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OF INTEGERS WITH SPECIAL PROPERTIES*

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ORTHOGONAL LATIN SYSTEMS

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Dedicated to the memory of our friend Vern E. Hoggatt

I. INTRODUCTION

A *Latin square of order n* can be interpreted as a multiplication table for a binary operation on n objects $0, 1, \ldots, n-1$ with both a right and a left cancellation law. That is, if we denote the operation by $*$, then

$$\begin{array}{c} a * b = a * c \Rightarrow b = c \\ b * a = c * a \Rightarrow b = c. \end{array} \tag{1.1}$$

In a completely analogous manner, a *Latin k-cube of order n* is a k-ary operation on n objects with a cancellation law in every position. That is, for the operation $(\ )_*$,

$$(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k)_* = (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_k)_* \tag{1.2}$$

implies $b = c$ for all choices of $i = 1, 2, \ldots, k$ and all choices of

$$\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\} \subset \{0, 1, \ldots, n-1\}.$$

We permit 1-cubes which are just permutations of $\{0, 1, \ldots, n-1\}$.

Two Latin squares are orthogonal if the simultaneous equations

$$x * y = a, \quad x \circ y = b \tag{1.3}$$

have a unique solution x, y for every pair a, b. *A set of Latin squares is orthogonal* if every pair of squares in the set is orthogonal.

In an analogous manner, a *k-tuple of Latin k-cubes is orthogonal* if the simultaneous equations

$$\begin{array}{c} (x_1, x_2, \ldots, x_k)_1 = a_1 \\ (x_1, x_2, \ldots, x_k)_2 = a_2 \\ \vdots \\ (x_1, x_2, \ldots, x_k)_k = a_k \end{array} \tag{1.4}$$

have a unique solution $x_1, \ldots, x_k$ for all choices of $a_1, \ldots, a_k$.

A *set of Latin k-cubes is orthogonal* if every k-tuple of the set is orthogonal.

In earlier papers, [1] and [2], we showed that the existence of a pair of orthogonal Latin squares can be used for the construction of a quadruple of orthogonal Latin cubes (3-cubes) and for the construction of orthogonal k-tuples of Latin k-cubes for every $k \geq 3$. In this note, we examine in greater detail what sets of orthogonal Latin k-cubes can be constructed by composition from cubes of lower dimensions.

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II. COMPOSITION OF LATIN CUBES

Let $C = (a_1, \ldots, a_s)$ be a Latin s-cube and let $C_i = (b_{i1}, b_{i2}, \ldots, b_{ik_i})_i$ be Latin k_i-cubes $i = 1, 2, \ldots, s$. Then

$$C^* = (C_1, C_2, \ldots, C_s)$$

is a Latin k-cube, where $k = k_1 + k_2 + \cdots + k_s$.

To see this we need only check that the cancellation law (1.2) holds. Now let all the entries be fixed except for the entry b_{ij} in the jth place of C_i. Since C is a Latin cube it follows that, if the values of C^* are equal for two different entries of b_{ij} then the values of C_i must be equal for those two entries. This contradicts the fact that C_i is a Latin cube.

This composition, while algebraically convenient, is not intuitive and we refer the reader to [1] where we explicitly constructed a quadruple of 3-cubes starting from a pair of orthogonal Latin squares of order 3. In the present notation, starting from $a * b$ and $a \circ b$ as orthogonal Latin squares, we constructed the quadruples

$$(a * b) * c, \quad (a * b) \circ c, \quad (a \circ b) * c, \quad (a \circ b) \circ c$$

or, equivalently,

$$a * (b * c), \quad a * (b \circ c), \quad a \circ (b * c), \quad a \circ (b \circ c)$$

as orthogonal quadruples of cubes.

Similarly, if $(\ )_1, \ldots, (\ )_k$ denote an orthogonal set of Latin k-cubes, then

$$(a_1, \ldots, a_k)_1 \circ a_{k+1}, \quad (a_1, \ldots, a_k)_2 \circ a_{k+1}, \ldots, (a_1, \ldots, a_k)_k \circ a_{k+1},$$
$$(a_1, \ldots, a_k)_i * a_{k+1}$$

is an orthogonal $(k + 1)$-tuple of Latin $(k + 1)$-cubes for any $i \in \{1, \ldots, k\}$. To see this, consider the system of equations

$$(x_1, \ldots, x_k)_j \circ x_{k+1} = a_j, \quad 1 \le j \le k$$
$$(x_1, \ldots, x_k)_i * x_{k+1} = a_{k+1}.$$

Then the two simultaneous equations

$$(x_1, \ldots, x_k)_i \circ x_{k+1} = a_i, \quad (x_1, \ldots, x_k)_i * x_{k+1} = a_{k+1}$$

have a unique solution $(x_1, \ldots, x_k)_i$ and x_{k+1}. Once x_{k+1} is determined, the equations

$$(x_1, \ldots, x_k)_j \circ x_{k+1} = a_j$$

determine $(x_1, \ldots, x_k)_j$ for all $j = 1, \ldots, i - 1, i + 1, \ldots, k$. Now by the orthogonality of the k-cubes the values of $x_1, \ldots, x_k$ are determined.

Since pairs of orthogonal Latin squares exist for all orders $n \neq 2, 6$, it follows that there exist orthogonal k-tuples of Latin k-cubes for all k provided the order n is different from 2 or 6. It is obvious that there are no orthogonal k-tuples of Latin k-cubes of order 2 for any $k \ge 2$. For order $n = 6$ and dimension $k > 2$, neither the existence nor the nonexistence of orthogonal k-tuples of k-cubes is known. It is therefore worth mentioning the following conditional fact.

Theorem II-1: If there exists a k-tuple of orthogonal Latin k-cubes of order n then there exists an ℓ-tuple of orthogonal Latin ℓ-cubes of order n for every $\ell = 1 + s(k - 1)$, $s = 0, 1, 2, \ldots$.

Proof: By induction on s. The statement is obvious for $s = 0$. So assume the statement true for ℓ and let $(\ )_1^k, \ldots, (\ )_k^k$ denote the orthogonal k-cubes and let $(\ )_1^\ell, \ldots, (\ )_\ell^\ell$ denote the orthogonal ℓ-cubes. Then we construct the following set of Latin $(\ell + k - 1)$-cubes.

$$\begin{aligned}
(a_1, \ldots, a_{\ell+k-1})_1^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k \\
(a_1, \ldots, a_{\ell+k-1})_2^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_2^k \\
&\vdots \\
(a_1, \ldots, a_{\ell+k-1})_k^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_k^k \\
(a_1, \ldots, a_{\ell+k-1})_{k+1}^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_2^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k \\
(a_1, \ldots, a_{\ell+k-1})_{k+2}^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_3^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k \\
&\vdots \\
(a_1, \ldots, a_{\ell+k-1})_{\ell+k-1}^{\ell+k-1} &= ((a_1, \ldots, a_\ell)_\ell^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k.
\end{aligned}$$

From the orthogonality of $(\ )_1^k, \ldots, (\ )_k^k$ it follows that the equations

$$(x_1, \ldots, x_{\ell+k-1})_i^{\ell+k-1} = a_i; \quad i = 1, \ldots, k$$

determine $(x_1, \ldots, x_\ell)_1^\ell, x_{\ell+1}, \ldots, x_{\ell+k-1}$. Once $x_{\ell+1}, \ldots, x_{\ell+k-1}$ are determined, then the equations

$$(x_1, \ldots, x_{\ell+k-1})_{k+j}^{\ell+k-1} = a_{k+j}; \quad j = 1, \ldots, \ell - 1$$

determine $(x_1, \ldots, x_\ell)_{j+1}^\ell$. Now, by the orthogonality of $(\ )_1^\ell, \ldots, (\ )_\ell^\ell$, this determines $x_1, \ldots, x_\ell$.

III. ORTHOGONAL $(k + 1)$-TUPLES OF LATIN k-CUBES

The above construction yielded a set of 4 orthogonal 3-cubes constructed with the help of a pair or orthogonal Latin squares $a \circ b$ and $a * b$. It is natural to ask whether analogous constructions exist for higher dimensions. At the moment we have only succeeded in doing this for dimensions 4 and 5.

Theorem III-1: The 4-cubes

$$\begin{aligned}
(abcd)_1^4 &= (a \circ b) \circ (c \circ d) \\
(abcd)_2^4 &= (a \circ b) * (c \circ d) \\
(abcd)_3^4 &= (a * b) \circ (c * d) \\
(abcd)_4^4 &= (a * b) * (c * d) \\
(abcd)_5^4 &= (a \circ b) \circ (c * d)
\end{aligned}$$

form an orthogonal set.

Proof: We need to show that the equations

$$(xyzw)_i^4 = a_i$$

determine x, y, z, w when i runs through any four of the five values. Consider first the case $i = 1, 2, 3, 4$. Then the first two equations determine $x \circ y$, $z \circ w$ and the next two equations determine $x * y$, $z * w$. Now $x \circ y$ and $x * y$ determine x, y and $z \circ w$, $z * w$ determine z, w.

Now assume that one of the first four values of i is omitted. By symmetry we may assume $i \neq 4$. Then the first two equations still determine $x \circ y$, $z \circ w$. Once $x \circ y$ is determined, the last equation determines $z * w$ and once $z * w$ is determined, the third equation determines $x * y$. The rest is as before.

Theorem III-2: Let $(\ )_1^3$, $(\ )_2^3$, $(\ )_3^3$ denote an orthogonal set of 3-cubes. Then the 5-cubes

$$
\begin{aligned}
(abcde)_1^5 &= (abc)_1^3 \circ (d \circ e)\\
(abcde)_2^5 &= (abc)_1^3 * (d \circ e)\\
(abcde)_3^5 &= (abc)_2^3 \circ (d * e)\\
(abcde)_4^5 &= (abc)_2^3 * (d * e)\\
(abcde)_5^5 &= (abc)_3^3 \circ (d \circ e)\\
(abcde)_6^5 &= (abc)_3^3 \circ (d * e)
\end{aligned}
$$

form an orthogonal set.

Proof: Consider the set of equations

$$(xyzuv)_i = a_i$$

where i runs through five of the six values. If $i \neq 5$ or 6 then the first two equations determine $(xyz)_1^3$ and $u \circ v$ and the second two equations determine $(xyz)_2^3$ and $u * v$. Thus, u, v are determined and, therefore, the last equation determines $(xyz)_3^3$ and thus x, y, z are determined.

If i omits one of the first four values, we may assume by symmetry $i \neq 4$. Then the first two equations determine $(xyz)_1^3$, and $u \circ v$. Now $i = 5$ determines $(xyz)_3^3$ and thereby $i = 6$ determines $u * v$. Finally, $i = 3$ determines $(xyz)_2^3$, and thus x, y, z, u, v are determined.

Applying these results to the lowest order, $n = 3$, we get the surprising result that there exists a 3 x 3 x 3 cube with 4-digit entries to the base 3, so that each digit runs through the values 0, 1, 2 on every line parallel to an edge of the cube and so that each triple from 000 to 222 occurs exactly once in every position as a subtriple of a quadruple. Similarly, there exists a 3 x 3 x 3 x 3 cube with 5-digit entries, and all quadruples from 0000 to 2222 occur exactly once in every position as subquadruples of the quintuples. Finally, there exists a 3 x 3 x 3 x 3 x 3 cube with 6-digit entries, every digit running through 0, 1, 2 on every line parallel to an edge and every quintuple occurring exactly once in every position as a subquintuple.

There does not appear to exist an obvious extension of Theorems III-1 and III-2 to dimensions greater than 5.

It is possible to use the case $n = 3$ to show that the existence of two orthogonal Latin squares of order n does not imply the existence of more than 4 orthogonal 3-cubes or 5 orghogonal 4-cubes of order n.

Theorem III-3: There do not exist 5 orthogonal 3-cubes of order 3.

Proof: Since relabelling the entries in the cube affects neither Latinity nor orthogonality, we may assume that $(i00)_j = i$ for all the 3-cubes $(\ )_j$. So the entries $(010)_j$ are all 1 or 2. If there are 5 orthogonal 3-cubes, then no 3 of them can have the same entry in the position $(010)_j$, since these triples occur already in the positions $(i00)_j$. But in 5 entries 1 or 2, there must be three equal ones.

Theorem III-4: There do not exist 6 orthogonal 4-cubes of order 3.

Proof: As before, assume $(i00)_j = i$, $j = 1, \ldots, 6$. Since all entries $(010)_j$ are either 1 or 2 and no four of them are equal, we may assume that the entries are 111222 as $j = 1, \ldots, 6$. Hence, the entries $(020)_j$ are 222111 in the same order. Now the entries $(001)_j$ and $(002)_j$ must also be three 1's and three 2's and cannot agree with 111222 or 222111 in four positions. But the agreement is always in an even number of positions, and if the agreement with 111222 is in

$2k$ positions, then the agreement with 222111 is in $6 - 2k$ positions and one of these numbers is at least 4.

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ON THE "QX + 1 PROBLEM," Q ODD—II

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In [1] we studied the functions

$$f(n) = \begin{cases} (5n+1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

and

$$g(n) = \begin{cases} (7n+1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

and proved:

1. The only nontrivial circuit of f which is a cycle is

$$13 \xrightarrow{3} 208 \xrightarrow{4} 13.$$

2. The function g has no nontrivial circuits which are cycles.

In this note, we consider briefly the general case for this problem and present the tables generated for the computation of $\log_2(5/2)$ and $\log_2(7/2)$ for the two cases presented in [1].

Let

$$h(n) = \begin{cases} (qn+1)/2 & n \text{ odd}, n > 1, q \text{ odd} \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

Then, as in [1], we have

Theorem 1: Let $v_2(m)$ be the highest power of 2 dividing m, $m \in Z$, and let n be an odd integer > 1, then

$$n < h(n) < \cdots < h^k(n), \text{ and } h^{k+1}(n) < h(n),$$

where $k = v_2((q-2)n+1)$.

Also, the equation corresponding to Eq. (1) in [1] is

(1) $$2^j((q-2)nj+1) = q^j((q-2)n+1).$$

Again, we write

$$n \xrightarrow{k} m \xrightarrow{\ell} n^*$$

where $\ell = v_2(m)$, $n^* = m/2^\ell$, $k = v_2((q-2)n+1)$ and

$$2^k((q-2)m+1) = q^k((q-2)n+1)$$

and obtain our usual definition of a circuit.

Finally, we can prove

Theorem 2: There exists n such that $T(n) = n$ only if there are positive integers k, ℓ, h satisfying

$$(2^{k+\ell} - q^k)h = 2^\ell - 1. \tag{2}$$

It would be of great interest to determine those q for which solutions of (2) give rise to a cyclic circuit under h. At present the only q known to do this is $q = 5$. As for those q which give rise to multiple circuit cycles, the only one known besides $q = 5$ is $q = 181$, which has two *double* circuit cycles:

$$27 \xrightarrow{1} 2444 \xrightarrow{2} 611$$
$$611 \xrightarrow{1} 55296 \xrightarrow{11} 27$$

and

$$35 \xrightarrow{1} 3168 \xrightarrow{5} 99$$
$$99 \xrightarrow{1} 8160 \xrightarrow{8} 35.$$

TABLES

TABLE 1. $\log_2 \frac{5}{2}$ *to 1200 Decimal Places*

1.132192809488736234787031942948939017586483139302458061205475639581
593477660862521585013974335937015509965737171025025182682409698426 35
268882753027729986553938519513526575055686430176091900248916669414 33
374011903124187375109715866467540179189655806735830779688432725883 27
499252244890238355997641739413792800977275668635547790148674505784 58
847802710422545609722346579569554153701915764117177924716513500239 21
127147339361440723397211574851007094987891658880831322194806793298 23
232593119506713995078370033673424807066352750084069176263862535468 80
153686216184188608589948353813214998930270441792078659226018229653 71
575367239660695116486836846623858508486062990542699469279116273206 13
400644670484763407043735233674221283089670364579092167721909021421 96
214245744465852453594844881548345925142954093735390654944863277929 84
242515911811311632981257694501981575037921855384878203551601973782 77
288881759874332866072712393825202213332805255124882743444884245316 54
650612414891822867932526642928116599228516273450818600714468395588 04
633121279264003631201457736887904048271052865203359481532478070748 32
712590336282976999102881681040419750373558623804925499672086216775 48
1010883457989804214485844199738212065312511525

TABLE 2. *The Continued Fraction Expansion of* $\log_2 \frac{5}{2}$

1	3	9	2	2	4	6	2	1	1	3
	1	18	1	6	1	2	1	1	4	1
	42	6	1	4	2	3	1	2	6	1
	3	4	1	8	1	4	1	2	2	7
	1	4	1	1	3	3	1	3	1	1
	7	6	1	5	10	2	2	1	8	1
	2	16	24	1	6	1	8	1	1	5
	1	1	1	1	1	2	1	1	3	7
	1	1	10	3	2	1	3	1	3	1
	2	1	3	11	1	1	1	5	1	5
	3	3	2	2	4	7	1	4	1	1
	2	7	1	3	3	2	32	1	119	1
	2	1	8	17	4	16	1	5	6	13

TABLE 2—Continued

1	2	1	2	1	5393	1	1	2	1	2
	3	3	10	2	1	2	1	1	7	32
	1	6	1	5	1	8	6	1	2	3
	17	1	1	1	4	1	2	12	1	27
	2	1	2	3	2	1	1	7	4	9
	10	1	4	1	5	6	2	3	2	3
	9	1	1	1	2	237	1	2	1	15
	1	1	17	1	1	1	2	3	2	6
	1	2	2	1	5	1	1	1	1	1
	2	1	5	1	23	5	1	1	1	1
	9	2	3	1	14	2	1	1	16	2
	1	1	1	2	1	1	8	1	1	3
	1	1	2	3	33	1	1	2	1	2
	3	3	2	5	12	1	13	1	11	23
	1	2	5	2	3	2	10	4	3	4
	1	1	1	6	4	1	8	5	1	1
	10	6	29	1	3	4	9	1	24	1
	3	8	38	3	1	1	1	1	6	2
	1	3	1	10	1	1	5	1	1	1
	1	1	1	6	3	3	3	9	1	3
	3	1	1	1	7	1	2	1	8	1
	1	16	1	2	1	1	5	1	4	2
	1	228	2	13	2	1	1	9	5	1
	28	1	4	1	1	4	3	1	2	1
	3	3	1	1	2	2	3	1	4	4
	5	2	11	2	1	1	2	1	3	6
	7	6	2	1	78	1	8	28	15	1
	1	1	1	2	1	2	172	2	3	1
	1	1	6	1	105	4	23	--		

TABLE 3. $Log_2 \frac{7}{2}$ *to 1200 Decimal Places*

1.180735492205760410744196931723183080864102662596614078367729172407
03208488621929864978609991702107851073605018893255730459733550189744
35783948545697421659367034036223711232893039172839880533054596558987
42842044049863242710660517715603594755455847742935680180016993525932
50632889709207655100521356641486039729352404730419795633055279942802
67077276110778204971932513254550267027235235681504586808823722107156
62259311528345703426110256015571456055227154958021504336696505010023
34988294495656908806896861271221799915017038085074366218220796188044
13300641248483810021757003214687292291654022734173979996398717392556
21657012062442265868128541719793524331738795293960080126504099080050
86143891504372773197711929325509449755438097944662727688654466455056
66144962718917439479811201832195534767729368027362015384968426483404
28194862620856744723428655525118561153949628390912550087758014235589
14221613005965234270525279790176286862630931786330372331743548294140
06377868059095886491534576253156671578606520583005556279536710386799
55857731719085677755305180653144090746707963928688620808186866798569
85299653671553315728082138583329807569231547710021897097214157437559
6833249986877724904346722049673575206749869960

TABLE 4. The Continued Fraction Expansion of $\log_2 \frac{7}{2}$

1	1	4	5	4	5	4	1	29	1	4
	8	1	1	2	1	31	10	1	2	2
	6	2	3	1	1	197	1	4	5	5
	5	1	10	1	4	4	1	3	14	3
	1	1	1	6	5	1	5	1	1	3
	59	1	11	13	11	1	85	1	5	1
	1	1	2	1	14	1	2	5	1	4
	24	20	1	1	1	1	3	1	1	2
	3	1	1	2	2	1	40	53	3	1
	1	1	7	3	1	1	2	1	1	8
	2	4	4	1	1	3	1	6	7	1
	1	1	1	1	1	1	2	3	1	4
	4	2	1	3	2	3	2	3	68	1
	6	1	3	1	4	1	1	1	5	2
	2	7	2	1	3	163	1	6	2	2
	1	1	1	20	1	21	1	1	6	1
	1	6	44	1	3	1	1	1	91	3
	2	1	1	59	2	2	18	1	5	6
	3	2	2	2	1	1	1	1	12	1
	2	3	2	14	1	2	1	1	3	1
	3	2	1	10	1	1	1	1	5	2
	2	1	6	3	1	4	3	4	1	1
	7	7	9	1	1	5	1	11	1	3
	1	2	1	3	1	4	1	4	1	2
	1	1	1	2	1	1	1	1	1	1
	7	2	3	9	2	4	10	4	1	1
	3	11	2	1	2	1	5	3	2	7
	1	3	1	1	4	1	1	1	3	2
	1	1	1	19	10	7	1	1	2	1
	2	3	8	3	3	26	2	1	12	1
	1	13	1	4	1	7	3	7	1	1
	2	1	1	1	4	2	1	1	1	1
	3	1	2	1	1	4	4	7	8	4
	1	1	6	4	1	3	1	1	1	1
	29	6	1	1	44	16	8	3	2	5
	1	2	15	2	2	3	2	4	1	1
	42	2	2	1	1	2	56	1	1	1
	1	3	2	1	4	1	7	1	5	2
	1	1	5	5	1	8	1	31	1	2
	4	1	4	2	1	1	4	2	4	1
	1	2	1	1	1	4	2	3	4	1
	1	1	1	4	1	1	1	67	5	1
	13	--								

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BINARY WORDS WITH MINIMAL AUTOCORRELATION AT OFFSET ONE

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In various communications problems, it has been found to be advantageous to make use of binary words with the property that at offset one they have aucorrelation value as small in magnitude as possible. The purpose of this paper is to derive the means and variances for the autocorrelations of these words at all possible offsets. The derivations are combinatorial in nature, and several new combinatorial identities are obtained.

1. AUTOCORRELATIONS AT VARIOUS OFFSETS

We consider two methods of autocorrelating binary words. The *cyclic* autocorrelation at offset d is defined to be the number of agreements minus the number of disagreements between the original word and a cyclic shift of itself by d places. If the word is $\underline{v} = (v_0, v_1, \ldots, v_{L-1})$, and if subscripts are reduced modulo L, then this autocorrelation is given by

$$\tau_d(\underline{v}) = \sum_{i=0}^{L-1} (-1)^{v_i + v_{i+d}}. \tag{1.1}$$

The *truncated* autocorrelation at offset d is the number of agreements minus the number of disagreements between the last $L-d$ bits of the original word and a right cyclic shift of the word by d places. The formula for this autocorrelation is

$$\tau_d^*(\underline{v}) = \sum_{i=0}^{L-d-1} (-1)^{v_i + v_{i+d}}. \tag{1.2}$$

Note that $\tau_d(\underline{v}) = \tau_d^*(\underline{v}) + \tau_{L-d}^*(\underline{v})$. By symmetry, $\tau_d(\underline{v}) = \tau_{L-d}(\underline{v})$. Thus, for $d > L/2$, we can compute $E(\tau_d)$, $E(\tau_d^2)$, and $E(\tau_d^*)$ from their values with $d \le L/2$. $E(\tau_d^{*2})$ needs special treatment. Therefore, unless stated otherwise, we assume $d \le L/2$.

Our principal result is

Theorem 1.1: Let L be a positive integer, and let $\underline{v}$ range over the binary L-tuples with minimal cyclic autocorrelation at offset 1. Then, for $1 \le d \le [L/2]$,

$$E(\tau_d) = \begin{cases} L\left(\dfrac{1+(-1)^d}{2}\right)(-1)^{d/2}\dbinom{L/2}{d/2}\Big/\dbinom{L}{d}, & L \equiv 0 \pmod 4 \\[2ex] L(-1)^{[d/2]}\dbinom{(L-1)/2}{[d/2]}\Big/\dbinom{L}{d}, & L \equiv 1 \pmod 4 \\[2ex] (L-2d)\left(\dfrac{1+(-1)^d}{2}\right)(-1)^{d/2}\dbinom{L/2}{d/2}\Big/\dbinom{L}{d}, & L \equiv 2 \pmod 4 \\[2ex] L(-1)^{[(d+1)/2]}\dbinom{(L-1)/2}{[d/2]}\Big/\dbinom{L}{d}, & L \equiv 3 \pmod 4 \end{cases}$$

If $L \equiv 2 \pmod 4$ and d is odd, and if we restrict $\underline{v}$ to range over those binary L-tuples satisfying $\tau_1(\underline{v}) = 2c$, $c = \pm 1$, then

$$E(\tau_d) = 2c(d+1)(-1)^{(d-1)/2}\binom{L/2}{(d+1)/2}\Big/\binom{L}{d}.$$

Also,

$$E(\tau^2) = \begin{cases} \dfrac{L^2}{L+2}\left[(-1)^d(L - 2d + 1)\dbinom{L/2}{d}\Big/\dbinom{L}{2d} + 1\right], & \text{if } L \equiv 0 \pmod 4; \\ \dfrac{1}{L+3}\left[\dfrac{(-1)^d(L - 2d)(L^2 - 2dL + 2L - 2d - 1)\dbinom{(L-1)/2}{d}}{\dbinom{L-1}{2d}} + L(L+1)\right], & \text{if } L \text{ is odd}; \\ \dfrac{1}{L+4}\left[(-1)^d(L - 2d + 1)(L^2 - 4dL + 2L - 8d - 4)\dbinom{L/2}{d}\Big/\dbinom{L}{2d} + L^2 + 2L + 4\right], & \text{if } L \equiv 2 \pmod 4). \end{cases}$$

If $\underline{v}$ ranges over the binary L-tuples with minimal truncated autocorrelations at offset 1, then, for $1 \le d \le [L/2]$,

$$E(\tau_d^*) = (-1)^{d/2}\left(\frac{1 + (-1)^d}{2}\right)(L - d)\binom{[(L-1)/2]}{d/2}\Big/\binom{L-1}{d}.$$

If L is even and $\underline{v}$ is constrained to range over the binary L-tuples satisfying $\tau_1^*(\underline{v}) = c$, $c = \pm 1$, then

$$E(\tau_d^*) = c^d(-1)^{[d/2]}(L - d)\binom{(L-2)/2}{[d/2]}\Big/\binom{L-1}{d}.$$

Also,

$$E(\tau_d^{*2}) = \begin{cases} \dfrac{1}{(L+2)(L+4)}\Bigg[(-1)^d(L - 2d + 1)(L^3 - 2dL^2 + 4L^2 - 8dL - 4d - 2) \\ \qquad \dbinom{L/2}{d}\Big/\dbinom{L}{2d} + L^3 - dL^2 + 4L^2 - 4dL + 2L + 2\Bigg], & \text{if } L \text{ is even}; \\ \dfrac{1}{(L+1)(L+3)}\Bigg[(-1)^d L(L^3 - 2dL^2 + 3L^2 - 4dL + L + 2d + 1) \\ \qquad \dbinom{(L+1)/2}{d}\Big/\dbinom{L+1}{2d} + L^3 - dL^2 + 2L^2 - 2dL - L + 3d\Bigg], & \text{if } L \text{ is odd}. \end{cases}$$

If $d > L/2$,

$$E(\tau_d^{*2}) = \begin{cases} \dfrac{1}{(L+2)(L+4)}\Bigg[(-1)^d 2(L + 1)(-2L + 2d - 1)\dbinom{(L+2)/2}{L-d}\Big/\dbinom{L+2}{2(L-d)} \\ \qquad + L^3 - dL^2 + 4L^2 - 4dL + 2L + 2\Bigg], & \text{if } L \text{ is even}; \\ \dfrac{1}{(L+1)(L+3)}\Bigg[(-1)^{d+1} 2L(-2L + 2d - 1)\dbinom{(L+1)/2}{L-d}\Big/\dbinom{L+1}{2(L-d)} \\ \qquad + L^3 - dL^2 + 2L^2 - 2dL - L + 3d\Bigg], & \text{if } L \text{ is odd}. \end{cases}$$

2. MINIMIZING AUTOCORRELATION AT OFFSET ONE

Suppose $\underline{v} = (v_0, v_1, \ldots, v_{L-1})$. If we change one bit, say v_i, to obtain $\hat{\underline{v}} = (v_0, \ldots, 1 - v_i, \ldots, v_{L-1})$, then $\tau_d(\hat{\underline{v}}) = \tau_d(\underline{v})$ or $\tau_d(\underline{v}) \pm 4$, because the sign of the two terms, $(-1)^{v_i + v_{i+d}}$ and $(-1)^{v_{i-d} + v_i}$, in the sum $\tau_d(\underline{v})$ have been changed. Since any binary L-tuple may be obtained from any other by changing

$k \leq L$ bits, one at a time, it follows that

$$\tau_d(\underline{v}) \equiv L \pmod 4.$$

A similar argument shows

$$\tau_d^*(\underline{v}) \equiv L - d \pmod 2.$$

Now, if L is odd, the sum (1.1) contains an odd number of terms; thus, $|\tau_d(\underline{v})| \geq 1$. In particular, $|\tau_1(\underline{v})| \geq 1$. The sum (1.2) contains an even number of terms if $d = 1$, so $\tau_1^*(\underline{v})$ may be 0. If $L \equiv 2 \pmod 4$, then $|\tau_1(\underline{v})| \geq 2$, $|\tau_1^*(\underline{v})| \geq 1$. If $L \equiv 0 \pmod 4$, $\tau_1(\underline{v})$ can be 0, while $|\tau_1^*(\underline{v})| \geq 1$.

Let $a_i = v_i \oplus v_{i+1}$ ("$\oplus$" denotes addition modulo 2), $0 \leq i \leq L - 1$, and let $\underline{a} = (a_0, a_1, \ldots, a_{L-1})$. It follows that

$$w(a) = \sum_{i=0}^{L-1} a_i \equiv 0 \pmod 2,$$

so that a_{L-1} is not independent of $a_0, a_1, \ldots, a_{L-2}$. Also, given v_r and a_0, $a_1, \ldots, a_{L-2}$, the vector $\boldsymbol{v}$ is completely determined by the relation

$$v_j = v_r + \sum_{i=\min(j,r)}^{\max(j,r)-1} a_i \pmod 2.$$

In particular,

$$v_i + v_{i+d} \equiv \sum_{k=1}^{d} a_{i+k-1} \pmod 2,$$

so that

$$\tau_d(\underline{v}) = \sum_{i=0}^{L-1} (-1)^{\sum_{k=1}^{d} a_{i+k-1}},$$

$$\tau_d^*(\underline{v}) = \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=1}^{d} a_{i+k-1}}.$$

The case $d = 1$ reduces to

$$\tau_1(\underline{v}) = c = \sum_{i=0}^{L-1} (-1)^{a_i} = L - 2\sum_{i=0}^{L-1} a_i,$$

$$\sum_{i=0}^{L-1} a_i = (L - c)/2;$$

that is, $\underline{a}$ has density $(L - c)/2$. This allows us to count the number $N(c)$ of vectors $\boldsymbol{v}$ with $\tau_1(\underline{v}) = c$:

$$N(c) = \begin{cases} 2\dbinom{L}{(L-c)/2} & \text{if } L \equiv c \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$$

The factor 2 before the binomial coefficient $\dbinom{L}{(L-c)/2}$ appears because both $\underline{v}$ and $\overline{\underline{v}}$ (the mod 2 complement of $\underline{v}$) give rise to the same vector $\underline{a}$. Likewise,

$$\tau_1^*(\underline{v}) = c = \sum_{i=0}^{L-2} (-1)^{a_i} = L - 1 - 2\sum_{i=0}^{L-2} a_i,$$

$$\sum_{i=0}^{L-2} a_i = (L - c - 1)/2.$$

Now, $\underline{a}$ has density $w(\underline{a}) = (L - c - 1)/2 + a_{L-1}$, and, since $w(\underline{a})$ is even,

$$w(\underline{a}) = (L - c - 1)/2 + a_{L-1} = \begin{cases} (L - c + 1)/2 & \text{if } (L - c - 1)/2 \text{ is odd;} \\ (L - c - 1)/2 & \text{if } (L - c - 1)/2 \text{ is even;} \end{cases}$$

$$= 2\left[\frac{\frac{L - c - 1}{2} + 1}{2}\right] = 2\left[\frac{L - c + 1}{4}\right].$$

Therefore, the number $N^*(c)$ of vectors $\underline{v}$ satisfying $\tau_1^*(\underline{v}) = c$ is given by

$$N^*(c) = \begin{cases} 2\binom{L}{2[(L - c + 1)/4]} & \text{if } L \equiv c + 1 \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

3. THE DISTRIBUTION OF THE CYCLIC AUTOCORRELATIONS τ_d

We now derive the quantities $E(\tau_d(\underline{v}))$ (E: = expected value) and $E(\tau_d^2(\underline{v}))$ when $\underline{v}$ is restricted to the set

$$S(c) = \{\underline{v}: \ \tau_1(\underline{v}) = c\}.$$

Various identities used in the derivation may be found in the Appendix with their proofs. We assume throughout that the binary vectors $\underline{v}$ have length L, and that $L \equiv c \pmod 4$. Of special interest, of course, are the cases $|c| \le 2$, corresponding to vectors with minimal autocorrelation at offset 1. Therefore, we assume that $|c|$ is minimal.

We have shown that, in studying the quantities $\tau_d(\underline{v})$ with $|\tau_1(\underline{v})|$ least possible, we may restrict our attention to the set of vectors

$$R = \left\{\underline{a} = (a_0, \ldots, a_{L-1}): \ \sum_{i=0}^{L-1} a_i = (L - c)/2\right\},$$

where

$$c = \begin{cases} 0 & \text{if } L \equiv 0 \pmod 4 \\ 1 & \text{if } L \equiv 1 \pmod 4 \\ \pm 2 & \text{if } L \equiv 2 \pmod 4 \\ -1 & \text{if } L \equiv 3 \pmod 4). \end{cases}$$

Note that $|R| = \binom{L}{(L - c)/2}$. Let

$$U(d, L) = \sum_{a \in R} \sum_{j=0}^{L-1} (-1)^{\sum_{k=1}^{d} a_{j+k-1}} = \sum_{j=0}^{L-1} \sum_{r=0}^{d} (-1)^r \binom{d}{r}\binom{L - d}{(L - c)/2 - r}$$

$$= L \sum_{r=0}^{d} (-1)^r \binom{d}{r}\binom{L - d}{(L - c)/2 - r},$$

since, for any j, $0 \le j \le L - 1$, $\binom{d}{r}\binom{L - d}{(L - c)/2 - r}$ is the number of d-tuples $(a_j, \ldots, a_{j+d-1})$ of density r. To obtain $E(\tau_d)$, we must divide $U(d, L)$ by $|R|$. We now proceed to determine the quantities $U(d, L)$ for $1 \le d \le L/2$.

