0).$$

Applying (2) to $f(x, L(n))$ for $n - L(n)$ times, we may treat the two parts independently and

$$f(x, n) = x^{L(n)} f(x, n - L(n)) + f(x, n - L(n)).$$

Consequently,

$$f(x, n) \bmod (x^{L(n)}) = \left\lfloor \frac{f(x, n)}{x^{L(n)}} \right\rfloor = f(x, n - L(n)).$$

We are now in a position to prove Conjecture 3.

Theorem 2 (Conjecture 3): $f(x, 2n + 1) = (x + 1)f(x, 2n)$.

Proof: Since $x + 1 = (1, 1)_x$, the theorem will follow from elementary rules of multiplication in the base x if we can prove that when $f(x, 2n)$ is expressed in the base x, no pair of consecutive digits are 1's. We will prove this property by induction. This is certainly true for $f(x, 0) = (1)_x$. For arbitrary $n > 0$, let

$$f(x, 2n) = (a_{2L(2n)-1}, \ldots, a_0)_x.$$

By Lemma 2, each half of this number is equal to $f(x, 2n - L(2n))$ which, by the induction hypothesis, does not have two consecutive 1's when expressed in the base x. It remains to be shown that $a_{L(2n)-1} = 0$. But, by Lemma 2,

$$a_{L(2n)-1} = a_{2L(2n)-1},$$

and $a_{2L(2n)-1}$ cannot be equal to 1 because $f(x, 2n) < x^{2L(2n)-1}$.

We conclude with a final observation on the sequence of numbers $f(x, n)$. Examining the $2^m \times 2^m$ binary matrix in which the entry a_{ij} is the jth digit of $f(x, i - 1)$, we note that the matrix is symmetric about its major diagonal.

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SOLUTIONS FOR GENERAL RECURRENCE RELATIONS

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1. STATEMENT OF THE PROBLEM

In a recent article [1], the author obtained representations for the solutions of certain r,s recurrence relations. In this paper we shall give representations for the solutions of general recurrence relations. In Section 4 we shall show that the results in [1] are a special case of the results of Sections 2 and 3 of this paper.

We first of all characterize all decompositions of an integer n, restricted to the first m positive integers. We define a multinomial from this that satisfies an mth-order recurrence relation with special initial conditions. Next the set of m positive integers is restricted to a subset A containing m, and a second multinomial that satisfies a recurrence relation with special initial conditions is defined.

In Section 3, we obtain solutions for comparable recurrence relations with general initial conditions. The final result gives us a solution for the general recurrence relation:

$$H_p = r_{a_1} H_{p-a_1} + \cdots + r_{a_t} H_{p-a_t};\ H_0,\ \ldots,\ H_{1-a_t} \text{ arbitrary.}$$

2. BASIC mth-ORDER RECURRENCE RELATIONS

One of the classic concepts in the theory of numbers is that of partitions of the positive integers. One of the subcases considered is for the component integers to be the set of integers from 1 to m. In this case we denote the set of all partitions of n as $P(n;m)$. The number of elements in this set is $P_m(n)$. A given partition can be characterized by a set of integers k_i. That is,

$$n = 1k_1 + \cdots + mk_m.$$

The integers k_i are referred to as the frequency of i in the given partitions. We refer to this given partition as $p(k,n;m)$.

For a given $p(k,n;m)$, we can represent n as a sum of integers from 1 to m in

$$\frac{(k_1 + \cdots + k_m)!}{k_1! \ \ldots \ k_m!}$$

ways. Each such representation is called a "decomposition of n" (some authors call them "compositions"). We denote this expression as $d_m(k,n)$. It is the number of decompositions of the partition $p(k,n;m)$.

This expression has a property that we shall find useful:

$$\frac{(k_1 + \cdots + k_m)!}{k_1! \ \ldots \ k_m!} = \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \ \ldots \ k_m!} \sum_{s=1}^{m} k_s$$

$$= \sum_{s=1}^{m} \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \ \ldots \ (k_s - 1)! \ \ldots \ k_m!}. \tag{2.1}$$

Symbolically we have

$$d_m(k,n) = \sum_{s=1}^{m} d_m(k(s),\ n - s),$$

where $d_m(k(s), n - s) = 0$ if $k_s = 0$. Otherwise, it is the number of decompositions for the partition of $n - s$ where all the k_i are the same as for the k partition of n except that k_s is reduced by 1.

We use this number of decompositions to define a multinomial. We then show that it is the solution for a special recurrence relation. Let

$$U_n = \sum_{P(n;m)} d_m(k, n) r_1^{k_1} \ldots r_m^{k_m},$$

that is, we sum over all partitions of n, a multinomial in $r_1, \ldots, r_m$ whose coefficients are the number of decompositions of the given partition. We can now prove our first theorem.

Theorem 2.1: The multinomial U_n satisfies the recurrence relation

$$U_t = \sum_{s=1}^{m} r_s U_{t-s}; \; U_0 = 1, \; U_{-1} = \cdots = U_{1-m} = 0.$$

By applying property (1) to the definition of U_n, we have

$$\begin{aligned} U_n &= \sum_{P(n;m)} d_m(k, n) r_1^{k_1} \ldots r_m^{k_m} \\ &= \sum_{P(n;m)} \sum_{s=1}^{m} d_m(k(s), n - s) r_1^{k_1} \ldots r_m^{k_m} \\ &= \sum_{s=1}^{m} r_s \sum_{P(n-s;m)} d_m(k(s), n - s) r_1^{k_1} \ldots r_s^{k_s - 1} \ldots r_m^{k_m} \\ &= \sum_{s=1}^{m} r_s U_{n-s}. \end{aligned}$$

We have used the fact that decreasing the frequency of s by 1 gives the restricted partitions of $n - s$. If s has a frequency of 0 for a given partition, then the corresponding term in the summation on s is 0.

For $n < m$, the frequencies for the integers $n + 1$ to m would all be zero. Hence the summation can be terminated at n. However, if we choose $U_{-1} = \ldots = U_{1-m} = 0$, then we do not need any restriction. This gives $m - 1$ initial conditions. For the mth one, we shall choose $U_0 = 1$. This is logical, since all factorials are 0! and all exponents of the r_i are 0. This would give a value of 1. Hence the U_n does satisfy the prescribed recurrence relation.

What we have just proved for the case of the restricted partitions of n can be specialized for a proper subset $A = \{a_1, \ldots, a_j\}$ of the integers from 1 to m. For convenience, we assume m is in A. The set of all partitions of n restricted to the set A we label $P(n; A)$. The number of elements in this set is $P_A(n)$. A given partition can be characterized by a set of frequencies k_i, so that

$$n = a_1 k_{a_1} + \cdots + a_j k_{a_j}.$$

We refer to this given partition as $p(k, n; a)$.

For each such partition, we can represent n as a sum of integers in A in

$$\left(\sum_{i=1}^{j} k_{a_i}\right)! \Big/ \prod_{i=1}^{i} k_{a_i}!$$

ways. We denote this number as $d_A(k, n)$, that is, there are this many decompositions of the given partition, restricted to A. We can define the following multinomial

$$V_n = \sum_{P(n;A)} d_A(k,\ n) \prod_{q \in A} r_q^{k_q}.$$

We then have the following theorem.

Theorem 2.2: The multinomial V_n satisfies the recurrence relation

$$V_t = \sum_{s \in A} r_s V_{t-s};\ V_0 = 1,\ V_{-1} = \cdots = V_{1-m} = 0.$$

This theorem is a special case of Theorem 2.1. First of all, the restriction to the set A means that the frequencies $k_i = 0$ if $i \in A$. This means that for each partition of n there is no s corresponding to each such i in the solution. Hence s is summed only on A. Furthermore, since the corresponding r_i is always to the zero power, we drop these r_i in the multinomial. The number of initial conditions is dependent only on the largest integer in A, which is assumed to be m.

3. GENERAL RECURRENCE RELATIONS

Using the results of the last section, we can obtain solutions for recurrence relations with arbitrary initial conditions. We shall consider two cases that are comparable to those in the last section. Our solutions will involve the U_n and V_n, respectively.

Theorem 3.1: The solution for the recurrence relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_0,\ \ldots,\ G_{1-m} \text{ arbitrary}, \tag{3.1}$$

is given by

$$G_n = \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{n-j} G_{j-q}. \tag{3.2}$$

For $n = 1$ in (3.2) the $U_{n-j} = U_{i-j}$ is zero except for $j = 1$. In this case $U_0 = 1$. The double summation reduces to

$$G_1 = \sum_{q=1}^{m} r_q G_{1-q},$$

which is (3.1) for $t = 1$ and $q = 2$.

For $n = 2$ in (3.2) the $U_{2-j} = 0$ for $j > 2$. We then have

$$G_2 = U_1 \sum_{q=1}^{m} r_q G_{1-q} + U_0 \sum_{q=2}^{m} r_q G_{2-q}.$$

From the previous section, we have that $U_0 = 1$ and $U_1 = r_1$. Also, by (3.1) the first sum is G_1. Hence we have

$$G_2 = r_1 G_1 + \sum_{q=2}^{m} r_q G_{2-q} = \sum_{q=1}^{m} r_q G_{2-q},$$

which is (3.1) for $t = 2$ and $s = q$.

We assume that (3.2) is a valid solution for $n = 1,\ \ldots,\ i - 1$. For $t = i$ in (3.1),

$$G_i = \sum_{s=1}^{m} r_s G_{i-s}.$$

We have assumed solutions for all the G_{i-s} in this summation. Hence on substitution into this expression, we obtain

$$\begin{aligned} G_i &= \sum_{s=1}^{m} r_s \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{i-s-j} G_{j-q} \\ &= \sum_{j=1}^{m} \sum_{q=j}^{m} r_q \left(\sum_{s=1}^{m} r_s U_{i-s-j} \right) G_{j-q} \\ &= \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{i-j} G_{j-q}. \end{aligned}$$

At the last step we use the fact that U_n satisfies a recurrence relation. This final result is (3.2) for $n = i$.

We are now ready to present the solution to a general recurrence relation. We assume that set A has the properties of the last section.

Theorem 3.2: The solution for the recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s}; \; H_0, \; \ldots, \; H_{1-m} \text{ arbitrary}, \tag{3.3}$$

is given by

$$H_n = \sum_{q \in A} \sum_{j=1}^{q} r_q V_{n-j} H_{j-q}. \tag{3.4}$$

This theorem follows from Theorem 3.1, just as Theorem 2.2 followed from Theorem 2.1. For convenience, we have interchanged the order of summations in the solution so that it is easier to adapt to the restriction on q.

4. SOME SPECIAL CASES

In this section we shall consider some special cases of the results of Sections 2 and 3. They are for both the U_n and G_n relations for $m = 2$.

