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# A GENERALIZED EXTENSION OF SOME FIBONACCI-LUCAS IDENTITIES TO PRIMITIVE UNIT IDENTITIES 

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This paper originated from an attempt to extend many of the elementary Fibo-nacci-Lucas identities, whose subscripts had a common odd or even difference to, first, other Type I real quadratic fields and, then, to the other three types of real quadratic field fundamental units. For example, the Edouard Lucas identity $F_{n+1}^{3}+F_{n}^{3}-F_{n-1}^{3}=F_{3 n}$ becomes, in the Type I real quadratic field,

$$
\left(\sqrt{61} ; \alpha=\frac{39+5 \sqrt{61}}{2}\right) F_{n+1}^{3}+39 F_{n}^{3}-F_{n-1}^{3}=(5)(195) F_{3 n}
$$

This suggests the Type $I$ extension identity $F_{n+1}^{3}+L_{1} F_{n}^{3}-F_{n-1}^{3}=F_{1} F_{2} F_{3 n}$ and the Type I generalization: $F_{n+2 r+1}^{3}+L_{2 r+1} F_{n}^{3}-F_{n-2 r-1}^{3}=F_{2 r+1} F_{4 r+2} F_{3 n}$. The Ezekiel Ginsburg identity $F_{n+2}^{3}-3 F_{n}^{3}+F_{n-2}^{3}=3 F_{3 n}$ becomes, in the Type I real quadratic field,

$$
(\sqrt{61}) F_{n+2}^{3}-1523 F_{n}^{3}+F_{n-2}^{3}=(195)(296985) F_{3 n}
$$

This suggests the Type $I$ identity extension $F_{n+2}^{3}-L_{2} F_{n}^{3}+F_{n-2}^{3}=F_{2} F_{4} F_{3 n}$ and the Type I generalization: $F_{n+2 r}^{3}-L_{2 r} F_{n}^{3}+F_{n-2 r}^{3}=F_{2 r} F_{4 r} F_{3 n}$ 。

The transformation from these Type I identities to Type III identities can be represented as

$$
\text { (I) } F_{n} \leftrightarrow \text { (III) } 2 F_{n} \quad \text { or (I) } L_{n} \leftrightarrow \text { (III) } 2 L_{n}
$$

The transformation from Type I to Type II and Type III to Type IV for identities in which there is a common even subscript difference $2 r$ can be represented as
(I, III) $F_{2 r} \leftrightarrow$ (II, IV) $F_{r}, L_{2 r} \leftrightarrow L_{r}, F_{n+2 r} \leftrightarrow F_{n+r}$, and $L_{n+2 r} \leftrightarrow L_{n+r}$.
I. Type I primitive units are given by

$$
\begin{aligned}
& \left.\alpha=\frac{a+b \sqrt{D}}{2}, \beta=\frac{a-b \sqrt{D}}{2}, \alpha \beta=-1, D \equiv 5 \text { (modulo } 8\right) \\
& a^{2}-b^{2} D=-4, a \text { and } b \text { are odd. } \\
& \left(\frac{a+b \sqrt{D}}{2}\right)^{n}=\frac{L_{n}+F_{n} \sqrt{D}}{2}, F_{n}=\frac{1}{\sqrt{D}}\left(\alpha^{n}-\beta^{n}\right), L_{n}=\alpha^{n}+\beta^{n}
\end{aligned}
$$

$F_{n}$ and $L_{n}$ are also given by the finite difference sequences:

$$
\begin{aligned}
& F_{n+2}=a F_{n+1}+F_{n}, F_{1}=b, F_{2}=a b ; \\
& L_{n+2}=a L_{n+1}+L_{n}, L_{1}=a, L_{2}=a^{2}+2
\end{aligned}
$$

II. Type II primitive units are given by

$$
\begin{aligned}
& \alpha=\frac{a+b \sqrt{D}}{2}, \beta=\frac{a-b \sqrt{D}}{2}, \alpha \beta=1, D \equiv 5 \text { (modulo } 8 \text { ), } \\
& a^{2}-b^{2} D=4, a^{2}-b^{2} D \neq-4, a \text { and } b \text { are odd. } \\
& \left(\frac{a+b \sqrt{D}}{2}\right)^{n}=\frac{L_{n}+F_{n} \sqrt{D}}{2}, F_{n}=\frac{1}{\sqrt{D}}\left(\alpha^{n}-\beta^{n}\right), L_{n}=\alpha^{n}+\beta^{n} .
\end{aligned}
$$

$F_{n}$ and $L_{n}$ are also given by the finite difference sequences:

$$
\begin{aligned}
& F_{n+2}=a F_{n+1}-F_{n}, F_{1}=b, F_{2}=a b ; \\
& L_{n+2}=a L_{n+1}-L_{n}, L_{1}=a, L_{2}=a^{2}-2 .
\end{aligned}
$$

III. Type III primitive units are given by

$$
\begin{aligned}
& \alpha=\alpha+b \sqrt{D}, \beta=\alpha-b \sqrt{D}, \alpha \beta=-1, \alpha^{2}-b \sqrt{D}=-1 . \\
& (\alpha+b \sqrt{D})^{n}=L_{n}+F_{n} \sqrt{D}, F_{n}=\frac{1}{2 \sqrt{D}}\left(\alpha^{n}-\beta^{n}\right), L_{n}=\frac{1}{2}\left(\alpha^{n}+\beta^{n}\right) .
\end{aligned}
$$

$F_{n}$ and $L_{n}$ are also given by the finite difference sequences:

$$
\begin{aligned}
& F_{n+2}=2 a F_{n+1}+F_{n}, F_{1}=b, F_{2}=2 a b ; \\
& L_{n+2}=2 a L_{n+1}+L_{n}, L_{1}=a, L_{2}=2 a^{2}+1 .
\end{aligned}
$$

IV. Type IV primitive units are given by
$\alpha=a+b \sqrt{D}, \beta=a-b \sqrt{D}, \alpha \beta=1, a^{2}-b^{2} D=1, a^{2}-b^{2} D \neq-1$.
$(\alpha+b \sqrt{D})^{n}=L_{n}+F_{n} \sqrt{D}, F_{n}=\frac{1}{2 \sqrt{D}}\left(\alpha^{n}-\beta^{n}\right), L_{n}=\frac{1}{2}\left(\alpha^{n}+\beta^{n}\right)$.
$F_{n}$ and $L_{n}$ are also given by the finite difference sequences:

$$
\begin{aligned}
& F_{n+2}=2 \alpha F_{n+1}-F_{n}, F_{1}=b, F_{2}=2 \alpha b ; \\
& L_{n+2}=2 \alpha L_{n+1}-L_{n}, L_{1}=a, L_{2}=2 a^{2}-1 .
\end{aligned}
$$

1. (a) Fibonacci-Lucas identity used: $F_{n}+L_{n}=2 F_{n+1}$
(b) Type I extension: $a F_{n}+b L_{n}=2 F_{n+1}$
(c) Generalizations:

Types I \& II

$$
\begin{aligned}
& L_{m} F_{n}+F_{m} L_{n}=2 F_{m+n} \\
& L_{m} F_{n}+F_{m} L_{n}=F_{m+n}
\end{aligned}
$$

Types III \& IV
2. (a) Fibonacci-Lucas identity used: $L_{n}-F_{n}=2 F_{n-1}$
(b) Type I extension: $b L_{n}-\alpha F_{n}=2 F_{n-1}$
(c) Generalizations:

Type I
Type II

$$
\begin{aligned}
& F_{m} L_{n}-L_{m} F_{n}=2(-1)^{m+1} F_{n-m} \\
& F_{n} L_{m}-F_{m} L_{n}=2 F_{n-m} \\
& F_{m} L_{n}-L_{m} F_{n}=(-1)^{m+1} F_{n-m} \\
& F_{n} L_{m}-F_{m} L_{n}=F_{n-m}
\end{aligned}
$$

Type IV
3. (a) Fibonacci-Lucas identity used: $F_{n+3}^{2}+F_{n}^{2}=2\left(F_{n+2}^{2}+F_{n+1}^{2}\right)$
(b) Type I extension: $b\left(F_{n+3}^{2}+F_{n}^{2}\right)=F_{3}\left(F_{n+2}^{2}+F_{n+1}^{2}\right)$
(c) Generalizations:

Types I \& III $F_{2 m-1}\left(F_{n+4 m-1}^{2}+F_{n}^{2}\right)=F_{4 m-1}\left(F_{n+2 m+r-1}^{2}+F_{n+2 m-r}^{2}\right)$

$$
F_{2 r-1}\left(L_{n+4 m-1}^{2}+L_{n}^{2}\right)=F_{4 m-1}\left(L_{n+2 m+r-1}^{2}+L_{n+2 m-r}^{2}\right)
$$

Types II \& IV

$$
F_{2 r-1}\left(F_{n+4 m-1}^{2}-F_{n}^{2}\right)=F_{4 m-1}\left(F_{n+2 m+r-1}^{2}-F_{n+2 m-r}^{2}\right)
$$

$$
L_{2 r-1}\left(L_{n+4 m-1}^{2}-L_{n}^{2}\right)=F_{4 m-1}\left(L_{n+2 m+r-1}^{2}-L_{n+2 m-r}^{2}\right)
$$

4. (a) Fibonacci-Lucas identity used:

$$
F_{n+3} F_{n+4}+F_{n} F_{n+1}=2\left(F_{n+2} F_{n+3}+F_{n+1} F_{n+2}\right)
$$

(b) Type I extension:

$$
b\left(F_{n+3} F_{n+4}+F_{n} F_{n+1}\right)=F_{3}\left(F_{n+2} F_{n+3}+F_{n+1} F_{n+2}\right)
$$

(c) Generalizations:

## Types I \& III

$$
\begin{aligned}
& F_{2 r-1}\left(F_{n+4 m-1} F_{n+4 m}+F_{n} F_{n+1}\right)=F_{4 m-1}\left(F_{n+2 m+r-1} F_{n+2 m+r}+F_{n+2 m-r} F_{n+2 m-r+1}\right) \\
& F_{2 r-1}\left(L_{n+4 m-1} L_{n+4 m}+L_{n} L_{n+1}\right)=F_{4 m-1}\left(L_{n+2 m+r-1} L_{n+2 m+r}+L_{n+2 m-r} L_{n+2 m-r+1}\right)
\end{aligned}
$$

## Types II \& IV

$$
\begin{aligned}
& F_{2 r-1}\left(F_{n+4 m-1} F_{n+4 m}-F_{n} F_{n+1}\right)=F_{4 m-1}\left(F_{n+2 m+r-1} F_{n+2 m+r}-F_{n+2 m-r} F_{n+2 m-r+1}\right) \\
& F_{2 r-1}\left(L_{n+4 m-1} L_{n+4 m}-L_{n} L_{n+1}\right)=F_{4 m-1}\left(L_{n+2 m+r-1} L_{n+2 m+r}-L_{n+2 m-r} L_{n+2 m-r+1}\right)
\end{aligned}
$$

5. (a) Fibonacci-Lucas identity used: $F_{2 m}+F_{m}^{2}=2 F_{m} F_{m+1}$
(b) Type I extension: $b F_{2 m}+\alpha F_{2 m}^{2}=2 F_{m} F_{m+1}$
(c) Generalizations:

Type I

$$
\begin{aligned}
F_{r} F_{2 m}+L_{r} F_{m}^{2} & =2 F_{m} F_{m+r} \\
D F_{r} F_{2 m}+L_{r} L_{m}^{2} & =2 L_{m} L_{m+r}
\end{aligned}
$$

Type II

Type III

$$
\begin{aligned}
F_{r} F_{2 m}+L_{r} F_{m}^{2} & =2 F_{m} F_{m+r} \\
D F_{r} F_{2 m}+L_{r} L_{m}^{2} & =2 L_{m} L_{m+r}
\end{aligned}
$$

$$
F_{r} F_{2 m}+2 L_{r} F_{m}^{2}=2 F_{m} F_{m+r}
$$

$$
D F_{r} F_{2 m}+2 L_{r} L_{m}^{2}=2 L_{m} L_{m+r}
$$

Type IV

$$
\begin{aligned}
F_{r} F_{2 m}+2 L_{r} F_{m}^{2} & =2 F_{m} F_{m+r} \\
D F_{r} F_{2 m}+2 L_{r} L_{m}^{2} & =2 L_{m} L_{m+r}
\end{aligned}
$$

6. (a) Fibonacci-Lucas identity used: $F_{2 m}-F_{m}^{2}=2 F_{m} F_{m-1}$
(b) Type I extension: $b F_{2 m}-\alpha F_{m}^{2}=2 F_{m} F_{m-1}$
(c) Generalizations:

Type I

$$
\begin{gathered}
F_{r} F_{2 m}-L_{r} F_{m}^{2}=2(-1)^{r+1} F_{m} F_{m-r} \\
D F_{r} F_{2 m}-L_{r} L_{m}^{2}=2(-1)^{r+1} L_{m} L_{m-r}
\end{gathered}
$$

Type II

$$
\begin{aligned}
F_{r} F_{2 m}-L_{r} L_{m}^{2} & =-2 L_{m} L_{m-r} \\
D F_{r} F_{2 m}-L_{r} L_{m}^{2} & =-2 L_{m} L_{m-r} \\
F_{r} F_{2 m}-2 L_{r} F_{m}^{2} & =2(-1)^{r+1} F_{m} F_{m-r} \\
D F_{r} F_{2 m}-2 L_{r} L_{m}^{2} & =2(-1)^{r+1} L_{m} L_{m-r}
\end{aligned}
$$

Type IV

$$
\begin{aligned}
F_{r} F_{2 m}-2 L_{r} F_{m}^{2} & =-2 F_{m} F_{m-r} \\
D F_{r} F_{2 m}-2 L_{r} L_{m}^{2} & =-2 L_{m} L_{m-r}
\end{aligned}
$$

7. (a) Fibonacci-Lucas identity used: $L_{n}^{2}-F_{n}^{2}=4 F_{n-1} F_{n+1}$
(b) Type I extension: $b^{2} L_{n}^{2}-a^{2} F_{n}^{2}=4 F_{n-1} F_{n+1}$

Types I \& III

$$
\begin{aligned}
F_{r}^{2} L_{n}^{2}-L_{r}^{2} F_{n}^{2} & =4(-1)^{r+1} F_{n+r} F_{n-r}, \text { I; }(-1)^{r+1} F_{n+r} F_{n-r}, \text { III } \\
D^{2} F_{r}^{2} F_{n}^{2}-L_{r}^{2} L_{n}^{2} & =4(-1)^{r+1} L_{n+r} L_{n-r}, \text { I; }(-1)^{r+1} L_{n+r} L_{n-r}, \text { III }
\end{aligned}
$$

Types II \& IV

$$
\begin{aligned}
F_{r}^{2} F_{n}^{2}-L_{r}^{2} F_{n}^{2} & =-4 F_{n+r} F_{n-r}, \text { II; }-F_{n+r} F_{n-r}, \text { IV } \\
D^{2} F_{r}^{2} F_{n}^{2}-L_{r}^{2} L_{n}^{2} & =-4 L_{n+r} L_{n-r}, \text { II; }-L_{n+r} L_{n-r}, \text { IV }
\end{aligned}
$$

8. (a) Fibonacci-Lucas identity used: $L_{2 n} L_{2 n+2}-5 F_{2 n+1}^{2}=1$
(b) Type I extension: $L_{2 n} L_{2 n+2}-D F_{2 n+1}^{2}=a^{2}$
(c) Generalizations:

All Types

$$
\begin{aligned}
& L_{2 n} L_{2 n+2 r}-D F_{2 n+r}^{2}=L_{r}^{2} \\
& L_{2 n+r}^{2}-D F_{2 n} F_{2 n+2 r}=L_{r}^{2}
\end{aligned}
$$

9. (a) Fibonacci-Lucas identity used:

$$
F_{r+m+n}=F_{m+1} F_{n+1} F_{r+1}+F_{m} F_{n} F_{r}-F_{m-1} F_{n-1} F_{r-1}
$$

(b) Type I extension:

$$
a b^{2} F_{r+m+n}=F_{m+1} F_{n+1} F_{r+1}+a F_{m} F_{n} F_{r}-F_{m-1} F_{n-1} F_{r-1}
$$

(c) Generalizations:

Type I $\quad F_{m+2 t+1} F_{n+2 t+1} F_{r+2 t+1}+L_{2 t+1} F_{m} F_{n} F_{r}-F_{m-2 t-1} F_{n-2 t-1} F_{r-2 t-1}$

$$
=\frac{1}{D}\left(L_{6 t+3}+L_{2 t+1}\right) F_{m+n+r}=F_{2 t+1} F_{4 t+2} F_{m+n+r}
$$

$$
L_{m+2 t+1} L_{n+2 t+1} L_{r+2 t+1}+L_{2 t+1} L_{m} L_{n} L_{r}-L_{m-2 t-1} L_{n-2 t-1} L_{r-2 t-1}
$$

$$
=\left(L_{6 t+3}+L_{2 t+1}\right) L_{m+n+r}=D F_{2 t+1} F_{4 t+2} F_{m+n+r}
$$

$$
F_{m+2 t} F_{n+2 t} F_{r+2 t}-L_{2 t} F_{m} F_{n} F_{r}+F_{m-2 t} F_{n-2 t} F_{r-2 t}=\frac{1}{D}\left(L_{6 t}-L_{2 t}\right) F_{m+n+r}
$$

$$
L_{m+2 t} L_{n+2 t} L_{r+2 t}-L_{2 t} L_{m} L_{n} L_{r}+L_{m-2 t} L_{n-2 t} L_{r-2 t}=\left(L_{6 t}-L_{2 t}\right) L_{m+n+r}
$$

$$
=D F_{2 t} F_{4 t} L_{m+n+r}
$$

Type II

$$
F_{m+t} F_{n+t} F_{r+t}-L_{t} F_{m} F_{n} F_{r}+F_{m-t} F_{n-t} F_{r-t}=\frac{1}{D}\left(L_{3 t}-L_{t}\right) F_{m+n+r}
$$

$$
L_{m+t} L_{n+t} L_{r+t}-L_{t} L_{m} L_{n} L_{r}+L_{m-t} L_{n-t} L_{r-t}=\left(L_{3 t}-L_{t}\right) L_{m+n+r}
$$

Type III $F_{m+2 t+1} F_{n+2 t+1} F_{r+2 t+1}+2 L_{2 t+1} F_{m} F_{n} F_{r}-F_{m-2 t-1} F_{n-2 t-1} F_{r-2 t-1}$

$$
=\frac{1}{2 D}\left(L_{6 t+3}+L_{2 t+1}\right) F_{m+n+r}=F_{2 t+1} F_{4 t+2} F_{m+n+r}
$$

$$
L_{m+2 t+1} L_{n+2 t+1} F_{r+2 t+1}+2 L_{2 t+1} L_{m} L_{n} L_{r}-L_{m-2 t-1} L_{n-2 t-1} L_{r-2 t-1}
$$

$$
=\frac{1}{2}\left(L_{6 t+3}+L_{2 t+1}\right) L_{m+n+r}=D F_{2 t+1} F_{4 t+2} L_{m+n+r}
$$

$$
F_{m+2 t} F_{n+2 t} F_{r+2 t}-2 L_{2 t} F_{m} F_{n} F_{r}+F_{m-2 t} F_{n-2 t} F_{r-2 t}=\frac{1}{2 D}\left(L_{6 t}-L_{2 t}\right) F_{m+n+r}
$$

$$
=F_{2 t} F_{4 t} F_{m+n+r}
$$

$$
L_{m+2 t} L_{n+2 t} L_{r+2 t}-2 L_{2 t} L_{m} L_{n} L_{r}+L_{m-2 t} L_{n-2 t} L_{r-2 t}=\frac{1}{2}\left(L_{6 t}-L_{2 t}\right) F_{m+n+r}
$$

Type IV
$F_{m+t} F_{n+t} F_{r+t}-2 L_{t} F_{m} F_{n} F_{r}+F_{m-t} F_{n-t} F_{r-t}=\frac{1}{2 D}\left(L_{3 t}-L_{t}\right) F_{m+n+r}$
$L_{m+t} L_{n+t} L_{r+t}-2 L_{t} L_{m} L_{n} L_{r}+L_{m-t} L_{n-t} L_{r-t}=\frac{1}{2}\left(L_{3 t}-L_{t}\right) L_{m+n+r}$
10. (a) Fibonacci-Lucas identity used:

$$
F_{n+1}^{2}+F_{n}^{2}+F_{n-1}^{2}=2\left(F_{n+1}^{2}-F_{n} F_{n-1}\right)
$$

(b) Type I extension:

$$
F_{n+1}^{2}+a^{2} F_{n}^{2}+F_{n-1}^{2}=2\left(F_{n+1}^{2}-a F_{n} F_{n-1}\right)
$$

(c) Generalizations:

Type I $\quad F_{n+2 r+1}^{2}+L_{2 r+1}^{2} F_{n}^{2}+F_{n-2 r-1}^{2}=2\left(F_{n+2 r+1}^{2}-L_{2 r+1} F_{n-2 r-1} F_{n}\right)$ $L_{n+2 r+1}^{2}+L_{2 r+1}^{2} L_{n}^{2}+L_{n-2 r-1}^{2}=2\left(L_{n+2 r+1}^{2}-L_{2 r+1} L_{n-2 r-1} L_{n}\right)$ $F_{n+2 r}^{2}+L_{2 r}^{2} F_{n}^{2}+F_{n-2 r}^{2}=2\left(F_{n+2 r}^{2}+L_{2 r} F_{n-2 r} F_{n}\right)$ $L_{n+2 r}^{2}+L_{2 r}^{2} L_{n}^{2}+L_{n-2 r}^{2}=2\left(L_{n+2 r}^{2}+L_{2 r} L_{n-2 r} L_{n}\right)$
Type II $\quad F_{n+r}^{2}+L_{r}^{2} F_{n}^{2}+F_{n-r}^{2}=2\left(F_{n+r}^{2}+L_{r} F_{n} F_{n-r}\right)$ $L_{n+r}^{2}+L_{r}^{2} L_{n}^{2}+L_{n-r}^{2}=2\left(L_{n+r}^{2}+L_{r} L_{n} L_{n-r}\right)$
Type III $F_{n+2 r+1}^{2}+4 L_{2 r+1}^{2} F_{n}^{2}+F_{n-2 r-1}^{2}=2\left(F_{n+2 r+1}^{2}-2 L_{2 r+1} F_{n-2 r-1} F_{n}\right)$ $L_{n+2 r+1}^{2}+4 L_{2 r+1}^{2} L_{n}^{2}+L_{n-2 r-1}^{2}=2\left(L_{n+2 r+1}^{2}-2 L_{2 r+1} L_{n-2 r-1} L_{n}\right)$
$F_{n+2 r}^{2}+4 L_{2 r}^{2} F_{n}^{2}+F_{n-2 r}^{2}=2\left(F_{n+2 r}^{2}+2 L_{2 r} F_{n-2 r} F_{n}\right)$ $L_{n+2 r}^{2}+4 L_{2 r}^{2} L_{n}^{2}+L_{n-2 r}^{2}=2\left(L_{n+2 r}^{2}+2 L_{2 r} L_{n-2 r} L_{n}\right)$

Type IV
$F_{n+r}^{2}+4 L_{r}^{2} F_{n}^{2}+F_{n-r}^{2}=2\left(F_{n+r}^{2}+2 L_{r} F_{n} F_{n-r}\right)$
$L_{n+r}^{2}+4 L_{r}^{2} L_{n}^{2}+L_{n-r}^{2}=2\left(L_{n+r}^{2}+2 L_{r} L_{n} L_{n-r}\right)$
11. (a) Fibonacci-Lucas identity used:

$$
F_{n+2}^{3}=F_{n}^{3}+F_{n+1}^{3}+3 F_{n} F_{n+1} F_{n+2}
$$

(b) Type I extension:

$$
F_{n+2}^{3}=F_{n}^{3}+a^{3} F_{n+1}^{3}+3 a F_{n} F_{n+1} F_{n+2}
$$

(c) Generalizations:

Type I $\quad F_{n+2 r+1}^{3}=F_{n-2 r-1}^{3}+L_{2 r+1}^{3} F_{n}^{3}+3 L_{2 r+1} F F_{n+2 r+1} F_{n-2 r-1}$ $L_{n+2 r+1}^{3}=L_{n-2 r-1}^{3}+L_{2 r+1}^{3} L_{n}^{3}+3 L_{2 r+1} L L_{n+2 r+1} L_{n-2 r-1}$ $F_{n+2 t}^{3}=L_{2 t}^{3} F_{n}^{3}-F_{n-2 t}^{3}-3 L_{2 t} F_{n-2 t} F_{n} F_{n+2 t}$ $L_{n+2 t}^{3}=L_{2 t}^{3} L_{n}^{3}-L_{n-2 t}^{3}-3 L_{2 t} L_{n-2 t} L_{n} L_{n+2 t}$

Type II
$F_{n+r}^{3}=L_{r}^{3} F_{n}^{3}-F_{n-r}^{3}-3 L_{r} F_{n} F_{n-r} F_{n+r}$
$L_{n+r}^{3}=L_{r}^{3} L_{n}^{3}-L_{n-r}^{3}-3 L_{r} L_{n} L_{n-r} L_{n+r}$
Type III $F_{n+2 r+1}^{3}=F_{n-2 r-1}^{3}+8 L_{2 r+1}^{3} F_{n}^{3}+6 L_{2 r+1} F_{n} F_{n+2 r+1} F_{n-2 r-1}$ $L_{n+2 r+1}^{3}=L_{n-2 r-1}^{3}+8 L_{2 r+1}^{3} L_{n}^{3}+6 L_{2 r+1} L_{n} L_{n+2 r+1} L_{n-2 r-1}$
$F_{n+2 t}^{3}=8 L_{2 t}^{3} F_{n}^{3}-F_{n-2 t}^{3}-6 L_{2 t} F_{n-2 t} F_{n} F_{n+2 t}$
$L_{n+2 t}^{3}=8 L_{2 t}^{3} L_{n}^{3}-L_{n-2 t}^{3}-6 L_{2 t} L_{n-2 t} L_{n} L_{n+2 t}$
[Dec.

Type IV $F_{n+r}^{3}=8 L_{r}^{3} F_{n}^{3}-F_{n-r}^{3}-6 L_{r} F_{n} F_{n-r} F_{n+r}$
$L_{n+r}^{3}=8 L_{r}^{3} L_{n}^{3}-L_{n-r}^{3}-6 L_{r} L_{n} L_{n-r} L_{n+r}$
12. (a) Fibonacci-Lucas identity used:

$$
F_{n+1}^{4}+F_{n}^{4}+F_{n-1}^{4}=2\left[F_{n+1}^{2}-F_{n} F_{n-1}\right]^{2}
$$

(b) Type I extension:

$$
F_{n+1}^{4}+a^{4} F_{n}^{4}+F_{n-1}^{4}=2\left[F_{n+1}^{2}-a F_{n} F_{n-1}\right]^{2}
$$

(c) Generalizations:

Type I $\quad F_{n+2 r+1}^{4}+L_{2 r+1}^{4} F_{n}^{4}+F_{n-2 r-1}^{4}=2\left[F_{n+2 r+1}^{2}-L_{2 r+1} F_{n} F_{n-2 r-1}\right]^{2}$
$L_{n+2 r+1}^{4}+L_{2 r+1}^{4} L_{n}^{4}+L_{n-2 r-1}^{4}=2\left[L_{n+2 r+1}^{2}-L_{2 r+1} L_{n} L_{n-2 r-1}\right]^{2}$
$F_{n+2 t}^{4}+L_{2 t}^{4} F_{n}^{4}+F_{n-2 t}^{4}=2\left[F_{n+2 t}^{2}+L_{2 t} F_{n} F_{n-2 t}\right]^{2}$
$L_{n+2 t}^{4}+L_{2 t}^{4} L_{n}^{4}+L_{n-2 t}^{4}=2\left[L_{n+2 t}^{2}+L_{2 t} L_{n} L_{n-2 t}\right]^{2}$
Type II
$F_{n+r}^{4}+L_{p}^{4} F^{4}+F_{n-r}^{4}=2\left[F_{n+r}^{2}+L_{r} F_{n} F_{n-r}\right]^{2}$
$L_{n+r}^{4}+L_{r}^{4} L^{4}+L_{n-r}^{4}=2\left[L_{n+r}^{2}+L_{r} L_{n} L_{n-r}\right]^{2}$
Type III $F_{n+2 r+1}^{4}+16 L_{2 r+1}^{4} F_{n}^{4}+\dot{F}_{n-2 r-1}^{4}=2\left[F_{n+2 r+1}^{2}-2 L_{2 r+1} F_{n} F_{n-2 r-1}\right]^{2}$
$L_{n+2 r+1}^{4}+16 L_{2 r+1}^{4} L_{n}^{4}+L_{n-2 r-1}^{4}=2\left[L_{n+2 r+1}^{2}-2 L_{2 r+1} L_{n} L_{n-2 r-1}\right]^{2}$
$F_{n+2 t}^{4}+16 L_{2 t}^{4} F_{n}^{4}+F_{n-2 t}^{4}=2\left[F_{n+2 t}^{2}+2 L_{2 t} F_{n} F_{n-2 t}\right]^{2}$
$L_{n+2 t}^{4}+16 L_{2 t}^{4} L_{n}^{4}+L_{n-2 t}^{4}=2\left[L_{n+2 t}^{2}+2 L_{2 t} L_{n} L_{n-2 t}\right]^{2}$
Type IV
$F_{n+r}^{4}+16 L_{r}^{4} F_{n}^{4}+F_{n-r}^{4}=2\left[F_{n+r}^{2}+2 L_{r} F_{n} F_{n-r}\right]^{2}$
$L_{n+r}^{4}+16 L_{r}^{4} L_{n}^{4}+L_{n-r}^{4}=2\left[L_{n+r}^{2}+2 L_{r} L_{n} L_{n-r}\right]^{2}$
13. (a) Fibonacci-Lucas identity used:

$$
F_{n+1}^{5}-F_{n}^{5}-F_{n-1}^{5}=5 F_{n} F_{n-1} F_{n+1}\left(F_{n+1}^{2}-F_{n-1} F_{n}\right)
$$

(b) Type I extension:

$$
F_{n+1}^{5}-\alpha^{5} F_{n}^{5}-F_{n-1}^{5}=5 \alpha F_{n} F_{n-1} F_{n+1}\left(F_{n+1}^{2}-\alpha F_{n-1} F_{n}\right)
$$

(c) Generalizations:

Type I

$$
\begin{aligned}
F_{n+2 r+1}^{5}- & L_{2 r+1}^{5} F_{n}^{5}-F_{n-2 r-1}^{5}=5 L_{2 r+1} F_{n} F_{n-2 r-1} F_{n+2 r+1}\left(F_{n+2 r+1}^{2}-L_{2 r+1} F_{n} F_{n-2 r-1}\right) \\
L_{n+2 r+1}^{5}- & L_{2 r+1}^{5} L_{n}^{5}-L_{n-2 r-1}^{5}=5 L_{2 r+1} L_{n} L_{n-2 r-1} L_{n+2 r+1}\left(L_{n+2 r+1}^{2}-L_{2 r+1} L_{n} L_{n-2} \quad 1\right) \\
& L_{2 t}^{5} F_{n}^{5}-F_{n+2 t}^{5}-F_{n-2 t}^{5}=5 L_{2 t} F_{n} F_{n-2 t} F_{n+2 t}\left(F_{n+2 t}^{2}+L_{2 t} F_{n} F_{n-2 t}\right) \\
& L_{2 t}^{5} L_{n}^{5}-L_{n+2 t}^{5}-L_{n-2 t}^{5}=5 L_{2 t} L_{n} L_{n-2 t} L_{n+2 t}\left(L_{n+2 t}^{2}+L_{2 t} L_{n} L_{n-2 t}\right)
\end{aligned}
$$

Type II $L_{r}^{5} F_{n}^{5}-F_{n+r}^{5}-F_{n-r}^{5}=5 L_{r} F_{n} F_{n-} F_{n+r}\left(F_{n+r}^{2}+L_{r} F_{n} F_{n-r}\right)$ $L_{r}^{5} L_{n}^{5}-L_{n+r}^{5}-L_{n-r}^{5}=5 L_{r} L_{n} L_{n-} L_{n+r}\left(L_{n+r}^{2}+L_{r} L_{n} L_{n-r}\right)$

## Type III

$F_{n+2 r+1}^{5}-32 L_{2 r+1}^{5} F_{n}^{5}-F_{n-2 r-1}^{5}=10 L_{2 r+1} F_{n} F_{n-2 r-1} F_{n+2 r+1}\left(F_{n+2 r+1}^{2}-2 L_{2 r+1} F_{n} F_{n-2 r-1}\right)$
$L_{n+2 r+1}^{5}-32 L_{2 r+1}^{5} L_{n}^{5}-L_{n-2 r-1}^{5}=10 L_{2 r+1} L_{n} L_{n-2 r-1} L_{n+2 r+1}\left(L_{n+2 r+1}^{2}-2 L_{2 r+1} L_{n} L_{n-2 r-1}\right)$

$$
\begin{aligned}
& 32 L_{2 t}^{5} F_{n}^{5}-F_{n+2 t}^{5}-F_{n-2 t}^{5}=10 L_{2 t} F_{n} F_{n-2 t} F_{n+2 t}\left(F_{n+2 t}^{2}+2 L_{2 t} F_{n} F_{n-2 t}\right) \\
& 32 L_{2 t}^{5} L_{n}^{5}-L_{n+2 t}^{5}-L_{n-2 t}^{5}=10 L_{2 t} L_{n} L_{n-2 t} L_{n+2 t}\left(L_{n+2 t}^{2}+2 L_{2 t} L_{n} L_{n-2 t}\right)
\end{aligned}
$$

Type IV

$$
\begin{aligned}
& 32 L_{r}^{5} F_{n}^{5}-F_{n+r}^{5}-F_{n-r}^{5}=10 L_{r} F_{n} F_{n-r} F_{n+r}\left(F_{n+r}^{2}+2 L_{r} F_{n} F_{n-r}\right) \\
& 32 L_{r}^{5} L_{n}^{5}-L_{n+r}^{5}-L_{n-r}^{5}=10 L_{r} L_{n} L_{n-r} L_{n+r}\left(L_{n+r}^{2}+2 L_{r} L_{n} L_{n-r}\right)
\end{aligned}
$$

14. (a) Fibonacci-Lucas identity used: $L_{n}^{3}=2 F_{n-1}^{3}+F_{n}^{3}+6 F_{n+1}^{2} F_{n-1}$
(b) Type I extension: $b^{3} L_{n}^{3}=2 F_{n-1}^{3}+\alpha^{3} F_{n}^{3}+6 F_{n+1}^{2} F_{n-1}$
(c) Generalizations:

Type I

$$
\begin{aligned}
& F_{2 r+1}^{3} L_{n}^{3}=2 F_{n-2 r-1}^{3}+L_{2 r+1}^{3} F_{n}^{3}+6 F_{n+2 r+1}^{2} F_{n-2 r-1} \\
& D^{3} F_{2 r+1}^{3} F_{n}^{3}=2 L_{n-2 r-1}^{3}+L_{2 r+1}^{3} L_{n}^{3}+6 L_{n+2 r+1}^{2} L_{n-2 r-1} \\
& F_{2 r}^{3} L_{n}^{3}=L_{2 r}^{3} F_{n}^{3}-2 F_{n-2 r}^{3}-6 F_{n+2 r}^{2} F_{n-2 r} \\
& D^{3} F_{2 r}^{3} F_{n}^{3}=L_{2 r}^{3} L_{n}^{3}-2 L_{n-2 r}^{3}-6 L_{n+2 r}^{2} L_{n-2 r}
\end{aligned}
$$

Type II

$$
\begin{array}{ll}
\text { Type II } & F_{r}^{3} L_{n}^{3}=L_{r}^{3} F_{n}^{3}-2 F_{n-r}^{3}-6 F_{n+r}^{2} F_{n-r} \\
& D^{3} F_{r}^{3} F_{n}^{3}=L_{r}^{3} L_{n}^{3}-2 L_{n-r}^{3}-6 L_{n+r}^{2} L_{n-r} \\
\text { Type III } \quad & 4 F_{2 r+1}^{3} L_{n}^{3}=F_{n-2 r-1}^{3}+4 L_{2 r+1}^{3} F_{n}^{3}+3 F_{n+2 r+1}^{2} F_{n-2 r-1} \\
& 4 D^{3} F_{2 r+1}^{3} F_{n}^{3}=L_{n-2 r-1}^{3}+4 L_{2 r+1}^{3} L_{n}^{3}+3 L_{n+2 r+1}^{2} L_{n-2 r-1} \\
& 4 F_{2 r}^{3} L_{n}^{3}=4 L_{2 r}^{3} F_{n}^{3}-F_{n-2 r}^{3}-3 F_{n+2 r}^{2} F_{n-2 r} \\
& 4 D^{3} F_{2 r}^{3} F_{n}^{3}=4 L_{2 r}^{3} L_{n}^{3}-L_{n-2 r}^{3}-3 L_{n+2 r}^{2} L_{n-2 r} \\
& 4 F_{r}^{3} L_{n}^{3}=4 L_{r}^{3} F_{n}^{3}-F_{n-r}^{3}-3 F_{n+r}^{2} F_{n-r} \\
\text { Type IV } & 4 D^{3} F_{r}^{3} F_{n}^{3}=4 L_{r}^{3} L_{n}^{3}-L_{n-r}^{3}-3 L_{n+r}^{2} L_{n-r}
\end{array}
$$

Type IV

## Concluding Remarks

Following the suggestions of the referee and the editor, the proofs of the 14 identity sets have been omitted. They are tedious and do involve complicated, albeit fairly elementary, calculations. For some readers, the proofs would involve the use of composition algebras which are not developed in the article and which may not be well known.

The author has completed a supplementary paper giving, with indicated proof, the Type I, Type II, Type III, and Type IV composition algebras. After each composition albegra the corresponding identities using that algebra have been stated and proved. Copies of this paper may be obtained by request from the author.

## 

## A FORMULA FOR TRIBONACCI NUMBERS

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In a recent paper [2], Scott, Delaney, and Hoggatt discussed the Tribonacci numbers $T_{n}$ defined by

$$
T_{0}=1, T_{1}=1, T_{2}=2 \text { and } T_{n}=T_{n-1}+T_{n-2}+T_{n-3} \text {, for } n \geq 3 \text {, }
$$

and found its generating function, which is written here in terms of the complex variable $z$, to be

$$
\begin{equation*}
f(z)=\frac{1}{1-z-z^{2}-z^{3}}=\sum_{n=0}^{\infty} T_{n} z^{n} . \tag{1}
\end{equation*}
$$

In this brief note, a formula for $T_{n}$ is found by means of an analytic method similar to that used by Hagis [1].

Observe that

$$
\begin{equation*}
z^{3}+z^{2}+z-1=(z-r)(z-s)(z-\bar{s}) \tag{2}
\end{equation*}
$$

where $r=.5436890127$,
$s=-.7718445064+1.115142580 i$,
$|s|=1.356203066$,
and
$|r-s|=1.724578573 ;$
thus $f(z)$ is meromorphic with simple poles at the points $z=r, z=s$, and $z=\bar{s}$, all of which lie within an annulus centered at the origin with inner radius of .5 and outer radius of 2 .

By the Cauchy integral theorem,

$$
T_{n}=\frac{f^{(n)}(0)}{n!}=\frac{1}{2 \pi i} \int_{|z|=.5} \frac{f(z) d z}{z^{n+1}}
$$

and by the Cauchy residue theorem,

$$
\begin{equation*}
T_{n}=\frac{1}{2 \pi i} \int_{|z|=R} \frac{f(z) d z}{z^{n+1}}-\left(R_{1}+R_{2}+R_{3}\right) \tag{3}
\end{equation*}
$$

where $R \geq 2$ and $R_{1}, R_{2}$, and $R_{3}$ are the residues of $f(z) / z^{n+1}$ at the poles $r, s$, and $\bar{s}$, respectively.

