## THE FIBONACCI QUARTERLY

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# THE FIBONACCI QUARTERLY 

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## A PROPERTY OF FIBONACCI HUMBERS

## R. L. GRAHAM

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## 1. INTRODUCTION

Let $A=\left(a_{1}, a_{2}, \cdots\right)$ denote a (possibly finite) sequence of integers. We shall let $P(A)$ denote the set of all integers of the form $\sum_{k=1}^{\infty} \epsilon_{k} a_{k}$ where $\epsilon_{k}$ is 0 or 1 . If all sufficiently large integers belong to $P(A)$ then $A$ is said to be complete. For example, if $F=\left(F_{1}, F_{2}, \cdots\right)$, where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number, i.e., $F_{0}=0, F_{1}=1$ and $F_{n+2}=F_{n+1}+F_{n}$ for $n \geq 0$, then F is complete (cf. [1]). More generally, it can be easily shown that $F$ satisfies the following conditions:
(A) If any one term is removed from $F$ then the resulting sequence is complete.
(B) If any two terms are removed from $F$ then the resulting sequence is not complete.
(A simple proof of (A) is given in [1] ; (B) will be proved in Section 2.)
In this paper it will be shown that a "slight" modification of $F$ produces a rather startling change in the additive properties of $F$. In particular, the sequence $S$ which has $F_{n}-(-1)^{n}$ as its $n^{\text {th }}$ term has the following remarkable properties:
(C) If any finite subsequence is deleted from $S$ then the resulting sequence is complete.
(D) If any infinite subsequence is deleted from S then the resulting sequence is not complete.

## 2. THE MAIN RESULTS

We first prove (B). Suppose $\mathrm{F}_{\mathrm{r}}$ and $\mathrm{F}_{\mathrm{S}}$ are removed from F to form $\mathrm{F}^{*}$ (where $\mathrm{r}<\mathrm{s}$ ). We show by induction that $\mathrm{F}_{\mathrm{S}+2 \mathrm{k}+1}-1 \notin \mathrm{P}\left(\mathrm{F}^{*}\right)$ for $\mathrm{k}=$ $0,1,2, \cdots$. We first note that the sum of all terms of $F^{*}$ which do not exceed $F_{s+1}-1$ is just

$$
\sum_{\mathrm{k}=1}^{\mathrm{s}-1} \mathrm{~F}_{\mathrm{k}}-\mathrm{F}_{\mathrm{r}}=\sum_{\mathrm{k}=1}^{\mathrm{s}-1}\left(\mathrm{~F}_{\mathrm{k}+2}-\mathrm{F}_{\mathrm{k}+1}\right)-\mathrm{F}_{\mathrm{r}}=\mathrm{F}_{\mathrm{s}+1}-1-\mathrm{F}_{\mathrm{r}}<\mathrm{F}_{\mathrm{S}+1}-1
$$

and hence $F_{S+1}-1 \notin P\left(F^{*}\right)$. Now assume that $F_{S+2 t+1}-1 \oint P\left(F^{*}\right)$ for some $t \geq 0$ and consider the integer $\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+3}-1$. The sum of all terms of $\mathrm{F}^{*}$ which are less than $\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+2}$ is just

$$
\sum_{\mathrm{k}=1}^{\mathrm{s}+2 t+1} \mathrm{~F}_{\mathrm{k}}-\mathrm{F}_{\mathrm{r}}-\mathrm{F}_{\mathrm{S}}=\mathrm{F}_{\mathrm{s}+2 \mathrm{t}+3}-1-\mathrm{F}_{\mathrm{r}}-\mathrm{F}_{\mathrm{s}}<\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+3}-1
$$

Thus, in order to have $\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+3}-1 \in \mathrm{P}\left(\mathrm{F}^{*}\right)$ we must have $\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+3}-1=$ $\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+2}+\mathrm{m}$, where $\mathrm{m} \in \mathrm{P}\left(\mathrm{F}^{*}\right)$. But $\mathrm{m}=\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+3}-\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+2}-1=\mathrm{F}_{\mathrm{S}+2 \mathrm{t}+1}$ - 1 which does not belong to $P\left(F^{*}\right)$ by assumption. Hence $F_{S+2 t+3}-1 \ddagger$ $\mathrm{P}\left(\mathrm{F}^{*}\right)$ and proof of (B) is completed.

We now proceed to the main result of the paper.
Theorem: Let $S=\left(s_{1}, S_{2}, \cdots\right)$ be the sequence of integers defined by $s_{n}=F_{n}-(-1)^{n}$. Then $S$ satisfies (C) and (D).

Proof: The proof of (D) will be given first. Let the infinite subsequence $\mathrm{s}_{\mathrm{i}_{1}}<\mathrm{s}_{\mathrm{i}_{2}}<\mathrm{s}_{\mathrm{i}_{3}}<\cdots$ be deleted from S and denote the remaining sequence by $S^{*}$. In order to prove (D) it suffices to show that

$$
s_{i_{n}+1}-1 \notin P\left(S^{*}\right) \text { for } n \geq 4
$$

We first note that

$$
s_{i_{1}}+s_{i_{2}} \geq s_{1}+s_{2}=2
$$

Therefore, we have (cf. Eq. (1) )

$$
\sum_{\substack{j=1 \\ j \neq i_{1}, i_{2}}}^{i_{n}-1} s_{j}<s_{i_{n}+1}-s_{i_{1}}-s_{i_{2}} \leq s_{i_{n}+1}-2
$$

Hence, to represent $s_{i_{n}+1}-1$ in $P\left(S^{*}\right)$ we must use some term of $S^{*}$ which exceeds $\mathrm{s}_{\mathrm{in}_{\mathrm{n}}-1}$ (since by above, the sum of all terms of $\mathrm{S}^{*}$ not exceeding $\mathrm{s}_{\mathrm{i}_{\mathrm{n}}}-1$ is less than $\mathrm{si}_{\mathrm{n}}+1-1$ for $\mathrm{n} \geq 4$ ). Since $\mathrm{si}_{\mathrm{n}}$ is missing from $\mathrm{S}^{*}$, then the smallest term of $S^{*}$ which exceeds $s_{i_{n}-1}$ is $s_{i_{n^{+}}}$(which, of course, is greater than $\mathrm{s}_{\mathrm{i}_{\mathrm{n}}+1}-1$ ). Thus

$$
\mathrm{s}_{\mathbf{i}_{\mathrm{n}}+1}-1 \notin \mathrm{P}\left(\mathrm{~S}^{*}\right) \text { for } \mathrm{n} \geq 4
$$

and (D) is proved.
To prove (C), let $k>4$ and let $S^{\prime}$ denote the sequence $\left(s_{k}, s_{k+1}\right.$, $s_{k+2},^{\circ \cdot}$ ). For non-negative integers $w$ and $x, P\left(S^{\prime}\right)$ is said to have no gaps of length greater than $w$ beyond $x$ provided there do not exist $w+1$ consecutive integers exceeding $x$ which do not belong to $P\left(S^{\prime}\right)$. The proof of (C) is now a consequence of the following two lemmas.

Lemma 1: There exists $v$ such that $P\left(S^{\prime}\right)$ has no gaps of length greater than v beyond $\mathrm{s}_{\mathrm{k}}$.

Lemma 2: If $\mathrm{w}>0$ and $\mathrm{P}\left(\mathrm{S}^{\prime}\right)$ has no gaps of length greater than w beyond $s_{h}$ then there exists $i$ such that $P\left(S^{\prime}\right)$ has no gaps of length greater than $w-1$ beyond $s_{i}$.

Indeed, by Lemma 1 and repeated application of Lemma 2 it follows that there exists $j$ such that $P\left(S^{\eta}\right)$ has no gaps of length greater than 0 beyond $\mathbf{S}_{\mathrm{j}}$. That is, $\mathrm{S}^{\prime}$ is complete, which proves (C).

Proof of Lemma 1: First note that
$s_{2 n}+s_{2 n+1}=F_{2 n}-(-1)^{2 n}+F_{2 n+1}-(-1)^{2 n+1}=F_{2 n}+F_{2 n+1}=F_{2 n+2}=$

$$
\mathrm{s}_{2 \mathrm{n}+2}+1
$$

Similarly,

$$
\begin{aligned}
s_{2 n+1}+s_{2 n+2} & =F_{2 n+1}-(-1)^{2 n+1}+F_{2 n+2}-(-1)^{2 n+2} \\
& =F_{2 n+1}+F_{2 n+2}=F_{2 n+3}=s_{2 n+3}-1
\end{aligned}
$$

Also, we have
(1)

$$
\left\{\begin{aligned}
s_{1}+s_{2}+\cdots+s_{n} & =\left(F_{1}+1\right)+\left(F_{2}-1\right)+\cdots+\left(F_{n}-(-1)^{n}\right) \\
& =\sum_{j=1}^{n} F_{j}+\epsilon_{n}=\sum_{j=1}^{n}\left(F_{j+2}-F_{j+1}\right)+\epsilon_{n} \\
& =F_{n+2}-1+\epsilon_{n} \\
& =s_{n+2}-\epsilon_{n}
\end{aligned}\right.
$$

where

$$
\epsilon_{\mathrm{n}}= \begin{cases}0 & \text { for } \mathrm{n} \text { even } \\ 1 & \text { for } \mathrm{n} \text { odd }\end{cases}
$$

Thus

$$
\sum_{j=m}^{n} s_{j}=s_{n+2}-s_{m+1}-\epsilon_{n}+\epsilon_{m-1} \text { for } n \geq m
$$

Now, let $h>k+1$ and let

$$
\mathrm{P}^{\prime}=P\left(\left(s_{k^{\prime}}, s_{k+1}, \cdots, s_{h}\right)\right)=\left\{p_{1}^{\prime}: p_{2}^{\prime}, \cdots, p_{n}^{\prime}\right\}
$$

where $\mathrm{p}_{1}^{\prime}<\mathrm{p}_{2}^{\prime}<\cdots<\mathrm{p}_{\mathrm{n}}^{\prime}$. Let

$$
v=\max _{1 \leq r \leq n-1}\left(p_{r+1}^{\prime}-p_{r}^{\prime}\right)
$$

Then

$$
\begin{aligned}
h>k+1> & \Rightarrow s_{h} \geq s_{k+1}+2 \\
& \Longrightarrow s_{h} \geq s_{k+1}+\epsilon_{h}-\epsilon_{k+1}+1 \\
& \Longrightarrow s_{h+2}-s_{h+1} \geq s_{k+1}+\epsilon_{h}-\epsilon_{k+1} \\
& \Longrightarrow s_{h+1} \leq s_{h+2}-s_{k+1}-\epsilon_{h}+\epsilon_{k+1}=\sum_{j=1}^{h} s_{j}
\end{aligned}
$$

Since

$$
\max _{1 \leq \mathrm{r} \leq \mathrm{n}-1}\left(\left(\mathrm{p}_{\mathrm{r}+1}^{\prime}+\mathrm{s}_{\mathrm{h}+1}\right)-\left(p_{\mathrm{r}}^{1}+\mathrm{s}_{\mathrm{h}+1}\right)\right)=\mathrm{v}
$$

then in

$$
\begin{aligned}
P^{\prime \prime} & =P\left(\left(s_{k}, \cdots, s_{h}, s_{h+1}\right)\right) \\
& =P\left(\left(s_{k^{\prime}}, \cdots, s_{h}\right)\right) \cup\left\{q+s_{h+1}: q \in P\left(\left(s_{k^{\prime}}, \cdots, s_{h}\right)\right)\right\} \\
& =\left\{p_{1}^{\prime \prime}, p_{2}^{\prime \prime} \cdots, p_{n^{\prime}}^{\prime \prime}\right\}
\end{aligned}
$$

where $\mathrm{p}_{1}^{\prime \prime}<\mathrm{p}_{2}^{\prime \prime}<\ldots<\mathrm{p}_{\mathrm{n}^{\prime \prime}}^{\prime \prime}$, we have

$$
\max _{1 \leq r \leq n^{\prime}-1}\left(p_{r+1}^{\prime \prime}-p_{r}^{\prime \prime}\right) \leq v
$$

Similarly, since

$$
h>k+1>5 \Rightarrow s_{h+2} \leq \sum_{j=k}^{h+1} s_{j}
$$

then in

$$
P^{\prime \prime \prime}=p\left(\left(s_{k^{\prime}}, \cdots, s_{h+2}\right)\right)=\left\{p_{1}^{\prime \prime \prime}, p_{2}^{\prime \prime \prime}, \cdots, p_{n^{\prime \prime}}^{\prime \prime \prime}\right\}
$$

where $p_{1}^{\prime \prime \prime}<p_{2}^{\prime \prime \prime}<\cdots<p_{n^{\prime \prime}}^{\prime \prime \prime}$, we have

$$
\max _{1 \leq r \leq n^{\prime \prime}-1}\left(p_{r+1}^{\prime \prime \prime}-p_{r}^{\prime \prime \prime}\right) \leq v, \text { etc. }
$$

By continuing in this way, Lemma 1 is proved.
The proof of Lemma 2 is a consequence of the following two results:
(a) For any $r \geq 0$ there exists $t$ such that $m>t$ implies all the integers

$$
s_{m}+y, \quad y=0, \pm 1, \pm 2, \cdots, \pm(r-1)
$$

belong to $\mathrm{P}\left(\mathrm{S}^{\top}\right)$.
(b) There exists $r^{\prime}$ such that for all sufficiently large $h^{\prime}, P\left(S^{\prime}\right)$ has no gaps of length greater than $w-1$ between $s_{h^{\prime}}+r^{\prime}$ and $s_{h^{\prime}+1}-$ $r^{p}$ (i, e., there do not exist $w$ consecutive integers exceeding $s_{h}$, $+r^{\prime}$ and less than $s_{h^{\prime}+1}-r^{\prime}$ which are missing from $P\left(S^{\prime}\right)$ ).
Therefore, for $s_{i}$ sufficiently large, $P\left(S^{\prime}\right)$ has no gaps of length greater than w-1 beyond $s_{i}$, which proves Lemma 2 。

Proof of (a): Choose p such that

$$
2 p-3 \geq k \quad \text { and } \quad s_{2 p-2} \geq r
$$

and choose $n$ such that

$$
\mathrm{n} \geq \mathrm{s}_{2 \mathrm{p}-2}+\mathrm{p} \quad \text { and } \quad \mathrm{n} \geq \mathrm{r}+\mathrm{k}
$$

Then
$\sum_{i=n-m}^{n} s_{2 i-1}+\sum_{j=2 p-3}^{2 n-2 m-4} s_{j}=\sum_{i=1}^{n} s_{2 i-1}-\sum_{i=1}^{n-m-1} s_{2 i-1}+\sum_{j=2 p-3}^{2 n-2 m-4} s_{j}$

$$
\begin{aligned}
& =n+\sum_{i=1}^{n} F_{2 i-1}-n+m+1-\sum_{i=1}^{n-m-1} F_{2 i-1}+\sum_{j=2 p-3}^{2 n-2 m-4} s_{j} \\
& =m+1+F_{2 n}-F_{2 n-2 m-2}+s_{2 n-2 m-2}+0-s_{2 p-2}-0 \\
& =s_{2 n}-\left(s_{2 p-2}-m-1\right), \text { for } 0 \leq m \leq n-p-1
\end{aligned}
$$

Since $2 p-3 \geq k$, then all the summands used on the left-hand side are in $S^{\prime}$ 。 Hence, all the integers

$$
s_{2 n}-\left(s_{2 p-2}-m-1\right), \quad 0 \leq m \leq n-p-1
$$

belong to $P\left(S^{\prime}\right)$. Since $n \geq s_{2 p-2}+p$, then

$$
n-p-1 \geq s_{2 p-2}-1
$$

Therefore, all the integers

$$
s_{2 n}-\left(s_{2 p-2}-m-1\right), \quad 0 \leq m \leq s_{2 p-2}-1
$$

belong to $\mathrm{P}\left(\mathrm{S}^{\prime}\right)$, i. e., all the integers

$$
s_{2 n}-m^{\prime}, \quad 0 \leq m^{\prime} \leq s_{2 p-2}-1
$$

But $s_{2 p-2} \geq r$, so that we finally see that all the integers

$$
s_{2 n}-m^{\prime}, \quad 0 \leq m^{t} \leq r-1
$$

belong to $\mathrm{P}\left(\mathrm{S}^{\prime}\right)$.
To obtain sums which exceed $s_{2 n}$, note that for $1 \leq m \leq n-k$ we have

$$
\begin{aligned}
\sum_{j=n-m+1}^{n} s_{2 j-1}+s_{2 n-2 m} & =\sum_{j=1}^{n} s_{2 j-1}-\sum_{j=1}^{n-m} s_{2 j-1}+s_{2 n-2 m} \\
& =n+F_{2 n}-(n-m)-F_{2 n-2 m}+s_{2 n-2 m} \\
& =m+F_{2 n}-1 \\
& =m+s_{2 n}
\end{aligned}
$$

Since the sums

$$
\sum_{j=n-m+1}^{n} s_{2 j-1}+s_{2 n-2 m} \text { for } m=1,2, \ldots, n-k
$$

are all elements of $P\left(S^{\prime}\right)$, and since $n-k \geq r$, then all the integers

$$
\mathrm{s}_{2 \mathrm{n}}+\mathrm{m}, \quad 1 \leq \mathrm{m} \leq \mathrm{r},
$$

belong to $P\left(S^{\prime}\right)$.
Arguments almost identical to this show that for all sufficiently large $n$, all the integers

$$
s_{2 n+1}+m, \quad m=0, \pm 1, \cdots, \pm(r-1)
$$

belong to $P\left(S^{\prime}\right)$. This proves (a).
Proof of (b): We first give a definition. Let $A=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ be a finite sequence of integers. The point of symmetry of $P(A)$ is defined to be the number $\frac{1}{2} \sum_{k=1}^{n} a_{k}$. The reason for this terminology arises from the fact that if $P(A)$ is consideredas a subset of the real line, then $P(A)$ is symmetric about the point $\frac{1}{2} \sum_{k=1}^{n} a_{k}$. For we have
$p=\sum_{k=1}^{n} \epsilon_{k} a_{k} \in P(A) \Longleftrightarrow \sum_{k=1}^{n}\left(1-\epsilon_{k}\right) a_{k}=\sum_{k=1}^{n} a_{k}-p \epsilon P(A)$
and the points $p$ and $\sum_{k=1}^{n} a_{k}-p$ are certainly equidistant from $\frac{1}{2} \sum_{k=1}^{n} a_{k}$.