The restricted partitions of n for $m = 2$ would be of the form $n = k_1 + 2k_2$. The summation over all such partitions can be represented by a summation on j when $j = k_2$. Then $k_i = n - 2j$, and the summation is from 0 to $[n/2]$. The number of decompositions for a given partition would be given by

$$d_2(k, n) = \frac{(n - 2j + j)!}{(n - 2j)!j!} = \binom{n-j}{j}.$$

The solution for U_n in this case is

$$U_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n-j}{j} r_1^{n-2j} r_2^j.$$

For the more general G_n relation we have

$$\begin{aligned} G_n &= \sum_{j=1}^{2} \sum_{q=j}^{2} r_q U_{n-j} G_{j-q} = (r_1 U_{n-1} G_0 + r_2 U_{n-1} G_{-1}) + (r_2 U_{n-2} G_0) \\ &= (r_1 U_{n-1} + r_2 U_{n-2}) G_0 + r_2 U_{n-1} G_{-1} = U_n G_0 + r_2 U_{n-1} G_{-1}. \end{aligned}$$

Substituting in the solution for U_n and U_{n-1},

$$G_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n-j}{j} r_1^{n-2j} r_2^j G_0 + \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-1-j}{j} r_1^{n-1-2j} r_2^{j+1} G_{-1}.$$

We change the second index of summation by replacing $j + 1$ by j, as follows:

$$G_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n-j}{j} r_1^{n-2j} r_2^j G_0 + \sum_{j=1}^{[\frac{n+1}{2}]} \binom{n-j}{j-1} r_1^{n+1-2j} r_2^j G_{-1}.$$

The author gave representations for some special recurrence relations in a previous paper [1]. We shall now show that these were particular cases of the U_n and G_n relations for $m = 2$.

The first relation presented was a generalized Fibonacci sequence,

$$G_k = rG_{k-1} + sG_{k-2};\ G_0 = 0,\ G_1 = 1,$$

which has the solution

$$G_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j.$$

We observe that both our indexing and the constants of the relations are different. To reconcile them, we replace n by $k - 1$, r_1 by r, and r_2 by s in the U_n solution. This gives us the desired result.

As a special case, when $r = s = 1$ we have the Fibonacci sequence. The general term would be given by

$$F_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j},$$

which is the number of decompositions of $k - 1$ restricted to 1 and 2.

Another sequence presented in [1] is the generalized Lucas sequence M_k, for which

$$M_k = rM_{k-1} + sM_{k-2};\ M_0 = 2,\ M_1 = r.$$

To obtain the solution we specialize the G_n for $m = 2$. We replace n by $k - 1$, r_1 by r, r_2 by s, G_0 by r, and G_{-1} by 2. We have

$$M = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j r + \sum_{j=1}^{[\frac{k-1}{2}]} \binom{k-1-j}{j-1} r^{k-2j} s^j 2.$$

We observe that the powers of r and s in both sums are the same. Hence we combine them into a single sum. It can be verified that this yields

$$M_k = \sum_{j=0}^{[\frac{k}{2}]} \frac{k}{k-j} \binom{k-j}{j} r^{k-2j} s^j,$$

which is the solution given in [1].

The third relation discussed in [1] is

$$U_k = rU_{k-1} + sU_{k-2};\ U_1,\ U_0 \text{ arbitrary}.$$

We can identify this with our G_n relation if we let $n = k - 1$, $r_1 = r$, $r_2 = s$, $G_0 = U_1$, and $G_{-1} = U_0$. This gives

$$U_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j U_1 + \sum_{j=1}^{[\frac{k}{2}]} \binom{k-1-j}{j-1} r^{k-2j} s^j U_0.$$

Applying some algebra to combine the two sum yields the following solution:

$$U_k = \sum_{j=0}^{[\frac{k}{2}]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j} s^j.$$

This can also be verified directly.

In a future paper we shall show that there are generating functions for the four recurrence relations given in this paper. These can also be used for the special cases of this section. We can use them to generate with a computer as many terms in a given recurrence relation as desired.

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ON GENERATING FUNCTIONS AND DOUBLE SERIES EXPANSIONS

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1. INTRODUCTION

Recently, Weiss *et al.* [9] gave a direct proof of a result due to Narayana [8] and Kreweras [6]:

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} u^r v^s = \frac{1}{2}[1-u-v-(1-2(u+v)+(u-v)^2)^{1/2}]. \tag{1.1}$$

A special case of Theorem 1a of this pàper is a five-parameter generalization of (1.1):

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha+1+gk+hp)}\binom{\alpha+gk+k+hp}{k}\binom{\beta+gck+hcp+p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\beta+1}}{(\alpha+1)} {}_2F_1\left[\begin{matrix}1,\ 1+\beta-c-\alpha c, \\ (\alpha+1+h)/h,\end{matrix}\ -y\right], \tag{1.2}$$

where

$$u = \frac{z}{(1+z)^{g+1}(1+y)^{gc}},\quad v = \frac{y}{(1+z)^h(1+y)^{hc+1}}.$$

See Luke [7, Sec. 6.10] for a discussion of Padé approximation for the hypergeometric function on the right-hand side of (1.2). Letting

$$g = -1,\ h = -1,\ c = 1,\ \alpha = -2, \text{ and } \beta = -2$$

in (1.2) and some manipulation will give (1.1).

Equation (1.2) also appears to be an extension of the important equation (6.1) of Gould [5], to which it reduces for $z = 0$.

An interesting simplification of (1.2) is the case $\beta = \alpha c + c - 1$, giving:

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha+1+gk+hp)}\binom{\alpha+gk+k+hp}{k}\binom{\alpha c+c-1+gck+hcp+p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\alpha c+c}}{(1+\alpha)}. \tag{1.3}$$

The importance of these types of expansions is the connection with Jacobi polynomials. Now, Carlitz [1] gave the important generating function

$$\sum_{n=0}^{\infty} \frac{\alpha}{n+\alpha} P_n^{(\alpha,-1)}(x) r^n = 2^{\alpha}(1 - r + R)^{-\alpha}, \tag{1.4}$$

where

$$R = (1 - 2xr + r^2)^{1/2} \quad \text{and} \quad P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix} -n,\ n+\alpha+\beta+1, \\ \alpha+1, \end{matrix} \ \frac{1-x}{2}\right].$$

A special case of Theorem 1a in this paper gives

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+\tau+1+an+bn+n)} P_n^{(\sigma+an,\tau+bn)}(w) \tag{1.5}$$

$$= \frac{1}{(\sigma+\tau+1)}(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma} {}_2F_1\left[\begin{matrix} 1,\ \{(1+a)(1+\tau) - \sigma b\}/(1+a+b), \\ (\sigma+\tau+a+b+2)/(a+b+1), \end{matrix} \ y\right],$$

where y and z are defined by

$$(1-w)/2 = z(1-y)/[y(1-z)],$$

$$\xi = y(1-y)^a/(1-z)^{a+b+1},$$

and

$$|\xi| < 1,\ |y| < 1,\ |z| < 1.$$

By letting $a = b = 0$ and $\sigma = -1$, (1.5) reduces to (1.4). See [3] for another generalization of (1.4), and some discussion regarding its importance.

A special case of interest occurs for $\tau = (\sigma b - a - 1)/(1 + a)$, giving:

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+\tau+1+an+bn+n)} P_n^{(\sigma+an,\tau+bn)}(w) = \frac{(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma}}{\sigma+\tau+1}. \tag{1.6}$$

Equation (1.5) is also a three-parameter extension of another equation of Carlitz [1, Eq. 8]. Letting $a = 0$ in (1.5) gives equation (1) of Cohen [4].

A special case of Theorem 1b of this paper yields the expression:

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{u^k v^p}{(\beta+1+gck+hcp)} \binom{\alpha+gk+k+hp}{k}\binom{\beta+gck+hcp+p}{p}$$

$$= \frac{(1+z)^{\alpha+1}(1+y)^{\beta+1}}{(\beta+1)} {}_2F_1\left[\begin{matrix} \dfrac{c-\beta-1+\alpha c}{c},\ 1, \\ \dfrac{\beta+1+gc}{gc}, \end{matrix} \ -z\right] \tag{1.7}$$

where

$$u = \frac{z}{(1+z)^{g+1}(1+y)^{gc}}, \quad v = \frac{y}{(1+z)^k(1+y)^{hc+1}}.$$

The analogous expression for the Jacobi polynomial takes the form

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+an+n)} P_n^{(\sigma+an,\tau+bn)}(w) \tag{1.8}$$

$$= \sigma^{-1}(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma} {}_2F_1\left[\begin{matrix} 1,\ (1+\tau)(1+a) - \sigma b/(1+a), \\ (\sigma+a+1)/a, \end{matrix} \ z\right].$$

Letting $\tau = -1$, $a = b = 0$, the Carlitz formula given by our equation (1.4) presents itself. Also, letting $b = 0$ in (1.8) gives essentially a main result in [2, Eq. (1.1)]. [The variables y and z are defined in (1.5).]

The statement and proof of Theorem 1 follow in the next section.

2. STATEMENT AND PROOF OF THEOREM 1

Theorem 1: For a, b, c, α, and β complex numbers and ℓ, ℓ', and j nonnegative integers:

a.

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\xi_1^k\xi_2^p\binom{\alpha+ak+k+bp}{k}\binom{\beta+ack+bcp+p}{p}}{(\alpha+ak+bp+\ell+1)\binom{(\alpha+ak+bp+\ell+1+j\ell')/j}{\ell'}}$$

$$= (1-z)^{\alpha+\ell+1}(1-y)^{\beta+1} \times$$

$$\sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\frac{(-1)^r(z)^k(1-z)^{jr-k}}{(\alpha+jr+ak+\ell+1)}\binom{\ell'}{r}\binom{\ell+jr}{k}{}_2F_1\left[\begin{matrix}1,\ 1-c-\quad c-\quad c+\quad -jcr, \\ (\alpha+1+jr+ak+b+\ell)/b,\end{matrix}\ y\right]. \tag{2.1}$$

b.