In particular, since $f(z)=-1 /((z-r)(z-s)(z-\bar{s}))$,
and

$$
\begin{align*}
R_{1} & =\lim _{z \rightarrow r}(z-r) f(z) / z^{n+1}=-1 /\left((r-s)(r-\bar{s}) r^{n+1}\right)  \tag{4}\\
& =-1 /\left(|r-s|^{2} r^{n+1}\right) \\
R_{2} & =\lim _{z \rightarrow s}(z-s) f(z) / z^{n+1}=-1 /\left((s-r)(s-\bar{s}) s^{n+1}\right)  \tag{5}\\
R_{3} & =\lim _{z \rightarrow s}(z-\bar{s}) f(z) / z^{n+1}=-1 /\left((s-r)(s-s) \bar{s}^{n+1}\right)=\bar{R}_{2} .
\end{align*}
$$

Along the circle $|z|=R \geq 2$ we have

$$
|f(z)|=\frac{1}{\left|z^{3}+z^{2}+z-1\right|} \leq \frac{1}{\left||z|^{3}-\left|z^{2}+z-1\right|\right|} \leq \frac{1}{R^{3}-R^{2}-R-1}
$$

hence

$$
\begin{equation*}
\left|\frac{1}{2 \pi i} \int_{|z|=R} \frac{f(z) d z}{z^{n+1}}\right| \leq \frac{1}{R\left(R^{3}-R^{2}-R-1\right)} \tag{7}
\end{equation*}
$$

Now, if $R$ is taken arbitrarily large, then from (3) and (7) it follows that

$$
\begin{equation*}
T_{n}=-\left(R_{1}+R_{2}+R_{3}\right) . \tag{8}
\end{equation*}
$$

One final estimate is needed to obtain the desired formula. From (5) we have for $n \geq 0$,

$$
\left|R_{2}\right|=\frac{1}{|s-r||s-\bar{s}||s|^{n+1}}=\frac{1}{2|s-r||\operatorname{In} s||s|^{n+1}}<.26 /|s|^{n+1}<.2
$$

which along with (8) and (6) implies
so

$$
T_{n}+R_{1}=-R_{2}-R_{3},
$$

hence

$$
\left|T_{n}+R_{1}\right|=\left|R_{2}+R_{3}\right| \leq 2\left|R_{2}\right|<.4
$$

or, equivalently,

$$
\begin{aligned}
& T_{n}-.4<-R_{1}<T_{n}+.4 \\
& T_{n}<-R_{1}+.4<T_{n}+1 .
\end{aligned}
$$

Substituting the value of $R_{1}$ from (4) into (9) we may rewrite (9) in terms of the greatest integer function and obtain the desired formula:

$$
T_{n}=\left[\frac{1}{|r-s|^{2} r^{n+1}}+.4\right]
$$

REFERENCES

1. P. Hagis. "An Analytic Proof of the Formula for $F_{n}$," The Fibonacci Quarterly 2 (1964):267-68.
2. A. Scott, T. Delaney, \& V.E. Hoggatt, Jr. "The Tribonacci Sequence." The Fibonacci Quarterly 15 (1977):193-200.

## *****

POLYNOMIALS ASSOCIATED WITH GEGENBAUER POLYNOMIALS
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1. INTRODUCTION

Chebyshev polynomials $T_{n}(x)$ of the first kind and $U_{n}(x)$ of the second kind are, respectively, defined as follows:

$$
\begin{array}{ll}
T_{n}(x)=\cos \left(n \cos ^{-1} x\right) & (|x| \leq 1) \\
U_{n}(x)=\frac{\sin \left[(n+1) \cos ^{-1} x\right]}{\sin \left(\cos ^{-1} x\right)} & (|x| \leq 1)
\end{array}
$$

In 1974 Jaiswal [6] investigated polynomials $p_{n}(x)$ related to $U_{n}(x)$. In 1977 Horadam [5] obtained similar results for polynomials $q_{n}(x)$, associated with $T_{n}(x)$. The polynomials $p_{n}(x)$ and $q_{n}(x)$ are defined as follows:

$$
\left\{\begin{array}{l}
p_{n}(x)=2 x p_{n-1}(x)-p_{n-3}(x) \quad(n \geq 3) \text { with }  \tag{1}\\
p_{0}(x)=0, p_{1}(x)=1, p_{2}(x)=2 x
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
q_{n}(x)=2 x q_{n-1}(x)-q_{n-3}(x) \quad(n \geq 3) \text { with }  \tag{2}\\
q_{0}(x)=0, q_{1}(x)=2, q_{2}(x)=2 x
\end{array}\right.
$$

Chebyshev's polynomials of both kinds are special cases of Gegenbauer polynomials ([1], [2], [3], [8], [9]) $C_{n}^{\lambda}(x)\left(\lambda>-\frac{1}{2},|x| \leq 1\right)$ defined by

$$
C_{0}^{\lambda}(x)=1, C_{1}^{\lambda}(x)=2 \lambda x,
$$

with the recurrence relation

$$
n C_{n}^{\lambda}(x)=2(\lambda+n-1) x C_{n-1}^{\lambda}(x)-(2 \lambda+n-2) C_{n-2}^{\lambda}(x), \quad n \geq 2
$$

Polynomials $C_{n}^{\lambda}(x)$ are related to $T_{n}(x)$ and $U_{n}(x)$ by the relations
and

$$
T_{n}(x)=\frac{n}{2} \lim _{\lambda \rightarrow 0} \frac{C_{n}^{\lambda}(x)}{\lambda} \quad(n \geq 1)
$$

$$
U_{n}(x)=C_{n}^{1}(x)
$$

In Jaiswal [6] and Horadam [5], it was established that $x=1$ in (1) and (2) yields simple relationships with the Fibonacci numbers $F_{n}$ defined by

$$
F_{0}=0, F_{1}=1, \text { and } F_{n}=F_{n-1}+F_{n-2} \quad(n \geq 2),
$$

namely,

$$
\begin{aligned}
& p_{n}(1)=F_{n+2}-1 \\
& q_{n}(1)=2 F_{n} .
\end{aligned}
$$

These results prompt the thought that some generalized Fibonacci connection might exist for $C_{n}^{\lambda}(x)$.

In the following sections, we define the polynomials $p_{n}^{\lambda}(x)$ related to $C_{n}^{\lambda}(x)$, determine their generating function, investigate a few properties, and exhibit the connection between these polynomials and Fibonacci numbers.

$$
\text { 2. THE POLYNOMIALS } p_{n}^{\lambda}(x)
$$

## Letting

$$
(\lambda)_{0}=1 \quad \text { and } \quad(\lambda)_{n}=\lambda(\lambda+1) \ldots(\lambda+n-1), n=1,2, \ldots,
$$

we find that the first few Gegenbauer polynomials are

$$
\begin{equation*}
C_{0}^{\lambda}(x)=1, C_{1}^{\lambda}(x)=2 \lambda x, \quad C_{2}^{\lambda}(x)=\frac{(\lambda)_{2}}{2!}(2 x)^{2}-\lambda . \tag{4}
\end{equation*}
$$

Listing the polynomials of (4) horizontally and taking sums along the rising diagonals, we get the resulting polynomials denoted by $p_{n}^{\lambda}(x)$. The first few polynomials $p_{n}^{\lambda}(x)$ are given by

$$
\begin{equation*}
p_{1}^{\lambda}(x)=1, p_{2}^{\lambda}(x)=2 \lambda x, p_{3}^{\lambda}(x)=\frac{(\lambda)_{2}}{2!}(2 x)^{2}, p_{4}^{\lambda}(x)=\frac{(\lambda)_{3}}{3!}(2 x)^{3}-\lambda . \tag{5}
\end{equation*}
$$

We define $p_{0}^{\lambda}(x)=0$.
3. GENERATING FUNCTION

Theorem 1: The generating function $G^{\lambda}(x, t)$ of $p_{n}^{\lambda}(x)$ is given by

$$
G^{\lambda}(x, t)=\sum_{n=1}^{\infty} p_{n}^{\lambda}(x) t^{n-1}=\left(1-2 x t+t^{3}\right)^{-\lambda}
$$

Proof: Putting $2 x=y$ in (4) we obtain the following figure.


It is clear from Figure 1 that the generating function for the $k$ th column is

$$
\frac{(-1)^{k}(\lambda)_{k}}{k!}(1-t y)^{-(\lambda+k)}
$$

Since $p_{n}^{\lambda}(x)$ are obtained by summing along the rising diagonals of Figure 1 , the row-adjusted generating function for the kth column becomes

$$
h_{k}^{\lambda}(y)=\frac{(-1)^{k}(\lambda)_{k}}{k!}(1-t y)^{-(\lambda+k)} t^{3 k}
$$

Since

$$
\sum_{k=0}^{\infty} h_{k}^{\lambda}(y)=\sum_{k=0}^{\infty} \frac{(-1)^{k}(\lambda)_{k}}{k!}\left(\frac{t^{3}}{1-t y}\right)^{k}(1-t y)^{-\lambda}=\left(1-t y+t^{3}\right)^{-\lambda}
$$

the generating function of $p_{n}^{\lambda}(x)$ is given by

$$
\begin{equation*}
G^{\lambda}(x, t)=\sum_{n=1}^{\infty} p_{n}^{\lambda}(x) t^{n-1}=\left(1-2 t x+t^{3}\right)^{-\lambda} \tag{6}
\end{equation*}
$$

Expanding the right-hand side of (6), we obtain

$$
\begin{equation*}
p_{n+1}^{\lambda}(x)=\sum_{k=0}^{[n / 3]} \frac{(-1)^{k}(\lambda)_{n-2 k}}{(n-2 k)!}(n-2 k)(2 x)^{n-3 k} \tag{7}
\end{equation*}
$$

Observe from (1), (5), (6), and (7) that $p_{n}^{1}(x)=p_{n}(x), n=0,1, \ldots$.

## 4. RECURRENCE RELATION

Theorem 2: The recurrence relation is given by

$$
\begin{equation*}
p_{n}^{\lambda}(x)=\frac{(2 x)(\lambda+n-2)}{n-1} p_{n-1}^{\lambda}(x)-\frac{3 \lambda+n-4}{n-1} p_{n-3}^{\lambda}(x), \quad(n \geq 3) . \tag{8}
\end{equation*}
$$

Proof: From (7), the $k$ th term on the right-hand side of (8) is

$$
\left.\begin{array}{rl} 
& (-1)^{k} \frac{(\lambda+n-2)}{n-1} \frac{(\lambda)_{n-2-2 k}}{(n-2-2 k)!}(n-2-2 k \\
k
\end{array}\right)(2 x)^{n-3 k-1}, ~(-1)^{k-1} \frac{(3 \lambda+n-4)}{n-1} \frac{(\lambda)_{n-4-2(k-1)}}{(n-4-2(k-1))!}\binom{n-4-2(k-1)}{k-1}(2 x)^{n-3 k-1} .
$$

After simplification, this becomes

$$
\frac{(-1)^{k}(\lambda)_{n-1-2 k}(2 x)^{n-3 k-1}}{k!(n-1-3 k)!}
$$

which is the $k$ th term on the left-hand side of (8).
Ordinary Fibonacci numbers $F_{n}$ are expressible in two equivalent forms:

$$
\left\{\begin{array}{l}
F_{n}=F_{n-1}+F_{n-2} \cdots  \tag{9}\\
F_{n}=2 F_{n-1}-F_{n-3} \cdots
\end{array}\right.
$$

Observe that expression (8) in Theorem 2 is of the form ( $\beta$ ) in $p_{n}^{\lambda}(x)$. An attempt to obtain the recurrence relation in the corresponding form ( $\alpha$ ), namely,

$$
p_{n}^{\lambda}(x)=A p_{n-1}^{\lambda}(x)+B p_{n-2}^{\lambda}(x),
$$

where $A$ and $B$ are functions of $\lambda$, leads to an intractable cubic. Perhaps the form (8) that follows the patterns of the forms for $p_{n}(x)$ and $q_{n}(x)$ is the best available.

The following recurrence relation involving the derivatives of $p_{n}^{\lambda}(x)$ is easily proved.
Theorem 3:
(10)

$$
2 x\left(p_{n+2}^{\lambda}(x)\right)^{\prime}-3\left(p_{n}^{\lambda}(x)\right)^{\prime}=2(n+1) p_{n+2}^{\lambda}(x)
$$

Equation (10) corresponds to the similar results satisfied by $p_{n}(x)$ and $q_{n}(x)$.

$$
\text { 5. THE POLYNOMIALS } S_{n}(x)
$$

Define
(11)

$$
\left\{\begin{aligned}
& S_{0}(x)=0, S_{1}(x)=3, \text { and } \\
& S_{n}^{\lambda}(x) \equiv S_{n}(x)=(n-1) \lim _{\lambda \rightarrow 0}\left[\frac{p_{n}^{\lambda}(x)}{\lambda}\right] \\
&=\sum_{k=0}^{\left[\frac{n-1}{3}\right]} \frac{(-1)^{k}(n-1)}{n-2 k-1}(n-2 k-1) y^{n-1-3 k} \\
& k(y=2 x), n \geq 2
\end{aligned}\right.
$$

From (5) and (11) we obtain

$$
\left\{\begin{array}{l}
S_{2}(x)=2 x, S_{3}(x)=(2 x)^{2}, S_{4}(x)=(2 x)^{3}-3  \tag{12}\\
S_{5}(x)=(2 x)^{4}-4(2 x), S_{6}(x)=(2 x)^{5}-5(2 x)^{2}, \ldots
\end{array}\right.
$$

Using (7) and (11) and following the argument of Theorem 2, we have
Theorem 4: $\quad S_{n}(x)=2 x S_{n-1}(x)-S_{n-3}(x) \quad(n \geq 3)$.
We readily observe the similarity of the form for $S_{n}(x)$ in Theorem 4 with the forms for $p_{n}(x)$ and $q_{n}(x)$ in (1) and (2).

Letting $\lambda=1$ in (7), using (11), and comparing $k$ th terms, we have
Theorem 5: $\quad S_{n}(x)=p_{n}(x)-2 p_{n-3}(x) \quad(n \geq 3)$.
Theorem 6: $S_{n}(x)=2 q_{n}(x)-p_{n}(x) \quad(n \geq 0)$.
Proo6: From Horadam [5, Eq. 6],

$$
\begin{equation*}
p_{n}(x)=q_{n}(x)+p_{n-3}(x) \tag{i}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
S_{n}(x) & =p_{n}(x)-2\left(p_{n}(x)-q_{n}(x)\right) \quad \text { from Theorem } 5 \text { and (i) } \\
& =2 q_{n}(x)-p_{n}(x),
\end{aligned}
$$

which proves the Theorem.
Letting $x=1$, we have by (3)

$$
S_{n}(1)=2 q_{n}(1)-p_{n}(1)=2 F_{n}-F_{n-1}+1
$$

Using the known generating functions for $p_{n}(x)$ and $q_{n}(x)$ given in [6] and [5], respectively, we can readily deduce the generating function for $S_{n}(x)$ from Theorem 6.

Theorem 2 is valid for all $x$. Hence Theorem 4 also follows from Theorem 2 on dividing throughout by $\lambda$ and letting $\lambda \rightarrow 0$.

$$
\text { 6. THE POLYNOMIALS } q_{n}^{\lambda}(x)
$$

Instead of examining $p_{n}^{\lambda}(x)$ as obtained in (7), suppose one investigates the rising diagonal functions $q_{n}^{\lambda}(x)$ of

$$
\begin{equation*}
n \lim _{\lambda \rightarrow 0} \frac{e_{n}^{\lambda}(x)}{\lambda} \quad(n \geq 1) \tag{13}
\end{equation*}
$$

An explicit formulation of $q_{n}^{\lambda}(x)$ is

$$
\begin{equation*}
q_{n}^{\lambda}(x)=\sum_{k=0}^{[n / 3]]} \frac{(-1)^{k}(n-k)(\lambda)_{n-2 k}^{\prime}}{(n-2 k)!}\binom{n-2 k}{k} y^{n-3 k} \quad(y=2 x) \tag{14}
\end{equation*}
$$

where
(15)

$$
(\lambda)_{n-2 k}^{\prime}=\lambda(\lambda)_{n-2 k} .
$$

Writing

$$
\begin{equation*}
r_{n}^{\lambda}(x)=p_{n+1}^{\lambda}(x)-q_{n}^{\lambda}(x) \tag{16}
\end{equation*}
$$

and using (7) and (14), we obtain

$$
\begin{equation*}
r_{n}^{\lambda}(x)=\sum_{k=0}^{[n / 3]} \frac{(-1)^{k}\left(\lambda^{-1}-n+k\right)}{k!(n-3 k)!}(\lambda)_{n-2 k}^{\prime} y^{n-3 k} \tag{17}
\end{equation*}
$$

Results similar to those obtained for $p_{n}^{\lambda}(x)$ may be obtained for $q_{n}^{\lambda}(x)$. At this stage, it is not certain just how useful a study of $q_{n}^{\lambda}(x)$ and $r_{n}^{\lambda}(x)$ might be.

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* $\because$ * * *


## ENUMERATION OF PERMUTATIONS BY SEQUENCES-II

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1. André [1] discussed the enumeration of permutations by number of sequences; his results are reproduced in Netto's book [5, pp. 105-12]. Let $P(n, s)$ denote the number of permutations of $Z_{n}=\{1,2, \ldots, n\}$ with $s$ ascending or descending sequences. It is convenient to put

$$
\begin{equation*}
P(0, s)=P(1, s)=\delta_{0, s} \tag{1.1}
\end{equation*}
$$

André proved that $P(n, s)$ satisfies

$$
\begin{equation*}
P(n+1, s)=s P(n, s)+2 P(n, s-1)+(n-s+1) P(n, s-2), \tag{1.2}
\end{equation*}
$$

$$
(n \geq 1)
$$

The following generating function for $P(n, s)$ was obtained in [2]:

$$
\begin{equation*}
\sum_{s=0}^{\infty}\left(1-x^{2}\right)^{-n / 2} \frac{z^{n}}{n!} \sum_{s=0}^{n} P(n+1, s) x^{n-s}=\frac{1-x}{1+x}\left(\frac{\sqrt{1-x^{2}}+\sin z}{x-\cos z}\right)^{2} \tag{1.3}
\end{equation*}
$$

However, an explicit formula for $P(n, s)$ was not found.
In the present note, we shall show how an explicit formula for $P(n, s)$ can be obtained. We show first that the polynomial
satisfies

$$
\begin{equation*}
p_{n}(x)=\sum_{s=0}^{n} P(n+1, x)(-x)^{n-s} \tag{1.4}
\end{equation*}
$$

$$
\begin{equation*}
p_{2 n}(x)=\frac{1}{2^{n-1}}(1-x)^{n-1}\left\{2 \sum_{k=1}^{n}(-1)^{n+k} A_{2 n+1, k} T_{n-k+1}(x)-A_{2 n+1}, n+1\right\} \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
p_{2 n-1}(x)=\frac{1}{2^{n-2}}(1-x)^{n-2} \sum_{k=0}^{n-1}(-1)^{k-1}\left(A_{2 n, k}+A_{2 n, k+1}\right) T_{n-k}(x) \tag{1.6}
\end{equation*}
$$

where the $A_{n, k}$ are the Eulerian numbers [3], [7, p. 240] defined by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{z^{n}}{n!} \sum_{k=0}^{n} A_{n, k} x^{k}=\frac{1-x}{1-x e^{z(1-x)}} \tag{1.7}
\end{equation*}
$$

and $T_{n}(x)$ is the Chebychev polynomial of the first kind defined by [6, p. 301] (1.8)

$$
T_{n}(x)=\cos n \phi, x=\cos \phi
$$

Making use of (1.5) and (1.6), explicit formulas for $P(n, s)$ are obtained. For the final results, see (3.7), (3.8), and (4.2), (4.3).
2. In (1.3) take $x=-\cos \phi$, so that
(2.1) $\sum_{n=0}^{\infty}(\sin \phi)^{-n} \frac{z^{n}}{n!} \sum_{s=0}^{n} P(n+1, s)(-\cos \phi)^{n-s}=\frac{1+\cos \phi}{1-\cos \phi}\left(\frac{\sin \phi+\sin z}{\cos \phi+\cos z}\right)^{2}$.
We have $\left(\frac{\sin \phi+\sin z}{\cos \phi+\cos z}\right)^{2}=\tan ^{2} \frac{1}{2}(z+\phi)=\frac{1-\cos (z+\phi)}{1+\cos (z+\phi)}$.
Hence, if we put

$$
\begin{equation*}
\left(\frac{\sin \phi+\sin z}{\cos \phi+\cos z}\right)^{2}=\sum_{n=0}^{\infty} f_{n}(\cos \phi) \frac{z^{n}}{n!} \tag{2.2}
\end{equation*}
$$

it is clear that

$$
\begin{equation*}
f_{n}(\cos \phi)=\frac{d^{n}}{d \phi^{n}} \frac{1-\cos \phi}{1+\cos \phi} . \tag{2.3}
\end{equation*}
$$

To evaluate this derivative, write

$$
-\frac{1-\cos \phi}{1+\cos \phi}=\left(\frac{e^{\phi i}-1}{e^{\phi i}+1}\right)^{2}=1-\frac{4}{e^{\phi i}+1}+\frac{4}{\left(e^{\phi i}+1\right)^{2}}
$$

Then

$$
-\frac{1}{4} \frac{d}{d \phi} \frac{1-\cos \phi}{1+\cos \phi}=\frac{i e^{\phi i}}{\left(e^{\phi i}+1\right)^{2}}-\frac{2 i e^{\phi i}}{\left(e^{\phi i}+1\right)^{3}}=\frac{i}{e^{\phi i}+1}-\frac{3 i}{\left(e^{\phi i}+1\right)^{2}}+\frac{2 i}{\left(e^{\phi i}+1\right)^{3}}
$$

and

$$
-\frac{1}{4} \frac{d^{2}}{d \phi^{2}} \frac{1-\cos \phi}{1+\cos \phi}=\frac{i^{2}}{e^{\phi i}+1}-\frac{7 i^{2}}{\left(e^{\phi i}+1\right)^{2}}+\frac{12 i^{2}}{\left(e^{\phi i}+1\right)^{3}}-\frac{6 i^{2}}{\left(e^{\phi i}+1\right)^{4}} .
$$

The general formula is

$$
\begin{equation*}
(-1)^{n} \frac{1}{4} \frac{d^{n-2}}{d \phi^{n-2}} \frac{1-\cos \phi}{1+\cos \phi}=i^{n-2} \sum_{k=1}^{n}(-1)^{k-1} \frac{(k-1)!S(n, k)}{\left(e^{\phi i}+1\right)^{k}}, \quad(n>2), \tag{2.4}
\end{equation*}
$$

where $S(n, k)$ is the Stirling number of the second kind [7, Ch. 2]:

$$
\sum_{n=k}^{\infty} S(n, k) \frac{z^{n}}{n!}=\frac{1}{k!}\left(e^{z}-1\right)^{k}
$$

The proof of (2.4) by induction is simple. The derivative of the right-hand side is equal to

$$
\begin{aligned}
i^{n-1} \sum_{k=1}^{n}(-1)^{k} \frac{k!S(n, k) e^{\phi i}}{\left(e^{\phi i}+1\right)^{k+1}} & =i^{n-1} \sum_{k=1}^{n}(-1)^{k}\left\{\frac{k!S(n, k)}{\left(e^{\phi i}+1\right)^{k}}-\frac{k!S(n, k)}{\left(e^{\phi i}+1\right)^{k+1}}\right\} \\
& =i^{n-1} \sum_{k=1}^{n+1}(-1)^{k} \frac{(k-1)!}{\left(e^{\phi i}+1\right)^{k}}\{(k S(n, k)+S(n, k-1)\}
\end{aligned}
$$

Since $k S(n, k)+S(n, k-1)=S(n+1, k)$, this evidently completes the induction.
We may rewrite (2.4) in the following form:

$$
\begin{equation*}
\frac{1}{4} \frac{d^{n-2}}{d \phi^{n-2}} \frac{1-\cos \phi}{1+\cos \phi}=\frac{(-1)^{n} i^{n-2}}{\left(e^{\phi i}+1\right)^{n}} \sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n, k)\left(e^{\phi i}+1\right)^{n-k}, \quad(n>2) \tag{2.5}
\end{equation*}
$$

In the next place, we require the identity

$$
\begin{equation*}
\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n, k)(x+1)^{n-k}=\sum_{k=0}^{n-1}(-1)^{n-k-1} A_{n-1, k} x^{k}, \quad(n \geq 1), \tag{2.6}
\end{equation*}
$$

where $A_{n-1, k}$ is the Eulerian number defined by (1.7).
To prove (2.6), take

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{z^{n}}{n!} \sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n, k)(x+1)^{n-k} \\
& \quad=\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!(x+1)^{-k} \sum_{n=k}^{\infty} S(n, k)^{z^{n}(x+1)^{n}} \\
& n! \\
& \\
& =\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x+1)^{-k}\left(e^{z(x+1)}-1\right)^{k} \\
& \quad=\log \left(1+\frac{e^{z(x+1)}-1}{x+1}\right)=\log \frac{x+e^{z(x+1)}}{x+1}
\end{aligned}
$$

Differentiating with respect to $z$, we get

$$
\sum_{n=\infty}^{\infty} \frac{z^{n-1}}{(n-1)!} \sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n, k)(x+1)^{n-k}=\frac{(x+1) e^{z(x+1)}}{x+e^{z(x+1)}}=\frac{1+x}{1+x e^{-z(1+x)}}
$$

On the other hand, by (1.7),

$$
\sum_{n=0}^{\infty}(-1)^{n} \frac{z^{n}}{n!} \sum_{k=0}^{n}(-1)^{k} A_{n, k} x^{k}=\frac{1+x}{1+x e^{-z(1+x)}}
$$

Hence,

$$
\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n, k)(x+1)^{n-k}=\sum_{k=0}^{n-1}(-1)^{n-k-1} A_{n-1, k} x^{k}
$$

3. By (2.5) and (2.6) we have, on replacing $n$ by $n+2$,

$$
\frac{1}{4} \frac{d}{d \phi} \frac{1-\cos \phi}{1+\cos \phi}=\frac{(-1)^{n}}{\left(e^{\phi i}+1\right)^{n-2}} \sum_{k=1}^{n+1}(-1)^{n-k+1} A_{n+1, k} e^{k \phi i}
$$

since $A_{n+1,0}=0$. Moreover, since [3]

$$
\begin{equation*}
A_{n+1, k}=A_{n+1, n-k+2} \quad(1 \leq k \leq n+1) \tag{3.1}
\end{equation*}
$$

we have

$$
\begin{aligned}
\frac{1}{4} \frac{d^{n}}{d \phi^{n}} \frac{1-\cos \phi}{1+\cos \phi} & =\frac{1}{2}(-1)^{n} \sum_{k=1}^{n+1}(-1)^{k+1} A_{n+1, k} \frac{e^{(n-k+2) \phi i}+(-1)^{n} e^{k \phi i}}{\left(e^{\phi i}+1\right)^{n+2}} \\
& =\frac{1}{2}(-i)^{n} \sum_{k=1}^{n+1}(-1)^{k+1} A_{n+1, k} \frac{e^{\frac{1}{2}(n-2 k+2) \phi i}+(-1)^{n} e^{-\frac{1}{2}(n-2 k+2) \phi i}}{\left(e^{\frac{1}{2} \phi i}+e^{-\frac{1}{2} \phi i}\right)^{n+2}}
\end{aligned}
$$

Therefore, in view of (2.3), we get

$$
\begin{equation*}
f_{n}(\cos \phi)=2(-i)^{n} \sum_{k=1}^{n+1}(-1)^{k+1} A_{n+1, k} \frac{e^{\frac{1}{2}(n-2 k+2) \phi i}+(-1)^{n} e^{-\frac{1}{2}(n-2 k+2) \phi i}}{\left(2 \cos \frac{1}{2} \phi\right)^{n+2}} \tag{3.2}
\end{equation*}
$$

It is convenient to consider $n$ even and $n$ odd separately, so that

$$
\begin{equation*}
f_{2 n}(\cos \phi)=\frac{1}{2^{2 n}} \sum_{k=1}^{2 n+1}(-1)^{n+k+1} A_{2 n+1, k} \frac{\cos (n-k+1) \phi}{\left(\cos \frac{1}{2} \phi\right)^{2 n+2}} \tag{3.3}
\end{equation*}
$$

and

$$
f_{2 n-1}(\cos \phi)=\frac{1}{2^{2 n-1}} \sum_{k=1}^{2 n}(-1)^{n+k} A_{2 n, k} \frac{\sin \frac{1}{2}(2 n-2 k+1) \phi}{(\cos \phi)^{2} 1}
$$

By (1.3), (1.4), and (2.2),

$$
\begin{align*}
p_{n}(\cos \phi) & =\frac{1+\cos \phi}{1-\cos \phi} \sin ^{n} \phi f_{n}(\cos \phi) \\
& =2^{n} \cos ^{n+2} \frac{1}{2} \phi \sin ^{n-2} \frac{1}{2} \phi f_{n}(\cos \phi) . \tag{3.5}
\end{align*}
$$

In particular

$$
p_{2 n}(\cos \phi)=2^{2 n} \cos ^{2 n+2} \frac{1}{2} \phi \sin ^{2 n-2} \frac{1}{2} \phi f_{2 n}(\cos \phi),
$$

so that, by (3.3),

$$
\begin{equation*}
p_{2 n}(\cos \phi)=\frac{1}{2^{n-1}}(1-\cos \phi)^{n-1} \cdot \sum_{k=1}^{2 n+1}(-1)^{n+k+1} A_{2 n+1, k} \cos (n-k+1) \phi \tag{3.6}
\end{equation*}
$$

Using (3.1) and (.18), (3.6) gives

$$
\begin{equation*}
p_{2 n}(x)=\frac{1}{2^{n-1}}(1-x)^{n-1}\left\{2 \sum_{k=1}^{n}(-1)^{n+k+1} A_{2 n+1, k} T_{n-k+1}(x)+A_{2 n+1, n+1}\right\} \tag{3.7}
\end{equation*}
$$

This proves (1.5).
Next, replacing $n$ by $2 n-1$ in (3.5), we get

$$
\begin{aligned}
p_{2 n-1}(\cos \phi) & =2^{2 n-1} \cos ^{2 n+1} \frac{1}{2} \phi \sin ^{2 n-3} \frac{1}{2} \phi f_{2 n-1}(\cos \phi) \\
& =\sin ^{2 n-3} \frac{1}{2} \phi \sum_{k=1}^{2 n}(-1)^{n+k} A_{2 n, k} \sin \frac{1}{2}(2 n-2 k+1) \phi
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{2} \sin ^{2 n-4} \frac{1}{2} \phi \sum_{k=1}^{2 n}(-1)^{n+k} A_{2 n, k}\{\cos (n-k) \phi-\cos (n-k+1) \phi\} \\
& =\frac{1}{2^{n-1}}(1-\cos \phi)^{n-2} \sum_{k=1}^{2 n}(-1)^{n+k}\left(A_{2 n, k}+A_{2 n, k+1}\right) \cos (n-k) \phi \\
& =\frac{1}{2^{n-2}}(1-\cos \phi)^{n-2}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}\left(A_{2 n, k}+A_{2 n, k+1}\right) \cos (n-k) \phi+A_{2 n, n}\right\} .
\end{aligned}
$$

Finally, therefore, by (1.8),

$$
\begin{equation*}
p_{2 n-1}(x)=\frac{1}{2^{n-2}}(1-x)^{n-2}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}\left(A_{2 n, k}+A_{2 n, k+1}\right) T_{n-k}(x)+A_{2 n, n}\right\} \tag{3.8}
\end{equation*}
$$

4. We recall that

$$
\begin{align*}
T_{n}(x) & =\frac{1}{2} \sum_{0<2 j \leq n}(-1)^{j}\left\{\binom{n-j}{j}+\binom{n-j-1}{j-1}\right\}(2 x)^{n-2 j}  \tag{4.1}\\
& =2^{n-1} x^{n}+\frac{1}{2} \sum_{0<2 j \leq n}(-1)^{j} \frac{n}{j}\binom{n-j-1}{j-1}(2 x)^{n-2 j}, \quad(n \geq 1)
\end{align*}
$$

Thus (3.7) becomes

$$
\begin{aligned}
P_{2 n}(x)= & \frac{1}{2^{n-1}} \sum_{t=0}^{n-1}(-1)^{t}\binom{n-1}{t} x^{t} \cdot\left\{\sum_{k=1}^{n}(-1)^{n+k-1} A_{2 n+1, k} \sum_{2 j \leq n-k+1}(-1)^{j}\right. \\
& \left.\cdot\left[\binom{n-k-j+1}{j}+\binom{n-k-j}{j-1}\right](2 x)^{n-k-2 j+1}+A_{2 n+1, n+1}\right\} \\
= & \frac{1}{2^{n-1}} \sum_{s=0}(-x)^{2 n-s} \sum_{k=1}^{n} A_{2 n+1, k} \sum_{s=n+k+2 j-t-1}(-1)^{j}\left\{\binom{n-k-j+1}{j}\right. \\
& \left.+\binom{n-k-j}{j-1}\right\}\binom{n-1}{t} 2^{n-k-2 j+1}+\frac{1}{2^{n-1} A_{2 n+1, n+1} \sum_{s=n+1}^{2 n}\binom{n-1}{2 n-s}(-x)^{2 n-s}} .
\end{aligned}
$$

Comparison with (1.4) gives

$$
\begin{align*}
P(2 n+1, s)= & \frac{1}{2^{n-1}} \sum_{k=1}^{n} A_{2 n+1, k} \sum_{s=n+k+2 j-t-1}(-1)^{j}\left\{\binom{n-k-j+1}{j}\right.  \tag{4.2}\\
& \left.+\binom{n-k-j}{j-1}\right\}\binom{n-1}{t} 2^{n-k-2 j+1}+\frac{1}{2^{n-1}}\binom{n-1}{2 n-s} A_{2 n+1, n+1}
\end{align*}
$$

Similarly, it follows from (3.8) that

$$
\left.\begin{array}{rl}
P(2 n, s)= & \frac{1}{2^{n-2}} \sum_{k=0}^{n-1}\left(A_{2 n, k}+A_{2 n, k+1}\right) \sum_{s=n+k+2 j-t-1}(-1)^{j}  \tag{4.3}\\
& \cdot\left\{\binom{n-k-j}{j}\right.
\end{array}+\binom{n-k-j-1}{j-1}\right\}\binom{n-2}{t} 2^{n-k-2 j}, ~\binom{n-2}{2^{n-2}(2 n-s-1} A_{2 n, n} .
$$

5. For numerical checks of the above results, it is probably easier to use (3.7) and (3.8) rather than the explicit formulas (4.2) and (4.3).

It is convenient to recall the following tables for $P(n, s)$ and $A_{n, k}$, respectively:

TABLE 1

| $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 |  |  |  |  |  |  |
| 2 |  | 2 |  |  |  |  |  |
| 3 |  | 2 | 4 |  |  |  |  |
| 4 |  | 2 | 12 | 10 |  |  |  |
| 5 |  | 2 | 28 | 58 | 32 |  |  |
| 6 |  | 2 | 60 | 236 | 300 | 122 |  |
| 7 |  | 2 | 124 | 836 | 1852 | 1682 | 544 |

TABLE 2

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 |  |  |  |  |  |  |
| 2 | 1 | 1 |  |  |  |  |  |
| 3 | 1 | 4 | 1 |  |  |  |  |
| 4 | 1 | 11 | 11 | 1 |  |  |  |
| 5 | 1 | 26 | 66 | 26 | 1 |  |  |
| 6 | 1 | 57 | 302 | 302 | 57 | 1 |  |
| 7 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 |

We first take (3.7) with $n=2$. Then

$$
\begin{aligned}
p_{4}(x) & =\frac{1}{2}(1-x)\left\{2 A_{5,1} T_{2}(x)-2 A_{5,2} T_{1}(x)+A_{5,3}\right\} \\
& =\frac{1}{2}(1-x)\left\{2\left(2 x^{2}-1\right)+52 x+66\right\} \\
& =2 x^{3}-28 x^{2}+58 x-32 .
\end{aligned}
$$

[Dec.

