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# The Fibonacci Quarterly 

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the official Journal of the fibonacci association<br>DEVOTED TO THE STUDY<br>OF INTEGERS WITH SPECIAL PROPERTIES

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# HAIL TO THEE, BLITHE SPIRIT! 

HOWARD EVES<br>University of Maine

It was in the mid-1940s that I left the Department of Applied Mathematics at Syracuse University in New York State to chair a small Department of Mathematics at the College of Puget Sound ${ }^{1}$ in Tacoma, Washington. Among my first teaching assignments at the new location was a beginning class in college algebra and trigonometry. At the first meeting of this class I noticed, among the twenty-some assembled students, a bright-looking and somewhat roundish fellow who paid rapt attention to the introductory lecture.

As time passed. I learned that the young fellow was named Verner Hoggatt, fresh from a hitch in the army and possessed of an unusual aptitude and appetite for mathematics. Right from the start there was little doubt in my mind that in Vern I had found the mathematics instructor's dream-a potential future mathematician. He so enjoyed discussing things mathematical that we soon came to devote late afternoons, and occasional evenings, to rambling around parts of Tacoma, whilst talking on mathematical matters. On these rambles I brought up things that I thought would particularly capture Vern's imagination and that were reasonably within his purview of mathematics at the time.

Since Vern seemed to possess a particular predilection and intuitive feeling for numbers and their beautiful properties, I started with the subject of Pythagorean triples, a topic that he found fascinating. I recall an evening, shortly after this initial discussion, when I thought I would test Vern's ability to apply newly acquired knowledge. I had been reading through Volume I of Jakob Bernou11i's Opera of 1744, and had come upon the alluring little problem: "Titius gave his friend, Sempronius, a triangular field of which the sides, in perticas, were 50,50 , and 80 , in exchange for a field of which the sides were 50, 50, and 60. I call this a fair exchange." I proposed to Vern that, in view of the origin of this problem, we call two noncongruent isosceles triangles a pair of Bernoullian triangles if the two triangles have integral sides, common legs, and common areas. I invited Vern to determine how we might obtain pairs of Bernoullian triangles. He immediately saw how such a pair can be obtained from any given Pythagorean triangle, by first putting together two copies of the Pythagorean triangle with their shorter legs coinciding and then with their longer legs coinciding. He pointed out that from his construction, the bases of such a Bernoullian pair are even, whence the common area is an integer, so that these Bernoullian triangles are Heronian.

On another ramble I mentioned the problem of cutting off in a corner of a room the largest possible area by a two-part folding screen. I had scarcely finished stating the problem when Vern came to a halt, his right arm at the same time coming up to a horizontal position, with an extended forefinger. "There's the answer," he said. I followed his pointing finger, and there, at the end of the block along which we were walking, was an octagonal stop sign.

There was a popular game at the time that was, for amusement, engaging many mathematicians across the country. It had originated in a problem in The American Mathematical Monthly. The game was to express each of the numbers from 1 through 100 in terms of precisely four 9s, along with accepted mathematical symbols of operation. For example

$$
1=9 / 9+9-9=99 / 99=(9 / 9)^{9 / 9},
$$

[^0]\[

$$
\begin{aligned}
& 2=9 / 9+9 / 9=. \dot{9}+. \dot{9}+9-9 \\
& 3=\sqrt{\sqrt{9} \sqrt{9}}+9-9=(\sqrt{9} \sqrt{9} \sqrt{9}) / 9
\end{aligned}
$$
\]

The next day Vern showed me his successful list. In this list were the expressions

$$
\begin{aligned}
& 67=(. \dot{9}+. \dot{9})^{\sqrt{9}!}+\sqrt{9}=(\sqrt{9}+. \dot{9})^{\sqrt{9}}+\sqrt{9}, \\
& 68=\sqrt{9} i(\sqrt{9}!\sqrt{9}!-\sqrt{9} i), \\
& 70=(9-. \dot{9}) 9-\sqrt{9} i=(. \dot{9}+. \dot{9})^{\sqrt{9}!}+\sqrt{9!},
\end{aligned}
$$

where the inverted exclamation point, i, indicates subfactorial. ${ }^{2}$ For all the other numbers from 1 through 100 , Vern had been able to avoid both exponents and subfactorials, and so he now tried to do the same with 67,68 , and 70 , this time coming up with

$$
\begin{aligned}
& 67=\sqrt{9!/(9 \times 9)+9}, \\
& 68=(\sqrt{9!})!/ 9-\sqrt{9!}-\sqrt{9!}, \\
& 70=(9+. \dot{9})(\sqrt{9!}+. \dot{9}) .
\end{aligned}
$$

It would take too much space to pursue further the many many things we discussed in our Tacoma rambles, but, before passing on to later events, I should point out Vern's delightful wit and sense of humor. I'11 give only one example. The time arrived in class when I was to introduce the concept of mathematical induction. Among some preliminary examples, I gave the following. "Suppose there is a shelf of 100 books and we are told that if one of the books is red then the book just to its right is also red. We are allowed to peek through a a vertical slit, and discover that the sixth book from the left is red. What can we conclude?" Vern's hand shot up, and upon acknowledging him, he asked, "Are they all good books?" Not realizing the trap I was walking into, I agreed that we could regard all the books as good ones. "Then," replied Vern, "all the books are red." "Why?" I asked, somewhat start1ed. "Because all good books are read," he replied, with a twinkle in his eye.

It turned out that I stayed only the one academic year at the College of Puget Sound, for I received an attractive offer from Professor Milne of Oregon State College ${ }^{3}$ to join his mathematics staff there. The hardest thing about the move was my leave-taking of Vern. We had a last ramble, and I left for Oregon.

I hadn't been at Oregon State very long when, to my great joy and pleasure, at the start of a school year I found Vern sitting in a couple of my classes. He had decided to follow me to Oregon. We soon inaugurated what became known as our "oscillatory rambles." Frequently, after our suppers, one of us would call at the home of the other (we lived across the town of Corvallis from one another), and we would set out for the home of the caller. Of course, by the time we reached that home, we were in the middle of an interesting mathematical discussion, and so returned to the other's home, only to find that a new topic had taken over which needed further time to conclude. In this way, until the close of a discussion happened to coincide with the reaching of one of our homes, or simply because of the lateness of the hour, we spent the evening in oscillation.

Our discussions now were more advanced than during our Tacoma rambles. I recall that one of our earliest discussions concerned what we called well-defined

[^1]Euclidean constructions. Suppose one considers a point of intersection of two loci as ill-defined if the two loci intersect at the point in an angle less than some given small angle $\theta$, that a straight line is ill-defined if the distance between the two points that determine it is less than some given small distance $d$, and that a circle is ill-defined if its radius is less than $d$; otherwise, the construction will be said to be well-defined. We proved that any Euclidean construction can be accomplished by a well-defined one. This later constituted our first jointly published paper (in The Mathematics Teacher). Another paper (published in The American Mathematical Monthly) that arose in our rambles, and an expansion of which became Vern's master's thesis, concerned the derivation of hyperbolic trigonometry from the Poincaré model. We researched on many topics, such as Schick's theorem, nonrigid polyhedra, new matrix products, vector operations as matrices, a quantitative aspect of linear independence of vectors, trihedral curves, Rouquet curves, and a large number of other topics in the field of differential geometry.

We did not forego our former interest in recreational mathematics. The number game of the Tacoma days had now evolved into what seemed a much more difficult one, namely, to express the numbers from 1 through 100 by arithmetic expressions that involve each of the ten digits $0,1, \ldots, 9$ once and only once. This game was completely and brilliantly solved when Vern discovered that, for any nonnegative integer $n$,

$$
\log _{(0+1+2+3+4) / 5}\left\{\log _{\sqrt{ } / \ldots /(-6+7+8)} 9\right\}=n
$$

where there are $n$ square roots in the second logarithmic base. Notice that the ten digits appear in their natural order, and that, by prefixing a minus sign if desired, Vern had shown that any integer, positive, zero, or negative, can be represented in the required fashion. ${ }^{4}$

A little event that proved very important in Vern's life took place during our Oregon association. When I was first invited to address the undergraduate mathematics club at Oregon State, I chanced to choose for my topic, "From rabbits to sunflowers," a talk on the famous Fibonacci sequence of numbers. Vern, of course, attended my address, and it reawakened in him his first great mathematical interest, the love of numbers and their endless fascinating properties. For weeks after the talk, Vern played assiduously with the beguiling Fibonacci numbers. The pursuit of these and associated numbers became, in time, Vern's major mathematical activity, and led to his eventual founding of The Fibonacci Quarterly, devoted chiefly to the study of such numbers. During his subsequent long and outstanding tenure at San Jose State University, Vern directed an enormous number of master's theses in this area, and put out an amazing number of attractive papers in the field, solo or jointly with one or another of his students. He became the authority on Fibonacci and related numbers.

After several years at Oregon State College, I returned east, but Vern continued to inundate me with copies of his beautiful findings. When $I$ wrote my Mathematical Circles Squared (Prindle, Weber \& Schmidt, 1972), I dedicated the volume

TO VERNER E. HOGGATT, JR.
who, over the years, has sent me more
mathematical goodies than anyone else

[^2]The great geographical distance between us prevented us from seeing one another very often. I did, on my way to lecturing in Hawaii, stop off to see Vern, and I spent a few days with him a couple of years later when I lectured along the California coast. He once visited me at the University of Maine, when, representing his university, he came as a delegate to a national meeting of Phi Kappa Phi (an academic honorary that was founded at the University of Maine). For almost four decades I had the enormous pleasure of Vern's friendship, and bore the flattering title, generously bestowed upon me by him, of his "mathematical mentor."

In mathematics, Vern was a skylark, and I regret, far more than I can possibly express, the sad fact that we now no longer will hear further songs by him. But, oh, on the other hand, how privileged I have been; I heard the skylark when he first started to sing.

Hail to thee, blithe Spirit!
Bird thou never wert,
That from Heaven, or near it, Pourest thy full heart
In profuse strains of unpremeditated art.

## * ※\#\#

DIAGONAL SUMS IN THE HARMONIC TRIANGLE
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Dedicated to the memory of my colleague and friend, Verner Hoggatt
Leibniz's harmonic triangle is related to reciprocals of the elements of Pascal's triangle, and was developed in summing infinite series by a telescoping process as discussed by Kneale [1] and Price [2], among others. Here, we find row sums and rising diagonal sums for the harmonic triangle.

1. PROPERTIES OF THE HARMONIC TRIANGLE

The harmonic triangle of Leibniz

$$
\begin{aligned}
& \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \cdots \\
& \frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{42} \cdots \\
& \frac{1}{3} \quad \frac{1}{12} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{105} \cdots \\
& \frac{1}{4} \quad \frac{1}{20} \quad \frac{1}{60} \quad \frac{1}{140} \cdots \\
& \frac{1}{5} \quad \frac{1}{30} \quad \frac{1}{105} \cdots \\
& \frac{1}{6} \cdots \quad \cdots
\end{aligned}
$$

is formed by taking successive differences of terms of the harmonic series.

After the first row, each entry is the difference of the two elements immediately above it, as well as being the sum of the infinite series formed by the entries in the row below and to the right. Also, each element is the sum of the element to its right and the element below it in the array. For example, for $1 / 6$ circled above, $1 / 2-1 / 3=1 / 6$, and

$$
\begin{aligned}
\frac{1}{6} & =\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\cdots \\
& =\left(\frac{1}{6}-\frac{1}{12}\right)+\left(\frac{1}{12}-\frac{1}{20}\right)+\left(\frac{1}{20}-\frac{1}{30}\right)+\left(\frac{1}{30}-\frac{1}{42}\right)+\cdots
\end{aligned}
$$

Notice that each row has the first element in the row above it as its sum.
Each rising diagonal contains elements which are $1 / n$ times the reciprocal of the similarly placed elements in Pascal's triangle


In contrast to the harmonic triangle, each element in any row after the first is the sum of all terms in the row above it and to the left, while it is also the difference of the two terms in the row beneath it, and the sum of the element to its left and the element above it.

Since the $n$th row in the harmonic triangle has sum $1 /(n-1)$, if we multiply the row by $n$, we can immediately write the sum of the reciprocals of elements found in the columns of Pascal's triangle written in left-justified form as

$$
\begin{equation*}
\frac{n}{n-1}=\sum_{i=n}^{\infty}\binom{i}{n}^{-1}, n>1 \tag{1.1}
\end{equation*}
$$

or we can begin by summing after $k$ terms, as

$$
\begin{equation*}
\frac{n+1}{n} \cdot\binom{n+k}{n}^{-1}=\sum_{i=n+1}^{\infty}\binom{i+k}{n+1}^{-1} \tag{1.2}
\end{equation*}
$$

As a corollary, we can easily sum the reciprocals of the triangular numbers $T=n(n+1) / 2$ by taking $n=2$ in (1.1), or we could simply multiply the second row of the harmonic triangle by 2 .

## 2. ROW SUMS OF THE HARMONIC TRIANGLE

We rewrite the harmonic triangle in left-justified form as

| $1 / 1$ |  |  |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $1 / 2$ | $1 / 2$ |  |  |  |  |
| $1 / 3$ | $1 / 6$ | $1 / 3$ |  |  |  |
| $1 / 4$ | $1 / 12$ | $1 / 12$ | $1 / 4$ |  |  |
| $1 / 5$ | $1 / 20$ | $1 / 30$ | $1 / 20$ | $1 / 5$ |  |
| $1 / 6$ | $1 / 30$ | $1 / 60$ | $1 / 60$ | $1 / 30$ | $1 / 6$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

[Aug.

We number the rows and columns to begin with zero. Then the element in the $i$ th row and $n$th column is

$$
1 /\left[(i+1)\binom{i}{n}\right], n=0,1,2, \ldots, i=0,1,2, \ldots .
$$

The row sums are $1,1,5 / 6,2 / 3,8 / 15,13 / 30,151 / 420, \ldots$, which sequence is the convolution of the harmonic sequence $1,1 / 2,1 / 3, \ldots, 1 / n, \ldots$, and the sequence $1,1 / 2,1 / 4,1 / 8, \ldots, 1 / 2^{n}, \ldots$, , which can be derived [3] as follows.

Let $G_{n}(x)$ be the generating function for the elements in the $n$th column of the harmonic triangle written in left-justified form. Then

$$
\begin{equation*}
G_{0}(x)=\ln [1 /(1-x)]=1+x / 2+x^{2} / 3+\cdots+x^{n} /(n+1)+\cdots \tag{2.1}
\end{equation*}
$$

and generally,

$$
\begin{equation*}
G_{n+1}(x)=(x-1) G_{n}(x)+x^{n} /(n+1) \tag{2.2}
\end{equation*}
$$

Consider the display

$$
\begin{array}{ccc}
G_{0}(x) & = & G_{0}(x) \\
G_{1}(x) & =(x-1) G_{0}(x)+1 \\
G_{2}(x) & =(x-1) G_{1}(x)+x / 2 \\
\ldots & \cdots & \ldots \\
G_{n+1}(x) & =(x-1) G_{n}(x)+x^{n} /(n+1)
\end{array}
$$

Let $S$ be the infinite sum of the column generators, and sum vertically:

$$
S=(x-1) S+2 G_{0}(x)
$$

Solving for $S$, we have

$$
S=G_{0}(x) /(1-x / 2)=(\ln [1 /(1-x)]) \cdot\left(\frac{1}{1-x / 2}\right)
$$

the product of the generating functions for the harmonic sequence and for the sequence of powers of $1 / 2$. Thus, the row sums are the convolution between the harmonic sequence $\{1 / n\}_{n=1}^{\infty}$ and the sequence $\left\{1 / 2^{n}\right\}_{n=0}^{\infty}$.

What we have found is

$$
\begin{align*}
\sum_{n=0}^{\infty}\left(\frac{1}{n+1} \sum_{k=0}^{n}\binom{n}{k}^{-1}\right) x^{n} & =\frac{\ln [1 /(1-x)]}{1-x / 2}  \tag{2.3}\\
\frac{1}{n+1} \sum_{k=0}^{n}\binom{n}{k}^{-1} & =\sum_{k=0}^{n} \frac{1}{k+1} \cdot \frac{1}{2^{n-k}} \tag{2.4}
\end{align*}
$$

We can also write the generating function $S^{*}(x)$ for the sums of elements appearing on the successive rising diagonals formed by beginning in the leftmost column and proceeding up $p$ elements and right one element throughout the array:

$$
\begin{equation*}
S^{*}(x)=\frac{G_{0}(x)+x^{p} G_{0}\left(x^{p+1}\right)}{1+x^{p}-x^{p+1}} \tag{2.5}
\end{equation*}
$$

By way of comparison, the Fibonacci numbers with negative subscripts are generated by

$$
\begin{equation*}
\frac{1}{1+x-x^{2}}=\sum_{n=1}^{\infty} F_{-n} x^{n-1} \tag{2.6}
\end{equation*}
$$

To contrast with the harmonic triangle, we write Pascal's triangle in leftjustified form, and number the rows and columns to begin with zero:

| 1 |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 |  |  |  |  |  |
| 1 | 2 | 1 |  |  |  |  |
| 1 | 3 | 3 | 1 |  |  |  |
| 1 | 4 | 6 | 4 | 1 |  |  |
| 1 | 5 | 10 | 10 | 5 | 1 |  |
| $\ldots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\ldots$ |

Then the generating function for the $n$th column is $G_{n}(x)=x^{n} /(1-x)^{n+1}$, which has been used to write the diagonal sums for Pascal's triangle [4], [7]. We recall the numbers $u(n ; p, q)$ of Harris and Styles [5], [6], formed as the sum of the element in the leftmost column and $n$th row and the elements obtained by taking steps $p$ units up and $q$ units right throughout the array. These numbers are generated by [4], [7],

$$
\begin{equation*}
\sum_{n=0}^{\infty} u(n ; p, q) x=\frac{(1-x)^{q-1}}{(1-x)^{q}-x^{p+q}}, p+q \geq 1, q \geq 0 \tag{2.7}
\end{equation*}
$$

A1so, the row sums of Pascal's triangle are given by $2^{n}=u(n ; 0,1)$, while the Fibonacci numbers are the sums on the rising diagonals, or, $u(n ; 1,1)=F_{n}$. If we extend $u(n ; p, q)$ to negative subscripts [3], we have

$$
\begin{equation*}
\frac{1}{1+x^{p}-x^{p+1}}=\sum_{n=0}^{\infty} u(-n ; p, 1) x^{n}, \tag{2.8}
\end{equation*}
$$

which has a form similar to (2.5) and becomes (2.6) when $p=1$.

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# SOME GENERALIZATIONS OF A BINOMIAL IDENTITY <br> <br> CONJECTURED BY HOGGATT 

 <br> <br> CONJECTURED BY HOGGATT}

## L. CARLITZ

Duke University, Durham NC 27706
To the memory of Verner Hoggatt

1. INTRODUCTION

In November 1979 Hoggatt sent me the following conjectured identity. Put

$$
\begin{equation*}
S_{n, r}=\frac{1}{r+1}\binom{n-r}{r}\binom{n-r-1}{r} \quad(n \geq 2 r+1 ; r \geq 0) \tag{1.1}
\end{equation*}
$$

Then

$$
\begin{equation*}
S_{n+1, r}=S_{n, r}+\sum_{k=0}^{r-1} \sum_{j=1}^{n-1} S_{j, k} S_{n-j-1, r-k-1} \quad(n \geq 2 r+1 ; r \geq 1) \tag{1.2}
\end{equation*}
$$

I was able to send him a proof of (1.2) that made use of various properties of special functions.

In this note we first sketch this proof. Next, using a different method, we obtain some generalizations of (1.2). In particular, if we put

$$
\begin{equation*}
S_{n, r}^{p}=\frac{1}{r+1}\binom{n-r}{r}\binom{n-r-p}{r} \quad(n \geq 2 r+p ; r \geq 0) \tag{1.3}
\end{equation*}
$$

where $p$ is a nonnegative integer, we show that

$$
\begin{equation*}
(p+q) S_{n, r}^{(p+q)}=p S_{n-q, r}^{(p)}+q S_{n-p, r}^{(q)}+p q \sum_{j=0}^{n-2} \sum_{s=0}^{r-1} S_{j, s}^{(p)} S_{n-j-2, r-s-1}^{(q)} \tag{1.4}
\end{equation*}
$$

and

$$
\begin{array}{r}
(r+1) S_{n, r}^{(p+q)}=(r+1) S_{n, r}^{(q)}+p \sum_{j=0}^{n-2} \sum_{s=0}^{r-1}(r-s) S_{j, s} S_{n-j-2, r-s-1}  \tag{1.5}\\
(p>0, q \geq 0, r>0)
\end{array}
$$

We remark that (1.4) is implied by (1.5).
The special case $p=1, q=0$, of (1.5) may be noted:

$$
\begin{equation*}
(r+1) S_{n, r}=\binom{n-r}{p}^{2}+\sum_{j=0}^{n-2} \sum_{s=0}^{r-1}\binom{j-s}{s}^{2} S_{n-j-2, r-s-1} \tag{1.6}
\end{equation*}
$$

For additional results, see (7.7) and (7.8) below.
Remark: The close relationship between the identities of this paper and ultraspherical polynomials suggests that even more general identities can be found that are related to the general Jacobi polynomials. This is indeed the case; however, we leave this for another paper.

SECTION 2
Put

$$
\begin{equation*}
f_{r}(x)=\frac{1}{r+1} \sum_{n=2 r+1}^{\infty}\binom{n-r}{r}\binom{n-r-1}{r} x^{n}=\sum_{n=2 r+1}^{\infty} S_{n, r} x^{n} \tag{2.1}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
f_{r}(x)=\frac{x^{2 r+1}}{(1-x)^{2 r+1}} \phi_{r}(x), \tag{2.2}
\end{equation*}
$$

where $\phi_{r}(x)$ is a polynomial in $x$. To get an explicit formula for $\phi_{r}(x)$, rewrite (2.2) in the form

$$
x^{2 r+1} \phi_{r}(x)=(1-x)^{2 r+1} f_{r}(x) .
$$

Thus

$$
\begin{align*}
\phi_{r}(x) & =\frac{1}{r+1}(1-x)^{2 r+1} \sum_{n=0}^{\infty}\binom{n+r}{r}\binom{n+r+1}{r} x^{n} \\
& =\frac{1}{r+1} \sum_{j=0}^{2 r+1}(-1)^{j}\binom{2 r+1}{j} x^{j} \sum_{n=0}^{\infty}\binom{n+r}{r}\binom{n+r+1}{r} x^{n} \\
& =\frac{1}{r+1} \sum_{m=0}^{\infty} x^{m} \sum_{\substack{j=0 \\
j \leq m}}^{2 r+1}(-1)^{j}\binom{2 r+1}{j}\binom{m-j+r}{r}\binom{m-j+r+1}{r} . \tag{2.3}
\end{align*}
$$

Since the product

$$
\binom{m-j+r}{p}\binom{m-j+r+1}{p}
$$

is of degree $2 r$ in $j$, it follows that the inner sum in (2.3) vanishes for

$$
m \geq 2 r+1
$$

Thus we need only consider $m \leq 2 r$. Hence the sum is equal to

$$
\binom{m+r}{r}\binom{m+r+1}{r} \sum_{j=0}^{m} \frac{(-2 r-1)_{j}(-m)_{j}(-m-1)_{j}}{j!(-m-r)_{j}(-m-r-1)_{j}},
$$

where

$$
(\alpha)_{j}=a(\alpha+1) \ldots(\alpha+j-1)
$$

App1ying Saalschütz' theorem [1, p. 87], we get

$$
\binom{m+r}{r}\binom{m+r+1}{r} \frac{(-r+1)_{m}(-m+r+1)_{m}}{(-m-r)_{m}(r+2)_{m}}=\frac{r+1}{m+1}\binom{r}{m}\binom{r-1}{m}
$$

We have, therefore,

$$
\begin{equation*}
\phi_{r}(x)=\sum_{m=0}^{r-1} \frac{1}{m+1}\binom{r}{m}\binom{r-1}{m} x^{m} \quad(r \geq 1) . \tag{2.4}
\end{equation*}
$$

For $r=0$, it is clear that

$$
\begin{equation*}
\phi_{0}(x)=1 \tag{2.5}
\end{equation*}
$$

In hypergeometric notation, (2.4) becomes

$$
\begin{equation*}
\phi_{r}(x)={ }_{2} F_{1}[-r+1,-r ; 1 ; x] . \tag{2.6}
\end{equation*}
$$

On the other hand [1, p. 254, Eq. (2)],

$$
P_{n}^{(1,1)}(x)=\frac{(2)_{n}}{n!}\left(\frac{x+1}{2}\right)^{n}{ }_{2} F_{1}\left[-n,-n-1 ; 2 ; \frac{x-1}{x+1}\right] .
$$

If we put

$$
y=\frac{x-1}{x+1}, x=\frac{1+y}{1-y},
$$

[Aug.
this becomes

$$
{ }_{2} F_{1}[-n,-n-1 ; 2 ; y]=\frac{1}{n+1}(1-y)^{n} P_{n}^{(1,1)}\left(\frac{1+y}{1-y}\right) .
$$

Thus, by (2.6),

$$
\begin{equation*}
\phi_{r+1}(x)=\frac{1}{r+1}(1-x)^{r} P_{r}^{(1,1)}\left(\frac{1+x}{1-x}\right) \tag{2.7}
\end{equation*}
$$

We have also the generating function [1, p. 271, Eq. (6)]
where

$$
\sum_{n=0}^{\infty} P_{n}^{(1,1)}(x) t^{n}=2^{2} \rho^{-1}(1+t+\rho)^{-1}(1-t+\rho)^{-1}
$$

$$
\rho=\left(1-2 x t+t^{2}\right)^{1 / 2}
$$

Thus

$$
(1+t+\rho)(1-t+\rho)=2(1-x t+\rho)
$$

so that

$$
\begin{equation*}
\sum_{n=0}^{\infty} P^{(1,1)}(x) t^{n}=2 \rho^{-1}(1-x t+\rho)^{-1} \tag{2.8}
\end{equation*}
$$

It can be verified that if

$$
\Phi=\frac{1-x t+\rho}{t}
$$

then

$$
\frac{d \Phi}{d t}=\frac{x^{2}-1}{\rho(1-x t+\rho)}
$$

Comparison with (2.8) gives

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{1}{n+1} P_{n}^{(1,1)}(x) t^{n+1}=\frac{2}{x^{2}-1} \frac{1-x t-\rho}{t} \tag{2.9}
\end{equation*}
$$

Now replace $x$ by $(1+x) /(1-x)$ and replace $t$ by $(1-x) z$. The result is $\sum_{n=0}^{\infty} \frac{1}{n+1}(1-x)^{n} P_{n}^{(1,1)}\left(\frac{1+x}{1-x}\right) z^{n}=\frac{1-(1+x) z-\sqrt{1-2(1+x) z+(1-x)^{2} z^{2}}}{2 x z^{2}}$. Thus, by (2.7), we get

$$
\begin{equation*}
\sum_{r=0}^{\infty} \phi_{r+1}(x) z^{r}=\frac{1-(1+x) z-\sqrt{1-2(1+x) z+(1-x)^{2} z^{2}}}{2 x z^{2}} \tag{2.10}
\end{equation*}
$$

## SECTION 3

We now rewrite the identity (1.2) in terms of the polynomial $\phi_{r}(x)$. To begin with, (1.2) can be replaced by

$$
\begin{aligned}
S_{n+1, r} & =S_{n, r}+\sum_{k=0}^{\dot{r}-1} \sum_{j=1}^{n-2} S_{j, k} S_{n-j-1, r-k-1}+\sum_{k=0}^{r-1} S_{n-1, k} S_{0, r-k-1} \\
& =S_{n, r}+S_{n-1, r-1}+\sum_{k=0}^{r-1} \sum_{j=1}^{n-2} S_{j, k} S_{n-j-1, r-k-1} .
\end{aligned}
$$

Then multiplying both sides by $x^{n+1}$ and summing over $n$ we get

$$
\begin{aligned}
\sum_{n=2 r+1}^{\infty} S_{n+1, r} x^{n+1}= & x \sum_{n=2 n+1}^{\infty} S_{n, r} x^{n}+x^{2} \sum_{n=2 p}^{\infty} S_{n, r-1} x^{n} \\
& +x^{2} \sum_{k=0}^{n-1} \sum_{j=2 k+1}^{\infty} S_{j, k} x^{j} \sum_{n=2 r-2 k-1}^{\infty} S_{n, r-k-1} x^{n} .
\end{aligned}
$$

In view of (2.1), this becomes

$$
(1-x) f_{r}(x)=x^{2} f_{r-1}(x)+x^{2} \sum_{k=0}^{r-1} f_{k}(x) f_{r-k-1}(x)
$$

Hence by (2.2) we get

$$
\begin{equation*}
\phi_{r}(x)=(1-x) \phi_{r-1}(x)+x \sum_{k=0}^{r-1} \phi_{k}(x) \phi_{r-k-1}(x) \quad(r \geq 1) . \tag{3.1}
\end{equation*}
$$

For example, we have
$\phi_{1}(x)=1, \phi_{2}(x)=1+x, \phi_{3}(x)=1+3 x+x^{2}, \phi_{4}(x)=1+6 x+6 x^{2}+x^{3}$ in agreement with (2.4).

Next put

$$
F=F(x, z)=\sum_{r=0}^{\infty} \phi_{r}(x) z^{r}
$$

then it is easily verified that (3.1) gives

$$
\begin{equation*}
F=1+(1-x) z F+x z F^{2} \tag{3.2}
\end{equation*}
$$

The solution of (3.2) such that $F(x, 0)=1$ is

$$
F=\frac{1-(1-x) z-\sqrt{1-2(1+x) z+(1-x)^{2} z^{2}}}{2 x z} .
$$

Since

$$
\frac{F-1}{z}=\sum_{r=0}^{\infty} \phi \quad{ }_{1}(x) z^{r},
$$

we get

$$
\begin{equation*}
\sum_{r=0}^{\infty} \phi_{r+1}(x) z^{r}=\frac{1-(1+x) z-\sqrt{1-2(1+x) z+(1-x)^{2} z^{2}}}{2 x z^{2}} \tag{3.3}
\end{equation*}
$$

Comparison of (3.3) with (2.10) evidently completes the proof of the desired result.

## SECTION 4

To generalize the above, we take

$$
\begin{equation*}
S_{n, r}^{(p)}=\frac{1}{p+1}\binom{n-p}{p}\binom{n-p-p}{p} \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{r}^{(p)}(x)=\sum_{n=2 r+p}^{\infty} S_{n, r}^{(p)} x^{n} \tag{4.2}
\end{equation*}
$$

where $p$ is a fixed nonnegative integer. Clearly

$$
\begin{equation*}
f_{r}^{(p)}(x)=\frac{x^{2 r+p}}{(1-x)^{2 r+1}} \phi_{r}^{(p)}(x), \tag{4.3}
\end{equation*}
$$

where $\phi_{r}^{(p)}(x)$ is a polynomial $x$. It is evident that
[Aug.

$$
S_{n, r}=S_{n, r}^{(1)}, f_{r}(x)=f_{r}^{(1)}(x), \phi_{r}(x)=\phi_{r}^{(1)}(x)
$$

Exactly as in the proof of (2.4) we find that
and

$$
\begin{equation*}
\phi_{r}^{(p)}(x)=\frac{(r+p)!}{(r+1)!} \sum_{m=0}^{r-p} \frac{1}{(m+1)_{p}}\binom{r}{m}\binom{r-p}{m} x^{m} \quad(p \leq r) \tag{4.4}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{r}^{(p)}(x)=\frac{(r+p)!}{(p+1)!} \sum_{m=0}^{r} \frac{(-1)^{m}}{(m+1)_{p}}\binom{r}{m}\binom{p-r+m-1}{m} x^{m} \quad(p>r) \tag{4.5}
\end{equation*}
$$

In hypergeometric notation, both (4.4) and (4.5) become

$$
\begin{equation*}
\phi_{r}^{(p)}(x)=\frac{(r+p)!}{(r+1)!p!} F[-r+p,-r ; p+1 ; x] . \tag{4.6}
\end{equation*}
$$

Note that $\phi_{r}^{(p)}(x)$ is of degree $r-p$ for $p \leq r$ and of degree $r$ for $p>r$.
Since [1, p. 254, Eq. (2)]

$$
P_{n}^{(p, p)}(x)=\frac{(p+1)_{n}}{n!}\left(\frac{x+1}{2}\right)^{n} F\left[-n,-n-p ; p+1 ; \frac{x-1}{x+1}\right]
$$

it follows that

$$
\begin{equation*}
\phi_{r+p}^{(p, p)}(x)=\frac{r!}{(p+1)_{r}}(1-x)^{r} P_{r}^{(p, p)}\left(\frac{1+x}{1-x}\right) . \tag{4.7}
\end{equation*}
$$

## SECTION 5

We shall now obtain a generating function for $S_{n, r}^{(p)}$ in the following way. We have

$$
\begin{aligned}
\sum_{r=0}^{\infty} z^{r} \sum_{m=2 r}^{\infty} \sum_{n=2 r+p}^{\infty}\binom{m-p^{p}}{p}\binom{n-r-p}{r} x^{m} y^{n} & =\sum_{r=0}^{\infty} \frac{x^{2 r} y^{2 r+p} z^{p}}{(1-x)^{-r-1}(1-y)^{-r-1}} \\
& =y^{p}\left((1-x)(1-y)-x^{2} y^{2} z\right)^{-1} \\
& =y^{p}\left(1+x y-x^{2} y^{2} z-(x+y)\right)^{-1}
\end{aligned}
$$

Replacing $x$ by $x y^{-1}$, we have

$$
\begin{aligned}
& \sum_{r=0}^{\infty} z^{r} \sum_{m=2 r}^{\infty} \sum_{n=2 r+p}^{\infty}\binom{m-r}{r}\binom{n-r-p}{r} x^{m} y^{n-m}=y^{p}\left(1+x-x^{2} z-\left(x y^{-1}+y\right)\right)^{-1} \\
&=y^{p}\left(1+x-x^{2} z\right)^{-1} \sum_{k=0}^{\infty} \frac{\left(x y^{-1}+y\right)^{k}}{\left(1+x-x^{2} z\right)^{k}} \\
&=\left(1+x-x^{2} z\right)^{-1} \sum_{j, k=0}^{\infty}\binom{j+k}{k} \frac{x^{j} y^{k-j+p}}{\left(1+x-x^{2} z\right)^{j+k}}
\end{aligned}
$$

Since we want only the terms on the right that are free of $y$, we take $j=k+p$. Thus

$$
\sum_{r=0}^{\infty} z^{r} \sum_{n=2 r+p}^{\infty}\binom{n-r}{p}\binom{n-r-p}{p} x^{n}=\left(1+x-x^{2} z\right)^{-p-1} \sum_{k=0}^{\infty}\binom{2 k+p}{k} \frac{x^{k+p}}{\left(1+x-x^{2} z\right)^{2 k}}
$$

Since [1, p. 70, Ex. 10]

$$
\sum_{k=0}^{\infty}\binom{2 k+p}{k} z^{k}=F\left[\frac{p+1}{2}, \frac{p+2}{2} ; p+1 ; 4 z\right]=(1-4 z)^{-1 / 2}\left\{\frac{2}{1+(1-4 z)^{1 / 2}}\right\}^{p}
$$

it follows easily that

$$
\begin{equation*}
\sum_{r=0}^{\infty} z^{r} \sum_{n=2 r+p}^{\infty}(r+1) S_{n, r}^{(p)} x^{n-2 r}=\frac{1}{R}\left(\frac{1+x-z-R}{2}\right)^{p} \quad(p \geq 0) \tag{5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
R=R(x, z)=\left(\left(1-x^{2}-2(1+x) z+z^{2}\right)^{1 / 2}\right. \tag{5.2}
\end{equation*}
$$

Since

$$
\frac{\partial R}{\partial z}=\frac{z-1-x}{R}
$$

it is easily verified that

$$
\frac{\partial}{\partial z}\left(\frac{1+x-z-R}{2}\right)^{p}=\frac{p}{R}\left(\frac{1+x-z-R}{2}\right)^{p}
$$

Hence (5.1) yields

$$
\begin{equation*}
\frac{x^{p}}{p}+\sum_{r=0}^{\infty} z^{r+1} \sum_{n=2 r+p}^{\infty} S_{n, r}^{(p)} x^{n-2 r}=\frac{1}{p}\left(\frac{1+x-z-R}{2}\right)^{p} \quad(p>0) . \tag{5.3}
\end{equation*}
$$

In the next place, by (4.2) and (4.3),

$$
\sum_{n=2 r+p} S_{n, r}^{(p)} x^{n-2 r}=x^{-2 r} f_{r}^{(p)}(x)=\frac{x^{p}}{(1-x)^{2 r+1}} \phi_{r}^{(p)}(x) .
$$

Thus (5.3) becomes

$$
1+p \sum_{r=0} \frac{z^{r+1}}{(1-x)^{2 r+1}} \phi_{r}^{(p)}(x)=\left(\frac{1+x-z-R}{2 x}\right)^{p}
$$

Replacing $z$ by $(1-x)^{2} z$, we get
$1+p(1-x) \sum_{r=0} \phi_{r}^{(p)}(x) z^{r+1}=\left(\frac{1+x-(1-x)^{2} z-(1-x) R_{0}}{2 x}\right)^{p} \quad(p>0)$,
where

$$
\begin{equation*}
R_{0}=\left(1-2(1+x) z-(1-x)^{2} z^{2}\right)^{1 / 2} \tag{5.5}
\end{equation*}
$$

For $p=1$, (5.4) reduces to

$$
\begin{equation*}
1+(1-x) \sum_{r=0}^{\infty} \phi_{r}(x) z^{r+1}=\frac{1+x-(1-x)^{2} z-(1-x) R_{0}}{2 x} \tag{5.6}
\end{equation*}
$$

It is easily verified that (5.6) is in agreement with (3.3).
Returning to (5.1), we have

$$
\sum_{r=0}^{\infty}(p+1) \phi^{(p)}(x) \frac{z^{r}}{(1-x)^{2 r+1}}=\frac{1}{R}\left(\frac{1+x-z-R}{2 x}\right)^{p},
$$

so that

$$
\begin{equation*}
\sum_{r=0}^{\infty}(r+1) \phi_{r}^{(p)}(x) z^{r}=\frac{1}{R_{0}}\left(\frac{1+x-(1-x)^{2} z-(1-x) R_{0}}{2 x}\right)^{p} \quad(p \geq 0) \tag{5.7}
\end{equation*}
$$

Note that (5.7) holds for $p \geq 0$.