Case 1: $c = 0$, $L \equiv 0 \pmod 4$

We make use of Identity 1 (Appendix) to write

$$U(d, L) = L(-1)^{d/2}\left(\frac{1 + (-1)^d}{2}\right)\binom{L/2}{d/2}\binom{L}{L/2}\bigg/\binom{L}{d}.$$

Case 2: $c = 1$, $L \equiv 1 \pmod 4$

Identity 2 (Appendix) gives us

$$U(d,\,L) = L(-1)^{[d/2]}\binom{(L-1)/2}{[d/2]}\binom{L}{(L-1)/2}\Big/\binom{L}{d}.$$

Case 3: $c = -1$, $L \equiv 3 \pmod 4$

We reverse the order of summation to obtain

$$\sum_{r=0}^{d}(-1)^r\binom{d}{r}\binom{L-d}{(L+1)/2-r} = \sum_{s=0}^{d}(-1)^{d-s}\binom{d}{s}\binom{L-d}{(L-1)/2-s},$$

the same sum we considered in Case 2, except for the factor $(-1)^d$. Therefore,

$$U(d,\,L) = L(-1)^{[(d+1)/2]}\binom{(L-1)}{[d/2]}\binom{L}{(L-1)/2}\Big/\binom{L}{d}.$$

Case 4: $c = 2$, $L \equiv 2 \pmod 4$

Identity 3 (Appendix) yields

$$U(d,\,L) = \frac{(-1)^{[d/2]}\binom{L}{L/2}\binom{L/2}{[d/2]}}{\binom{L}{2[d/2]}}\cdot\frac{[(L/2)(1+(-1)^d) - 2d(-1)^d]}{L+1}.$$

Case 5: $c = -2$, $L \equiv 2 \pmod 4$

Again, reversing the order of summation in the sum of Case 4 yields

$$U(d,\,L) = \frac{(-1)^{[(d+1)/2]}\binom{L}{L/2}\binom{L/2}{[d/2]}}{\binom{L}{2[d/2]}}\quad\frac{[(L/2)(1+(-1)^d) - 2d(-1)^d]}{L+1}.$$

Combining the results of Cases 1-5 gives $E(\tau_d)$ in Theorem 2.1. We now proceed with the computation of $E(\tau_d^2)$. Let

$$S(d,\,L) = \sum_{\underline{a}\in R}\tau_d^2(\underline{v}).$$

Then

$$S(d,\,L) = \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{i+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\left\{\sum_{i=0}^{d-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}} + \sum_{i=d}^{L-d}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}}\right.$$

$$\left. + \sum_{i=L-d+1}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}}\right\}.$$

This decomposition of the innermost summation is made to facilitate the summing of the expressions involved. We change the order of summation in each of the three triple sums so that, in each case, the innermost sum is over all $\underline{a} \in R$. Then, for each i and j, we group together all those vectors $\underline{a}$ satisfying

$$r = \sum_{k=0}^{d-1} (a_{j+k} \oplus a_{i+j+k}).$$

If $0 \le i \le d-1$, then the number of vectors $\underline{a} \in R$ satisfying this condition is

$$\binom{2i}{r}\binom{L-2i}{(L-c)/2-r} = \binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\Big/\binom{L}{2i}.$$

Analogous results hold when $d \le i \le L-d$ and $L-d+1 \le i \le L-1$. Thus, we have

$$\begin{aligned} S(d, L) = \sum_{j=0}^{L-1}\Bigg\{ &\sum_{i=0}^{d-1}\sum_{r=0}^{2i} (-1)^r \binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\Big/\binom{L}{2i} \\ &+ \sum_{i=d}^{L-d}\sum_{r=0}^{2d} (-1)^r \binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2d-r}\Big/\binom{L}{2d} \\ &+ \sum_{i=L-d+1}^{L-1}\sum_{r=0}^{2L-2i} (-1)^r \binom{L}{(L-c)/2}\binom{(L-c)/2}{r} \\ &\qquad\cdot \binom{(L+c)/2}{2L-2i-r}\Big/\binom{L}{2L-2i}\Bigg\}. \end{aligned}$$

The summand is independent of j, so

$$\begin{aligned} S(d, L) = L\binom{L}{(L-c)/2}\Bigg\{ &\sum_{i=0}^{d-1}\left[\sum_{r=0}^{2i} (-1)^r \binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\right]\Big/\binom{L}{2i} \\ &+ (L-2d+1)\left[\sum_{r=0}^{2d}(-1)^r\binom{(L-c)/2}{r}\binom{(L+c)/2}{2d-r}\right]\Big/\binom{L}{2d} \\ &+ \sum_{i=L-d+1}^{L-1}\left[\sum_{r=0}^{2L-2i}(-1)^r\binom{(L-c)/2}{r}\binom{(L+c)/2}{2L-2i-r}\right]\Big/\binom{L}{2L-2i}\Bigg\}. \end{aligned}$$

As above, we divide our calculation into five cases.

Case 1: $c = 0$, $L \equiv 0 \pmod 4$

Applying Identity 6 (Appendix), we obtain

$$\begin{aligned} S(d, L) = L\binom{L}{L/2}\Bigg\{ &\sum_{i=0}^{d-1}(-1)^i\binom{L/2}{i}\Big/\binom{L}{2i} + (L-2d+1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d} \\ &+ \sum_{i=L-d+1}^{L-1}(-1)^{L-i}\binom{L/2}{L-1}\Big/\binom{L}{2L-2i}\Bigg\} \end{aligned}$$

(continued)

$$= L\binom{L}{L/2}\left\{\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d}+1\right]\right.$$

$$+ (L-2d+1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d}$$

$$\left.+\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d}-1/(L+1)\right]\right\}$$

by Identity 4 (Appendix). Thus,

$$S(d, L) = L^2\binom{L}{L/2}\left\{(-1)^d(L-2d+1)\binom{L/2}{d}\Big/\binom{L}{2d}+1\right\}/(L+2).$$

Case 2: $c = 1$, $L \equiv 1 \pmod 4$

Applying Identity 7 (Appendix), we obtain

$$S(d, L) = L\binom{L}{(L-1)/2}\left\{\sum_{i=0}^{d-1}(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L}{2i}+(L-2d+1)(-1)^d\right.$$

$$\left.\binom{(L-1)/2}{d}\Big/\binom{L}{2d}+\sum_{i=L-d+1}^{L-1}(-1)^{L-i}\binom{(L-1)/2}{L-i}\Big/\binom{L}{2L-2i}\right\}$$

$$= L\binom{L}{(L-1)/2}\left\{\frac{L+2}{L+3}\left[(-1)^{d-1}\binom{(L+3)/2}{d}\Big/\binom{L+3}{2d}+1\right]\right.$$

$$+ (L-2d+1)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L}{2d}$$

$$\left.+\frac{L+2}{L+3}\left[(-1)^{d-1}\binom{(L+3)/2}{d}\Big/\binom{L+3}{2d}-1/(L+2)\right]\right\}$$

by Identity 5 (Appendix). Hence,

$$S(d, L) = \binom{L}{(L-1)/2}\left\{(-1)^d(L-2d)[L^2+(-2d+2)L-2d-1]\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right.$$
$$\left.+L(L+1)\right\}/(L+3).$$

Case 3: $c = -1$, $L \equiv 3 \pmod 4$

Reversing the order of summation on r in all three sums reduces this to the previous case.

Case 4: $c = 2$, $L \equiv 2 \pmod 4$

Apply Identity 8 (Appendix) to obtain

$$S(d, L) = L\binom{L}{L/2-1}\left\{\sum_{i=0}^{d-1}(-1)^i\binom{L/2}{i}\Big/\binom{L}{2i}-2\sum_{i=0}^{d-1}(-1)^i\binom{L/2-1}{i-1}\Big/\binom{L}{2i}\right.$$

$$+ (L-2d+1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d}-2(L-2d+1)(-1)^d\binom{L/2-1}{d-1}\Big/\binom{L}{2d}$$

$$\left.+\sum_{i=L-d+1}^{L-1}(-1)^{L-1}\binom{L/2}{L-i}\Big/\binom{L}{2L-2i}-2\sum_{i=L-d+1}^{L-1}(-1)^{L-1}\binom{L/2-1}{L-i-1}\Big/\binom{L}{2i}\right\}$$

(continued)

$$= L\binom{L}{L/2-1}\left\{\frac{L+1}{L+2}(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} + 1 - 2\left(\frac{2(L+1)}{L(L+2)(L+4)}\right)\right.$$

$$\cdot\left[(-1)^{d-1}\{(L+2)d-1\}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1\right]$$

$$+ (L-2d+1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d} - 2(L-2d+1)(-1)^d\binom{L/2-1}{d-1}\Big/\binom{L}{2d}$$

$$+\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1/(L+1)\right] - 2\left(\frac{2(L+1)}{L(L+2)(L+4)}\right)$$

$$\cdot\left[(-1)^{d-1}\{(L+2)d-1\}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1\right]$$

using Identities 4 and 9 (Appendix). Therefore,

$$S(d, L) = \binom{L}{L/2-1}\left\{(-1)^d(L-2d+1)(L^2+[-4d+2]L-8d-4)\right.$$

$$\left.\cdot\binom{L/2}{d}\Big/\binom{L}{2d} + L^2 + 2L + 4\right\}/(L+4).$$

Case 5: $c = -2$, $L \equiv 2 \pmod 4$)

Reversing the order on summation on r in all three sums reduces this to the previous case.

Combining the results of cases 1-5 gives $E(\tau_d^2)$ in Theorem 1.1.

4. THE DISTRIBUTION OF THE TRUNCATED AUTOCORRELATION τ_d^*

We now derive the quantities $E(\tau_d^*(\underline{v}))$ and $E(\tau_d^{*2}(\underline{v}))$ when $\underline{v}$ is restricted to the set

$$S^*(c) = \{\underline{v}:\ \tau_1^*(\underline{v}) = c\}.$$

Various identities used in this derivation may be found in the Appendix, with their proofs. We again assume that the binary vectors $\underline{v}$ have length L, but now $L \equiv c + 1 \pmod 2$. Of special interest, of course, are the cases $|c| \le 1$, corresponding to vectors with minimal autocorrelation at offset 1. Again, we assume that $|c|$ is minimal.

We have shown that, in studying the quantities $\tau_d^*(\underline{v})$ with $|\tau_1^*(\underline{v})|$ least possible, we may restrict our attention to the set of vectors

$$R^* = \left\{\underline{a} = (a_0, a_1, \ldots, a_{L-1}):\ \sum_{i=0}^{L-2} a_i = (L-c-1)/2\right\},$$

where

$$c = \begin{cases} 0 & \text{if } L \text{ is odd,} \\ \pm 1 & \text{if } L \text{ is even.}\end{cases}$$

Note that

$$|R^*| = \binom{L-1}{(L-c-1)/2}.$$

Let

$$U^*(d, L) = \sum_{\underline{a}\in R^*}\sum_{j=0}^{L-d-1}(-1)^{\sum_{k=0}^{d-1} a_{j+k}}$$

Since there are $\binom{d}{r}\binom{L-d-1}{(L-c-1)/2-r}$ d-tuples $(a_j, \ldots, a_{j+d-1})$ of density r, we have

$$U^*(d, L) = \sum_{j=0}^{L-d-1} \sum_{r=0}^{d} (-1)^r \binom{d}{r} \binom{L-d-1}{(L-d-1)/2 - r}$$

$$= (L-d) \sum_{r=0}^{d} (-1)^r \binom{d}{r} \binom{L-d-1}{(L-c-1)/2 - r}$$

$$= \begin{cases} (L-d)(-1)^{d/2} \dfrac{1+(-1)^d}{2} \dbinom{(L-1)/2}{d/2} \dbinom{L-1}{(L-1)/2} \Big/ \dbinom{L-1}{d} \\ \qquad \text{if } c = 0, \text{ by Identity 1;} \\ (L-d)(-1)^{[d/2]} \dbinom{(L-2)/2}{[(d+1)/2]} \dbinom{L-1}{(L-2)/2} \Big/ \dbinom{L-1}{2[(d+1)/2]} \\ \qquad \text{if } c = 1, \text{ by Identity 2;} \\ (L-d)(-1)^{[(d+1)/2]} \dbinom{(L-2)/2}{[(d+1)/2]} \dbinom{L-1}{(L-2)/2} \Big/ \dbinom{L-1}{2[(d+1)/2]} \\ \qquad \text{if } c = -1, \text{ by Identity 2.} \end{cases}$$

To obtain $E(\tau_d^*)$ of Theorem 1.1, we divide $U^*(d, L)$ by $|R^*|$, and combine the cases $c = \pm 1$, to obtain

$$E(\tau_d^*) = (-1)^{d/2} \frac{1+(-1)^d}{2} (L-d) \binom{[(L-1)/2]}{d/2} \Big/ \binom{L-1}{d}.$$

We now proceed with the computation of $E(\tau_d^{*2})$. Let

$$\begin{aligned} S^*(d, L) &= \sum_{\underline{a} \in R^*} \tau_d^{*2}(\underline{v}) \\ &= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k}} \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{i+k}} \\ &= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k} + a_{i+k}} \\ &= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} \sum_{i=-j}^{L-d-j-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k} + a_{i+j+k}}. \end{aligned}$$

We split this sum into three parts, depending on the degree of overlap of the two d-tuples $(a_j, \ldots, a_{j+d-1})$ and $(a_{i+j}, \ldots, a_{i+j+d-1})$: complete, partial, or none.

$$\begin{aligned} S^*(d, L) = \binom{L-1}{(L-c-1)/2} \Bigg\{ &\sum_{i=0}^{0} \sum_{j=0}^{L-d-1} 1 + 2 \sum_{i=1}^{d-1} \sum_{j=0}^{L-d-i-1} \sum_{r=0}^{2i} (-1)^r \\ &\cdot \binom{(L-c-1)/2}{r} \binom{(L+c-1)/2}{2i-r} \Big/ \binom{L-1}{2i} \\ &+ 2 \sum_{i=d}^{L-d-1} \sum_{j=0}^{L-d-i-1} \sum_{r=0}^{2d} (-1)^r \binom{(L-c-1)/2}{r} \binom{(L+c-1)/2}{2d-r} \Big/ \binom{L-1}{2d} \Bigg\}. \end{aligned}$$

We now consider three cases, depending on the value of c:

Case 1: $c = 0$, L odd. Applying Identity 6 (Appendix), we obtain,

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\left\{(L-d) + 2\sum_{i=1}^{d-1}(L-d-i)(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$\left.+ 2\binom{L-2d+1}{2}(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right\}$$
$$= \binom{L-1}{(L-1)/2}\left\{L - d + 2(L-d)\sum_{i=1}^{d-1}(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$- (L-1)\sum_{i=1}^{d-1}(-1)^i\binom{(L-3)/2}{i-1}\Big/\binom{L-1}{2i}$$
$$\left.+ (L-2d+1)(L-2d)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L+1}{2d}\right\}$$
$$= \binom{L-1}{(L-1)/2}\left\{L - d + 2(L-d)\frac{L}{L+1}\left[(-1)^{d-1}\binom{(L+1)/2}{d}\Big/\binom{L+1}{2}-1/L\right]\right.$$
$$- (L-1)\frac{2}{(L-1)(L+1)(L+3)}\left[(-1)^{d-1}(dL+d-1)\right.$$
$$\left.\binom{(L+1)/2}{d}\Big/\binom{L+1}{2d} - 1\right]$$
$$\left.+ (L-2d+1)(L-2d)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right\}$$

by Identities 4 and 9. Therefore,

$$S^*(d, L) = \frac{1}{(L+1)(L+3)}\binom{L-1}{(L-1)/2}\left\{(-1)^dL[L^3 + (3-2d)L^2 + (1-4d)L\right.$$
$$\left.+ (1+2d)]\binom{(L+1)/2}{d}\Big/\binom{L+1}{2d} + [L^3 + (2-d)L^2 + (-1-2d)L + (3d)]\right\}.$$

Case 2: $c = 1$, L even. Applying Identity 7 (Appendix), we obtain,

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\left\{L - d + 2\sum_{i=1}^{d-1}(L-d-i)(-1)^i\binom{(L-2)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$\left.+ 2\binom{L-2d+1}{2}(-1)^d\binom{(L-2)/2}{d}\Big/\binom{L-1}{2d}\right\}$$
$$= \binom{L-1}{(L-2)/2}\left\{L - d + 2(L-d)\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{(L+2)/2}{d}\Big/\binom{L-2}{2d}\right.\right.$$
$$\left.- 1/(L+1)\right] - 2\left(\frac{L+1}{(L+2)(L+4)}\right)[(-1)^{d-1}(dL+2d-1)$$
$$\left.- (L+2)/(L+1)] + (L-2d+1)(L-2d)(-1)^d\binom{(L-2)/2}{d}\Big/\binom{L-1}{2d}\right\}$$

sing Identities 5 and 10. Consequently,

$$S^*(d, L) = \frac{1}{(L+2)(L+4)}\binom{L-1}{(L-2)/2}\Big\{(-1)^d(L-2d+1)[L^3+(4-2d)L^2$$

$$+ (-8d)L + (-2-4d)]\binom{L/2}{d}\Big/\binom{L}{2d} + [L^3 + (4-d)L^2 + (2-4d)L+2]\Big\}.$$

Case 3: $c = -1$, L even. Reversing the order of summation on r in both sums reduces this to the previous case.

Combining the results of Cases 1–3 gives $E(\tau_d^{*2})$ for $d \leq L/2$ in Theorem 1.1. The case when $d > L/2$ is similarly handled, with the result:

$$S^*(d, L) = \binom{L-1}{(L-c-1)/2}\Big\{\sum_{i=0}^{0}\sum_{j=0}^{L-d-1} 1 + 2\sum_{i=1}^{L-d-1}\sum_{j=0}^{L-d-i-1}\sum_{r=0}^{2i}(-1)^r$$

$$\cdot \binom{(L-c-1)/2}{r}\binom{(L+c-1)/2}{2i-r}\Big/\binom{L-1}{2i}\Big\}.$$

Once again we consider three cases, depending on the value of c:

Case 1: $c = 0$, L odd. Applying Identity 6, we obtain

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\Big\{L - d + 2\sum_{i=1}^{L-d-1}(L-d-i)(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}$$

$$= \binom{L-1}{(L-1)/2}\Big\{L - d + 2(L-d)\frac{L}{L+1}\Big[(-1)^{L-d-1}$$

$$\cdot\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)} - 1/L\Big]$$

$$- \frac{2L}{(L+1)(L+3)}\Big[(-1)^{L-d-1}\{(L+1)(L-d)-1\}$$

$$\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)} - 1\Big]\Big\}$$

using Identities 5 and 9. This yields

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\Big\{(-1)^{L-d}2L(-2L+2d-1)\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)}$$

$$+ L^3 + (2-d)L^2 + (-1-2d)L + 3d\Big\}\Big/(L+1)(L+3).$$

Case 2: $c = 1$, L even. Applying Identity 7, we obtain

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\Big\{L - d + 2\sum_{i=1}^{L-d-1}(-1)^i(L-d-i)\binom{(L-2)/2}{i}\Big/\binom{L-1}{2i}\Big\}$$

$$= \binom{L-1}{(L-2)/2}\Big\{L - d + 2(L-d)\frac{L+1}{L+2}\Big[(-1)^{L-d-1}\binom{(L+2)/2}{L-d}\Big/\binom{L+2}{2(L-d)}$$

$$- 1/(L+1)\Big] - \frac{2(L+1)}{(L+2)(L+4)}\Big[(-1)^{L-d-1}\{(L+2)(L-d)-1\}$$

(continued)

$$\cdot \binom{(L+2)/2}{L-d} \Big/ \binom{L+2}{2(L-d)} - 1\Big]\Big\}$$

using Identities 6 and 10. As a result,

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\Big\{(-1)^{L-d}2(L+1)(-2L+2d-1)\binom{(L+2)/2}{L-d}\Big/\binom{L+2}{2(L-d)}$$

$$+ L^3 + (4-d)L^2 + (2-4d)L + 2\Big\}\Big/(L+2)(L+4).$$

Case 3: $c = -1$, L even. Reversing the order of summation on r reduces this to the previous case.

Combining the results of Cases 1-3 gives $E(\tau_d^{*2})$ for $d > L/2$ in Theorem 1.1.

5. VARIANCES

The variances of τ_d and τ_d^* may be obtained from the above results by noting that the variance σ^2 of any statistic x is given by

$$\sigma^2(x) = E(x^2) - E(x)^2.$$

These numbers are tabulated along with $E(\tau_d)$, etc., in Table 1.

TABLE 1. Expected Values for Selected Values of L

L	d	$E(\tau_d)$	$E(\tau_d^2)$	$E(\tau_d^*)$	$E(\tau_d^{*2})$	σ^2	σ^{*2}
4	1	.00000	.00000	.00000	1.00000	.00000	1.00000
4	2	-1.33333	5.33333	-.66667	1.33333	3.55556	.88889
4	3	.00000	.00000	.00000	1.00000	.00000	1.00000
5	1	1.00000	1.00000	.00000	.00000	.00000	.00000
5	2	-1.00000	5.00000	-1.00000	3.66667	4.00000	2.66667
5	3	-1.00000	5.00000	.00000	1.33333	4.00000	1.33333
5	4	1.00000	1.00000	1.00000	1.00000	.00000	.00000
6	1	.00000	4.00000	.00000	1.00000	4.00000	1.00000
6	2	-.40000	4.00000	-.80000	4.00000	3.84000	3.36000
6	3	.00000	10.40000	.00000	2.60000	10.40000	2.60000
6	4	-.40000	4.00000	.40000	1.60000	3.84000	1.44000
6	5	.00000	4.00000	.00000	1.00000	4.00000	1.00000
7	1	-1.00000	1.00000	.00000	.00000	.00000	.00000
7	2	-1.00000	7.40000	-1.00000	5.80000	6.40000	4.80000
7	3	.60000	4.20000	.00000	1.60000	3.84000	1.60000
7	4	.60000	4.20000	.60000	2.60000	3.84000	2.24000
7	5	-1.00000	7.40000	.00000	1.60000	6.40000	1.60000
7	6	-1.00000	1.00000	-1.00000	1.00000	.00000	.00000
8	1	.00000	.00000	.00000	1.00000	.00000	1.00000
8	2	-1.14286	9.14286	-.85714	6.28571	7.83673	5.55102
8	3	.00000	3.65714	.00000	3.51429	3.65714	3.51429
8	4	.68571	12.80000	.34286	3.20000	12.32980	3.08245
8	5	.00000	3.65714	.00000	2.60000	3.65714	2.60000
8	6	-1.14286	9.14286	-.28571	1.71429	7.83673	1.63265
8	7	.00000	.00000	.00000	1.00000	.00000	1.00000

(continued)

TABLE 1 (continued)

L	d	$E(\tau_d)$	$E(\tau_d^2)$	$E(\tau_d^*)$	$E(\tau_d^{*2})$	σ^2	σ^{*2}
9	1	1.00000	1.00000	.00000	.00000	.00000	.00000
9	2	-1.00000	9.57143	-1.00000	7.85714	8.57143	6.85714
9	3	-.42857	6.14286	.00000	3.54286	5.95918	3.54286
9	4	.42857	9.00000	-.42857	5.80000	8.81633	5.61633
9	5	.42857	9.00000	.00000	3.20000	8.81633	3.20000
9	6	-.42857	6.14286	.42857	2.60000	5.95918	2.41633
9	7	-1.00000	9.57143	.00000	1.71429	8.57143	1.71429
9	8	1.00000	1.00000	1.00000	1.00000	.00000	.00000
16	1	.00000	.00000	.00000	1.00000	.00000	1.00000
16	2	-1.06667	17.06667	-.93333	14.66667	15.92889	13.79556
16	3	.00000	13.12821	.00000	10.96923	13.12821	10.96923
16	4	.24615	14.91841	.18462	11.10676	14.85782	11.07268
16	5	.00000	13.52603	.00000	9.61414	13.52603	9.61414
16	6	-.11189	15.31624	-.06993	9.25128	15.30372	9.24639
16	7	.00000	11.37778	.00000	7.75556	11.37778	7.75556
16	8	.08702	28.44444	.04351	7.11111	28.43687	7.10922
16	9	.00000	11.37778	.00000	6.33333	11.37778	6.33333
16	10	-.11189	15.31624	-.04196	5.42222	15.30372	5.42046
16	11	.00000	13.52603	.00000	4.54188	13.52603	4.54188
16	12	.24615	14.91841	.06154	3.64755	14.85782	3.64377
16	13	.00000	13.12821	.00000	2.76410	13.12821	2.76410
16	14	-1.06667	17.06667	-.13333	1.86667	15.92889	1.84889
16	15	.00000	.00000	.00000	1.00000	.00000	1.00000

APPENDIX

In this section we give identities used in the proof of Theorem 1.1. Some are merely stated, and others, previously unknown to the authors, are proved.

Identity 1:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k} = (-1)^{n/2}\,\frac{1+(-1)^n}{2}\binom{x}{n/2}\binom{2x}{x}\Big/\binom{2x}{n}$$

Identity 2:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+1-n}{x-k} = (-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x}\Big/\binom{2x+1}{n}$$

Identity 3:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+2-n}{x-k}$$

$$= \frac{(-1)^{[n/2]}\binom{2x+2}{x+1}\binom{x+1}{[n/2]}}{\binom{2x+2}{2[n/2]}} \cdot \frac{[(x+1)(1+(-1)^n) - 2n(-1)^n]}{2(x+2)}$$

Identity 4:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k}\Big/\binom{2x}{2k} = \frac{2x+1}{2x+2}\left[(-1)^n\binom{x+1}{n+1}\Big/\binom{2x+2}{2n+2} + (-1)^a\binom{x+1}{a}\Big/\binom{2x+2}{2a}\right]$$

Identity 5:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k}\Big/\binom{2x+1}{2k} = \frac{2x+3}{2x+4}\left[(-1)^n\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2} + (-1)^a\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right]$$

Identity 6:

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k} = (-1)^{n/2}\,\frac{1+(-1)^n}{2}\binom{x}{n/2}$$

Identity 7:

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x+1}{n-k} = (-1)^{[n/2]}\binom{x}{[n/2]}$$

Identity 8:

$$\sum_{k=0}^{2n}(-1)^k\binom{x}{k}\binom{x+2}{2n-k} = (-1)^n\left[\binom{x}{n} - \binom{x}{n-1}\right] = (-1)^n\left[\binom{x+1}{n} - 2\binom{x}{n-1}\right]$$

Identity 9:

$$\sum_{k=a}^{n}(-1)^k\binom{x-1}{k-1}\Big/\binom{2x}{2k}$$

$$= \frac{2(2x+1)}{(2x)(2x+2)(2x+4)}\left\{(-1)^n[2(x+1)(n+1)-1]\binom{x+1}{n+1}\Big/\binom{2x+2}{2n+2} + (-1)^a[2a(x+1)-1]\binom{x+1}{a}\Big/\binom{2x+2}{2a}\right\}$$

Identity 10:

$$\sum_{k=a}^{n}(-1)^k k\binom{x}{k}\Big/\binom{2x+1}{2k}$$

$$= \frac{2x+3}{(2x+4)(2x+6)}\left\{(-1)^n[2(x+2)(n+1)-1]\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2} + (-1)^a[2a(x+2)-1]\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right\}$$

Proof of Identity 1: See [1, 3.58].

Proof of Identity 2: Let

$$f(x, n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+1-n}{x-k}$$

$$= \sum_{k=0}^{n}(-1)^k\left[\binom{n}{k}\binom{2x-n}{x-k} + \binom{n+1}{k+1}\binom{2x-n}{x-1-k} - \binom{n}{k+1}\binom{2x-n}{x-1-k}\right]$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\binom{2x+1-(n+1)}{x-k} + \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k}$$

$$= (-1)^{n/2}\,[1+(-1)^n]\binom{x}{n/2}\binom{2x}{x}\Big/\binom{2x}{n} - f(x, n+1) \quad \text{by Identity 1.}$$

$$f(x,\ 2m) = \sum_{k=1}^{m} [f(x,\ 2k) - f(x,\ 2k-2)] + f(x,\ 0)$$

$$= -2\binom{2x}{x}\sum_{k=1}^{m}(-1)^{k-1}\binom{x}{k-1} \Big/ \binom{2x}{2k-2} + \binom{2x+1}{x}$$

$$= (-1)^m\binom{x}{m}\binom{2x+1}{x} \Big/ \binom{2x+1}{2m} \quad \text{by Identity 4.}$$

Thus

$$f(x,\ 2m-1) = (-1)^{m-1}\binom{x}{m-1}\binom{2x+1}{x} \Big/ \binom{2x+1}{2m-1}$$

$$f(x,\ n) = (-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x} \Big/ \binom{2x+1}{n} \qquad \text{Q.E.D.}$$

Proof of Identity 3: Let

$$g(x,\ n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+2-n}{x-k}$$

$$= \sum_{k=0}^{n}(-1)^k\left(\binom{n}{k}\binom{2x+1-n}{x-k} + \binom{n+1}{k+1}\binom{2x+1-n}{x-1-k} - \binom{n}{k+1}\binom{2x+1-n}{x-1-k}\right)$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+1-n}{x-k} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\binom{2x+2-(n+1)}{x-k}$$

$$+ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+1-n}{x-k}$$

$$= 2(-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x} \Big/ \binom{2x+1}{n} - g(x,\ n+1) \quad \text{by Identity 2.}$$

$$g(x,\ 2m) = \sum_{k=1}^{m}[g(x,\ 2k) - g(x,\ 2k-2)] + g(x,\ 0)$$

$$= -2\binom{2x+1}{x}\left[\sum_{k=1}^{m}(-1)^k\binom{x+1}{k} \Big/ \binom{2x+2}{2k} + \sum_{k=0}^{m-1}(-1)^k\binom{x}{k} \Big/ \binom{2x+1}{k}\right]$$

$$+ \binom{2x+2}{x}$$

$$= -2\binom{2x+1}{x}\left[\frac{2x+3}{2x+4}\left\{(-1)^m\binom{x+2}{m} \Big/ \binom{2x+4}{2m+2} - 1/(2x+3)\right\}\right.$$

$$\left. + \frac{2x+3}{2x+4}\left\{(-1)^{m-1}\binom{x+2}{m} \Big/ \binom{2x+4}{2m} + 1\right\}\right] \quad \text{by Identities 4 and 5.}$$

$$g(x,\ 2m) = (-1)^m(x-2m+1)\binom{2x+4}{x+2}\binom{x+2}{m} \Big/ \binom{2x+4}{2m}2(2x-2m+3)$$

Thus

$$g(x,\ 2m-1) = (-1)^{m-1}(2m-1)\binom{2x+4}{x+2}\binom{x+2}{m-1} \Big/ \binom{2x+4}{2m-2}2(2x-2m+5)$$

and

$$g(x,\ n) = (-1)^{[n/2]}\ \{[x + 1][1 + (-1)^n] - 2n(-1)^n\}\binom{2x + 2}{x + 1}\binom{x + 1}{[n/2]} \Big/ \binom{2x + 2}{2[n/2]}2(x + 2)$$

Q.E.D.

Proof of Identity 4:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x}{2k} = \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\sum_{k=a}^{n}\binom{k - 1/2}{x} \qquad \text{by [1, Z.55]}$$

$$= \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\left[\binom{n + 1/2}{x + 1} - \binom{a - 1/2}{x + 1}\right] \qquad \text{by [1, 1.48]}$$

$$= \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\left[(-1)^{x-n}2^{-2x-2}\binom{2x + 2}{x + 1}\binom{x + 1}{n + 1} \Big/ \binom{2x + 2}{2n + 2}\right.$$

$$\left. -(-1)^{x-a+1}2^{-2x-2}\binom{2x + 2}{x + 1}\binom{x + 1}{a} \Big/ \binom{2x + 2}{2a}\right] \qquad \text{by [1, Z.55]}$$

$$= \frac{2x + 1}{2x + 2}\left[(-1)^n\binom{x + 1}{n + 1} \Big/ \binom{2x + 2}{2n + 2} + (-1)^a\binom{x + 1}{a} \Big/ \binom{2x + 2}{2a}\right]$$

Q.E.D.

Proof of Identity 5:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x + 1}{2k} = \sum_{k=a}^{n}(-1)^k\binom{x + 1}{k} \Big/ \binom{2x + 2}{2k}$$

$$= \frac{2x + 3}{2x + 4}\left[(-1)^n\binom{x + 2}{n + 1} \Big/ \binom{2x + 4}{2n + 2} + (-1)^a\binom{x + 2}{a} \Big/ \binom{2x + 4}{2a}\right]$$

by Identity 4.

Proof of Identity 6: See [1, 3.32].

Proof of Identity 7: Apply [1, 3.31] with $y = x + 1$.

Proof of Identity 8: Apply [1, 3.31] with $y = x + 2$.

Proof of Identity 9:

$$\sum_{k=a}^{n}(-1)^k\binom{x - 1}{k - 1} \Big/ \binom{2x}{2k} = \frac{2n + 1}{2n}\sum_{k=a}^{n}(-1)^k\binom{n + 1}{k + 1} \Big/ \binom{2x + 2}{2k + 2} - \frac{1}{2x}\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x}{2k}$$

$$= -\frac{2x + 1}{2x} \cdot \frac{2x + 3}{2x + 4}\left[(-1)^{n+1}\binom{x + 2}{n + 2} \Big/ \binom{2x + 4}{2n + 4}\right.$$

$$\left. + (-1)^{a+1}\binom{x + 2}{a + 1} \Big/ \binom{2x + 4}{2a + 2}\right)$$

$$-\frac{1}{2x} \cdot \frac{2x + 1}{2x + 2}\left[(-1)^n\binom{x + 1}{n + 1} \Big/ \binom{2x + 2}{2n + 2}\right.$$

$$\left. + (-1)^a\binom{x + 1}{a} \Big/ \binom{2x + 2}{2a}\right] \qquad \text{by Identity 4}$$

(continued)

$$= \frac{2(2x + 1)}{2x(2x + 2)(2x + 4)}\left\{(-1)^n[2(n + 1)(x + 1) - 1]\binom{x + 1}{n + 1}\bigg/\binom{2x + 2}{2n + 2}\right.$$

$$\left. + (-1)^a[2a(x + 1) - 1]\binom{x + 1}{a}\bigg/\binom{2x + 2}{2a}\right\} \qquad \text{Q.E.D.}$$

Proof of Identity 10:

$$\sum_{k=a}^{n}(-1)^k k\binom{x}{k}\bigg/\binom{2x + 1}{2k} = (x + 1)\sum_{k=a}^{n}(-1)^k\binom{x}{k - 1}\bigg/\binom{2x + 2}{2k}$$

$$= \frac{2x + 3}{(2x + 4)(2x + 6)}\left\{(-1)^n[2(n + 1)(x + 2) - 1]\right.$$

$$\binom{x + 2}{n + 1}\bigg/\binom{2x + 4}{2n + 2}$$

$$\left. + (-1)^a[2a(x + 2) - 1]\binom{x + 2}{a}\bigg/\binom{2x + 4}{2a}\right\} \text{ by Identity 9}$$

Q.E.D.