Now note that if $r$ is sufficiently large then
and

$$
\begin{gathered}
s_{r-1}>3>-s_{k+1}+3 \\
s_{r+1}-s_{r}>-s_{k+1}+2 \\
s_{r}+1-s_{k+1}<s_{r+1}-1 \\
s_{r+2}-s_{r+1}-s_{k+1}<s_{r+1}+\epsilon_{r}-\epsilon_{k+1} \\
s_{r+2}-s_{k+1}<2 s_{r+1}+\epsilon_{r}-\epsilon_{k+1}
\end{gathered}
$$

Therefore

$$
\frac{1}{2} \sum_{j=k}^{r} s_{j}=\frac{1}{2}\left(s_{r+2}-s_{k+1}-\epsilon_{r}+\epsilon_{k+1}\right)<s_{r+1}
$$

and

$$
\frac{1}{2}\left(s_{r+2}-s_{k+1}-\epsilon_{r}+\epsilon_{k+1}\right)>s_{h}
$$

for all sufficiently large $r$. In other words, for all sufficiently large $r$, the point of symmetry of $P\left(\left(s_{k}, \cdots, s_{r}\right)\right)$ lies between $s_{h}$ and $s_{r+1}$. By hypothesis no gaps of length greater than $w$ occur in $P\left(S^{\prime}\right)$ beyond $s_{h}$. Since $h>k$ $>4$ implies

$$
s_{h}<s_{h+1}<s_{h+2}<\cdots,
$$

then no gaps of length greater than $w$ can occur in $P\left(\left(s_{k}, \cdots, s_{r}\right)\right)$ between $s_{h}$ and $s_{r+1}$. (For if they did, then they would remain in $P\left(S^{\prime}\right)$ since $s_{r+1}$ $<\mathrm{s}_{\mathrm{r}+2}<\cdots$.) But

$$
s_{r+1}>\frac{1}{2} \sum_{j=k}^{r} s_{j}
$$

and $\frac{1}{2} \sum_{j=k}^{r} s_{j}$ is the point of symmetry of $P\left(\left(s_{k}, \cdots, s_{r}\right)\right)$. Therefore,

$$
\sum_{j=k}^{r} s_{j}-s_{r+1}<\frac{1}{2} \sum_{j=k}^{r} s_{j}
$$

and by symmetry no gaps of length greater than $w$ occur in $P\left(\left(s_{k}, \cdots, s_{r}\right)\right)$ between

$$
\sum_{j=k}^{r} s_{j}-s_{r+1} \text { and } \sum_{j=k}^{r} s_{j}-s_{h}
$$

Thus, no gaps of length greater than $w$ occur between $s_{h}$ and

$$
\sum_{j=k}^{r} s_{j}-s_{h}=s_{r+2}-s_{k+1}-\epsilon_{h}+\epsilon_{k+1}-s_{h}
$$

provided that $r$ is sufficiently large. Now consider $P\left(\left(s_{k}, \cdots, s_{r+3}\right)\right)$. Since

$$
\mathrm{s}_{\mathrm{r}+1}+\mathrm{s}_{\mathrm{r}+2}=\mathrm{s}_{\mathrm{r}+3}+(-1)^{\mathrm{r}+1}
$$

then $s_{r+1}+s_{r+2}+p$ and $s_{r+3}+p$ are elements of $P\left(\left(s_{k}, \ldots, s_{r+3}\right)\right)$ which differ by 1 whenever $p$ is an element of $P\left(\left(s_{k}, \cdots, s_{r}\right)\right)$. Hence, since in $P\left(\left(s_{k^{\prime}}, \cdots, s_{r}\right)\right)$ there are no gaps of length greater than $w$ between $s_{h}$ and $\sum_{j=k}^{r} s_{j}-s_{h}$, then in $P\left(\left(s_{k}, \cdots, s_{r+3}\right)\right)$ there are no gaps of greater length than w-1 between

$$
s_{h}+s_{r+3} \text { and } \sum_{j=k}^{r} s_{j}-s_{h}+s_{r+3}
$$

Similarly, consider $P\left(\left(s_{k}, \cdots, s_{r+4}\right)\right)$. Since

$$
\mathrm{s}_{\mathrm{r}+2}+\mathrm{s}_{\mathrm{r}+3}=\mathrm{s}_{\mathrm{r}+4}+(-1)^{\mathrm{r}+2}
$$

and there are no gaps in $\left.\mathrm{P}\left(\mathrm{s}_{\mathrm{k}^{\prime}} \cdots, \mathrm{s}_{\mathrm{r}+1}\right)\right)$ of length greater than w between $s_{h}$ and $\sum_{j=k}^{r+1} s_{j}-s_{h}$, then there are no gaps in $P\left(\left(s_{k}, \cdots, s_{r+4}\right)\right)$ of length greater than w-1 between

$$
s_{h}+s_{r+4} \text { and } \sum_{j=k}^{r+1} s_{j}-s_{h}+s_{r+4}
$$

In general, for $\mathrm{q}>0$ since $\mathrm{s}_{\mathrm{r}+\mathrm{q}}+\mathrm{s}_{\mathrm{r}+\mathrm{q}+1}=\mathrm{s}_{\mathrm{r}+\mathrm{q}+2}+(-1)^{\mathrm{r}+\mathrm{q}}$ and there are no gaps in $P\left(\left(s_{k}, \cdots, s_{r+q-1}\right)\right)$ of length greater than $w$ between $s_{h}$ and $\mathrm{r}+\mathrm{q}-1$
$\sum_{j=k} s_{j}-s_{h}$, then there are no gaps $\operatorname{in~}_{r+q-1} P\left(s_{k}, \cdots, s_{r+q+2}\right)$ ) of length greater than $w-1$ between $\mathrm{s}_{\mathrm{h}}+\mathrm{s}_{\mathrm{r}+\mathrm{q}+2}$ and $\sum_{\mathrm{j}=\mathrm{k}} \mathrm{s}_{\mathrm{j}}-\mathrm{s}_{\mathrm{h}}+\mathrm{s}_{\mathrm{r}+\mathrm{q}+2}$. But

$$
\begin{aligned}
\sum_{j=k}^{r+q-1} s_{j}-s_{h}+s_{r+q-2} & =s_{r+q+1}-s_{k+1}-\epsilon_{r+q+1}+\epsilon_{k+1}-s_{h}+s_{r+q+2} \\
& =s_{r+q+3}+(-1)^{r+q+1}-s_{k+1}-s_{h}-\epsilon_{r+q+1}+\epsilon_{k+1} \\
& \geq s_{r+q+3}-s_{k+1}-s_{h}-2 .
\end{aligned}
$$

Therefore, if we let

$$
\mathrm{r}^{\prime}=\mathrm{s}_{\mathrm{k}+1}+\mathrm{s}_{\mathrm{h}}+2
$$

then for all sufficiently large $z$, there are no gaps in $P\left(\left(s_{k}, \cdots, s_{z}\right)\right)$ of length greater than $w-1$ between $s_{z}+r^{\prime}$ and $s_{z+1}-r^{\prime}$ (since the preceding argument is valid for $q>0$ and all sufficiently large $r$ ). This completes the proof of (b) and the theorem.

## 3. CONCLUDING REMARKS

Examples of sequences of positive integers which satisfy both (C) and (D) are rather elusive. It would be interesting to know if there exists such a sequence, say $T=\left(t_{1}, t_{2}, \cdots\right)$, which is essentially different from $S$, e.g., such that

$$
\lim _{n \rightarrow \infty} \frac{t_{n+1}}{t_{n}} \neq \frac{1+\sqrt{5}}{2}
$$

The author wishes to express his gratitude to the referee for several suggestions which made the paper considerably more readable.

## REFERENCE

1. J. L. Brown, "On Complete Sequences of Integers," Amer. Math. Monthly, 68 (1961) pp. 557-560.


## on square lucas numbers

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Among the first dozen members of the Lucas sequence $(1,3,4,7,11,18$, $\cdots$ ) there are two squares, $L_{1}=1$ and $L_{3}=4$. Are there any other squares in the Lucas sequence?

Since the period of the Lucas sequence modulo 8 is 12 , it follows that $\mathrm{L}_{12 \mathrm{k}+\lambda} \equiv \mathrm{L}_{\lambda}(\bmod 8)$, so that all possible residues are represented in the following table.

| $\lambda$ | $\mathrm{L}_{12 \mathrm{k}+\lambda}(\bmod 8)$ |
| ---: | :---: |
| 0 | 2 |
| 1 | 1 |
| 2 | 3 |
| 3 | 4 |
| 4 | 7 |
| 5 | 3 |
| 6 | 2 |
| 7 | 5 |
| 8 | 7 |
| 9 | 4 |
| 10 | 3 |
| 11 | 7 |

It follows that the only Lucas numbers which may be squares are $L_{12 k+\lambda}$ with $\lambda=1,3$ or 9 , since the other residues modulo 8 are quadratic non-residues of 8 .

From the general relation

$$
2 \mathrm{~L}_{\mathrm{a}+\mathrm{b}}=5 \mathrm{~F}_{\mathrm{a}} \mathrm{~F}_{\mathrm{b}}+\mathrm{L}_{\mathrm{a}} \mathrm{~L}_{\mathrm{b}}
$$

it follows if $t=2^{r}, r \geq 1$, that

$$
\begin{aligned}
2 L_{\lambda+2 t} & =5 F_{\lambda} F_{2 t}+L_{\lambda} L_{2 t} \\
& =5 F_{\lambda} F_{t} L_{t}+L_{\lambda}\left(L_{t}^{2}-2\right)
\end{aligned}
$$

so that

$$
2 \mathrm{~L}_{\lambda+2 \mathrm{t}} \equiv-2 \mathrm{~L}_{\lambda}\left(\bmod \mathrm{L}_{\mathrm{t}}\right)
$$

But $\left(L_{t}, 2\right)=1$. Hence

$$
L_{\lambda+2 t} \equiv-L_{\lambda}\left(\bmod L_{t}\right)
$$

We can use this relation to advantage by writing

$$
L_{12 k+\lambda} \text { as } L_{\lambda+2 m t}
$$

where $m$ is odd and $t=2^{r}, r \geq 1$.
Then

$$
\begin{aligned}
L_{\lambda+2 m t} \equiv & -L_{\lambda+2(m-1) t}\left(\bmod L_{t}\right) \\
\equiv & +L_{\lambda+2(m-2) t}\left(\bmod L_{t}\right) \\
& \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
\equiv & (-1)^{m} L_{\lambda}\left(\bmod L_{t}\right)
\end{aligned}
$$

For $\lambda=1$,

$$
\mathrm{L}_{12 \mathrm{k}+1} \equiv-\mathrm{L}_{1} \equiv-1\left(\bmod \mathrm{~L}_{\mathrm{t}}\right), \quad \mathrm{t}=2^{\mathrm{r}}, \mathrm{r} \geq 1
$$

But

$$
\left(\frac{-1}{L_{t}}\right)^{*}=-1, \text { since } L_{t} \equiv 3(\bmod 4)
$$

Therefore $L_{12 k+1}$ may not be a perfect square except for $L_{1}=1$. Similarly, $L_{12 k+3}$ can be shown to be ruled out by entirely the same argument except for $L_{3}=4$.

Finally,

$$
\mathrm{L}_{12 \mathrm{k}+9}=\mathrm{L}_{4 \mathrm{k}+3}\left[\mathrm{~L}_{4 \mathrm{k}+3}^{2}+\theta\right]
$$

The $\theta$ in the bracket may be either 3 or 1 . But since only Lucas numbers $L_{4 k+2}$ are divisible by 3 , it follows that $L_{4 k+3}$ and $L_{4 k+3}^{2}+3$ are relatively prime. Therefore, if $L_{12 k+9}$ is to be a perfect square, both factors must be such. It is clear that $L_{4 k+3}$ is not a perfect square for $k=1$ or 2 . For other values, $k$ equals either $3 k^{\prime}, 3 k^{\prime}+1$ or $3 k^{\prime}+2$ with $k^{\imath} \geq 1$. But this gives us Lucas numbers $L_{12 k^{\prime}+3}, L_{12 k^{\prime}+7}$, and $L_{12 k^{\prime}+11}$ respectively and it has already been shown that these cannot be squares.

Thus the only squares in the Lucas sequence are $L_{1}=1$ and $L_{3}=4$.

* Legendre's symbol.


# LATTICE PATHS AND FHBONACCI NUMRERS 

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    L. Moser and W. Zayachkowski [1] considered lattice paths from (0,0) to ( $m, n$ ) where the possible moves were of three types: (i) horizontal moves from $(\mathrm{x}, \mathrm{y})$ to $(\mathrm{x}+1, \mathrm{y})$; (ii) vertical moves from $(\mathrm{x}, \mathrm{y})$ to $(\mathrm{x}, \mathrm{y}+1)$, and (iii) diagonal moves from $(x, y)$ to $(x+1, y+1)$. A special case of some interest arises when $m=n$.

Consider now a much more restricted set of paths. Require: (a) that the path be symmetric about the line $x+y=n$, (b) that prior to arriving or touching the above line that one use only horizontal and diagonal moves (and symmetry after now requires that vertical and diagonal moves be used to arrive at ( $n, n$ )), and (c) that all of the paths be "below" or on the line $\mathrm{y}=\mathrm{x}$ (also required by the previous conditions).

For small values of $n$ one can enumerate the possible paths. Thus for $\mathrm{n}=1$, one need only consider the three points $(0,0),(1,0)$ and $(1,1)$, and there are two paths as pictured in Fig. 1. For $n=2$, there will be three paths. For $n=3$ there will be five paths. See Figs. 2 and 3, respectively.

This suggests that the collection of path numbers may be closely related to the Fibonacci sequence, with appropriate renumbering to bring the two sequences into step. Thus, letting $h(n)$ be the number of paths for $(0,0),(n, n)$ case, one has the tabulation

$$
\begin{array}{l|lll}
\mathrm{n} & 1 & 2 & 3 \\
\hline \mathrm{~h}(\mathrm{n}) & 2 & 3 & 5
\end{array}
$$

Also, beginning at $(0,0)$, there are only two initial moves, to $(1,0)$ and to $(1,1)$. Due to symmetry imposed by requirement (a) the last move in the path is also determined so that one has the choices schematically portrayed in Fig. 4. The two path schemes depicted in Fig. 4 are mutually exclusive andcollectively they exhaust all of the allowable paths satisfy:ng conditions (a), (b) and (c). Hence, $h(n)=h(n-1)+h(n-2)$ for $n=3,4, \cdots$. Thus the sequence of path numbers is a Fibonacei sequence with appropriate relabelling and identification of $h(0)$ and $h(-1)$ as unity.