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\xi_1^k\xi_2^p\binom{\alpha+ak+k+bp}{k}\binom{\beta+ack+bcp+p}{p}}{(\beta+ack+bcp+\ell+1)\binom{(\beta+1+\ell+ack+bcp+j\ell')/j}{\ell'}}$$

$$= (1-y)^{\beta+1+\ell}(1-z)^{\alpha+1} \times$$

$$\sum_{r=0}^{\ell'}\sum_{p=0}^{\ell+jr}\frac{(-1)^r(y)^p(1-y)^{jr-p}}{(\beta+\ell+1+jr+bcp)}\binom{\ell'}{r}\binom{\ell+jr}{p}{}_2F_1\left[\begin{matrix}1,\ (c-\beta-1+ac-jr-\ell)/c, \\ (\ell+\beta+1+jr+bcp)/(ac),\end{matrix}\ z\right], \tag{2.2}$$

where $\binom{a}{n} = \dfrac{\Gamma(a+1)}{n!\Gamma(a-n+1)}$, and y and z are defined through

$$\xi_1 = \frac{-z}{(1-z)^{a+1}(1-y)^{ac}},\quad \xi_2 = \frac{-y}{(1-z)^b(1-y)^{bc+1}}$$

$$|y| < 1,\ |z| < 1,\ |\xi_1| < 1, \text{ and } |\xi_2| < 1.$$

Corollary 1a: Reduction of Theorem 1a for the Jacobi polynomial gives:

$$\sum_{n=0}^{\infty}\frac{\xi^n}{(\sigma+\tau+1+\ell+an+bn+n)\left(\dfrac{\sigma+\tau+1+\ell+an+bn+n+j}{j}\right)_{\ell'}}P_n^{(\sigma+an,\,\tau+bn)}(w)$$

$$= \sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\frac{(-z)^k(1-z)^{\ell+jr-k+\sigma+\tau+1}(1-y)^{-\sigma}(-\ell')_r(-\ell-jr)_k}{k!r!\ell'!(\ell+jr+(1+a+b)k+\sigma+\tau+1)} \times$$

$${}_2F_1\left[\begin{matrix}1,\ \dfrac{(1+a)(1+\ell+\tau+jr)-\sigma b}{1+a+b}, \\ \dfrac{\ell+jr+(1+a+b)k+\sigma+\tau+a+b+2}{1+a+b},\end{matrix}\ y\right]. \tag{2.3}$$

$|y| < 1$, $|z| < 1$, and $|\xi| < 1$, where $(a+1)_n/n! = \binom{a+n}{n}$, and y and z are defined in (1.5).

Corollary 1b: Reduction of Theorem 1b for the Jacobi polynomial gives:

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma + n(1+a) - \ell)\left(\dfrac{\ell - \sigma - (1+a) + j}{j}\right)_{\ell'}} P_n^{(\sigma+an,\, \tau+bn)}(w)$$

$$= \frac{(1-z)^{\sigma+\tau+1}}{\ell'!}(1-y)^{-\sigma+\ell} \sum_{r=0}^{\ell'} \sum_{p=0}^{\ell+jr} \frac{(-y)^p (1-y)^{jr+p} (-\ell')_r (-\ell - jr)_p}{p!\,r!\,(\sigma + p(1+a) - \ell - jr)} \times$$

$${}_2F_1\left[\begin{matrix} 1, \dfrac{(1+a)(1+\tau+\ell+jr) - b(\sigma - \ell - jr)}{(1+a)}, \\ \dfrac{\sigma + (1+a)p - \ell - jr + a + 1}{(1+a)}, \end{matrix} \; z\right], \tag{2.4}$$

where y and z are defined in (1.5).

Proof of Theorem 1a: Now consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \int_0^1 x^{c\ell}(1 - x^{jc})^{\ell'}\, \delta^n [x^{\alpha c + nc - \beta} D^m \{(1 - x^{ac})^n (1 - x^{bc})^n x^{\beta+m}\}]\, dF \tag{2.5}$$

$$\left(\text{where } F \equiv x^c,\; D \equiv \frac{d}{dF},\; \delta \equiv \frac{d}{dF}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \sum_{r=0}^{\ell'} \frac{(-\ell')_r (-\ell - jr)_n}{r!} \int_0^1 x^{c\ell + jcr + \alpha c - \beta} D^m [(1 - x^{ac})^n (1 - x^{bc})^m x^{\beta+m}]\, dF \tag{2.6}$$

where $(a)_m = \Gamma(a+m)/\Gamma(a)$, quotient of gamma functions. Equation (2.6) is deduced from (2.5) by expanding $(1 - x^{cj})^{\ell'}$ and integrating the resulting equation by parts n times. Equation (2.6) may, in turn, by reduced to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \sum_{r=0}^{\ell'} \frac{(-\ell')_r (-\ell - jr)_n (-n)_k (-m)_p (\beta + 1 + ack + bcp)_m}{r!\,k!\,p!\,(\ell + 1 + jr + \alpha + ak + bp)} \tag{2.7}$$

The evaluation is achieved through a further integration by parts m times, expansions of $(1 - x^{ac})^n$ and $(1 - x^{bc})^m$, and subsequent integration. By applying the double series transform to (2.7), one obtains

$$\sum_{r=0}^{\ell'}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-z)^k (-y)^p z^n y^m (-\ell')_r (-\ell - jr + k)_n (-\ell - ar)_k (\beta + 1 + p + bcp + ack)_m}{n!\,m!\,k!\,p!\,r!\,(\ell + \alpha + 1 + jr + ak + bp)} \tag{2.8}$$

$$= \sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\sum_{p=0}^{\infty} \frac{(-z)^k (-y)^p (-\ell')_r (-\ell - jr)_k \Gamma(\beta + 1 + p + bcp + ack)(1 - z)^{\ell + jr - k}}{r!\,k!\,p!\,\Gamma(\beta + 1 + ack + bcp)(\ell + \alpha + 1 + jr + ak + bp)(1 - y)^{\beta + 1 + p + bcp + ack}} \tag{2.9}$$

We now return to our original expression (2.5) and proceed with its evaluation through a modified approach. Consider the operator and its expansion:

$$\delta^n [x^{\alpha c + nc - \beta} D^m \{(1 - x^{ac})^n (1 - x^{bc})^n x^{\beta+m}\}]$$

$$= \sum_{p=0}^{m}\sum_{k=0}^{n} \frac{(-n)_k (-m)_p \Gamma(\beta + m + 1 + ack + bcp)\Gamma(\alpha + n + 1 + ak + bp)}{k!\,p!\,\Gamma(\beta + 1 + ack + bcp)\Gamma(\alpha + 1 + ak + bp)}. \tag{2.10}$$

With the aid of (2.10), (2.5) may be reduced to give:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n y^m}{n!m!}\sum_{m=0}^{p}\sum_{k=0}^{n}\frac{(-n)_k(-m)_p\Gamma(\beta+m+1+ack+bcp)\Gamma(\alpha+n+1+ak+bp)}{k!p!\Gamma(\beta+1+ack+bcp)\Gamma(\alpha+1+ak+bp)}$$

$$\times\frac{\ell'!\Gamma[(\alpha+ak+bp+\ell+1)/j]}{(j)\Gamma[(\alpha+ak+bp+\ell+1+j\ell'+j)/j]}. \qquad (2.11)$$

Using the double series transformation and reducing the subsequent series over n and m gives:

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-z)^k(-y)^p\Gamma(\beta+1+ack+bcp+p)\Gamma(\alpha+1+ak+k+bp)\Gamma[(\alpha+ak+bp+\ell+1)/j]\ell'!}{k!p!(1-z)^{\alpha+1+ak+k+bp}(1-y)^{\beta+1+ack+bcp+p}\Gamma(\alpha+1+ak+bp)\Gamma(\beta+1+ack+bcp)}$$

$$\times\frac{1}{(j)\Gamma[(\alpha+ak+bp+\ell+1+j\ell'+j)/j]}. \qquad (2.12)$$

Now equating the expressions (2.9) and (2.12) together with some simple transformations yields the required Theorem 1a.

Proof of Theorem 1b: The procedure adopted is similar to that for Theorem 1a. The modified integral is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n y^m}{n!m!}\int_0^1 x^{\ell+\beta+1-\alpha c-c}(1-x^j)^{\ell'}\,\delta^n[x^{\alpha c+nc-\beta}D^m\{(1-x^{ac})^n(1-x^{bc})^m x^{\beta+m}\}]dF,$$

where the previous definitions are in effect. The details of the proof follow the proof of Theorem 1a to give expression (2.2).

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AN EQUIVALENT FORM OF BENFORD'S LAW

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Benford's law states that the probability of a positive integer having 1st digit d is given by

$$Pr(j = d) = \log_{10}(1 + 1/d). \tag{1}$$

In terms of the cumulative probability distribution, (1) is restated as

$$Pr(j < d) = \log_{10}d.$$

This result was first noted by Benford [1] in 1938 and has since been extended to counting bases other than 10 as well as to certain subsets, called Benford sequences, of the positive integers. Geometric progressions or, more generally, integer solutions of finite difference equations are examples of Benford sequences that have received considerable attention in the literature, e.g., [2]. This interest is due, in part, to the fact that the Fibonacci and Lucas numbers are obtained as solutions of the finite difference equation

$$x_{n+2} = x_{n+1} + x_n.$$

We refer the reader to [3] for an extensive bibliography concerning this and other aspects of the 1st-digit problem.

Since the consideration of varying counting bases will be of concern to us here, we introduce the following notation. We write $Pr(j < d)_b$ for the probability of $j < d$ when numbers are represented as digits in base $b \geq 2$. In this notation, Benford's law states that

$$Pr(j < d)_b = \log_b d, \text{ for } d \leq b. \tag{2}$$

The purpose of this paper is to establish that, for the set of positive integers, (2) is equivalent to the following "monotonicity statement":

$$\underline{\text{If } b \leq b', \text{ then } Pr(j < d)_b \geq Pr(j < d)_{b'}.}$$

While this statement still makes sense for $b < d \leq b'$, we confine our attention to $d \leq b$. In so doing, it follows immediately that the monotonicity statement is implied by Benford's law as given in (2).

To reverse the above implication for the positive integers, we need two lemmas. Both of these results could be established via the functional equation

$$Pr(j < a) + Pr(j < c) = Pr(j < ac),$$

which is valid whenever the positive integers a and c as well as their product divide b. Instead of this approach, we present arguments based on a counting machine that randomly generates numbers in varying counting bases. The idea is as follows. It is clear that in binary ($b = 2$) the 1st digit must be 1. Consequently, if we represent numbers in oct 1 ($b = 8$) where each digit is denoted by a string of three binary symbols, then the 1st digit is determined by simply ascertaining the length of the binary representation modulo 3. Since the possible lengths (mod 3) of the binary representation of a randomly chosen number are equally likely, we obtain some probabilities. More generally, we have the following.

<u>*Lemma 1*</u>: Let $m, n \geq 0$, $a \geq 2$ denote integers. If randomly chosen positive integers are represented in base $b = a^n$, then

$$Pr(j < a^m) = m/n, \; m \leq n. \tag{3}$$

Proof: We denote by $b_1 b_2 \ldots b_k$ the random number as represented in base b. Thus, $0 \le b_i < b$ for $i = 1, 2, \ldots, k$ and $b_1 \ne 0$. Rewrite each b_i as $a_{1i} a_{2i} \ldots a_{ni}$, where the a_i's represent digits in base a. This yields a string of nk digits each of which is less than a. Removing the 0 digits occurring at the beginning of this, we obtain the base a representation of the random number. Suppose this base a representation contains x digits. We solve the congruence relation $x \equiv y \pmod{n}$ where $0 \le y < n$. If $y = 0$, the 1st digit j (in base b) satisfies $a^{n-1} \le j < a^n = b$. For any other value of y, the 1st digit satisfies $a^{y-1} \le j < a^y$. Since each value of y is equally likely, we obtain

$$Pr(a^{y-1} \le j < a^y) = Pr(a^{n-1} \le j < a^n) = 1/n. \tag{4}$$

Equation (3) follows immediately from (4). This completes the proof.