## SECTION 6

As an immediate consequence of (5.4), we have

$$
\begin{gathered}
\left\{1+p(1-x) \sum_{s=0}^{\infty} \phi_{s}^{(p)}(x) z^{s+1}\right\}\left\{1+q(1-x) \sum_{t=0}^{\infty} \phi_{t}^{(q)}(x) z^{t+1}\right\} \\
=1+(p+q)(1-x) \sum_{r=0}^{\infty} \phi_{r}^{(p+q)}(x) z^{r+1}
\end{gathered}
$$

Comparison of coefficients of $z^{r+1}$ yields the convolution formula

$$
\begin{align*}
(p+q) \phi_{r}^{(p+q)}(x)= & p \phi_{r}^{(p)}(x)+q \dot{\phi}_{r}^{(q)}(x) \\
& +p q(1-x) \sum_{s=0}^{r-1} \phi_{s}^{(p)}(x) \phi_{r-s-1}^{(q)}(x) \quad(p>0, q>0) \tag{6.1}
\end{align*}
$$

Similarly, by (5.4) and (5.7),

$$
\begin{align*}
(r+1) \phi_{r}^{(p+q)}(x)= & (p+1) \phi_{r}^{(q)}(x) \\
& +p(1-x) \sum_{s=0}^{p-1}(r-s) \phi_{s}^{(p)}(x) \phi_{r-s-1}^{(q)}(x) \quad(p>0, q \geq 0) . \tag{6.2}
\end{align*}
$$

In the next place, it is evident from (5.4) and (5.6) that

$$
\begin{equation*}
1+p(1-x) \sum_{r=0}^{\infty} \phi_{r}^{(p)}(x) z^{r+1}=\left\{1+(1-x) \sum_{r=0}^{\infty} \phi_{r}(x) z^{r+1}\right\}^{p} \quad(p>0) . \tag{6.3}
\end{equation*}
$$

For $p=q=1$, (6.1) reduces to

$$
2 \phi_{r}^{(2)}(x)=2 \phi_{r}(x)+(1-x) \sum_{s=0}^{r-1} \phi_{s}(x) \phi_{r-s-1}(x) .
$$

However, by (3.1), we have

$$
\phi_{r}(x)=(1-x) \phi_{r-1}(x)+x \sum_{s=0}^{r-1} \phi_{s}(x) \phi_{r-s-1}(x) .
$$

It follows that

$$
\begin{equation*}
2 x \phi_{r}^{(2)}(x)=(1+x) \phi_{r}(x)-(1-x)^{2} \phi_{r-1}(x) \quad(r>0) . \tag{6.4}
\end{equation*}
$$

This formula can be generalized by means of the easily proved identity

$$
\begin{equation*}
2\binom{m}{p}\binom{m-p-1}{p}=\binom{m}{p}\binom{m-p}{p}+\binom{m-1}{p}\binom{m-p-1}{p}-\frac{p+p}{p}\binom{m-1}{p-1}\binom{m-p-1}{p-1} . \tag{6.5}
\end{equation*}
$$

Multiplying both sides of (6.5) by $x^{m}$ and summing over $m$, we get

$$
2(r+1) f_{r}^{(p+1)}(x)=(r+1)(1+x) f_{r}^{(p)}(x)-(r+p) x^{2} f_{r-1}^{(p)}(x)
$$

and therefore
$2(p+1) x \phi_{r}^{(p+1)}(x)=(x+1)(1+x) \phi_{r}^{(p)}(x)-(r+p)(1-x)^{2} \phi_{r-1}^{(p)}(x)$.
For example, for $p=2$, we get
$4(r+1) x^{2} \phi_{r}^{(3)}(x)=(r+1)(1+x)^{2} \phi_{r}(x)-(2 r+3)(1+x)(1-x)^{2} \phi_{r-1}(x)$ $+(r+2)(1-x)^{4} \phi_{r_{-2}}(x) \quad(r>1)$.

Repeated application of (6.6) leads to a result of the form

$$
\begin{equation*}
(2 x)^{p} \psi_{r}^{(p+1)}(x)=\sum_{s=0}^{p}(-1)^{s} c(p, r, s)(1+x)^{p-s}(1-x)^{2 s} \psi_{r-s}(x) \quad(r \geq 0) \tag{6.8}
\end{equation*}
$$

where

$$
\psi_{r}^{(p)}(x)=(r+1)!\phi_{r}^{(p)}(x), \psi_{r}(x)=(r+1)!\phi_{r}(x)
$$

and the coefficients $c(p, r, s)$ are independent of $x$.

## SECTION 7

We shal1 now state the binomial identities implied by (6.1) and (6.2). In terms of $f_{p}^{(p)}(x),(6.1)$ and (6.2) become

$$
\begin{align*}
(p+q) f_{r}^{(p+q)}(x)= & p x^{q} f_{r}^{(p)}(x)+q x^{p} f_{r}^{(q)}(x) \\
& +p q x^{2} \sum_{s=0}^{p-1} f_{s}^{(p)}(x) f_{r-s-1}^{(q)}(x) \quad(p>0, q>0) \tag{7.1}
\end{align*}
$$

and

$$
\begin{align*}
(r+1) f_{r}^{(p+q)}(x)= & (r+1) f_{r}^{(q)}(x) \\
& +p \sum_{s=0}^{r-1}(r-s) f_{s}^{(p)}(x) f_{r-s-1}^{(q)}(x) \quad(p>0, q \geq 0), \tag{7.2}
\end{align*}
$$

respectively. Using (4.2) and equating coefficients of $x^{n}$, we obtain the following identities.

$$
\begin{align*}
(p+q) S_{n, r}^{(p+q)}= & p S_{n-q, r}^{(p)}+q S_{n-p, r}^{(p)} \\
& +p q \sum_{j=0}^{n-2} \sum_{s=0}^{r-1} S_{j, s}^{(p)} S_{n-j-2, r-s-1}^{(q)} \quad(p>0, q>0) \tag{7.3}
\end{align*}
$$

$$
\begin{align*}
(r+1) S^{(p+q)}= & (r+1) S^{(q)} \\
& +p \sum_{j=0}^{n-2} \sum_{s=0}^{r-1}(r-s) S_{j, s}^{(p)} S_{n-j-2, r-s-1}^{(q)} \quad(p>0, q \geq 0) . \tag{7.4}
\end{align*}
$$

In particular, since

$$
(r+1) S^{(0)}=\binom{n-r}{r}^{2}
$$

it is evident that, for $q=0$, (7.4) reduces to

$$
(r+1) S_{n, r}^{(p)}=\binom{n-r}{r}^{2}+p \sum_{j=0}^{n-2} \sum_{s=0}^{r-1}\binom{j-s}{s}^{2} S_{n-j-2, r-s-1}^{(p)} \quad(p>0)
$$

The special case, $p=1$, was stated in the Introduction.
A second pair of identities is also implied by (6.1) and (6.2). Put

$$
\begin{equation*}
T_{r, m}^{(p)}=\frac{(p+p)!}{(p+1)!(m+1)_{p}}\binom{p}{m}\binom{p-p}{m}=\frac{1}{r+1}\binom{p+p}{m+p}\binom{r-p}{m} \tag{7.5}
\end{equation*}
$$

Then by (4.4) we have

$$
\begin{equation*}
\phi_{r}^{(p)}(x)=\sum_{m=0}^{r} T_{r, m}^{(p)} x^{m} \quad(p \geq 0) . \tag{7.6}
\end{equation*}
$$

Note that，by（4．4）and（4．5），（7．6）holds for all nonnegative $p$ ．Substituting from（7．6）in（6．1）and（6．2）and evaluating coefficients of $x^{m}$ ，we obtain the following two identities．

$$
\begin{align*}
(p+q) T_{r, m}^{(p+q)}= & p T_{r, m}^{(p)}+q T_{r, m}^{(q)}+p q \sum_{s=0}^{r-1} \sum_{j=0}^{m} T_{s, j}^{(p)} T_{r-s-1, m-j}^{(q)} \\
& -p q \sum_{s=0}^{r-1} \sum_{j=0}^{m-1} T_{s, j}^{(p)} T_{r-s-1, m-j-1}^{(q)} \quad(p>0, q>0),  \tag{7.7}\\
(p+1) T_{r, m}^{(p+q)}= & (p+1) T_{r, m}^{(q)}+p \sum_{s=0}^{r-1} \sum_{j=0}^{m}(p-s) T_{s, j}^{(p)} T_{r-s-1, m-j}^{(q)}  \tag{7.8}\\
& -p \sum_{s=0}^{r-1} \sum_{j=0}^{m-1}(r-s) T_{s, j}^{(p) T_{r-s-1, m-j-1}^{(q)} \quad(p>0) .}
\end{align*}
$$

In particular，for $q=0$ ，（7．8）reduces to

$$
\begin{aligned}
(r+1) T_{r, m}^{(p)}= & \binom{p}{m}^{2}+p \sum_{s=0}^{r-1} \sum_{j=0}^{m}\binom{s}{j}^{2} T_{r-s-1, m-j}^{(p)} \\
& -p \sum_{s=0}^{r-1} \sum_{j=0}^{m-1}\binom{s}{j}^{2} T_{r-s-1, m-j-1}^{(p)} \quad(p>0) .
\end{aligned}
$$

We remark that（6．1）is implied by（6．2）．To see this，multiply both sides of（6．2）by $q$ ，interchange $p$ and $q$ ，and then add corresponding sides of the two equations．Similarly，it can be verified that（7．3）is implied by（7．4）and （7．7）is implied by（7．8）．

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莫和为为落

## SOME EXTREMAL PROBLEMS ON DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS

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Dedicated to the memory of my friend Vern Hoggatt
A sequence of integers $A=\left\{\alpha_{1}<\alpha_{2}<\cdots<\alpha_{k} \leq n\right\}$ is said to have property $P_{r}(n)$ if no $a_{i}$ divides the product of $r$ other $a^{\prime} s$ ．Property $P(n)$ means that no $a_{i}$ divides the product of the other $a^{\prime} s$ ．A sequence has property $Q(n)$ if the products $a_{i} \alpha_{j}$ are all distinct．

Many decades ago I proved the following theorems［2］：
Let $A$ have property $P_{1}$（i．e．，no $\alpha_{i}$ divides any other）．Then

$$
\max k=\left[\frac{n+1}{2}\right] .
$$

The proof is easy．

Let $A$ have property $P_{2}$ then [ $\pi(n)$ is the number of primes not exceeding $n$ ]

$$
\begin{equation*}
\pi(n)+c_{1} n^{2 / 3}(\log n)^{-2}<\max k \leq \pi(n)+c_{2} n^{2 / 3}(\log n)^{-2} \tag{1}
\end{equation*}
$$

The $c^{\prime}$ s will denote positive absolute constants not necessarily the same at each occurrence. We will write $P_{r}$ instead of $P_{r}(n)$ if there is no danger of confusion.

Probably there is a $c$ for which

$$
\begin{equation*}
\max k=\pi(n)+(c+0(1)) n^{2 / 3}(\log n)^{-2} \tag{2}
\end{equation*}
$$

but I could never prove (2).
Assume next that $A$ has property $Q$. Then

$$
\begin{equation*}
\pi(n)+c_{3} n^{3 / 4}(\log n)^{-3 / 2}<\max k<\pi(n)+c_{4} n^{3 / 4}(\log n)^{-2 / 3} \tag{3}
\end{equation*}
$$

Here too I conjectured

$$
\begin{equation*}
\max k=\pi(n)+(c+0(1)) n^{3 / 4}(\log n)^{-3 / 2} \tag{4}
\end{equation*}
$$

I could never prove (4), which seems more difficult than (2).
In this note I consider slightly different problems. Denote by $S_{n}$ the set of positive integers not exceeding $n$. Observe that $S_{n}$ can be decomposed into $1+\left[\frac{\log n}{\log 2}\right]$ sets having property $P_{1}$. To see this, let $S$ consist of the integers $\left[\frac{n}{2^{i}}\right]<\alpha \leq\left[\frac{n}{2^{i-1}}\right]$. The powers of 2 show that $1+\left[\frac{\log n}{\log 2}\right]$ is best possible.

Denote by $f_{r}(n)$ the smallest integer for which $S_{n}$ can be decomposed as the union of $f_{r}(n)$ sets having property $P_{r}$ and $g(n)$ is the smallest integer for which $S_{n}$ can be decomposed into $g(n)$ sets having property $Q$. We just observed $f_{1}(n)=1+\left[\frac{\log n}{\log 2}\right]$. We prove
Theorem 1:

$$
\begin{align*}
& c \frac{n^{1 / 2}}{\log n}<f_{2}(n)<2 n^{1 / 2}  \tag{5}\\
& c \frac{n^{1 / 3}}{\log n}<g(n)<2 n^{1 / 2} \tag{6}
\end{align*}
$$

The upper bound in (5) and (6) follows immediately from the fact that

$$
m \nmid\left(m+i_{1}\right)\left(m+i_{2}\right) \quad \text { if } \quad 1 \leq i_{1} \leq i_{2}<m^{1 / 2} .
$$

Now we prove the lower bound in (5). The proof will be similar to the proof in [2]. Let $S^{\prime}$ be the integers of the form

$$
\begin{equation*}
p u, u<\frac{1}{2} n^{1 / 2}, n^{1 / 2}<p<2 n^{1 / 2} . \tag{7}
\end{equation*}
$$

Clearly

$$
\begin{equation*}
\left|S_{n}^{\prime}\right|>c \frac{n}{\log n} \tag{8}
\end{equation*}
$$

Now let $a_{1}<a_{2}<\cdots<\alpha_{k}$ be a subset of $S_{n}^{\prime}$ which satisfies property $P_{2}$. We prove that then

$$
\begin{equation*}
k<\frac{n^{1 / 2}}{2}+c \frac{n^{1 / 2}}{\log n}<n^{1 / 2} . \tag{9}
\end{equation*}
$$

(8) and (9) clearly complete the proof of (5).

Thus we only have to prove (9). Put $\alpha_{i}=p_{i} u_{i}$ where $p_{i}$ and $u_{i}$ satisfy (7). Now make correspond to the set $\alpha_{1}<\cdots<\alpha_{k}$ a bipartite graph where the white
vertices are the $u^{\prime} s$ and whose black vertices are the primes $p_{i}$. To $a_{i}=p_{i} u_{i}$ corresponds the edge joining $p_{i}$ and $u_{i}$. This graph clearly cannot contain a path of length three. To see this, observe that if $\alpha_{1}=p_{1} u_{1}, \alpha_{2}=u_{1} p_{2}$, and $a_{3}=p_{2} u_{2}$ is a path of length three then $\alpha_{2} \mid \alpha_{1} \alpha_{3}$, which is impossible. A bipartite graph which contains no path of length three is a forest and hence it is well known and easy to see that the number of its edges is less than the number of its vertices. This proves (9) and completes the proof of (5).

By a more judicious choice of the black and white vertices the lower bound of (5) can be improved considerably. A well known and fairly deep theorem of mine states that the number of integers $m<n$ of the forms $u \cdot v$, where both $u$ and $v$ are not exceeding $n^{1 / 2}$ is greater than

$$
\frac{n}{(\log n)^{\alpha+\varepsilon}}, \alpha=1-\frac{1+\log \log 2}{\log 2}
$$

for $n>n_{0}(\varepsilon)$, and that this choice of $\alpha$ is the best possible [3]. This immediately gives, by our method,

$$
f_{2}(n)>\frac{n^{I / 2}}{(\log n)^{\alpha+\varepsilon}}
$$

We do not pursue this further, since we cannot at present decide whether

$$
f_{2}(n)=0\left(n^{1 / 2}\right)
$$

is true. The following extremal problem, which I believe is new, is of interest in this connection: Let $1 \leq \alpha_{1}<\cdots<\alpha_{n} \leq n$ and $1 \leq b_{1}<\cdots<b_{s} \leq n$ be two sequences of integers. Denote by $1 \leq u_{1}<\cdots<u_{t} \leq n$ the integers not exceeding $n$ of the form $\alpha_{i} b_{j}$. Put

$$
h(n)=\max \frac{t}{p+s}
$$

where the maximum is extended over all possible choices of the $a^{\prime} s$ and $b^{\prime} s$. Our proof immediately gives $f_{2}(n) \geq h(n)$. I can prove

$$
h(n)<\frac{n^{I / 2}}{(\log n)^{\beta}} \text { for some } \beta>0
$$

It would be interesting if it would turn out that for some $\beta<\alpha$,

$$
h(n)>\frac{n^{1 / 2}}{(\log n)^{\beta}} .
$$

The upper bound of (6) is obvious, thus to complete the proof of Theorem 1 we only have to prove the lower bound in (6). The proof will again be similar to that of [2]. Let $S_{n}^{\prime \prime}$ be the integers of the form

$$
\begin{equation*}
p u<n, u<\left|\frac{1}{2}\right| n^{1 / 3}, n^{2 / 3}<p<2 n^{2 / 3} . \tag{10}
\end{equation*}
$$

Clearly (by the prime number theorem or a more elementary theorem)

$$
\begin{equation*}
\left|S_{n}^{\prime \prime}\right|>\frac{c n}{\log n} \tag{11}
\end{equation*}
$$

Now let $a_{1}<\cdots<\alpha_{k}$ be a subset of $S_{n}^{\prime \prime}$ having property $Q$ (i.e., all the products $\alpha_{i} \alpha_{j}$ are distinct). We prove

$$
\begin{equation*}
k<n^{2 / 3}+c \frac{n^{2 / 3}}{\log n} \tag{12}
\end{equation*}
$$

(11) and (12) clearly give the lower bound of (6); thus to complete the proof of our Theorem we only have to prove (12). Consider a bipartite graph whose
white vertices are the primes $n^{2 / 3}<p<2 n^{2 / 3}$ and whose black vertices are the integers

$$
u<\frac{1}{2} n^{1 / 3}
$$

To each $a=p u$, we make correspond the edge joining $p$ to $u$. This graph cannot contain a $C_{4}$, i.e., a circuit of size four. To see this, observe that if $p_{1}$, $p_{2}, u_{1}$, and $u_{2}$ are the vertices of this $C_{4}$ then $p_{1} u_{1}, p_{1} u_{2}, p_{2} u_{1}$, and $p_{2} u_{2}$ are all members of our sequence and

$$
p_{1} u_{1} \cdot p_{2} u_{2}=p_{1} u_{2} \cdot p_{2} u_{1},
$$

or the products $a_{i} a_{j}$ are not all distinct, which is impossible.
Now let $v_{i}$ be the valency (or degree) of $p_{i}\left(n^{2 / 3}<p_{i}<2 n^{2 / 3}\right)$. We now estimate $k$, the number of the edges of our graph, as follows: The $p_{i}$ 's with $v_{i}=1$ contribute to $k$ at most

$$
s<c \frac{n^{2 / 3}}{\log n}
$$

Now let $p_{1}, \ldots, p_{r}$ be the primes whose valency $v_{i}$ is greater than 1 . Observe that

$$
\sum_{i=1}^{r}\binom{v_{i}}{2} \leq\left(\left[\begin{array}{c}
\frac{1}{2} n^{1 / 3}  \tag{13}\\
2
\end{array}\right) \leq \frac{1}{8} n^{2 / 3}\right.
$$

$\left[\frac{1}{2} n^{1 / 3}\right]$ is the number of $u^{\prime} s$. If $p_{i}$ is joined to $v_{i} u^{\prime} s$, form the $\binom{v_{i}}{2}$ pairs of $u$ 's joined to $p_{i}$. Now, if (13) would not hold, then by the box principle there would be two $p$ 's joined to the same two $u^{\prime} s$, i.e., our graph would contain a $C_{4}$, which is impossible. Thus (13) is proved.

From (13) we immediately have

$$
\begin{equation*}
\sum_{i=1}^{r} v_{i}<\frac{n^{2 / 3}}{\min \left(v_{i}-1\right)} \leq n^{2 / 3} \tag{14}
\end{equation*}
$$

(14) clearly implies (12) and hence the proof of our Theorem is complete.

I expect $g(n)<n^{(1 / 3+\varepsilon)}$ but have not even been able to prove $g(n)=O\left(n^{1 / 2}\right)$.
Recall that $f_{r}(n)$ is the smallest integer for which $S_{n}$ can be decomposed into $f_{r}(n)$ sets having property $P_{r}$. We have
Theorem 2: For every $\varepsilon>0$,

$$
n^{1-\frac{1}{r}-\varepsilon}<f_{r}(n)<c_{r} n^{1-\frac{1}{r}}
$$

The proof of Theorem 2 is similar to that of Theorem 1 and will not be given here. Perhaps

$$
f_{r}(n)=o\left(n^{1-\frac{1}{r}}\right) .
$$

Finally, denote by $F(n)$ the smallest integer for which $S_{n}$ can be decomposed into $F(n)$ sets $\left\{A_{i}\right\}, 1 \leq i \leq F(n)$, having property $P$.

Using certain results of de Bruijn [1], I can prove that for a certain absolute constant $c$

$$
\begin{equation*}
F(n)=n \exp \left((-c+0(1))(\log n \log \log n)^{1 / 2}\right) \tag{15}
\end{equation*}
$$

We do not give the proof of (15) here.
Now I discuss some related results and conjectures. Let $a_{1}<\alpha_{2}<\ldots<a_{k}$ be the largest subset of $S_{n}$ for which the sums $\alpha_{i}+\alpha_{j}$ are all distinct. Turán
[Aug.
and I proved that [4]

$$
\max k=\left(1+o(1) n^{1 / 2}\right.
$$

and we in fact conjectured

$$
\begin{equation*}
\max k=n^{1 / 2}+O(1) \tag{16}
\end{equation*}
$$

(16) is probably deep, and $I$ offer $\$ 500$ for a proof or disproof.

I conjectured more than 15 years ago that if $b_{1}<\cdots<b_{n}$ is any sequence of integers then there always is a subsequence

$$
b_{i_{1}}<\cdots<b_{i_{s}}, s \geq(1+o(1)) n^{1 / 2}
$$

so that all the sums $b_{i_{j_{1}}}+b_{i_{j_{2}}}$ are distinct. Komlós, Sulyok and Szemerédi [5] proved a much more general theorem from which they deduced a slightly weaker form of my conjecture, namely $s>c n^{1 / 2}$ for some $c<1$. Denote by $m(n)$ the largest integer so that for every set of $n$ integers $b_{1}<\cdots<b_{n}$ one can find a subsequence of $m(n)$ terms so that the sum of any two terms of the subsequence are distinct. Perhaps $m(n)$ is assumed for $S$.

Recently I conjectured that if $b_{1}<b_{2}<\cdots<b_{n}$ is any sequence of $n$ integers, one can always select a subsequence $b_{i_{1}}<\cdots<b_{i_{s}}, s>(1+o(1)) n^{1 / 2}$ so that the product of any two $b_{i_{j}}$ 's is distinct. Straus observed that with $s>c n^{1 / 2}$ this follows from the Komlós, Sulyok and Szemerédi theorem by a method which he often used. One can change the multiplicative problem to an additive one by taking logarithms and then, by using Hamel bases, one can easily deduce $s>c n^{1 / 2}$ from the theorem of Komlós, Sulyok and Szemerédi.

Let $1 \leq \alpha_{1}<\cdots<\alpha_{k} \leq n$ be any sequence of $k$ integers, not exceeding $n$. Denote by $\bar{F}(k, n)$ the largest integer so that there always is a subsequence of the $\alpha$ 's having $F(k, n)$ terms and property $P_{1}$. It is easy to see that

$$
\begin{equation*}
F(k, n) \geq \frac{k}{1+\log n} \tag{17}
\end{equation*}
$$

and the powers of 2 show that (17) in general is best possible. It is not difficult to see that if $k \geq c n$ then $F(k, n) \geq g(c) n$ and the best value of $g(c)$ would be easy to determine although I have not done so. It is further easy to see that $g(c) / c \rightarrow 0$ if $c \rightarrow 0$. If $k<n^{1-\varepsilon}$, then (17) gives the correct order of magnitude except for a constant factor $c$, and in general the determination of $F(k, n)$ is not difficult.

Many further questions of this type could be asked. For example, denote by $F_{2}(k, n)$ the largest integer so that our sequence always has a subsequence of $F_{2}(k, n)$ terms having property $P_{2} . F_{2}(k, n)$ seems to be more difficult to handie than $F(k, n)$. It is easy to see that

$$
F_{2}(k, n)>k\left(2 n^{1 / 2}\right)^{-1}
$$

but perhaps this can be improved and quite possibly for every $c>0$

$$
F_{2}(c n, n) / n^{1 / 2} \rightarrow \infty
$$

The following question seems of some interest to me: Let

$$
1 \leq a_{1}<\cdots<a_{k} \leq n .
$$

What is the smallest value of $k$ that forces the existence of three (or $s$ ) $\alpha^{\prime} s$, so that the product of every two is a multiple of the others? In particular, is it true that if $k>$ en there always are three $a^{\prime}$ s so that the product of any two is a multiple of the third? At the moment I cannot answer this question, but perhaps I overlooked a trivial argument.

To end our paper, we state one more question: What is the smallest $k=k_{n}$ for which $F_{2}(k, n) \geq 3$ ? In other words: Determine or estimate the smallest
$k=k_{n}$ for which for every $1 \leq \alpha_{1}<\cdots<\alpha_{k} \leq n$ there are three $a^{\prime} s, a_{i_{1}}, a_{i_{2}}$, $a_{i_{3}}$ so that the product of two is not a multiple of the third. I have no satisfactory answer, but perhaps again I overlooked a trivial argument.

On the other hand, I can get a reasonably satisfactory answer to a slightly modified question.
Theorem 3: Let $1 \leq a_{1}<\cdots<a_{k} \leq n$ be such that the product of every two $a^{\prime}$ s is a multiple of all the others. Then (exp $z=e^{z}$ )

$$
\begin{equation*}
\max k=\exp \left((1+o(1)) \log 2 \cdot \frac{2}{3} \log n(\log \log n)^{-1}\right) \tag{18}
\end{equation*}
$$

We only outline the proof of Theorem 3. Let $2,3, \ldots, p_{s}$ be the primes not exceeding $(1-\varepsilon) \frac{2}{3} \log n$. Let the $a^{\prime}$ 's be the integers of the form

$$
\begin{equation*}
u \prod_{i=1}^{s} p_{i} \tag{19}
\end{equation*}
$$

where $u$ runs through the integers that are the product of [s/2] or fewer of the $p^{\prime} s$. From the prime number theorem, we easily obtain that all the $\alpha$ 's are not exceeding $n$. To see this, observe that by the prime number theorem

$$
\prod_{i=1}^{s} p_{i}=\exp \left((1+o(1))(1-\varepsilon) \frac{2}{3} \log n\right)
$$

and

$$
u<\left(\prod_{i=1}^{s} p_{i}\right)^{\frac{1}{2}+o(1)}<\exp \left((1+o(1)) \frac{\log n}{3}\right)
$$

Further, by the prime number theorem,

$$
s>(1-\varepsilon) \frac{2}{3} \log n(\log \log n)^{-1}
$$

and the number of $u^{\prime} s$ is not less than $2^{s-1}$, which proves the lower bound in (18).

Now we outline the proof of the upper bound of (18). Let $p_{1}, \ldots, p_{s}$ be the prime factors of

$$
\prod_{i=1}^{k} a_{i}
$$

Since $\alpha_{i} \alpha_{j}$ is a multiple of all the other $\alpha^{\prime} s$, all but one of the $\alpha^{\prime} s$, say $\alpha^{(j)}$, are multiples of $p_{j}, 1 \leq j \leq s$. Disregarding these $\alpha^{(j)}$ 's, we assume that all the $a^{\prime}$ s are multiples of all the $p_{j}^{\prime}$ s. By the same argument we can assume that for every $p_{j}$ there is an $\alpha_{j}$ so that every $a_{i}$ divides $p_{j}$ with an exponent $x_{i}, j$, $\alpha_{j} \leq x_{i, j} \leq 2 \alpha_{j}$. From this and the prime number theorem we obtain by a simple computation, the details of which I suppress, the upper bound in (18).

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# hommage à archimède 

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Verner Hoggatt and I were friends and colleagues for twenty years. He was a person with special properties who studied mathematical objects with special properties. In addition to his incomparable knowledge of all things Fibonacci was his remarkable storehouse of appreciation for geometrical matters of all kinds. I'm glad that he very much liked my "1:2:3" result connected with Figure 1, below; he gave me several of the references which appear in the brief, annotated bibliography. Of course, he liked the fact that his old friend

$$
x^{2}-x-1=0
$$

had motivated my pleasant little discovery; and he, too, did not at all hesitate to show his students some simple things that opened up interesting, broader and longer, avenues for them to pursue in the literature and in their private studies. Those of you who are teachers are invited to show these few paragraphs to your students. In some ways, the best way to remember a friend is to try to emulate him. So, in that spirit, I offer this leisurely little essay.

In the central Quad at San Jose State University, across from the landmark Tower, there now stands a seven foot bronze abstract sculpture, Hormage $\grave{\alpha}$ Archimède, which provides an already pleasant place with an additional pleasant intellectual sweep. This handsome bronze tribute to Archimedes, made possible by contributions from friends of the School of Science of the University, incorporates several noteworthy scientific and artistic design ideas which are dealt with below.

For nearly two decades I had entertained the hope of placing some abstract sculpture on campus that would involve the Archimedes-related design in Figure 1. This hope was known to Kathleen Cohen, Chairman of our Art Department, who introduced me to Robert J. Knight, a sculptor who was spending some time last year on our campus.


Fig. 1
Knowing that I did not want some heroic-bearded-Russian-heavy-General-type figure, the sculptor cooperatively modified one of his existing graceful abstract models to accommodate that Archimedean design, the only element of the whole piece which will make any immediate sense to a knowledgeable observer. Concerning the sculpture as a whole, I can only offer this quotation from a page in an Art Department brochure: "It is an exploration of the figurative formula and displacement of space as it relates to the human form."

There are many who think of Archimedes (287-212 B.C.) as the greatest intellect of antiquity. He is always listed among the top four or five mathematicians who have ever lived. In his time, and even much later, mathematicians were natural philosophers, natural scientists, if you please. Only in times much closer to our own do we find a greater scientific abstraction and consequent apparent separation of many mathematicians from other parts of the quest to understand nature. I do not know if Archimedes ever studied botany, say, but the existence of his significant interest and work in important areas of science other than pure mathematics is well documented. He wrote numerous mas-terpieces-on optics, hydrostatics, theoretical mechanics, astronomy, and mathematics, for example. It is true that, although he was surely a very good engineer, he did not regard very highly his own dramatic mechanical contrivances.

As indicated by his wishes regarding what should appear on his gravestone, Archimedes did most highly value a figure something like Figure 2, which refers to some beautiful geometry connected with his fundamental work as a primary and


Fig. 2
impeccable forerunner of those who much later established modern integral calculus. (The Roman statesman Cicero wrote about and restored the Archimedes gravestone when it was rediscovered long after Archimedes had died.) That figure, a square with an inscribed circle, refers to this result of his: If the figure is rotated in space about that central vertical axis, the resulting sphere and cylinder have volumes which are as 2 is to 3 ; and their surface areas are also as 2:3. Archimedes, having discovered and appreciated much beautiful geometry, would certainly have understood what Edna St. Vincent Millay was saying (years later): "Euclid alone has looked on Beauty bare."

Democritus, who lived before Archimedes, knew the following result about Figure 3: The volumes of the cone and cylinder which are generated when the triangle and rectangle are rotated as indicated are as 1: 3.

Over the years, some calculus students have been told to consider Figure 4 -which involves a square, not just any rectangle-and to use (modern) elementary calculus tools to find this beautiful result:


Fig. 4

Fig. 3
Some years after my teacher, George Pólya, had shown me that result, I read about the researches in aesthetics conducted by Gustav Theodor Fechner (in Germany), whose experimental results in 1876 -later confirmed in varying degrees by others-indicated that 75.6 percent (!) of his popular observers of rectangles found that rectangle to be most pleasant whose proportions are about $8: 5$, as in Figure 1. The "about 8:5" refers to what Renaissance writers referred to as the "divine proportion," the "golden mean" or "golden section" of Greek geometers, used by da Vinci and others, by Salvador Dali in our time, and still making its appearance in some contemporary design. (The façade of the ancient Parthenon, if one includes the face of the roof, fits into such a rectangle. Dali's "Last Supper" painting has exactly these proportions.)

Now $8 \div 5=1.60$, while the "golden mean" is actually

$$
(1+\sqrt{5}) \div 2=1.6180339 \ldots
$$

a number which solves the equation $x^{2}-x-1=0$ and is well known to Fibonacci people. This equation arises when geometers divide a line segment in "extreme and mean ratio"; i.e., so that its length is divided into parts of length $x$ and 1 such that $(x+1) / x=x / 1$.

Figure 5 shows how easily we can construct a rectangle which possesses that "most pleasant shape." We simply start with a square, $A B C D$, and locate $M$, the midpoint of its base. With the length $M C$ from $M$ to an opposite corner $C$ as a radius, we locate the point $P$ on the extension of $A B$ shown. The sides of the resulting big rectangle $A P Q D$ have lengths which are as $1+\sqrt{5}$ is to 2 .

Well, after having read about conclusions such as Fechner's, it once (quite long ago now) occurred to me to draw what $I$ have here shown as Figure 1 , involving that most pleasant rectangular shape, and to calculate the volumes of revolution generated by spinning this figure about its vertical bisecting axis. This can very quickly be done with the powerful elementary tools of calculus which have been bequeathed to us. I was then privileged to encounter the following beautiful result about these volumes:

Cone : Ellipsoid : Cylinder = $1: 2: 3$.
(If the cone holds one liter, then the ellipsoid holds two liters, and the cylindrical can will hold precisely 3 liters.)


I don't think Archimedes knew this theorem. I am sure he would have treasured it. And if he had not been killed by a Roman soldier while he was absorbed in studying some circles in his home (in Syracuse, in southern Sicily, during the second Punic war), he might well have observed it and recorded it at a later time.
[Here are the calculations which yield the result ( $\beta$ ). We refer to Figure 1 , where we let the radius of revolution be $r$, and the height of the cylinder be $h$. Then

$$
\text { CONE volume }=\frac{(\text { BASE area) } \times(\text { height })}{3}=\pi r^{2} h / 3,
$$

and, of course, the volume of the CYLINDER, $\pi r^{2} h$, is exactly three times this number.

The volume of an ellipsoid with minor axes of length $\alpha, b$, and $c$, is

$$
(4 / 3) \pi a b c .
$$

Our ELLIPSOID from Figure 1 thus has volume equal to

$$
(4 / 3) \pi(r)(r)(h / 2)=(2 / 3) \pi r^{2} h .
$$

Putting all of this together, we have this relation among the three volumes of revolution:

$$
\text { CONE : ELLIPSOID : CYLINDER }=1 / 3: 2 / 3: 3 / 3=1: 2: 3
$$

These calculations show that, actually, this beautiful result ( $\beta$ ) holds for any encompassing rectangular shape-not just one with divine proportions. Furthermore, it should be recorded here that this result, once we have guessed it, is directly derivable from ( $\alpha$ ) by an application of the powerful (modern) theory of "affine transformations."]