It is also possible to "count" the paths so as to have

$$
h(n)=\sum_{k=0}\binom{n+1-k}{k}
$$

but the grouping of paths with summation index $k$ seems slightly artificial and lends little or nothing to the general theory.


Figure 1 (two cases)


Figure 2 (three cases)


Figure 3 (five cases, smaller scale)


There are $h(n-1)$ paths from $(1,0)$ to ( $n, n-1$ ) inside dashed triangle.

There are $h(n-2)$ paths from ( 1,1 ) to ( $n-1, n-1$ ) insided dashed triangle.

Figure 4

## REFERENCE

1. L. Moser and W. Zayachkowski, "Lattice Paths with Diagonal Steps," Scripta Mathematica, Vol. XXVI, No. 3, pp. 223-229.

## A NOTE ON FIBONACCI NUMBERS

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We shall employ the notation

$$
\begin{aligned}
& u_{0}=0, u_{1}=1, u_{n+1}=u_{n}+u_{n-1} \quad(n \geq 1) \\
& v_{0}=2, \quad v_{1}=1, \quad v_{n+1}=v_{n}+v_{n-1} \quad(n \geq 1)
\end{aligned}
$$

Thus

$$
\begin{equation*}
u_{\mathrm{n}}=\frac{\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}}{\alpha-\beta}, \mathrm{v}_{\mathrm{n}}=\alpha^{\mathrm{n}}+\beta^{\mathrm{n}} \tag{1}
\end{equation*}
$$

where

$$
\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}, \alpha+\beta=1, \alpha \beta=-1 .
$$

The first few values of $u_{n}, v_{n}$ follow.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{u}_{\mathrm{n}}$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| $\mathrm{v}_{\mathrm{n}}$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 |

It follows easily from the definition of (1) that

$$
\begin{align*}
& u_{n}=u_{n-k+1} u_{k}+u_{n-k} u_{k-1} \quad(n \geq k \geq 1)  \tag{2}\\
& v_{n}=u_{n-k+1} v_{k}+u_{n-k} v_{k-1} \quad(n \geq k \geq 1)
\end{align*}
$$

It is an immediate consequence of (1) that
(6)

$$
\begin{gather*}
\left.\mathrm{u}_{\mathrm{k}}\right|^{\mathrm{u}_{\mathrm{mk}}}  \tag{4}\\
\left.\mathrm{v}_{\mathrm{k}}\right|^{\mathrm{u}_{2 \mathrm{mk}}}  \tag{5}\\
\left.\mathrm{v}_{\mathrm{k}}\right|^{\mathrm{v}}(2 \mathrm{~m}-1) \mathrm{k}
\end{gather*}
$$

*Supported in part by National Science Foundation Grant G16485.
where $m$ and $k$ are arbitrary positive integers. It is perhaps not sofamiliar that, conversely,
$(4)^{\prime}$

$$
\mathrm{u}_{\mathrm{k}} \mid \mathrm{u}_{\mathrm{n}} \Longrightarrow \mathrm{n}=\mathrm{mk} \quad(\mathrm{k}>2)
$$

$(5)^{\prime}$
$u_{k} \mid u_{n} \Longrightarrow n=2 m k \quad(k>1)$,
$(6){ }^{\prime}$
$\mathrm{v}_{\mathrm{k}} \mid \mathrm{v}_{\mathrm{n}} \Longrightarrow \mathrm{n}=(2 \mathrm{~m}-1) \mathrm{k}(\mathrm{k}>1)$.
These results can be proved rapidly by means of (1) and some simple results about algebraic numbers. If we put

$$
\begin{equation*}
\mathrm{n}=\mathrm{mk}+\mathrm{r} \quad(0 \leq \mathrm{r}<\mathrm{k}) \tag{7}
\end{equation*}
$$

then

$$
\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}=\alpha^{\mathrm{r}}\left(\alpha^{\mathrm{mk}}-\beta^{\mathrm{mk}}\right)+\beta^{\mathrm{mk}}\left(\alpha^{\mathrm{r}}-\beta^{\mathrm{r}}\right)
$$

so that

$$
u_{\mathrm{n}}=\alpha^{\mathrm{r}} \mathrm{u}_{\mathrm{mk}}+\beta^{\mathrm{mk}} \mathrm{u}_{\mathrm{r}}
$$

If $u_{k} \mid u_{n}$ it therefore follows that $u_{k} \mid \beta^{m k} u_{r}$. Since $\beta$ is a unit of the field $R(\sqrt{ } 5), u_{k} \mid u_{r}$, which requires $r=0$. This proves (4)'.

Similarly if

$$
\mathrm{n}=2 \mathrm{mk}+\mathrm{r} \quad(0 \leq \mathrm{r}<2 \mathrm{k})
$$

then

$$
u_{\mathrm{n}}=\alpha^{\mathrm{r}} \mathrm{u}_{2 \mathrm{mk}}+\beta^{2 m k_{\mathrm{r}}}
$$

Hence if $v_{k} \mid u_{n}$ it follows that $v_{k} \mid u_{r}$. If then $r>0$ we must have $r>k$ and the identity

$$
(\alpha-\beta) \mathrm{u}_{\mathrm{r}}=\alpha^{\mathrm{r}-\mathrm{k}_{\mathrm{v}_{\mathrm{k}}}-\beta \mathrm{v}_{\mathrm{r}-\mathrm{k}}}
$$

gives $\mathrm{v}_{\mathrm{k}} \mid \mathrm{v}_{\mathrm{r}-\mathrm{k}}$, which is impossible. The proof of (6) ${ }^{\text {r }}$ is similar.

If we prefer, we can prove (4)', (5)', (6)' without reference to algebraic numbers. For example if $u_{k} \mid u_{n}$, then (2) implies $u_{k} \mid u_{n-k} u_{k-1}$. Since $u_{k}$ and $u_{k-1}$ are relatively prime we have $u_{k} \mid u_{n-k}$. Continuing in this way we get $u_{k} \mid u_{r}$, where $r$ is defined by (7). The proof is now completed as above. In the same way we can prove (5)' and (6)'.

In view of the relation

$$
\begin{equation*}
u_{2 n}=u_{n} v_{n} \tag{8}
\end{equation*}
$$

it is natural to ask for the general solution of the equation

$$
\begin{equation*}
u_{n}=u_{m} v_{k}(m>2, k>1) \tag{9}
\end{equation*}
$$

It is easily verified, using (1), that (9) can be replaced by

$$
\begin{equation*}
u_{\mathrm{n}}=u_{\mathrm{m}+\mathrm{k}}+(-1)^{\mathrm{k}} \mathrm{u}_{\mathrm{m}-\mathrm{k}} \quad(\mathrm{~m} \geq \mathrm{k}) \tag{10}
\end{equation*}
$$

or

$$
\begin{equation*}
u_{n}=u_{m+k}-(-1)^{k} u_{k-m} \quad(k>m) \tag{11}
\end{equation*}
$$

Now the equation

$$
\begin{equation*}
u_{r}=u_{s}+u_{t} \quad(s>t>1) \tag{12}
\end{equation*}
$$

is satisfied only when $r-1=s=t+1$. Indeed if $1<t<s-1$, then

$$
u_{s}+u_{t}<u_{s}+u_{s-1}=u_{s+1}
$$

so that (12) is impossible; if $t=s-1$, then clearly $r=s+1$. If $t=1$ in (12) we have the additional solution $r=4, \mathrm{~s}=3$.

Returning to (10) and (11) we first dispose of the case $m-k=1$. For k even (10) will be satisfied only if $\mathrm{m}+\mathrm{k}=3$, which implies $\mathrm{k}=1$; for k odd we get $\mathrm{n}=2, \mathrm{~m}+\mathrm{k}=3$ or $\mathrm{n}=3, \mathrm{~m}+\mathrm{k}=4$, which is impossible. Equation (11) with $k-m=1$ is disposed of in the same way.

We may therefore assume in (10) and (11) that $|\mathrm{m}-\mathrm{k}|>1$. Then if k is even, it is evident from the remark concerning (12) that (10) is impossible. If k is odd, we have

$$
u_{m+k}=u_{n}+u_{m-k}
$$

so that $k=1, m=n$. As for (11), if $m$ is odd we get

$$
u_{n}=u_{m+k}+u_{k-m}
$$

which is impossible. However, if $m$ is even, we get

$$
u_{m+k}=u_{n}+u_{k-m}
$$

so that $\mathrm{m}+\mathrm{k}=\mathrm{n}+1=\mathrm{k}-\mathrm{m}+2$; this requires $\mathrm{m}=1$, $\mathrm{k}=\mathrm{n}$.
This completes the proof of
Theorem 1. The equation

$$
\mathrm{u}_{\mathrm{n}}=\mathrm{u}_{\mathrm{m}} \mathrm{v}_{\mathrm{k}} \quad(\mathrm{~m}>2, \mathrm{k}>1)
$$

has only the solutions $n=2 \mathrm{~m}=2 \mathrm{k}$.
The last part of the above proof suggests consideration of the equation

$$
\begin{equation*}
u_{n}=v_{k} \quad(k>1) \tag{13}
\end{equation*}
$$

Since (13) is equivalent to

$$
u_{n}=u_{k+1}+u_{k-1}
$$

it follows at once that the only solution of (13) is $n=4, k=2$.
The equation

$$
\begin{equation*}
u_{n}=v_{m} v_{k} \quad(m \geq k>1) \tag{14}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
u_{\mathrm{n}}=\mathrm{v}_{\mathrm{m}+\mathrm{k}}+(-1)^{\mathrm{k}^{2}} \mathrm{v}_{\mathrm{m}-\mathrm{k}} \tag{15}
\end{equation*}
$$

If $k$ is even it is clear that $n>m+k$; indeed since $v_{m+k}=u_{m+k+1}+u_{m+k-1}$ we must have $n>m+k+1$. Then (15) implies

$$
u_{m+k+2} \leq u_{m+k+1}+u_{m+k-1}+v_{m-k}
$$

which simplifies to

$$
\begin{equation*}
u_{\mathrm{m}+\mathrm{k}-2} \leq \mathrm{v}_{\mathrm{m}-\mathrm{k}} \tag{16}
\end{equation*}
$$

If $\mathrm{m}=\mathrm{k}$, (16) holds only when $\mathrm{m}=2$; however this does notlead to a solution of (14). If $m>k$, (16) may be written as

$$
u_{m+k-2} \leq u_{m-k+1}+u_{m-k-1}<u_{m-k}
$$

which holds only when $\mathrm{m}=4, \mathrm{k}=2$.
If $k$ is odd, (15) becomes

$$
\begin{equation*}
u_{n}+v_{m-k}=v_{m+k} \tag{17}
\end{equation*}
$$

If $\mathrm{m}=\mathrm{k}$ this reduces to

$$
u_{n}+2=u_{2 k+1}+u_{2 k-1}
$$

which implies $2 \mathrm{k}-1=3, \mathrm{k}=2$. If $\mathrm{m}=\mathrm{k}+1$ (17) gives

$$
u_{n}+1=u_{2 k+2}+u_{2 k}
$$

which is clearly impossible. For $m>k+1$ we get

$$
u_{m+k+1}+u_{m+k-1} \geq u_{n}+2 u_{m-k}
$$

so that $n \leq m+k+1$. Since

$$
u_{m+k}+2 u_{m-k}<u_{m+k+1}+u_{m+k-1}
$$

we must have $\mathrm{n}=\mathrm{m}+\mathrm{k}+1$. Hence (17) becomes

$$
\mathrm{v}_{\mathrm{m}-\mathrm{k}}=\mathrm{u}_{\mathrm{m}+\mathrm{k}-1} ;
$$

[Feb.
as we have seen above, this implies

$$
m-k=2, m+k-1=4,
$$

so that we do not get a solution.
We may state
Theorem 2. The equation

$$
u_{\mathrm{n}}=\mathrm{v}_{\mathrm{m}} \mathrm{v}_{\mathrm{k}} \quad(\mathrm{~m} \geq \mathrm{k}>1)
$$

has the unique solution $\mathrm{n}=8, \mathrm{~m}=4, \mathrm{k}=2$.
It is clear from (4)' that the equation

$$
\begin{equation*}
u_{n}=c u_{k} \quad(k>2) \tag{18}
\end{equation*}
$$

where c is a fixed integer $>1$ is solvable only when $\mathrm{k} \mid \mathrm{n}$. Moreover the number of solutions is finite. Indeed (18) implies

$$
\mathrm{cu}_{\mathrm{k}} \geq \mathrm{u}_{2 \mathrm{k}} \geq \mathrm{u}_{\mathrm{k}} \mathrm{v}_{\mathrm{k}}, \mathrm{c} \geq \mathrm{v}_{\mathrm{k}}
$$

moreover if $\mathrm{n}=\mathrm{rk}$ then for fixed $\mathrm{k}, \mathrm{r}$ is uniquely determined by (18).
This observation suggests two questions: For what values of $c$ is (18) solvable and, secondly, can the number of solutions exceed one? In connection with the first question consider the equation

$$
\begin{equation*}
u_{n}=2 u_{k} \quad(k>2) \tag{19}
\end{equation*}
$$

Since for $n>3$

$$
2 u_{n-2}<u_{n}=2 u_{n-2}+u_{n-3}<2 u_{n-1}
$$

we get

$$
u_{n-2}<u_{k}<u_{n-1}
$$

which is clearly impossible. Similarly, since for $n>4$

$$
3 u_{n-3}<u_{n}=3 u_{n-3}+2 u_{n-4}<3 u_{n-2}
$$

it follows that the equation
(20)

$$
u_{n}=3 u_{k} \quad(k>2)
$$

has no solution.
Let us consider the equation

$$
\begin{equation*}
u_{n}=u_{m} u_{k} \quad(m \geq k>2) \tag{21}
\end{equation*}
$$

We take

$$
u_{n}=u_{n-m+1} u_{m}+u_{n-m} u_{m-1}
$$

so that

$$
u_{n-m+1} u_{m}<u_{n}<u_{n-m+2} u_{m},
$$

provided $\mathrm{n}>\mathrm{m}$. Then clearly (21) is impossible.
For the equation

$$
\begin{equation*}
\mathrm{v}_{\mathrm{n}}=\mathrm{u}_{\mathrm{m}} \mathrm{v}_{\mathrm{k}}(\mathrm{~m}>2, \mathrm{k}>1) \tag{22}
\end{equation*}
$$

we use

$$
v_{n}=u_{m} v_{n-m+1}+u_{m-1} v_{n-m}
$$

Then

$$
u_{m} v_{n-m+1}<v_{n}<u_{m} v_{n-m+2}
$$

so that (22) is impossible.
This proves
Theorem 3. Each of the equations (21), (22) possesses no solutions.