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FIBONACCI CUBATURE

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Korobov [1] developed procedures for integration over an N-dimensional cube which are referred to in the literature [2, 3, 4] as number-theoretical methods or the method of optimal coefficients. These methods involve summation over a lattice of nodes defined by a single index instead of N nested summations. For the two-dimensional case, a particularly simple form involving the Fibonacci numbers is obtained. Designating the Nth Fibonacci number by F_N, k/F_N by x_k, and $\{F_{N-1}x_k\}$ by y_k, where { } denotes the fractional part, the cubature rule is

$$\int_0^1\int_0^1 f(x, y)\, dxdy = \frac{1}{F_N}\sum_{k=1}^{F_N} f(x_k, y_k). \tag{1}$$

The summation can also be taken as running from 0 to $F_N - 1$, which replaces a node 1, 0 by 0, 0 while leaving the rest unchanged. This cubature rule was also given by Zaremba [5].

The investigators have been interested primarily in the higher-dimensional cases and very little has been published on the two-dimensional case. An examination of the nodes for the two-dimensional case suggested an interesting conjecture about their symmetry properties and a modification which improves the accuracy significantly.

Conjecture: If x_k, y_k is a node for $1 \leq k \leq F_N - 1$ and if N is $\binom{\text{odd}}{\text{even}}$, then $\begin{pmatrix} y_k, & 1 - x_k \\ y_k, & x_k \end{pmatrix}$ is also a node.

Perhaps a reader can supply a proof.

One would expect the nodes of an efficient cubature rule to be symmetric about the center of the square so as to give identical results for $f(x, y)$, $f(x, 1 - y)$, $f(1 - x, y)$, and $f(1 - x, 1 - y)$. This suggests modifying (1) to

$$\int_0^1 \int_0^1 f(x, y)\, dxdy = \frac{f(0, 0) + f(0, 1) + \sum_{k=1}^{F_N} f(x_k, y_k) + f(x_k, 1 - y_k)}{2(F_N + 1)}. \tag{2}$$

Essentially, we have completed the square on the nodes. Some preliminary calculations* indicated that this gain in accuracy more than compensated for doubling the number of function evaluations.

The performance of the method is reasonably good, although it is not competitive with a high-order-product Gauss rule using a comparable number of nodes. It might be a useful alternative for use on programmable hand calculators which do not have the memory to store tables of weights and nodes and where the use of only one loop in the algorithm is a significant advantage.

I also plan to investigate the effect of the symmetrization in higher-dimensional calculations, but in such cases the number of nodes increases very rapidly with the dimensionality.

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*I am indebted to Mr. Robert Harper, a graduate student in the Department of Chemical Engineering for programming the procedure on a T159.

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ON A PROBLEM OF S. J. BEZUSZKA AND M. J. KENNEY ON CYCLIC DIFFERENCE OF PAIRS OF INTEGERS

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Begin with four nonnegative integers, for example, a, b, c, and d. Take cyclic difference of pairs of integers (the smaller integer from the larger), where the fourth difference is always the difference between the last integer

d and the first integer a. Repeat this process on the differences. If we start with 8, 3, 5, 6 and follow the procedure described above, then the process terminates in the sixth row with all zeros. Now we have the following problem due to S. J. Bezuszka and M. J. Kenney [1].

Problem: Is there a selection procedure that will yield sets of four starting integers which terminate with all zeros on the 7th row, the 8th row, ..., the nth row?

Bezuszka and Kenney are of the opinion that the solution to this problem is an interesting application of Tribonacci numbers.

First we note the following easy facts, which we shall use later.

1. If we start with a set of four nonnegative integers—a, b, c, d—that terminates with all zeros on the ith row, then the set of four integers $a + x$, $b + x$, $c + x$, $d + x$, where x is a positive integer, also terminates with all zeros on the ith row.

2. The set of four nonnegative integers a, b, c, d gives the same number of rows as the set na, nb, nc, nd, where n is a positive integer.

3. The set $x - a$, $x - b$, $x - c$, $x - d$ yields the same number of rows as a, b, c, d, provided none of $x - a$, $x - b$, $x - c$, $x - d$ is a negative integer. Again, the set $x - a$, $x - b$, $x - c$, $x - d$ yields the same number of rows as a, b, c, d. We can take the integer x big enough to make each of $x - a$, $x - b$, $x - c$, and $x - d$ nonnegative.

4. If in place of a, b, c, d any cyclic or reverse cyclic order of a, b, c, d is taken as the set of four starting numbers, we again get the same number of rows. For example, 0, 0, a, b being the reverse cycle of 0, 0, b, a will terminate in the same number of steps.

From the above, it is clear that any set of four nonnegative integers a, b, c, d can be replaced by the set 0, u, v, w, which yields the same number of rows as a, b, c, d.

Let a, b, c, d be the four starting numbers. Denote a_1, b_1, c_1, d_1 as r_2; a_2, b_2, c_2, d_2 as r_3; ...; and A_1, B_1, C_1, D_1 as R_2; A_2, B_2, C_2, D_2 as R_3; ... . For example

$$\begin{array}{c} \cdots\cdots\cdots \\ A_1,\ B_1,\ C_1,\ D_1 \\ a,\ b,\ c,\ d \\ a_1,\ b_1,\ c_1,\ d_1 \\ \cdots\cdots\cdots \end{array}$$

Suppose we are given four nonnegative integers a, b, c, d. Is it always possible to find R_2? That is, can we find four nonnegative integers t, u, v, w that will yield a, b, c, d in the second row?

If we start with four nonnegative integers t, u, v, w as our first row, where $t+u+v+w$ is either odd or even, and get a, b, c, d in the second row, then it is easy to see that $a+b+c+d$ is always even. So a, b, c, d with an odd total can never be the second row of any set of four nonnegative integers t, u, v, w. Hence, R_2 is not possible if $a+b+c+d$ is odd. Again, R_3 is not possible if a, b, c, d are such that a and b are odd (even) and c and d are even (odd), for then, if R_2 exists, R_2 will have three odd and one even or one odd and three even, thereby making $A_1 + B_1 + C_1 + D_1$ odd and R_3 impossible.

If the four starting numbers are a, b, c, d and R_2 exists for this set of numbers, then after a little calculation it can be seen that we must have one of the following situations:

(i)	$a = b + c + d$	(v)	$a + b = c + d$
(ii)	$b = a + c + d$	(vi)	$a + c = b + d$
(iii)	$c = a + b + d$	(vii)	$a + d = b + c$
(iv)	$d = a + b + c$		

Hence, if we are given a, b, c, d where none of the above seven cases holds, then R_2 is impossible.

Since any set of four nonnegative integers t, u, v, w can be replaced by 0, a, b, c ($c \geq a$) without changing the number of steps, from now on, we take 0, a, b, c ($c \geq a$) as our starting numbers.

In case the four starting numbers are 0, a, b, c ($c \geq a$), then R_2 is possible if either $b = c + a$ or $c = a + b$. If we have 0, a, $a+c$, c, we can take R_2 as

(i) $a, a, 0, a + c$ (iii) $a + c, a + c, c, a + 2c$ or
(ii) $c, c, a + c, 0$ (iv) $a + c, a + c, 2a + c, a$

If we have 0, a, b, $a+b$, we can take R_2 as

(i) $0, 0, a, a + b$ (iii) $a, a, 2a, 2a + b$ or
(ii) $a + b, a + b, b, 0$ (iv) $b, b, a + b, a + 2b$

The two sets of four starting numbers a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2 are said to be complements of each other if $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = d_1 + d_2$. If two sets of four starting numbers are complements of each other, they terminate on the same number of rows. Now $a, a, 0, a + c$ and $c, c, a + c, 0$ are complements of each other and $0, 0, a, a + b$ and $a + b, a + b, b, 0$ are complements of each other.

Theorem 1: If the set of four nonnegative integers 0, a, b, c, where $c \geq a + b$ terminates in k steps, then the set of four integers 0, $c - b$, $2c - b$, $4c - b - a$ terminates in $k + 3$ steps.

Proof: Let the four starting numbers be 0, $c - b$, $2c - b$, $4c - b - a$. They are clearly nonnegative. Then we have

$$\begin{array}{cccc} 0, & c - b, & 2c - b, & 4c - b - a \\ c - b, & c, & 2c - b, & 4c - b - a \\ b, & c - a, & 2c - b, & 3c - a \\ c - a - b, & c + a - b, & c + b - a, & 3c - a - b \end{array}$$

The fourth row can be rewritten as x, $2a+x$, $2b+x$, $2c+x$ where $x = c - a - b$, a nonnegative integer. Now, the four starting integers x, $2a+x$, $2b+x$, $2c+x$ will take the same number of steps as 0, $2a$, $2b$, $2c$ for termination. Again, 0, $2a$, $2b$, $2c$ will yield the same number of steps as 0, a, b, c. Thus the set 0, $c - b$, $2c - b$, $4c - b - a$ needs three steps more than 0, a, b, c for termination. Hence, the theorem is proved.

Since $4c - b - a \geq (c - b) + (2c - b) - 3c - 2b$, taking 0, $c - b$, $2c - b$, $4c - b - a$ as 0, a_1, b_1, c_1, where $c_1 \geq a_1 + b_1$, we can get four nonnegative integers 0, $c_1 - b_1$, $2c_1 - b_1$, and $4c_1 - b_1 - a_1$ which will yield three steps more than 0, $c - b$, $2c - b$, $4c - b - a$. We can continue this process n times to get $3n$ steps more than the number of steps given by 0, a, b, c.

If we have 0, a, b, c, where $c < a + b$ but greater than each of a and b, then we consider the reverse cycle of its complement c, $c - a$, $c - b$, 0, that is, 0, $c - b$, $c - a$, c. Now Theorem 1 can be applied to 0, $c - b$, $c - a$, c for $c > (c - b) + (c - a)$.

Theorem 2: If the set of four nonnegative integers 0, a, 0, b, where $b > a$, terminates in k steps, then the set of four integers 0, $a + b$, $a + 2b$, $a + 4b$ terminates in $k + 3$ steps. If $a > b$, we can take 0, $2b$, $3b$, $a + 4b$.

Proof: The proof is easy and is left to the reader.

Since $a + 4b > (a + b) + (a + 2b)$ for $b > a$, we can apply Theorem 1 to the new set. Hence, if we start with 0, a, 0, b, $b > a$, which terminates on the 5th row, we get two different sets of four starting numbers, one from Theorem 1 and the other from Theorem 2, each of which terminates on the 8th row. They are given by $0, b, 2b, 4b - a$ and $0, a + b, a + 2b, a + 4b$. Their reversed complements, given by $0, 2b - a, 3b - a, 4b - a$ and $0, 2b, 3b, 4b + a$ will also terminate on the 8th row.

Since $0, a, 0, b$ and $0, a + x, x, b + 2x$ $(b > a)$ have the same number of steps, we get another set $0, b + x, 2b + 3x, 4b - a + 6x$, by Theorem 1, which also terminates on the 8th row. Again, $0, b, 2b, 4b - a$ and $0, b + x, 2b + x, 4b - a + 2x$ have the same number of steps.

We give examples of some sets of four integers that terminate on the 3rd, 4th, 5th, 6th, and 7th row. We have not included their complements in our list.

1. $0, 0, 0, a$ $(a > 0)$ .... five rows

2. $0, 0, a, a$ $(a > 0)$ .... four rows
   $0, 0, a, a + b$; $0, 0, a + x, a + b + 2x$ $(0 < b \leq a)$ ...... five rows
   $0, 0, a, 2a + x$ $(x \quad 0)$; $0, 0, a, na + x$ $(n \geq 3)$ .... seven rows

3. $0, a, 0, a$ $(a > 0)$; $0, a, 2a, a$ $(a > 0)$ .... three rows
   $0, a, a + x, x$ $(x \neq a)$; $0, a + x, x, a + 2x$ $(x > 0)$;
   $0, a + x, a, 2a + x$ $(a > 0)$ .... four rows
   $0, a, a, 2a + x$ $(x > 0)$; $0, a, a, 2a - x$ $(x \leq a)$ .... five rows
   $0, a, 0, b$ $(a \neq b$, not both zero);
   $0, a + x, x, b + 2x$ $(b \neq a$, not both zero)
   $0, a + x, a, a + 2x$ .... five rows

4. $0, a, a + x, 2a + x$; $0, x, a + x, a + 2x$ $(a, x > 0)$ .... six rows
   $0, a, 2a, 5a$ $(a > 0)$; $0, 3a, 4a, 5a$ $(a > 0)$ .... six rows
   $0, a + x, 2a + x, 3a + x$ $(a \neq 0, x > 2a)$;
   $0, a - x, 2a - x, 3a - x$ $(a \leq x < a)$ .... six rows
   $0, a, a + x, a + x$ $(x \geq a > 0)$ .... five rows
   $0, a, a + x, a + x$ $(x < a)$; $0, 3a, 5a, 4a$ $(a > 0)$ .... seven rows

The above list contains many sets of four nonnegative integers $0, a, b, c$ where $c \geq a + b$. Hence, Theorem 1 can be applied to any of these sets to get three rows more than the particular set of four numbers has. For example,

(i) $0, a, 0, a \to 0, a, 2a, 3a \to 0, a, 4a, 9a \to \ldots$ can be continued $n - 1$ times to get $3n$ steps.

(ii) $0, 0, a, a$ $(a > 0) \to 0, 0, a, 3a \to 0, 2a, 5a, 11a \to \ldots$ can be continued $n - 1$ times to get $3n + 1$ steps.

(iii) $0, 0, 0, a$ $(a > 0) \to 0, a, 2a, 4a \to 0, 2a, 6a, 13a \to \ldots$ can be continued $n - 1$ times to get $3n + 2$ steps.

Hence, we have a selection procedure that will yield sets of four starting numbers that will terminate with all zeros on the nth row, $n = 6, 7, 8, \ldots$.

Below we note some interesting facts:

1. $0, a, a, a$ $(a > 0)$; $0, a, a, a + x$ $(x \neq a)$; $0, a, a + x, a$ $(x \neq a)$; and $0, a, a + x, a + x$ $(x \geq a > 0)$ have five rows.

2. $0, b, 2b, 4b - a$ $(b > 0)$ and $0, b + x, 2b + x, 4b + 2x$ have eight steps.

3. $0, x, a + x, a + b + 2x$ gives three steps more than $0, 0, a, a + b$ $(0 < b \leq a, x > a)$.
   $0, 1, 1 + a, 1 + a + a^2$ $(a > 2)$ gives three steps more than $1, 0, a, a + 1$.

We know that $0, s, 0, s$ $(s \neq 0)$ terminates on the 3rd row. We can write $s = b - a$ in many ways. Then $0, b - a, 2b + 2x, 3b \pm a + 4x$ gives three steps more than $0, b - a, 0, b - a$. Similarly, $0, m, 2m - \ell, 5m - 3\ell + x$ gives three steps more than $s, 0, s, 2s$, where $s = m - \ell$. Again, $0, a, 2a, 5a$ and $0, 3a,$

$4a$, $5a$ gives three steps more than a, 0, a, $2a$. Hence, we can have many sets of four numbers of the form 0, a, b, c having the same number of steps.

However, we can tell the number of steps of the reduced set 0, a, b, c in the following cases:

0, 0, 0, a ($a > 0$) five rows; 0, 0, a, a ($a > 0$) four rows;
0, 0, a, b ($a < b \le 2a$) five rows; 0, 0, a, $2a + x$ ($x > 0$) seven rows;
0, 0, a, $na + x$ ($n \ge 3$) seven rows; 0, a, 0, a ($a > 0$) three rows;
0, a, 0, b ($a \ne b$) five rows; 0, a, b, c ($b = a + c$, $a = c > 0$) three rows;
0, a, b, c ($b = a + c$, $a \ne c$) four rows;
0, a, b, c ($c = a + b$, $a = b > 0$) four rows;
0, a, b, c ($c = a + b$, $a < b$) six rows; and
0, a, b, c ($c = a + b$, $a > b$) four rows.

From the above, it is clear that the only case which presents difficulty in deciding the number of steps without actual calculation is

$$0,\ a,\ b,\ c\ (abc \ne 0,\ b \ne a + c,\ c \ne a + b),$$

where we can assume $a < c$.

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ASYMPTOTIC BEHAVIOR OF LINEAR RECURRENCES

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In general, it is difficult to predict at a glance the ultimate behavior of a linear recurrence sequence. For example, in some problems where the sequence represents the value of a physical quantity at various times, we might want to know if the sequence is always positive, or at least positive from some point on.

Consider the two sequences:

$$w_0 = 3,\ w_1 = 3.01,\ w_2 = 3.0201$$

and

$$w_{n+3} = 3.01w_{n+2} - 3.02w_{n+1} + 1.01w_n \quad \text{for } n \ge 0;$$

$$v_0 = 3,\ v_1 = 3.01,\ v_2 = 3.0201$$

and

$$v_{n+3} = 3v_{n+2} - 3.01v_{n+1} + 1.01v_n \quad \text{for } n \ge 0.$$

The sequence $\{w_n\}$ is always positive, but the sequence $\{v_n\}$ is infinitely often positive and infinitely often negative. This last fact is not obvious from looking at the first few terms of $\{v_n\}$ since the first negative term is v_{735}.

Clearly, the behavior of a recurrence sequence depends on the roots of its characteristic polynomial. We will prove some results which make this dependence precise.

Let

$$(1) \qquad u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}, \; a_i \in R, \; 1 \le i \le k,$$

denote a kth-order linear recurrence with corresponding characteristic polynomial

$$p(x) = x^k - a_1x^{k-1} - \cdots - a_k.$$

For simplicity we shall assume $p(x)$ has distinct roots (although possibly complex). All the results stated here carry through in the case that $p(x)$ has multiple roots and we invite the interested reader to verify such cases in order to obtain a more complete understanding.

The terms of the sequence $\{u_n\}_{n=0}^{\infty}$ defined by (1) can be expressed in terms of the roots of $p(x)$ by use of the Binet formula as follows:

$$(2) \qquad u_n = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} (c_{\alpha_i}\alpha_i^n + \overline{c}_{\alpha_i}\overline{\alpha}_i^n) = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} 2\mathrm{Re}(c_{\alpha_i}\alpha_i^n)$$

where r_i, $1 \le i \le s$, denote the real roots of $p(x)$ and α_i, $s+1 \le i \le t$ denote the roots with nonzero imaginary parts. It is assumed c_{r_i} and c_{α_i} are nonzero.

We are now ready to determine under what conditions the tail of the sequence $\{u_n\}_{n=0}^{\infty}$ will contain only positive terms. We begin with a definition.

Definition: A sequence $\{u_n\}_{n=0}^{\infty}$ is said to be asymptotically positive (denoted a.p.) if there exists $N \in Z$ such that for all $n \ge N$ we have $u_n > 0$.

We first prove a lemma that will shed light on the effects of a complex root of $p(x)$ on the behavior of the sequence $\{u_n\}_{n=0}^{\infty}$.

Lemma 1: If $\theta \not\equiv 0 \bmod \pi$, then the sequence $\{\cos(\lambda + n\theta)\}_{n=0}^{\infty}$, $\lambda, \theta \in \mathbb{R}$, has infinitely many positive and infinitely many negative terms.

Proof: *Case 1.*—θ is a rational multiple of 2π. Then there exist integers s and t, $(s,t) = 1$, such that $\frac{s}{t}2\pi = \theta$. Since $\theta \not\equiv 0 \bmod \pi$, we have $t \ge 3$. Observing that

$$\cos(\lambda + n\theta) = \mathrm{Re}\{e^{i(\lambda + n\theta)}\},$$

we turn our attention to the points $\{e^{i(\lambda+n\theta)}\}_{n=1}^{t}$ in $\mathbb{C}$. The image points in $\mathbb{C}$ differ in argument by at most $\frac{2}{3}\pi$ radians for any two neighboring points. Thus there is always at least one point in each of the half planes $\mathrm{Re}\{z\} > 0$ and $\mathrm{Re}\{z\} < 0$. Since $\cos(\lambda + n\theta)$ is periodic with period t, and in every t consecutive terms there must be at least one positive and one negative term, the lemma holds.

Case 2.—θ is an irrational multiple of 2π. The sequence $\{\lambda + n\theta\}_{n=0}^{\infty}$ is dense mod 2π. (Indeed, it is uniformly distributed mod 2π [2].) As the cosine is continuous, the image of $\{\lambda + n\theta\}_{n=0}^{\infty}$ under the cosine is dense in $[-1, 1]$. This completes the proof.

We are now ready to state and prove the main result.

Theorem 1: Let u_n be a kth-order linear recurrence as in (1) whose characteristic polynomial $p(x)$ has distinct roots. Let Γ be a root of $p(x)$ such that $|\Gamma| > |\gamma|$ where γ is any other root of $p(x)$ with the exception of $\gamma = \overline{\Gamma}$ when Γ is not real.

If $\Gamma > 0$ and $c > 0$, then $\{u_n\}_{n=0}^{\infty}$ is a.p., and $\{u_n\}_{n=0}^{\infty}$ has infinitely many negative terms otherwise.

Proof: From (2), we have

$$u_n = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} 2\mathrm{Re}(c_{\alpha_i}\alpha_i^n)$$

or, assuming $\Gamma \in \mathbb{R}$, and letting $c = c_\Gamma$,

$$u_n = c\Gamma^n(1 + o(1)). \tag{3}$$

It is clear from (3) that $\Gamma > 0$, $c > 0$, will insure that $\{u_n\}_{n=0}^{\infty}$ is a.p. and that $c < 0$ or $\Gamma < 0$ will produce infinitely many negative terms.

If Γ is not real, we obtain from (3) by use of Euler's formula

$$u_n = |c||\Gamma|^n(\cos(\arg c + n \arg \Gamma))(1 + o(1)). \tag{4}$$

From Lemma 1, we conclude that $\{u_n\}$ has infinitely many negative terms.

The examples at the beginning of the article serve as a simple illustration. The sequence $\{w_n\}$ has as its Binet formula $w_n = 1^n + 1^n + (1.01)^n$, which is clearly positive for all $n \geq 0$. However, the roots associated with $\{u_n\}$ are $1, 1 \pm \sqrt{-1}/10$. Thus the root called Γ in Theorem 1 is $1 + \sqrt{-1}/10$ which is not real. Therefore $\{u_n\}$ has infinitely many positive and infinitely many negative terms.

We now discuss the case of $p(x)$ having s distinct roots of greatest magnitude $|\Gamma|$. Again appealing to the Binet formula, we have

$$\begin{aligned} u_n = |\Gamma|^n\{c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots \\ + c_s \cos(\lambda_s + n\theta_s) + o(1)\}, \; c_i \in \mathbb{R}. \end{aligned} \tag{5}$$

By letting $f(n) = n$ or $n + 1$ we may assume $c_2 \geq 0$. Also, as the cosine is an even periodic function, $\cos(\lambda + n\theta) = -\cos((\lambda + \pi) + n\theta)$. Thus when necessary, we may replace $\cos(\lambda_i + n\theta_i)$ by $\cos((\lambda_i + \pi) + n\theta_i)$ and thereby allow us to assume $c_i \geq 0$, $3 \leq i \leq s$.

Theorem 2: Let u_n be as in (5). If $c_1 - c_2 - \cdots - c_s > 0$, then $\{u_n\}_{n=0}^{\infty}$ is a.p.

Proof: Let $\eta > 0$ be such that

$$c_1 - \sum_{i=2}^{s} c_i > \eta > 0.$$

We have

$$u_n = |\Gamma|^n(c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots + c_3(\cos(\lambda_s + n\theta_s) + g(n))$$

where $g(n)$ is $o(1)$. Choose N so large that for $n > N$, $|g(n)| < \eta/2$. Then for $n > N$,

$$\begin{aligned} u_n = |\Gamma|^n(c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots + c_3 \cos(\lambda_3 + n\theta_3) \\ + g(n)) \geq |\Gamma|^n(\eta - \eta/2) = |\Gamma|^n(\eta/2) > 0. \end{aligned}$$

Thus $\{u_n\}_{n=0}^{\infty}$ is a.p.

It may be noted that Theorem 2 is the best possible in the following sense: Let

$$u_n = -\frac{1}{2}u_{n-1} + u_{n-2} - \frac{1}{2}u_{n-3}.$$

We have

$$u_n = c_1(1)^n + c_2(-1)^n + c_3\left(-\frac{1}{2}\right)^n \text{ where } c_3\left(-\frac{1}{2}\right)^n \text{ is } o(1).$$

If we choose $c_1 = c_2 = 1$, then $c_1 - c_2 = 0$. As every other term of u_n is negative, $\{u_n\}_{n=0}^{\infty}$ is not a.p. Thus the condition $c_1 - c_2 - \cdots - c_3 > 0$ may not in general be relaxed.

However, upon examining specific cases, it is often possible to improve Theorem 2. For example, if θ_i is a rational multiple of 2π, then

$$\{\cos(\lambda_i + n\theta_i)\}_{n=0}^{\infty}$$

is periodic. Letting

$$\alpha_i = \left|\min_n\{\cos(\lambda_i + n\theta_i)\}_{i=0}^{n}\right|$$

we may use, in Theorem 2, the condition

$$c_1 - c_2 - \cdots - \alpha_i c_i - \cdots - c_s > 0.$$

Thus it is evident how improvements of Theorem 2 can be made when more is known about the roots of the characteristic polynomial.

We now consider the special case of second-order linear recurrences which are completely characterized by the following theorem.

Theorem 3: Let $u_n = au_{n-1} + bu_{n-2}$, $a,b \in \mathbb{R}$, be a second-order linear recurrence. Let

$$\alpha_1 = \frac{a+\delta}{2},\ \alpha_2 = \frac{a-\delta}{2}$$

be the roots of $p(x) = x^2 - ax - b$ where $\delta = \sqrt{a^2 + 4b}$. $\{u_n\}_{n=0}^{\infty}$ is a.p. if and only if $\delta \in \mathbb{R}$ and either

(i) $a = 0$, $u_0 > 0$, $u_1 > 0$

or

(ii) $a > 0$, $2u_1 > (a - \delta)u_0$

where u_0, u_1 are the initial values.

Proof: *Case 1.*—Suppose that δ is not real. Since $\alpha_2 = \overline{\alpha}_1$, Theorem 1 applies with $\Gamma = \alpha_1 \not> 0$. Thus $\{u_n\}_{n=0}^{\infty}$ is not a.p.

Case 2.—$a < 0$. The root of largest absolute value is α_2, and $\alpha_2 < 0$. By Theorem 1, $\{u_n\}_{n=0}^{\infty}$ has infinitely many negative terms.

Case 3.—$a = 0$. The recurrence becomes $u_n = bu_{n-2}$ and the roots of $p(x)$ are $\pm\delta/2$. From the Binet formula, we have

$$u_n = c_1\left(\frac{\delta}{2}\right)^n + c_2\left(-\frac{\delta}{2}\right)^n = \left(\frac{\delta}{2}\right)^n(c_1 + (-1)^n c_2).$$

If suffices to show $c_1 + c_2 > 0$ and $c_1 - c_2 > 0$. $u_0 = c_1 + c_2$ so we must have $u_0 > 0$. $u_1 = \frac{\delta}{2}(c_1 - c_2)$ so that $\frac{2}{\delta}u_1 = c_1 - c_2 > 0$. As $\frac{2}{\delta} > 0$ we have $u_1 > 0$.

Case 4.—$a > 0$. The largest root in absolute value is α_1, and $\alpha_1 > 0$. From Theorem 1 it suffices to show $c_1 > 0$. Using Cramer's Rule, we have

$$c_1 = \frac{u_1 - u_0\alpha_2}{\alpha_1 - \alpha_2}.$$

Since $\alpha_1 - \alpha_2 > 0$, we require that $u_1 - u_0\alpha_2 > 0$ or $2u_1 > u_0(a - \delta)$.

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ANOMALIES IN HIGHER-ORDER CONJUGATE QUATERNIONS: A CLARIFICATION

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1. INTRODUCTION

In a previous paper [3], brief mention was made of the conjugate quaternion $\overline{P}_n$ of the quaternion P_n. Following the definitions given by Horadam [2], Iyer [6], and Swamy [7], we have

(1) $$P_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3},$$

and consequently, its conjugate $\overline{P}_n$ is given by

(2) $$\overline{P}_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$
$$jk = -kj = i, \quad ki = -ik = j.$$

In [3], T_n was defined to be a quaternion with quaternion components P_{n+r} ($r = 0, 1, 2, 3$), that is,

(3) $$T_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$

and the conjugate of T_n was defined as

(4) $$\overline{T}_n = P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}$$

which, with (1), yields

(5) $$\overline{T}_n = W_n + W_{n+2} + W_{n+4} + W_{n+6}.$$

Here the matter of conjugate quaternions was laid to rest without investigating further the inconsistency that had arisen, namely, the fact that the conjugate for the quaternion T_n [defined in (4) analogously to the standard conjugate quaternion form (2)] was a scalar (5) and not a quaternion as normally defined. This inconsistency, however, made attempts to derive expressions for conjugate quaternions of higher order similar to those of higher-order quaternions established in [4] and [5], rather difficult. The change in notation from that used in [3] to the operator notation adopted in [4] and [5], added further complications. Given that $\Omega W_n \equiv P_n$ and $\Omega^2 W_n \equiv T_n$, the introduction of this operator notation created a whole new set of possible conjugates for each of the higher-order quaternions. For example, for quaternions with quaternion components (quaternions of order 2), we could apparently define the conjugate of $\Omega^2 W_n$ in several ways, viz. (6)-(9):

(6) $$\Omega\overline{\Omega}W_n = \overline{\Omega}W_n + i\overline{\Omega}W_{n+1} + j\overline{\Omega}W_{n+2} + k\overline{\Omega}W_{n+3};$$

(7) $$\overline{\Omega}\Omega W_n = \Omega W_n - i\Omega W_{n+1} - j\Omega W_{n+2} - k\Omega W_{n+3};$$

(8) $$\overline{\Omega}^2 W_n = \overline{\Omega}W_n - i\overline{\Omega}W_{n+1} - j\overline{\Omega}W_{n+2} - k\overline{\Omega}W_{n+3};$$

(9) $$\overline{\Omega^2}W_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} - 2iW_{n+1} - 2jW_{n+2} - 2kW_{n+3}.$$

It is clear that the difficulties which have arisen are due, in part, to the choice of the defining notation. It is the purpose of this paper to redefine higher-order conjugate quaternions using the more descriptive nomenclature provided by the operator notation as outlined in [4]. We are thus concerned with determining the unique conjugate of a general higher-order quaternion.

2. SECOND-ORDER CONJUGATE QUATERNIONS

We begin by defining the conjugate of ΩW_n as $\overline{\Omega} W_n$ ($\equiv \overline{P}_n$, c.f. (6) in [3]), where

$$\overline{\Omega} W_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}. \tag{10}$$

Consider (6) and (7) above. If we expand these expressions using (10) and (1) with $\Omega W_n = P_n$, respectively, we find that

$$\Omega\overline{\Omega} W_n = \overline{\Omega}\Omega W_n = W_n + W_{n+2} + W_{n+4} + W_{n+6}, \tag{11}$$

which is the same as (5). Since the right-hand side of (5) and (11) are independent of the quaternion vectors i, j, and k, $\Omega\overline{\Omega} W_n$, $\overline{\Omega}\Omega W_n$, and $\overline{T}_n$ are not quaternions and, therefore, cannot be defined as the conjugate of $\Omega^2 W_n$ ($= T_n$). We emphasize that $\overline{T}_n$, as defined by (4), 9(a) of [3], is not the conjugate of T_n.

Since the expanded expression for $\Omega^2 W_n$ ($\equiv T_n$, c.f. 8(a) in [3]) is

$$\Omega^2 W_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} + 2iW_{n+1} + 2jW_{n+2} + 2kW_{n+3}, \tag{12}$$

it follows that the conjugate of $\Omega^2 W_n$ must be $\overline{\Omega^2} W_n$ as given by (9). If we now take (8) and expand the right-hand side, we see that it is identical to the right-hand side of (9), so that the conjugate of $\Omega^2 W_n$ can also be denoted $\overline{\Omega}^2 W_n$.

By taking the product of $\Omega^2 W_n$ and $\overline{\Omega}^2 W_n$, we obtain

$$\begin{aligned} \Omega^2 W_n \overline{\Omega}^2 W_n = {} & W_n^2 + W_{n+2}^2 + W_{n+4}^2 + W_{n+6}^2 \\ & + 4W_{n+1}^2 + 4W_{n+2}^2 + 4W_{n+3}^2 \\ & - 2W_n W_{n+2} - 2W_n W_{n+4} - 2W_n W_{n+6} \\ & + 2W_{n+2}W_{n+4} + 2W_{n+2}W_{n+6} + 2W_{n+4}W_{n+6}, \end{aligned} \tag{13}$$

and we observe that the right-hand side of this equation is a scalar. Thus $\overline{\Omega}^2 W_n$ preserves the basic property of a conjugate quaternion.

We note in passing that as $\overline{P}_n \equiv \overline{\Omega} W_n$, the conjugate quaternion $\overline{T}_n$ should have been defined as [c.f. (8)],

$$\overline{T}_n = \overline{P}_n - i\overline{P}_{n+1} - j\overline{P}_{n+2} - k\overline{P}_{n+3}. \tag{14}$$

3. THE GENERAL CASE

In Section 2 above, the conjugate $\overline{\Omega^2} W_n$ of $\Omega^2 W_n$ was determined by expanding the quaternion $\Omega^2 W_n$ and conjugating in the usual way. It was established that $\overline{\Omega}^2 W_n \equiv \overline{\Omega^2} W_n$. We now seek to prove that this relationship is generally true, i.e., for any integer λ, $\overline{\Omega}^\lambda W_n \equiv \overline{\Omega^\lambda} W_n$.

First, we need to derive a Binet form for the generalized conjugate quaternion of arbitrary order.

