The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 19
NUMBER 2

APRIL
1981

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980),
Br. Alfred Brousseau, and I.D. Ruggles

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
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OF INTEGERS WITH SPECIAL PROPERTIES*

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PERFECT MAGIC CUBES OF ORDER $4m$

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ABSTRACT

It has long been known that there exists a perfect magic cube of order n where $n \neq 3, 5, 7, 2m$, and $4m$ with m odd and $m \geq 7$. That they do not exist for orders 2, 3, and 4 is not difficult to show. Recently, several authors have constructed perfect magic cubes of order 7. We shall give a method for constructing perfect magic cubes of orders $n = 4m$ with m odd and $m \geq 7$.

1. INTRODUCTION

A magic square of order n is an $n \times n$ arrangement of the integers $1, 2, \ldots, n^2$ so that the sum of the integers in every row, column and the two main diagonals is $n(n^2+1)/2$: the magic sum. Magic squares of orders 5 and 6 are shown in Figure 1.

20	22	4	6	13
9	11	18	25	2
23	5	7	14	16
12	19	21	3	10
1	8	15	17	24

1	34	33	32	9	2
29	11	18	20	25	8
30	22	23	13	16	7
6	17	12	26	19	31
10	24	21	15	14	27
35	3	4	5	28	36

Fig. 1

It is a well-known and long established fact that there exists a magic square of every order n, $n \neq 2$. For details of these constructions, the reader is referred to W. S. Andrews [2], Maurice Kraitchik [13], and W. W. Rouse Ball [6].

We can extend the concept of magic squares into three dimensions. A *magic cube* of order n is an $n \times n \times n$ arrangement of the integers $1, 2, \ldots, n^3$ so that the sum of the integers in every row, column, file and space diagonal (of which there are four) is $n(n^3+1)/2$: the magic sum. A magic cube of order 3 is exhibited in Figure 2.

10	26	6	23	3	16	9	13	20
24	1	17	7	14	21	11	27	4
8	15	19	12	25	5	22	2	18

Fig. 2

Magic cubes can be constructed for every order n, $n \neq 2$ (see W. S. Andrews [2]). A *perfect magic cube* of order n is a magic cube of order n with the additional property that the sum of the integers in the main diagonals of every layer parallel to a face of the cube is also $n(n^3+1)/2$. In 1939 Barkley Rosser and R. J. Walker [15] showed that there exists a perfect magic cube of order n, $n \neq 3, 5, 7, 2m$, or $4m$, m odd. In fact, they constructed diabolic magic cubes of order n and showed that they exist only when $n \neq 3, 5, 7, 2m$, or $4m$, m odd. A *diabolic* (or *pandiagonal*) *magic cube* of order n is a magic cube of order n in which the sum of the integers in every diagonal, both broken and unbroken, is $n(n^3+1)/2$. Clearly, a diabolic magic cube is also a perfect magic cube. We shall prove that there do not exist perfect magic cubes of orders 3 and 4. These proofs are due to Lewis Myers, Jr. [9] and Richard Schroeppel [9]. Perfect magic cubes of order 7 are known to have been constructed by Ian P. Howard, Richard Schroeppel, Ernst G. Straus, and Bayard E. Wynne. In this paper, we shall present a construction for perfect magic cubes of orders $n = 4m$, m odd, $m \geq 7$, leaving only the orders $n = 5$, 12, 20, and $2m$ for m odd to be resolved. We remark here that Schroeppel has shown that if a perfect magic cube of order 5 exists, then its center must be 63.

2. *DEFINITIONS and CONSTRUCTIONS*

As far as possible, the definitions will be in accord with those given in J. Dénes and A. D. Keedwell [7].

An $n \times n \times n$ three-dimensional matrix comprising n files, each having n rows and n columns, is called a *cubic array* of order n. We shall write this array as $A = (a_{ijk})$, $i, j, k \in \{1, 2, \ldots, n\}$ where a_{ijk} is the element in the ith row, jth column, and kth file of the array. When we write $a_{i+r,\, j+s,\, k+t}$ we mean for the indices $i + r$, $j + s$, and $k + t$ to be calculated modulo n on the residues $1, 2, \ldots, n$.

The set of n elements $\{a_{i+\ell,\, j,\, k}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a column; $\{a_{i,\, j+\ell,\, k}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a row; $\{a_{i,\, j,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ constitutes a file; and $\{a_{i+\ell,\, j+\ell,\, k}\colon\ \ell = 1, 2, \ldots, n\}$, $\{a_{i+\ell,\, j,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$, $\{a_{i,\, j+\ell,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ and $\{a_{i+\ell,\, j+\ell,\, k+\ell}\colon\ \ell = 1, 2, \ldots, n\}$ constitute the diagonals. Note that a diagonal is either broken or unbroken; being unbroken if all n of its elements lie on a straight line. The unbroken diagonals consist of the main diagonals, of which there are two in every layer parallel to a face of the cube, and the four space diagonals.

We shall distinguish three types of layers in a cube. There are those with fixed row, fixed column, or fixed file. The first we shall call the CF-layers, the ith CF-layer consisting of the n^2 elements $\{a_{ijk}\colon\ 1 \leq j,\ k \leq n\}$. The secone are the RF-layers in which the jth RF-layer consists of the n^2 elements $\{a_{\ jk}\colon\ 1 \leq i,\ k \leq n\}$. And finally, the RC-layers, the kth consisting of the elements $\{a_{ijk}\colon\ 1 \leq i,\ j \leq n\}$.

A cubic array of order n is called a *Latin cube* of order n if it has n distinct elements each repeated n^2 times and so arranged that in each layer parallel to a face of the cube all n distinct elements appear, and each is repeated exactly n times in that layer. In the case when each layer parallel to a face of the cube is a Latin square, we have what is called a *permutation cube* of order n. From this point on, we shall be concerned only with Latin cubes (and permutation cubes) based on the integers $1, 2, \ldots, n$.

Three Latin cubes of order n, $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$, are said to be *orthogonal* if among the n^3 ordered 3-tuples of elements $(a_{ijk}, b_{ijk}, c_{ijk})$ every distinct ordered 3-tuple involving the integers $1, 2, \ldots, n$ occurs exactly once. Should A, B, and C be orthogonal permutation cubes, they are said to form a *variational cube*. We shall write $D = (d_{ijk})$ where $d_{ijk} = (a_{ijk}, b_{ijk}, c_{ijk})$ to be the cube obtained on superimposing the Latin cubes A, B, and C and will denote it by $D = (A, B, C)$.

A cubic array of order n in which each of the integers $1, 2, \ldots, n^3$ occurs exactly once and in which the sum of the integers in every row, column, file, and unbroken diagonal is $n(n^3 + 1)/2$ is called a *perfect magic cube*.

We shall give two methods for constructing perfect magic cubes. These methods form the basis on which the perfect magic cubes of order $n = 4m$, m odd, $m \geq 7$, of Section 3 will be constructed.

Construction 1: Let $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$ be three orthogonal Latin cubes of order n with the property that in each cube the sum of the integers in every row, column, file, and unbroken diagonal is $n(n+1)/2$. Then the cube $E = (e_{ijk})$ where $e_{ijk} = n^2(a_{ijk} - 1) + n(b_{ijk} - 1) + (c_{ijk} - 1) + 1$ is a perfect magic cube of order n. This is verified by checking that each of the integers $1, 2, \ldots, n^3$ appears in E and that the sum of the integers in every row, column, file, and unbroken diagonal is $n(n^3 + 1)/2$. It is clear that each of $1, 2, \ldots, n^3$ appears exactly once in E. We shall show that the sum of the integers in any row of E is $n(n^3 + 1)/2$. The remaining sums can be checked in a similar manner.

$$\begin{aligned}\sum_{\ell=1}^{n} e_{i,j+\ell,k} &= \sum_{\ell=1}^{n} (n^2(a_{i,j+\ell,k} - 1) + n(b_{i,j+\ell,k} - 1) + (c_{i,j+\ell,k} - 1) + 1) \\ &= n^2 \sum_{\ell=1}^{n} a_{i,j+\ell,k} + n\sum_{\ell=1}^{n} b_{i,j+\ell,k} + \sum_{\ell=1}^{n} c_{i,j+\ell,k} - \sum_{\ell=1}^{n}(n^2 + n) \\ &= (n^2 + n + 1)(n(n+1)/2) - n(n^2 + n) \\ &= n(n^3 + 1)/2.\end{aligned}$$

Construction 2: Let $A = (a_{ijk})$ and $B = (b_{ijk})$ be perfect magic cubes of orders m and n, respectively. Replace b_{ijk} in B by the cube $C = (c_{rst})$ where $c_{rst} = a_{rst} + m^3(b_{ijk} - 1)$. This results in a perfect magic cube $E = (e_{ijk})$ of order nm. Each of the integers

$$1, 2, \ldots, m^3, m^3 + 1, m^3 + 2, \ldots, 2m^3, \ldots,$$
$$(n^3 - 1)m^3 + 1, (n^3 - 1)m^3 + 2, \ldots, n^3m^3$$

appears exactly once in E. As in the first construction, we shall show that the row sum in E is $nm((nm)^3 + 1)/2$; the remaining sums are similarly verified.

$$\begin{aligned}\sum_{\ell=1}^{nm} e_{u,v+\ell,w} &= n\sum_{\ell=1}^{m} a_{r,s+\ell,t} + m\sum_{\ell=1}^{n} m^3(b_{i,j+\ell,k} - 1) \\ &= nm(m^3 + 1)/2 + m^4(n(n^3 + 1)/2 - n) \\ &= nm((nm)^3 + 1)/2.\end{aligned}$$

It will be seen in Theorem 3.6 that it is not necessary that A and B should both be perfect magic cubes in order for E to be a perfect magic cube.

3. PERFECT MAGIC CUBES

The first result is stated without proof and is due to Barkley Rosser and R. J. Walker [15].

Theorem 3.1: There exists a perfect magic cube of order n provided $n \neq 3, 5, 7, 2m$, or $4m$ for m odd.□

The following three theorems are the only known nonexistence results for perfect magic cubes. For the first, the proof is trivial. The proof of the second theorem is that of Lewis Myers, Jr. (see [9]) and of the third is that of Richard Schroeppel (also see [9]).

Theorem 3.2: There is no perfect magic cube of order 2.□

Theorem 3.3: There is no perfect magic cube of order 3.

Proof: Let $A = (a_{ijk})$ be a perfect magic cube of order 3; the magic sum is 42. The following equations must all hold:

$$a_{11k} + a_{22k} + a_{33k} = a_{13k} + a_{22k} + a_{31k} = a_{12k} + a_{22k} + a_{32k} = 42$$

and

$$a_{11k} + a_{12k} + a_{13k} = a_{31k} + a_{32k} + a_{33k} = 42.$$

But together these imply that $a_{22k} = 14$ for $k = 1, 2$, and 3, a contradiction.□

Theorem 3.4: There is no perfect magic cube of order 4.

Proof: Let $A = (a_{ijk})$ be a perfect magic cube of order 4; the magic sum is 130.

First, we shall show that in any layer of such a cube the sum of the four corner elements is 130. Consider the kth RC-layer. The following equations must hold in A:

$$\begin{aligned} a_{11k} + a_{12k} + a_{13k} + a_{14k} &= a_{11k} + a_{22k} + a_{33k} + a_{44k} \\ &= a_{11k} + a_{21k} + a_{31k} + a_{41k} = 130, \\ a_{14k} + a_{23k} + a_{32k} + a_{41k} &= a_{14k} + a_{24k} + a_{34k} + a_{44k} \\ &= a_{41k} + a_{42k} + a_{43k} + a_{44k} = 130. \end{aligned}$$

These imply that

$$2(a_{11k} + a_{14k} + a_{41k} + a_{44k}) + \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ijk} = 6 \cdot 130$$

and as

$$\sum_{i=1}^{4} \sum_{j=1}^{4} a_{ijk} = 4 \cdot 130,$$

then

$$a_{11k} + a_{14k} + a_{41k} + a_{44k} = 130.$$

Since the same argument holds for any type of layer in the cube, we have that the sum of the four corner elements in any layer is 130. A similar argument shows that $a_{111} + a_{114} + a_{441} + a_{444} = 130$. Thus we have

$$\begin{aligned} a_{111} + a_{114} + a_{144} + a_{141} &= a_{141} + a_{144} + a_{444} + a_{441} \\ &= a_{111} + a_{114} + a_{444} + a_{441} = 130, \end{aligned}$$

from which it follows that

$$a_{111} + a_{114} + a_{144} + a_{141} + 2(a_{444} + a_{441}) = 260,$$

and hence $a_{444} + a_{441} = 65$. Similarly, we can show that $a_{141} + a_{441} = 65$. Combining these two results, we have $a_{141} = a_{444}$, a contradiction. □

Using an argument similar to that of Theorem 3.4 Schroeppel has shown that if there exists a perfect magic cube of order 5 its center is 63.

For some time it was not generally known whether or not there existed a perfect magic cube of order 7 but when, in 1976, Martin Gardner [9] asked for such a cube, it appeared that they had been constructed without difficulty by many authors including Schroeppel, Ian P. Howard, Ernst G. Straus, and Bayard E. Wynne [17].

Theorem 3.5: There exists a perfect magic cube of order 7.

Proof: We shall construct a variational cube of order 7 from which a perfect magic cube of order 7 can be obtained via Construction 1. Let the three cubes forming the variational cube be A, B, and C; the first RC-layer of each being shown in Figure 3. Complete A, B, and C using the defining relations

$$a_{i,j,k+1} = a_{ijk} + 1,\ b_{i,j,k+1} = b_{ijk} + 1$$

and

$$c_{i,j,k+1} = c_{ijk} + 2,$$

where the addition is modulo 7 on the residues 1, 2, ..., 7. Now, in A and B, exchange the integers 4 and 7 throughout each cube. The variational cube of order 7 now has the properties required by Construction 1 and so we can construct a perfect magic cube of order 7. This can easily be checked. □

1	4	7	3	6	2	5	1	6	4	2	7	5	3	1	3	5	7	2	4	6
6	2	5	1	4	7	3	4	2	7	5	3	1	6	7	2	4	6	1	3	5
4	7	3	6	2	5	1	7	5	3	1	6	4	2	6	1	3	5	7	2	4
2	5	1	4	7	3	6	3	1	6	4	2	7	5	5	7	2	4	6	1	3
7	3	6	2	5	1	4	6	4	2	7	5	3	1	4	6	1	3	5	7	2
5	1	4	7	3	6	2	2	7	5	3	1	6	4	3	5	7	2	4	6	1
3	6	2	5	1	4	7	5	3	1	6	4	2	7	2	4	6	1	3	5	7

Fig. 3

We shall now proceed to the main theorem.

Theorem 3.6: There exists a perfect magic cube of order $4m$ for m odd and $m \geq 7$.

Proof: We know that there exists a perfect magic cube of order m, m odd and $m \geq 7$. This follows from Theorems 3.1 and 3.5. Since there does not exist a perfect magic cube of order 4 (Theorem 3.4), we cannot simply appeal to Construction 2 and obtain the desired perfect magic cubes. However, we can use Construction 2 and by a suitable arrangement of cubes of order 4 obtain a perfect magic cube of order $4m$. The construction is as follows.

Let $A = (a_{ijk})$ be a perfect magic cube of order m, m odd and $m \geq 7$. Let $B = (b_{ijk})$ be a cubic array of order m in which each b_{ijk} is some cubic array D_{ijk} of order 4 whose entries are ordered 3-tuples from the integers 1, 2, 3, 4 with every such 3-tuple appearing exactly once. The D_{ijk} are to be chosen in such a way that in the cubic array B the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$. It is now a simple matter to produce a perfect magic cube of order 4 . In D_{ijk} replace the 3-tuple (r, s, t) by the integer

$$(16(r - 1) + 4(s - 1) + (t - 1) + 1) + 64(a_{ijk} - 1).$$

The cubic array $E = (e_{ijk})$ of order $4m$ so constructed is, by considering Constructions 1 and 2, a perfect magic cube.

It remains then to determine the order 4 cubic arrays D_{ijk}.

Consider the four Latin cubes X_1, X_2, X_3, and X_4 as shown in Figure 4 where from left to right we have the first to the fourth RC-layers. It is not difficult to check that X_1, X_2, and X_3 are orthogonal, as are X_1, X_2, and X_4. We shall write X_i^* for the Latin cube X_i in which the integers 1 and 4 have been exchanged as have 2 and 3. Also $(X_1, X_2, X_3)'$ means that the cubic array (X_1, X_2, X_3) has been rotated forward through $90°$ so that RC-layers have become CF-layers, CF-layers have become RC-layers, and the roles of rows and files have interchanged in RF-layers.

X_1:	1	1	4	4	3	3	2	2	4	4	1	1	2	2	3	3
	4	4	1	1	2	2	3	3	1	1	4	4	3	3	2	2
	2	2	3	3	4	4	1	1	3	3	2	2	1	1	4	4
	3	3	2	2	1	1	4	4	2	2	3	3	4	4	1	1
X_2:	1	4	2	3	4	1	3	2	2	3	1	4	3	2	4	1
	1	4	2	3	4	1	3	2	2	3	1	4	3	2	4	1
	4	1	3	2	1	4	2	3	3	2	4	1	2	3	1	4
	4	1	3	2	1	4	2	3	3	2	4	1	2	3	1	4
X_3:	4	4	1	1	2	2	3	3	3	3	2	2	1	1	4	4
	1	1	4	4	3	3	2	2	2	2	3	3	4	4	1	1
	2	2	3	3	4	4	1	1	1	1	4	4	3	3	2	2
	3	3	2	2	1	1	4	4	4	4	1	1	2	2	3	3

(continued)

$$X_4: \begin{matrix} 3 & 3 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 4 & 4 \\ 4 & 4 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 4 & 4 \\ 4 & 4 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{matrix} \qquad \begin{matrix} 4 & 4 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \end{matrix} \qquad \begin{matrix} 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 4 & 4 & 1 & 1 \\ 1 & 1 & 4 & 4 \end{matrix}$$

Fig. 4

We can now define the cubic arrays D_{ijk}.

$$D_{1,\frac{m+1}{2},\frac{m+1}{2}} = D_{m,\frac{m+1}{2},\frac{m+1}{2}} = D_{i,1,m} = D_{i,m,m} = (X_1, X_2, X_3)', \; i = 2, 3, \ldots, m-1$$

$$D_{1,1,m} = D_{1,m,m} = D_{1,1,1} = D_{1,m,1} = (X_1, X_2, X_3)$$

$$D_{2,2,m-1} = D_{2,m-1,m-1} = (X_2^*, X_3, X_1^*)$$

$$D_{2,2,2} = D_{2,m-1,2} = (X_2, X_4, X_1)$$

$$D_{3,3,m-2} = D_{3,m-2,m-2} = (X_3^*, X_1^*, X_2^*)$$

$$D_{3,3,3} = D_{3,m-2,3} = (X_3, X_1, X_2)$$

$$D_{i,m+1-i,i} = D_{i,m+1-i,m+1-i} = D_{i,i,m+1-i} = D_{i,i,i} = (X_1, X_2, X_3), \quad i = 4, 5, \ldots, \frac{m+3}{2}$$

$$D_{i,m+1-i,i} = D_{i,m+1-i,m+1-i} = D_{i,i,m+1-i} = D_{i,i,i} = (X_1^*, X_2^*, X_3^*), \quad i = \frac{m+5}{2}, \frac{m+7}{2}, \ldots, m.$$

In every CF-layer of B, except for the second and third, replace the remaining b_{ijk} in each unbroken diagonal by either

$$D_{ijk} = (X_1, X_2, X_3) \text{ or } D_{ijk} = (X_1^*, X_2^*, X_3^*)$$

so that in each diagonal there are $(m-1)/2$ arrays (X_1, X_2, X_3) and $(m-1)/2$ arrays (X_1^*, X_2^*, X_3^*). In the second and third layers do the same but here there are to be only $(m-3)/2$ of each type of array as already three arrays in each diagonal are determined. All remaining b_{ijk} are to be replaced by

$$D_{ijk} = (X_1, X_2, X_3).$$

We must now verify that in this cubic array the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$.

Since the sum of the integers in every row, column, and file of X_i and X_i^*, $i = 1, 2, 3, 4$, is 10, then in B the componentwise sum of the integers in every row, column, and file is $(10m, 10m, 10m)$. Also, as the sum of the integers in every unbroken diagonal in the RC-layers and RF-layers of X_i and X_i^*, $i = 1, 2, 3, 4$, is 10, and as $(X_1, X_2, X_3)'$ does not occur on any of these unbroken diagonals in B, then the componentwise sum of the integers in these unbroken diagonals of B is $(10m, 10m, 10m)$. So we now have only to check the sums on the unbroken diagonals of the CF-layers and the sums on the four space diagonals of B.

The unbroken diagonals in the CF-layers of B are $D_{i11}, D_{i22}, \ldots, D_{imm}$ and $D_{im1}, D_{i,m-1,2}, \ldots, D_{i1m}$, $i = 1, 2, \ldots, m$. Let us write $S_r(D_{ijk})$ for the componentwise sum of the integers in the relevant diagonal in the rth CF-layer of D_{ijk}. We want to show that

$$\sum_{j=1}^{m} S_r(D_{ijj}) = \sum_{j=1}^{m} S_r(D_{i,m+1-j,j}) = (10m, 10m, 10m), \; r = 1, 2, 3, 4.$$

If $i \neq 2, 3$, then

$$\sum_{j=1}^{m} S_r(D_{ijj}) = \frac{m-1}{2}S_r((X_1, X_2, X_3)) + \frac{m-1}{2}S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-1}{2}(8, 4, 12) + \frac{m-1}{2}(12, 16, 8) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-1}{2}(12, 4, 8) + \frac{m-1}{2}(8, 16, 12) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-1}{2}(12, 16, 12) + \frac{m-1}{2}(8, 4, 8) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-1}{2}(8, 16, 8) + \frac{m-1}{2}(12, 4, 12) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m).$$

Also,

$$\sum_{j=1}^{m} S_r(D_{2jj}) = \frac{m-3}{2}S_r((X_1, X_2, X_3)) + \frac{m-3}{2}S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_2, X_4, X_1)) + S_r((X_2^*, X_3, X_1^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-3}{2}(8, 4, 12) + \frac{m-3}{2}(12, 16, 8) + (4, 8, 8) + (16, 12, 12) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-3}{2}(12, 4, 8) + \frac{m-3}{2}(8, 16, 12) + (4, 12, 12) + (16, 8, 8) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-3}{2}(12, 16, 12) + \frac{m-3}{2}(8, 4, 8) + (16, 8, 12) + (4, 12, 8) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-3}{2}(8, 16, 8) + \frac{m-3}{2}(12, 4, 12) + (16, 12, 8) + (4, 8, 12) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m)$$

and

$$\sum_{j=1}^{m} S_r(D_{3jj}) = \frac{m-3}{2}S_r((X_1, X_2, X_3)) + \frac{m-3}{2}S_r((X_1^*, X_2^*, X_3^*)) + S_r((X_3, X_1, X_2)) + S_r((X_3^*, X_1^*, X_2^*)) + S_r((X_1, X_2, X_3)')$$

$$= \begin{cases} \frac{m-3}{2}(8, 4, 12) + \frac{m-3}{2}(12, 16, 8) + (12, 8, 4) + (8, 12, 16) + (10, 10, 10) & \text{when } r = 1 \\ \frac{m-3}{2}(12, 4, 8) + \frac{m-3}{2}(8, 16, 12) + (8, 12, 4) + (12, 8, 16) + (10, 10, 10) & \text{when } r = 2 \\ \frac{m-3}{2}(12, 16, 12) + \frac{m-3}{2}(8, 4, 8) + (12, 12, 16) + (8, 8, 4) + (10, 10, 10) & \text{when } r = 3 \\ \frac{m-3}{2}(8, 16, 8) + \frac{m-3}{2}(12, 4, 12) + (8, 8, 16) + (12, 12, 4) + (10, 10, 10) & \text{when } r = 4 \end{cases}$$

$$= (10m, 10m, 10m).$$

Similarly, one can check that

$$\sum_{j=1}^{m} S_r(D_{i,m+1-j,j}) = (10m,\ 10m,\ 10m)$$

and so the componentwise sum of the integers in the unbroken diagonals in the CF-layers of B is $(10m, 10m, 10m)$.

The four space diagonals of B are

$$D_{iii},\ i = 1, 2, \ldots, m;\ D_{i,m+1-i,i},\ i = 1, 2, \ldots, m;$$

$$D_{m+1-i,i,i},\ i = 1, 2, \ldots, m;\ D_{m+1-i,m+1-i,i},\ i = 1, 2, \ldots, m.$$

Write $S(D_{ijk})$ to be the sum of the integers in the relevant space diagonal of D_{ijk}. We want to show that

$$\sum_{i=1}^{m} S(D_{iii}) = \sum_{i=1}^{m} S(D_{i,m+1-i,i}) = \sum_{i=1}^{m} S(D_{i,m+1-i,m+1-i}) = \sum_{i=1}^{m} S(D_{i,i,m+1-i})$$
$$= (10m,\ 10m,\ 10m).$$

Consider each of the space diagonals in turn.

$$\begin{aligned}\sum_{i=1}^{m} S(D_{iii}) &= S((X_1, X_2, X_3)) + S((X_2, X_4, X_1)) + S((X_3, X_1, X_2)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (6, 10, 14) + (10, 14, 6) + (14, 6, 10) + \frac{m-3}{2}(6, 10, 14) \\ &\qquad + \frac{m-3}{2}(14, 10, 6) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,m+1-i,i}) &= S((X_1, X_2, X_3)) + S((X_2, X_4, X_1)) + S((X_3, X_1, X_2)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (14, 10, 6) + (10, 6, 14) + (6, 14, 10) + \frac{m-3}{2}(14, 10, 6) \\ &\qquad + \frac{m-3}{2}(6, 10, 14) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,m+1-i,m+1-i}) &= S((X_1, X_2, X_3)) + S((X_2^*, X_3, X_1^*)) + S((X_3^*, X_1^*, X_2^*)) \\ &\qquad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (14, 10, 14) + (10, 14, 6) + (6, 6, 10) + \frac{m-3}{2}(14, 10, 14) \\ &\qquad + \frac{m-3}{2}(6, 10, 6) \\ &= (10m,\ 10m,\ 10m)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} S(D_{i,i,m+1-i}) &= S((X_1, X_2, X_3)) + S((X_2^*, X_3, X_1^*)) + S((X_3^*, X_1^*, X_2^*)) \\ &\quad + \frac{m-3}{2}S((X_1, X_2, X_3)) + \frac{m-3}{2}S((X_1^*, X_2^*, X_3^*)) \\ &= (6, 10, 6) + (10, 6, 14) + (14, 14, 10) + \frac{m-3}{2}(6, 10, 6) \\ &\quad + \frac{m-3}{2}(14, 10, 14) \\ &= (10m, 10m, 10m).\end{aligned}$$

Thus we have found a way of arranging order 4 cubic arrays, in which each of the ordered 3-tuples on 1, 2, 3, 4 appears exactly once, in an order m cubic array B so that in B the componentwise sum of the integers in every row, column, file, and unbroken diagonal is $(10m, 10m, 10m)$. Therefore, as previously stated, we can construct a perfect magic cube of order $4m$ for m odd and $m \geq 7$. □

4. EXTENSIONS AND PROBLEMS

We know now that there exists a perfect magic cube of order n provided $n \neq$ 3, 4, 5, 12, 20, $2m$, for m odd, and that they do not exist when n = 2, 3, or 4. So the question remaining is whether or not there exist perfect magic cubes of orders n = 5, 12, 20, and $2m$, for m odd and $m \geq 3$. It seems probable that such cubes of orders 12 and 20 can be constructed along the lines of Theorem 3.6 using cubic arrays of orders 3 and 5 that are *close* to being perfect magic cubes and arranging in them order 4 cubic arrays composed from X_i and X_i^*, i = 1, 2, 3, 4, as before. It may also be possible that by arranging order 2 cubic arrays in order m cubic arrays, m odd and $m \geq 7$, one can obtain perfect magic cubes of order $2m$. As for order 5, all we know is that if there is a perfect magic cube of order 5 its center is 63.

A more recent problem in the study of magic cubes is that of extending them into k dimensions. For details on this problem and the related problem of constructing variational cubes in k dimensions, the reader is referred to [1], [3], [4], [5], [7], [8], [12], and [16].

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GENERATING FUNCTIONS FOR RECURRENCE RELATIONS

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1. INTRODUCTION

In a previous paper [3] the author gave explicit solutions for four recurrence relations. The first was a basic relation with special initial conditions. The solution was shown to be related to the decompositions of the integer n relative to the first m positive integers. The second basic relation then restricted the first so that the solution was related to the decomposition of n relative to a subset of the first m positive integers. Then the initial conditions for both were extended to any arbitrary values.

In the next section we shall give the generating functions for all four of these cases, starting with the initial condition of highest index. We also note the form of the function for arbitrary indices for the initial conditions. Finally, we give a second function that generates all the initial conditions.

In Section 3 we give a simple example of the fourth kind of relation. We determine the first few terms of this relation and then compute its generating function. Then we consider relations given in [1] and [2] and determine their generating functions.

2. THE BASIC GENERATING FUNCTION

We shall consider a recurrence relation defined by

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1-m},\ \ldots,\ G_0 \text{ arbitrary.}$$

For notation, we shall refer to its generating function as $R_m(G; x)$. The first term generated will be G_0. Later, we shall give a second function that will start with G_{1-m}.

Theorem 2.1: The generating function for the recurrence relation G_n is as follows:

$$R_m(G;\ x) = \left(G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

To prove that this does generate G_n, we set this equal to $\sum_{n=0}^{\infty} G_n x^n$ and then multiply by the factor with the negative exponent. This gives

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_s x^s - \sum_{s=1}^{m} \sum_{n=0}^{\infty} r_s G_n x^{s+n}.$$

In the last summation, we replace n by $n - s$ and transpose it to the left side so that

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n + \sum_{s=1}^{m} \sum_{n=s}^{\infty} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

We now break the second sum at $n = m$ and then interchange the orders of summation. We have

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n + \sum_{n=1}^{m} \sum_{s=1}^{n} r_s G_{n-s} x^n + \sum_{n=m+1}^{\infty} \sum_{s=1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

Note that for the first sum, if $n = m$, there would be no second sum, so we can combine the first three summations to give

$$G_0 + \sum_{n=1}^{\infty} \sum_{s=1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

It remains only to observe that the inner sum on the left is just G_n, so we have the desired result.

We now specialize this result for the U_n relation.

Corollary 2.2: The generating function for the relation

$$U_t = \sum_{s=1}^{m} r_s U_{t-s};\ U_{1-m} = \cdots = U_{-1} = 0,\ U_0 = 1,$$

is given by

$$R_m(U;\ x) = \left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

In Theorem 2.1 the double summation of the numerator is zero since all initial conditions involved are zero. The other initial condition is 1, so the first factor is 1.

An implication of this result is that the generating function for G_n is obtained from that of U_n by multiplication by a polynomial of degree $m - 1$.

In [3] we generalized both the U_n and the G_n relations to the V_n and the H_n relations. This was accomplished by taking a subset A of the integers from 1 to m, including m. The solutions then were obtained by replacing r_i with 0 if $i \notin A$. We shall do that for their generating functions.

Corollary 2.3: The generating function for the relation

$$V_t = \sum_{s \in A} r_s V_{t-s};\ V_{1-m} = \cdots = V_{-1} = 0,\ V_0 = 1,$$

is given by

$$R_A(V;\ x) = \left(1 - \sum_{s \in A} r_s x^s\right)^{-1}.$$

This follows directly from Corollary 2.2 by replacing r_i with 0 if $i \notin A$.

The most general recurrence relation is the H_n. Its generating function is given in the next corollary.

Corollary 2.4: The recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s};\ H_{1-m},\ \ldots,\ H_0 \text{ arbitrary},$$

is given by

$$R_A(H;\ x) = \left(H_0 + \sum_{s \in A'} \sum_{n=1}^{s-1} r_s H_{n-s} x^n\right)\left(1 - \sum_{s \in A} r_s x^s\right)^{-1},$$

where A' is A with 1 deleted if $1 \in A$; otherwise $A' = A$.

For the proof of this, we first need to interchange the order of summation in the numerator of the function of Theorem 2.1. Then we replace r_i with 0 for $i \notin A$.

The theorem together with the three corollaries start generating the given relation with the initial condition of highest order. In all our cases, this was the one with index 0. We can modify the notation to obtain a generating function with any indices for the initial conditions.