It is this result ( $\beta$ ), then, which is built into the sculpture now in our Quad, installed as a tribute back over the ages to Archimedes. (The rectangle in our San Jose State sculpture is about 1.5 feet across, by the way, and the number associated with its proportions is about 1.61 , which is, we think, close enough for anybody riding by on a horse!)

Finally, one of the speakers at the January 19, 1981 dedication ceremonies was Professor Gerald Alexanderson (a friend of Dr. Hoggatt's and mine, and Chairman of the Mathematics Department at the nearby University of Santa Clara),
who spoke on behalf of Dr. Dorothy Bernstein, President of The Mathematical Association of America. Here are a few excerpts from his remarks about "this powerful piece of sculpture honoring Archimedes."
"As one who has spent much time and money trying to locate and visit mathematical shrines, often in the form of statuary and monuments to mathematicians, I am particularly happy to be here today. Let us review what some of those monuments are. There's the statue of Simon Stevin in Bruges. (For those whose history of science is a little rusty: he gave us decimals.) Then there are those great cenotaphs for the Bernoullis in the Peterskirche in Base1, with wonderful ladies in marble doing geometry with golden compasses. Of course, the best part of a visit to the Peterskirche is that one walks up the Eulerstrasse to get there. There's the Gauss-Weber monument in Göttingen. Actually, I think they're shown doing physics, but never mind. Gauss was certainly a mathematician. (They are discussing their invention of the telegraph.) And a favorite of many is Roubiliac's statue of Newton outside the chapel at Trinity College, Cambridge. But the best of all is the romantic, heroic statue of Abel in the Royal Park in Oslo. He is shown standing erect, head thrown back, with hair caught in the wind, and he's standing on two vanquished figures, obviously beaten in battle. One is the elliptic function and the other is the fifth-degree polynomial equation. Actually, I cannot tell which is which, because they're not terribly good likenesses.
"Now right here in San Jose we have a monument to Archimedes. I am grateful. When the urge comes on to visit a mathematical monument, it will be much more convenient (and cheaper) to make a pilgrimage to San Jose, than to Syracuse."

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## THE UBIQUITOUS RATIONAL SEQUENCE

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## DEDICATION

Vern Hoggatt has been the inspiration of many papers that have appeared in this journal. He shared his enthusiasm and curiosity about mathematics with a notable generosity. His students, friends, and pen pals were enriched by the problems he posed and often helped to solve. My own interest in sequences was greatly influenced by the correspondence we started when I was a graduate student at the university of Alberta. Some of my first papers written at that
time were solutions to research problems he posed. To this day, rational sequences permeate my research work. It seems appropriate to present my own view of the theory of finite differences which has evolved over the years. This paper will be useful for the beginner, the sort of person Vern Hoggatt helped so much, and it should have some novelty for others as well. In it, I hope to show how rational sequences fit into some parts of mathematics-linear algebra and elementary calculus in particular. The exposition will be brief with plenty of gaps to be filled in by the reader.

## 1. RATIONAL SEQUENCES

What is a rational sequence? A mapping $f$ from $\mathbb{N}=\{0,1,2, \ldots\}$ into a field $\mathfrak{F}$ is rational if and only if there exist elements $c_{1}, \ldots, c_{k} \varepsilon \mathcal{F}$ with $c_{k} \neq 0$, and there exists $h \in \mathbb{N}$ with $k \leq h$ such that

$$
\begin{equation*}
f(n)=c_{1} f(n-1)+\cdots+c_{k} f(n-k) \quad(n \in \mathbb{N}, h<n) \tag{1}
\end{equation*}
$$

Sometimes a rational sequence is said to satisfy a linear homogeneous difference equation with constant coefficients. This long phrase is usually shortened to "difference equation" or "linear recurrence." We refer to (1) as the difference equation form, meaning it is one way of presenting a rational sequence. The term "rationa1" is short, and it describes a characteristic feature of such sequences. Namely, the generating function of $f$ is rational (the quotient of two polynomials); in fact, the generating function is
(2) $\sum_{n=0}^{\infty} f(n) z^{n}=\frac{f(0)+\left\{f(1)-c_{1} f(0)\right\} z+\cdots+\left\{f(h)-\cdots-c_{k} f(h-k)\right\} z^{h}}{1-c_{1} z-\cdots-c_{k} z^{k}}$.

We refer to (2) as the generating function form of the definition of $f$. For example, the difference equation form of the Fibonacci sequence is

$$
F_{0}=0, F_{1}=1, \text { and } F_{n}=F_{n-1}+F_{n-2} \text { for all } n \geq 2
$$

This is equivalent to the generating function form

$$
\begin{equation*}
\sum_{n=0}^{\infty} F_{n} z^{n}=\frac{z}{1-z-z^{2}} \tag{3}
\end{equation*}
$$

Perhaps it should be emphasized that (2) and (3) have purely algebraic interpretations. We are merely using the formal sum as a convenient notation for a sequence. For example, (3) only means that the Cauchy product of the sequences ( $1,-1,-1,0,0, \ldots$ ) and $\left(F_{0}, F_{1}, F_{2}, \ldots\right)$ is equal to ( $0,1,0,0, \ldots$ ). In terms of formal power series, this means

$$
\begin{equation*}
\left(1-z-z^{2}\right) \sum_{n=0}^{\infty} F_{n} z^{n}=z \tag{4}
\end{equation*}
$$

We are not concerned with the fact that the power series on the left-hand side in (3) represents the rational function on the right-hand side for certain values of $z$. Such a discussion would have to be given to justify putting $z=\frac{1}{2}$ in (3) to conclude

$$
\begin{equation*}
\frac{F_{1}}{2}+\frac{F_{2}}{4}+\cdots+\frac{F_{n}}{2^{n}}+\cdots=2 \tag{5}
\end{equation*}
$$

but this is not the sort of application we have in mind. The algebraic basis can be found in [1], for example.

Rational sequences may be recognized as such in other ways than by the difference equation or rational generating function. Next most important after
these is the exponential form. To get to the heart of the matter, suppose $P(z) / Q(z)$ is the generating function of the rational sequence $f$ with $P, Q$ polynomials over $\mathcal{F}$ such that $Q(0)=1$, and $P, Q$ have no common zeroes. (That is, $P / Q$ is a "reduced fraction.") Also, there is no loss in generality to assume $P$ has degree less than that of $Q$. (Otherwise, write $P / Q=R+S / Q$ where $R, S$ are polynomials with the degree of $S$ less than that of $Q$. ) Also, it can be supposed that the zeroes of $Q$ are elements of $\mathfrak{F}$ (otherwise, just extend $\mathfrak{F}$ by these zeroes). Suppose the distinct zeroes of $Q$ are $1 / \theta_{1}, \ldots, 1 / \theta_{t}\left[\theta_{1}, \ldots\right.$, $\theta_{t} \neq 0$ because $\left.Q(0)=1\right]$, and let $d_{i}$ denote the multiplicity of $1 / \theta_{i}$ for $i=1$, $\ldots, t$. Then $Q(z)=\left(1-\theta_{1} z\right)^{d_{1}} \ldots\left(1-\theta_{t} z\right)^{d_{t}}$, and it can be shown that there exist polynomials $P_{1}, \ldots, P_{t}$ over $\mathcal{F}$ with the degree of $P_{i}$ less than $d_{i}$ for $i=1$, ..., $t$ such that

$$
\begin{equation*}
\frac{P(z)}{Q(z)}=\frac{P_{1}(z)}{\left(1-\theta_{1} z\right)^{d_{1}}}+\cdots+\frac{P_{t}(z)}{\left(1-\theta_{t} z\right)^{d_{t}}} . \tag{6}
\end{equation*}
$$

Rather than give an explicit formula for the coefficients of $P_{i}(z)$, we will just show how to compute them. To do this, it is enough to consider the case $i=1$. Start with

$$
\begin{equation*}
P_{1}(z)+\left(1-\theta_{1} z\right)^{d_{1}} \sum_{i=2}^{t} \frac{P_{i}(z)}{\left(1-\theta_{i} z\right)^{d_{i}}}=\frac{P(z)}{\sum_{i=2}^{t}\left(1-\theta_{i} z\right)^{d_{i}}}, \tag{7}
\end{equation*}
$$

and differentiate $d_{1}-1$ times with respect to $z$ to obtain $d_{1}$ equations involving the various derivatives of the polynomial $P_{1}(z)$. Then put $z=1 / \theta_{1}$ in each of these equations to get

$$
\begin{equation*}
D^{j}\left\{P_{1}(z)\right\}_{z=1 / \theta_{1}}=D^{j}\left\{P(z) \sum_{i=2}^{t}\left(1-\theta_{2} z\right)^{-d_{i}}\right\}_{z=1 / \theta_{1}} \quad\left(j=0, \ldots, d_{1}-1\right), \tag{8}
\end{equation*}
$$

where $D$ denotes the differential with respect to $z$. (All of this can be done in an algebraic manner by introducing a formal operation on sequences; calculus is not actually required.) Note that by putting $z=1 / \theta_{1}$ in the $j$ th differential equation, all of the terms involving $P_{2}, \ldots, P_{t}$ have a factor ( $1-\theta_{1} z$ ), so these terms drop out of the computation. This gives rise to a linear system of $d_{1}$ equations in the $d_{1}$ coefficients of $P_{1}$. This system can be solved because the matrix of the system is upper triangular and has a nonzero diagonal. Once we have $P_{1}, \ldots, P_{t}$ in (6), we can develop each of the $t$ rational functions on the right into a power series using the binomial theorem. In fact, the full force of the binomial theorem is not needed. One only needs

$$
\begin{equation*}
\frac{1}{(1-z)^{d}}=\sum_{n=0}^{\infty}\binom{n+d-1}{d-1} z^{n} \tag{9}
\end{equation*}
$$

and this can be established by induction on $d$. Thus, if

$$
P_{i}(z)=p_{0}+p_{1} z+\cdots+p_{d_{i}-1} z^{d_{i}-1}
$$

then

$$
\begin{equation*}
\frac{P_{i}(z)}{\left(1-\theta_{i} z\right)}=\sum_{n=0}^{\infty}\left\{p_{0}\binom{n+d_{i}-1}{d_{i}-1}+\frac{p_{1}}{\theta_{i}}\binom{n+d_{i}-2}{d_{i}-1}+\cdots\right\} \theta_{i}^{n} z^{n} \tag{10}
\end{equation*}
$$

for $i=1, \ldots, t$. Since $\binom{n+d-1}{d-1}$ is a polynomial in $n$ with degree $d-1$, the coefficient of $z^{n}$ in the right member of (10) has the form $\pi_{i}(n) \theta_{i}^{n}$, where
$\pi_{i}(n)$ is a polynomial in $n$ whose degree is $d_{i}-1$. Summing over $i$, we can conclude that

$$
\begin{equation*}
f(n)=\pi_{1}(n) \theta_{1}^{n}+\cdots+\pi_{t}(n) \theta_{t}^{n} \quad(n \varepsilon \mathbb{N}), \tag{11}
\end{equation*}
$$

where $1 / \theta_{1}, \ldots, 1 / \theta_{t}$ are the distinct zeroes of $Q(z)$ with multiplicities $d_{1}$, $\ldots, \alpha_{t}$, respectively, and $\pi_{1}, \ldots, \pi_{t}$ are polynomials over $\mathcal{F}$ where $\pi_{i}$ has degree less than $d_{i}$ for $i=1, \ldots, t$. We call (11) the exponential form for the rational sequence $f$. We derived the exponential form from the rational form, but it is important to note that given any one of the forms (1), (2), or (11), the other two can be derived from it.

Continuing the example dealing with the Fibonacci sequence, note that 1 -$z-z^{2}$ has zeroes $1 / \alpha, 1 / \beta$ where $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2$. Hence

$$
1-z-z^{2}=(1-\alpha z)(1-\beta z)
$$

and using the method outlined above, we find

$$
\begin{equation*}
\sum_{n=0}^{\infty} F_{n} z^{n}=\frac{z}{1-z-z^{2}}=\frac{\alpha z}{1-\alpha z}-\frac{\beta z}{1-\beta z}=\sum_{n=0}^{\infty} \frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} z^{n} \tag{12}
\end{equation*}
$$

Thus, the Fibonacci sequence has the exponential form

$$
\begin{equation*}
F_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \quad(n \varepsilon \mathbb{N}) \tag{13}
\end{equation*}
$$

where $\alpha=(1+\sqrt{5}) / 2, \beta=(1-\sqrt{5}) / 2$.
There is still another useful presentation of a rational sequence called the matrix form. Let
(14) $\quad M=\left[\begin{array}{cccccc}0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_{k} & c_{k-1} & c_{k-2} & c_{k-3} & \cdots & c_{1}\end{array}\right], v_{n}=\left[\begin{array}{l}f(n) \\ f(n+1) \\ \vdots \\ f(n+k-1)\end{array}\right] \quad(n \varepsilon \mathbb{V})$,
where $M$ is a $k \times k$ matrix having $\left[c_{k}, \ldots, c_{1}\right]$ as its bottom row and the ( $k-1$ ) $\times(k-1)$ identity matrix as the minor of the ( $k, 1$ )-entry. It is easy to verify that $M v_{n}=v_{n+1}$ for all $n \in \mathbb{N}$, and hence that $M^{n} v_{0}=v_{n}$ for all $n \varepsilon \mathbb{N}$. Since $M^{n}$ can be computed in about $\log n$ matrix multiplications,it follows that $f(n)$ can be computed in $0(\log n)$ basic steps instead of the $0(n)$ steps one might guess. Note that the eigenvalues of $M$ are $\theta_{1}, \ldots, \theta_{t}$ with multiplicities $d_{1}$, $\ldots, d_{t}$, respectively, because the characteristic polynomial of $M$ is

$$
\operatorname{det}(M-z I)=(-z)^{k} Q(1 / z)=(-1)^{k}\left(z^{k}-c_{1} z^{k-1}-\cdots-c_{k}\right)
$$

(This can be shown by a simple induction proof on $k$.)
In the case of the Fibonacci sequence, we have

Hence

$$
M=\left[\begin{array}{ll}
0 & 1  \tag{15}\\
1 & 1
\end{array}\right], \quad v_{n}=\left[\begin{array}{l}
F_{n} \\
F_{n+1}
\end{array}\right] \quad(n \in \mathbb{N}) .
$$

Hence

$$
\left[\begin{array}{ll}
0 & 1  \tag{16}\\
1 & 1
\end{array}\right]^{n}\left[\begin{array}{l}
0 \\
1
\end{array}\right]\left[\begin{array}{l}
F_{n} \\
F_{n+1}
\end{array}\right] \quad\left(\begin{array}{llll} 
& \varepsilon & \mathbb{V}) .
\end{array}\right.
$$

The difference equation

$$
\begin{equation*}
x_{n}=c_{1} x_{n-1}+\cdots+c_{k} x^{n-k} \tag{17}
\end{equation*}
$$

with $c_{1}, \ldots, c_{k} \in \mathfrak{F}, c_{k} \neq 0$, has order $k$. The order of a rational sequence is the minimum order of all difference equations it satisfies. A rational sequence $f$ of order $k$ satisfies a unique difference equation of order $k$. [The uniqueness depends on the standard form given in (17); after all, nothing is changed by multiplying through (17) with a nonzero element of $\mathfrak{F}$.] In general, a rational sequence of order $k$ satisfies as many difference equations of order $k+d$ as there are polynomials $R$ over $\mathcal{F}$ with degree $d$ and $R(0)=1$. To see that the difference equation of lowest order satisfied by $f$ is unique, suppose for the moment there are two. Say $f$ satisfies (1) and

$$
\begin{equation*}
f(n)=b_{1} f(n-1)+\cdots+b_{k} f(n-k) \quad(h \leq n) . \tag{18}
\end{equation*}
$$

Taking the difference of equations (1) and (18) leads to a new difference equation with order less than $k$ satisfied by $f$ if $b \neq c$ for some $i$ with $1 \leq i \leq k$. So $k$ cannot be the order of $f$ as was assumed. If $f$ has order $k$ and (17) is the unique difference equation of order $k$ satisfied by $f$ (this is called the minimal equation, then the generating function of $f$ has the form (2). Let $P$ and $Q$ denote the numerator and denominator, respectively, in the right member of (2), and note that $Q(0)=1$, and $P$ and $Q$ have no common zeroes. (Otherwise, $g$ would satisfy a difference equation of order $k-1$.) We call the rational function $P / Q$ the canonical generating function of $f$, and note that it is unique. For each polynomial $R$ with degree $d$ over $\mathcal{F}$ with $R(0)=1$, we have $P / Q=P R / Q R$, so $f$ satisfies a difference equation of order $k+d$ with coefficients equal to the coefficients of $Q R$. All difference equations of order $k+d$ satisfied by $f$ are obtained in this way, because each difference equation of order $k+d$ satisfied by $f$ gives rise to polynomials $U, V$ over $\mathcal{F}$ with $V(0)=1$ such that $P / Q=U / V$. But this means $P R=U$ and $Q R=V$ with $R$ a polynomial over $\mathcal{F}$ with degree $d$ and $R(0)=1$. We conclude this discussion of the order of a sequence by observing that the order of $f$ can be deduced from its exponential form by adding $t$ to the sum of the degrees of $\pi_{1}, \ldots, \pi_{t}$.

## 2. SITUATIONS IN WHICH RATIONAL SEQUENCES ARISE


#### Abstract

Sometimes rational sequences are formed in terms of other rational sequences. For example, if $f, g$ are rational sequences over the field $\mathcal{F}$ and $a, b \in \mathcal{F}$, then we can form a new sequence $h=a f+b g$ defined by


$$
\begin{equation*}
h(n)=a f(n)+b g(n) \quad(n \in \mathbb{N}) \tag{19}
\end{equation*}
$$

Let $F, G, H$ denote the generating functions of $F, G, H$, respectively, then $H=$ $\alpha F+b G$. This means that $H$ is a rational function because $F$ and $G$ are, so $h$ is a rational sequence. It is easy to check that the set of all rational sequences over $\mathfrak{F}$ forms a subspace of the vector space of all sequences over $\mathcal{F}$. Furthermore, the sequences which satisfy equation (17) form a $k$-dimensional subspace. If $\theta_{1}, \ldots, \theta_{t}$ denote the zeroes of $z^{k}-c_{1}^{k-1} \ldots . .-c_{k}$ with multiplicities $d_{1}, \ldots, d_{t}$, respectively, it is easy to check that each of the sequences

$$
\left(n^{j} \theta_{i}^{n}: n \varepsilon \mathbb{N}\right) \text { for all } j \in \mathbb{N} \text { and } i=1, \ldots, t
$$

satisfies (17). Using the exponential form for any sequence $f$ which satisfies (17), it follows that the $k$ sequences

$$
\begin{equation*}
\left(n^{j} \theta_{i}^{n}: n \varepsilon \mathbb{V}\right) \quad 0 \leq j<d, i=1, \ldots, t \tag{20}
\end{equation*}
$$

form a basis for the vector space of all sequences which satisfy (17). That this actually is a basis depends heavily on the proof that every solution of (17) has the exponential form given in (11).

There are ways other than forming linear combinations to build new rational sequences from those on hand. For example, consider the Cauchy product $f \times g$
or the termwise product $f \cdot g$ defined, respectively,

$$
\begin{array}{ll}
(f \times g)(n)=\sum_{i=0}^{n} f(i) g(n-i) & \\
(n \in \mathbb{N})  \tag{22}\\
(f \cdot g)(n)=f(n) g(n) & \\
(n \in \mathbb{V})
\end{array}
$$

Let $h=f \times g$, and let $F, G, H$ denote the generating functions of $f, g$, $h$, respectively. Then $H=F G$, and $H$ is a rational function if the same is true of $F$ and $G$. Hence, $h$ is a rational sequence if $f$ and $g$ are. To see that $f \cdot g$ is rational whenever $f$ and $g$ are, use the exponential form of $f$ and $g$. It is fairly easy to check that the product of two exponential forms is again an exponential form, so this approach gives a proof. The generating function of $f \cdot g$ can be given in terms of $F$ and $G$ by means of a contour integral as was shown in [2]. The fact that the termwise product of two rational sequences is again rational seems to be due to Vaidyanathaswamy [10].

The termwise product can be used to produce all sorts of unexpected results. For example, since the Fibonacci sequence is rational, it follows that

$$
\left(F_{n}^{j}: n \in \mathbb{N}\right)
$$

is rational for all $j \in \mathbb{P}$. The minimal equation for the $j$ th powers of the Fi bonacci sequence were given in [9]. Also, the sequence $p$ defined by $p(n)=n$ ( $n \in \mathbb{N}$ ) satisfies

$$
p(n)=2 p(n-1)-p(n-2), 2 \leq n,
$$

so $p^{j}=\left(n^{j}: n \varepsilon \mathbb{N}\right)$ is rational for all $j \in \mathbb{P}$. Hence, the linear combination $q=a_{0}+a_{1} p+\cdots+a_{j} p^{j}$ is also rational, and so is $g \cdot f$ for any rational $f$. For example, again using the Fibonacci sequence,

$$
\left(n^{2} F_{n}-n+2: n \varepsilon \mathbb{N}\right)
$$

is rational. A little subtler use of the termwise product involves periodic sequences. Suppose $s$ is a sequence such that $s(n)=s(n-m)$ for all $n \geq h$ for some $h, m \in \mathbb{N}$; that is, $s$ is eventually periodic and has period $m$. By definition, $s$ satisfies a difference equation, so $s$ is rational. In particular, let $\alpha, m \in \mathbb{N}$ with $0<m$, and define $s(n)=1$ whenever $\alpha \leq n, n \equiv \alpha(\bmod m)$, and $s(n)=0$ otherwise. Since $s$ is eventually periodic, $s \cdot f$ is rational whenever $f$ is; furthermore, the generating function of $s$ • $f$ has the form $z^{a} P\left(z^{m}\right) / Q\left(z^{m}\right)$ with $P, Q$ polynomials over $\mathcal{F}$ and $Q(0)=1$. Hence, the sequence $g$ defined by $g(n)=f(m n+\alpha)$ for all $n \varepsilon \mathbb{N}$ has $P(z) / Q(z)$ as its generating function, so $g$ is rational. For example, the subsequence $\left(F_{2}, F_{7}, F_{12}, \ldots\right)=\left(F_{5 n+2}: n \in \mathbb{N}\right)$ of the Fibonacci sequence is rational (the difference equation is

$$
\left.x_{n}=11 x_{n-1}+x_{n-2}, 2 \leq n\right) .
$$

Interwoven rational sequences are also rational. More precisely, suppose $f_{0}$, ..., $f_{m-1}$ are rational sequences, and define $f$ by

$$
\begin{equation*}
f(n)=f_{r}(n) \quad[\text { where } n \equiv r(\bmod m), n \in \mathbb{N}] . \tag{23}
\end{equation*}
$$

Let $F, F_{0}, \ldots, F_{m-1}$ denote the generating functions of $f, f_{0}, \ldots, f_{m-1}$, respectively, then

$$
\begin{equation*}
F(z)=F_{0}\left(z^{m}\right)+z F_{1}\left(z^{m}\right)+\cdots+z^{m-1} F_{m-1}\left(z^{m}\right) . \tag{24}
\end{equation*}
$$

Since $F_{0}, \ldots, F_{m-1}$ are rational functions, so is $F$; therefore, $f$ is a rational sequence.

Sometimes a finite set of sequences is defined by means of some initial conditions and a finite set of difference equations. It turns out that each of the sequences is rational in this case. This can be formulated more precisely as follows: Let $f_{1}, \ldots, f_{m}$ be sequences, and suppose for each $i, 1 \leq i \leq m$,
there exists $h_{i} \in \mathbb{N}$, together with finite sets $S_{i 1}, \ldots, S_{i m} \subseteq \mathbb{N}$ and constants $c_{i j k}$ corresponding to each $j \in S_{i k}$ such that

$$
\begin{equation*}
f_{i}(n)=\sum_{k=1}^{m} \sum_{j \in S_{i k}} e_{i j k} f_{k}(n-j) \quad\left(n \varepsilon \mathbb{N}, h_{i}<n\right) \tag{25}
\end{equation*}
$$

for $i=1$, ..., m. Also, suppose $f_{i}(n)$ is given for all $n$ with $n \leq h_{i}$ for $i=$ $1, \ldots, m$, and suppose that this boundary condition together with the system (25) gives an unambiguous algorithm to compute the sequences $f_{1}, \ldots, f_{m}$. Then each of $f_{1}, \ldots, f_{m}$ is rational. To see this, convert the system (25) to a system of linear equations in the generating functions $F_{1}, \ldots, F_{m}$. The coefficients in this system are polynomials in $z$ over the field $\mathcal{F}$. This system can be solved using Cramer's Rule to deduce that each of $F_{1}, \ldots, F_{m}$ is a rational function. In fact, $F_{i}$ has the form $P_{i} / Q$ where $Q$ is the determinant of the system, and $P_{i}$ is a polynomial computed in a similar fashion.

A particular case of the foregoing situation involves matrices. Suppose $M=\left[c_{i j}\right]$ is an $m \times m$ matrix over the field $\mathcal{F}$, and let $v_{0}=\left[f_{1}(0), \ldots, f_{m}(0)\right]^{T}$ (where $T$ denotes the transpose operator). Define $v_{n}$ for all $n \varepsilon \mathbb{N}$ by $v_{n+1}=M v_{n}$. This is equivalent to the system of difference equations

$$
\begin{equation*}
f_{i}(n+1)=c_{i 1} f_{1}(n)+\cdots+c_{i m} f_{m}(n) \quad(n \in \mathbb{N}) \tag{26}
\end{equation*}
$$

for $i=1, \ldots, m$. In terms of generating functions, this becomes

$$
\begin{equation*}
M F=v_{0} \tag{27}
\end{equation*}
$$

where $F=\left[F_{1}, \ldots, F_{m}\right]$. The determinant of this system is the characteristic polynomial of $M$; that is, $\operatorname{det}(M-z I)$. This gives information about the denominator polynomials in the generating functions $F_{1}, \ldots, F_{m}$. This observation can be taken a little further to deduce the Cayley-Hamilton Theorem as was done in [3].

One might get the impression that the rational sequence $f_{l}$ (defined in the previous paragraph) has order $m$, and that the minimal equation is given by the characteristic polynomial of $M$. But this is not always the case, and then Krylov's method may be useful. (See [11].) The idea here is to look for a linear dependency among the vectors $M^{0} v_{0}, M^{1} v_{0}, \ldots, M^{k} v_{0}$ for $k=1,2, \ldots$. Once one has $c_{0}, \ldots, c_{k} \in \mathfrak{F}$ for some minimal $k$ such that

$$
\begin{equation*}
c_{0} M^{0} v_{0}+\cdots+c_{k} M^{k} v_{0}=M^{k+1} v_{0} \tag{28}
\end{equation*}
$$

multiply through (28) with $M^{n}$ to deduce that $f_{1}$ satisfies

$$
\begin{equation*}
c_{0} x_{n}+c_{1} x_{n+1}+\cdots+c_{k} x_{n+k}=x_{n+k+1} \quad(n \in \mathbb{N}) \tag{29}
\end{equation*}
$$

## 3. SOME APPLICATIONS

This section gives brief descriptions of some recent results obtained by the author which involve rational functions. We start with domino tilings of rectangles with fixed width [4]. The idea here is based on an old, wel1-known observation about the number of paths of fixed length in a directed graph. Let $V=\{1, \ldots, m\}$, let $E \subseteq V \times V$, and let $M=\left[e_{i j}\right]$ be an $m \times m$ matrix defined by $e_{i j}=1$ if $(i, j) \varepsilon E$ and $e_{i j}=0$ otherwise. Elements of $V$ are vertices, elements of $E$ are directed edges, and $M$ is the matrix of the directed graph $(V, E)$. A sequence $\left(v_{0}, \ldots, v_{k}\right)$ is a path of length $k$ in ( $V, E$ ) just when $\left(v_{i-1}, v_{i}\right) \varepsilon$ $E$ for $i=1, \ldots, k$. It is well known that the number of paths ( $v_{0}, \ldots, v_{k}$ ) of length $k$ in ( $V, E$ ) with $v_{0}=i$ and $v_{k}=j$ is the ( $i, j$ )-entry in $M^{k}$. Suppose we are only interested in paths which begin and end with vertex 1 . Then let $c^{(k)}$ denote the first column of $M^{k}$ for all $k \in \mathbb{N}$, and observe that $M_{c}(k)=$ $c^{(k+1)}$ for all $k \in \mathbb{N}$. We want the top element $c_{11}^{(k)}$ of $c^{(k)}$ for all $k \in \mathbb{N}$, so
the method outlined in the last paragraphs of Section 2 can be applied. In particular, it follows that $\left(c_{11}^{(k)}: k \in \mathbb{N}\right)$ is rational, and Krylov's method can be used to find a difference equation. Now let us see how this applies to domino tilings. Let $t(m, n)$ denote the number of tilings of an $m \times n$ rectangle with dominoes for all $m, n \in \mathbb{N}$. We fix the width $m$ and concentrate on the computation of the sequence $(t(m, n): n \varepsilon \mathbb{N})$. To do this, we create a graph whose vertices are cross-sections of tilings, and two cross-sections form a directed edge in the graph just when one can immediately follow the other in some tiling. A cross-section is a grid line parallel to the end of width $m$ which cuts across some dominoes and passes others. Cross-sections can be encoded as binary sequences: 1 denotes a cut domino, and 0 denotes a crack between. For example, the $5 \times 6$ tiling shown in Figure 1 is encoded by the columns of the $0-1$ matrix shown to its right. If we make the all-zero cross-section vertex 1 , the $m \times n$ domino tilings correspond one-to-one with paths of length $n$ beginning and ending at vertex l. More details can be found in [4].


Fig. 1. A $5 \times 6$ domino tiling with its binary cross-section encoding
Now we give an example illustrating how a rational sequence can arise in a system of difference equations. Let $A$ denote a finite set called an alphabet, and let $A^{*}$ denote the set of all finite sequences of elements of $A$. Such sequences are called words, and in particular $\Lambda$ denotes the empty word. Let $F$ denote a finite subset of $A^{*}$ and let $A^{*} / F$ denote the set of all elements of $A^{*}$ which do not have any elements of $F$ as subwords. Elements of $A^{*}$ belonging to $A^{*} / F$ are called good and others are called bad. Let $w$ denote a weight function defined on $A^{*}$ such that $w(u v)=w(u) w(v)$ for all $u, v \varepsilon A^{*}$. Suppose further that for each $u \in A^{*} / F$ the sum

$$
G_{u}=\sum_{\left(u v \in A^{* / F}\right)} w(u v)
$$

is also a weight. The problem is to compute

$$
G=G_{\Lambda}=\sum_{\left(u \in A^{\star} / F\right)} w(u) .
$$

It was shown in [5] that $G$ is a rational function in the weights of elements of A. This follows from two equations:

$$
\begin{equation*}
G=w(\Lambda)+\sum_{a \in A} G a, \tag{30}
\end{equation*}
$$

(31)

$$
G_{u}=w(u)\left\{G-\sum G_{v}\right\},
$$

where the sum in the right member of (31) is over all basic words $u v$ with $v \varepsilon$ $A^{*} / F$. A word is basic if it is bad but no proper initial subword is bad. Note that if $u$ is good, and $u v$ is basic, then $v$ is not longer than $n$ where $n+1$ is the length of the longest word in $F$. (A terminal subword of $u v$ is an element of $F$ and must overlap $u_{0}$ ) Together (30) and (31) give rise to a linear system involving $G_{u}$ for all good words $u$ not longer than $n$. A procedure may be followed to keep this system small. First, write down (30). Then in subsequent stages write down expressions for those $G_{u}$ which have appeared on the right side of earlier expressions obtained from (31). Since the length of $u$ is bounded by $n$, this procedure terminates leaving us with a system linear in certain $G_{u}$, $u \in A^{*} / F$. We may conclude from the general argument given in Section 2 that $G_{u}$ is rational in the weights $w(\alpha), \alpha \varepsilon A$; in particular, this is true of $G$.

The result just described was used in [5] to treat a special case of the following unsolved problem. Let $\alpha_{i}(x)=m_{i} x+\alpha_{i}$ be an affine function defined on the integers with $\left.m_{i}, a_{i} \varepsilon \mathbb{N}, m_{i}\right\rangle 1$, for $i=1, \ldots, k$. Let $\langle A\rangle$ denote the semigroup generated by $A=\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}$ under composition of functions. Note that an element $\alpha \in\langle A\rangle$ has the form $\alpha(x)=m x+\alpha$ with $m$ a product of the numbers $m_{1}, \ldots, m_{k}$. Let $p_{1}, \ldots, p_{h}$ denote the distinct prime divisors of $m_{1}, \ldots$, $m_{k}$, and for each $\alpha \varepsilon\langle A\rangle$ with $\alpha(x)=p_{1}^{i_{1}} \ldots p_{h}^{i_{h}} x+\alpha$, let $w(\alpha)=x_{1}^{i_{1}} \ldots x_{h}^{i_{h}}$. It is easy to check that $w(\alpha \beta)=w(\alpha) w(\beta)$ for all $\alpha, \beta \varepsilon\langle A\rangle$ where $\alpha \beta(x)=$ $\alpha(\beta(x))$. Is it true that

$$
\sum_{u \in\langle A\rangle} w(u)
$$

is a rational function? This problem has been solved when $m_{i}=m^{e_{i}}$ for some $e_{i}, m \in \mathbb{Z}, i=1, \ldots, k$; the case when $e_{i}=\ldots=e_{k}=1$ is treated in [6], and the systems of difference equations play an important role.

We conclude with an example which illustrates a frequently used formula from combinatorics. Let $A$ denote a finite alphabet, let $A^{*}$ denote the set of words over $A$, and let $w$ denote a weight function on $A^{*}$ which satisfies $w(u v)=$ $w(u) w(v)$ for all $u, v \in A^{*}$. Then

$$
\begin{equation*}
\sum_{u \in A^{*}} w(u)=\frac{1}{1-\sum_{a \in A} w(a)} . \tag{32}
\end{equation*}
$$

Thus, rational functions arise. For example, this simple formula together with the inclusion-exclusion formula were used in [7] and [8] to show that the sequence of forms assumed by growing crystals is rational. more precisely, let $H, D \in \mathbb{Z}^{k}$ be finite sets, and consider the sequence of crystals

$$
H, H+D, H+D+D, \ldots
$$

formed by starting with the initial hub $H$, and adding increments equal to $D$ in subsequent stages. Such a sequence is indicated in Figure 2 with $k=2, H=$ $\{(0,0)\}, D=\{(0,0),(1,0),(0,1)\}$.


Fig. 2. A growing crystal

Give an element $i=\left(i_{1}, \ldots, i_{k}\right) \varepsilon \mathbb{Z}^{k}$ weight $w(i)=x_{1}^{i_{1}} \ldots x_{k}^{i_{k}}$ and define the weight $\omega(S)$ of $S \subseteq \mathbb{Z}^{k}$ to be the sum of the weights of the elements of $S$. The main result is that

$$
\begin{equation*}
w(H)+w(H+D) z+w(H+D+D) z^{2}+\cdots \tag{33}
\end{equation*}
$$

is a rational function in $x_{1}, \ldots, x_{k}$ and $z$. A consequence of this is that the sequence of volumes $(|H|,|H+D|,|H+D+D|, \ldots)$ forms a rational sequence.

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## ON THE PROBABILITY THAT $n$ AND $\Omega(n)$ ARE RELATIVELY PRIME <br> KRISHNASWAMI ALLADI <br> The University of Michigan, Ann Arbor MI 48109

To the memory of V. E. Hoggatt Jr. -my teacher and friend
It is a well-known result due to Chebychev that if $n$ and $m$ are randomly chosen positive integers, then $(n, m)=1$ with probability $6 / \pi^{2}$. It is the purpose of this note to show that if $\Omega(n)$ is the number of prime factors of $n$ counted with multiplicity, then the probability that $(n, \Omega(n))=1$ is also $6 / \pi^{2}$. Thus, as far as common factors are concerned, $\Omega(n)$ behaves randomly with respect to $n$.