As in [5], we introduce the extended Binet form for the generalized quaternion of order λ:

$$\Omega^\lambda W_n = A\alpha^n \underline{\alpha}^\lambda - B\beta^n \underline{\beta}^\lambda \quad (A, B \text{ constants}) \tag{15}$$

where α and β are defined as in Horadam [1] and

$$\begin{cases} \underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \\ \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3. \end{cases} \tag{16}$$

We now define the conjugates $\overline{\underline{\alpha}}$ and $\overline{\underline{\beta}}$ so that

$$\begin{cases} \overline{\underline{\alpha}} = 1 - i\alpha - j\alpha^2 - k\alpha^3 \\ \overline{\underline{\beta}} = 1 - i\beta - j\beta^2 - k\beta^3. \end{cases} \tag{17}$$

Substituting the Binet forms, as given by (1.6) in Horadam [1], for the terms on the right-hand side of (10), we obtain

$$\begin{aligned}\overline{\Omega}W_n &= A\alpha^n - B\beta^n - i(A\alpha^{n+1} - B\beta^{n+1}) - j(A\alpha^{n+2} - B\beta^{n+2}) - k(A\alpha^{n+3} - B\beta^{n+3})\\ &= A\alpha^n(1 - i\alpha - j\alpha^2 - k\alpha^3) - B\beta^n(1 - i\beta - j\beta^2 - k\beta^3),\end{aligned}$$

i.e.,

$$\overline{\Omega}W_n = A\alpha^n\overline{\underline{\alpha}} - B\beta^n\overline{\underline{\beta}}, \tag{18}$$

which is the Binet form for the conjugate quaternion $\overline{\Omega}W_n$. This result can easily be generalized by induction, so that, for λ an integer,

$$\overline{\Omega}^\lambda W_n = A\alpha^n\overline{\underline{\alpha}}^\lambda - B\beta^n\overline{\underline{\beta}}^\lambda. \tag{19}$$

Lemma: For some integer λ,

$$\overline{\underline{\alpha}}^\lambda = \overline{\underline{\alpha}^\lambda} \quad \text{and} \quad \overline{\underline{\beta}}^\lambda = \overline{\underline{\beta}^\lambda}.$$

Proof: We will prove only the result for the quaternion $\underline{\alpha}$, as the proof of the result for $\underline{\beta}$ is identical. From (16) above, it follows that

$$\left\{\begin{aligned}\underline{\alpha}^2 &= (1 + i\alpha + j\alpha^2 + k\alpha^3)^2\\ &= 1 - \alpha^2 - \alpha^4 - \alpha^6 + 2i\alpha + 2j\alpha^2 + 2k\alpha^3\\ &= 2\underline{\alpha} - (1 + \alpha^2 + \alpha^4 + \alpha^6).\end{aligned}\right. \tag{20}$$

Letting

$$S_\alpha = 1 + \alpha^2 + \alpha^4 + \alpha^6, \tag{21}$$

we have

$$\underline{\alpha}^2 = 2\underline{\alpha} - S_\alpha. \tag{22}$$

Hence, on multiplying both sides of this equation by $\underline{\alpha}$, we obtain

$$\underline{\alpha}^3 = 2\underline{\alpha}^2 - \underline{\alpha}S_\alpha,$$

which, by (22) becomes

$$\underline{\alpha}^3 = (4 - S_\alpha)\underline{\alpha} - 2S_\alpha.$$

If we continue this process, a pattern is discernible from which we derive a general expression for $\underline{\alpha}^\lambda$ given by

$$\begin{aligned}\underline{\alpha}^\lambda = &\left\{\sum_{r=0}^{\left[\frac{\lambda-1}{2}\right]}\binom{\lambda - 1 - r}{r}2^{\lambda-1-2r}(S_\alpha)^r(-1)^r\right\}\underline{\alpha}\\ &- \left\{\sum_{r=0}^{\left[\frac{\lambda-2}{2}\right]}\binom{\lambda - 2 - r}{r}2^{\lambda-2-2r}(S_\alpha)^r(-1)^r\right\}S_\alpha,\end{aligned} \tag{23}$$

where $\left[\frac{\lambda - 1}{2}\right]$ refers to the integer part of $\frac{\lambda - 1}{2}$.

From equations (20) and (22), it is evident that

$$\overline{\underline{\alpha}^2} = 2\overline{\underline{\alpha}} - S_\alpha. \tag{24}$$

Since S_α is a scalar, and the only quaternion in the right-hand side of (23) is $\underline{\alpha}$, it follows that the conjugate $\overline{\underline{\alpha}^\lambda}$ must be

(25)
$$\overline{\underline{\alpha}^{\lambda}} = \left\{\sum_{r=0}^{\left[\frac{\lambda-1}{2}\right]} \binom{\lambda - 1 - r}{r} 2^{\lambda-1-2r}(S_\alpha)^r(-1)^r\right\}\overline{\underline{\alpha}} - \left\{\sum_{r=0}^{\left[\frac{\lambda-2}{2}\right]} \binom{\lambda - 2 - r}{r} 2^{\lambda-2-2r}(S_\alpha)^r(-1)^r\right\}S_\alpha.$$

We now employ the same procedure as we used above to obtain a general expression for $\underline{\alpha}^\lambda$ [c.f. (23)] to secure a similar result for $\overline{\underline{\alpha}}^\lambda$.

From (17) it ensues that

$$\begin{aligned}\overline{\underline{\alpha}}^2 &= (1 - i\alpha - j\alpha^2 - k\alpha^3)^2\\ &= 1 - \alpha^2 - \alpha^4 - \alpha^6 - 2i\alpha - 2j\alpha^2 - 2k\alpha^3,\end{aligned}$$

i.e.,

(26)
$$\overline{\underline{\alpha}}^2 = 2\overline{\underline{\alpha}} - S_\alpha,$$

and we note that this equation is identical to (24). Multiplying both sides of (26) by $\overline{\underline{\alpha}}$ gives us

$$\overline{\underline{\alpha}}^3 = 2\overline{\underline{\alpha}}^2 - \overline{\underline{\alpha}}S_\alpha,$$

which, by (26), yields

$$\overline{\underline{\alpha}}^3 = (4 - S_\alpha)\overline{\underline{\alpha}} - 2S_\alpha.$$

It is obvious from the emerging pattern that, by repeated multiplication of both sides by $\overline{\underline{\alpha}}$ and subsequent substitution for $\overline{\underline{\alpha}}^2$ by the right-hand side of (26), the expression derived for $\overline{\underline{\alpha}}^\lambda$ will be precisely (25). Hence, $\overline{\underline{\alpha}}^\lambda = \overline{\underline{\alpha}^\lambda}$. Similarly, it can be shown that $\overline{\underline{\beta}}^\lambda = \overline{\underline{\beta}^\lambda}$.

Theorem: For λ an integer,
$$\overline{\Omega}^\lambda W = \overline{\Omega^\lambda W}\,.$$

Proof: Taking the conjugate of both sides of (15) gives us

$$\begin{aligned}\overline{\Omega^\lambda W_n} &= \overline{A\alpha^n\underline{\alpha}^\lambda - B\beta^n\underline{\beta}^\lambda}\\ &= A\alpha^n\overline{\underline{\alpha}^\lambda} - B\beta^n\overline{\underline{\beta}^\lambda}\\ &= A\alpha^n\overline{\underline{\alpha}}^\lambda - B\beta^n\overline{\underline{\beta}}^\lambda \qquad \text{(Lemma)}\\ &= \overline{\Omega}^\lambda W_n \qquad [\text{c.f. } (19)]\end{aligned}$$

as desired.

We have thus established that the conjugate for a generalized quaternion of order λ can be determined by taking λ operations on the conjugate quaternion operator $\overline{\Omega}$. This provides us with a rather simple method of finding the conjugate of a higher-order quaternion.

Finally, let us again consider the conjugate quaternion $\overline{\Omega}W_n$. It readily follows from (10) that

$$\overline{\Omega}W_n = 2W_n - W_n - iW_{n+1} - jW_{n+2} - kW_{n+3},$$

i.e.,

$$\overline{\Omega}W_n = 2W_n - \Omega W_n.$$

This equation relates the conjugate quaternion $\overline{\Omega}W_n$ to the quaternion ΩW_n. If we rewrite (27) as

$$\overline{\Omega}W_n = (2 - \Omega)W_n.$$

it is possible to manipulate the operators in the ensuing fashion:

$$\overline{\Omega}^2 W_n = (2 - \Omega)^2 W_n = (4 - 4\Omega + \Omega^2) W_n = 4W_n - 4\Omega W_n + \Omega^2 W_n.$$

This result can be verified directly through substitution by (1), (9), and (12), recalling that $P_n = \Omega W_n$ and $\overline{\Omega}^2 W_n = \overline{\Omega^2} W_n$. Once again, by induction on λ, it is easily shown that

$$\overline{\Omega}^\lambda W_n = (2 - \Omega)^\lambda W_n. \tag{28}$$

It remains open to conjecture whether an examination of various permutations of the operators Ω and $\overline{\Omega}$, together with the operator Δ (defined in [4]) and its conjugate $\overline{\Delta}$, will lead to further interesting relationships for higher-order quaternions.

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ON THE CONVERGENCE OF ITERATED EXPONENTIATION—II*

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In a previous paper [1], we have discussed the properties of the function $f(x)$ defined as:

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \tag{1}$$

and a generalization of $f(x)$, namely [2, 3],

$$F_n(x) = g_1(x)^{g_2(x)^{g_3(x)^{\cdot^{\cdot^{\cdot^{g_n(x)}}}}}} = \mathop{\Xi}_{j=1}^{n} g_j(x), \tag{2}$$

where the $g_j(x)$ are functions of a positive real variable x, and the symbol Ξ is used to denote the iterated exponentiation [4]. For both (1) and (2), the

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ordering of the exponentiations is important; here and throughout this paper, we mean a bracketing order "from the top down" e.g., for (2), g_{n-1} raised to the power g_n, followed by g_{n-2} raised to the resulting power, all the way down to g_1. It was shown in [1] that f converges as the number of x's in (1) increases for x from $e^{-e} \cong 0.065988\ldots$ to $e^{1/e} \cong 1.444668\ldots$. For $x > e^{1/e}$, f is divergent, and for $x < e^{-e}$, the function f is "dual convergent," i.e., it converges to two different values according as the number of x's [or n in (2)] is even or odd. If the number of x's is even, one obtains a curve of $f(x)$ which increases from $1/e$ at $x = e^{-e}$ to $f = 1$ at $x = 0$, and if the number of x's is odd, one obtains a second curve of $f(x)$ which decreases from the unique value $f(e^{-e}) = 1/e = 0.36788$ to $f(0) = 0$ at $x = 0$. Typical values of the limiting $f(x)$ in the region $0 < x < e^{-e}$ are: $f(0.02) \cong 0.03146$ (odd number n of x's) and $f(0.02) \cong 0.88419$ (even n); also $f(0.04) \cong 0.08960$ (odd n), 0.74945 (even n); $f(0.06) \cong 0.21690$ (odd n), 0.54323 (even n). The property of dual convergence has been shown in [1] and [3] to be a general property of the function $F_n(x)$ of (2), when $g_j(x)$ is a decreasing function of j for fixed x, e.g., the function $g_j(x) = x/j^2$, for which $F_n(x)$ is shown in Fig. 3 of [1].

In the present paper we consider a particularly simple generalization of the function $f(x)$, namely the function $F(x, y)$ defined as:

$$F(x, y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}}} \tag{3}$$

where an infinite number of exponentiations is understood, and x is at the bottom of the "ladder." Thus, $F(x, y)$ corresponds to the limit of $F_n(x)$ as $n \to \infty$ in (2), where $g_j(x) \equiv x$ for j = odd, and $g_j(x) \equiv y$ for j = even. Both x and y are assumed to be positive (real) quantities. Depending upon the values of x and y, $F(x, y)$ can be monoconvergent, dual convergent, or divergent. For the special case $x = y$, $F(x, x) = f(x)$ of (1), which is monoconvergent in the range $e^{-e} < x < e^{1/e}$, as discussed above. Also, we have $F(x, 1) = x$, $F(1, y) = 1$; $F(x, 0) = 1$, $F(0, y) = 0$, for finite x and y. We now consider the case where $x > 1$. We also expand the definition of $F(x, y)$ to include the function

$$F(y, x) \equiv F'(x, y) = y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}} \tag{4}$$

where y is at the bottom of the "ladder."

By enlarging the definition of $F(x, y)$ to include the function $F(y, x)$, we obtain the following three convergence possibilities:

1. *Dual convergence*, when $F(x, y)$ converges to a well-defined value regardless of whether the number of x's in the "ladder" is even or odd. In this case $F(y, x)$ also converges to a well-defined value. Because of the total of two values involved [$F(y, x) \neq F(x, y)$], we have called this possibility "dual convergence."

2. *Quadriconvergence*, when $F(x, y)$ converges to *two* well-defined values depending upon whether the number of x's in the "ladder" is even or odd. In this case $F(y, x)$ also converges to two well-defined values, again depending upon whether the number of x's and y's in the "ladder" is even or odd. Because of the total of four values of the functions $F(x, y)$ and $F(y, x)$, we have called this possibility "quadriconvergence." However, it should be realized that the quadriconvergence corresponds to the dual convergence of both $F(x, y)$ and $F(y, x)$ in the sense defined in [1] and [3].

3. *Divergence*, in which case both $F(x, y)$ and $F(y, x)$ diverge as the number of x's and y's in (3) and (4) is increased indefinitely. In Figs. 1 and 3 and in Table 1, we have abbreviated dual convergence as D.C., quadriconvergence as Q.C., and divergence as Div.

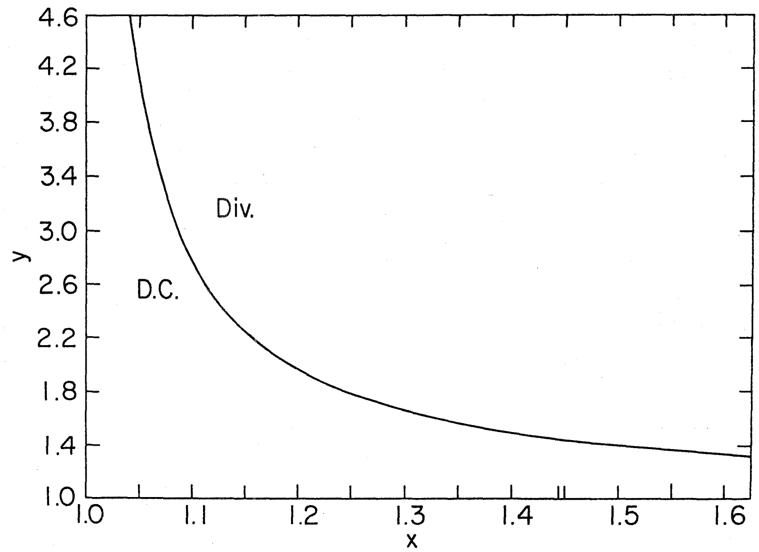

Fig. 1. The curve of the limiting y value $y_{\lim}$ as a function of x for $x > 1$, such that for $y > y_{\lim}$, the function $F(x, y)$ is divergent and for $y \le y_{\lim}$, $F(x, y)$ is dual convergent, i.e., it converges to two values F_1 and F_2 depending upon whether x or y is at the bottom of the "ladder" in (3) and (4). The point $x = e^{1/e} = 1.444668$, for which $y_{\lim} = x$ has been marked on the abscissa axis.

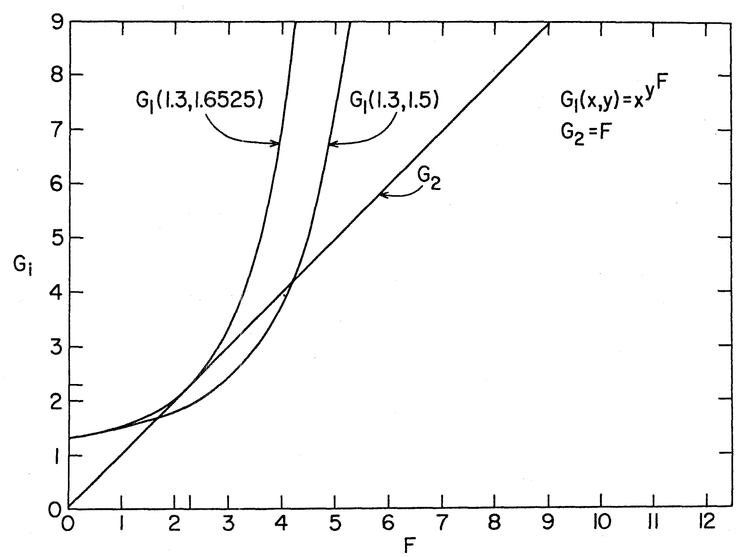

Fig. 2. The functions $G_1 = x^{y^F}$ and $G_2 = F$ plotted vs F. The two curves of G_1 pertain to $x = 1.3$, $y = 1.5$, and $x = 1.3$, $y = 1.6525$, respectively. The curve of $G_1(1.3, 1.5)$ intersects the 45° line $G_2 = F$ at the two points $F^{(1)} = 1.679$ and $F^{(2)} = 4.184$, whose significance is explained in the text. The curve of $G_1(1.3, 1.6525)$ is tangent to the $G_2 = F$ line at $F = 2.304$. Note that 1.6525 is the value of $y_{\lim}$ pertaining to $x = 1.3$.

Table 1. A listing of the values of $F(x, y)$ for several illustrative choices of x and y. The third column indicates whether the function $F(x, y)$ is dual convergent or quadriconvergent. For dual convergence, the two values of F_1 and F_2 are listed, which correspond to F of (3) and F' of (4), with x at the bottom of the "ladder" and y at the bottom of the "ladder," respectively. Thus we have $F_1 = x^{F_2}$ and $F_2 = y^{F_1}$. For the cases of quadriconvergence, four values F_1, F_2, F_3, and F_4 are listed, where the relations between the F_i are given by (23). The last column of the table lists the value of y_{lim} for the x value considered. For $0 < x < 1$, y_{lim} defines the boundary between the regions of dual convergence and quadriconvergence (see Fig. 3). For $x > 1$, y_{lim} defines the boundary between the dual convergence region and the region where $F(x, y)$ is divergent (see Fig. 1).

x	y	Conv.	F_1	F_2	F_3	F_4	y_{lim}
0.2	60	D.C.	0.09398	1.4693			107.0
0.2	150	Q.C.	0.14901	2.1099	0.03352	1.1829	107.0
0.2	10,000	Q.C.	0.19988	6.3028	3.93×10^{-5}	1.00036	107.0
0.4	20	D.C.	0.19414	1.7889			24.02
0.4	30	Q.C.	0.31046	2.8747	0.07179	1.2766	24.02
0.4	1,000	Q.C.	0.40000	15.849	4.93×10^{-7}	1.0000	24.02
0.7	10	D.C.	0.40447	2.5379			15.16
0.7	25	Q.C.	0.65509	8.2371	0.05297	1.1859	15.16
0.7	1,000	Q.C.	0.70000	125.89	3.16×10^{-20}	1.0000	15.16
0.9	15	D.C.	0.59224	4.9719			21.55
0.9	30	Q.C.	0.82743	16.681	0.17248	1.7979	21.55
0.9	1,000	Q.C.	0.90000	501.19	1.167×10^{-23}	1.0000	21.55
1.05	3.80	D.C.	1.3379	5.9658			4.1232
1.10	2.40	D.C.	1.3732	3.3274			2.7497
1.20	1.80	D.C.	1.5914	2.5482			1.9514
1.30	1.50	D.C.	1.6792	1.9756			1.6527
1.40	1.46	D.C.	2.1154	2.2267			1.4940

In this connection, it should be pointed out that for $x \neq y$, if there is convergence, the minimum number of values obtained is two, namely F and F', and we have the following obvious relations:

$$F(x, y) = x^{F'(x,y)}, \tag{5}$$

$$F'(x, y) = y^{F(x,y)}. \tag{6}$$

The curve of y_{lim} vs x for $x > 1$ is shown in Fig. 1, where y_{lim} is the limiting value of y for convergence. This curve was obtained from the following equation derivable directly from (3):

$$F(x, y) = x^{y^{F(x,y)}}. \tag{7}$$

To obtain y_{lim} as a function of x, the following procedure was employed using a Hewlett-Packard calculator. Consider the plane (F, G), with F along the abscissa and G along the ordinate. For a given value of x and a trial value of y, the curve $G_1 = x^{y^F}$ was plotted as a function of F. This is an increasing function of F, since $x > 1$ and $y > 1$. Thus, for $F = 0$, $y^F = 1$, $G_1 = x$, and the curve is concave upward as F is increased to positive values. The intersection of this upward curve with the straight line $G_2 = F$ is then searched for. If y is too large and, hence, if x^y is too large, the curve G_1 will not intersect the 45° line $G_2 = F$ (which starts at zero for $F = 0$). Thus, this value of y will be larger than $y_{\lim}$, and the function $F(x, y)$ diverges, and of course also $F'(x, y)$. If y is made appreciably smaller, the curve of G_1 will rise more slowly and will generally intersect the 45° line $G_2 = F$ at two values of F. It can be shown that the lower value of F gives the correct F as obtained by continued exponentiation, and the corresponding value of F' is given by

$$F' = y^F.$$

Finally, for a certain intermediate value of y, the curve x^y vs F will be just tangent to the 45° line $G_2 = F$. This value of y is the limiting value $y_{\lim}$, which we have plotted in Fig. 1 as a function of x. An illustration of the possible relationships in the G vs F plane is shown in Fig. 2, for the case $x = 1.3$, for which $y_{\lim} = 1.6525$. Thus, Fig. 2 shows that the derivative of G_1 at the tangent point must be +1. Thus:

$$\left.\frac{dx^{y^F}}{dF}\right|_F = +1. \tag{8}$$

This condition, together with the equation

$$x^{y^F} = F, \tag{9}$$

can be used to derive equations for x and y, given the assumed value of F. We obtain, from (8),

$$\frac{d}{dF}x^{y^F} = \frac{d}{dF}\exp\{\log x[\exp(F \log y)]\} = F\frac{d}{dF}\{\log x[\exp(F \log y)]\} = +1, \tag{10}$$

whence:

$$\frac{1}{F} = \log x \frac{d}{dF}[\exp(F \log y)] = \log x \log y \exp(F \log y). \tag{11}$$

But from (9), we find

$$F = x^{y^F} = x^{\exp(F \log y)} = \exp[\log x \exp(F \log y)], \tag{12}$$

so that

$$\log F = \log x \exp(F \log y). \tag{13}$$

Upon dividing (11) by (13), we obtain

$$\frac{1}{F \log F} = \log y, \tag{14}$$

which gives

$$y = \exp(1/F \log F). \tag{15}$$

In order to obtain the corresponding equation for x, we note that from (12) and (15),

$$\log F = \log x\, y^F = \log x \exp(1/\log F), \tag{16}$$

which gives:

$$\log x = \log F \exp(-1/\log F), \tag{17}$$

$$x = \exp[\log F \exp(-1/\log F)] = \exp[\log F/\exp(1/\log F)]. \tag{18}$$

For the case where one of the quantities, say x, is less than 1, but where y can be large, and still keeping $y > 1$, we have a somewhat different situation. In this case, the function $G_1 = x^{y^F}$ is a decreasing function of F, starting at $G_1 = x$ for $F = 0$ and going down to x^y ($< x$) at $F = 1$. Thus, the curve of G_1 vs F will always intersect the 45° line $G_2 = F$ at a value of $F < 1$. It can then be shown that the functions F and F' must be quadriconvergent if the negative slope dx^{y^F}/dF at $x^{y^F} = F$ is algebraically smaller than -1. Thus, the limiting curve of y_{lim} vs x which separates the regions of dual and quadriconvergence is obtained from the following pair of equations:

$$\left.\frac{dx^{y^F}}{dF}\right|_F = -1, \tag{19}$$

$$x^{y^F} = F. \tag{20}$$

Thus, if the slope $\left(dx^{y^F}/dF\right) < -1$, we will have quadriconvergence, whereas for $\left(dx^{y^F}/dF\right) > -1$, we will have dual convergence.

Now we note that (19) and (20) are remarkably similar to (8) and (9), the only difference being the change of sign in (19) as compared to (8). We thus obtain the following equations for x and y for the limiting curve (i.e., $y = y_{\text{lim}}$):

$$x = \exp[\log F \ \exp(1/\log F)], \tag{21}$$

$$y = \exp(-1/F \ \log F). \tag{22}$$

By means of these equations, we have obtained the plot of y vs x of Fig. 3.

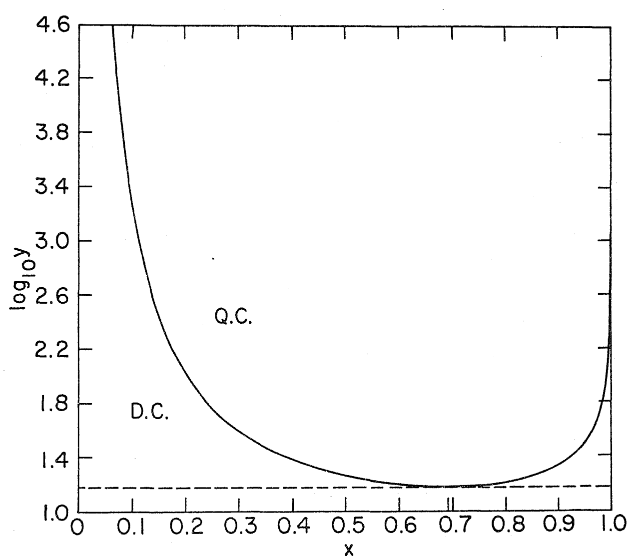

Fig. 3. The curve of $\log_{10} y_{\text{lim}}$ as a function of x for $0 < x < 1$. For $y \le y_{\text{lim}}$, the function $F(x, y)$ is dual convergent, i.e., it converges to two values F_1 and F_2, depending on whether x or y is at the bottom of the "ladder" in (3) and (4). For $y > y_{\text{lim}}$, $F(x, y)$ is quadriconvergent, i.e., it converges to two values each for both x and y at the bottom of the "ladder" in (3) and (4); thus, it converges to four values altogether [see (23) and (24)]. The dashed horizontal line $\log_{10} y = \log_{10}(e^e) = \log_{10} 15.15421$ is tangent to the curve at the point $x = e^{-1/e} = 0.692201$.

By letting $F' = y^F$ vary from $F' = 1$ to large F', we cover the range $x = 0$ to $x = 1$. (Note that $y > 1$ is assumed.) The regions of dual convergence and quadriconvergence are indicated as D.C. and Q.C., respectively. We note that regardless of x in the range 0 to 1 the functions F and F' will each converge to a single value, provided that $y < e^e \cong 15.154$. The line $y = e^e$ is marked as a dashed line and the curve of y vs x is tangent to this line at the point $x = e^{-1/e} \cong 0.6922$. This value of x is just the reciprocal of the value $x' = e^{1/e}$ which is the limit of convergence of the function $f(x) = F(x, x)$ which has been discussed in [1] - [3]. We also note that the minimum value of y_{lim} for $x < 1$, namely $y_{\text{lim}} = e^e$, is just the reciprocal of the value $x = e^{-e} = 0.065988$, below which the function $f(x)$ becomes dual convergent, as has been shown in [1]. The value of $f(x = e^{-e})$ is $1/e$. The curve of y_{lim} vs x is asymptotic to the vertical lines $x = 0$ and $x = 1$ in Fig. 3.

Values of the functions $F(x,y)$ and $F'(x,y)$ have been calculated by means of iterated exponentiation on a Hewlett-Packard calculator. We have considered a large number of combinations (x, y), both on the limiting curve (x, y_{lim}) where the convergence is slow and away from the limiting curve (x, y_{lim}) where the convergence is much faster. (The computing program was designed to carry out up to 1600 exponentiations, if necessary.) A few typical values exhibiting both dual and quadriconvergence have been tabulated in Table 1. For the reader's convenience, we have listed the value of y_{lim} pertaining to the x value in each entry. Also, the notation D.C. or Q.C. has been included.

For the case of quadriconvergence, we have listed in Table 1 four values denoted by F_1, F_2, F_3, and F_4. In order to make the identification of the F_i $(i = 1 - 4)$ with the functions $F(x, y)$ and $F'(x, y)$ introduced above in (3) and (4), we note that we have the following relations:

$$y^{F_1} = F_2,\ x^{F_2} = F_3,\ y^{F_3} = F_4,\ x^{F_4} = F_1, \tag{23}$$

so that we can write

$$F_1 = F_a,\ F_2 = F'_a,\ F_3 = F_b,\ F_4 = F'_b. \tag{24}$$

Both F_1 and F_3 are functions of the type F with x at the bottom of the "ladder" [see (3)], and they are therefore denoted by F_a and F_b, respectively. Similarly, F_2 and F_4 are functions of the type F' with y at the bottom of the "ladder" [see (4)], and they are therefore denoted by F'_a and F'_b, respectively. In view of (23) and (24), we see that the quadriconvergence for $y > y_{\text{lim}}$ (and $x < 1$) is actually the analog of the dual convergence observed in [1] and [3] for functions of one variable (x) only, since the functions F_a and F_b which have the same definition take on two different values, and similarly for F'_a and F'_b.

For the case of dual convergence of $F(x,y)$ and $F'(x,y)$ which occurs when $y \leq y_{\text{lim}}$, the two functions F_1 and F_2 of Table 1 can be simply identified as $F_1 = F$ and $F_2 = F'$ of (3) and (4).

In Table 1, we have included a few cases with y very large (for $x < 1$), namely, $y = 10{,}000$ for $x = 0.2$ and $y = 1{,}000$ for $x = 0.4$, 0.7, and 0.9. The reason is that, in the limiting case of large y, the following equations hold to a very high accuracy, as is shown by the entries in Table 1:

$$F_1 \approx x,\ F_2 \approx y^x,\ F_3 \approx 0,\ F_4 \approx 1. \tag{25}$$

The above equations can be derived very simply by noting that starting with a value $F_1 = x$, we have $F_2 = y^x$, and if y^x is large enough, $F_3 = x^{(y^x)}$ will be very small (i.e., ≈ 0) for $x < 1$, and hence, $F_4 \cong y^0 = 1$, and the next value to be denoted by F_5 is: $F_5 \approx x^1 = x$, i.e., F has the value assumed above for F_1, so that the four equations of (25) are mutually consistent, provided that $y^x >> 1$, so that $x^{(y^x)} \approx 0$.

Before leaving this discussion of the functions f and F, we wish to point out an interesting property. First, considering the function F at the tangency point $x = e^{-1/e}$ (see Fig. 3), for the two values of F at $x = e^{-1/e} = 0.692200$, $y = e^e = 15.1542$, we find $F'(x, y) = e$, and $F(x, y) = x^{F'} = 1/e$. Furthermore, for the function $f(x)$ at the point $x = e^{-e} = 0.065988$, we find the single value $f(x = e^{-e}) = 1/e$, whereas at the other extreme of the region of convergence, namely, $x = e^{1/e}$, we find $f(x) = e$. Thus, the six quantities

$$e,\ 1/e,\ e^{1/e},\ e^{-1/e},\ e^e, \text{ and } e^{-e}$$

are directly involved in the results obtained for the functions $f(x)$ and $F(x, y)$ at certain special points x and y.

Finally we will consider a generalization of the functions $f(x)$ and $F(x, y)$ to be denoted $f_N(x)$ and $F_N(x, y)$, respectively. We first define $f_N(x)$ by the equation

$$f_N(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x^N}}}}}}, \tag{26}$$

where N is an arbitrary positive quantity, and we are interested in the limit of an infinite number of x's in the "ladder." Again, the bracketing order is as usual "from the top down." Now for $N = x$, we find $f_x(x) = f(x)$ as before. It can be shown that for $x > 1$, if N is too large, the function $f_N(x)$ diverges even though x lies in the range $1 < x < e^{1/e}$ for which the simpler function $f(x)$ converges. In order to obtain the limitation on N, we consider the plane of G vs f as shown in Fig. 4. The line $G_2 = f$ is the 45° straight line in this figure. In addition, we have plotted the function $G_1(x) = x^f$ for two different values of x, namely, $x = 1.35$ and $x = e^{1/e} = 1.444668$. For $x = e^{1/e}$, $G_1(x)$ is just tangent to the straight line $G_2 = f$ at $f = e$. However, for $x = 1.35$, $G_1(x)$ intersects the line $G_2 = f$ at two values of f, namely, $f^{(1)} = 1.6318$ and $f^{(2)} = 5.934$. The value $f^{(1)}$ corresponds to the simple function $f(x = 1.35)$. We now note that in the region of f, $1.6318 < f < 5.934$, we have $1.35^f < f$, as shown by Fig. 4. It is therefore easy to show that if $N \leq 5.934$, the function $f_N(x)$ of (26) converges simply to the value $f(x = 1.35) = 1.6318$. On the other hand, for $N > 5.934$, we have $1.35^N > N$, so that as we go down the "ladder" of (26), progressively larger results are obtained and the function $f_N(1.35)$ diverges in this case even though $f(1.35)$ converges, since $x < e^{1/e}$. The value $f^{(2)}$, which is the limiting value for N, corresponds to the dashed part of the curve of x vs $f(x)$ in Fig. 1 of [1], which we had labeled at that time as "not meaningful" for the function $f(x)$. As can be seen from this figure, $f^{(2)}(x)$ increases rapidly with decreasing x until it becomes infinite as $x \to 1$. Typical values of $f^{(2)}(x)$, as obtained from the equations

$$x^f = f \tag{27}$$

$$\log x = \log f/f, \tag{28}$$

are as follows:

$$f^{(2)}(1.4) = 4.41,\ f^{(2)}(1.3) = 7.86,\ f^{(2)}(1.2) = 14.77,\ f^{(2)}(1.15) = 22.17,$$
$$f^{(2)}(1.1) = 38.2,\ f^{(2)}(1.05) = 92.95.$$

Thus, for $x = 1.1$, we have

$$f_N(1.1) = f^{(1)}(1.1) = 1.112, \text{ for } N \leq 38.2, \tag{29}$$

while $f_N(1.1)$ diverges for $N > 38.2$.

It can be easily shown that for $x < 1$, we have $f_N(x) = f(x)$, regardless of the (positive) value of N, and, correspondingly, the curve of x vs $f(x)$ in Fig. 2 of [1] does not have a second branch similar to that of Fig. 1.

We now define the function $F_N(x, y)$ as follows:

$$F_N(x,\ y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^{y^N}}}}}}}} \quad . \tag{30}$$

We will examine this function first for the case that both x and y are larger than 1. We assume that $x \leq y$. The situation is then very similar to that for $f_N(x)$. As an illustration, we consider the case where $x = 1.3$, and consider the plane G_i vs F, where $G_2 = F$ (45° straight line) and $G_1 = x^{y^F} = 1.3^{y^F}$. For $y = y_{\text{lim}} = 1.6525$, we are at the border between the regions of dual convergence and divergence in Fig. 1. Correspondingly, the curve of $G_1 = 1.3^{1.6525^F}$ is just tangent to the line $G_2 = F$ at the point $F = 2.304$ (see Fig. 2). Now consider the curve $G_1 = 1.3^{1.5^F}$, which has two points of intersection $F^{(1)}$ and $F^{(2)}$ with the line $G_2 = F$. We have:

$$F^{(1)} = 1.679,\ F^{(2)} = 4.184.$$

For $F^{(1)} < F < F^{(2)}$, we find that $G_1(1.3, 1.5) = 1.3^{1.5^F} < F$. Therefore, it can be concluded in the same manner as for $f_N(x)$ that $F_N(1.3, 1.5)$ converges to the value $F(1.3,\ 1.5)$ for $N \leq 4.184$, while for $N > 4.184$, $F\ (1.3,\ 1.5)$ diverges. Thus, for $(x,\ y)$ with $y < y_{\text{lim}}$, the roots of the equation

$$x^{y^F} - F = 0, \tag{31}$$

determine both the value of F $(=F^{(1)})$ and of $N_{\max}$, such that for $N \leq N_{\max}$, the modified function $F_N(x, y)$ converges to the value of $F(x, y)$. Here $N_{\max} = F^{(2)}$. Of course, for $y = y_{\text{lim}}$, we have $F^{(1)} = F^{(2)}$ (point of tangency), and $N_{\max} = F^{(1)} = F^{(2)}$. As an example, for $x = 1.3$, $y = 1.6525$, the tangency occurs at $F = 2.304$ in Fig. 4, and we have convergence of $F\ (1.3,\ 1.6525)$ to the value $F = 2.304$, provided that $N \leq 2.304$.