<u>*Theorem 2.5:*</u> The recurrence relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1+p},\ \ldots,\ G_{m+p} \text{ arbitrary},$$

has for its generating function

$$\left(G_{m+p} x^{m+p} + \sum_{n=m+1+p}^{2m+1+p} \sum_{s=n-m+1}^{m} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

This reduces to Theorem 2.1 when $p = -m$, as can be verified.

The only change we have for the U_n and V_n relations is to have as the numerator $U_{m+p}x^{m+p}$ and $V_{m+p}x^{m+p}$, respectively. The change for the H_n relation is given in the next corollary.

<u>*Corollary 2.6:*</u> The recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s};\ H_{1-p},\ \ldots,\ H_{m+p} \text{ arbitrary},$$

has for its generating function

$$\left(H_{m+p} x^{m+p} + \sum_{s \in A'} \sum_{n=m+p+1}^{m+p+s-1} r_s H_{n-s} x^n\right)\left(1 - \sum_{s \in A} r_s x^s\right)^{-1}.$$

Once more, this reduces to the result of Corollary 2.4 for $p = -m$.

If it were desired to generate all the initial conditions, the generating function is given in the next theorem.

<u>*Theorem 2.7:*</u> A generating function for the relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1+p},\ \ldots,\ G_{m+p} \text{ arbitrary},$$

is given by

$$\left(\sum_{n=1+p}^{m+p} G_n x^n - \sum_{n=2+p}^{m+p} \sum_{s=1}^{n-1-p} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

If we set this equal to $\sum_{n=1+p}^{\infty} G_n x^n$ and clear the negative exponent, we have

$$\sum_{n=1+p}^{m+p} G_n x^n - \sum_{n=2+p}^{m+p} \sum_{s=1}^{n-1-p} r_s G_{n-s} x^n = \sum_{n=1+p}^{\infty} G_n x^n - \sum_{s=1}^{m} \sum_{n=1+p}^{\infty} r_s G_n x^{n+s}.$$

To simplify this expression, we use the first term on the left to reduce the first term on the right. We transpose the second sum on the right. Further, we change the summation on n by replacing n by $n - s$, and then break the sum at $n = m + p$. This gives

$$\left(\sum_{s=1}^{m}\sum_{n=1+s+p}^{m+p} r_s G_{n-s}x^n - \sum_{n=2+p}^{m+p}\sum_{s=1}^{n-1+p} r_s G_{n-s}x^n\right) + \sum_{s=1}^{m}\sum_{n=m+1+p}^{\infty} r_s G_{n-s}x^n = \sum_{n=m+1+p}^{\infty} G_n x^n.$$

If $s = m$ in the first sum, we would have no second sum; thus we need sum only to $m - 1$. It can be verified that these two summations are the same. Finally, interchanging the summation on the last term on the left will give the right side from the definition of the G_n relation.

For the U_n and V_n relations, this gives the same generating function we had before.

3. EXAMPLES OF THE GENERATING FUNCTIONS

A simple example of an H_n relation will illustrate the results of the last section. Let $A = \{2, 5\}$ so $m = 5$ and $H_t = r_2H_{t-2} + r_5H_{t-5}$ with H_{-4}, H_{-3}, H_{-2}, H_{-1}, H_0 all arbitrary. It can be readily verified that the application of the definition of the relation yields, for the first seven terms,

$$\begin{aligned}
H_1 &= r_2H_{-1} + r_5H_{-4}\\
H_2 &= r_2H_0 + r_5H_{-3}\\
H_3 &= r_2^2H_{-1} + r_2r_5H_{-4} + r_5H_{-2}\\
H_4 &= r_2^2H_0 + r_2r_5H_{-3} + r_5H_{-1}\\
H_5 &= r_2^3H_{-1} + r_2^2r_5H_{-4} + r_2r_5H_{-2} + r_5H_0\\
H_6 &= r_2^3H_0 + r_2^2r_5H_{-3} + 2r_2r_5H_{-1} + r_5^2H_{-4}\\
H_7 &= r_2^4H_{-1} + r_2^3r_5H_{-4} + r_2^2r_5H_{-2} + 2r_2r_5H_0 + r_5^2H_{-3}.
\end{aligned}$$

The generating function is given by

$$(H_0 + r_2H_{-1}x + r_5(H_{-4}x + H_{-3}x^{-2} + H_{-2}x^3 + H_{-1}x^4))(1 - r_2x^2 - r_5x^5)^{-1}.$$

For the corresponding V_n relation, we have

$$V_1 = 0,\ V_2 = r_2,\ V_3 = 0,\ V_4 = r_2^2,\ V_5 = r_5,\ V_6 = r_2^3,\ V_7 = 2r_2r_5.$$

The generating function that gives all the initial conditions has for its numerator

$$H_{-4}x^{-4} + H_{-3}x^{-3} + (H_{-2} - r_2H_{-4})x^{-2} + (H_{-1} - r_2H_{-3})x^{-1} + (H_0 - r_2H_{-2}).$$

We shall list the five relations given in [2] and the one in [1, p. 4], and note their generating functions below them.

1. $G_k = rG_{k-1} + sG_{k-2};\ G_0 = 0,\ G_1 = 1$

$$x(1 - rx - sx^2)^{-1} = x + rx^2 + (r^2 + s)x^3 \ldots$$

2. $F_k = F_{k-1} + F_{k-2};\ F_0 = 0,\ F_1 = 1$

$$x(1 - x - x^2)^{-1} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$$

(This is the famous Fibonacci sequence.)

3. $M_k = rM_{k-1} + sM_{k-2};\ M_0 = 2,\ M_1 = r$

$$(rx + 2sx^2)(1 - rx - sx^2)^{-1} = r + (r^2 + 2s)x^2 + \cdots$$

or

$$(2 - rx)(1 - rx - sx^2)^{-1} = 2 + rx + (r^2 + 2s)x^2 + \cdots$$

4. $L_k = L_{k-1} + L_{k-2}$; $L_0 = 2$, $L_1 = 1$

$$(x + 2x^2)(1 - x - x^2)^{-1} = x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

or

$$(2 - x)(1 - x - x^2)^{-1} = 2 + x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

(This is the Lucas sequence.)

5. $U_k = rU_{k-1} + sU_{k-2}$; U_0, U_1 arbitrary

$$(U_1x + U_0sx^2)(1 - rx - sx^2)^{-1} = U_1x + (rU_1 + sU_0)x^2 + \cdots$$

or

$$(U_0 + (U_1 - U_0)x)(1 - rx - sx^2)^{-1} = U_0 + U_1x + (rU_1 + sU_0)x^2 + \cdots$$

6. $T_n = rT_{n-1} + sT_{n-2} - rsT_{n-3}$; T_0, T_1, T_2 arbitrary

$$(T_2x^2 + (sT_1 - rsT_0)x^3 - rsT_1x^4)(1 - rx - sx^2 + rsx^3)^{-1}$$
$$= T_2x^2 + (rT_2 + sT_1 - rsT_0)x^3 + \cdots$$

or

$$(T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2)(1 - rx - sx^2 + r2x^3)^{-1}$$
$$= T_0 + T_1x + T_2x^2 + (rT_2 + sT_1 - r\ T_0)x^3 + \cdots$$

From the solutions given in [2] and [1], it can be verified that we obtain the terms generated above.

The generating function given in Section 2 can be used to generate terms of any given recurrence relation. With specified values for the r_i and the initial conditions, the problem becomes a division of one polynomial by another.

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THE RESIDUES OF n^n MODULO p

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SUMMARY

In this paper we investigate the residues of $n^n \pmod p$, where $1 \leq n \leq p-1$ and p is an odd prime. We find new upper bounds for the number of distinct residues of $n^n \pmod p$ that can occur. We also give lower bounds for the number of quadratic nonresidues and primitive roots modulo p that do not appear among the residues of $n^n \pmod p$. Further, we prove that given any arbitrarily large positive integer M, there exist sets of primes $\{p_i\}$ and $\{q_j\}$, both with positive density in the set of primes, such that the congruences

$$x^x \equiv 1 \pmod{p_i},\ 1 \leq x \leq p_i - 1 \tag{1}$$

and

$$x^x \equiv -1 \pmod{q_j},\ 1 \leq x \leq q_j - 1 \tag{2}$$

both have at least M solutions.

1. INTRODUCTION

Roger Crocker [4] and [5] first examined the residues of n^n modulo p. It is clear that if $n \geq 1$, then the sequence $\{n^n\}$ reduced modulo p is periodic with a period of $p(p - 1)$. This follows from the facts that $(p-1, p) = 1$ and that if

$$n_1 \equiv n_2 \pmod{p} \text{ and } n_1 \equiv n_2 \pmod{p-1},$$

then

$$n_1^{n_1} \equiv n_2^{n_2} \pmod{p}.$$

The following theorem shows that every residue appears among the residues of n^n modulo p, where $1 \leq n \leq p(p - 1)$, and counts the number of times a particular residue occurs.

Theorem 1: Consider the residues of n^n modulo p, where $1 \leq n \leq p(p - 1)$. Then the residue 0 appears $p - 1$ times. If $r \not\equiv 0 \pmod{p}$ and the exponent of r modulo p is d, then the number of times the residue r appears is

$$\sum_{d|d'|p-1} \phi(d')((p - 1)/d'). \tag{3}$$

Proof: First, it is clear that the residue 0 appears $p - 1$ times. Now consider any fixed nonzero residue n. It is raised to the various powers $n + kp$, where $0 \leq k \leq p - 2$. These powers form a complete residue system modulo $p - 1$. Thus n is raised to each power m, where $1 \leq m \leq p - 1$. Now, the congruence

$$n^x \equiv r \pmod{p} \tag{4}$$

is solvable for x if and only if

$$(p - 1, \text{Ind } n) \,|\, (p - 1, \text{Ind } r), \tag{5}$$

where Ind a is the index of a (mod p) with respect to a fixed primitive root. This can occur only if

$$\frac{p - 1}{(p - 1, \text{Ind } r)} \,\Bigg|\, \frac{p - 1}{(p - 1, \text{Ind } n)}. \tag{6}$$

but

$$\frac{p - 1}{(p - 1, \text{Ind } r)} = d$$

is the exponent of r (mod p) and

$$\frac{p - 1}{(p - 1, \text{Ind } r)} = d'$$

is the exponent of n (mod p). Thus congruence (4) has solutions if and only if d divides d'. It is evident that the number of solutions to (4) is then

$$(p - 1/d').$$

However, there are exactly $\phi(d')$ residues belonging to the expondnt d' (mod p). The theorem now follows.

From here on, we restrict n so that $1 \leq n \leq p - 1$. Then not every nonzero residue of p can appear among the residues of n^n (mod p). This follows from the fact that the residue 1 appears at least twice, since

$$1^1 \equiv 1 \quad \text{and} \quad (p - 1)^{p-1} \equiv 1 \pmod{p}.$$

We shall now address ourselves to determining how many and what types of residues modulo p can appear among the residues of n^n (mod p), where $1 \leq n \leq p - 1$.

2. A NEW UPPER BOUND FOR THE NUMBER OF DISTINCT RESIDUES OF n^n

Let $A(p)$ be the number of distinct residues of n^n (mod p), $1 \le n \le p - 1$. Roger Crocker [5] showed that

$$\sqrt{(p-1)/2} \le A(p) \le p - 4.$$

We obtain a much better upper bound for $A(p)$ in the following theorem.

Theorem 2: Let p be an odd prime. Let $A(p)$ be the number of distinct residues of n^n (mod p), where $1 \le n \le p - 1$. Then

$$A(p) < 3p/4 + C_1(\varepsilon)p^{1/2+\varepsilon}$$

where ε is any positive real number and $C_1(\varepsilon)$ is a constant depending solely on ε.

To establish Theorem 2 we shall estimate the number $N(p)$ of quadratic nonresidues not appearing among the residues of n^n (mod p), where $1 \le n \le p - 1$. We will in fact show that

$$N(p) > p/4 + C_2(\varepsilon)p^{1/2+\varepsilon} \tag{7}$$

where ε is any positive real number and $C_2(\varepsilon)$ is a constant dependent only on ε. It is easily seen that Theorem 2 then immediately follows.

The only way that n^n, $1 \le n \le p - 1$, can be a quadratic nonresidue is if n is odd. However, if n is odd and n is a quadratic residue, then n^n is not a quadratic nonresidue of p. Let $N_1(p)$ be the number of odd quadratic residues modulo p. Then

$$N(p) \ge N_1(p), \tag{8}$$

since the number of odd integers in the interval $(0, p)$ and the number of quadratic nonresidues are both equal to $(p-1)/2$. We refine inequality (8) by the following lemma.

Lemma 1: Let p be an odd prime. Let $1 \le n \le p - 1$. Let $N_1(p)$ be the number of integers in the interval $(0, p)$ for which n is an odd quadratic residue modulo p.

(i) At least $N_1(p)$ quadratic nonresidues do not appear among the residues of n^n (mod p).

(ii) If $p > 5$ and $p \equiv 5$ (mod 8) or $p > 7$ and $p \equiv 7$ (mod 8), then at least $N_1(p) + 1$ quadratic nonresidues (mod p) do not appear.

Proof: The proof of (i) follows from our discussion preceding the lemma. To prove (ii), first assume $p \equiv 5$ (mod 8). Then $(p + 1)/2$ and $p - 2$ are both odd quadratic nonresidues. Now, using Euler's criterion

$$((p+1)/2)^{(p+1)/2} \equiv 1^{(p+1)/2}/2^{(p+1)/2} \equiv 1/(2)2^{(p-1)/2} \equiv -1/2 \pmod{p}. \tag{9}$$

Also

$$(p-2)^{p-2} \equiv (-2)^{p-1}/(-2) \equiv -1/2 \pmod{p}. \tag{10}$$

Thus the quadratic nonresidues $((p + 1)/2)^{(p+1)/2}$ and $(p - 2)^{p-2}$ are identical. Now n^n can be a quadratic nonresidue only if n is already a quadratic nonresidue (in fact odd) and two such residues repeat. Thus, by part (i), at least $N_1(p) + 1$ residues do not appear among $\{n^n\}$ modulo p, where $1 \le n \le p - 1$.

Now suppose $p \equiv 7$ (mod 8). Then $(3p - 1)/4$ and $p - 2$ are both odd quadratic nonresidues modulo p. Further,

$$\begin{aligned}((3p-1)/4)^{(3p-1)/4} &\equiv -1/4^{(3p-1)/4} \equiv -1/2^{(3p-1)/2} \\ &\equiv -1/2^{3((p-1)/2)+1} \equiv -1/2 \pmod{p}.\end{aligned} \tag{11}$$

Again,

$$(p-2)^{p-2} \equiv -1/2 \pmod{p}. \tag{12}$$

The result now follows as before.

According to Lemma 1, we now need a determination of $N_1(p)$ to establish an upper bound for $A(p)$. Lemmas 2, 3, and 4 will provide this information.

Lemma 2: If $p \equiv 1 \pmod 4$, then $N_1(p) = (p - 1)/4$.

Proof: Let r be a quadratic nonresidue modulo p. Then $p - r$ is also a quadratic nonresidue. But exactly one of p and $p - r$ is odd. Hence exactly half of the $(p - 1)/2$ quadratic nonresidues of p are odd and $N_1(p) = (p - 1)/4$.

Lemma 3: If $p \equiv 7 \pmod 8$, then $N_1(p) = (p - 1 - 2h(-p))/4$, where $h(-p)$ is the class number of the algebraic number field $Q(\sqrt{-p})$.

Proof: It is known (see [3]) that

$$h(-p) = V - T,$$

where V and T denote the number of quadratic residues and quadratic nonresidues in the interval $(0, p/2)$, respectively. To evaluate $V - T$, we will make use of the sum of Legendre symbols

$$S = \sum_{0<n<p/2} (n/p).$$

We partition S in two different ways as

$$S = S_1 + S_2 = S' + S'', \tag{13}$$

where

$$S_1 = \sum_{0<n<p/4} (n/p), \quad S_2 = \sum_{p/4<n<p/2} (n/p)$$

$$S' = \sum_{\substack{0<n<p/2 \\ n \text{ even}}} (n/p), \quad S'' = \sum_{\substack{0<n<p/2 \\ n \text{ odd}}} (n/p).$$

It is known (see [2]) that $S_2 = 0$. Then

$$S = S' + S'' = (2/p) \sum_{0<j<p/4} (j/p) + S'' = (1)S_1 + S'' = S_1 + S_2. \tag{14}$$

Hence $S'' = S_2 = 0$.

Now let V_o and T_o denote the number of odd quadratic residues and nonresidues in $(0, p/2)$, respectively. Let V_e and T_e be the number of even quadratic residues and nonresidues in $(0, p/2)$, respectively. Inspection shows that

$$V_o + T_o = (p + 1)/4 \quad \text{and} \quad V_e + T_e = (p - 3)/4.$$

Since $S'' = 0$,

$$V_o = T_o = (p + 1)/8. \tag{15}$$

Further,

$$h(-p) = V - T = (V_o - T_o) + (V_e - T_e) = V_e - T_e. \tag{16}$$

Also,

$$(p - 3)/4 = V_e + T_e. \tag{17}$$

Solving (16) and (17) for T_e, we obtain

$$T_e = (p - 3 - 4h(-p))/8.$$

Finally,

$$N_1(p) = V_o + T_e = (p - 1 - 2h(-p))/4,$$

since the number of odd quadratic residues in $(p/2, p)$ equals the number of even quadratic nonresidues in $(0, p/2)$.

Lemma 4: If $p \equiv 3 \pmod 8$, then $N_1(p) = (p - 1 + 6h(-p))/4$.

Proof: We shall use the same notation as in the proof of Lemma 3. By [3],

$$h(-p) = 1/3(V - T).$$

As in the proof of Lemma 3, we now evaluate the sum of Legendre symbols S.

$$S = S' + S'' = (2/p) \sum_{0<j<p/4} (j/p) + S'' = (-1)S_1 + S'' = S_1 + S_2. \tag{18}$$

However, it is known (see [2]) that $S_1 = 0$. Hence, $S_2 = S'' = S$ and $S_1 = S'$. Examination shows that

$$V_e + T_e = (p - 3)/4 \quad \text{and} \quad V_o + T_o = (p + 1)/4.$$

Since $S' = 0$,

$$V_e = T_e = (p - 3)/8. \tag{19}$$

Thus,

$$h(-p) = (1/3)(V - T) = (1/3)[(V_o - T_o) + (V_e - T_e)] = (1/3)(V_o - T_o) \tag{20}$$

and

$$(p + 1)/4 = V_o + T_o.$$

Solving (20) and (21) for V_o, we obtain

$$V_o = (12h(-p) + p + 1)/8.$$

Hence

$$N_1(p) = V_o + T_e = (p - 1 + 6h(-p))/4.$$

We utilize our results of Lemmas 1-4 in estimating $N(p)$ in the following theorem.

Theorem 3: Let p be an odd prime. Let $N(p)$ be the number of quadratic nonresidues not appearing among the residues n^n, where $1 \le n \le p - 1$.

(i) $N(p) \ge (p - 1)/4$ if $p \equiv 1 \pmod 8$.
(ii) $N(p) \ge (p + 3)/4$ if $p > 5$ and $p \equiv 5 \pmod 8$.
(iii) $N(p) \ge (p - 1 + 6h(-p))/4$ if $p \equiv 3 \pmod 8$.
(iv) $N(p) \ge (p + 3 - 2h(-p))/4$ if $p > 7$ and $p \equiv 7 \pmod 8$.

Proof: This follows from Lemmas 1-4.

We are now ready for the proof of our main theorem.

Proof of Theorem 2: By Siegel's theorem [1],

$$h(-p) < C_2(\varepsilon)p^{1/2+\varepsilon}$$

where ε is a positive real number and $C_2(\varepsilon)$ is a constant dependent solely on ε. Note that

$$A(p) \le p - 1 - N(p).$$

The theorem now follows from Theorem 3.

3. PRIMITIVE ROOTS NOT APPEARING AMONG THE RESIDUES OF n^n

In Section 2 we determined lower bounds for the number of quadratic nonresidues not appearing among the residues of n^n modulo p. In this section we determine lower bounds for the number of primitive roots (mod p) that do not appear among the residues of n^n (mod p), where $1 \le n \le p - 1$. Crocker [4] has shown that n^n can be congruent to a primitive root (mod p) only if $(n, p - 1) = 1$, where $1 \le n \le p - 1$. Using this criterion, we shall prove Theorem 4.

Theorem 4: Let p be an odd prime. Let $1 \le n \le p - 1$.

(i) At least one primitive root does not appear among the residues of n^n (mod p).

(ii) If $p \equiv 1$ (mod 8) or $p \equiv 3$ (mod 8) and $p > 3$, then at least three primitive roots do not appear among the residues of n^n (mod p).

Proof of (i): Note that 1^1 is not congruent to a primitive root (mod p). Now, n^n can be congruent to a primitive root (mod p) only if $(n, p - 1) = 1$. Certainly $(1, p - 1) = 1$. Hence at least one primitive root does not appear among the residues of n^n (mod p), since n^n can be a primitive root only if n already is.

Proof of (ii): Suppose $p \equiv 1$ (mod 8). Then $((p + 1)/2, p - 1) = 1$. But

$$((p + 1)/2)^{(p+1)/2} \equiv 1/2 \pmod{p}. \tag{22}$$

However, $(2^{-1}/p) = 1$. Thus $((p + 1)/2)^{(p+1)/2}$ is not congruent to a primitive root (mod p). Also, $(p - 2, p - 1) = 1$ and

$$(p - 2)^{p-2} \equiv -1/2 \pmod{p}. \tag{23}$$

Again, $((-2)^{-1}/p) = 1$ and $(p - 2)^{p-2}$ is not congruent to a primitive root (mod p). Hence at least three primitive roots do not appear.

Now suppose $p \equiv 3$ (mod 8). As before, $(p - 2, p - 1) = 1$, and $(p - 2)^{p-2}$ is not congruent to a primitive root (mod p). Further, $((p + 1)/4, p - 1) = 1$ and

$$((p + 1)/4)^{(p+1)/4} \equiv -1/2 \pmod{p}. \tag{24}$$

However, $((-2)^{-1}/p) = 1$ and consequently $((p + 1)/4)^{(p+1)/4}$ is not congruent to a primitive root (mod p). Thus at least three primitive roots do not appear among the residues of n^n (mod p) if $p \equiv 3$ (mod 8) and $p > 3$.

4. THE NUMBER OF TIMES THE RESIDUES 1 AND -1 APPEAR

Theorems 5 and 6 in this section will show that there is no upper bound for the number of times that the residues 1 or -1 can appear among the residues of n^n (mod p), $1 \le n \le p - 1$, where p is allowed to vary among all the primes.

Theorem 5: Let M be any positive integer. Let $\{p_i\}$ be the set of primes such that

$$x^x \equiv 1 \pmod{p_i}, \tag{25}$$

where $1 \le x \le p_i - 1$, has at least M solutions. Then $\{p_i\}$ has positive density in the set of primes.

Proof: Let $N = M - 1$. Let $p \equiv 1$ (mod 2^N) be a prime. Suppose that 2 is a 2^Nth power (mod p). Then, if $0 \le k \le N - 1$, 2^k is a 2^kth power (mod p). Further, if $0 \le k \le N - 1$, $(p - 1)/2^k$ is an even integer. Now, if x is a dth power (mod p) and $p \equiv 1$ (mod d), then

$$x^{(p-1)/d} \equiv 1 \pmod{p}.$$

Hence, if $0 \le k \le N - 1$,

$$((p - 1)/2^k)^{(p-1)/2^k} \equiv (-1)^{(p-1)/2^k}/(2^k)^{(p-1)/2^k} \equiv 1/1 \equiv 1 \pmod{p}. \tag{26}$$

Thus we now have M solutions to congruence (25); namely, 1 and $(p - 1)/2^k$ for $0 \le k \le N - 1$.

We now show that the set of primes p_i such that $p_i \equiv 1$ (mod 2^N) and 2 is a 2^Nth power (mod p) indeed has positive density t in the set of primes. Let ζ be a primitive 2^Nth root of unity. Let L be the algebraic number field

$$Q(2^{1/2^N}, \zeta).$$

Let $p \equiv 1$ (mod 2^N) be a rational prime. Suppose that 2 is a 2^Nth power (mod p).

By Kummer's theorem, this occurs if and only if, in the field L, p splits completely in each of the subfields $Q(\zeta^k \cdot 2^{1/2^N})$, where $1 \leq k \leq 2^N$. Let P be a prime ideal of L dividing the principal ideal (p). Let Z_P be the decomposition field of P. Then $Z_P \supseteq Q(\zeta^k \cdot 2^{1/2^N})$ for $1 \leq k \leq 2$, since p splits completely in each of these subfields. Hence $Z_P \supseteq Q(\zeta, 2^{1/2^N}) = L$, the compositum of the subfields $Q(\zeta^k \cdot 2^{1/2^N})$, where $1 \leq k \leq 2^N$. Let D_P be the decomposition group of P. Then $D_P = \langle 1 \rangle$ for all prime ideals P dividing (p). Thus, by the Tchebotarev density theorem, the density

$$t = 1/[L:Q] = 1/2^{2N-2} = 1/2^{2M-4} > 0. \tag{27}$$

Theorem 6: Let M be any positive integer. Let $\{p_i\}$ be the set of primes such that the congruence

$$x^x \equiv -1 \pmod{p_i}, \tag{28}$$

where $1 \leq x \leq p_i - 1$, has at least M solutions. Then $\{p_i\}$ has positive density in the set of primes.

Proof: Let $N = M - 1$. Let p be a prime and suppose that $p \equiv 1 \pmod{2 \cdot 3^N}$ and $p \equiv 7 \pmod 8$. Suppose further that both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p). Note that if $p \equiv 7 \pmod 8$, $(2/p) = 1$, and it is possible that 2 is a $(2 \cdot 3^N)$th power (mod p). Then, if $1 \leq k \leq N$, $2 \cdot 3^k$ is a $(2 \cdot 3^k)$th power (mod p). Moreover, if $1 \leq k \leq N$, $(p - 1)/(2 \cdot 3^k)$ is an odd integer. Hence, if $1 \leq k \leq N$,

$$\begin{aligned}((p-1)/(2 \cdot 3^k))^{(p-1)/(2\cdot 3^k)} &\equiv (-1)^{(p-1)/(2\cdot 3^k)} / (2 \cdot 3\ )^{(p-1)/(2\cdot 3^k)} \\ &\equiv -1/1 \equiv -1 \pmod p. \end{aligned} \tag{29}$$

Thus we now have M solutions to congruence (28).

I now claim that the set of primes $\{p_i\}$ such that $p_i \equiv 1 \pmod{2 \cdot 3\ }$, $p_i \equiv 7 \pmod 8$, and both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p_i) has positive density u in the set of primes. Let ζ be a primitive $(4 \cdot 3^N)$th root of unity. Let L be the algebraic number field

$$Q(\zeta, 2^{1/(2\cdot 3^N)}, 3^{2/(2\cdot 3^N)}).$$

Suppose that p is a rational prime and that $p \equiv 1 \pmod{2 \cdot 3^N}$ and $p \equiv 7 \pmod 8$. Assume that both 2 and 3 are $(2 \cdot 3^N)$th powers (mod p). Then, by Kummer's theorem, p splits completely in each of the subfields

$$Q(\zeta^{2k} \cdot 2^{1/(2\cdot 3^N)}) \quad \text{and} \quad Q(\zeta^{2k} \cdot 3^{1/(2\cdot 3^N)}),$$

where $1 \leq k \leq 2 \cdot 3^N$. Hence p splits completely in

$$K = Q(\zeta^2, 2^{1\ (2\cdot 3^N)}, 3^{1\ (2\cdot 3^N)}),$$

the compositum of these subfields. Let P be a prime ideal in L dividing (p). Then, if Z_P is the decomposition field of P, $Z_P \supseteq K$. Furthermore, since $p \equiv 7 \pmod 8$, $(-1/p) = -1$, and p does not split in the subfield $Q(\sqrt{-1})$ of L. Consequently, $Z_P \not\supseteq Q(\sqrt{-1})$. Let σ be the automorphism of Gal (L/Q) such that

$$\sigma(\zeta) = -\zeta, \tag{30}$$

$$\sigma(2^{1/(2\cdot 3^N)}) = 2^{1/(2\cdot 3^N)},$$

and

$$\sigma(3^{1/(2\cdot 3^N)}) = 3^{1/(2\cdot 3^N)}.$$

Then $\langle\sigma\rangle$ is the subgroup of Gal (L/Q) fixing K. It follows that the decomposition group $D_P = \langle\sigma\rangle$ for all prime ideals P dividing (p). By the Tchebotarev density theorem, the density

$$u = 1/[L:Q] = 1/(8 \cdot 3^{3N-1}) = 1/(8 \cdot 3^{3M-4}) > 0. \tag{31}$$

5. CONCLUDING REMARK

Further problems concerning the residues of n^n (mod p), where $1 \leq n \leq p-1$, are obtaining better upper and lower bounds for the number of distinct residues appearing among $\{n^n\}$ and determing estimates for the number of times that residues other than ±1 may occur.

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A GENERALIZATION OF A PROBLEM OF STOLARSKY

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For fixed positive integer $k \geq 1$, we set

$$a_k = [\overline{k}] = \frac{k + \sqrt{k^2 + 4}}{2},$$

a real number with completely periodic continued fraction expansion and period of length one. For all integers $n \geq 1$, we use $f_k(n)$ to denote the nearest integer to na_k.

Using this notation, we define an array $(b_{i,j}^{(k)})$ as follows. The first row has

$$b_{1,1}^{(k)} = 1 \quad \text{and} \quad b_{1,j}^{(k)} = f\ (b_{1,j-1}^{(k)}), \text{ for all } j \geq 2.$$

After inductively setting $b_{i,1}^{(k)}$ to be the smallest integer that has not occurred in a previous row, we define the remainder of the ith row by

$$b_{i,j}^{(k)} = f_k(b_{i,j-1}^{(k)}), \text{ for all } j \geq 2.$$

K. Stolarsky [4] developed this array for $k = 1$, showed that each positive integer occurs exactly once in the array, and proved that any three consecutive entries of each row satisfy the Fibonacci recursion. The latter result can be viewed as a generalization of a result of V. E. Hoggatt, Jr. [3, Theorem III]. In Theorem 1, we prove an analogous result for general k.

<u>*Theorem 1*</u>: Each positive integer occurs exactly once in the array $(b_{i,j}^{(k)})$. Moreover, the rows of the array satisfy

$$b_{i,j+2}^{(k)} = kb_{i,j+1}^{(k)} + b_{i,j}^{(k)}, \text{ for all } i,\ j \geq 1.$$

<u>*Proof*</u>: By construction, each positive integer occurs at least once. For $m \neq n$ we have $|(n-m)a_k| > 1$ and so $f_k(m) \neq f_k(n)$. Since the first column entry is the smallest in any row, every positive integer occurs exactly once.

Since

$$b_{i,j+2}^{(k)} = f_k(b_{i,j+1}^{(k)}) = f_k(f_k(b_{i,j}^{(k)})),$$

it suffices to show that $f_k(f_k(m)) = kf_k(m) + m$, for all $m \geq 1$. For any $m \geq 1$, $f_k(m) = ma_k + r$ for some $|r| \leq 1/2$. Hence

$$(a_k - k)f_k(m) = \frac{1}{a_k}(ma_k + r) = m + \frac{r}{a_k}, \text{ where } \left|\frac{r}{a_k}\right| < \frac{1}{2a_k} < \frac{1}{2}.$$

This implies that $f_k(f_k(m)) = kf_k(m) + m$, completing the proof.

For all integers $i \geq 1$, we set

$$b_{i,0}^{(k)} = b_{i,2}^{(k)} - kb_{i,1}^{(k)}.$$

For $k = 1$, Stolarsky [4] considered the sequence $\{b_{i,0}^{(1)}\}$ of differences and asked whether or not the sequence was a subset of the union of the first and second columns of $(b_{i,j}^{(1)})$. The following theorem shows that, for $k \geq 2$, no analogous result can hold.

Theorem 2: For $k \geq 2$, every positive integer occurs at least $k - 1$ times in the sequence of differences. For $k = 1$, D is a difference if and only if D occurs twice in the sequence $\{f_1(n) - n\}$.

Proof: Fix $k \geq 1$. If we append the difference $b_{i,0}^{(k)}$ to the beginning of the ith row of the array, this new augmented array contains (with the same multiplicity) all the elements of the sequence with general term $f_k(n) - kn$; that is, the nearest integer to $n(a_k - k)$.