Results of this type for fairly general additive functions have been proved by Hall [2], and in [1] and [3] he looks closely at the situation regarding the special additive function $g(n)$, the sum of the distinct prime factors of $n$. Hall's results do not apply to either $\Omega(n)$ or $\omega(n)$, the number of distinct prime factors of $n$, and so our result is of interest. Our proof, which is of an
analytic nature, proceeds along classical lines, and so must surely be known to specialists in the field. In any case, it never seems to have been stated in the literature and so we felt it was worthwhile to prove it,particularly since it is interesting when viewed in the context of several celebrated results on the distribution of $\Omega(n)$ (see [5]) such as those of Hardy-Ramanujan and ErdösKac. By a slight modification of our proof, the same result can be established for $\omega(n)$; we have concentrated on $\Omega(n)$ for the sake of simplicity. Throughout, implicit constants are absolute unless otherwise indicated and $p$ always denotes a prime number.
Theorem: Let

$$
Q(x)=\sum_{1 \leq n \leq x,(n, \Omega(n))=1} 1
$$

$$
Q(x)=\frac{6 x}{\pi^{2}}+0\left(x(\log \log \log x)^{-1 / 3} \cdot(\log \log \log \log x)^{-1}\right)
$$

To prove the theorem, we need a few auxiliary results.
Lemma 1: Let $x>20$, and $k$ be a positive integer such that

$$
k \leq\{\log \log x / \log 1 \log \log x\}^{1 / 3}
$$

Then for all integers $j$,

$$
\sum_{\substack{1 \leq n \leq x \\ \Omega(n) \equiv j(\bmod k)}} 1=\frac{x}{k}+0\left(x \exp \left\{-(\log 1 \log x)^{1 / 3}\right\}\right) .
$$

Proof: Let $z$ be a complex number with $|z|=1$. Then it follows from a result due to Selberg [6] that

$$
\begin{equation*}
S_{z}(x)=\sum_{1 \leq n \leq x} z^{\Omega(n)}=\frac{A(z) x}{(\log x)^{1-z}}+0\left(\frac{1}{\left|(\log x)^{2-z}\right|}\right) \tag{1}
\end{equation*}
$$

where $A(z)$ is analytic for $|z|<2$. Note that

$$
\begin{align*}
\sum_{\substack{1 \leq n \leq x \\
\Omega(n) \equiv j(\bmod k)}} 1-\frac{S_{1}(x)}{k} & =\sum_{1 \leq n \leq x} \frac{1}{k} \sum_{\ell=1}^{k-1} e^{2 \pi i(\Omega(n)-j) \ell / k} \\
& =\frac{1}{k} \sum_{\ell=1}^{k-1} e^{-2 \pi i j \ell / k} \cdot S_{\rho \ell}(x) \tag{2}
\end{align*}
$$

where

$$
\rho=\exp \{2 \pi i / k\}
$$

From (1) we deduce that the largest term on the right of (2) arises out of the root of unity with largest real part. Since $S_{1}(x)=[x]$, the largest integer $\leq x$, we get from (2)

$$
\begin{equation*}
\sum_{\substack{1 \leq n \leq x \\(n) \leqq j(\bmod k)}} 1=\frac{x}{k}+0\left(x(\log x)^{\cos (2 \pi / k)-1}\right) \tag{3}
\end{equation*}
$$

Lemma 1 follows from (3) with a little computation.
Lemma 2: Let $x>20$ and $k$ be a positive integer satisfying

$$
k \leq \frac{3}{2} \log \log x
$$

Then for all integers $j$, we have

$$
\sum_{\substack{1 \leq n \leq x \\ \Omega(n) \equiv j(\bmod k)}} 1 \ll \frac{x}{k^{1 / 3}} .
$$

Proof: We may assume that $j \geq 1$. We rewrite the sum in the lemma as

$$
\begin{equation*}
\sum_{\substack{1 \leq n \leq x \\ \Omega(n) \equiv j(\bmod k)}} 1=\sum_{\substack{v \equiv j(\bmod k) \\ 0<v \leq(3 / 2) \log \log x}} \sum_{\substack{1 \leq n n \leq x \\ \Omega(n)=v}} 1+\theta \sum_{\substack{1 \leq n \leq x \\ \Omega(n)>(3 / 2) \log \log x}} 1, \tag{4}
\end{equation*}
$$

where $0 \leq \theta \leq 1$.
To estimate the first term on the right of (4) we use the following result due to Sathe and Selberg (see [6]):

$$
\sum_{1 \leq n \leq x, \Omega(n)=\nu} 1=0\left(\frac{x(\log \log x)^{\nu-1}}{\log x \cdot(\nu-1)!}\right) \text { for } 1 \leq \nu \leq \frac{3}{2} \log \log x .
$$

So, the term is

$$
\begin{equation*}
\ll \frac{x}{\log x} \sum_{\substack{m=0 \\ m \equiv j(\bmod k)}}^{\infty} \frac{(\log \log x)^{m}}{m!} . \tag{5}
\end{equation*}
$$

Set $y=\log \log x$. So the sum in (5) is

$$
\begin{equation*}
\frac{x}{\log x} \cdot \frac{1}{k} \sum_{w^{k}=1} \frac{e^{w y}}{w^{j-1}} \ll \frac{x}{\log x} \cdot \frac{1}{k} \sum_{w^{k}=1} e^{u y}, \tag{6}
\end{equation*}
$$

where $w$ ranges over all $k$ th roots of unity and $w=u+i v$.
First, we assume that

$$
\{\log \log x / \log \log \log x\}^{1 / 3} \leq k \leq \frac{3}{2} \log \log x
$$

Then

$$
\begin{equation*}
\sum_{w^{k}=1} e^{u y}=\sum_{\ell=0}^{k-1} \exp \left\{\left(\cos \frac{2 \pi \ell}{k}\right) y\right\}=\sum_{\substack{\ell \leq k^{2 / 3} \text { or } \\ k-k^{2 / 3} \leq \ell \leq k}}+\sum_{k^{2 / 3}<\ell<k-k^{2 / 3}}=S_{1}+S_{2} . \tag{7}
\end{equation*}
$$

C1early

$$
\begin{equation*}
S_{1} \leq 2 k^{2 / 3} e^{y} \ll k^{2 / 3} \cdot \log x . \tag{8}
\end{equation*}
$$

To estimate $S_{2}$, write

$$
\begin{equation*}
\exp \left\{\left(\cos \frac{2 \pi \ell}{k}\right) y\right\}=e^{y} \exp \left\{\left(\left(\cos \frac{2 \pi \ell}{k}\right)-1\right) y\right\} \tag{9}
\end{equation*}
$$

and observe that

$$
\begin{equation*}
-\left\{\left(\cos \frac{2 \pi l}{k}\right)-1\right\} \gg \frac{1}{k^{2 / 3}} . \tag{10}
\end{equation*}
$$

From (7), (9), and (10) we deduce that

$$
\begin{equation*}
S_{2} \ll k^{2 / 3} \log x . \tag{11}
\end{equation*}
$$

If we combine (6), (7), (8), and (11), we see that the first term on the right of (4) is $\ll x / k^{1 / 3}$ if

$$
\begin{equation*}
\{\log \log x / \log \log \log x\}^{1 / 3} \leq k \leq \frac{3}{2} \log \log x \tag{12}
\end{equation*}
$$

The last term in (4) is easily bounded by appealing to the following theorem of Turán (see [4, pp. 356-358]):

$$
\begin{equation*}
\sum_{1 \leq n \leq x}\{\Omega(n)-\log \log x\}^{2} \ll x \log \log x \tag{13}
\end{equation*}
$$

That is, if $N(x)$ is the number of $n \leq x$ for which

$$
\Omega(n)>\frac{3}{2} \log \log x
$$

then (13) shows that
whence

$$
\begin{align*}
& N(x) \cdot(\log \log x)^{2} \ll x \log \log x, \\
& N(x) \ll x / \log \log x \ll x / k^{1 / 3} . \tag{14}
\end{align*}
$$

Thus, we have established Lemma 2, for $k$ satisfying (12). On the other hand, if $k \leq\{\log \log x / \log \log \log x\}^{1 / 3}$, then Lemma 2 follows from Lemma 1.

Proof of the Theorem: For $\eta>0$, define

$$
k(n, \eta)=\prod_{p \leq \eta, p \mid(n, \Omega(n))} p \quad \text { and } \quad N_{\eta}=\prod_{p \leq \eta} p .
$$

Then

$$
\begin{align*}
Q(x) & =\sum_{\substack{1 \leq n \leq x \\
k(n, n)=1}} 1+\theta^{\prime} \sum_{3 p>n, p \mid(n, \Omega(n))} 1  \tag{15}\\
& =S_{3}+S_{4}, \text { respectively, }
\end{align*}
$$

where $-1 \leq \theta^{\prime} \leq 0$. But

$$
\begin{align*}
S_{3}=\sum_{1 \leq n \leq x} \sum_{d \mid k(n, n)} \mu(d) & =\sum_{d \mid N_{n}} \mu(d) \sum_{\substack{1 \leq \leq n \leq x \\
d \mid(n, \bar{\Omega}(n))}} 1  \tag{16}\\
& =\sum_{d \mid N_{n}} \mu(d) \sum_{\substack{1 \leq m \leq x / d \\
\Omega(m) \equiv-\Omega(d)(\bmod d)}} 1
\end{align*} .
$$

In (15) we will choose $\eta$ such that the integers $d$ in (16) satisfy

$$
d \leq\left\{\left(\log \log \frac{x}{d}\right) / \log \log 1 \log \left(\frac{x}{d}\right)\right\}^{1 / 3} .
$$

The Prime Number Theorem (see [4, p. 9]) shows that

$$
\begin{equation*}
\eta=\frac{1}{4} \log \log \log x \tag{17}
\end{equation*}
$$

is a permissible choice.
With this choice of $\eta$ in (16), Lemma 1 shows that

$$
\begin{align*}
S_{3} & =\sum_{d \mid N_{n}} \mu(d)\left\{\frac{x}{d^{2}}+0\left(\frac{x}{d \log 1 \log x}\right)\right\}  \tag{18}\\
& =x \sum_{d \mid N_{n}} \frac{\mu(d)}{d^{2}}+0\left(\frac{x}{\log 1 \log x} \sum_{d \mid N_{n}} \frac{1}{d}\right)
\end{align*}
$$

First

$$
\begin{equation*}
\sum_{d \mid N_{n}} \frac{\mu(d)}{d^{2}}=\sum_{d \leq n} \frac{\mu(d)}{d^{2}}+0\left(\sum_{d \leq n} \frac{1}{d^{2}}\right)=\frac{6}{\pi^{2}}+0\left(\frac{1}{n}\right) . \tag{19}
\end{equation*}
$$

Also

$$
\begin{equation*}
\sum_{d \mid N_{n}} \frac{1}{d}=\prod_{p \leq n}\left(1+\frac{1}{p}\right) \ll \log \eta . \tag{20}
\end{equation*}
$$

From (17), (18), (19), and (20), we see that

$$
\begin{equation*}
S_{3}=\frac{6 x}{\pi^{2}}+0\left(\frac{x}{\eta}\right) \tag{21}
\end{equation*}
$$

To estimate $S_{4}$, we note that

$$
\begin{align*}
\left|S_{4}\right| \leq & \sum_{\substack{1 \leq n \leq x \\
3 p>n, p \leq(3 / 2) \log 1 \mathrm{og} x}} 1
\end{align*}+\theta^{\prime \prime} \sum_{\substack{1 \leq n \leq x  \tag{22}\\
p \mid(n, \Omega(n))}} 1
$$

where $0 \leq \theta^{\prime \prime} \leq 1$. Lemma 2 shows that

$$
\begin{equation*}
S_{5} \leq \sum_{n<p \leq(3 / 2) \log \log x} \sum_{\substack{m \leq x / p \\ \Omega(m) \equiv-1(\bmod p)}} 1 \ll \sum_{p>n} \frac{x}{p^{4 / 3}} . \tag{23}
\end{equation*}
$$

From the Prime Number Theorem and (23), we deduce that

$$
\begin{equation*}
S_{5} \ll \frac{x}{\eta^{1 / 3} \log \eta} \tag{24}
\end{equation*}
$$

With regard to $S_{6}$, note that

$$
\begin{equation*}
S_{6} \leq \quad \sum_{\substack{1 \leq n \leq x \\ \Omega(n)>(3 / 2) \log \log x}} 1 \quad \ll \frac{x}{\log \log x} \tag{25}
\end{equation*}
$$

by the use of (14).
Finally, by combining (15), (21), (22), (24), and (25), we arrive at

$$
\begin{equation*}
Q(x)=\frac{6 x}{\pi^{2}}+0\left(\frac{x}{\eta^{1 / 3} \log n}\right) . \tag{26}
\end{equation*}
$$

The theorem follows from (26) and (17).
Remarks: With a little more care, our theorem can be improved to

$$
Q(x)=\frac{6 x}{\pi^{2}}+0_{\varepsilon}\left(x(\log \log \log x)^{-1 / 2+\varepsilon}\right),
$$

where $\varepsilon>0$ is arbitrarily small.
If $n>0$ is a randomly chosen square-free integer, and $m$ a randomly chosen positive integer, then ( $n, m$ ) = 1 with probability

$$
c=\prod_{p}\left(1-\frac{1}{p^{2}+p}\right)
$$

By suitably modifying the proof of our theorem, we can show that if $n$ is square free, then $(n, \Omega(n))=1$ with probability $c$. Thus, $\Omega(n)$ behaves randomly with respect to $n$, even in the square-free case.

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PROPORTIONAL ALLOCATION IN INTEGERS

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The U.S. Constitution mandates that "Representatives shall be apportioned among the several states according to their respective numbers. . . . The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative." Implementation is left to Congress.

Controversy arose over the first reapportionment. Congress passed a bill based on a method supported by Alexander Hamilton. President George Washington used his first veto to quash this bill, and an apportionment using Thomas Jefferson's method of "greatest divisors" was adopted. This matter is still controversial. Analyses, reviews of the history, and proposed solutions are contained in the papers [3], [4], and [5] in the American Mathematical Monthly.

The purpose of this paper is to cast new light on various methods of proportional allocation in natural numbers by moving away from the application to reapportionment of the House of Representatives after a census and instead considering the application to division of delegate positions among presidential candidates based on a primary in some district.

1. THE MATHEMATICAL PROBLEM

Let $N=\{0,1,2, \ldots\}$ and let $W$ consist of all vectors $V=\left(v_{1}, \ldots, v_{n}\right)$ with components $v_{i}$ in $N$ and dimension $n \geq 2$. Let the size of such a $V$ be

$$
|v|=v_{1}+\cdots+v_{n} .
$$

An allocation method is a function $F$ from $N \times W$ into $W$ such that

$$
F(s, V)=S=\left(s_{1}, \ldots, s_{n}\right) \text { with }|S|=s
$$

We will sometimes also write $F(s, V)$ as $F\left(s ; v_{1}, \ldots, v_{n}\right) . S=F(s, V)$ should be the vector in $W$ with size $s$ and the same dimension as $V$ which in some sense is most nearly proportional to $V$.

A property common to all methods discussed below is the faimess property that

$$
\begin{equation*}
s_{i} \geq s_{j} \text { whenever } v_{i}>v_{j} \tag{1}
\end{equation*}
$$

Note that $s_{i}>s_{j}$ can occur with $v_{i}=v_{j}$ since the requirement that each $s_{i}$ be an integer may necessitate use of tie-breaking (e.g., when all $v_{i}$ are equal and $s / n$ is not an integer).

## 2. TYPES OF APPLICATIONS

For reapportionment of the U.S. House of Representatives, at present $n=50$, $s=435$, the $v_{i}$ are the populations of the states (say in the 1980 census), and $s_{i}$ is the number of seats in the House to be alloted by the method to the $i$ th state. Proportional allocation could also be used to divide congressional committee positions among the parties or to allot Faculty Senate positions to the various colleges of a university.

We want to get away from the relatively fixed nature of the dimension $n$ and the constitutional requirement that each $s_{i} \geq 1$ in the reapportionment of the House problem and therefore, in the main, will use language and examples appropriate for the application to presidential primaries.

## 3. TWO EXTREME METHODS

The "plurality takes all" method $P$ has

$$
P\left(s ; v_{1}, \ldots, v_{n}\right)=\left(s_{1}, \ldots, s_{n}\right)
$$

with $s_{k}=s$ if $v_{k}$ is the largest of the $v_{i}$ and $s_{i}=0$ for all other $i$. This method is used in elections in which $s=1$, e.g., elections for mayor or governor. It is also used in allocating the total electoral vote of a state based on the vote for president in general elections. This method is certainly not one of "proportional" allocation.

Perhaps at the other extreme is the "leveling" method

$$
L\left(s ; v_{1}, \ldots, v_{n}\right)=\left(s_{1}, \ldots, s_{n}\right)
$$

in which the $s_{i}$ are as nearly equal as possible. That is, if $s=q n+r$ with $q$ and $r$ integers such that $0 \leq r<n$, then $s_{i}=q+1$ for the $r$ values of $i$ with the largest components $v_{i}$ and $s_{i}=q$ for the other values of $i$. This is the method used to allocate the 100 seats in the U.S. Senate among the 50 states. It too is not a method of proportional allocation.

## 4. ONE PERSON, ONE EFFECTIVE VOTE

We find it helpful to preface our discussion of proportional allocation with the consideration of a proportional representation election to choose people for a city council, or a school board, or to represent the electorate in some other way. As a means of achieving proportional representation it is decided to give each voter only one vote; the $s$ candidates with the highest votes will be declared elected.

Each voter has a favorite candidate but a vote for the favorite may be a wasted vote because that candidate is so strong as not to need the vote in order to be elected, or is too weak to be in contention. If enough electors change their votes in fear of such wastage, the results may be a serious distortion of their wishes and may involve an even greater wasting of votes.

But there are methods which provide near optimum effectiveness for the total vote. They involve a preferential ballot on which each voter places the number 1 next to the voter's first choice, 2 next to the second choice, etc. Then, a very sophisticated system is used to transfer a vote when necessary to the highest indicated choice who has not yet been declared elected or been eliminated due to lack of support. Such a method is used to select members of the Nominating Committee of the American Mathematical Society (see [6]) and to elect members of the Irish Parliament (see [2]).

The following arithmetical question arises in such single vote, multiposition elections: If there are $v$ voters and $s$ positions to be filled, what is the
smallest integer $q$ such that $q$ votes counted for a candidate will guarantee election under all possibilities for the other ballots? Clearly, the answer is the smallest $q$ such that $(s+1) q>v$ or

$$
\begin{equation*}
q=[(v+1) /(s+1)] \text {, where }[x] \text { is the greatest integer in } x . \tag{2}
\end{equation*}
$$

If Americans in general were more educated politically and mathematically, such a method might be used to elect delegates to presidential nominating conventions. Then an elector could number choices based on whatever criteria were considered most important, such as the presidential candidate backed, major issues, or confidence in a specific candidate for delegate.

At best, our primaries allow electors to express one choice for president. What "one person, one vote" mechanism could we use to assign each vote to a candidate for delegate to make best use of the only information we have, that is, the number $v_{i}$ of votes for presidential candidate $C_{i}$ ? If the $v_{i}$ people voting for $C_{i}$ knew that they could maximize the number of delegates allocated to $C_{i}$ by dividing into $s_{i}$ equal-sized subsets with each subset voting for a different delegate candidate pledged to $C_{i}$, the only information we have indicates that they would do so. The result would be the allocation of the Jefferson method, which we discuss in the next section.

## 5. JEFFERSON'S GREATEST DIVISOR METHOD

In a given primary, let there be $n$ presidential candidates $C_{i}$, let $v_{i}$ be the number of votes for $C_{i}$, and let $s$ be the total number of delegate seats (in a given political party) at stake in that primary. Suppose that the presidential candidates have submitted disjoint lists of preferred candidates for delegate positions with the names on each list ranked in order of preference.

Now let us fix $i$ and consider each individual vote for $C_{i}$ as a single transferable vote which is to be assigned to one of the delegate candidates on the $C_{i}$ list, with the assignment process designed to maximize the number $s_{i}$ of people on this list winning delegate seats. Below we show inductively that the following algorithm performs this optimal assignment and determines all the $s_{i}$.

From the $n s$ ordered pairs ( $i, j$ ) with $1 \leq i \leq n$ and $1 \leq j \leq s$, choose the $s$ ordered pairs for which $v_{i} / j$ is largest. This may require a tie-breaking scheme (as is true of all allocation methods). Then the allocation $s_{a}$ to candidate $C_{a}$ is the number of ( $i, j$ ) with $i=a$ among these $s$ chosen pairs.

## 6. FIRST EXAMPLE

Here let $n=4$ and the votes for four presidential candidates $C_{i}$ in a given primary be given by the vector

$$
\left(v_{1}, v_{2}, v_{3}, v_{4}\right)=V=(3110,2630,2620,1640) .
$$

The necessary calculations and ordering for the Jefferson method $J$ of allocating a total of $s$ delegate positions among the four contending campaign organizations is shown for $1 \leq s \leq 22$ in the following table:

|  |  | $C_{1}$ |  | $C_{2}$ |  | $C_{3}$ |  | $C_{4}$ |
| :--- | ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v=$ vote received | $(1)$ | 3110 | $(2)$ | 2630 | $(3)$ | 2620 | $(4)$ | 1640 |
| $v / 2$ | $(5)$ | 1550 | $(6)$ | 1315 | $(7)$ | 1310 | $(11)$ | 820 |
| $v / 3$ | $(8)$ | $1036+$ | $(9)$ | $876+$ | $(10)$ | $873+$ | $(16)$ | $546+$ |
| $v / 4$ | $(12)$ | $777+$ | $(13)$ | $657+$ | $(14)$ | 655 |  | 410 |
| $v / 5$ | $(15)$ | 622 | $(17)$ | 526 | $(18)$ | 524 |  |  |
| $v / 6$ | $(19)$ | $518+$ | $(21)$ | $438+$ | $(22)$ | $436+$ |  |  |
| $v / 7$ | $(20)$ | $444+$ |  |  |  |  |  |  |
| $v / 8$ |  | $388+$ |  |  |  |  |  |  |

The position indicators in parentheses before various quotients $v_{i} / j$ show the positions of these quotients when all quotients for all candidates are merged together in decreasing order. The number $s_{i}$ of delegate seats to be alloted to presidential candidate $C_{i}$ is the number of position indicators from the range $1,2, \ldots, s$ which appear in the column for $C_{i}$. For examp1e, when $s=20$, the allotment to $C_{2}$ is $s_{2}=5$ since the five position numbers $2,6,9,13,17$ from the range $1,2, \ldots, 20$ appear prior to quotients $v_{2} / j$ in the $C_{2}$ column. The complete allocations for $s=20,21$, and 22 are

$$
\begin{aligned}
& J(20, V)=(7,5,5,3) \\
& J(21, V)=(7,6,5,3) \\
& J(22, V)=(7,6,6,3)
\end{aligned}
$$

This example helps us in discussing the rationale for the method $J$. Let the total number of delegate seats to be alloted be 21 . Think of 20 of the 21 spots as having already been alloted with the distribution

$$
J(20, V)=(7,5,5,3)
$$

and ask to whom the 21 st spot should be given. Clearly, $C_{3}$ is not entitled to a 6 th spot before $C_{2}$ obtains a 6 th spot. To see if $C_{1}$ should get an 8 th, or $C_{2}$ a 6 th, or $C_{4}$ a 4 th , one looks at the largest quotient among $v_{1} / 8, v_{2} / 6$, and $v_{4} / 4$. The 21st spot goes to $C_{2}$ on this basis. The result is the same as what would happen if we considered each vote for $C_{2}$ as a single transferable ballot which should be assigned so as to maximize the number of delegates pledged to $C_{2}$. Then the 2630 votes for $C_{2}$ could be assigned in six batches of at least 438 for delegate candidates pledged to $C_{2}$ and it would be impossible to assign the votes for the other presidential candidates in batches of at least 438 to more than 7 people pledged to $C_{1}, 5$ to $C_{3}$, and 3 to $C_{4}$.

Now, let us alter the above example by introducing new presidential candidates $C_{5}, C_{6}, \ldots, C_{n}$ (some of whom may be mythical write-in names) with votes $v_{5}, \ldots, v_{n}$. We keep the total number of spots at $s=20$, and note that the 20th largest $v_{i} / j$ among the original candidates $C_{1}, C_{2}, C_{3}, C_{4}$ is 444+. Hence the allocation will be ( $7,5,5,3,0,0, \ldots, 0$ ) unless some new $v_{i}$ is at least 445. Thus the method $J$ has a built-in mechanism for distinguishing "real candidates" from "ego-trippers" and recipients of small batches of write-in votes deliberately wasted as a form of protest.

## 7. HAMILTON'S ROUNDING METHOD

Let us continue to use the votes vector $v=(3110,2630,2620,1640)$ of our example above. Let the number $s$ of delegate positions available be 20. Hamilton's reasoning was similar to the following:

The "ideal" allocation of 20 positions in exact proportion to the $v_{i}$, but dropping the requirement of allocating in whole numbers, would be

$$
\begin{array}{cccc}
C_{1} & C_{2} & C_{3} & C_{4} \\
6.22 & 5.26 & 5.24 & 3.28
\end{array}
$$

If we have to change these to whole numbers, then clearly $C_{1}$ is entitled to at least $6, C_{2}$ and $C_{3}$ are entitled to at least 5, and $C_{4}$ to at least 3. That disposes of 19 of the 20 positions. Who should get the 20th? Hamilton's method $H$, also called the Vinton Method, would give it to $C_{4}$ on the ground that his "ideal" allotment has the largest fractional remainder. (In Europe, this method is called the "greatest remainders" method.)

We note that the Hamilton allotment $H(20, V)=(6,5,5,4)$, while the Jefferson allotment is $J(20, V)=(7,5,5,3)$. Before we decide on which is "more nearly proportional," let us use the same votes vector $V=(3110,2630,2620$,
1640) but change from $s=20$ to $s=22$. Then the "ideal" decimal allocation becomes

| $C_{1}$ | $C_{2}$ | $C_{3}$ | $C_{4}$ |
| :---: | :---: | :---: | :---: |
| 6.842 | 5.786 | 5.764 | 3.608 |

and the Hamilton allotment $H(22, V)$ is (7, 6, 6, 3). The surprise is that adding two new positions, while keeping the votes the same, results in $C_{4}$ losing one spot and each of the others gaining one. This phenomenon using the method $H$ is called the "Alabama Paradox" and was first noticed when the Census Office chief clerk, C. W. Seaton, showed that the apportionment under $H$ of seats to Alabama in the House after the 1880 census would decrease from 8 to 7 if the House size were increased from 299 to 300 (with the same population figures).

Balinski and Young [3, p. 705] quote Seaton, after discovering the paradox, as writing that "Such a result as this is to me conclusive proof that the process employed in obtaining it is defective. . . . [The] result of my study of this question is the strong conviction that an entirely different process should be employed" and also quote [3, p. 704] Representative John C. Ball of Colorado as saying that "This atrocity which [mathematicians] have elected to call a 'paradox' . . . this freak [which] presents a mathematical impossibility."

Since Seaton's observation, Hamilton's method has not been used for reapportionment of the House. However, it is perhaps the most widely used method in elections. The 1980 delegate selection rules of one of our major political parties required that "this atrocity" be used.

Since the Jefferson method $J$ allocates spots one by one as $s$ increases, it is trivial to show that the Alabama Paradox cannot occur under J. (No parenthetic position number is erased when $s$ increases by one.)

## 8. OTHER QUOTA METHODS

The discovery of the Alabama Paradox inspired a number of mathematicians to seek quota methods which are "house monotone," that is, quota methods which do not allow this particular type of paradox. These variations on $H$ maintain the insistence that the "ideal" decimal allotments can be changed only through rounding up or down but they use other criteria than size of the decimal remainder to decide on which way to round. Such "quota" methods are described and justified in [3], [4], and [5]. One should note that these papers deal only with the application to reapportionment of the House. For this application, the mathematicians of the National Academy of Sciences are available and could use sophisticated mathematics such as that of [5].

Our contention is that, at least in applications to primaries, all quota methods exhibit other anomalies, and that the criticisms of Jefferson's method are not very relevant. For additional ammunition to bolster these assertions, we consider new examples.

## 9. A NEW PARADOX

For the remaining examples, we fix the number of delegate spots at $s=20$ and vary the number $n$ of presidential candidates. Using the same votes vector $V=(3110,2630,2620,1640)$ as above, one finds that a sophisticated quota method $Q$, such as those in [3], [4], and [5], has in effect been forced to agree with the allotment (7, 5, 5, 3) of the Jefferson method to avoid the Alabama Paradox. Now we introduce five new presidential candidates $C_{5}, \ldots, C_{9}$ with $C_{9}$ a write-in candidate (my favorite is Kermit the Frog). Let the new votes vector be $V=(3110,2630,2620,1640,99,97,86,84,1)$.

Under the Jefferson method, the 20th quotient remains at $444+$. Hence those $v_{i}$ which are less than 445 do not influence the results and the allocation is
(7, $5,5,3,0,0,0,0,0)$. However, no quota method $Q$ can give the same result. The reason is that the one vote "wasted" on $C_{9}$ has reduced the "ideal" decimal allotment for $C_{1}$ below 6 and, thus, all quota (i.e., rounding) methods bar $C_{1}$ from having more than 6 delegates. This means that, under $Q$, one of the 367 people who voted for $C_{5}, \ldots, C_{9}$ took a delegate spot away from $C_{1}$ and gave it to $C_{2}$ or $C_{4}$. In this example, the write-in for $C_{9}$ is the vote that forced this anomaly.

The present author feels that such an effect is also "an atrocity" and is still paradoxical. Jefferson's method avoids this anomaly since under $J$ a vote can take a spot away from $C_{i}$ only by adding a spot for the candidate $C_{j}$ for whom the vote was cast.

Altering $Q$, as long as it remains a quota method, can only make us change our example. No quota method is immune to this anomaly.

> 10. INTERNAL CONSISTENCY

Let $F$ be an allocation method, $V=\left(v_{1}, \ldots, v_{n}\right)$, and

$$
S=F(s, V)=\left(s_{1}, \ldots, s_{n}\right)
$$

Let $A$ be any proper subset of $\{1,2, \ldots, n\}, s^{\prime}$ be the sum of the $s_{i}$ for $i$ in $A$, and $V^{\prime}$ and $S^{\prime}$ be the vectors resulting from the deletions of the components $v_{j}$ of $V$ and $s_{j}$ of $S$, respectively, for all $j$ not in $A$. If under all such situations we have $F\left(s^{\prime}, V^{\prime}\right)=S^{\prime}$, we say that the method $F$ is internally consistent. The discussion in the previous section indicates why no quota method can be internally consistent.

The Jefferson method $J$ is easily seen to be internally consistent. So is the Huntington "Method of Equal Proportions," which is the one used in recent reapportionments of the House. This method $E$ is the variation on $J$ in which the quotients $v_{i} / j$ are replaced by the functions $v_{i} / \sqrt{j}(j-1)$. Note that this function is infinite for $j=1$ and is finite for $j>1$. Hence in the application to apportionment of the House, one could interpret $E$ as requiring that each state must be given one seat in the House before any state can receive two seats. Since this is required by the U.S. Constitution, $E$ is a method that has this mandated bias toward states with very small populations and gradually decreases this bias as the population grows. References [3], [4], and [5] take the position that an acceptable apportionment method must be a quota method; they therefore reject $J$ and $E$ and all methods which we call internally consistent. Neither of these references mentions the fact that $E$ "naturally" satisfies the constitutional requirement that each state must have at least one Representative. Despite this naturalness in using $E$ for apportionment of the House, it seems to be an absurdity to use $E$ in a presidential preference primary since single write-ins for enough names to make $n \geq s$ would force all allocations $s_{i}$ to be in $\{0,1\}$.

Balinski and Young [3, p. 709] ask: "Why choose one stability criterion rather than another? Why one rank-index than another? Why one divisor criterion than another." Later on the same page they quote a Feb. 7, 1929, report of the National Academy of Sciences "signed by lions of the mathematical community, G. A. Bliss, E. W. Brown, L. P. Eisenhart, and Raymond Pear1" as containing the statement that "Mathematically there is no reason for choosing between them." The word "them" refers to a number of methods which are internally consistent.

In the application to presidential primaries, one reason for choosing $J$ over other methods if that $J$ achieves the same results as the "single transferable ballot" method if one considers each vote for a presidential candidate $C_{i}$ to be a ballot marked with perfect strategy solely for delegate candidates pledged to $C$.

## 11. DIVIDE AND CONQUER

The Hamilton method (and other quota methods) may allow a group to round an "ideal" allotment of 4.2 into 7 by the group breaking up into seven equal subgroups, each with an "ideal" allotment of 0.6 . Thus, quota methods can reward fragmentation and seem especially inappropriate in selecting just one person to lead a political party (and perhaps the nation). Under Jefferson's method, no group can gain by dividing into subsets and no collection of groups can lose by uniting into one larger group.

When $J$ was originally proposed for reapportionment, it was criticized for not being biased toward small states. The criticism by mathematicians, such as in [3], [4], and [5], is that it is not a quota (i.e., rounding) method.
12. UNDERLYING CAUSES OF ANOMALIES

Why does the Hamilton method allow the "Alabama Paradox" and why are the other, more sophisticated quota methods subject to regarding a vote for $Z$ as a vote for $Y$ and/or a vote against $X$ ? Basically, the trouble with all quota methods is that they mix the multiplicative operation with addition and subtraction. For example, they allow 0.1 to be rounded up to 1 but do not allow 8.99 to be "rounded" to 10 . Thus, they allow the actual allotment to be ten times the "ideal" for one candidate while not allowing it to be 1.2 times the "ideal" for another. The characteristic feature of quota methods is the insistence that there be no integer strictly between the actual and the "ideal" allotment. Thus, there is a bound of 1 on this difference, although there is no bound on the corresponding ratio. A method that claims to give "most proportional" results should give more importance to the ratio than to the difference.

This author also feels that quota methods (for primaries) are wrong in insisting that $v_{i}$ be at least $|V| / s$ to guarantee at least one delegate for $C_{i}$. The discussion of "single transferable ballot" methods (Section 4 above) indicates that this should be $[(|V|+1) /(s+1)]$ instead of $|V| / s$. Also, quota methods ignore the fact that many votes in a primary may unavoidably be just wasted votes. Using these wasted votes to determine "ideal" allotments allows a vote cast for $Z$ to have the effect of a vote against $X$ and/or a vote for $Y$. A minimal step in the right direction would be to delete the $v_{i}$ for candidates who receive zero allocations from the total vote size $|V|$ in determining the "ideal" allotment. (This might entail iteration of some process.)

## 13. THE REAL LIFE EXPERIENCE

The 1980 delegate selection rules of one of our major parties for the national presidential nominating convention required that the "paradoxical atrocity" $H$ be used. However, the paradoxes illustrated above could not occur because these rules also stated that candidates who received less than 15 percent of the vote in some primary were not eligible for delegate allocations. In reaction to the "plurality takes all" procedures of previous years, these rules also said that no candidate who received less than 90 percent of the total vote could be alloted all the delegate positions at stake in a given primary. If there were three candidates and their percentages of the total vote were 75, 14, and 11 percent, then any allocation under this patched up version of $H$ would contradict some provision of these rules. So patches were added onto the patches described above. Contradictions were being discovered and patches added until all the delegates were selected and the issue became moot.

The Jefferson method is much simpler to use and would have achieved more or less the same overall result．At least one state recognizes the Jefferson method in its presidential primary act．

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## ※れれまれ

## IRRATIONAL SEQUENCE－GENERATED FACTORS OF INTEGERS

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Dedicated in respectful and affectionate remembrance to the memory of our good friend，Vern Hoggatt，a man and a mathematician of high quality．

## 1．INTRODUCTION

In Horadam，Loh，and Shannon［5］，a generalized Fibonacci－type sequence $\left\{A_{n}(x)\right\}$ was defined by

$$
\left\{\begin{array}{l}
A_{0}(x)=0, A_{1}(x)=1, A_{2}(x)=1, A_{3}(x)=x+1, \text { and }  \tag{1.1}\\
A_{n}(x)=x A_{n-2}(x)-A_{n-4}(x)
\end{array} \quad(n \geq 4) .\right.
$$

The notion of a proper divisor was there extended as follows：
Definition：For any sequence $\left\{U_{n}\right\}, n \geq 1$ ，where $U_{n} \varepsilon \mathbb{Z}$ or $U_{n}(x) \varepsilon \mathbb{Z}(x)$ ，the proper divisor $w_{n}$ is the quantity implicitly defined，for $n \geq 1$ ，by $w_{1}=U_{1}$ and $w_{n}=\max \left\{d: d \mid U_{n}, \operatorname{g.c.d} .\left(d, w_{m}\right)=1\right.$ for every $\left.m<n\right\}$ ．

It was then shown that
and

$$
\begin{equation*}
A_{n}(x)=\prod_{d \mid n} w_{d}(x) \tag{1.2}
\end{equation*}
$$

$$
\begin{equation*}
w_{n}(x)=\prod_{d \mid n}\left(A_{d}(x)\right)^{\mu(n / d)} \tag{1.3}
\end{equation*}
$$

where $\mu(n / d)$ are Möbius functions．

Elsewhere [8], Shannon, Horadam, and Loh have proved (with $n$ replaced by $2 n$ ) that

$$
\begin{equation*}
A_{4 n}(x)=\sum_{j=0}^{\left[n-\frac{1}{2}\right]}(-1)^{j}\binom{2 n-j-1}{j} x^{2 n-2 j-1} \tag{1.4}
\end{equation*}
$$

The background to this paper is that the authors were shown (Wilson [9], [10]) several numerical results relating to the sets of numbers in Table 2, and asked to establish a theoretical basis for these results. In the process, some useful further properties of (1.4) were developed.