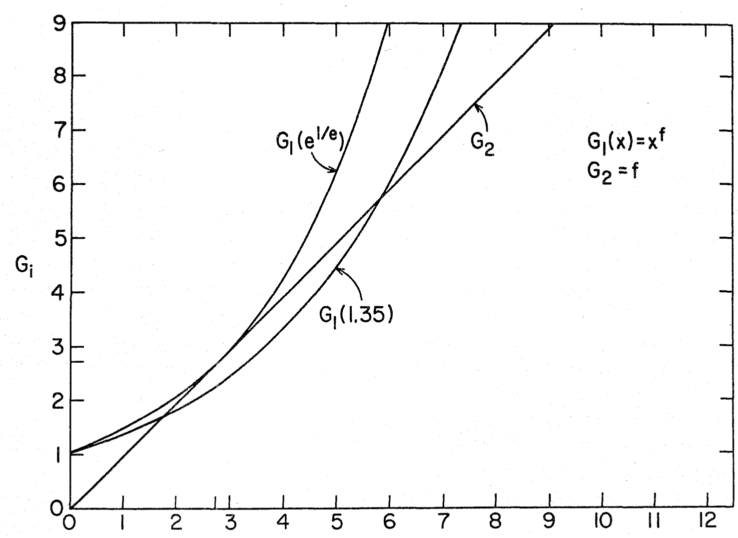

Fig. 4. The functions $G_1 = x^f$ and $G_2 = f$ plotted vs f. The two curves of G_1 pertain to the x values $x = 1.35$ and $x = e^{1/e} = 1.444668$. The curve of $G_1(1.35)$ intersects the 45° line $G_2 = f$ at the two points $f^{(1)} = 1.6318$ and $f^{(2)} = 5.934$, whose significance is explained in the text. The curve of $G_1(e^{1/e})$ is tangent to the $G_2 = f$ line at $f = e$ (see [1]).

When either x or $y < 1$ (or both x and $y < 1$), it is easily shown that the function $F_N(x,\ y) = F(x,\ y)$, regardless of the value of N. Thus, assume that

$x < 1$, but $y > 1$. Then, if N is arbitrarily large, y^N will be still larger, i.e., $y^N = N'$ where $N' > N$. The next step in the calculation of $F(x, y)$ involves raising x to the power N'. For N' very large, we find $x^{N'} \sim 0$, followed by $y^0 = 1$, and $x^1 = x$. This proves that $F_N(x, y) = F(x, y)$ regardless of the value of N. Note that for N very small, we have $y^N \sim 1$, followed by $x^{y^N} \cong x^1 = x$, independently of N.

The preceding argument involving F_N can also be used to prove the following theorem, when a similar function H of more than two variables is involved. Here we assume that H is a function of the type of F of Eqs. (2) and (3). As an example, we define $H(x, y, z)$ as follows:

$$H(x, y, z) = x^{y^{z^{\cdot^{\cdot^{\cdot^{x^{y^z}}}}}}}, \tag{32}$$

where x, y, z are arbitrary positive quantities. It can be easily shown that if one of the three numbers x, y, or z is $\leqq 1$, then $H(x, y, z)$ will not diverge (although it may converge to two values for any given value of x, y, or z at the bottom of the ladder, by virtue of the property of dual convergence introduced in [1] and [3]). To prove the theorem, we assume that $x \leqq 1$, but y and $z > 1$. At the top of the ladder, we obtain $x^{(y^z)}$, where y^z may be arbitrarily large. We will write $y^z = M$. Now $x^M \sim 0$ for $x < 1$ and large M. The next step calls for the calculation of $z^{x^M} \sim z^0 = 1$, followed by $y^{z^0} = y$, and so on. It is easily seen that the sequence $H(x, y, z)$ will never diverge provided that x, y, or z is $\leqq 1$. For the case where x, y, z are all larger than 1, but do not exceed $e^{1/e}$, we may use the result of [1] to prove that

$$H(x, y, z) \leqq f(e^{1/e}) = e,$$

and thus $H(x, y, z)$ is convergent. On the other hand, if at least one of the triplet x, y, z is larger than $e^{1/e}$, say $x > e^{1/e}$, whereas the other two lie in the range $1 < (y, z) < e^{1/e}$, then $H(x, y, z)$ will converge or diverge depending on the values of x, y, z relative to $e^{1/e}$, in the same manner as for $F(x, y)$ (see Fig. 1).

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SUMMATION OF SECOND-ORDER RECURRENCE TERMS AND THEIR SQUARES*

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Consider the linear recurrence sequence $\{R_n\}$ defined by $R_n = pR_{n-1} + qR_{n-2}$ for all n, where p and q are real. Initial conditions for any two consecutive terms completely define the sequence. We are interested in finding sums of the form

$$\sum_{x \le i \le y} R_i \quad \text{and} \quad \sum_{x \le i \le y} R_i^2$$

for arbitrary values of p and q.

Theorem 1:

$$\sum_{x \le i \le y} R_i = \left[\frac{1}{p+q-1}(qR_n + R_{n+1})\right]_{n=x-1}^{n=y}, \qquad \text{if } p+q-1 \ne 0. \tag{1}$$

$$\sum_{x \le i \le y} R_i = \left[\frac{1}{q+1}(qR_n + n(qR_0 + R_1))\right]_{n=x-1}^{n=y}, \quad \text{if } p+q-1 = 0,\ q+1 \ne 0. \tag{2}$$

$$\sum_{x \le i \le y} R_i = \left[\frac{n}{2}(R_n + R\ )\right]_{n=x-1}^{n=y}, \qquad \text{if } p+q-1 = 0,\ q+1 = 0. \tag{3}$$

The solution to the recurrence relation is determined by the roots of the characteristic equation $x^2 - px - q = 0$ and by the initial conditions.

If the two roots α and β of the characteristic equation are distinct and different from 1, then the solution of the recurrence is $R_n = a\alpha^n + b\beta^n$, where a and b are constants determined by the initial conditions. The sum may be calculated easily from the formula for the sum of a geometric series and from the equation

$$(\alpha - 1)(q + \alpha) = (p + q - 1)\alpha. \tag{4}$$

If α is a double root of the characteristic equation and $\alpha \ne 1$, then the solution of the recurrence is $R_n = a\alpha^n + bn\alpha^n$, where again a and b are constants determined by the initial constants. Multiplying (4) by $\alpha^n/(\alpha - 1)$, and taking the derivative with respect to α gives the following equation:

$$qn\alpha^{n-1} + (n+1)\alpha^n = \frac{(p+q-1)[n\alpha^{n+1} - (n+1)\alpha^n]}{(\alpha - 1)^2}; \tag{5}$$

the appropriate summation formula can be simplified with (5) to give (1).

Equations (2) and (3) apply to the degenerate cases where the roots of the characteristic equation are $(p - 1, 1)$ and $(1, 1)$, respectively. The corresponding summations have nongeometric terms in them and simplify to different forms.

The results of Theorem 1 are well known, particularly equation (1) (see, for example, [2] and [3]). Often, however, the need for separate proofs for the cases of a double root and a root equal to 1 is not recognized. In the special case that $p = q = 1$, equation (1) applies, and we have, as simple corollaries, formulas for the summation of Fibonacci and Lucas numbers:

**This work was performed in part while the author was with the Computer Science Department of the University of Southern California in Los Angeles.*

$$\sum_{1 \le i \le n} F_i = F_{n+2} - 1, \qquad \sum_{1 \le i \le n} L_i = L_{n+2} - 3.$$

In Theorem 1 the results depend on the roots of the characteristic equation. If we consider the sum of the *squares* of the recurrence terms, the results depend on the possible values for the *products* of two roots.

Theorem 2: Let $\{R_n\}$ satisfy

$$R_n = pR_{n-1} + qR_{n-2}$$

and let $\{S_n\}$ satisfy

$$S_n = pS_{n-1} + qS_{n-2}$$

for all n and all real p, q.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2(1-q)R_nS_n + pqR_nS_{n+1} + pqR_{n+1}S_n + (1-q)R_{n+1}S_{n+1}}{(q+1)(p+q-1)(p-q+1)}\right]_{n=x-1}^{n=y}, \tag{6}$$

if $q + 1 \ne 0$, $p + q - 1 \ne 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2}{q^2 - 1}R_nS_n - \frac{q}{q^2 - 1}(bS_n + dR_n) + bdn\right]_{n=x-1}^{n=y}, \tag{7}$$

where $b = (qR_0 + R_1)/(q + 1)$, $d = (qS_0 + S_1)/(q + 1)$,
if $q + 1 \ne 0$, $p + q - 1 = 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2}{q^2 - 1}R_nS_n - \frac{q}{q^2 - 1}(-1)^n(bS_n + dR_n) + bdn\right]_{n=x-1}^{n=y}, \tag{8}$$

where $b = (qR_0 - R_1)/(q + 1)$, $d = (qS_0 - S_1)/(q + 1)$,
if $q + 1 \ne 0$, $p + q - 1 \ne 0$, $p - q + 1 = 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{R_0S_0 + R_1S_1}{2}n + \frac{R_0S_0 - R_1S_1}{2} \cdot \frac{(-1)^n}{2}\right]_{n=x-1}^{n=y}, \tag{9}$$

if $q + 1 \ne 0$, $p + q - 1 = 0$, $p - q + 1 = 0$.

$$\sum_{x \le i \le y} R_i S_i = \frac{1}{(\alpha^2 - 1)^2}\left[\alpha c\frac{\alpha^{2n+2}}{\alpha^2 - 1} + (bc + ad)n + bd\frac{1}{1 - \alpha^2}\left(\frac{1}{\alpha}\right)^{2n}\right]_{n=x-1}^{n=y}, \tag{10}$$

where $\alpha = \frac{1}{2}(p + (p^2 - 4)^{\frac{1}{2}})$, and
$a = (\alpha R_1 - R_0)$,
$b = \alpha(\alpha R_0 - R_1)$,
$c = (\alpha S_1 - S_0)$,
$d = \alpha(\alpha S_0 - S_1)$,
if $q + 1 = 0$, $p + q - 1 \ne 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[R_0S_0 + \tfrac{1}{2}(R_1S_1 - R_{-1}S_{-1})\frac{n(n+1)}{2} + (R_1 - R_0)(S_1 - S_0)\frac{n(n+1)(2n+1)}{6}\right]_{n=x-1}^{n=y}, \tag{11}$$

if $q + 1 = 0$, $p + q - 1 = 0$, $p - q + 1 \ne 0$.

$$\sum_{x\le i\le y} R_iS_i = \left[R_0S_0 + \tfrac{1}{2}(R_1S_1 - R_{-1}S_{-1})\frac{n(n+1)}{2} + (R_1 + R_0)(S_1 + S_0)\frac{n(n+1)(2n+1)}{6}\right]_{n=x-1}^{n=y}, \tag{12}$$

$$\text{if } q + 1 = 0,\ p + q - 1 \ne 0,\ p - q + 1 = 0.$$

Proof: The key relation, analogous to (4), is the following, where α and β are roots of $x^2 - px - q = 0$, $\alpha\beta \ne 1$, $\alpha^2 \ne 1$, $\beta^2 \ne 1$:

$$\frac{\alpha^{n+1}\beta^{n+1}}{\alpha\beta - 1} = \frac{q^2(1-q)\alpha^n\beta^n + pq\alpha^n\beta^{n+1} + pq\alpha^{n+1}\beta^n + (1-q)\alpha^{n+1}\beta^{n+1}}{(q+1)(p-q+1)(p+q-1)}. \tag{13}$$

This is proved by considering the following equation (recall that $\alpha^2 = p\alpha + q$ and $\beta^2 = p\beta + q$):

$$\begin{aligned}
(\alpha\beta - 1)&[q^2(1-q) + pq\beta + pq\alpha + (1-q)\alpha\beta]\\
&= \alpha\beta q^2(1-q) + \alpha\beta^2 pq + \alpha^2\beta pq + \alpha^2\beta^2(1-q) - q^2(1-q) - pq\beta - pq\alpha - (1-q)\alpha\beta\\
&= \alpha\beta q^2(1-q) + \alpha(p\beta + q)pq + (p\alpha + q)\beta pq + (p\alpha + q)(p\beta + q)(1-q)\\
&\qquad - q^2(1-q) - pq\beta - pq\alpha - (1-q)\alpha\beta\\
&= \alpha\beta[q^2(1-q) + p^2q + p^2q + p^2(1-q) - (1-q)]\\
&= \alpha\beta[p^2(q+1) - (q^2-1)(q-1)]\\
&= \alpha\beta(q+1)(p+q-1)(p-q+1).
\end{aligned} \tag{14}$$

Now $\alpha\beta = 1$ is possible if and only if (1) $p + q - 1 = 0$ (the roots are $p - 1$ and 1); (2) $p - q + 1 = 0$ (the roots are $p + 1$ and -1); or (3) $q + 1 = 0$ (the roots are reciprocals). Thus we can divide both sides of (14) by

$$(\alpha\beta - 1)(q + 1)(p + q - 1)(p - q + 1);$$

multiplying by $\alpha^n\beta^n$ gives (13).

In the remainder of the proof, we use α and β to represent roots of

$$x^2 - px - q = 0,$$

we use a, b, c, d to represent constants determined by initial conditions of the recurrences, and we let

$$\Delta = (q + 1)(p + q - 1)(p - q + 1).$$

If omitted, the limits of summation are understood to be x and y; the right-hand sides are to be evaluated at $n = y$ and $n = x - 1$.

Suppose that $\alpha \ne \beta$. Then the solutions to the recurrences are

$$R_n = a\alpha^n + b\beta^n \quad\text{and}\quad S_n = c\alpha^n + d\beta^n.$$

$$\begin{aligned}
\Delta\Sigma R_iS_i &= \Delta\Sigma(a\alpha^i + b\beta^i)(c\alpha^i + d\beta^i)\\
&= \Delta\Sigma(ac\alpha^{2i} + ad\alpha^i\beta^i + bc\alpha^i\beta^i + bd\beta^{2i})\\
&= \Delta\left(\frac{ac\alpha^{2n+2}}{\alpha^2 - 1} + \frac{ad(\alpha\beta)^{n+1}}{\alpha\beta - 1} + \frac{bc(\alpha\beta)^{n+1}}{\alpha\beta - 1} + \frac{bd\beta^{2n+2}}{\beta^2 - 1}\right).
\end{aligned}$$

Since $q + 1 \ne 0$, $p + q - 1 \ne 0$, and $p - q + 1 \ne 0$, we know that $\alpha^2 \ne 1$, $\beta^2 \ne 1$, and $\alpha\beta \ne 1$. Equation (13) can thus be applied to each term individually; when terms are collected the desired result is obtained:

$$\begin{aligned}
\Delta\Sigma R_iS_i &= q^2(1-q)[ac\alpha^n\alpha^n + ad\alpha^n\beta^n + bc\alpha^n\beta^n + bd\beta^n\beta^n]\\
&\quad + pq[ac\alpha^n\alpha^{n+1} + ad\alpha^n\beta^{n+1} + bc\alpha^n\beta^{n+1} + bd\beta^n\beta^{n+1}]\\
&\quad + pq[ac\alpha^{n+1}\alpha^n + ad\alpha^{n+1}\beta^n + bc\alpha^{n+1}\beta^n + bd\beta^{n+1}\beta^n]\\
&\quad + (1-q)[ac\alpha^{n+1}\alpha^{n+1} + ad\alpha^{n+1}\beta^{n+1} + bc\alpha^{n+1}\beta^{n+1} + bd\beta^{n+1}\beta^{n+1}]\\
&= q^2(1-q)R_nS_n + pqR_nS_{n+1} + pqR_{n+1}S_n + (1-q)R_{n+1}S_{n+1}.
\end{aligned}$$

If α is a double root of $x^2 - px - q = 0$, then the sum takes the following form:

$$\begin{aligned} \Delta\Sigma R_i S_i &= \Delta\Sigma(a\alpha^i + bi\alpha^i)(c\alpha^i + di\alpha^i) \\ &= \Delta\Sigma(ac\alpha^{2i} + adi\alpha^{2i} + bci\alpha^{2i} + bdi^2\alpha^{2i}). \end{aligned} \tag{15}$$

By taking various derivatives of (13), it is easy to show that the following expressions hold:

$$\begin{aligned} \Delta\Sigma i\alpha^i\beta^i &= q^2(1 - q)n\alpha^n\beta^n + pq(n + 1)\alpha^n\beta^{n+1} + pqn\alpha^{n+1}\beta^n + (1 - q)(n + 1)\alpha^{n+1}\beta^{n+1} \\ &= q^2(1 - q)n\alpha^n\beta^n + pqn\alpha^n\beta^{n+1} + pq(n + 1)\alpha^{n+1}\beta^n + (1 - q)(n + 1)\alpha^{n+1}\beta^{n+1}, \end{aligned}$$

$$\begin{aligned} \Delta\Sigma i^2\alpha^i\beta^i &= q^2(1 - q)n^2\alpha^n\beta^n + pqn(n + 1)\alpha^n\beta^{n+1} + pqn(n + 1)\alpha^{n+1}\beta^n \\ &\quad + (1 - q)(n + 1)^2\alpha^{n+1}\beta^{n+1}. \end{aligned}$$

Substitution into (15) and simplification complete the proof of (6).

Equations (7-12) apply in various degenerate cases where the product of some two roots of the characteristic equation is 1, and there is a nongeometric term in the corresponding summation:

- in equation (7) the roots are $(\alpha, 1)$, $\alpha \neq 1, -1$;
- in equation (8) the roots are $(\alpha, -1)$, $\alpha \neq 1, -1$;
- in equation (9) the roots are $(1, -1)$;
- in equation (10) the roots are (α, α^{-1});
- in equation (11) the roots are $(1, 1)$;
- in equation (12) the roots are $(-1, -1)$.

The results of Theorem 2 correct and complete the discussion of Hoggatt [1]. Note that if $q = 1$ and $p \neq 0$ the following special cases are derived (see also Russell [5]):

$$\sum_{x \le i \le y} R_i = \left[\frac{R_n + R_{n+1}}{p}\right]_{n=x-1}^{n=y},$$

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{R_{n+1}S_n + R_n S_{n+1}}{2p}\right]_{n=x-1}^{n=y}.$$

Nothing in the derivations has precluded the possibility that $q = 0$. In this case the recurrences are first-order recurrences and the solutions are readily seen to reduce to the appropriate sums.

The method of this paper can be extended to other sums involving products of terms from recurrence sequences. The "most pleasing" sums derived are those that can be expressed as linear combinations of terms "similar" to the summand, without multiplications by functions of n. Such sums, as in equations (1) and (6), have been called *standard sums* in [4], where they are more precisely defined. It seems clear from the proofs of Theorems 1 and 2 that such standard sums do not exist if there is a set of values $\{\alpha_i \mid \alpha_i$ is a root of the characteristic equation of the ith recurrence sequence in the product being summed$\}$ such that $\Pi\alpha_i = 1$. When such a standard sum does exist, it can be found directly, without knowing the roots of the characteristic equations, by the method described in [4]. The "key formulas" (4) and (13) were, in fact, first found in this way.

The sums found are, of course, not unique. For instance, using the relation

$$R_{n+2}S_{n+2} = p^2R_{n+1}S_{n+1} + pqR_{n+1}S_n + pqR_nS_{n+1} + q^2R_nS_n,$$

equation (6) can also be written as follows:

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{pq^2(R_n S_{n+1} + R_{n+1}S_n) + (1-q)[R_{n+2}S_{n+2} + (1-p^2)R_{n+1}S_{n+1}]}{(1-q)(p+q-1)(p-q+1)}\right]_{n=x-1}^{n=y},$$

$$\text{if } q+1 \ne 0,\ p+q-1 \ne 0,\ p-q+1 \ne 0. \tag{16}$$

In closing, we note that the expressions of this paper can be used to derive some identities among recurrence terms. As an example consider $\Sigma R_i S_i$ with R_i and S_i identical sequences, $R_0 = S_0 = 0$, $R_1 = S_1 = 1$, $p = 1$, $q = 2 + \varepsilon$, and limits of summation $0 \le i \le n$. As $\varepsilon \to 0$, the sum approaches a well-defined value, and thus the right-hand side of (16) must also have a finite limit. Since the denominator goes to 0, so must the numerator. We conclude that the following must be true:

$$\left[8R_y R_{y+1} - R_{y+2}^2\right]_{y=-1}^{y=n} = 8R_n R_{n+1} - R_{n+2}^2 + 1 = 0$$

or

$$8R_n R_{n+1} = (R_{n+2} + 1)(R_{n+2} - 1)$$
$$\text{if } p = 1,\ q = 2,\ R_0 = 0,\ R_1 = 1.$$

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ITERATING THE PRODUCT OF SHIFTED DIGITS

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1. INTRODUCTION

Let t be a fixed nonnegative integer. For positive integers n written in decimal as

$$n = \sum_{i=0}^{k} d_i \cdot 10^i,$$

with $0 \le d_i \le 9$ and $d_k > 0$, we define

$$f_t(n) = \prod_{i=0}^{k} (t + d_i).$$

Also define $f_0(0) = 0$. Erdös and Kiss [1] have asked about the behavior of the sequence of iterates n, $f_t(n)$, $f_t(f_t(n))$, ... They noted that $f_4(120) = 120$. For $t = 0$, every such sequence eventually reaches a one-digit number. Sloane

[2] has considered this case. For $t = 1$, we prove that the sequence of iterates from any starting point n remains bounded, and we list the two possible cycles. For $t \geq 10$, it is clear that $f_t(n) > n$ for every n so that the sequence always tends to infinity. We discuss the cases $2 \leq t \leq 9$ and present numerical evidence and a heuristic argument which conclude that every sequence remains bounded when $t \leq 6$, while virtually every sequence tends to infinity for $t \geq 7$. In Table 1 we give the known cycles in which these sequences may be trapped when $0 \leq t \leq 6$. See also [3] for the case $t = 0$.

TABLE 1. Some Data on the Cycles of f_t for $0 \leq t \leq 6$

t	Least Term of Cycle	Cycle Length	First Start Leading to it	# of Starts $\leq$ 100000 Leading to it
0	0	1	10	82402
	1	1	1	5
	2	1	2	3213
	3	1	3	15
	4	1	4	894
	5	1	5	607
	6	1	6	6843
	7	1	7	15
	8	1	8	5971
	9	1	9	35
1	2	9	1	92043
	18	1	18	7957
2	6	3	2	9927
	9	2	1	6
	12	1	12	29105
	24	1	16	60105
	35	1	35	2
	56	1	56	811
3	24	10	1	47955
	648	2	134	52045
4	96	5	1	6793
	112	16	37	70677
	120	1	29	20
	315	1	135	6
	1280	2	589	4798
	2688	3	1289	6971
	4752	1	1157	90
	7744	1	4477	185
	15840	2	4779	9992
	24960	1	10489	378
	57915	1	15579	90
5	50	1	50	1
	210	1	57	6
	450	1	3	222
	780	1	158	10
	1500	1	4	35726

(continued)

TABLE 1 (continued)

t	Least Term of Cycle	Cycle Length	First Start Leading to it	# of Starts $\leq$ 100000 Leading to it
5	1600	3	228	7058
	3920	1	22	91
	16500	1	1339	146
	16800	4	1	4927
	32760	4	368	51483
	91728	1	11899	300
	1293600	1	38899	30
6	90	1	34	3
	840	1	4	40
	4320	1	3	329
	9360	2	35	550
	51744	5	18	2626
	59400	1	7899	300
	60480	1	6	3300
	917280	1	7777	493
	2419200	1	26778	12
	533744640	62	38	10968
	1556755200	21	1	25484
	139089000960	85	5	5895

2. THE CASE $t = 1$

This is the only nontrivial case in which we can prove that every sequence of iterates is bounded.

Theorem: Let n be a positive integer. Then $f_1(n) = n$ if and only if $n = 18$. Also $f_1(n) > n$ if and only if $n = d \cdot 10^k - 1$, where $k \geq 0$ and $2 \leq d \leq 10$. In In the latter case, $f_1(n) = n + 1$. Iteration of f_1 from a positive starting number eventually leads either to the fixed point 18 or to the cycle (2, 3, 4, 5, 6, 7, 8, 9, 10).

Proof: If $n = d \cdot 10^k - 1$ with $2 \leq d \leq 10$, then the digits of n are $d - 1$ and k nines. Thus $f_1(n) = d \cdot 10^k = n + 1$. Now suppose $k \geq 1$ and n has $k + 1$ digits, but n is not of the form $d \cdot 10^k - 1$. Then the low-order k digits are not all nines. Write

$$n = \sum_{i=0}^{k} d_i \cdot 10^i,$$

and let j be the greatest subscript such that $j < k$ and $d_j < 9$. Then

$$\begin{aligned} (1) \qquad f_1(n) &\leq (d_k + 1)(d_j + 1)10^{k-1} \\ &= d_k \cdot 10^k + d_j \cdot 10^{k-1} + (1 + d_k(d_j - 9))10^{k-1}. \end{aligned}$$

Now $d_j - 9 \leq -1$ and $d_k \geq 1$. Hence the last term of (1) is nonpositive, and it vanishes if and only if $d_k = 1$ and $d_j = 8$. Hence $f_1(n) < n$ if either $d_k > 1$ or $d_j < 8$. If $d_k = 1$ and $d_j = 8$ and $j < k - 1$, then also $f_1(n) < n$. Otherwise, either $n = 18$ [and $f_1(18) = 18$] or n has at least three digits, the first two of which are 18. If any lower-order digit were nonzero, the inequality in (1) would be strict and give $f_1(n) < n$. Finally, if $n = 1800...0$, clearly

$$f_1(n) = (1 + 1)(8 + 1) = 18 < n.$$

The last statement of the theorem follows easily from the earlier ones by induction on n. For $n > 18$, either $f_1(n) < n$ or $f_1(f_1(n)) < n$.

3. THE CASES $t = 2$ THROUGH 6

These five cases are alike in that there is compelling evidence that all the sequences are bounded, but we cannot prove it. In Table 1 we gave some data on the known cycles of f for $0 \le t \le 6$. Table 2 lists the cycles of length > 1. For $t \le 5$, every starting number up to 100000 eventually reaches one of these cycles. For $t = 6$, the same is true up to 50000.

TABLE 2. Cycles of at Least Two Terms

t	Cycle
1	(2, 3, 4, 5, 6, 7, 8, 9, 10)
2	(6, 8, 10)
2	(9, 11)
3	(24, 35, 48, 77, 100, 36, 54, 56, 72, 50)
3	(648, 693)
4	(96, 130, 140, 160, 200)
4	(112, 150, 180, 240, 192, 390, 364, 560, 360, 280, 288, 864, 960, 520, 216, 300)
4	(1280, 1440)
4	(2688, 8640, 3840)
4	(15840, 17280)
5	(1600, 1650, 3300)
5	(16800, 21450, 18900, 27300)
5	(32760, 36960, 67760, 87120)
6	(9360, 9720)
6	(51744, 100100, 63504, 71280, 61152)
6	(533744640, 833976000, 573168960, 1634592960, 10777536000, 23678246592, 199264665600, 1034643456000, 1163973888000, 5504714691840, 6992425440000, 2463436800000, 1015831756800, 2466927695232, 20495794176000, 36428071680000, 14379662868480, 27960455776000, 654872648601600, 703005740236800, 94421561794560, 119870150400000, 28834219814400, 41821194240000, 5974456320000, 2642035968000, 2483144294400, 3048192000000, 296284262400, 445906944000, 384912000000, 49380710400, 22289904000, 20901888000, 17923368960, 160487308800, 349505694720, 1100848320000, 322620641280, 187280916480, 906125875200, 383584481280, 1150082841600, 920066273280, 391283343360, 499979692800, 4776408000000, 794794291200, 919900800000, 92588832000, 56330588160, 69709102848, 138692736000, 385169541120, 451818259200, 401616230400, 65840947200, 62270208000, 8695185408, 25101014400, 3911846400, 4000752000)

TABLE 2 (continued)

t	Cycle
6	(1556755200, 4604535936, 12702096000, 8151736320, 4576860288, 27122135040, 11623772160, 28848089088, 325275955200, 473609410560, 420323904000, 60466176000, 24455208960, 70253568000, 24659002368, 68976230400, 61138022400, 10241925120, 10431590400, 9430344000, 1574640000)
	(139089000960, 277766496000, 984031027200, 142655385600, 486857226240, 1239869030400, 2222131968000, 983224811520, 438126796800, 998587699200, 4903778880000, 4868115033600, 2661620290560, 2648687247360, 19781546803200, 38445626419200, 48283361280000, 15485790781440, 106051785840000, 84580378122240, 45565186867200, 118144020234240, 47795650560000, 37781114342400, 18931558464000, 40663643328000, 18284971622400, 41422897152000, 16273281024000, 6390961274880, 14978815488000, 87214615488000, 39869538508800, 219583673971200, 642591184435200, 309818234880000, 203251004006400, 14898865766400, 256304176128000, 105450861035520, 112464019261440, 119489126400000, 80655160320000, 5736063320064, 3112798740480, 6310519488000, 2218016908800, 2007417323520, 1165698293760, 16476697036800, 100144080691200, 32262064128000, 6742112993280, 6657251328000, 2761808265216, 7290429898752, 37777259520000, 38697020144640, 42796615680000, 37661021798400, 38944920268800, 92177326080000, 13352544092160, 19916886528000, 82805964595200, 97371445248000, 42499416960000, 35271936000000, 5447397795840, 45218873700000, 14279804098560, 91537205760000, 14425516385280, 53013342412800, 7604629401600, 2445520896000, 2529128448000, 2503581696000, 2390026383360, 2742745743360, 9020284416000, 877879296000, 2009063347200, 943272345600, 480370176000)

Some cycles may by reached from only finitely many starting numbers. For example, it is easy to see that $f_5(n) = 50$ only when $n = 50$. The cycle (9, 11) for f_2 may be reached only from the odd numbers below 12. Only 35 and 53 lead to the fixed point 35 of f_2. It is a ten-minute exercise to discover all twenty starting numbers which lead to the fixed point 120 of f_4. The fixed point 90 for f_6 may be reached only from the starting numbers 34, 43, and 90.

Given a cycle, what is the asymptotic density of the set of starting numbers which lead to it? We cannot answer this question even for the two cycles for $t = 1$. Some relevant numerical data is shown in the last column of Table 1. Since the digit 0 occurs in almost all numbers, the answer to the question is clear in case $t = 0$.

4. THE CASES $t = 7$ THROUGH 9

The starting number 5 leads to the fixed point 31746120037632000 of f_7. We found no other cycles in these three cases. Every sequence with starting number up to 1000 rises above 10^{14}. Every sequence starting below 17 (except 5 and 12 for f_7) rises above 10^{300}. These observations, together with the heuristic argument below, suggest that nearly every sequence diverges to infinity.

When $7 \leq t \leq 9$, it usually happens that $f_t(n) > n$. The least n with $f_t(n) \leq n$ is 700, 9000, 90000000, for $t = 7, 8, 9$, respectively.

The sequences show a strong tendency to merge. We conjecture that there is a finite number of sequences such that every sequence merges with one of them.

5. THE HEURISTIC ARGUMENT

Let t be a fixed positive integer. Consider a positive number n of k digits, where k is large. For $0 \leq d \leq 9$ and most n, about $k/10$ of the digits will be d. Thus

$$f_t(n) \approx \prod_{d=0}^{9} (d+t)^{k/10} = (p_t)^k, \text{ where } p_t = \left(\prod_{d=0}^{9}(d+t)\right)^{1/10}.$$

This means that $f_t(n)$ will have about $k \cdot \log_{10} p_t$ digits. From Table 3, it is clear that this implies that $f_t(n) < n$ for most large n when $1 \leq t \leq 5$, and that $f_t(n) > n$ for most large n when $6 \leq t \leq 9$.

It is tempting to apply the same reasoning to the subsequent terms of the sequence. Note, however, that $f_t(n)$ cannot be just any number. About one-fifth of the digits of n are $\equiv -t \pmod 5$ and about half of them have the same parity as t. Hence the highest power of 10 that divides $f_t(n)$ is usually about $10^{k/5}$, so that $f_t(n)$ will have many more zero digits than other numbers of comparable size. It is plausible that, after several iterations, the fraction of digits which are low-order zeros will stabilize. Furthermore, it is likely that the significant digits will take on the ten possible values with equal frequency. Suppose we reach a number m of k digits. Assume there are constants a, b, s, which depend on t but not on m or k, so that (i) m has about ak low-order zeros, (ii) each of the ten digits occurs about bk times as a significant digit of m, and (iii) $f_t(m)$ has about sk digits, of which approximately ask are low-order zeros. Then $a + 10b = 1$ and

$$ask \approx \min(\operatorname{ord}_2(f_t(m)), \operatorname{ord}_5(f_t(m))), \tag{2}$$

where $\operatorname{ord}_p(w)$ denotes the ordinal of w at the prime p. By hypotheses (i) and (ii), we have

$$f_t(m) \approx (0+t)^{ak+bk}(1+t)^{bk} \ldots (9+t)^{bk} = (t^a \pi_t^b)^k,$$

where

$$\pi_t = \prod_{d=0}^{9}(d+t).$$

Since $sk \approx \log_{10} f_t(m)$, we find

$$s \approx a \log_{10} t + b \log_{10} \pi_t. \tag{3}$$

When $t = 5$, equation (2) becomes

$$ask \approx \min(8bk, ak + 2bk)$$

because $8 = \operatorname{ord}_2 \pi_5$. Hence $as \approx 8b$, so

$$a \approx \frac{8}{8+10s} \quad \text{and} \quad b \approx \frac{s}{8+10s}.$$

Substitution in (3) gives a quadratic equation in s whose positive root is shown in Table 3, together with a and b.

If $1 \leq t \leq 9$ and $t \neq 5$, then (2) becomes

$$ask \approx \min(gak + hbk, bk + bk),$$

where $g \geq 0$ and $h \geq 7$. Hence $as \approx 2b$, and we find

$$a \approx \frac{2}{2+10s} \quad \text{and} \quad b \approx \frac{s}{2+10s}.$$

Using (3) produces a quadratic equation in s whose positive root is given in Table 3.

TABLE 3. Values of p_t and s for $1 \leq t \leq 9$

t	p_t	$\log_{10} p_t$	a	b	s
1	4.5	0.66	0.30	0.070	0.46
2	5.8	0.76	0.23	0.077	0.65
3	6.9	0.84	0.21	0.079	0.76
4	8.0	0.90	0.19	0.081	0.84
5	9.0	0.96	0.49	0.051	0.83
6	10.086	1.0037	0.17	0.083	0.965
7	11.1	1.05	0.16	0.084	1.013
8	12.2	1.08	0.16	0.084	1.06
9	13.2	1.12	0.15	0.085	1.09

We may defend the third hypothesis this way: If we had assumed that $f_t(m)$ had about the same number of digits as m, i.e., that $s = 1$, and followed the remainder of the argument above, we would have concluded that the sequence forms an approximate geometric progression, which is the essence of (iii). There is no other simple assumption for the change in the number of digits from one term to the next.