Since $a_k - k < 1/k$, every positive integer must occur at least k times in the sequence $\{f_k(n) - kn\}$. On the other hand, in Theorem 1 we showed that the unaugmented array contains each positive integer exactly once. Therefore, for $k \geq 2$, every positive integer occurs at least $k - 1$ times in $\{b_{i,0}^{(k)}\}$.

From $a_1 - 1 > 1/2$, we know that any integer can occur at most twice in the sequence $\{f_1(n) - n\}$. Then, since $\{b_{i,0}^{(1)}\}$ is the modification of $\{f_1(n) - n\}$ obtained by deleting one copy of the set of positive integers, D is a difference for $k = 1$ if and only if D occurs twice in $\{f_1(n) - n\}$.

J. C. Butcher [1] and M. D. Hendy [2] have independently proved Stolarsky's conjecture and also have shown that the sequence of differences, for $k = 1$, is exactly the union of the first and second columns. In the remainder of this note we give another proof of these facts.

The following is an extension of Lemma 1 in [1].

Lemma 1: For any positive integers i, k, $j + 1$,

$$\frac{1}{2a_k^j} < \left|b_{i,j}^{(k)}a_k - b_{i,j+1}^{(k)}\right| < \frac{1}{2a_k^{j-1}}.$$

Proof: By definition of $b_{i,1}^{(k)}$, the left inequality holds for $j = 0$. Also, since $b_{i,2}^{(k)}$ is the nearest integer of $b_{i,1}^{(k)}a_k$, we have

$$\left|b_{i,0}^{(k)}a_k - b_{i,1}^{(k)}\right| = a_k\left|b_{i,0}^{(k)} - b_{i,1}^{(k)}\frac{1}{a_k}\right| = a_k\left|b_{i,2}^{(k)} - b_{i,1}^{(k)}a_k\right| < \frac{a_k}{2},$$

proving the right inequality for $j = 0$.

From the recursion formula to Theorem 1 we obtain

$$b_{i,j}^{(k)}a_k - b_{i,j+1}^{(k)} = a_k(b_{i,j+2}^{(k)} - a_kb_{i,j+1}^{(k)}).$$

Therefore, the lemma follows by induction from our verification for $j = 0$.

We henceforth only consider $k = 1$. Therefore, we suppress the index k and let $a = a_1 = \dfrac{1 + \sqrt{5}}{2}$ for the remainder of this paper.

Definition: Let D be a positive integer. We call D *early* if either $D = b_{i,1}$ or $D = b_{i,2}$ for some $i \geq 1$. D is called *late* if $D = b_{i,j}$ for some $i \geq 1$, $j \geq 3$.

The following corollary is a consequence of Lemma 1 and Theorem 1.

Corollary: Let D be a positive integer. Then D is early if and only if

$$\min_n |D - na| > \frac{1}{2a};$$

D is late if and only if

$$\min_n |D - na| < \frac{1}{2a},$$

where each minimum is taken over the set of integers.

Proof: By Theorem 1 there exist $i, j \geq 1$ for which $D = b_{i,j}$. By definition, $b_{i,j+1}$ is the nearest integer to Da. Therefore, from Lemma 1,

$$\frac{1}{2a^j} < |b_{i,j}a - b_{i,j+1}| = \min_n |Da - n| < \frac{1}{2a^{j-1}}. \tag{1}$$

Since D is an integer,

$$\min_n |Da - n| = \min_n |D(a - 1) - n| = \frac{1}{a} \min_n |D - na|.$$

Hence, (1) implies that

$$\frac{1}{2a^{j-1}} < \min_n |D - na| < \frac{1}{2a^{j-2}},$$

completing the proof.

Lemma 2: For any three (two) consecutive integers, at most two (at least one) are differences. Also, if both $N \pm 1$ are not differences, then both $N - 2$ and $N - 3$ are differences.

Proof: First, we suppose that the three consecutive integers N, $N \pm 1$ are differences. Then by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} > N - 1 \quad \text{and} \quad \frac{b + 5}{a} + \frac{1}{2} < N + 2;$$

that is,

$$\frac{b + 5}{a} - \frac{3}{2} < N < \frac{b}{a} + \frac{3}{2},$$

which contradicts $a < 5/3$. Since $a > 3/2$, the alternative statement follows similarly from Theorem 3.

By the alternative statements, if neither $N \pm 1$ is a difference, then both N and $N - 2$ are differences. Therefore, if $N - 3$ were to occur only once we would contradict $a > 8/5$.

Theorem 3: D is a difference if and only if D is early.

Proof: We suppose, on the contrary, that there exists a smallest integer D for which the theorem fails.

First, we assume that D is early but not a difference. By Lemma 2, $D - 1$ must be a difference and, by our assumption on D, is therefore early. Let na be the smallest multiple of a greater than $D - 1$. Since both $D - 1$ and D are early and $a = 1 + 1/a$, the corollary implies that

$$D - 1 + \frac{1}{2a} < na < D - \frac{1}{2a}. \tag{2}$$

Hence

$$|(D - 2) - (n - 1)a| < \frac{1}{2a},$$

and so, by the corollary, $D - 2$ is late. From our assumption on D, we thus obtain that $D - 2$ is not a difference. Because neither D nor $D - 2$ is a difference, by Lemma 2 both $D - 3$ and $D - 4$ are differences; thus $D - 3$ and $D - 4$ are early, and (2) implies that

$$D - 4 + \frac{1}{2a} < (n - 2)a < D - 3 - \frac{1}{2a}.$$

Combining this with (2) and successively manipulating, we obtain

$$D + \frac{5}{2}a - \frac{9}{2} = D - 4 + 2a + \frac{1}{2a} < na < D - \frac{1}{2a} = D - \frac{a}{2} + \frac{1}{2},$$

$$na + \frac{a}{2} - \frac{1}{2} < D < na - \frac{5}{2}a + \frac{9}{2},$$

$$n + 1 - \frac{a}{2} = n + \frac{1}{2} - \frac{1}{2a} < \frac{D}{a} < n - \frac{5}{2} + \frac{9}{2a} = n + \frac{9}{2}a - 7. \tag{3}$$

On the other hand, by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} < D - 2 \quad \text{and} \quad \frac{b + 5}{a} + \frac{1}{2} > D + 1;$$

that is,

$$b - D + \frac{5}{2}a < Da - D = \frac{D}{a} < b - D + 5 - \frac{a}{2}.$$

Comparing this with (3), we have

$$n + 1 - \frac{a}{2} < b - D + 5 - \frac{a}{2} \quad \text{and} \quad n + \frac{9}{2}a - 7 > b - D + \frac{5}{2}a;$$

that is, the integer $b - d$ satisfies

$$n - 4 < b - D < n - 7 + 2a < n - 3,$$

a contradiction.

Therefore, D is a late difference. By the corollary there exists an integer n for which

$$|na - D| < \frac{1}{2a}.$$

Hence

$$|(n - 1)a - (D - 1)| > \frac{1}{2a}$$

and, by the corollary, $D - 1$ is early and so is a difference. Since $D - 1$ and D are both differences, Lemma 2 implies that $D - 2$ cannot be a difference and so is late. Therefore,

$$|(n - 1)a - (D - 2)| < \frac{1}{2a}.$$

Combining this with the previous inequality

$$|na - D| < \frac{1}{2a},$$

we have

$$D - \frac{1}{2a} < na < D - 2 + \frac{1}{2a} + a.$$

We manipulate this to get the following two sets of inequalities:

$$n - 4 + \frac{5}{2}a < \frac{D}{a} < n + 1 - \frac{a}{2}, \tag{4}$$

and

$$D - 4 + \frac{1}{2a} < D - \frac{1}{2a} - 2a < (n - 2)a < D - 2 + \frac{1}{2a} - a = D - 3 - \frac{1}{2a}.$$

The latter implies that both $D - 3$ and $D - 4$ are early and so, by assumption, are differences. Therefore, by Theorem 2 there exists an integer b for which

$$\frac{b}{a} + \frac{1}{2} > D - 4 \quad \text{and} \quad \frac{b + 8}{a} + \frac{1}{2} < D + 1;$$

that is,

$$b - D + 8 - \frac{a}{2} < \frac{D}{a} < b - D + \frac{9}{2}a.$$

Comparing this with (4), we obtain

$$n - 8 < n - 4 - 2a < b - D < n - 7,$$

which is contrary to the fact that $b - D$ and n are integers.

ACKNOWLEDGMENT

The author would like to thank Gerald Bergum for his aid in increasing the readability of this paper.

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INITIAL DIGITS IN NUMBER THEORY

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INTRODUCTION

It has been observed empirically by various authors (cf. Raimi [5] and his references) that the numbers in "random" tables of physical or other data tend to begin with low digits more frequently than one might on first consideration expect. In fact, in place of the plausible-looking frequency of 1/9, it is found that for the numbers with first significant digit equal to

$$a \in \{1, 2, \ldots, 9\}$$

in any particular table the observed proportion is often approximately equal to

$$\log_{10}\left(1 + \frac{1}{a}\right).$$

A variety of explanations have been put forward for this surprising phenomenon.

Although more general cases have also been considered, most people might agree that it should suffice to consider only sets of positive integers, since empirical data are normally listed in terms of finite lists of numbers with finite decimal expansions (for which the signs or positions of decimal points are immaterial here). On accepting this simplification, the common tendency would probably then be to seek an explanation in terms of the concept of *natural density* of a set T of positive integers, i.e.,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x,\, n \in T} 1.$$

Unfortunately, this density simply does not exist for the immediately relevant set $N(a)$ of all positive integers beginning with the digit a as above, and this fact seems to have led both to a search for alternative explanations and to a certain amount of controversy as to what should actually constitute a satisfactory "explanation." Ignoring the latter difficulty for the moment (regarding which some further comments are offered in Section 3 below), the situation may be summarized by noting that various explanations have been suggested in terms of extensions of the density concepts that do exist and take tne experimentally observed value of $\log_{10}(1 + 1/a)$ for the set $N(a)$; the most general and convincing of such approaches is perhaps that of Cohen [1].

The main purpose of this note is to add to these explanations by showing that the same type of initial-digit phenomenon occurs in a variety of number-theoretical situations. A notable investigation of this phenomenon of specific number-theoretical interest is that of Whitney [7] regarging the set P of all *prime* numbers. Whitney employs perhaps the most commonly used extension of the density concept, *logarithmic* (or Dirichlet) density, and this will also be used below. His discussion uses a corollary of one of the deeper forms of the Prime Number Theorem.

Here, using only elementary methods, it will be shown that, for quite a wide class of sets T of positive integers possessing a natural density, the subset $T(a) = T \cap N(a)$ has the relative logarithmic density $\log_{10}(1 + 1/a)$ in T. More generally, for quite a wide class of arithmetical functions, f, the logarithmic average value of f over all positive integers compared with that over $N(a)$ is shown to be weighted in the ratio $1{:}\log_{10}(1 + 1/a)$. In the actual discussion below, 10 is replaced by an arbitrary base $q \geq 2$, and a is replaced by an arbitrary initial sequence $a_1, a_2, \ldots, a_r$ of digits $a_i \in \{0, 1, \ldots, q - 1\}$ with $a_1 \neq 0$.

1. LOGARITHMIC AVERAGES AND DENSITIES

In order to cover a variety of specific examples of arithmetical functions and sets of positive integers in a fairly wide setting, first consider any fixed integers $q \geq 2$ and

$$A = a_1 q^{r-1} + a_2 q^{r-2} + \cdots + a_r,$$

with $a_i \in \{0, 1, \ldots, q - 1\}$ and $a_1 \neq 0$. Let $N(A)$ denote the set of all positive integers whose canonical q-adic expansions begin with the sequence of digits $a_1, a_2, \ldots, a_r$. We first wish to present the following theorem.

Theorem 1.1: Let f denote a nonnegative, real-valued function of the positive integers such that

$$\sum_{n \leq x} f(n) = Bx^{\delta} + O(x^{\eta}) \text{ as } x \to \infty,$$

where B, δ, and η are constants with $0 < \delta$, $\eta < \delta$. Then

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x,\, n \in N(A)} f(n) n^{-\delta} = \delta B \log_q\left(1 + \frac{1}{A}\right).$$

Before proving Theorem 1.1, we need the following lemma.

Lemma 1.2: Under the hypothesis of Theorem 1.1, there exists a constant $\gamma = \gamma_f$ such that

$$\sum_{n \le x} f(n)n^{-\delta} = \delta B \log x + \gamma + O(x^{\delta - \eta}) \text{ as } x \to \infty.$$

Proof: This lemma is actually a special case of a result discussed in [3, p. 86]. However, for the reader's convenience, we outline a direct proof here. Let

$$F(x) = \sum_{n \le x} f(n).$$

Then by partial summation (cf. [2, Theorem 421]), one obtains

$$\begin{aligned}\sum_{n \le x} f(n)n^{-\delta} &= F(x)x^{-\delta} + \delta \int_1^x F(t)t^{-\delta-1}dt \\ &= [Bx^\delta + O(x^\eta)]x^{-\delta} + \delta \int_1^x [Bt^\delta + O(t^\eta)]t^{-\delta-1}dt \\ &= B + \delta B \log x + I(x) + O(x^{\eta-\delta}),\end{aligned}$$

where

$$\begin{aligned}I(x) &= \delta\left(\int_1^\infty - \int_x^\infty\right)[F(t) - Bt^\delta]t^{-\delta-1}dt \\ &= I - O\left(\int_x^\infty t^{\eta-\delta-1}dt\right) = I - O(x^{\eta-\delta}),\end{aligned}$$

for some constant I. The lemma follows, with $\gamma = B + I$.

Proof of Theorem 1.1: In order to deduce Theorem 1.1, first consider

$$x_m = (A + 1)q^m.$$

By Lemma 1.2 (using the convention that $Aq^0 - 1$ be replaced by 1 if $A = 1$), we have

$$\begin{aligned}\sum_{n < x_m,\, n \in N(A)} f(n)n^{-\delta} &= \sum_{t=0}^{m} \sum_{Aq^t \le n (A+1)q^t} f(n)n^{-\delta} \\ &= \sum_{t=0}^{m} \left\{\delta B \log \frac{(A+1)q^t - 1}{Aq^t - 1} + O(q^{t(\eta-\delta)})\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n < x_m,\, n \in N(A)} f(n)n^{-\delta} &= \delta B \sum_{t=0}^{m} \log \frac{(A+1)q^t - 1}{Aq^t - 1} + O(1) \\ &= \delta B \sum_{t=0}^{m} \left\{\log\left(1 + \frac{1}{A}\right) + \log\left(1 - \frac{1}{(A+1)q^t}\right) \right. \\ &\qquad \left. - \log\left(1 - \frac{1}{Aq^t}\right)\right\} + O(1) \\ &= (m+1)\delta B \log\left(1 + \frac{1}{A}\right) + O(1),\end{aligned}$$

since (for $c \ge 1$),

$$\sum_{t=1}^{m} \log\left(1 - \frac{1}{cq^t}\right) = \sum_{t=1}^{m} O(q^{-t}) = O(1).$$

Now let $x_{m-1} \le x \le x_m$. Then $\log x \sim m \log q$ as $m, x \to \infty$, and [for $g(n) \ge 0$],

$$\sum_{n < x_{m-1},\, n \in N(A)} g(n) \le \sum_{n \le x,\, n \in N(A)} g(n) \le \sum_{n < x_m,\, n \in N(A)} g(n).$$

The asymptotic formula implies that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x,\, n\in N(A)} f(n)n^{-\delta} = \delta B\, \frac{\log(1 + 1/A)}{\log q} = \delta B \log_q\left(1 + \frac{1}{A}\right),$$

and Theorem 1.1 is proved.

We say that a function f of positive integers has *mean-value* (respectively, *logarithmic* mean-value) α over a set T of positive integers if and only if

$$\alpha = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x,\, n\in T} f(n)$$

$$\left(\text{respectively, } \alpha = \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x,\, n\in T} \frac{f(n)}{n}\right).$$

It is shown by Wintner [8, p. 52] that the existence of the logarithmic mean-value over a set T follows from that of the mean-value over T and the values are equal. The converse is false. Applying Theorem 1.1, we have

Corollary 1.3: Let f denote a nonnegative, real-valued function that possesses the mean-value B over all positive integers in the strong sense that there exists a constant $\eta < 1$ such that

$$\sum_{n\le x} f(n) = Bx + O(x^{\eta}) \text{ as } x \to \infty.$$

Then f possesses the logarithmic mean-value $B \log_q(1 + 1/A)$ over $N(A)$.

A subset S of a set T of positive integers is said to have the relative *logarithmic density* Δ in T if and only if

$$\Delta = \lim_{x\to\infty}\left(\sum_{n\le x,\, n\in S} \frac{1}{n}\right)\Big/\left(\sum_{n\le x,\, n\in T} \frac{1}{n}\right).$$

If the function f of Corollary 1.3 is replaced by the characteristic function of the set T in the set N of all natural numbers, we obtain

Corollary 1.4: Let T denote a set of positive integers having natural density B in the strong sense that there exists a constant $\eta < 1$ such that

$$\sum_{n\le x,\, n\in T} 1 = Bx + O(x^{\eta}) \text{ as } x \to \infty.$$

Then the set $T(A) = T \cap N(A)$ has the relative logarithmic density $\log_q(1 + 1/A)$ in T.

2. APPLICATIONS TO SPECIFIC SETS AND FUNCTIONS

In addition to the set N of all natural numbers, the following natural examples of sets T satisfying the hypothesis and hence the conclusion of Corollary 1.4 may be noted:

(2.1) Let $T_{m,r}$ denote the arithmetical progression

$$r,\ r + m,\ r + 2m,\ \ldots \quad (0 \le r < m).$$

Then clearly

$$\sum_{n\le x,\, n\in T_{m,r}} 1 = \sum_{r+km\le x} 1 = \left[\frac{x - r}{m}\right] = \frac{x}{m} + O(1) \text{ as } x \to \infty.$$

(2.2) Given any integer $k \ge 2$, let $N_{[k]}$ denote the set of all *k-free* positive integers, i.e., integers not divisible by any kth power $r^k \neq 1$. (Thus $N_{[2]}$ is the familiar set of all *square-free* numbers.) Then it is known (see, e.g., [3, p. 108]) that

$$\sum_{n \leq x,\, n \in N_{[k]}} 1 = \frac{x}{\zeta(k)} + 0(x^{1/k}) \text{ as } x \to \infty,$$

where

$$\zeta(k) = \sum_{n=1}^{\infty} n^{-k}.$$

(2.3) Let $T_{m,r,k} = T_{m,r} \cap N_{[k]}$, where $T_{m,r}$ and $N_{[k]}$ are the sets defined above. If m, r are coprime, it is known (see, e.g., [4, p. 112]) that as $x \to \infty$,

$$\sum_{n \leq x,\, n \in T_{m,r,k}} 1 = \frac{x}{m\zeta(k)} \prod_{\text{prime } p|m} (1 - p^{-k})^{-1} + 0(x^{1/k}),$$

where $\zeta(k)$ is as before.

Many naturally occurring arithmetical functions f satisfy the hypothesis and hence the conclusion of Corollary 1.3. Out of examples of such functions treated in books, we mention only two:

(2.4) Let $r(n)$ denote the number of lattice points (a, b) such that $a^2 + b^2 = n$. Then (see, e.g., [2, Theorem 339]),

$$\sum_{n \leq x} r(n) = \pi x + 0(x^{1/2}) \quad \text{as } x \to \infty.$$

(2.5) Let $a(n)$ denote the total number of nonisomorphic abelian groups of finite order n. A theorem of Erdös and Szekeres (see, e.g., [3, p. 117]) states that

$$\sum_{n \leq x} a(n) = x \prod_{k=2}^{\infty} \zeta(k) + 0(x^{1/2}) \quad \text{as } x \to \infty.$$

Next we mention a few examples of concrete arithmetical functions f satisfying the slightly more general hypothesis of Theorem 1.1:

(2.6) The Euler function

$$\phi(n) = \sum_{r \leq n (r,n) = 1} 1$$

has the property that

$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + 0(x \log x) \text{ as } x \to \infty$$

(see, e.g., [2, Theorem 330]).

(2.7) The divisor-sum function

$$\sigma(n) = \sum_{d|n} d$$

has the property that

$$\sum_{n \leq x} \sigma(n) = \frac{1}{12}\pi^2 x^2 + 0(x \log x) \text{ as } x \to \infty$$

(cf. [2, Theorem 324]).

(2.8) Given any positive integer k, let $T_{m,r}^k$ denote the set of all kth powers of numbers in the arithmetical progression $T_{m,r}$ of (2.1). Then,

$$\sum_{n \leq x,\, n \in T_{m,r}^k} 1 = \sum_{n \leq x^{1/k},\, n \in T_{m,r}} 1 = \frac{1}{m} x^{1/k} + 0(1),$$

by (2.1). Thus Theorem 1.1 applies to the characteristic function of the set $T^k_{m,r}$ in N.

Finally, it may be remarked that the applicability of Theorem 1.1 carries over to the restrictions to $T_{m,r}$ of arithmetical functions of the above kinds, when m, r are coprime. (For preliminary theorems that make such applications possible, see, e.g., [3, Ch. 9], [4, Ch. II], and Smith [6].)

3. "SCIENTIFIC" VERSUS MATHEMATICAL EXPLANATIONS

In [5] Raimi expresses some reservations about purely mathematical explanations of the initial-digit phenomenon in numerical tables of empirical data and calls for a more "scientific" discussion (e.g., in terms of statistical distribution functions). However, in this direction, general agreement does not seem to have been reached or even to be imminent. By way of contrast, even if it is theoretically correct to have done so, one might query whether such a problem would ever have been seriously raised in practice if it had not been for the nonexistence of certain desired natural densities.

For, suppose that a detailed examination of "random" tables of numerical data was found to show that, in most cases, approximately 1/10 of the numbers considered *end* in a particular digit $b \in \{0, 1, \ldots, 9\}$, or approximately 10 of them end in a particular sequence of digits $b_1, b_2, \ldots, b \in \{0, 1, \ldots, 9\}$. In view of the elementary example (2.1) above, surely very few people would be surprised by this or be led to call seriously for a "scientific" explanation, even though it is theoretically as legitimate to do so here as in the original problem.

Although the nonexistence of natural densities does on first consideration seem to lend an element of confusion to the initial-digit problem, the preceding remarks suggest that (unless overwhelming experimental evidence* warrants otherwise) it is perhaps nevertheless adequate for most purposes to accept an explanation in terms of one or more reasonable mathematical substitutes for natural density. In showing the quite widespread nature of this phenomenon in number theory, the earlier theorem and various mathematical examples perhaps lend further weight to this suggestion. After all, what can be scientifically interesting about the purely *numerological* properties of a list of street addresses, or areas of rivers, and so on?

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**The status of Raimi's anomalous population data $PP(n)$ [5, p. 522] is difficult to evaluate without further investigation, but his anomalous data $V(n)$ do not seem surprising if one remembers that telephone numbers normally have favoured initial digits.*

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FIBONACCI NUMBERS AND STOPPING TIMES

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For each integer $k \geq 2$, let $\{a_{n,k}\}$ and $\{b_{n,k}\}$ be two sequences of integers defined by $a_{n,k} = 0$ for all $n = 1, \ldots, k - 1$, $a_{k,k} = 1$, and

$$a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}$$

for all $n \geq k$; $b_{1,k} = 0$, and

$$b_{n,k} = a_{n,k} + \sum_{j=1}^{n-1} a_{j,k} b_{n-j,k}$$

for all $n \geq 2$.

Let $\{Y_n\}$ be the fair coin-tossing sequence, i.e.,

$$P(Y_j = 0) = \frac{1}{2} = P(Y_j = 1)$$

for all $j = 1, 2, \ldots,$ and $Y_1, Y_2, \ldots$ are independent. With respect to the sequence $\{Y_n\}$, for each integer $k \geq 1$, let $\{R_{n,k}\}$ and $\{N_{n,k}\}$ be two sequences of stopping times defined by

$$R_{1,k}(Y_1, Y_2, \ldots) = \inf \{m \mid Y_m = \cdots = Y_{m-k+1} = 0\},$$

$= \infty$ if no such m exists, and for all $n \geq 2$,

$$R_{n,k}(Y_1, Y_2, \ldots) = \inf \{m \mid m \geq R_{n-1,k} + k \text{ and } Y_m = \cdots = Y_{m-k+1} = 0\},$$

$= \infty$ if no such m exists; $N_{1,k} = R_{1,k}$ and $N_{n,k} = R_{n,k} - R_{n-1,k}$ for all $n \geq 2$.

In this note, we shall prove the following interesting theorems.

<u>*Theorem 1*</u>: For each integer $k \geq 2$,

$$a_{n,k} = 2^n P(N_{1,k} = n) \quad \text{and} \quad b_{n,k} = 2^n P(R_{m,k} = n \text{ for some integer } m \geq 1).$$

<u>*Theorem 2*</u>: For each integer $k \geq 2$,

$$b_{n,k} = 2b_{n-1,k} + 1 \quad \text{or} \quad 2b_{n-1,k} - 1 \quad \text{or} \quad 2b_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k - 1$.

<u>*Theorem 3*</u>: For each integer $k \geq 2$, let

$$\mu_k = \sum_{n=1}^{\infty} n 2^{-n} a_{n,k} = E(N_{1,k}),$$

then

$$b_{nk,k} = \{2^{nk} + 2^k - 2\}/\mu_k \quad \text{and} \quad b_{nk+j,k} = 2^{j-1}\{2^{nk+1} - 2\}/\mu_k$$

for all $n \geq 1$ and $j = 1, 2, \ldots, k - 1$.

We start with the following elementary lemmas.

<u>*Lemma 1*</u>: For each integer $k \geq 1$, let

$$\Phi_k(t) = E(t^{N_{1,k}}) \text{ if } E(|t|^{N_{1,k}}) < \infty;$$

then

$$\Phi_k(t) = \left(\frac{t}{2}\right)^k \Big/ \left\{1 - \sum_{j=1}^{k} \left(\frac{t}{2}\right)^j\right\} \quad \text{for all } -1 \leq t \leq 1.$$

Proof: For $k = 1$, it is well known that

$$\Phi_1(t) = \left(\frac{t}{2}\right) \Big/ \left\{1 - \left(\frac{t}{2}\right)\right\}$$

for all t in $[-1, 1]$. For $k \geq 2$, it is easy to see that $N_{1,k} = N_{1,k-1} + Z$, where Z is a random variable such that

$$P(Z = 1) = P(Z = 1 + N_{1,k}) = \frac{1}{2}$$

and Z is independent of $N_{1,k-1}$. Hence

$$\Phi_k(t) = \frac{t}{2}\{\Phi_{k-1}(t) + \Phi_k(t)\}$$

for all $-2 < t < 2$. Therefore, for each integer $k \geq 1$,

$$\Phi_k(t) = \left(\frac{t}{2}\right) \Big/ \left\{1 - \sum_{j=1}^{k}\left(\frac{t}{2}\right)\right\}$$

for all $-1 \leq t \leq 1$.

Lemma 2: For each integer $k \geq 2$, let

$$G_k(t) = \sum_{n=1}^{\infty} t^n a_{n,k}$$

for all t such that

$$\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty;$$

then

$$G_k(t) = \Phi_k(2t) \text{ for all } -\frac{1}{2} < t < \frac{1}{2} \text{ and } k \geq 2.$$

Proof: Since $a_{n,k} = 0$ for all $n = 1, 2, \ldots, k - 1$, $a_{k,k} = 1$, and

$$a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}$$

for all $n > k$,

$$G_k(t) = \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i G_k(t).$$

Therefore,

$$G_k(t) = t^k \Big/ \left\{1 - \sum_{j=1}^{k} t^j\right\}$$

for all $k \geq 2$ and all t such that

$$\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty.$$

Since $a_{n,k} \leq 2^n$ for all $n \geq 1$ and all $k \geq 2$, $G_k(t)$ exists for all $-\frac{1}{2} < t < \frac{1}{2}$. By Lemma 1, we have

$$G_k(t) = \Phi_k(2t) \text{ for all } t \text{ in the interval } \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and all } k \geq 2.$$

For each integer $k \geq 1$, let $u_{0,k} = 1$, and for all $n \geq 1$, let

$$u_{n,k} = P\{R_{m,k} = n \text{ for some integer } m \geq 1\}$$

and $f_{n,k} = P\{N_{1,k} = n\}$. Since $\{Y_n\}$ is a sequence of i.i.d. random variables, and $u_{0,k} = 1$, it is easy to see that

$$u_{n,k} = \sum_{j=1}^{n} f_{j,k} u_{n-j,k} \text{ for all } n \geq 1 \text{ and all } k \geq 1.$$

Hence we have the following theorem.

Theorem 1': For each integer $k \geq 2$, $2^n u_{n,k} = 2^n P\{R_{m,k} = n$ for some integer $m \geq 1\} = b_{n,k}$ and $2^n f_{n,k} = 2^n P\{N_{1,k} = n\} = a_{n,k}$ for all $n \geq 1$.

Let $A = \{(w_1, w_2, \ldots, w_n) | w_i = 0$ or 1 for all $i = 1, 2, \ldots, n$ and $w_j = 1 \neq w_{j+1} = \cdots = w_n = 0$ for some $j = n - Jk$ and some integer $J \geq 1\}$.

Let $B = \{(v_1, v_2, \ldots, v_{n-1}) | v_i = 0$ or 1 for all $i = 1, 2, \ldots, n - 1$ and $v_{j-1} = 1 \neq v_j = \cdots = v_{n-1} = 0$ for some $j = n - Jk$ for some integer $J \geq 1\}$.

Lemma 3: For each integer $k \geq 2$,

$$2^n u_{n,k} = 2^n u_{n-1,k} + 1 \quad \text{or} \quad 2^n u_{n-1,k} - 1 \quad \text{or} \quad 2^n u_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k - 1$.

Proof: By the definition of $\{u_{n,k}\}$, for each integer $k \geq 2$,

$$2^n u_{n,k} = \text{the number of elements in } A$$

and

$$2^{n-1} u_{n-1,k} = \text{the number of elements in } B.$$

(i) If $n = mk$ for some integer $m \geq 1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if $(v_1, v_2, \ldots, v_{n-1})$ is in B, and $(0, 0, \ldots, 0)$, n-tuple, is also in A even $(0, 0, \ldots, 0)$, $(n - 1)$-tuple, is not in B. Hence the number of elements in $A \geq 2 \cdot$ the number of elements in $B + 1$. Since each element $(w_1, w_2, \ldots, w_n)$ in A such that $w_j \neq w_{j+1}$ for some $1 \leq j \leq n - 1$ is a form of $(0, v_1, v_2, \ldots, v_{n-1})$ or a form of $(1, v_1, v_2, \ldots, v_{n-1})$ for some element $(v_1, v_2, \ldots, v_{n-1})$ in B. Hence the number of elements in $A \leq 2 \cdot$ the number of elements in $B + 1$. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in $B + 1$.

(ii) If $n = mk + 1$ for some integer $m \geq 1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if $(v_1, v_2, \ldots, v_{n-1})$ is in B and $v_j \neq v_{j+1}$ for some $1 \leq j \leq n - 2$ and $(1, 0, 0, \ldots, 0)$, n-tuple, is also in A [$(0, 0, \ldots, 0)$, $(n - 1)$-tuple, is in B]. Hence the number of elements in $A \geq 2 \cdot$ the number of elements in $B - 1$. Since each element $(w_1, w_2, \ldots, w_n)$ in A such that $w_j \neq w_{j+1}$ for some $2 \leq j \leq n - 1$ is a form of $(0, v_1, v_2, \ldots, v_{n-1})$ or a form of $(1, v_1, v_2, \ldots, v_{n-1})$ for some element $(v_1, v_2, \ldots, v_{n-1})$ in B. Hence the number of elements in $A \leq 2 \cdot$ the number of elements in $B - 1$. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in $B - 1$.

(iii) If $n = mk + j$ for some integers $m \geq 1$ and $2 \leq j \leq k - 1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if and only if $(v_1, v_2, \ldots, v_{n-1})$ is in B. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in B.

By (i), (ii), and (iii), the proof of Lemma 3 is now complete.

Theorem 2': For each integer $k \geq 2$,

$$b_{n,k} = 2b_{n-1,k} + 1 \quad \text{or} \quad 2b_{n-1,k} - 1 \quad \text{or} \quad 2b_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k - 1$.

Proof: By Theorem 1' and Lemma 3.