Taking $n=3$ in (3.7), we get

$$
\begin{aligned}
P_{6}(x) & =\frac{1}{4}(1-x)^{2}\left\{-2 A_{7,1} T_{3}(x)+2 A_{7,2} T_{2}(x)-2 A_{7,3} T_{1}(x)+A_{7,4}\right\} \\
& =\frac{1}{4}(1-x)^{2}\left\{-2\left(4 x^{3}-3 x\right)+2 \cdot 120\left(2 x^{2}-1\right)-2 \cdot 1191 x+2416\right\} \\
& =(1-x)^{2}\left(544-1188 x+120 x^{2}-2 x^{3}\right) \\
& =544-1682 x+1852 x^{2}-836 x^{3}+124 x^{4}-2 x^{5} .
\end{aligned}
$$

Next, taking $n=2$ in (3.8), we get

$$
\begin{aligned}
p_{3}(x) & =\sum_{k=0}^{1}(-1)^{k}\left(A_{4, k}+A_{4, k+1}\right) T_{2-k}(x)+A_{4,2} \\
& =A_{4,1} T_{2}(x)-\left(A_{4,1}+A_{4,2}\right) T_{1}(x)+A_{4,2} \\
& =\left(2 x^{2}-1\right)-12 x+11 \\
& =2 x^{2}-12 x+10 .
\end{aligned}
$$

Similarly, taking $n=3$ in (3.8), we get

$$
\begin{aligned}
p_{5}(x) & =\frac{1}{2}(1-x)\left\{\sum_{k=0}^{2}(-1)^{3+k}\left(A_{6, k}+A_{6, k+1}\right) T_{3-k}(x)+A_{6,3}\right\} \\
& =\frac{1}{2}(1-x)\left\{-A_{6,1} T_{3}(x)+\left(A_{6,1}+A_{6,2}\right) T_{2}(x)-\left(A_{6,2}+A_{6,3}\right) T_{1}(x)+A_{6,3}\right\} \\
& =\frac{1}{2}(1-x)\left\{-\left(4 x^{3}-3 x\right)+58\left(2 x^{2}-1\right)-359 x+302\right\} \\
& =2 x^{4}-60 x^{3}+236 x^{2}-300 x+122 .
\end{aligned}
$$

Another partial check is furnished by taking $x=-1$ in (3.7) and (3.8). Since $T_{n}(-1)=\cos n \pi=(-1)^{n}$, it is easily verified that (3.7) and (3.8) reduce to
and

$$
p_{2 n}(-1)=2 \sum_{k=1}^{n}\left(A_{2 n+1, k}+A_{2 n+1, n+1}\right)=\sum_{k=1}^{2 n+1} A_{2 n+1, k}=(2 n+1)!
$$

$$
p_{2 n-1}(-1)=\sum_{k=0}^{n-1}\left(A_{2 n, k}+A_{2 n, k+1}\right)+A_{2 n, n}=\sum_{k=1}^{2 n} A_{2 n, k}=(2 n)!,
$$

respectively.
On the other hand, for $x=1$, it is evident from (3.7) and (3.8) that

$$
\begin{equation*}
p_{n}(1)=0 \quad(n \geq 4) \tag{5.1}
\end{equation*}
$$

Moreover, since $T_{n}(1)=1$, it follows from (3.7) and (3.1) that

$$
\begin{aligned}
p_{2 n}^{(n+1)}(1) & =(-1)^{n-1} \frac{(n-1)!}{2^{n-1}}\left\{2 \sum_{k=1}^{n}(-1)^{n+k+1} A_{2 n+1, k}+A_{2 n+1, n+1}\right\} \\
& =\frac{(n-1)!}{2^{n-1}} \sum_{k=1}^{2 n+1}(-1) A_{2 n+1, k}
\end{aligned}
$$

By (1.7), we have

$$
1+\sum_{n=1}^{\infty} \frac{z^{n}}{n!} \sum_{k=1}^{n}(-1)^{k} A_{n, k}=\frac{2}{e^{2 z}+1}=\sum_{n=0}^{\infty} C_{n} \frac{z^{n}}{n!}
$$

in the notation of Nörlund [5, p. 27]. Hence

$$
\begin{equation*}
p_{2 n}^{(n-1)}(1)=\frac{(n-1)!}{2^{n-1}} C_{2 n+1} . \tag{5.2}
\end{equation*}
$$

For example

$$
p_{6}^{\prime \prime}(1)=3704-5016+1488-40=136 ;
$$

since $C_{7}=272$, this is in agreement with (5.2).
As for $p_{2 n-1}(x)$, it follows from (3.8) that

$$
\begin{aligned}
p_{2 n-1}^{(n-2)}(1) & =(-1)^{n-2} \frac{(n-2)!}{2^{n-2}}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}\left(A_{2 n, k}+A_{2 n, k+1}\right)+A_{2 n, n}\right\} \\
& =\frac{(n-2)!}{2^{n-2}}\left\{\sum_{k=1}^{n}(-1)^{k-1} A_{2 n, k}+\sum_{k=1}^{n-1}(-1)^{k} A_{2 n-k+1}+(-1)^{n} A_{2 n, n}\right\} \\
& =\frac{(n-2)!}{2^{n-2}} \sum_{k=1}^{2 n}(-1)^{k-1} A_{2 n, k},
\end{aligned}
$$

so that

$$
\begin{equation*}
p_{2 n-1}^{(n-2)}(1)=0 \quad(n \geq 2) \tag{5.3}
\end{equation*}
$$

Next, we have

$$
p_{2 n-1}^{(n-1)}(1)=(-1)^{n-2} \frac{(n-1)!}{2^{n-2}}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}\left(A_{2 n, k}+A_{2 n, k+1}\right)\right\} T_{n-k}^{\prime}(1)
$$

By (1.8),

$$
T_{n}^{\prime}(x)=\frac{n \sin n \phi}{\sin \phi} \quad(x=\cos \phi)
$$

which gives $T_{n}^{\prime}(1)=n^{2}$. Thus

$$
p_{2 n-1}^{(n-1)}(1)=\frac{(n-1)!}{2^{n-2}} \sum_{k=0}^{n-1}(-1)^{k}(n-k)^{2}\left(A_{2 n, k}+A_{2 n, k+1}\right)
$$

After some manipulation, we get

$$
\begin{align*}
p_{2 n-1}^{(n-1)}(1) & =\frac{(n-1)!}{2^{n-2}} \sum_{k=1}^{n}(-1)^{k-1}(2 n-2 k+1) A_{2 n, k}  \tag{5.4}\\
& =\frac{(n-1)!}{2^{n-2}} \sum_{k=1}^{2 n}(-1)^{k} k A_{2 n, k}
\end{align*}
$$

Making use of (1.7), it can be proved that

$$
\begin{equation*}
(1-x) A_{n}^{\prime}(x)=A_{n+1}(x)-(n+1) x A_{n}(x) \tag{5.5}
\end{equation*}
$$

where

Hence

$$
A_{n}(x)=\sum_{k=1}^{n} A_{n k} x^{k} \quad(n \geq 1)
$$

$$
2 A_{2 n}^{\prime}(-1)=A_{2 n+1}(-1)+(2 n+1) A_{2 n}(-1)=C_{2 n+1}
$$

where $C_{2 n+1}$ has the same meaning as above. Thus (5.4) reduces to

$$
\begin{equation*}
p_{2 n-1}^{(n-1)}(1)=\frac{(n-1)!}{2^{n-1}} C_{2 n+1} . \tag{5.6}
\end{equation*}
$$

For example,

$$
p_{5}^{\prime \prime}(1)=24-360+472=136
$$

in agreement with (5.6).
(Please turn to page 465.)

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HOW TO FIND THE "GOLDEN NUMBER" WITHOUT REALLY TRYING

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". . . I wish . . . to point out that the use of the golden section . . . has apparently burst out into a sudden and devastating disease which has shown no signs of stopping .. ." [2, p. 521]

Most of the papers involving claims concerning the "golden number" deal with distinct items such as paintings, basing their assertions on measurements of these individual objects. As an example, we may cite the article by Hedian [13]. However measurements, no matter how accurate, cannot be used to reconstruct the original system of proportions used to design an object, for many systems may give rise to approximately the same set of numbers; see [6, 7] for an example of this. The only valid way of determining the system of proportions used by an artist is by means of documentation. A detailed investigation of three cases [8, 9, 10, 11] for which it had been claimed in the literature that the artist in question had used the "golden number" showed that these assertions were without any foundation whatsoever.

There is, however, another class of papers that seeks to convince the reader via statistical data applied to a whole class of related objects. The earliest examples of these are Zeising's morphological works, e.g., [17]. More recently we have Duckworth's book [5] on Vergil's Aeneid and a series of papers by Benjafield and his coauthors involving such things as interpersonal relationships (see e.g. [1], which gives a partial listing of some of these papers).

Mathematically we may approach the question in the following way. Suppose we have a certain length which is split into two parts, the larger being $M$ and the smaller $m$. If the length is divided according to the golden section, then it does not matter which of the quantities, $m / M$ or $M /(M+m)$, we use, for they are equal. But now suppose we have a collection of lengths and we are trying to determine statistically if the data are consistent with a partition according to the golden
section. Authors invariably use $M /(M+m)$, but we may reasonably ask which of the two we should really use or whether or not it matters.

Our starting point is a remark by Dalzell in his review of Duckworth's book: "But Professor Duckworth always uses the more complex ratio $M /(M+m)$, which he describes as 'slightly more accurate.' Just the reverse is true. In the relatively few instances when the quotient is exactly . 618 then $m / M=M /(M+m)$ and it does not matter which ratio is used. But in all other cases the more complex ratio is less sensitive to deviations from the perfect figure of .618" [4].

Let us designate $m / M$ by $x$, then $M /(M+m)$ becomes $1 /(1+x)$. The golden number is $\Phi=(1+\sqrt{5}) / 2$, and we let $\varphi=1 / \Phi$. We then have Dalzell's Theorem: For all $x$ in $[0,1] ;|1 /(1+x)-\varphi| \leq|x-\varphi|$.

Why should this result be true? Intuitively, we might reason that in writing $1 /(1+x)$ we are starting to form the continued fraction expansion of $\varphi$. We shall see later that in a sense our intuition is correct, but that there are limits to its validity. As a direct proof via continued fractions seems difficult, we use a roundabout approach.
Lemma: Let $f$ be differentiable on $[a, b]$ with $\left|f^{\prime}\right| \leq M$. If $\alpha$ is a root of $f(x)=x$ (i.e., a fixed point of $f$ ), then $|f(x)-\alpha| \leq M|x-\alpha|$ on $[\alpha, b]$.

Proof: Mean value theorem.
Corollary: Dalzell's theorem.
Alternatively, we can obtain the estimate $|f(x)-f(y)| \leq\left[1 /(1+\alpha)^{2}\right]|x-y|$ with $f(x)=1 /(1+x), 0 \leq \alpha \leq x, y$, by simple computation. This shows directly that $f$ is a contraction operator with fixed point $\varphi$. In particular, when we restrict ourselves to an interval bounded away from 0 , we see that the distortion caused by using $M /(M+m)$ instead of $m / M$ is larger than that indicated by Dalzell's theorem.
Corollary: On the interval $[\alpha, 1]$ where $\alpha=\sqrt{2}-1=.414 \ldots$,

$$
\left|\frac{1}{(1+x)}-\varphi\right| \leq \frac{1}{2}|x-\varphi| .
$$

Note in fact that as $x$ ranges from . 5 to $.75,1 /(1+x)$ only ranges from .667 to . 571 .
Corollary: For $x$ close to $\varphi$,

$$
|1 /(1+x)-\varphi| \approx\left(\varphi^{2}\right)|x-\varphi| \quad\left(\varphi^{2}=1-\varphi=.381 \ldots\right)
$$

Because of its independent interest we now make a slight digression into continued fractions. We restrict ourselves to the unit interval and therefore write $\left[0, \alpha_{1}, \alpha_{2}, \ldots\right]$ for $1 /\left(\alpha_{1}+1 / \alpha_{2}+\cdots\right)$.
Theorem: Let $\alpha \varepsilon[0,1]$ have a periodic continued fraction expansion of the form $\alpha=\left[0, \overline{a_{1}}, a_{2}, \ldots, a_{k}\right]$. Then for any number $x$ in $[0,1]$,

$$
\left|\left[0, a_{1}, a_{2}, \ldots, a_{k}, x\right]-\alpha\right| \leq|x-\alpha|
$$

Proof: Define $f$ by $f(x)=\left[\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}, x\right]$, then, by the periodicity, $f(\alpha)=\alpha$. Furthermore, $f(x)=\frac{A x+B}{C x+D}$, where the coefficients are integers which
do not depend on $x$ and satisfy $|A D-B C|=1(A / C$ and $B / D$ are, respectively, the ( $k-1$ )st and $k$ th convergents to $\alpha$; see [12, Th. 175] and [15, Th. 7.3]). From this we obtain $\left|f^{\prime}(x)\right|=1 /|(C x+D)|^{2} \leq 1$ on $[0,1]$. The proof is concluded by use of the lemma.

Corollary: Dalzell's theorem.

$$
\text { Proof: } \varphi=[0,1,1,1, \ldots] ;[0,1, x]=1 /(1+x) .
$$

Remark: This theorem justifies our earlier intuitive remark as to why Dalzell's theorem should hold; however, our intuition will lead us into difficulties unless we stop at the end of a period. Indeed, if $\alpha=\left[0, \overline{b_{1}}, \ldots, b_{k}\right]$ and $j<k$, then for $x=\alpha, x-\alpha$ is zero, whereas $\left[0, b_{1}, \ldots, b_{j}, x\right]-x$ is not zero.
Remark: The above approach can be used to place some results involving continued fractions in the domain of attraction of fixed points and contraction operators, but we shall not pursue this path here.
Remark: It is known that every periodic continued fraction is a quadratic surd, i.e., an irrational root of a quadratic equation with integral coefficients, and conversely ([10, Ths. 176, 177] and [15, Th. 7.19]). In the case of $\alpha=\varphi$, the corresponding equation is $x^{2}+x=1$ or $x=1 /(1+x)$. One would thus be tempted to treat the general periodic case as follows: Suppose $\alpha$ satisfies $A x^{2}+B x=C$. We rewrite this as $x=f(x)=C /(A x+B)$, and would like to conclude that

$$
|f(x)-\alpha| \leq|x-\alpha|
$$

as above. However, we run into difficulty because we no longer have a control on $f^{\prime}$.

Let us now turn our attention to the statistical aspects. We denote random variables by capital letters, expectation by $E$, variance by $\sigma^{2}$, and standard deviation by SD. We restrict ourselves to distributions with continuous densities concentrated on the unit interval. By the second corollary above (p. 407) and the Mean value theorem for integrals, we have immediately-
Theorem: If $X$ is a random variable taking values in a small interval near, then the ratios $r_{1}=|E(X)-\varphi| /|E(Y)-\varphi|$ and $r_{2}=\operatorname{SD}(X) / \operatorname{SD}(Y)$ are both near $\Phi^{2}$.

Now consider a general "aesthetic" situation involving lengths of various sizes. We should not be surprised that, rather than being controlled by some mystical numerical force, our ratios $m / M$ occur randomly. Furthermore, in situations such as the lengths of sections in a poem, there will be a tendency to avoid the two extremes of complete asymmetry and equality, i.e., we can expect values relatively far away from 0 and bounded away from 1 .

Thus we are led to consider the situation where $X$ is uniformly distributed on a subinterval $[a, b]$ of the unit interval. In this case, $E(X)=(a+b) / 2$ and $\sigma^{2}(X)=(b-a)^{2} / 12[14, \operatorname{pp} .74,101,111]$ and straightforward calculations [14, p. 78] now show that the distribution functions of $Y=1 /(1+X)$ assigns weight $(1 / c-1 / d) /(b-a)$ to a subinterval $[c, d]$ of $[1 /(1+b), 1 /(1+a)]$. Furthermore,

$$
E(Y)=\left(\frac{1}{b-a}\right) \cdot \ln \left(\frac{1+b}{1+a}\right) \quad \text { and } \quad \sigma^{2}(Y)=\frac{1}{(1+a)(1+b)}-[E(Y)]^{2} .
$$

Note that if $[c, d]$ is contained in $[1 /(1+b), 1 /(1+\alpha)]$ and also in $[a, b]$ then the distribution function of $y$ assigns $1 / c d$ times more weight to $[c, d]$ than does the unifrom distribution on $[a, b]$. Under these conditions, if [ $c, d]$ is a small subinterval about $1 / \Phi$, then this ratio is approximately $\Phi^{2}=2.618$, i.e., for a large sample over two and one-half times as many values of the transformed data as of the untransformed values will lie in the interval. Also note that the
weight assigned by the distribution of $Y$ to an interval [ $c, d]$ depends only on the length of the interval $[a, b]$ and not on the actual values of the endpoints.

In fact, numerical computation shows that even for large intervals relatively far away from 0 and bounded away from 1 the ratios $r_{1}$ and $r_{2}$ as well as the probability . ratios will not be too far from 2.6. To illustrate this situation, let us suppose that our ratios are uniformly distributed on [.45, .70] so that the average value is . 575 and the standard deviation . 072 . For a large sample, only $16 \%$ of the values will fall in the subinterval [.60, .64]. If we now transform the data, the mean is . 636 and the standard deviation only .029. This means that for a sample size of 20 or so it is almost sure that the mean will lie in the interval [.607, .665]. Furthermore, for a large sample, $42 \%$ of the actual values of $1 /(1+x)$ will lie in our subinterval [.60, .64]. If we look at [.59, .65], then the probabilities are $24 \%$ and $62 \%$.

Finally, to support our claim that the various seemingly impressive results in the literature are really due to an invalid transformation of data from a more or less uniform distribution, we mention two case studies.

The first is due to Shiffman and Bobko [16] who considered linear portionings and concluded that a uniform distribution of preferences was indeed the most likely hypothesis.

The other, a study on Duckworth's data, was done by the present author in connection with a historical study [3] of the numerical treatment of $\Phi$ by Hero of Alexandria who lived soon after Vergil. If we consider the first hundred entries in Duckworth's Table $I$, then the range of the $m / M$ values is from 4/7 = . 571 (four times) to $2 / 3=.667$ (twelve times). If this range is split up into five equal parts, then the five subintervals contain $10,25,33,15$, and 17 values, respectively. When we look at the actual values, we note that the Fibonacci ratios 3/5, $5 / 8$, and $13 / 21$ appear 15,16 , and 2 times, respectively. In other words, $2 / 3$ of the ratios are not Fibonacci approximations to the "golden number." If we compute means and standard deviations, then for the $m / M$ ratios we obtain the values . 621 and .025 as opposed to the values . 616 and .010 for the $M /(M+m)$ ratios, which only range from . 600 to .637. It is interesting to note that if Vergil had used the end values $4 / 7$ and $2 / 3$ fifty times each, then the average would have been

$$
\frac{1}{2}\left(\frac{4}{7}+\frac{2}{3}\right)=\frac{13}{21}
$$

which is a good Fibonacci approximation to $\varphi$. This only proves once more how deceiving averages can be. A similar study of the sixteen values in Duckworth's Table IV-the main divisions-reveals that not a single Fibonacci ratio appears. The $m / M$ values range from .594 to .663 with a mean of .625 and standard deviation of .021 as opposed to values of .615 and .008 for the $M /(M+m)$ values.

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## EXTENDED BINET FORMS FOR GENERALIZED QUATERNIONS OF HIGHER ORDER

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In a prior article [4], the concept of a higher-order quaternion was established and some identities for these quaternions were then obtained. In this paper we introduce a "Binet form" for generalized quaternions and then proceed to develop expressions for extended Binet forms for generalized quaternions of higher order. The extended Binet formulas make possible an approach for generating results which differs from that used in [4].

We recall from Horadam [1] the Binet form for the sequence $W_{n}(a, b ; p, q)$, viz.,

$$
W_{n}=A \alpha^{n}-B \beta^{n}
$$

where

$$
\begin{aligned}
W_{0} & =a, & W_{1} & =b \\
A & =\frac{b-\alpha \beta}{\alpha-\beta}, & B & =\frac{b-\alpha \alpha}{\alpha-\beta}
\end{aligned}
$$

and where $\alpha$ and $\beta$ are the roots of the quadratic equation

$$
x^{2}-p x+q=0
$$

We define the vectors $\underline{\alpha}$ and $\underline{\beta}$ such that

$$
\underline{\alpha}=1+i \alpha+j \alpha^{2}+k \alpha^{3} \quad \text { and } \quad \underline{\beta}=1+i \beta+j \beta^{2}+k \beta^{3},
$$

where $i, j, k$ are the quaternion vectors as given in Horadam [2].
Now, as in [4], we introduce the operator $\Omega$ :

$$
\begin{aligned}
\Omega W_{n} & =W_{n}+i W_{n+1}+j W_{n+2}+k W_{n+3} \\
& =A \alpha^{n}-B \beta^{n}+i\left(A \alpha^{n+1}-B \beta^{n+1}\right)+j\left(A \alpha^{n+2}-B \beta^{n+2}\right)+k\left(A \alpha^{n+3}-B \beta^{n+3}\right) \\
& =A \alpha^{n}\left(1+i \alpha+j \alpha^{2}+k \alpha^{3}\right)-B \beta^{n}\left(1+i \beta+j \beta^{2}+k \beta^{3}\right)
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\Omega W_{n}=A \alpha^{n} \underline{\alpha}-B \beta^{n} \underline{\beta} . \tag{1}
\end{equation*}
$$

This is the Binet formula for the generalized quaternion of order one. Consider

$$
\begin{aligned}
\Delta W_{n} & =W_{n}+i q W_{n-1}+j q^{2} W_{n-2}+k q^{3} W_{n-3} \\
& =A \alpha^{n}-B \beta^{n}+i q\left(A \alpha^{n-1}-B \beta^{n-1}\right)+j q^{2}\left(A \alpha^{n-2}-B \beta^{n-2}\right)+k q^{3}\left(A \alpha^{n-3}-B \beta^{n-3}\right) \\
& =A \alpha^{n}\left(1+i q \alpha^{-1}+j q^{2} \alpha^{-2}+k q^{3} \alpha^{-3}\right)-B \beta^{n}\left(1+i q \beta^{-1}+j q^{2} \beta^{-2}+k q^{3} \beta^{-3}\right)
\end{aligned}
$$

but

$$
\alpha \beta=q
$$

i.e., $\alpha=q \beta^{-1}$ and $\beta=q \alpha^{-1}$; hence,

$$
\Delta W_{n}=A \alpha^{n}\left(1+i \beta+j \beta^{2}+k \beta^{3}\right)-B \beta^{n}\left(1+i \alpha+j \alpha^{2}+k \alpha^{3}\right)
$$

Therefore,

$$
\begin{equation*}
\Delta W_{n}=A \alpha^{n} \underline{\beta}-B \beta^{n} \underline{\alpha} . \tag{2}
\end{equation*}
$$

Thus we see that the quaternion formed by the $\Delta$ operator, that proved so useful in [3] and [4], has a Binet form which is a simple permutation of result (1) above.

We now examine quaternions of order $\lambda$ (for $\lambda$ an integer) and prove by induction that

$$
\begin{equation*}
\Omega^{\lambda} W_{n}=A \alpha^{n} \underline{\alpha}^{\lambda}-B \beta^{n} \underline{\beta}^{\lambda} . \tag{3}
\end{equation*}
$$

Proo f: When $\lambda=1$, the result is true because

$$
\Omega^{1} W_{n}=\Omega W_{n}=A \alpha^{n} \underline{\alpha}-B \beta^{n} \underline{\beta} .
$$

Assume that the result is true for $\lambda=m$, i.e.,

$$
\Omega^{m} W_{n}=A \alpha^{n} \underline{\alpha}^{m}-B \beta^{n} \underline{\beta}^{m} .
$$

Now, for $\lambda=m+1$,

$$
\begin{aligned}
\Omega^{m+1} W_{n}= & \Omega^{m} W_{n}+i \Omega^{m} W_{n+1}+j \Omega^{m} W_{n+2}+k \Omega^{m} W_{n+3} \\
= & A \alpha^{n} \underline{\alpha}^{m}-B \beta^{n} \underline{\beta}^{m}+i\left(A \alpha^{n+1} \underline{\alpha}^{m}-B \beta^{n+1} \underline{\beta}^{m}\right)+j\left(A \alpha^{n+2} \underline{\alpha}^{m}-B \beta^{n+2} \underline{\beta}^{m}\right) \\
& +k\left(A \alpha^{n+3} \underline{\alpha}^{m}-B \beta^{n+3} \underline{\beta}^{m}\right) \\
= & A \alpha^{n}\left(1+i \alpha+j \alpha^{2}+k \alpha^{3}\right) \underline{\alpha}^{m}-B \beta^{n}\left(1+i \beta+j \beta^{2}+k \beta^{3}\right) \underline{\beta}^{m} \\
= & A \alpha^{n} \underline{\alpha} \underline{\alpha}^{m}-B \beta^{n} \underline{\beta} \underline{\beta}^{m} \\
= & A \alpha^{n} \underline{\alpha}^{m+1}-B \beta^{n} \underline{\beta}^{m+1} .
\end{aligned}
$$

Since the result is true for $\lambda=1$ and also true for $\lambda=m+1$ whenever the result holds for $\lambda=m$, it follows from the principle of induction that the result is true for all integral $\lambda$. Similarly, it can be shown that

$$
\begin{equation*}
\Delta^{\lambda} W_{n}=A \alpha^{n} \underline{\beta}^{\lambda}-B \beta^{n} \underline{\alpha}^{\lambda} . \tag{4}
\end{equation*}
$$

Since
and

$$
\begin{aligned}
& \Omega \Delta W_{n}=\Delta W_{n}+i \Delta W_{n+1}+j \Delta W_{n+2}+k \Delta W_{n+3} \\
& \Delta \Omega W_{n}=\Omega W_{n}+i q \Omega W_{n-1}+j q^{2} \Omega W_{n-2}+k q^{3} \Omega W_{n-3},
\end{aligned}
$$

we secure, using equations (2) and (1), respectively,

$$
\begin{align*}
& \Omega \Delta W_{n}=A \alpha^{n} \underline{\alpha} \underline{\beta}-B \beta^{n} \underline{\beta} \underline{\alpha}  \tag{5}\\
& \Delta \Omega W_{n}=A \alpha^{n} \underline{\underline{\alpha}} \underline{\alpha}-B \beta^{n} \underline{\alpha} \underline{\beta}
\end{align*}
$$

If we let $\lambda=2$ in equations (3) and (4) and also use equations (5) and (6), we can derive the six permutations for quaternions of order 3 involving both $\Omega$ and $\Delta$ operators, namely

$$
\begin{align*}
& \Omega^{2} \Delta W_{n}=A \alpha^{n} \underline{\alpha}^{2} \underline{\beta}-B \beta^{n} \underline{\beta}^{2} \underline{\alpha}  \tag{7}\\
& \Delta^{2} \Omega W_{n}=A \alpha^{n} \underline{\beta}^{2} \underline{\alpha}-B \beta^{n} \underline{\alpha}^{2} \underline{\beta}  \tag{8}\\
& \Omega \Delta^{2} W_{n}=A \alpha^{n} \underline{\alpha} \underline{\beta}^{2}-B \alpha^{n} \underline{\beta} \underline{\alpha}^{2}  \tag{9}\\
& \Delta \Omega^{2} W_{n}=A \alpha^{n} \underline{\beta} \underline{\alpha}^{2}-B \beta^{n} \underline{\alpha} \underline{\beta}^{2}  \tag{10}\\
& \Omega \Delta \Omega W_{n}=A \alpha^{n} \underline{\alpha} \underline{\beta} \underline{\alpha}-B \beta^{n} \underline{\beta} \underline{\alpha} \underline{\beta}  \tag{11}\\
& \Delta \Omega \Delta W_{n}=A \alpha^{n} \underline{\beta} \underline{\alpha} \underline{\beta}-B \beta^{n} \underline{\alpha} \underline{\beta} \underline{\alpha} \tag{12}
\end{align*}
$$

We now pause to investigate the effects of operators $\Omega^{*}$ and $\Delta^{*}$ on the Binet forms. Note from [4] that

$$
\Omega^{*} \Delta W_{n}=\Delta W_{n}+\Delta W_{n+1} \cdot i+\Delta W_{n+2} \cdot j+q^{3} \Delta W_{n+3} \cdot k=\Delta \Omega W_{n}
$$

and

$$
\Delta^{*} \Omega W_{n}=\Omega W_{n}+q \Omega W_{n-1} \cdot i+q^{2} \Omega W_{n-2} \cdot j+q^{3} \Omega W_{n-3} \cdot k=\Omega \Delta W_{n}
$$

and thus the operators $\Omega^{*}$ and $\Delta^{*}$ provide no new results for quaternions of order 2. Since equations (7) to (12) and equations (3) and (4) for $\lambda=3$ provide every possible triad of combination of $\underline{\alpha}$ and $\underline{\beta}$, it is unlikely that quaternions of order 3 involving the starred operators will produce any Binet form distinct from those given. A close inspection of the modus operandi of $\Omega^{*}$ and $\Delta^{*}$ verifies that this is indeed the case. For example, it is easily calculated that

$$
\Omega^{*} \Delta \Omega W_{n}=A \alpha^{n} \underline{\beta} \underline{\alpha}^{2}-B \beta^{n} \underline{\alpha}^{2} \underline{\beta}^{2}
$$

which is the same expression as that for $\Delta \Omega^{2} W_{n}$.
We can generalize these statements to say that the operators $\Omega^{*}$ and $\Delta^{*}$ yield no results that cannot be obtained solely by manipulating the operators $\Omega$ and $\Delta$.

From equations (3) and (4), it can be readily shown that, for $\mu$ an integer,

$$
\begin{align*}
& \Delta^{\lambda} \Omega^{\mu} W_{n}=A \alpha^{n} \underline{\beta}^{\lambda} \underline{\alpha}^{\mu}-B \beta^{n} \underline{\alpha}^{\lambda} \underline{\beta}^{\mu}  \tag{13}\\
& \Omega^{\lambda} \Delta^{\mu} W_{n}=A \alpha^{n} \underline{\alpha}^{\lambda} \underline{\beta}^{\mu}-B \beta^{n} \underline{\underline{\beta}}^{\lambda} \underline{\alpha}^{\mu}
\end{align*}
$$

The pattern between the higher-order quaternions and their related Binet forms being clearly established, we deduce, for integral $\lambda_{i}, i=1, \ldots, m$, the ensuing extended Binet formulas of finite order:

$$
\begin{align*}
& \Omega^{\lambda_{1}} \Delta^{\lambda_{2}} \ldots \Omega^{\lambda_{m}} W_{n}=A \alpha^{n} \underline{\alpha}^{\lambda_{1}} \underline{\beta}^{\lambda_{2}} \cdots \underline{\alpha}^{\lambda_{m}}-B \beta^{n} \underline{\beta}^{\lambda_{1}} \underline{\alpha}^{\lambda_{2}} \cdots \underline{\beta}^{\lambda_{m}}  \tag{15}\\
& \Omega^{\lambda_{1}} \Delta^{\lambda_{2}} \ldots \Delta^{\lambda_{m}} W_{n}=A \alpha^{n} \underline{\alpha}^{\lambda_{1}} \underline{\beta}^{\lambda_{2}} \cdots \underline{\beta}^{\lambda_{m}}-B \underline{\beta}^{n} \underline{\underline{\beta}}^{\lambda_{1}} \underline{\alpha}^{\lambda_{2}} \cdots \underline{\alpha}^{\lambda_{m}}  \tag{16}\\
& \Delta^{\lambda_{1}} \Omega^{\lambda_{2}} \ldots \Omega^{\lambda_{m} W_{n}}=A \alpha^{n} \underline{\beta}^{\lambda_{1}} \underline{\alpha}^{\lambda_{2}} \cdots \underline{\alpha}^{\lambda_{m}}-B \underline{\beta}^{n} \underline{\alpha}^{\lambda_{1}} \underline{\beta}^{\lambda_{2}} \cdots \underline{\beta}^{\lambda_{m}}  \tag{17}\\
& \Delta^{\lambda_{1}} \Omega^{\lambda_{2}} \ldots \Delta^{\lambda_{m}} W_{n}=A \alpha^{n} \underline{\underline{\beta}}^{\lambda_{1}} \underline{\alpha}^{\lambda_{2}} \cdots \underline{\beta}^{\lambda_{m}}-B \beta^{n} \underline{\alpha}^{\lambda_{1}} \underline{\underline{\beta}}^{\lambda_{2}} \cdots \underline{\alpha}^{\lambda_{m}} \tag{18}
\end{align*}
$$

From equations (2.6) and (2.7) of Horadam [1], we derive the following Binet formulas:

$$
\begin{align*}
& \Omega^{\lambda} U_{n}=\left[\alpha^{n+1} \underline{\alpha}^{\lambda}-\beta^{n+1} \underline{\beta}^{\lambda}\right] / \alpha  \tag{19}\\
& \Delta^{\lambda} U_{n}=\left[\alpha^{n+1} \underline{\beta}^{\lambda}-\beta^{n+1} \underline{\alpha}^{\lambda}\right] / d  \tag{20}\\
& \Omega^{\lambda} V_{n}=\alpha^{n} \underline{\alpha}^{\lambda}+\beta^{n} \underline{\beta}^{\lambda}  \tag{21}\\
& \Delta^{\lambda} V_{n}=\alpha^{n} \underline{\beta}^{\lambda}+\beta^{n} \underline{\alpha}^{\lambda} \tag{22}
\end{align*}
$$

We now use the extended Binet formulas to establish some identities. As an example, consider a simple generalization of equation (28) in [4]:

$$
\begin{aligned}
\Omega^{\lambda} V_{r} \Omega^{\mu} V_{n} & =\left(\alpha^{r} \underline{\alpha}^{\lambda}+\beta^{r} \underline{\beta}^{\lambda}\right)\left(A \alpha^{n} \underline{\alpha}^{\mu}-B \beta^{n} \underline{\beta}^{\mu}\right) \\
& =A \alpha^{n+r} \underline{\alpha}^{\lambda+\mu}+A \alpha^{n} \underline{\beta}^{r} \underline{\beta}^{\lambda} \underline{\alpha}^{\mu}-B \alpha^{r} \beta^{n} \underline{\alpha}^{\lambda} \underline{\beta}^{\mu}-B \beta^{n+r} \underline{\beta}^{\lambda+\mu} \\
& =A \alpha^{n+r} \underline{\alpha}^{\lambda+\mu}-B \beta^{n+r} \underline{\beta}^{\lambda+\mu}+\alpha^{r} \beta^{r}\left(A \alpha^{n-r} \underline{\beta}^{\lambda} \underline{\alpha}^{\mu}-B \beta^{n-r} \underline{\alpha}^{\lambda} \underline{\beta}^{\mu}\right)
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\Omega^{\lambda} V_{r} \Omega^{\mu} W_{n}=\Omega^{\lambda+\mu_{W_{n+r}}+q^{r} \Delta^{\lambda} \Omega^{\mu} W_{n-r}} \tag{23}
\end{equation*}
$$

This, in turn, can easily be further extended to provide a most generalized formula, viz.,

$$
\begin{equation*}
\Omega^{\lambda_{1}} \Delta^{\lambda_{2}} \ldots \Omega^{\lambda_{m}} V_{r} \Omega^{\mu_{1}} \Delta^{\mu_{2}} \ldots \Omega^{\mu_{m}} W_{n} \tag{24}
\end{equation*}
$$

$$
=\Omega^{\lambda_{1}} \Delta^{\lambda_{2}} \ldots \Omega^{\lambda_{m}+\mu_{1}} \Delta^{\mu_{2}} \ldots \Omega^{\mu_{m}} W_{n+r}+q^{r} \Delta^{\lambda_{1}} \Omega^{\lambda_{2}} \ldots \Delta^{\lambda_{m}} \Omega^{\mu_{1}} \Delta^{\mu_{2}} \ldots \Omega^{\mu_{m}} W_{n-r}
$$

It is obvious to the reader that other similar generalizations of the results in [4] can be procured by this method.

We now look at an equation not contained in [3] or [4]. Consider

$$
\begin{aligned}
\left(\frac{\Omega^{\lambda} V_{n}+d \Omega^{\lambda} U_{n-1}}{2}\right)^{m} & =\left(\frac{\alpha^{n} \underline{\alpha}^{\lambda}+\beta^{n} \underline{\beta}^{\lambda}+\alpha^{n} \underline{\alpha}^{\lambda}-\beta^{n} \underline{\beta}^{\lambda}}{2}\right)^{m}=\left(\alpha^{n} \underline{\alpha}^{\lambda}\right)^{m}=\alpha^{m n} \underline{\alpha}^{\lambda m} \\
& =\frac{2 \alpha^{m n} \underline{\alpha}^{\lambda m}}{2}=\frac{\alpha^{m n} \underline{\alpha}^{\lambda m}+\beta^{m n} \underline{\beta}^{\lambda m}-\beta^{m n} \underline{\beta}^{\lambda m}+\alpha^{m n} \underline{\alpha}^{\lambda m}}{2}
\end{aligned}
$$

Thus

$$
\begin{equation*}
\left(\frac{\Omega^{\lambda} V_{n}+d \Omega^{\lambda} U_{n-1}}{2}\right)^{m}=\frac{\Omega^{\lambda m} V_{m n}+d \Omega^{\lambda m} U_{m n-1}}{2} \tag{25}
\end{equation*}
$$

This is a De Moivre type identity for higher-order quaternions.
Thus we see that the extended Binet formulas not only permit direct verification of the identities contained in [3] and [4], and extensions of these as we have shown, but also facilitate the attainment of new results.

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## A COMPLETE CHARACTERIZATION OF THE DECIMAL FRACTIONS THAT CAN BE REPRESENTED AS $\Sigma 10^{-k(i+1)} F_{\alpha i}$, WHERE $F_{\alpha i}$ IS THE $\alpha i$ TH FIBONACCI NUMBER <br> RICHARD H. HUDSON* University of South Carolina, Columbia, SC 29208 <br> C. F. WINANS <br> 1106 Courtleigh Drive, Akron, OH 44313 <br> 1. INTRODUCTION

In 1953 Fenton Stancliff [2] noted (without proof) that

$$
\Sigma 10^{-(i+1)} F_{i}=\frac{1}{89}
$$

where $F_{i}$ denotes the $i$ th Fibonacci number. Until recently this expansion was regarded as an anomalous numerical curiosity, possibly related to the fact that 89 is a Fibonacci number (see Remark in [2]), but not generalizing to other fractions in an obvious manner.

Recently, the second of us showed that the sums $\sum 10^{-(i+1)} F_{\alpha i}$ approximate $1 / 71$, $2 / 59$, and $3 / 31$ for $\alpha=2,3$, and 4, respectively. Moreover, Winans showed that the sums $\sum 10^{-2(i+1)} F_{\alpha i}$ approximate $1 / 9899,1 / 9701,2 / 9599$, and $3 / 9301$ for $\alpha=1,2$, 3 , and 4, respectively.

In this paper, we completely characterize all decimal fractions that can be approximated by sums of the type

$$
\frac{1}{F_{\alpha}}\left(\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}\right), \alpha \geq 1, k \geq 1
$$

In particular, all such fractions must be of the form

$$
\begin{equation*}
\frac{1}{10^{2 k}-10^{k}-1-10^{k}\left(\sum_{j=1}^{(\alpha-1) / 2} L_{2 j}\right)} \tag{1.1}
\end{equation*}
$$

when $\alpha$ is odd, and of the form

$$
\begin{equation*}
\frac{1}{10^{2 k}-3\left(10^{k}\right)+1-10^{k}\left(\sum_{j=1}^{(\alpha+1) / 2} L_{2 j+1}\right)} \tag{1.2}
\end{equation*}
$$

when $\alpha$ is even [ $L_{j}$ denotes the $j$ th Lucas number and the denominators in (1.1) and (1.2) are assumed to be positive].

Recalling that the $i$ th term of the Fibonacci sequence is given by

$$
\begin{equation*}
F_{i}=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i}-\left(\frac{1-\sqrt{5}}{2}\right)^{i}, \tag{1.3}
\end{equation*}
$$

it is straightforward to prove that the sums $\frac{1}{F_{\alpha}}\left(\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}\right)$ converge to the

[^0]fractions indicated in (1.1) and (1.2) provided that $((1+\sqrt{5}) / 2)^{\alpha}<10^{k}$. For example, we have $((1+\sqrt{5}) / 2)^{2}=(3+\sqrt{5}) / 2$ and $(3+\sqrt{5}) / 2<10$. Hence, appealing to the formula for the sum of a convergent geometric series, we have
\[

$$
\begin{align*}
\sum_{i=1}^{\infty} \frac{F_{2 i}}{10^{i+1}} & =\frac{1}{10 \sqrt{5}}\left(\frac{1}{1-(3+\sqrt{5}) / 20}-\frac{1}{1-(3-\sqrt{5}) / 20}\right)  \tag{1.4}\\
& =\frac{2 \sqrt{5}}{5}\left(\frac{17+\sqrt{5}}{284}-\frac{17-\sqrt{5}}{284}\right)=\frac{1}{71}
\end{align*}
$$
\]

The surprising fact, indeed the fact that motivates the writing of this paper, is that the fractions given by (1.1) and (1.2) are completely determined by values in the Lucas sequence, totally independent of any consideration regarding Fibonacci numbers. The manner in which this dependence on Lucas numbers arises seems to us thoroughly remarkable.

$$
\text { 2. THE SUMS } \Sigma 10^{-k(i+1)} F_{\alpha i}, k=1
$$

Case 1: $\alpha=1$.
Using Table 1 (see Section 6 below), we have
(2.1) $\sum_{i=1}^{60} 10^{-(i+1)} F_{i}$
$=.0112359550561797752808988764044943820224719101123296681836230$.
It is easily verified that $1 / 89$ repeats with period 44 and that

$$
\begin{equation*}
\frac{1}{89}=.01123595505617977528089887640449438202247191011235 \ldots \tag{2.2}
\end{equation*}
$$

The approximation $\sum_{i=1}^{60} 10^{-(i+1)} F_{i} \approx \frac{1}{89}$ is accurate only to 49 places, solely because we have used only the first 60 Fibonacci numbers. A good ballpark estimate of the accuracy of the approximation $\sum_{i=1}^{s} 10^{-k(i+1)} F_{\alpha i} \approx \frac{p}{q}$ may be obtained by looking at the number of zeros preceding the first nonzero entry in the expansion

$$
\begin{equation*}
\frac{F_{\alpha s}}{10^{k(s+1)}}=.000 \ldots a_{n} \cdot a_{n+1} \ldots a_{\ell} \tag{2.3}
\end{equation*}
$$

$\alpha_{n}$ is the first nonzero entry and $\ell=k(s+1)$.
Thus, e.g.,

$$
\begin{equation*}
\frac{F_{60}}{10^{61}}=.000 \ldots 1548008755920 \tag{2.4}
\end{equation*}
$$

The number of zeros preceding $a_{n}$ above is 48 , so that the 49-place accuracy found is to be expected.