The few sequences we studied with $7 \leq t \leq 9$ behaved roughly in accordance with the three hypotheses and the data in Table 3.

In summary, for most large n, $f_t(n)$ will have many fewer digits than n for $1 \leq t \leq 5$, about 0.37% more digits when $t = 6$, and substantially more digits for $7 \leq t \leq 9$. However, after several iterations, when we reach a number m, say, it will usually happen that $f_t(m)$ has many fewer digits than m for $1 \leq t \leq 6$ and many more digits for $7 \leq t \leq 9$. Thus if we iterate f_t, the sequence almost certainly will diverge swiftly to infinity for $7 \leq t \leq 9$, but remain bounded for $1 \leq t \leq 6$.

Numbers in the image of f_t not only are divisible by a high power of 10, but all their prime factors are below $10 + t$. How this property affects the distribution of digits in such numbers is unclear. There are only $O(\log^r x)$ of them up to x, where r is the number of primes up to $9 + t$.

Let $1 \leq t \leq 6$, and suppose that iteration of f_t from any starting number does lead to a cycle. How many iterations will be required to reach the cycle? The above heuristic argument predicts that about

$$(\log_{10}\log_{10} n)/(-\log_{10} s) + O(1)$$

iterations will be needed, which is very swift convergence indeed. In the case $t = 0$, Sloane [2] has conjectured that a one-digit number will be reached in a bounded number of iterations. The sequence for $t = 6$ starting at $n = 5$ does not enter the 85-term cycle until the 121st iterate.

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ON MAXIMIZING FUNCTIONS BY FIBONACCI SEARCH

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1. INTRODUCTION

The search for a local maximum of a function $f(x)$ involves a sequence of function evaluations, i.e., observations of the value of $f(x)$ for a fixed value of x. A sequential search scheme allows us to evaluate the function at different points, one after the other, using information from earlier evaluations to decide where to locate the next ones. At each stage, the smallest interval in which a maximum point of the function is known to lie is called the *interval of uncertainty*.

Most of the theoretical search procedures terminate the search when either the interval of uncertainty is reduced to a specific size or two successive estimates of the maximum are closer than some predetermined value. However, an additional termination rule which surprisingly has not received much attention by theorists exists in most practical search codes, namely the number of function evaluations cannot exceed a predetermined number, which we denote by N.

A well-known procedure designed for a fixed number of function evaluations is the so-called Fibonacci search method. This method can be applied whenever the function is unimodal and the initial interval of undertainty is finite. In this paper, we propose a two-stage procedure which can be used whenever these requirements do not hold. In the first stage, the procedure tries to bracket the maximum point in a finite interval, and in the second it reduces this interval using the Fibonacci search method or a variation of it developed by Witzgall.

2. THE BRACKETING ALGORITHM

A function f is *unimodal* on $[a, b]$ if there exists $a \le \bar{x} \le b$ such that $f(x)$ is strictly increasing for $a \le x < \bar{x}$ and strictly decreasing for $\bar{x} < x \le b$. It has been shown (Avriel and Wilde [2], Kiefer [6]) that the Fibonacci search method guarantees the smallest final interval of uncertainty among all methods requiring a fixed number of function evaluations. This method and its variations (Avriel and Wilde [3], Beamer and Wilde [4], Kiefer [6], Oliver and Wilde [7], Witzgall [10]) use the following idea:

Suppose y and z are two points in $[a, b]$ such that $y < z$, and f is unimodal, then

$$\begin{aligned} f(y) < f(z) &\quad \text{implies} \quad y \le \bar{x} \le b, \\ f(y) > f(z) &\quad \text{implies} \quad a \le \bar{x} \le z, \text{ and} \\ f(y) = f(z) &\quad \text{implies} \quad y \le \bar{x} \le z. \end{aligned}$$

Thus the property of unimodality makes it possible to obtain, after examining $f(y)$ and $f(z)$, a smaller new interval of uncertainty. When it cannot be said in advance that f is unimodel, a similar idea can be used.

Suppose that $f(x_1)$, $f(x_2)$, and $f(x_3)$ are known such that

$$x_1 < x_2 < x_3 \quad \text{and} \quad f(x_2) \ge \max\{f(x_1),\ f(x_3)\}, \tag{1}$$

then a local maximum of f exists somewhere between x_1 and x_3. Evaluation of the function at a new point x_4 in the interval (x_1, x_3) will reduce the interval of uncertainty and form a new set of three points x_1', x_2', x_3' satisfying equation (1):

Suppose $x_1 < x_4 < x_2$, then if $f(x_4) \geq f(x_2)$ let $x_1' = x_1$, $x_2' = x_4$, $x_3' = x_2$, and if $f(x_4) < f(x_2)$ let $x_1' = x_4$, $x_2' = x_2$, and $x_3' = x_3$. Similarly for $x_2 < x_4 < x_3$, if $f(x_4) > f(x_2)$ then let $x_1' = x_2$, $x_2' = x_4$, $x_3' = x_3$, and if $f(x_4) < f(x_2)$ then let $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_4$.

When applying quadratic approximation methods, the new point x_4 is chosen as the maximum point of a quadratic function which approximates f. The assumption behind this method is that f is nearly quadratic, at least in the neighborhood of its maximum. However, when the number of function evaluations is fixed in advance, this method may terminate with an interval of uncertainty which is long relative to the initial one.

The quadratic approximation algorithm of Davies, Swann, and Campey [5] includes a subroutine that finds three equally spaced points satisfying equation (1). A more general method developed by Rosenbrock [8] can serve as a preparatory step for a quadratic approximation algorithm (Avriel [1]).

We now describe the search for points satisfying equation (1) in a general form that allows further development of our algorithm. The input data includes the function f, the number of evaluations N, and a set of positive numbers α_i, $i = 3, \ldots, N$.

Bracketing Algorithm:

Step 1. Evaluate f at two distinct points. Denote these points by x_1 and x_2 so that $f(x_1) \leq f(x_2)$. Set $k = 3$.

Step 2. Evaluate f at $x_k = x_{k-1} + \alpha_k(x_{k-1} - x_{k-2})$.
If $f(x_k) \leq f(x_{k-1})$, stop. (A local maximum exists between x_{k-2} and x_k.)
If $f(x_k) > f(x_{k-1})$, set $k \leftarrow k + 1$.

Step 3. If $k = N + 1$, stop. (The search failed to bracket a local maximum.)
If $k \leq N$, return to Step 2.

If the algorithm terminates in Step 2, then the function was evaluated $k \leq N$ times and a local maximum was bracketed between x_{k-2} and x_k. The interval of uncertainty may now be further reduced by evaluating the function at $N - k$ new points $x_{k+1}, \ldots, x_N$. Notice that there is already one point, x_{k-1}, in the interval of uncertainty, for which f is known.

3. REDUCTION OF THE INTERVAL OF UNCERTAINTY

In this section, we propose and analyze alternatives for selecting the increment multipliers α_k. Let $F_0 = F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$, $n = 2, 3, \ldots$, denote the Fibonacci numbers. If either

$$x_{k-1} = x_{k-2} + \frac{F_{N-k}}{F_{N-k+1}}(x_k - x_{k-2}) \tag{2}$$

or

$$x_{k-1} = x_k - \frac{F_{N-k}}{F_{N-k+1}}(x_k - x_{k-2}) \tag{3}$$

then x_{k-1} is one of the two first evaluations in a Fibonacci search with $N - k + 1$ evaluations, on the interval bounded by x_{k-2} and x_k. In this case, $x_{k+1}, \ldots, x_N$ can be chosen as the next points in this Fibonacci search. This choice guarantees the smallest final interval of uncertainty among all other methods requiring $N - k$ additional evaluations.

If both (2) and (3) do not hold, the next points can be chosen according to Witzgall's algorithm [10]. This algorithm guarantees the smallest final interval of uncertainty in a fixed number of function evaluations when, for some reason, the first evaluation took place at some argument other than the two optimal ones.

We now show how to choose the increment multipliers α_i, $i = 3, \ldots, N - 1$, so that equations (2) or (3), according to our preference, will hold when the bracketing algorithm terminates after $k < N$ evaluations.

Equation (2) implies that

$$x_{k-1} - x_{k-2} = \frac{F_{N-k}}{F_{N-k+1}}[(x_k - x_{k-1}) + (x_{k-1} - x_{k-2})]$$

or

$$\frac{F_{N-k+1}}{F_{N-k}} = \frac{x_k - x_{k-1}}{x_{k-1} - x_{k-2}} + 1 = \alpha_k + 1.$$

Denote the value of α_k which satisfies the above equation by $\alpha_k^{(1)}$, then

$$\alpha_k^{(1)} = \frac{F_{N-k+1}}{F_{N-k}} - 1 = \frac{F_{N-k-1}}{F_{N-k}} \le 1. \tag{4}$$

Equation (2) holds for $k < N$ if and only if $\alpha_k = \alpha_k^{(1)}$.

Similarly, equation (3) implies that

$$x_k - x_{k-1} = \frac{F_{N-k}}{F_{N-k+1}}[(x_k - x_{k-1}) + (x_{k-1} - x_{k-2})]$$

or

$$\frac{F_{N-k+1}}{F_{N-k}} = 1 + \frac{x_{k-1} - x_{k-2}}{x_k - x_{k-1}} = 1 + \frac{1}{\alpha_k}.$$

Denote the value of α_k which satisfies this equation by $\alpha_k^{(2)}$, then

$$\alpha_k^{(2)} = \frac{1}{\alpha_k^{(1)}} = \frac{F_{N-k}}{F_{N-k-1}} \ge 1. \tag{5}$$

Equation (3) holds for $k < N$ if and only if $\alpha_k = \alpha_k^{(2)}$.

Let $d_k = |x_k - x_{k-1}|$, $k = 1, \ldots, N$, denote the search increments, then

$$\begin{aligned} &d_2 = |x_2 - x_1| \quad \text{and} \\ &d_k = \alpha_k d_{k-1} = \alpha_k \cdot \alpha_{k-1} \cdot \cdots \cdot \alpha_3 |x_2 - x_1|, \; k = 3, \ldots, N. \end{aligned} \tag{6}$$

Denote the search increments by $d_k^{(1)}$ and $d_k^{(2)}$ when $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$ are chosen, respectively, for $k = 2, \ldots, N - 1$. Then equations (4) and (6) yield

$$d_k^{(1)} = \frac{F_{N-k-1}}{F_{N-k}} \cdot \frac{F_{N-k}}{F_{N-k+1}} \cdot \cdots \cdot \frac{F_{N-4}}{F_{N-3}} \cdot |x_2 - x_1| = \frac{F_{N-k-1}}{F_{N-3}}|x_2 - x_1|,$$

$k = 2, \ldots, N - 1$.

If the bracketing algorithm terminates after $k < N$ evaluations, then the maximum is located in an interval of length

$$|x_k - x_{k-2}| = d_k^{(1)} + d_{k-1}^{(1)} = \frac{F_{N-k-1} + F_{N-k}}{F_{N-3}}|x_2 - x_1| = \frac{F_{N-k+1}}{F_{N-3}}|x_2 - x_1|.$$

This interval is further searched by a Fibonacci search with $N - k + 1$ evaluations (including the one in x_{k-1}) which reduces its length by a factor $(F_{N-k+1})^{-1}$. Consequently, the length of the final interval is

$$\frac{|x_2 - x_1|}{F_{N-3}},$$

independent of k. This length is satisfactorily small in comparison with

$$\frac{|x_2 - x_1|}{F_{N-2}},$$

which can be achieved by $N - 2$ evaluations if f is known to be unimodal with a maximum between x_1 and x_2.

Suppose, however, that the bracketing algorithm terminates after N evaluations without bracketing a local maximum. The total size of the searched interval is

$$\sum_{k=2}^{N-1} d_k^{(1)} = \frac{|x_2 - x_1|}{F_{N-3}}(F_{N-3} + F_{N-4} + \cdots + F_0) = \frac{F_{N-1}}{F_{N-3}}|x_2 - x_1|$$
$$= \left(1 + \frac{F_{N-2}}{F_{N-3}}\right)|x_2 - x_1| \le 3|x_2 - x_1|.$$

In fact, when N is large, this sum approaches $(1 + \tau)|x_2 - x_1|$ where $\tau \cong 1.618$ satisfies $\tau^2 = 1 + \tau$. The cost of obtaining a small final interval in case of success is in searching a relatively small interval and thus increasing the chances that the bracketing algorithm will fail.

This default can be overcome by using $\alpha_k^{(2)}$ rather than $\alpha_k^{(1)}$. In this case,

$$d_k^{(2)} = \frac{F_{N-3}}{F_{N-k-1}} \quad \frac{F_{N-k+1}}{F_{N-k}} \cdot \cdots \cdot \frac{F_{N-3}}{F_{N-4}}|x_2 - x_1| = \frac{F_{N-3}}{F_{N-k-1}}|x_2 - x_1|,$$
$$k = 2, \ldots, N - 1.$$

The sequence $d_k^{(2)}$ increases with k so that a larger interval is scanned, and it is less likely that the bracketing algorithm will fail. In practice, some of the last increments $d_k^{(2)}$ may be replaced by smaller increments, possibly by $d_k^{(1)}$.

4. SUMMARY

We suggest a two-stage search procedure for maximizing functions by a fixed number of evaluations. The first stage is a quite standard bracketing subroutine and the second is either the regular Fibonacci search or the modified method of Witzgall. During the first stage, the kth evaluation is at the point x_k calculated from $x_k = x_{k-1} + \alpha_k(x_{k-1} - x_{k-2})$. We suggest three alternatives:

A. Let $\alpha_k = \alpha_k^{(1)} \le 1$. In case of success, proceed by Fibonacci search to obtain a small final interval.

B. Let $\alpha_k = \alpha_k^{(2)} > 1$. In case of success, proceed by Fibonacci search.

The chances for success are better than in case A, but the final interval is longer.

C. Let $\alpha_k > 0$ be arbitrary and proceed by Witzgall's method.

We note that different alternatives may be chosen for different values of k.

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2,3 SEQUENCE AS BINARY MIXTURE

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The integer sequence formed by multiplying integral powers of the numbers 2 and 3 can be viewed as a binary sequence. The numbers 2 and 3 are the component factors of this binary. This paper explores the combination of these components to form the properties of the integers in the binary. Properties considered are: value, ordinality (position in the sequence), and exponents of the factors of each integer in the binary sequence.

Questions related to the properties of integer sequences with irregular nth differences are notoriously hard to answer [1]. The integers in the 2,3 sequence produce irregular nth differences. These integers can be related to the graphs constructed in the study of 2,3 trees [2, 3]. It is shown in this paper that the ordinality property of the integers in the 2,3 sequence can be derived from the irrational number log 3/log 2. This number also finds application in the derivation of a discontinuous spatial pattern found in the study of fractal dimension [4].

In Table 1, the first fifty-one numbers in the 2,3 sequence are listed according to their ordinality with respect to value. Since the 2,3 sequence consists of numbers which are integral multiples of the factors 2 and 3, it is convenient to plot the information in Table 1 in the form of a two-dimensional lattice, as shown in Figure 1. In this figure, the horizontal axis represents integral powers of 2 and the vertical axis represents integral powers of 3. The ordinality of each number is printed next to its corresponding lattice point. For example, the number $2592 = 2^5 3^4$ and $\text{Ord}(2^5 3^4) = 50$; therefore, at the coordinates 2^5, 3^4, the number "50" is printed.

TABLE 1. Value, Ordinality, and Factors of the First Fifty-one Numbers in the 2,3 Sequence

Value	Ordinality	Factors	Value	Ordinality	Factors
1	0	$2^0 3^0$	243	26	$2^0 3^5$
2	1	$2^1 3^0$	256	27	$2^8 3^0$
3	2	$2^0 3^1$	288	28	$2^5 3^2$
4	3	$2^2 3^0$	324	29	$2^2 3^4$
6	4	$2^1 3^1$	384	30	$2^7 3^1$
8	5	$2^3 3^0$	432	31	$2^4 3^3$
9	6	$2^0 3^2$	486	32	$2^1 3^5$
12	7	$2^2 3^1$	512	33	$2^9 3^0$
16	8	$2^4 3^0$	576	34	$2^6 3^2$
18	9	$2^1 3^2$	648	35	$2^3 3^4$
24	10	$2^3 3^1$	729	36	$2^0 3^6$
27	11	$2^0 3^3$	768	37	$2^8 3^1$
32	12	$2^5 3^0$	864	38	$2^5 3^3$
36	13	$2^2 3^2$	972	39	$2^2 3^5$
48	14	$2^4 3^1$	1024	40	$2^{10} 3^0$
54	15	$2^1 3^3$	1152	41	$2^7 3^2$
64	16	$2^6 3^0$	1296	42	$2^4 3^4$
72	17	$2^3 3^2$	1458	43	$2^1 3^6$
81	18	$2^0 3^4$	1536	44	$2^9 3^1$
96	19	$2^5 3^1$	1728	45	$2^6 3^3$
108	20	$2^2 3^3$	1944	46	$2^3 3^5$
128	21	$1^7 3^0$	2048	47	$2^{11} 3^0$
144	22	$2^4 3^2$	2187	48	$2^0 3^7$
162	23	$2^1 3^4$	2304	49	$2^8 3^2$
192	24	$2^6 3^1$	2592	50	$2^5 3^4$
216	25	$2^3 3^3$			

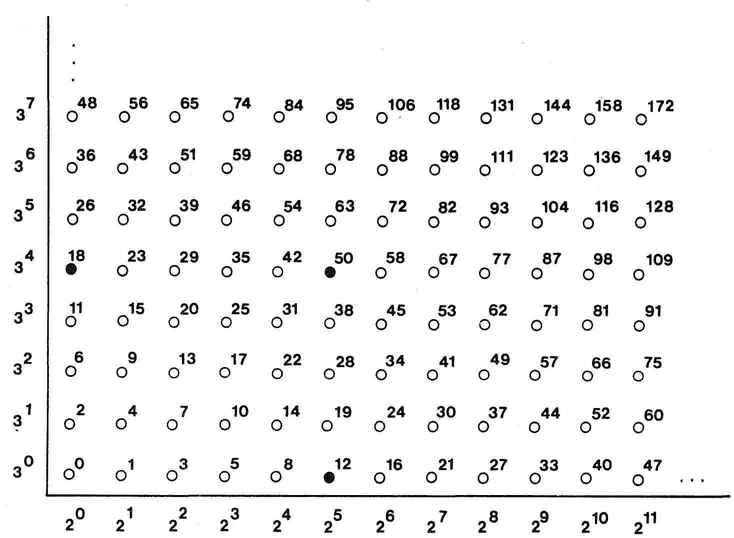

FIGURE 1

We shall now develop a theorem that will condense the information in Figure 1.

Theorem 1: $\mathrm{Ord}(2^a3^b) = ab + \mathrm{Ord}(2^a3^0) + \mathrm{Ord}(2^03^b)$.

This theorem states that the ordinality of any point in the 2,3 lattice can be determined from the exponents of the coordinates of the point, and a knowledge of the ordinalities of the projections of the point onto the horizontal and vertical baselines. For example, in the case of the number 2^53^4, this theorem takes the form

$$\begin{array}{ccccccc} \mathrm{Ord}(2^53^4) & = & (5)(4) & + & \mathrm{Ord}(2^53^0) & + & \mathrm{Ord}(2^03^4). \\ 50 & = & 20 & + & 12 & + & 18 \end{array}$$

Point 50 and its projections onto the horizontal and vertical baselines (i.e., points 12 and 18, respectively) can be seen in Figure 1 as the blacked-in points.

Since an ordinality of 50 means there are fifty points of lower value, and hence, lower ordinality in the lattice, it will be useful to examine in detail the locations of these points. In Figure 2, the three polygons enclose all the points with ordinalities less than 50.

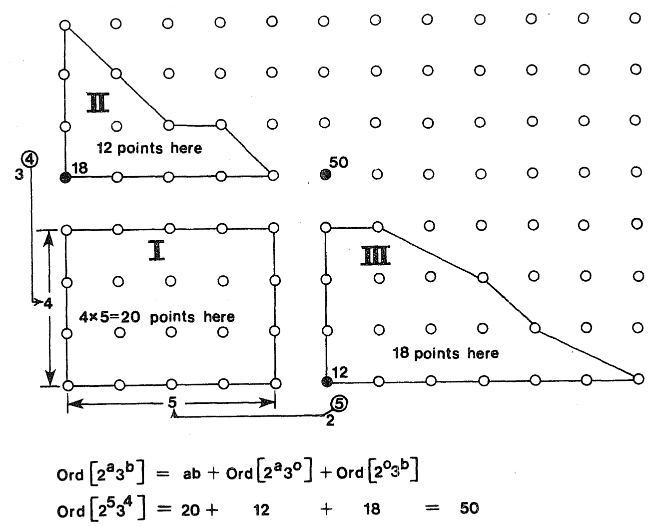

FIGURE 2

Polygon:

I. Those points with $a < 5$ and $b < 4$ (since both a and b are smaller in these points than in point 50, the ordinalities of these points must be less than 50).

II. Those points with ordinalities less than 50, with $a < 5$ and $b \geq 4$.

III. Those points with ordinalities less than 50, with $b < 4$ and $a \geq 5$.

Since ordinality is determined with respect to value, the fifty points in polygons I, II, and III must represent numbers whose values are less than 2^53^4.

The reason that the ordinality of point 12 is exactly equal to the number of lattice points in polygon II can be seen from Figures 3 and 4, with the help of the following discussion. By the definition of "ordinality 12" and the fact that point 12 lies on the horizontal baseline, there must be twelve points of lower value west and northwest of point 12 (since there are no points south of the horizontal baseline, and all points north, northeast, or east are larger). But the relative values of all points in the 2,3 lattice are related to each other according to relative position. For example, take any lattice point, the point directly above it is three times greater in value, the point directly below it is one-third as great in value, the point directly to the right is twice as large in value, and the point directly to the left is half as large in value. If we normalize the *value* of point 12 to the relative value 1, the relative value of all points west and northwest that are lower in value can be seen in Figure 4. This relative value relationship holds for the points west and northwest of point 12 in exactly the same way that it holds for the points west and northwest of point 50, since the relative values of all points are related to each other according to their relative position to each other. Thus, the ordinality of point 12 is identical to the number of points in polygon II and the ordinality of point 18 is identical to the number of points in polygon III (this can be seen with the help of Figures 5 and 6).

To the west and northwest of point 12 there are twelve points of lower value. And to the west and northwest of point 50 there are twelve points of lower value.

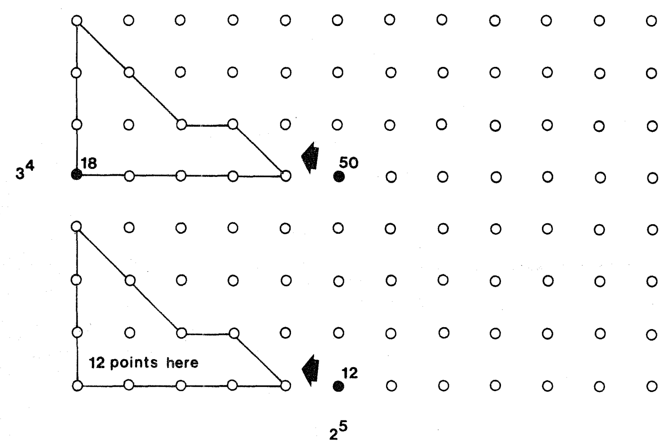

FIGURE 3

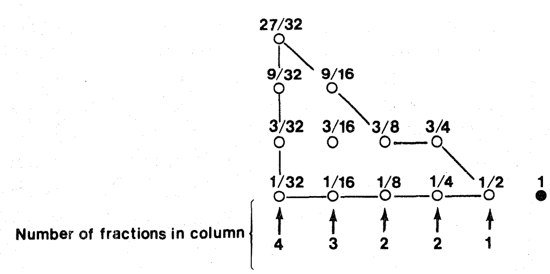

FIGURE 4

To the south and southeast of point 18 there are eighteen points of lower <u>value</u>. And to the south and southeast of point 50 there are eighteen points of lower <u>value</u>.

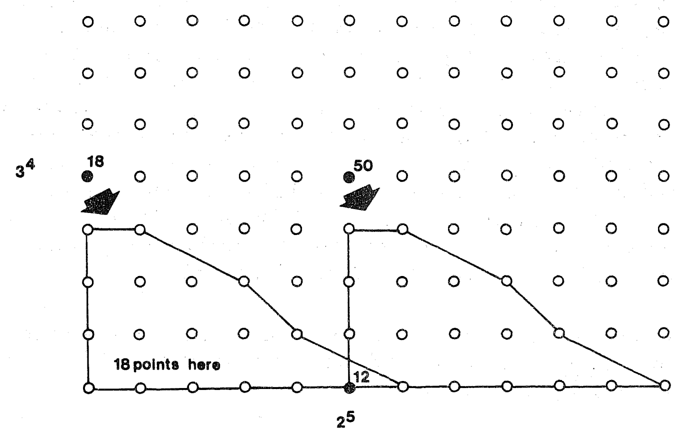

FIGURE 5

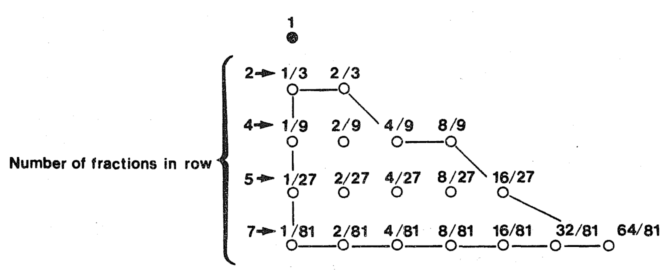

FIGURE 6

If the baseline ordinalities could be computed without recourse to any knowledge of non-baseline ordinalities, a considerable computational effort could be saved. A theorem that will allow us to compute baseline ordinalities directly will now be developed. However, before this new theorem is presented, it will be necessary to expand our nomenclature.

Up to this point, we have been concerned with only one sequence, the 2,3 sequence. All ordinalities were of 2,3 sequence numbers with respect to the 2,3 sequence. However, it is possible to conceive of ordinalities (with respect to the 2,3 sequence) of numbers that are not in this sequence. Take the number 5 as an example. In Table 1, we see that the 2,3 sequence skips from value 4 to value 6. The question "What is the ordinality of 5 with respect to the 2,3 sequence?" is written as: $\text{Ord}(5)_{2,3} = ?$ Please note that the subscripts 2, 3 are written outside of the parentheses, whereas when we previously wrote $\text{Ord}(2^5 3^4)$ there were no subscripts. We could have written $\text{Ord}(2^5 3^4)_{2,3}$ but in order to make the notation more compact, the reference sequence will be specified only when it is different from the enclosed factors or when an ambiguity exists. The convention is also adopted that when the ordinality of a number that is not in a sequence is to be determined with respect to the sequence, the ordinality of the next highest number in the sequence (with respect to the number whose ordinality is to be determined) is the ordinality chosen. For example,

$\text{Ord}(5)_{2,3}$	=	$\text{Ord}(6)_{2,3}$	=	$\text{Ord}(2^1 3^1) = 4$
The ordinality of 5 with respect to the 2,3 sequence.		The next highest number in the 2,3 sequence is 6. That is, 5 "rounds up" to 6 in the 2,3 sequence.		And the ordinality of 6 in the 2,3 sequence is 4, as found in Table 1.

But, $\text{Ord}(4)_{2,3} = \text{Ord}(2^2 3^0) = 3$.

No round up, since the number 4 is found in the 2,3 sequence.

Instead of rounding up in the binary 2,3 sequence, as the example above illustrates, we shall be concerned with rounding up between the two unary sequences: the 2 sequence and the 3 sequence. Thus, from Table 2, we learn that

$\text{Ord}(2^0)_3 = 0$, $\text{Ord}(2^1)_3 = 1$, $\text{Ord}(2^2)_3 = 2$, $\text{Ord}(2^3)_3 = 2$, $\text{Ord}(2^4)_3 = 3$, $\text{Ord}(2^5)_3 = 4$, $\text{Ord}(3^0)_2 = 0$, $\text{Ord}(3^1)_2 = 2$, $\text{Ord}(3^2)_2 = 4$, $\text{Ord}(3^3)_2 = 5$, $\text{Ord}(3^4)_2 = 7$.

TABLE 2

2 Sequence			3 Sequence		
Value	Ordinality	Factors	Value	Ordinality	Factors
1	0	2^0	1	0	3^0
2	1	2^1			
4	2	2^2	3	1	3^1
8	3	2^3			
16	4	2^4	9	2	3^2
32	5	2^5	27	3	3^3
64	6	2^6			
128	7	2^7	81	4	3^4

With this nomenclature in mind, we can proceed to the next theorem.

<u>*Theorem 2:*</u> $\text{Ord}(2^a 3^0) = \sum_{k=0}^{a} \text{Ord}(2^k)_3$.

This theorem states that the ordinality of any point on the horizontal baseline of the 2,3 lattice can be determined from a knowledge of the ordinality of terms in the 3 sequence. And since the ordinality of any term in the 3 sequence is simply its exponent (as can be seen from Table 2), the determination of baseline ordinalities is straightforward.

For example, in the case of the number $2^5 3^0$, this theorem takes the form

$$\mathrm{Ord}(2^5 3^0) = \sum_{k=0}^{5} \mathrm{Ord}(2^k)_3.$$

$$12 = 0 + 1 + 2 + 2 + 3 + 4$$

Table 3 should help clarify this result.

TABLE 3

$k =$	$2^k =$		"Rounded Up" to the Next Highest Number in the 3 Sequence	$\mathrm{Ord}(2^k)_3 =$
0	1	$\to$	$1 = 3^0$	0
1	2	$\to$	$3 = 3^1$	1
2	4	$\to$	$9 = 3^2$	2
3	8	$\to$	$9 = 3^2$	2
4	16	$\to$	$27 = 3^3$	3
5	32	$\to$	$81 = 3^4$	4
			Total =	12

The origin of this result can also be seen in Figure 4. If we list the number of fractions in each *column* to the left of the blacked-in point, we obtain (going right to left), 1, 2, 2, 3, 4. Since each fraction in these columns is less than one and consists of a numerator that is a power of 3 and a denominator that is a power of 2, the question "What is the highest power of 3 in the numerator, for a given power of 2 in the denominator, consistent with a fraction less than one?" can be seen to be related to the question

$$\mathrm{Ord}(2^k)_3 = ?$$

For example, let $k = 5$, then, as previously developed, $\mathrm{Ord}(2^5)_3 = 4$. But the highest power of 3 in the numerator consistent with 32 in the denominator, and a fraction whose overall value is less than one, is 3. That is,

$$3^4/2^5 > 1 > 3^3/2^5, \text{ or } 3^4 > 2 > 3^3.$$

Counting 27/32 and the three fractions beneath it in the leftmost column of Figure 4 gives

$$1 + 3 = 4 \text{ fractions: } 27/32,\ 9/32,\ 3/32,\ 1/32.$$

Thus we see that a numerator power of 3 gives four fractions, since the fraction with the numerator 3^0 must be counted. Therefore, "rounding up" counts this zero exponent term.

The next theorem applies to the vertical baseline.

<u>*Theorem 3:*</u> $\mathrm{Ord}(2^0 3^b) = \sum_{k=0}^{b} \mathrm{Ord}(3^k)_2.$

This theorem states that the ordinality of any point on the vertical baseline of the 2,3 lattice can be determined from a knowledge of the ordinality of terms in the 2 sequence. And since the ordinality of any term in the 2 sequence is simply its exponent (as can be seen from Table 2), the determination of these ordinalities is straightforward.

For example, in the case of the number $2^0 3^4$, this theorem takes the form

$$\mathrm{Ord}(2^0 3^4) = \sum_{k=0}^{4} \mathrm{Ord}(3^k)_2.$$

$$18 = 0 + 2 + 4 + 5 + 7$$

Table 4 should help clarify this result.

TABLE 4

k =	3^k =		"Rounded Up" to the Next Highest Number in the 2 Sequence	$\mathrm{Ord}(3^k)_2$ =
0	1	→	$1 = 2^0$	0
1	3	→	$4 = 2^2$	2
2	9	→	$16 = 2^4$	4
3	27	→	$32 = 2^5$	5
4	81	→	$128 = 2^7$	7
			Total =	18

The origin of this result can be seen in Figure 6. If we list the number of fractions in each *row* beneath the blacked-in point in Figure 6, we obtain (from top to bottom) 2, 4, 5, 7. Since each fraction in these rows is less than one and consists of a numerator that is a power of 2 and a denominator that is a power of 3, the question "What is the highest power of 2 in the numerator, for a given power of 3 in the denominator, consistent with a fraction less than one?" can be seen to be related to the question

$$\mathrm{Ord}(3^k)_2 = ?$$

For example, let $k = 4$, then, as previously developed, $\mathrm{Ord}(3^4)_2 = 7$. But the highest power of 2 in the numerator consistent with 81 in the denominator, and a fraction whose overall value is less than one, is 6. That is,

$$2^7/3^4 > 1 > 2^6/3^4, \text{ or } 2^7 > 3^4 > 2^6.$$

Counting 64/81 and the six fractions to its left, in the southmost row of Figure 6 gives

$$1 + 6 = 7 \text{ fractions: } 64/81,\ 32/81,\ 16/81,\ 8/81,\ 4/81,\ 2/81,\ 1/81.$$

Thus we see that a numerator power of 6 gives seven fractions, since the fraction with numerator 2^0 must be counted. Therefore "rounding up" counts this zero exponent term.

The combination of Theorems 1-3 gives Theorem 4.

Theorem 4: $\mathrm{Ord}(2^a 3^b) = ab + \sum_{k=0}^{a} \mathrm{Ord}(2^k)_3 + \sum_{k=0}^{b} \mathrm{Ord}(3^k)_2.$

This is the mathematical equivalent of describing a binary mixture in terms of its pure components.

Evaluating $\mathrm{Ord}(2^5)_3$ has been shown to be equivalent to finding the integral power of 3 (i.e., 3^k) such that

$$3^{k+1} > 2^5 > 3^k.$$

The ordinality was then shown to be one more than k (i.e., ordinality $= k + 1$), since the fraction with zero power in the numerator had to be counted. This problem can be simplified to a linear problem if the logarithms of the terms involved are used. For example, take the above problem. If $2^5 > 3^k$, then

$$5 \log 2 > k \log 3 \quad \text{or} \quad k < 5 \log 2/\log 3.$$

The term on the right of the last inequality must have an integral and a non-integral part (since log 2 and log 3 are independent irrationals). To five places, $5 \log 2/\log 3 = 3.15465$. Since 3^{k+1} was constrained to be greater than 2^5, we can write

$$(k + 1)\log 3 > 5 \log 2.$$

Also, since k was specified to be an integer, we evaluate k as the integral part of $5 \log 2/\log 3$. Therefore, 1 + integral part of $5 \log 2/\log 3$ is the same as the round up of $5 \log 2/\log 3$ to the next positive integer. Since this is also $k + 1$, and $k + 1$ is equal to the ordinality, we can write Theorem 5.