For each integer $k \geq 1$, let

$$\mu_k = E\{N_{1,k}\} = \sum_{n=1}^{\infty} nP(N_{1,k} = n) = \sum_{n=1}^{\infty} nf_{n,k}.$$

By Theorem 1',

$$\mu_k = \sum_{n=1}^{\infty} n 2^{-n} a_{n,k} \text{ for each integer } k \geq 2.$$

Since

$$\left(\frac{1}{2}\right)^k = \sum_{j=0}^{k-1} u_{n-j,k}\left(\frac{1}{2}\right)^j \text{ for all } n \geq k \text{ and } k \geq 1,$$

$$\left(\frac{1}{2}\right)^k = \lim_{n \to \infty} \sum_{j=0}^{k-1} u_{n-j,k}\left(\frac{1}{2}\right)^j.$$

By the Renewal Theorem (see [1, p. 330]), we have

$$\left(\frac{1}{2}\right)^k = \{E(N_{1,k})\}^{-1} \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^j \text{ for all } k \geq 1.$$

Hence

$$\mu_k = E\{N_{1,k}\} = \sum_{n=1}^{\infty} n f_{n,k} = \sum_{n=1}^{\infty} n 2^{-n} a_{n,k} = \sum_{j=1}^{k} 2^j = 2^{k+1} - 2.$$

Theorem 3': For each integer $k \geq 2$, let

$$\mu_k = \sum_{n=1}^{\infty} n 2^{-n} a_{n,k} = 2^{k+1} - 2;$$

then

$$b_{mk,k} = \{2^{mk} + 2^k - 2\}/\mu_k \quad \text{and} \quad b_{mk+j,k} = 2^{j-1}\{2^{mk+1} - 2\}/\mu_k$$

for all integers $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

Proof: By the definition of $\{b_{n,k}\}$, $b_{k,k} = 1$. Hence, by Theorem 2', Theorem 3' holds when $m = 1$. Suppose that Theorem 3' holds for $m = 1, 2, \ldots, M-1$ and $j = 1, 2, \ldots, k-1$, where M is an integer ≥ 2. Now, let $m = M$, then, by Theorem 2',

$$\begin{aligned} b_{Mk,k} = 2b_{Mk-1,k} + 1 &= 2^{k-1}\{2^{(M-1)k+1} - 2\}/\mu_k + 1 = (2^{Mk} - 2^k + 2^{k+1} - 2)/\mu_k \\ &= (2^{Mk} + 2^k - 2)/\mu_k, \end{aligned}$$

since $\mu_k = 2^{k+1} - 2$.

$$b_{Mk+1,k} = 2b_{Mk,k} - 1 = (2^{Mk+1} + 2^{k+1} - 4)/\mu_k - 1 = (2^{Mk+1} - 2)/\mu_k.$$

$$b_{Mk+j,k} = 2^{j-1} b_{Mk+1,k} = 2^{j-1}(2^{Mk+1} - 2)/\mu_k, \text{ for all } j = 2, 3, \ldots, k-1.$$

Hence Theorem 3' holds for $m = M$ and $j = 1, 2, \ldots, k-1$. Therefore, Theorem 3' holds for all $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

Corollary to Theorem 3': For each integer $k \geq 2$,

$$u_{mk,k} = \mu_k^{-1}\{1 + 2^{-mk+k} - 2^{-mk+1}\} = (2^{k+1} - 2)^{-1}\{1 + 2^{-mk+k} - 2^{-mk+1}\}$$

and

$$u_{mk+j,k} = \mu_k^{-1}\{1 - 2^{-mk} = (2^{k+1} - 2)^{-1}\{1 - 2^{-mk}\}$$

for all integers $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

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A NON-FIBONACCI SEARCH PLAN WITH FIBONACCI-LIKE RESULTS

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ABSTRACT

This article describes a nondeterministic search plan, hereinafter called the *mid-point technique*. While not optimal in the minimax sense, the plan offers several possible advantages over the Fibonacci technique. Further, the expected value of the reduction ratio at each stage is identical to the reduction ratio achieved by the minimax optimal Fibonacci method.

INTRODUCTION

Search techniques often use the minimax criterion as the assumed measure of effectiveness. As a result of Kiefer's pioneering work [4] demonstrating the minimax optimality of the Fibonacci search technique, a number of authors have focused attention on this particular search method. See, for instance, [1], [3], [5], [6], [7], and [8].

Unfortunately, in the authors' opinions, there are three disadvantages associated with the Fibonacci technique. First, the plan requires that the final reduction ratio be specified prior to beginning the search. Second, the Fibonacci search is one of the more complex unimodal sequential search techniques available, and this complexity may cause some potential users to avoid the Fibonacci technique in favor of a simpler method such as the dichotomous search or the golden section search [6]. Finally, if there is an upper bound on the number of experiments permitted, it may be impossible to achieve the required reduction ratio, i.e., the Fibonacci method does not provide the user with the option to gamble.

On a more fundamental level, the minimax criterion of optimality itself is open to challenge. The extremely pessimistic and jaundiced view of nature inherent within the minimax criterion may not represent a desirable framework from which to view the search procedure. While possibly valid for cases of warfare or for investors with extreme risk aversion, the minimax assumption of a malevolent opponent capable of altering the probabilities inherent within any gamble should be looked at with some skepticism. Murphy's Law notwithstanding, it is not reasonable to assume that all gambles taken by the searcher will necessarily be losing ones.

ASSUMPTIONS OF THE MID-POINT TECHNIQUE

The mid-point technique utilizes five assumptions. The first four are readily recognizable as being common ones often employed in search procedures. The fifth represents a significant departure from the minimax optimal Fibonacci method.

1. The response variable (y) is a function of the independent variable (x) and has a maximum (y^*) at $x = x^*$. The purpose of the search is to determine or approximate the value of x^*.

2. The function is unimodal; that is, given two experiments x_1 and x_2 with $x_1 < x_2$, let their outcomes by y_1 and y_2, respectively. Then $x_2 < x^*$ implies $y_1 < y_2$ and $x_1 > x^*$ implies $y_1 > y_2$.

3. The minimum separation distance (ε) between experiments is negligible.

4. The original interval of uncertainty for x can be scaled to $[0, 1]$.

5. A priori, any given interval of finite length is assumed to have the same probability of containing x^* as any other interval of the same length (where both intervals lie within the remaining interval of uncertainty).

MECHANICS OF THE MID-POINT TECHNIQUE

The first experiment, x_1, is placed at the center of the interval, and the second experiment, x_2, is placed at ε (the minimum possible separation) to the right of x_1. If $y_1 > y_2$, the interval $[x_2, 1]$ is dropped from further consideration. If $y_2 > y_1$, the interval $[0, x_1]$ is discarded. Under the assumptions of unimodality and negligible separation distance, this will necessarily reduce the interval of uncertainty to one-half of its original length. The third experiment, x_3, is then placed at the center of the remaining interval. The third experiment will either halve the interval of uncertainty or reduce it by ε (the distance between x_1 and x_2). Under assumption 5, each of these mutually exclusive and exhaustive events is assumed to occur with probability 0.5. In the former case, the fourth experiment is again placed at the center of the remaining interval of uncertainty with the outcome of x_4 determining whether or not the remaining interval is significantly reduced. For the latter case (x_3 having negligible effect on the reduction ratio), x_4 is placed a distance of ε from x_3, and the fourth experiment necessarily reduces the interval of uncertainty to one-half its pervious length.

Figure 1 represents two of the six possible sets of experimental outcomes leading to a reduction ratio of 4.

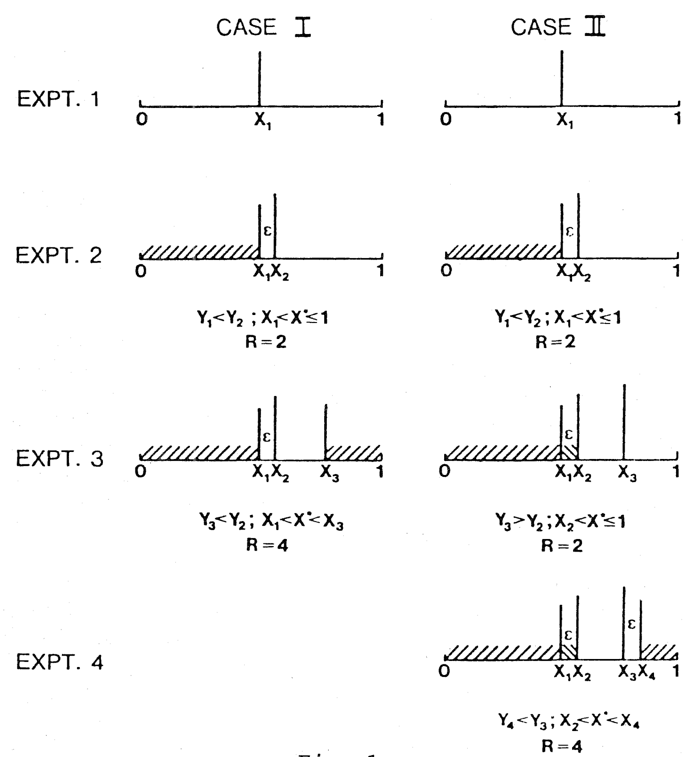


Fig. 1

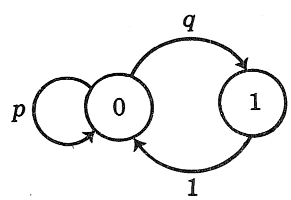
This search procedure continues until either a satisfactory reduction ratio has been attained or until the maximum number of experiments permitted has been run.

MODELING THE MID-POINT SEARCH TECHNIQUE

Each experiment of the mid-point technique results in exactly one of two possible outcomes: (1) the remaining interval of uncertainty is significantly reduced (by half), or (2) the interval of uncertainty is not significantly reduced. Clearly x_1 (by itself) has no effect, while the result of x_2 necessarily reduced the original interval by half. For $n \geq 3$, if x_n is placed in the center of the remaining interval, it will significantly reduce the interval (with probability 0.5) or it will fail to do so (also with probability 0.5). If, however, x_n is placed a distance ε from x_{n-1}, it will significantly reduce the interval (with probability 1) or will fail to do so (with probability 0).

It is thus natural to describe each experimental outcome as resulting in either a "success" (a significant reduction of the interval of uncertainty) or a "failure" (no significant reduction achieved). Further, the probabilities for achieving success or failure on each experimental trial depend exclusively on where the experiment is placed (either in the center of the remaining interval, or a distance ε from the last experiment), where placement depends upon the information derived from the previous experiment.

This suggests the use of a Markov chain to model the process (see Figure 2). Transition to State 0 represents a success in the terminology described above, while a transition to State 1 represents a failure.



$$P = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$$

$$\pi = \pi P = \left(\frac{1}{1+q}, \frac{q}{1+q}\right)$$

Fig. 2

The first two transitions of the chain are deterministic. The first experiment results in the occurrence of State 1 and the second experiment results in State 0 with a probability of 1. In this particular application, $p = q = 0.5$. States 0 and 1 form an irreducible recurrent set. The process of interest is the return times to State 0, which clearly forms a renewal process.

In terms of the mid-point technique, each transition of the Markov chain represents one experiment. The result of the first experiment necessarily results in a failure, that is, the chain making the transition to State 1 (with probability 1). Since the second experiment is placed at a distance ε from x_1, the result of x_2 is necessarily a success, that is, the chain making the transition from State 1 to State 0 (also with probability 1). This first visit to State 0 is called the first renewal, and the first visit and all subsequent returns to State 0 result in a halving of the remaining interval of uncertainty. Equivalently, each time the chain undergoes a renewal, the reduction ratio is effectively doubled.

Obtaining the probability mass function for the number of renewals in a fixed number of transitions is a relatively straightforward matter (see Appendix). From this mass function, the exact probability for the number of renewals can be computed. If the random variable N_n represents the number of visits to State 0 after n transitions of the chain, then the reduction ratio R_n after n transitions (experiments) is simply expressed as $R_n = 2^{N_n}$. Since the probability mass function for N_n has been completely specified, this also specifies the mass function for the various values that the random variable R_n takes. From this, the expected value of R_n immediately follows.

Table 1 lists the expected value of the reduction ratio after n transitions or experiments, for values of n ranging from one to ten. Readers of this journal will immediately recognize the Fibonacci sequence.

Table 1

Number of Experiments	Number of Renewals	Equivalent Reduction Ratio	Probability of Occurrence	Expected Value of Reduction Ratio
0	0	1	1.0	1
1	0	1	1.0	1
2	1	2	1.0	2
3	1	2	0.5	3
	2	4	0.5	
4	2	4	0.75	5
	3	8	0.25	
5	2	4	0.25	
	3	8	0.625	8
	4	16	0.125	
6	3	8	0.500	
	4	16	0.4375	13
	5	32	0.0625	
7	3	8	0.125	
	4	16	0.5625	21
	5	32	0.28125	
	6	64	0.03125	
8	4	16	0.3125	
	5	32	0.500	34
	6	64	0.171875	
	7	128	0.015625	
9	4	16	0.0625	
	5	32	0.4375	
	6	64	0.390625	55
	7	128	0.1015625	
	8	256	0.0078125	
10	5	32	0.1875	
	6	64	0.46875	
	7	128	0.28125	89
	8	256	0.05859375	
	9	512	0.00390625	

ADVANTAGES OF THE MID-POINT TECHNIQUE

1. When a search point, x_i, falls sufficiently close to x^*, the subsequent experiments, x_{i+k}, will all be successes with consequent rapid convergence. As an extreme example, the case where x^* is located at the center of the original interval of uncertainty can be considered. In this case, x^* will lie in the interval $[x_1, x_2]$. For x_2 and all subsequent experiments, the interval of uncertainty will be halved.

2. In many situations involving a direct search, the marginal cost of additional experiments is constant, while the marginal value of information rapidly decreases with the time required to obtain the information. The expected profit of the search under these circumstances may be larger when using the mid-point

technique than when using the Fibonacci method. For example, if a reduction ratio of at least 30 is required and if the cost of placing experiments is $10 per experiment while revenues are $[100 - 0.1n^2]$, the Fibonacci search requires 9 experiments and gives a profit of $1.90. The mid-point technique has an expected profit of $13.50 and requires 6 to 10 experiments.

3. If the desired reduction ratio must be accomplished within a specified number of search points, the Fibonacci search may be incapable of meeting the requirement. Under this circumstance, a rational choice is to gamble and the mid-point technique does allow gambling, although it does not insure a winning gamble.

4. The mid-point technique is easier to use than the Fibonacci search.

ACKNOWLEDGMENT

The authors are grateful to Dr. Richard Feldman for his valuable assistance in dealing with the probabilistic aspects of the problem.

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APPENDIX

The first two transitions of the Markov chain are strictly deterministic. The chain goes to State 1 and then to State 0 (all with probability 1). Therefore, for the purposes of this analysis, we can ignore the first two transitions and take State 0 (our renewal state) as the initial state of the chain. Diagrammatically, the chain appears as in Figure 2.

Let $N(t)$ be the number of renewals in $[0, t]$ where t represents time, and let m be the number of transitions that occur in this interval. If each transition is assumed to require one time unit, then m is the integer part of t. We are interested in the probability distribution for the number of renewals in a finite number of transitions, i.e., $P\{N(t) = k\}$ for the various values of k. Counting the initial state of the chain as a renewal, the total number of renewals is clearly equal to one plus the number of returns to State 0.

Let $f(\cdot)$ be the probability mass function of inter-renewal times. Then

$$f(1) = p \quad \text{and} \quad f(2) = q.$$

Let $f^k(k + n)$ be the probability of obtaining the kth return, the $(k + 1)$th renewal, on the $(k + n)$th transition for $n = 0, 1, \ldots, k$. Note that f^k is the k-fold convolution of $f(\cdot)$. A little algebra quickly reveals that

$$f\ (k + n) = \binom{k}{n} p^{k-n} q^n \text{ for } n = 0, 1, \ldots, k.$$

For purposes of algebraic simplicity, let $m = k + n$. Then

$$f^k(m) = \binom{k}{m-k} p^{2k-m} q^{m-k}; \; m = k, \; k+1, \; \ldots, \; 2k,$$

where $f^k(m)$ is the probability of obtaining the kth return on the mth transition. Similarly,

$$f^{k-1}(m) = \binom{k-1}{m-k+1} p^{2k-m-2} q^{m-k+1}; \; m = k-1, \; k, \; \ldots, \; 2k-2.$$

Let

$$F^k(t) = \sum_{m \le t} f^k(m); \; m = k, \; k+1, \; \ldots, \; 2k.$$

Note that F^k is the probability of obtaining the kth return, the $(k+1)$th renewal, at or prior to time t, where the maximum value of m is the largest integer less than or equal to t. It follows immediately that

$$F^k(t) = P\{N(t) \ge k+1\}.$$

Similarly,

$$F^{k-1}(t) = \sum_{m \le t} f^{k-1}(m) = P\{N(t) \ge k\}.$$

With F^k and F^{k-1} completely specified as above, and using

$$P\{N(t) = k\} = F^{k-1}(t) - F^k(t)$$

(see [2, Ch. 9]), the distribution of $N(t)$ can be determined.

Algebraic manipulation and simplification results in the following:

$$P\{N(t) = k\} = F^{k-1}(t) - F^k(t)$$

$$= \begin{cases} 0; & t < k-1 \\ p^{k-1}; & k-1 \le t < k \\ p^{k-1} + \displaystyle\sum_{m=k}^{[t]} p^{2k-m} q^{m-k}\left[p^{-2} q \binom{k-1}{m-k+1} - \binom{k}{m-k}\right]; & k \le t < 2k-2 \\ 1 - \displaystyle\sum_{m=k}^{[t]} \binom{k}{m-k} p^{2k-m} q^{m-k}; & 2k-2 \le t < 2k \\ 0; & t \ge 2k \end{cases}$$

where $[t]$ is the integer part of t.

A short computer program was written in FORTRAN to calculate these probabilities as well as the mean, variance, standard deviation, and skew for the number of renewals and its equivalent reduction ratio. The number of transitions was varied from one to twenty in increments of one. The program was compiled and executed under WATFIV and run on an AMDAHL 470/V6 computer in well under 0.5 seconds. Copies of this program are available on request.

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CONGRUENCES FOR BELL AND TANGENT NUMBERS

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1. INTRODUCTION

The Bell numbers B_n defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}$$

and the tangent numbers T_n defined by

$$\sum_{n=0}^{\infty} T_n \frac{x^n}{n!} = \tan x$$

are of considerable importance in combinatorics, and possess interesting number-theoretic properties. In this paper we show that for each positive integer n, there exist integers $a_0, a_1, \ldots, a_{n-1}$ and $b_1, b_2, \ldots, b_{n-1}$ such that for all $m \geq 0$,

$$B_{m+n} + a_{n-1}B_{m+n-1} + \cdots + a_0 B_m \equiv 0 \pmod{n!}$$

and

$$T_{m+n} + b_{n-1}T_{m+n-1} + \cdots + b_1 T_{m+1} \equiv 0 \pmod{(n-1)!n!}.$$

Moreover, the moduli in these congruences are best possible. The method can be applied to many other integer sequences defined by exponential generating functions, and we use it to obtain congruences for the derangement numbers and the numbers defined by the generating functions $e^{x+x^2/2}$ and $(2 - e^x)^{-1}$.

2. THE METHOD

A *Hurwitz series* [5] is a formal power series of the form

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

where the a_n are integers. We will use without further comment the fact that Hurwitz series are closed under multiplication, and that if f and g are Hurwitz series and $g(0) = 0$, then the composition $f \circ g$ is a Hurwitz series. In particular, $g^k/k!$ is a Hurwitz series for any nonnegative integer k. We will work with Hurwitz series in two variables, that is, series of the form

$$\sum_{m,n=0}^{\infty} a_{mn} \frac{x^m}{m!} \frac{y^n}{n!},$$

where the a_{mn} are integers. The properties of these series that we will need follow from those for Hurwitz series in one variable.

The exact procedure we follow will vary from series to series, but the general outline is as follows: The kth derivative of the Hurwitz series

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \quad \text{is} \quad f^{(k)}(x) = \sum_{n=0}^{\infty} a_{n+k} \frac{x^n}{n!}$$

Our goal is to find some linear combination with integral coefficients of

$$f(x),\ f'(x),\ \ldots,\ f^{(n)}(x)$$

all of whose coefficients are divisible by $n!$ (or in some cases a larger number). To do this we use Taylor's theorem

$$f(x + y) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{y^k}{k!}.$$

We then make the substitution $y = g(z)$ and multiply by some series $h(z)$ to get

$$h(z)f[x + g(z)] = \sum_{k=0}^{\infty} f^{(k)}(x)h(z)\frac{[g(z)]^k}{k!}.$$

If $h(z)$ and $g(z)$ are chosen appropriately, the coefficient of $\frac{x^m}{m!}z^n$ on the left will be integral. Then the coefficient of $\frac{x^m}{m!}\frac{z^n}{n!}$ on the right is divisible by $n!$, and we obtain the desired congruence.

3. BELL NUMBERS

We define the *exponential polynomials* $\phi_n(t)$ by

$$\sum_{n=0}^{\infty} \phi_n(t)\frac{x^n}{n!} = e^{t(e^x - 1)}.$$

Thus

$$\phi_n(1) = B_n \quad \text{and} \quad \phi_n(t) = \sum_{k=0}^{\infty} S(n, k)t^k,$$

where $S(n, k)$ is the Stirling number of the second kind. We will obtain a congruence for the exponential polynomials that for $t = 1$ reduces to the desired Bell number congruence.

We set

$$f(x) = e^{t(e^x - 1)} = \sum_{n=0}^{\infty} \phi_n(t)\frac{x^n}{n!}.$$

Then

$$\begin{aligned} f(x + y) &= \exp\,[t(e^{x+y} - 1)] = \exp\,[t(e^x - 1) + t(e^y - 1)e^x] \\ &= f(x)\exp\,[t(e^y - 1)e^x]. \end{aligned}$$

Now set $y = \log(1 + z)$. We then have

$$\sum_{k=0}^{\infty} f^{(k)}(x)\frac{[\log(1 + z)]^k}{k!} = f(x)e^{tze^x}.$$

Multiplying both sides by e^{-tz}, we obtain

$$\sum_{k=0}^{\infty} f^{(k)}(x)e^{-tz}\frac{[\log(1 + z)]^k}{k!} = f(x)e^{tz(e^x-1)} = \sum_{n=0}^{\infty} z^n t^n f(x)\frac{(e^x - 1)^n}{n!}. \tag{1}$$

Now define polynomials $D_{n,k}(t)$ by

$$e^{-tz}\,\frac{[\log(1 + z)]^k}{k!} = \sum_{n=k}^{\infty} D_{n,k}(t)\frac{z^n}{n!}. \tag{2}$$

[Note that $D_{n,n}(t) = 1$.] Then the left side of (1) is

$$\sum_{k=0}^{\infty} f^{(k)}(x)\sum_{n=0}^{\infty} D_{n,k}(t)\frac{z^n}{n!} = \sum_{m,n=0}^{\infty} \frac{x^m}{m!}\frac{z^n}{n!}\sum_{k=0}^{n} D_{n,k}(t)\phi_{m+k}(t). \tag{3}$$

Since

$$\frac{[\log(1+z)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k)\frac{z^n}{n!},$$

where $s(n, k)$ is the Stirling number of the first kind, we have the explicit formula

$$D_{n,k}(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} s(n-j, k)t^j. \tag{4}$$

Since

$$\frac{(e^x - 1)^n}{n!} = \sum_{m=n}^{\infty} S(m, n)\frac{x^m}{m!},$$

we have

$$f(x)\frac{(e^x-1)^n}{n!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{j=0}^{m} \binom{m}{j} S(m-j, n)\phi_j(t),$$

hence the right side of (1) is

$$\sum_{m,n=0}^{\infty} \frac{x^m}{m} z^n t^n \sum_{j=0}^{m} \binom{m}{j} S(m-j, n)\phi_j(t). \tag{5}$$

Equating coefficients of $\frac{x^m}{m!}\frac{z^n}{n!}$ in (3) and (5) we have

Proposition 1: For all $m, n \geq 0$,

$$\sum_{k=0}^{n} D_{n,k}(t)\phi_{m+k}(t) = n!t^n \sum_{j=0}^{m} \binom{m}{j} S(m-j, n)\phi_j(t),$$

where

$$D_{n,k}(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} s(n-j, k)t^j.$$

Now let $D_{n,k} = D_{n,k}(1)$. Setting $t = 1$ in Proposition 1, we obtain

Proposition 2: For $m, n \geq 0$,

$$\sum_{k=0}^{n} D_{n,k}B_{m+k} = n! \sum_{j=0}^{m} \binom{m}{j} S(m-j, n)B_j, \tag{6}$$

where

$$D_{n,k} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} s(n-j, k).$$

A recurrence for the numbers $D_{n,k}$ is easily obtained. From (2), we have

$$\sum_{n=k}^{\infty} D_{n,k}\frac{z^n}{n!} = e^{-z}\frac{[\log(1+z)]^k}{k!},$$

hence

$$D(u, z) = \sum_{n \geq k} D_{n,k}u^k\frac{z^n}{n!} = e^{-z}(1+z)^u. \tag{7}$$

From (7), we obtain

$$\frac{\partial}{\partial z}D(u, z) = -e^{-z}(1+z)^u + ue^{-z}(1+z)^{u-1},$$

thus

$$\begin{aligned}(1+z)\frac{\partial}{\partial z}D(u, z) &= -(1+z)D(u, z) + uD(u, z)\\ &= (u-1-z)D(u, z).\end{aligned} \tag{8}$$

Equating coefficients of $u^k \frac{z^n}{n!}$ in (8), we have

$$D_{n+1,\,k} = D_{n,\,k-1} - (n+1)D_{n,\,k} - nD_{n-1,\,k} \text{ for } n,\ k \geq 0,$$

with $D_{0,\,0} = 1$ and $D_{n,\,k} = 0$ for $k > n$ or $k < 0$. Here are the first few values of $D_{n,\,k}$:

Table 1

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	-1	1						
2	1	-3	1					
3	-1	8	-6	1				
4	1	-24	29	-10	1			
5	-1	89	-145	75	-15	1		
6	1	-415	814	-545	160	-21	1	
7	-1	2372	-5243	4179	-1575	301	-28	1

Thus the first few instances of (6) yield

$$B_{m+2} + B_{m+1} + B_m \equiv 0 \pmod 2$$
$$B_{m+3} + 2B_{m+1} - B_m \equiv 0 \pmod 6$$
$$B_{m+4} - 10B_{m+3} + 5B_{m+2} + B \equiv 0 \pmod{24}.$$

If we set

$$D_n(u) = \sum_{k=0}^{n} D_{n,\,k} u^k,$$

then from (7) we have

$$D_n(u) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (u)_j,$$

where $(u)_j = u(u-1)\ldots(u-j+1)$. It can be shown that for prime p, $D_n(u)$ satisfies the congruence $D_{n+p}(u) \equiv (u^p - u - 1)D_n(u) \pmod p$. In particular, $D_p(u) \equiv u^p - u - 1 \pmod p$, and we recover Touchard's congruence [8]

$$B_{n+p} \equiv B_n + B_{n+1} \pmod p.$$

Touchard later [9] found the congruence

$$B_{2p} - 2B_{p+1} - 2B_p + p + 5 \equiv 0 \pmod{p^2},$$

which is a special case of

$$B_{n+2p} - 2B_{n+p+1} - 2B_{n+p} + B_{n+2} + 2B_{n+1} + (p+1)B \equiv 0 \pmod{p^2},$$

but these congruences do not seem to follow from Proposition 2.

We now show that in a certain sense the congruence obtained from Proposition 2 cannot be improved.

Proposition 3: Let $A_0, A_1, A_2, \ldots$ be a sequence of integers and let $a_0, a_1, \ldots, a_n$ be integers such that

$$\sum_{k=0}^{n} a_k A_{m+k} = \begin{cases} 0 & \text{if } 0 \leq m < n \\ N & \text{if } m = n \end{cases}.$$

Let $b_0, b_1, \ldots, b_n$ be integers such that $\sum_{k=0}^{n} b_k A_{m+k}$ is divisible by R for all $m \geq 0$. Then R divides $b_n N$.

Proof: Let
$$S = \sum_{i,j=0}^{n} a_i b_j A_{i+j}.$$
Since
$$S = \sum_{i=0}^{n} a_i \left[\sum_{j=0}^{n} b_j A_{i+j} \right],$$
R divides S. But
$$S = \sum_{j=0}^{n} b_j \left[\sum_{i=0}^{n} a_i A_{j+i} \right] = b_n N.$$

Corollary: If for some integers $b_0, b_1, \ldots, b_{n-1}$, we have
$$B_{m+n} + b_{n-1}B_{m+n-1} + \cdots + b_0 B_m \equiv 0 \pmod{R} \text{ for all } m \geq 0,$$
then R divides $n!$.

Proof: Since $S(n, k) = 0$ if $n < k$ and $S(n, n) = 1$, the right side of (6) is zero for $0 \leq m < n$ and $n!$ for $m = n$. Thus Proposition 3 applies, with $b_n = 1$.

For other Bell number congruences to composite module, see Barsky [1] and Radoux [7].

4. TANGENT NUMBERS

We have
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \tan x + \sum_{n=1}^{\infty} \sec^2 x \tan^{n-1} x \tan^n y.$$
Now set $y = \arctan z$. Then
$$\tan(x + \arctan z) = \tan x + \sum_{n=1}^{\infty} z^n \sec^2 x \tan^{n-1} x, \tag{9}$$
and by Taylor's theorem,
$$\tan(x + \arctan z) = \sum_{k=0}^{\infty} \tan^{(k)} x \frac{(\arctan z)^k}{k!}, \tag{10}$$
where $\tan^{(k)} x = \dfrac{d^k}{dx^k} \tan x$.

Now let us define integers $T(n, k)$ and $t(n, k)$ by
$$\frac{\tan^k x}{k!} = \sum_{n=k}^{\infty} T(n, k) \frac{x^n}{n!} \quad \text{and} \quad \frac{(\arctan x)^k}{k!} = \sum_{n=k}^{\infty} t(n, k) \frac{x^n}{n!}.$$
Tables of $T(n, k)$ and $t(n, k)$ can be found in Comtet [3, pp. 259-260]. Note that
$$\frac{d}{dx} \frac{\tan^k x}{k!} = \sec^2 x \frac{\tan^{k-1} x}{(k - 1)!},$$
so
$$\sec^2 x \tan^{n-1} x = (n - 1)! \sum_{m=n-1}^{\infty} T(m + 1, n) \frac{x^m}{m!} \text{ for } n \geq 1.$$
Then from (9) and (10), we have
$$\sum_{m,n=0}^{\infty} \frac{x^m}{m!} \frac{z^n}{n!} \sum_{k=0}^{n} t(n, k) T_{m+k} = \tan x + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{x^m}{m!} z^n n! (n - 1)! T(m + 1, n).$$

Then by equating coefficients of $\dfrac{x^m}{m!} \dfrac{z^n}{n!}$ we have

Proposition 4: For $m \geq 0$, $n \geq 1$,

$$\sum_{k=0}^{n} t(n, k)T_{m+k} = n!(n-1)!T(m+1, n). \tag{11}$$

From Proposition 4, we obtain the congruence

$$\sum_{k=0}^{n} t(n, k)T_{m+k} \equiv 0 \pmod{n!(n-1)!}.$$

The first few instances are

$$\begin{aligned}
&T_{m+2} \equiv 0 \pmod{2}\\
&T_{m+3} - 2T_{m+1} \equiv 0 \pmod{12}\\
&T_{m+4} - 8T_{m+2} \equiv 0 \pmod{144}\\
&T_{m+5} - 20T_{m+3} + 24T_{m+1} \equiv 0 \pmod{2880}\\
&T_{m+6} - 40T_{m+4} + 184T_{m+2} \equiv 0 \pmod{86400}.
\end{aligned}$$

Note that the right side of (11) is zero for $m < n - 1$ and $n!(n - 1)!$ for $m = n - 1$. Proposition 3 does not apply directly, but if we observe that

$$t(n, 0) = 0 \text{ for } n > 0,$$

and write T'_n for T_{n+1}, then (11) becomes

$$\sum_{k=0}^{n-1} t(n, k+1)T'_{m+k} = n!(n-1)!T(m+1, n),$$

to which Proposition 3 applies: if for some integers $b_1, b_2, \ldots, b_{n-1}$, we have

$$T_{m+n} + b_{n-1}T_{m+n-1} + \cdots + b_1T_{m+1} \equiv 0 \pmod{R} \text{ for all } m \geq 0,$$

then R divides $n!(n - 1)!$.

Proposition 3 does not preclude the possibility that a better congruence may hold with $m \geq M$ replacing $m \geq 0$, for some M. In fact, this is the case, since the tangent numbers are eventually divisible by large powers of 2; more precisely, $x \tan x/2$ is a Hurwitz series with odd coefficients (the Genocchi numbers).