Consider next the equation

$$
\begin{equation*}
\mathrm{v}_{\mathrm{n}}=\mathrm{v}_{\mathrm{m}} \mathrm{v}_{\mathrm{k}} \quad(\mathrm{~m} \geq \mathrm{k}>1) \tag{23}
\end{equation*}
$$

This is equivalent to

$$
\begin{equation*}
\mathrm{v}_{\mathrm{n}}=\mathrm{v}_{\mathrm{m}+\mathrm{k}}+(-1)^{\mathrm{k}} \mathrm{v}_{\mathrm{m}-\mathrm{k}} \tag{24}
\end{equation*}
$$

For k even, (24) is obviously impossible. For k odd we may write

$$
\mathrm{v}_{\mathrm{m}+\mathrm{k}}=\mathrm{v}_{\mathrm{n}}+\mathrm{v}_{\mathrm{m}-\mathrm{k}}
$$

which requires $\mathrm{m}+\mathrm{k}=\mathrm{n}+1=\mathrm{m}-\mathrm{k}+2$, so that $\mathrm{k}=1$. This proves
Theorem 4. The equation (23) possesses no solutions.
The remaining type of equation is

$$
\begin{equation*}
v_{n}=u_{m} u_{k} \quad(m \geq k>2) \tag{25}
\end{equation*}
$$

This is equivalent to

$$
\begin{equation*}
5 \mathrm{v}_{\mathrm{n}}=\mathrm{v}_{\mathrm{m}+\mathrm{k}}+(-1)^{\mathrm{k}} \mathrm{v}_{\mathrm{m}-\mathrm{k}} \tag{26}
\end{equation*}
$$

Clearly $\mathrm{n}<\mathrm{m}+\mathrm{k}$. Then since

$$
\mathrm{v}_{\mathrm{m}+\mathrm{k}}=5 \mathrm{v}_{\mathrm{m}+\mathrm{k}-4}+3 \mathrm{v}_{\mathrm{m}+\mathrm{k}-5}
$$

(26) implies

$$
\begin{equation*}
5 \mathrm{v}_{\mathrm{n}}=5 \mathrm{v}_{\mathrm{m}+\mathrm{k}-4}+3 \mathrm{v}_{\mathrm{m}+\mathrm{k}-5}+(-1)^{\mathrm{k}} \mathrm{v}_{\mathrm{m}-\mathrm{k}} \tag{27}
\end{equation*}
$$

Consequently $n \geq m+k-3$, while the right member of (27) is less than

$$
5 \mathrm{v}_{\mathrm{m}+\mathrm{k}-4}+4 \mathrm{v}_{\mathrm{m}+\mathrm{k}-5}<5 \mathrm{v}_{\mathrm{m}+\mathrm{k}-3} .
$$

This evidently proves

Theorem 5. The equation (25) possesses no solution.
Next we discuss the equations
(29)

$$
\begin{align*}
& u_{m}^{2}+u_{n}^{2}=u_{k}^{2} \quad(0<m \leq n)  \tag{28}\\
& v_{m}^{2}+v_{n}^{2}=v_{k}^{2} \quad(0 \leq m \leq n)
\end{align*}
$$

We shall require the following
Lemma. The following inequalities hold.

$$
\begin{align*}
& \frac{u_{n+1}}{u_{n}} \geq \frac{3}{2} \quad(n \geq 2)  \tag{30}\\
& \frac{v_{n+1}}{v_{n}} \geq \frac{3}{2} \quad(n \geq 3) \tag{31}
\end{align*}
$$

Proof. Since $u_{n} \leq 2 u_{n-1}$ for $n \geq 2$, we have

$$
\frac{u_{n+1}}{u_{n}}=1+\frac{u_{n-1}}{u_{n}} \geq \frac{3}{2}
$$

The proof of (31) is exactly the same.
Returning to (28) it is evident that

$$
u_{\mathrm{n}}^{2}<\mathrm{u}_{\mathrm{k}}^{2}<2 \mathrm{u}_{\mathrm{n}}^{2}
$$

so that

$$
u_{n}<u_{k}<u_{n} \sqrt{2}
$$

Then $\mathrm{k}>\mathrm{n}$ and by the lemma

$$
u_{\mathrm{k}} \geq \mathrm{u}_{\mathrm{n}+1} \geq \frac{3}{2} \mathrm{u}_{\mathrm{n}}
$$

Since $\sqrt{2}<3 / 2$, we have a contradiction. The same argument applies to (29). The lemma requires that $\mathrm{n} \geq 2$ or 3 but there is of course no difficulty about
the excluded values. This proves
Theorem 6. Each of the equations (28), (29), possesses no solutions. More generally, each of the equations

$$
\begin{aligned}
& u_{m}^{r}+u_{n}^{r}=u_{k}^{r} \quad(0<m \leq n) \\
& v_{m}^{r}+v_{n}^{r}=v_{k}^{r} \quad(0 \leq m \leq n)
\end{aligned}
$$

where $r \geq 2$ has no solutions.
Remark. The impossibility of (29) can also be inferred rapidly from the easily proved fact that no $\mathrm{v}_{\mathrm{n}}$ is divisible by 5 . Indeed since

$$
\alpha^{5} \equiv \beta^{5} \equiv \frac{1}{2} \quad(\bmod \sqrt{5})
$$

it follows that

$$
\mathrm{v}_{\mathrm{n}+5}=\alpha^{\mathrm{n}+5}+\beta^{\mathrm{n}+5} \equiv \frac{1}{2}\left(\alpha^{\mathrm{n}}+\beta^{\mathrm{n}}\right)=\frac{1}{2} \mathrm{v}_{\mathrm{n}} \quad(\bmod \sqrt{5})
$$

so that $\mathrm{v}_{\mathrm{n}+5} \equiv \mathrm{v}_{\mathrm{m}}(\bmod \sqrt{5})$. Moreover none of $\mathrm{v}_{0}, \mathrm{v}_{1}, \mathrm{v}_{3}, \mathrm{v}_{4}$ is divisible by 5 . The mixed equation

$$
\begin{equation*}
\mathrm{v}_{\mathrm{m}}^{2}+\mathrm{v}_{\mathrm{n}}^{2}=\mathrm{u}_{\mathrm{k}}^{2} \quad(0 \leq \mathrm{m} \leq \mathrm{n}) \tag{32}
\end{equation*}
$$

has the obvious solution $m=2, \mathrm{n}=3, \mathrm{k}=5$; the equation

$$
\begin{equation*}
u_{\mathrm{m}}^{2}+\mathrm{v}_{\mathrm{n}}^{2}=\mathrm{u}_{\mathrm{k}}^{2} \quad(\mathrm{~m}>0) \tag{33}
\end{equation*}
$$

has the solution $\mathrm{m}=4, \mathrm{n}=3, \mathrm{k}=5$.
Clearly (32) implies

$$
\mathrm{v}_{\mathrm{n}}<\mathrm{u}_{\mathrm{k}}<\mathrm{v}_{\mathrm{n}} \sqrt{2}
$$

This inequality is not sufficiently sharp to show that (32) has no solutions although it does suffice for the equation

$$
\mathrm{v}_{\mathrm{m}}^{\mathrm{r}}+\mathrm{v}_{\mathrm{n}}^{\mathrm{r}}=\mathrm{u}_{\mathrm{k}}^{\mathrm{r}}
$$

with r sufficiently large.
However (32) is equivalent to

$$
\begin{equation*}
\mathrm{v}_{2 \mathrm{~m}}+(-1)^{\mathrm{m}} 2+\mathrm{v}_{2 \mathrm{n}}+(-1)^{\mathrm{n}} 2=\frac{1}{5}\left\{\mathrm{v}_{2 \mathrm{k}}-(-1)^{\mathrm{k}} 2\right\} \tag{34}
\end{equation*}
$$

If $m+n \equiv 1(\bmod 2)$, this reduces to

$$
\mathrm{v}_{2 \mathrm{~m}}=\mathrm{v}_{2 \mathrm{n}}=\frac{1}{5}\left\{\mathrm{v}_{2 \mathrm{k}}-(-1)^{\mathrm{k}} 2\right\}
$$

There is no loss in generality in assuming $k \geq 5$. Then since

$$
\mathrm{v}_{2 \mathrm{k}}=5 \mathrm{v}_{2 \mathrm{k}-4}+3 \mathrm{v}_{2 \mathrm{k}-5}
$$

we get

$$
\mathrm{v}_{2 \mathrm{~m}}+\mathrm{v}_{2 \mathrm{n}}=\mathrm{v}_{2 \mathrm{k}-4}+\frac{1}{5}\left\{3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2\right\} .
$$

Since $m<n$ and

$$
\frac{1}{5}\left\{3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2\right\}<\mathrm{v}_{2 \mathrm{k}-5}
$$

we must have $2 \mathrm{n}=2 \mathrm{k}-4$ and

$$
5 \mathrm{v}_{2 \mathrm{~m}}=3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2=6 \mathrm{v}_{2 \mathrm{k}-7}+3 \mathrm{v}_{2 \mathrm{k}-8}-(-1)^{\mathrm{k}} 2
$$

It is therefore necessary that $2 \mathrm{~m}=2 \mathrm{k}-6$ and we get

$$
5 \mathrm{v}_{2 \mathrm{~m}}=6 \mathrm{v}_{2 \mathrm{~m}-1}+3 \mathrm{v}_{2 \mathrm{~m}-2}+(-1)^{\mathrm{m}_{2}}
$$

which simplifies to

$$
\mathrm{v}_{2 \mathrm{~m}-4}=(-1)^{\mathrm{m}} 2
$$

Hence $\mathrm{m}=2, \mathrm{k}=5, \mathrm{n}=3$ (a solution of (22)).
Next if $m \equiv \mathrm{n}(\bmod 2)$, (34) reduces to

$$
\mathrm{v}_{2 \mathrm{~m}}+\mathrm{v}_{2 \mathrm{n}}+(-1)^{\mathrm{n}} 4=\frac{1}{5}\left\{\mathrm{v}_{2 \mathrm{k}}-(-1)^{\mathrm{k}} 2\right\}
$$

and as above we get

$$
\begin{equation*}
\mathrm{v}_{2 \mathrm{~m}}+\mathrm{v}_{2 \mathrm{n}}+(-1)^{\mathrm{m}} 4=\mathrm{v}_{2 \mathrm{k}-4}+\frac{1}{5}\left\{3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2\right\} \tag{35}
\end{equation*}
$$

It is necessary that $2 \mathrm{n}=2 \mathrm{k}-4$, so that (35) reduces to

$$
\begin{equation*}
5 \mathrm{v}_{2 \mathrm{~m}}+(-1)^{\mathrm{m}} 20=3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2 \tag{36}
\end{equation*}
$$

Clearly $2 \mathrm{~m} \leq 2 \mathrm{k}-6$. If $2 \mathrm{~m}<2 \mathrm{k}-6$ we get

$$
3 \mathrm{v}_{2 \mathrm{k}-5}-(-1)^{\mathrm{k}} 2 \leq 5 \mathrm{v}_{2 \mathrm{k}-7}+(-1)^{\mathrm{m}} 20
$$

or

$$
\mathrm{v}_{2 \mathrm{k}-6}+2 \mathrm{v}_{2 \mathrm{k}-8} \leq{(-1)^{\mathrm{m}} 20+(-1)^{\mathrm{k}} 2, ~, ~}^{\mathrm{m}}
$$

which is not possible. Thus $2 \mathrm{~m}=2 \mathrm{k}-6$ and (36) becomes

$$
5 \mathrm{v}_{2 \mathrm{~m}}+(-1)^{\mathrm{m}} 20=3 \mathrm{v}_{2 \mathrm{~m}+1}+(-1)^{\mathrm{m}} 2
$$

This reduces to

$$
\mathrm{v}_{2 \mathrm{~m}-4}=(-1)^{\mathrm{m}-1} 18
$$

which is satisfied by $m=5$. Then $k=8, n=6$ but this does not lead to $a$ solution of (32).

This completes the proof of
Theorem 7. The equation

$$
\mathrm{v}_{\mathrm{m}}^{2}+\mathrm{v}_{\mathrm{n}}^{2}=\mathrm{u}_{\mathrm{k}}^{2} \quad(0 \leq \mathrm{m} \leq \mathrm{n})
$$

has the unique solution $\mathrm{m}=2, \mathrm{n}=3, \mathrm{k}=5$.
The equation

$$
\begin{equation*}
u_{\mathrm{m}}^{2}+\mathrm{v}_{\mathrm{n}}^{2}=\mathrm{u}_{\mathrm{k}}^{2} \quad(\mathrm{~m}>0) \tag{37}
\end{equation*}
$$

can be treated in a less tedious manner. Suppose first that $\mathrm{v}_{\mathrm{n}} \leqslant \mathrm{u}_{\mathrm{m}}$. Then (37) implies

$$
u_{\mathrm{m}}^{2}<\mathrm{u}_{\mathrm{k}}^{2}<2 \mathrm{u}_{\mathrm{m}}^{2}
$$

and as we have seen above this is impossible. Next let $u_{m}<v_{n}$. If $k>n+2$ then

$$
\begin{array}{r}
u_{k}^{2} \geq u_{n+3}^{2}=\left(2 u_{n+1}+u_{n}\right)^{2}=2\left(u_{n+1}+u_{n-1}\right)^{2}+2 u_{n+1}^{2}+2 u_{n+1} u_{n-2} \\
+u_{n}^{2}-u_{n-1}^{2}>2 v_{n}^{2}
\end{array}
$$

so that (37) is certainly not satisfied. Since $k>n+1$ it follows that $k=n+2$. Thus (37) becomes

$$
\begin{equation*}
u_{m}^{2}=u_{n+2}^{2}-v_{n}^{2}=3\left(u_{n}^{2}-u_{n-1}^{2}\right) \tag{38}
\end{equation*}
$$

as is easily verified. If $m>n+2$ then

$$
u_{m}^{2} \geq u_{n+2}^{2}=\left(2 u_{n}+u_{n-1}\right)^{2}>3\left(u_{n}^{2}-u_{n-1}^{2}\right)
$$

contradicting (38). Since for $n>3$

$$
3\left(u_{n}^{2}-u_{n-1}^{2}\right)-u_{n}^{2}=2 u_{n}^{2}-3 u_{n-1}^{2}>\frac{9}{2} u_{n-1}^{2}-3 u_{n-1}^{2}>0
$$

it follows that $\mathrm{m}>\mathrm{n}$. Thus $\mathrm{m}=\mathrm{n}+1$ and (38) becomes

$$
u_{n+1}^{2}=3\left(u_{n}^{2}-u_{n-1}^{2}\right)
$$

This implies $u_{n}+u_{n-1}=3, n=3$, which leads to the solution $n=3, m=4$, $\mathrm{k}=5$ of (37). As for the excluded values $\mathrm{n}=1,2$ it is obvious that they do not furnish a solution. This proves

Theorem 8. The equation

$$
u_{\mathrm{m}}^{2}+\mathrm{v}_{\mathrm{n}}^{2}=\mathrm{u}_{\mathrm{k}}^{2} \quad(\mathrm{~m}>0)
$$

has the unique solution $\mathrm{m}=4, \mathrm{n}=3, \mathrm{k}=5$ 。


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## SOME MEW FIBORACCI IDENTITIES

## VERNER E. HOGGATT, JR. and MARJORIE BICKNELL <br> San Jose State College, San Jose, California

In this paper, some new Fibonacci and Lucas identities are generated by matrix methods.

The matrix

$$
R=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{array}\right)
$$

satisfies the matrix equation

$$
R^{3}-2 R^{2}-2 R+I=0
$$

Multiplying by $R^{n}$ yields

$$
\begin{equation*}
R^{n+3}-2 R^{n+2}-2 R^{n+1}+R^{n}=0 \tag{1}
\end{equation*}
$$

It has been shown by Brennan [1] and appears in an earlier article [2] and as Elementary Problem B-16 in this quarterly that
(2) $\quad R^{n}=\left(\begin{array}{lcc}F_{n-1}^{2} & F_{n-1} F_{n} & F_{n}^{2} \\ 2 F_{n} F_{n-1} & F_{n+1}^{2}-F_{n-1} F_{n} & 2 F_{n} F_{n+1} \\ F_{n}^{2} & F_{n} F_{n+1} & F_{n+1}^{2}\end{array}\right)$
where $F_{n}$ is the $n^{\text {th }}$ Fibonacci number.
By the definition of matrix addition, corresponding elements of $R^{n+3}$, $R^{n+2}, R^{n+1}$ and $R^{n}$ must satisfy the recursion formula given in Equation (1). That is, for example,

$$
\mathrm{F}_{\mathrm{n}+3}^{2}-2 \mathrm{~F}_{\mathrm{n}+2}^{2}-2 \mathrm{~F}_{\mathrm{n}+1}^{2}+\mathrm{F}_{\mathrm{n}}^{2}=0
$$

and

$$
F_{n+3} F_{n+4}-2 F_{n+2} F_{n+3}-2 F_{n+1} F_{n+2}+F_{n} F_{n+1}=0
$$

Returning again to

$$
R^{3}-2 R^{2}-2 R+I=0,
$$

this equation can be rewritten as

$$
(R+I)^{3}=R^{3}+3 R^{2}+3 R+I=5 R(R+I)
$$

In general, by induction, it can be shown that

$$
\begin{equation*}
R^{p}(R+I)^{2 n+1}=5^{n} R^{n+p}(R+I) \tag{3}
\end{equation*}
$$

Equating the elements in the first row and third column of the above matrices, by means of Equation (2), we obtain

$$
\begin{equation*}
\sum_{i=0}^{2 n+1}\binom{2 n+1}{i} F_{i+p}^{2}=5^{n} F_{2(n+p)+1} \tag{4}
\end{equation*}
$$

It is not difficult to show that the Lucas numbers and members of the Fibonacci sequence have the relationship

$$
\mathrm{L}_{\mathrm{n}}^{2}-5 \mathrm{~F}_{\mathrm{n}}^{2}=(-1)^{\mathrm{n}} 4
$$

Since also

$$
\sum_{i=0}^{2 n+1}\binom{2 n+1}{i}(-1)^{i+p}=0
$$

we can derive the following sum of squares of Lucas numbers,

$$
\sum_{i=0}^{2 n+1}\binom{2 n+1}{i} L_{i+p}^{2}=5^{n+1} F_{2(n+p)+1}
$$

by substitution of the preceding two identities in Equation (4).
Upon multiplying Equation (3) on the right by $(R+I)$, we obtain

$$
\begin{equation*}
R^{p}(R+1)^{2 n+2}=5^{n} R^{n+p}(R+I)^{2} \tag{5}
\end{equation*}
$$