A particular aim of this investigation is to use the generalized Fibonaccitype sequence to show that any integer $n>0$ can be expressed as the product of (mostly) irrational numbers in an infinite number of ways according to a specific pattern.

Besides expressing our appreciation of the stimulation provided by Wilson ([9], [10]), we wish to register our thanks to A. Hartman and R. B. Eggleton [4] for their valuable comments, and to Professor G. E. Andrews, University of Pennsylvania, for the Hancock reference [3].

## 2. FACTORS, PROPER DIVISORS, AND TRIGONOMETRY

From (1.4) we observe that
so that

$$
\begin{equation*}
\operatorname{deg} \cdot\left(\frac{A_{4 n}(x)}{x}\right)=2 n-2 \tag{2.1}
\end{equation*}
$$

has $n-1$ squares of roots

$$
\alpha_{1}^{2}, \alpha_{2}^{2}, \ldots, \alpha_{n-1}^{2}
$$

For notational convenience write

$$
\begin{equation*}
\beta_{i}=\alpha_{i}^{2} \quad i=1,2, \ldots, n-1 \tag{2.3}
\end{equation*}
$$

Since the constant term in (2.2) is $(-1)^{n-1} n$, we have, from the theory of equations, that

$$
\begin{equation*}
n=\prod_{i=1}^{n-1} \beta_{i} \tag{2.4}
\end{equation*}
$$

and also, with $j=1$ in the left-hand side of (2.2) that

$$
\begin{equation*}
2 n-2=\sum_{i=1}^{n-1} \beta_{i} . \tag{2.5}
\end{equation*}
$$

Thus, to find the factors of any integer $n$, we seek the $n-1 \beta_{i}$ of (2.2), which by (1.2) can be obtained from the proper divisors of $A_{4 n}(x) / x$. The first few of the $A_{4 n}(x) / x$ are listed in Table 1 along with their factors and proper divisors.

For example, from Table 1, [5], and (1.2), $A_{20}(x)$ has as its factors

$$
\begin{aligned}
& w_{20}(x)=x^{4}-5 x^{2}+5, w_{10}(x)=x^{2}-x-1, w_{5}(x)=x^{2}+x-1, \\
& w_{4}(x)=1, w_{2}(x)=1, \text { and } w_{1}(x)=1
\end{aligned}
$$

trivially.
In the search for proper divisors, the (provable) result

$$
\text { deg. } w_{n}(x)=\frac{1}{2} \phi(n)
$$

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TABLE 1. Factors and Proper Divisors of $A_{4 n}(x) / x$ for $n=2,3, \ldots, 12$

| $n$ | $A_{4 n}(x) / x$ | $w_{4 n}(x)$ | Other factor (s) |
| :---: | :---: | :---: | :---: |
| 2 | $x^{2}-2$ | $x^{2}-2$ | $A_{4}(x) / x$ |
| 3 | $x^{4}-4 x^{2}+3$ | $x^{2}-3$ | $A_{6}(x)$ |
| 4 | $x^{6}-6 x^{4}+10 x^{2}-4$ | $x^{4}-4 x^{2}+2$ | $A_{8}(x) / x$ |
| 5 | $x^{8}-8 x^{6}+21 x^{4}-20 x^{2}+5$ | $x^{4}-5 x^{2}+5$ | $A_{10}(x)$ |
| 6 | $x^{10}-10 x^{8}+36 x^{6}-56 x^{4}+35 x^{2}-6$ | $x^{4}-4 x^{2}+1$ | $w_{8}(x) \cdot A_{12}(x) / x$ |
| 7 | $\begin{aligned} x^{12} & -12 x^{10}+55 x^{8}-70 x^{6}+126 x^{4} \\ & -56 x^{2}+7 \end{aligned}$ | $x^{6}-7 x^{4}+14 x^{2}-7$ | $A_{14}(x)$ |
| 8 | $\begin{aligned} x^{14} & -14 x^{12}+78 x^{10}-220 x^{8} \\ & +330 x^{6}-252 x^{4}+84 x^{2}-8 \end{aligned}$ | $\begin{aligned} x^{8} & -8 x^{6}+20 x^{4} \\ & -16 x^{2}+2 \end{aligned}$ | $A_{16}(x) / x$ |
| 9 | $\begin{aligned} x^{16} & -16 x^{14}+105 x^{12}-364 x^{10} \\ & +715 x^{8}-792 x^{6}+462 x^{4} \\ & -120 x^{2}+9 \end{aligned}$ | $x^{6}-6 x^{4}+9 x^{2}-3$ | $w_{12}(x) \cdot A_{18}(x)$ |
| 10 | $\begin{aligned} x^{18} & -18 x^{16}+136 x^{14}-560 x^{12} \\ & +1365 x^{10}-2002 x^{8}+1716 x^{6} \\ & -792 x^{4}+165 x^{2}-10 \end{aligned}$ | $\begin{aligned} x^{8} & -8 x^{6}+19 x^{4} \\ & -12 x^{2}+1 \end{aligned}$ | $w_{8}(x) \cdot A_{20}(x) / x$ |
| 11 | $\begin{aligned} x^{20} & -20 x^{18}+171 x^{16}-816 x^{14} \\ & +2380 x^{12}-4368 x^{10}+5005 x^{8} \\ & -3432 x^{6}+1287 x^{4}-220 x^{2}+11 \end{aligned}$ | $\begin{aligned} x^{10} & -11 x^{8}+44 x^{6} \\ & -77 x^{4}+55 x^{2} \\ & -11 \end{aligned}$ | $A_{22}(x)$ |
| 12 | $\begin{aligned} x^{22} & -22 x^{20}+210 x^{18}-1140 x^{16} \\ & +3876 x^{14}-8568 x^{12} \\ & +12376 x^{10}-11440 x^{8}+6435 x^{6} \\ & -2002 x^{4}+286 x^{2}-12 \end{aligned}$ | $\begin{aligned} x^{8} & -8 x^{6}+20 x^{4} \\ & -16 x^{2}+1 \end{aligned}$ | $w_{16}(x) \cdot A_{24}(x) / x$ |

```
TABLE 2. List of Factors for }n=2,3,\ldots., 14 (9 decimal places
    from Wilson [9]
```


where $\phi(n)$ is Euler's $\phi$-function, is useful. E.g., deg. $w_{20}(x)=4=\frac{1}{2} \phi(20)$.
From (1.2) and (2.3),

$$
\begin{aligned}
\frac{A_{4 n}(x)}{x} & =\prod_{d \mid 4 n} w_{d}(x) \quad n \geq 2, \text { since } w_{4}(x)=x \\
& =\prod_{j=1}^{n-1}\left(x^{2}-\beta_{j}\right)
\end{aligned}
$$

whence

$$
\begin{align*}
\left.\frac{A_{4 n}(x)}{x}\right|_{x=0} & =\prod_{d \mid 4 n} w_{d}(0) & & n \geq 2 \\
& =(-1)^{n-1} \prod_{j=1}^{n-1} \beta_{j} & & \text { from (2.6) }  \tag{2.7}\\
& =(-1)^{n-1} n & & \text { from (2.4). }
\end{align*}
$$

Consider, as an example, the case $n=5$, i.e.,

$$
5=\left.\frac{A_{20}(x)}{x}\right|_{x=0}=\prod_{j=1}^{4} \beta_{j},
$$

from (2.7). Then the factors of 5 are given by the $\beta_{i}$ of

$$
\begin{aligned}
x^{4}-5 x^{2}+5 & =\left(x^{2}-\frac{1}{2}(5+\sqrt{5})\right)\left(x^{2}-\frac{1}{2}(5-\sqrt{5})\right)=w_{20}(x) \\
& =\left(x^{2}-3.618033989\right)\left(x^{2}-1.381966010\right) \\
& =\left(x^{2}-\beta_{1}\right)\left(x^{2}-\beta_{3}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(x^{2}-x-1\right)\left(x^{2}+x-1\right) & =x^{4}-3 x^{2}+1=w_{10}(x) \omega_{5}(x) \\
& =\left(x^{2}-\frac{1}{2}(3+\sqrt{5})\right)\left(x^{2}-\frac{1}{2}(3-\sqrt{5})\right) \\
& =\left(x^{2}-2.618033989\right)\left(x^{2}-0.381966010\right) \\
& =\left(x^{2}-\beta_{2}\right)\left(x^{2}-\beta_{4}\right),
\end{aligned}
$$

that is,

$$
5=\beta_{1} \beta_{2} \beta_{3} \beta_{4}
$$

where the subscript labelling of the irrational $\beta^{\prime}$ s has been chosen to correspond to the decreasing order of magnitude given by Wilson [9], and where numerical calculations have been computed by pocket calculator to nine decimal places.

Our $\beta_{i}$ have a simple trigonometrical expression. From [8] and (1.4),

$$
\begin{equation*}
A_{2 n}(2 x)=U_{n-1}(x) \quad n \geq 2, U_{0}=1 \tag{2.8}
\end{equation*}
$$

where $U_{n}(x)$ is the Chebyshev polynomial of the second kind (Magnus, Oberhettinger, and Soni [7]). That is,

$$
\begin{equation*}
\frac{A_{4 n}(2 x)}{x}=\frac{U_{2 n-1}(x)}{x} \quad n \geq 1 . \tag{2.9}
\end{equation*}
$$

Solving

$$
\begin{equation*}
U_{n}(\cos \theta)=\frac{\sin (n+1) \theta}{\sin \theta}=0 \tag{2.10}
\end{equation*}
$$

for $\theta$ gives

$$
\theta=\frac{k \pi}{n+1} \quad(k=0,1,2, \ldots, n)
$$

Therefore, the $n-1 \beta_{i}$ of (2.2) are simply

$$
\begin{equation*}
\beta_{i}=4 \cos ^{2} \frac{i \pi}{2 n} \quad(i=1,2, \ldots, n-1) \tag{2.11}
\end{equation*}
$$

Of course, (2.4) with the $\beta_{i}$ given by (2.11), is a known result (see, e.g., Durell and Robson [1]).

In the example following (2.7), where $n=5$, we have

$$
\beta_{1}=4 \cos ^{2} \frac{\pi}{10}, \beta_{2}=4 \cos ^{2} \frac{\pi}{5}, \beta_{3}=4 \cos ^{2} \frac{3 \pi}{10}, \beta_{4}=4 \cos ^{2} \frac{2 \pi}{5}
$$

Wilson's $a, b, c, \ldots$ in Table 2 are $\beta_{1}, \beta_{2}, \beta_{3}, \ldots$.
Clearly, from (2.11),

$$
\begin{equation*}
\beta_{i}+\beta_{n-i}=4 \tag{2.12}
\end{equation*}
$$

Polynomials $A_{2 n-1}(x)$ satisfy the identity previously established in [5], namely,

$$
\begin{equation*}
A_{2 n+1}(x)=A_{2 n+2}(x)+A_{2 n}(x) \tag{2.13}
\end{equation*}
$$

so the polynomials $A_{n}(x)$ for $n$ odd are the sum of two consecutive Chebyshev polynomials.

Moreover,

$$
\begin{equation*}
A_{2 n+1}(x)=\bar{f}_{n}(x) \tag{2.14}
\end{equation*}
$$

in the notation of Hancock [3], about which further comments will be made later.
3. GENERATION OF IRRATIONAL FACTORS OF INTEGERS

One of our main results is Theorem 1 (below) relating to the system of equations satisfied by the $\beta_{i}\left(=\alpha_{i}^{2}\right)$.
Lemma 1:
(3.1)
in which

$$
n \delta(2, n)=\sum_{j=0}^{n-2}(-1)^{j}\binom{2 n-j-1}{j} 2^{2 n-2 j-3}
$$

$$
\delta(2, n)= \begin{cases}1 & \text { if } 2 \mid n  \tag{3.2}\\ 0 & \text { if } 2 \nmid n\end{cases}
$$

Proof: Equation (1.72) of Gould [2] states that

$$
\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^{k}\binom{n-k}{k} 2^{n-2 k}=n+1
$$

Algebraic manipulation of this equation yields
so

$$
2 n=\sum_{j=0}^{n-1}(-1)^{j}(2 n-j-1) 2^{2 n-2 j-1}
$$

that is,

$$
\begin{aligned}
n & =\sum_{j=0}^{n-1}(-1)^{j}\binom{2 n-j-1}{j} 2^{2 n-2 j-2} \\
& =(-1)^{n-1} n+\sum_{j=0}^{n-2}(-1)^{j}\binom{2 n-j-1}{j} 2^{2 n-2 j-2}
\end{aligned}
$$

$$
n+(-1)^{n} n=\sum_{j=0}^{n-2}(-1)^{j}(2 n-j-1) 2^{2 n-2 j-2}
$$

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whence

$$
\begin{aligned}
& n(2, n)=\sum_{j=0}^{n-2}(-1)^{j}\binom{2 n-j-1}{j} 2 \\
= & 2 n \delta(2, n) .
\end{aligned}
$$

since $n+(-1)^{n} n=2 n \delta(2, n)$.
Theorem 1: The $n-1 \alpha^{2}$ of $\frac{A_{4 n}(x)}{x}=0$ satisfy the system of equations

$$
\left\{\begin{array}{c}
\alpha_{1}^{2}-\alpha_{2}^{2}+\cdots+(-1) \alpha_{n-1}^{2}=2  \tag{3.3}\\
\alpha_{1}^{4}-\alpha_{2}^{4}+\cdots+(-1) \alpha_{n-1}^{4}=2^{3} \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\alpha_{1}^{2 n-2}-\alpha_{2}^{2 n-2}+\cdots+(-1)^{n} \alpha_{n-1}^{2 n-2}=2^{2 n-3} .
\end{array}\right.
$$

Proof: To solve (2.2), consider the $n-1 \alpha_{i}^{2}(i=1,2, \ldots, n-1)$. Then $0=\sum_{i=1}^{n-1}(-1)^{i-1} A_{4 n}\left(\alpha_{i}\right) / \alpha_{i}$
$=\sum_{i=1}^{n-1} \sum_{j=0}^{n-1}(-1)^{i+j-1}\binom{2 n-j-1}{j} \alpha_{i}^{2 n-2 j-2} \quad$ from (1.4)
$=\sum_{j=0}^{n-1} \sum_{i=0}^{n-1}(-1)^{i+j-1}(2 n-j-1) \alpha_{j}^{2 n-2(j+1)}$
$=\sum_{j=0}^{n-2} \sum_{i=1}^{n-1}(-1)^{i+j-1}\binom{2 n-j-1}{j} \alpha_{i}^{2 n-2(j+1)}+\sum_{i=1}^{n-1}(-1)^{i+n-2}\binom{n}{n-1} \alpha_{i}^{0}$
$=\sum_{j=0}^{n-2}(-1)^{j}(2 n-j-1) \sum_{j=1}^{n-1}(-1)^{i-1} \alpha_{i}^{2 n-2(j+1)}-n \delta(2, n) \quad$ by (3.2)
$=\sum_{j=0}^{n-2}(-1)^{j}(2 n-j-1) \sum_{j=1}^{n-1}(-1)^{i-1} \alpha_{i}^{2 n-2(j+1)}-\sum_{j=0}^{n-2}(-1)^{j}\binom{2 n-j-1}{j} 2^{2 n-2 j-3}$
by Lemma 1
from which it follows that, with a slight variation in the set of values of $j$,

$$
\sum_{i=1}^{n-1}(-1)^{i-1} \alpha_{i}^{2 n-2 j}=2^{2 n-2 j-1} \quad j=1,2, \ldots, n-1,
$$

which is the system of equations (3.3).
Illustration of Theorem 1: Theorem 1 tells us that there are $2 \beta_{i}$ of

$$
\frac{A_{12}(x)}{x}=x^{4}-4 x^{2}+3=0
$$

which satisfy (3.3) when $n=3$, i.e., $\beta_{1}-\beta_{2}=2, \beta_{1}^{2}-\beta_{2}^{2}=2^{3}$, namely,

$$
\begin{equation*}
\beta_{1}=3=4 \cos ^{2} \frac{\pi}{6}, \beta_{2}=1=4 \cos ^{2} \frac{\pi}{3} \tag{3.4}
\end{equation*}
$$

Also, there are $5 \beta_{i}$ of $\frac{A_{24}(x)}{x}=0$ which satisfy (3.3) when $n=6$, i.e.,

$$
\sum_{i=1}^{5}(-1)^{i-1} \beta_{i}^{j}=2^{2 j-1} \quad j=1,2, \ldots, 5
$$

namely,

$$
\left\{\begin{array}{l}
\beta_{1}=3.732050807=2+\sqrt{3}=4 \cos ^{2} \frac{\pi}{12}, \beta_{2}=3.000000000,  \tag{3.5}\\
\beta_{3}=2.000000000, \beta_{4}=1.000000000, \beta_{5}=0.267949192=2-\sqrt{3},
\end{array}\right.
$$

as can be seen in the entry for $n=6$ in Table 2 .
i.e. Similarly, there are $8 \beta_{i}$ of $\frac{A_{36}(x)}{x}=0$ which satisfy (3.3) when $n=9$,

$$
\sum_{i=1}^{8}(-1)^{i-1} \beta_{i}^{j}=2^{2 j-1} \quad j=1,2, \ldots, 8
$$

namely,

$$
\left\{\begin{array}{l}
\beta_{1}=3.8793385241=4 \cos ^{2} \frac{\pi}{18}, \beta_{2}=3.532088884, \beta_{3}=3.000000000,  \tag{3.6}\\
\beta_{4}=2.347296348, \beta_{5}=1.652703651, \beta_{6}=1.000000000, \\
\beta_{7}=0.467911115, \beta_{8}=0.120614758,
\end{array}\right.
$$

as can be seen in the entry for $n=9$ in Table 2 .
From (3.4), (3.5), and (3.6), we observe that

$$
\begin{aligned}
3 & =\beta_{1} \beta_{2} & & n=3 \\
& =\beta_{1} \beta_{2} \beta_{4} \beta_{5} & & n=6 \\
& =\beta_{1} \beta_{2} \beta_{4} \beta_{5} \beta_{7} \beta_{8} & & n=9
\end{aligned}
$$

(and so on). Notice that every $\beta_{i}$, for which $3 \mid i$, does not occur in the products. This is the gist of (3.9). (Other combinations are possible, e.g.,

$$
\begin{aligned}
3 & =\beta_{2} \beta_{4} & & n=6 \\
& =\beta_{3} \beta_{6} & & n=9 \\
& =\beta_{4} \beta_{8} & & n=12
\end{aligned}
$$

and so on.)
Elementary trigonometry with (2.6) and (2.11) may be used to show that

$$
\begin{equation*}
A_{4 n}(x) / x+(-1)^{n}\left\{A_{4 n}\left(\left(4-x^{2}\right)^{\frac{1}{2}}\right)\right\} / x=0 \tag{3.7}
\end{equation*}
$$

where, by the second term in (3.7) is meant the expression for $A_{4 n}(x) / x$ when $x^{2}$ is replaced by $4-x^{2}$.

If (3.7) is treated from a combinatorial number theory point of view, we have, on using (1.4) and the binomial expansion for $\left(4-x^{2}\right)^{n-1-j}$ and then considering the coefficient of $x^{2 n-2-2 p}$, the result

$$
\begin{equation*}
\sum_{j=0}^{n-1}(-1)^{j} 2^{2(p-j)}\binom{2 n-1-j}{j}\binom{n-1-j}{p-j}=\binom{2 n-1-p}{p} \tag{3.8}
\end{equation*}
$$

for every $p \leq n-1$.
This identity is very similar to result (3.44) in Gould [2].
The next (known) result is important for Table 2:

$$
\begin{equation*}
\prod_{\substack{i=1 \\ r \nless i}}^{n-1} \beta_{i}=r \quad r \mid n, \beta_{j+1}<\beta_{j}(j=1, \ldots, n-2) . \tag{3.9}
\end{equation*}
$$

To prove (3.9), divide (2.4) by $\prod_{i=1}^{k-1} \beta_{r_{i}}=\prod_{i=1}^{k-1} \beta_{i}^{*}=k$ where $\beta_{i}^{*}=4 \cos ^{2} \frac{i \pi}{2 k}$ and
$n=r k$, i.e., $r \mid n$, i.e., $r \mid 2 n$. E.g., $n=8$ in Table 2 gives

$$
2=\prod_{\substack{i=1 \\ 2 \nless i}}^{7} \beta_{i}=\beta_{1} \beta_{3} \beta_{5} \beta_{7} \quad \text { with } 2=\beta_{2} \beta_{6}=\beta_{4}
$$

Refer also to the Illustration of Theorem 1 on page 244 above.
From (3.9), and, earlier, (3.4), (3.5), and (3.6), it is clear that the sequence (1.1) shows how any integer ( $>0$ ) may be expressed as a product of (mostly) irrational numbers in an infinite number of ways, in accordance with a pattern of generation.

## 4. MISCELLANEOUS RESULTS

Results (4.1)-(4.5), which are stated without proof, may be derived from (1.2) and (2.11).

$$
\frac{A_{4 n}(x)}{x}= \begin{cases}A_{2 n}(x) \cdot B_{4 n}(x) & n \text { odd }  \tag{4.1}\\ \frac{A_{2 n}(x)}{x} \cdot B_{4 n}(x) & n \text { even }\end{cases}
$$

where $B_{4 n}(x)=w_{4 n}(x) \times$ (some product of proper divisors depending on the factors of $n$ ).

Some particular instances of (4.1) are shown in Table 1.
Consider again the transformation $x^{2} \rightarrow 4-x^{2}$. This has the following effects:
$n$ odd
(4.2) $\quad \beta_{2 i} \rightarrow \beta_{2 i-1} \quad$ in reverse order (and conversely), so

$$
\begin{equation*}
A_{2 n}(x) \leftrightarrow B_{4 n}(x) \tag{4.3}
\end{equation*}
$$

n even

$$
\begin{align*}
& \frac{A_{2 n}(x)}{x} \leftrightarrow \frac{A_{2 n}(x)}{x}  \tag{4.4}\\
& B_{4 n}(x) \leftrightarrow B_{4 n}(x) . \tag{4.5}
\end{align*}
$$

Previously, in (2.14), we mentioned the connection between our $A_{2 n+1}(x)$ and $\bar{f}_{n}(x)$ in Hancock [3]. It is instructive to compare in detail our treatment, where the motivation originated from combinatorial and number theoretic considerations, with Hancock's approach to somewhat similar material through cyclotomy and trigonometry.

However, to conserve space, we merely indicate without justification some comparisons of interest as well as some fresh properties of $A_{n}(x)$. Familiarity with Hancock's notation is assumed.

Observe, firstly that our

$$
A_{2 n}(x), A_{4 n+2}(x), B_{4(2 n+1)}(x), \text { and } x B_{4(2 n+1)}(x)+2
$$

are, respectively, Hancock's

$$
A_{n-1}(x), \psi_{2 n}(x), \Phi_{2 n}(x), \text { and } F_{2 n+1}(x)
$$

Further, we note that

$$
\left\{\begin{align*}
A_{2 n+2}(x)-A_{2 n}(x) & =f_{n}(x)=(-1)^{n} \bar{f}_{n}(-x)  \tag{4.6}\\
A_{2 n}(x) & =\frac{1}{2}\left(f_{n-1}(x)+\bar{f}_{n-1}(x)\right) \\
\sum_{k=1}^{n} f_{k}(x) & =A_{2 n+2}(x)-1
\end{align*}\right\}
$$

while some fresh results are

$$
\left\{\begin{align*}
A_{4 n+2}(x) & =A_{2 n+2}^{2}(x)-A_{2 n}^{2}(x)  \tag{4.7}\\
A_{2 n}(2) & =n \\
A_{2 n+1}(2) & =n+1
\end{align*}\right.
$$

## 5. CONCLUDING COMMENTS

Newton's iteration can be used to solve the system of equations (3.3). A1ternatively, the problem may be approached through the theory of recurring sequences.

Using the notation of Jarden [6], we may consider equation (2.2), with $x$ replaced by $\sqrt{y}$, as the auxiliary equation of the homogeneous linear recurrence relation of order $n-1$ :

$$
\begin{equation*}
0=\sum_{j=0}^{n-1}(-1)^{j}(2 n-j-1) w_{m-j}^{(n-1)}, \tag{5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
w_{m}^{(n-1)}=\sum_{i=1}^{n-1}(-1)^{i-1} \beta_{i}^{m} \tag{5.2}
\end{equation*}
$$

is the general term of the recurring sequence $\left\{\omega_{m}^{(n-1)}\right\}$ defined by (5.1) with the initial conditions (3.3). Thus, when $n=3$, (2.2) becomes

$$
x^{4}-4 x^{2}+3=0
$$

which can be rewritten as

$$
y^{2}-4 y+3=0
$$

i.e., the auxiliary equation for (5.1) in the form

$$
\omega_{m}^{(2)}=4 \omega_{m-1}^{(2)}-3 \omega_{m-2}^{(2)} .
$$

Initial conditions are
and

$$
w_{1}^{(2)}=\beta_{1}-\beta_{2}=2
$$

$$
w_{2}^{(2)}=\beta_{1}^{2}-\beta_{2}^{2}=2^{3} .
$$

Finally, it is worth noting that the theoretical foundations for the ideas implicit in [9] and [10] have by no means been fully exploited.

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## *****

## A HISTORY OF THE FIBONACCI $Q$-MATRIX AND A HIGHER-DIMENSIONAL PROBLEM

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To the memory of Verner E. Hoggatt, Ir.
One of the most popular and recurrent recent methods for the study of the Fibonacci sequence is to define the so-called Fibonacci $Q$-matrix

$$
Q=\left(\begin{array}{ll}
1 & 1  \tag{1}\\
1 & 0
\end{array}\right)
$$

so that

$$
Q^{n}=\left(\begin{array}{ll}
F_{n+1} & F_{n}  \tag{2}\\
F_{n} & F_{n-1}
\end{array}\right)
$$

where $F_{n+1}=F_{n}+F_{n-1}$, with $F_{1}=1, F_{0}=0$.
Theorems may then be cited from linear algebra so as to give speedy proofs of Fibonacci formulas. Write $|A|$ for the determinant of a matrix $A$. Then it is well known that $|A B|=|A| \cdot|B|$, and in general $\left|A^{n}\right|=|A|^{n}$. The Fibonacci $Q$-matrix method then gives at once the famous formula

$$
\begin{equation*}
F_{n+1} F_{n-1}-F_{n}^{2}=(-1)^{n} \tag{3}
\end{equation*}
$$

which was first given by Robert Simson in 1753. Formula (3) is the basis for the well-known geometrical paradox attributed to Lewis Carroll in which a unit of area mysteriously appears or disappears upon dissecting a suitable square and reassembling into a rectangle.

Where did this $Q$-matrix method originate? The object of the present paper is to give a tentative answer to this question, and present a reasonably complete bibliography of papers bearing on the use of such a matrix for the study of Fibonacci numbers. An unsolved problem is included.

The phrase "Q-matrix" seems to have originated in the master's thesis of Charles King [10]. At least, Basin and Hoggatt [16] cite this source, and from then on the idea caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in our Fibonacci Quarterly authored by Hoggatt and/or his students and other collaborators where the $Q$-matrix method became a central tool in the analysis of Fibonacci properties. Vern Hoggatt carried on a farranging correspondence in which he jotted down ideas and made innumerable suggestions for further research. For example, his letters to me make up a foothigh stack of paper very nearly, representing creative thinking going on for 20
years. His contagious enthusiasm for research and the properties of numbers infected all whom he met or wrote to, and it seems to me that Vern must have been a major force for popularizing the $Q$-matrix method. Vern wrote me many letters, beginning in 1962, about using the $Q$-matrix method to study the Fibonacci polynomials and other related systems. He was very modest about claiming any credit for ideas and would often outline some method to me and then say "but this is probably pretty well known to you." Sometimes it was; more often not.

However, an early place that the Fibonacci matrix seems to appear in the form we know it is in an abstract by Joel Brenner [6], which I shall quote in detail for its historic significance:

$$
\begin{aligned}
& \text { "The } n \text {-th power of the matrix }\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) \text { is } \\
& \qquad\left(\begin{array}{ll}
u_{n+1} & u_{n} \\
u_{n} & u_{n-1}
\end{array}\right),
\end{aligned}
$$

where $u_{n}$ is Fibonacci's number. More generally, the $n$-th power of $\left(\begin{array}{cc}a-b & -a b \\ \text { is } & 0\end{array}\right)$

$$
\left(\begin{array}{ll}
u_{n+1} & -a b u_{n} \\
u_{n} & -a b u_{n-1}
\end{array}\right),
$$

where $u_{n}=\frac{a^{n}-b^{n}}{a}$ is Lucas' number. From these facts it is easy to deduce a part of the general theory of these numbers.
"The sequences $u_{n}=A_{1} u_{n-1}+\ldots+A_{r} u_{n-r}$ have properties some of which are quickly obtained from the study of a matrix of dimension $r$ which generalizes the matrices above."

In copying the abstract I have corrected several misprints. Vern and I used to discuss the history of the $Q$-matrix, and he published a 'belated acknowledgement' in our Quarterly [28] which appears as a note that was never listed in the volume index and thus has remained hard to locate. I shall quote the acknowledgement here in full:
"The first use of the $Q$-matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title 'Lucas' Matrix.' This abstract appeared in the March, 1951 Amewican Mathematical Monthly on pages 221 and 222. The basic exploitation of the $Q$-matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title 'Some Further Properties of the Fibonacci Numbers.' Further utilization of the $Q$-matrix appears in the Fibonacci Primer sequence parts I-V."

To show that there was an active undercurrent of Fibonacci matrix activity around 1949-51, I wish next to quote an abstract by David DeVol [5] which appears, curiously, in the issue just preceding that in which Brenner's abstract turns up:

[^3]Besides this there was a paper by J. Sutherland Frame [4] in 1949 that used matrices to study continued fractions, and the matrix

$$
M_{1}=\left(\begin{array}{ll}
a & 1 \\
1 & 0
\end{array}\right)
$$

appears, but no mention is ever made of the Fibonacci numbers per se. Matrix analysis of continued fractions is an old story also.

Rosenbaum [8] uses the matrix

$$
R=\left(\begin{array}{ll}
p & q \\
1 & 0
\end{array}\right)
$$

to get an explicit formula for $F_{n}$, but does not consider $R^{n}$.
Miles [9] uses the matrix

$$
A_{n}=\left(\begin{array}{ll}
f_{n} & f_{n+1} \\
f_{n+1} & f_{n+2}
\end{array}\right)
$$

but does not consider it as a power of a matrix.
Waddill's doctoral thesis [12] uses the matrix

$$
\binom{11}{10}
$$

and also uses the third-order extensions. His later papers [26], [41] exploit the matrix further.

A remarkable insight is gained by examination of the well-known book of Schwerdtfeger [11]. On pages $104-105$ he discusses Fibonacci polynomials and matrix methods due to Jacobsthal[1]. Schwerdtfeger uses a German gothic B for the matrix involved. Changing the lettering slightly we can summarize part of what Schwerdtfeger says as follows. Let

$$
B=\left(\begin{array}{ll}
1 & b \\
1 & 0
\end{array}\right)
$$

Then

$$
B^{n}=\left(\begin{array}{ll}
f_{n}(b) & b f_{n-1}(b)  \tag{4}\\
f_{n-1}(b) & b f_{n-2}(b)
\end{array}\right)
$$

where the Fibonacci polynomials are defined by $f_{n+1}(x)=f_{n}(x)+x f_{n-1}(x)$, with $f_{0}(x)=1$ and $f_{-1}(x)=0$. Explicitly

Let

$$
f_{n}(x)=\sum_{0 \leq k \leq n / 2}\binom{n-k}{k} x^{k}
$$

$$
H=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right), t=a+d=\text { trace } \neq 0
$$

Then there exists a matrix $T$ such that $T H T^{-1}=q B$. In fact

$$
T=\left(\begin{array}{cc}
c & d \\
0 & a+d
\end{array}\right)
$$

Finally
(5)

$$
H^{n}=q^{n} T^{-1} B^{n} T \text {, }
$$

where $b=-(a d-b c) / t^{2}$. This is an interesting result, since it shows how to express the $n$-th power of a 2 by 2 matrix in terms of powers of the " $Q$ " matrix of a Fibonacci polynomial.

The only other reference I have noted in our Quarterly which cited Jacobsthal was the paper by Paul Byrd [14] in the very first issue of our journal. None cites Schwerdtfeger.

But the concept of a Fibonacci polynomial antedates Jacobsthal by a good many years. In fact, as Byrd [14] notes, a kind of Fibonacci polynomial was introduced as early as 1883 by E. Catalan, however, we shall not discuss this here. It is not entirely clear when in the pages of histroy a matrix was first used for such work.

Robinson [15] gives an extended discussion of matrix methods, citing many references, such as Bell [2], Ward [3], Brenner [7], and Rosenbaum [8]. He writes the matrix as

$$
U=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

and has

$$
U^{n}=\left(\begin{array}{ll}
u_{n-1} & u_{n} \\
u_{n} & u_{n+1}
\end{array}\right)
$$

He calls $U$ the Fibonacci matrix. Contrast this with Brenner who calls his matrix the Lucas matrix. I have not been able to ascertain whether Edouard Lucas himself used the matrix method.

Brennan [20] writes

$$
Q_{1}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right), Q_{2}=\left(\begin{array}{lll}
1 & 1 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

and higher-order extensions. But in [21] he writes

$$
P=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

and cites Basin and Hoggatt.
A novel application to group theory is afforded by the paper of White [22] who uses

$$
A=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \quad \text { and } \quad B=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)
$$

to generate $G L(2, Z)$. See also Gale [27].
Bicknell finds the square root of the $Q$ matrix [23], and goes on to fractional powers.

Lind [25] exhibits two matrices

$$
R=\left(\begin{array}{rr}
-1 & 0 \\
0 & 1
\end{array}\right), S=\left(\begin{array}{rr}
-1 & -1 \\
1 & 0
\end{array}\right)
$$

such that $R^{2}=S^{3}=I$, hence $R$ and $S$ are of finite order. However,

$$
R S=Q=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) \text {, and }(R S)^{n}=Q^{n}
$$

which is easily seen never to equal $I$, so that $R S$ is of infinite order. This
does not happen in an abelian group, of course, where the product of two elements of finite order must again be of finite order.

Ivie [31] considers a general $Q$-matrix. He defines and uses the $r$ by $r$ matrix

$$
Q_{r}=\left(\begin{array}{cccccc}
1 & 1 & 0 & & \cdots & 0 \\
1 & 0 & 1 & & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & 0 \\
1 & 0 & 0 & & 0 & 0 \\
1 & 0 & 0 & \ldots & 0
\end{array}\right)
$$

which is well known as associated with higher-order linear recursions.
Serkland [33], in his master's thesis, uses a matrix analogous to $Q$ in his study of the Pell sequence, which is therefore just a variant of the same consideration. See also a detailed report on this by Bicknell [37]. Here, of course,

$$
M=\left(\begin{array}{ll}
2 & 1 \\
1 & 0
\end{array}\right) \quad \text { and } \quad M^{n}=\left(\begin{array}{ll}
P_{n+1} & P_{n} \\
P_{n} & P_{n-1}
\end{array}\right)
$$

where the Pell sequence is defined by

$$
P_{n}=2 P_{n-1}+P_{n-2}, P_{1}=1, P_{2}=2
$$

The Pell sequence is again studied by Ercolano [43].
Hoggatt and Bicknell-Johnson [42] use what have been called Morgan-Voyce polynomials $b_{n}, B_{n}$ and they find the following. Let

$$
A=\left(\begin{array}{ll}
x & 1 \\
1 & 0
\end{array}\right), B=\left(\begin{array}{ll}
y & 1 \\
1 & 0
\end{array}\right) .
$$

Then

$$
(A B)^{n}=\left(\begin{array}{ll}
b_{n}(x y) & x B_{n-1}(x y)  \tag{6}\\
y B_{n-1}(x y) & b_{n-1}(x y)
\end{array}\right)
$$

Pollin and Schoenberg [45] turn the $Q$-matrix upside down in the form

$$
A=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

and use $A^{n}$ in their study of the converse of the congruence $p=$ prime implies $L_{p} \equiv 1(\bmod p)$, where $L_{p}$ is the Lucas number.