5. OTHER NUMBERS

We give here congruences for other sequences of combinatorial interest, omitting some of the details of their derivation.

The numbers g_n defined by

$$\sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = (2 - e^x)^{-1}$$

count "preferential arrangements" or ordered partitions of a set. They have been studied by Touchard [8], Gross [4], and others.

If we set $G(x) = (2 - e^x)^{-1}$, then

$$G(x + y) = e^{-y}\sum_{n=0}^{\infty} \frac{2^n(1 - e^{-y})^n}{(2 - e^x)^{n+1}}. \tag{12}$$

Substituting $y = -\log(1 - z)$ in (12), we have

$$G[x - \log(1 - z)] = (1 - z)\sum_{n=0}^{\infty} \frac{2^n z^n}{(2 - e^x)^{n+1}}. \tag{13}$$

Proceeding as before, we obtain from (13) the congruence

$$\sum_{k=0}^{n} c(n, k)g_{m+k} \equiv 0 \pmod{2^{n-1}n!}, \ m \geq 0, \tag{14}$$

where $c(n, k) = |s(n, k)|$ is the unsigned Stirling number of the first kind,

$$\sum_{n=0}^{\infty} c(n, k)\frac{z^n}{n!} = \frac{[-\log(1 - z)]^k}{k!}.$$

The first few instances of (14) are

$$g_{m+2} + g_{m+1} \equiv 0 \pmod 4$$
$$g_{m+3} + 3g_{m+2} + 2g_{m+1} \equiv 0 \pmod{24}$$
$$g_{m+4} + 6g_{m+3} + 11g_{m+2} + 6g_{m+1} \equiv 0 \pmod{192}.$$

The *derangement numbers* $d(n)$ may be defined by

$$\sum_{n=0}^{\infty} d(n)\frac{x^n}{n!} = \frac{e^{-x}}{1 - x}.$$

It will be convenient to consider the more general numbers $d(n, s)$ defined by

$$D_s(x) = \sum_{n=0}^{\infty} d(n, s)\frac{x^n}{n!} = \frac{e^{-x}}{(1 - x)^s}.$$

Then

$$D_s(x + y) = \frac{e^{-x}}{(1 - x)^s}\,\frac{e^{-y}}{[1 - y/(1 - x)]^s} = e^{-y}\sum_{n=0}^{\infty} y^n\binom{n + s - 1}{n}\frac{e^{-x}}{(1 - x)^{n+s}}. \tag{15}$$

Multiplying both sides of (15) by e^y and equating coefficients, we obtain

$$\sum_{k=0}^{n}\binom{n}{k} d(m + k, s) = n!\binom{n + s - 1}{n} d(m, n + s). \tag{16}$$

In particular, we find from (16) that for prime p,

$$d(m + p, s) + d(m, s) \equiv 0 \pmod p.$$

The numbers t defined by

$$T(x) = \sum_{n=0}^{\infty} t_n\frac{x^n}{n!} = e^{x + \frac{x^2}{2}}$$

have been studied by Chowla, Herstein, and Moore [2], Moser and Wyman [6], and others, and count partitions of a set into blocks of size one and two. We have $T(x + y) = T(x)T(y)e^{x+y}$; hence

$$T(y)^{-1}T(x + y) = T(x)e^{x+y}. \tag{17}$$

Let

$$W(y) = \sum_{n=0}^{\infty} w_n\frac{y^n}{n!} = T(y)^{-1} = e^{-y - \frac{y^2}{2}}.$$

Then from (17) we obtain

$$\sum_{k=0}^{n}\binom{n}{k} w_{n-k}t_{m+k} = n!\binom{m}{n} t_{m-n}, \tag{18}$$

where we take $t_n = 0$ for $n < 0$. We note that (18) satisfies the hypothesis of Proposition 3, so we obtain here a best possible congruence.

The numbers w_n have been studied by Moser and Wyman [6]. From the differential equation $W'(y) = -(1 + y)W(y)$, we obtain the recurrence

$$w_{n+1} = -(w_n + nw_{n-1}),$$

from which the w_n are easily computed. The first few instances of (18) are

$$t_{m+1} - t_m = mt_{m-1}$$

$$t_{m+2} - 2t_{m+1} = 2\binom{m}{2}t_{m-2}$$

$$t_{m+3} - 3t_{m+2} + 2t_m = 6\binom{m}{3}t_{m-3}$$

$$t_{m+4} - 4t_{m+3} + 8t_{m+1} - 2t_m = 24\binom{m}{4}t_{m-4}.$$

A natural question is: To what series does this method apply? In other words, we want to characterize those Hurwitz series $f(x)$ for which there exist Hurwitz series $h(z)$ and $g(z)$, with $h(0) = 1$, $g(0) = 0$, and $g'(0) = 1$, such that for all $m, n \geq 0$, the coefficient of $(x^m/m!)z^n$ in $h(z)f[x + g(z)]$ is integral.

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A QUADRATIC PROPERTY OF CERTAIN LINEARLY RECURRENT SEQUENCES

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In [1] one of the authors proved the following result.

Let u be a real number such that $u > 1$, and let $\{x_n\}_{n\geq 0}$ be a sequence of nonnegative real numbers such that

$$x_{n+1} = ux_n + \sqrt{(u^2 - 1)(x_n^2 - x_0^2) + (x_1 - ux_0)^2}$$

for every $n \geq 0$. Then

$$x_{n+2} = 2ux_{n+1} - x_n$$

for every $n \geq 0$; and, in particular, if u, x_0, x_1 are integers, then x_n is an integer for every $n \geq 0$.

In this note we shall show that, under certain conditions, the preceding result admits a converse.

We begin with the following general preliminary proposition.

Proposition: Let R be a commutative ring with unit element; let $t, u \in R$, and define a polynomial $f \in R[X, Y]$ by

$$f(X, Y) = tX^2 - 2uXY + Y^2.$$

If $\{r_n\}_{n \geq 0}$ is a sequence of elements of R such that

$$r_{n+2} = 2ur_{n+1} - tr_n$$

for every $n \geq 0$, then

$$f(r_n, r_{n+1}) = t^n f(r_0, r_1)$$

for every $n \geq 0$.

Proof: We shall prove this result by induction. The conclusion holds identically for $n = 0$. Assume now that it holds for some $n \geq 0$. Then

$$\begin{aligned} f(r_{n+1}, r_{n+2}) &= tr_{n+1}^2 - 2ur_{n+1}r_{n+2} + r_{n+2}^2 \\ &= tr_{n+1}^2 - 2ur_{n+1}(2ur_{n+1} - tr_n) + (2ur_{n+1} - tr_n)^2 \\ &= t(tr_n^2 - 2ur_n r_{n+1} + r_{n+1}^2) \\ &= tf(r_n, r_{n+1}) \\ &= tt^n f(r_0, r_1) \\ &= t^{n+1} f(r_0, r_1), \end{aligned}$$

which shows that the conclusion also holds for $n + 1$. ■

This proposition can be applied in some familiar particular cases:

If we take

$$R = Q,\ t = -1,\ u = 1/2,$$

we find that the Fibonacci and Lucas sequences satisfy

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$$
$$L_{n+1}^2 - L_{n+1}L_n - L_n^2 = 5(-1)^{n+1}$$

for every $n \geq 0$.

And if we take

$$R = Z,\ t = -1,\ u = 1,$$

we also find that the Pell sequence satisfies

$$P_{n+1}^2 - 2P_{n+1}P_n - P_n^2 = (-1)^n$$

for every $n \geq 0$.

We are now in a position to state and prove our result.

Theorem: Let t, u be real numbers such that

$$t^2 = 1 \quad \text{and} \quad u > \max(t, 0),$$

and let $\{x_n\}_{n \geq 0}$ be a sequence of real numbers such that

$$x_1 \geq \max(ux_0,\ (t/u)x_0,\ 0)$$

and satisfying

$$x_{n+2} = 2ux_{n+1} - tx_n$$

for every $n \geq 0$. We then have:

(i) $ux_{n+1} \geq \max(tx_n, 0)$ for every $n \geq 0$; and

(ii) $x_{n+1} = ux_n + \sqrt{(u^2 - t)(x_n^2 - t^n x_0^2) + t^n(x_1 - ux_0)^2}$ for every $n \geq 0$.

Proof: We shall prove (i) by induction. Our assumptions clearly imply that the stated inequality holds when $n = 0$. Now suppose that $ux_{n+1} \geq \max(tx_n, 0)$ for some $n \geq 0$. As the given conditions on t, u imply that $u > 0$ and $u^2 > t$, we first deduce that $x_{n+1} \geq 0$, and then that

$$\begin{aligned} ux_{n+2} &= u(2ux_{n+1} - tx_n) \\ &= u^2 x_{n+1} + u(ux_{n+1} - tx_n) \geq u^2 x_{n+1} \geq \max(tx_{n+1}, 0), \end{aligned}$$

as required.

Since $x_1 - ux_0 \geq 0$, it is clear that, in order to prove (ii), we need only consider the case where $n > 0$. In view of the proposition, we have

$$tx_{n-1}^2 - 2ux_{n-1}x_n + x_n^2 = t^{n-1}(tx_0^2 - 2ux_0x_1 + x_1^2).$$

Since $t^2 = 1$, we also have

$$x_{n-1}^2 - 2tux_{n-1}x_n + tx_n^2 = t^n(tx_0^2 - 2ux_0x_1 + x_1^2),$$

and hence

$$-2tux_{n-1}x_n + x_{n-1}^2 = -tx_n^2 + t^{n+1}x_0^2 - 2t^n ux_0 x_1 + t^n x_1^2;$$

it then follows that

$$\begin{aligned} (ux_n - tx_{n-1})^2 &= u^2x_n^2 - 2tux_{n-1}x_n + x_{n-1}^2 \\ &= u^2x_n^2 - tx_n^2 + t^{n+1}x_0^2 + t^n x_1^2 - 2t^n ux_0x_1 \\ &= (u^2 - t)(x_n^2 - t^n x_0^2) + t^n(x_1 - ux_0)^2. \end{aligned}$$

By virtue of (i), we now conclude that

$$\begin{aligned} x_{n+1} &= 2ux_n - tx_{n-1} \\ &= ux_n + (ux_n - tx_{n-1}) \\ &= ux_n + \sqrt{(ux_n - tx_{n-1})^2} \\ &= ux_n + \sqrt{(u^2 - t)(x_n^2 - t^n x_0^2) + t^n(x_1 - ux_0)^2}, \end{aligned}$$

which is what was needed. ■

Applying this theorem to the three special sequences considered above, we obtain the following formulas for every $n \geq 0$:

$$\begin{aligned} F_{n+1} &= \tfrac{1}{2}(F_n + \sqrt{5F_n^2 + 4(-1)^n}) \\ L_{n+1} &= \tfrac{1}{2}(L_n + \sqrt{5L_n^2 + 20(-1)^{n+1}}) \\ P_{n+1} &= P_n + \sqrt{2P_n^2 + (-1)^n}. \end{aligned}$$

These formulas, of course, can also be derived directly from the quadratic equalities established previously.

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THE DETERMINATION OF CERTAIN FIELDS OF INVARIANTS

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It is often difficult to determine Galois groups and their fields of invariants by elementary methods. The objective of this note is to illustrate how some basic considerations on fields and polynomials can be used to determine the fields of invariants of certain groups of field automorphisms.

We shall be concerned with a field K and the field $K(X)$ of rational functions in one variable. The Galois group of $K(X)$ over K will be denoted by

$$\mathrm{Gal}(K(X)/K);$$

for every subgroup Γ of $\mathrm{Gal}(K(X)/K)$, the field of invariants of Γ will be denoted by

$$\mathrm{Inv}(\Gamma).$$

For each $u \in K - \{0\}$, let ρ_u denote the K-automorphism of $K(X)$ such that $X \to uX$; and for each $u \in K$, let τ_u denote the K-automorphism of $K(X)$ such that $X \to u + X$.

Now we are in a position to state and prove the following assertions.

$\mathcal{A}$. If M is an infinite subgroup of the multiplicative group of nonzero elements of K, then the mapping $u \to \rho_u$ from M to $\mathrm{Gal}(K(X)/K)$ is an injective group homomorphism, and K is the field of invariants of its image.

$\mathcal{B}$. If A is an infinite subgroup of the additive group of K, then the mapping $u \to \tau_u$ from A to $\mathrm{Gal}(K(X)/K)$ is an injective group homomorphism, and K is the field of invariants of its image.

A quick proof of these assertions can be obtained from the following two results: (1) $K(X)$ is a finite algebraic extension of each of its subfields properly containing K; and (2) Artin's theorem on the field of invariants of a finite group of field automorphisms. These results are discussed in [1, p. 158] and [2, p. 69], respectively. We shall now prove $\mathcal{A}$ and $\mathcal{B}$ by using only very elementary properties of polynomials.

In the discussion that follows, we shall consider an element Y of $K(X)$ not belonging to K, and write it in the form $Y = f(X)/g(X)$, where $f(X)$ and $g(X)$ are relatively prime polynomials in $K[X]$. Put $m = \deg(f(X))$ and $n = \deg(g(X))$; then write

$$f(X) = \sum_{i=0}^{m} a_i X^i \quad \text{and} \quad g(X) = \sum_{j=0}^{n} b_j X^j,$$

where $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in K$ and $a_m \neq 0 \neq b_n$.

Proof of $\mathcal{A}$: If $u, v \in M$, then $(uv)X = u(vX)$, whence $\rho_{uv} = \rho_u\rho_v$. It follows that the mapping $u \to \rho_u$ from M to $\mathrm{Gal}(K(X)/K)$ is a group homomorphism. Its injectivity is evident: indeed, if $u \in A$ and ρ_u is the identity mapping on $K(X)$, then $X = \rho_u(X) = uX$, which implies that $u = 1$.

Let Γ denote the image of this homomorphism. We shall now prove that the condition $Y \in \mathrm{Inv}(\Gamma)$ leads to a contradiction.

Now assume that $Y \in \mathrm{Inv}(\Gamma)$. Then, for every $u \in M$, we have $Y = \rho_u(Y)$, which means that

$$f(X)/g(X) = \rho_u(f(X))/\rho_u(g(X)) = f(uX)/g(uX),$$

and, hence, $f(X)g(uX) = f(uX)g(X)$; since $f(X)$ and $g(X)$ are relatively prime in $K[X]$, and since

$$\deg(f(X)) = m = \deg(f(uX))$$

and
$$\deg(g(X)) = n = \deg(g(uX)),$$
it now follows that
$$f(uX) = u^m f(X) \quad \text{and} \quad g(uX) = u^n g(X),$$
which means that $u^{m-i}a_i = a_i$ for $0 \leq i \leq m$ and $u^{n-j}b_j = b_j$ for $0 \leq j \leq n$.

If $0 \leq i < m$ and $a \neq 0$, then $u^{m-i} = 1$ for every $u \in M$; hence, every element of M is an $(m - i)$th root of unity in K. As M is infinite, this is impossible. Similarly, the conditions $0 \leq j < n$ and $b_j \neq 0$ imply an absurd conclusion. We conclude that $a_i = 0 = b_j$ for $0 \leq i < m$ and $0 \leq j < n$.

Consequently, we have $f(X) = a_m X^m$ and $g(X) = b_n X^n$. If we put $c = a_m/b_n$ and $r = m - n$, then $c \in K - \{0\}$ and $Y = cX^r$; since $Y \notin K$, we see that $r \neq 0$. For every $u \in M$, we now have
$$cX^r = Y = \rho_u(Y) = c(uX)^r = cu^r X^r,$$
which implies that $u^r = 1$. Thus, every element of M is an $|r|$th root of unity in K. This contradicts the infiniteness of M, and completes the proof of A.

Proof of B: If $u, v \in A$, then
$$(u + v) + X = u + (v + X),$$
which implies that $\tau_{u+v} = \tau_u \tau_v$. Thus the mapping $u \to \tau_u$ from A to $\mathrm{Gal}(K(X)/K)$ is a group homomorphism. To verify that it is injective, note that if $u \in A$ and τ_u is the identity mapping on $K(X)$, then $X = \tau_u(X) = u + X$, whence $u = 0$.

Let Δ denote the image of this homomorphism. Assume, by way of contradiction, that $Y \in \mathrm{Inv}(\Delta)$. For each $u \in A$, we have $Y = \tau_u(Y)$, which implies that
$$f(X)/g(X) = \tau_u(f(X))/\tau_u(g(X)) = f(u + X)/g(u + X),$$
and hence
$$f(X)g(u + X) = f(u + X)g(X);$$
taking into account that $f(X)$ and $g(X)$ are relatively prime in $K[X]$, and that
$$\deg(f(X)) = m = \deg(f(u + X))$$
and
$$\deg(g(X)) = n = \deg(g(u + X)),$$
we conclude that
$$f(u + X) = f(X) \quad \text{and} \quad g(u + X) = g(X).$$

It now follows that $f(u) = f(0) = a_0$ and $g(u) = g(0) = b_0$ for every $u \in A$. This means that every element of A is a root of the polynomials $f(X) - a_0$ and $g(X) - b_0$, which is incompatible with the assumption that M is infinite. This completes the proof of B.

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FIBONACCI NUMBER IDENTITIES FROM ALGEBRAIC UNITS

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1. INTRODUCTION

In several recent papers L. Bernstein [1], [2] introduced a method of operating with units in cubic algebraic number fields to obtain combinatorial identities. In this paper we construct kth degree ($k \geq 2$) algebraic fields with the special property that certain units have Fibonacci numbers for coefficients. By operating with these units we will obtain our main result, an infinite class of identities for the Fibonacci numbers. The main result is given in Theorem 1 and illustrated in Figure 1.

2. MAIN RESULT

Theorem 1: For each posotive integer k let A_k be a $(2k - 1) \times (2k - 1)$ determinant, $A_k = \det(a_{ij})$, see Figure 1, where a_{ij} is given by

$$a_{ij} = \begin{cases} (-1)^{n+1}F_{n+2} & \text{if } i = j \text{ and } j < k \\ (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2} & \text{if } i = j \text{ and } j \geq k \\ (-1)^n F_{n+1} & \text{if } i = j - k \text{ and } i < k \\ & \text{or } i = j + k \text{ and } i > k \\ 0 & \text{otherwise} \end{cases} \quad (k > 1).$$

For $k = 1$, we define A_1 to be the middle entry in Figure 1, i.e.,

$$A_1 = F_{n+2} - F_{n+1}.$$

Then, for all $k \geq 1$, we have $F_n = |A_k|$.

$$A_k = \begin{array}{ccccccccc}
(-1)^{n+1}F_{n+2} & 0 & \cdots & 0 & (-1)^n F_{n+1} & 0 & \cdots & \cdots & 0 \\
0 & \ddots & & & & \ddots & & & \vdots \\
\vdots & & (-1)^{n+1}F_{n+2} & & & & & & 0 \\
\vdots & & & \ddots & & & & & (-1)^n F_{n+1} \\
0 & \cdots & \cdots & 0 & (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2} & 0 & \cdots & \cdots & 0 \\
(-1)^n F_{n+1} & & & & & & & & \vdots \\
0 & \ddots & & & & \ddots & & & \vdots \\
\vdots & & & & & & & & 0 \\
0 & \cdots & 0 & (-1)^n F_{n+1} & 0 & \cdots & 0 & & (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2}
\end{array}$$

Fig. 1 (2k - 1) × (2k - 1) Determinant

Proof: Throughout the entire ensuing discussion, k will be a fixed positive integer. Consider the following $2k$ recursion formulas with the accompanying $2k$ initial conditions. For each fixed j, $j = 0, 1, \ldots, 2k - 1$, let

$$a_j(n + 2k) = a_j(n + k) + a_j(n) \qquad (n \geq 0) \tag{1}$$

and

$$a_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In particular, for $k = j = 1$, we obtain

$$a_1(n + 2) = a_1(n + 1) + a_1(n) \qquad (n \geq 0)$$

and

$$a_1(0) = 0, \quad a_1(1) = 1,$$

that is, $\{a_1(n)\}_{n=1}^{\infty}$ is the Fibonacci sequence. In general, one can verify that for any fixed k and any j, $j = 0, 1, \ldots, 2k - 1$, the nonzero terms of the sequence $\{a_j(n)\}_{n=1}^{\infty}$ are the Fibonacci numbers. More precisely, from (1) and (2) one can obtain the equations:

$$\begin{aligned} a_j(k - 1 + kn) &= 0 \text{ if } j \neq 2k - 1 \text{ or } k - 1 \\ a_{k-1}(k - 1 + kn) &= F_{n-1} \\ a_{2k-1}(k - 1 + kn) &= F_n. \end{aligned} \tag{3}$$

Now consider the algebraic number field $Q(w)$ where $w^{2k} = 1 + w^k$. We claim that the nonnegative powers of w are given by the equation

$$w^n = a_0(n) + a_1(n)w + \cdots + a_{2k-1}(n)w^{2k-1}, \tag{4}$$

where the $a_j(n)$, $0 \leq j \leq 2k - 1$, satisfy (1) and (2). From (4) we obtain

$$\begin{aligned} w^{n+1} = a_{2k-1}(n) + a_0(n)w + a_1(n)w^2 + \cdots + (a_{k-1}(n) + a_{2k-1}(n))w^k \\ + \cdots + a_{2k-2}(n)w^{2k-1}. \end{aligned} \tag{5}$$

Comparison of the coefficients in (4) and (5) yields the following $2k$ equations:

$$\begin{aligned} a_0(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{2k-1}(n) \\ a_1(n + 1) &= 1 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 0 \cdot a_{2k-1}(n) \\ a_2(n + 1) &= 0 \cdot a_0(n) + 1 \cdot a_1(n) + \cdots + 0 \cdot a_{2k-1}(n) \\ &\vdots \\ a_k(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{k-1}(n) + \cdots + 1 \cdot a_{2k-1}(n) \\ &\vdots \\ a_{2k-1}(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{2k-2}(n) + 0 \cdot a_{2k-1}(n). \end{aligned} \tag{6}$$

This system of equations can be written more simply in matrix form as follows. Let C be the coefficient matrix of the $a_j(n)$. Explicitly, $C = (c_{ij})$ is a $(2k)$ by $(2k)$ matrix, where

$$\begin{aligned} c_{1,2k} &= 1 \\ c_{k+1,2k} &= 1 \\ c_{ij} &= 1 \text{ if } i = 1 + j \\ c_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

Let T_n denote the following column matrix:

$$T_n = \begin{bmatrix} a_0(n) \\ \vdots \\ a_{2k-1}(n) \end{bmatrix} \qquad (n \geq 0) \tag{7}$$

The system (6) can now be written as

$$T_{n+1} = CT_n.$$

More generally, if I denotes the identity matrix, then

$$\begin{aligned} T_n &= IT_n \\ T_{n+1} &= CT_n \\ &\vdots \\ T_{n+2k} &= C^{2k}T_n. \end{aligned} \tag{8}$$

The characteristic equation of C is found to be

$$\det(C - \lambda I) = \lambda^{2k} - \lambda^k - 1 = 0. \tag{9}$$

The Hamilton-Cayley theorem states that every square matrix satisfies its characteristic equation. Hence,

$$C^{2k} - C^k - I = 0$$
$$(C^{2k} - C^k - I)T_n = 0,$$

and from (8)

$$T_{n+2k} = T_{n+k} + T_n. \tag{10}$$

From (7) and (10) we have

$$a_j(n + 2k) = a_j(n + k) + a_j(n), \; j = 0, \ldots, 2k - 1.$$

Thus (1) of our claim is established. The initial conditions for (10) can be obtained from (4) and are given by the $2k$ column matrices

$$T_j = (t_{i1}), \; j = 0, 1, \ldots, 2k - 1,$$

where

$$t_{i1} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad i = 0, 1, \ldots, 2k - 1. \tag{11}$$

From (7) we have that $t_{i1} = a_i(n)$. Hence, $a_i(n) = 1$ if and only if $i = j = n$, and (2) is established, thus completing the proof of our claim.

From $w(w^{2k-1} - w^{k-1}) = 1$, we see that

$$w^{-1} = w^{2k-1} - w^{k-1}.$$

If we denote the negative powers of w by

$$w^{-n} = b_0(n) + b_1(n)w + \cdots + b_{2k-1}w^{2k-1} \qquad (n \geq 0), \tag{12}$$

then by calculations analogous to those used for the coefficients of the positive powers of w, we obtain the following results. The coefficients satisfy the recursion formulas,

$$b_j(n + 2k) = b_j(n) - b_j(n + k), \; j = 0, 1, \ldots, 2k - 1.$$

The initial conditions that are not zero are given by

$$\begin{aligned}
b_0(0) &= 1 \\
b_0(k) &= -1 \\
b_j(k-j) &= -1 \\
b_j(2k-j) &= 2 && j = 1, 2, \ldots, k-1, \\
b_j(j) &= 1 && j = k, \\
b_j(2k-j) &= 1 \\
b_j(3k-j) &= -1. && j = k+1, \ldots, 2k-1,
\end{aligned} \tag{13}$$

The result analogous to (3) is given by

$$\begin{aligned}
b_1(k-1+kn) &= (-1)^{n+1}F_{n+2} \\
b_{k+1}(k-1+kn) &= (-1)^n F_{n+1} \\
b_j(k-1+kn) &= 0, \qquad \text{if } j \neq 1 \text{ or } k+1.
\end{aligned} \tag{14}$$

If we employ (4), (12), and (14), then omitting the argument $(k - 1 + kn)$ from the a_j and b_j, we can write

$$\begin{aligned}
1 &= w^{k-1+kn}w^{-(k-1+kn)} \\
&= (a_0 + a_1 w + \cdots + a_{2k-1}w^{2k-1})(b_1 w + b_{k+1}w^{k+1}).
\end{aligned}$$

Multiplying out the right-hand side and comparing coefficients, we obtain the $2k$ equations:

$$\begin{aligned}
a_{k-1}b_{k+1} + a_{2k-1}(b_1 + b_{k+1}) &= 1 \\
a_0 b_1 + a_k b_{k+1} &= 0 \\
a_1 b_1 + a_{k+1}b_{k+1} &= 0 \\
\vdots \qquad \vdots \qquad &\vdots \\
a_{k-2}b_1 + a_{2k-2}b_{k+1} &= 0 \\
a_{k-1}(b_1 + b_{k+1}) + a_{2k-1}(b_1 + 2b_{k+1}) &= 0 \\
a_0 b_{k+1} + a_k(b_1 + b_{k+1}) &= 0 \\
a_1 b_{k+1} + a_{k+1}(b_1 + b_{k+1}) &= 0 \\
\vdots \qquad \vdots \qquad &\vdots \\
a_{k-2}b_{k+1} + a_{2k-2}(b_1 + b_{k+1}) &= 0.
\end{aligned}$$

We will consider the $a_0, \ldots, a_{2k-1}$ as the unknowns and solve for a_{2k-1} by Cramer's rule. If we denote the coefficient matrix by D and use (3) and (14) to replace b_1, b_{k+1}, and a_{2k-1}, then Cramer's rule yields

$$F_n = \pm\frac{A_k}{\det D}.$$

We will complete the proof of the theorem by showing that $\det D = \pm 1$.

The norm of $e = b_1 w + b_{k+1}w^{k+1}$ is given by the determinant of the matrix whose entries are the coefficients of w^j, $j = 0, \ldots, 2k - 1$, in the following equations:

$$\begin{aligned}
e &= b_1 w + b_{k+1}w^{k+1} \\
ew &= b_1 w^2 + b_{k+1}w^{k+2} \\
\vdots \quad & \quad \vdots \qquad \vdots \\
ew^{k-1} &= b_{k+1} + (b_1 + b_{k+1})w^k
\end{aligned}$$

(continued)

$$ew^k = b_{k+1}w + (b_1 + b_{k+1})w^{k+1} \tag{15}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$ew^{2k-1} = b_1 + b_{k+1} + (b_1 + 2b_{k+1})w^k.$$

The norm of e is ± 1 since $e = w^{-(k-1+kn)}$ and w is a unit. We observe, however, that D is just the transpose of the matrix from which the norm of e was calculated. Hence, det $D = \pm 1$, and our theorem is proved.

As a concluding note we remark that, if $k = 2$, then the theorem yields—with the appropriate choice of the plus/minus signs—the identity

$$F_n = (-1)^{n+1}F_{n+2}^3 + 2(-1)^n F_{n+1}F_{n+2}^2 + (-1)^{n+1}F_{n+1}^3. \tag{16}$$

This can also be verified as follows: Replace F_{n+2} by $F_n + F_{n+1}$ in (16) and simplify to obtain

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \tag{17}$$

Finally, compare (17) with the known [6, p. 57] identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

to complete the verification of (16).

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POINTS AT MUTUAL INTEGRAL DISTANCES IN S^n

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In radio-astronomy circles, it is sometimes jokingly speculated whether it is possible to place infinitely many in-phase, nonaligned antennas in a plane (say, vertical dipoles in a horizontal plane). Geometrically, this means placing infinitely many nonaligned points in R^2, with integral pairwise distances; and naturally the mathematician wants to generalize to R^3 and R^n. In R^3 there is still a physical meaning for acoustic radiators, but not for electromagnetic radiators, since none exists with a spherical symmetry radiation pattern (for more serious questions on antenna configurations, see [2]).

A slightly different problem is that of placing a receiving antenna in a point P, where it receives in phase from transmitting antennas placed in nonaligned coplanar points A_1, A_2, ... (in phase with each other); geometrically, this means that the distances A_iA_j are integral, and that all the differences PA_i-PA_j are also integral. We shall prove that the first question has a negative answer, and that only finitely many P's satisfy the second condition.

Our proof of these facts, set out in Paragraph 1, is only the first step in an inductive demonstration (given in Sections 1 to 4) of the following

Theorem: In a Euclidean space of dimension $n \geq 2$ there exist only finite sets of noncollinear points all of whose mutual distances are integers.

If in all the reasoning used in Paragraphs 1-4 to prove the theorem, one requires that m be a positive real number rather than a positive integer, then one obtains the following result.

Theorem (bis): If one places $n + 1$ antennas in phase at the vertices of a nondegenerate $(n+1)$-hedron in a Euclidean space of dimension $n \geq 2$, then the set of points of the space from which the signals are received in phase is a finite set.

Remark: Two antennas are in phase if their distance is a multiple of the wavelength; a point P receives in phase from two antennas A and B if the differences AP-BP is a multiple of the wavelength.

The last section describes two methods (one due to Euler) to construct systems of points in the plane with integral mutual distances.

By PQ we denote, as usual, the distance between the points P and Q. The phrase "points at integral distance" will be abbreviated to "points at ID."

1. Let O and A be two points of the plane having distance $OA = a$, an integer. We show that the points of the plane for which OP and AP are both integers must all lie on a distinct hyperbolas (one of which is degenerate). As our coordinate system, we take the orthogonal axes with O as origin and the line through O and A as x-axis. Let P be a point at ID from O and from A, assume $P \neq O$, A, and set $OP = m > 0$, $AP = m - k$ with m and k integers.

Note that by the triangle inequality we have $AP \leq OA + OP$ and $OP \leq OA + AP$, which imply that $-a \leq k \leq a$. It is immediate that P lies on the hyperbola $\mathfrak{A}_k$ with foci at O and A defined by the equation

$$(a^2 - k^2)(x - a/2)^2 - k^2y^2 = (k^2/4)(a^2 - k^2); \tag{1}$$

its center is $A' = (a/2, 0)$, and its axes are the lines $y = 0$ and $x = a/2$. It intersects the x-axis at the points with abscissas $x = a/2 \pm k/2$. We conclude that any point that has integral distance from both O and A must lie on one of the hyperbolas $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_a$.

Note that for $k = 0$ and $k = \pm a$, Eq. (1) defines a degenerate parabola, and that there are $a+1$ curves in all. However, all of the curves will be hyperbolas, exactly a in number, if we take $(x - a/2)y = 0$ as $\mathfrak{A}_a$. See Figure 1.

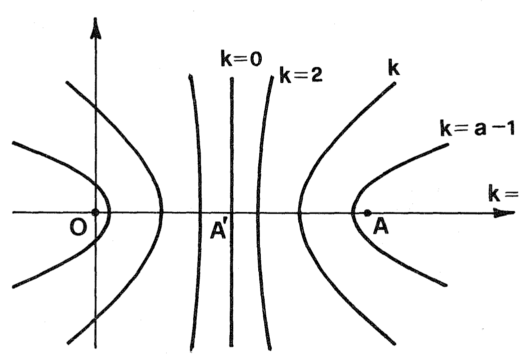

Fig. 1

Now let B be a point at distance $OB = b$ from O and noncollinear with O and A. Repeating the discussion above for A we find that the points at integral distance from O and from B all must lie on b hyperbolas $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_b$. All these hyperbolas have as center the midpoint B' of the segment OB and as axes the line through O and B and the line through B' perpendicular to it. Hence it is clear that none of these hyperbolas coincides with any $\mathfrak{A}_i$, since they have different sets of axes. The possible points at integral distance from O, A, and B are found in the union with respect to i and j of the $\mathfrak{A}_i \cap \mathfrak{B}_j$, and hence are in number at most $4ab$.