Then, using the expression for $R^{n}$ given in Equation (2) and the identity

$$
L_{k}=F_{k-1}+F_{k+1}
$$

we find that

$$
\begin{aligned}
\left(R^{n+1}+R^{n}\right)(R+I) & =\left(\begin{array}{ccc}
F_{2 n-1} & F_{2 n} & F_{2 n+1} \\
2 F_{2 n} & 2 F_{2 n+1} & 2 F_{2 n+2} \\
F_{2 n+1} & F_{2 n+2} & F_{2 n+3}
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 1 \\
0 & 2 & 2 \\
1 & 1 & 2
\end{array}\right) \\
& =\left(\begin{array}{ccc}
L_{2 n} & L_{2 n+1} & L_{2 n+2} \\
2 L_{2 n+1} & 2 L_{2 n+2} & 2 L_{2 n+3} \\
L_{2 n+2} & L_{2 n+3} & L_{2 n+4}
\end{array}\right)
\end{aligned}
$$

Finally, by equating the elements in the first row and third column of the matrices of Equation (5), we derive the two identities

$$
\sum_{i=0}^{2 n+2}\binom{2 n+2}{i} F_{i+p}^{2}=5^{n} L_{2(n+p)}
$$

and

$$
\sum_{i=0}^{2 n+2}\binom{2 n+2}{i} L_{i+p}^{2}=5^{n+1} L_{2(n+p)}
$$

By similar steps, by equating the elements appearing in the first row and second column of the matrices of Equations (3) and (5), we can write the additional identities,

$$
\sum_{i=0}^{2 n+1}\binom{2 n+1}{i} F_{i-1+p} F_{i+p}=5^{n} F_{2(n+p)}
$$

and

$$
\sum_{i=0}^{2 n+2}\binom{2 n+2}{i} F_{i-1+p} F_{i+p}=5^{n} L_{2(n+p)+1}
$$

## REFERENCES

1. From the unpublished notes of Terry Brennan.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, 1 (1963), April, pp. 47-52.
 TWO CORRECTIONS, VOL. 12 NO. 4

Page 73: In proposal B-26, the last equation should read

$$
B_{n}(x)=(x+1) B_{n-1}(x)+b_{n-1}(x)
$$

Page 74: In proposal B-27, the line for $\cos n \phi$ should read

$$
\cos n \phi=P_{n}(x)=\sum_{j=1}^{N} A_{j n} x^{n+2-2 j}(N=[(n+2) / 2]
$$

## PRIMES WHICHARE FACTORS OF ALL FIBOMACCISEQUENEES

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In studying the Fibonacci and Lucas sequences, one of the striking differences observed is the fact that ALL primes are factors of some positive term of the Fibonacci sequence while for the Lucas sequence many primes are excludedas factors. This difference raises some interesting questions regarding Fibonacci sequences in general.
(1) For a given Fibonacci sequence, how do we find which primes are factors and which are non-factors of its terms?
(2) Are there certain primes which are factors of all Fibonacci sequences? It is this latter question which will be given attention in this paper.

We are considering Fibonacci sequences in which there is a series of positive terms with successive terms relatively prime to each other. For any sequence we can find two consecutive terms $a \geq 0, b>0$, $a<b$, and take these as

$$
\mathrm{f}_{0}=\mathrm{a}, \quad \mathrm{f}_{1}=\mathrm{b},
$$

the defining relation for the sequence being

$$
f_{n+1}=f_{n}+f_{n-1}, \quad(n \geq 2)
$$

The particular sequence with $a=0$ and $b=1$ is known as the Fibonacci sequence and will have its terms designated by $F_{0}=0, F_{i}=1$, and so on.

Theorem: The only Fibonacci sequence having all primes as factors of some of its positive terms is the sequence with $\mathrm{a}=0$ and $\mathrm{b}=1$.

Proof: Since zero is an element of the sequence, the fact that all primes divide some positive terms of the sequence follows from the periodicity of the series relative to any given modulus.

To prove the converse, we note that each sequence is characterized by a quantity $D=b^{2}-a(a+b)$. For if $f_{n}$ is the $n^{\text {th }}$ term of the sequence,

$$
f_{n}=F_{n-1} b+F_{n-2}
$$

Then

$$
f_{n}^{2}-f_{n-1} f_{n+1}=\left(F_{n-1} b+F_{n-2} a\right)^{2}-\left(F_{n-2} b+F_{n-3} a\right)\left(F_{n} b+F_{n-1} a\right)
$$

which equals

$$
b^{2}\left(F_{n-1}^{2}-F_{n-2} F_{n}\right)+a b\left(F_{n-1} F_{n-2}-F_{n} F_{n-3}\right)+a^{2}\left(F_{n-2}^{2}-F_{n-3} F_{n-1}\right)
$$

or

$$
(-1)^{n}\left(b^{2}-a b-a^{2}\right)
$$

so that the values are successively +D and -D .
Now $D$ is equal to 1 in the case of the sequence $0,1,1,2,3, \cdots$ and in no other Fibonacci sequences. For if $a$ is kept fixed, the quantity $b(b-a)-$ $a^{2}$ increases with $b$ 。 Therefore its minimum value is found for $b=a+1$ 。 But then $b(b-a)-a^{2}$ becomes $a+1-a^{2}$. Now if $a=0,1$ or $2,\left|a+1-a^{2}\right|$ $=1$ and we have the Fibonacci sequence. If $a \geq 3,\left|a+1-a^{2}\right| \geq 5$.

Thus, apart from the Fibonacci sequence properly so-called, D > 1 . Furthermore, D must be odd if $a$ and $b$ are relatively prime. Hence if $f_{n} \equiv 0$ modulo some prime factor $p$ of $D$, we would then have

$$
f_{n-1} f_{n+1} \equiv 0 \quad(\bmod p)
$$

from the relation

$$
f_{n}^{2}-f_{n-1} f_{n+1}=(-1)^{n} D
$$

so that either $f_{n-1}$ or $f_{n+1} \equiv 0(\bmod p)$. Thus two successive terms of the series would be divisible by $p$ and consequently all terms would be divisible by $p$ which would lead to the conclusion that $p \mid(a, b)$, contrary to hypothesis.

Therefore, the only Fibonacci sequence having all primes as divisors one or the other of its terms is the one Fibonacci sequence with a zero element, namely: $0,1,1,2,3,5,8,13, \cdots$.

## CONGRUENTIAL FIBONACCI SEQUENCES

For a given prime modulus, such as eleven, there are eleven possible residues modulo 11: $0,1,2,3, \cdots, 10$. These may be arranged in ordered pairs repetitions being allowed in $11^{2}$ or 121 ways. Each such pair of residues can be made the starting point of a congruential Fibonacci sequence modulo 11, though of course various pairs will give rise to the same sequence. The one pair that needs to be excluded as trivial is $0-0$ since all the terms of the sequence would then be 0 and we have assumed throughout that no two successive terms have a common factor. Hence there are 120 possible sequence pairs. A complete listing of these congruential sequences modulo 11 is displayed below.

| (A) | 1 | 1 | 2 | 3 | 5 | 8 | 2 | 10 | 1 | 0 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (B) | 2 | 2 | 4 | 6 | 10 | 5 | 4 | 9 | 2 | 0 |
| (C) | 3 | 3 | 6 | 9 | 4 | 2 | 6 | 8 | 3 | 0 |
| (D) | 4 | 4 | 8 | 1 | 9 | 10 | 8 | 7 | 4 | 0 |
| (E) | 5 | 5 | 10 | 4 | 3 | 7 | 10 | 6 | 5 | 0 |
| (F) | 6 | 6 | 1 | 7 | 8 | 4 | 1 | 5 | 6 | 0 |
| (G) | 7 | 7 | 3 | 10 | 2 | 1 | 3 | 4 | 7 | 0 |
| (H) | 8 | 8 | 5 | 2 | 7 | 9 | 5 | 3 | 8 | 0 |
| (I) | 9 | 9 | 7 | 5 | 1 | 6 | 7 | 2 | 9 | 0 |
| (J) | 10 | 10 | 9 | 8 | 6 | 3 | 9 | 1 | 10 | 0 |
| (K) | 1 | 8 | 9 | 6 | 4 | 10 | 3 | 2 | 5 | 7 |
| (L) | 1 | 4 | 5 | 9 | 3 |  |  |  |  |  |
| (M) | 2 | 8 | 10 | 7 | 6 |  |  |  |  |  |

That all possible sequence-pairs are covered is shown in the following table where the number in the column at the left is the first term of the pair and the number in the row at the top is the second.

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | X | A | B | C | D | E | F | G | H | I | J |
| 1 | A | A | A | G | L | F | I | F | K | D | J |
| 2 | B | G | B | A | B | K | C | H | M | I | A |
| 3 | C | L | K | C | G | A | C | E | H | J | G |
| 4 | D | F | C | E | D | L | B | G | D | B | K |
| 5 | E | I | H | H | B | E | F | K | A | L | E |
| 6 | F | F | M | J | K | E | F | I | C | C | B |
| 7 | G | K | I | G | D | I | M | G | F | H | E |
| 8 | H | D | A | C | F | H | J | D | H | K | M |
| 9 | I | J | B | L | C | H | K | I | J | I | D |
| 10 | J | A | G | K | E | B | E | M | D | J | J |

We shall now consider various categories so as to cover all primes.
(A) $\mathrm{p}=2$

If either $a$ or $b$ is even, 2 is a factor of terms of the series; if both are odd, then $\mathrm{a}+\mathrm{b} \equiv 0(\bmod 2)$. Thus, 2 is a factor of all Fibonacci sequences.
(B) $\mathrm{p}=5$

Since 5 is not a factor of terms of the Lucas series, it cannot be a divisor of all Fibonacci sequences.
(C) $p=10 x \pm 1$

For $p$ of the form $10 x \pm 1$, the period $h(p)$ for any Fibonacci sequence is a divisor of $p-1$. Since there are $p^{2}-1$ sequence pairs of residues, the number of congruential sequences modulo $p$ would have to be

$$
\geq \frac{p^{2}-1}{p-1} \text { or } p+1
$$

But since there are only $p-1$ residues other than zero, sequence triples a-0-a can only be p-1 in number. Thus there cannot be one per sequence. Hence no prime of the form $10 \mathrm{x} \pm 1$ canbe a divisor of all Fibonacci sequences.
(D) $\mathrm{p}=10 \mathrm{x} \pm 3$

For $p$ of the form $10 x \pm 3$, the situation is as follows:
(1) The period is a factor of $2 p+2$.
(2) $2 p+2$ is divisible by 4 .
(3) The period contains all power of 2 found in $2 p+2$ 。
(4) The period is the same as the period of the Fibonacci sequence, $\mathrm{F}_{\mathrm{n}}$. [ 1 ]

Accordingly, if the period is less than $2 p+2$, it will also be less than p-1 and hence as before there will not be enough sequence pairs with zeros to cover all the sequences. Thus a necessary condition is that the period be $2 p+2$ if a prime is to be found as a factor of all Fibonacci sequences.

Two cases may be distinguished: (a) The case in which the period $h(p)$ $=2^{2}(2 r+1) ;$ (b) The case in which the period $h(p)=2^{m}(2 r+1), m \geq 3$.

1964] PRIMES WHICH ARE FACTORS OF ALL FIBONACCI SEQUENCES 37
(a) $h(p)=2^{2}(2 r+1)$

In this instance, if a sequence has a zero at $k$, it will also have zeros at $k / 4, k / 2$, and $3 k / 4$ or four zeros per sequence. The number of sequences is

$$
\frac{p^{2}-1}{2 p+2}=\frac{p-1}{2}
$$

To provide 4 zeros per sequence there would have to be

$$
\frac{4(p-1)}{2}=2(p-1) \text { zeros, }
$$

whereas there are only $p-1$.
(b) $k=2^{m}(2 r+1), m \geq 3$.

For a period of this form, if there is a zero at $k$, there will also be a zero at $k / 2$, but not at $k / 4$ or $3 k / 4$. The number of zeros required for ( $\mathrm{p}-1$ ) $/ 2$ sequences would be

$$
2(p-1) / 2=p-1,
$$

which is the exact number available. Thus the primes which divide all Fibonacci sequences are primes of the form $10 x \pm 3$ for which $2 p+2$ is equal to $2^{m}$ ( $2 r$ +1 ), $\mathrm{m} \geq 3$. In other words,

$$
\begin{aligned}
& p \equiv \pm 3(\bmod 10) \\
& p=2^{m-1}(2 r+1)-1 \text { or } p \equiv-1(\bmod 4)
\end{aligned}
$$

These congruences lead to the solution $p \equiv 3,7(\bmod 20)$ 。

## CONCLUSION

The primes which are factors of all Fibonacci sequences are:
(1) The prime 2
(2) Primes of the form $20 \mathrm{k}+3,7$, having a period $2 \mathrm{p}+2$.

38 PRIMES WHICH ARE FACTORS OF ALL FIBONACCI SEQUENCES
LIST OF PRIMES WHICH DIVIDE
ALL FIBONACCI SEQUENCES $(p<3000)$

| 2 | 383 | 787 | 1327 | 1783 | 2383 |
| :--- | ---: | ---: | ---: | :--- | :--- |
| 3 | 443 | 823 | 1367 | 1787 | 2423 |
| 7 | 463 | 827 | 1423 | 1847 | 2467 |
| 23 | 467 | 863 | 1447 | 1867 | 2503 |
| 43 | 487 | 883 | 1487 | 1907 | 2543 |
| 67 | 503 | 887 | 1543 | 1987 | 2647 |
| 83 | 523 | 907 | 1567 | 2003 | 2683 |
| 103 | 547 | 983 | 1583 | 2063 | 2707 |
| 127 | 587 | 1063 | 1607 | 2083 | 2767 |
| 163 | 607 | 1123 | 1627 | 2087 | 2803 |
| 167 | 643 | 1163 | 1663 | 2143 | 2843 |
| 223 | 647 | 1187 | 1667 | 2203 | 2887 |
| 227 | 683 | 1283 | 1723 | 2243 | 2903 |
| 283 | 727 | 1303 | 1747 | 2287 | 2927 |
| 367 |  |  |  | 2347 | 2963 |

## REFERENCE

1. D. D. Wall, "Fibonacci Series Modulo m," The American Mathematical Monthly, June-July, 1960, p. 529.


SOME CORRECTIONS TO VOLUME 1, NO. 4

Pages 45-46: $\quad \mathrm{D}=31$ should read (2,7), (3,8).
There was an omission in the Table of "D's" as follows:

| D |  | D |  |
| ---: | :--- | :---: | :--- |
| 305 | $(1,18)(16,33)$ | 361 | $(8,25)(9,26)$ |
| 311 | $(5,21)(11,27)$ | 379 | $(1,20)(18,37)$ |
| 319 | $(2,19)(7,23)(9,25)(15,32)$ | 389 | $(5,23)(13,31)$ |
| 331 | $(3,20)(14,31)$ | 395 | $(2,21)(17,36)$ |
| 341 | $(1,19)(4,21)(13,30)(17,35)$ |  |  |
| 349 | $(5,22)(12,29)$ |  |  |
| 355 | $(6,23)(11,28)$ |  |  |
| 359 | $(7,24)(10,27)$ |  |  |

## A FIBONACCI TEST FOR CONVERGENCE

## J. H. JORDAN

Washington State University, Pullman, Wash.

Let $g(n)$ be a non-increasing positive function defined on the positive integers. There are many available tests to determine whether or not $\sum_{n=1}^{\infty} g(n)$ converges. It is the purpose of this paper to exhibit a test for convergence which utilizes the Fibonacci numbers.

THE FIBONACCI TEST
$\sum_{n=1}^{\infty} g(n)$ converges if and only if $\sum_{n=1}^{\infty} f_{n} g\left(f_{n}\right)$ converges, where $f_{n}$ is the nth Fibonacci number.