By a simple variation of the combinatoric argument used in the proof of Lemma 1, we next obtain a result that permits the comparison of the distribution of the 1st digit with respect to two different bases.

Lemma 2: Using the notation introduced above, we have

$$Pr(j < d)_b = mPr(j < d)_{b^m}.$$

Proof: A random number represented by k digits in base b^m is rewritten as a string of km digits in base b. As in Lemma 1, we delete all consecutive zeros from the left-hand side of the km digits. This yields a base b representation of the number. For $j < d$, in base b, there are m equally likely possible values for the position of j in the base b^m representation. Since the position of j is independent of its value, we conclude that the probability of $j < d$ in base b^m is $1/m$ times the corresponding probability in base b. This is equivalent to the statement of Lemma 2 and completes the proof.

To deduce Benford's law from the lemmas, we proceed as follows. According to Lemma 2,

$$Pr(j < d)_b = mPr(j < d)_{b^m}. \tag{5}$$

The monotonicity statement and Lemma 1 yield the inequality

$$\frac{1}{n} = Pr(j < d)_{d^n} \ge Pr(j < d)_{b^m} \ge Pr(j < d)_{d^{n+1}} = \frac{1}{n+1} \tag{6}$$

whenever

$$d^n \le b^m \le d^{n+1}. \tag{7}$$

By the euclidean algorithm, (7) is always satisfied by some $n \ge 0$ for any given values of $b > d > 1$ and $m \ge 0$. Combining (5) and (6), we obtain

$$\frac{m}{n} \ge Pr(j < d)_b \ge \frac{m}{n+1}.$$

Now let $m \to \infty$ and choose n so as to maintain the validity of (7). Taking logarithms in (7), this implies that

$$\frac{m}{n+1} \le \log_b d \le \frac{m}{n}.$$

To show that $m/n \to \log_b d$ as $m \to \infty$, we simply note that

$$\frac{m}{n} - \frac{m}{n+1} = \frac{1}{n}\left(\frac{m}{n+1}\right) \le \frac{1}{n} \log_b d \to 0.$$

This establishes (2).

The proofs presented here rely heavily upon properties of the set of positive integers which are not shared by other Benford sequences. As such, it is worth commenting on the more general situation. By definition, any Benford sequence satisfies (2) and, as noted above, this implies the monotonicity statement. The lemmas are also valid although the proofs given above are not. To give a more interesting example, consider the geometric progression $\{a^k\}$ which constitutes a Benford sequence in base b if and only if $a \neq b^{p/q}$ (p, q integers). Setting $a = 3$ and $b' = 9$, we obtain a subset of the positive integers which is not a Benford sequence. Moreover, $Pr(j < 4)_9 = 1$ for the geometric progression $\{3^k\}$. Since $\{3^k\}$ is a Benford sequence in base $b = 8$, we may apply Lemma 1 with $a = 2$, $m = 2$, $n = 3$ to yield $Pr(j < 4)_8 = 2/3$. A comparison of the above probabilities for $b = 8$ and $b' = 9$ shows that the monotonicity statement is false for this example.

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A NEW TYPE MAGIC LATIN 3-CUBE OF ORDER TEN

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A Latin 3-cube of order n is an $n \times n \times n$ cube (n rows, n columns, and n files) in which the numbers $0, 1, 2, \ldots, n-1$ are entered so that each number occurs exactly once in each row, column, and file. A magic Latin 3-cube of order n is an arrangement of n^3 integers in three orthogonal Latin 3-cubes, each of order n (where every ordered triple 000, 001, ..., $n-1$, $n-1$, $n-1$ occurs) such that the sum of the entries in every row, every column, and every file, in each of the four major diagonals (diameters) and in each of the n^2 broken major diagonals is the same; namely, $\frac{1}{2}n(n^3+1)$. We shall list the cubes in terms of n squares of order n that form its different levels from the top square 0 down through (inclusively) square 1, square 2, ..., square $n-1$. We define a broken major diagonal as a path (route) which begins in square 0 and goes through the n different levels (square 0, square 1, ..., square $n-1$) of the cube and passes through precisely one cell in each of the n squares in such a way that no two cells the broken major diagonal traverses are ever in the same file.

The sum of the entries in the n cells that make up a broken major diagonal equals $\frac{1}{2}n(n^3+1)$. A complete system consists of n^2 broken major diagonals, where each broken major diagonal emanates from a cell in square 0, and thus the n^2 broken major diagonals traverse each of the n^3 cells of the cube in n^2 distinct routes. The cube is initially constructed as a Latin 3-cube in which the numbers are expressed in the scale of n ($0, 1, 2, \ldots, n-1$). However, after adding 1 throughout and converting the numbers to base 10, we have the n^3 numbers $1, 2, \ldots, n^3$ where the sum of the entries in every row, every column, and every file in each of the four major diagonals, and in each of the n^2 broken major diagonals is the same; namely, $\frac{1}{2}n(n^3+1)$.

In this paper, for the first time in mathematics, we construct a magic Latin 3-cube of order ten. In this case, the sum of the numbers in every row, every column, and every file in each of the four major diagonals, and in each of the 10^2 broken major diagonals is the same; namely, $\frac{1}{2}(10)(10^3 + 1) = 5005$.

In Chart 1 we list (by columns) the coordinates of the cells through which 10 broken major diagonals pass. It should be noted that the first digit of the coordinates denotes the row, the second digit the column, and at the right side of each row is the square number in which each cell is to be found. Each one of the 10 broken major diagonals is found under one of the 10 columns, that is, in Chart 1 we find listed by columns 10 broken major diagonals, where each column denotes one broken major diagonal. For example, under column 0, we find the coordinates 00, 99, 55, 66, 11, 88, 77, 22, 44, and 33. These cells determine one broken major diagonal. After adding 1 to each number found in the corresponding 10 cells in the 10 squares, we get

$$764 + 373 + 791 + 588 + 707 + 026 + 445 + 340 + 612 + 359 = \tfrac{1}{2}(10)(10^3 + 1) = 5005.$$

Now, in order to find the remaining 90 broken major diagonals that emanate from square 0, we must construct nine more charts to get Chart 1, Chart 2, ..., Chart X. We need only show (as an example) how to construct Chart 2 from Chart 1 and the Key Chart, since the remaining eight charts (Chart 2, ..., Chart X) are constructed in exactly the same way.

In the Key Chart under column I are the numbers in the same order that are found in Chart 1 under column 0.

In Chart 1, we define the rows as follows:

a (00) row =	00	16	29	35	42	53	64	71	87	98	square 0
a (99) row =	99	41	62	56	84	75	23	10	08	37	square 1
....											
a (33) row =	33	28	45	04	79	86	90	57	61	12	square 9

Thus, in the Key Chart we have under column I a (00) row, a (99) row, ..., and a (33) row, which is, of course, a restatement in a shorter form of the entire Chart 1.

Now, in the Key Chart, each number under column II which is identical to a number under column I (in the Key Chart) represents the identical row found in Chart 1. Therefore, Chart 2 is written as:

(11) row:	11	89	76	27	58	94	32	03	40	65	square 0
(33) row:	33	28	45	04	79	86	90	57	61	12	square 1
....											
(99) row:	99	41	62	56	84	75	23	10	08	37	square 9

Then the columns of Chart 2 give 10 more broken major diagonals.

We can find the remaining eight charts—Chart 3, Chart 4, ..., Chart X—in exactly the same way as Chart 2, using the Key Chart in conjunction with Chart 1. (The charts are presented on the following pages.)

It should be noted here that Chart 1 is constructed by superposing two orthogonal Latin squares of order ten. Now, since it is impossible to superpose two Latin squares of order n when $n = 2$ or 6, we may state that this type of magic Latin 3-cube is impossible for order 2 and for order 6.

In the near future, we shall present a more comprehensive general paper in which we consider the general order $4m + 2$ and the powers of prime numbers.

CHART 1

0	1	2	3	4	5	6	7	8	9	
00	16	29	35	42	53	64	71	87	98	square 0
99	41	62	56	84	75	23	10	08	37	square 1
55	97	38	19	60	02	81	24	73	46	square 2
66	50	93	82	07	21	15	48	39	74	square 3
11	89	76	27	58	94	32	03	40	65	square 4
88	72	01	43	95	67	59	36	14	20	square 5
77	34	80	68	13	49	06	92	25	51	square 6
22	05	54	70	31	18	47	69	96	83	square 7
44	63	17	91	26	30	78	85	52	09	square 8
33	28	45	04	79	86	90	57	61	12	square 9

KEY CHART FOR 100 BROKEN MAJOR DIAGONALS

I	II	III	IV	V	VI	VII	VIII	IX	X	
00	11	22	33	44	55	66	77	88	99	square 0
99	33	88	44	22	00	77	55	11	66	square 1
55	22	33	00	77	11	88	99	66	44	square 2
66	44	77	55	00	88	99	11	33	22	square 3
11	00	99	77	66	44	33	22	55	88	square 4
88	55	11	66	99	33	22	44	00	77	square 5
77	88	66	99	11	22	55	00	44	33	square 6
22	66	44	88	55	99	00	33	77	11	square 7
44	77	00	22	33	66	11	88	99	55	square 8
33	99	55	11	88	77	44	66	22	00	square 9

MAGIC LATIN 3-CUBE OF ORDER TEN

Square Number 0

	0	1	2	3	4	5	6	7	8	9
0	763	886	540	979	015	428	601	354	232	197
1	279	963	097	654	832	301	728	186	440	515
2	897	340	463	201	579	632	154	915	028	786
3	140	454	901	063	628	715	879	297	586	332
4	932	228	754	815	163	086	597	401	379	640
5	328	697	132	740	486	563	215	079	954	801
6	554	032	286	128	701	997	363	840	615	479
7	415	779	828	532	397	240	986	663	101	054
8	686	501	315	497	254	179	040	732	863	928
9	001	115	679	386	940	854	432	528	797	263

Square Number 1

	0	1	2	3	4	5	6	7	8	9
0	472	138	264	085	793	616	947	821	359	500
1	385	072	700	921	159	847	416	538	664	293
2	100	864	672	347	285	959	521	093	716	438
3	564	621	047	772	916	493	185	300	238	859
4	059	316	421	193	572	738	200	647	885	964
5	816	900	559	464	638	272	393	785	021	147
6	221	759	338	516	447	000	872	164	993	685
7	693	485	116	259	800	364	038	972	547	721
8	938	247	893	600	321	585	764	459	172	016
9	747	593	985	838	064	121	659	216	400	372