2. We now give the proof of the theorem stated in the introduction for the case of a space of dimension 3, since the proof still has an intuitive geometric meaning that sheds light on the more general situation.

Let X be a set of points in the space that are noncollinear and mutually at ID. We wish to show that X has finite cardinality.

If the points of X all lie in a plane, we are already done. Otherwise, X contains four noncoplanar points O, A, B, and C. We set $OA = a$, $OB = b$, and $OC = c$. We fix an orthogonal coordinate system having origin at O, the line OA as x-axis, and the plane determined by OAB as xy plane. With an argument similar to that used in the case of the plane, one sees immediately that the points P at ID from O and A, and thus in particular the points of the set X, must all lie on the $a + 1$ quadrics $S_{A,k}$ defined by the equations

$$S_{A,k}\colon\quad (a^2 - k^2)(x - a/2)^2 - k^2y^2 - k^2z^2 = (k^2/4)(a^2 - k^2),\ k = 0, 1, \ldots, a.$$

If $0 < k < a$, the quadric $S_{A,k}$ is an elliptic hyperboloid of revolution around the line OA; the point A' (midpoint of the segment OA) is its center, and each plane of the pencil through OA is a plane of symmetry; among these there is the xy plane. For $k = 0$ the quadric $S_{A,0}$ is the plane $x = a/2$ (counted twice), and for $k = a$ the real points of $S_{A,a}$ are the real points of the line that passes through O and A.

The points at ID from O and B are all to be found on the $b + 1$ quadrics $S_{B,h}$ with $h = 0, 1, \ldots, b$. For $0 < h < b$, the quadric $S_{B,h}$ is an elliptic hyperboloid of revolution around the line OB; its center is at B' (the midpoint of the segment OB), and it has as planes of symmetry all the planes of the pencil through the axis of revolution, among which there is the xy plane. For $h = 0$ the quadric $S_{B,0}$ is a double plane; for $h = b$ the real points of $S_{B,b}$ are the real points on the axis of revolution.

By analogy, the points at ID from O and C are found on $c + 1$ quadrics $S_{C,\ell}$ with $\ell = 0, 1, \ldots, c$. For $0 < \ell < c$, the quadrics $S_{C,\ell}$ are elliptic hyperboloids of revolution around the line OC, and they certainly do not have the xy plane as a plane of symmetry. For $\ell = 0$ the quadric $S_{C,0}$ is the plane through C' (midpoint of the segment OC) which is orthogonal to the line OC, this plane being counted twice. For $\ell = c$ the real points of the quadric $S_{C,c}$ are the points of the line OC.

Since the points of X are at ID from A, B, C, and O, we have

$$X \subseteq \cup(S_{A,k} \cap S_{B,h} \cap S_{C,\ell}) \text{ for } k = 0, \ldots, a;\ h = 0, \ldots, b;\ \ell = 0, \ldots, c.$$

Now if one of the three quadrics that appear in $S_{A,k} \cap S_{B,h} \cap S_{C,\ell}$ is degenerate ($k = 0, a$, or $h = 0, b$, or $\ell = 0, c$), it is clear that the real points of the intersection either are finitely many or lie in a plane. Therefore, the points of X contained in the intersection are finitely many. If none of those quadrics is degenerate, let $\gamma_{k,h}$ be the real intersection of $S_{A,k}$ with $S_{B,h}$, with k and h fixed.

In view of the facts that the $S_{A,k}$ and $S_{B,h}$ are real quadrics, that there are no real lines contained in them, and that we are considering only the real points of the intersection, there are only two possible cases to discuss:

a. $\gamma_{k,h}$ has real points and is irreducible.

b. $\gamma_{k,h}$ splits into two nondegenerate conics with real points.

In case b, a conic being a plane curve, we see that there can be only a finite number of points of the set X which lie on $\gamma_{k,h} \cap S_{C,\ell}$ for every ℓ. In case a, if we show that $\gamma_{k,h}$ cannot lie entirely on any $S_{C,\ell}$, then $\gamma_{k,h} \cap S_{C,\ell}$ is a finite set of points for each ℓ, and that will complete our proof.

So suppose that $\gamma_{k,h}$ is irreducible. Since $S_{C,\ell}$ is not symmetric with respect to the xy plane, we can write its equation in the form

$$(\alpha x + \beta y + \delta)z + F(x, y, z^2) = 0, \text{ with } \alpha, \beta, \delta \text{ not all } 0.$$

A simple calculation shows that the pairs of points symmetric with respect to the xy plane and which lie on the quadric are either in the xy plane itself or in the plane $\alpha x + \beta y + \delta = 0$. Since $\gamma_{k,h}$ is symmetric with respect to the xy plane, were it contained entirely in $S_{C,\ell}$ it would have to be contained in one of the two planes just mentioned. But, a quadric does not contain irreducible plane quartic curves.

3. In R^n the proof is similar and is based on induction on the dimension n of the space. Here we give only a sketch of the demonstration.

Let X be a set of points in R^n that are all mutually at ID.

a. It is evident that the points that have integral distance from a point O and from another point P are located on a finite number (equal to $OP + 1$) of quadrics $S_{P,k}$ with $k = 0, 1, \ldots, OP$. The quadrics $S_{P,k}$ for $0 < k < OP$ are hyperboloids: for $k = 0$ the real points of $S_{P,0}$ span an R^{n-1}; for $k = OP$ the real points of $S_{P,OP}$ are the points of the line passing through O and P.

b. If X does not contain $n+1$ independent points, it follows that $X \subseteq R^{n-1}$ and the induction holds. Otherwise, let $O, P_1, \ldots, P_n$ be $n + 1$ independent points of X. We fix a cartesian coordinate system with origin at O and the first $n - 1$ coordinate axes in the R^{n-1} determined by $O, P_1, \ldots, P_{n-1}$.

c. From a it follows that the points of X are contained in the union (with respect to the k_i) of the intersection (with respect to i) of the quadrics S_{P_i,k_i} obtained from the pairs of points $OP_1, OP_2, \ldots, OP_n$. We can write

$$X \subseteq \bigcap_{i=1}^{n} \left(\bigcup_{k_i=1}^{OP_i-1} S_{P_i,k_i} \right) \cup \left\{ \begin{array}{l} \text{the points of } X \text{ that come from the intersections in} \\ \text{which a quadric is degenerate, that is, for } k_i = 0, \\ OP_i. \end{array} \right\}$$

If an S_{P_i,k_i} is degenerate, it is immediate that the intersection either consists of a finite set of points or is contained in an R^{n-1}, so that its contribution to the cardinality of X is a finite number of points.

d. We consider the real intersection of $n - 1$ nondegenerate quadrics

$$\bigcap_{i=1}^{n-1} S_{P_i,k_i} \quad \text{with } k = (k_1, \ldots, k_{n-1}) \text{ fixed.}$$

This is either a finite set of points or else is a curve γ_k of order 2^n with real points and symmetric with respect to the hyperplane $x_n = 0$, since all the quadrics that appear in the intersection possess this symmetry.

e. We intersect the curve γ_k with a quadric S_{P_n,k_n} ($k_n = 1, \ldots, OP_n - 1$). This last quadric is certainly not symmetric with respect to the hyperplane $x_n = 0$.

f. If γ_k is irreducible, it cannot lie entirely on any S_{P_n,k_n} (the proof is analogous to the case $n = 3$). Hence, the real intersection is a finite set of points. If γ_k is reducible and $\gamma_k \cap S_{P_n,k_n}$ is not a finite set, then an irreducible component $\overline{\gamma}_k$ of the curve γ_k lies in S_{P_n,k_n}, and the order of $\overline{\gamma}_k$ is less than the order of γ_k.

g. If no point of X lies on $\overline{\gamma}_k$, the proof is finished. Otherwise, let P_{n+1} be a point of X lying on the curve $\overline{\gamma}_k$. In this case, it is required that the other points of X also be at integral distance from P_{n+1}, and, hence, that they lie in the intersection of $\overline{\gamma}_k$ with the $OP_{n+1} + 1$ quadrics $S_{P_{n+1}, k_{n+1}}(k_{n+1} = 0, 1, \ldots, OP_{n+1})$.

We can immediately exclude the case in which the quadrics are degenerate (see *c*). Now, either the real intersection $\overline{\gamma}_k \cap S_{P_{n+1}, k_{n+1}}$ is a finite set of points for every $k_{n+1} = 1, \ldots, OP_{n+1} - 1$, in which case the proof is already completed, or the real intersection of $\overline{\gamma}_k$ with a quadric is a curve whose order is lower than that of $\overline{\gamma}_k$.

By repeating the procedure outlined in *g*, we shall surely stop after a finite number of steps, because we find that the real intersection either is a finite set of points, or it contains no points of X, or it is a curve of order at most $n - 1$. In this last case it is known that the curve must lie in a subspace of dimension at most $n - 1$, and, hence, in particular, $X \subseteq R^{n-1}$.

4. We now give another demonstration of the result of Paragraph 1, which does not, however, give any idea of how the possible points must be distributed in the plane.

Given a triangle OAB in the plane, we fix a system of coordinates as in Paragraph 1. Let $OA = a$, $OB = b$, and $OC = c$, and let φ = angle AOB. We wish to find the points of the plane at integral distance from the vertices of the triangle. Let P be such a point, and set $OP = m$, $AP = m - k$, and $BP = m - h$, with m a positive integer and h, k integers. By the triangle inequality (see Paragraph 1), we have $|k| \le a$, $|h| \le b$, and, hence, if we denote the integral part of a by α and the integral part of b by β, we see that k can take only the values $0, \pm 1, \ldots, \pm\alpha$, and h only the values $0, \pm 1, \ldots, \pm\beta$. The coordinates x, y of P are solutions of the system of equations:

$$\begin{cases} x^2 + y^2 = m^2 \\ (x - a)^2 + y^2 = (m - k)^2 \\ (x - b\cos\varphi)^2 + (y - b\sin\varphi)^2 = (m - h)^2. \end{cases} \tag{2}$$

Substituting m^2 in place of $x^2 + y^2$ in each of the last two equations one finds

$$x = \frac{a^2 - k^2}{2a} + \frac{km}{a}$$

$$y = \frac{b^2 - h^2}{2b \sin\varphi} + \frac{hm}{b \sin\varphi} - \frac{\cos\varphi \; x}{\sin\varphi}$$

which shows that for every integral triple (k, h, m) there is a point (x, y). Now, the first equation of (2) gives a second-degree, nonidentical equation in m, whose coefficients are functions of h, k, a, b, $\cos\varphi$, $\sin\varphi$. (If a, b, and AB are integers, $\cos\varphi$ is rational and m is an integral solution of a diophantine equation.) Since as k and h vary one obtains $(2\alpha + 1)(2\beta + 1)$ such equations, we find at most $2(2\alpha + 1)(2\beta + 1)$ integral values for m and a like number of points at ID from the vertices of the triangle.

The generalization to R^n is analogous. Hence, we may state the following

Theorem: Given an $n + 1$-hedron in R^n with vertices $O, P_1, P_2, \ldots, P_n$, then there are at most

$$2\prod_{i=1}^{n}(2\alpha_i + 1)$$

points of R^n that have integral distance from the vertices of the $n + 1$-hedron; here α_i is the integral part of OP_i.

5. At this point, it is natural to ask if for any given integer n there is a configuration of n points of the plane that are mutually at ID. The answer is affirmative, and here we give a first method to construct such configurations.

Euler gave a construction (recorded, naturally, by Dickson) of polygons having sides, chords, and area all rational numbers, inscribed in a circle of radius $R = 1$: one selects n "Heron angles" $\alpha_1, \alpha_2, \ldots, \alpha_n$, that is, angles with rational sine and cosine, whose sum is less than π, and then, having fixed a point P_0 on the circumference of the circle with center at O and radius $R = 1$, one places P_1 on the circumference in a way such that $P_0OP_1 = 2\alpha_1$; then P_2 so that $P_1OP_2 = 2\alpha_2$; and so on. It is evident that the sides are $P_0P_1 = 2R \sin \alpha_1$, $\ldots$, $P_{i-1}P_i = 2R \sin \alpha_i, \ldots, P_nP_0 = 2R \sin(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, that the chords are $P_iP_j = 2R \sin(\alpha_{i+1} + \cdots + \alpha_j)$ for $i < j$, and that the area is

$$A = (R/2)(P_0P_1 \cos \alpha_1 + P_1P_2 \cos \alpha_2 + \cdots + P_nP_0 \cos \alpha_{n+1});$$

here $\alpha_{n+1} = \pi - (\alpha_1 + \cdots + \alpha_n)$ is obviously a Heron angle. By the addition formula for the sine and the cosine, all sides and chords are rational numbers, and so is the area.

Set $t_i = \tan(\alpha_i/2) = p_i/q_i$ with p_i, q_i relatively prime integers. Then

$$\sin \alpha_i = 2p_iq_i/(p_i^2 + q_i^2) \quad \text{and} \quad \cos \alpha_i = (q_i^2 - p_i^2)/(p_i^2 + q_i^2).$$

Hence, it is clear that it suffices to take a circle with radius

$$4R = \prod_{i=1}^{n} (p_i^2 + q_i^2)$$

in order to obtain a similar polygon with sides and chords all integral numbers.

Let us see how it is possible to "improve" on the construction of Euler. Let $P_0P_1 \ldots P_n$ be a polygon, with rational sides and chords, inscribed in a circumference with center O and radius R, not necessarily rational. Set

$$P_{i-1}OP_i = 2\alpha_i \quad (i = 1, \ldots, n).$$

Since the angle $P_{i-1}P_{i+1}P_i = \alpha_i$, α_i is an angle of the rational-sided triangle $P_{i-1}P_iP_{i+1}$; hence, $\cos \alpha_i$ is rational, and also

$$\tan^2(\alpha_i/2) = (1 - \cos \alpha_i)/(1 + \cos \alpha_i)$$

is a rational number, for $i = 1, 2, \ldots, n$ (here $P_{n+1} = P_0$). Set

$$\tan(\alpha_i/2) = (p_i/q_i)d_i^{1/2}$$

with d_i a positive square-free integer and p_i, q_i integers for each i. Since $P_{i-1}P_i$ is rational, we must have $R = c_id_i^{1/2}$ with c_i rational ($i = 1, \ldots, n$). But then $d_1 = d_2 = \cdots = d_n = d$. In conclusion, we must have

$$\tan(\alpha_i/2) = (p_i/q_i)d^{1/2} \quad \text{and} \quad R = cd^{1/2}$$

with c rational.

Conversely, consider a circle of radius $R = cd^{1/2}$, with c rational and d a square-free integer; it is then possible to inscribe in it a polygon $P_0 \ldots P_n$, for any given n, with rational sides and chords. To achieve this, just select angles α_i such that

$$\alpha_1 + \cdots + \alpha_n < \pi \quad \text{and} \quad \tan(\alpha_i/2) = (p_i/q_i)d^{1/2} \quad (p_i,\ q_i \text{ integers}),$$

and recall that

$$P_iP_j = 2R \sin(\alpha_{i+1} + \cdots + \alpha_j) \text{ for } i < j.$$

Hence, we have established the following theorem.

Theorem: A necessary and sufficient condition in order that a circle of radius R may circumscribe a polygon with rational sides and chords is that $R = cd^{1/2}$, with c rational and d a square-free positive integer. The area of the polygon is rational if and only if $d = 1$.

6. Another method ("the kite") for constructing a configuration of $2n + 3$ noncollinear points mutually at ID, with n fixed in advance, is the following. One selects n Pythagorean triples x_i, y_i, z_i, that is, integral solutions to the equation $x^2 + y^2 = z^2$. Let

$$a = \prod_{i=1}^{n} x_i,\quad b_j = y_j \prod_{i \neq j} x_i,\quad c_j = z_j \prod_{i \neq j} x_i.$$

For each i, we have $a^2 + b_i^2 = c_i^2$. Fix a point O in the plane, and let A be a point at distance a from O.

One can place n points $P_1, \ldots, P_n$ on the line through O perpendicular to OA, with P_i at distance b_i from O. Let A' be the point symmetric to A with respect to O and let $Q_1, \ldots, Q_n$ be the points symmetric to $P_1, \ldots, P_n$ (see Figure 2). The points $O, A, P_1, \ldots, P_n, A', Q_1, \ldots, Q_n$ are $2n + 3$ noncollinear points of the plane mutually at ID, and more precisely,

$$P_iP_j = Q_iQ_j = |b_i - b_j|,$$
$$AP_i = AQ_i = A'P_i = A'Q_i = c_i,$$
$$OP_i = OQ_i = b_i,\ OA = OA' = a.$$

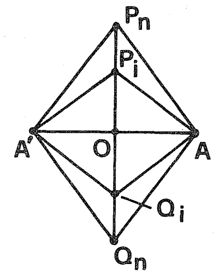

Fig. 2

Remark: It is not necessary that the angle P_iOA be a right angle. It must, however, be an angle φ with $\cos\varphi = p/q \in Q$. Then one has $\sin\varphi = d^{1/2}/q$ with d a positive integer. Let O, A, and P be as in Figure 3, a, b, c are integral solutions of the equation

$$x^2 + y^2 - 2xy\cos\varphi = z^2. \tag{4}$$

Set

$$\begin{cases} X = x\cos\varphi - y \\ Y = x\sin\varphi \\ Z = z \end{cases} \quad \text{so that} \quad \begin{cases} x = Y/\sin\varphi = Yq/d^{1/2} \\ y = Y\cos\varphi/\sin\varphi - X = Yp/d^{1/2} - X \\ z = Z \end{cases}$$

Equation (4) becomes

$$X^2 + Y^2 = Z^2, \tag{5}$$

and then

$$X = h^2 - k^2d$$
$$Y = 2hkd^{1/2}$$
$$Z = h^2 + k^2d$$

are the solutions of (5) as h and k range over Z (the ring of integers); hence,

$$x = 2hkq$$
$$y = 2hkp - (h^2 - k^2d)$$
$$z = h^2 + k^2d$$

are integral solutions of (4), as h and k range over Z. Having selected n solutions of (4), by picking n pairs (h, k), the kite method outlined at the beginning of this section supplies $n + 2$ noncollinear points mutually at ID (see Figure 3).

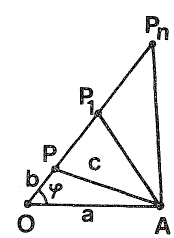

Fig. 3

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MEANS, CIRCLES, RIGHT TRIANGLES, AND THE FIBONACCI RATIO

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In looking for a convenient way to graph the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) of two positive numbers, I came across a connection between Kepler's "two great treasures" of geometry, the Pythagorean Theorem and the Golden Ratio, as well as several attractive geometric patterns.

Let us take a and b as the two positive numbers to be averaged and let

$$a + b = k. \tag{1}$$

The three means are defined as

$$\mathrm{AM}(a, b) = \frac{a+b}{2} = \frac{k}{2} \tag{2}$$

$$\mathrm{GM}(a, b) = \sqrt{ab} \tag{3}$$

$$\mathrm{HM}(a, b) = \frac{2ab}{a+b} = \frac{2ab}{k}. \tag{4}$$

To graph the three means, recall that a perpendicular line from a point on a circle to a diameter of the circle is the mean proportional (i.e., geometric mean) of the two segments of the diameter created by the line. In Figure 1, diameter AB, of length k, is composed of line segment $AD = a$ and line segment $DB = b$. The perpendicular DE is the geometric mean. When O is the center of the circle, the AM is equal to any radius, e.g., AO and OB. To find the harmonic mean, we proceed in the following manner. Construct a perpendicular to the diameter at the center O of height equal to DE, say line OP. Next, construct the perpendicular bisector of AP that meets diameter AB at C. Let Q be the center of a circle passing through A, B, and point C on AB. Since OP is the geometric mean of AO and OC, we have $OC = 2ab/k$, and thus the desired HM is line segment OC.

$$\frac{AD}{AB} = \frac{3}{4}$$

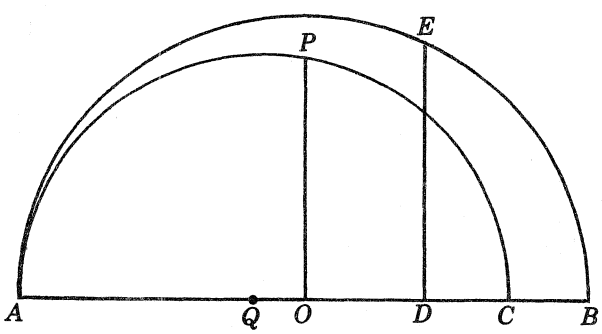

Fig. 1 Constructing the Arithmetic, Geometric, and Harmonic Means

Now, what if points D and C in Figure 1 were the same point? The graph is shown in Figure 2, and it can be seen that

$$AC = a = \text{AM} + \text{HM} = \frac{k}{2} + \frac{2ab}{k}. \tag{5}$$

Replacing b by $k - a$ and solving for a in terms of k, we have

$$a = \frac{1+\sqrt{5}}{4}k = \frac{\phi}{2}k \tag{6}$$

and

$$b = \frac{k}{2\phi^2} \tag{7}$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the Golden Ratio. The difference between a and b is k/ϕ. In Figure 2, the AM remains $\frac{1}{2}k$, but

$$\text{HM} = \frac{k}{2\phi} \tag{8}$$

and

$$\text{GM} = \frac{k}{2\sqrt{\phi}}. \tag{9}$$

In right triangle POC,

$$\overline{PC}^2 = \overline{OP}^2 + \overline{OC}^2 = \text{GM}^2 + \text{HM}^2 = \frac{k^2}{4\phi^2} + \frac{k^2}{4\phi} = k^2\frac{(\phi+1)}{4\phi^2} = \frac{k^2}{4} = \text{AM}^2, \tag{10}$$

since $\phi^2 - \phi - 1 = 0$. Hence, PC is equal to the AM and right triangle POC has sides whose lengths can be expressed in terms of the Golden Ratio.

$$\frac{AC}{AB} = \frac{\phi}{2}$$

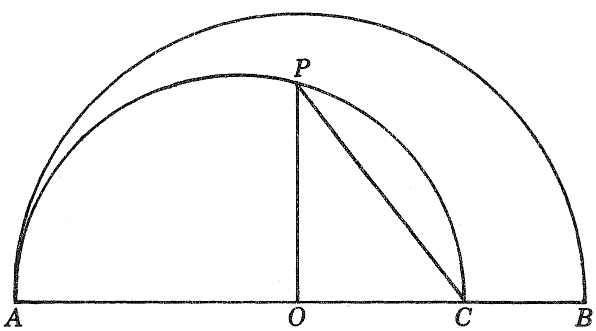

Fig. 2 The Arithmetic, Geometric, and Harmonic Means of Fibonacci Related Numbers Forming a Right Triangle

Since AM is larger than HM and GM, the AM must be the hypotenuse of a right triangle whose sides are AM, HM, and GM. Using the Pythagorean Theorem for that right triangle, we have $\text{AM}^2 = \text{HM}^2 + \text{GM}^2$ or

$$\left(\frac{a+b}{2}\right)^2 = \frac{4a^2b^2}{(a+b)^2} + ab. \tag{11}$$

Clearing of fractions and solving for a in terms of b, we obtain

$$a = b\sqrt{9 + 4\sqrt{5}}.$$

But $9 + 4\sqrt{5} = (2 + \sqrt{5})^2 = (\phi^3)^2$, hence

$$a = b\phi^3. \tag{12}$$

Therefore, the arithmetic, harmonic, and geometric means of positive numbers a and b can form the sides of a right triangle if and only if $a = b\phi^3$. When $b = 1$, the hypotenuse of that right triangle is ϕ^2, and the legs are $\phi^{3/2}$ and ϕ. Sequences of such triangles and a discussion of their relationships to Fibonacci sequences can be found in [1].

Expanding upon Figure 2, using the same values for a and b (i.e., from Eqs. (6) and (7)], we have the elegant picture of Figure 3. The diameters of both inner circles lie on AB and are of length a = AM + HM = $\frac{1}{2}\phi k$. Line segment FC is twice the harmonic mean (or k/ϕ), PR is twice the geometric mean, and FP, FR, CP, and CR are equal to the arithmetic mean. The ratio of the area of each inner circle to the area of the outer circle is $\phi^2/4$. The ratio of the area of the overlap between the two inner circles to the area of each inner circle is $[2w/\pi + 4/\pi\phi^{4.5}]$, while the ratio of the area of the overlap to the area of the outer circle is $[w\phi^2/2\pi - 1/\pi\phi^{2.5}]$, where $\tan w = 2\phi^{1.5}$, with w measured in radians. While those latter ratios are a bit complex, the image of Figure 3 remains one of unity and harmony.

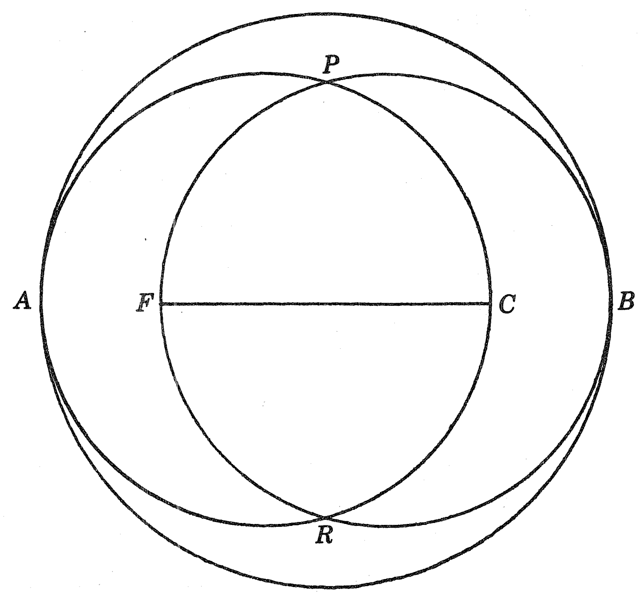

Fig. 3 A Harmonious Blending of Means

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A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS—III

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1. INTRODUCTION

The simple continued fraction expansions of rational multiples of quadratic surds of the form $[a, \dot{b}]$ and $[a, \dot{b}, \dot{c}]$ where the notation is that of Hardy and Wright [1, Ch. 10] were studied in some detail in the first two papers [2] and [3] in this series. Of course, for $a = b = c = 1$, the results concerned the golden ratio, $(1+\sqrt{5})/2$, and the Fibonacci and Lucas numbers since, as is well known, $(1+\sqrt{5})/2 = [\dot{1}]$ and the nth convergent to this fraction is F_{n+1}/F_n where F_n denotes the nth Fibonacci number.

In this paper, we consider the simple continued fraction expansions of powers of the surd $\xi = [\dot{a}]$ and of some related surds. We also consider the special case $(1+\sqrt{5})/2 = [\dot{1}]$ since statements can be made about this surd that are not true in the more general case.

2. PRELIMINARY CONSIDERATIONS

Let a be a positive integer and let the integral sequences

$$\{f_n\}_{n\geq 0} \quad \text{and} \quad \{g_n\}_{n \geq 0}$$

be defined as follows:

$$f_0 = 0,\ f_1 = 1,\ f_n = af_{n-1} + f_{n-2},\ n \geq 2, \tag{1}$$

and

$$g_0 = 2,\ g_1 = a,\ g_n = ag_{n-1} + g_{n-2},\ n \geq 2. \tag{2}$$

These difference equations are easily solved to give

$$f_n = \frac{\xi^n - \overline{\xi}^n}{\sqrt{a^2+4}}, \quad n \geq 0, \tag{3}$$

and

$$g_n = \xi^n + \overline{\xi}^n, \quad n \geq 0, \tag{4}$$

where

$$\xi = (a + \sqrt{a^2+4})/2 \quad \text{and} \quad \overline{\xi} = (a - \sqrt{a^2+4})/2$$

are the two irrational roots of the equation

$$x^2 - ax - 1 = 0. \tag{5}$$

Of course, these results are entirely analogous to those for the Fibonacci and Lucas sequences, $\{F_n\}$ and $\{L_n\}$, and many of the Fibonacci and Lucas results translate immediately into corresponding results for $\{f_n\}$ and $\{g_n\}$. For example, if we solve (3) and (4) for f_n and g_n in terms of ξ^n and $\overline{\xi}^n$, we obtain

$$\xi^n = \frac{g_n + f_n\sqrt{a^2+4}}{2} \tag{6}$$

and

$$\overline{\xi}^n = \frac{g_n - f_n\sqrt{a^2+4}}{2}. \tag{7}$$

Also, since

$$\xi\overline{\xi} = \frac{a+\sqrt{a^2+4}}{2} \quad \frac{a-\sqrt{a^2+4}}{2} = \frac{a^2-(a^2+4)}{4} = -1,$$

it follows that

$$(-1)^n = \xi^n\overline{\xi}^n = \frac{g_n^2 - (a^2+4)f_n^2}{4} \tag{8}$$

and also that

$$\bar{\xi}^n = \frac{(-1)^n}{\xi^n}. \tag{9}$$

We exhibit the first few terms of $\{f_n\}$ and $\{g_n\}$ in the following table and note that both sequences are strictly increasing for $n \geq 2$.

n	0	1	2	3	4	5
f_n	0	1	a	$a^2 + 1$	$a^3 + 2a$	$a^4 + 3a^2 + 1$
g_n	2	a	$a^2 + 2$	$a^3 + 3a$	$a^4 + 4a^2 + 2$	$a^5 + 5a^3 + 5a$

The following lemmas, of some interest in their own right, will prove useful in obtaining the main results.

Lemma 1: For $n > 1$,

(a) $[f_{2n}\sqrt{a^2 + 4}] = g_{2n} - 1$,

(b) $[f_{2n-1}\sqrt{a^2 + 4}] = g_{2n-1}$.

Proof of (a): By (8),

$$(a^2 + 4)f_{2n}^2 = g_{2n}^2 - 4 > g_{2n}^2 - 2g_{2n} + 1$$

since $2g_{2n} - 1 > 4$ for $n > 1$. Therefore,

$$f_{2n}\sqrt{a^2 + 4} > g_{2n} - 1 \tag{10}$$

for $n > 1$. On the other hand

$$g_{2n}^2 > g_{2n}^2 - 4 = (a^2 + 4)f_{2n}^2,$$

so that

$$g_{2n} > f_{2n}\sqrt{a^2 + 4} \tag{11}$$

for all n. But (10) and (11) together imply that

$$[f_{2n}\sqrt{a^2 + 4}] = g_{2n} - 1$$

for $n > 1$ as claimed.

Proof of (b): Again by (8),

$$(a^2 + 4)f_{2n-1}^2 = g_{2n-1}^2 + 4$$

so that

$$f_{2n-1}\sqrt{a^2 + 4} = \sqrt{g_{2n-1}^2 + 4} > g_{2n-1}. \tag{12}$$

Also, for $n > 1$,

$$(g_{2n-1} + 1)^2 = g_{2n-1}^2 + 2g_{2n-1} + 1 > g_{2n-1}^2 + 4 = (a^2 + 4)f_{2n-1}^2$$

so that

$$g_{2n-1} + 1 > f_{2n-1}\sqrt{a^2 + 4}. \tag{13}$$

Thus, from (12) and (13),

$$[f_{2n-1}\sqrt{a^2 + 4}] = g_{2n-1}$$

and the proof is complete.

Lemma 2: For $n > 1$,

(a) $[g_{2n}\sqrt{a^2 + 4}] = (a^2 + 4)f_{2n}$,

(b) $[g_{2n-1}\sqrt{a^2 + 4}] = (a^2 + 4)f_{2n-1} - 1$.

Proof: The argument here is quite similar to that for Lemma 1 and is thus omitted.

3. THE GENERAL CASE

The first two theorems give the simple continued fraction expansions of ξ^n and $\bar{\xi}^n$.

Theorem 3: For $n \geq 1$,

(a) $\xi^{2n-1} = [\dot{g}_{2n-1}]$

(b) $\xi^{2n} = [g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$.