Proof: Assume $\sum_{n=1}^{\infty} g(n)$ converges.

$$
\begin{array}{ll}
\frac{1}{2} g(1) & =\frac{1}{2} f_{1} g\left(f_{2}\right) \\
\frac{1}{2}\{g(1)+g(2)\} & \geq \frac{1}{2} f_{2} g\left(f_{3}\right) \\
\frac{1}{2}\{g(2)+g(3)+g(4)+g(5)\} & \geq \frac{1}{2} f_{3} g\left(f_{4}\right) \\
\frac{1}{2}\{g(3)+g(4)+g(5)+g(6)+g(7)+g(8)\} & \geq \frac{1}{2} f_{4} g\left(f_{5}\right) \\
& \cdot \\
& \cdot \\
\frac{1}{2}\left\{g\left(f_{n-2}+1\right)+\cdots+\left(g\left(f_{n}\right)\right)\right\} &
\end{array}
$$

The sum of all terms on the left side of this array is $\sum_{n=1}^{\infty} g(n)$. The sum of all terms on the right side of the array is

$$
\frac{1}{2} \sum_{n=2}^{\infty} f_{n-1} g\left(f_{n}\right) \geq \frac{1}{4} \sum_{n=2}^{\infty} f_{n} g\left(f_{n}\right)
$$

Since the left side dominates the right side ${\underset{n}{\infty}}_{\infty}^{\infty} f_{n} g\left(f_{n}\right)$ converges.

$$
\text { Assume that } \sum_{n=1}^{\infty} g(n) \text { diverges. }
$$

$$
\begin{aligned}
\mathrm{f}_{2} \mathrm{~g}\left(\mathrm{f}_{1}\right) & =\mathrm{g}(1) \\
\mathrm{f}_{3} \mathrm{~g}\left(\mathrm{f}_{2}\right) & \geq \mathrm{g}(1)+\mathrm{g}(2) \\
\mathrm{f}_{4} \mathrm{~g}\left(\mathrm{f}_{3}\right) & \geq \mathrm{g}(2)+\mathrm{g}(3)+\mathrm{g}(4) \\
& \vdots \\
\mathrm{f}_{\mathrm{n}+1} \mathrm{~g}\left(\mathrm{f}_{\mathrm{n}}\right) & \geq \mathrm{g}\left(\mathrm{f}_{\mathrm{n}}\right)+\cdots+\mathrm{g}\left(\mathrm{f}_{\mathrm{n}+2}-1\right)
\end{aligned}
$$

The sum of all terms on the right side of this array is $2 \Sigma^{\infty} \mathrm{g}(\mathrm{n})$. The sum of all terms on the left side of this array is $\sum_{n=1}^{\infty} f_{n+1} g\left(f_{n}\right) \stackrel{n=1}{\leq} \sum_{n=1} f_{n} g\left(f_{n}\right)$. Since $\sum_{n=1} g(n)$ diverges so does $\sum_{n=1} f_{n} g\left(f_{n}\right)$.

## FURTHER REMARKS

It should be noticed that this result can be generalized to the following: Theorem: If $1 \leq c_{k}=\left[H \cdot a^{k}\right]^{*}$ where $a>1$ and $H$ is a fixed positive constant then $\sum_{n=1}^{\infty} g(n)$ converges if and only if $\sum_{k=1}^{\infty} c_{k} g\left(c_{k}\right)$ converges.

The proof is quite similar.
It seems unlikely that this Fibonacci test for convergence will ever become widely used. To designate some of its useful qualities the following results are exhibited.

Corollary 1: $\sum_{n=1}^{\infty} n^{a}$ converges or diverges as $\sum_{n=2}^{\infty} n^{-1} \ln ^{a} n$ does, $a<0$.
Proof: Let $r=1$ be the golden ratio, i.e., $r=(1+\sqrt{5}) / 2$ and notice that $f_{n-1}=\left[r^{n} / \sqrt{5}\right]^{*}$. Now $\sum_{n=1}^{\infty} n^{a}$ converges or diverges as $\sum_{n=1}^{\infty} n^{a} \ln ^{a}{ }_{\infty}$ does. But $n^{a} \ln ^{a} r=\left(\ln r^{n}\right)$ a $\begin{aligned} & n=1 \\ & \infty\end{aligned}$ converges or diverges as $\sum_{n=1}^{\infty}\left(\ln \sqrt{5} f_{n}\right)^{a}$ does. Now

$$
\sum_{n=1}^{\infty}\left(\ln \sqrt{5} f_{n}\right)^{a}=\sum_{n=1}^{\infty} f_{n} \cdot f_{n}^{-1}\left(\ln \sqrt{5} f_{n}\right)^{a}
$$

and appealing to the Fibonacci test this converges or diverges as
*[x] is greatest integer in $x$.

1964]

$$
\sum_{n=1}^{\infty} n^{-1}(\ln \sqrt{5} n)^{a}
$$

does, which is essentially the desired result.
This corollary tells us for example that since the harmonic series diverges then $\sum_{n=2}^{\infty} n^{-1} \ln n$ diverges and since $\sum_{n=1}^{\infty} n^{-1-\epsilon}$ converges then $\sum^{\infty} n^{-1}(\ln n)^{-1-\epsilon}$ converges. $\mathrm{n}=2$

Corollary 2: $\sum_{n=2}^{\infty} n^{-1} \mathrm{ln}^{\text {a }} n$ converges or diverges as

$$
\sum_{n=3}^{\infty}(n \ln n)^{-1}(\ln \ln n)^{a}
$$

does.
The proof is quite similar.
Corollary $\mathbf{j}$ : $\mathbf{j}=3,4,5, \ldots$ are likewise provable.
The Fibonacci test is an effective substitute for the integral test in each of these corollaries.

Consider the following example that is handled easily by the Fibonacci test. Let $\mathrm{g}(\mathrm{n})=\mathrm{f}_{\mathrm{m}}^{\mathrm{a}}$ for $\mathrm{f}_{\mathrm{m}-1}<\mathrm{n} \leq \mathrm{f}_{\mathrm{m}}$, where $\mathrm{a}<0$. Thus

$$
\sum_{n=1}^{\infty} g(n)=1^{a}+2^{a}+3^{a}+5^{a}+5^{a}+8^{a}+8^{a}+8^{a}+13^{a}+\cdots
$$

Applying the Fibonacci test one obtains

$$
\sum_{n=1}^{\infty} f_{n} g\left(f_{n}\right)=\sum_{n=1}^{\infty} f_{n}^{a+1}=\sum_{n=1}^{\infty}\left[\frac{x^{n+1}}{\sqrt{5}}\right]^{(a+1)}
$$

which converges or diverges as $\sum_{n=1}^{\infty}\left(r^{n+1}\right)^{a+1}=\sum_{n=1}^{\infty}\left(r^{a+1}\right)^{n+1}$, but this is a geometric progression and converges provided $r^{a+1}<1$ or when $a<-1$.

## EXPLORING FIBONACCI RESIDUES

BROTHER U. ALFRED
St. Mary's College, California

Mathematicians have developed a simple and powerful method of relating numbers to each other from the standpoint of division. They say that two numbers $a$ and $b$ are congruent to each other modulo $m$ when the difference $a$ - $b$ is divisible by $m$. It can be readily shown that this is equivalent to the statement that on dividing $a$ and $b$ by $m$, they will both give the same "remainder" - which the mathematician calls the least positive residue.

What remainder is obtained when we divided one Fibonacci number by another? The remainder could of course be zero, but as is well known, zero is one of the Fibonacci numbers. Do we always obtain a Fibonacci number for the least positive residue? If not, will we obtain a Fibonacci number if we allow the use of either the least positive or the least negative residue?

This is the general line of investigation. In cases in which Fibonacci numbers are the result, the investigator should seek to find some type of regularity and thus formulate a mathematical theorem. Once this has been done a proof is a desideratum.

Just to start the process let us find a fewleast positive and least negative residues. Using $\mathrm{F}_{20}(6765)$ as the dividend and various Fibonacci numbers as divisors, we find that
$6765 \equiv 33(\bmod 34) \quad 6765 \equiv 8 \quad(\bmod 233) \quad 6765 \equiv 843(\bmod 987)$
$6765 \equiv 0(\bmod 55) \quad 6765 \equiv 356(\bmod 377) \quad 6765 \equiv 377(\bmod 1597)$
$6765 \equiv 141(\bmod 144) \quad 6765 \equiv 55(\bmod 610)$
It seems that in some cases the least positive residue is a Fibonacci number whereas in others apparently it is not. In the latter, we go to the least negative residue, we apparently get Fibonacci numbers in these cases as well. Thus $6765 \equiv-21(\bmod 377) ; 6765 \equiv-144(\bmod 987)$.

Those making discoveries in regard to this problem are encouraged to send their findings to Brother U. Alfred, St. Mary's College, Calif., by July 31st so that they may be published in the October, 1964, issue of the Quarterly.

## 

# TRANSCENDENTAL NUMBERS bASED ON THE FIBOMACCI SEQUENCE 

```
            DONALD KNUTH
California Institute of Technology, Pasadena, California
```

A well-known theorem due to Liouville states that if $\xi$ is an irrational algebraic number of degree $n$, then the equation

$$
\begin{equation*}
\left|\xi-\frac{p}{q}\right|<\frac{1}{q^{n+\epsilon}} \tag{1}
\end{equation*}
$$

has only finitely many solutions for integers $p, q$, given any $\epsilon>0$. Therefore, an irrational number $\xi$, for which

$$
\begin{equation*}
\left|\xi-\frac{\mathrm{p}}{\mathrm{q}}\right|<\frac{1}{\mathrm{q}^{\mathrm{t}}} \tag{2}
\end{equation*}
$$

has solutions for arbitrarily large $t$, must be transcendental. Numbers of this type have been called Liouville numbers.

In 1955, Roth published his celebrated improvement of Liouville's theorem, replacing " n " by " 2 " in equation (1). Let us call an irrational number $\xi$, for which
(3)

$$
\left|\xi-\frac{p}{q}\right|<\frac{1}{q^{2+\epsilon}}
$$

his infinitely many solutions for some $\epsilon>0$, a Roth number. Roth numbers are also transcendental, and they include many more numbers than the Liouville numbers.

Let b be an integer greater than 1 . Then we define $\xi_{\mathrm{b}}$ to be the continued fraction

$$
\begin{equation*}
\xi_{\mathrm{b}}=\frac{1}{\mathrm{~b}^{\mathrm{F}_{0}}}+\frac{1}{\mathrm{~b}_{1}}+\frac{1}{\mathrm{~b}_{2}}+\cdots \tag{4}
\end{equation*}
$$

Theorem: $\xi_{\mathrm{b}}$ is a Roth number, hence $\xi_{\mathrm{b}}$ is transcendental.

Proof: From the elementary theory of continued fractions, it is well known that if $p_{n} / q_{n}$ is the $n^{\text {th }}$ convergent to $\xi_{b}$, then

$$
\begin{equation*}
\left|\xi_{b}-\frac{p_{n}}{q_{n}}\right|<1 / q_{n} q_{n+1} \tag{5}
\end{equation*}
$$

In this case, $q_{0}=1, q_{1}=b^{F_{0}}$, and $q_{n+1}=b^{F_{n}} q_{n}+q_{n-1}$. We can therefore easily verify by induction that

$$
\begin{equation*}
q_{n}=\frac{b^{F_{n+1}}-1}{b-1} \tag{6}
\end{equation*}
$$

 1) $\left.q_{n}\right]^{\phi}$ where $\phi=.618 \cdots$ is the golden ratio. Therefore for large $n$ we have approximately

$$
\left|\xi_{b}-\frac{p_{n}}{q_{n}}\right|<\frac{1}{q_{n}^{2+\phi}}
$$

and this completes the proof of the theorem.
Remarks. It can be easily shown that the set of Roth numbers is of measure zero, but it is uncountable. For example, the number $\sum_{n=1}^{\infty} b^{-c_{n}}$, where $\left\{c_{n}\right\}$ is a strictly increasing sequence of positive integers, is a Roth number if $\lim _{\mathrm{n} \rightarrow \infty} \sup \left(c_{\mathrm{n}+1} / \mathrm{c}_{\mathrm{n}}\right)>2$, and it is a Liouville number if this $\lim$ sup is infinite. In terms of continued fractions, the number

$$
\frac{1}{a_{1}}+\frac{1}{a_{2}}+\frac{1}{a_{3}}+\ldots
$$

is a Roth number if and only if

$$
\limsup _{n \rightarrow \infty}\left(\log a_{n} / \log q_{n}\right)>0
$$

where $q_{n}$ are the denominators as in the proof of the above theorem.
The rapid convergence of (4) allows us to evaluate $\xi_{b}$ easily with high precision, e.g.,

$$
\begin{aligned}
& \xi_{2}=.709803444861291 \ldots \\
& \xi_{3}=.768597562593155 \ldots
\end{aligned}
$$

Reference to this article on p .52 .


# STREMGTHEMED INEQUALITES FOR FIBONACOI AMI LUCAS MUMBERS 

$$
\begin{gathered}
\text { DOV JARDEN } \\
\text { Jerusalem, Israel }
\end{gathered}
$$

In a paper entitled "On the Greatest Primitive Divisors of Fibonacci and Lucas Numbers" (henceforth referred to as P), published in The Fibonacci Quarterly, Volume 1, Number 3, pages 15-20, I have proved for the Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ the following inequalities:

$$
\begin{equation*}
\mathrm{F}_{\mathrm{n}^{\mathrm{x}+1}}>\mathrm{F}_{\mathrm{n}^{\mathrm{X}}}^{2} \quad(\mathrm{n} \geq 2, \mathrm{x} \geq 1) \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
\mathrm{F}_{5^{\mathrm{X}}+1}>5 \mathrm{~F}_{5}^{2} \mathrm{X} \quad(\mathrm{x} \geq 1) \tag{5}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{L}_{\mathrm{n}^{\mathrm{X}+1}}>\mathrm{L}_{\mathrm{n}^{\mathrm{X}}}^{2}(\mathrm{n}>2, \mathrm{x} \geq 1) \tag{*}
\end{equation*}
$$

The aim of this note is to strengthen (4), (5), and (4*) as follows:

$$
\begin{align*}
& F_{n^{x+1}}>n_{n^{x}}^{n} \quad(n \geq 2, x \geq 1)  \tag{A}\\
& L_{n^{x+1}}>L_{n^{x}}^{n-1} \quad(n \geq 2, x \geq 1)
\end{align*}
$$

For the proof of (A), (B) we shall use the well-known formulae
(C) $\quad \mathrm{F}_{\mathrm{n}}=\frac{1}{\sqrt{ } 5}\left\{\alpha^{\mathrm{n}}-(-1)^{\mathrm{n}} \alpha^{-\mathrm{n}_{7}}\right\}$
(D) $\quad L_{n}=\alpha^{n}+(-1)^{n} \alpha^{-n}$

$$
\alpha=\frac{1+\sqrt{5}}{2}>\frac{3}{2}, \alpha^{\mathrm{n}+2}=\alpha^{\mathrm{n}+1}+\alpha^{\mathrm{n}}
$$

as well as the following inequalities:
(E)

$$
\begin{array}{r}
\frac{1}{\sqrt{5}} \cdot \frac{6}{7} \cdot 2^{n}>n \quad(n \geq 3) \\
\frac{1}{2} \alpha^{n}>F_{n}(n \geq 2) \\
\frac{6}{5} \alpha^{n}>L_{n} \quad(n \geq 2)  \tag{G}\\
45
\end{array}
$$

Proof of (E) (by induction). (E) is equivalent to
( $\mathrm{E}^{\prime}$ )

$$
6 \cdot 2^{n}>7 \mathrm{n} \sqrt{5}
$$

$\left(\mathrm{E}^{\prime}\right)$ is valid for $\mathrm{n}=3$. If ( $\mathrm{E}^{\prime}$ ) is valid for n , then:
$6 \cdot 2^{n+1}=6 \cdot 2^{n}+6 \cdot 2^{n}>7 n \sqrt{5}+7 n \sqrt{5}>7 n \sqrt{5}+7 \sqrt{5}=7 \sqrt{5}(n+1)$.