Square Number 2

	0	1	2	3	4	5	6	7	8	9
0	190	924	771	313	808	565	289	637	446	052
1	413	390	852	237	946	689	165	024	571	708
2	952	671	590	489	713	246	037	308	865	124
3	071	537	389	890	265	108	913	452	724	646
4	346	465	137	908	090	824	752	589	613	271
5	665	252	046	171	524	790	408	813	337	989
6	737	846	424	065	189	352	690	971	208	513
7	508	113	965	746	652	471	324	290	089	837
8	224	789	608	552	437	013	781	146	990	365
9	889	008	213	624	371	937	546	765	152	490

Square Number 3

	0	1	2	3	4	5	6	7	8	9
0	987	250	823	431	649	002	794	575	118	366
1	131	487	666	775	218	594	902	350	023	849
2	266	523	087	194	831	718	375	449	602	950
3	323	075	494	687	702	949	231	166	850	518
4	418	102	975	249	387	650	866	094	531	723
5	502	766	318	923	050	887	149	631	475	294
6	875	618	150	302	994	466	587	223	749	031
7	049	931	202	818	566	123	450	787	394	675
8	750	894	549	066	175	331	623	918	287	402
9	694	349	731	550	423	275	018	802	966	187

Square Number 4

	0	1	2	3	4	5	6	7	8	9
0	606	541	355	727	434	999	010	162	883	278
1	827	706	478	062	583	110	699	241	955	334
2	578	115	906	810	327	083	262	734	499	641
3	255	962	710	406	099	634	527	878	341	183
4	783	899	662	534	206	441	378	910	127	055
5	199	078	283	655	941	306	834	427	762	510
6	362	483	841	299	610	778	106	555	034	927
7	934	627	599	383	178	855	741	006	210	462
8	041	310	134	978	862	227	455	683	506	799
9	410	234	027	141	755	562	983	399	678	806

Square Number 5

	0	1	2	3	4	5	6	7	8	9
0	525	069	488	842	157	230	376	903	691	714
1	642	825	114	303	091	976	530	769	288	457
2	014	988	225	676	442	391	703	857	130	569
3	788	203	786	125	330	557	042	614	469	991
4	891	630	503	057	725	169	414	276	942	388
5	930	314	791	588	269	425	657	142	803	076
6	403	191	669	730	576	814	925	088	357	242
7	257	542	030	491	914	688	869	325	776	103
8	369	476	957	214	603	742	188	591	025	830
9	176	757	342	969	888	003	291	430	514	625

Square Number 6

	0	1	2	3	4	5	6	7	8	9
0	044	312	136	698	961	777	453	280	505	829
1	598	644	929	480	305	253	077	812	736	161
2	329	236	744	553	198	405	880	661	977	012
3	836	780	653	944	477	061	398	529	112	205
4	605	577	080	361	844	912	129	753	298	436
5	277	429	805	036	712	144	561	998	680	353
6	180	905	512	877	053	629	244	336	461	798
7	761	098	377	105	229	536	612	444	853	980
8	412	153	261	729	580	989	936	005	344	677
9	953	861	498	212	636	380	705	177	029	544

Square Number 7

	0	1	2	3	4	5	6	7	8	9
0	239	773	617	104	582	351	868	096	920	445
1	904	139	545	986	720	068	251	473	317	682
2	745	017	339	968	604	820	496	182	551	273
3	417	396	168	539	851	282	704	945	673	020
4	120	951	296	782	439	573	645	368	004	817
5	051	845	420	217	373	639	982	504	196	768
6	696	520	973	451	268	145	039	717	882	304
7	382	204	751	620	045	917	173	839	468	596
8	873	668	082	345	996	404	517	220	739	151
9	568	482	804	073	117	796	320	651	245	939

Square Number 8

	0	1	2	3	4	5	6	7	8	9
0	311	495	909	556	270	884	122	748	067	633
1	056	511	233	148	467	722	384	695	809	970
2	433	709	811	022	956	167	648	570	284	395
3	609	848	522	211	184	370	456	033	995	767
4	567	084	348	470	611	295	933	822	756	109
5	784	133	667	309	895	911	070	256	548	422
6	948	267	095	684	322	533	711	409	170	856
7	870	356	484	967	733	009	595	111	622	248
8	195	922	770	833	048	656	209	367	411	584
9	222	670	156	795	509	448	867	984	333	011

Square Number 9

	0	1	2	3	4	5	6	7	8	9
0	858	607	092	260	326	143	535	419	774	981
1	760	258	381	519	674	435	843	907	192	026
2	681	492	158	735	060	574	919	226	343	807
3	992	119	235	358	543	826	660	781	007	474
4	274	743	819	626	958	307	081	135	460	592
5	443	581	974	892	107	058	726	360	219	635
6	019	374	707	943	835	281	458	692	526	160
7	126	860	643	074	481	792	207	558	935	319
8	507	035	426	181	719	960	392	874	658	243
9	335	926	560	407	292	619	174	043	881	758

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COMPLEX FIBONACCI NUMBERS

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1. INTRODUCTION

In this note, a new approach is taken toward the significant extension of Fibonacci numbers into the complex plane. Two differing methods for defining such numbers have been considered previously by Horadam [4] and Berzsenyi [2]. It will be seen that the new numbers include Horadam's as a special case, and that they have a symmetry condition which is not satisfied by the numbers considered by Berzsenyi.

The latter defined a set of complex numbers at the Gaussian integers, such that the characteristic Fibonacci recurrence relation is satisfied at any horizontal triple of adjacent points. The numbers to be defined here will have the symmetric condition that the Fibonacci recurrence occurs on any horizontal or vertical triple of adjacent points.

Certain recurrence equations satisfied by the new numbers are outlined, and using them, some interesting new Fibonacci identities are readily obtained. Finally, it is shown that the numbers generalize in a natural manner to higher dimensions.

2. THE COMPLEX FIBONACCI NUMBERS

The numbers, to be denoted by $G(n, m)$, will be defined at the set of Gaussian integers $(n, m) = n + im$, where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. By direct analogy with the classical Fibonacci recurrence

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1, \tag{2.1}$$

the numbers $G(n, m)$ will be required to satisfy the following two-dimensional recurrence

$$G(n + 2, m) = G(n + 1, m) + G(n, m), \tag{2.2}$$

$$G(n, m + 2) = G(n, m + 1) + G(n, m), \tag{2.3}$$

where $$G(0, 0) = 0,\ G(1, 0) = 1,\ G(0, 1) = i,\ G(1, 1) = 1 + i. \tag{2.4}$$

The conditions (2.2), (2.3), and (2.4) are sufficient to specify the unique value of $G(n, m)$ at each point (n, m) in the plane, and the actual value of $G(n, m)$ will now be obtained.

From (2.2), the case $m = 0$ gives

$$G(n + 2, 0) = G(n + 1, 0) + G(n, 0);\ G(0, 0) = 0,\ G(1, 0) = 1$$

and hence that

$$G(n,\ 0) = F_n, \tag{2.5}$$

the classical Fibonacci sequence.

The case $m = 1$ gives the recurrence

$$G(n+2,\ 1) = G(n+1,\ 1) + G(n,\ 1);\ G(0,\ 1) = i,\ G(1,\ 1) = 1 + i,$$

which is an example of the well-known generalized Fibonacci sequence considered by Horadam [3] that satisfies

$$G(n,\ 1) = F_{n-1}G(0,\ 1) + F_n G(1,\ 1).$$

By substitution

$$G(n,\ 1) = iF_{n-1} + (1 + i)F_n = F_n + i(F_{n-1} + F_n),$$

and so by (2.1)

$$G(n,\ 1) = F_n + iF_{n+1}. \tag{2.6}$$

Recurrence (2.3) together with initial values (2.5) and (2.6) specify another generalized Fibonacci sequence, so that

$$\begin{aligned} G(n,\ m) &= F_{m-1}G(n,\ 0) + F_m G(n,\ 1) \\ &= F_{m-1}F_n + F_m(F_n + iF_{n+1}) = (F_{m+1} + F_m)F_n + iF_m F_{n+1}, \end{aligned}$$

and so by (2.1) the complex Fibonacci numbers $G(n,\ m)$ are given by

$$G(n,\ m) = F_n F_{m-1} + iF_{n+1}F_m. \tag{2.7}$$

It can be noted at once that along the horizontal axis $G(n,\ 0) = F_n$, and that on the vertical axis $G(0,\ m) = iF_m$. Also, the special case $n = 1$ corresponds to the complex numbers considered by Horadam [4].

3. RECURRENCE EQUATIONS AND IDENTITIES

Combining (2.2) and (2.3), it follows that

$$G(n+2,\ m+2) = G(n+1,\ m+1) + G(n+1,\ m) + G(n,\ m+1) + G(n,\ m), \tag{3.1}$$

which is an interesting two-dimensional version of the Fibonacci recurrence relation and gives the growth-characteristic of the numbers in a diagonal direction: any complex Fibonacci number $G(n,\ m)$ is the sum of the four previous numbers at the vertices of a square diagonally below and to the left of that number's position on the Gaussian lattice.

From (2.7) and (2.1), it follows that

$$\begin{aligned} G(n+1,\ m+1) &= F_{n+1}F_{m+2} + iF_{n+2}F_{m+1} \\ &= F_{n+1}(F_{m+1} + F_m) + i(F_{n+1} + F_n)F_{m+1} \\ &= F_{n+1}F_{m+1}(1 + i) + F_m F_{n+1} + iF_{m+1}F_n, \end{aligned}$$

and so by (2.7) again, the following recurrence equation is obtained:

$$G(n+1,\ m+1) = (1 + i)F_{n+1}F_{m+1} + G(n,\ m). \tag{3.2}$$

By repetition of equation (3.2), it follows that

$$G(n+2,\ m+2) = (1 + i)(F_{n+2}F_{m+2} + F_{n+1}F_{m+1}) + G(n,\ m), \tag{3.3}$$

and by repeated application of (3.2) and (3.3) the following even and odd cases result:

$$G(n+2k,\ m+2k) = (1 + i)\sum_{j=1}^{2k} F_{n+j}F_{m+j} + G(n,\ m), \tag{3.4}$$

$$G(n + 2k + 1,\ m + 2k + 1) = (1 + i)\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} + G(m,\ n). \tag{3.5}$$

From (3.4),

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = (1 + i)^{-1}[G(n + 2k,\ m + 2k) - G(n,\ m)],$$

and so by (2.7),

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = \frac{1}{2}(1 - i)[F_{n+2k}F_{m+2k+1} - F_nF_{m+1} + iF_{n+2k+1}F_{m+2k} - iF_{n+1}F_m],$$

and, equating real and imaginary parts

$$F_{n+2k}F_{m+2k+1} - F_{n+2k+1}F_{m+2k} + F_{n+1}F_m - F_nF_{m+1} = 0, \tag{3.6}$$

and

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = \frac{1}{2}[F_{n+2k}F_{m+2k+1} - F_nF_{m+1} + F_{n+2k+1}F_{m+2k} - F_{n+1}F_m]. \tag{3.7}$$