Our bibliography does not summarize all of the literature, but does give a good idea of what has been done with the $Q$-matrix and its extensions.

Now we wish to close with some remarks about problems that remain unsolved. These problems involve higher-dimensional determinants and matrices.

In my paper [19], I studied an operator I called a Turán operator, defined by

$$
\begin{equation*}
T f=T_{x} f(x)=T_{x, a, b} f(x)=f(x+a) f(x+b)-f(x) f(x+a+b) \tag{7}
\end{equation*}
$$

It is easy to show that

$$
\begin{equation*}
T_{x} \sin x=T_{x} \cos x=\sin a \sin b \tag{8}
\end{equation*}
$$

and, as an extension of (3), it is possible to prove that

$$
\begin{equation*}
T_{n} F_{n}=F_{n+a} F_{n+b}-F_{n} F_{n+a+b}=(-1)^{n} F_{a} F_{b}, \tag{9}
\end{equation*}
$$

so that (3) occurs when $a=1$ and $b=-1$.

My paper obtained extensions of this formula by exploring some possible extensions of determinants to three and four dimensions. Thus it was found that

$$
\begin{align*}
F_{n+a} F_{n+b} F_{n+c} & -F_{n} F_{n+a} F_{n+b+c}+F_{n} F_{n+b} F_{n+a+c}-F_{n} F_{n+c} F_{n+a+b}  \tag{10}\\
& =(-1)^{n}\left(F_{a} F_{b} F_{n+c}-F_{c} F_{a} F_{n+b}+F_{b} F_{c} F_{n+a}\right),
\end{align*}
$$

with further reductions, and yet the trouble is that there is no unique way to go about defining higher-dimensional determinants.

Since it is possible to prove (9) by means of skillful manipulations with a two-dimensional $Q$-matrix, one naturally desires to extend the idea to (10) and related formulas using a three-dimensional Q-matrix. Again, there seems to be difficulty in defining three-dimensional matrices. It would be necessary to see how to extend the property mentioned at the outset of this paper,

$$
\begin{equation*}
|A \cdot B|=|A| \cdot|B| \tag{11}
\end{equation*}
$$

for square two-dimensional matrices. How can this be extended, if indeed at all, to three-dimensional matrices? We leave this unsolved problem for the reader.

If Vern Hoggatt had worked on this problem we might have a solution already. Such was the enthusiasm he had for the $Q$-matrix, but he never got around to exploring this higher-dimensional direction.

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## SOME DIVISIBILITY PROPERTIES OF PASCAL's TRIANGLE

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This paper is dedicated to the memory of Professor V. E. Hoggatt, Ir., whose happy enthusiasm for mathematics has been an inspiration to all who knew him and whose friendship has enormously enriched the lives of so many, including, in particular, the present author.

1. INTRODUCTION

Let $p$ denote a prime and let $m, n, h, k$, and $\alpha$ denote integers with

$$
0 \leq k \leq n, 1 \leq h \leq n, m \geq 1, \text { and } \alpha \geq 1
$$

Let $\Delta_{n, k}$ denote the triangle of entries

$$
\begin{array}{cc} 
& \cdot\binom{n m}{k m} \\
\binom{n m+m-1}{k m} & \cdot \\
& \cdot
\end{array}
$$

from Pascal's triangle. And let $\nabla_{n, h}$ denote the triangle of entries from Pascal's triangle indicated by

$$
\begin{array}{cc}
\binom{n m}{h m-m+1} & \cdot
\end{array}\binom{n m}{n m-1}
$$

For $m=p^{\alpha}$, we showed in [2] that all elements of Pascal's triangle not contained in some $\Delta_{n, k}$ (i.e., those contained in some $\nabla_{n, h}$ ) are congruent to 0 modulo $p$, that, modulo $p$, there are precisely $p$ distinct triangles $\Delta_{n, k}$, and that these triangles can be put in one-to-one correspondence with the residues $0,1,2, \ldots, p-1$ in such a way that the triangle of triangles

\[

\]

is "isomorphic" to the original Pascal triangle in the sense that

$$
\Delta_{n, k}+\Delta_{n, k+1}=\Delta_{n+1, k+1}
$$

where the addition is elementwise modulo $p$. We also showed that if $D$ is the greatest common divisor of the three corner elements of $\nabla_{n, 1}$ and $d$ is the greatest common divisor of all the elements of $\nabla_{n, 1}$, then $d=p$ and $D=p^{\alpha}$ if $m=p^{\alpha}$ and $d=1$ and $D=m$ for all other integers $m \geq 2$. In the present paper we obtain, for $m=p^{2}$, a result similar to the first result for $\Delta_{n, k}$ and, for $m=p^{\alpha}$, we extend the second result to $\nabla_{n, h}$ for $1 \leq h \leq n$. Finally, we obtain a number of interesting properties of the $p$-index triangle of Pascal's triangle, which is simply the triangle of numbers $\left[\begin{array}{l}n \\ k\end{array}\right]$ that indicate the exponent $e$ to which a given prime $p$ divides $\binom{n}{k}$; i.e., such that $p^{e} \left\lvert\,\binom{ n}{k}\right.$ and $p^{e+1} \nmid\binom{n}{k}$.

$$
\text { 2. THE ITERATED TRIANGLE MODULO } p^{2}
$$

To extend the first result mentioned above to $p^{2}$, we set $\alpha=2 \beta$ where $\beta \geq 1$ is an integer. Thus, the $\Delta_{n, k}$ are equilateral triangles with $p^{2 \beta}$ elements per side. Furthermore, we say that two such triangles are equivalent provided that their top $p$ rows are identical, and it is clear that this is an equivalence relation in the technical sense. Let $\delta_{n, k}$ denote the class of all triangles equivalent to $\Delta_{n, k}$. Then, again, we claim that there exist precisely $p^{2}$ equivalence classes of triangles, and that there exists a one-to-one correspondence between these classes and the residues $0,1, \ldots, p^{2}-1$ such that the triangular array
is "isomorphic" to the original Pascal triangle in the sense that

$$
\delta_{n, k}+\delta_{n, k+1}=\delta_{n+1, k+1}
$$

where the addition is defined by the elementwise addition modulo $p^{2}$ of the top $p$ rows of any two representatives of $\delta_{n, k}$ and $\delta_{n, k+1}$. All of this follows from Theorem 5 below, but first we need several lemmas. The first is well known (see, for example, [1, problem 16, p. 57]).
Lemma 1: Let $p$ be a prime and let $n$ and $k$ be integers with $0 \leq k \leq n$. Then

$$
p^{e} \|\binom{ n}{k}
$$

if and only if $e$ is the number of carries made in adding $k$ to $n-k$ in base $p$. Equivalently, $e$ is the number of carries made in subtracting $k$ from $n$ in base $p$.
Lemma 2: Let $p$ be a prime and let $k, h$, and $\alpha$ be integers with $k \geq 1, \alpha \geq 1$, and $0 \leq h \leq k p^{\alpha}$. If $p \nmid h$, then

$$
\binom{k p^{\alpha}}{h} \equiv 0\left(\bmod p^{\alpha}\right) .
$$

Proof: Since plh, the units digit in the base $p$ representation of $h$ is not zero. Thus, it is clear that the subtraction of $h$ from $k p^{\alpha}$ in base $p$ requires $\alpha$ carries and the result follows from Lemma 1.

We note in passing that the converse of Lemma 2 is false since, for example,

$$
\binom{16}{2} \equiv 0(\bmod 4)
$$

and yet $2 \mid 2$.
Lemma 3: Let $p$ be a prime and let $k, h, \alpha$, and $\beta$ be integers with $\beta \geq 1, \alpha=2 \beta$, and $0 \leq h \leq k$. Then

$$
\binom{k p^{\alpha}}{h p^{\alpha}} \equiv\binom{k}{h}\left(\bmod p^{2}\right) .
$$

Proof: In [4], J. H. Smith proves that

$$
\binom{k p}{h p} \equiv\binom{k}{h}\left(\bmod p^{2}\right)
$$

Thus, the result claimed follows immediately by induction.
Lemma 4: Let $p$ be a prime and let $k, h, r, s, \alpha$, and $\beta$ be integers with

$$
0 \leq h \leq k, 0 \leq s \leq r<p, \beta \geq 1, \text { and } \alpha=2 \beta .
$$

Then

$$
\binom{k p^{\beta}+r}{h p^{\beta}+s} \equiv\binom{k}{h}\binom{r}{s}\left(\bmod p^{2}\right)
$$

Proof: This is an immediate consequence of Lemma 3 and the fact that, by Lemma 2, the $p-1$ coefficients on either side of $\binom{k p^{\beta}}{h p^{\beta}}$ must all be congruent
to 0 modulo $p^{2}$. to 0 modulo $p^{2}$.
Theorem 5: Let $p$ be a prime and let $\alpha, \beta, k$, and $n$ be integers with $\beta \geq 1, \alpha=$ $\overline{2 \beta \text {, and } 0} \leq k \leq n$. Then the first $p$ rows of $\Delta_{n, k}$ modulo $p^{2}$ are

$$
\binom{n}{k}\binom{0}{0}
$$

$$
\begin{aligned}
& \binom{n}{k}\binom{1}{0} \quad\binom{n}{k}\binom{1}{1} \\
& \binom{n}{k}\binom{p-1}{0} \quad \cdots \quad\binom{n}{k}\binom{p-1}{p-1}
\end{aligned}
$$

Also,

$$
\delta_{n, k}+\delta_{n, k+1}=\delta_{n+1, k+1}
$$

where $\delta_{n, k}$ and this addition are defined above.
Proof: The elements in the first $p$ rows of $\Delta_{n, k}$ are the binomial coefficients

$$
\binom{n p^{\beta}+r}{k p^{\beta}+s}, 0 \leq s \leq r<p
$$

and, by Lemma 4,

$$
\binom{n p^{\beta}+r}{k p^{\beta}+s} \equiv\binom{n}{k}\binom{r}{s}\left(\bmod p^{2}\right)
$$

This gives the first assertion of the theorem and implies the second since

$$
\begin{aligned}
\binom{n p^{\beta}+r}{k p^{\beta}+s}+\binom{n p^{\beta}+r}{(k+1) p^{\beta}+s} & \equiv\binom{n}{k}\binom{r}{s}+\binom{n}{k+1}\binom{r}{s} \\
& \equiv\binom{n+1}{k+1}\binom{r}{s} \equiv\binom{(n+1) p^{\beta}+r}{(k+1) p^{\beta}+s}\left(\bmod p^{2}\right)
\end{aligned}
$$

[Aug.

Of course, the fact that every entry in Pascal's triangle not contained in $\Delta_{n, k}$ for $m=p^{\alpha}=p^{2 \beta}$ is congruent to zero modulo $p$ follows immediately from Theorem l of [2] with $\alpha=2 \beta$. One might have guessed that all these elements were in fact congruent to zero modulo $p^{2}$, but this is easily seen not to be the case. In particular, $\binom{4}{2}$ is not contained in any $\Delta_{n, k}$ for $p=2$ and $\alpha=2$, and

$$
0 \not \equiv\binom{4}{2} \equiv 2(\bmod 4)
$$

## 3. SOME GREATEST COMMON DIVISOR PROPERTIES

Recall that for integers $m, n$, and $h$ with $1 \leq m$ and $1 \leq h \leq n, \nabla_{n, h}$ denotes the triangle of binomial coefficients

$$
\begin{aligned}
\binom{n m}{n m-m+1} & \cdot \cdot
\end{aligned}\binom{n m}{n m-1}
$$

that $d$ denotes the greatest common divisor of all the coefficients of $\nabla_{n, h}$, and that $D$ denotes the greatest common divisor of the corner coefficients. In [2], we completely determined $d$ and $D$ for $\nabla_{n, 1}$ and we extend those results in this section. The increased generality, however, makes for somewhat weaker results as seen in the following theorems.
Theorem 6: Let $\alpha, D$, and $\nabla_{n, h}$ be as above where $m=p^{\alpha}$ with $p$ a prime and $\alpha$ a positive integer. If $p^{e} \| n$, then $p^{e+1} \| d$ and $p^{e+\alpha} \| D$.

Proof: We first prove that $p^{e+\alpha} \| D$. The upper left coefficient in $\nabla_{n, h}$ is

$$
L=\binom{n p^{\alpha}}{(h-1) p^{\alpha}+1}
$$

Since $p^{e} \| n, n p^{\alpha}=N p^{e+\alpha}$ where $p \nmid N$. Also, the units digit in the base $p$ representation of $(h-1) p+1$ is 1 and this clearly implies that $e+\alpha$ carries are required in subtracting $(h-1) p^{\alpha}+1$ for $n p^{\alpha}$ in base $p$. Thus, $p^{e+\alpha} \| L$. The upper right coefficient in $\nabla_{n, h}$ is

$$
R=\binom{n p^{\alpha}}{h p^{\alpha}-1}
$$

Since $p \nmid h p^{\alpha}-1$, the units digit in the base $p$ representation of $h p^{\alpha}-1$ is not 0 and again $e+\alpha$ carries are required in subtracting $h p^{\alpha}-1$ from $n p^{\alpha}$ in base $p$. Thus, $p^{e+\alpha} \| R$. Finally, the bottom coefficient of $\nabla_{n, h}$ is

$$
B=\binom{(n+1) p^{\alpha}-2}{h p^{\alpha}-1}
$$

Here,

$$
\begin{aligned}
(n+1) p^{\alpha}-2 & =n p^{\alpha}+p^{\alpha}-2 \\
& =N p^{e+\alpha}+(p-1) p^{\alpha-1}+\cdots+(p-1) p+(p-2) \\
h p^{\alpha}-1 & =(h-1) p^{\alpha}+p^{\alpha}-1 \\
& =(h-1) p^{\alpha}+(p-1) p^{\alpha-1}+\cdots+(p-1) p+(p-1),
\end{aligned}
$$

and, since $p-2<p-1$, it follows that the subtraction of $h p^{\alpha}-1$ from $(n+$ 1) $p^{\alpha}-2$ necessitates $e+\alpha$ carries. Thus, $p^{e+\alpha} \| B$ and it follows that $p^{e+\alpha} \| D$ as claimed.

To show that $p^{e+1} \| d$, it suffices to show that $p^{e+1}$ divides each element in the top row of $\nabla_{n, h}$ and that $p^{e+1}$ exactly divides one of these elements. The elements in the top row of $\nabla_{n, h}$ are

$$
\binom{n p^{\alpha}}{n p^{\alpha}-p^{\alpha}+s}, 1 \leq s \leq p^{\alpha}-1 .
$$

Again $n p^{\alpha}=N p^{e+\alpha}$ where $p \nmid N$. Since $1 \leq s \leq p^{\alpha+1}$, the base $p$ representation of $s$ must contain at least one nonzero digit in some position prior to the $\alpha-$ th. Thus, the subtraction of $h p^{\alpha}-p^{\alpha}+s$ from $n p^{\alpha}$ requires carries from the ( $e+\alpha$ ) column of the base $p$ representation of $n p^{\alpha}$ and these must be at least $e+1$ in number. Thus, $p^{e+1}$ divides every element in the top row of $\nabla_{n, n}$. Now consider the element

$$
M=\binom{n p^{\alpha}}{h p^{\alpha}-p^{\alpha}+p^{\alpha-1}}
$$

Again, since $n p^{\alpha}=N p^{e+\alpha}$ as above, the carrying in the subtraction of $(h-1) p^{\alpha}+$ $p^{\alpha-1}$ from $n p^{\alpha}$ is precisely from the $e+\alpha$ column to the $\alpha-1$ column for a total of exactly $e+1$ carries. Therefore, $p^{e+1} \| M$ and $p^{e+1} \| d$ as claimed.

Note that for $p=3$ and $\alpha=2, \nabla_{4,2}$ is such that

$$
d=16,182=2 \cdot 3^{2} \cdot 29 \cdot 31 \text { and } D=48,546=3 d
$$

and this suggests that Theorem 6 might be considerably strengthened. However, $d=3$ and $D=3^{2}$ in $\nabla_{4,1}$. Also, if $n=h=1, p$ is a prime, and $\alpha$ is a positive integer, then $d=p$ and $D=p^{\alpha}$ by Theorem 2 of [2]. Thus, in a sense, Theorem 6 is best possible for prime powers.

In case $m$ is composite but not a prime power, our best result is as follows. Theorem 7: Let $d, D$, and $\nabla_{n, h}$ be as above with $m$ composite and not a prime power. Then $m \mid D$.

Proof: Let $p^{\alpha} \| m$ so that $m=M p^{\alpha}$ and $p \nmid M$. Then the argument of Theorem 6 can be repeated exactly to thow that $p^{\alpha} \mid D$. Thus, if

$$
m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}
$$

is the canonical representation of $m$, it follows that $p_{i}^{\alpha_{i}} \mid D$ for each $i$ and hence that $m \mid D$ as claimed.

Several examples suffice to show that Theorem 7 is also, in a sense, best possible. Consider the triangles $\nabla_{n, h}$ with $m=6$. That $d$ is not necessarily equal to 1 even when $6 \nmid n$ is shown by $\nabla_{5,2}$ where $d=870$. Also, the fact that $m[n$ does not necessarily increase the power of $m$ that divides $d$ and $D$ is shown by $\nabla_{6,1}$ where $d=3$ so that $6 \nmid d$ and by $\nabla_{6,2}$ where $6 \mid D$ but $6^{2} \nmid D$.
4. THE $p$-INDEX TRIANGLE

For a given prime $p$, let $\left[\begin{array}{l}n \\ k\end{array}\right]$ denote the exponent of the highest power of $p$ that divides $\binom{n}{k}$. The triangle of entries $\left[\begin{array}{l}n \\ k\end{array}\right], 0 \leq k \leq n$, is called the $p$ index triangle of Pascal's triangle and seems to have been studied first by K. R. McLean [3]. Quite apart from their attractiveness as kind of mathematical
art, the $p$-index triangles exhibit interesting patterns that reveal additional structure in Pascal's triangle. Some of the more interesting of these properties are detailed in the following theorems.

Theorem 8: Let $p$ be a prime, then

$$
\left[\begin{array}{c}
m p^{\alpha}-1 \\
k
\end{array}\right]=0 \text { for } 1 \leq m<p \text { and } 0 \leq k \leq m p^{\alpha}-1
$$

Proof: Since $0 \leq k \leq m p^{\alpha}-1$ and $1 \leq m<p$,

$$
k=\sum_{i=0}^{\alpha} k_{i} p^{i}
$$

with $0 \leq k_{i}<p$ for all $i$ and $k_{\alpha}<m$. But

$$
m p^{\alpha}-1=(m-1) p^{\alpha}+\sum_{i=0}^{\alpha-1}(p-1) p^{i}
$$

Thus, there are no carries in subtracting $k$ from $m p^{\alpha}-1$ in base $p$ and

$$
\left[\begin{array}{c}
m p^{\alpha}-1 \\
k
\end{array}\right]=0
$$

as claimed.
Theorem 9: Let $p$ be a prime, then $\left[\begin{array}{c}p^{\alpha} \\ k\end{array}\right] \geq 1$ for $1 \leq k<p^{\alpha}$. Of course,

$$
\left[\begin{array}{c}
p^{\alpha} \\
0
\end{array}\right]=\left[\begin{array}{c}
p^{\alpha} \\
p^{\alpha}
\end{array}\right]=0
$$

Proo 6: Since $1 \leq k<p^{\alpha}$,

$$
k=\sum_{i=0}^{\alpha-1} k_{i} p^{i}
$$

with $0 \leq k_{i}<p$ for all $i$ and $k_{i} \neq 0$ for at least one $i$. Therefore, there is at least one carry in subtracting $k$ from $p^{\alpha}$ and the result follows.

Theorem 10: Let $p$ be a prime and let $m$ and $n$ be positive integers with $1 \leq$ $m<p$ and $1 \leq n<p$. Then

Proo f: Note that

$$
m p^{\alpha}+n p^{\alpha-1}-1=m p^{\alpha}+(n-1) p^{\alpha-1}+\sum_{i=0}^{\alpha-2}(p-1) p^{i}
$$

Thus, if $0 \leq k \leq m p^{\alpha}+n p^{\alpha-1}-1$, the only time a carry will be required in subtracting $k$ from $m p^{\alpha}+n p^{\alpha-1}-1$ in base $p$ is when $k$ has a digit $k_{\alpha-1} \geq n$ in the $\alpha-1$ position. But this occurs precisely when

$$
n p^{\alpha-1} \leq k<p^{\alpha} \text {, or } p^{\alpha}+n p^{\alpha-1} \leq k<2 p^{\alpha} \text {, or } \cdots \text {, }
$$

or $(m-1) p^{\alpha}+n p^{\alpha-1} \leq k<m p^{\alpha}$ as claimed.
As in Pascal's triangle modulo $p$, the $p$-index triangle naturally decomposes into an array of interesting subtriangles. Thus, we have the following results.

Theorem 11：Let $p$ be a prime and let $\alpha \geq 1$ be an integer．For integers $n$ and $\bar{k}$ with $0 \leq k \leq n$ ，let $T_{n, k}$ denote the subtriangle of entries from the $p$－index triangle indicated by

$$
\begin{gathered}
\quad\left[\begin{array}{l}
n p^{\alpha} \\
k p^{\alpha}
\end{array}\right] \\
{\left[\begin{array}{c}
(n+1) p^{\alpha}-1 \\
k p^{\alpha}
\end{array}\right] \quad \cdot \cdot}
\end{gathered} \cdot\left[\begin{array}{l}
(n+1) p^{\alpha}-1 \\
(k+1) p^{\alpha}-1
\end{array}\right] .
$$

Then，$T_{n, k}=\left[\begin{array}{l}n \\ k\end{array}\right]+T_{0,0}$ where this is understood to mean that $\left[\begin{array}{l}n \\ k\end{array}\right]$ is added to each element of $T_{0,0}{ }^{\circ}$

Proob：Note that $T_{0,0}$ is the triangle of entries

$$
\left[\begin{array}{l}
r \\
s
\end{array}\right], 0 \leq s \leq r<p^{\alpha} .
$$

Similarly，$T_{n, k}$ is the triangle of entries

$$
\left[\begin{array}{l}
n p^{\alpha}+r \\
k p^{\alpha}+s
\end{array}\right], 0 \leq s \leq r<p^{\alpha} .
$$

Since $s \leq r$ ，the number of carries required in subtracting $k p^{\alpha}+s$ from $n p^{\alpha}+r$ is just the number required in subtracting $k$ from $n$ plus those required in sub－ tracting $s$ from $r$ ．That is

$$
\left[\begin{array}{l}
n p^{\alpha}+r \\
k p^{\alpha}+s
\end{array}\right]=\left[\begin{array}{l}
n \\
k
\end{array}\right]+\left[\begin{array}{l}
r \\
s
\end{array}\right]
$$

Thus，$T_{n, k}=\left[\begin{array}{l}n \\ k\end{array}\right]+T_{0,0}$ as claimed．
Corollary 12：Consider the infinite array of triangles $T_{n, k}, 0 \leq k \leq n$ ，as in Theorem 11．The array consisting of the top vertex element of each of these triangles is just the original $p$－index triangle．Thus，the p－index triangle contains a $p$－index triangle which contains a $p$－index triangle，and so on with－ out end．

Proof：The triangle of top elements of $T_{n, k}$ is the triangle $\left[\begin{array}{l}n p^{\alpha} \\ k p^{\alpha}\end{array}\right], 0 \leq$ $k \leq \bar{n}$ ．But，by Theorem 11，

$$
\left[\begin{array}{l}
n p^{\alpha} \\
k p^{\alpha}
\end{array}\right]=\left[\begin{array}{l}
n \\
k
\end{array}\right]
$$

and the result follows．

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# FIBONACCI INDUCED GROUPS AND THEIR HIERARCHIES 

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Not only was the late Probessor Hoggatt a dedicated teacher and a prolific scholar, he was also a fine colleague and a reliable co-worker. In the summer of 1980 we met often, both of us being involved in Santa Clara's Undergraduate Research Participation program in mathematics. I showed him a rough draft of the few ideas expressed in this paper, and, while he was encouraging me to prepare it for The Fibonacci quarterly, at the same time he asked me urgently to make it readable for a great variety of readers, "our readers" as he called them not without afbection. In trying to comply with his request, I discovered that the paper became more than the communication of some results; it began to tell the story of how they were obtained, from simple well-known beginnings, through some redundant complications, toward a simple ending. May the effort be a stone in the monument to the memory of Dr. Verner E. Hoggatt, Jr.

## 1. INTRODUCTION

Sequences of integers give rise to algebraic structures and sequence hierarchies in various ways, some trivially, others by more sophisticated methods. A glance at a trivial example may help to grasp readily the subject matter of this paper.

Let $N$ be the set of positive integers, $Z$ the set of all integers. The function $s: N \rightarrow Z$, somehow defined, constitutes the sequence $s_{1}, s_{2}, s_{3}, \ldots$, where the arguments are written as subscripts. For every $z \varepsilon Z$, a function $t_{z}: N \rightarrow Z$ can be defined by $t_{z}(n)=z s_{n}$, thus constituting the sequence

$$
t_{z 1}=z s_{1}, t_{z 2}=z s_{2}, t_{z 3}=z s_{3}, \ldots .
$$

Let $Z_{t}=\left\{t_{z}: Z \varepsilon Z\right\}$. On $Z_{t}$ one defines addition and multiplication by

$$
t_{a}+t_{b}=t_{a+b} \quad \text { and } \quad t_{a} t_{b}=t_{a b}
$$

These definitions are not arbitrary or ad hoc; they amount to the usual pointwise addition and multiplication of functions, e.g.,

$$
\begin{aligned}
\left(t_{a}+t_{b}\right)(n) & =t_{a}(n)+t_{b}(n)=t_{a n}+t_{b n}=a s_{n}+b s_{n} \\
& =(a+b) s_{n}=t_{(a+b) n}=t_{a+b}(n)
\end{aligned}
$$

The result is an algebraic structure

$$
\left(Z_{t},+, \cdot, t_{0}, t_{1}\right)
$$

where $t_{0}$ and $t_{1}$ are additive and multiplicative identities, respectively; $t_{0}$ is the sequence with all terms 0 and $t_{1}$ is $s$. This algebraic structure is clearly isomorphic to $(Z,+, \cdot, 0,1)$, the familiar integral domain of the integers, by $\phi: Z \rightarrow Z_{t}$ with $\phi(z)=t_{z}$, thus being itself an integral domain.

The integral domain $Z_{t}$ is a trivial example of an algebraic structure induced by the sequence $s$. As to the hierarchy involved, let the function $S: N \rightarrow Z_{t}$ be defined by $S(n)=t_{s(n)}$. The result is a sequence $S$ with

$$
S_{1}=t_{s(1)}, S_{2}=t_{s(2)}, S_{3}=t_{s(3)}, \ldots,
$$

a sequence of sequences, such that each term of $S$ is the element of $Z_{t}$ that has the corresponding term of $s$ as index; as a sequence, $S$ is completely patterned after $s$. One could call $S$ the second level of a hierarchy of which $s$ is the first and lowest level. Starting with the sequence $S$ one arrives in a similar way at the third leve1. For every $y, z \varepsilon Z$, let $y t_{z}$ be the function $y t_{z}: N \rightarrow Z_{t}$
with $\left(y t_{z}\right)(n)=t_{y z}(n)$. For every $y \varepsilon Z$, let $T_{y}: N \rightarrow Z_{t}$ with $T_{y}(n)=y S_{n},(=$ $\left.y t_{s(n)}=t_{y s(n)}\right)$. Further, let $Z_{T}=\left\{T_{y}: y \in Z\right\}$. On $Z_{T}$ one defines addition and multiplication pointwise, thus obtaining $Z_{T}$ as an integral domain. The function $\delta: N \rightarrow Z_{T}$ with $\delta(n)=T_{s(n)}$ constitutes the sequence

$$
\mathcal{S}_{1}=T_{s(1)}, S_{2}=T_{s(2)}, \cdots
$$

the third level in the hierarchy. This construction can be repeated indefinitely.

As a concrete example of the above one could take any sequence of integers, but in this context one can as well take the Fibonacci sequence $f: N \rightarrow Z$ with $f_{1}=1, f_{2}=1$, and, for $n>2, f_{n}=f_{n-1}+f_{n-2}$, yielding the well-known sequence

$$
f_{1}=1, f_{2}=1, f_{3}=2, f_{4}=3, f_{5}=5, f_{6}=8, \ldots .
$$

For every $z \varepsilon Z$, a function $g_{z}: N \rightarrow Z$ can be defined by $g_{z}(n)=z f_{n}$, thus constituting the sequence

$$
g_{z 1}=z f_{1}, g_{z 2}=z f_{2}, g_{z 3}=z f_{3}, \ldots .
$$

Let $Z_{g}=\left\{g_{z}: \approx \in Z\right\}$. On $Z_{g}$ one defines $g_{a}+g_{b}=g_{a+b}$ and $g_{a} g_{b}=g_{a b}$. The result is the integral domain ( $Z_{g},+, \cdot, g_{0}, g_{1}$ ), isomorphic with the integral domain of the integers, induced by the sequence $f$. Let $F: N \rightarrow Z_{g}$ be defined by $F(n)=g_{f(n)}$. This yields the sequence $F$ with terms

$$
F_{1}=g_{1}, F_{2}=g_{1}, F_{3}=g_{2}, F_{4}=g_{3}, F_{5}=g_{5}, F_{6}=g_{8}, \ldots .
$$

It should be noticed that again, for $n>2, F_{n}=F_{n-1}+F_{n-2}$. The sequence $F$ could be called a Fibonacci sequence of Fibonacci sequences, the second level of a hierarchy of which $f$ is the first and lowest level. Continuing, for every $y, z \varepsilon Z$, let $y g_{z}$ be the function $y g_{z}: N \rightarrow Z_{g}$ with $\left(y g_{z}\right)(n)=g_{y z}(n)$, $\left(=y z f_{n}\right)$. For every $z \varepsilon Z$, let $G_{z}$ be the function $G_{z}: N \rightarrow Z_{g}$ with $G(n)=z F_{n},\left(=z g_{f(n)}=\right.$ $\left.g_{z f(n)}\right)$. Let $Z_{G}=\left\{G_{z}: z \in Z\right\}$. Again introducing pointwise addition and multiplication on $Z_{G}$, one obtains $Z_{G}$ as an integral domain. Let $\mathcal{F}: N \rightarrow Z_{G}$ with $\mathcal{F}(n)=G_{f(n)}$, then $\mathcal{F}$ constitutes the sequence

$$
\mathscr{F}_{1}=G_{1}, \mathfrak{F}_{2}=G_{1}, \mathfrak{F}_{3}=G_{2}, \mathcal{F}_{4}=G_{3}, \mathfrak{F}_{5}=G_{5}, \mathfrak{F}_{6}=G_{8}, \ldots,
$$

the third level of the infinite hierarchy. The sequence is a Fibonacci sequence of Fibonacci sequences of Fibonacci sequences.

## 2. GENERATION OF A HIERARCHY OF GROUPS

One way of generalizing the Fibonacci sequence consists in extending its domain from $N$ to $Z$. In this section let $f$ denote the function $f: Z \rightarrow Z$ defined by $f_{0}=0, f_{1}=1$, and $f_{n}=f_{n-2}+f_{n-1}$, or, $f_{n-2}=f_{n}-f_{n-1}$, where the arguments are again written as subscripts. Clearly, $f \mid N$, the restriction of $f$ to $N$, yields the original Fibonacci sequence. The set of values $\left\{f_{n}: n \varepsilon Z\right\}$ can be pictured as an extension to the left of the original sequence:

$$
\begin{aligned}
\ldots, f_{-5} & =5, f_{-4}=-3, f_{-3}=2, f_{-2}=-1, f_{-1}=1, \\
f_{0} & =0, f_{1}=1, f_{2}=1, f_{3}=2, f_{4}=3, f_{5}=5, \ldots .
\end{aligned}
$$

The restriction $f \mid N$ is nearly injective, spoiled only at the very beginning by $f_{1}=f_{2}$. The extended $f$ is only half as nice due to the identities formulated in Lemma 1.
Lemma 1: If $n \varepsilon N \cup\{0\}$ is even, then $f_{-n}=-f_{n}$,
if $n \varepsilon N$ is odd, then $f_{n}=f_{-n}$ 。

## Proof:

Base step. -Trivially $f_{-0}=f_{0}=0=-f_{0}$ and obviously $f_{1}=1=f_{-1}$.
Induction step. - For $m>1$, let it be assumed that the lemma holds for all $k \in N$ such that $k<m$. By definition,

$$
f_{-m}=f_{-m+2}-f_{-m+1}=f_{-(m-2)}-f_{-(m-1)}
$$

If $m$ is even, then $m-2$ is even and $m-1$ is odd, and hence, by the induction hypothesis,

$$
f_{-m}=-f_{m-2}-f_{m-1}=-\left(f_{m-2}+f_{m-1}\right)=-f_{m} .
$$

If $m$ is odd, then $m-2$ is odd, $m-1$ is even, and the induction hypothesis yields

$$
f_{-m}=f_{m-2}-\left(-f_{m-1}\right)=f_{m-2}+f_{m-1}=f_{m}
$$

The next lemma is an extension of a well-known lemma. The proof is extended to all the integers.
Lemma 2: For every $n \in Z$, let $D_{n}$ be the determinant of the matrix

$$
\left(\begin{array}{ll}
f_{n-1} & f_{n} \\
f_{n} & f_{n+1}
\end{array}\right)
$$

If $n$ is even, then $D_{n}=1$; if $n$ is odd, then $D_{n}=-1$.
Proof: Obviously $D_{-1}=-1, D_{0}=1$, and $D_{1}=-1$. Moreover, for every $m \in Z$,

$$
\begin{aligned}
D_{m} & =\left|\begin{array}{ll}
f_{m-1} & f_{m} \\
f_{m} & f_{m+1}
\end{array}\right|=\left|\begin{array}{ll}
f_{m-1} & f_{m-1}+f_{m-2} \\
f_{m} & f_{m}+f_{m-1}
\end{array}\right| \\
& =0+\left|\begin{array}{ll}
f_{m-1} & f_{m-2} \\
f_{m} & f_{m-1}
\end{array}\right|=\left|\begin{array}{ll}
f_{m-3}+f_{m-2} & f_{m-2} \\
f_{m-2}+f_{m-1} & f_{m-1}
\end{array}\right| \\
& =\left|\begin{array}{ll}
f_{m-3} & f_{m-2} \\
f_{m-2} & f_{m-1}
\end{array}\right|+0=D_{m-2} .
\end{aligned}
$$

Hence, if $n$ is even, $\frac{1}{2}|n|$ applications of the rule $D_{m}=D_{m-2}$, upward or downward, according to whether $n$ is negative or positive, respectively, yield $D_{n}=$ $D_{0}=1$. And if $n$ is odd, $\frac{1}{2}(|n|-1)$ applications of the rule, upward or downward, yield $D_{n}=D_{-1}=-1$ or $D_{n}=D_{1}=-1$.

As is well known, the invertible $2 \times 2$ matrices with real entries have determinants $\neq 0$ and form a group under matrix multiplication with $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ as identity and $\left(\begin{array}{cc}\frac{a}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{\alpha}{D}\end{array}\right)$ as the inverse of $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, where $D$ is the determinant of $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$. Let $q=\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ and let $\langle q\rangle$ be the cyclic subgroup generated by $q$.