Case 2: $\alpha=2$.
Look at every second Fibonacci number; then, using Table 1, we have

$$
\begin{equation*}
\sum_{i=1}^{25} 10^{-(i+1)} F_{2 i}=.01408450704225347648922085 \tag{2.5}
\end{equation*}
$$

Now,

$$
\begin{equation*}
\frac{1}{71}=.0140845070422535 \ldots \tag{2.6}
\end{equation*}
$$

Note that
(2.7)

$$
\frac{F_{50}}{10^{26}}=.000 \ldots 12586269025
$$

where the number of zeros preceding $a_{n}=1$ is 15 .
Case 3: $\alpha=3$.
Looking at every third Fibonacci number, we have

$$
\begin{equation*}
\sum_{i=1}^{16} 10^{-(i+1)} F_{3 i}=.03389826975294276 \tag{2.8}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\frac{2}{59}=.0338983 \ldots \tag{2.9}
\end{equation*}
$$

The six place accuracy is to be expected in light of the fact that

$$
\begin{equation*}
\frac{F_{48}}{10^{17}}=.00000004807526976 \tag{2.10}
\end{equation*}
$$

Case 4: $\alpha=4$.
Looking at every fourth Fibonacci number up to $F_{100}$, we have

$$
\begin{equation*}
\sum_{i=1}^{25} 10^{-(i+1)} F_{4 i}=.09676657589472715467557065 \tag{2.11}
\end{equation*}
$$

Now

$$
\begin{equation*}
\frac{3}{31}=.096774 \ldots \tag{2.12}
\end{equation*}
$$

The convergence of (2.11) is very slow, as can be seen by the fact that $\frac{F_{100}}{10^{26}}$

$$
\begin{equation*}
\frac{F_{100}}{10^{26}}=.00000354224638179261842845 \tag{2.13}
\end{equation*}
$$

Case 5: $\alpha \geq 5$.
Consider $\Sigma 10^{-(i+1)} F_{5 i}$. The sum is of the form

$$
\begin{align*}
& .05  \tag{2.14}\\
&+ .055 \\
&+ .0610 \\
&+ .06765 \\
&+\quad . \\
& \hline
\end{align*}
$$

Clearly this sum does not converge at all and, a fortiori, $\sum 10^{-(i+1)} F_{\alpha i}$ does not converge for any $\alpha \geq 5$.

Summary of Section 2:

$$
\begin{equation*}
\sum_{i=1}^{n} 10^{-(i+1)} F_{i} \approx \frac{1}{89} \quad \sum_{i=1}^{n} 10^{-(i+1)} F_{2 i} \approx \frac{1}{71} \tag{2.15}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i=1}^{n} 10^{-(i+1)} F_{3 i} \approx \frac{2}{59} . \quad \sum_{i=1}^{n} 10^{-(i+1)} F_{4 i} \approx \frac{3}{31} \tag{2.16}
\end{equation*}
$$

$$
\begin{align*}
& \sum_{i=1}^{n} 10^{-(i+1)} F_{\alpha i} \rightarrow \infty \text { as } n \rightarrow \infty \text { if } \alpha \geq 5  \tag{2.17}\\
& \text { THE SUMS } \Sigma 10^{-k(i+1)} F_{\alpha i}, k=2
\end{align*}
$$

If $\alpha=10$, the sum $\sum 10^{-2(i+1)} F_{\alpha i}$ is of the form

$$
\begin{gather*}
.0055  \tag{3.1}\\
+. .006765 \\
+. .00832040 \\
+.0102334155 \\
+\quad \ldots \\
\hline
\end{gather*}
$$

and this clearly does not converge. There are, consequently, exactly nine fractions with four-digit denominators that are approximated by sums of the type

$$
\sum_{i=1}^{n} 10^{-2(i+1)} F_{\alpha i}
$$

Henceforth, for brevity, we denote $\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}$ by $S_{\alpha i}(k)$. Then, for $\alpha=1$, 2,..., 9, we have, respectively, $S_{\alpha i}(2) \approx 1 / 9899,1 / 9701,2 / 9599,3 / 9301,5 / 8899$, $8 / 8201,13 / 7099,21 / 5301$, and $34 / 2399$.

We indicate the computation for $S_{4 i}(2)$, leaving the reader to check the remaining values. To compute $\sum_{i=1}^{12} 10^{-2(i+1)} F_{4 i}$, we must perform the addition:

$$
\begin{equation*}
.0003 \tag{3.2}
\end{equation*}
$$

$$
.000021
$$

$$
.00000144
$$

987
6765
.00032254596279969541950276
Now

$$
\begin{equation*}
\frac{3}{9301}=.000322545962799698 \ldots \tag{3.3}
\end{equation*}
$$

Notice that the approximation is considerably more accurate for small $n$ than the analogous approximation given by (2.11). Of course, this is because, from the point of rapidity of convergence (or lack thereof), $S_{4 i}$ (1) is more closely analogous to $S_{8 i}$ (2) -each represents the largest value of $\alpha$ for which convergence is possible for the respective value of $k$.

The reader may well wonder how we arrived at fractions such as $21 / 5301$ and $34 / 2399$, since $S_{8 i}(2)$ and $S_{9 i}(2)$ converge so slowly that it is not obvious what
fractions they are approximating. The values for $S_{\alpha i}(2), \alpha=1, \ldots, 6$, were obtained from empirical evidence. The pattern for the numerators is obvious. After looking at the denominators for some time, the first of us noted (with some astonishment) the following pattern governing the first two digits of the denominators:

$$
\begin{align*}
& 98-95=3  \tag{3.4}\\
& 97-93=4 \\
& 95-88=7 \\
& 93-82=11 \\
& 88-70=18
\end{align*}
$$

Subsequent empirical evidence revealed what poetic justice required, namely that the eighth and ninth denominators must be 5301 and 2301, for

$$
\begin{equation*}
82-53=29 \text { and } 70-23=47 \tag{3.5}
\end{equation*}
$$

The indicated differences are, of course, precisely the Lucas numbers beginning with $L_{2}=3$. Notice that entirely apart from any numerical values for the Fibonacci numbers, the existence of a value for $S_{10 i}(2)$ is outlawed by the above pattern. For the first two digits of the denominator of such a fraction would be (on the basis of the pattern) $53-76<0$, presumably an absurdity.

Naturally, the real value of recognizing the pattern is that values can easily be given for $S_{\alpha i}(k)$ for every $k$ and every $\alpha$ for which it is possible that these sums converge. Moreover, values of $\alpha$ for which convergence is an obvious impossibility (because terms in the sum are increasing), and the denominators of the fractions which these sums approximate for the remaining $\alpha$, may be determined by consideration of the Lucas numbers alone.

We may proceed at once to the general case, but for the sake of illustration we briefly sketch the case $k=3$ employing the newly discovered pattern.

$$
\text { 4. THE SUMS } \Sigma 10^{-k(i+3)} F_{\alpha i}, k=3
$$

In analogy to the earlier cases it is not difficult to obtain and empirically check that $1 / 998999$ and $1 / 997001$ are fractions that are approximated by $S_{i}(3)$ and $S_{2 i}$ (3), respectively.

Now, using Table 2 (see Section 6 below),

$$
\begin{align*}
& 998-3=995,997-4=993,995-7=988,993-11=982,  \tag{4.1}\\
& 988-18=970,982-29=953,970-47=923,953-76=877, \\
& 923-123=800,877-199=678,800-322=478, \\
& 678-521=157, \text { and } 478-843<0
\end{align*}
$$

Therefore, we expect that $S_{\alpha i}(3)$ is meaningful if $\alpha \leq 14$ and if the fourteen fractions corresponding to these $\alpha^{\prime}$ s are precisely:

$$
\begin{align*}
& \frac{1}{998999}, \frac{1}{997001}, \frac{2}{995999}, \frac{3}{993001}, \frac{5}{988999},  \tag{4.2}\\
& \frac{8}{982001}, \frac{13}{970999}, \frac{21}{953001}, \frac{34}{923999}, \frac{55}{877001}, \\
& \frac{89}{800999}, \frac{144}{678001}, \frac{233}{478999}, \frac{377}{157001}
\end{align*}
$$

We leave for the reader the aesthetic satisfaction of checking that $\alpha=15$ is, indeed, the smallest value of $\alpha$ such that the terms of $S_{\alpha i}(3)$ are not decreasing. Example:

$$
\text { Consider } \sum_{i=1}^{7} 10^{-3(i+1)} F_{9 i}
$$

This sums as follows:
(4.3)

$$
\begin{aligned}
& .000034 \\
& .000002584 \\
& .000000196418 \\
& 14930352 \\
& 1134903170 \\
& 86267571272 \\
& 6557470319842 \\
& \hline .000036796576080211591842
\end{aligned}
$$

On the other hand, the ninth fraction in (4.2) is

$$
\begin{equation*}
\frac{34}{923999}=.00003679657 \ldots \tag{4.4}
\end{equation*}
$$

## 5. THE GENERAL CASE

All that has gone before can be summarized succinctly as follows. The totality of decimal fractions that can be approximated by sums of the form

$$
\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}, \quad \alpha \geq 1, k \geq 1
$$

are given by

$$
\begin{equation*}
\frac{F_{\alpha}}{10^{2 k}-10^{k}-1-10^{k}\left(\sum_{j=1}^{(\alpha-1) / 2} L_{2 j}\right)} \tag{5.1}
\end{equation*}
$$

when $\alpha$ is odd and the denominator is positive, and by

$$
\begin{equation*}
\frac{F_{\alpha}}{10^{2 k}-3\left(10^{k}\right)+1-10^{k}\left(\sum_{j=1}^{(\alpha-2) / 2} L_{2 j+1}\right)} \tag{5.2}
\end{equation*}
$$

when $\alpha$ is even and the denominator is positive.
Remark: The appearance of $F_{\alpha}$ in the numerator of the above fractions is not essential to the analysis. One can just as well look at sums of the form

$$
\frac{1}{F_{\alpha}} \sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}
$$

These approximate fractions identical with those in (5.1) and (5.2), except that their numerators are always 1. These fractions are determined, then, only by Lucas numbers with no reference at all to the Fibonacci sequence.

Example 1: Let $k=4$. The smallest positive value of the denominators in (5.1), (5.2) is

$$
10^{8}-10^{4}-1-10^{4}\left(\sum_{j=1}^{(19-1) / 2} L_{2 j}\right)=6509999 .
$$

This means that there are exactly nineteen fractions arising in the case $k=4$ and

$$
\begin{equation*}
S_{19 i}(4) \approx \frac{4184}{6509999}, \tag{5.3}
\end{equation*}
$$

although it will be necessary to sum a large number of terms to get a good approximation (or even to get an approximation that remotely resembles 4184/6509999). However, if one looks at the nineteenth fraction arising when $k=5$, one obtains

$$
\begin{equation*}
\frac{4184}{9065099999}=.0000004612 \ldots \tag{5.4}
\end{equation*}
$$

On the other hand, $\sum_{i=1}^{5} 10^{-5(i+1)} F_{19 i}$ equals

$$
\begin{array}{r}
.0000004181  \tag{5.5}\\
+.000000039088169 \\
365435296162 \\
3416454622906707 \\
31940414634990093395 \\
\hline .000000461216107838545660793395
\end{array}
$$

which restores one's faith in (5.3) with much less pain than employing direct computation.

Example 2: Let $k=8$ and let $\alpha=32$ so that (5.2) must be used. From Table 1, we have

$$
\sum_{i=1}^{3} 10^{-8(i+1)} F_{32 i}
$$

$$
\begin{array}{r}
.0000000002178309  \tag{5.6}\\
+.000000000010610209857723 \\
\hline .00000000000051680678854858312532 \\
\hline .00000000022895791664627158312532
\end{array}
$$

On the other hand, from (5.2) and Tables 1 and 2 we have that the thirtysecond fraction arising when $k=8$ is:

a good approximation considering that only three Fibonacci numbers ( $F_{32}, F_{64}$, and $F_{96}$ ) are used in (5.6).

## 6. TABLES OF FIBONACCI AND LUCAS NUMBERS

TABLE 1

| $F_{1}$ | 1 | $F_{14}$ | 377 |
| :--- | ---: | ---: | ---: |
| $F_{2}$ | 1 | $F_{15}$ | 610 |
| $F_{3}$ | 2 | $F_{16}$ | 987 |
| $F_{4}$ | 3 | $F_{17}$ | 1597 |
| $F_{5}$ | 5 | $F_{18}$ | 2584 |
| $F_{6}$ | 8 | $F_{19}$ | 4184 |
| $F_{7}$ | 13 | $F_{20}$ | 6765 |
| $F_{8}$ | 21 | $F_{21}$ | 10946 |
| $F_{9}$ | 34 | $F_{22}$ | 17711 |
| $F_{10}$ | 55 | $F_{23}$ | 28657 |
| $F_{11}$ | 89 | $F_{24}$ | 46368 |
| $F_{12}$ | 144 | $F_{25}$ | 75025 |
| $F_{13}$ | 233 | $F_{26}$ | 121393 |


| $F_{27}$ | 196418 |
| :--- | ---: |
| $F_{28}$ | 317811 |
| $F_{29}$ | 514229 |
| $F_{30}$ | 832040 |
| $F_{31}$ | 1346269 |
| $F_{32}$ | 2178309 |
| $F_{33}$ | 3524578 |
| $F_{34}$ | 5702889 |
| $F_{35}$ | 9227465 |
| $F_{36}$ | 14930352 |
| $F_{37}$ | 24157817 |
| $F_{38}$ | 39088169 |
| $F_{39}$ | 63245986 |

$F_{40}$
$F_{41}$
$F_{42}$
$F_{4}$
$F_{44}$
$F_{45}$
$F_{46}$
$F_{47}$
$F_{48}$
$F_{49}$
$F_{50}$
$F_{51}$
$F_{52}$

102334155
165580141
267914296
433494437
701408733
1134903170
1836311903
2971215073
4807526976
7778742049
12586269025
20365011074
32951280099

TABLE 1 (continued)

| $F_{53}$ | 53316291173 | $F_{77}$ | 5527939700884757 |
| :--- | ---: | ---: | ---: |
| $F_{54}$ | 86267571272 | $F_{78}$ | 8944394323791464 |
| $F_{55}$ | 139583862445 | $F_{79}$ | 14472334024676221 |
| $F_{56}$ | 225851433717 | $F_{80}$ | 23416728348467685 |
| $F_{57}$ | 365435296162 |  |  |
| $F_{58}$ | 591286729879 |  |  |
| $F_{59}$ | 956722026041 |  |  |
| $F_{60}$ | 548008755920 |  |  |
| $F_{61}$ | 2504730781961 |  |  |
| $F_{62}$ | 4052739537881 |  |  |
| $F_{63}$ | 6557470319842 |  |  |
| $F_{64}$ | 10610209857723 |  |  |
| $F_{65}$ | 17167680177565 |  |  |
| $F_{66}$ | 27777890035288 |  |  |
| $F_{67}$ | 44945570212853 |  |  |
| $F_{68}$ | 72723460248141 |  |  |
| $F_{69}$ | 117669030460994 |  |  |
| $F_{70}$ | 190392490709135 |  |  |
| $F_{71}$ | 308061521170129 |  |  |
| $F_{72}$ | 498454011879264 |  |  |
| $F_{73}$ | 806515533049393 |  |  |
| $F_{74}$ | 1304969454928657 |  |  |
| $F_{75}$ | 2111485077978050 |  |  |
| $F_{76}$ | 3416454622906707 |  |  |

TABLE 2

| $L_{1}$ | 1 | $L_{11}$ | 199 | $L_{21}$ | 24476 | $L_{31}$ | 3010349 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $L_{2}$ | 3 | $L_{12}$ | 322 | $L_{22}$ | 39603 | $L_{32}$ | 4870847 |
| $L_{3}$ | 4 | $L_{13}$ | 521 | $L_{23}$ | 64079 | $L_{33}$ | 7881196 |
| $L_{4}$ | 7 | $L_{14}$ | 843 | $L_{24}$ | 103682 | $L_{34}$ | 12752043 |
| $L_{5}$ | 11 | $L_{15}$ | 1364 | $L_{25}$ | 167761 | $L_{35}$ | 20633239 |
| $L_{6}$ | 18 | $L_{16}$ | 2207 | $L_{26}$ | 271443 | $L_{36}$ | 33385282 |
| $L_{7}$ | 29 | $L_{17}$ | 3571 | $L_{27}$ | 439204 | $L_{37}$ | 54068521 |
| $L_{8}$ | 47 | $L_{18}$ | 5778 | $L_{28}$ | 710647 | $L_{38}$ | 87483803 |
| $L_{9}$ | 76 | $L_{19}$ | 9349 | $L_{29}$ | 1149851 | $L_{39}$ | 141552324 |
| $L_{10}$ | 123 | $L_{20}$ | 15127 | $L_{30}$ | 1860498 | $L_{40}$ | 228826127 |

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## ON SOME EXTENSIONS OF THE MEIXNER-WEISNER GENERATING FUNCTIONS

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I. INTRODUCTION

With the aid of group theory, Weisner [10] derived the Bilinear generating function for the ultraspherical polynomial:

$$
\begin{align*}
& \sum_{n=0}^{\infty} \frac{n!t^{n}}{(2 \alpha)_{n}} C_{n}^{\alpha}(\cos x) C_{n}^{\alpha}(\cos y) \\
& \quad=\left\{1-2 t \cos (x+y)+t^{2}\right\}^{-\alpha}{ }_{2} F_{I}\left[\begin{array}{l}
\left.\alpha, \alpha ; \frac{4 t \sin x \sin y}{1-2 t \cos (x+y)+t^{2}}\right]
\end{array}\right. \tag{1.1}
\end{align*}
$$

See [5] for definition and properties. (1.1) had also been proved by Meixner [6], Ossicini [7], and Watson [8], and was recently investigated by Carlitz [2], [3]. (1.1) is seen to be a special case of Theorem 1 in this paper, as are the formulas (1.2), (1.4), and (1.5), which appear to be new. Note that the expressions given below are generating functions for the ultraspherical polynomial of type $C_{n+\ell}^{\lambda}(x)$. See Cohen [4] for the single Jacobi polynomial.

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{t^{n}(n+\ell)!}{(2 u+2 \ell+1)_{n}} C_{n}^{u+\ell+\frac{1}{2}}(x) C_{n+\ell}^{u}(y) \\
& \quad=\frac{2^{\ell+1} \Gamma(u+\ell+1) \ell!}{\Gamma(u) \Gamma(2 u+\ell)\left[t^{2}\left(x^{2}-1\right)\right]^{u+\frac{1}{2} \ell}} C_{\ell}^{u}\left[\left\{\frac{2(y-x t)^{2}}{2 x^{2} t^{2}-2 x y t-t^{2}+1+\rho}\right\}^{\frac{1}{2}}\right]_{\ell}^{u}
\end{aligned}
$$

$$
\begin{equation*}
\left[\left\{\frac{2 x^{2} t^{2}-2 x y t-t^{2}+1+p}{2 t^{2}\left(x^{2}-1\right)}\right\}^{\frac{1}{2}}\right] \tag{1.2}
\end{equation*}
$$

where $\rho=\left[\left(1-2 x y t+t^{2}\right)^{2}-4 t^{2}\left(1-x^{2}\right)\left(1-y^{2}\right)\right]^{\frac{1}{2}},|t|<1,|x t / y|<1$, $\ell$ is a nonnegative integer, and $D_{l}^{u}$ is the Gegenbauer function defined by Watson $[9$, p. 129] as

$$
D_{\ell}^{u}(z)=\frac{\Gamma(u) \Gamma(2 u+\ell) z^{-2 u-\ell}}{2^{\ell+1} \Gamma(u+\ell+1)}{ }_{2} F_{1}\left[\begin{array}{l}
u+\frac{1}{2} \ell, \quad u+\frac{1}{2} \ell+\frac{1}{2} ; \frac{1}{z^{2}}  \tag{1.3}\\
u+\ell+1 ;
\end{array}\right]
$$

A special case of (1.2) is deduced for $x=0, y=\cos \phi$, and $t$ suitably modified:

$$
\sum_{n=0}^{\infty} \frac{t^{n}(2 n+\ell)!}{2^{2 n} n!(v+\ell+1)_{n}} C_{2 n+\ell}^{v}(\cos \phi)
$$

$$
\begin{equation*}
=\frac{\ell!\Gamma(v+\ell+1) 2^{\ell+1}}{\Gamma(v) \Gamma(2 v+\ell) t^{v+\frac{1}{2} \ell}} C_{\ell}^{v}\left[\left\{\frac{2 \cos ^{2} \phi}{1+t+\sigma}\right\}^{\frac{1}{2}}\right] D_{\ell}^{v}\left[\left\{\frac{1+t+\sigma}{2 t}\right\}^{\frac{1}{2}}\right] \tag{1.4}
\end{equation*}
$$

where $\sigma=\left(1-2 t \cos (2 \phi)+t^{2}\right)^{\frac{1}{2}}$, and $|t|<1$.
$\sum_{n=0}^{\infty} \frac{t^{n} C_{n}^{u+\frac{1}{2}}(\cos \theta) C_{n+1}^{u}(\cos \phi)}{C_{n+1}^{u}(0)}$

$$
=(\cos \phi-t \cos \theta)^{-1-2 u}(1-\xi \eta)^{-1}(1-\xi)^{u+\frac{1}{2}}(1-\eta)^{u+1}
$$

where $|t \cos \theta / \cos \phi|<1,|t|<1$, $\xi=\left\{1-2 t \cos \theta \cos \phi+t^{2}\right.$
$\left.-\left[\left(1-2 t \cos (\theta-\phi)+t^{2}\right)\left(1-2 t \cos (\theta+\phi)+t^{2}\right)\right]^{\frac{1}{2}}\right\} / 2 \sin ^{2} \phi$, and
$\eta=\left\{1-2 t \cos \theta \cos \phi+t^{2}\right.$
$\left.-\left[\left(1-2 t \cos (\theta-\phi)+t^{2}\right)\left(1-2 t \cos (\theta+\phi)+t^{2}\right)\right]^{\frac{1}{2}}\right\} / 2 t^{2} \sin ^{2} \theta$.
Theorem 2 yields the new finite expansions

$$
\begin{align*}
\sum_{n=0}^{\ell} & \frac{(-1)^{n} t^{n}}{(2 v)_{n}(1-2 \ell-2 v)_{\ell-n}} C_{n}^{v}(x) C_{\ell-n}^{\frac{1}{2}-v-\ell}(y) \\
& =\frac{\ell!\left(y^{2}-1\right)^{\frac{1}{2} \ell}}{2^{\ell}(v)_{\ell}(2 v)_{\ell}} C_{l}^{v}\left[\left\{\frac{2(y-x t)^{2}}{2 y^{2}-2 x y t+t^{2}-1+\rho}\right\}^{\frac{1}{2}}\right] C_{l}^{v}\left[\left\{\frac{2(y-x t)^{2}}{2 y^{2}-2 x y t+t^{2}-1-\rho}\right\}^{\frac{1}{2}}\right] \tag{1.6}
\end{align*}
$$

where $\rho$ is defined in equation (1.2).
Equation (1.6) may also be expressed as

$$
\sum_{n=0}^{\ell} \frac{t^{\prime n} 2^{n}}{(2 v)_{n}(v)_{\ell-n}} C_{n}^{v}(x) C_{\ell-n}^{v+n}\left(y^{\prime}\right)
$$

(1.7)

$$
=\frac{\ell!}{(v)_{\ell}(2 v)_{l}} C_{l}^{v}\left[\left\{\frac{y^{\prime 2}+1+2 x y^{\prime} t^{\prime}+t^{\prime 2}-\rho^{\prime}}{2}\right\}^{\frac{1}{2}}\right] C_{\ell}^{v}\left[\left\{\frac{y^{\prime 2}+1+2 x y^{\prime} t^{\prime}+t^{\prime 2}+\rho^{\prime}}{2}\right\}^{\frac{1}{2}}\right]
$$

where $\rho^{\prime}=\left\{\left(y^{\prime 2}-1+2 x y^{\prime} t^{\prime}+t^{\prime 2}\right)^{2}+4 t^{\prime 2}\left(1-x^{2}\right)\right\}^{\frac{1}{2}}$.
A special case of (1.6) is the relation

$$
\sum_{n=0}^{[\ell / 2]} \frac{t^{n}}{2^{2 n}(2 v)_{\ell-2 n}(1-\ell-v)_{n} n!} C_{\ell-2 n}^{v}(\cos \phi)
$$

$$
\begin{equation*}
=\frac{\ell!t^{\frac{1}{2} \ell}}{2^{\ell}(v)_{l}(2 v)_{l}} C_{l}^{v}\left[\left\{\frac{2 \cos ^{2} \phi}{1+t+\sigma}\right\}^{\frac{1}{2}}\right] C_{l}^{v}\left[\left\{\frac{2 \cos ^{2} \phi}{1+t-\sigma}\right\}^{\frac{1}{2}}\right] \tag{1.8}
\end{equation*}
$$

where $\sigma$ is defined in equation (1.4).
Equation (1.8) is deduced from (1.6) by putting $x=0$, and rearranging the parameters. Also, if $y=1$ in (1.6), one obtains a known expression [5, p. 227, last formula].

## SECTION II

Theorem 1: For $u$ and $v$ arbitrary complex numbers and $\ell$ a nonnegative integer,

$$
\sum_{n=0}^{\infty} \frac{t^{n}(n+l)!}{(2 v)_{n}} C_{n}^{v}(x) C_{n+l}^{u}(y)
$$

$$
\begin{equation*}
=(2 u)_{\ell}(y-x t)^{-2 u-\ell} F_{4}\left[\frac{1}{2}(2 u+\ell), \frac{1}{2}(2 u+\ell+1) ; v+\frac{1}{2}, u+\frac{1}{2} ; \frac{t^{2}\left(x^{2}-1\right)}{(y-x t)^{2}}, \frac{y^{2}-1}{(y-x t)^{2}}\right] \tag{2.1}
\end{equation*}
$$

where $|x t / y|<1,|t|<1$, and $F_{4}$ denotes the fourth type of Appe11's [1, p. 14] hypergeometric function of two variables defined by

$$
F_{4}\left[a, b ; c, d ; x_{1}, y_{1}\right]=\sum_{k_{1}, k_{2}}^{\infty} \frac{(a)_{k_{1}+k_{2}}(b)_{k_{1}+k_{2}}}{k_{1}!k_{2}!(c)_{k_{1}}(d)_{k_{2}}} x_{1}^{k_{1}} x_{2}^{k_{2}}
$$

Proob: The left-hand side of (2.1) may be expressed as

$$
\sum_{n=0}^{\infty} \frac{(2 u)_{n+\ell} t^{n} x^{n} y^{n+\ell}(n+\lambda)}{n!}{ }_{2} F_{1}\left[\begin{array}{l}
-\frac{1}{2} n,-\frac{1}{2} n+\frac{1}{2} ; \frac{x^{2}-1}{x^{2}} \\
v+\frac{1}{2} ;
\end{array}\right]
$$

$$
\cdot{ }_{2} F_{1}\left[\begin{array}{l}
-\frac{1}{2}(n+\ell),-\frac{1}{2}(n+\ell)+\frac{1}{2} ; \frac{y^{2}-1}{y^{2}}  \tag{2.2}\\
u+\frac{1}{2} ;
\end{array}\right]
$$

$=\sum_{n=0}^{\infty} \frac{(2 u)_{n+\ell} t^{n} x^{n} y^{n+\ell}(n+\lambda)}{n!}{ }_{2} F_{1}\left[\begin{array}{l}-\frac{1}{2} n,-\frac{1}{2} n+\frac{1}{2} ; \frac{x^{2}-1}{x^{2}} \\ v+\frac{1}{2} ;\end{array} y^{-2 u-2 n-2 \ell}\right.$

$$
\cdot{ }_{2} F_{1}\left[\begin{array}{l}
u+\frac{1}{2}(n+\ell), u+\frac{1}{2}(n+\ell)+\frac{1}{2} ; \frac{y^{2}-1}{y^{2}}  \tag{2.3}\\
u+\frac{1}{2} ;
\end{array}\right]
$$

$$
=\sum_{k=0}^{\infty} \sum_{p=0}^{\infty}\left[\frac{t^{2}\left(x^{2}-1\right)}{4 y^{2}}\right]^{k}\left[\frac{y^{2}-1}{4 y^{2}}\right]^{p} \frac{(2 u)_{\ell}(2 u+\ell)_{2 p+2 k}(n+\lambda+2 k)}{y^{2 u+l} p!k!\left(u+\frac{1}{2}\right)_{p}\left(v+\frac{1}{2}\right)_{k}}
$$

$$
\begin{equation*}
\cdot \sum_{n=0}^{\infty} \frac{(x t / y)^{n}(2 u+\ell+2 k+2 p)_{n}}{n!} \tag{2.4}
\end{equation*}
$$

From (2.2) to (2.3) we have used the Kummer transformation. Going from (2.3) to (2.4) entails the use of the following:

$$
\begin{gather*}
2^{2 k}\left(-\frac{1}{2} n\right)_{k}\left(-\frac{1}{2} n+\frac{1}{2}\right)_{k}=\frac{n!}{(n-2 k)!}  \tag{2.5}\\
\sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} f(n, k)=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n+2 k, k) \tag{2.6}
\end{gather*}
$$

and

$$
(2 u)_{n+\ell+2 k}(2 u+n+\ell+2 k)_{2 p}=(2 u)_{\ell}(2 u+\ell)_{2 p+2 k}(2 u+\ell+2 p+2 k)_{n} .
$$

Now

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{(x t / y)^{n}(2 u+\ell+2 k+2 p)_{n}}{n!}=[1-x t / y]^{-2 u-\ell-2 k-2 p} \tag{2.8}
\end{equation*}
$$

Hence, (2.4) reduces to

$$
\begin{equation*}
(y-x t)^{-2 u-\ell}(2 u)_{\ell} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty}\left[\frac{t^{2}\left(x^{2}-1\right)}{(y-x t)^{2}}\right]^{k}\left[\frac{\left(y^{2}-1\right)}{(y-x t)^{2}}\right]^{p} \tag{2.9}
\end{equation*}
$$

$$
\cdot \frac{\left(\frac{1}{2}(2 u+\ell)\right)_{p+k}\left(\frac{1}{2}(2 u+\ell+1)\right)_{p+k}}{k!p!\left(v+\frac{1}{2}\right)_{k}\left(u+\frac{1}{2}\right)_{p}}
$$

By definition, (2.9) is the right-hand side of Theorem 1.
Theorem 2: For $u$ and $v$ arbitrary complex numbers and $\ell$ a nonnegative integer,

$$
\sum_{n=0}^{\ell} \frac{(-1)^{n} t^{n}}{(2 v)_{n}(2 u)_{\ell-n}} C_{n}^{v}(x) C_{\ell-n}^{v}(y)
$$

$$
\begin{equation*}
=\frac{(y-x t)^{\ell}}{\ell!} F_{4}\left[-\frac{1}{2} \ell,-\frac{1}{2} \ell+\frac{1}{2} ; v+\frac{1}{2}, u+\frac{1}{2} ; \frac{t^{2}\left(x^{2}-1\right)}{(y+x t)^{2}}, \frac{y^{2}-1}{(y-x t)^{2}}\right] . \tag{2.10}
\end{equation*}
$$

Proof: The left-hand side of (2.10) is put in the form

$$
\sum_{n=0}^{\infty} \frac{(-1)^{n} t^{n} x^{n} y^{\ell-n}}{n!(\ell-n)!}{ }_{2} F_{1}\left[\begin{array}{l}
-\frac{1}{2} n,-\frac{1}{2} n+\frac{1}{2} ; \\
v+\frac{x^{2}-1}{x^{2}}
\end{array}\right]
$$

$$
\text { - }{ }_{2} F_{1}\left[\begin{array}{l}
-\frac{1}{2}(\ell-n),-\frac{1}{2}(\ell-n)+\frac{1}{2} ; \frac{y^{2}-1}{y^{2}}  \tag{2.11}\\
u+\frac{1}{2} ;
\end{array}\right] .
$$

Following a procedure analogous to that in the proof of Theorem 1, with appropriate changes, (2.11) is simplified to yield the right-hand side of (2.10).

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# ALMOST ARITHMETIC SEQUENCES AND COMPLEMENTARY SYSTEMS 

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What about the sequence $3,6,9,12,15, \ldots$ ? If this is simply the arithmetic sequence $\{3 n\}$, then its study would be essentially that of the positive integers. However, suppose the $n$th term is $[(3+1 / \sqrt{29}) n]$, or perhaps $[(4-5 / \sqrt{37}) n]$, where $[x]$ means the greatest integer $\leq x$. In these sequences, 15 is followed by 19 rather than 18. Such almost arithmetic sequences have many interesting properties which have been discovered only in recent years. Of special interest are complementary systems of such sequences.

The adjective "complementary" means that every positive integer occurs exactly once in exactly one of a given set of sequences. Consider, for example, the three sequences

$$
\begin{equation*}
1,4,6,8,10,13, \ldots ; 2,5,9,12,16, \ldots ; 3,7,11,14, \ldots ; \tag{1}
\end{equation*}
$$

which can be accounted for as follows: If the positive integers that are squares, twice squares, or thrice squares are all arranged in increasing order, we find at the beginning

$$
\begin{equation*}
1,2,3,4,8,9,12,16,18,25,27,32,36 . \tag{2}
\end{equation*}
$$

Each of these numbers occupies a position in the arrangement. In particular, the squares $1,4,9,16,25,36, \ldots$ occupy positions numbered $1,4,6,8,10,13, \ldots$, the first sequence in (1). This line of reasoning can be extended to show that the three sequences in (1) are given, respectively, by the formulas

$$
n+\left[\frac{n}{\sqrt{2}}\right]+\left[\frac{n}{\sqrt{3}}\right],[n \sqrt{2}]+n+\left[\frac{n \sqrt{2}}{\sqrt{3}}\right],[n \sqrt{3}]+\left[\frac{n \sqrt{3}}{\sqrt{2}}\right]+n .
$$

The three sequences in (1) may be compared with the sequences

$$
1,4,7,10,13,16, \ldots ; 2,5,8,11,14, \ldots ; 3,6,9,12, \ldots ;
$$

which form a complementary system of arithmetic sequences given by $3 n+1,3 n+2$, and $3 n+3$. Each has a common difference, or slope, equal to 3 . Similarly, the sequences in (1) have slopes $s=1+1 / \sqrt{2}+1 / \sqrt{3}, \sqrt{2} s$, and $\sqrt{3} s$, as shown by formulas ( $1^{\prime}$ ). Here the similarity ends, however. Writing $a_{n}=3 n+1$, we call to mind the very simple recurrence relation $\alpha_{n+1}-a_{n}=3$. On the other hand, writing $b_{n}=n+[n / \sqrt{2}]+[n / \sqrt{3}]$, we find $b_{n+1}-b_{n} \neq s$ for all $n$. Instead, $b_{n+1}-b_{n}$ takes values 1,2 , and 3 , depending on $n$. Moreover, $a_{n+2}-a_{n}=6$ for all $n$, whereas $b_{n+2}-b_{n}$ takes values 4, 5, and 6 .

We are now in a position to state the purpose of this note: first, to introduce a definition of "almost arithmetic" that covers sequences as in (1), and then to present some theorems about almost arithmetic sequences and complementary systems.

One more thought before defining the general almost arithmetic sequence $\left\{a_{n}\right\}$ is that there should be a real number $u$ such that $\alpha_{n}$ must stay close to the arithmetic sequence $n u$. Specifically, $a_{n}-n u$ should stay bounded as $n$ goes through the positive integers, and this could be used as the defining property for "almost arithmetic" sequences. However, this property depends on the existence of a real number $u$, and since the $\alpha_{n}$ are positive integers, a definition which refers only to positive integers is much to be preferred. From such a definition, we should be able to determine the number $u$. The following definition meets these requirements.

Suppose $\ell \leq k$ are nonnegative integers and $\left\{\alpha_{n}\right\}$ is a strictly increasing sequence of positive integers satisfying

$$
\begin{equation*}
0 \leq a_{m+n}-a_{m}-a_{n}+\ell \leq k, \text { for all } m, n \geq 1 \tag{3}
\end{equation*}
$$

The sequence $\left\{a_{n}\right\}$ is almost arithmetic, or, more specifically, ( $k, \ell$ )-arithmetic.
It is fairly easy to check that for any positive real numbers $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}$ and $\beta_{1}, \beta_{2}, \ldots, \beta_{k}$ the sequence with $n$th term

$$
\begin{equation*}
a_{n}=\left[\alpha_{1} n\right]+\left[\alpha_{2} n\right]+\cdots+\left[\alpha_{k} n\right] \tag{4}
\end{equation*}
$$

is ( $k, 0$ )-arithmetic, and the sequence with $n$th term

$$
\begin{equation*}
\alpha_{n}=\left[\alpha_{1} n+\beta_{1}\right]+\left[\alpha_{2} n+\beta_{2}\right]+\cdots+\left[\alpha_{k} n+\beta_{k}\right] \tag{5}
\end{equation*}
$$

is ( $k^{\prime}, \ell$ )-arithmetic for some $\ell$ and some $k^{\prime} \geq k$.
For example, the sequence $\{3 n\}$ is ( 0,0 )-arithmetic; $\{3 n+1\}$ is ( 0,1 )-arithmetic, and $\{n+[n / \sqrt{2}]+[n / \sqrt{3}]\}$ is (2, 0)-arithmetic.