Substitution for $F_{n+2k}F_{m+2k+1}$ from (3.6) into (3.7) gives

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = F_{n+2k+1}F_{m+2k} - F_{n+1}F_m. \tag{3.8}$$

Similarly, for the odd case,

$$\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} = F_{n+2k+2}F_{m+2k+1} - F_nF_{m+1}. \tag{3.9}$$

Identities (3.8) and (3.9) unify and generalize certain identities of Berzsenyi [1] and provide interesting examples as special cases. For example, $n = m = 0$ yields the well-known identity:

$$F_1^2 + F_2^2 + \cdots + F_N^2 = F_NF_{N+1}.$$

From (3.8), the case $m = 0$, $n = 1$ gives

$$F_1F_2 + F_2F_3 + \cdots + F_{2k}F_{2k+1} = F_{2k}F_{2k+2},$$

and from (3.9), $n = 0$, $m = 1$ gives the identity

$$F_1F_2 + F_2F_3 + \cdots + F_{2k+1}F_{2k+2} = F_{2k+2}^2.$$

Many other interesting identities can be specified in this way by suitable choice of parameters. For example, equation (3.8) with $m = 0$, $n = 2$ gives

$$F_1F_3 + F_2F_4 + \cdots + F_{2k}F_{2k+2} = F_{2k}F_{2k+3},$$

and for $m = 2$, $n = 0$, equation (3.9) gives

$$F_1F_3 + F_2F_4 + \cdots + F_{2k+1}F_{2k+3} = F_{2k+2}F_{2k+3}.$$

Identity (3.6) has the following counterpart for the case $2k + 1$:

$$F_{n+2k+1}F_{m+2k+2} - F_{n+2k+2}F_{m+2k+1} = F_mF_{n+1} - F_{m+1}F_n, \tag{3.10}$$

and together (3.6) and (3.10) constitute a generalization of some well-known classical identities. For example, if $n = 1$, $m = 0$, they give

$$F_{N-1}F_{N+1} - F_N^2 = (-1)^N,\ N \geq 1.$$

As another example, equations (3.6) and (3.10) with $n = 1$ and $m = -2$ yield the identity

$$F_{N-1}F_{N+1} - F_{N-2}F_{N+2} = 2(-1)^N.$$

4. HIGHER DIMENSIONS

The above development of complex Fibonacci numbers naturally extends to higher dimensions and, in order to illustrate, the three-dimensional case will be outlined.

The number $G(\ell, m, n)$ will be required to satisfy

$$G(\ell + 2, m, n) = G(\ell + 1, m, n) + G(\ell, m, n), \tag{4.1}$$

$$G(\ell, m + 2, n) = G(\ell, m + 1, n) + G(\ell, m, n), \tag{4.2}$$

and

$$G(\ell, m, n + 2) = G(\ell, m, n + 1) + G(\ell, m, n), \tag{4.3}$$

where

$$G(0, 0, 0) = (0, 0, 0);\ G(1, 0, 0) = (1, 0, 0);\ G(0, 1, 0) = (0, 1, 0);$$

$$G(0, 0, 1) = (0, 0, 1);\ G(1, 1, 0) = (1, 1, 0);\ G(1, 0, 1) = (1, 0, 1);$$

$$G(1, 1, 1) = (1, 1, 1).$$

Thus, G has a Fibonacci recurrence in each of the three coordinate directions.

Each of (4.1), (4.2), and (4.3) is a generalized Fibonacci sequence; thus, from (4.1),

$$G(\ell, 0, 0) = F_{\ell-1}(0, 0, 0) + F_\ell(1, 0, 0) \tag{4.4}$$

and from (4.1) again,

$$G(\ell, 1, 0) = F_{\ell-1}(0, 1, 0) + F_\ell(1, 1, 0). \tag{4.5}$$

From (4.2), it follows that

$$G(\ell, m, 0) = F_{m-1}G(\ell, 0, 0) + F_m G(\ell, 1, 0). \tag{4.6}$$

From (4.1) again

$$G(\ell, 0, 1) = F_{\ell-1}(0, 0, 1) + F_\ell(1, 0, 1), \tag{4.7}$$

and

$$G(\ell, 1, 1) = F_{\ell-1}(0, 1, 1) + F_\ell(1, 1, 1). \tag{4.8}$$

Equation (4.2) then gives

$$G(\ell, m, 1) = F_{m-1}G(\ell, 0, 1) + F_m G(\ell, 1, 1), \tag{4.9}$$

and from (4.3),

$$G(\ell, m, n) = F_{n-1}G(\ell, m, 0) + F_n G(\ell, m, 1). \tag{4.10}$$

Combining equations (4.4)-(4.10), and using the classical Fibonacci recurrence to reduce the expressions obtained, one finally gets

$$G(\ell, m, n) = (F_\ell F_{m+1} F_{n+1},\ F_{\ell+1} F_m F_{n+1},\ F_{\ell+1} F_{m+1} F_n),$$

which is the three-dimensional version of Fibonacci numbers. This form readily generalizes to higher dimensions in the obvious fashion.

It is interesting to note that if (4.1), (4.2), and (4.3) are combined directly, then it follows that the value of $G(\ell + 2, m + 2, n + 2)$ is given by the sum of the values of G at the eight vertices of the cube diagonally below that point—a generalization of (3.1).

The structure provided by the complex Fibonacci numbers was seen in Section 3 to result in some interesting classical identities involving products. It is conjectured that the above three-dimensional numbers may lead to identities involving triple products.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-442 *Proposed by P. L. Mana, Albuquerque, NM*

The identity

$$2\cos^2\theta = 1 + \cos(2\theta)$$

leads to the identity

$$8\cos^4\theta = 3 + 4\cos(2\theta) + \cos(4\theta).$$

Are there corresponding identities on Lucas numbers?

B-443 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For all integers n and w with w odd, establish the following

$$L_{n+2w}L_{n+w} - 2L_wL_{n+w}L_{n-w} - L_{n-w}L_{n-2w} = L_n^2(L_{3w} - 2L_w).$$

B-444 *Proposed by Herta T. Freitag, Roanoke, VA*

In base 10, the palindromes (that is, numbers reading the same forward or backward) 12321 and 112232211 are converted into new palindromes using

$$99[10^3 + 9(12321)] = 11077011,$$
$$99[10^5 + 9(112232211)] = 100008800001.$$

Generalize on these to obtain a method or methods for converting certain palindromes in a general base b to other palindromes in base b.

B-445 *Proposed by Wray G. Brady, Slippery Rock State College, PA*

Show that

$$5F_{2n+2}^2 + 2L_{2n}^2 + 5F_{2n-2}^2 = L_{2n+2}^2 + 10F_{2n}^2 + L_{2n-2}^2$$

and find a simpler form for these equal expressions.

B-446 *Proposed by Jerry M. Metzger, University of N. Dakota, Grand Forks, ND*

It is familiar that a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3. The same is true for 9. For 27, this is false since, for example, 27 divides $1 + 8 + 9 + 9$ but does not divide 1899. However, $27|1998$.

Prove that 27 divides the sum of the digits of n if and only if 27 divides one of the integers formed by permuting the digits of n.

B-447 *Based on the previous proposal by Jerry M. Metzger.*

Is there an analogue of B-446 in base 5?

SOLUTIONS

Consequence of the Euler-Fermat Theorem

B-418 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $n^{15} - n^3$ is an integral multiple of $2^{15} - 2^3$ for all integers n.

Solution by Lawrence Somer, Washington, D.C.

The assertion is correct. First, note that

$$n^{15} - n^3 = n^3(n^{12} - 1).$$

Further,

$$2^{15} - 2^3 = 2^3(2^6 - 1)(2^6 + 1) = 8(9)(7)(5)(13).$$

By Euler's generalization of Fermat's theorem,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

if $(a, n) = 1$, where ϕ is Euler's totient function. It follows that $a^{k\phi(d)} \equiv 1 \pmod{d}$ for integral k. Now

$$\phi(8) = 4,\ \phi(9) = 6,\ \phi(7) = 6,\ \phi(5) = 4, \text{ and } \phi(13) = 12.$$

Thus, it follows in each instance that if $(n, d) = 1$, where $d = 8, 9, 7, 5$, or 13, then $n^{12} - 1 \equiv 0 \pmod{d}$, since $\phi(d)|12$ for each d. Further, if $(n, d) \neq 1$ for $d = 8, 9, 7, 5$, or 13, then $d|n^3$, since $d|p^3$ for some prime p. Since $(8, 9, 7, 5, 13) = 1$, it now follows that

$$n^3(n^{12} - 1) \equiv 0 \pmod{8\cdot 9\cdot 7\cdot 5\cdot 13}.$$

Thus, $2^{15} - 2^3$ divides $n^{15} - n^3$.

Also solved by Paul S. Bruckman, Duane A. Cooper, M. J. DeLeon, Robert M. Giuli, Bob Prielipp, C. B. Shields, Sahib Singh, Gregory Wulczyn, and the proposer.

NOTE: DeLeon generalized to show that for $k \in \{2, 3, 4\}$, $2^k(2^{12} - 1)$ divides $n^k(n^{12} - 1)$ for all positive integers n.

Symmetric Congruence

B-419 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For i in $\{1, 2, 3, 4\}$, establish a congruence

$$F_n L_{5k+i} \equiv a_i n L_n F_{5k+i} \pmod{5}$$

with each a_i in $\{1, 2, 3, 4\}$.

Solution by Sahib Singh, Clarion State College, Clarion, PA

We know that $nL_n \equiv F_n \pmod{5}$. (See the solution to Problem B-368 in the December 1978 issue.) Thus

$$F_n = nL_n \pmod{5}, \tag{1}$$

and

$$(5k + i)L_{5k+i} \equiv F_{5k+i} \pmod{5} \text{ or } L_{5k+i} \equiv (i)^{-1}F_{5k+i} \pmod{5}. \tag{2}$$

Multiply (1) and (2) to get

$$F_nL_{5k+i} \equiv (i)^{-1}nL_nF_{5k+i} \pmod 5.$$

Thus, $a_i = (i)^{-1}$ where $(i)^{-1}$ is the multiplicative inverse of i in Z_5. Therefore, $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, and $a_4 = 4$.

Also solved by Paul S. Bruckman, M. J. DeLeon, Bob Prielipp, and the proposer.

Finding Fibonacci Factors

B-420 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let

$$g(n, k) = F^4_{n+10k} + F_n^4 - (L_{4k} + 1)(F^4_{n+8k} + F^4_{n+2k}) + L_{4k}(F^4_{n+6k} + F^4_{n+4k}).$$

Can one express $g(n, k)$ in the form $L_rF_sF_tF_uF_v$ with each of r, s, t, u, and v linear in n and k?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The answer to the question stated above is "yes."