Proof of (F), (G) (by induction on $n$ and $n+1$ ).
(F) is valid for $n=2,3$, since

$$
\begin{aligned}
& \alpha^{2}=1+\alpha=1+\frac{1+\sqrt{5}}{2}=\frac{3+\sqrt{5}}{2}>\frac{3+\sqrt{4}}{2}>2=2 \mathrm{~F}_{2}, \\
& \alpha^{3}=\alpha+\alpha^{2}=\frac{1+\sqrt{5}}{2}+\frac{3+\sqrt{ } 5}{2}=2+\sqrt{ } 5>2+\sqrt{ } 4=4=2 \mathrm{~F}_{3} .
\end{aligned}
$$

If

$$
\begin{aligned}
\alpha^{n} & >2 F_{n} \\
\alpha^{n+1} & >2 F_{n+1}
\end{aligned}
$$

then also:

$$
\alpha^{\mathrm{n}+2}=\alpha^{\mathrm{n}}+\alpha^{\mathrm{n}+1}>2\left(\mathrm{~F}_{\mathrm{n}}+\mathrm{F}_{\mathrm{n}+1}\right)=2 \mathrm{~F}_{\mathrm{n}+2}
$$

(G) may be proven analogously, noting that, by arguments employed in the proof of (F), (G) is valid for $n=2,3$, since

$$
\begin{aligned}
& \frac{6}{5} \alpha^{2}>\frac{6}{5} \cdot \frac{3+\sqrt{4}}{2}=3=\mathrm{L}_{2} \\
& \frac{6}{5} \alpha^{3}>\frac{6}{5} \cdot 4>4=\mathrm{L}_{3}
\end{aligned}
$$

Proof of (A).
(1) For $\mathrm{n}=2$ we have, by (C):

$$
\begin{aligned}
& \mathrm{F}_{2^{\mathrm{X}+1}}=\frac{1}{\sqrt{5}}\left\{\alpha^{2^{\mathrm{X}+1}}-\alpha^{-2^{\mathrm{x}+1}}\right\}=\frac{\sqrt{ } 5}{5}\left\{\alpha^{2^{\mathrm{x}+1}}-\alpha^{-2^{\mathrm{x}+1}}\right\}>\frac{2}{5}\left\{\alpha^{2^{x+1}}-\alpha^{-2^{\mathrm{x}+1}}\right\}> \\
& \frac{2}{5}\left\{\alpha^{2^{x+1}}-\left(2-\alpha^{-2^{x+1}}\right)\right\}=\frac{2}{5}\left\{\alpha^{2^{x+1}}-2+\alpha^{-2^{x+1}}\right\}=2\left\{\frac{1}{\sqrt{5}}\left(\alpha^{2^{x}}-\alpha^{-2^{x}}\right)\right\}^{2}=2 F_{2^{x}}^{2}
\end{aligned}
$$

(2) For $n \geq 3$ we have, by arguments employed in the proof of (F),

$$
\alpha^{\mathrm{n}+1} \geq \alpha^{3^{2}}=\left(\alpha^{3}\right)^{3}>4^{3}>7,
$$

i.e.,

$$
\frac{\alpha^{\mathrm{x}+1}}{7}>1
$$

Hence, by (C), (E):

$$
\begin{array}{r}
\mathrm{F}_{\mathrm{n}^{\mathrm{X}+1}}=\frac{1}{\sqrt{5}}\left\{\alpha^{\mathrm{n}^{\mathrm{x}+1}}-(-1)^{\mathrm{n}} \alpha^{-\mathrm{n}^{\mathrm{x}+1}}\right\}>\frac{1}{\sqrt{5}}\left\{\alpha^{\mathrm{n}^{\mathrm{x}+1}}-\frac{\alpha^{\mathrm{n}^{\mathrm{x}+1}}}{7}\right\}= \\
\frac{1}{\sqrt{5}} \cdot \frac{6}{7} \alpha^{\mathrm{n}^{\mathrm{x}+1}}=\frac{1}{\sqrt{5}} \cdot \frac{6}{7} \cdot 2^{\mathrm{n}}\left(\frac{\alpha^{\mathrm{n}}}{2}\right)^{\mathrm{n}}>\mathrm{nF}_{\mathrm{n}^{\mathrm{x}}}^{\mathrm{n}}
\end{array}
$$

Proof of (B). For $n \geq 2$ we have $\left(n^{x}-1\right) /(n-1)=n^{x-1}+n^{x-2}+\cdots$ $+1 \geq n^{x-1} \geq(n-1)^{x-1}$, whence: $n^{x}-1 \geq(n-1)^{x}$. Hence, by (D), (G), and noting that (by arguments employed in the proof of (A), part (2)) $-\alpha^{-\mathrm{n}^{\mathrm{x}+1}}>-\frac{1}{7}$ we have:

$$
\begin{aligned}
& \mathrm{L}_{\mathrm{n}^{\mathrm{X}+1}}=\alpha^{\mathrm{n}^{\mathrm{x}+1}}+(-1)^{\mathrm{n}} \alpha^{-\mathrm{n}^{\mathrm{x}+1}} \geqslant \alpha^{\mathrm{n}^{\mathrm{x}+1}}-\alpha^{-\mathrm{n}^{\mathrm{x}+1}}>\alpha^{\mathrm{n}^{\mathrm{x}+1}}-\frac{1}{7}> \\
& \alpha^{\mathrm{n}^{\mathrm{X}+1}}-\frac{1}{3} \alpha^{\mathrm{n}^{\mathrm{x}+1}}=\frac{2}{3}\left(\alpha^{\mathrm{nx}}\right)^{\mathrm{n}}>\frac{1}{\alpha}\left(\alpha^{\mathrm{X}}\right)^{\mathrm{n}}=\alpha^{\mathrm{n}^{\mathrm{X}}-1}\left(\alpha^{\mathrm{n}}\right)^{\mathrm{n}-1}> \\
& \alpha^{(\mathrm{n}-1) \mathrm{x}}\left(\alpha^{\mathrm{n}^{\mathrm{x}}}\right)^{\mathrm{n}-1} \geq \alpha^{\mathrm{n}-1}\left(\alpha^{\mathrm{n}^{\mathrm{x}}}\right)^{\mathrm{n}-1}>\left(\frac{6}{5}\right)^{\mathrm{n}-1}\left(\alpha^{\mathrm{n}^{\mathrm{x}}}\right)^{\mathrm{n}-1}= \\
& \left(\frac{6}{5} \alpha^{n^{x}}\right)^{\mathrm{n}-1}>\mathrm{L}_{\mathrm{n}^{\mathrm{x}}}^{\mathrm{n}-1} .
\end{aligned}
$$

Remark. In proving the inequalities (A), (B), I was assisted by my son, Moshe, who also noted that (B) cannot be strengthened, analogously to (A), to: $\mathrm{L}_{\mathrm{n}_{\mathrm{X}+1}}>\mathrm{L}_{\mathrm{n}^{\mathrm{x}}}^{\mathrm{n}}$. Indeed, for $\mathrm{n}=4$, $\mathrm{x}=1$, we have: $\mathrm{L}_{4^{2}}=2207<2401=7^{4}$ $=L_{4}^{4}$.

It may also easily be seen, by (C), (D), that
(H)

$$
\lim _{x \rightarrow \infty} \frac{F_{n x+1}}{\mathrm{nF}_{\mathrm{n}^{x}}^{\mathrm{n}}}=\infty
$$

$$
\lim _{x \rightarrow \infty} \frac{L_{n^{x+1}}}{L_{n^{x}}^{n-1}}=\infty
$$

which shows that, for any given $n \geq 2$, there exists an $X$ such that, for any $\mathrm{x}>\mathrm{X}, \mathrm{F}_{\mathrm{n}^{\mathrm{X}+1}}>\mathrm{nF}_{\mathrm{n}^{\mathrm{X}}}^{\mathrm{n}}, \mathrm{L}_{\mathrm{n}^{\mathrm{X}+1}}>\mathrm{L}_{\mathrm{n}^{\mathrm{X}}}^{\mathrm{n}-1}$.

By means of (A), (B), and employing the same reasoning as in the proof of (3), $\left(3^{*}\right)$ in $P$, we have, for the greatest primitive divisors $F_{n}^{\prime}$ of $F_{n}$ and $L_{n}^{\prime}$ of $L_{n}$, the following generalized inequalities:

$$
\text { SOME CORRECTIONS TO VOLUME } 1, \text { NO. } 3
$$

Page 16: In Equation ( $4^{*}$ ), replace $n \geq 2$ by $n>2$.
The last line should read:
$\ldots$ for any positive integer $n \geq 2, n>2$, respectively.

Page 17: On line 6, add $>$ to read:

$$
\alpha=\frac{1+\sqrt{5}}{2}>\frac{1+\sqrt{4}}{2}=\frac{3}{2}
$$

Line 8, Equation (7), should be corrected to read:

$$
\alpha>\frac{3}{2}
$$

On Line 11, add = to read:

$$
\beta=\frac{1-\sqrt{5}}{2}<\frac{1-\sqrt{4}}{2}=-\frac{1}{2}
$$

$$
\begin{align*}
& \mathrm{F}_{\mathrm{p}^{\mathrm{x}+1}}>\mathrm{pF}_{\mathrm{p}^{\mathrm{x}}}^{\mathrm{p-1}} \quad(\mathrm{p}-\text { a prime } \neq 5, \mathrm{p} \geq 2, \mathrm{x} \geq 1)  \tag{J}\\
& \mathrm{F}_{5^{\mathrm{X}+1}}>\mathrm{F}_{5^{4}}^{4} \quad(\mathrm{x} \geq 1)  \tag{K}\\
& L_{p^{\prime}+1}>{\underset{p}{x}}_{p-2}^{x} \quad(p-\text { a prime, } \quad p \geq 2, x \geq 1) .
\end{align*}
$$

## ADVANGED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-29 Proposed by Brotber U. Alfred, St. Mary's College, California.
Find the value of a satisfying the relation

$$
n^{n}+(n+a)^{n}=(n+2 a)^{n}
$$

in the limit as $n$ approaches infinity.

H-30 Proposed by J. A. H. Hunter, Toronto, Ontario, Canada
Find all non-zero integral solutions to the two Diophantine equations,
(b)

$$
\begin{align*}
& X^{2}+X Y+X-Y^{2}=0  \tag{a}\\
& X^{2}-X Y-X-Y^{2}=0
\end{align*}
$$

H-31 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California

Prove the following:
Theorem: Let $a, b, c, d$ be integers satisfying $a>0, d>0$ and $a d-$ $\mathrm{bc}=1$, and let the roots of $\lambda^{2}-\lambda-1=0$ be the fixed points of

$$
w=\frac{a z+b}{c z+d} .
$$

Then it is necessary and sufficient for all integral $n \neq 0$, that $a=F_{2 n+1}$, $\mathrm{b}=\mathrm{c}=\mathrm{F}_{2 \mathrm{n}}$, and $\mathrm{d}=\mathrm{F}_{2 \mathrm{n}-1}$, where $\mathrm{F}_{\mathrm{n}}$ is the $\mathrm{n}^{\text {th }}$ Fibonacci number. $\quad\left(\mathrm{F}_{1}\right.$ $=1, \mathrm{~F}_{2}=1$ and $\mathrm{F}_{\mathrm{n}+2}=\mathrm{F}_{\mathrm{n}+1}+\mathrm{F}_{\mathrm{n}}$ for all integral $\mathrm{n}_{\mathrm{o}}$ )

H-32 Proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, N. J.
Prove the following:
Given a positive integer $n$, if there exist $m$ line segments $L_{i}$ having lengths $a_{i}, 1 \leq a_{i} \leq n$, for all $1 \leq i \leq m$, such that no three $L_{i}$ can be used to form a non-degenerate triangle then $F_{m} \leq n$, where $F_{m}$ is the $m^{\text {th }}$ Fibonacci number.

H-33 Proposed by Malcolm Tallman, Brooklyn, N. Y.
If a Lucas number is a prime number and its subscriptis composite, then the subscript must be of the form $2^{\mathrm{m}}, \mathrm{m} \geq 2$.

## SOLUTIONS

## A TOUGH PROBLEM

H-1 Proposed by H. W. Gould, West Virginia University, Morgantown, w. Va.
Find a formulafor the $\mathrm{n}^{\text {th }}$ non-Fibonacci number, thatis, for the sequence $4,6,7,9,10,11,12,14,15,16,17,18,19,20,22,23, \cdots$ 。 (See paper by L. Moser and J. Lambek, American Mathematical Monthly, vol. 61 (1954), pp. 454-458.)

A paper by the proposer will soon appear in the Fibonacci Quarterly, which will discuss this problem.

## A WORLD-FAMOUS PROBLEM

H-2 Proposed by L. Moser and L. Carlitz, University of Alberta, Edmonton, Alberta,
and Duke University, Durham, N. C.

Resolve the conjecture: There are no Fibonacci numbers which are integral squares except 0,1 , and 144 .

See "Lucas Squares," by Brother U. Alfred in this issue. A discussion by J. H. E. Cohn on Fibonacci Squares will be in the next issue.

## AN UNSOLVED PROBLEM

H-15 Proposed by Malcolm H. Tallman, Brooklyn, N. Y.
Do there exist integers $N_{1}, N_{2}$, and $N_{3}$ for which the following expressions cannot equal other Fibonacci numbers?
(iii)

$$
\begin{array}{ll}
F_{n}^{3}-F_{n}^{2} F_{m}-F_{m}^{3} & m, n \geq N_{1} \\
F_{n}^{3}+F_{n}^{2} F_{m}+F_{n} F_{m}^{2}  \tag{ii}\\
F_{n}^{2}-3 F_{m}^{3} & m, n \geq N_{2}
\end{array}
$$

No discussion of any kind has been received on this problem.
AN INSPIRING PROBLEM

H-17 Proposed by Brother U. Alfred, St. Mary's College, California
Sum

$$
\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{k}^{3} \mathrm{~F}_{\mathrm{k}}
$$

(Editorial Comment: There will be three different approaches to the solution of the general case of the above problem which will appear soon in the Fibonacci Quarterly.)
Solution by Joseph Erbacher and John Allen Fuchs, University of Santa Clara, Calif.

Let $L(E)=\left(E^{2}-E-1\right)^{4}=\sum_{i=0}^{8} a_{i} E^{i}$ where $E$ is the linear operator such that $E^{i} F_{k}=F_{k+i}$. Then $L(E) k^{3} F_{k}=0$. (This follows from a result of James A. Jeske, "Linear Recurrence Relations - Part I, " Fibonacci Quarterly, April, 1963, p. 72 , Equation (4.8).) Let $\mathrm{S}=\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{k}^{3} \mathrm{~F}_{\mathrm{k}}$. Since $\sum_{\mathrm{i}=0}^{8} \mathrm{a}_{\mathrm{i}}=1, \quad \mathrm{~S}=$ $\sum_{i=0}^{8} a_{i} S=\sum_{i=0}^{8}\left\{a_{i} \sum_{k=i+1}^{n+i} k^{3} F_{k}+a_{i} \sum_{j=1}^{i}\left[j^{3} F_{j}-(n+j)^{3} F_{n+j}\right]\right\}=R+T$, where $R$ is the first double summation and $T$ is the second double summation. Reversing the order of summation in $R$, we have $R=\sum_{8}^{n} \sum_{i=1}^{8} a_{i}(i+k)^{3} F_{i+k}$. Since $\sum_{i=0}^{8} a_{i}(i+k)^{3} F_{i+k}=\sum_{i=0}^{8} a_{i} E^{i} k^{3} F_{k}=L(E) k^{3} F_{k}=0$, it follows that $R=0$ and $\mathrm{S}=\mathrm{T}$. Using the relation $\mathrm{F}_{\mathrm{n}+2}=\mathrm{F}_{\mathrm{n}+1}+\mathrm{F}_{\mathrm{n}}$ in T , one can transform the solution into the form $S=50+\left(n^{3}+6 n=12\right) F_{n+2}+\left(-3 n^{2}+9 n=19\right) F_{n+3}$.

Generalizing on the above technique one sees that $\sum_{k=1}^{n} p_{k} F_{k}=u(n) F_{n+2}$
$+v(n) F_{n+3}+A_{p}$, where $u$ and $v$ are polynomials in $n$ of degree $p$ and $A_{p}$ is a constant independent of $n$. It can be shown that the coefficients of $u$ and $v$ may be found by solving the $2 p+2$ equations obtained by letting $n$ take on any $2 p+2$ consecutive values.

Also solved by Zvi Dresner and Marjorie Bicknell

## A CLASSICAL SOLUTION

H-16 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.
Define the ordinary Hermite polynomials by $H_{n}=(-1)^{n} e^{x^{2}} D^{n}\left(e^{-x^{2}}\right)$.
(i)

$$
\sum_{n=0}^{\infty} H_{n}(x / 2) \frac{x^{n}}{n!}=1
$$

Show that:
(ii)

$$
\sum_{n=0}^{\infty} H_{n}(x / 2) \frac{x^{n}}{n!} F_{n}=0,
$$

(iii)

$$
\sum_{n=0}^{\infty} H_{n}(x / 2) \frac{x^{n}}{n!} L_{n}=2 e^{-x^{2}}
$$

where $F_{n}$ and $L_{n}$ are the $n^{\text {th }}$ Fibonacci and $n^{\text {th }}$ Lucas numbers, respectively.

We recall that $\sum_{n=0}^{\infty} H_{n}(t) \frac{x^{n}}{n!}=e^{2 t x-x^{2}}$. For $t=\frac{x}{2}$ this reduces to $\sum_{n=0}^{\infty} H_{n}\left(\frac{x}{2}\right) \frac{x^{n}}{n!}=1$.
$\mathrm{n}=0 \quad$ Put $\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}$. Then $(\alpha-\beta) \sum_{\mathrm{n}=0}^{\infty} H_{\mathrm{n}}\left(\frac{\mathrm{x}}{2}\right) \frac{\mathrm{x}^{\mathrm{n}}}{\mathrm{n}!} \mathrm{F}_{\mathrm{n}}=\mathrm{e}^{\left(\alpha-\alpha^{2}\right) \mathrm{x}^{2}}-$ $e^{\left(\beta=\beta^{2}\right) x^{2}=0}$ since $\alpha-\alpha^{2}=\beta-\beta^{2}=-1$.