As we shall soon see, there are many almost arithmetic sequences $\left\{a_{n}\right\}$ for which no formula in closed form for $a_{n}$ is known. Nevertheless, our first theorem will show that every almost arithmetic sequence $\left\{a_{n}\right\}$ must have a slope $u$, and $a_{n}$ must stay close to $n u$.
Theorem 1: If $\left\{a_{n}\right\}$ is a ( $k, \ell$ )-arithmetic sequence, then the number $u=\lim _{n \rightarrow \infty} \frac{a_{n}}{n}$, hereinafter referred to as the slope of $\left\{a_{n}\right\}$, exists, and

$$
\begin{equation*}
a_{n} \leq r u+\ell \leq a_{n}+k, \text { for } n=1,2, \ldots . \tag{6}
\end{equation*}
$$

Proof: Let $\varepsilon>0$, and let $m$ be so large that

$$
\max \left\{\frac{\ell}{m}, \frac{k-\ell}{m}\right\}<\varepsilon .
$$

for any $n>m$, we have $n=q m+r$ where $q=[n / m]$ and $0 \leq r<m$. By (3),
and

$$
a_{m}-\ell \leq a_{n}-a_{n-m} \leq a_{m}-\ell+k
$$

$$
\begin{array}{lcc} 
& a_{m}-\ell \leq a_{n-m}-a_{n-2 m} \leq a_{m}-\ell+k \\
\vdots & \vdots & \vdots \\
\text { and finally } & a_{m}-\ell \leq a_{n-(q-1) m}-a_{r} \leq a_{m}-\ell+k . \\
\text { Adding these }: & q\left(a_{m}-\ell\right) \leq a_{n}-a_{r} \leq q\left(a_{m}-\ell+k\right) .
\end{array}
$$

Adding these:

Now adding $\alpha_{r}-q \alpha_{m}$ and dividing by $n$ yields

$$
\frac{a_{r}}{n}-\frac{\ell q}{n} \leq \frac{a_{n}}{n}-\frac{q \alpha_{m}}{n} \leq \frac{q}{n}(k-\ell)+\frac{a_{r}}{n} .
$$

When this is added to the easily verified

$$
-\frac{a_{m}}{n} \leq \frac{q \alpha_{m}}{n}-\frac{a_{m}}{m} \leq 0
$$

one obtains

$$
\begin{aligned}
\frac{a_{r}-a_{m}}{n}-\varepsilon<\frac{a_{r}-a_{m}}{n}-\frac{\ell}{m} & \leq \frac{a_{r}-a_{m}-\ell q}{n} \leq \frac{a_{n}}{n}-\frac{\alpha_{m}}{m} \leq \frac{q(k-\ell)+a_{r}}{n} \\
& \leq \frac{k-\ell}{m}+\frac{a_{r}}{n}<\varepsilon+\frac{a_{r}}{n}
\end{aligned}
$$

As $n \rightarrow \infty$ we see that $\left|\frac{a_{n}}{n}-\frac{a_{m}}{m}\right| \leq \varepsilon$, so that $\left\{\frac{a_{n}}{n}\right\}$, as a Cauchy sequence, converges.
Now as a first step in an induction argument,

$$
a_{n}-\ell \leq a_{2 n}-a_{n} \leq a_{n}-\ell+k
$$

Assume for arbitrary $j>3$ that

$$
(j-2)\left(a_{n}-\ell\right) \leq a_{(j-1) n}-a_{n} \leq(j-2)\left(a_{n}-\ell+k\right)
$$

Adding this with $a_{n}-\ell \leq a_{j n}-a_{(j-1) n} \leq a_{n}-\ell+k$ gives

$$
(j-1)\left(a_{n}-\ell\right) \leq a_{j n}-a_{n} \leq(j-1)\left(a_{n}-\ell+k\right)
$$

which concludes the induction argument. This set of inequalities is equivalent to

$$
j a_{n}-(j-1) \ell \leq a_{j n} \leq j a_{n}+(j-1)(k-\ell)
$$

Dividing by $j n$,

$$
\frac{a_{n}}{n}-\frac{1}{n} \frac{j-1}{j} \ell \leq \frac{a_{j n}}{j n} \leq \frac{a_{n}}{n}+\frac{1}{n} \frac{j-1}{j}(k-\ell) .
$$

Since $\lim _{j \rightarrow \infty} \frac{a_{j n}}{j n}=u$, we have

$$
\frac{a_{n}}{n}-\frac{\ell}{n} \leq u \leq \frac{a_{n}}{n}+\frac{1}{n}(k-\ell)
$$

and (6) follows.
Theorem 1 should be compared with similar results in Pólya and Szegö [7, pp. 23-24].

Note the contrast between the defining inequality (3) and Theorem 1. The former is entirely combinatorial, whereas the notion of slope is analytic. Specifically, when $\ell$ is the least integer such that

$$
a_{m+n}-a_{m}-a_{n}+\ell \geq 0, \text { for all } m, n \geq 1
$$

and if $\dot{k}$ is the least integer such that

$$
a_{m+n}-a_{m}-a_{n}+\ell \leq k, \text { for all } m, n \geq 1,
$$

then $k$ counts the extent to which the sequence $\left\{a_{n}-l\right\}$ deviates from the rule

$$
c_{m+n}-c_{m}-c_{n}=0
$$

that is, from being an arithmetic sequence. On the other hand, the slope $u$ gives the average growth rate of $\left\{\alpha_{n}\right\}$. With this analytic notion in mind, we may predict that if $\left\{a_{n}\right\}$ has slope $u$ and $\left\{b_{n}\right\}$ slope $v$, then the composite sequences $\left\{a_{b_{n}}\right\}$ and $\left\{b_{a_{n}}\right\}$ will have slope $u v$. Or, if the given sequences are disjoint, we can combine them in increasing order, thus getting a sequence with slope $\left(u^{-1}+v^{-1}\right)^{-1}$, the harmonic mean of $u$ and $v$. Then returning to a combinatorial attitude, we may ask about the bounding numbers $k$ and $l$ for these new sequences. Our first theorem of the sort just suggested shows how to make almost arithmetic sequences from a given real $u \geq 1$.
Theorem 2: If $u \geq 1$ is a real number and $\left\{a_{n}\right\}$ is an increasing sequence of positive integers satisfying $0 \leq n u-a_{n}+\ell \leq k$ for $0 \leq l \leq k$ and for $n=1,2, \ldots$, then $\left\{a_{n}\right\}$ is a $(3 k, k+l)$-arithmetic sequence with slope $u$.

Proof: Subtracting $0 \leq(m+n) u-a_{m+n}+\ell \leq k$ from $0 \leq m u-a_{m}+\ell \leq k$ gives $-k \leq a_{m+n}-a_{m}-n u \leq k$. This implies $n u \leq a_{m+n}-a_{m}+k \leq n u+2 k$. Bounds for $n u$ come from $0 \leq n u-a_{n}+\ell \leq k$, namely $a_{n}-\ell \leq n u \leq a_{n}-\ell+k$. Thus

$$
a_{n}-\ell \leq a_{m+n}-a_{m}+k \leq a_{n}-\ell+3 k
$$

or equivalently,

$$
0 \leq a_{m+n}-a_{m}-a_{n}+\ell+k \leq 3 k,
$$

as required.

As an example, let $a_{n}=2 n$ if $n$ is prime and $2 n+1$ otherwise. Then $k=\ell=1$ in Theorem 2, and $\left\{a_{n}\right\}$ is a (3, 2)-arithmetic sequence. Actually, $\left\{a_{n}\right\}$ is also a (2, 2)-arithmetic sequence, which is saying more. This example shows that the $k$ and $\ell$ in Theorem 2 need not be the least values for which (3) holds. This same observation holds for the theorems that follow.

Consider next $a_{n}=10 n+2$ and $b_{n}=10+5$ for $n=0,1,2, \ldots$. We combine these to form the sequence $\left\{c_{n}\right\}$ given by $2,5,12,15,22,25, \ldots$, and ask if this is an almost arithmetic sequence. If so, what numbers $k$, $l$ describe the maximal spread which $c_{n}$ has away from $5 n$ ? The question leads to the following theorem about disjoint unions of almost arithmetic sequences.
Theorem 3: Suppose $\left\{a_{n}\right\}$ is a ( $k, \ell$ )-arithmetic sequence and $\left\{b_{n}\right\}$ is a ( $k^{\prime}, l^{\prime}$ )arithmetic sequence, disjoint from $\left\{a_{n}\right\}$ in the sense that $b_{n} \neq a_{m}$ for all $m$ and $n$. Let $\left\{c_{n}\right\}$ be the union of $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$. Then $\left\{c_{n}\right\}$ is a ( $\left.\mathcal{K}, \mathcal{L}\right)$-arithmetic sequence for some $\mathcal{K}$ and $\mathcal{L}$ (given in the proof). If $\left\{a_{n}\right\}$ has slope $u$ and $\left\{b_{n}\right\}$ has slope $v$, then $\left\{c_{n}\right\}$ has slope $\left(u^{-1}+v^{-1}\right)^{-1}$.

Proof: Let $n$ be a positive integer.
Case 1. Suppose $c_{n}=a_{N}$ for some $N$. Let $x=N u / v$. By Theorem 1,
$x v-k+\ell \leq a_{N} \leq x v+\ell$ and $i v-k^{\prime}+\ell^{\prime} \leq b_{i} \leq i v+\ell^{\prime}, i=1,2, \ldots$. The inequality $i v+\ell^{\prime} \leq x v-k+\ell$ shows that $b_{i} \leq a_{N}$ whenever

$$
i \leq x+\left(\ell-\ell^{\prime}-k\right) / v
$$

Similarly, $b_{i} \geq a_{N}$ for

$$
i \geq x+\left(\ell-\ell^{\prime}+k^{\prime}\right) / v
$$

Thus, the number of $b_{i}$ which are $\leq a_{N}$ is $x+\delta$, where

$$
\left(\ell-\ell^{\prime}-k\right) / v \leq \delta \leq\left(\ell-\ell^{\prime}+k^{\prime}\right) / v
$$

so that

$$
n=\left(\# a_{i} \leq a_{N}\right)+\left(\# b_{i} \leq a_{N}\right)=N+N u / v+\delta .
$$

Multiplying by $w=u v /(u+v)$ gives $N u=(n-\delta) w$. Now, substituting this and $a_{N}=c_{n}$ into $N u-k+\ell \leq \alpha_{N} \leq N u+\ell$, we obtain

$$
(n-\delta) w-k+\ell \leq c_{n} \leq(n-\delta) w+\ell
$$

Case 2. Suppose $c_{n}=b_{N}$ for some $N$. As in Case 1, there exists $\delta^{\prime}$ satisfying $\left(\ell^{\prime}-\ell-k^{\prime}\right) / u \leq \delta^{\prime} \leq\left(\ell^{\prime}-\ell+k\right) / u$ such that

$$
\left(n-\delta^{\prime}\right) w-k^{\prime}+\ell^{\prime} \leq c_{n} \leq\left(n-\delta^{\prime}\right) w+\ell^{\prime}
$$

To accommodate both cases, let

$$
\mathscr{L}^{\prime \prime}=\min \left\{\begin{array}{l}
\ell-k-\delta w \\
\ell^{\prime}-k^{\prime}-\delta^{\prime} w
\end{array} \quad \text { and } \quad x^{\prime \prime}=\max \left\{\begin{array}{l}
\ell-\delta w \\
\ell^{\prime}-\delta^{\prime} w,
\end{array}\right.\right.
$$

and then let

$$
\mathscr{L}^{\prime}=\left\{\begin{array}{l}
\mathscr{L}^{\prime \prime} \text { if } \mathscr{L}^{\prime \prime} \text { is an integer } \\
{\left[\mathcal{L}^{\prime \prime}\right]+1 \text { otherwise }}
\end{array} \text { and } \mathscr{K}^{\prime}=\left[\mathscr{K}^{\prime \prime}\right]\right.
$$

Now $n w+\mathcal{L}^{\prime} \leq c_{n} \leq n w+\mathcal{K}^{\prime}$, so that $0 \leq n w-c_{n}+\mathcal{K}^{\prime} \leq \mathcal{K}^{\prime}-\mathcal{L}^{\prime}$. By Theorem 2, $\left\{c_{n}\right\}$ is ( $\mathcal{K}, \mathcal{L})$-arithmetic, where $\mathcal{K}=3\left(\mathcal{K}^{\prime}-\mathcal{L}^{\prime}\right)$ and $\mathcal{L}=2 \mathcal{K}^{\prime}-\mathcal{L}^{\prime}$.

The theorem just proved has an interesting application to complementary systems, as follows.
Theorem 4: Suppose $\left\{a_{1 n}\right\},\left\{\alpha_{2 n}\right\}, \ldots,\left\{a_{m n}\right\}$ are almost arithmetic sequences that comprise a complementary system. Let $u_{i}=\lim _{n \rightarrow \infty} \frac{a_{i n}}{n}$ for $i=1,2, \ldots, m$. Then

$$
\frac{1}{u_{1}}+\frac{1}{u_{2}}+\cdots+\frac{1}{u_{m}}=1 .
$$

Proof: As members of a complementary system, $\left\{\alpha_{1 n}\right\}$ and $\left\{\alpha_{2 n}\right\}$ are disjoint. By Theorem 3, their union is an almost arithmetic sequence with slope $w$ satisfying $1 / w=1 / u_{1}+1 / u_{2}$. Assume for arbitrary $k \leq m-1$ that the union

$$
\left\{a_{1 n}\right\} \cup\left\{a_{2 n}\right\} \cup \cdots \cup\left\{a_{k n}\right\}
$$

is almost arithmetic with slope $u$ satisfying $1 / u=1 / u_{1}+\cdots+1 / u_{k}$. Theorem 3 applies. By mathematical induction on $k$, we have $1 / v=1 / u_{1}+\cdots+1 / u_{m}$, where $v$ is the slope of the union of all the given sequences, that is, 1.

In case $m=2$, the identity

$$
1=\sum_{i=1}^{m} 1 / u_{i}
$$

is the subject of the famous Beatty Problem [1] of 1926. An extensive bibliography on results stemming from Beatty's Problem and other research on sequences of the form \{[un]\} is given in Stolarsky [8]; the interested reader should also consult Fraenkel, Mushkin, and Tassa [3]. A generalization of Beatty's Problem by Skolem [7] is that sequences $\{[u n]\}$ and $\{[v n]\}$, where $u$ and $v$ are positive irrationals, are disjoint if and only if $a / u+b / v=1$ for some integers $a$ and $b$. Skolem's generalization suggests a still more general question, which we state here hoping that an answer will someday by found: What criteria exist for disjointness of two sequences of the form (4), for $k \geq 2$ ?

We turn next to composites of almost arithmetic sequences.
Theorem 5: Composites of almost arithmetic sequences are almost arithmetic. Spe$\overline{\text { cifically }}$, if $\left\{a_{n}\right\}$ is ( $k, \ell$ )-arithmetic with slope $u$ and $\left\{b_{n}\right\}$ is ( $k^{\prime}, \ell^{\prime}$ )-arithmetic with slope $v$, then the sequence $\left\{c_{n}\right\}$ defined by $c_{n}=b_{a_{n}}$ is ( $b_{\ell}+b_{k-\ell}+3 k$ ' $\left.-2 \ell^{\prime}, b_{l}+k^{\prime}\right)$-arithmetic with slope $u v$. (Here $b_{0} \equiv 0$.)

Proof: We must show that
(7)

$$
\begin{gathered}
0 \leq c_{m+n}-c_{m}-c_{n}+b_{l}+k^{\prime} \\
c_{m+n}-c_{m}-c_{n}+b_{\ell}+k^{\prime} \leq b_{\ell}-b_{k-\ell}+3 k^{\prime}-2 \ell^{\prime}
\end{gathered}
$$

and
(8)

Now

$$
\begin{aligned}
0 & \leq b_{a_{m}+a_{n}}-b_{a_{m}}-b_{a_{n}}+\ell^{\prime} \text { by (3) } \\
& \leq b_{a_{m+n}+\ell}-b_{a_{m}}-b_{a_{n}}+\ell^{\prime} \text { since } a_{m}+a_{n} \leq a_{m+n}+\ell \\
& \leq\left(b_{a_{m+n}}+b_{\ell}+k^{\prime}-\ell^{\prime}\right)-b_{a_{m}}-b_{a_{n}}+\ell^{\prime} \text { by (3). }
\end{aligned}
$$

This proves (7). To prove (8),

$$
\begin{aligned}
b_{a_{m+n}}-b_{a_{m}}-b_{a_{n}}+b_{\ell}+k^{\prime} & \leq b_{a_{m}+a_{n}+k+\ell-b_{a_{m}}-b_{a_{n}}+b_{\ell}+k^{\prime}} \\
& \leq b a_{m}+a_{n}+b_{k-\ell}+k^{\prime}-\ell^{\prime}-b_{a_{m}}-b_{a_{n}}+b_{\ell}+k^{\prime} \\
& \leq b_{a_{m}}+b_{a_{n}}+k^{\prime}-\ell^{\prime}+b_{k-\ell}+k^{\prime}-\ell^{\prime} \\
& -b_{a_{m}}-b_{a_{n}}+b_{\ell}+k^{\prime} \\
& =b_{\ell}+b_{k-\ell}+3 k^{\prime}-2 \ell^{\prime},
\end{aligned}
$$

as required.
For slopes we have $a_{n} \sim u n$ and $b_{n} \sim v n$, where the symbol $\sim$ abbreviates the relationship indicated in (6). Consequently, $b a_{n} \sim v a_{n} \sim v u n$.

To illustrate Theorem 5, let $a_{n}=[\sqrt{2 n}]$ and $b_{n}=[\sqrt{3} n]$. Each provides a (1, 0)arithmetic sequence. The composite $b_{a_{n}}=[\sqrt{3}[\sqrt{2} n]]$ has slope $\sqrt{6}$ and is (4, 1)arithmetic. The same is true for $a_{b_{n}}=[\sqrt{2}[\sqrt{3} n]]$.

Theorem 6: The complement of a ( $k, \ell$ )-arithmetic sequence $\left\{a_{n}\right\}$ having slope $u>1$ is a $\left(\left[\frac{3(u+k)}{u-1}\right],\left[\frac{u+2 k-\ell}{u-1}\right]\right)$-arithmetic sequence with slope $u /(u-1)$.

Proof: The complement of $\left\{a_{n}\right\}$ is the increasing sequence $\left\{a_{n}^{*}\right\}$ of all positive integers missing from $\left\{a_{n}\right\}$. By (6) we can write

$$
a_{n}=n u+\delta, \text { where } \ell-k \leq \delta=\delta(n) \leq \ell .
$$

Then the inequality $\alpha_{i}<a_{n}^{*}$ can be expressed as $i<\left(a_{n}^{*}-\delta\right) / u$, and the greatest such $i$ is $\left[\left(a_{n}^{*}-\delta\right) / u\right]$. Now $a_{n}^{*}=n+f\left(a_{n}^{*}\right)$, where $f(x)$ is the number of terms $a_{i}$ satisfying $a_{i}<x$. Thus $a_{n}^{*}=n+\left[\left(a_{n}^{*}-\delta\right) / u\right]$, and

$$
n+\left(a_{n}^{*}-\delta\right) / u-1 \leq a_{n}^{*} \leq n+\left(a_{n}^{*}-\delta\right) / u
$$

This readily leads to
so that

$$
\delta \leq u n-(u-1) a_{n}^{*} \leq u+\delta,
$$

$$
0 \leq \frac{u n}{u-1}-a_{n}^{*}+\frac{k-\ell}{u-1} \leq \frac{u+k}{u-1}
$$

and we conclude, by the method of proof of Theorem 2 , that $\left\{a_{n}^{*}\right\}$ is an almost arithmetic sequence of the required sort.

Theorem 6 shows, for example, that the set of all positive integers not of the form $[\sqrt{7} n+\sqrt{3}]+[\sqrt{7} n-\sqrt{3}]=\alpha_{n}$ forms an almost arithmetic sequence. Suppose that, given a sequence such as $\left\{\alpha_{n}\right\}$, we remove a subsequence which is almost arithmetic, for example $\left\{\alpha_{[\sqrt{7} n]}\right\}$. Will the remaining terms of $\left\{\alpha_{n}\right\}$ still form an almost arithmetic sequence? We call such remaining terms the relative complement (of $\left\{\alpha_{[\sqrt{7} n]}\right\}$ in $\left\{a_{n}\right\}$ ), and have the following strengthening of Theorem 6 .
Theorem 7: The relative complement of an almost arithmetic subsequence of an almost arithmetic sequence is almost arithmetic.

Proob: Suppose $\left\{a_{n_{i}}\right\}$ is an almost arithmetic subsequence of an almost arithmetic sequence $\left\{a_{n}\right\}$. By Theorem 1 , there exist positive real $u$ and $v$ and nonnegative integers $\ell, k, \ell^{\prime}, k^{\prime}$ such that

$$
\begin{gather*}
a_{n_{i}} \leq n_{i} u+l \leq a_{n_{i}}+k, \ell \leq k, i=1,2, \ldots,  \tag{9}\\
a_{n_{i}} \leq i v+l^{\prime} \leq a_{n_{i}}+k^{\prime}, \ell^{\prime} \leq k^{\prime}, i=1,2, \ldots . \tag{10}
\end{gather*}
$$

Dividing by $u$ in (9) and (10) leads to

$$
n_{i}+\frac{\ell}{u}-\frac{k}{u} \leq \frac{a_{n_{i}}}{u} \leq i \frac{v}{u}+\frac{\ell^{\prime}}{u} \leq \frac{a_{n_{i}}}{u}+\frac{k^{\prime}}{u} \leq n_{i}+\frac{\ell}{u}+\frac{k^{\prime}}{u},
$$

so that

$$
0 \leq i \frac{v}{u}-n_{i}+\frac{l^{\prime}+k-l}{u} \leq \frac{k+k^{\prime}}{u}
$$

Thus, by Theorem 2, the sequence $\left\{n_{i}\right\}$ is almost arithmetic. By Theorem 7, the complementary sequence $\left\{n_{i}^{*}\right\}$, consisting of all positive integers which are not terms of $\left\{n_{i}\right\}$, is almost arithmetic. By Theorem 6, the sequence $\left\{a_{n_{i}^{*}}\right\}$, which consists of all the $a_{n}$ 's missing from $\left\{a_{n_{i}}\right\}$, is almost arithmetic, as was to be proved.

Corollary to the Proof of Theorem 7: Suppose $\left\{\dot{a}_{n_{i}}\right\}$ is an almost arithmetic subsequence of an almost arithmetic sequence $\left\{\alpha_{n}\right\}$. Then the sequence $\left\{n_{i}\right\}$ is almost arithmetic.

We now return to the complementary system
(1) $1,4,6,8,10,13, \ldots ; 2,5,9,12,16, \ldots ; 3,7,11,14, \ldots$.

Writing these sequences as $\left\{a_{n}\right\},\left\{b_{n}\right\},\left\{c_{n}\right\}$, we list all the positive integers as follows:

$$
a_{1}, b_{1}, c_{1}, a_{2}, b_{2}, a_{3}, c_{2}, a_{4}, b_{3}, a_{5}, c_{3}, \ldots
$$

Removing all the $c_{i}$ leaves

$$
\begin{equation*}
a_{1}, b_{1}, a_{2}, b_{2}, a_{3}, a_{4}, b_{3}, a_{5}, \ldots \tag{1"}
\end{equation*}
$$

Now let $\left\{a_{n}\right\} \oplus\left\{c_{n}\right\}$ and $\left\{b_{n}\right\} \oplus\left\{c_{n}\right\}$ represent, respectively, the number of the position of $a_{n}$ and $b_{n}$ in ( $1^{\prime \prime}$ ), counting from the left. These two sequences form a complementary system of almost arithmetic sequences. In fact, for comparison with formulas ( $1^{\prime}$ ), one may easily check that

$$
\begin{aligned}
& \left\{a_{n}\right\} \oplus\left\{c_{n}\right\}=\left\{n+\left[\frac{n}{\sqrt{2}}\right]\right\}=\{1,3,5,6,8,10,11,13,15,17,18,20, \ldots\} \\
& \left\{b_{n}\right\} \oplus\left\{c_{n}\right\}=\{n+[\sqrt{2} n]=\{2,4,7,9,12,14,16,19,21, \ldots\} .
\end{aligned}
$$

We define $\oplus$ in general as follows: For disjoint strictly increasing sequences $\left\{a_{n}\right\}$ and $\left\{c_{n}\right\}$ of positive integers, let $\left\{d_{n}\right\}$ be the sequence obtained by writing all the $a_{i}$ and $a_{i}^{*}$ in increasing order and then removing all the $c_{i}$. Then

$$
\left\{\alpha_{n}\right\} \oplus\left\{a_{n}\right\}
$$

is the sequence whose $n$th term is the position of $a_{n}$ in the sequence $\left\{d_{n}\right\}$.
Even if $\left\{a_{n}\right\}$ and $\left\{c_{n}\right\}$ are not disjoint, we define a second operation $\Theta$ as follows: Construct a sequence $\left\{e_{n}\right\}$ by putting $c_{n}$ at position $c_{n}$ for all $n$ and filling all the remaining positions with the $a_{i}$ and $\alpha_{i}^{*}$ written in increasing order. Then

$$
\left\{a_{n}\right\} \Theta\left\{c_{n}\right\}
$$

is the sequence whose $n$th term is the position of $a_{n}$ in the sequence $\left\{e_{n}\right\}$.
One relationship between $\oplus$ and $\Theta$ is indicated by the identity

$$
\begin{aligned}
& \left(\left\{a_{n}\right\} \Theta\left\{c_{n}\right\}\right) \oplus\left\{c_{n}\right\}=\left\{a_{n}\right\} \\
& \left(\left\{a_{n}\right\} \Theta\left\{c_{n}\right\}\right) \Theta\left\{c_{n}\right\}=\left\{a_{n}\right\}
\end{aligned}
$$

in case $\left\{a_{n}\right\}$ and $\left\{c_{n}\right\}$ are disjoint.
Both operations $\oplus$ and $\Theta$ can be used on any given complementary system of sequences $\left\{a_{1 n}\right\},\left\{a_{2 n}\right\}, \ldots,\left\{a_{m n}\right\}, m \geq 2$, to produce new complementary systems whose sequences remain almost arithmetic in case the original sequences were so, as we shall see in Theorems 8 and 9. Specifically,

$$
\left\{a_{1 n}\right\} \oplus\left\{a_{m n}\right\}, \ldots,\left\{a_{m-1, n}\right\} \oplus\left\{a_{m n}\right\}
$$

is a complementary system of $m$ - 1 sequences, and for any strictly increasing sequence $\left\{c_{n}\right\}$ of positive integers, the collection

$$
\left\{a_{1 n}\right\} \Theta\left\{c_{n}\right\}, \ldots,\left\{a_{m n}\right\} \Theta\left\{c_{n}\right\}
$$

together with $\left\{c_{n}\right\}$ itself, is a complementary system of $m+1$ sequences.
What about slopes and formulas for the $n$th terms of sequences arising from $\oplus$ and $\Theta$ ? We have the following two theorems.
Theorem 8: Suppose $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are disjoint almost arithemtic sequences having slopes $u$ and $v$, respectively. Let $c_{n}=b_{n}+n-1$, then

$$
\left\{a_{n}\right\} \oplus\left\{b_{n}\right\}=\left\{2 a_{n} \Theta c_{a_{n}}^{\star}\right\}
$$

is an almost arithmetic sequence having slope $u-u / v$.

Proof：Let $(\# ⿰ ⿰ 三 丨 ⿰ 丨 三 \leq n)$ denote the number of $b_{i}$ that are $\leq n$ ．Using the formula $C F(n)=n+f^{\dagger}(n)$ on p .457 of Lambek and Moser［6］，we find

$$
\begin{aligned}
n+\left(\# b_{i} \leq n\right)= & n \text {th positive integer not } \\
& \text { in the sequence }\left\{b_{n}+n-1\right\},
\end{aligned}
$$

so that

$$
\left(\# b_{i} \leq a_{n}\right)=-a_{n}+a_{n} \text { th term of the complement of }\left\{b_{n}+n-1\right\},
$$

whence the $n$th term of $\left\{a_{n}\right\} \oplus\left\{b_{n}\right\}$ ，which is clearly $a_{n}-\left(\# b_{i} \leq a_{n}\right)$ ，must equal $2 a_{n}-c_{a_{n}}^{\star}$ ．Since $\left\{c_{n}\right\}$ is almost arithmetic with slope $v+1$ ，$\left\{c_{n}^{\star}\right\}$ is almost arith－ metic with slope $1+1 / v$ ，by Theorem 6 ．Then $\left\{c_{\alpha_{n}}^{*}\right\}$ is almost arithmetic with slope $u(1+1 / v)$ ，by Theorem 5．Thus，$\left\{2 \alpha_{n}-c_{\alpha_{n}}^{\alpha_{n}}\right\}$ is almost arithmetic with slope $2 u-$ $u(1+1 / v)$ ．

Theorem 9：Suppose $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are almost arithmetic sequences having slopes $u$ and $v$ ，respectively．Then

$$
\left\{a_{n}\right\} \Theta\left\{b_{n}\right\}=\left\{b_{a_{n}}^{*}\right\}
$$

is an almost arithmetic sequence with slope $u v /(v-1)$ ．
Prook：By definition，the $n$th term of $\left\{a_{n}\right\} \Theta\left\{b_{n}\right\}$ is the $a_{n}$ th positive inte－ ger not one of the $b_{i}$ ，as claimed．As a composite of a complement，this is an almost arithmetic sequence with slope $u v /(v-1)$ ，much as in the proof of Theorem 8.

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## ＊＊＊＊＊

## SUMS OF THE INVERSES OF BINOMIAL COEFFICIENTS

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In this note，we discuss several sums of inverses of binomial coefficients． We evaluate these sums by application of a fundamental recurrence relation in much the same manner as sums of binomial coefficients may be treated．As an applica－ tion，certain iterated integrals of the logarithm are evaluated．

Let $n \geq k$ be positive integers. One of the basic recurrence relations of binomial coefficients
is that

$$
\binom{n}{k}=\frac{n!}{(n-k)!k!}
$$

$$
\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}
$$

For the inverse of the binomial coefficient,

$$
\binom{n}{k}^{-1}=\frac{(n-k)!k!}{n!},
$$

we observe that

$$
\begin{aligned}
\binom{n}{k}^{-1} & =\frac{(n-k)!(k-1)!}{n!} \cdot(n-(n-k)) \\
& =\frac{((n-1)-(k-1))!(k-1)!}{n!} \cdot n-\frac{((n-(k-1))(n-k)!(k-1)!}{n!} \cdot \frac{(n-k)}{(n-(k-1))}
\end{aligned}
$$

and so

$$
\begin{equation*}
\binom{n}{k}^{-1}=\binom{n-1}{k-1}^{-1}-\frac{(n-k)}{(n-k+1)}\binom{n}{k-1}^{-1} \tag{*}
\end{equation*}
$$

This relation is studied from a different viewpoint in [5, Ch. 1, Prob. 5]. For a similar sum formula not to be discussed here, see [4, n. 21].

Using mathematical induction on $n$ and the identity (*), we find

$$
\binom{n+m}{m}^{-1}=1-\frac{n}{n+1} \sum_{k=1}^{m}\binom{n+k}{k-1}^{-1}
$$

for any two positive integers $n$ and $m$ (for the corresponding relation for binomial coefficients, see [2, p. 200]).

Theorem 1: Let $I_{n}=\sum_{k=0}^{n}\binom{n}{k}^{-1}$. Then $I_{n}$ satisfies the recursion relation

$$
I_{n}=\frac{n+1}{2 n} I_{n-1}+1
$$

and

$$
I_{n}=\frac{(n+1)}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^{k}}{k}
$$

This corrects a slight error in [3].
Proof by Induction on $n$ : For $n=1$, we have $I_{1}=2$ from the definition and from the formula. We now show that the formula for $n+1$ follows from the formula for $n$ and the relation (*).

$$
I_{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}^{-1}=\binom{n+1}{0}^{-1}+\sum_{k=1}^{n+1}\binom{n+1}{k}^{-1} .
$$

Applying (*) to each term of the sum, we have

$$
\begin{aligned}
I_{n+1} & =1+\sum_{k=1}^{n+1}\left(\binom{n}{k-1}^{-1}-\frac{(n+1)-k}{(n+1-k)+1} \cdot\binom{n+1}{k-1}^{-1}\right) \\
& =1+I_{n}-\sum_{k=0}^{n} \frac{n-k}{(n+1)-k}\binom{n+1}{k}^{-1} .
\end{aligned}
$$

Since

$$
\frac{n-k}{(n+1)-k}=1-\frac{1}{(n+1)-k},
$$

we may rewrite our last expression as two sums:

$$
\begin{aligned}
I_{n+1} & =1+I_{n}-\sum_{k=0}^{n}\binom{n+1}{k}^{-1}+\sum_{k=0}^{n} \frac{1}{(n+1)-k}\binom{n+1}{k}^{-1} \\
& =2+I_{n}-I_{n+1}+\frac{1}{n+1} I_{n}
\end{aligned}
$$

so that

$$
I_{n+1}=\frac{n+2}{2(n+1)} I_{n}+1
$$

and the recursion relation is established. Applying the induction hypothesis for $I_{n}$ yields

$$
I_{n+1}=\frac{n+2}{2(n+1)}\left(\frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^{k}}{k}\right)+\frac{n+2}{2^{n+2}} \frac{2^{n+2}}{n+2}=\frac{(n+1)+1}{2^{(n+1)+1}} \sum_{k=1}^{(n+1)+1} \frac{2^{k}}{k}
$$

as required.
Theorem 2: For $n \geq 2, \sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{n-1}$.
Proof by Induction: For $n=2$, the sum is

$$
2 \sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)
$$

and the terms pairwise cance1. For $n>2$, we observe that

$$
\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\binom{n+0}{0}^{-1}+\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}=1+\sum_{k=0}^{\infty}\binom{n+(k+1)}{k+1}^{-1}
$$

Applying (*) to each term of the sum, we have

$$
\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=1+\sum_{k=0}^{\infty}\left(\binom{n+k}{k}^{-1}-\frac{n}{n+1}\binom{n+(k+1)}{k}^{-1}\right)
$$

Assuming $\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{n-1}$ and hence is finite, we obtain

$$
\frac{n}{n+1} \sum_{k=0}^{\infty}((n+1)+k)^{-1}=1
$$

completing our proof.
$\frac{\text { Theorem 3: For } n \geq 1 \text {, let } J_{n}=\sum_{k=0}^{\infty}(-1)^{k}\binom{n+k}{k}^{-1} \text {. Then } J_{n} \text { satisfies the recur- }- \text { relation }}{\text { sion }}$

$$
J_{n+1}=\frac{n+1}{n}\left(2 J_{n}-1\right)
$$

and

$$
J_{n}=\frac{n}{2}\left(2^{n} \ln (2)-\sum_{k=1}^{n-1} \frac{2^{k}}{n-k}\right)
$$

Proof by Induction: For $n=1$, we have $J_{1}=\ln (2)$. For $n>1$, we follow the method of proof of Theorem 1.

$$
\begin{aligned}
J_{n} & =1+\sum_{k=0}^{\infty}(-1)^{k+1}\binom{n+(k+1)}{k+1}^{-1} \\
& =1+\sum_{k=0}^{\infty}(-1)^{k+1}\left(\binom{n+k}{k}^{-1}-\frac{n}{n+1}\binom{n+(k+1)}{k}^{-1}\right), \quad \text { by } \quad(*), \\
& \left.\left.=1-\sum_{k=0}^{\infty}(-1)^{k}\binom{n+k}{k}^{-1}+\frac{n}{n+1} \sum_{k=0}^{\infty}(-1)^{k}\binom{(n+1}{k}+k\right)\right)^{-1} \\
& =1-J_{n}+\frac{n}{n+1} J_{n+1}
\end{aligned}
$$

and the recursion relation follows. Thus

$$
\begin{aligned}
J_{n+1} & =(n+1) \frac{2}{n}\left(\frac{n}{2}\left(2^{n} \ln (2)-\sum_{k=1}^{n-1} \frac{2^{k}}{n-k}\right)\right)-\frac{n+1}{n} \\
& =\frac{n+1}{2}\left(2^{n+1} \ln (2)-\sum_{k=1}^{n} \frac{2^{k}}{(n+1)-k}\right) .
\end{aligned}
$$

As an application of these last two results, we use them and a theorem of Abel (see [1]) to evaluate an iterated integral of the logarithmic function.

Let $f_{0}(x)=(1-x)^{-1}$ and, for $n>0$, let

$$
f_{n}(x)=\int_{0}^{x} f_{n-1}(t) d t
$$

Recall that integration by parts gives the formula

$$
\int x^{n} \ln (x) d x=\frac{x^{n+1}}{n+1} \ln (x)-\frac{x^{n+1}}{(n+1)^{2}} \text { for } n \geq 0
$$

Since $f_{1}(x)=-\ln (1-x)$, we see that

$$
f_{2}(x)=\int_{0}^{x}-\ln (1-t) d t=(1-x) \ln (1-x)-(1-x)+1
$$

and by induction on $n$ we find

$$
f_{n}(x)=\frac{(-1)^{n}}{(n-1)!}(1-x)^{n-1} \ln (1-x)+A(n) \cdot(1-x)^{n-1}+\sum_{k=0}^{n-2} B(n, k) \cdot x^{k}
$$

for $n \geq 2$ and $x$ in the open interval ( $-1,1$ ). Here $A(n)$ is given by $A(1)=0$ and for $n \geq 2$,

$$
A(n)=\frac{-1}{n-1}\left(A(n-1)+\frac{(-1)^{n}}{(n-1)!}\right)
$$

and $B(n, k)$ is given by $B(n, 0)=-A(n)$ for $n \geq 1$, while for $n \geq 2$ and $k \geq 1$,

$$
B(n, k)=\frac{1}{k} B(n-1, k-1) .
$$

Notice that repeated application of this last relation gives

$$
B(n, k)=\frac{1}{k!} B(n-k, 0) \text { for } k \leq n-2
$$

and so

Since

$$
B(n, 0)=\frac{(-1)^{n}}{(n-1)!} \sum_{k=1}^{n-1} \frac{1}{k}
$$

$$
\sum_{k=1}^{m} \frac{1}{k}=\sum_{k=1}^{m} \frac{(-1)^{k-1}}{k}\binom{m}{k}
$$

we see that each $B(n, 0)$ may be regarded as a binomial sum．
On the other hand，

$$
f_{0}(x)=(1-x)^{-1}=\sum_{k=0}^{\infty} x^{k}
$$

and term by term integration of this power series gives

$$
f_{n}(x)=x^{n} \sum_{k=0}^{\infty} \frac{x^{k}}{(k+1) \cdot \cdots \cdot(k+n)} .
$$

For $n \geq 2$ ，this series converges at $x= \pm 1$ and is uniformly convergent on the closed interval［ $-1,1]$ ．By Abel＇s theorem for power series，the values of our functions at the endpoints of the interval of convergence are given by the power series

$$
\lim _{x+1} f_{n}(x)=\sum_{k=0}^{\infty} \frac{1}{(k+1) \cdot \cdots \cdot(k+n)}=\frac{1}{n!} \sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\frac{1}{n!} \cdot \frac{n}{n-1}
$$

by out Theorem 2，while our Theorem 3 gives

$$
\lim _{x+-1} f_{n}(x)=(-1)^{n} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k+1) \cdot \cdots \cdot(k+n)}=\frac{(-1)^{n}}{n!} \sum_{k=0}^{\infty}(-1)^{k}\binom{n+k}{k}^{-1}=\frac{(-1)^{n}}{n!} J_{n} .
$$

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## TIling the plane with incongruent regular polygons

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Professor Michael Edelstein asked me how to tile the Euclidean plane with squares of integer side lengths all of which are incongruent．The question can be answered in a way that involves a perfect squared square and a geometric applica－ tion of the Fibonacci numbers．

A perfect squared square is a square of integer side length which is tiled with more than one（but finitely many）component squares of integer side lengths all of which are incongruent．For more information，see the survey articles［3］ and［5］．A perfect squared square is simple if it contains no proper subrectangle
formed from more than one component square; otherwise it is compound. It is known ([3], p. 884) that a compound perfect squared square must have at least 22 components. Duijvestijn's simple perfect squared square [2] (see Fig. 1) thus has the least possible number of components (21).


FIGURE 1
The Fibonacci numbers are defined recursively by $f_{1}=1, f_{2}=1$, and

$$
f_{n+2}=f_{n}+f_{n+1} \quad(n \geq 1) .
$$

They are used in connection with the tiling shown in Figure 2. Its nucleus is a 21 component Duijvestijn square, indicated by diagonal hatching, having side length $s=f_{1} \cdot s=112$, as in Figure 1 .


FIGURE 2
On top of this square we tile a one-component square $s$ of side length $f_{2} \cdot s=$ $s=112$, forming an overall rectangle of dimensions $2 s$ by $s$. On the left side of this rectangle (the longer edge) we tile a square $2 s$ of side length $f_{3} \cdot s=2 s=$ 224 , forming an overall rectangle of dimensions $3 s$ by $2 s$. We now proceed counterclockwise as shown, each time tiling a square $f_{n} s$ onto the required longer edge of the last overall rectangle of dimensions $f_{n} s$ by $f_{n-1} s$, forming a new overall rectangle of dimensions $f_{n+1} s$ by $f_{n} s$-this follows from (*). The tiling can continue indefinitely in this way at each stage, because $f_{n} s=f_{n-1} s+f_{n-4} s+f_{n-3} s$ [this is used for $n \geq 5$ and also follows from (*)]. A closely related Fibonacci tiling for a single quadrant of the plane (but beginning with two congruent squares) occurs in [1, p. 305, Fig. 3].