On pp. 376-377 of the December 1979 issue (see solution to Problem H-279) Paul Bruckman established that

$$F^4_{n+6k} - (L_{4k} + 1)(F^4_{n+4k} - F^4_{n+2k}) - F_n^4 = F_{2k}F_{4k}F_{6k}F_{4n+12k}.$$

Substituting $n + 4k$ for n yields

$$F^4_{n+10k} - (L_{4k} + 1)(F^4_{n+8k} - F^4_{n+6k}) - F^4_{n+4k} = F_{2k}F_{4k}F_{6k}F_{4n+28k}.$$

Thus, $g(n, k) =$

$$\begin{aligned}
&[F^4_{n+10k} - (L_{4k} + 1)(F^4_{n+8k} - F^4_{n+6k}) - F^4_{n+4k}]\\
&\quad - [F^4_{n+6k} - (L_{4k} + 1)(F^4_{n+4k} - F^4_{n+2k}) - F^4_n]\\
&= F_{2k}F_{4k}F_{6k}F_{4n+28k} - F_{2k}F_{4k}F_{6k}F_{4n+12k}\\
&= F_{2k}F_{4k}F_{6k}[F_{(4n+20k)+8k} - F_{(4n+20k)-8k}]\\
&= F_{2k}F_{4k}F_{6k}F_{8k}L_{4n+20k},
\end{aligned}$$

because $F_{s+t} - F_{s-t} = F_tL_s$, t even (see p. 115 of the April 1975 issue of this journal).

Also solved by Paul S. Bruckman and the proposer.

Unique Representation

B-421 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $\{u_n\}$ be defined by the recursion $u_{n+3} = u_{n+2} + u_n$ and the initial conditions $u_1 = 1$, $u_2 = 2$, and $u_3 = 3$. Prove that every positive integer N has a unique representation

$$N = \sum_{i=1}^{n} c_iu_i,$$

with $c_n = 1$, each $c_i \in \{0, 1\}$, $c_ic_{i+1} = 0 = c_ic_{i+2}$ if $1 \le i \le n - 2$.

Solution by Paul S. Bruckman, Concord, CA

We first observe that the condition "$c_ic_{i+1} = 0 = c_ic_{i+2}$ for $1 \le i \le n - 2$" should be replaced by

$$c_ic_{i+1} = 0 \text{ for } 1 \le i \le n - 1 \quad\text{and}\quad c_ic_{i+2} = 0 \text{ for } 1 \le i \le n - 2. \tag{1}$$

Let $U = (u_n)_{n=1}^{\infty}$. We call a representation $(N)_U \equiv c_n c_{n-1} \ldots c_1$ of N a *U-nary* representation of N if

$$N = \sum_{i=1}^{n} c_i u_i,$$

with the c_i's satisfying the given conditions, as modified by (1). It is not assumed a priori that such a representation is necessarily unique. In any U-nary representation of N, any two consecutive "1's" appearing must be separated by at least two zeros. Without the modification given in (1), the representations are certainly not unique; examples:

$$(3)_U = 100 = 11 \quad \text{and} \quad (11)_U = 11001 = 100010,$$

ignoring (1) and substituting the given condition of the published problem.

We require a pair of preliminary lemmas.

Lemma 1:

$$\sum_{k=0}^{m} u_{n-3k-1} = u_{\ } - 1, \ (n = 2, 3, 4, \ldots), \text{ where } m = \left[\frac{n-2}{3}\right]. \tag{2}$$

Proof: Using the recursion satisfied by the u_n's,

$$\sum_{k=0}^{m} u_{n-3k-1} = \sum_{k=0}^{m} (u_{n-3k} - u_{n-3k-3}) = \sum_{k=0}^{m} u_{n-3k} - \sum_{k=1}^{m+1} u_{n-3k} = u_n - u_{n-3m-3}.$$

Note that $n - 3m - 3 = -1$, 0, or 1 for all n. We may extend the sequence U to nonpositive indices k of u_k by using the initial values and the recursion satisfied by the elements of U; we then obtain:

$$u_{-1} = u_0 = u_1 = 1.$$

This establishes the lemma.

Lemma 2: If $(u_n)_U = c_m c_{m-1} \ldots c_1$, then $m = n$ and $c_i = \delta_{ni}$ (Kronecker delta).

Proof: By definition,

$$c_m = 1 \quad \text{and} \quad u_n = \sum_{i=1}^{m} c_i u_i.$$

Since $u_n \geq u_m$, thus $m \leq n$. On the other hand, since any two consecutive "1's" in a U-nary representation are separated by at least two zeros, it follows that

$$u_n \leq \sum_{i=0}^{h} u_{m-3k}, \text{ where } h = \left[\frac{m-1}{3}\right].$$

Substituting $n = m + 1$ in Lemma 1, it follows that $u_n \leq u_{m+1} - 1$, or $u_n < u_{m+1}$. Since $u_m \leq u_n < u_{m+1}$, it follows that $m = n$. Hence $c_n = 1$, from which it follows that the remaining c_i's vanish. Q.E.D.

Now, define S to be the set of all positive integers N that have a *unique* U-nary representation. We will find it convenient to extend S to include the number zero. Note that zero certainly satisfies all the conditions of "U-naryness," *except* for $c_n = 1$; for this exceptional element of S only, we waive this condition. Note that $u_k = k \in S$, $k = 1, 2, 3, 4$.

We seek to establish that S consists of all nonnegative integers, and our proof is by induction on k. Assume that $K \in S$, $0 \leq K < u_k$, where $k \geq 4$. In particular, $M \in S$, where $0 \leq M < u_{k-2}$. Then $(M)_U = c_r c_{r-1} \ldots c_1$, for some r, where $c_r = 1$. Since $M < u_{k-2}$, thus $r \leq k - 3$; otherwise, $r \geq k - 2$, which implies $M \geq u_{k-2}$, a contradiction. Let

$$N = M + u_k. \tag{3}$$

Then

$$N = \sum_{i=1}^{k} c_i u_i, \text{ with } c_k = 1,\ c_i = 0, \text{ if } r < i < k.$$

Since $r \leq k - 3$, we see that the foregoing expression yields a U-nary representation of N, namely $(N)_U = c_k c_{k-1} \ldots c_1$, though not necessarily unique. Suppose that $(N)_U = d_t d_{t-1} \ldots d_1$ is another U-nary representation of $N = M + u_k$. Then (since $M \in S$) $d_i = c_i$, $1 \leq i \leq r$. Moreover, $u_k = N - M$ has a unique U-nary representation, by Lemma 2; hence, $t = k$, which implies that $N \in S$.

Since $0 \leq M < u_{k-2}$, thus $u_k \leq N < u_{k-2} + u_k = u_{k+1}$. The inductive step is:

$$S \supset \{0, 1, 2, \ldots, u_k - 1\} \Rightarrow S \supset \{0, 1, 2, \ldots, u_{k+1} - 1\}.$$

By induction, S consists of all nonnegative integers. Q.E.D.

Also solved by Sahib Singh and the proposer.

Lexicographic Ordering of Coefficients

B-422 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

With representations as in B-421, let

$$N = \sum_{i=1}^{n} c_i u_i,\quad N + 1 = \sum_{i=1}^{m} d_i u_i.$$

Show that $m \geq n$ and that if $m = n$ then $d_k > c_k$ for the largest k with $c_k \neq d_k$.

Solution by Paul S. Bruckman, Concord, CA

We refer to the notation and solution of B-421 above. Given

$$(N)_U = c_n c_{n-1} \ldots c_1 \quad\text{and}\quad (N + 1)_U = d_m d_{m-1} \ldots d_1,$$

which we now know are the unique U-nary representations of N and $N + 1$, respectively.

Since $u_n \leq N < u_{n+1}$ and $u_m \leq N + 1 < u_{m+1}$, thus $u_m - u_{n+1} < 1 < u_{m+1} - u_n$. Now $u_{m+1} > u_n + 1 > u_n \Rightarrow m + 1 > n$, since U is an increasing sequence. On the other hand, $u_m < u_{n+1} + 1 \leq u_{n+2} \Rightarrow m < n + 2$. Hence,

$$m = n \quad\text{or}\quad m = n + 1. \tag{1}$$

Note that (1) is somewhat stronger than the desired result: $m \geq n$.

Now, suppose $m = n$, and let k be the largest integer i such that $c_i \neq d_i$. Then $c_i = d_i$, $k < i \leq n$. Hence,

$$\sum_{i=k+1}^{n} c_i u_i = \sum_{i=k+1}^{n} d_i u_i.$$

This, in turn, implies

$$N - \sum_{i=k+1}^{n} c_i u_i = N + 1 - 1 - \sum_{i=k+1}^{n} d_i u_i,$$

or

$$1 + \sum_{i=1}^{k} c_i u_i = \sum_{i=1}^{k} d_i u_i.$$

Suppose $c_k = 1$, $d_k = 0$. Then the left member of (2) is $\geq 1 + u_k$. On the other hand, the right member of (2) is

$$\leq \sum_{i=0}^{p} u_{k-1-3i} = u_k - 1, \text{ where } p = \left[\frac{k-2}{3}\right],$$

using the properties of the U-nary representation and Lemma 1 of the solution to B-421. This contradiction establishes the only remaining possibility, i.e., $c_k = 0$, $d_k = 1$. This establishes the desired result.

Also solved by Sahib Singh and the proposer.

Telescoping Infinite Product

B-423 *Proposed by Jeffery Shallit, Palo Alto, CA*

Here let F_n be denoted by $F(n)$. Evaluate the infinite product

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{13}\right)\left(1 + \frac{1}{610}\right) \cdots = \prod_{n=1}^{\infty}\left[1 + \frac{1}{F(2^{n+1} - 1)}\right].$$

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Let L_n also be written as $L(n)$ and $A_n = 1 + [1/F(2^{n+1} - 1)]$. It is easily seen (for example, from the Binet formulas) that

$$L(2)L(4)L(8) \ldots L(2^n) = F(2^{n+1}) \quad \text{and} \quad 1 + F(2^{n+1} - 1) = F(2^n - 1)L(2^n).$$

Hence, $A_n = F(2^n - 1)L(2^n)/F(2^{n+1} - 1)$ and

$$\prod_{i=1}^{\infty} A_n = \lim_{n\to\infty} \frac{F(1)F(3)F(7)F(15) \cdots F(2^n - 1)L(2)L(4)L(8) \cdots L(2^n)}{F(3)F(7)F(15) \cdots F(2^{n+1} - 1)}$$

$$= \lim_{n\to\infty} \frac{F(2^{n+1})}{F(2^{n+1} - 1)},$$

and the desired limit is $\alpha = (1 + \sqrt{5})/2$.

Also solved by Paul S. Bruckman, Bob Prielipp, and the proposer.