Theorem 5: $\operatorname{Ord}(2^k)_3 = \operatorname{Ord}(k \log 2/\log 3)_1$.

The subscript 1 in Theorem 5 represents a round up process that rounds up to the next highest integer (i.e., We call the sequence of positive integers the 1 sequence. In this sequence, the ordinality of an integer is defined to be its value).

Evaluating $\operatorname{Ord}(3^4)_2$ has been shown to be equivalent to finding the integral power of 2 (i.e., 2^k) such that

$$2^{k+1} > 3^4 > 2^k.$$

The ordinality was then shown to be one more than k (i.e., ordinality $= k + 1$), since the fraction with zero power in the numerator had to be counted. This problem can be simplified to a linear problem if the logarithms of the terms involved are used. For example, take the above problem. If $3^4 > 2^k$, then

$$4 \log 3 > k \log 2 \quad \text{or} \quad k < 4 \log 3/\log 2.$$

The term on the right of the last inequality must have an integral and non-integral part (since log 2 and log 3 are independent irrationals). To five places, $4 \log 3/\log 2 = 6.33985$. Since 2^{k+1} was constrained to be greater than 3^4, we can write

$$(k + 1)\log 2 > 4 \log 3.$$

Also, since k was specified to be an integer, we evaluate k as the integral part of $4 \log 3/\log 2$. Therefore, 1 + integral part of $4 \log 3/\log 2$ is the same as the round up of $4 \log 3/\log 2$ to the next positive integer. Since this is also $k + 1$, and $k + 1$ is equal to the ordinality, we can write Theorem 6.

Theorem 6: $\operatorname{Ord}(3^k)_2 = \operatorname{Ord}(k \log 3/\log 2)_1$.

The combination of Theorems 4-6 gives Theorem 7.

Theorem 7: $\operatorname{Ord}(2^a 3^b) = ab + \sum_{k=0}^{a} \operatorname{Ord}\left(\frac{k}{\log 3/\log 2}\right)_1 + \sum_{k=0}^{b} \operatorname{Ord}\left(\frac{k}{\log 2/\log 3}\right)_1.$

The lattice for the 2,3 sequence is not unique to numbers of the form $2^a 3^b$, a, b integers, $a \geq 0$, $b \geq 0$. Instead, it represents the ordinality sequence of all numbers of the form

$$(2^x)^a (3^x)^b, \; a, b \text{ integers}, \; a \geq 0, \; b \geq 0, \; x > 0.$$

$2^a 3^b$ is seen as the special case in which $x = 1$. However, the right side of

Theorem 7 applies to the ordinality of any number in the $2^x, 3^x$ sequence, since

$$\operatorname{Ord}((2^x)^a (3^x)^b)_{2^x,\,3^x} = ab + \sum_{k=0}^{a} \operatorname{Ord}\left(\frac{k}{x \log 3/x \log 2}\right)_1 + \sum_{k=0}^{b} \operatorname{Ord}\left(\frac{k}{x \log 2/x \log 3}\right)_1 .$$

And since the x's cancel, we obtain the terms on the right side of the equals sign in Theorem 7.

Therefore, all sequences with component terms of the form $(2^x)^a(3^x)^b$ have in common the fact that their lattice representations are identical. If a lattice does not uniquely specify a sequence, is there anything that it does specify uniquely? The answer lies in Theorem 7. From this theorem, we see that the number log 3/log 2 (and its reciprocal) are uniquely specified by the lattice representation of the $2^x, 3^x$ sequence. Therefore, to generate the lattice associated with any real number N, we generalize the results of Theorem 7, to give

Theorem 8: $\operatorname{Ord}(a, b) = ab + \sum_{k=0}^{a} \operatorname{Ord}\left(\frac{k}{N}\right)_1 + \sum_{k=0}^{b} \operatorname{Ord}\left(\frac{k}{1/N}\right)_1 .$

In Theorem 8, $\operatorname{Ord}(a, b)$ is defined as the ordinality of the point at coordinates a, b. Since Theorem 8 is derived from Theorem 1, we can combine the two theorems to obtain

Theorem 9: $\operatorname{Ord}(a, b) = ab + \operatorname{Ord}(a, 0) + \operatorname{Ord}(0, b)$.

ACKNOWLEDGMENT

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IDENTITIES FOR CERTAIN PARTITION FUNCTIONS AND THEIR DIFFERENCES*

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1. INTRODUCTION

If $i \geq 0$ and $n \geq 1$, let $q_i^e(n)$ $(q_i^o(n))$ denote the number of partitions of into an even (odd) number of parts, where each part occurs at most i times; $q_i^e(0) = 1$ $(q_i^o(0) = 0)$. If $i \geq 0$ and $n \geq 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

We note that for $i \geq 0$ and $n \geq 0$, $q_i^e(n) + q_i^o(n) = p_i(n)$, where $p_i(n)$ denotes the number of partitions of n where each part occurs at most i times.

The purpose of this paper is to give identities for $q_i^e(n)$, $q_i^o(n)$, and $\Delta_i(n)$. The function $\Delta_i(n)$ has been studied by Hickerson [3] and [4], and by Alder and Muwafi [1]. They have given formulas to determine $\Delta_i(n)$, for $i > 1$, in terms of certain restricted partition functions. The case $i = 1$ is a well known result due to Euler [2, p. 285]. Another result of this type, the Sylvester-Euler theorem [5, p. 264], states

$$\Delta(n) = (-1)^n Q(n), \tag{1}$$

where $\Delta(n)$ is the difference function with the restriction on the number of times a part may occur removed, and $Q(n)$ is the number of partitions of n into distinct odd parts.

Here we first obtain identities for $\Delta_i(n)$, some of which are recursive. We then find several identities for $q_i^e(n)$ and $q_i^o(n)$ which also give us some new results for $\Delta_i(n)$. Our identities not only demonstrate relationships between these functions and other partition functions, but many of them are also useful computationally.

We will make use of the following partition functions in addition to those already defined. For $n \geq 1$:

(i) $p(n)$ $(q(n))$ denotes the number of (distinct) partitions of n.

(ii) $p_{a_1, \ldots, a_r; b}(n)$ $(q_{a_1, \ldots, a_r; b}(n))$ denotes the number of (distinct) partitions of n into parts $\equiv a_j \pmod{b}$, $1 \leq j \leq r$.

(iii) $Q_k(n)$ denotes the number of partitions of n into distinct odd multiples of k.

(iv) $q^i(n)$; $p_{0;2}^i(n)$ denote, respectively, the number of partitions of n into distinct parts and even parts, where no part is divisible by i.

By convention, when $n = 0$, each of these partition functions assumes the value 1.

We let $[x]$ denote the greatest integer function and $\sum\limits_r$ denote the sum over all nonnegative r such that the summands are defined. Finally, we let m be an integer ≥ 1 unless otherwise specified.

**The material in this paper is part of the author's doctoral dissertation, written under the direction of Professor L. M. Chawla at Kansas State University.*

2. IDENTITIES FOR THE DIFFERENCE FUNCTION

We will base our proofs in this section on the generating function of Δ_i, which is given by

$$\sum_{n=0}^{\infty}\Delta_i(n)x^n = \prod_{j=1}^{\infty}\frac{1+(-1)^i x^{(i+1)j}}{1+x^j} \tag{2}$$

Theorem 1: (i) $\Delta_{2m}(n) = \sum_{r=0}^{n}\Delta(r)q_{0;\,2m+1}(n-r)$,

(ii) $\Delta_{2m-1}(n) = \sum_{r}(-1)^r\Delta(n-(3r^2 \pm r)m)$.

Proof: Since

$$\sum_{n=0}^{\infty}\Delta(n)x^n = \prod_{j=1}^{\infty}\frac{1}{1+x^j} \tag{3}$$

and

$$\sum_{n=0}^{\infty}q_{0;\,a}(n)x^n = \prod_{j=1}^{\infty}(1+x^{aj}), \tag{4}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_{2m}(n)x^n &= \prod_{j=1}^{\infty}\frac{1+x^{(2m+1)j}}{1+x^j} = \prod_{j=1}^{\infty}\frac{1}{1+x^j}\prod_{j=1}^{\infty}(1+x^{(2m+1)j})\\ &= \sum_{n=0}^{\infty}\Delta(n)x^n\sum_{n=0}^{\infty}q_{0;\,2m+1}(n)x^n\\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\Delta(r)q_{0;\,2m+1}(n-r)\right)x^n,\end{aligned}$$

and equating coefficients proves (i). On the other hand,

$$\sum_{n=0}^{\infty}\Delta_{2m-1}(n)x^n = \prod_{j=1}^{\infty}\frac{1-x^{2mj}}{1+x^j} = \prod_{j=1}^{\infty}\frac{1}{1+x^j}\prod_{j=1}^{\infty}(1-x^{2mj}).$$

Now Euler's Pentagonal Number Theorem [2, p. 284] states

$$\prod_{j=1}^{\infty}(1-x^{aj}) = \sum_{r=-\infty}^{\infty}(-1)^r x^{\frac{1}{2}(3r^2+r)a}. \tag{5}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_{2m-1}(n)x^n &= \sum_{n=0}^{\infty}\Delta(n)x^n\sum_{r=0}^{\infty}(-1)^r x^{(3r^2\pm r)m}\\ &= \sum_{n=0}^{\infty}\left(\sum_{r}(-1)^r\Delta(n-(3r^2\pm r)m)\right)x^n,\end{aligned}$$

and equating coefficients gives (ii).

Using the Sylvester-Euler identity (1) for Δ in Theorem 1 yields the following result.

Corollary 1: (i) $\Delta_{2m}(n) = \sum_{r=0}^{n} (-1)^r Q(r) q_{0;\,2m+1}(n - r)$,

(ii) $\Delta_{2m-1}(n) = (-1)^n \sum_{r} (-1)^r Q(n - (3r^2 \pm r)m)$.

Theorem 2: (i) $\Delta_{4m-1}(n) = \sum_{r=0}^{n} \Delta_{2m-1}(r) q_{0;\,2m}(n - r)$,

(ii) $\Delta_{4m+1}(n) = \sum_{r} (-1)^r \Delta_{2m}(n - \tfrac{1}{2}(3r^2 \pm r)(2m + 1))$,

where (ii) also holds for $m = 0$.

Proof: From (2) we have

$$\sum_{n=0}^{\infty} \Delta_{4m-1}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{4mj}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 + x^j} \prod_{j=1}^{\infty} (1 + x^{2mj}).$$

Thus, applying (2) and (4),

$$\sum_{n=0}^{\infty} \Delta_{4m-1}(n)x^n = \sum_{n=0}^{\infty} \Delta_{2m-1}(n)x^n \sum_{n=0}^{\infty} q_{0;\,2m}(n)x^n,$$

and (i) follows. Now

$$\sum_{n=0}^{\infty} \Delta_{4m+1}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(4m+2)j}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 + x^j} \prod_{j=1}^{\infty} (1 - x^{(2m+1)j})$$
$$= \sum_{n=0}^{\infty} \Delta_{2m}(n)x^n \sum_{r=0}^{n} (-1)^r x^{\frac{1}{2}(3r^2 \pm r)(2m+1)}$$

from (2) and (5), and (ii) follows immediately.

Theorem 3:

$$\sum_{r=0}^{n} \Delta_i(n)q(n - r) = \begin{cases} q_{0;\,2m+1}(n) \quad \text{for } i = 2m, \\ \left.\begin{cases} (-1)^r \ \text{ if } n = (3r^2 \pm r)m \text{ for } r = 0, 1, 2, \ldots, \\ 0 \quad \text{otherwise,} \end{cases}\right\} \text{for } i = 2m - 1. \end{cases}$$

Proof: Using (2) and (4) we have

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \sum_{n=0}^{\infty} q(n)x^n = \prod_{j=1}^{\infty} (1 + (-1)^i x^{(i+1)j}).$$

Thus,

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_i(r)q(n - r)\right)x^n = \begin{cases} \prod_{j=1}^{\infty} (1 + x^{(2m+1)j}) = \sum_{n=0}^{\infty} q_{0;\,2m+1}(n)x^n \ \text{ for } i = 2m, \\ \prod_{j=1}^{\infty} (1 - x^{2mj}) = \sum_{r=0}^{\infty} (-1)^r x^{(3r^2 \pm r)m} \ \text{ for } i = 2m - 1, \end{cases}$$

From (4) and (5). Equating coefficients, the theorem is proved.

Theorem 4:

$$\sum_{r=0}^{n} \Delta_i(r)p_i(n - r) = \begin{cases} p_{0;\,2}^{4m+2}(n) \ \text{ for } i = 2m, \\ \sum_{r} (-1)^r p_{0;\,2}^{2m}(n - (3r^2 \pm r)m) \ \text{ for } i = 2m - 1. \end{cases}$$

Proof: The generating function of p_i is given by

$$\sum_{n=0}^{\infty} p_i(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(i+1)j}}{1 - x^j}, \tag{6}$$

and so using this and (2):

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \sum_{n=0}^{\infty} p_i(n)x^n = \prod_{j=1}^{\infty} \frac{1 + (-1)^i x^{(i+1)j}}{1 - x^{2j}}(1 - x^{(i+1)j}).$$

Now if $i = 2m$,

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_{2m}(r)p_{2m}(n - r)\right)x^n &= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)}}{1 - x^{2j}}(1 - x^{(2m+1)j}) \\ &= \prod_{j=1}^{\infty} \frac{1 - x^{2(2m+1)j}}{1 - x^{2j}} \\ &= \prod_{\substack{j \geq 1 \\ 2m+1 \nmid j}} \frac{1}{1 - x^{2j}} = \sum_{n=0}^{\infty} p_{0;\,2}^{4m+2}(n)x^n.\end{aligned}$$

Likewise, if $i = 2m - 1$,

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_{2m-1}(r)p_{2m-1}(n - r)\right)x^n &= \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^{2j}}(1 - x^{2mj}) \\ &= \sum_{n=0}^{\infty} p_{0;\,2}^{2m}(n)x^n \sum_{r=0}^{\infty} (-1)^r x^{(3r^2 \pm r)m} \\ &= \sum_{n=0}^{\infty} \left(\sum_{r} (-1)^r p_{0;\,2}^{2m}(n - (3r^2 \pm r)m)\right)x^n,\end{aligned}$$

where we use (5) to obtain the second equation. Thus the theorem is proved.

Corollary 2: If n is odd,

$$\sum_{r=0}^{n} \Delta_i(r)p_i(n - r) = 0.$$

Proof: First we note that both $p_{0;\,2}^{4m+2}(n) = 0$ and $p_{0;\,2}^{2m}(n) = 0$ for $n \equiv 1 \pmod 2$, and since $n - (3r^2 \pm r)m \equiv n \pmod 2$ the corollary follows from Theorem 4.

Theorem 5: For $n \geq 1$,

(i) $\displaystyle\sum_{r=0}^{n} \Delta_{2m}(r)q^{2m+1}(n - r) = 0,$

(ii) $\displaystyle\sum_{r=0}^{n} \Delta_{2m-1}(r)p_{0,1,3,\ldots,2m-1;\,2m}(n - r) = 0.$

Proof: From (2) we have

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \prod_{j=1}^{\infty} \frac{1 + x^j}{1 + (-1)^i x^{(i+1)j}} = 1,$$

where

$$\prod_{j=1}^{\infty} \frac{1 + x^j}{1 + x^{(2m+1)j}} = \prod_{\substack{j \geq 1 \\ 2m+1 \nmid j}} (1 + x^j) = \sum_{n=0}^{\infty} q^{2m+1}(n)x^n,$$

and

$$\prod_{j=1}^{\infty} \frac{1 + x^j}{1 - x^{2mj}} = \prod_{j=1}^{\infty} \frac{1}{(1 - x^{2mj})(1 - x^{2j-1})}$$

$$= \prod_{j=0}^{\infty} \frac{1}{(1 - x^{2mj+1})(1 - x^{2mj+3}) \cdots (1 - x^{2mj+(2m-1)})(1 - x^{2mj+2m})}$$

$$= \sum_{n=0}^{\infty} p_{0,1,3,\ldots,2m-1;\,2m}(n)x^n,$$

and so the theorem follows.

3. IDENTITIES FOR THE DEFINING PARTITION FUNCTIONS

We will base the proofs in this section on the generating functions of q_i^{e} and q_i^{o}, which we construct in the following two lemmas.

Lemma 1: (i) $$\sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty}(-1)^{r+1}x^{(2m+1)(r+1)^2} + \sum_{r=0}^{\infty}(-1)^r x^{r^2}\right),$$

(ii) $$\sum_{n=0}^{\infty} q_{2m}^{o}(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty}(-1)^r x^{(2m+1)r^2} - \sum_{r=0}^{\infty}(-1)^r x^{r^2}\right).$$

Proof: First we recall that $p_i(n) = q_i^{e}(n) + q_i^{o}(n)$. Thus, using the definition of $\Delta_i(n)$, we have $2q_i^{e}(n) = p_i(n) + \Delta_i(n)$. Hence

$$2\sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n = \sum_{n=0}^{\infty} p_{2m}(n)x^n + \sum_{n=0}^{\infty} \Delta_{2m}(n)x^n,$$

and so, from (2) and (6), we have

$$2\sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 - x^j} + \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 + x^j}$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 + x^{(2m+1)j}} + \prod_{j=1}^{\infty} \frac{1 - x^j}{1 + x^j}\right).$$

Now

$$\prod_{j=1}^{\infty} \frac{1 - x^{aj}}{1 + x^{aj}} = \sum_{r=-\infty}^{\infty} (-1)^r x^{ar^2},$$

which is a special case of Jacobi's identity [2, p. 283]. Using this result twice yields,

$$2\sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=-\infty}^{\infty} (-1)^r x^{(2m+1)r^2} + \sum_{r=-\infty}^{\infty} (-1)^r x^{r^2}\right)$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(2 + 2\sum_{r=1}^{\infty}(-1)^r x^{(2m+1)r^2} + 2\sum_{r=1}^{\infty}(-1)^r x^{r^2}\right),$$

from which (i) follows immediately. To prove (ii), we note that

$$q_{2m}^{o}(n) = p_{2m}(n) - q_{2m}^{e}(n),$$

and so

$$\sum_{n=0}^{\infty} q_{2m}^{o}(n)x^n = \sum_{n=0}^{\infty} p_{2m}(n)x^n - \sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n$$

$$= \prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 - x^j} - \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^{r+1} x^{(2m+1)(r+1)^2} + \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right)$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 + x^{(2m+1)j}} - \sum_{r=0}^{\infty} (-1)^{r+1} x^{(2m+1)(r+1)^2} - \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right)$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=-\infty}^{\infty} (-1)^r x^{(2m+1)r^2} - \sum_{r=1}^{\infty} (-1)^r x^{(2m+1)r^2} - \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right).$$

Simplifying the right-hand side of this equation yields (ii), and so the lemma is proved.

Using the same method of proof as in Lemma 1, with several minor alterations, proves the following result.

Lemma 2: (i) $$\sum_{n=0}^{\infty} q_{2m-1}^{e}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^r x^{r^2}\right),$$

(ii) $$\sum_{n=0}^{\infty} q_{2m-1}^{o}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^r x^{(r+1)^2}\right).$$

We now give identities for q_i^e and q_i^o, and then combine these results to obtain formulas for Δ_i. First note that in Lemma 1, using (4),

$$\prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j} = \sum_{n=0}^{\infty} q_{0;\,2m+1}(n)x^n \sum_{n=0}^{\infty} p(n)x^n,$$

and in Lemma 2, from (6),

$$\prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j} = \sum_{n=0}^{\infty} p_{2m-1}(n)x^n.$$

Thus, using these two results, the following two theorems follow directly from the lemmas.

Theorem 6:

(i) $$q_{2m}^{e}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^{r+1}\big(p(k - (2m+1)(r+1)^2) - p(k - r^2)\big), \\ \displaystyle\sum_{k=0}^{n} p(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^{r+1}\big(q_{0;\,2m+1}(k - (2m+1)(r+1)^2) - q_{0;\,2m+1}(k - r^2)\big), \end{cases}$$

$$\text{(ii)}\quad q_{2m}^{\circ}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^r (p(k-(2m+1)r^2) - p(k-r^2)), \\ \displaystyle\sum_{k=0}^{n} p(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^r (q_{0;\,2m+1}(k-(2m+1)r^2) \\ \qquad\qquad - q_{0;\,2m+1}(k-r^2)), \end{cases}$$

where $p(n) = 0$ and $q_{0;\,2m+1}(n) = 0$ for $n < 0$.

Theorem 7: (i) $\displaystyle q_{2m-1}^{e}(n) = \sum_{r=0}^{[\sqrt{n}]} (-1)^r p_{2m-1}(n-r^2)$,

(ii) $\displaystyle q_{2m-1}^{\circ}(n) = \sum_{r=1}^{[\sqrt{n}]} (-1)^{r+1} p_{2m-1}(n-r^2)$.

Using the results of Theorems 6 and 7 and the definition of Δ_i proves the following corollary.

Corollary 3:

$$\text{(i)}\quad \Delta_{2m}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k)\left(p(k) + 2\sum_{r=1}^{[\sqrt{k}]} (-1)^r p(k-r^2)\right), \\ \displaystyle\sum_{k=0}^{n} p(n-k)\left(q_{0;\,2m+1}(k) + 2\sum_{r=1}^{[\sqrt{k}]} (-1)^r q_{0;\,2m+1}(k-r^2)\right), \end{cases}$$

$$\text{(ii)}\quad \Delta_{2m-1}(n) = p_{2m-1}(n) + 2\sum_{r=1}^{[\sqrt{n}]} (-1)^r p_{2m-1}(n-r^2).$$

Corollary 4: $\displaystyle \Delta(n) = p(n) + 2\sum_{r=1}^{[\sqrt{n}]} (-1)^r p(n-r^2)$.

Proof: This follows from the results of Theorem 1(i) and Corollary 3(i).

Multiplying both sides of the generating functions in Lemma 1 by

$$\prod_{j=1}^{\infty} (1 - x^j),$$

and using (5), yields the following identities.

Theorem 8:

$$\text{(i)}\quad \sum_k (-1)^k q_{2m}^{e}(n - \tfrac{1}{2}(3k^2 \pm k)) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r+1} (q_{0;\,2m+1}(n-(2m+1)(r+1)^2) - q_{0;\,2m+1}(n-r^2)),$$

$$\text{(ii)}\quad \sum_k (-1)^k q_{2m}^{\circ}(n - \tfrac{1}{2}(3k^2 \pm k)) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r} (q_{0;\,2m+1}(n-(2m+1)r^2) - q_{0;\,2m+1}(n-r^2)),$$

where $q_{0;\,2m+1}(n) = 0$ when $n < 0$.

Corollary 5:

$$\sum_k (-1)^k \Delta_{2m}(n - \tfrac{1}{2}(3k^2 \pm k)) = q_{0;\,2m+1}(n) + 2\sum_{r=1}^{[\sqrt{n}]} (-1)^r q_{0;\,2m+1}(n-r^2).$$

Theorem 9:

$$\text{(i)}\quad \sum_{r=0}^{n} (-1)^{n-r} q_{2m}^{e}(r) Q_{2m+1}(n-r) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r+1} (p(n-(2m+1)(r+1)^2) - p(n-r^2)),$$

$$\text{(ii)}\quad \sum_{r=0}^{n}(-1)^{n-r}q_{2m}^{o}(r)Q_{2m+1}(n-r) = \sum_{r=0}^{[\sqrt{n}]}(-1)^{r}(p(n-(2m+1)r^2)-p(n-r^2)),$$

where $p(n) = 0$ when $n < 0$.

Proof: This follows from Lemma 1 if we multiply the generating functions on both sides by

$$\prod_{j=1}^{\infty}\frac{1}{1+x^{(2m+1)j}} = \prod_{j=1}^{\infty}(1-x^{(2m+1)(2j-1)}) = \sum_{n=0}^{\infty}(-1)^{n}Q_{2m+1}(n)x^{n}.$$

Theorem 10: (i) $\displaystyle\sum_{r=0}^{n}q_{2m-1}^{e}(r)p_{0;\,2m}(n-r) = \sum_{r=0}^{[\sqrt{n}]}(-1)^{r}p(n-r^2),$

(ii) $\displaystyle\sum_{r=0}^{n}q_{2m-1}^{o}(r)p_{0;\,2m}(n-r) = \sum_{r=1}^{[\sqrt{n}]}(-1)^{r+1}p(n-r^2).$

Proof: Here we multiply the generating functions of Lemma 2 by

$$\prod_{j=1}^{\infty}\frac{1}{1-x^{2mj}} = \sum_{n=0}^{\infty}p_{0;\,2}(n)x^{n} \tag{7}$$

on both sides, and the theorem follows.

Corollary 6:

$$\left.\begin{array}{l}\displaystyle\sum_{r=0}^{n}(-1)^{n-r}\Delta_{2m}(r)Q_{2m+1}(n-r)\\ \displaystyle\sum_{r=0}^{n}\Delta_{2m-1}(r)p_{0;\,2m}(n-r)\end{array}\right\} = \Delta(n).$$

Proof: Using the results of Theorems 9 and 10, we have

$$\left.\begin{array}{l}\displaystyle\sum_{r=0}^{n}(-1)^{n-r}\Delta_{2m}(r)Q_{2m+1}(n-r)\\ \displaystyle\sum_{r=0}^{n}\Delta_{2m-1}(r)p_{0;\,2m}(n-r)\end{array}\right\} = p(n) + 2\sum_{r=1}^{[\sqrt{n}]}(-1)^{r}p(n-r^2),$$

and so this result follows from Corollary 4.

Theorem 11:

$$\text{(i)}\quad \sum_{k=0}^{n}p_{0;\,2m}(n-k)\sum_{r}(-1)^{r}q_{2m-1}^{e}(k-\tfrac{1}{2}(3r^2\pm r)) = \begin{cases}(-1)^{n} \text{ if } n = t^2,\\ \quad\text{for } t = 0, 1, 2, \ldots,\\ 0 \text{ otherwise,}\end{cases}$$

$$\text{(ii)}\quad \sum_{k=0}^{n}p_{0;\,2m}(n-k)\sum_{r}(-1)^{r}q_{2m-1}^{o}(k-\tfrac{1}{2}(3r^2\pm r)) = \begin{cases}(-1)^{n} \text{ if } n = t^2,\\ \quad\text{for } t = 1, 2, \ldots,\\ 0 \text{ otherwise.}\end{cases}$$

Proof: These identities follow from Lemma 2 if we multiply both sides of the generating functions by

$$\prod_{j=1}^{\infty}\frac{1-x^{j}}{1-x^{2mj}},$$

and use (5) and (7).

Corollary 7:

$$\sum_{k=0}^{n} p_{0;\,2m}(n-k) \sum_{r} (-1)^r \Delta_{2m-1}(k - \tfrac{1}{2}(3r^2 \pm r)) = \begin{cases} 1 \text{ if } n = 0 \\ (-1)^n 2 \text{ if } n = t^2, \\ \quad \text{for } t = 1, 2, \ldots, \\ 0 \text{ otherwise.} \end{cases}$$

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FIBONACCI AND LUCAS NUMBERS OF THE FORMS $w^2 - 1$, $w^3 \pm 1$

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INTRODUCTION

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. All such numbers of the forms w^2, w^3, $w^2 + 1$ have been determined by J. H. E. Cohn [2], H. London and R. Finkelstein [8], R. Finkelstein [4] and [5], J. C. Lagarias and D. P. Weisser [7], R. Steiner [10], and H. C. Williams [11]. In this article, we find all Fibonacci and Lucas nubmers of the forms $w^2 - 1$, $w^3 \pm 1$.

PRELIMINARIES

(1) $L_n = w^2 \to n = 1$ or 3

(2) $L_n = 2w^2 \to n = 0$ or ± 6

(3) $L_n = w^3 \to n = \pm 1$

(4) $L_n = 2w^3 \to n = 0$

(5) $L_n = 4w^3 \to n = \pm 3$

(6) $L_{-n} = (-1)^n L_n$

(7) $(F_n, F_{n-1}) = (L_n, L_{n-1}) = 1$

(8) $3 | F_n$ iff $4 | n$

(9) $L_{2n} = L_n^2 - 2(-1)^n$

(10) $L_{2n+1} = L_n L_{n+1} - (-1)^n$

(11) If $(x, y) = 1$ and $xy = w^n$, then $x = u^n$, $y = v^n$, with $(u, v) = 1$ and $uv = w$.

(12) $F_{4n \pm 1} = F_{2n \pm 1} L_{2n} - 1$

(13) $F_{4n} = F_{2n-1} L_{2n+1} - 1$

(14) $F_{4n-2} = F_{2n-2} L_{2n} - 1$

(15) $F_{4n\pm1} = F_{2n}L_{2n\pm1} + 1$
(16) $F_{4n} = F_{2n+1}L_{2n-1} + 1$
(17) $F_{4n-2} = F_{2n-2}L_{2n} + 1$
(18) $L_{m+n} = F_{m-1}L_n + F_m L_{n+1}$
(19) The Diophantine equation $y^2 - D = x^3$, with $y \geq 0$, has precisely the solutions: $(-1, 0)$, $(0, 1)$, $(2, 3)$ if $D = 1$; $(1, 2)$ if $D = 3$; $(1, 0)$ if $D = -1$; no solution if $D = -3$.

Remarks: (1) and (2) are Theorems 1 and 2 in [2]. (3) is Theorem 4 in [8], modified by (6). (4) and (5) follow from Theorem 5 in [7]. (6) through (11) are elementary and/or well known. (12) through (17) appear in Theorem 1 of [3]. (18) is a special case of 1.6, p. 62 in [1]. (19) is excerpted from the tables on pp. 74–75 of [6].

THE MAIN THEOREMS

Theorem 1: $(F_m, L_{m\pm n})|L_n$.

Proof: By (6), it suffices to show that $(F_m, L_{m+n})|L_n$. Let $d = (F_m, L_{m+n})$. (18) $\to d|F_{m-1}L_n$; (7) $\to d|L_n$.

Corollary 1: $(F_m, L_{m\pm2}) = 1$ or 3.

Proof: Let $n = 2$ in Theorem 1.

Corollary 2: $(F_{2n\pm1}, L_{2n\mp1}) = 1$.

Proof: (8) $\to 3 \nmid F_{2n\pm1}$. The conclusion now follows from Corollary 1.

Lemma 1: Let $(F_i, L_j) = 1$ and $F_i L_j = w^k \neq 0$. Then $k = 2$ implies $j = 1$ or 3; $k = 3$ implies $j = \pm1$.

Proof: Hypothesis and (11) imply $F_i = u^k$, $L_j = v^k$. The conclusion follows from (1) and (3).

Consider the following equations:

$$\text{(i)} \quad F_m = w^k - 1$$

$$\text{(ii)} \quad F_m = w^k + 1$$

$$\text{(iii)} \quad L_m = w^k - 1$$

$$\text{(iv)} \quad L_m = w^k + 1$$

For given k, a solution is a pair: (m, w). If $|w| \leq 1$, we say the solution is trivial.

Lemma 2: The trivial solutions of (i) through (iv) are as follows:

(i) $(0, 1)$, $(-2, 0)$ for all k; $(0, \pm1)$ for k even.
(ii) $(\pm1, 0)$, $(2, 0)$, $(\pm3, 1)$ for all k; $(0, -1)$ for k odd.
(iii) $(-1, 0)$ for all k.
(iv) $(0, 1)$, $(1, 0)$ for all k.

Proof: Obvious.

Theorem 2: If $k = 2$, the nontrivial solutions of (i) are $(4, 2)$ and $(6, 3)$.

Proof: *Case 1*.—Let $m = 4n \pm 1$. Hypothesis and (12) $\to F_{2n\pm1}L_{2n} = w^2 \neq 0$. Theorem 1 $\to (F_{2n\pm1}, L_{2n}) = 1$. Lemma 1 $\to 2n = 1$ or 3, an impossibility.

Case 2.—Let $m = 4n$. Hypothesis and (13) $\to F_{2n-1}L_{2n+1} = w^2 \neq 0$.

Case 2.—continued

Corollary 2 and (11) $\to L_{2n-1} = v^2$.
Now (1) $\to 2n + 1 = 1$ or $3 \to n = 0$ or 1.
Hypothesis $\to m \neq 0 \to n \neq 0 \to n = 1 \to m = 4 \to w = 2$.

Case 3.—Let $m = 4n - 2$. Hypothesis and (14) $\to F_{2n}L_{2n-2} = w^2 \neq 0$. Let $d = (F_{2n}, L_{2n-2})$. If $d = 1$, we have a contradiction, as in Case 1. If $d \neq 1$, then Corollary 1 $\to d = 3$. Hence, $(F_{2n}/3)(L_{2n-2}/3) = (w/3)^2$. Now (11) $\to F_{2n} = 3u^2$, $L_{2n-2} = 3v^2$. But $F_{2n} = 3u^2 \to n = 0$ or 2 by a result of R. Steiner [10, pp. 208-10].
Hypothesis $\to m \neq -2 \to n \neq 0 \to n = 2 \to m = 6 \to w = 3$.

Theorem 3: If $k = 3$, then (i) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n \pm 1$. As in the proof of Theorem 2, Case 1, we have Lemma 1 $\to 2n = \pm 1$, an impossibility.

Case 2.—Let $m = 4n$. As in the proof of Theorem 2, Case 2, we have $L_{2n+1} = v^3$. Now (3) $\to 2n + 1 = \pm 1 \to n = 0$ or -1.
Hypothesis $\to n \neq 0$ $n = -1 \to m = -4 \to F_{-4} = -3 = w^3 - 1$, an impossibility.

Case 3.—Let $m = 4n - 2$. As in the proof of Theorem 2, Case 3, we have $F_{2n}L_{2n-2} = w^3 \neq 0$, $(F_{2n}, L_{2n-2}) = 3$, so $F_{2n} = 3u^3$, $L_{2n-2} = 3v^3$. Now Theorem 2 of [7] $\to n = 2 \to m = 6 \to F_6 = 8 = w^3 - 1$, an impossibility.

Theorem 4: If $k = 3$, then (ii) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n \pm 1$. Hypothesis and (15) $\to F_{2n}L_{2n\pm1} = w^3 \neq 0$. Theorem 1 and Lemma 1 $\to 2n \pm 1 = \pm 1 \to n = 0$ or $\pm 1 \to m = \pm 1, \pm 3, \pm 5$. But $F_{\pm 5} = 5 \neq w^3 + 1$. Therefore, $m = \pm 1, \pm 3$ (trivial solutions).

Case 2.—Let $m = 4n$. Hypothesis and (16) $\to F_{2n+1}L_{2n-1} = w^3 \neq 0$, $n \neq 0$.
Theorem 1 and Lemma 1 $\to 2n - 1 = \pm 1 \to n = 1 \to m = 4 \to F_4 = 3 = w^3 + 1$, an impossibility.