If we consider the center of the nuclear hatched square as the origin， 0 ，of the plane，it is clear that the tiling eventually covers an arbitrary disc centered at $O$ and thus covers the whole plane．Finally，note that all the component squares used in the tiling have integer side lengths and are incongruent．

The tiling described above may be called static，since the tiles remain fixed where placed，and the outward growth occurs at the periphery．It is also inter－ esting to consider a dynamic tiling．Start with a Duijvestijn square．Its small－ est component has side length 2．Enlarge it by a factor of 56．The smallest com－ ponent in the resulting square has side length 112．Replace it by a Duijvestijn square．Now enlarge the whole configuration again by a factor of 56．Repeat this process indefinitely，thus obtaining the tiling．Here no tile remains fixed，out－ ward growth occurs everywhere，and it is impossible to write down a sequence of side lengths of squares used in the tiling．

The three－dimensional version of this tiling problem（due to D．F．Daykin）is still unsolved：Can 3 －space be filled with cubes，all with integer side lengths， no two cubes being the same size？（［4］，p．11）．

The plane can also be tiled with incongruent regular triangles and a single regular hexagon，all having integer side lengths．

Begin with regular hexagon $I$（see Fig．3）and tile regular triangles with side lengths $1,2,3,4$ ，and 5 counterclockwise around it as shown．Now tile a regular triangle with side length 7 along the sixth side of the hexagon．This counter－ clockwise tiling can be continued indefinitely to cover the plane．The recursion formula for the side lengths of the triangles is

$$
s_{i}=i \text { for } 1 \leq i \leq 5, s_{6}=7, s_{i}=s_{i-1}+s_{i-5} \text { for } i \geq 7
$$



FIGURE 3
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# A NEW DEFINITION OF DIVISION IN RINGS OF QUOTIENTS OF EUCLIDEAN RINGS 

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INTRODUCTION
It is known that notions such as that of divisibility and greatest common divisor can be defined in any Euclidean ring. Such notions can be defined similarly in the corresponding ring of quotients, and there these notions, in general, become trivial. In this paper, we show that minor alterations to some of these definitions lead to many interesting results concerning divisibility and greatest common divisors as well as primes and congruences. In each case these results generalize ones that hold in the original ring.

The set of integers $Z$, the set of finite polynomials $P[x]$ over a field, and the set of complex numbers $Z[i]$, with integer real and imaginary parts, form Euclidean rings. The results we obtain on rings of quotients then apply to rational numbers, quotients of polynomials, and complex numbers with real and imaginary parts which are rationals (or square roots of rationals, depending on the definition).

## QUOTIENTS OF EUCLIDEAN RINGS

Throughout this paper, $R$ will denote a Euclidean ring with unity, as defined in [1]. The norm function associated with $R$ will be denoted by $g$, and the set of divisors of zero in $R$ by $\Theta$. If $g$, in addition to its two commonly accepted properties, also satisfies

$$
g(a b)=g(a) g(b) \text { for all } a, b, a b \varepsilon R-\{0\}
$$

then $R$ will be called a Euclidean ${ }^{+}$ring.
In $R$, we use the standard definitions, as found in [1], for divides, greatest common divisor, mutually prime, unit, prime, congruence modulo $c$, and $\leq$.

The ring of quotients of $R$, as defined in [1], will be denoted here by $R^{\prime}$ and the elements of $R^{\prime}$ by $(a, b)$ where $b \notin \Theta$. The zero of $R^{\prime}$ will be denoted by ( 0,1 ) and the unity by ( 1,1 ).

If $R$ is a Euclidean domain, so that $\theta=\{0\}$, then it is obvious that for $(c, d) \neq(0,1)$ we have

$$
(a, b)=(a d, b c) \cdot(c, d)+(0,1)
$$

so that with norm function $g^{\prime}$ given by

$$
g^{\prime}(a, b)=g^{\prime}(1,1)=g(1)
$$

$R^{\prime}$ is a Euclidean ring.
If $\Theta$ is larger than $\{0\}$ it may not be possible to define a $g^{\prime}$ on $R^{\prime}$ which extends $g$.

Since the division algorithm given above is a trivial one, we now give definitions that will lead to a nontrivial division algorithm which applies to any ring of quotients of a Euclidean+ ring.
Definition 1: (a) $\quad(a, b)<(c, d)$ if $g(a) g(d)<g(c) g(b)$,
(b) $\quad(a, b) \leq(c, d)$ if $(a, b)<(c, d)$ or $(a, b)=(c, d)$.

The symbol < can easily be shown to be irreflexive, asymmetric, and transitive, while the symbol $\leq$ is a partial ordering of $R^{\prime}$.

Definition 2: If $(a, b) \neq(0,1)$, we say that $(a, b)$ divides $(c, d)$, that is, $(a, b) \mid(c, d)$,
if there is a $q \in R$ such that $(c, d)=(q, 1)(a, b)$; in other words, if $a d \mid b c$.
Note that the $q$ in Definition 2 is unique if $a \notin \theta$ and that this definition is a generalization of division as defined in $R$. We can now prove
Theorem 1: If $a, b, c, d$ are elements of a Euclidean ${ }^{+}$ring $R$, and $(a, b) \mid(c, d)$, then $(a, b)<(c, d)$ or $g(a) g(d)=g(b) g(c)$.

Proo6: If $(a, b) \mid(c, d)$, then for some $q \in R$, $q \alpha d=b c$.
When $g(q)=1$, we have $g(a) g(d)=g(b) g(c)$; otherwise $g(b) g(c)>g(a) g(d)$, so the theorem holds. We can define units and primes in $R^{\prime}$ just as we did in $R$. Definition 3: $(a, b)$ is a unit if for some $(c, d) \varepsilon R^{\prime}$,

$$
(a, b) \cdot(c, d)=(1,1)
$$

Definition 4: $(a, b)$ is a prime if it is not a unit and if

$$
(a, b)=(c, d) \cdot(e, f)
$$

implies that $(c, d)$ or $(e, f)$ is a unit.
If $a \notin \theta$, we have $(a, b) \cdot(b, a)=(1,1)$, so $(a, b)$ is a unit and hence not a prime.

If $a \varepsilon \in$ and $(a, b) \cdot(c, d)=(1,1)$, then $b d a^{\prime}=0$, where $a \alpha^{\prime}=0$. Now, as $b \notin \theta, d a^{\prime}=0$, and so $d \varepsilon \theta$, which is impossible. Hence we have:
Theorem 2: $a \varepsilon \theta$ if and only if ( $a, b$ ) is not a unit.
Suppose $a \varepsilon \theta$ and $a=\alpha_{1} \alpha_{2} \alpha_{3}$, with $\alpha_{1}, \alpha_{2} \varepsilon \theta$, then

$$
(a, b)=\left(a_{1}, b\right) \cdot\left(a_{2} a_{3}, 1\right)
$$

where $\left(a_{1}, b\right)$ and ( $a_{2} a_{3}, 1$ ) are not units, so $(a, b)$ is not prime.
If $\alpha \varepsilon \theta$ and $\alpha=\alpha_{1} \alpha_{2}$, where $\alpha_{1} \varepsilon \theta$ is prime, $\alpha_{2} \nexists \theta$, and
$(a, b)=(c, d) \cdot(e, f)$, with $a, b, c, d, e$, and $f$ mutually prime,
then $\alpha_{1} a_{2} d f=c e b$. When $a_{1}|c, e| a_{2} d f$, so that $e \neq \theta$ and $(e, f)$ is a unit.
Similarly, if $a_{1} \mid e,(c, d)$ is a unit. Hence in this case $(a, b)$ is prime.
We have therefore proved
Theorem 3: If $a \varepsilon \theta$, then ( $a, b$ ), where $a$ and $b$ are mutually prime, is prime if and only if $a=\alpha_{1} \alpha_{2}$, where $\alpha_{1} \varepsilon \theta$ is prime in $R$ and $\alpha_{2} \nsubseteq \theta$.

In addition to the above, we can prove the following version of the fundamental theorem of arithmetic, which connects primes in $R$ with elements of $R^{\prime}$.
Theorem 4: If $a$ and $b$ are unequal elements of $R$, then ( $a, b$ ) can be expressed as
$(u, 1) \cdot\left(p_{1}, 1\right) \cdot\left(p_{2}, 1\right) \ldots\left(p_{k}, 1\right) \cdot\left(1, q_{1}\right) \cdot\left(1, q_{2}\right) \ldots\left(1, q_{m}\right)$,
where $p_{1}, p_{2}, \ldots, p_{k}, q_{1}, q_{2}, \ldots, q_{m}$ are primes of $R$ and $u$ is a unit of $R$. This representation is unique except for the order of the factors. (In the case where $a$ and $b$ are units, $k=m=0$.)

Proof: Let $(a, b)=\left(a_{1}, b_{1}\right)$ where $\alpha_{1}$ and $b_{1}$ are mutually prime.
Any non-unit can be represented uniquely as a unit times a product of primes of $R$ (see [2]). For a unit this holds as well, but the number of primes is zero. Thus

$$
\begin{aligned}
& \alpha_{1}=u_{1} p_{1} p_{2} \cdots p_{k} \\
& b_{1}=u_{2} q_{1} q_{2} \cdots q_{m}
\end{aligned}
$$

where $p_{1}, p_{2}, \ldots, p_{k}, q_{1}, q_{2}, \ldots, q_{m}$ are primes and $u_{1}$ and $u_{2}$ are units.
Then

$$
\begin{aligned}
& \left(\alpha_{1}, b_{1}\right)=\left(u_{1}, u_{2}\right) \cdot\left(p_{1}, 1\right) \cdot\left(p_{2}, 1\right) \ldots\left(p_{k}, 1\right) \cdot\left(1, q_{1}\right) \cdot\left(1, q_{2}\right) \ldots\left(1, q_{m}\right) . \\
& \quad \text { If } u_{2} v=1 \text { and } u=u_{1} v, \text { this becomes } \\
& (\alpha, b)=(u, 1) \cdot\left(p_{1}, 1\right) \cdot\left(p_{2}, 1\right) \ldots\left(p_{k}, 1\right) \cdot\left(1, q_{1}\right) \cdot\left(1, q_{2}\right) \ldots\left(1, q_{m}\right)
\end{aligned}
$$

We now state the new division algorithm.
Theorem 5: If $R$ is a Euclidean ${ }^{+}$ring and $(a, b),(c, d) \neq(0,1)$, then there is a $q \varepsilon R$ and $(r, s) \varepsilon R^{\prime}$ such that

$$
(a, b)=(q, 1) \cdot(c, d)+(r, s)
$$

where $(r, s)<(c, d)$ or $(r, s)=(0,1)$.
Proof: Since $b c \neq 0$ and $a d \neq 0$, there exist $q, r \in R$ such that

$$
a d=q c b+r
$$

with $r=0$ or $g(r)<g(c b)$.
Thus

$$
(a, b)=(q c b+r, b d)=(q, 1) \cdot(c, d)+(r, b d)
$$

If $r=0$, then $(r, b d)=(0,1)$. If $r \neq 0$, then $g(r) g(d)<g(c) g(b d)$.
Letting $s=b d$, we have

$$
(a, b)=(q, 1) \cdot(c, d)+(r, s)
$$

where $(r, s)=(0,1)$ or $(r, s)<(c, d)$.
We will show later that this algorithm allows us to find a greatest common divisor of $(a, b)$ and $(c, d)$ as defined below.
Definition 5: ( $e, f$ ) is a g.c.d. of $(a, b)$ and $(c, d)$ if $(e, f)|(a, b),(e, f)|(c, d)$, and $(i, j) \mid(a, b)$
and

$$
(i, j) \mid(c, d) \text { imp1ies }(i, j) \mid(e, f)
$$

In $R$, if $d_{1}$ and $d_{2}$ are both g.c.d.s of $a$ and $b$, then $g\left(d_{1}\right)=g\left(d_{2}\right)$. Similarly here, if $\left(e_{1}, f_{1}\right)$ and $\left(e_{2}, f_{2}\right)$ are g.c.d.s of $(a, b)$ and $(c, d)$, we have

$$
g\left(e_{1} f_{2}\right)=g\left(e_{2} f_{1}\right)
$$

The following theorem relates g.c.d.s in $R$ with g.c.d.s in $R^{\prime}$.
Theorem 6: If $i$ is a g.c.d. of $a$ and $c$ and $j$ is a g.c.d. of $b$ and $d$, then ( $i j, b d$ ) is a g.c.d. of $(a, b)$ and ( $c, d)$.

Proob: We can assume without loss of generality that $a$ and $b$ and $c$ and $d$ are mutually prime.

Let $i$ be a g.c.d. of $a$ and $c$ and $j$ be a g.c.d. of $b$ and $d$ and $a=i a_{1}$ and $d=$ $j d_{1}$, then

$$
(i j, b d) \cdot\left(a_{1} d_{1}, 1\right)=(a d, b d)=(a, b),
$$

so
Similarly,

$$
\begin{aligned}
& (i j, b d) \mid(a, b) \\
& (i j, b d) \mid(c, d)
\end{aligned}
$$

If $(r, s) \mid(a, b)$ and $(r, s) \mid(c, d)$, where again we assume that $r$ and $s$ are mutually prime, we have, for some $t, u \varepsilon R$,

$$
t r b=s a \quad \text { and } \quad u r d=s c .
$$

Thus $r \mid a$ and $r \mid c$ so $r \mid i$, and $b \mid s$ and $a \mid s$ so $b a \mid j s$. Therefore, $r b a \mid i j s$ and $(r, s) \mid(i j, b d)$.
Thus ( $i j, b d$ ) is a g.c.d. of $(a, b)$ and ( $c, d$ ).
Corollary: If the only units of $R$ are 1 and -1 , then any two g.c.d.s of two elements of $R^{\prime}$ are equal or are additive inverses of each other.

Several other standard theorems on g.c.d.s and divisibility hold in $R^{\prime}$ :
Theorem 7: If ( $e, f$ ) and ( $e^{\prime}, f^{\prime}$ ) are g.c.d.s of $(a, b)$ and $(c, d)$, where $e \notin \theta$, then there is a unit $u$ of $R$ such that $(e, f)=(u, 1) \cdot\left(e^{\prime}, f^{\prime}\right)$.

Proof: Assuming that $e$ and $f$ and $e^{\prime}$ and $f^{\prime}$ are mutually prime, we have

$$
(e, f) \mid\left(e^{\prime}, f^{\prime}\right) \quad \text { and } \quad\left(e^{\prime}, f^{\prime}\right) \mid(e, f)
$$

Thus, for some $m, n \varepsilon R$, $e f^{\prime}=m e^{\prime} f$ and $e^{\prime} f=n e f^{\prime}$. Therefore,

$$
e e^{\prime} f f^{\prime}(1-m n)=0
$$

Since $e \nsubseteq \Theta, e^{\prime} \nsubseteq \Theta$, and $m$ and $n$ are units of $R$, the theorem holds.
Theorem 8: If ( $e, f$ ) is a g.c.d. of ( $\alpha, b$ ) and ( $c, d$ ), there exist $m, n \varepsilon R$ such that

$$
(e, f)=(m, 1)(a, b)+(n, 1)(c, d)
$$

Proof: In the notation of the proof of Theorem 6, (ij, bd) is a g.c.d. of $(a, \bar{b})$ and $(c, d)$, where $i$ is a g.c.d. of $a$ and $c$ and $j$ is a g.c.d. of $b$ and $d$.

Then $i j$ will be a g.c.d. of $\alpha d$ and $b c$. Hence, by a property for $R$, there are elements $k$ and $h$ of $R$ such that

$$
i j=k a d+h b c .
$$

Thus

$$
(i j, b d)=(k a d+h b c, b d)=(k, 1)(a, b)+(h, 1)(c, d)
$$

If ( $e, f$ ) is any g.c.d. of $(a, b)$ and $(c, d)$, then

$$
\begin{aligned}
(e, f) & =(t, 1)(i j, b d), \text { for some } t \varepsilon R, \\
& =(m, 1)(a, b)+(n, 1)(c, d),
\end{aligned}
$$

where $m=t k$ and $n=t h$.
Theorem 9: Any g.c.d. of $(a, b) \cdot(c, d)$ and $(a, b) \cdot(e, f)$ can be written as ( $a, b$ ) times a g.c.d. of $(c, d)$ and ( $e, f$ ).

Proof: Any g.c.d. of $(a, b) \cdot(c, d)$ and $(a, b) \cdot(e, f)$ will, by Theorems 6 and 7 , take the form ( $u k h$, $b d b f$ ), where $k$ is a g.c.d. of $a c$ and $a e, h$ is a g.c.d. of $b d$ and $b f$, and $u$ is a unit.

Then $k=a i$ and $h=b j$, where $i$ is a g.c.d. of $c$ and $e$, and $j$ is a g.c.d. of $f$ and $d$. Thus

$$
(u k h, b d b f)=(u a b i j, b d b f)=(a, b) \cdot(u i j, d f),
$$

which is the form required by the theorem.
Theorem 10: If $(a, b)=(q, 1)(c, d)+(r, s)$, then any g.c.d. of $(a, b)$ and $(c, d)$ is a g.c.d. of $(c, d)$ and $(r, s)$.

Proof: Similar to that for $R$.

Theorem 10 and part of the proof of Theorem 5 give us a technique for finding the g.c.d.s of two elements of $R^{\prime}$ where $R$ is Euclidean ${ }^{+}$.

Given ( $a, b$ ) and ( $c, d$ ) in $R^{\prime}$, we have, by the proof of Theorem 5, $q, p \varepsilon R$ such that

$$
(a, b)=(q, 1) \cdot(c, d)+(r, b d),
$$

where $g(r)<g(b c)$ or $r=0$.
Now if $r \neq 0$, as $c b \neq 0$, there are $q_{1}$ and $r_{1}$ in $R$ such that

$$
c b=q_{1} r+r_{1}
$$

where $g\left(r_{1}\right)<g(r)$ or $r_{1}=0$.
Therefore,

$$
c b d=q_{1} r d+r_{1} d
$$

and so

$$
(c, d)=\left(q_{1}, 1\right) \cdot(r, b d)+\left(r_{1}, b d\right),
$$

where $g\left(r_{1}\right)<g(r)$ or $r_{1}=0$.
Again, if $r_{1} \neq 0$, we can obtain $q_{2}, r_{2} \varepsilon R$ such that

$$
(r, b d)=\left(q_{2}, 1\right)\left(r_{1}, b d\right)+\left(r_{2}, b d\right),
$$

where $g\left(r_{2}\right)<g\left(r_{1}\right)$ or $r_{2}=0$, etc.
As each $g\left(r_{i}\right)$ is a positive integer, this process terminates, and for some $r_{k}$ we have
and

$$
\left(r_{k-2}, b d\right)=\left(q_{k}, 1\right)\left(r_{k-1}, b d\right)+\left(r_{k}, b d\right)
$$

$$
\left(r_{k-1}, b d\right)=\left(q_{k+1}, 1\right)\left(r_{k}, b d\right)
$$

Then $\left(r_{k}, b d\right)$ and ( $r_{k-1}, b d$ ) have $\left(r_{k}, b d\right)$ as a g.c.d. and this, by repeated use of Theorem 9, can be seen to be a g.c.d. of ( $a, b$ ) and ( $c, d$ ).

If $a, b \notin \theta$, the g.c.d. is, by Theorem 7, unique except for a factor ( $u, 1$ ), where $u$ is a unit of $R$.

Using our unique representation of elements of $R^{\prime}$ given by Theorem 4 and writing all factors of the form ( $p, 1$ ) and ( $1, q$ ) for both ( $a, b$ ) and ( $c, d$ ), using zero exponents where necessary, it is clear that any g.c.d. of

$$
\left.\begin{array}{l}
\quad(u, 1)\left(p_{1}, 1\right)^{i_{1}}\left(p_{2}, 1\right)^{i_{2}} \ldots\left(p_{e}, 1\right)^{i_{e}}\left(p_{e+1}, 1\right)^{i_{e+1}} \ldots\left(p_{f}, 1\right)^{i_{f}}\left(1, q_{1}\right)^{j_{1}}\left(1, q_{2}\right)^{j_{2}} \\
\\
\text { and } \\
\\
\quad \ldots\left(1, q_{m}\right)^{j_{m}}\left(1, q_{m+1}\right)^{j_{m+1}} \ldots\left(1, q_{g}\right)^{j_{g}} \\
(v, 1)\left(p_{1}, 1\right)^{r_{1}}\left(p_{2}, 1\right)^{r_{2}}
\end{array} \quad \ldots\left(p_{e}, 1\right)^{r_{e}}\left(p_{e+1}, 1\right)^{r_{e+1}} \ldots\left(p_{f}, 1\right)^{r_{f}}\left(1, q_{1}\right)^{s_{1}}\left(1, q_{2}\right)^{s_{2}}\right)
$$

where all powers are integers $\geq 0$, is

$$
(w, 1)\left(p_{1}, 1\right)^{t_{1}}\left(p_{2}, 1\right)^{t_{2}} \ldots\left(p_{f}, 1\right)^{t_{f}}\left(1, q_{1}\right)^{n_{1}}\left(1, q_{2}\right)^{n_{2}} \ldots\left(1, q_{g}\right)^{n_{g}}
$$

where $t_{k}=\min \left(i_{k}, r_{k}\right)$ and $n_{k}=\max \left(j_{k}, s_{k}\right)$ and $\omega$ is an arbitrary unit of $R$.
If a g.c.d. of $(a, b)$ and $(c, d)$ is $(1,1)$, it follows that $(a, b)=(e, 1)$ and $(c, d)=(f, 1)$ for some $e, f \in R$ which are mutually prime.

The following definition extends the notion of mutually prime elements of $R$ to $R^{\prime}$.
Definition 6: If $a$ and $b$ as well as $c$ and $d$ are mutually prime and $a$ and $b$ are not both zero, then ( $a, b$ ) and ( $c, d$ ) are mutually prime if ( $1, b d$ ) is a g.c.d. of $(a, b)$ and ( $c, d$ ).
The special case where $b=d=1$ conforms to the definition for $R$
The property:
If $x \mid y z$ and $x$ and $y$ are mutually prime, then $x \mid z$, which holds in $R$ for $y, z \notin \Theta$, fails in $R^{\prime}$.

For example, if $R=2,3 \left\lvert\, 4 \cdot \frac{3}{4}\right.$ in $R^{\prime}(=Q)$ and 3 and 4 are mutually prime, but $3 \times \frac{3}{4}$.

The following seems to be the most general replacement for the above that we can prove.
Theorem 11: If $(a, b) \mid(c, d) \cdot(e, f)$, where $(a, b)$ and $(c, d)$ as we11 as $f$ and $c$ are mutually prime, then $(a, b) \mid(e, f)$.

Proo 6: Assume that $a$ and $b, c$ and $d, f$ and $c$ and $e$ and $f$ are mutually prime and that $(a, b) \mid(c, d) \cdot(e, f)$. Then $a d f \mid b c e$.

Now, if $(a, b)$ and $(c, d)$ are mutually prime, so are $a$ and $c$. Therefore, $a \mid e$ and $f \mid b$, and hence $a f \mid b e$.

We define congruence in $R^{\prime}$ as follows.
Definition 7: $(a, b) \equiv(c, d) \bmod (e, f)$, if $(e, f) \mid\{(a, b)-(c, d)\}$.
Alternatively, $(a, b) \equiv(c, d) \bmod (e, f)$, if $b d e \mid(a d f-b c f)$. Congruence mod ( $e, f$ ) is clearly an equivalence relation over $R^{\prime}$.

The equivalence class of $(c, d), \bmod (e, f)$, will consist of all elements of the form ( $c f+d k e, d f$ ), it will include elements of the form ( $h, 1$ ) only if $d \mid f$.

From our division algorithm,

$$
(a, b)=(q, 1)(e, f)+(r, s),
$$

it follows that ( $a, b$ ) and the remainder ( $r, s$ ) upon division by ( $e, f$ ) are in the same equivalence class, mod ( $e, f$ ). Also, all the elements in the equivalence class of ( $a, b$ ) mod ( $e, f$ ), will have common g.c.d.s with ( $a, b$ ) and ( $e, f$ ).

Each equivalence class, mod ( $e, f$ ), can therefore be uniquely determined by a particular divisor ( $\omega, t$ ) of ( $e, f$ ); the elements of the class will all be of the form ( $k w, t$ ).

If all remainders ( $r, s$ ) obtained upon division by ( $e, f$ ) in a particular $R^{\prime}$ are unique, the set of all such remainders can be said to form a set of least residues mod ( $e, f$ ). If when such remainders are not unique they always form a "positive" and "negative" pair, the positive remainders can be said to be least positive residues mod ( $e, f$ ).

The usual elementary theorems about residues can be summed up as follows.
Theorem 12: If $(a, b) \equiv(c, d) \bmod (e, f),\left(a^{\prime}, b^{\prime}\right) \equiv\left(c^{\prime}, d^{\prime}\right) \bmod (e, f), \ldots$ and $\phi$ is any polynomial in several variables with integer coefficients, then

$$
\phi\left((a, b),\left(a^{\prime}, b^{\prime}\right), \ldots\right) \equiv \phi\left((c, d),\left(c^{\prime}, d^{\prime}\right), \ldots\right) \bmod (e, f) .
$$

The following cancellation theorem:
If $d$ is a g.c.d. of $e$ and $c, e \notin \theta$, and $a e \equiv b e \bmod c$, then $a \equiv b \bmod \frac{c}{d}$, which holds in $R$, fails in $R^{\prime}$. For example, in $Z^{\prime}$, the set of rationals
but

$$
2 \frac{1}{3} \cdot 4 \equiv 2 \frac{1}{3} \cdot \frac{6}{7} \bmod 3 \frac{2}{3}
$$

$4 \not \equiv \frac{6}{7} \bmod 11$.
We can prove the following more restricted generalization of the above theorem for $R$.
Theorem 13: If $a$ and $b, c$ and $d, e$ and $f$ and $k$ and $h$ are mutually prime pairs of elements of $R, k \notin \Theta, m$ is a g.c.d of $k$ and $e, n$ a g.c.d. of $f$ and $h, e=e_{1} m$, $k=k_{1} m$, and $f=f_{1} n$, where $k_{1}$ is mutually prime to $b$ and $d$, and if

$$
(k, h) \cdot(a, b) \equiv(k, h) \cdot(c, d) \bmod (e, f),
$$

then

$$
(a, b) \equiv(c, d) \bmod \left(e_{1}, f_{1}\right)
$$

Proof: If the conditions of the theorem hold, then

$$
b \hbar d e \mid(a d-b c) k f
$$

Letting $h=h_{1} n$, we have $m, n \notin \theta$ and $b h_{1} d e_{1} \mid(a d-b c) k_{1} f_{1}$. Then, as $e_{1} b d$ and $k_{1}$ are mutually prime and $k_{1} \nsubseteq \theta$,

$$
b d e_{1} \mid(a d-b c) f_{1}
$$

and so

$$
(a, b) \equiv(c, d) \bmod \left(e_{1}, f_{1}\right)
$$

Under the conditions of the theorem, we can also obtain, from the proof:
and

$$
(a, b) \equiv(c, d) \bmod (e h, k f)
$$

$$
(a, b) \equiv(c, d) \quad \bmod \left(e_{1} h_{1}, k_{1} f_{1}\right) .
$$

We now consider the solution of the linear congruence

$$
(a, b) \cdot(x, y) \equiv(c, d) \bmod (e, f)
$$

Clearly if $a \notin \theta,(x, y)=(b c, a d)+(t e b, f a)$ is a solution for every $t \in R$. It is therefore of more interest to find solutions with $y=1$.

Conditions for the existence of such solutions are given in the next theorem. Theorem 14: (i) If $i$ is a g.c.d. of $a$ and $e$ and $j$ is a g.c.d. of $b$ and $f$ and

$$
\begin{equation*}
(a, b) \cdot(x, 1) \equiv(c, d) \bmod (e, f) \tag{1}
\end{equation*}
$$

has a solution, then $(i j, b j) \mid(c, d)$.
(ii) If $b=b_{1} j$ and $e=e_{1} i$, the solution is unique $\bmod b_{1} e_{1}$.

Proof: (i) If (1) has a solution, ( $a, b$ ), ( $c, d$ ) and ( $e, f$ ), by our earlier work on the division algorithm, clearly have a common g.c.d. Thus, if $i$ and $j$ are defined as in the theorem, $(i j, b f) \mid(c, d)$.
(ii) If we have a solution to (1), we also have a solution to

$$
\begin{equation*}
d f a x \equiv b c f \bmod \text { bed. } \tag{2}
\end{equation*}
$$

Let $a=a_{1} i, e=e_{1} i, b=b_{1} j$, and $f=f_{1} j$. Assume that $a$ and $b, e$ and $f$ and $c$ and $d$ are mutually prime. Since (2) has a solution, $d i \mid b_{1} c f$ so that $i \mid c$ and $a \mid b_{1} f$.

Let $c=c_{1} i$ and $k d=b_{1} f$, then (2) becomes

$$
f_{1} a_{1} x \equiv k c_{1} \quad \bmod b_{1} e_{1}
$$

If also $f_{1} a_{1} x^{\prime} \equiv k c_{1} \bmod b_{1} e_{1}$, we have

$$
f_{1} a_{1}\left(x-x^{\prime}\right) \equiv 0 \bmod b_{1} e_{1}
$$

Since $f_{1} a_{1}$ and $b_{1} e_{1}$ are mutually prime,

$$
x \equiv x^{\prime} \bmod b_{1} e_{1}
$$

Thus the solution $x$ is unique $\bmod b_{1} e_{1}$.
Corollary: If $(k, h)$ is a g.c.d. of $(a, b)$ and $(e, f)$, then
$(a, b) \cdot(x, 1) \equiv(c, d) \bmod (e, f)$,
if and only if $(k, h) \mid(c, d)$.
Proob: By the fact that $(k, h) \mid(i j, b f)$ and ( $i j, b f) \mid(k, h)$ in the notation of the above proof.

In the case where the ring $R$ is $Z$ ，the set of integers，we can determine the total number of different solutions $\bmod (e, f)$ ，or $\frac{e}{f}$ ．

This number of solutions will be the smallest positive integer $n$ such that $\left(n b_{1} e_{1}, 1\right) \equiv 0 \bmod (e, f)$ ，
i．e．，such that $e \mid n b_{1} e_{1} f$ ．
Now，as we can assume that $e$ and $f$ and $a$ and $b$ are mutually prime，this reduces to $i \mid n$ ，so the smallest $n$ is $i$ ．

Thus in the ring of integers，the number of noncongruent solutions mod（ $e, f$ ） of（1）is $i$ ．

Take，as an example，

$$
15 \frac{5}{39} x \equiv \frac{5}{6} \bmod 20 \frac{5}{52} .
$$

Clearly，g．c．d． $\left.\left(15 \frac{5}{39}, 20 \frac{5}{52}\right)=\frac{5}{156} \right\rvert\, \frac{5}{6}$ ，and we can obtain $x=-89$ as a solution to

$$
4(15.39+5) x \equiv 26.5 \bmod (60.52+15) .
$$

Now $b_{1}$ comes to 3 and $e_{1}$ to 209 ，so the simplest noncongruent positive integer solutions， $\bmod 20 \frac{5}{52}$ ，are $194,821,1448,2075$ ，and 2702.

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## A RECURSION－TYPE FORMULA FOR SOME PARTITIONS <br> AMIN A．MUWAFI <br> The American University of Beirut，Beirut，Lebanon

If $p(n)$ denotes the number of unrestricted partitions of $n$ ，the following re－ currence formula，known as Euler＇s identity，permits the computation of $p(n)$ if $p(k)$ is already known for $k<n$ ．
$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-+++\cdots$

$$
=\sum_{j \neq 0}(-1)^{j+1} p\left(n-\frac{1}{2}\left(3 j^{2}+j\right)\right),
$$

where the sum extends over all integers $j$ ，except $j=0$ ，for which the arguments of the partition function are nonnegative．

Hickerson［1］gave a recursion－type formula for $q(n)$ ，the number of partitions of $n$ into distinct parts，in terms of $p(k)$ for $k \leq n$ ，as follows，

$$
\begin{equation*}
q(n)=\sum_{j=-\infty}^{\infty}(-1)^{j} p\left(n-\left(3 j^{2}+j\right)\right) \tag{2}
\end{equation*}
$$

where the sum extends over all integers $j$ for which the arguments of the partition function are nonnegative．

Alder and Muwafi［2］gave a recursion－type formula for $p^{\prime}(0, k-r, 2 k+a ; n)$ ， the number of partitions of $n$ into parts $\not \equiv 0, \pm(k-r) \bmod 2 k+a$ ，where $0 \leq r \leq$ $k-1$ ．

$$
\begin{equation*}
p^{\prime}(0, k-r, 2 k+a ; n)=\sum_{j=-\infty}^{\infty}(-1)^{j} p\left(n-\frac{(2 k+\alpha) j^{2}+(2 r+\alpha) j}{2}\right) \tag{3}
\end{equation*}
$$

where the sum extends over all integers $j$ for which the arguments of the partition function are nonnegative. Letting $k=a=1$ and $r=0$, formula (3) reduces to Euler's identity; and letting $k=a=2$ and $r=0$, formula (3) reduces to Hickerson's formula (2).

Ewell [3] gave two recurrence formulas for $q(2 \ell)$ and $q(2 \ell+1)$ for nonnegative integers $\ell$ in a slightly different, but equivalent, form to that in formula (2).

This paper presents a recursion-type formula for $p_{k}^{*}(n)$, the number of partitions of $n$ into parts not divisible by $k$, where $k$ is some given integer $\geq 1$. It is shown that formulas (1) and (2) are special cases of formula (4) below.
Theorem: If $n \geq 0, k \geq 1$, and $p_{k}^{*}(n)$ is the number of partitions of $n$ into parts not divisible by $k$, where $p_{k}^{*}(0)=1$, then

$$
\begin{equation*}
p_{k}^{*}(n)=\sum_{j=-\infty}^{\infty}(-1)^{j} p\left(n-\frac{k\left(3 j^{2}+j\right)}{2}\right) \tag{4}
\end{equation*}
$$

where the sum extends over all integers $j$ for which the arguments of the partition function are nonnegative.

Proof: The generating function for $p_{k}^{*}(n)$ is given by

$$
\sum_{n=0}^{\infty} p_{k}^{*}(n) x^{n}=\frac{\prod_{j=1}^{\infty}\left(1-x^{k j}\right)}{\prod_{j=1}^{\infty}\left(1-x^{j}\right)}=\sum_{r=0}^{\infty} p(r) x^{r} \prod_{j=1}^{\infty}\left(1-x^{k j}\right)
$$

By Euler's product formula, we have

Hence

$$
\begin{aligned}
& \prod_{j=1}^{\infty}\left(1-x^{k j}\right)=\sum_{j=-\infty}^{\infty}(-1)^{j} x^{\frac{k j(3 j+1)}{2}} \\
& \sum_{n=0}^{\infty} p_{k}^{*}(n) x^{n}=\sum_{n=0}^{\infty} p(r) x^{r} \sum_{j=-\infty}^{\infty}(-1)^{j} x^{\frac{k j(3 j+1)}{2}} \\
&=\sum_{n=0}^{\infty}\left\{\sum_{j=-\infty}^{\infty}(-1)^{j} p\left(n-\frac{k j(3 j+1)}{2}\right)\right\} x^{n} .
\end{aligned}
$$

Equating coefficients on both sides of this equation, and noticing that $j=0$ when $n=0$, we get the required result in (4).
Corollary 1: If in Eq. (4) we let $k=1$, then $p_{1}^{*}(n)=0$, so that Eq. (4) becomes

$$
0=\sum_{j=-\infty}^{\infty}(-1)^{j} p\left(n-\frac{3 j^{2}+j}{2}\right)
$$

from which Eq. (1) follows by moving the term corresponding to $j=0$ to the lefthand side. Thus Eq. (1) becomes a special case of the theorem.
Corollary 2: If in Eq. (4) we let $k=2$, then $p_{2}^{*}(n)$ denotes the number of partitions of $n$ into parts not divisible by 2 , and hence it is equal to the number of partitions of $n$ into odd or distinct parts. Thus $p_{2}^{*}(k)=q(n)$, and Eq. (4) reduces to (2). Hence Eq. (2) is a special case of the theorem.

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## PRIMITIVE PYTHAGOREAN TRIPLES AND THE INFINITUDE OF PRIMES <br> DELANO P. WEGENER <br> Central Michigan University, Mt. Pleasant, MI 48859

A primitive Pythagorean triple is a triple of natural numbers ( $x, y, z$ ) such that $x^{2}+y^{2}=z^{2}$ and $(x, y)=1$. It is well known $[1, \mathrm{pp} .4-6]$ that all primitive Pythagorean triples are given, without duplication, by

$$
x=2 m n, y=m^{2}-n^{2}, z=m^{2}+n^{2}
$$

where $m$ and $n$ are relatively prime natural numbers which are of opposite parity and satisfy $m>n$. Conversely, if $m$ and $n$ are relatively prime natural numbers which are of opposite parity and $m>n$, then the above formulas yield a primitive Pythagorean triple. In this note I will refer to $m$ and $n$ as the generators of the triple ( $x, y, z$ ) and I will refer to $x$ and $y$ as the legs of the triple.

A study of the sums of the legs of primitive Pythagorean triples leads to the following interesting variation of Euclid's famous proof that there areminfinitely many primes.

Suppose there is a largest prime, say $p_{k}$. Let $m$ be the product of this finite list of primes and let $n=1$. Then $(m, n)=1, m>n$, and they are of opposite parity. Thus $m$ and $n$ generate a primitive Pythagorean triple according to the above formulas. If $x+y$ is prime, it follows from

$$
x+y=2 m n+m^{2}-n^{2}=2\left(2 \cdot 3 \cdot \cdots \cdot p_{k}\right)+\left(2 \cdot 3 \cdot \cdots \cdot p_{k}\right)^{2}-1>p_{k}^{2}
$$

that $x+y$ is a prime greater than $p_{k}$. If $x+y$ is composite, it must have a prime divisor greater than $p_{k}$. This last statement follows from the fact that every prime $q \leq p_{k}$ divides $m$ and hence divides $x$. If $q$ divides $x+y$, then it divides $y$, which contradicts the fact that ( $x, y, z$ ) is a primitive Pythagorean triple. Thus the assumption that $p_{k}$ is the largest prime is false.

By noting that

$$
\begin{aligned}
y-x & =\left(2 \cdot 3 \cdot \cdots \cdot p_{k}\right)^{2}-1-2\left(2 \cdot 3 \cdot \cdots \cdot p_{k}\right) \\
& =2\left(2 \cdot 3 \cdot \cdots \cdot p_{k}\right)\left(3 \cdot \cdots \cdot p_{k}-1\right)-1>p_{k}
\end{aligned}
$$

a similar proof can be constructed by using the difference of the legs of the primitive Pythagorean triple ( $x, y, z$ ).