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(Continued from page 6)

Hence

$$u_{n-1} = x_1 u_n - Dy_1 v_n = (x_1 - 1)u_n - Dy_1 v_n + u_n \geq u_n.$$

Thus $n = 0$.

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ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-322 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $f_n^{(k)}$ by

$$f_0^{(k)} = 0, \; f_1^{(k)} = 1, \text{ and}$$

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k+1. \end{cases}$$

Show the following:

(a) $f_n^{(k)} = 2^{n-2}$ if $2 \leq n \leq k+1$;

(b) $f_n^{(k)} < 2^{n-2}$ if $n \geq k+2$;

(c) $\displaystyle\sum_{n=1}^{\infty} (f^{(k)}/2^n) = 2^{k-1}$.

H-323 *Proposed by Paul Bruckman, Concord, CA*

Let $(x_n)_0^\infty$ and $(y_n)_0^\infty$ be two sequences satisfying the common recurrence

$$p(E)z_n = 0, \tag{1}$$

where p is a monic polynomial of degree 2 and $E = 1 + \Delta$ is the unit right-shift operator of finite difference theory. Show that

$$x_n y_{n+1} - x_{n+1} y_n = (p(0))^n (x_0 y_1 - x_1 y_0), \; n = 0, 1, 2, \ldots . \tag{2}$$

Generalize to the case where p is of degree $e \geq 1$.

H-324 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Establish the identity

$$\begin{aligned} A &\equiv F_{14r}(F_{n+14r}^7 + F_n^7) - 7F_{10r}(F_{n+4r}^6 F_n + F_{n+4r} F_n^6) \\ &\quad + 21F_{6r}(F_{n+4r}^5 F_n^2 + F_{n+4r}^2 F_n^5) - 35F_{2r}(F_{n+4r}^4 F_n^3 + F_{n+4r}^3 F_n^4) \\ &= F_{4r}^7 F_{7n+14}. \end{aligned}$$

H-325 *Proposed by Leonard Carlitz, Duke University, Durham, NC*

For arbitrary a, b put

$$S_m(a, b) = \sum_{j+k=m} \binom{a}{j} \binom{b+k-1}{k} \qquad (m = 0, 1, 2, \ldots).$$

Show that

$$\sum_{m+n=p} S_m(a,\ b)S_n(c,\ d) = S_p(a + c,\ b + d) \tag{1}$$

$$\sum_{m+n=p} (-1)^n S_m(a,\ b)S_n(c,\ d) = S_p(a - d,\ b - c). \tag{2}$$

H-326 *Proposed by Larry Taylor, Briarwood, NY*

(A) If $p \equiv 7$ or $31 \pmod{36}$ is prime and $(p - 1)/6$ is also prime, prove that $32(1 \pm \sqrt{-3})$ is a primitive root of p.

(B) If $p \equiv 13$ or $61 \pmod{72}$ is prime and $(p - 1)/12$ is also prime, prove that $32(\sqrt{-1} \pm \sqrt{3})$ is a primitive root of p.

For example:

$$11 \equiv \sqrt{-3} \pmod{31},\ 12 \text{ and } 21 \text{ are primitive roots of } 31;$$

$$11 \equiv \sqrt{-1} \pmod{61},\ 8 \equiv \sqrt{3} \pmod{61},\ 59 \text{ and } 35 \text{ are primitive roots of } 61.$$

SOLUTIONS

Vandermonde

H-299 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA (Vol. 17, No. 2, April 1979)*

(A)

$$\text{Evaluate } \Delta = \begin{vmatrix} F_{2r} & F_{6r} & F_{10r} & F_{14r} & F_{18r} \\ F_{4r} & F_{12r} & F_{20r} & F_{28r} & F_{36r} \\ F_{6r} & F_{18r} & F_{30r} & F_{42r} & F_{54r} \\ F_{8r} & F_{28r} & F_{40r} & F_{56r} & F_{72r} \\ F_{10r} & F_{30r} & F_{50r} & F_{70r} & F_{90r} \end{vmatrix}$$

(B)

$$\text{Evaluate } D = \begin{vmatrix} 1 & L_{2r+1} & L_{4r+2} & L_{6r+3} & L_{8r+4} \\ 1 & -L_{6r+3} & L_{12r+6} & L_{18r+9} & L_{24r+12} \\ 1 & L_{10r+5} & L_{20r+10} & L_{30r+15} & L_{40r+20} \\ 1 & -L_{14r+7} & L_{28r+14} & -L_{42r+21} & L_{56r+28} \\ 1 & L_{18r+9} & L_{36r+18} & L_{54r+27} & L_{72r+36} \end{vmatrix}$$

(C)

$$\text{Evaluate } D_1 = \begin{vmatrix} 1 & L_{2r} & L_{4r} & L_{6r} & L_{8r} \\ 1 & L_{6r} & L_{12r} & L_{18r} & L_{24r} \\ 1 & L_{10r} & L_{20r} & L_{30r} & L_{40r} \\ 1 & L_{18r} & L_{36r} & L_{54r} & L_{72r} \end{vmatrix}$$

Solution by the proposer

(A) Taking out the common column factors

$$F_{2r},\ F_{6r},\ F_{10r},\ F_{14r},\ \text{and } F_{18r}$$

and simplifying, we obtain:

$$\Delta = F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ L_{2r} & L_{6r} & L_{10r} & L_{14r} & L_{18r} \\ L_{4r} & L_{12r} & L_{20r} & L_{28r} & L_{36r} \\ L_{6r} & L_{18r} & L_{30r} & L_{42r} & L_{56r} \\ L_{8r} & L_{24r} & L_{40r} & L_{56r} & L_{72r} \end{vmatrix}$$

$$\begin{aligned} &= F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}(L_{6r} - L_{2r})(L_{10r} - L_{2r})(L_{14r} - L_{2r})(L_{18r} - L_{2r}) \\ &\qquad (L_{10r} - L_{6r})(L_{14r} - L_{6r})(L_{18r} - L_{6r}) \\ &\qquad (L_{14r} - L_{10r})(L_{18r} - L_{10r}) \\ &\qquad (L_{18r} - L_{14r}) \\ &= 5^{10}F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}F_{2r}^4F_{4r}^4F_{6r}^3F_{8r}^3F_{10r}^2F_{12r}^2F_{14r}F_{16r} \\ &= 5^{10}F_{2r}^5F_{4r}^4F_{6r}^4F_{8r}^3F_{10r}^3F_{12r}^2F_{14r}^2F_{16r}F_{18r}. \end{aligned}$$

(B) The solution is as follows:

(1) $L_{6r+3} + L_{2r+1} = 5F_{4r+2}F_{2r+1}$ ... (5) $L_{14r+7} - L_{6r+3} = 5F_{10r+5}F_{4r+2}$
(2) $L_{12r+6} + L_{4r+2} = 5F_{8r+4}F_{4r+2}$ ... (6) $L_{28r+14} - L_{12r+6} = 5F_{20r+10}F_{8r+4}$
(3) $L_{18r+9} + L_{6r+3} = 5F_{12r+6}F_{6r+3}$ ... (7) $L_{42r+21} - L_{18r+9} = 5F_{30r+15}F_{12r+6}$
(4) $L_{24r+12} + L_{8r+4} = 5F_{16r+8}F_{8r+4}$ ... (8) $L_{56r+27} - L_{24r+12} = 5F_{40r+20}F_{16r+8}$

(1) divides (2), (3), (4), ... (5) divides (6), (7), (8).

$$\begin{aligned} D &= (L_{6r+3} + L_{2r+1})(L_{10r+5} - L_{2r+1})(L_{14r+7} + L_{2r+1})(L_{18r+9} - L_{2r+1}) \\ &\qquad (L_{10r+5} + L_{6r+3})(L_{14r+7} - L_{6r+3})(L_{18r+9} + L_{6r+3}) \\ &\qquad (L_{14r+7} + L_{10r+5})(L_{18r+9} - L_{10r+5}) \\ &\qquad (L_{18r+9} + L_{14r+7}) \\ &= 5^{10}F_{2r+1}^4F_{4r+2}^4F_{6r+3}^3F_{8r+4}^3F_{10r+5}^2F_{12r+6}^2F_{14r+7}F_{16r+8}. \end{aligned}$$

(C) The solution is as follows:

(1) $L_{r(4t+2)} - L_{r(4s+2)} = 5F_{r(2s+2t+2)}F_{r(2t-2s)}$ (3)
(2) $L_{rk(4t+2)} - L_{rk(4s+2)} = 5F_{rk(2s+2t+2)}F_{rk(2t-2s)}$ (4)

Since (3) divides (4), (1) divides (2). Checking for proper degree and sign, the sum of the subscripts in the main diagonal, we have

$$\begin{aligned} D_1 &= (L_{6r} - L_{2r})(L_{10r} - L_{2r})(L_{14r} - L_{2r})(L_{18r} - L_{2r}) \\ &\qquad (L_{10r} - L_{6r})(L_{14r} - L_{6r})(L_{18r} - L_{6r}) \\ &\qquad (L_{14r} - L_{10r})(L_{18r} - L_{10r}) \\ &\qquad (L_{18r} - L_{14r}) \end{aligned}$$

or $D_1 = 5^{10}F_{2r}^4F_{4r}^4F_{6r}^3F_{8r}^3F_{10r}^2F_{12r}^2F_{14r}F_{16r}$.

Sum Difference

H-301 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 17, No. 2, April 1979)*

Let $A_0, A_1, A_2, \ldots, A_n, \ldots$ be a sequence such that the nth differences are zero (that is, the diagonal sequence terminates). Show that, if

$$A(x) = \sum_{i=0}^{\infty} A_i x^i,$$

then

$$A(x) = 1/(1 - x)D(x/(1 - x)),$$

where

$$D(x) = \sum_{i=0}^{\infty} d_i x^i.$$

Solution by Paul Bruckman, Concord, CA

It is assumed that the d_i's, which are not explicitly defined, are in fact, defined as $d_i \equiv \Delta^i A_0$. Then,

$$\frac{1}{(1-x)} D\left(\frac{x}{(1-x)}\right) = \sum_{i=0}^{\infty} d_i x^i (1 - x)^{-i-1} = \sum_{i=0}^{\infty} d_i x^i \sum_{k=0}^{\infty} \binom{-i-1}{k} (-x)^k$$

$$= \sum_{i=0}^{\infty} d_i \sum_{k=0}^{\infty} \binom{i+k}{i} x^{i+k} = \sum_{i=0}^{\infty} d_i \sum_{k=i}^{\infty} \binom{k}{i} x^k$$

$$= \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} \binom{k}{i} d_i = \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} \binom{k}{i} \Delta^i A_0$$

$$= \sum_{k=0}^{\infty} x^k (1 + \Delta)^k A_0 = \sum_{k=0}^{\infty} x^k E^k A_0$$

$$= \sum_{k=0}^{\infty} A_k x^k = A(x). \quad \text{Q.E.D.}$$

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Effective August 1, 1980