Lemma 3: The cyclic group $\langle q\rangle$ is of infinite order and for every $n \varepsilon Z$,

$$
q^{n}=\left(\begin{array}{ll}
f_{n-1} & f_{n} \\
f_{n} & f_{n+1}
\end{array}\right)
$$

Proof: (a) For $n \in N \cup\{0\}$ by straightforward induction.

$$
q^{0}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{ll}
f_{-1} & f_{0} \\
f_{0} & f_{1}
\end{array}\right)
$$

Next, for $m \in N$, if $q^{m}=\left(\begin{array}{ll}f_{m-1} & f_{m} \\ f_{m} & f_{m+1}\end{array}\right)$, then

$$
q^{m+1}=q^{m} q=\left(\begin{array}{ll}
f_{m-1} & f \\
f_{m} & f_{m+1}
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)=\left(\begin{array}{ll}
f_{m} & f_{m-1}+f_{m} \\
f_{m+1} & f_{m}+f_{m+1}
\end{array}\right)=\left(\begin{array}{ll}
f_{m} & f_{m+1} \\
f_{m+1} & f_{m+1}
\end{array}\right)
$$

(b) For $n<0$, let $n=-m$. Then $m \in N$ and by (a) above,

$$
q^{m}=\left(\begin{array}{ll}
f_{m-1} & f_{m} \\
f_{m} & f_{m+1}
\end{array}\right)
$$

If $m$ is even, then $D_{m}=1$ and

$$
q^{n}=q^{-m}=\left(q^{m}\right)^{-1}=\left(\begin{array}{ll}
f_{m+1} & -f_{m} \\
-f_{m} & f_{m-1}
\end{array}\right)=\left(\begin{array}{ll}
f_{-m-1} & f_{-m} \\
f_{-m} & f_{-m+1}
\end{array}\right)=\left(\begin{array}{ll}
f_{n-1} & f_{n} \\
f_{n} & f_{n+1}
\end{array}\right)
$$

by Lemmas 1 and 2. Similarly, if $m$ is odd, then $D_{m}=-1$ and

$$
q^{n}=q^{-m}=\left(q^{m}\right)^{-1}=\left(\begin{array}{ll}
-f_{m+1} & f_{m} \\
f_{m} & -f_{m-1}
\end{array}\right)=\left(\begin{array}{ll}
f_{-m-1} & f_{-m} \\
f_{-m} & f_{-m+1}
\end{array}\right)=\left(\begin{array}{ll}
f_{n-1} & f_{n} \\
f_{n} & f_{n+1}
\end{array}\right)
$$

(c) The infinite order of $\langle q\rangle$ is now obvious; for every $n>0$, $q^{n} \neq q^{0}$, since $f_{n} \neq f_{0}$.

Since $\langle q\rangle$ is a cyclic group of infinite order, $q^{m}=q^{n}$ if and only if $m=n$. The function $s: Z \rightarrow\langle q\rangle$ with $s(n)=q^{n}$ is bijective. Moreover,

$$
s(n+m)=q^{n+m}=q^{n} \cdot q^{m}=s(n) \cdot s(m)
$$

Hence, the multiplicative group $\langle q\rangle$ is isomorphic to ( $Z,+$ ), the additive group of the integers. Since the elements of $\langle q\rangle$ are $2 \times 2$ matrices, they can be added by the usual matrix addition, but $\langle q\rangle$ is not closed under that addition. However, the following lemma holds.
Lemma 4: For every $n \in Z, q^{n}=q^{n-2}+q^{n-1}$, where + is the usual matrix addition.

Proof:

$$
\begin{aligned}
q^{n} & =\left(\begin{array}{ll}
f_{n-1} & f_{n} \\
f_{n} & f_{n+1}
\end{array}\right)=\left(\begin{array}{ll}
f_{n-3}+f_{n-2} & f_{n-2}+f_{n-1} \\
f_{n-2}+f_{n-1} & f_{n}+f_{n-1}
\end{array}\right) \\
& =\left(\begin{array}{ll}
f_{n-3} & f_{n-2} \\
f_{n-2} & f_{n-1}
\end{array}\right)+\left(\begin{array}{ll}
f_{n-2} & f_{n-1} \\
f_{n-1} & f_{n}
\end{array}\right)=q^{n-2}+q^{n-1} .
\end{aligned}
$$

The function $s$ can be seen as a two-sided sequence with $s_{n}=q^{n}$ for every $n \in Z$, with $s_{n} \neq s_{m}$ for all $n, m \varepsilon Z$ such that $n \neq m$, and with $s_{n}=s_{n-2}+s_{n-1}$. Moreover, the elements of the sequence form an abelian group under multiplication.

Once $s$ is established, one may as well dispense with the matrices. For every $n \in Z, s_{n}$ is uniquely determined by the ordered triple $\left(f_{n-1}, f_{n}, f_{n+1}\right)$, and conversely. Let $T$ be the set of all ordered triples of consecutive members of the sequence $f: Z \rightarrow Z$ as defined above, ordered from left to right. Let $t:\langle q\rangle \rightarrow T$ with $t\left(q^{n}\right)=\left(f_{n-1}, f_{n}, f_{n+1}\right)$. Let "multiplication" be defined on $T$ by

$$
\left(f_{m-1}, f_{m}, f_{m+1}\right)\left(f_{n-1}, f_{n}, f_{n+1}\right)=\left(f_{m+n-1}, f_{m+n}, f_{m+n+1}\right)
$$

Then

$$
\begin{aligned}
t\left(q^{m} q^{n}\right) & =t\left(q^{m+n}\right)=\left(f_{m+n-1}, f_{m+n}, f_{m+n+1}\right) \\
& =\left(f_{m-1}, f_{m}, f_{m+1}\right)\left(f_{n-1}, f_{n}, f_{n+1}\right)=t\left(q^{m}\right) t\left(q^{n}\right)
\end{aligned}
$$

Thus, $t$ establishes an isomorphism between $\langle q\rangle$ and $T$. Putting $F=$ ts, the composition of $s$ and $t$, one obtains $F: Z \rightarrow T$ with $F_{n}=\left(f_{n-1}, f_{n}, f_{n+1}\right)$. Since both $s$ and $t$ are isomorphisms, so is $F$, and ( $T, \cdot$ ) is a multiplicative group isomorphic to the additive group of the integers. Moreover, using the familiar addition for ordered triples of numbers,

$$
(a, b, c)+\left(a^{\prime}, b^{\prime}, c^{\prime}\right)=\left(a+a^{\prime}, b+b^{\prime}, c+c^{\prime}\right)
$$

one obtains, as in Lemma 4, $F_{n}=F_{n-2}+F_{n-1}$. Summarizing these results, one obtains the following lemma.
Lemma 5: Let $f: Z \rightarrow Z$ with $f_{0}=0, f_{1}=1$, and $f_{n}=f_{n-2}+f_{n-1}$. Let

$$
T=\left\{\left(f_{n-1}, f_{n}, f_{n+1}\right): n \in z\right\}
$$

and let $F: Z \rightarrow T$ with $F_{n}=\left(f_{n-1}, f_{n}, f_{n+1}\right)$. Let $F_{n} F_{m}=F_{n+m}$ and let

$$
F_{n}+F_{m}=\left(f_{n-1}+f_{m-1}, f_{n}+f_{m}, f_{n+1}+f_{m+1}\right)
$$

Then $F$ is a bijective function constituting a two-sided sequence with terms $F_{n}, n \in Z$, and the property $F_{n}=F_{n-2}+F_{n-1}$. Moreover, the terms of $F$ form an abelian group under multiplication, isomorphic to the additive group of the integers.

The group ( $T, \cdot$ ) may be called a Fibonacci induced group. The sequences $f$ and $F$ form the first and second levels of an infinite hierarchy. Next, one may consider the set of all ordered triples of consecutive members of the sequence $F: Z \rightarrow T$, ordered from left to right, say $\mathcal{J}=\left\{\left(F_{n-1}, F_{n}, F_{n+1}\right): n \varepsilon Z\right\}$. Let $\mathfrak{F}: Z \rightarrow \mathcal{J}$ with $\mathfrak{F}_{n}=\left(F_{n-1}, F_{n}, F_{n+1}\right)$. Further, let $\mathfrak{F}_{n} \mathfrak{F}_{m}=\mathfrak{F}_{n+m}$ and

$$
\mathfrak{F}_{n}+\mathfrak{F}_{m}=\left(F_{n-1}+F_{m-1}, F_{n}+F_{m}, F_{n+1}+F_{m+1}\right)
$$

Although $\mathfrak{J}$ is not closed under addition, one still has $\mathfrak{F}_{n}=\mathfrak{F}_{n-2}+\mathfrak{F}_{n-1}$, because

$$
\begin{aligned}
\mathcal{F}_{n} & =\left(F_{n-1}, F_{n}, F_{n+1}\right)=\left(F_{n-3}+F_{n-2}, F_{n-2}+F_{n-1}, F_{n-1}+F_{n}\right) \\
& =\left(F_{n-3}, F_{n-2}, F_{n-1}\right)+\left(F_{n-2}, F_{n-1}, F_{n}\right)=F_{n-2}+F_{n-1} .
\end{aligned}
$$

The terms of $\mathcal{F}$ again form an abelian group under multiplication, isomorphic to the additive group of the integers. The identity element is $\mathcal{F}_{0}=\left(F_{-1}, F_{0}, F_{1}\right)$ and the inverse of $\mathfrak{F}_{n}$ is $\mathfrak{F}_{-n}$. Associativity and commutativity are inherited from the integers that serve as indices.

## 3. CONCLUSION: FROM TRIPLES TO q-TUPLES

The previous section resulted in groups of triples and their hierarchy. The group operation was induced by the multiplication of $2 \times 2$ matrices. Discarding the matrices, one can define this operation as well on ordered $q$-tuples ( $q \in N, q>1$ ) of consecutive terms of the extended, two-sided Fibonacci sequence, with the ordering from left to right.

The ordered pairs are the first to be considered. Again let $f: Z \rightarrow Z$ with $f_{0}=0, f_{1}=1$, and $f_{n}=f_{n-2}+f_{n-1}$. Let $P=\left\{\left(f_{n-1}, f_{n}\right): n \varepsilon Z\right\}$, the set of al1 ordered pairs of consecutive terms of $f$, ordered from left to right. Let $F^{(2)}$ be the function $F^{(2)}: Z \rightarrow P$ with $F_{n}^{(2)}=\left(f_{n-1}, f_{n}\right)$.
Lemma 6: The function $F^{(2)}$ is bijective.
Proof: The set $\left\{F_{n}^{(2)}: n \in Z\right\}$ is partitioned into the sets

$$
A=\left\{F_{n}^{(2)}: n<0\right\},
$$

$$
B=\left\{F_{0}^{(2)}, F_{I}^{(2)}\right\}=\{(1,0),(0,1)\},
$$

and

$$
C=\left\{F_{n}^{(2)}: n>1\right\} .
$$

The three sets are disjoint, because: (i) every pair in $A$ contains a negative number (Lemma 1), no pair in $B$ or $C$ contains a negative number; (ii) every pair in $B$ contains 0, no pair in $C$ contains 0 . Moreover, if $m \neq n$, then $F_{m}^{(2)} \neq F_{n}^{(2)}$; in $B$, trivially; in $C$, because the second coordinates of the pairs form the set $\left\{f_{n}: n>1\right\}$ and for $n>1$ the Fibonacci numbers are all different; in $A$, because the absolute values of the first coordinates of the pairs form the set $\left\{f_{n}: n>1\right\}$, and hence are all different. Thus $F^{(2)}$ is injective. Obviously, $F^{(2)}$ is also surjective, and therefore bijective.

On $\left\{F_{n}^{(2)}: n \in Z\right\}$, let multiplication be defined by $F_{m}^{(2)} F_{n}^{(2)}=F_{m+n}^{(2)}$ and let addition be the usual addition of ordered pairs,

$$
F_{m}^{(2)}+F_{n}^{(2)}=\left(f_{m-1}+f_{n-1}, f_{m}+f_{n}\right)
$$

Clearly, the set is closed under multiplication but not under addition. However, $F_{n}^{(2)}=F_{n-2}^{(2)}+F_{n-1}^{(2)}$ because

$$
\begin{aligned}
F_{n}^{(2)}=\left(f_{n-1}, f_{n}\right) & =\left(f_{n-3}+f_{n-2}, f_{n-2}+f_{n-1}\right)=\left(f_{n-3}, f_{n-2}\right)+\left(f_{n-2}, f_{n-1}\right) \\
& =F_{n-2}^{(2)}+F_{n-1}^{(2)} .
\end{aligned}
$$

The terms of $F^{(2)}$ form an abelian group under multiplication, with

$$
F_{0}^{(2)}=\left(f_{-1}, f_{0}\right)=(1,0)
$$

as identity element and $F_{n}^{(2)}$ as the inverse of $F_{n}^{(2)}$. Associativity and commutativity are again inherited from the addition of the integers that serve as indices.

Passing to ordered $q$-tuples, let $q$ be a fixed positive integer $>1$. Let $Q=\left\{\left(f_{n-1}, f_{n}, \ldots, f_{n+q-2}\right): n \varepsilon Z\right\}$, the set of all ordered $q$-tuples of consecutive terms of $f$, ordered from left to right. Further, let $F^{(q)}: Z \rightarrow Q$ with $F_{n}^{(q)}=\left(f_{n-1}, \ldots, f_{n+q-2}\right)$, (the choice of indices is for the sake of the previous ordered triples).
Lemma 7: For every $q \in N-\{1\}$, the function $F^{(q)}$ is bijective.
Proof: Obviously, $F^{(q)}$ is surjective. The proof that $F^{(q)}$ is injective is by straightforward induction on $q$.

Base step. -Given by Lemma 6.
Induction step. - Assume that the lemma holds for $m$, i.e., all ordered $m$ tuples of consecutive terms of $f$ are different. Then clearly all ordered
$(m+1)$-tuples are different also, since

$$
\left(f_{n-1}, f_{n}, \ldots, f_{n+m-1}\right)=\left(\left(f_{n-1}, \ldots, f_{n+m-2}\right), f_{n+m-1}\right)
$$

and ordered pairs with different first coordinates are different.
On $\left\{F_{n}^{(q)}: n \in Z\right\}$, let multiplication be defined by $F_{m}^{(q)} F_{n}^{(q)}=F_{m+n}^{(q)}$ and let addition be the usual addition of ordered $q$-tuples

$$
F_{m}^{(q)}+F_{n}^{(q)}=\left(f_{m-1}+f_{n-1}, \ldots, f_{m+q-2}+f_{n+q-2}\right) .
$$

Again, there is closure under multiplication, but not under addition. Still $F_{n}^{(q)}=F_{n-2}^{(q)}+F_{n-1}^{(q)}$ because

$$
\begin{aligned}
F_{n}^{(q)} & =\left(f_{n-1}, \ldots, f_{n+q-2}\right)=\left(f_{n-3}+f_{n-2}, \ldots, f_{n+q-4}+f_{n+q-3}\right) \\
& =\left(f_{n-3}, f_{n-2}, \ldots, f_{n+q-4}\right)+\left(f_{n-2}, f_{n-1}, \ldots, f_{n+q-3}\right) \\
& =F_{n-2}^{(q)}+F_{n-1}^{(q)} .
\end{aligned}
$$

The terms of $F^{(q)}$ form an abelian group under multiplication with

$$
F_{0}^{(q)}=\left(f_{1}, f_{0}, \ldots, f_{q-2}\right)
$$

as identity element and $F_{-n}^{(q)}$ as the inverse of $F_{n}^{(q)}$. Associativity and commutativity are again inherited from the integers. All this results in a generalization of Lemma 5.
Theorem 1: Let $f: Z \rightarrow Z$ with $f_{0}=0, f_{1}=1$, and $f_{n}=f_{n-2}+f_{n-1}$. For any fixed $q \varepsilon N-\{1\}$, let

$$
Q=\left\{\left(f_{n-1}, f_{n}, \ldots, f_{n+q-2}\right): n \varepsilon Z\right\}
$$

and let $F^{(q)}: Z \rightarrow Q$ with

$$
F_{n}^{(q)}=\left(f_{n-1}, f_{n}, \ldots, f_{n+q-2}\right)
$$

Further, let $F_{m} F_{n}=F_{m+n}$ and

$$
F_{m}+F_{n}=\left(f_{m-1}+f_{n-1}, f_{m}+f_{n}, \ldots, f_{m+q-2}+f_{n+q-2}\right)
$$

Then $F^{(q)}$ is a bijective function constituting a two-sided sequence with terms $F_{n}^{(q)}, n \in Z$, and the property $F_{n}^{(q)}=F_{n-2}^{(q)}+F_{n-1}^{(q)}$. Moreover, the terms of $F^{(q)}$ form an abelian group under multiplication.

The hierarchy is now more complicated. Again calling $f$ the first level, one obtains a second level which contains an infinity of sequences $F^{(q)}$, one for every $q \in N-\{1\}$. Each $F^{(q)}$ in its turn contributes infinitely many sequences $\mathcal{F}(q, r), q, r \in N-\{1\}$, to the third level of the hierarchy, where the terms of $\mathfrak{F}(q, r)$ consist of ordered $r$-tuples (from left to right) of consecutive terms of $F^{(q)}$. This construction can be repeated indefinitely. One can picture the hierarchy as a partial ordering, as follows:


One may avoid running out of letter types by noticing that the level number is one more than the number of coordinates in the tuples that form the superscripts. This way one can use capital letters for all levels, e.g., $F(q, r, s)$
refers to a sequence of the fourth level, whose terms consist of ordered stuples of consecutive terms of $F^{(q, r)}$; this $F^{(q, r)}$ in its turn is a sequence of the third level, whose terms consist of ordered $r$-tuples of consecutive terms of $F^{(q)}$; and $F^{(q)}$ is a sequence of the second level whose terms consist of ordered $q$-tuples of consecutive terms of $f$.

The hierarchy results in a generalization of Theorem 1.
Theorem 2: Let $f: Z \rightarrow Z$ with $f_{0}=0, f_{1}=1$, and $f_{n}=f_{n-2}+f_{n-1}$. For every $m \in N$, let $q_{1}, \ldots, q_{m} \varepsilon N-\{1\}$. Let

$$
Z^{\left(q_{1}\right)}=\left\{\left(f_{n-1}, f_{n}, \ldots, f_{n+q_{1}-2}\right): n \varepsilon Z\right\}
$$

the set of all ordered $q_{1}$-tuples of consecutive terms of $f$. Let $F^{\left(q_{1}\right)}: Z \rightarrow Z^{\left(q_{1}\right)}$ with

$$
F^{\left(q_{1}\right)}=\left(f_{n-1}, f_{n}, \ldots, f_{n+q_{1}-2}\right)
$$

Further, let

$$
Z^{\left(q_{1}, \ldots, q_{m}\right)}=\left\{\left(F_{n-1}^{\left(q_{1}, \ldots, q_{m-1}\right)}, F_{n}^{\left(q_{1}, \ldots, q_{m-1}\right)}, \ldots, F_{n+q_{m}-2}^{\left(q_{1}, \ldots, q_{m-1}\right)}\right): n \varepsilon \Omega\right\}
$$

the set of all ordered $q_{m}$-tuples of consecutive terms of $F^{\left(q_{1}, \ldots, q_{m-1}\right)}$. Let

$$
F_{n}^{\left(q_{1}, \ldots, q_{m}\right)}: Z \rightarrow Z^{\left(q_{1}, \ldots, q_{m}\right)}
$$

with

$$
F_{n}^{\left(q_{1}, \ldots, q_{m}\right)}=\left(F_{n-1}^{\left(q_{1}, \ldots, q_{m-1}\right)}, F_{n}^{\left(q_{1}, \ldots, q_{m-1}\right)}, \ldots, F_{n+q_{m}-2}^{\left(q_{1}, \ldots, q_{m-1}\right)}\right) .
$$

Then $F^{\left(q_{1}, \ldots, q_{m}\right)}$ constitutes a two-sided sequence with terms $F_{n}^{\left(q_{1}, \ldots, q_{m}\right)}, n \in Z$, and the property

$$
F_{n}^{\left(q_{1}, \ldots, q_{m}\right)}=F_{n-2}^{\left(q_{1}, \ldots, q_{m}\right)}+F_{n-1}^{\left(q_{1}, \ldots, q_{m}\right)} .
$$

Moreover, the terms of $F^{\left(q_{1}, \cdots, q_{m}\right)}$ form an abelian group under the multiplication

$$
F_{n}^{\left.\left(q_{1}, \ldots, q_{m}\right)_{p}^{\left(q_{1}\right.}, \ldots, q_{m}\right)}=F_{n+p}^{\left(q_{1}, \ldots, q_{m}\right)} .
$$

$$
\begin{gathered}
* * * * * \\
\text { EXPLORING AN ALGORITHM } \\
\text { DMITRI THORO and HUGH EDGAR } \\
\text { San Jose State University, San Jose CA } 95192
\end{gathered}
$$

Dedicated to the memory of our dear friend and colleague, Vern

## 1. INTRODUCTION

We start with a simple algorithm for generating pairs $L$ (left column) and $R$ (right column) of Fibonacci numbers. In a slightly modified version we wish to investigate the ratios $L / R$ as the number of iterations $n \rightarrow \infty$. This, it turns out, involves (ancient) history, geometry, number theory, linear algebra, numerical analysis, etc.!

## 2. THE BASIC ALGORITHMS

Let us consider a "computer project" (appropriate for the first assignment in an Introduction to Programming course):

Given a suitable positive integer $N$, write a program which generates the sum of the first $N$ Fibonacci numbers with even subscripts.
Of course one can generate $F_{2 i}$ and form a cumulative sum. A more imaginative student, however, might use the following algorithm.
Algorithm I:
(a) Input $N$
(b) $L \leftarrow 1, R \leftarrow 0$
(c) $L \leftarrow L+R, R \leftarrow L+R, N \leftarrow N-1$
(d) If $N \neq 0$, go to step (c); else output $L+R-1$ and stop. [" $\leftarrow$ " means "is replaced by."]
Thus in BASIC PLUS we would write
10 INPUT N

| 20 | $L=1 \mid \quad R=0$ |
| :--- | :--- |
| 30 | $L=L+R\|\quad R=L+R\| \quad N=N-1$ |
| 40 | IF N $<>0$ THEN 30 ELSE PRINT $A+B-1$ |
| 999 | END |

Or, on the TI 59 Programmable Calculator, we could enter $N$, press $A$, and execute:

LBL A STO 001 STO 01 O STO 02
LBL B RCL 02 SUM 01 RCL 01 SUM 02
DSZ 0 B RCL $01+$ RCL $02-1=R / S$
[Here LBL, STO, RCL, SUM, DSZ, and R/S are codes for labe1, store, recall, sum, decrement-and-skip-on-zero, and run-stop, respectively. In particular, $N$ is placed in memory location 00 and, after each pair of consecutive Fibonacci numbers is generated, the contents of loc. 00 is decreased by 1 ; if the result $\neq$ 0 , we repeat by going back to "LBL B".]

The reader is invited to guess (or determine) the values of $N$ for which our output doesn't exceed the 10 digits which are displayed on the TI 59.

If we started with $L=R=1$, then the pairs $L, R$ would have ratios $L / R$ approaching the golden mean $(1+\sqrt{5}) / 2$. Given a pair $L, R$ let us now generate the next pair by slightly modifying the preceding algorithm.
Algorithm II: Given $N(N>0)$
(a) $L \leftarrow 1, R \leftarrow 1$
(b) $T \leftarrow L+N R, R \leftarrow L+R, L \leftarrow T$
(c) Repeat step (b) if desired; else output $L / R$.

We wish to investigate the ratios $L / R$ as the number of iterations $n \rightarrow \infty$.

## 3. PRELIMINARY OBSERVATIONS

Algorithm II can be described by the equations

$$
\left\{\begin{array}{l}
L_{k+1}=L_{k}+N R_{k} \\
R_{k+1}=L_{k}+R_{k}, k=0,1,2, \ldots,
\end{array}\right.
$$

where $L_{0}, R_{0}$, and $N>0$ are given real numbers. We will use the matrix form

$$
\binom{L_{k+1}}{R_{k+1}}=A\binom{L_{k}}{R_{k}} \text { where } A=\left(\begin{array}{ll}
1 & N \\
1 & 1
\end{array}\right) .
$$

Two examples are:

| $L_{0}=R_{0}$ | $N=2$ | $L_{0}=R_{0}=1, N=3$ |  |
| :---: | :---: | :---: | :---: |
|  | (2) |  | (3) |
| 1 | 1 | 1 | 1 |
| 3 | 2 | 4 | 2 |
| 7 | 5 | 10 | 6 |
| 41 | 29 | 28 | 16 |
| : | : |  | : |

The ratios $L / R$ in each row are, indeed, the convergents of the continued fraction expansions of $\sqrt{2}$ and $\sqrt{3}$, respectively. (The reader is invited to try $L_{0}=R_{0}=1, N=7$.) Will this ever happen again?

## 4. AN ATTEMPT TO ACCELERATE CONVERGENCE

After consideration of additional examples, it becomes evident that for large $N$ the values $L / R \rightarrow \sqrt{N}$ slowly. This suggests that we might be able to accelerate convergence by applying the algorithm to $1 / N$ and then taking the reciprocal of the final approximation.

Unfortunately, this doesn't help. E.g., for $L_{0}=R_{0}=1, N=5$, we get ratios $1,3,2,7 / 3,11 / 5,9 / 4, \ldots$, while $N=1 / 5$ yields ratios $1,3 / 5,1 / 2$, $7 / 15,5 / 11,9 / 20, \ldots$.

In general,

$$
R_{2 k}(1 / N)=1 / R_{2 k}(N) \quad \text { and } \quad R_{2 k+1}(1 / N)=R_{2 k+1}(N) / N .
$$

Thus, if $R_{i}(N) \rightarrow \sqrt{N}$, then

$$
R_{2 k}(1 / N) \rightarrow 1 / \sqrt{N} \quad \text { and } \quad R_{2 k+1}(1 / N) \rightarrow \sqrt{N} / N=1 / \sqrt{N} ;
$$

i.e., $R_{i}(1 / N) \rightarrow 1 / \sqrt{N}$ as $i \rightarrow \infty$ and, moreover, convergence is "at the same rate."

As we will later see, it is the size of the ratio

$$
|g(N)|=\left|\frac{1-\sqrt{N}}{1+\sqrt{N}}\right|
$$

which determines the rate of convergence; the smaller the ratio, the faster the convergence! Since $|g(1 / T)|=|g(T)|$ the above idea is fruitless. Put another way, the closer $N$ is to 1 , rather than 0 , the faster the convergence.
5. A MATRIX PROOF
(a) We start with the characteristic polynomial

$$
f(\lambda)=\operatorname{det}(A-\lambda I)=\lambda^{2}-2 \lambda+1-N
$$

associated with the matrix

$$
A=\left(\begin{array}{ll}
1 & N \\
1 & 1
\end{array}\right)
$$

Solving $f(\lambda)=0$ we get eigenvalues $\lambda_{1}=1+\sqrt{N}$ and $\lambda_{2}=1-\sqrt{N}$.
(b) Applying the division algorithm to the polynomials $\lambda^{k}$ and $f(\lambda)$ (in the Euclidean domain $R[\lambda]$ of polynomials with real coefficients) yields

$$
\lambda^{k}=\left(\lambda^{2}-2 \lambda+1-N\right) g(\lambda)+r(\lambda)
$$

where $r(\lambda)=\beta_{1}+\beta_{2} \lambda$.
(c) Setting $\lambda=\lambda_{1}, \lambda_{2}$ we get

$$
\lambda_{1}^{k}=\beta_{1}+\beta_{2} \lambda_{1} \quad \text { and } \quad \lambda_{2}^{k}=\beta_{1}+\beta_{2} \lambda_{2}
$$

When solved simultaneously, one finds

$$
\beta_{1}=\left(\lambda_{2} \lambda_{1}^{k}-\lambda_{1} \lambda_{2}^{k}\right) /\left(\lambda_{2}-\lambda_{1}\right) \quad \text { and } \quad \beta_{2}=\left(\lambda_{2}^{k}-\lambda_{1}^{k}\right) /\left(\lambda_{2}-\lambda_{1}\right) .
$$

(d) Invoking the Cayley-Hamilton theorem produces $A^{k}=\beta_{1} I+\beta_{2} A$ (where $I$ is, as usual, the $2 \times 2$ identity matrix).
(e) Our original matrix equation can easily be written in the form

$$
\binom{L_{k}}{R_{k}}=A^{k}\binom{L_{0}}{R_{0}}, k=1,2,3, \ldots .
$$

Using (c), (d), and a little algebra, we get

$$
\binom{L_{k}}{R_{k}}=\binom{\left(\beta_{1}+\beta_{2}\right) L_{0}+\beta_{2} N R_{0}}{\beta_{2} L_{0}+\left(\beta_{1}+\beta_{2}\right) R_{0}}
$$

or

$$
\frac{L_{k}}{R_{k}}=\frac{\left(\frac{\beta_{1}}{\beta_{2}}+1\right) L_{0}+N R_{0}}{L_{0}+\left(\frac{\beta_{1}}{\beta_{2}}+1\right) R_{0}}
$$

However, $\beta_{1} / \beta_{2}=\frac{\lambda_{2}-\lambda_{1}\left(\lambda_{2} / \lambda_{1}\right)^{k}}{\left(\lambda_{2} / \lambda_{1}\right)^{k}-1} \rightarrow-\lambda_{2}$ as $k \rightarrow \infty$ (since $\left|\lambda_{1}\right|>\left|\lambda_{2}\right|$ ). Thus

$$
\frac{L_{k}}{R_{k}} \rightarrow \frac{\sqrt{N} L_{0}+N R_{0}}{L_{0}+\sqrt{N R_{0}}}=\sqrt{N} \text { as } k \rightarrow \infty .
$$

6. SOME ACCIDENTS

Consider $Q \equiv \frac{\sqrt{N} L+N R}{L+\sqrt{N} R}$.
(a) Illegal Cancellation 1.1: "Erasing" the first term in the numerator and denominator of $Q$ yields $Q=N R /(\sqrt{N} R)=\sqrt{N}$.
(b) Illegal Cancellation 1.2: "Erasing" the second term in the numerator and denominator yields $Q=\sqrt{N} L / L=\sqrt{N}$.
(c) Illegal Simplification 1.3: Setting $L=R=1$, we get

$$
Q=(\sqrt{N}+N) /(1+\sqrt{N})=\sqrt{N}
$$

(d) Of course, even without multiplying numerator and denominator of $Q$ by $L-\sqrt{N} R$,

$$
Q=N\left(\frac{L+(N / \sqrt{N}) R}{L+(N / \sqrt{N}) R}\right)=\sqrt{N}
$$

Moreover, $(s L+t R) /(L+s R)=s$ implies $s L+t R=s L+s^{2} R$ or $s=\sqrt{t}$; thus in one sense our accidents are unique!

## 7. ANOTHER MODIFICATION

Instead of considering the ratios $L_{k} / R_{k}$, suppose we now look at $L_{k+1} / L_{k}$. E.g., when $N=2$ we get $L_{6} / L_{5}=239 / 99 \approx 2.41414$. Thus, in general, one might guess $L_{k+1} / L_{k} \rightarrow 1+\sqrt{N}$ as $k \rightarrow \infty$. Not only is this the case in general, but in numerical analysis consideration of "ratios of corresponding components" yields the so-called Power Method for computing the numerically largest eigenvalue of a matrix.

To see the essential notions, let $T$ be a $2 \times 2$ matrix with eigenvalues

$$
\left|\lambda_{1}\right|>\left|\lambda_{2}\right| \geq 0
$$

and linearly independent eigenvectors $\mathbf{x}_{1}, \mathbf{x}_{2}$. If $\mathbf{V}^{(0)}$ is an arbitrary vector, then suppose

$$
\mathbf{V}^{(0)}=c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}, \text { where } c_{1} \neq 0
$$

As before, define $\mathbf{V}^{(m)}=A \mathbf{V}^{(m-1)}, m=1,2, \ldots$. This yields

$$
\mathbf{V}^{(m)}=c_{1} \lambda_{1}^{m} \mathbf{x}_{1}+c_{2} \lambda_{2}^{m} \mathbf{x}_{2}=\lambda_{1}^{m}\left(c_{1} \mathbf{x}_{1}+c_{2}\left(\frac{\lambda_{2}}{\lambda_{1}}\right)^{m} \mathbf{x}_{2}\right)
$$

(since $A \mathbf{x}_{i}=\lambda \mathbf{x}_{i}$ ) with the second term $\rightarrow 0$ as $m \rightarrow \infty$. Thus if $\mathbf{V}^{(m)} \approx \lambda_{1}^{m} c_{1}\binom{a}{b}$, then the ratio of, say, first components

$$
\frac{\lambda_{1}^{m+1} c_{1} a}{\lambda_{1}^{m} c_{1} a}
$$

approximates $\lambda_{1}$; moreover, $\binom{a}{b}$ is a corresponding eigenvector.
In actual practice this version of the Power Method is usually improved by an appropriate scaling (such as normalization) to avoid overflow. Modifications for the case of a symmetric matrix and deflation techniques (for approximating nondominant eigenvalues) are discussed in [1].
8. CONCLUSION

It is somewhat amusing that for many years one of the authors asked students to investigate Algorithm II without being aware that its probable origins go back some nineteen centuries. An interesting discussion of its relationship to Pell's Equation as well as to the geometry of the ancient Greeks may be found in [3].

We leave the reader with at least two possible excursions. Suppose $N$ is a positive (nonsquare) integer with continued fraction convergents $p_{k} / q_{k}$ [2].
(a) If $L_{k} / R_{k}=p_{k} / q_{k}$ for $k=0$ and 1 , what can you say about $N$ ?
(b) If the equation in (a) holds for $k=0,1$, and 2 , what can be said about $N$ ? (E.g., it holds when $N=7$.)

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## AN IMPLICIT TRIANGLE OF NUMBERS

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Sketch reprinted from California MathematiCs 5(2), October, 1980.
To Vern Hoggatt, whose common sense, plain language, and energetic enthusiasm brought real mathematics into the
lives of diverse people throughout the world.
This elementary note introduces a new triangle of numbers that is implicitly defined in Pascal's Triangle. It shares many properties with Pascal's Triangle, including the generation of Fibonacci numbers. It differs from Pascal's Triangle in that it is not symmetrical (and therefore is not a special case of the Fontené-Ward Triangle [1]). When I asked Vern Hoggatt-who seemed to know everything there is to know about Pascal's Triangle-about the Implicit Triangle, he surprised me by replying that he did not know of either the triangle or any of its properties. Therefore, the following may add to our readers' list of "Neat Little Facts about Integers."

The question that led to the discovery of the Implicit Triangle is: "How do we get the squares out of Pascal's Triangle?" One fairly well-known way is to note that

$$
0+1=1,1+3=4,3+6=9, \ldots,\binom{n+1}{2}+\binom{n}{2}=n^{2}
$$

This can be generalized using Eulerian numbers, so that

$$
\begin{gathered}
0+4(0)+1=1,0+4(1)+4=8,1+4(4)+10=27, \ldots, \\
\binom{n}{3}+4\binom{n+1}{3}+\binom{n+2}{3}=n,\binom{n}{4}+11\binom{n+1}{4}+11\binom{n+2}{4}+\binom{n+3}{4}=n^{4},
\end{gathered}
$$

etc. See [2], for example. But there is another way to get the squares out of Pascal's Triangle, and this is not so well known:

$$
\begin{aligned}
& \binom{0}{1}+\binom{0}{2}+\binom{1}{1}+\binom{1}{2}=1,\binom{1}{1}+\binom{1}{2}+\binom{2}{1}+\binom{2}{2}=4, \\
& \binom{2}{1}+\binom{2}{2}+\binom{3}{1}+\binom{3}{2}=9, \ldots,\binom{n-1}{1}+\binom{n-1}{2}+\binom{n}{1}+\binom{n}{2}=n^{2} .
\end{aligned}
$$

These squares are generated by adding rhombuses of entries from Pascal's Triangle. By adding other rhombuses, we generate our new triang1e:

$$
\begin{aligned}
& / 2 / 1 / 0-1-1 \\
& / 2 / 3 / 1 / 0 / 0 \\
& 12 / 5 / 4 / 1 / 0 / \\
& \text { /2/7/9/5/1-1/0/1 } \\
& 0-1-3-3-1-0-0-0
\end{aligned}
$$

$$
\begin{aligned}
& \text { / } 1 / 13 / 36 / 55 / 55 / 27 / 8 / 10-10-0-1 \\
& 0-1-6-15-20-15-6-1-0-0-0 \\
& 0-1 \text { - } 7-21-35-35-21-7-1-0-0-0 \\
& \text { / } 2 / 17 / 64 / 140 / 196 / 182 / 112 / 44 / 10 / 1 / 0 / \\
& 0-1-8-28-56-70-56-28-8-1-0-0-0 \\
& 12 / 19 / 81 / 204 / 336 / 378 / 294 / 156 / 54 / 11 / 1 / 0 / 0
\end{aligned}
$$

Suppressing the entries from Pascal's Triangle, we get the (almost) triangular array:

$$
\begin{aligned}
& 21 \\
& \begin{array}{lll}
2 & 3 & 1
\end{array} \\
& \begin{array}{llll}
2 & 5 & 4 & 1
\end{array} \\
& \begin{array}{lllllllll} 
& 2 & 7 & & 9 & 5 & 1 & \\
2 & & 9 & 16 & 14 & & 6 & & 1
\end{array} \\
& \begin{array}{lllllll}
2 & 11 & 25 & 30 & 20 & 7 & 1
\end{array} \\
& \begin{array}{llllllllllll} 
& 2 & 13 & 36 & 55 & 50 & 27 & 8 & 1 & \\
2 & 15 & 49 & 91 & 105 & 77 & 35 & 9 & 1
\end{array} \\
& \begin{array}{llllllllll}
2 & 17 & 64 & 140 & 196 & 182 & 112 & 44 & 10 & 1
\end{array} \\
& \begin{array}{lllllllllll}
2 & 19 & 81 & 204 & 336 & 378 & 294 & 156 & 54 & 11 & 1
\end{array}
\end{aligned}
$$

This Implicit Triangle has the generating formula

$$
I(n, k)=\binom{n-2}{k-1}+\binom{n-2}{k}+\binom{n-1}{k-1}+\binom{n-1}{k},
$$

where $I(n, k)$ is the Implicit Triangle entry in the $n$th row, $k$ th diagonal,

$$
n=1,2,3, \ldots, k=0,1,2,3, \ldots .
$$

(The zeroth row is missing from this new triangle.)