Proof: Since it is well known that $[\dot{g}_{2n-1}]$ converges, we may set

$$x = [\dot{g}_{2n-1}] = g_{2n-1} + \frac{1}{x}.$$

Thus,

$$x^2 - xg_{2n-1} - 1 = 0$$

and hence, using (8) and (6),

$$x = \frac{g_{2n-1} + \sqrt{g_{2n-1}^2 + 4}}{2} = \frac{g_{2n-1} + f_{2n-1}\sqrt{a^2 + 4}}{2} = \xi^{2n-1},$$

and this proves (a). Also, set

$$y = [\dot{1}, \dot{g}_{2n} - 2] = 1 + \frac{1}{g_{2n} - 2 + 1/y}$$

so that

$$y^2(g_{2n} - 2) - y(g_{2n} - 2) - 1 = 0.$$

Then,

$$y = \frac{g_{2n} - 2 + \sqrt{(g_{2n} - 2)^2 + 4(g_{2n} - 2)}}{2(g_{2n} - 2)} = \frac{g_{2n} - 2 + \sqrt{g_{2n}^2 - 4}}{2(g_{2n} - 2)}$$

and, again using (8) and (6),

$$[g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2] = g_{2n} - 1 + \frac{1}{y} = g_{2n} - 1 + \frac{2(g_{2n} - 2)}{g_{2n} - 2 + \sqrt{g_{2n}^2 - 4}}$$

$$= \frac{g_{2n} + \sqrt{g_{2n}^2 - 4}}{2} = \frac{g_{2n} + f_{2n}\sqrt{a^2 + 4}}{2} = \xi^{2n}$$

as claimed.

Theorem 4: For $n \geq 1$,

(a) $\bar{\xi}^{2n-1} = [-1, 1, g_{2n-1} - 1, \dot{g}_{2n-1}]$,

(b) $\bar{\xi}^{2n} = [0, g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$.

Proof: From (9) we have immediately that

$$\bar{\xi}^{2n} = \frac{1}{\xi^{2n}} \text{ and } \bar{\xi}^{2n-1} = -\frac{1}{\xi^{2n-1}}.$$

Since $\xi^{2n} = [g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$ from the preceding theorem, it follows that $\bar{\xi}^{2n} = [0, g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$ as claimed. We also have from the preceding theorem that

$$\xi^{2n-1} = [\dot{g}_{2n-1}]$$

so that

$$\frac{1}{\xi^{2n-1}} = [0, \dot{g}_{2n-1}].$$

But it is well known that if α is real, $\alpha = [a_0, a_1, a_2, \ldots]$ and $a_1 > 1$, then $-\alpha = [-(a_0 + 1), 1, a_1 - 1, a_2, \ldots]$. Thus, it follows that

$$\overline{\xi}^{2n-1} = -\frac{1}{\xi^{2n-1}} = [-1, 1, g_{2n-1} - 1, \dot{g}_{2n-1}]$$

and the proof is complete.

Recall that two real numbers α and β are said to be equivalent if there exist integers A, B, C, and D such that $|AD - BC| = 1$ and

$$\alpha = \frac{A\beta + B}{C\beta + D}.$$

We indicate this equivalency by writing $\alpha \sim \beta$. Recall too that $\alpha \sim \beta$ if and only if the simple continued fraction expansions of α and β are identical from some point on. With this in mind we state the following corollary, which follows immediately from the two preceding theorems.

Corollary 5: If n is any positive integer, then $\xi^n \sim \overline{\xi}^n$.

Noting the form of the surds

$$\xi^n = \frac{g_n + f_n\sqrt{a^2 + 4}}{2} \quad \text{and} \quad \overline{\xi}^n = \frac{g_n - f_n\sqrt{a^2 + 4}}{2},$$

it seemed reasonable also to investigate the simple continued fraction expansions of surds of the form

$$\frac{ag_m \pm f_n\sqrt{a^2 + 4}}{2}, \frac{af_m \pm g_n\sqrt{a^2 + 4}}{2},$$

and so on. It turned out to be impossible to give explicit general expansions of these surds valid for all a, m, and n, but it was possible to obtain the following more modest results.

Theorem 6: Let a be as above and let m, n, and r be positive integers with $m \equiv r \equiv 0 \pmod 3$ or $mr \not\equiv 0 \pmod 3$ if a is odd. Also, let $\{u_n\}$ be either of the sequences $\{f_n\}$ or $\{g_n\}$ and similarly for $\{v_n\}$ and $\{w_n\}$. Then

$$\frac{au_m + w_n\sqrt{a^2 + 4}}{2} \sim \frac{av_r + w_n\sqrt{a^2 + 4}}{2}$$

and

$$\frac{au_m + w_n\sqrt{a^2 + 4}}{2} \sim \frac{av_r - w_n\sqrt{a^2 + 4}}{2}.$$

Proof: We first note that, if a is odd, $f_n \equiv g_n \equiv 0 \pmod 2$ if $n \equiv 0 \pmod 3$ and $f_n \equiv g_n \equiv 1 \pmod 2$ if $n \not\equiv 0 \pmod 3$. Thus $u_m \pm v_r \equiv 0 \pmod 2$ if and only if $m \equiv r \equiv 0 \pmod 3$ or $mr \not\equiv 0 \pmod 3$. To show the first equivalence, let $A = 1$, $B = a(u_m - v_r)/2$, $C = 0$, and $D = 1$. Then B is an integer, since either a or $u_m - v_r$ is divisible by 2 by the above. Moreover,

$$\frac{A \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + B}{C \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + D} = \frac{1 \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + \dfrac{a(u_m - v_r)}{2}}{0 \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + 1}$$

$$= \frac{au_m + w_n\sqrt{a^2 + 4}}{2},$$

and this shows the first equivalence claimed, since $|AD - BC| = 1$. Since the proof of the second equivalence is the same, it is omitted here.

Corollary 7: If m and n are positive integers, then the surds in the following two sets are equivalent:

$$(a)\quad \frac{af_m + g_n\sqrt{a^2+4}}{2},\ \frac{af_m - g_n\sqrt{a^2+4}}{2},$$
$$\frac{ag_m + g_n\sqrt{a^2+4}}{2},\ \frac{ag_m - g_n\sqrt{a^2+4}}{2},$$

and

$$(b)\quad \frac{ag_m + f_n\sqrt{a^2+4}}{2},\ \frac{ag_m - f_n\sqrt{a^2+4}}{2},$$
$$\frac{af_m + f_n\sqrt{a^2+4}}{2},\ \frac{af_m - f_n\sqrt{a^2+4}}{2}.$$

Proof: The first of the above equivalences follows immediately from the second equivalence in Theorem 6 by setting $r = m$, $u_m = f_m$, and $w_n = g_n$ and the others are obtained similarly.

Theorem 8: Let a be as above and let $m > 0$ and $n > 2$ denote integers. Also, let $x = af_m + (a^2 + 4)f_n$ and $y = ag_m + (a^2 + 4)f_n$. Then

$$\frac{af_m + g_n\sqrt{a^2+4}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r] \text{ and } \frac{ag_m + g_n\sqrt{a^2+4}}{2} = [b_0, \dot{a}_1, \ldots, \dot{a}_r]$$

where the vector $(a_1, a_2, \ldots, a_{r-1})$ is symmetric and

$$a_r = 2a_0 - af_m = 2b_0 - ag_m.$$

Also

$$a_0 = \frac{af_m + (a^2+4)f_n - b}{2} = \frac{x-b}{2} \text{ and } b_0 = \frac{ag_m + (a^2+4)f_n - c}{2} = \frac{y-c}{2}$$

where

$$\begin{aligned}
b &= 0 \text{ if } n \equiv x \equiv 0 \pmod 2,\\
b &= 1 \text{ if } x \equiv 1 \pmod 2,\\
b &= 2 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2,\\
c &= 0 \text{ if } n \equiv y \equiv 0 \pmod 2,\\
c &= 1 \text{ if } y \equiv 1 \pmod 2, \text{ and}\\
c &= 2 \text{ if } n - 1 \equiv y \equiv 0 \pmod 2.
\end{aligned}$$

Proof: Let $\nu = (af_m + g_n\sqrt{a^2+4})/2$. Then, by Lemma 2,

$$\begin{aligned}
a_0 = [\nu] &= \left[\frac{af_m + g_n\sqrt{a^2+4}}{2}\right] = \left[\frac{af_m + [g_n\sqrt{a^2+4}]}{2}\right]\\
&= \begin{cases} \left[\dfrac{af_m + (a^2+4)f_n}{2}\right], & n \text{ even}, n > 2\\[2ex] \left[\dfrac{af_m + (a^2+4)f_n - 1}{2}\right], & n \text{ odd}, n > 2 \end{cases}\\
&= \frac{af_m + (a^2+4)f_n - b}{2},
\end{aligned}$$

where it is clear that

$$b = 0 \text{ if } n \equiv x \equiv 0 \pmod 2,$$
$$b = 1 \text{ if } x \equiv 1 \pmod 2, \text{ and}$$
$$b = 2 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2.$$

Thus a_0 is as claimed. Moreover, $0 < \nu - a_0 < 1$, so if we set $\nu_1 = 1/(\nu - a_0)$, it follows that

$$\nu_1 > 1. \tag{14}$$

Taking conjugates, we have that

$$\overline{\nu}_1 = \frac{1}{\dfrac{af_m - g_n\sqrt{a^2+4}}{2} - \dfrac{af_m + (a^2+4)f_n - b}{2}} = \frac{-2}{(a^2+4)f_n - b + g_n\sqrt{a^2+4}} \tag{15}$$

and it is clear that

$$-1 < \overline{\nu}_1 < 0, \tag{16}$$

since a and n are both positive. But (14) and (16) together show that ν_1 is reduced and so, by [4, p. 101], for example, has a purely periodic simple continued fraction expansion $[\dot{a}_1, a_2, \ldots, \dot{a}_r]$. Thus

$$\nu = \frac{af_m + g_n\sqrt{a^2+4}}{2} = [a_0, \nu_1] = [a_0, \dot{a}_1, a_2, \ldots, \dot{a}_r]. \tag{17}$$

On the other hand, again by [4, p. 93],

$$-\frac{1}{\overline{\nu}_1} = [\dot{a}_r, a_{r-1}, \ldots, \dot{a}_1]. \tag{18}$$

But then

$$\begin{aligned}-\frac{1}{\overline{\nu}_1} &= \frac{(a^2+4)f_n - b + g_n\sqrt{a^2+4}}{2}\\ &= \frac{af_m + g_n\sqrt{a^2+4}}{2} + \frac{af_m + f_n(a^2+4) - b}{2} - \frac{2af_m}{2}\\ &= \nu + a_0 - af_m = [2a_0 - af_m, \dot{a}_1, a_2, \ldots, \dot{a}_r].\end{aligned}$$

Comparing (18) and (19), we immediately have that $2a_0 - af_m = a_r$, $a_1 = a_{r-1}$, $a_2 = a_{r-2}, \ldots, a_{r-1} = a_1$. This completes the proof for ν. The proof for $\mu = (ag_m + g_n\sqrt{a^2+4})/2$ is similar and is omitted.

The following theorem is similar to Theorem 8 and is stated without proof.

Theorem 9: Let a be as above and let $m > 0$ and $n > 2$ denote integers. Also, let $x = af_m + g_n$ and $y = ag_m + g_n$. Then

$$\frac{af_m + f_n\sqrt{a^2+4}}{2} = [c_0, \dot{c}_1, \ldots, \dot{c}_r] \text{ and } \frac{ag_m + f_n\sqrt{a^2+4}}{2} = [d_0, \dot{c}_1, \ldots, \dot{c}_r]$$

where the vector $(c_1, c_2, \ldots, c_{r-1})$ is symmetric and

$$c_r = 2c_0 - af_m = 2d_0 - ag_m.$$

Also

$$c_0 = \frac{af_m + g_n - b}{2} = \frac{x - b}{2} \text{ and } d_0 = \frac{ag_m + g_n - c}{2} = \frac{y - c}{2}$$

where

$$b = 0 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2,$$
$$b = 1 \text{ if } x \equiv 1 \pmod 2,$$
$$b = 2 \text{ if } n \equiv x \equiv 0 \pmod 2,$$

$c = 0$ if $n - 1 \equiv y \equiv 0 \pmod 2$,
$c = 1$ if $y \equiv 1 \pmod 2$, and
$c = 2$ if $n \equiv y \equiv 0 \pmod 2$.

Theorem 10: Let m, n, and a denote positive integers and let $\{u_n\}$ and $\{v_n\}$ be as in Theorem 6. Also, let

$$\frac{au_m + v_n\sqrt{a^2+4}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r].$$

(*a*) If $a_1 > 1$, then

$$\frac{au_m - v_n\sqrt{a^2+4}}{2} = [-a_0 + au_m - 1, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1].$$

(*b*) If $a_1 = 1$, then

$$\frac{au_m - v_n\sqrt{a^2+4}}{2} = [-a_0 + au_m - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_2, \dot{a}_1].$$

Proof of (a): Let $\eta = (au_m + v_n\sqrt{a^2+4})/2$. Then by hypothesis,

$$\eta = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$$

and

$$\frac{1}{\dfrac{1}{\eta - a_0} - a_1} = [\dot{a}_2, \ldots, a_r, \dot{a}_1].$$

But then

$$\begin{aligned}
&[-a_0 + au_m - 1, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1] \\
&= -a_0 + au_m - 1 + \cfrac{1}{1 + \cfrac{1}{a_1 - 1 + \cfrac{1}{\cfrac{1}{\cfrac{1}{\eta - a_0} - a_1}}}} \\
&= au_m - \eta \\
&= \frac{au_m - u_n\sqrt{a^2+4}}{2}
\end{aligned}$$

as claimed.

Proof of (b): If $a_1 = 1$, the above analysis still holds except that $a_1 - 1 = 0$, so that we no longer have a simple continued fraction. But then, we immediately have that

$$\begin{aligned}
\frac{au_m - v_n\sqrt{a^2+4}}{2} &= [-a_0 + au_m - 1, 1, 0, \dot{a}_2, \ldots, a_r, \dot{a}_1] \\
&= [-a_0 + au_m - 1, 1, 0, a_2, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2] \\
&= [-a_0 + au_m - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2]
\end{aligned}$$

and the proof is complete.

Interestingly, it appears that the integer r in the above results is always even but we have not been able to show this. Also, while it first seemed that r was bounded for all a, m, and n, this now appears not to be the case. For example, if $a = 4$ and we consider the related surd, $f_m + g_n\sqrt{5}$, r is sometimes quite large and appears to grow with n without bound. On the other hand, if $a = 2$, and we consider the related surds, $f_m + g_n\sqrt{2}$ and $g_m + g_n\sqrt{2}$, it can no doubt be shown that r equals 2 or 4 according as n is even or odd, and that for $f_m + f_n\sqrt{2}$ and $g_m + f_n\sqrt{2}$, r equals 1 or 2 as n is odd or even.

4. SPECIAL RESULTS WHEN $a = 1$

Of course, all the preceding theorems hold when $a = 1$, in which case

$$\xi = (1 + \sqrt{5})/2, \; f_n = F_n, \text{ and } g_n = L_n$$

for all n. On the other hand, in this special case, far more specific results can be obtained as the following theorems show. Note especially that throughout the remainder of the paper we use m and k to denote a positive integer and a nonnegative integer, respectively.

Theorem 11: If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n}{2}, \dot{F_n}, 5\dot{F_n}\right]$$

and

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n}{2}, \dot{F_n}, 5\dot{F_n}\right].$$

Proof: It is immediate from the hypotheses and Theorem 8 that

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n}{2}, \dot{a}_1, \ldots, \dot{a}_r\right]$$

and that

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n}{2}, \dot{a}_1, \ldots, \dot{a}_r\right].$$

Let

$$x = \cfrac{1}{F_n + \cfrac{1}{5F_n + x}}.$$

Then

$$x^2 + 5F_n x - 5 = 0,$$

and, since x is clearly positive and $5F_n^2 + 4 = L_n^2$ is a special case of (8),

$$x = \frac{-5F_n + \sqrt{25F_n^2 + 20}}{2} = \frac{-5F_n + L_n\sqrt{5}}{2}.$$

But then,

$$\left[\frac{F_m + 5F_n}{2}, \dot{F_n}, 5\dot{F_n}\right] = \frac{F_m + 5F_n}{2} + \frac{-5F + L_n\sqrt{5}}{2} = \frac{F_m + L_n\sqrt{5}}{2},$$

and similarly,

$$\left[\frac{L_m + 5F_n}{2}, \dot{F_n}, 5\dot{F_n}\right] = \frac{L_m + L_n\sqrt{5}}{2}$$

as claimed.

Theorem 12: If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5L_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F_n} - 2\right]$$

and

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F_n} - 2\right].$$

Proof: Again it is immediate from the hypotheses and Theorem 8 that

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n - 2}{2}, \dot{a}_1, \ldots, \dot{a}_r\right]$$

and that

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n - 2}{2}, \dot{a}_1, \ldots, \dot{a}_r\right].$$

Then, since n is odd, we have from Theorem 3 of [2] that

$$x = [\dot{1}, F_n - 2, 1, 5\dot{F}_n - 2] = \frac{L_n + L_n\sqrt{5}}{2} - L_{n+1} + 1.$$

Thus,

$$\begin{aligned}\left[\frac{F_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right] &= \frac{F_m + 5F_n - 2}{2} + x \\ &= \frac{F_m + 5F_n - 2}{2} + \frac{L_n + L_n\sqrt{5} - 2L_{n+1} + 2}{2} \\ &= \frac{F_m + 5F_n - 2}{2} + \frac{-5F_n + L_n\sqrt{5} + 2}{2} \\ &= \frac{F_m + L_n\sqrt{5}}{2}.\end{aligned}$$

Similarly,

$$\left[\frac{L_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right] = \frac{L_m + L_n\sqrt{5}}{2}$$

and the proof is complete.

Theorem 13: If $3\nmid m$ and $n = 6 + 6k$ or $9 + 6k$, or if $3|m$ and $n = 4 + 6k$, $5 + 6k$, $7 + 6k$, or $8 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r] \quad \text{and} \quad \frac{L_m + L_n\sqrt{5}}{2} = [b_0, \dot{a}_1, \ldots, \dot{a}_r]$$

with $a_0 = (F_m + 5F_n - 1)/2$, $b_0 = (L_m + 5F_n - 1)/2$, $a_r = 5F_n - 1$, and where the vector $(a_1, \ldots, a_{r-1})$ is symmetric.

Proof: This is an immediate consequence of Theorem 8.

The only surds of the form $(F_m + L_n\sqrt{5})/2$ and $(L_m + L_n\sqrt{5})/2$ not treated by the above theorems are when $3\nmid m$ and $n = 1$ or 3, and when $3|m$ and $n = 1$ or 2. For these cases, the results are as follows.

Theorem 14:

(*a*) If $3\nmid m$, then

$$\frac{F_m + L_1\sqrt{5}}{2} = \left[\frac{F_m + 1}{2}, \dot{1}\right],$$

$$\frac{L_m + L_1\sqrt{5}}{2} = \left[\frac{L_m + 1}{2}, \dot{1}\right],$$

$$\frac{F_m + L_3\sqrt{5}}{2} = \left[\frac{F_m + 7}{2}, \dot{1}, 34. 1, \dot{7}\right],$$

and

$$\frac{L_m + L_3\sqrt{5}}{2} = \left[\frac{L_m + 7}{2}, \dot{1}, 34, 1, \dot{7}\right].$$

(*b*) If $3|m$, then

$$\frac{F_m + L_1\sqrt{5}}{2} = \left[\frac{F_m + 2}{2}, \dot{8}, \dot{2}\right],$$

$$\frac{L_m + L_1\sqrt{5}}{2} = \left[\frac{L_m + 2}{2}, \dot{8}, \dot{2}\right],$$

$$\frac{F_m + L_2\sqrt{5}}{2} = \left[\frac{F_m + 6}{2}, \dot{2}, 1, 4, 1, 2, \dot{6}\right],$$

and

$$\frac{L_m + L_2\sqrt{5}}{2} = \left[\frac{L_m + 6}{2}, \dot{2}, 1, 4, 1, 2, \dot{6}\right].$$

Theorem 15:

(*a*) If $3\nmid m$ and $n = 4 + 6k$ or $n = 8 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - F_n - 2}{2}, 1, F_n - 1, 5\dot{F}_n, \dot{F}_n\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - F_n - 2}{2}, 1, F_n - 1, 5\dot{F}_n, \dot{F}_n\right].$$

(*b*) If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 9 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n}{2}, F_n - 1, \dot{1}, 5F_n - 2, 1, \dot{F_n - 2}\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n}{2}, F_n - 1, \dot{1}, 5F_n - 2, 1, \dot{F_n - 2}\right].$$

(*c*) Let $(F_m + L_n\sqrt{5})/2 = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$ as is always the case from Theorem 8. If $3\nmid m$ and $n = 6 + 6k$, or if $3|m$ and $n = 4 + 6k$ or $8 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n - 1}{2}, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n - 1}{2}, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2\right].$$

And if $3\nmid m$ and $n = 9 + 6k$, or if $3|m$ and $n = 5 + 6k$ or $7 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n - 1}{2}, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n - 1}{2}, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1\right].$$

The preceding theorem omits the cases when $n = 1, 2,$ or 3. These cases are treated in the following result, which is also stated without proof.

Theorem 16:

(*a*) If $3\nmid m$, then

$$\frac{F_m - L_1\sqrt{5}}{2} = \left[\frac{F_m - 3}{2},\ 2,\ \dot{1}\right],$$

$$\frac{L_m - L_1\sqrt{5}}{2} = \left[\frac{L_m - 3}{2},\ 2,\ \dot{1}\right],$$

$$\frac{F_m - L_2\sqrt{5}}{2} = \left[\frac{F_m - 7}{2},\ 6,\ \dot{1},\ \dot{5}\right],$$

$$\frac{L_m - L_2\sqrt{5}}{2} = \left[\frac{L_m - 7}{2},\ 6,\ \dot{1},\ \dot{5}\right],$$

$$\frac{F_m - L_3\sqrt{5}}{2} = \left[\frac{F_m - 9}{2},\ 35,\ \dot{1},\ 7,\ 1,\ \dot{34}\right],$$

and

$$\frac{L_m - L_3\sqrt{5}}{2} = \left[\frac{L_m - 9}{2},\ 35,\ \dot{1},\ 7,\ 1,\ \dot{34}\right].$$

(b) If $3|m$, then

$$\frac{F_m - L_1\sqrt{5}}{2} = \left[\frac{F_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right],$$

$$\frac{L_m - L_1\sqrt{5}}{2} = \left[\frac{L_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right],$$

$$\frac{F_m - L_2\sqrt{5}}{2} = \left[\frac{F_m - 8}{2},\ 1,\ 1,\ \dot{1},\ 4,\ 1,\ 2,\ 6,\ \dot{2}\right],$$

$$\frac{L_m - L_2\sqrt{5}}{2} = \left[\frac{L_m - 8}{2},\ 1,\ 1,\ \dot{1},\ 4,\ 1,\ 2,\ 6,\ \dot{2}\right],$$

$$\frac{F_m - L_3\sqrt{5}}{2} = \left[\frac{F_m - 10}{2},\ 1,\ 1,\ \dot{8},\ \dot{2}\right],$$

and

$$\frac{L_m - L_3\sqrt{5}}{2} = \left[\frac{L_m - 10}{2},\ 1,\ 1,\ \dot{8},\ \dot{2}\right].$$

We close with two theorems which give the expansions for $(F_m \pm F_n\sqrt{5})/2$ and $(L_m \pm F_n\sqrt{5})/2$ for all positive integers m and n. Again, these theorems are stated without proof.

Theorem 17

(a) If $3\nmid m$ and $n = 1 + 6k$ or $5 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n}{2},\ \dot{L_n}\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n}{2},\ \dot{L_n}\right].$$

(b) If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n - 2}{2},\ \dot{1},\ \dot{L_n - 2}\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n - 2}{2}, \dot{1}, \dot{L_n} - 2\right].$$

(c) Let $(F_m + F_n\sqrt{5})/2 = [a_0, a_1, \ldots, a\ ]$. If $3\nmid m$ and $n = 3 + 6k$ or $6 + 6k$, or if $3|m$ and $n = 2 + 6k$, $4 + 6k$, $5 + 6k$, or $7 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n - 1}{2}, \dot{a}_1, \ldots, a_{r-1}, \dot{L_n} - 1\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n - 1}{2}, \dot{a}_1, \ldots, a_{r-1}, \dot{L_n} - 1\right]$$

and the vector $(a_1, \ldots, a_{r-1})$ is symmetric.

(d) If $3|m$, then

$$\frac{F_m + F_1\sqrt{5}}{2} = \left[\frac{F_m + 2}{2}, \dot{8}, \dot{2}\right]$$

and

$$\frac{L_m + F_1\sqrt{5}}{2} = \left[\frac{L_m + 2}{2}, \dot{8}, \dot{2}\right].$$

Theorem 18

(a) If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m - F_n\sqrt{5}}{2} = \left[\frac{F_m - L_n - 2}{2}, 1, L_n - 1, \dot{L}_n\right]$$

and

$$\frac{L_m - F_n\sqrt{5}}{2} = \left[\frac{L_m - L_n - 2}{2}, 1, L_n - 1, \dot{L}_n\right].$$

(b) If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m - F_n\sqrt{5}}{2} = \left[\frac{F_m - L_n}{2}, L_n - 1, \dot{1}, \dot{L_n} - 2\right]$$

and

$$\frac{L_m - F_n\sqrt{5}}{2} = \left[\frac{L_m - L_n}{2}, L_n - 1, \dot{1}, \dot{L_n} - 2\right].$$

(c) Let

$$(F_m + F_n\sqrt{5})/2 = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$$

and let

$$(L_m + L_n\sqrt{5})/2 = [b_0, \dot{a}_1, \ldots, \dot{a}_r].$$

If $3\nmid m$ and $n = 3 + 6k$, or if $3|m$ and $n = 5 + 6k$ or $7 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = [a_0 - a_r - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2]$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = [b_0 - a_r - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2].$$

If $3\nmid m$ and $n = 6 + 6k$ or if $3|m$ and $n = 4 + 6k$ or $8 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = [a_0 - a_r - 1, 1, 1, \dot{a}_2, \ldots, a_r, \dot{a}_1]$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = [b_0 - a_r - 1,\ 1,\ 1,\ \dot{a}_2,\ \ldots,\ a_r,\ \dot{a}_1].$$

(*d*) If $3\nmid m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \left[\frac{F_m - 3}{2},\ 2,\ \dot{1}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \left[\frac{L_m - 3}{2},\ 2,\ \dot{1}\right].$$

If $3|m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{F_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{L_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right].$$

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BENFORD'S LAW FOR FIBONACCI AND LUCAS NUMBERS

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Benford's law states that the probability that a random decimal begins (on the left) with the digit p is $\log_{10}(p + 1)/p$. Recent computations by J. Wlodarski [3] and W. G. Brady [1] show that the Fibonacci and Lucas numbers tend to obey both this law and its natural extension: the probability that a random decimal in base b begins with p is $\log_b(p + 1)/p$. By using the fact that the terms of the Fibonacci and Lucas sequences have exponential growth, we prove the following result.

Theorem: The Fibonacci and Lucas numbers obey the extended Benford's law. More precisely, let $b \geq 2$ and let p satisfy $1 \leq p \leq b - 1$. Let $A_p(N)$ be the number of Fibonacci (or Lucas) numbers F_n (or L_n) with $n \leq N$ and whose first digit in base b is p. Then

$$\lim_{N\to\infty} \frac{1}{N} A_p(N) = \log_b\left(\frac{p + 1}{p}\right).$$

Proof: We give the proof for the Fibonacci sequence. The proof for the Lucas sequence is similar.

Throughout the proof, log will mean $\log_b$. Also, $\langle x\rangle = x - [x]$ will denote the fractional part of x.

Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$, so $F_n = (\alpha^n - (-\alpha)^{-n})/\sqrt{5}$. We first need the following:

Lemma: The sequence $\{\langle n \log \alpha\rangle\}_{n=1}^{\infty}$ is uniformly distributed mod 1.

Recall that a sequence $\{a_n\}$ of real numbers satisfying $0 \le a_n < 1$ is uniformly distributed mod 1 if for every pair of numbers c, d with $0 \le c < d \le 1$ we have

$$\lim_{N\to\infty} \frac{1}{N}(\text{number of } a_n \text{ with } n \le N \text{ and } c \le a_n < d) = d - c.$$

In other words, the fraction of a_n in the interval $[c, d)$ is $d - c$.

Since no power of α is an integer it is easy to see that $\log \alpha$ is irrational (otherwise $b^{m/n} = \alpha$, so $\alpha^n = b^m$, which is integral). In fact, $\log \alpha$ is transcendental, but we shall not need this rather deep fact.

A famous theorem of Weyl [2] states the following: If β is irrational, then the sequence $\{\langle n\beta\rangle\}$ is uniformly distributed mod 1. Letting $\beta = \log \alpha$, we obtain the lemma.

We now continue with the proof of the theorem. Let $\varepsilon > 0$ be small. Let p satisfy $1 \le p \le b - 1$. With the above notation, let

$$c = \log \sqrt{5} + \log p + \varepsilon,\ d = \log \sqrt{5} + \log(p + 1) - \varepsilon.$$

Then $[c, d)$ is an interval of length $\log\left(\frac{p+1}{p}\right) - 2\varepsilon$. Therefore, the fraction of n such that $\langle n \log \alpha\rangle$ lies in $[c, d)$ is $\log\left(\frac{p+1}{p}\right) - 2\varepsilon$. For uniformity of exposition, we have used the convention that all intervals are considered mod 1, so an interval such as $[0.7, 1.2)$ is to be considered as the union of the two intervals $[0.7, 1)$ and $[0, 0.2)$. This technicality occurs only when d is greater than 1 and we leave it to the reader to check that our argument may be extended to cover this case. In particular, the interval $[c, d)$ may be broken into two parts and each part may be treated separately.

Let $m = [n \log \alpha]$ = integer part of $n \log \alpha$. If

$$\log \sqrt{5} + \log p + \varepsilon \le \langle n \log \alpha\rangle = n \log \alpha - m,$$

then

$$pb^m b^\varepsilon \le \alpha^n/\sqrt{5}.$$

If n, hence m, is sufficiently large, then

$$pb^m \le pb^m p^\varepsilon - 1 \le \frac{\alpha^n - (-\alpha)^{-n}}{\sqrt{5}} = F_n,$$

since $p^\varepsilon > 1$ and $|\alpha^{-n}| < 1$. Similarly, if $\langle n \log \alpha\rangle < d$ and n is large, then

$$F_n < (p + 1)b^m.$$

But these last two inequalities simply state that F_n in base b begins with the digit p. Therefore, we have shown that if n is large and $\langle n \log \alpha\rangle$ lies in $[c, d)$ then F_n begins with p. Therefore, the fraction of n such that F_n begins with p is at least

$$\log\left(\frac{p+1}{p}\right) - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we find that this fraction is at least $\log(p + 1)/p$. However, this is true for each p, and

$$\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{b}{b-1}\right) = 1.$$

Therefore, the fraction with first digit p can be no larger than $\log(p + 1)/p$; otherwise, these fractions would have sum greater than 1. Thus, the answer is exactly $\log(p + 1)/p$ and the proof of the theorem is complete.

Finally, we will mention one technicality that we have ignored in the above proof. Since we do not know a priori that $\lim \frac{1}{N}A_p(N)$ exists, it is slightly inaccurate to discuss the fraction of F with first digit p. However, what we proved was that $\liminf \frac{1}{N} A_p(N) \geq \log\left(\frac{p+1}{p}\right)$. By the remark at the end of the proof, it is then easy to see that it is impossible to have $\limsup \frac{1}{N} A_p(N)$ greater than $\log\left(\frac{p+1}{p}\right)$ for any p. Therefore, lim sup = lim inf and the limit exists.

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A SIMPLE DERIVATION OF A FORMULA FOR $\sum_{k=1}^{n} k^r$

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The formula for

$$\sum_{k=1}^{n} k^r \quad (r \text{ and } n \text{ being positive integers})$$

is known (see Barnard & Child [1] and Jordan [2]). However, most undergraduate texts in algebra and calculus give these formulas only for $r = 1$, 2, and 3. Perhaps the reason is that the known formula for general integral r is a bit involved and requires some background in the theory of polynomials and Bernoulli numbers. In this note we give a very simple derivation of this formula and no background beyond the knowledge of binomial theorem (integral power) and some elementary facts from calculus are needed. Consequently, the author hopes that the general formula can be exposed to undergraduates at some proper level.