The following lemma will be useful in proving that there are infinitely many primes of the form $8 t \pm 1$.
Lemma: If $(x, y, z)$ is a primitive Pythagorean triple and $p$ is a prime divisor of $\overline{x+y}$ or $|x-y|$, then $p$ is of the form $8 t \pm 1$.

$$
\begin{gathered}
\text { Proof: Suppose } p \text { divides } x+y \text { or }|x-y| \cdot \text { Note that this implies } \\
\qquad(x, p)=(y, p)=1, \quad \text { and } x \equiv \pm y(\bmod p)
\end{gathered}
$$

so that

$$
2 x^{2} \equiv x^{2}+y^{2} \equiv z^{2}(\bmod p)
$$

By definition，$x^{2}$ is a quadratic residue of $p$ ．The above congruence implies $2 x^{2}$ is also a quadratic residue of $p$ ．If $p$ were of the form $8 t \pm 3$ ，then 2 would be a quadratic nonresidue of $p$ and since $x^{2}$ is a quadratic residue of $p, 2 x^{2}$ would be a quadratic nonresidue of $p$ ，a contradiction．Thus $p$ must be of the form $8 t \pm 1$ ．

Now，if we assume that there is a finite number of primes of the form $8 t \pm 1$ ， and if we let $m$ be the product of these primes，then we obtain a contradiction by imitating the above proof that there are infinitely many primes．

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\＃\＃れみ茖

## AN APPLICATION OF PELL＇S EQUATION

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The following problem solution is a good classroom presentation or exercise following a discussion of Pell＇s equation．

Statement of the Problem
Find all natural numbers $a$ and $b$ such that

$$
\frac{a(a+1)}{2}=b^{2}
$$

An alternate statement of the problem is to ask for all triangular numbers which are squares．

> Solution of the Problem

$$
\begin{aligned}
\frac{a(a+1)}{2}=b^{2} \Longleftrightarrow & a^{2}+a=2 b^{2} \Longleftrightarrow a^{2}+a-2 b^{2}=0 \Longleftrightarrow a=\frac{-1 \pm \sqrt{1+8 b^{2}}}{2} \Longleftrightarrow \exists \\
& \text { an odd integer } t \text { such that } t^{2}-2(2 b)^{2}=1 .
\end{aligned}
$$

This is Pell＇s equation with fundamental solution［1，p．197］$t=3$ and $2 b=2$ or，equivalently，$t=3$ and $b=1$ ．Note that $t=3$ implies

$$
a=\frac{-1 \pm 3}{2},
$$

but，according to the following theorem，we may discard $a=-2$ ．Also note that $t$ is odd．
Theorem 1：If $D$ is a natural number that is not a perfect square，the Diophantine equation $x^{2}-D y^{2}=1$ has infinitely many solutions $x, y$ ．

All solutions with positive $x$ and $y$ are obtained by the formula

$$
x_{n}+y_{n} \sqrt{D}=\left(x_{1}+y_{1} \sqrt{D}\right)^{n}
$$

where $x_{1}, y_{1}$ is the fundamental solution of $x^{2}-D y^{2}=1$ and where $n$ runs through all natural numbers．

A comparison of $\left(x_{n}+y_{n} \sqrt{2}\right)(3+2 \sqrt{2})$ and $\left(\begin{array}{ll}3 & 4 \\ 2 & 3\end{array}\right)\binom{x_{n}}{y_{n}}$ shows that all solutions
of $t^{2}-2(2 b)^{2}=1$ are obtained by

$$
\left(\begin{array}{ll}
3 & 4 \\
2 & 3
\end{array}\right)\binom{t_{n}}{2 b_{n}}=\binom{t_{n+1}}{2 b_{n+1}}
$$

and hence all solutions of $\frac{\alpha(\alpha+1)}{2}=b^{2}$ are obtained from $a_{n}=\frac{t_{n}-1}{2}, b_{n}=\frac{2 b_{n}}{2}$. Note that $t_{n}$ is odd for all $n$ so $a_{n}$ is an integer.


CENTRAL FACTORIAL NUMBERS AND RELATED EXPANSIONS
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1. INTRODUCTION

The central factorials have been introduced and studied by Stephensen; properties and applications of these factorials have been discussed among others and by Jordan [3], Riordan [5], and recently by Roman and Rota [4].

For positive integer $m$,

$$
x^{[m, b]}=x\left(x+\frac{1}{2} m b-b\right)\left(x+\frac{1}{2} m b-2 b\right) \cdots\left(x-\frac{1}{2} m b+b\right)
$$

defines the generalized central factorial of degree $m$ and increment $b$. This definition can be extended to any integer $m$ as follows:

$$
\begin{aligned}
x^{[0, b]} & =1 \\
x^{[-m, b]} & =x^{2} / x^{[m+2, b]}, m \text { a positive integer } .
\end{aligned}
$$

The usual central factorial $(b=1)$ will be denoted by $x^{[m]}$. Note that these factorials are called "Stephensen polynomials" by some authors.

Carlitz and Riordan [1] and Riordan [5, p. 213] studied the connection constants of the sequences $x^{[m]}$ and $x^{n}$, that is, the central factorial numbers $t(m, n)$ and $T(m, n)$ :

$$
x^{[m]}=\sum_{n=0}^{m} t(m, n) x^{n}, x^{m}=\sum_{n=0}^{m} T(m, n) x^{[n]} ;
$$

these numbers also appeared in the paper of Comtet [2]. In this paper we discuss some properties of the connection constants of the sequences $x^{[m, g]}$ and $x^{[n, h], ~} h \neq$ $g$, of generalized central factorials, that is, the numbers $K(m, n, s)$ :

$$
x^{[m, g]}=\sum_{n=0}^{m} g^{m} h^{-n} K(m, n, s) x^{[n, h]}, s=h / g .
$$

## 2. EXPANSIONS OF CENTRAL FACTORIALS

The central difference operator with increment $\alpha$, denoted by $\delta_{a}$, is defined by

$$
\delta_{a} f(x)=f(x+a / 2)-f(x-a / 2)
$$

Note that

$$
\begin{equation*}
\delta_{a}=E_{a}^{\frac{1}{2}}-E_{a}^{-\frac{1}{2}}=E_{a}^{-\frac{1}{2}} \Delta_{a} \tag{2.1}
\end{equation*}
$$

where $E_{a}$ and $\Delta_{a}$ denote the displacement and difference operators with increment $a$, respectively. Therefore,

$$
\begin{equation*}
\delta_{a}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} E^{n / 2-k} \tag{2.2}
\end{equation*}
$$

When the increment $a=1$, we write $\delta_{1} \equiv \delta, E_{1} \equiv E$, and $\Delta_{1} \equiv \Delta$.
The central factorial of degree $m$ and increment $b$, denoted by $x^{[m, b]}$, is defined by

Note that

$$
x^{[m, b]}=x\left(x+\frac{1}{2} m b-b\right)\left(x+\frac{1}{2} m b-2 b\right) \cdots\left(x-\frac{1}{2} m b+b\right) .
$$

where

$$
\begin{equation*}
x^{[m, b]}=x\left(x+\frac{1}{2} m b-b\right)_{m-1, b}, \tag{2.3}
\end{equation*}
$$

is the falling factorial of degree $m$ and increment $b$.
It is not difficult to verify that

Using the relation

$$
\begin{equation*}
x^{[m, b]}=\left[x^{2}-\left(\frac{1}{2} m-1\right)^{2} b^{2}\right] x^{[m-2, b]} \tag{2.4}
\end{equation*}
$$

$$
\begin{equation*}
(y)_{-m, b}=\frac{1}{(y+m b)_{m, b}}, \tag{2.5}
\end{equation*}
$$

and, by (2.3), we get

$$
\begin{equation*}
x^{[-m, b]}=\frac{x^{2}}{x^{[m+2, b]}} \tag{2.6}
\end{equation*}
$$

When the increment $b=1$, we write

Note also that

$$
\begin{align*}
& x^{[m, 1]} \equiv x^{[m]},(y)_{m, 1} \equiv(y)_{m} \\
& (b x)^{[m]}=b^{m} x^{[m, h]}, \quad h=1 / b \tag{2.7}
\end{align*}
$$

From formula (2.8) (see Riordan [5, p. 147]),

$$
\begin{equation*}
u^{\alpha}=1+\sum_{n=1}^{\infty} \frac{\alpha}{n}\binom{\alpha+\beta n-1}{n-1} v^{n}=\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n}\binom{\alpha+\beta n}{n} v^{n}, v=(1-u) u^{-\beta}, \tag{2.8}
\end{equation*}
$$

with $\alpha=b x, \beta=1 / 2, u=E, v=(E-1) E^{-\frac{1}{2}}=\delta$, we get the symbolic formula

$$
E^{b x}=\sum_{n=0}^{\infty}(b x)^{[n]} \frac{1}{n!} \delta^{n}
$$

Since $\left[E^{b x}(s z)^{[m]}\right]_{z=0}=(a x)^{[m]}, s=a / b$, we obtain

$$
(a x)^{[m]}=\sum_{n=0}^{m}\left[\frac{1}{n!} \delta^{n}(s x)^{[m]}\right]_{x=0} \cdot(b x)^{[n]}
$$

Denoting the number in brackets by

$$
\begin{equation*}
K(m, n, s)=\left[\frac{1}{n!} \delta^{n}(s x)^{[m]}\right]_{x=0} \tag{2.9}
\end{equation*}
$$

we have

$$
\begin{equation*}
(a x)^{[m]}=\sum_{n=0}^{m} K(m, n, s)(b x)^{[n]}, s=a / b \tag{2.10}
\end{equation*}
$$

Using (2.7), (2.10) may be rewritten in the form

$$
\begin{equation*}
x^{[m, g]}=\sum_{n=0}^{m} g^{m} h^{-n} K(m, n, s) x^{[n, h]}, s=h / g . \tag{2.11}
\end{equation*}
$$

Note also that

$$
\begin{equation*}
K(m, n, s)=\left[\frac{1}{n!b^{m}} \delta_{a}^{n} x^{[m, b]}\right]_{x=0}, s=a / b . \tag{2.12}
\end{equation*}
$$

From the definition (2.9), we may deduce an explicit expression for the numbers $K(m, n, s)$. Indeed, from the symbolic formula (2.2) with $a=1$, and since

$$
\left[E^{n / 2-k}(s x)^{[m]}\right]_{x=0}=\left(s\left[\frac{1}{2} n-k\right]\right)^{[m]}
$$

we get

$$
\begin{align*}
K(m, n, s) & =\frac{1}{n!} \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(s\left[\frac{1}{2} n-k\right]\right)^{[m]} \\
& =\frac{1}{n!} \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(\frac{1}{2} s n-s k\right)\left(\frac{1}{2} s n+\frac{1}{2} m-s k-1\right)_{m-1} \tag{2.13}
\end{align*}
$$

A recurrence relation for the numbers $K(m, n, s)$, useful for tabulation purposes, may be obtained from (2.10) and (2.4) as follows:

Hence

$$
\begin{aligned}
(s k)^{[m+2]} & =\sum_{n=0}^{m+2} K(m+2, n, s) x^{[n]}=\left(s^{2} x^{2}-\frac{1}{4} m^{2}\right) \sum_{n=0}^{m} K(m, n, s) x^{[n]} \\
& =\sum_{n=0}^{m} K(m, n, s)\left[s^{2} x^{[n+2]}+\frac{1}{4}\left(s^{2} n^{2}-m^{2}\right) x^{[n]}\right] .
\end{aligned}
$$

$$
\begin{equation*}
K(m+2, n, s)=\frac{1}{4}\left(s^{2} n^{2}-m^{2}\right) K(m, n, s)+s^{2} K(m, n-2, s) . \tag{2.14}
\end{equation*}
$$

The initial conditions are

$$
K(0,0, s)=1, K(0, n, s)=0, n>0, K(m, 0, s)=0, m>0
$$

Moreover,

$$
K(2 m, 2 n+1, s)=0, K(2 m+1,2 n, s)=0 .
$$

From the recurrence relation and the initial conditions, it follows that:

$$
\begin{aligned}
& \text { If } s \text { is an integer, the numbers } \\
& s^{-2 n} K(2 m, 2 n, s) \text { and } 4^{m-n} s^{-2 n-1} K(2 m+1,2 n+1, s)
\end{aligned}
$$

are positive integers and, moreover,
If $s$ is a negative integer, the numbers

$$
K(2 m, 2 n, s)=0, m<n, m>n|s|
$$

$$
K(2 m+1,2 n+1, s)=0, m<n, 2 m+1>(2 n+1)|s|
$$

Other properties of these numbers will be discussed in the next section.
We now proceed to determine the coefficients $A(n, m, s)$ in the expansion

$$
x^{[-m]}=\sum_{n=m}^{\infty} A(n, m, s)(s x)^{[-n]}
$$

Since $x^{[-m+2]}=\left(x^{2}-\frac{1}{4} m^{2}\right) x^{[-m]}$, we get

$$
\begin{aligned}
\sum_{n=n-2}^{\infty} A(n, m-2, s)(s x)^{[-n]} & =\left(x^{2}-\frac{1}{4} m^{2}\right) \sum_{n=m}^{\infty} A(n, m, s)(s x)^{[-n]} \\
& =\sum_{n=m}^{\infty} A(n, m, s)\left[s^{-2}(s x)^{[-n+2]}+\frac{1}{4}\left(s^{-2} n^{2}-m^{2}\right)(s x)^{[-n]}\right]
\end{aligned}
$$

Hence

$$
A(n+2, m, s)=\frac{1}{4}\left(s^{2} m^{2}-n^{2}\right) A(n, m, s)+s^{2} A(n, m-2, s)
$$

$$
A(0,0, s)=1, A(0, m, s)=0, \quad>0
$$

Comparing this recurrence with (2.14), we conclude that

$$
\begin{equation*}
x^{[-m]}=\sum_{n=m}^{\infty} K(n, m, s)(s x)^{[-n]}, \tag{2.15}
\end{equation*}
$$

which may be written in the form
or

$$
\begin{equation*}
(b x)^{[-m]}=\sum_{n=m}^{\infty} K(n, m, s)(\alpha x)^{[-n]} \tag{2.16}
\end{equation*}
$$

$$
\begin{equation*}
x^{[-m, g]}=\sum_{n=m}^{\infty} g^{n} h^{-m} K(n, m, s) x^{[-n, h]}, s=h / g \tag{2.17}
\end{equation*}
$$

## 3. SOME PROPERTIES OF THE CENTRAL FACTORIAL NUMBERS

Some other properties of the numbers $K(m, n, s)$, defined by (2.9) or, equivalently, by (2.12), will be discussed in this section.

From (2.10) we may easily get the relation

$$
\begin{equation*}
\sum_{k=n}^{m} K(m, k, \alpha / b) K(k, n, b / a)=\delta_{m n} \tag{3.1}
\end{equation*}
$$

where $\delta_{m n}$ denotes the Kronecker delta. This relation implies the pairs of inverse relation

$$
\begin{array}{ll}
a_{m}=\sum_{n=0}^{m} K(m, n, a / b) b_{n}, & b_{m}=\sum_{n=0}^{m} K(m, n, b / a) a_{n} \\
c_{n}=\sum_{m=n}^{\infty} K(m, n, \alpha / b) d_{m}, & d_{n}=\sum_{m=n}^{\infty} K(m, n, b / a) c_{m}
\end{array}
$$

For the central factorial numbers

$$
t(m, n)=\left[\frac{1}{n!} D^{n} x^{m}\right]_{x=0} \quad \text { and } \quad T(m, n)=\left[\frac{1}{n!} \delta^{n} x^{m}\right]_{x=0}
$$

we have (see Riordan [5, p. 213])

$$
\begin{align*}
x^{[m]} & =\sum_{n=0}^{m} t(m, n) x^{n}  \tag{3.2}\\
x^{m} & =\sum_{n=0}^{m} T(m, n) x^{[n]} \tag{3.3}
\end{align*}
$$

Expanding $(s x)^{[m]}$ into powers of $x$ by means of (3.2) and then the powers into central factorials by means of (3.3), we obtain
or

$$
(s x)^{[m]}=\sum_{k=0}^{m} s^{k} t(m, k) x^{k}=\sum_{k=0}^{m} \sum_{n=0}^{k} s^{k} t(m, k) T(k, n) x^{[n]}
$$

$$
(s x)^{[m]}=\sum_{n=0}^{m} \sum_{k=n}^{m} s^{k} t(m, k) T(k, n) x^{[n]}
$$

which, in virtue of (2.10) with $b=1, a=s$, gives
similarly, it can be shown that

$$
\begin{align*}
& K(m, n, s)=\sum_{k=n}^{m} s^{k} t(m, k) T(k, n) ;  \tag{3.4}\\
& \text { own that }
\end{align*}
$$

and

$$
\begin{equation*}
t(m, n)=s^{-n} \sum_{k=n}^{m} K(m, k, s) t(k, n) \tag{3.5}
\end{equation*}
$$

$$
\begin{equation*}
T(m, n)=s^{-m} \sum_{k=n}^{m} T(m, k) K(k, n, s) . \tag{3.6}
\end{equation*}
$$

Since $\lim _{s \rightarrow \pm \infty} s^{-m}(s x)^{[m]}=x^{m}$, we get, from (2.9),

$$
\begin{equation*}
\lim _{s \rightarrow \pm \infty} s^{-m} K(m, n, s)=\left[\frac{1}{n!} \delta^{n} x^{m}\right]_{x=0}=T(m, n) . \tag{3.7}
\end{equation*}
$$

From (2.12) with $b=1, a=s$, and noting that $\lim _{s \rightarrow 0} s^{-1} \delta_{s}=D$, we deduce

$$
\begin{equation*}
\lim _{s \rightarrow 0} s^{-n} K(m, n, s)=\left[\frac{1}{n!} D^{n} x^{m}\right]_{x=0}^{s \rightarrow 0}=t(m, n) . \tag{3.8}
\end{equation*}
$$

Turning to the generating function, we find, on using (2.13) and (2.8), with

$$
\alpha=\frac{1}{2} s n-s k, \beta=\frac{1}{2}, v=y,(u-1) u^{-\frac{1}{2}}=y,
$$

that

$$
\begin{aligned}
g_{n}(y ; s) & =\sum_{m=0}^{\infty} K(m, n, s) \frac{y^{m}}{m!} \\
& =\frac{1}{n!} \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left[\begin{array}{c}
\left.1+\sum_{m=1}^{\infty} \frac{\frac{1}{2} s n-s k}{m}\binom{\frac{1}{2} s n-s k+\frac{1}{2} m-1}{m-1} y^{m}\right] \\
\\
\end{array}=\frac{1}{n!}\left(u^{s / 2}-u^{-s / 2}\right),(u-1) u^{-1 / 2}=y .\right.
\end{aligned}
$$

Putting $u=e^{w}$ and $s=r$ to avoid mistakes in the hyperbolic formulas, we get
and
Therefore,

$$
\begin{aligned}
g_{n}(y ; r) & =\frac{1}{n!}\left[2 \sinh \left(\frac{1}{2} r w\right)\right]^{n} \\
y & =2 \sinh \left(\frac{1}{2} w\right) .
\end{aligned}
$$

$$
\begin{align*}
g_{n}(y ; r) & =\frac{1}{n!}\left[2 \sinh \left\{r \sinh ^{-1}\left(\frac{1}{2} y\right)\right\}\right]^{n} \\
& =\frac{1}{n!}\left[2 \sinh \left\{r \log \left(\frac{1}{2} y+\frac{1}{2} \sqrt{y^{2}+4}\right)\right\}\right]^{n} . \tag{3.9}
\end{align*}
$$

The corresponding generating functions for the Carlitz-Riordan central factorial numbers may be obtained as

$$
\begin{align*}
& \sum_{m=0}^{\infty} t(m, n) \frac{y^{m}}{m!}=\frac{1}{n!}\left[2 \sinh ^{-1}\left(\frac{1}{2} y\right)\right]^{n}  \tag{3.10}\\
& \sum_{m=0}^{\infty} T(m, n) \frac{y^{m}}{m!}=\frac{1}{n!}\left[2 \sinh \left(\frac{1}{2} y\right)\right]^{n} \tag{3.11}
\end{align*}
$$

Using formulas (3.10), (3.11), and (3.9), and since
$\delta_{a}^{n}=\left[2 \sinh \left(\frac{1}{2} a D\right)\right]^{n}, \quad a^{n} D^{n}=\left[2 \sinh ^{-1}\left(\frac{1}{2} \delta_{a}\right)\right]^{n}, \quad \delta_{a}^{n}=\left[2 \sinh \left\{r \sinh ^{-1}\left(\frac{1}{2} \delta_{b}\right)\right\}\right]^{n}$,
we get

$$
\begin{gathered}
\delta_{a}^{n}=\sum_{m=0}^{\infty} \frac{n!}{m!} T(m, n) a^{m} D^{m}, \quad a^{n} D^{n}=\sum_{m=0}^{\infty} \frac{n!}{m!} t(m, n) \delta_{a}^{m}, \\
\delta_{a}^{n}=\sum_{m=0}^{\infty} \frac{n!}{m!} K(m, n, r) \delta_{b}^{m}, r=a / b .
\end{gathered}
$$

Finally, let
and put

$$
Q_{m}(z ; s)=\sum_{x=0}^{z}(s x)^{[m]}
$$

$$
Q_{2 m}(z ; s)=\frac{2 z+1}{2} \sum_{n=0}^{m} \frac{Q_{m, n, s}}{2 n+1} \frac{(z+n)!}{(z-n)!}
$$

Then
and by (2.10),

$$
(s x)^{[2 m]}=\sum_{n=0}^{m} Q_{m, n, s} \frac{\dot{x}(x+n-1)!}{(x-n)!}=\sum_{n=0}^{m} Q_{m, n, s} x^{[2 m]},
$$

$$
Q_{m, n, s}=K(2 m, 2 n, s)
$$

A similar expression may be obtained for $Q_{2 m+1}(z ; s)$.

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## ON THE FIBONACCI NUMBERS MINUS ONE

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Let $A$ be the $n \times n$ matrix with elements defined by

$$
\alpha_{i j}=-1 \text { if } i=j-1 ; 1+\mu \text { if } i=j ;-\mu \text { if } i=j+2 \text {; }
$$

and 0 otherwise. If $n \geq 3$ and $\mu$ is a positive number, then $A$ is a special case of a matrix that was shown in [1] to be useful in the design of two-up, one-down ideal cascades for uranium enrichment. The purpose of this paper is to derive certain properties of the determinant $D_{n}$ of $A$ and to point out its relation to the Fibonacci numbers.

Expansion of the determinant of $A$ according to its first column leads to the recurrence relation
(1) $\quad D_{1}=1+\mu, D_{2}=(1-\mu)^{2}$, and $D_{n}=(1+\mu) D_{n-1}-\mu D_{n-3}$ for $n \geq 3$.

For convenience, set $D_{0}=1$.
By using standard techniques for generating functions, it can be shown that the generating function $D(x)$ for $\left\{D_{n}\right\}$ (with positive radius of convergence) is

$$
\begin{equation*}
D(x)=\left[1-(1+\mu) x+\mu x^{3}\right]^{-1}=\sum_{i=0}^{\infty} \sum_{j=0}^{i}(-1)^{j}\binom{i}{j} \mu^{j}(1+\mu)^{i-j} x^{i+2 j} \tag{2}
\end{equation*}
$$

Therefore, an explicit expression for $D_{n}$ is

$$
\begin{equation*}
D_{n}=\sum_{k=0}^{[n / 3]}(-1)^{k}\binom{n-2 k}{k} \mu^{k}(1+\mu)^{n-3 k} \tag{3}
\end{equation*}
$$

where $[n / 3]$ denotes the integral part of $n / 3$.
Adding the recurrence relations (1) for $n=3,4,5, \ldots, m$ leads, on simplification, to the alternative recurrence relation

$$
\begin{equation*}
D_{m}-D_{m-1}-D_{m-2}=1 \text { for } m \geq 3 \tag{4}
\end{equation*}
$$

The homogeneous equation corresponding to (4) has the linearly independent solutions

$$
g_{m}(\mu)=\left(\frac{\mu+\sqrt{\mu^{2}+4 \mu}}{2}\right)^{m}, h_{m}(\mu)=\left(\frac{\mu-\sqrt{\mu^{2}+4 \mu}}{2}\right)^{m}, \text { for all } m \geq 3
$$

and a particular solution of (4) is

$$
p_{m}(\mu)= \begin{cases}1 /(1-2 \mu) & \text { if } \mu \neq 1 / 2 \\ 2 m / 3 & \text { if } \mu=1 / 2\end{cases}
$$

Hence, the general solution of (4) is of the form

$$
\begin{equation*}
D_{m}=c_{1} g_{m}+c_{2} h_{m}+p_{m} \text { for } m \geq 3 \tag{5}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are constants that can be determined from (1).
In the special case when $\mu=1$, let $\Delta_{m}$ denote the determinant of the matrix $A$. Then (3), (4), and (5), respectively, become

$$
\begin{aligned}
\Delta_{n}= & \sum_{k=0}^{[n / 3]}(-1)^{k}\binom{n-2 k}{k} 2^{n-3 k}, n \geq 0 \\
& \Delta_{m}-\Delta_{m-1}-\Delta_{m-2}=1, m \geq 3
\end{aligned}
$$

and

$$
\Delta_{m}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+3}-\left(\frac{1-\sqrt{5}}{2}\right)^{m+3}\right]-1, m \geq 3
$$

It is clear that the members of the sequence $\left\{\Delta_{m}\right\}$ are the Fibonacci numbers minus one [2] and that the results for $\mu \neq 1$ generalize those for $\mu=1$.

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EDITOR'S NOTE:
Selecting the names of those individuals who were asked to submit manuscripts for the Memorial Issue was not an easy task on the part of the Board of Directors and Herta Hoggatt. Vern knew and worked with so many of you that it would have been impossible to ask all of you. As the editor, I apologize for any oversights. Furthermore, Mrs. Herta Hoggatt and family wish to express their sincere appreciation to all of those authors who contributed to the Memorial Issue.
-Gerald E. Bergum

## PASCAL's TRIANGLE MODULO

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## 1. INTRODUCTION

In "Mathematical Games" in the December 1966 issue of Scientific American, Martin Gardner made the following statement regarding Pascal's triangle: "Almost anyone can study the triangle and discover more properties, but it is unlikely that they will be new, for what is said here only scratches the surface of a vast literature." But, of course, many new results have been discovered since 1966 and we present some here that were even suggested by Gardner's article, although the more immediate stimulation was the recent brief article by S. H. L. Kung [3] concerning the parity of entries in Pascal's triangle.

## 2. THE ITERATED TRIANGLE

Consider Pascal's triangle with its entries reduced to their least nonnegative residues modulo $p$, where $p$ denotes a prime. Let $k$, $n$, and $m$ be integers with $0 \leq$ $k \leq n$ and $1 \leq m$, and let $\Delta_{n, k}$ denote the triangle of entries

$$
\begin{gathered}
\bullet\binom{n p^{m}}{k p^{m}} \cdot \\
\binom{n p^{m}+p^{m}-1}{k p^{m}} \cdot \cdot \cdot\binom{n p^{m}+p^{m}-1}{k p^{m}+p^{m}-1}
\end{gathered}
$$

For fixed $m$, we claim that all those elements not contained in one of these triangles are zeros, that there are precisely $p$ distinct triangles $\Delta_{n, k}$, and that these triangles are in one-to-one correspondence with the residues $0,1,2, \ldots$, $p-1$ in such a way that the triangle of triangles

$$
\begin{gathered}
\Delta_{0,0} \\
\Delta_{1,0} \Delta_{1,1} \\
\Delta_{2,0} \quad \Delta_{2,1} \quad \Delta_{2,2}
\end{gathered}
$$

is "isomorphic" to the original Pascal triangle. In particular, we claim that there is an element-wise addition of the triangles $\Delta_{n, k}$ which satisfies the equation

$$
\Delta_{n, k}+\Delta_{n, k+1}=\Delta_{n+1, k+1}
$$

where the addition is modulo $p$.
If we repeatedly iterate this process by mapping the triangles $\Delta_{n, k}$ onto the residues it follows that, modulo $p$, Pascal's triangle is a triangle that contains a Pascal triangle of triangles, that in turn contains a Pascal triangle of triangles, ..., ad infinitum. For example, let $m=1$ and consider Pascal's triangle, modulo 2.

1
11
$1 \quad 0 \quad 1$
$1_{0}^{1} 0_{0}^{1}{ }^{1}$
$\begin{array}{llllllll}1 & 1 & 0 & 0 & 1 & 1\end{array}$
$\begin{array}{llllllllllllll} & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & 1 & \\ 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\ 1\end{array}$

If we actually draw triangles around the $\Delta_{n, k}$ defined above, we obtain the following array:


And is we suppress the triangles with a single zero (with the points pointed downward) and make the substitution indicated by the one-to-one correspondence

we obtain

\[

\]

$$
\begin{array}{llll}
1 & 1 & 1 \\
& \cdot & . & \cdot
\end{array}
$$

which is simply the original Pascal triangle modulo 2. Also, using element-wise addition modulo 2, we note that

and similarly for the other "digit" sums.
Iterating a second time (or, equivalently, taking $m=2$ ) amounts to partitioning the original triangle as follows:


This time, suppressing the inverted triangles of zeros and making the replacement indicated by the correspondence

we obtain

which is again the original Pascal triangle modulo 2. Also, again adding elementwise modulo 2, we have

as required by the Pascal recurrence.
These results are summarized for any prime $p$ in the following theorem.
Theorem 1: Let $p$ be a prime and let $\Delta_{n, k}$ be defined as above for $0 \leq k \leq n$ and $1 \leq m$. Then $\Delta_{n, k}$ is the triangle

$$
\left.\begin{array}{c}
\binom{n}{k}\binom{0}{0} \\
\binom{n}{k}\binom{1}{0}\binom{n}{k}\binom{1}{1} \\
\cdot \\
\binom{n}{k}\left(p^{m}-1\right. \\
0
\end{array}\right) \cdot\binom{n}{k}\binom{p^{m}-1}{p^{m}-1} .
$$

with all the products reduced modulo $p$ and

$$
\Delta_{n, k}+\Delta_{n, k+1}=\Delta_{n+1, k+1}
$$

where the addition is element-wise addition modulo $p$. Finally, every element in Pascal's triangle and not in one of the $\Delta_{n, k}$ is congruent to zero modulo $p$.

Proof: The elements of $\Delta_{n, k}$ are the binomial coefficients

$$
\binom{n p^{m}+r}{k p^{m}+s}, 0 \leq s \leq r<p^{m}
$$

and, by Lucas' theorem for binomial coefficients [1], [5, p. 230],

$$
\binom{n p^{m}+r}{k p^{m}+s} \equiv\binom{n}{k}\binom{r}{s}(\bmod p) .
$$

This gives the first assertion of the theorem and also implies the second, since

$$
\begin{aligned}
\binom{n p^{m}+r}{k p^{m}+s}+\binom{n p^{m}+r}{(k+1) p^{m}+s} & \equiv\binom{n}{k}\binom{r}{s}+\binom{n}{k+1}\binom{r}{s} \\
& =\binom{n+1}{k+1}\binom{r}{s} \\
& \equiv\binom{(n+1) p^{m}+r}{(k+1) p^{m}+s}(\bmod p)
\end{aligned}
$$

Finally, the entries of Pascal's triangle not included in any of the $\Delta_{n, k}$ form triangles $\nabla_{n, k}$ of the form shown below.

$$
\begin{gathered}
\binom{n p^{m}}{k p^{m}+1} \cdot \cdot \cdot\binom{n p^{m}}{k p^{m}+p^{m}-1} \\
\bullet \cdot \\
\bullet \\
\binom{n p^{m}+p^{m}-2}{k p^{m}+p^{m}-1}
\end{gathered}
$$

with the elements reduced modulo $p$. Thus, every element in $\nabla_{n, k}$ is of the form

$$
\binom{n p^{m}+r}{k p^{m}+s}, 0 \leq r<s \leq p^{m}-1
$$

and, again from Lucas' theorem,

$$
\binom{n p^{m}+r}{k p^{m}+s} \equiv\binom{n}{k}\binom{r}{s} \equiv 0(\bmod p) .
$$

since $r<s$. This completes the proof.

## 3. A GREATEST COMMON DIVISION PROPERTY

In this section, we need the following remarkable lemma [4, p. 57, Prob. 16] which is readily derived from Lucas' theorem. Note that by $p^{f} \| n$ we mean that $p^{f} \mid n$ and $p^{f+1} \|_{n}$.
Lemma: Let $p$ be a prime and let $n$ and $k$ be integers with $0 \leq k \leq n$. If $p f \|\binom{ n}{k}$, then $f$ is the number of carries one makes when adding $k$ to $n-k$ in base $p$.

We now prove an interesting greatest common divisor property for the binomial coefficients in the triangular array

$$
\begin{gathered}
\binom{m}{1} \quad \cdot \cdot \cdot \cdot\binom{m}{m-1} \\
\binom{m+1}{2} \cdot\binom{m+1}{m-1} \\
\bullet \cdot \\
\binom{2 m-2}{m-1}
\end{gathered}
$$

which we denote by $\nabla_{m}$.
Theorem 2: Let $p$ be a prime, let $d$ be the greatest common divisor of all elements in $\nabla_{m}$, and let $D$ denote the greatest common divisor of the three corner elements

$$
\binom{m}{1},\binom{m}{m-1}, \text { and }\binom{2 m-2}{m-1}
$$

Then, (i) $d=D=p$ if $m=p$,
(ii) $d=p$ and $D=p$ if $m=p^{\alpha}$, where $\alpha>1$ is an integer, and
(iii) $d=1$ and $D=m$ for all other integers $m \geq 2$.

Proo6: (i) Since $\binom{m}{1}=\binom{p}{1}=p$ and $d|D|\binom{m}{1}$, it suffices to show that $p \mid d$. Consider an arbitrary element

$$
\binom{p+k}{h}, 0 \leq k \leq p-2, k+1 \leq h \leq p-1
$$

of $\nabla_{p}$. By Lucas' formula

$$
\binom{p+k}{h} \equiv\binom{k}{h} \equiv 0(\bmod p)
$$

since $k<h$. Thus, $p$ divides every element of $\nabla_{p}$ and so $p \mid d$ as required.
(ii) Here the elements of $\nabla_{p^{\alpha}}$ are the form

$$
\binom{p^{\alpha}+k}{h}, 0 \leq k \leq p-2, k+1 \leq h \leq p-1
$$

and, again by Lucas' theorem,

$$
\binom{p^{\alpha}+k}{h} \equiv\binom{k}{h} \equiv 0(\bmod p)
$$

since $k<h$. Thus, $p|d| D$. On the other hand, $p \|\binom{ p^{\alpha}}{p^{\alpha-1}}$, since the only carry you make in adding $p^{\alpha-1}$ to $p^{\alpha}-p^{\alpha-1}=(p-1) p^{\alpha-1}$ is just 1 . This implies that $d \mid p$, and hence that $d=p$. Furthermore,

$$
\binom{p^{\alpha}}{1}=\binom{p^{\alpha}}{p^{\alpha}-1}=p^{\alpha} \quad \text { and } \quad p^{\alpha} \|\binom{ 2 p^{\alpha}-2}{p^{\alpha}-1}
$$

since

$$
p^{\alpha}-1=\sum_{i=0}^{\alpha-1}(p-1) p^{i}
$$

so that you carry precisely $\alpha$ times when adding $p^{\alpha}-1$ to $p^{\alpha}-1$ in base $p$. Therefore, $D=p^{\alpha}$ as claimed.
(iii) In this case, $m$ is not a prime power. Since

$$
\binom{m}{1}=\binom{m}{m-1}=m,
$$

we have that $D \mid m$. Thus, to show that $D=m$, it suffices to show that $m \mid D$. This will clearly be the case if we show that $m \left\lvert\,\binom{ 2 m-2}{m-1}\right.$ and for this it suffices to show that

$$
p_{i}^{\alpha_{i}} \left\lvert\,\binom{ 2 m-2}{m-1}\right., 1 \leq i \leq r,
$$

where

$$
m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}
$$

is the canonical representation of $m$. Let $m=k p$, where $k$ is an integer and $p \nmid k$. Since

$$
k p^{\alpha}-1=(k-1) p^{\alpha}+p^{\alpha}-1=(k-1) p^{\alpha}+\sum_{i=0}^{\alpha-1}(p-1) p
$$

it is clear that the number of carries made in adding $k p^{\alpha}-1$ to $k p^{\alpha}-1$ in base $p$ is at least $\alpha$. Therefore,

$$
p^{\alpha} \left\lvert\,\binom{ 2 k p^{\alpha}-2}{k p^{\alpha}-1}\right.
$$

and the result follows.
We now show that $d=1$. Since
it suffices to show that

$$
\binom{m}{1}=m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}, r>1,
$$

$$
p_{i} \nmid\binom{m}{p_{i}^{\alpha_{i}}}, 1 \leq i \leq r .
$$

If we fix $i$, we may write $m=h p_{i}^{\alpha_{i}}$ with $h>1$ and $\left(h, p_{i}\right)=1$. The question will then be settled if we show that there are no carries when adding $p_{i}^{\alpha_{i}}$ to $m-p_{i}^{\alpha_{i}}=$ ( $h-1) p_{i}^{\alpha_{i}}$ in base $p$. Since the only nonzero digit in the representation of $p_{i}^{\alpha_{i}}$ to base $p_{i}{ }_{i}$ is the 1 that multiplies $p_{i}^{\alpha}$, we need consider only the digit that multiplies $p_{i}^{\alpha_{i}}$ in the base $p_{i}$ representation of (h-1) $p_{i}^{\alpha_{i}}$. Indeed, it is clear that we have a carry if and only if $h-1=q p_{i}+\left(p_{i}-1\right)$ for some integer $q$. But this is so if and only if $h=(q+1) p_{i}$, and this contradicts the fact that $\left(h, p_{i}\right)=1$. Thus,

$$
p_{i} \not \backslash\binom{m}{p_{i}^{\alpha_{i}}}
$$

for $1 \leq i \leq r$, and the proof is complete.

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## ON THE NUMBER OF FIBONACCI PARTITIONS OF A SET

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## 1. PARTITIONS OF $\bar{n}$ IN FIBONACCI SETS

Let $\bar{n}:=\{1,2, \ldots, n\}$. It is well known [1] that the number of sets $A \subseteq \bar{n}$, with

$$
\begin{equation*}
i, j \in A, i \neq j \text { imp1ies }|i-j| \geq 2, \tag{1}
\end{equation*}
$$

is the Fibonacci number $F_{n+1} . \quad\left(F_{0}=F_{1}=1, F_{n+2}=F_{n+1}+F_{n}\right)$
A set $A \subseteq \bar{n}$ with the property (1) will be called a Fibonacci set.
A partition of $\bar{n}$ is a family of disjoint (nonempty) subsets of $\bar{n}$ whose union is $\bar{n}$. The number of partitions of $\bar{n}$ is $B_{n}$, the $n$th Bell number [2].

In this section the number $C_{n}$ of partitions of $\bar{n}$ in Fibonacci subsets will be considered. There exists an interesting connection with $B_{n}$.
Theorem 1: $\quad C_{n}=B_{n-1}$.
Proo6: This will be proved by arguments analogous to Rota's in [2]. First, the number of functions $f: \bar{n} \rightarrow U$ ( $U$ has $u$ elements) with $f(i) \neq f(i+1)$ for all $i$ is determined: for $f(1)$ there are $u$ possibilities; for $f(2)$ there are $u-1$ possibilities; for $f(3)$ there are $u-1$ possibilities, and so on. The desired number of functions is $u(u-1)^{n-1}$.

These functions are partitioned with respect to their kernels. (Note that exactly those kernels appear which are Fibonacci sets!)

$$
\begin{equation*}
\sum(u)_{N(\pi)}=u(n-1)^{n-1}, \tag{2}
\end{equation*}
$$

the sum is extended over all kernels $\pi$, and $N(\pi)$ denotes the number of distinct subsets of $\pi$.