Although it lacks the symmetry of Pascal's Triangle, the Implicit Triangle shares many of its properties.
Theorem 1: $I(n-1, k-1)+I(n-1, k)=I(n, k)$.
Proof: This version of Pascal's Identity follows from that identity in Pascal's Triangle.

$$
\begin{aligned}
I(n-1, k-1)+I(n-1, k)= & \binom{n-3}{k-2}+\binom{n-3}{k-1}+\binom{n-2}{k-2}+\binom{n-2}{k-1} \\
& +\binom{n-3}{k-1}+\binom{n-3}{k}+\binom{n-2}{k-1}+\binom{n-2}{k} \\
= & \binom{n-2}{k-2}+2\binom{n-2}{k-1}+\binom{n-2}{k-1}+2\binom{n-2}{k} \\
= & \binom{n-2}{k-1}+\binom{n-2}{k}+\binom{n-1}{k-1}+\binom{n-1}{k} \\
= & I(n, k) .
\end{aligned}
$$

This is not really surprising, since the Implicit entries are linear combinations of Pascal entries, and these linear combinations carry along the properties of Pascal's Triangle.

Theorem 2 ("Christmas Stocking Theorem"):

$$
\sum_{n=k}^{k+r} I(n, k)=I(k+r+1, k+1)
$$

Theorem 3 ("Hockey Stick Theorem"):

$$
I(n, r)=\sum_{k=r+1}^{k=n}(-1)^{k-r-1} I(n+1, k)
$$

Theorem 4 ("Fibonacci Number Theorem"):

$$
\sum_{k=0}^{\infty} I(n-k, k)=F_{n+2}
$$

( $\infty$ exploits the fact that proceeding up a diagonal we eventually get all 0's.) Theorem 5 ("Alternating Row Sum Theorem"):

$$
\sum_{k=0}^{n+1}(-1)^{k} I(n, k)=0, n=2,3,4, \ldots
$$

Proofs: All of these theorems follow from the fact that the Implicit entries are linear combinations of the Pascal entries.

And then there are properties different from, but analogous to, properties of Pascal's Triangle. For examples,
Theorem 6 ("Lucas Number Theorem"):

$$
\sum_{k=0}^{\infty} I(n-k, n+1-2 k)=L_{k+1}
$$

Theorem 7 ("Row Sum Theorem"):

$$
\sum_{k=0}^{n} I(n, k)=2^{n-1}(3)
$$

Proofs: Both of these theorems may be proved just as their analogues are proved for Pascal's Triangle. Theorem 7 may be proved very easily with the aid of Theorem 8.
Theorem 8 ("Coefficient Theorem"): $I(n, k)$ is the coefficient of $x^{n-k}$ in the expansion of $(2 x+1)(x+1)^{n-1}$.

Proof: From the identity

$$
I(n, k)=\binom{n-1}{k-1}+2\binom{n-1}{k},
$$

we can see that the Implicit Triangle is formed from the binomial coefficients of two overlapping Pascal Triangles:

$$
I(n, k)=\binom{n}{k}+\binom{n-1}{k} .
$$

The theorem then follows from the fact that

$$
(2 x+1)(x+1)^{n-1}=x(x+1)^{n-1}+(x+1)^{n} .
$$

We are now in a position to look at a Generalized Implicit Triangle:

$$
\begin{aligned}
& \text { a } 1 \\
& a \quad(a+1) \quad 1 \\
& a(2 a+1) \quad(a+2) \quad 1 \\
& a(3 a+1) \quad(3 \alpha+3) \quad(a+3) \quad 1 \\
& a(4 \alpha+1)(6 a+4) \quad(4 \alpha+6) \quad(\alpha+4) \quad 1 \\
& a(5 a+1)(10 a+5)(10 \alpha+10)(5 a+10)(\alpha+5) 1 \\
& \text {. . . }
\end{aligned}
$$

Here the generating identity is

$$
G(n, k)=G(n-1, k-1)+G(n-1, k), G(n, 0)=\alpha, G(n, n)=1
$$

for $\alpha=1$, this is just Pascal's Identity.
Theorem 9 ("Generalized Coefoicient Theorem"): $G(n, k)$ is the coefficient of $x^{n-k}$ in the expansion of $(a x+1)(x+1)^{n-1}$.

Proof: The Generalized Implicit Triangle is again just the overlap of Pascal's Triangle and Pascal's Triangle with every entry multiplied by $a-1$. The theorem follows from the identity

$$
(a x+1)(x+1)^{n-1}=(x+1)^{n}+(\alpha-1) x(x+1)^{n-1}
$$

Since each entry of the Generalized Implicit Triangle is a linear combination of entries from Pascal's Triangle, those foregoing theorems whose proofs were based on linear combinations will hold in the general case, with appropriate modifications; for example, the row sums will be of the form $2^{n-1}(a+1)$.

Had Vern Hoggatt been able to coauthor this article he would no doubt have found many more results. Perhaps our readers will celebrate his memory by looking for further results themselves.

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## FRACTIONAL PARTS $(n r-s)$, ALMOST ARITHMETIC SEQUENCES, AND FIBONACCI NUMBERS

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To the memory of Vern Hoggatt, with gratitude and admiration.
Except where noted otherwise, sequences $\left\{a_{n}\right\},\left\{b_{n}\right\}$, and $\left\{c_{n}\right\}$ are understood to satisfy the following requirements, as stated for $\left\{a_{n}\right\}$ :
(i) the indexing set $\{n\}$ is the set of $\alpha Z Z$ integers;
(ii) $a_{n}$ is an integer for every $n$;
(iii) $\left\{\alpha_{n}\right\}$ is a strictly increasing sequence;
(iv) the least positive term of $\left\{\alpha_{n}\right\}$ is $\alpha_{1}$.

We call $\left\{a_{n}\right\}$ almost arithmetic if there exist real numbers $u$ and $B$ such that

$$
\begin{equation*}
\left|a_{n}-u n\right|<B \tag{1}
\end{equation*}
$$

for all $n$, and we write $a_{n} \sim$ un if (1) holds for some $B$ and all $n$.
Suppose $r$ is any irrational number and $s$ is any real number. Put
$c_{m}=[m r-s]=$ the greatest integer less than or equal to $m r-s$,
and let $b$ be any nonzero integer. It is easy to check that $c_{m+b}-c_{m}=[b r]$, if $(m r-s)<(-b r)$, and $=[b r]+1$, otherwise.

Let $a_{n}$ be the $n$th term of the sequence of all $m$ satisfying $c_{m+b}-c_{m}=[b r]$.
In the following examples, $r=(1+\sqrt{5}) / 2$, the golden mean, and $s=1 / 2$. Selected values of $m$ and $c_{m}$ are: $-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16$ $-9,-7,-6,-4,-3,-1,1,2,4,5,7,9,10,12,14,15,17,18,20,22,23,25$.
When $b=1$ we have $[b r]=1$, and selected values of $n$ and $a_{n}$ are:

$$
\begin{aligned}
& -1, ~ 0,1,2,3,4, ~ 5, ~ 6 \\
& -4,-2,1,3,6,9,11,14 .
\end{aligned}
$$

When $b=2$ we have $[b r]=3$, and selected values of $n$ and $a_{n}$ are:

$$
\begin{aligned}
& -4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10,11 \\
& -5,-4,-3,-2,0, \underline{1}, \underline{2}, \underline{3}, \underline{5}, 6, \underline{8}, 9,10,11,13,14 .
\end{aligned}
$$

Note here the presence of Fibonacci numbers among the $a_{n}$. Methods given in this note can be used to confirm that the Fibonacci sequence is a subsequence of $\left\{a_{n}\right\}$ in the present case.

When $b=-2$ we have $[b r]=-4$, and selected values of $n$ and $a_{n}$ are:

$$
\begin{array}{r}
0,1, \\
-4, \\
-4,
\end{array}, 3,9,14
$$

The main purpose of this note is to give an elementary and constructive method of proving, in general, that the jump-sequence $\left\{\alpha_{n}\right\}$ is almost arithmetic. To accomplish this, we must solve the inequality $(m r-s)<(-b r)$ for $m$. The method of solution, when applied to the case $r=(1+\sqrt{5}) / 2$, leads to a number of identities involving Fibonacci numbers, Lucas numbers, and the greatest integer function.
Lemma 1: Suppose $a_{n} \sim$ un, where $u>1$. Let $\left\{a_{n}^{*}\right\}$ be the complement of $\left\{a_{n}\right\}$, that is, the sequence of integers not in $\left\{a_{n}\right\}$, indexed according to requirements i-iv. Then

$$
a_{n}^{*} \sim \frac{u}{u-1} n .
$$

Lemma 2: Suppose $a_{n} \sim u n$ and $b_{n} \sim v n$. Then the composite $c_{n}=b_{a_{n}}$ satisfies $c_{n} \sim u v^{2}$.
Lemma 3: Suppose $a_{n} \sim u n$ and $b_{n} \sim v n$, where $a_{j} \neq b_{k}$ for all $j$ and $k$. Let $\left\{c_{n}\right\}$ be the union of $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$. Then

$$
c_{n} \sim\left(\frac{1}{u}+\frac{1}{v}\right)^{-1} n .
$$

Proofs of the three lemmas found in [7] for positive $n$ can be extended readily to the case of all integers $n$.
Theorem: Suppose $r$ is an irrational number, $s$ a real number, and $b$ a nonzero integer. Let $\left\{a_{n}\right\}$ be the sequence of integers $m$ satisfying ( $m r-s$ ) $\leq(b r)$. Then $a_{n} \sim n /(b r)$.

Proof: First, we note that $m r-s$ can be an integer for at most one value of $m$, and that whether the sequence $\left\{a_{n}\right\}$ is almost arithmetic does not depend on whether it contains such an $m$. Accordingly, we shall assume that all fractional parts which occur in this proof are positive. Also without loss we assume that $0<r<1$.

Suppose $b \geq 1$, and let $p=[b r]$. If

$$
\begin{equation*}
(m r-s) \leq(b r), \tag{2}
\end{equation*}
$$

then for $k=[m r-s]$, the integer $m$ must lie in the interval

$$
J_{k}=\left(\frac{k+s}{r}, \frac{k+s}{r}+b-\frac{p}{r}\right]
$$

Conversely, any $m$ in such an interval satisfies (2) with $k=[m r-s]$ 。
Now let $q=[(b r) / r]$, the greatest integer $i$ satisfying $i-b+\frac{p}{p}<0$. Then

$$
\left(\frac{k+s}{r}\right) \geq q-b+\frac{p}{r}
$$

for all integers $k$, so that for $q \geq 1$ and $i=1,2, \ldots, q$, the integers

$$
\begin{equation*}
m_{k}=\left[\frac{k+s}{r}\right]+i, \quad k=0, \pm 1, \pm 2, \ldots, \tag{3}
\end{equation*}
$$

satisfy (2). Each of these sequences $\left\{m_{k}\right\}$ is almost arithmetic with slope $1 / r$.
If $q=0$ then some interval $J_{k}$ contains no integer. In this case the solutions of (2) are the integers $\left[\frac{k+s}{r}\right]+1$ for which $\left(\frac{k+s}{r}\right) \geq 1-b+\frac{p}{r}$. If $q \geq 1$, then the integers $\left[\frac{k+s}{p}\right]+q+1$, where

$$
\begin{equation*}
\left(\frac{k+s}{r}\right) \geq q+1-b+\frac{p}{p} \tag{4}
\end{equation*}
$$

are solutions of (2), along with the solutions given in (3), and by definition of $q$ there are no other solutions. The two inequalities involving $\left(\frac{k+s}{r}\right)$ are each equivalent to

$$
\begin{equation*}
\left(\frac{k+s}{r}\right) \geq\left(\frac{p}{r}\right) \tag{5}
\end{equation*}
$$

By Lemma 2 the sequences $\left[\frac{k+s}{r}\right]+1$ and $\left[\frac{k+s}{r}\right]+q+1$ are almost arithmetic if the sequence of $k$ satisfying (5) is so. By Lemma 1 this is indeed the case if the complementary sequence, consisting of all integers $k$ satisfying

$$
\left(\frac{k+s}{r}\right)<\left(\frac{p}{r}\right)
$$

is almost arithmetic. Except for at most one $k$, this inequality is equivalent to
(6) $\left(k r^{\prime}-s^{\prime}\right) \leq\left(p r^{\prime}\right)$,
where $r^{\prime}=(1 / r)$ and $s^{\prime}=-s / r$.
As (6) is of the same form as (2), we note that with a finite number of applications of the process from (2) to (6) the integer $p$ decreases to 0 , since initially $0 \leq p \leq b-1$. When $p=0$, the number on the right-hand side of (4) is 1 , indicating that there are no further values of $k$ to be found. By forming the union of the (pairwise disjoint) solution sequences which have been found, we get an almost arithmetic sequence, by Lemma 3 .

Suppose now $b \leq-1$. Then the integers $m$ satisfying $(-m r+s) \leq(-b r)$ form an almost arithmetic sequence. Thus, by Lemma 1 , those integers $m$ satisfying $(-m r+s)>(-b r)=1-(b r)$, or equivalently, $(m r-s)=1-(-m r+s)<(b r)$, form an almost arithmetic sequence.

We have finished proving that $\left\{\alpha_{n}\right\}$ is almost arithmetic. It remains to see that the number $u$ in (1) is $1 /(b r)$.

If $b=1$, then the $a_{n}$ are the numbers $\left[\frac{k+s}{r}\right]+1, k=0, \pm 1, \pm 2, \ldots$, as already proved, and hence $a_{n} \sim n /(r)$. For an induction hypothesis, suppose, for $b \geq 2$, that for all $d \leq b-1$ the sequence $\left\{c_{n}\right\}$ of solutions $m$ of

$$
\begin{equation*}
\left(m r^{\prime}+s^{\prime}\right) \leq\left(d r^{\prime}\right) \tag{7}
\end{equation*}
$$

satisfies $c_{n} \sim n /\left(d r^{\prime}\right)$, for any given positive irrational $r^{\prime}$ and real $s^{\prime}$. Let $\left\{b_{n}\right\}$ be the sequence of solutions of (7) where $r^{\prime}=(1 / r), s^{\prime}=-s / r$, and $d=p=[b r] \leq b-1$.

Let $\left\{b_{n}^{*}\right\}$ be the complement of $\left\{b_{n}\right\}$, so that

$$
b_{n}^{*} \sim \frac{n}{1-(p(1 / r))}=\frac{n}{1-(p / r)},
$$

by Lemma 2. There are no other solutions if $q=0$, and if $q \geq 1$, the remaining solutions are simply

$$
f_{k, i}=\left[\frac{k+s}{r}\right]+i, \quad k=0, \pm 1, \pm 2, \ldots ; i=1,2, \ldots, q
$$

as already proved. Since $f_{k, i} \sim k / r$ for $i=1,2, \ldots, q$, we have, by Lemma 3, for $q \geq 1$ :

$$
\begin{aligned}
a_{n} \sim \frac{n}{q r+r-r(p / r)} & =\frac{n}{[(b r) / r] r+r-r([b r] / r)} \\
& =\frac{n}{[(b r) / r] r+r-r(-(b r) / r)}=n /(b r)
\end{aligned}
$$

In case $q=0$, we find similarly $a_{n} \sim \frac{n}{r-r(p / r)}=n /(b r)$.
Finally, suppose $b \leq-1$. The integers $m$ satisfying $(-m r+s)>(-b r)$ form a sequence $\left\{c_{n}\right\}$ satisfying $c_{n} \sim n /(-b r)$. For the complement $\left\{a_{n}\right\}=\left\{c_{n}^{*}\right\}$, we have $a_{n} \sim n /(b r)$, by Lemma 1 .
Corollary 1: Suppose $r$ is an irrational number and $s$ is a real number. Suppose $a$ and $b$ are nonzero integers such that $(a r)<(b r)$. Let $\left\{a_{n}\right\}$ be the sequence of integers $m$ satisfying (ar) $<(m r-s) \leq(b r)$. Then

$$
a_{n} \sim \frac{n}{(b r)-(a r)} .
$$

Proof: Let $\left\{f_{n}\right\}$ and $\left\{h_{n}\right\}$ be the solution sequences of the inequalities $(m r-s) \leq(a r)$ and $(m r-s) \leq(b r)$, respectively. The sequence $\left\{a_{n}\right\}$ is, in the terminology of [7], the relative complement of $\left\{f_{n}\right\}$ in $\left\{h_{n}\right\}$. Applying the method used in [7], we conclude that

$$
a_{n} \sim \frac{n}{(b r)-(a r)}
$$

Fraenkel, Mushkin, and Tassa [3] have obtained results indicated in their title, "Determination of [ $n \theta$ ] by Its Sequence of Differences." The theorem in this present note supplements those results. We may ask, for example, for a sequence $\left\{c_{n}\right\}$ whose consecutive differences are all 1 's and 2's, determined by the rule $c_{n+1}-c_{n}=1$ for exactly those $n$ of the form $[m \theta-\phi]$, where $\theta$ and $\phi$ are given. The question leads to the following corollary.
Corollary 2: Suppose $\theta$ is a positive irrational number and $\phi$ is a real number. Let $h=[\theta]$, and let $\left\{c_{n}\right\}$ be the sequence determined by $c_{n+1}-c_{n}=h$ for exactly those $n$ of the form $[m \theta-\phi]$ and $=h+1$ otherwise. Then

$$
c_{n}=[n+n h-n / \theta-\phi / \theta], \quad n=0, \pm 1, \pm 2, \ldots .
$$

Proot: We have $c_{n+1}-c_{n}=h$ for exactly those $n$ satisfying ( $n r-s$ ) $<(-r)$, where $r=1+h-1 / \theta$ and $s=\phi / \theta$. These $n$ are the integers of the form $[m /(-r)-s /(-r)]$, but this is $[m \theta-\phi]$.

The method of proof of the theorem readily shows that for any irrational $r$ and any real $s$, the sequences given by

$$
\begin{equation*}
\left[\frac{n+s}{(x)}\right] \quad \text { and } \quad\left[\frac{n-s}{(-x)}\right] \tag{8}
\end{equation*}
$$

are complementary (except that one term, and only one, can be common to the two sequences, as when $n=s=0$ ). This fact is a generalization of the well-known result by Beatty [1], obtained here by putting $s=0$ and restricting the sequences to positive integers and $r$ to the unit interval.

Corresponding to (8), the jump-sequence $\left\{a_{n}\right\}$ of indexes $m$ such that $[m r+$ $r-s]-[m r-s]=[r]$ is given by $m=\left[\frac{n+s}{(r)}\right]+1$, and the complementary jumps of size $[r]+1$ occur at $m$ of the form $\left[\frac{n-s}{(r)}\right]+1$.

Explicit results for $b=2$ and only positive terms are also easy to state, in two cases: If $(r)<1 / 2$, then the three sequences

$$
\left[\frac{n+s}{(r)}\right],\left[\frac{n+s}{(r)}\right]+1,\left[\frac{\left[\frac{n-s /(-r)}{(-1 /(-r))}\right]+1-s}{(-r)}\right]
$$

are complementary, and if $(r)>1 / 2$, then the sequences

$$
\begin{equation*}
\left[\frac{n-s}{(-r)}\right],\left[\frac{n-s}{(-r)}\right]+1,\left[\frac{\left[\frac{n+s /(r)}{(-1 /(r))}\right]+1+s}{(r)}\right] \tag{9}
\end{equation*}
$$

are complementary.
For $(r)<1 / 2$, the jump-sequence of $m$ such that $[m r+2 r-s]-[m r-s]=$ $[2 r]$ is given by the union of the sequences $\left[\frac{n-s}{(-r)}\right]+1$ and $\left[\frac{n-s}{(-r)}\right]+2$, and jumps of size $[2 r]+1$ occur at integers $m=\left[\frac{k+s}{(r)}\right]+1$, where $k$ has the form $\left[\frac{n-s /(-r)}{(-1 /(r))}\right]+1$.

It is of historical interest that Hecke [4] first proved the theorem of this note in the case $s=0$. That ( $b r$ ) must equal ( $j r$ ) for some integer $j$ in order for $\left\{c_{m}\right\}$ to be an almost arithmetic sequence, for the case $s=0$, was proved by Kesten [6].

Taking $r$ in (9) to be $(1+\sqrt{5}) / 2$ leads to a number of identities involving Fibonacci numbers. For example, with $s=0$, the three sequences in (9) may be written

$$
\left[\frac{2 n}{3-\sqrt{5}}\right],\left[\frac{2 n}{3-\sqrt{5}}\right]+1,\left[\frac{2\left[\frac{2 n}{\sqrt{5}-1}\right]+2}{\sqrt{5}-1}\right]
$$

It is easy to prove that the first two of these sequences contain all the Fibonacci numbers. In fact, the method of Bergum [2] can be used to show that

$$
F_{n+2}= \begin{cases}{\left[\frac{2 F_{n}}{3-\sqrt{5}}\right]} & \text { for odd } n \geq 1 \\ {\left[\frac{2 F_{n}}{3-\sqrt{5}}\right]+1} & \text { for even } n \geq 0\end{cases}
$$

and

$$
\left.\begin{array}{rl}
{\left[2\left[\frac{2 F_{n}}{\sqrt{5}-1}\right]+2\right.} \\
\sqrt{5}-1
\end{array}\right]= \begin{cases}F_{n+3}+1 & \text { for odd } n \geq 1 \\
F_{n+3}-1 & \text { for even } n \geq 0\end{cases}
$$

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ON THE " $Q X+1$ PROBLEM," $Q$ ODD
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## Dedicated to the memory of V. E. Hoggatt

INTRODUCTION
In a previous paper [3] we studied the function

$$
f(n)= \begin{cases}(3 n+1) / 2 & n \text { odd }>1 \\ n / 2 & n \text { even } \\ 1 & n=1\end{cases}
$$

and showed that there are no nontrivial circuits for this function which are cycles. In the present paper we shall consider the analogous problem for

$$
h(n)= \begin{cases}(q n+1) / 2 & n \text { odd }>1, q \text { odd } \\ n / 2 & n \text { even } \\ 1 & n=1\end{cases}
$$

for $q=5$ and 7 and shall find all circuits which are cycles for these functions.

$$
\text { 1. THE CASE } Q=5
$$

Let

$$
f(n)= \begin{cases}(5 n+1) / 2 & n \text { odd }>1 \\ n / 2 & n \text { even } \\ 1 & n=1\end{cases}
$$

Theorem 1: Let $v_{2}(m)$ be the highest power of 2 dividing $m, m \varepsilon Z$ and let $n$ be an odd integer $>1$. Then, $n<f(n)<\cdots<f^{k}(n)$, and $f^{k+1}(n)<f^{k}(n)$, where $k=v_{2}(3 n+1)$.

Proof: Let $n_{0}=n_{1}, n=f^{i}(n), i>1$. Suppose $n_{1}, n_{2}, \ldots, n_{j-1}$ are all odd. Then

$$
\begin{aligned}
5 n_{0}+1 & =2 n_{1} \\
5 n_{1}+1 & =2 n_{2} \\
\ldots+1 & =2 n_{j}
\end{aligned}
$$

By simple recursion, we get

$$
2^{j} n_{j}=5^{j} n+\frac{5^{j}-2^{j}}{3}
$$

Thus

$$
\begin{equation*}
2\left(3 n_{j}+1\right)=5^{j}(3 n+1) \tag{1}
\end{equation*}
$$

If $j<v_{2}(3 n+1)$, then $n_{j}$ is odd and we may extend the increasing sequence. If $j=v_{2}(3 n+1)$, then $n_{j}$ is even. The result follows.

Following [2], let us write $n \xrightarrow{k} m \xrightarrow{\ell} n^{*}$, where $\ell=v_{2}(m), n^{*}=m / 2^{\ell}$, $k=v_{2}(3 m+1)$ and $m$ is given by $2^{k}(3 m+1)=5^{k}(3 n+1)$.

If we let $A$ be the set of positive odd integers and define $T: A \rightarrow A$ by ( $(\mathbb{R})$ $=n^{*}$, the map from $n$ to $n^{*}$ is called a circuit. Our goal is to prove that

$$
13 \xrightarrow{3} 208 \xrightarrow{4} 13 \quad \text { and } \quad 1 \xrightarrow{2} 8 \xrightarrow{3} 1
$$

are the only circuits which are cycles under $F$. We shall accomplish this by reducing our problem to a Diophantine equation and then using a result of Baker [1] to solve the equation.
Theorem 2: There exists $n$ such that $T(n)=n$ only if there are positive integers $k, l$, and $h$ satisfying

$$
\begin{equation*}
\left(2^{k+\ell}-5^{k}\right) h=2^{\ell}-1 \tag{2}
\end{equation*}
$$

Proof: Suppose $T(n)=n$. Then $2^{k}(3 m+1)=5^{k}(3 n+1)$ and $2^{l} n=m$. If we write

$$
\begin{aligned}
& 3 m+1=5^{k} h \\
& 3 n+1=2^{k} h
\end{aligned}
$$

We get

$$
\left(2^{k+1}-5^{k}\right) h=2^{\ell}-1
$$

as required.
We note that the converse of this theorem is false: $k=1, \ell=2$, $h=1$ yields a solution of (2), but this solution does not yield integer values of $m$ and $n$.
Theorem 3: The only solutions of equation (2) in positive integers are

$$
\begin{aligned}
& k=1, l=2, h=1 \\
& k=2, l=3, h=1 \\
& k=3, l=2, h=1 .
\end{aligned}
$$

Proob: We reduce (2) to an inequality in the linear forms of algebraic numbers and then apply the following theorem of Baker [1, p. 45]: Theorem 3. If $\alpha_{1}, \ldots, \alpha_{n}, n \geq 2$, are nonzero algebraic numbers with degrees and heights at most $d(\geq 4)$ and $A(\geq 4)$ respectively, and if rational integers $b_{1}, \ldots, b_{n}$ exist with absolute values at most $B$ such that

$$
0<\left|b_{1} \ln \alpha_{1}+\cdots+b_{n} \ln \alpha_{n}\right|<-e^{-\delta B}
$$

where $0<\delta<1$ and the logarithms have their principal values, then

$$
B<\left(4^{n^{2}} \delta^{-1} d^{2 n} \log A\right)^{(2 n+1)^{2}} .
$$

Returning to (2) we find that the only solutions for $k<4$ are

$$
\begin{aligned}
& k=1, l=2, h=1 \\
& k=2, l=3, h=1 \\
& k=3, l=4, h=5 .
\end{aligned}
$$

Thus $k \geq 4$ and (2) yields

$$
\begin{equation*}
0<2^{k+l}-5^{k} \leq 2^{l}-1 \tag{3}
\end{equation*}
$$

Dividing both sides of (3) by $2^{k+l}$ and using the fact that

$$
\frac{1}{2^{k}-1}>\ln \frac{2^{k}}{2^{k}-1} \text { for } k \geq 1
$$

we get

$$
\begin{aligned}
& 0<|(k+\ell) \ln 2-k \ln 5|<\frac{1}{2^{k}-1}, \text { and hence } \\
& 0<\left|\frac{\ell}{k}-\log _{2} \frac{5}{2}\right|<\frac{1}{k \ln 2\left(2^{k}-1\right)}
\end{aligned}
$$

Since $\ln 2\left(2^{k}-1\right)>2 k$ for $k \geq 4$, we see that if $k \geq 4$, $l / k$ must be a convergent in the continued fraction expansion of $\log _{2}(5 / 2)$. With the aid of a computer, we find that the first 7 convergents of this continued fraction are

$$
\frac{1}{1}, \frac{4}{3}, \frac{37}{28}, \frac{78}{59}, \frac{193}{146}, \frac{850}{643}, \text { and } \frac{5293}{4004}
$$

and it is easily verified that if $k>4$, none of these convergents satisfy (4). Thus we may assume $k>4004$.

Now we derive a lower bound for the partial quotients in the continued fraction expansion of $\log _{2}(5 / 2)$, using the following theorem of Legendre.

Theorem 4: Let $\theta$ be a real number, $p_{n} / q_{n}$ a convergent in the continued fraction expansion of $\theta$, and $\alpha_{n}$ the corresponding partial quotient. Then

$$
\frac{1}{\left(a_{n+1}+2\right) q n^{2}}<\left|\theta-\frac{p_{n}}{q_{n}}\right|
$$

which yields

$$
\frac{1}{\left(a_{n+1}+2\right) k^{2}}<\left|\theta-\frac{p_{n}}{q_{n}}\right|<\frac{1}{k \ln 2\left(2^{k}-1\right)}
$$

Thus, since $k>4004$,

$$
a_{n+1}>\frac{2^{k}-1}{k} \ln 2-2>\frac{2^{4004}-1}{4004} \ln 2-2>10^{2750}
$$

Thus, any further solution of (4) corresponds to an extremely large partial quotient in the continued fraction expansion of $\theta$. Finally, we derive an upper bound for $k$. To this end, we note that if $k>4004$, we have $\ell / k<1.33$, so

$$
\ell+k<2.33 k \text { and } 2^{k}-1>e^{.00233 k}>e^{.001(\ell+k)} .
$$

Thus (4) becomes

$$
0<|(k+\ell) \ln 2-k \ln 5|<\frac{1}{2^{k-1}}<e^{-.001 B}
$$

where $B=\ell+k$.
Now we apply Theorem 3, with $n=2, d=4, A=4, \delta=.001$, and get

$$
B=\ell+k<\left(4^{4} \cdot 10^{3} \cdot 4^{4} \cdot \ln 5\right)^{25}<\left(10^{2.41} \cdot 10^{3} \cdot 10^{2.41} \cdot 10^{21}\right)^{25}<10^{201} .
$$

Thus $k<10^{201}$ also. With the aid of a computer and a multiple precision package designed by Ellison, we computed $\log _{2}(5 / 2)$ to 1200 decimal places and then computed the continued fraction expansion of $\log _{2}(5 / 2)$ until $q_{n}$ exceeded $10^{201}$. The largest partial quotient is found to be 5393 , so (2) has no solutions in positive integers other than $(1,2,1),(2,3,1)$, and ( $3,4,5$ ). Thus the only circuits which are cycles under $f$ are $1 \xrightarrow{2} 8 \xrightarrow{3} 1$ and $13 \xrightarrow{3} 208 \xrightarrow{4} 13$.
[Note: $n=17$ also gives rise to a cycle under $f$. But this cycle results from a double circuit: $17 \xrightarrow{2} 108 \xrightarrow{2} 27$ and $27 \xrightarrow{1} 68 \xrightarrow{2}$ 17.]

It would be of great interest to know if any $n$ other than 17 and 27 gives rise to a cycle under $f$ which is the result of a multiple circuit.

$$
\text { 11. THE CASE } Q=7
$$

For $n \varepsilon Z^{+}$, let

$$
g(n)= \begin{cases}\frac{7 n+1}{2} & n \text { odd, } n>1 \\ n / 2 & n \text { even } \\ 1 & n=1\end{cases}
$$

as in Case I. Then we can prove the following theorem.
Theorem 5: Let $v_{2}(m)$ be the highest power of 2 dividing $m, m \varepsilon Z$ and let $n$ be an odd integer $>1$. Then $n<g(n)<\cdots<g^{k}(n)$ and $g^{k+1}(n)<g^{k}(n)$, where $k=v_{2}(5 n+1)$.

Also, the equation corresponding to (1) is

$$
\begin{equation*}
2^{j}\left(5 n_{j}+1\right)=7^{j}(5 n+1) . \tag{5}
\end{equation*}
$$

Now we write $n \xrightarrow{k} m \xrightarrow{\ell} n^{*}$ where $\ell=v_{2}(m), n^{*}=m / 2 \ell, \quad k=v_{2}(5 n+1)$, and $2^{k}(5 m+1)=7^{k}(5 n+1)$.

Again, if we let $A$ be the set of positive odd integers and define $T: A \rightarrow A$ by $T(n)=n^{*}$, the map from $n$ to $n^{*}$ is called a circuit. Our goal is to prove:

Theorem 6: The only positive odd integer $n$ such that $T(n)=n$ is $n=1$.
Proof: As in Case I, we reduce this problem to solving

$$
\begin{equation*}
\left(2^{k+\ell}-7\right) h=2^{\ell}-1 \tag{6}
\end{equation*}
$$

where $k, l$, and $h$ are positive integers. We shall show that the only solutions of (6) in positive integers are

$$
k=1, \ell=2, h=3 \quad \text { and } \quad k=2, \ell=4, h=1 .
$$

First, if $k \leq 4$, we find that the only solutions of (6) are the ones stated in the theorem and that only the first gives rise to a cyclic circuit, namely, $1 \xrightarrow{1} 4 \xrightarrow{2} 1$.

So $k \geq 4$, and as in Case $I$, we find that

$$
\begin{equation*}
0<2^{k+l}-7^{k} \leq 2^{l}-1 \tag{7}
\end{equation*}
$$

$$
\begin{align*}
& 0<|(k+\ell) \ln 2-k \ln 7|<\frac{1}{2^{k}-1}  \tag{8}\\
& 0<\left|\frac{\ell}{k}-\log _{2} \frac{7}{2}\right|<\frac{1}{k \ln 2\left(2^{k}-1\right)} . \tag{9}
\end{align*}
$$

Since $\ln 2\left(2^{k}-1\right)>2 k$ for $k>4, l / k$ must be a convergent in the continued fraction expansion of $\log _{2}(7 / 2)$.

Again, we find that the first 7 convergents of $\log _{2}(7 / 2)$ are

$$
\frac{1}{1}, \frac{2}{1}, \frac{9}{5}, \frac{47}{26}, \frac{197}{109}, \frac{1032}{571}, \text { and } \frac{4325}{2393} .
$$

Of these, $9 / 5$ does not furnish a solution of (6) and the remainder do not satisfy (9). Thus $k>2393$. Further, by Theorem 4, we get

$$
a_{n+1}>\frac{2^{2393}-1 \ln 2}{2393}-2>10^{1500}
$$

Finally, we note that if $k>2393$, we have $\ell / k<1.81$. So $\ell+k<2.81 k$ and $2^{k}-1>e^{.00281 k}>e^{.001(\ell+k)}$. Thus $0<|(k+l) \ln 2-k \ln 7|<e^{-.001 B}$, where $B=\ell+k$. Again, by Theorem 3, with $n=2, d=4, A=4, \delta=.001$,

$$
B=l+k<\left(4^{4} \cdot 10^{3} \cdot 4^{4} \ln 7\right)^{25}<10^{203} .
$$

With the aid of Ellison's package, we computed $\log _{2}(7 / 2)$ to 1200 decimal places and then computed the continued fraction expansion of $\log _{2}(7 / 2)$ until $q$ exceeded $10^{203}$. The largest partial quotient is found to be 197 , so (6) has only the solutions stated in the theorem and the only circuit which is a cycle under $g$ is $1 \xrightarrow{1} 4 \xrightarrow{2} 1$.

In a subsequent paper to be published in this journal, we shall study the general case for this problem and present the tables generated during the computation of $\log _{2}(5 / 2)$ and $\log _{2}(7 / 2)$ for the two cases presented here.

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[^0]:    ${ }^{1}$ Now the University of Puget Sound.

[^1]:    ${ }^{2} n!=n!\left[1-1 / 1!+1 / 2!-1 / 3!+\cdots+(-1)^{n} n!\right]$.
    ${ }^{3}$ Now Oregon State University.

[^2]:    ${ }^{4}$ Another entertaining number game that we played was that of expressing as many of the successive positive integers as possible in terms of not more than three $\pi^{\prime} s$, along with accepted symbols of operation.

[^3]:    "Defining Fibonacci sequences by the property $u_{n+1}=u_{n}+u_{n-1}$, several relations between the terms are easily obtained by the manipulation of two-by-two matrices whose elements are terms of the sequence. The speaker concluded by pointing out a geometric connection between the Fibonacci sequences and the sequences of polygonal numbers."