Case 3.—Let $m = 4n + 2$. Hypothesis and (17) $\to F_{2n}L_{2n+2} = w^3 \neq 0$. As in the proof of Theorem 3, Case 3, we have $F_{2n} = 3u^3$, $L_{2n+2} = 3v^3$, an impossibility.

Theorem 5: If $k = 2$, then the nontrivial solutions of (iii) are $(\pm 2, \pm 2)$.

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\to L_{2n}^2 - 2 = w^2 - 1 \to L_{2n}^2 - w^2 = 1 \to L_{2n} - w = L_{2n} + w = \pm 1 \to w = 0 \to L_{2n} = \pm 1$, an impossibility.

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\to L_{2n+1}^2 + 2 = w^2 - 1 \to w^2 - L_{2n+1}^2 = 3 \to L_{2n+1} = \pm 1$, $w = \pm 2$ $m = \pm 2$.

Case 3.—Let $m = 4n + 1$. Hypothesis and (10) $\to L_{2n}L_{2n+1} = w^2$. (7) and (11) $\to L_{2n} = u^2$, $L_{2n+1} = v^2$, contradicting (1).

Case 4.—Let $m = 4n - 1$. Hypothesis and (10) $\to L_{2n}L_{2n-1} + 2 = w^2$.

(9) and (10) $\to \{L_n^2 - 2(-1)^n\}\{L_nL_{n-1} + (-1)^n\} + 2 = w^2$. We have:

$$L_n^3L_{n-1} + (-1)^nL_n^2 - 2(-1)^nL_nL_{n-1} = w^2.$$

Let $M_n = L_n^2L_{n-1} + (-1)^n(L_n - 2L_{n-1})$. Now, $L_nM_n = w^2$. Let p be an

Case 3.—continued

odd prime such that $p^e || L_n$. (7) $\rightarrow p \nmid M_n \rightarrow p^e || w^2 \rightarrow 2 | e$. Therefore, we must have $L_n = u^2$ or $2u^2$.
(1) and (2) $\rightarrow n = 0, 1, 3$, or $\pm 6 \rightarrow m = -1, 3, 11, 23, -25$. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = -1$ (trivial solution).

Theorem 6: If $k = 3$, then (iii) has the unique nontrivial solution (4, 2).

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\rightarrow L_{2n}^2 - 2 = w^3 - 1 \rightarrow L_{2n}^2 - 1 = w^3$.
Now (19) $\rightarrow L_{2n} = 0, 1$, or $3 \rightarrow L_{2n} = 3 \rightarrow 2n = 2 \rightarrow m = 4 \rightarrow w = 2$.

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\rightarrow L_{2n+1}^2 + 2 = w^3 - 1 \rightarrow L_{2n+1}^2 + 3 = w^3$, contradicting (19).

Case 3.—Let $m = 4n + 1$. Hypothesis and (10) $\rightarrow L_{2n}L_{2n+1} = w^3$. (7) and (11) $\rightarrow L_{2n} = u^3$, $L_{2n+1} = v^3$, contradicting (3).

Case 4.—Let $m = 4n - 1$. As in the proof of Theorem 5, Case 4, we have $L_nM_n = w^3$. If p is an odd prime such that $p^e || L_n$, then $p \nmid M_n$, so that $p^e || w^3 \rightarrow 3 | e$. Therefore, $L_n = u^3$, $2u^3$, or $4u^3$.

But (3), (4), and (5) $\rightarrow n = 0, \pm 1$, or $\pm 3 \rightarrow m = -1, 3, -5, 11$, or -13. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = -1$ (trivial solution).

Theorem 7: If $k = 3$, then (iv) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\rightarrow L_{2n}^2 - 2 = w^3 + 1 \rightarrow L_{2n}^2 - 3 = w^3$.
(19) $\rightarrow L_{2n} = 2$, $w = 1 \rightarrow n = 0 \rightarrow m = 0$ (trivial solution).

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\rightarrow L_{2n+1}^2 + 2 = w^3 + 1 \rightarrow L_{2n+1}^2 + 1 = w^3$.
(19) $\rightarrow L_{2n+1} = 0$, $w = 1$, an impossibility.

Case 3.—Let $m = 4n - 1$. Hypothesis and (10) $\rightarrow L_{2n}L_{2n-1} = w^3$. (7) and (11) $\rightarrow L_{2n} = u^3$, $L_{2n-1} = v^3$, contradicting (3).

Case 4.—Let $m = 4n + 1$. Hypothesis and (10) $\rightarrow L_{2n}L_{2n+1} - 2 = w^3$. (9) and (10) $\rightarrow \{L_n^2 - 2(-1)^n\}\{L_nL_{n+1} - (-1)^n\} - 2 = w^3$. We have:

$$L_n^3L_{n+1} - (-1)^nL_n^2 - 2(-1)^nL_nL_{n+1} = w^3.$$

Let $M_n = L_n^2L_{n+1} - (-1)^n(L_n + 2L_{n+1})$. Now, $L_nM_n = w^3$. As in the proof of Theorem 6, Case 4, $n = 0, \pm 1$, or ± 3. Therefore, $m = 1$, $-3, 5, -11, 13$. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = 1$ (trivial solution).

Remark: Cases 1 and 2 could also be disposed of by appeal to Theorem 13 in [9].

SUMMARY OF RESULTS

$$\begin{aligned}
F_m &= w^2 - 1 \rightarrow w = 0, \pm 1, \pm 2, \pm 3 \\
F_m &= w^3 - 1 \rightarrow w = 0, 1 \\
F_m &= w^3 + 1 \rightarrow w = -1, 0, 1 \\
L_m &= w^2 - 1 \rightarrow w = 0, \pm 2 \\
L_m &= w^3 - 1 \rightarrow w = 0, 2 \\
L_m &= w^3 + 1 \rightarrow w = 0, 1
\end{aligned}$$

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THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

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1. George E. Andrews [1] gave the following formulas for the Fibonacci numbers F_n ($F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$) in terms of binomial coefficients $\left((n;\ r) = \binom{n}{r}\right)$:

$$F_n = \sum_j (-1)^j \quad (n - 1;\ [(n - 1 - 5j)/2]), \tag{1.1}$$

$$F_n = \sum_j (-1)^j \quad (n;\ [(n - 1 - 5j)/2]). \tag{1.2}$$

Hansraj Gupta [2] has pointed out that (1.1) and (1.2) can be written, respectively, as

$$F_{2m+1} = S(2m,\ m) - S(2m,\ m - 2), \tag{1.3a}$$

$$F_{2m+2} = S(2m + 1,\ m) - S(2m + 1,\ m - 2) \tag{1.3b}$$

and

$$F_{2m+1} = S(2m + 1,\ m) - S(2m + 1,\ m - 1) \tag{1.4a}$$

$$F_{2m+2} = S(2m + 2,\ m) - S(2m + 1,\ m - 1), \tag{1.4b}$$

where $S(n,\ k) = \Sigma(n;\ j)$, the sum being taken over those j congruent to k modulo 5, and has given inductive proofs of (1.3) and (1.4).

The object of this note is to obtain (1.3) and (1.4) by first finding $S(n, k)$ explicitly in terms of such familiar numbers as

$$\alpha = \frac{1}{2}(1 + \sqrt{5}),\ \beta = \frac{1}{2}(1 - \sqrt{5}).$$

2. We begin by noting that

$$(1 + x)^n = \Sigma(n;\ j)x^j. \tag{2.1}$$

If we put $x = 1, \omega, \omega^2, \omega^3, \omega^4$ into (2.1) in turn (where $\omega = e^{\frac{2\pi i}{5}}$), add the resulting series, and divide by 5, we obtain

$$S(n,\ 0) = \frac{1}{5}(2^n + (1 + \omega)^n + (1 + \omega^2)^n + (1 + \omega^3)^n + (1 + \omega^4)^n). \tag{2.2a}$$

In similar fashion,

$$S(n,\ 1) = \frac{1}{5}(2^n + \omega^4(1 + \omega)^n + \omega^3(1 + \omega^2)^n + \omega^2(1 + \omega^3)^n + \omega(1 + \omega^4)^n), \tag{2.2b}$$

$$S(n,\ 2) = \frac{1}{5}(2^n + \omega^3(1 + \omega)^n + \omega(1 + \omega^2)^n + \omega^4(1 + \omega^3)^n + \omega^2(1 + \omega^4)^n), \tag{2.2c}$$

$$S(n,\ 3) = \frac{1}{5}(2^n + \omega^2(1 + \omega)^n + \omega^4(1 + \omega^2)^n + \omega(1 + \omega^3)^n + \omega^3(1 + \omega^4)^n), \tag{2.2d}$$

$$S(n,\ 4) = \frac{1}{5}(2^n + \omega(1 + \omega)^n + \omega^2(1 + \omega^2)^n + \omega^3(1 + \omega^3)^n + \omega^4(1 + \omega^4)^n). \tag{2.2e}$$

Now, $1 + \omega = 1 + e^{\frac{2\pi i}{5}} = 2\cos\frac{\pi}{5}\cdot e^{\frac{\pi i}{5}} = \alpha e^{\frac{\pi i}{5}}$, and similarly,

$$1 + \omega^2 = -\beta e^{\frac{2\pi i}{5}},$$
$$1 + \omega^3 = -\beta e^{-\frac{2\pi i}{5}},$$
$$1 + \omega^4 = \alpha e^{-\frac{\pi i}{5}},$$

so (2.2a) becomes

$$\begin{aligned} S(n,\ 0) &= \frac{1}{5}(2^n + \alpha^n e^{n\pi i/5} + (-\beta)^n e^{2n\pi i/5} + (-\beta)^n e^{-2n\pi i/5} + \alpha^n e^{-n\pi i/5}) \\ &= \frac{1}{5}(2^n + 2\alpha^n \cos n\pi/5 + 2(-\beta)^n \cos 2n\pi/5). \end{aligned} \tag{2.3a}$$

In similar fashion,

$$S(n,\ 1) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 2)\pi/5 + 2(-\beta)^n \cos(2n - 4)\pi/5), \tag{2.3b}$$

$$S(n,\ 2) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 4)\pi/5 + 2(-\beta)^n \cos(2n + 2)\pi/5), \tag{2.3c}$$

$$S(n,\ 3) = \frac{1}{5}(2^n + 2\alpha^n \cos(n + 4)\pi/5 + 2(-\beta)^n \cos(2n - 2)\pi/5), \tag{2.3d}$$

$$S(n,\ 4) = \frac{1}{5}(2^n + 2\alpha^n \cos(n + 2)\pi/5 + 2(-\beta)^n \cos(2n + 4)\pi/5). \tag{2.3e}$$

It follows that, for every k,

$$S(n,\ k) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 2k)\pi/5 + 2(-\beta)^n \cos(2n - 4k)\pi/5). \tag{2.4}$$

3. Now we are in a position to prove (1.3) and (1.4). We have

$$S(2m,\ m) = \frac{1}{5}(2^{2m} + 2\alpha^{2m} + 2\beta^{2m}),$$

$$S(2m,\ m-2) = \frac{1}{5}\left(2^{2m} + 2\alpha^{2m}\cos\frac{4\pi}{5} + 2\beta^{2m}\cos\frac{2\pi}{5}\right),$$

so

$$\begin{aligned} S(2m,\ m) - S(2m,\ m-2) &= \frac{2}{5}\alpha^{2m}\left(1 - \cos\frac{4\pi}{5}\right) + \frac{2}{5}\beta^{2m}\left(1 - \cos\frac{2\pi}{5}\right) \\ &= \frac{2}{5}\alpha^{2m}\cdot\frac{\sqrt{5}}{2}\alpha + \frac{2}{5}\beta^{2m}\cdot\frac{\sqrt{5}}{2}(-\beta) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2m+1} - \beta^{2m+1}) \\ &= F_{2m+1}, \end{aligned}$$

which is (1.3a). The derivations of (1.3b) and (1.4) from (2.4) are similar, and are omitted.

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SOME CONSTRAINTS ON FERMAT'S LAST THEOREM

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1. INTRODUCTION

The proof of "Fermat's Last Theorem," namely that there are no nontrivial integer solutions of $x^n + y^n = z^n$, where n is an integer greater than 2, is well known for the cases $n = 3$ and 4. We propose to look at some constraints on the values of x, y, and z, if they exist, when $n = p$, an odd prime. The history of the extension of the bounds on is interesting and illuminating [3], as is the development of the theory of ideals from Kummer's attempt to verify Fermat's result for all primes [2].

2. CONSTRAINT ON z

It can be readily established that there is no loss of generality in assuming that $0 < x < y < z$. Since $x \neq y$, $z - y \geq 1$ and $z - x \geq 2$. Following Guillotte [4], we consider $(x/z)^i + (y/z)^i = 1 + e_i$, where $e_0 = 1$, $e_p = 0$, and $e_i \in (0, 1)$ for $1 \leq i \leq p$. Summing over i from 0 to p, Guillotte further showed that

$$1/(1 - x/z) + 1/(1 - y/z) > p + 1 + \sum_{i=0}^{p} e_i,$$

from which we obtain

$$z(1/(z - x) + 1/(z - y)) > p + 2.$$

Since
$$z(1/(z - x) + 1/(z - y)) \leq z(\tfrac{1}{2} + 1);$$
$$3z/2 \geq z/(z - x) + z/(z - y) > p + 2.$$
Hence
$$z > 2(p + 2)/3.$$

Thus, if solutions in integers exist for the case when $p = 7$, we must have $z > 6$.

3. CONSTRAINT ON x

Now let $hx + y = z$. Since $z - y \geq 1$, $x \geq 1/h$. It has been shown in [1] that $h < 2^{1/p} - 1$, and so
$$x > 1/(2^{1/p} - 1).$$
Hence, if integer solutions exist for $p = 7$, we know that $x > 9.607$. Since $z \geq x + 2$, we know for $p = 7$ that $z \geq 11.607$, which is better than the bound found in Section 2.

4. CONSTRAINT ON y

Since $1/x < 2^{1/p} - 1$, we have $1 + p/x < 2$. Hence, $x > p$. For the case $p = 7$ we have that, if solutions exist, then $x > 7$, which is not an improvement on the result in Section 3. However, from [1], $z \to y + 1$ as $p \to \infty$, and thus if solutions in integers exist for very large values of p, then very large values of x and y are involved. We note also, since
$$2^{1/p} = \sum_{r=0}^{\infty}\left(\frac{1}{p}\ln 2\right)^r/r!,$$
that
$$2^{1/p} - 1 < \ln 2/(p - \ln 2),$$
which with the results from Sections 2 and 3 gives
$$y > p/\ln 2.$$
When $p = 7$, this yields $x > 9.099$ compared with $x > 9.607$ from Section 3. However, as p increases, the inequalities become closer, and the simpler $y > p/\ln 2$ is adequate. $y > 1.442695$ is also "sharper" than $x > p$.

Zeitlin [6] proved that no integer solutions exist for $x + ny \leq nz$. We note that for $n = 7$, $x > 9.607$, $y > 10.099$, $z \geq 11.607$ as above, $x + ny > 80.300$ and $nz \geq 81.249$. Perisastri [5] showed that in our notation
$$\sqrt{2}x > \sqrt{y}(1 + 1/(2p \ln 2p)).$$
For $p = 7$ and x, y, z as above, $\sqrt{2}x = 13.586$ and $\sqrt{y}(1 + 1/(2p \ln 2p)) = 3.264$.

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-454 *Proposed by Charles W. Trigg, San Diego, CA*

In the square array of the nine nonzero digits

$$\begin{matrix} 6 & 7 & 5 \\ 2 & 1 & 3 \\ 9 & 4 & 8 \end{matrix}$$

the sum of the four digits in each 2-by-2 corner array is 16. Rearrange the nine digits so that the sum of the digits in each such corner array is *seven* times the central digit.

B-455 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i} \quad \text{and} \quad T_m = 10S_m/(m + 2).$$

Prove that T_m is a sum of two Lucas numbers for $m = 0, 1, 2, \ldots$.

B-456 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that any two consecutive Fibonacci numbers are coprime (i.e., their gcd is 1). Prove or disprove: Two distinct Fibonacci numbers are coprime if each of them is the product of two distinct primes.

B-457 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that there exists a positive integer b such that the Pythagorean type relationship $(5F_n^2)^2 + b^2 \equiv (L_n^2)^2 \pmod{5m^2}$ holds for all m and n with $m|F_n$.

B-458 *Proposed by H. Klauser, Zürich, Switzerland*

Let T_n be the triangular number $n(n + 1)/2$. For which positive integers k do there exist positive integers n such that $T_{n+k} - T_n$ is a prime?

B-459 *Proposed by E. E. McDonnell, Palo Alto, CA, and J. O. Shallit, Berkeley, CA*

Let g be a primitive root of the odd prime p. For $1 \le i \le p-1$, let a_i be the integer in $S = \{0, 1, \ldots, p-2\}$ with $g^{a_i} \equiv i \pmod{p}$. Show that

$$a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_{p-1} - a_{p-2}$$

(differences taken mod $p-1$ to be in S), is a permutation of $1, 2, \ldots, p-2$.

SOLUTIONS

Double a Triangular Number

B-430 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switzerland*

For every positive integer a, prove that

$$(a^2 + a - 1)(a^2 + 3a + 1) + 1$$

is a product $m(m+1)$ of two consecutive integers.

Solution by Frank Higgins, Naperville, IL

Noting that

$$(a^2 + a - 1)(a^2 + 3a + 1) + 1 = (a^2 + 2a - 1)(a^2 + 2a),$$

the assertion follows with m the integer $a^2 + 2a - 1$.

Also solved by J. Annulis, Paul S. Bruckman, D. K. Chang, M. J. DeLeon, Charles G. Fain, Herta T. Freitag, Robert Girse, Graham Lord, John W. Milsom, F. D. Parker, Bob Prielipp, A. G. Shannon, Charles B. Shields, Sahib Singh, Lawrence Somer and the proposers.

Making it an Identity

B-431 *Proposed by Verner E. Hoggatt, Jr., San Jose, CA*

For which fixed ordered pairs (h, k) of integers does

$$F_n(L^2_{n+h} - F^2_{n+h}) = F_{n+4}(L^2_{n+k} - F^2_{n+k})$$

for all integers n?

Solution by Paul S. Bruckman, Concord, CA

For any integer m,

$$\begin{aligned} L_m^2 - F_m^2 = (L_m - F_m)(L_m + F_m) &= (F_{m+1} + F_{m-1} - F_m)(F_{m+1} + F_{m-1} + F_m) \\ &= 2F_{m-1} \cdot 2F_{m+1} = 4F_{m-1}F_{m+1}. \end{aligned}$$

Hence, the desired identity is equivalent to:

$$F_nF_{n+h+1}F_{n+h-1} = F_{n+4}F_{n+k+1}F_{n+k-1}, \tag{1}$$

which is to hold for some pair (h, k) of integers and for all n. In particular, (1) must hold for $n = 0$ and $n = -4$, which yields:

$$F_4F_{k+1}F_{k-1} = F_{-4}F_{h-3}F_{h-5} = 0.$$

Since the only term of the Fibonacci sequence that vanishes is F_0, we must have $h \in \{3, 5\}$ and $k \in \{-1, 1\}$, i.e.,

$$(h, k) \in \{(3, -1), (5, -1), (5, 1), (3, 1)\}.$$

Checking out these possibilities, one finds that the unique solution is

$$(h, k) = (3, 1).$$

Also solved by M. D. Agrawal, M. J. DeLeon, Herta T. Freitag, Frank Higgins, John W. Milsom, A. G. Shannon, Charles B. Shields, Sahib Singh, M. Wachtel and the proposer.

Alternating Signs

B-432 *Proposed by Verner E. Hoggatt, Jr., San Jose, CA*

Let

$$G_n = F_n F_{n+3}^2 - F_{n+2}^3.$$

Prove that the terms of the sequence $G_0, G_1, G_2, \ldots$ alternate in sign.

Solution by F. D. Parker, St. Lawrence University, Canton, NY

$$\begin{aligned} G_n &= F_n F_{n+3}^2 - F_{n+2}^3 = F_n(2F_{n+1} + F_n)^2 - (F_n + F_{n+1})^3 \\ &= F_n^3 + 4F_n^2 F_{n+1} + 4F_{n+1}^2 F_n - F_n^3 - 3F_n^2 F_{n+1} - 3F_n F_{n+1}^2 - F_{n+1}^3 \\ &= F_{n+1}^2 F_n + F_{n+1} F_n^2 - F_{n+1}^3 \\ &= F_{n+1}(F_n F_{n+2} - F_{n+1}^2) = (-1)^n F_{n+1}. \end{aligned}$$

Also solved by M. D. Agrawal, Stephan Andres, Paul S. Bruckman, L. Carlitz, M. J. DeLeon, Herta T. Freitag, Frank Higgins, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, Lawrence Somer and the proposer.

Alternate Definition of a Sequence

B-433 *Proposed by J. F. Peters and R. Pletcher, St. John's University, Collegeville, MN*

For each positive integer n, let q_n and r_n be the integers with

$$n = 3q_n + r_n \quad \text{and} \quad 0 \le r_n < 3.$$

Let $\{T(n)\}$ be defined by

$$T(0) = 1,\ T(1) = 3,\ T(2) = 4, \text{ and } T(n) = 4q_n + T(r_n), \text{ for } n \ge 3.$$

Show that there exist integers a, b, c such that

$$T(n) = \left[\frac{an + b}{c}\right],$$

where $[x]$ denotes the greatest integer in x.

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given arithmetic function $T(n)$ can be defined as

$$T(3t) = 4t + 1;\ T(3t + 1) = 4t + 3;\ T(3t + 2) = 4t + 4$$

or, equivalently,

$$T(n) = [(4n + 5)/3].$$

Hence, $a = 4$, $b = 5$, and $c = 3$.

Also solved by Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, Frank Higgins, H. Klauser, Graham Lord, A. G. Shannon and the proposers.

Never a Square

B-434 *Proposed by Herta T. Freitag, Roanoke, VA*

For which positive integers n, if any, is $L_{3n} - (-1)^n L_n$ a perfect square?

Solution by A. G. Shannon, New South Wales Institute of Technology, Sydney, Australia

$$L_{3n} - (-1)^n L_n = a^{3n} + b^{3n} - (ab)^n(a^n + b^n) = 5L_nF_n^2,$$

which would be a perfect square if and only if $5 | L_n$; but this is impossible for all n.

Also solved by Paul S. Bruckman, Frank Higgins, J. W. Milsom, F. D. Parker, Bob Prielipp, Sahib Singh, Lawrence Somer, M. Wachtel and the proposer.

Restricted Divisors of a Quadratic

B-435 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switzerland*

For every positive integer a, prove that no integral divisor of $a^2 + a - 1$ is congruent to 3 or 7 modulo 10.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We begin by observing that since $a^2 + a - 1$ is odd, each of its divisors must be odd. Suppose there is a divisor d of $a^2 + a - 1$ which is congruent to 3 or 7 modulo 10. Then d must have at least one prime divisor p which is congruent to 3 or 7 modulo 10. (If this were not the case, the only primes that could be divisors of d would be 5, primes congruent to 1 modulo 10, and primes congruent to 9 modulo 10. But then d would have to be congruent to 1, 5, or 9 modulo 10.) It follows that $a^2 + a - 1 \equiv 0 \pmod{p}$. Hence, $4a^2 + 4a \equiv 4 \pmod{p}$ so $(2a + 1)^2 \equiv 5 \pmod{p}$. Thus, 5 is a quadratic residue modulo p.

Let q be an odd prime such that $(q, 5) = 1$. Then, by the Law of Quadratic Reciprocity,

$$\left(\frac{5}{q}\right) = \left(\frac{q}{5}\right)$$

and

$$\left(\frac{q}{5}\right) = \begin{cases} \left(\frac{2}{5}\right) = -1 \text{ if } q \equiv 2 \pmod{5}, \\ \left(\frac{3}{5}\right) = -1 \text{ if } q \equiv 3 \pmod{5}. \end{cases}$$

Hence, 5 is a quadratic nonresidue of all odd primes which are congruent to 2 or 3 modulo 5, so 5 is a quadratic nonresidue of every prime congruent to 3 or 7 modulo 10.

This contradiction tells us that no divisor of $a^2 + a - 1$ is congruent to 3 or 7 modulo 10.

Also solved by Paul S. Bruckman, M. J. DeLeon, A. G. Shannon, Sahib Singh, and Lawrence Somer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-330 *Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose CA*

If θ is a positive irrational number and $1/\theta + 1/\theta^3 = 1$,

$$A_n = [n\theta],\ B_n = [n\theta^3],\ C_n = [n\theta^2],$$

then prove or disprove:

$$A_n + B_n + C_n = C_{B_n}.$$

H-331 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $\{f_n^{(k)}\}_{n=0}^{\infty}$ by $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k+1. \end{cases}$$

Letting $\alpha = [(1 + \sqrt{5})/2]$, show:

(a) $f^{(k)} > \alpha^{n-2}$ if $n \geq 3$;

(b) $\{f^{(k)}\}_{n=2}^{\infty}$ has Schnirelmann density 0.

H-332 *Proposed by David Zeitlin, Minneapolis, MN*

Let $\alpha = (1 + \sqrt{5})/2$. Let $[x]$ denote the greatest integer function. Show that after k iterations ($k \geq 1$), we obtain the identity

$$[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]] = F_{(2p+1)(2k+1)}/F_{2p+1},\ (p = 0, 1, \ldots).$$

Remarks: The special case $p = 0$ appears as line 1 in Theorem 2, p. 309, in the paper by Hoggatt and Bicknell-Johnson, *The Fibonacci Quarterly* 17(4):306-318. For $k = 2$, the above identity gives

$$[\alpha^{4p+2}[\alpha^{4p+2}]] = F_{5(2p+1)}/F_{2p+1} = L_{4(2p+1)} - L_{2(2p+1)} + 1.$$

SOLUTIONS

Con-Vergent

H-308 *Proposed by Paul S. Bruckman, Corcord, CA*
(Vol. 17, No. 4, Dec., 1979)

Let

$$[a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{p_n(a_1, a_2, \ldots, a_n)}{q_n(a_1, a_2, \ldots, a_n)}$$

denote the nth convergent of the infinite simple continued fraction

$$[a_1, a_2, \ldots], \; n = 1, 2, \ldots .$$

Also, define $p_0 = 1$, $q_0 = 0$. Further, define

$$(1) \qquad \begin{aligned} W_{n,k} &= p_n(a_1, a_2, \ldots, a_n)q_k(a_1, a_2, \ldots, a_k) \\ &\quad - p_k(a_1, a_2, \ldots, a_k)q_n(a_1, a_2, \ldots, a_n) \\ &= p_n q_k - p_k q_n, \; 0 \le k \le n. \end{aligned}$$

Find a general formula for $W_{n,k}$.

Solution by the proposer.

Recall that the p_n's and q_n's satisfy the basic recursion

$$(2) \qquad r_{n+1} = a_{n+1}r_n + r_{n-1}, \; n = 1, 2, \ldots .$$

Also, the following relations are either obvious or well known:

$$(3) \qquad W_{n,n} = 0;$$

$$(4) \qquad W_{n,n-1} = (-1)^n, \; n \ge 1;$$

$$(5) \qquad W_{n,n-2} = (-1)^{n-1}a_n, \; n \ge 2.$$

[See Niven and Zuckerman, *An Introduction to the Theory of Numbers*, 3rd ed. (New York: Wiley, 1972), Theorem 7.5, for a proof of (4) and (5).]

We show, by strong induction, that

$$(6) \qquad W_{n,k} = (-1)^{k+1}p_{n-k-1}(a_{k+2}, a_{k+3}, \ldots, a_n).$$

Let S denote the set of positive integers n such that (6) holds for $0 \le k < n$. Setting $n = 1$ in (4) yields $W_{1,0} = -1 = (-1)^{0+1}p_0$; hence, $1 \in S$. Suppose that for some integer $m \ge 2$, $1, 2, \ldots, m \in S$. By (4) and (5), we have:

$$(7) \qquad W_{m+1,m} = (-1)^{m+1} = (-1)^{m+1}p_0, \text{ and } W_{m+1,m-1} = (-1)^m a_{m+1}, \text{ or}$$

$$(8) \qquad W_{m+1,m-1} = (-1)^{m-1+1}p_1(a_{m+1}).$$

Also, if $0 \le k \le m - 2$,

$$\begin{aligned} W_{m+1,k} &= p_{m+1}q_k - p_k q_{m+1} = (a_{m+1}p_m + p_{m-1})q_k - p_k(a_{m+1}q_m + q_{m-1}) \\ &= a_{m+1}(p_m q_k - p_k q_m) + p_{m-1}q_k - p_k q_{m-1} = a_{m+1}W_{m,k} + W_{m-1,k} \end{aligned}$$

[using (1) and (2)]. Hence, by the inductive hypothesis and (2),

$$\begin{aligned} W_{m+1,k} &= (-1)^{k+1}a_{m+1}p_{m-k-1}(a_{k+2}, \ldots, a_m) + (-1)^{k+1}p_{m-k-2}(a_{k+2}, \ldots, a_{m-1}) \\ &= (-1)^{k+1}p_{m-k}(a_{k+2}, \ldots, a_{m+1}). \end{aligned}$$

Thus, using (7) and (8),

(9) $$W_{m+1,\,k} = (-1)^{k+1}p_{m-k}(a_{k+2}, \ldots, a_{m+1}),\ 0 \le k \le m,$$

which is equivalent to the statement $(m + 1) \in S$. Hence,

$$1, 2, \ldots, m \in S \Rightarrow (m + 1) \in S.$$

By induction, (6) is proved.

Fibonacci and Lucas Are the Greatest Integers

H-310 *Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 17, No. 4, Dec., 1979)*

Let $\alpha = (1 + \sqrt{5})/2$, $[n\alpha] = a_n$, and $[n\alpha^2] = b_n$. Clearly $a_n + n = b_n$.

(a) Show that if $n = F_{2m+1}$, then $a_n = F_{2m+2}$ and $b_n = F_{2m+3}$.

(b) Show that if $n = F_{2m}$, then $a_n = F_{2m+1} - 1$ and $b_n = F_{2m+2} - 1$.

(c) Show that if $n = L_{2m}$, then $a_n = L_{2m+1}$ and $b_n = L_{2m+2}$.

(d) Show that if $n = L_{2m+1}$, then $a_n = L_{2m+2} - 1$ and $b_n = L_{2m+3} - 1$.

Solution by Paul S. Bruckman, Corcord, CA

We begin by noting that

$$\begin{aligned} F_{n+1} - \alpha F_n &= \frac{1}{\sqrt{5}}\{\alpha^{n+1} - \beta^{n+1} - \alpha(\alpha^n - \beta^n)\} \\ &= \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} - \beta^{n-1}) \\ &= -\beta^n/\sqrt{5}(\beta - \alpha), \end{aligned}$$

or

(1) $$\beta^n = F_{n+1} - \alpha F_n.$$

Also, $\alpha L_n - L_{n+1} = \alpha(\alpha^n + \beta^n) - (\alpha^{n+1} + \beta^{n+1}) = -\beta^n(\beta - \alpha)$, or

(2) $$\beta^n\sqrt{5} = \alpha L_n - L_{n+1}.$$

Since $-1 < \beta < 0$, thus $0 < \beta^{2n} \le 1$ and $-1 < \beta^{2n+1} < 0$ $(n \ge 0)$. Hence, using (1)

$$0 < F_{2n+1} - \alpha F_{2n} \le 1 \quad \text{and} \quad -1 < F_{2n+2} - \alpha F_{2n+1} < 0;$$

note that equality is attained above if and only if $n = 0$. Therefore,

$$F_{2n+1} - 1 \le \alpha F_{2n} < F_{2n+1} \quad \text{and} \quad F_{2n+2} < \alpha F_{2n+1} < F_{2n+2} + 1 \ (n \ge 0).$$

It follows that

(3) $$[\alpha F_{2n}] = F_{2n+1} - 1, \text{ and}$$

(4) $$[\alpha F_{2n+1}] = F_{2n+2}\ (n \ge 0).$$

Now (3) implies $[\alpha^2 F_{2n}] = [(1 + \alpha)F_{2n}] = F_{2n} + [\alpha F_{2n}] = F_{2n} + F_{2n+1} - 1$, or

(5) $$[\alpha^2 F_{2n}] = F_{2n+2} - 1.$$

Also, $[\alpha^2 F_{2n+1}] = F_{2n+1} + [\alpha F_{2n+1}] = F_{2n+1} + F_{2n+2}$, or

(6) $$[\alpha^2 F_{2n+1}] = F_{2n+3}.$$

Note that (4) and (6) are equivalent to (a) of the original problem; also, (3) and (5) are equivalent to (b) of the original problem.

In order to prove (c) and (d), we proceed similarly, using the result in (2). We need only observe that $|\beta^n\sqrt{5}| < 1$ for $n \ge 2$. The desired results then follow, as before, for all values of n except for possibly $n = 0$; however, a quick inspection shows that the results also hold for $n = 0$, i.e.,

(7) $$[\alpha L_{2n}] = L_{2n+1},\ [\alpha L_{2n+1}] = L_{2n+2} - 1,$$

which imply the other two results.

Comment by *Bob Prielipp, University of Wisconsin-Oshkosh, WI*

Sharp-eyed readers will find that this problem can be solved easily by using the following four lemmas established in the article "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio" by Hoggatt and Bicknell-Johnson [*The Fibonacci Quarterly* 17(4):306-318].

Lemma 1 (p. 308): $[\alpha F_n] = F_{n+1}$, n odd, $n \geq 2$;
$[\alpha F_n] = F_{n+1} - 1$, n even, $n \geq 2$.

Lemma 2 (p. 308): $[\alpha^2 F_n] = F_{n+2}$, n odd, $n \geq 2$;
$[\alpha^2 F_n] = F_{n+2} - 1$, n even, $n \geq 2$.

Lemma 6 (p. 315): $[\alpha L_n] = L_{n+1}$ for n even, if $n \geq 2$;
$[\alpha L_n] = L_{n+1} - 1$ for n odd, if $n \geq 3$.

Lemma 7 (p. 315): $[\alpha^2 L_n] = L_{n+2}$ if n is even and $n \geq 2$;
$[\alpha^2 L_n] = L_{n+2} - 1$ if n is odd and $n \geq 1$.

Also solved by Bob Prielipp, G. Wulczyn, and the proposers.

CORRECTIONS

1. The problem solved in Vol. 18, No. 2, April 1980 is H-284 not H-285.
2. H-315 as it appeared in Vol. 18, No. 2, April 1980 had several misprints in it. A corrected version is given below.

H-315 *Proposed by D. P. Laurie, National Research Institute for Mathematical Sciences, Pretoria, South Africa*

Let the polynomial P be given by

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

and let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. The following iteration scheme for factorizing P has been suggested by Kerner [1]:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)};\ i = 1, 2, \ldots, n.$$

Prove that if $\sum\limits_{j=1}^{n} z_i = -a_{n-1}$, then also $\sum\limits_{i=1}^{n} \hat{z}_i = -a_{n-1}$.

Reference

1. I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen." *Numer. Math.* 8 (1966):290-94.

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