Now let $L$ be the functional defined by $(u)_{n} \rightarrow 1$ for all $n$. Then, from (2),

$$
\begin{equation*}
L\left(\sum(u)_{N(\pi)}\right)=C_{n}=L\left(u(u-1)^{n-1}\right) \tag{3}
\end{equation*}
$$

In [2] it is proved that $L(u \cdot p(u-1))=L(p(u))$ holds for all polynomials $p$. With $p(u)=u^{n-1}$,

$$
C_{n}=L\left(u(u-1)^{n-1}\right)=L\left(u^{n-1}\right)=B_{n-1}
$$

(The last equality is the essential result of [2].)
At this time it is legitimate to ask of a natural bijection $\varphi$ from the partitions of $\bar{n}$ to the Fibonacci partitions of $\overline{n+1}, \varphi$ and $\varphi^{-1}$ are given by the following algorithms (due to F. J. Urbanek).
Algorithm for $\varphi$ :
A1. $n+1$ is adjoined to the given partition in a new class.
A2. Do Step A3 for all classes except the one of $n+1$.
A3. Run through the class in decreasing order. If with the considered number $i$, $i+1$ is also in the same class, give $i$ in the class of $n+1$.
 $\rightarrow 135|47| 89|2610 \rightarrow 135| 47|9| 26810$.
Algorithm for $\varphi^{-1}$ : The number $n+1$ is erased; the other numbers in this class are to be distributed: If $i+1$ has its place and $i$ is to be distributed, give $i$ in the class of $i+1$.

Example: $138|24| 6|579 \rightarrow 1378| 24 \mid 56$.
It is not difficult to see that $\varphi$ and $\varphi^{-1}$ are inverse and that only $\varphi^{-1}$ preserves the partial order of partitions (with respect to refinement).
2. A GENERALIZATION: d-FIBONACCI SETS

Ad-Fibonacci set $A \subseteq \bar{n}$ has the property

$$
\begin{equation*}
i, j \in A, i \neq j \text { imp1ies }|i-j| \geq d \tag{4}
\end{equation*}
$$

Let $C_{n}^{(d)}$ be the number of $d$-Fibonacci partitions. $\left(C_{n}^{(2)}=C_{n}, C_{n}^{(1)}=B_{n}.\right)$
Theorem 2: $C_{n}^{(d)}=B_{n+1-d}$.
Proof: First the number of functions $f: \bar{n} \rightarrow U$ with
$|\{f(i), f(i+1), \ldots, f(i+d-1)\}|=d$ for all $i$
is considered. By the same argument as in Section 1 , this number is

$$
(u)_{d-1}(u-d+1)^{n+1-d} .
$$

Again

$$
\begin{equation*}
\sum(u)_{N(\pi)}=(u)_{d-1}(u-d+1)^{n+1-d} \tag{5}
\end{equation*}
$$

where the summation ranges over all $d$-Fibonacci partitions of $\bar{n}$. Applying the functional $L$ on (5) yields

$$
\begin{equation*}
C_{n}^{(d}=L\left((u)_{d-1}(u-k+1)^{n+1-d}\right) . \tag{6}
\end{equation*}
$$

As in [2],

$$
\begin{equation*}
L\left((u)_{d-1} p(u-d+1)\right)=L(p(u)) \tag{7}
\end{equation*}
$$

holds for all polynomials $p$. With $p(u)=u^{n+1-d}$ it follows from (6) and (7) that

$$
C_{n}^{(d)}=L\left((u)_{d-1}(u-d+1)^{n+1-d}\right)=L\left(u^{n+1-d}\right)=B_{n+1-d}
$$

It is possible to construct a bijection $\varphi$ from the partitions of $\bar{n}$ to the $d$ Fibonacci partitions of $\overline{n+d-1}$ in a way similar to that given in the previous section; however, this is more complicated to describe and therefore is omitted.

## 3. A GENERALIZATION OF THE FIBONACCI NUMBERS

The fact that $F_{n+1}$ is the number of Fibonacci subsets of $\bar{n}$ can be seen as the starting point to define the numbers $F_{n}^{(s)}(s \in N)$ :
$F_{n+1}^{(8)}$ is defined to be the number of ( $A_{1}, \ldots, A_{s}$ ) with $A_{i} \subseteq \bar{n}$ and $A_{i} \cap A_{j} \neq \emptyset$ for $i^{n+1} \neq j$. The recurrence

$$
F_{n+1}^{(s)}=s F_{n}^{(s)}+F_{n-1}^{(s)}, F_{1}^{(s)}=1, F_{2}^{(s)}=1+s
$$

can be established as follows:
First, $F_{n+1}^{(s)}$ can be expressed as the number of functions

$$
f: \bar{n} \rightarrow\left\{\varepsilon, a_{1}, \ldots, a_{s}\right\}
$$

with $f(i)=f(i+1)=a_{j}$ is impossible. If $f(n)=\varepsilon$, the contribution to $F_{n+1}^{(8)}$ is $F_{n}^{(s)}$. If $f(n)=a_{i}$, the contribution is $F_{n}^{(s)}$ minus the number of functions

$$
f: \overline{n-1} \rightarrow\left\{\varepsilon, a, \ldots, a_{s}\right\}
$$

with $f(n-1)=\alpha_{i}$. Taken all together,

$$
\begin{equation*}
F_{n+1}^{(s)}=F_{n}^{(s)}+s\left[F_{n}^{(s)}-F_{n-1}^{(s)}+F_{n-2}^{(s)}-+\cdots\right] . \tag{8}
\end{equation*}
$$

Also

$$
\begin{equation*}
F_{n+2}^{(s)}=F_{n+1}^{(s)}+s\left[F_{n+1}^{(s)}-F_{n}^{(s)}+F_{n-1}^{(s)}-+\cdots\right] \tag{9}
\end{equation*}
$$

Adding (8) and (9) gives the result. An explicit expression is

$$
F_{n}^{(s)}=\frac{1}{\sqrt{s^{2}+4}}\left[\left(\frac{s+\sqrt{s^{2}+4}}{2}\right)^{n+1}-\left(\frac{s-\sqrt{s^{2}+4}}{2}\right)^{n+1}\right] .
$$

## REFERENCES

1. L. Comtet. Advanced Combinatorics. Boston: Reidel, 1974.
2. .G.-C. Rota. "The Number of Partitions of a Set." Amer. Math. Monthly 71 (1964), reprinted in his Finite Operator Calculus. New York: Academic Press, 1975.

## (continued from page 406)

Added in proof. Other explicit formulas for $P(n, s)$ were obtained in the paper "Enumeration of Permutations by Sequences," The Fibonacci Quarterly 16 (1978): 259-68. See also L. Comtet, Advanced Combinatorics (Dordrecht \& Boston: Reidel, 1974), pp. 260-61.
L. Carlitz

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by<br>A. P. HILLMAN<br>University of New Mexico, Albuquerque, New Mexico 87131

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS
The Fibonacci numbers $F_{n}$ and Lucas numbers $L_{n}$ satisfy
and

$$
F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1
$$

$L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1$.
A1so, $a$ and $b$ designate the roots $(1+\sqrt{5}) / 2$ and $(1-\sqrt{5}) / 2$, respectively, of $x^{2}-x-1=0$.

## PROBLEMS PROPOSED IN THIS ISSUE

B-460 Proposed by Larry Taylor, Rego Park, NY
For all integers $j, k, n$, prove that

$$
F_{k} F_{n+j}-F_{j} F_{n+k}=(-1)^{j} F_{k-j} F_{n}
$$

B-461 Proposed by Larry Taylor, Rego Park, NY
For all integers $j, k, n$, prove or disprove that

$$
F_{k} L_{n+j}-F_{j} L_{n+k}=(-1)^{j} F_{k-j} L_{n}
$$

B-462 Proposed by Herta T. Freitag, Roanoke, VA
Let $L(n)$ denote $L_{n}$ and $T_{n}=n(n+1) / 2$. Prove or disprove:

$$
L(n)=(-1)^{T_{n-1}}\left[L\left(T_{n-1}\right) L\left(T_{n}\right)-L\left(n^{2}\right)\right]
$$

B-463 Proposed by Herta T. Freitag, Roanoke, VA
Using the notations of $\mathrm{B}-462$, prove or disprove:

$$
L(n) \equiv(-1)^{T_{n-1}} L\left(n^{2}\right) \quad(\bmod 5)
$$

B-464 Proposed by Gregory Wulcyzn, Bucknell University, Lewisburg, PA
Let $n$ and $\omega$ be integers with $\omega$ odd. Prove or disprove:

$$
F_{n+2 w} F_{n+w}-2 L_{w} F_{n+w} F_{n-w}-F_{n-w} F_{n-2 w}=\left(L_{3 w}-2 L_{w}\right) F_{n}^{2}
$$

B-465 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA
For positive integers $n$ and $k$, prove or disprove:

$$
\frac{F_{2 k}+F_{6 k}+F_{10 k}+\cdots+F_{(4 n-2) k}}{L_{2 k}+L_{6 k}+L_{10 k}+\cdots+L_{(4 n-2) k}}=\frac{F_{2 n k}}{L_{2 n k}}
$$

## SOLUTIONS

## Sequence Identified and Summed

B-436 Proposed by Sahib Singh, Clarion State College, Clarion, PA
Find an appropriate expression for the $n$th term of the following sequence and also find the sum of the first $n$ terms:

$$
4,2,10,20,58,146,388,1001, \ldots .
$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI
The general term $d_{k}$ of the above sequence is given by

$$
d_{k}=F_{k}^{2}+L_{k}^{2}, k=0,1,2, \ldots
$$

Let $S$ be the sum of its first $n$ terms. Then

$$
S=\sum_{k=0}^{n-1} F_{k}^{2}+\sum_{k=0}^{n-1} L_{k}^{2}=F_{n-1} F_{n}+L_{n-1} L_{n}+2
$$

[see $\left(\mathrm{I}_{3}\right)$ and $\left(\mathrm{I}_{4}\right)$ on p. 55 of Fibonacci and Lucas Numbers by V. E. Hoggatt, Jr.] Also solved by Wray G. Brady, Lars Brodin, Paul S. Bruckman, Scott St. Michel and James F. Peters, A.G. Shannon, Charles B. Shields, M. Wachtel and E. Schmutz and H. Klauser, Gregory Wulczyn, and the proposer.

## Hoggatt-Hanse11 Property

B-437 Proposed by G. Iommi Amunategui, Universidad Católica de Valparaíso, Chile Let $[m, n]=m n(m+n) / 2$ for positive integers $m$ and $n$. Show that:
(a) $[m+1, n][m, n+2][m+2, n+1]=[m, n+1][m+2, n][m+1, n+2]$.
(b) $\sum_{k=1}^{m}[m+1-k, k]=m(m+1)^{2}(m+2) / 12$.
(We note that part a is the Hoggatt-Hansell "Star of David" property for the [m, n].)

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, $P A$
(a) $[m+1, n][m, n+2][m+2, n+1]=\frac{1}{8} m(m+1)(m+2)(m+n+1)$

$$
(m+n+2)(m+n+3)
$$

$=[m, n+1][m+2, n][m+1, n+2]$.
(b) $\sum_{k=1}^{m}[m+1-k, k]=\frac{1}{2}(m+1) \sum_{k=1}^{m} k(m+1-k)$

$$
=\frac{m+1}{2}\left[\frac{(m+1) m(m+1)}{2}-\frac{m(m+1)(2 m+1)}{6}\right]
$$

$$
=\frac{m(m+1)^{2}}{12}[3 m+3-2 m-1]=\frac{m(m+1)^{2}(m+2)}{12}
$$

Also solved by Wray G. Brady, Paul S. Bruckman, D. K. Chang, Herta T. Freitag, Northern State College Problems Group, Bob Prielipp, A. G. Shannon, Sahib Singh, Lawrence Somer, Jonathan Weitzman, and the proposer.

## Problem Editor's Error <br> B-438 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Let $n$ and $w$ be integers with $w$ odd. Prove or disprove the proposed identity

$$
F_{n+2 w} F_{n+w}-2 L_{w} F_{n+w} F_{n-w}+F_{n-w} F_{n-2 w}=\left(L_{3 w}-2 L_{w}\right) F_{n}^{2} .
$$

Solution by Paul S. Bruckman, Concord, CA
The identity indicated in $\mathrm{B}-438$ is false. To see this, we only need to set $n=0, w=1$ in the left member of the proposed identity, which yields, for this expression,

$$
F_{2} F_{1}-2 L_{1} F_{1} F_{-1}+F_{-1} F_{-2}=1 \cdot 1-2 \cdot 1 \cdot 1 \cdot 1+1(-1)=1-2-1=-2 ;
$$

however, in the right member, we obtain $\left(L_{3}-2 L_{1}\right) F_{0}^{2}=0 \neq-2$. This also disposes of B-439.
Disproofs were also given by Herta T. Freitag, Bob Prielipp, Sahib Singh, and the cited proposer. For the proposer's version, see B-464 above.

## Companion Problem

B-439 Proposed by A. P. Hillman, University of New Mexico, Albuquerque, NM
Can the proposed identity of $\mathrm{B}-438$ be proved by mere verification for a finite set of ordered pairs ( $n, w$ )? If so, how few pairs suffice? Solution by Paul S. Bruckman contained in his solution to B-438.

Converse Does Not Hold
B-440 Proposed by Jeffrey Shallit, University of California, Berkeley, CA
(a) Let $n=x^{2}+y^{2}$, with $x$ and $y$ integers not both zero. Prove that there is a nonnegative integer $k$ such that $n \equiv 2^{k}\left(\bmod 2^{k+2}\right)$.
(b) If $n \equiv 2^{k}\left(\bmod 2^{k+2}\right)$, must $n$ be a sum of squares?

Solution by Paul S. Bruckman, Concord, CA
It is a well-known result of number theory that any positive integer $n$ is representable as the sum of two squares if and only if its prime factorization only contains even powers (possibly zero) of primes congruent to 3 (mod 4). In this case, the odd portion of $n$ ( $n$ itself, if odd) must be congruent to 1 (mod 4). Thus, $n=2^{k}(4 s+1)=s \cdot 2^{k+2}+2^{k}$, or $n \equiv 2^{k}\left(\bmod 2^{k+2}\right)$. This proves part (a) of the problem, where the desired integer $k$ is simply the greatest power of 2 in the prime factorization of $n$.

If $n=21 \cdot 2^{k}(k \geq 0)$, then $n=2^{k} \cdot 3 \cdot 7$, which cannot be a sum of two squares, because of the result quoted above. Nevertheless,

$$
n=2^{k}\left(5 \cdot 2^{2}+1\right) \equiv 2^{k}\left(\bmod 2^{k+2}\right) .
$$

Hence, the answer to part (b) of the problem is negative. In general, if $n$ contains an even number of odd powers of primes congruent to 3 (mod 4) in its prime factorization, it will satisfy the given congruence, but cannot be expressed as the sum of two squares.

Also solved by D. K. Chang, M. J. DeLeon, Herta T. Freitag, H. Klauser, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

## Sum of Base－b Palindrome Reciprocals

B－441 Proposed by Jeffrey Shallit，University of California，Berkeley，$C A$
A base－b palindrome is a positive integer whose base－b representation reads the same forward and backward．Prove that the sum of the reciprocals of all base－ $b$ palindromes converges for any given integer $b \geq 2$ ．
Solution by H．Klauser，Zurich，Switzerland
Among the $2 n$－digit numbers，there are $(b-1) b^{n-1}$ palindromes．A lower bound for them is $b^{2 n-1}$ and the sum of their reciprocals is

$$
S_{2 n}<(b-1) b^{n-1} b^{1-2 n}=(b-1) b^{-n}
$$

There are $(b-1) b^{n}$ palindromes with $2 n+1$ digits，a lower bound is $b^{2 n}$ ，and the sum of their reciprocals is

$$
S_{2 n+1}<(b-1) b^{-n}
$$

It follows that the sum $S$ of the reciprocals of all palindromes is less than

$$
2(b-1) \sum_{n=1}^{\infty} b^{-n}=2
$$

Also solved by Wray G．Brady，Paul S．Bruckman，Lawrence Somer，Jonathan Weitzman， and the proposer．

# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. Preference will be given to solutions that are submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE
H-333 Proposed by Paul S. Bruckman, Concord CA
The following problem was suggested by Problem 307 of 536 Puzzles \& Curious Problems, by Ernest Dudeney, edited by Martin Gardner (New York: Charles Scribner's Sons, 1967).


Leonardo and the pig he wishes to catch are at points $A$ and $B$, respectively, one unit apart (which we may consider some convenient distance, e.g., 100 yards). The pig runs straight for the gateway at the origin, at uniform speed. Leonardo, on the other hand, goes directly toward the pig at all times, also at a uniform speed, thus taking a curved course. What must be the ratio $r$ of Leonardo's speed to the pig's, so that Leonardo may catch the pig just as they both reach the gate?

H-334 Proposed by Lawrence Somer, Washington, D.C.
Let the Fibonacci-like sequence $\left\{H_{n}\right\}_{n=0}^{\infty}$ be defined by the relation

$$
H_{n+2}=a H_{n+1}+b H_{n}
$$

where $a$ and $b$ are integers, $(a, b)=1$, and $H_{0}=0, H_{1}=1$. Show that if $p$ is an odd prime such that $-b$ is a quadratic nonresidue of $p$, then $p \nmid H_{2 n+1}$ for any $n \geq 0$. (This is a generalization of Problem B-224, which appeared in the Dec. 1971 issue of The Fibonacci Quarterly.

## SOLUTIONS

## Convergents

H-311 Proposed by Paul S. Bruckman, Concord, CA (Vol. 18, No. 1, February 1980)

Let $a$ and $b$ be relatively prime positive integers such that $a b$ is not a perfect square. Let $\theta_{0}=\sqrt{b / a}$ have the continued fraction expansion

$$
\left[u_{1}, u_{2}, u_{3}, \ldots\right]
$$

with convergents $p_{n} / q_{n}(n=1,2, \ldots) ;$ also, define $p_{0}=1, q_{0}=0$, and $p_{-1}=0$. The process of finding the sequence $\left(u_{n}\right)_{n=1}^{\infty}$ may be described by the recursions:

$$
\begin{equation*}
\theta_{n}=u_{n+1}+1 / \theta_{n+1}=\frac{\sqrt{a b}+r_{n}}{d_{n}}, \text { where } r_{0}=0, d_{0}=\alpha, 0<\theta_{n}<1 \tag{1}
\end{equation*}
$$

$r_{n}$ and $d_{n}$ are positive integers, $n=1,2, \ldots$.
Prove:

$$
\begin{align*}
& r_{n}=(-1)^{n-1}\left(a p_{n} p_{n-1}-b q_{n} q_{n-1}\right)  \tag{2}\\
& d_{n}=(-1)^{n}\left(a p_{n}^{2}-b q_{n}^{2}\right), n=0,1,2, \ldots \tag{3}
\end{align*}
$$

Solution by the proposer
Proof: Let $S$ denote the set of nonnegative integers $n$ for which (2) and (3) both hold. Note that
and

$$
\begin{aligned}
& a p_{0} p_{-1}-b q_{0} q_{-1}=a \cdot 1 \cdot 0-b \cdot 0=0=r_{0} \\
& (-1)^{0}\left(a p_{0}^{2}-b q_{0}^{2}\right)=a \cdot 1-b \cdot 0=a=d_{0}
\end{aligned}
$$

hence, $0 \varepsilon S$. Suppose $m \varepsilon S$. Then

$$
\begin{aligned}
1 / \theta_{m+1} & =\theta_{m}-u_{m+1}=\frac{\sqrt{a b}+r_{m}}{d_{m}}-u_{m+1}=\frac{\sqrt{a b}-(-1)^{m}\left(a p_{m} p_{m-1}-b q_{m} q_{m-1}\right)}{(-1)^{m}\left(a p_{m}^{2}-b q_{m}^{2}\right)}-u_{m+1} \\
& =\frac{\sqrt{a b}-(-1)^{m}\left(a p_{m} p_{m-1}-b q_{m} q_{m-1}+a p_{m}^{2} u_{m+1}-b q_{m}^{2} u_{m+1}\right)}{(-1)^{m}\left(a p_{m}^{2}-b q_{m}^{2}\right)} \\
& =\frac{\sqrt{a b}-(-1)^{m}\left\{a p_{m}\left(u_{m+1} p_{m}+p_{m-1}\right)-b q_{m}\left(u_{m+1} q_{m}+q_{m-1}\right)\right\}}{(-1)^{m}\left(a p_{m}^{2}-b q_{m}^{2}\right)} \\
& =\frac{\sqrt{a b}-(-1)^{m}\left(a p_{m} p_{m+1}-b q_{m} q_{m+1}\right)}{(-1)^{m}\left(a p_{m}^{2}-b q_{m}^{2}\right)} \\
& =(-1)^{m}\left(a p_{m}^{2}-b q_{m}^{2}\right) \cdot \frac{\sqrt{a b}+(-1)^{m}\left(a p_{m+1} p_{m}-b q_{m+1} q_{m}\right)}{a b-\left(a p_{m+1} p_{m}-b q_{m+1} q_{m}\right)^{2}}
\end{aligned}
$$

However, $\quad a b-\left(a p_{m+1} p_{m}-b q_{m+1} q_{m}\right)^{2}$
$=a b-\left(a p_{m}^{2}-b q_{m}^{2}\right)\left(a p_{m+1}^{2}-b q_{m+1}^{2}\right)-a b\left(p_{m+1} q_{m}-q_{m+1} p_{m}\right)^{2}$
$=-\left(a p_{m}^{2}-b q_{m}^{2}\right)\left(a p_{m+1}^{2}-b q_{m+1}^{2}\right)$,

$$
p_{m+1} q_{m}-q_{m+1} p_{m}=(-1)^{m+1}
$$

since

Thus, using the inductive hypothesis,

$$
\theta_{m+1}=\frac{\sqrt{a b}+(-1)^{m}\left(a p_{m+1} p_{m}-b q_{m+1} q_{m}\right)}{(-1)^{m+1}\left(a p_{m+1}^{2}-b q_{m+1}^{2}\right)}
$$

which is the assertion of (2) and (3) for $n=m+1$. Hence, $m \varepsilon S \Longrightarrow(m+1) \varepsilon S$. By induction, (2) and (3) hold for all $n$.

## Sum Series

H-312 Proposed by L. Carlitz, Duke University, Durham, NC (Vol. 18, No. 1, February 1980)
Let $m, r, s$ be nonnegative integers. Show that

$$
\begin{gather*}
\sum_{j, k}(-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s} \frac{m!}{(m-j)!(m-k)!(j+k-m)!}=(-1)^{m-r}\binom{m}{r} \delta_{r s},  \tag{*}\\
\delta_{r s}= \begin{cases}1 & (r=s) \\
0 & (r \neq s)\end{cases}
\end{gather*}
$$

where
Solution by Paul S. Bruckman, Concord, CA
Make the following definition:

$$
\begin{equation*}
\theta(r, s, m) \equiv \sum_{j, k}(-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s} \frac{m!}{(m-j)!(m-k)!(j+k-m)!} . \tag{1}
\end{equation*}
$$

It may be noted, by symmetry, that

$$
\begin{equation*}
\theta(r, s, m)=\theta(s, r, m) \tag{2}
\end{equation*}
$$

Making the substitution $j+k=u$, we then obtain:

$$
\begin{aligned}
& \theta(r, s, m)=\sum_{j, u}(-1)^{u-r-s}\binom{j}{r}\binom{u-j}{s} \frac{m!}{(m-j)!(m+j-u)!(u-m)!} \\
& =\sum_{j, u}(-1)^{u-r-s}\binom{j}{p}\binom{u-j}{s}\binom{m}{j}\binom{j}{u-m} \\
& =\sum_{j}(-1)^{p+s}\binom{j}{p}\binom{m}{j} \sum_{u}(-1)^{m+j-u}\binom{m-u}{s}\binom{j}{j-u} \quad \begin{array}{l}
\text { (substituting } \\
m+j-u \text { for } u)
\end{array} \\
& =(-1)^{m+r+s} \sum_{j}(-1)^{j}\binom{m}{j}\binom{j}{r} \sum_{u}(-1)^{u}\binom{m-u}{m-s-u}\binom{j}{u} \\
& =(-1)^{m+r+s} \sum_{j}(-1)^{j}\binom{m}{j}\binom{j}{r} \sum_{u}\binom{-s-1}{m-s-u}\binom{j}{u}(-1)^{m-s} \quad \begin{array}{c}
\text { "negative } \\
\text { binomial" }
\end{array} \\
& \text { coefficient } \\
& \text { relationship). }
\end{aligned}
$$

Now employing the Vandermonde convolution formula, we find that

$$
\begin{equation*}
\theta(r, s, m)=(-1)^{r} \sum_{j}(-1)^{j}\binom{m}{j}\binom{j}{r}\binom{j-s-1}{m-s} \tag{3}
\end{equation*}
$$

Since $\theta(r, s, m)=\theta(s, r, m)$, we may without loss of generality assume $r \geq s$. In (3), note that $s \leq r \leq j \leq m$. Since $j-s-1<m-s$, the binomial coefficient $\binom{j-s-1}{m-s}$ vanishes whenever $j-s-1>0$. If $j=s+1,\binom{j-s-1}{m-s}=\delta_{m s}$; however, $s=m$ implies $j=s$, a contradiction, which implies that $\binom{j-s-1}{m-s}=0$ whenever $j>s$. The only remaining possibility is $j=s$, which implies $r=s=j$.

Hence, all terms in (3) vanish except if $r=s$, in which case (3) reduces to the single term obtained by setting $j=r=s$. In this exceptional case,

$$
\theta(r, r, m)=(-1)^{2 r}\binom{m}{r}\binom{p}{r}\binom{-1}{m-r}=(-1)^{m-r}\binom{m}{r}
$$

Hence, if $r \geq s$,

$$
\begin{equation*}
\theta(r, s, m)=(-1)^{m-r}\binom{m}{r} \delta_{r s} . \tag{4}
\end{equation*}
$$

Clearly, this expression is also true if $r \leq s$, by use of (2). Q.E.D.
Also solved by the proposer.

## Form Partitions!

H-313 Proposed by V.E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 18, No. 2, April 1980)
(A) Show that the Fibonacci numbers partition the Fibonacci numbers.
(B) Show that the Lucas numbers partition the Fibonacci numbers.

Solution by Paul Bruckman, Concord, CA
The following definition (paraphrased) is recalled from the source indicated in the statement of the problem.
Definition: If $U$ and $V$ are subsets of the natural numbers, $U$ is said to partition $\bar{V}$ into the subsets $V_{1}$ and $V_{2}$ if there exist subsets $V_{1}$ and $V_{2}$ of $V$ with the following properties:
(1) $V_{1} \cap V_{2}=\emptyset$;
(2) $V_{1} \cup V_{2}=V$;
(3) $x, y \in V_{i}$ with $x<y(i=1$ or 2$) \Longrightarrow(x+y) \notin U$.

We also say that $U$ partitions $V$ uniquely into the subsets $V_{1}$ and $V_{2}$ if it partitions $V$ into the subsets $V_{1}$ and $V_{2}$, and if such subsets are uniquely determined.

We will have recourse to the following theorem (see [1]):
Theorem: If $r, s$, and $t$ are integers with $2 \leq r<s, t \geq 0$, all Diophantine solutions ( $r, s, t$ ) of the equations indicated below are as follows:
(4) $F_{r}+F_{s}=F_{t} \longleftrightarrow(r, s, t)=(r, r+1, r+2)$;
(5) $F_{r}+F_{s}=L_{t} \Longleftrightarrow(r, s, t)=(r, r+2, r+1)$ or (2, 3, 2).

Proof of A: Set

$$
U=V=F \equiv\left(F_{n}\right)_{n=2}^{\infty}, \quad V_{1}=P_{1} \equiv\left(F_{2 n}\right)_{n=1}^{\infty}, V_{2}=P_{2} \equiv\left(F_{2 n+1}\right)_{n=1}^{\infty}
$$

Clearly, $P_{1}$ and $P_{2}$ satisfy (1) and (2), since $F$ is a strictly increasing sequence. Suppose $x, y \in P_{i}, x<y(i=1$ or 2$)$, and $(x+y) \varepsilon F$. Then there exist unique integers $r, s$, $t$, with $2 \leq r<s$ and $t \geq 4$ (since $F_{t} \geq F_{2}+F_{3}=F_{4}$ ), such that $x=F_{r}, y=F_{s}$, and $x+y=F_{t}$. Since $x$ and $y$ are in the same set $P_{1}$ or $P_{2}$, thus $s-r \geq 2$. This, however, contradicts (4), which implies that $s-r=1$. Thus, the supposed condition is impossible, and its negation must be true, i.e., (3), with the sets as designated. Hence, $F$ partitions $F$ into the subsets $P_{1}$ and $P_{2}$.

To show uniqueness, suppose that $F$ partitions $F$ into the subsets $P_{3}$ and $P_{4}$, which are distinct from $P_{1}$ and $P_{2}$. Then, there exists an integer $u \geq 2$ such that $F_{u}, F_{u+1} \varepsilon P_{i}(i=3$ or 4$)$. This, however, would imply $F_{u}+F_{u+1}=F_{u+2} \varepsilon F$, contradicting (3) and the supposition. Therefore, $F$ partitions $F$ uniquely into the subsets $P_{1}$ and $P_{2}$. Q.E.D.

Proof of B: Set

$$
\begin{gathered}
U=L \equiv\left(L_{n}\right)_{n=0}^{\infty}, V=F, V_{1}=Q_{1} \equiv\left\{F_{n}: n \geq 2, n \equiv 1 \text { or } 2(\bmod 4)\right\}, \\
V_{2}=Q_{2} \equiv\left\{F_{n}: n \geq 3, n \equiv 0 \text { or } 3(\bmod 4)\right\} .
\end{gathered}
$$

Clearly, $Q_{1}$ and $Q_{2}$ satisfy (1) and (2). Suppose $x, y \varepsilon Q_{i}, x<y$ ( $i=1$ or 2), and $(x+y) \varepsilon L$. Then there exist unique integers $r, s, t$ with $2 \leq r<s$ and $t \geq 2$ (since $L_{t} \geq F_{4}=3$, as before, and $L$ is an increasing sequence, after the first term), such that $x=F_{r}, y=F_{s}$, and $x+y=L_{t}$. A moment's reflection shows that, since $x$ and $y$ are in the same set $Q_{1}$ or $Q_{2}$, thus $s-r=1$ or 3 . Since $1=$ $F_{2} \varepsilon Q_{1}$ and $2=F_{3} \varepsilon Q_{2}$, we must not include the solution (2, 3, 2) of (5). However, for the other solutions of (5), $s-r=2$, which also excludes those solutions. Thus, the supposed condition is impossible, which implies (3), with the sets as designated. Hence, $L$ partitions $F$ into the subsets $Q_{1}$ and $Q_{2}$.

To show uniqueness, suppose that $L$ partitions $F$ into the subsets $Q_{3}$ and $Q_{4}$, which are distinct from $Q_{1}$ and $Q_{2}$. It is readily seen that, in this case, there exists an integer $u \geq 2$ such that $F_{u}, F_{u+2} \varepsilon Q_{i}$ ( $i=3$ or 4). This, however, would imply $F_{u}+F_{u+2}=L_{u+1} \varepsilon L$, contradicting (3) and the supposition. Therefore, $L$ partitions $F$ uniquely into the subsets $Q_{1}$ and $Q_{2}$. Q.E.D.
Reference: [1] Private correspondence with Professor Verner E. Hoggatt, Jr. (June $\overline{1980)}$, in which allusion is made to Fibonacci and Lucas Numbers by V. E. Hoggatt, Jr. (The Fibonacci Association, 1969), p. 74, and to Problem E 1424 in The American Mathematical Monthly proposed by V. E. Hoggatt, Jr. The Theorem follows from Zeckendorf's Theorem.
Also solved by the proposer.

## It's the Limit

H-314 Proposed by Paul S. Bruckman, Concord, CA (Vol. 18, No. 2, April 1980)
Given $x_{0} \varepsilon(-1,0)$, define the sequence $S=\left(x_{n}\right)_{n=0}^{\infty}$ as follows:

$$
\begin{equation*}
x_{n+1}=1+(-1)^{n} \sqrt{1+x_{n}}, n=0,1,2, \ldots . \tag{1}
\end{equation*}
$$

Find the limit point(s) of $S$, if any.
Solution by the proposer
We will show that $S$ has precisely two limit points and that

$$
\begin{equation*}
x_{2 n} \rightarrow \beta \quad \text { and } \quad x_{2 n+1} \rightarrow \alpha, \tag{2}
\end{equation*}
$$

where $\alpha$ and $\beta$ are the Fibonacci constants. We first prove, by induction, that

$$
\begin{equation*}
-1<x_{2 n}<0,1<x_{2 n+1}<2, n=0,1,2, \ldots . \tag{3}
\end{equation*}
$$

Let $T$ denote the set of nonnegative integers $n$ satisfying (3). Note that (1) implies:
(4) $\quad x_{2 n^{\prime}+1}=1+\sqrt{1+x_{2 n}}, x_{2 n+2}=1-\sqrt{1+x_{2 n+1}}, n=0,1,2, \ldots$.

Thus, since $-1<x_{0}<0$, we have: $1<x_{1}<2$. Hence, $0 \varepsilon T$. Assuming $k \in T$, by (4) we have:

$$
1-\sqrt{3}<x_{2 k+2}<1-\sqrt{2} \Longrightarrow-1<x_{2 k+2}<0 \Longrightarrow 1<x_{2 k+3}<2
$$

i.e., $k \in T \Longrightarrow(k+1) \varepsilon T$. By induction, (3) is proved. Now define

$$
\begin{equation*}
a_{n}=x_{2 n+1}-\alpha, b_{n}=x_{2 n}-\beta, n=0,1,2, \ldots . \tag{5}
\end{equation*}
$$

Then, using (4), we obtain:

$$
\begin{aligned}
\left|a_{n}\right|=\left|\beta+\sqrt{1+x_{2 n}}\right| & \left.=\left|\beta+\sqrt{\beta^{2}+b_{n}}=\alpha^{-1}\right| 1-\left(1+\alpha^{2} b_{n}\right)^{\frac{1}{2}} \right\rvert\, \\
& =\frac{\alpha^{-1}\left|-\alpha^{2} b_{n}\right|}{\left|1+\left(1+\alpha^{2} b_{n}\right)^{\frac{1}{2}}\right|}=\frac{\left|b_{n}\right| \alpha}{\left|1+\left(1+\alpha^{2} b_{n}\right)^{\frac{1}{2}}\right|} .
\end{aligned}
$$

However, using (3) and (5),

$$
-\beta^{2}=-1-\beta<b_{n}<-\beta .
$$

Hence,

$$
0<1+\alpha^{2} b_{n}<1+\alpha=\alpha^{2} \quad \text { and } \quad 1<\left|1+\left(1+\alpha^{2} b_{n}\right)^{\frac{1}{2}}\right|<\alpha
$$

Therefore,

$$
\begin{equation*}
\left|a_{n}\right|<\alpha\left|b_{n}\right| \tag{6}
\end{equation*}
$$

A1so,

$$
\begin{aligned}
\left|b_{n}\right|=\left|\alpha-\sqrt{1+x_{2 n-1}}\right| & =\left|\alpha-\sqrt{\alpha^{2}+a_{n-1}}\right|=\alpha\left|1-\left(1+\beta^{2} \alpha_{n-1}\right)^{\frac{1}{2}}\right| \\
& =\frac{\alpha\left|-\beta^{2} a_{n-1}\right|}{\left|1+\left(1+\beta^{2} \alpha_{n-1}\right)^{\frac{1}{2}}\right|}=\frac{\alpha^{-1}\left|\alpha_{n-1}\right|}{\left|1+\left(1+\beta^{2} \alpha_{n-1}\right)^{\frac{1}{2}}\right|}
\end{aligned}
$$

Again using (3) and (5), we have
hence

$$
\beta<a_{n-1}<\beta^{2}
$$

or

$$
2 \beta^{2}<1+\beta^{2} a_{n-1}<3 \beta^{2} \Longrightarrow 1+\sqrt{2} \alpha^{-1}<\left|1+\left(1+\beta^{2} a_{n-1}\right)^{\frac{1}{2}}\right|
$$

Thus,

$$
\left|b_{n}\right|<\frac{\left|a_{n-1}\right|}{\alpha+\sqrt{2}}
$$

since $\alpha+\sqrt{2}>3$, thus

$$
\begin{equation*}
\left|b_{n}\right|<\frac{\left|a_{n-1}\right|}{3} . \tag{7}
\end{equation*}
$$

It follows from (6) and (7) that
$\left|a_{n}\right|<\frac{\alpha}{3}\left|a_{n-1}\right|<.6\left|a_{n-1}\right|$ and $\left|b_{n}\right|<\frac{\alpha}{3}\left|b_{n-1}\right|<.6\left|b_{n-1}\right| \quad(n=1,2,3, \ldots)$.
Note that

$$
-1-\beta<x_{0}-\beta<-\beta \Longrightarrow-1<-\beta^{2}<b_{0}<\alpha^{1}<1 \Longrightarrow\left|b_{0}\right|<1
$$

also,

$$
1-\alpha<x_{1}-\alpha<2-\alpha \Longrightarrow \beta<a_{0}<\beta^{2} \Longrightarrow\left|a_{0}\right|<1
$$

Therefore, by an easy induction, $\left|a_{n}\right|<(.6)^{n}$ and $\left|b_{n}\right|<(.6)^{n}$, which implies

$$
\begin{array}{rlll}
\left|a_{n}\right| & \rightarrow 0 & \text { and } & \left|b_{n}\right| \rightarrow 0 \\
a_{n} \rightarrow 0 & \text { and } & b_{n} \rightarrow 0
\end{array}
$$

and hence

This, in turn, implies (2).
Note: The condition $x_{0} \varepsilon(-1,0)$ is sufficient but not necessary for the stated result to follow. It is only necessary that $x_{0} \varepsilon[-1,3]$.

## Factor

H－315 Proposed by D．P．Laurie，National Research Institute for Mathematical Sciences，Pretoria，South Africa（Vol．18，No．2，April 1980）

Let the polynomial $P$ be given by

$$
P(z)=z^{n}+a_{n-1} z^{n-1}+a_{n-2} z^{n-2}+\cdots+a_{1} z+a_{0}
$$

and let $z_{1}, z_{2}, \ldots, z_{n}$ be distinct complex numbers．The following iteration scheme for factorizing $P$ has been suggested by Kerner［1］：

$$
\hat{z}_{i}=z_{i}-\frac{P\left(z_{i}\right)}{\prod_{\substack{j=1 \\ j \neq i}}^{n}\left(z_{i}-z_{j}\right)} ; i=1,2, \ldots, n .
$$

Prove that if $\sum_{i=1}^{n} z_{i}=-a_{n-1}$ ，then also $\sum_{i=1}^{n} \hat{z}_{i}=-a_{n-1}$ ．
Reference：［1］I．Kerner．＂Ein Gesamtschrittverfahren zur Berechnung der Null－ stellen von Polynomen．＂Numer．Math． 8 （1966）：290－94．

Solution by the proposer
Let

$$
R(z)=P(z)-\prod_{i=1}^{n}\left(z-z_{i}\right)
$$

since $\sum_{i=1}^{n} z_{i}=-a_{n-1}, R(z)$ is a polynomial of degree $n-2$ ．We have

$$
\sum_{i=1}^{n} \hat{z}_{i}=\sum_{i=1}^{n} z_{i}-\sum_{i=1}^{n} \frac{P\left(z_{i}\right)}{\prod_{\substack{j=1 \\ j \neq i}}^{n}\left(z_{i}-z_{j}\right)}=\sum_{i=1}^{n} z_{i}-\sum_{i=1}^{n} \frac{R\left(z_{i}\right)}{\prod_{\substack{j=1 \\ j \neq i}}^{n}\left(z_{i}-z_{j}\right)}
$$

The second sum on the right is equal to the $n$th divided difference of $R$ at the points $z_{1}, z_{2}, \ldots, z_{n}$（see Davis［1］，p．40），and thus zero，since $R$ is only of degree $n-2$ ．
Reference：［1］P．Davis．Interpolation and Approximation．Blaisde11， 1963.
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