Let

$$S_r(n) = \sum_{k=1}^{n} k^r,$$

where $r = 0, 1, \ldots, n = 1, 2, \ldots,$ and note that $S_0(n) = n$. In order to find a formula for $S_r(n)$, we use the following identity: For any integer k we have

$$\begin{aligned}\int_{k-1}^{k} x^r dx &= \frac{1}{r+1}(k^{r+1} - (k-1)^{r+1}) \\ &= \frac{1}{r+1} \sum_{j=0}^{r} \binom{r+1}{j} (-1)^{r+2-j} k^j \\ &= \sum_{j=0}^{r} a_j(r) k^j,\end{aligned}$$

where $a_j(r) = (-1)^{r+2-j}\binom{r+1}{j} \Big/ (r+1)$. Hence,

$$\sum_{k=1} \int_{k-1}^{k} x^r dx = \int_0^n x^r dx = \sum_{j=0}^{r} a_j(r) S_j(n),$$

and

(1) $$\frac{n^{r+1}}{r+1} = \sum_{j=0}^{r} a_j(r) S_j(n).$$

Since $a_r(r) = 1$, it follows from (1) that

(2) $$S_r(n) = \frac{n^{r+1}}{r+1} - \sum_{j=0}^{r-1} a_j(r) S_j(n).$$

The numbers $a_j(r)$ can be easily evaluated. Here we list some of the $a_j(r)$'s:

$a_0(1) = -\frac{1}{2}$

$a_0(2) = \frac{1}{3},\ a_1(2) = -1$

$a_0(3) = -\frac{1}{4},\ a_1(3) = 1,\ a_2(3) = -\frac{3}{2}$

$a_0(4) = \frac{1}{5},\ a_1(4) = -1,\ a_2(4) = 2,\ a_3(4) = -2$

$a_0(5) = -\frac{1}{6},\ a_1(5) = 1,\ a_2(5) = -\frac{5}{2},\ a_3(5) = \frac{10}{3},\ a_4(6) = -\frac{5}{2}$

$a_0(6) = \frac{1}{7},\ a_1(6) = -1,\ a_2(6) = 3,\ a_3(6) = -5,\ a_4(6) = 5,\ a_5(6) = -3$, etc.

Using (2) we obtain

$$S_1(n) = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

$$S_2(n) = \frac{n^3}{3} - \left(\frac{n}{3} - \frac{n(n+1)}{2}\right) = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = \frac{n^4}{4} - \left(-\frac{n}{4} + \frac{n(n+1)}{2} - \frac{3}{2}\,\frac{n(n+1)(2n+1)}{6}\right) = \left(\frac{n(n+1)}{2}\right)^2.$$

Continuing in this fashion we obtain

$$S_4(n) = n(n+1)(2n+1)(3n^2+3n-1)/30$$
$$S_5(n) = n^2(n+1)^2(2n^2+2n-1)/12$$
$$S_6(n) = n(n+1)(2n+1)(3n^4+6n^3-3n-1)/42.$$

However, such evaluations get messy with higher values of r. An integral formula for $S_r(n)$ is known (cf. Barnard & Child [1]), but its evaluation depends on Bernoulli numbers. We derive this formula from (2) with an advantage that the required Bernoulli numbers satisfy a simple recurrence relation in terms of $a_j(r)$ which is a by-product of our derivation.

Treating n as a continuous variable and differentiating (2) with respect to n we have

(3) $$S_r'(n) = n^r - \sum_{j=0}^{r-1} a_j(r) S_j'(n),$$

where $S_j'(n) = \dfrac{dS_j(n)}{dn}$.

Since $S_j(n) - S_j(n-1) = n^j$, one obtains

$$\begin{aligned} S_j'(n) &= j\,n^{j-1} + S_j'(n-1) = \cdots \\ &= j\,n^{j-1} + j(n-1)^{j-1} + \cdots + j\,1^{j-1} + S_j'(0). \end{aligned}$$

Clearly, $S_j'(0)$ is the coefficient of n in $S_j(n)$, and writing $B_j = S_j'(0)$, where $B_0 = 1$ and $S_0(n) = n$, we obtain

$$S_j'(n) = j\sum_{k=1}^{n} k^{j-1} + B_j = j\,S_{j-1}(n) + B_j. \tag{4}$$

From (3) and (4) we obtain

$$S_r'(n) = n^r - \sum_{j=0}^{r-1} j\,a_j(r)S_{j-1}(n) - \sum_{j=0}^{r-1} a_j(r)B_j.$$

It is easy to verify that

$$j\,a_j(r) = r\,a_{j-1}(r-1),$$

and hence

$$\begin{aligned} S_r'(n) &= n^r - \sum_{j=1}^{r-1} r\,a_{j-1}(r-1)S_{j-1}(n) - \sum_{j=0}^{r-1} a_j(r)B_j \\ &= r\left[\frac{n^r}{r} - \sum_{j=0}^{r-2} a_j(r-1)S_j(n)\right] - \sum_{j=0}^{r-1} a_j(r)B_j. \end{aligned} \tag{5}$$

Thus it follows from (2) and (5) that

$$S_r'(n) = r\,S_{r-1}(n) + B_r, \tag{6}$$

where

$$B_r = -\sum_{j=0}^{r-1} a_j(r)B_j, \quad B_0 = 1. \tag{7}$$

The relation (6) immediately leads to

$$S_r(n) = r\int S_{r-1}(n)\,dn + n\,B_r. \tag{8}$$

The numbers B_r ($r = 0, 1, \ldots$) are Bernoulli numbers and can be generated from (7), and starting with $S_0(n) = n$ one obtains $S_r(n)$ from (8) for any desired r. Note that the relation (7) for Bernoulli numbers is a by-product of our derivation of (8) from (2). Consequently, no background in the theory of polynomials and Bernoulli numbers is needed to arrive at (8). Moreover, (7) and (8) together make it possible to evaluate $S_r(n)$ for any r, or one can use (8) to get an explicit expression for $S_r(n)$ (see Barnard & Child [1]).

To illustrate the preceding, from the list of $a_j(r)$ and (7) we easily obtain

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ \ldots,$$

and since $S_0(n) = n$, it follows from (8) that

$$\begin{aligned} S_1(n) &= \int n\,dn + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2}, \\ S_2(n) &= 2\int\left(\frac{n^2}{2} + \frac{n}{2}\right)dn + \frac{n}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \text{ etc.} \end{aligned}$$

Finally, we note the following interesting fact. Since

$$a_0(r) = \pm\frac{1}{r+1}$$

and

$$S_0(n) = n,$$

it follows from (2) that

$$S_r(n) = S_1(n)P_{r-1}(n),$$

where $P_{r-1}(n)$ is a polynomial in n of degree $r - 1$.

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A NOTE ON THE POLYGONAL NUMBERS

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1. INTRODUCTION

Polygonal numbers of order k ($k = 3, 4, 5, \ldots$) are the numbers

(1) $$P_{n,k} = \frac{1}{2}[(k-2)n^2 - (k-4)n] \quad (n = 1, 2, 3, \ldots).$$

If $k = 4$, they are reduced to the square numbers. It is clear that there are an infinite number of square numbers which are at a time the sum and difference and the product of such numbers, from the identity

$$\begin{aligned}(4m^2+1)^2 &= (4m)^2 + (4m^2-1)^2\\ &= (8m^4+4m^2+1)^2 - (8m^4+4m^2)^2,\end{aligned}$$

and since there are an infinite number of composite numbers of the form $4m^2+1$ (for example, if $m = 5j+1$, $4m^2+1$ is divisible by 5).

Sierpinski [1] proved that there are an infinite number of triangular numbers ($k = 3$) which are at a time the sum and the difference and the product of such numbers.

For $k = 5$, Hansen [2] proved that there are an infinite number of $P_{n,5}$ that can be expressed as the sum and the difference of such numbers.

O'Donnell [3] proved a similar result for $k = 6$, and conjectured that there will be a similar result for the general case.

In this paper it will be shown that their method of proof is valid for the general case, proving the following theorem.

Theorem: Let a and b be given integers such that $a \neq 0$ and $a \equiv b \pmod 2$, and let

(2) $$A_n = \frac{1}{2}(an^2 + bn) \quad (n = 1, 2, 3, \ldots).$$

There are an infinite number of A_n's which can be expressed as the sum and the difference of the numbers of the same type.

2. PROOF OF THE THEOREM

If $a < 0$, we obtain a set of integers whose elements are the negatives of the elements in the set obtained by using $-a$ and $-b$ instead of a and b. Hence we can assume $a > 0$ in the following.

Let

$$B_n = A_n - A_{n-r} = \frac{1}{2}[a(2nr - r^2) + br], \tag{3}$$

where n and r are positive integers, $n > r$, and r is odd unless a is even.

Lemma 1: For

$$m = ars + r, \tag{4}$$

where s is a positive integer such that

$$a^2 s + 2a > -\frac{b}{r}, \tag{5}$$

the equation

$$A_m = B_n = A_n - A_{n-r} \tag{6}$$

is satisfied by the integer

$$n = \frac{1}{2}s[r(a^2 s + 2a) + b] + r. \tag{7}$$

Proof: Solving

$$\frac{1}{2}[ar^2(as + 1)^2 + br(as + 1)] = \frac{1}{2}[a(2nr - r^2) + br]$$

for n, we have (7).

For any integer c, $c^2 \equiv c \pmod 2$, so that

$$s[r(a^2 s + 2a) + b] = ra^2 s^2 + 2ars + bs \equiv ras + as$$
$$= (r + 1)as \equiv 0 \pmod 2,$$

by the conditions for r and a, which ensures that n is an integer, and the lemma is proved.

For m and n of Lemma 1,

$$A_n = A_m + A_{n-r}. \tag{8}$$

In order to find a number of this type which is equal to some B_p, let $s = art$, for any positive integer t such that

$$a^3 r^2 t + b \geq 0. \tag{9}$$

Then (5) is satisfied and from (4) and (7) we have

$$m = a^2 r^2 t + r, \tag{10}$$

$$n = aru + r, \tag{11}$$

where

$$u = \frac{1}{2}t[r(a^3 rt + 2a) + b] \tag{12}$$

is an integer such that $u \geq s$ by the condition (9).

From Lemma 1, using u in place of s, for the integer

$$p = \frac{1}{2}u[r(a^2u + 2a) + b] + r, \tag{13}$$

we have $A_n = B_p$. This equation, together with equation (8), provides the following lemma, from which we can easily establish the theorem.

Lemma 2: Let a, r, and t be positive integers, where r is odd unless a is even and the condition (9) is satisfied. Then, m, n, u, and p, which are given by (10), (11), (12), and (13), respectively, are also positive integers, and

$$A_n = A_m + A_{n-r} = A_p - A_{p-r}.$$

3. THE CASE OF POLYGONAL NUMBERS

The result for the polygonal numbers of order k is given for

$$a = k - 2, \quad b = -(k - 4)$$

in Lemma 2. In this case, condition (9) is always satisfied for any positive integer t.

Example 1: For $r = 1$, we have

$$P_{n,k} = P_{m,k} + P_{n-1,k} = P_{p,k} - P_{p-1,k},$$

where

$$m = (k - 2)^2t + 1,$$

$$n = (k - 2)u + 1,$$

and

$$p = \frac{1}{2}u[(k - 2)^2u + k] + 1$$

for

$$u = \frac{1}{2}t[(k - 2)^3t + k].$$

Let T_n, Q_n, P_n, H_n, and S_n denote $P_{n,k}$ for $k = 3$, 4, 5, 6, and 7, respectively. Then we have

$$T_{\frac{1}{2}(t^2+3t)+1} = T_{t+1} + T_{\frac{1}{2}(t^2+3t)} = T_p - T_{p-1},$$

where $p = \frac{1}{8}(t^4 + 6t^3 + 15t^2 + 18t) + 1$,

$$Q_{8t^2+4t+1} = Q_{4t+1} + Q_{8t^2+4t} = Q_p - Q_{p-1},$$

where $p = 32t^4 + 32t^3 + 16t^2 + 4t + 1$,

$$P_{\frac{1}{2}(81t^2+15t)+1} = P_{9t+1} + P_{\frac{1}{2}(81t^2+15t)} = P_p - P_{p-1},$$

where $p = \frac{1}{8}(6561t^4 + 2430t^3 + 495t^2 + 50t) + 1$,

$$H_{128t^2+12t+1} = H_{16t+1} + H_{128t^2+12t} = H_p - H_{p-1},$$

where $p = 8192t^4 + 1536t^3 + 168t^2 + 9t + 1$,

$$S_{\frac{1}{2}(625t^2+35t)+1} = S_{25t+1} + S_{\frac{1}{2}(625t^2+35t)} = S_p - S_{p-1},$$

where $p = \frac{1}{8}(390625t^4 + 43750t^3 + 2975t^2 + 98t) + 1$.

Example 2: For the case $r = 3$, $t = 1$, we have

$$m = 9(k - 2)^2 + 3,$$
$$n = 3(k - 2)u + 3,$$

and

$$p = \frac{1}{2}u[3(k - 2)^2 u + 5k - 8] + 3,$$

where

$$u = \frac{1}{2}(9k^3 - 54k^2 + 113k - 80).$$

For $k = 6$, it gives

$$H_{3591} = H_{147} + H_{3588} = H_{2148916} - H_{2148913},$$

which is not covered by Theorem 2 of O'Donnell [3].

The generalized relation in Lemma 2, however, does not yield all such relations. For instance, the relation

$$H_{25} = H_{10} + H_{23} = H_{307} - H_{306}$$

cannot be deduced from our Lemma 2.

4. ACKNOWLEDGMENT

The author would like to thank Professor G. E. Bergum for correcting some errors and for suggesting a few improvements to this note.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-448 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} F_{5i+1}L_{5i} \equiv 0 \pmod{5}.$$

B-449 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} (-1)^{i+1} F_{8i+1}L_{8i} \equiv 0 \pmod{7}.$$

B-450 *Proposed by Lawrence Somer, Washington, D.C.*

Let the sequence $\{H_n\}_{n=0}^{\infty}$ be defined by $H_n = F_{2n} + F_{2n+2}$.

(a) Show that 5 is a quadratic residue modulo H_n for $n \geq 0$.

(b) Does H_n satisfy a recursion relation of the form $H_{n+2} = cH_{n+1} + dH_n$, with c and d constants? If so, what is the relation?

B-451 *Proposed by Keats A. Pullen, Jr., Kingsville, MD*

Let k, m, and p be positive integers with p an odd prime. Show that in base $2p$ the units digit of $m^{k(p-1)+1}$ is the same as the units digit of m.

B-452 *Proposed by P. L. Mana, Albuquerque, NM*

Let $c_0 + c_1x + c_2x^2 + \cdots$ be the Maclaurin expansion for $[1-ax)(1-bx)]^{-1}$, where $a \neq b$. Find the rational function whose Maclaurin expansion is

$$c_0^2 + c_1^2x + c_2^2x^2 + \cdots$$

and use this to obtain the generating functions for F_n^2 and L_n^2.

B-453 *Proposed by Paul S. Bruckman, Concord, CA*

Solve in integers r, s, t with $0 \leq r < s < t$ the Fifibonacci Diophantine equation

$$F_{F_r} + F_{F_s} = F_{F_t}$$

and the analogous Lulucas equation in which each F is replaced by an L.

SOLUTIONS

Counting Hands

B-424 *Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM*

Of the $\binom{52}{5}$ possible 5-card poker hands, how many form a:

(i) full house?
(ii) flush?
(iii) straight?

Solution by Paul S. Bruckman, Concord, CA

(i) The two denominations represented in a full house may be chosen in $2\binom{13}{2}$ ways, the coefficient "2" reflecting the fact that the three-of-a-kind may appear in either of two ways. The individual cards for these denominations can be chosen in $\binom{4}{3}\binom{4}{2}$ ways. Thus, the total number of possible full houses is $2\binom{13}{2}\binom{4}{3}\binom{4}{2} = 3{,}744$.

(ii) The suit represented in a flush may be chosen in 4 ways, and the 5 cards of the flush in that suit may be chosen in $\binom{13}{5}$ ways. Hence, the total number of possible flushes (including "straight flushes") is $4\binom{13}{5} = 5{,}148$.

(iii) With the ace being either high or low, there are 10 different ways to choose the denominations appearing in a straight. With each of these ways, there are 4^5 choices for the individual cards. Thus, the total number of possible straights (including "straight flushes") is $10 \cdot 4^5 = 10{,}240$.

NOTE: Since the total number of possible straight flushes is $10 \cdot 4 = 40$, the answers to (ii) and (iii) above *excluding* straight flushes would be reduced by 40, and so would equal 5,108 and 10,200, respectively.

Also solved by John W. Milsom, Bob Prielipp, Charles B. Shields, Lawrence Somer, Gregory Wulczyn, and the proposer.

Average in a Fixed Rank

B-425 *Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM*

Let k and n be positive integers with $k < n$, and let S consist of all k-tuples $X = (x_1, x_2, \ldots, x_k)$ with each x_j an integer and

$$1 \le x_1 < x_2 < \cdots < x_k \le n.$$

For $j = 1, 2, \ldots, k$, find the average value $\overline{x}_j$ of x_j over all X in S.

Solution by Graham Lord, Université Laval, Québec, Canada

The number of k-tuples X in which $x_j = m$ is $\binom{m-1}{j-1}\binom{n-m}{k-j}$. [Choose the $j-1$ smaller integers from among the first $m-1$ natural numbers and the $k-j$ larger ones from among $m+1, \ldots, n$.] Evidently the total number of k-tuples,

$$\sum_{m=j}^{n-k+j} \binom{m-1}{j-1}\binom{n-m}{k-j},$$

is simply the number of k-subsets from $\{1, 2, \ldots, n\}$, that is $\binom{n}{k}$. Hence,

$$\begin{aligned}\overline{x}_j &= \left\{\sum_{m=j}^{n-k+j} m\binom{m-1}{j-1}\binom{n-m}{k-j}\right\}\Big/\binom{n}{k}\\ &= \left\{j\sum_{m+1=j+1}^{n-k+j+1}\binom{m+1-1}{j+1-1}\binom{n+1-(m+1)}{k+1-(j+1)}\right\}\Big/\binom{n}{k}\\ &= j\cdot\binom{n+1}{k+1}\Big/\binom{n}{k}\\ &= \frac{j(n+1)}{k+1}.\end{aligned}$$

Also solved by Paul S. Bruckman and the proposer.

Fibonacci Pythagorean Triples

B-426 *Proposed by Herta T. Frietag, Roanoke, VA*

Is $(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2$ a perfect square for all positive integers n, i.e., are there integers c_n such that $(F_nF_{n+3}, 2F_{n+1}F_{n+2}, c_n)$ is always a Pythagorean triple?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The answer to the question posed above is "yes" and $c_n = F_{2n+3}$. To establish this result, we observe that

$$F_nF_{n+3} = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = F^2_{n+2} - F^2_{n+1}$$

so

$$\begin{aligned}(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2 &= (F^2_{n+2} - F^2_{n+1})^2 + 4F^2_{n+2}F^2_{n+1}\\ &= (F^2_{n+2} + F^2_{n+1})^2 = F^2_{2n+3}.\end{aligned}$$

[The last equality follows from the fact that $F^2_{n+1} + F^2_n = F_{2n+1}$ for each nonnegative integer n.]

Also solved by Paul S. Bruckman, M. J. DeLeon, A. F. Horadam, Graham Lord, John W. Milsom, A. G. Shannon, Charles B. Shields, Sahib Singh, Lawrence Somer, M. Wachtel, Gregory Wulczyn, and the proposer.

<u>*NOTE:*</u> Each of Horadam and Shannon pointed out that both B-402 and B-426 are special cases of general equation (2.2)′ in A. F. Horadam: "Special Properties of the Sequence $W(a, b, p, q)$," *The Fibonacci Quarterly* 5 (1967):425.

Closed Form, Ingeniously

B-427 *Proposed by Phil Mana, Albuquerque, NM*

Establish a closed form for $\displaystyle\sum_{k=1}^{n} k\binom{k}{2}\binom{n-k}{3}$.

Solution by Graham Lord, Université Laval, Québec, Canada

$$\begin{aligned}\text{The sum} &= \sum_{k=2}^{n-3}(k+1)\binom{k}{2}\binom{n-k}{3} - \sum_{k=2}^{n-3}\binom{k}{2}\binom{n-k}{3}\\ &= 3\cdot\sum_{k=2}^{n-3}\binom{k+1}{3}\binom{n-k}{3} - \binom{n+1}{6}\end{aligned}$$

(continued)

$$= 3\cdot\binom{n+2}{7} - \binom{n+1}{6} = \frac{3n-1}{7}\cdot\binom{n+1}{6}.$$

<u>*NOTE:*</u> As shown in my solution to B-425, $\sum_m \binom{m}{a}\binom{n-m}{b}$ counts the number of subsets of $a + b + 1$ elements chosen from a set of $n+1$ elements: this latter sum equals $\binom{n+1}{a+b+1}$.

Also solved by Paul S. Bruckman, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Closed Form, Industriously

B-428 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For odd positive integers w, establish a closed form for

$$\sum_{k=0}^{2s+1}\binom{2s+1}{k}F^2_{n+kw}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Since $F_j = (a^j - b^j)/\sqrt{5}$ [where $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$],

$$\sum_{k=0}^{2s+1}\binom{2s+1}{k}F^2_{n+kw} = \frac{1}{5}\sum_{k=0}^{2s+1}\binom{2s+1}{k}(a^{2n+2kw} - 2(-1)^{n+kw} + b^{2n+2kw})$$

$$= \frac{1}{5}[a^{2n}(1+a^{2w})^{2s+1} - 2(-1)^n(1+(-1)^w)^{2s+1} + b^{2n}(1+b^{2w})^{2s+1}] \quad \text{(by the Binomial Theorem)}$$

$$= \frac{1}{5}[a^{2n}(1+a^{2w})^{2s+1} + b^{2n}(1+b^{2w})^{2s+1}] \quad \text{(because } w \text{ is odd)}$$

$$= \frac{1}{5}[a^{2n+(2s+1)w}(a^{-w}+a^w)^{2s+1} + b^{2n+(2s+1)w}(b^{-w}+b^w)^{2s+1}]$$

$$= \frac{1}{5}[a^{2n+(2s+1)w}(a^w-b^w)^{2s+1} + b^{2n+(2s+1)w}(b^w-a^w)^{2s+1}]$$

$$= \frac{1}{5}[(a^{2n+(2s+1)w} - b^{2n+(2s+1)w})(a^w-b^w)^{2s+1}]$$

$$= 5^s F_{2n+(2s+1)w}F_w^{2s+1}.$$

Also solved by Paul S. Bruckman, A. G. Shannon, and the proposer.

Yes, When Boiled Down

B-429 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Is the function

$$F^4_{n+10r} + F^4_n - (L_{8r} + L_{4r} - 1)(F^4_{n+8r} + F^4_{n+2r}) + (L_{12r} - L_{8r} + 2)(F^4_{n+6r} + F^4_{n+4r})$$

independent of n? Here n and r are integers.

Solution by Paul S. Bruckman, Concord, CA and
Sahib Singh, Clarion State College, Clarion, PA, independently

Yes. It boils down to

$$12F_{2r}^2F_{4r}^2$$

or to

$$12(L_{12r} - 2L_{8r} - L_{4r} + 4)/25.$$

(The steps were deleted by the Elementary Problems editor.)

Also solved by Bob Prielipp and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-327 *Proposed by James F. Peters, St. John's University, Collegeville, MN*

The sequence

1,3,4,6,8,9,11,12,14,16,17,19,21,22,24,25,27,29,30,32,34,35, ...

was introduced by D. E. Thoro [Advanced Problem H-12, *The Fibonacci Quarterly* 1 (1963):54]. Dubbed "A curious sequence," the following is a slightly modified version of the defining relation for this sequence suggested by the Editor (*The Fibonacci Quarterly* 1 (1963):50):

If

$$T_0 = 1,\ T_1 = 3,\ T_2 = 4,\ T_3 = 6,\ T_4 = 8,\ T_5 = 9,\ T_6 = 11,\ T_7 = 12,$$

then

$$T_{8m+k} = 13m + T_k,\text{ where } k \geq 0,\ m = 1, 2, 3, \ldots .$$

Assume

$$F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}$$

and verify the following identities:

(1) $T_{F_n - 2} = F_{n+1} - 2$, where $n \geq 6$.

For example,

$$T_{F_6 - 2} = T_6 = 11 = F_7 - 2$$
$$T_{F_7 - 2} = T_{11} = 19 = F_8 - 2$$
etc.

(2) $T_{F_n - 2} - T_{F_{n-2} - 2} = F_n$, where $n \geq 6$.

(3) $T_{F_n - 2} = F_{n+1} - 2 + L_{n-12}$, where $n \geq 15$.

H-328 *Proposed by Verner E. Hoggatt, Jr.*

Let θ be a positive irrational number such that $1/\theta + 1/\theta^{j+1} = 1$ ($j \geq 1$ an integer) Further, let

$$A_n = [n\theta],\ B_n = [n\theta^{j+1}],\text{ and } C_n = [n\theta^j].$$

Prove: (a) $A_{C_n} + 1 = B_n$

(b) $A_{C_n+1} - A_{C_n} = 2$

$A_{m+1} - A_m = 1$ ($m \neq C_k$ for any $k > 0$)

(c) $B_n - n$ is the number of A_j's less than B_n.

H-329 *Proposed by Leonard Carlitz, Duke University, Durham, NC*

Show that, for s and t nonnegative integers,

$$(1)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{k!(s-k)!(t-k)!}.$$

More generally, show that

$$(2)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k+\alpha}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{(s-k)!t!}\binom{\alpha+t}{k}$$

and

$$(3)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k+\beta}{t} = \sum_k \frac{x^{s+t-k}}{s!(t-k)!}\binom{\beta+s}{k}.$$

SOLUTIONS

Determined

H-302 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX (Vol. 17, No. 3, October 1979)*

Let c be a constant and define the sequence $\langle a_n\rangle$ by $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} + ca_{n-2}$ for $n \geq 2$. Determine the sequence $\langle b_n\rangle$ for which

$$a_n = \sum_{k=0}^{n}\binom{n}{k}b_k.$$

Solution by the proposer.

The equation $a_n = \sum_{k=0}^{n}\binom{n}{k}b_k$ determines the sequence $\langle b_n\rangle$ uniquely as it is easily seen by letting $n = 0, 1, 2, \ldots$ in succession and solving the resulting equalities recursively for $b_0, b_1, b_2, \ldots$. The first few values are thus found to be

$$b_0 = 1,\ b_1 = 1,\ b_2 = c + 1,\ b_3 = c + 1,\ b_4 = (c + 1)^2,\ \ldots .$$

We will prove that the sequence $\langle b_n\rangle$ defined by $b_{2n} = b_{2n+1} = (c + 1)^n$ satisfies the given equation and envoke its unicity to solve the problem.

The generating functions $A(x)$ and $B(x)$ for the sequences $\langle a_n\rangle$ and $\langle b_n\rangle$, respectively, are easily shown to be

$$A(x) = \frac{1}{1 - 2x - cx^2} \quad\text{and}\quad B(x) = \frac{1 + x}{1 - x^2 - cx^2}.$$

Therefore, utilizing Hoggatt's approach [*The Fibonacci Quarterly* 9 (1971):122], one finds

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k x^n = \frac{1}{1-x}\sum_{n=0}^{\infty}b_n\left(\frac{x}{1-x}\right)^n = \frac{1}{1-x}\,\frac{1+\dfrac{x}{1-x}}{1-\left(\dfrac{x}{1-x}\right)^2 - c\left(\dfrac{x}{1-x}\right)^2}$$

$$= \frac{1}{1 - 2x - cx^2} = \sum_{n=0}^{\infty}a_n x^n,$$

implying the desired relationship between the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

Also solved by P. Bruckman, P. Byrd, D. Russell, and A. Shannon.

Zeta

H-303 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 17, No. 3, October 1979)

If $0 < s < 1$, and n is any positive integer, let

$$H_n(s) = \sum_{k=1}^{n} k^{-s}, \tag{1}$$

and

$$\theta_n(s) = \frac{n^{1-s}}{1-s} - H_n(s). \tag{2}$$

Prove that $\lim\limits_{n\to\infty} \theta_n(s)$ exists, and find this limit.

Solution by the proposer.

The following is Formula 23.2.9 in *Handbook of Mathematical Functions*, ed. by M. Abramowitz and I. A. Stegun. Ninth Printing. (Washington, D.C.: National Bureau of Standards, Nov. 1970 [with corrections]), p. 807:

$$\zeta(s) = \sum_{k=1}^{n} k^{-s} + (s-1)^{-1}n^{1-s} - s\int_n^\infty \frac{x-[x]}{x^{s+1}}dx,\ n = 1, 2, \ldots;\ s \neq 1,\ Re(s) > 0, \tag{3}$$

where ζ is the Riemann zeta function. If we let

$$I_n(s) = \int_n^\infty \frac{x-[x]}{x^{s+1}}dx, \tag{4}$$

we see that formula (3) reduces to

$$-\zeta(s) = \theta_n(s) + sI_n(s). \tag{5}$$

Note from (4) that $I_n(s) > 0$. Moreover,

$$I_n(s) < \int_n^\infty \frac{dx}{x^{s+1}} = \frac{1}{sn^s}.$$

Hence, $\lim\limits_{n\to\infty} sI_n(s) = \lim\limits_{n\to\infty} n^{-s} = 0$. We thus see from (5) that

$$\lim_{n\to\infty} \theta_n(s) = -\zeta(s). \tag{6}$$

Since $\zeta(s)$ is defined for $0 < s < 1$, this is the solution to the problem.

Like Fibonacci-like Sum

H-305 *Proposed by Martin Schechter, Swarthmore College, Swarthmore, PA*
(Vol. 17, No. 3, October 1979)

For fixed positive integers m and n, define a Fibonacci-like sequence as follows:

$$S_1 = 1,\ S_2 = m,\ S_k = \begin{cases} mS_{k-1} + S_{k-2} & \text{if } k \text{ is even,} \\ nS_{k-1} + S_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

(Note that for $m = n = 1$, one obtains the Fibonacci numbers.)

(a) Show the Fibonacci-like property holds that if j divides k then S_j divides S_k and in fact that $(S_q, S_r) = S_{(q,r)}$ where $(\ ,\ )$ = g.c.d.

(b) Show that the sequences obtained

when $[m = 1,\ n = 4]$ and when $[m = 1,\ n = 8]$,

respectively, have only the element 1 in common.

Partial solution by the proposer.

(a) It is convenient first to define a sequence of polynomials $\{Q_k\}_1^\infty$, where Q_k is a polynomial of k commuting variables, as follows:

$$Q_0 = 1,\ Q_1(a_1) = a_1,$$

and

$$Q_k(a_1, \ldots, a_k) = a_kQ_{k-1}(a_1, \ldots, a_{k-1}) + Q_{k-2}(a_1, \ldots, a_{k-2}).$$

It is easy to show by induction that for $j = 1, \ldots, k - 1$, Q_k has the expansion:

$$\begin{aligned} Q_k(a_1, \ldots, a_k) &= Q_j(a_1, \ldots, a_j)Q_{k-j}(a_{j+1}, \ldots, a_k) \\ &\quad - Q_{j-1}(a_1, \ldots, a_{j-1})Q_{k-j-1}(a_{j+2}, \ldots, a_k). \end{aligned}$$

Note that $S_k = Q_{k-1}\underbrace{(m, n, m, n, \ldots)}_{k-1}$

Associated to S_k is the sequence $\overline{S}_k$, which is obtained by interchanging the roles of m and n. The sequences S_k and $\overline{S}_k$ are easily shown to satisfy the relations:

$$S_k = \overline{S}_k \quad \text{if } k \text{ is odd,}$$
$$nS_k = m\overline{S}_k \quad \text{if } k \text{ is even.}$$

Note that if j is odd, $S_j = (mn + 1)S_{j-2} + nS_{j-3}$.

It follows from this equation, by induction, that if j is odd, then $(S_j, n) = 1$. It is also clear that for any j, $(S_j, S_{j+1}) = 1$.

Using the above polynomials, we may readily establish:

$$S_k = \begin{cases} S_{j+1}S_{k-j} + S_j\overline{S}_{k-j-1} & \text{if } j \text{ is even,} \\ S_{j+1}\overline{S}_{k-j} + S_jS_{k-j-1} & \text{if } j \text{ is odd.} \end{cases}$$

An easy induction argument now shows that $j|k$ implies $S_j|S_k$.

Finally, an indirect argument using induction shows that

$$(S_q, S_r) = S_{(q,r)}.$$

<u>*Late Acknowledgment*</u>: H-281 *solved by J. Shallit*, H-283 *solved by J. La Grange.*

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Effective August 1, 1980