Similarly,

$$
\sum_{n=0}^{\infty} H_{n}\left(\frac{x}{2}\right) \frac{x^{n}}{n!} L_{n}=e^{\left(\alpha-\alpha^{2}\right) x^{2}}+e^{\left(\beta-\beta^{2}\right) x^{2}}=2 e^{-x^{2}}
$$

See also the solution in the last issue by Zvi Dresner.

## 

Reference continued from page 44 .

1. K. F. Roth, "Rational Approximations to Algebraic Numbers," Mathematika 2 (1955) pp. 1-20, p. 168.

## Edited by DMITRI THORO <br> San Jose State College, San Jose, California <br> THE EUCLIDEAN ALGORITHM I

## 1. INTRODUCTION

Consider the problem of finding the greatest common divisor of 34 and 144. The factorizations $34=2 \cdot 17,144=2^{4} \cdot 3^{2}$ make this a trivial problem. However, this approach is discouraging when one deals with, say, "long" Fibonacci numbers. Fortunately in Prop. 2 of Book VII, Euclid gave an elegant algorithm. As usual, we shall designate the g.c.d. of $s$ and $t$ by $(s, t)$.

## 2. THE ALGORITHM

The algorithm may be defined by the following flow chart. $\mathrm{A} \rightarrow \mathrm{B}$ means $A$ replaces $B$, i.e., set $B=$ the current value of $A$.
$M$ represents the remainder in the division of $K$ by $L$.


Flow Chart for Computing the G. C. D of Positive Integers I and J

For $I=13$ and $J=8$, the successive values of $K, L$, and $M$ are:

| $\underline{\mathrm{K}}$ | $\underline{\mathrm{L}}$ | $\underline{\mathrm{M}}$ |
| ---: | ---: | ---: |
| 13 | 8 | 5 |
| 8 | 5 | 3 |
| 5 | 3 | 2 |
| 3 | 2 | 1 |
| 2 | 1 | 0 |

The last value of $L$ is the desired g.c.d. In the following computation, (10946, $2584)=$ the last non-zero remainder.


In this discussion we shall emphasize computational considerations. There are, however, numerous "theoretical" applications of the Euclidean Algorithm. As LeVeque [1] expresses it, "... it is the cornerstone of multiplicative number theory." For a related theorem see Glenn Michael [2], this issue.

## 3. A FORTRAN PROGRAIM

With an occasional glance at our flow chart, it is easy to decipher the following Fortran program. (Fortran is a problem-oriented language commonly used in conversing with electronic digital computers.)
(i) $\mathrm{A}=\mathrm{B}$ means A is replaced by B .
(ii) The READ and PUNCH statements refer to card input/output.
(iii) In this context, $N=K / L$ is an instruction to set $N$ equal to [K/L], i. e., the greatest integer not exceeding $K / L$ (sometimes called an integer or fixed point quotient). Thus if $\mathrm{K}=13$ and $\mathrm{L}=3, \mathrm{~N}$ will equal 4 .
(iv) The symbol for multiplication is an asterisk.
(v) A "conditional transfer" is achieved by using an IF statement: if M $\leq 0$, go to statement 3 for the next instruction; otherwise go to statement 4 .
(vi) The FORMAT and END statements are technical requirements (which may be ignored).

$$
\begin{gathered}
\text { READ 10, I, J } \\
10 \text { FORMAT (3I5) } \\
\mathrm{K}=\mathrm{I} \\
\mathrm{~L}=\mathrm{J} \\
2 \mathrm{~N}=\mathrm{K} / \mathrm{L} \\
\mathrm{M}=\mathrm{K}-\mathrm{L} * \mathrm{~N} \\
\text { IF (M) 3, 3, } 4 \\
4 \mathrm{~K}=\mathrm{L} \\
\mathrm{~L}=\mathrm{M} \\
\text { GO TO } 2 \\
3 \text { PUNCH 10, I, J, L } \\
\text { END }
\end{gathered}
$$

## 4. Length of the Algorithm

A natural question arises: What is the "length" of this algorithm? I. e., if $s$ and $t$ are given, how many divisions are required to compute ( $s, t$ ) via the Euclidean Algorithm?

Let us designate this number by $N(s, t)$. For convenience we may assume $\mathrm{s} \geq \mathrm{t}$. Thus for $\mathrm{n}>1, \mathrm{~N}(\mathrm{n}+1, \mathrm{n})=2$; the first division yields the remainder 1 , whereas the second results in a zero remainder - signifying termination of the algorithm. (As a byproduct we see that any two consecutive integers are relatively prime.)

In Part II we shall see how Fibonacci numbers $\left(F_{1}=F_{2}=1, F_{i+1}=F_{i}\right.$ $+\mathrm{F}_{\mathrm{i}-1}$ ) were used by Lamé to establish a remarkable result. Additional properties of $N(s, t)$ are suggested in the following exercises.

## 5. EXERCISES

E1. Note that the Euclidean Algorithm applied to the positive integers s and $t$ may be described by the equations

$$
\begin{array}{rlrl}
s & =t q_{1}+r_{1}, & & 0 \leq r_{1}<s \\
t & =r_{1} q_{2}+r_{2}, & 0 \leq r_{2}<r_{1} \\
r_{1} & =r_{2} q_{3}+r_{3}, & 0 \leq r_{3}<r_{2} \\
& \cdot & & \cdot \\
\cdot & & \cdot \\
r_{n-2} & =r_{n-1} q_{n}+r_{n}, & & 0 \leq r_{n}<r_{n-1} \\
r_{n-1} & =r_{n} q_{n+1}+0 & &
\end{array}
$$

Explain why we must reach a remainder ( $r_{n+1}$ ) which is zero in a finite number of steps. Hint: Look at the inequalities.

E2. In E 1 , show that $(\mathrm{s}, \mathrm{t})=\mathrm{r}_{\mathrm{n}}$ (the last non-zero remainder). Hint: Use repeated applications of Problem 1.3 [3].

E3. (a) Verify that $M=K-L * N$ is the remainder in the division represented by statement $2(\mathrm{~N}=\mathrm{K} / \mathrm{L})$ of the Fortran program.
(b) Can the Fortran program be used to compute (I, J) when $\mathrm{I} \leq \mathrm{J}$ ?

E4. Prove that if $n \geq 3$, then $N(n, 3)=1,2$, or 3 .
E5. Suppose that $n>5$ is chosen at random. Find the probability that $\mathrm{N}(\mathrm{n}, 5)>2$.

E6. Prove that for $n>3, N(n+3, n)=2,3$, or 4 .
E7. For what values of $n$ is $3 \leq N(2 n-5, n) \leq 6$ ?
E8. Express $\left(F_{n+1}, F_{n}\right)$ as a function of $n$.
E9. Investigate the following conjecture: If $a \leq F_{K}$, then $N(n, a) \leq K-1$. Can $n$ be any positive integer?

E10. Investigate the following conjecture: Let $\mathrm{F} \geq 2$ be any Fibonacci number. Then $\max _{\mathrm{n}} \mathrm{N}(\mathrm{n}, \mathrm{F})=1+\max _{\mathrm{n}}(\mathrm{n}, \mathrm{F}-1)$.

## REFERENCES

1. William J. LeVeque, Topics in Number Theory, Addison-Wesley, Reading, Mass., 1956, Vol. I, Chap. 2.
2. Glenn Michael, "A New Proof for an Old Property," Fibonacci Quarterly, Vol. 2, No. 1, Feb. 1964, p. 57.
3. D. E. Thoro, "Divisibility I," Fibonacci Quarterly, Vol. 1, No. 1, Feb. 1963, p. 51.

## A MEW PROOF FOR AN OLD PROPERTY*

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## 1. INTRODUCTION

The following theorem is certainly well known.
Theorem: If $m$ and $n$ are positive integers, then $\left(F_{m}, F_{n}\right)=F_{(m, n)}$. For example, proofs can be found in [1, pp. 30-32] and [2, pp. 148-149]. In this paper we give an alternative proof which is believed to be new.

## 2. PRELIMINARY RESULTS

In addition to elementary divisibility properties of integers, the proof depends on the following lemmas which may be found in [1, pp. 10, 30 and 29].

Lemma 1: For $n \geq 0$,

$$
F_{m+n}=F_{m-1} F_{n}+F_{m} F_{n+1}
$$

Lemma 2: For any $n,\left(F_{n}, F_{n+1}\right)=1$.
Lemma 3: For $n \neq 0, F_{n} \mid F_{m n}$.

## 3. PROOF OF THE THEOREM

For $m \geq 1, n \geq 1$, we show that $\left(F_{m}, F_{n}\right)=F_{(m, n)}$. Let

$$
\mathrm{c}=(\mathrm{m}, \mathrm{n})
$$

Then $\mathrm{c}|\mathrm{m}, \mathrm{c}| \mathrm{n}$ and, by Lemma 3, $\mathrm{F}_{\mathrm{c}} \mid \mathrm{F}_{\mathrm{m}}$ and $\mathrm{F}_{\mathrm{c}} \mid \mathrm{F}_{\mathrm{n}}$. Thus, $\mathrm{F}_{\mathrm{c}}$ is a common divisor of $F_{m}$ and $F_{n}$ and it follows that $F_{c} \mid d$ where $d=\left(F_{m}\right.$, $F_{n}$ ). Also, since $c=(m, n)$, there exist integers $a$ and $b$ such that

$$
c=a m+b n
$$

[^0] Long for presentation to the Washington State University Mathematics Club.

Since $c \leq m$ and $m$ and $n$ are positive, either $a \leq 0$ or $b \leq 0$ 。 Suppose $\mathrm{a} \leq 0$ and set $\mathrm{k}=-\mathrm{a}$ 。 Then

$$
\mathrm{bn}=\mathrm{c}+\mathrm{km}
$$

and, by Lemma 1 ,

$$
\begin{equation*}
F_{b n}=F_{c+k m}=F_{c-1} F_{k m}+F_{c} F_{k m+1} \tag{1}
\end{equation*}
$$

Now $d\left|F_{n}, d\right| F_{m}$ and, by Lemma 3, $F_{n} \mid F_{b n}$ and $F_{m} \mid F_{k m}$. Therefore, $d\left|F_{b n}, d\right| F_{k m}$ and it follows from (1) that $d \mid F_{k m+1} F_{c}$. But ( $d, F_{k m+1}$ ) $=1$ since $\mathrm{d} \mid \mathrm{F}_{\mathrm{km}}$ and by Lemma 2, $\left(\mathrm{F}_{\mathrm{km}}, \mathrm{F}_{\mathrm{km}+1}\right)=1$. Therefore, $\mathrm{d} \mid \mathrm{F}_{\mathrm{c}}$. But, as seen above, $\mathrm{F}_{\mathrm{c}} \mid \mathrm{d}$. Hence, since both are positive,

$$
\left(\mathrm{F}_{\mathrm{m}}, \mathrm{~F}_{\mathrm{n}}\right)=\mathrm{d}=\mathrm{F}_{\mathrm{c}}=\mathrm{F}_{(\mathrm{m}, \mathrm{n})}
$$

and the proof is complete.

## REFERENCES

1. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York and London, 1961.
2. G. H. Hardy and E. M. Wright, The Theory of Numbers, Oxford University Press, London, 1954.

SOME CORRECTIONS TO VOLUME 1, NO. 3
Page 19: On the third line from the bottom, put in $>$ for $=$ to read

$$
\left(5+\beta^{\mathrm{n}^{\mathrm{x}+1}}\right)>
$$

Page 24: Line 5 should read, instead of " $\mathrm{a} \alpha+2 \beta=0$,"

$$
\mathrm{a} \alpha+2 \mathrm{~b}=0
$$

Page 30: On line 4, change " $e_{i}$ " to " $e_{1}$ ".
On line 18, change "unit" to "limit."

# A PRIMER FOR THE FIBONACCI NUMBERS - PART V 

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## CORRECTION

Read the last displayed equation, on page 67 of Part IV, as

$$
\operatorname{Tan}\left\{\operatorname{Tan}^{-1} \frac{\mathrm{~F}_{\mathrm{n}}}{\mathrm{~F}_{\mathrm{n}+1}}-\operatorname{Tan}^{-1} \frac{\sqrt{5}-1}{2}\right\}=(-1)^{\mathrm{n}+1}\left(\frac{\sqrt{5}-1}{2}\right)^{2 \mathrm{n}+1}
$$

## 1. INTRODUCTION

In Section 8 of Part IV, we discussed an alternating series. This time we shall lay down some brief foundations of sequences and infinite series. This leads to some very interesting results in this issue and to the broad topic of generating functions in the next issue and to continued fractions in the issue after that. Many Fibonacci numbers shall appear.

## 2. SEQUENCES

Definition: An ordered set of numbers $a_{1}, a_{2}, a_{3}, \cdots, a_{n}, \cdots$ is called an infinite sequence of numbers. If there are but a finite number of the $a^{\prime} s, a_{1}, a_{2}$, $\cdots, a_{n}$ then it is a finite sequence of numbers.

A sequence of real numbers $\left\{a_{n}\right\}_{n=1}^{\infty}$ is said to have a real number, a, as a limit (written $\lim _{n \rightarrow \infty} a_{n}=a$ ) if for every positive real number $\epsilon,\left|a_{n}-a\right|<$ $\epsilon$ for all but a finite number of the members of the sequence $\left\{a_{n}\right\}$. If the sequence $\left\{a_{n}\right\}$ has a limit, this limit is unique and the sequence is said to converge to this limit. If the sequence $\left\{a_{n}\right\}$ fails to approach a limit, then the sequence is said to diverge. We now give examples of each kind.

If $a_{n}=1,\left\{a_{n}\right\}=1,1,1, \cdots$ converges since $\lim _{n \rightarrow \infty} a_{n}=1$.
If $a_{n}=1 / n,\left\{a_{n}\right\}=1,1 / 2,1 / 3, \cdots, 1 / n, \cdots$ converges to zero.
If $a_{n}=(-1)^{n},\left\{a_{n}\right\}=1,-1,+1,-1,+1, \cdots$ diverges by oscillation. That
is, it does not approach any limit.
If $a_{n}=n,\left\{a_{n}\right\}=1,2,3, \cdots$ diverges to plus infinity.

Finally if $a_{n}=\frac{n}{n+1}$, then $\left\{a_{n}\right\}=\frac{1}{2}, \frac{2}{3}, \cdots$ converges to one.
Some limit theorems for sequences are the following:
If $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are two sequences of real numbers withlimits $a$ and $b$, respectively, then

$$
\begin{aligned}
& \lim _{n \rightarrow \infty}\left(a_{n}+b_{n}\right)=a+b \\
& \lim _{n \rightarrow \infty}\left(a_{n}-b_{n}\right)=a-b \\
& \lim _{n \rightarrow \infty}\left(c a_{n}\right)=c a, \text { any real } c \\
& \lim _{n \rightarrow \infty} a_{n} b_{n}=a b \\
& \lim _{n \rightarrow \infty}\left(a_{n} / b_{n}\right)=a / b_{2} b \neq 0
\end{aligned}
$$

## 3. BOUNDED MONOTONE SEQUENCES

The sequence $\left\{a_{n}\right\}$ is said to be bounded if there exists a positive number, $K$, such that $\left|a_{n}\right|<K$ for all $n \geq 1$. If $a_{n+1} \geq a_{n}$, for $n \geq 1$, the sequence $\left\{a_{n}\right\}$ is said to be a monotone increasing sequence; if $a_{n} \geq a_{n+1}$ for $\mathrm{n} \geq 1$, the sequence is monotone decreasing sequence. If a sequence is such that it is either monotone increasing or monotone decreasing it will be called a monotone sequence.

The following useful and important theorem is stated without proof:
Theorem 1: A bounded monotone sequence converges.
As an example, consider the sequence $\left\{(1+1 / n)^{n}\right\}$, this sequence is monotone increasing and bounded above by 3 . The limit of this sequence is well known. We will use Theorem 1 in the material to come.

## 4. ANOTHER IMPORTANT THEOREM

The following sufficient conditions for the convergence of an alternating series are given below.

Theorem 2: If, for the sequence $\left\{\mathrm{s}_{\mathrm{n}}\right\}$,

1. $\mathrm{S}_{1}>0$,
2. $\left(\mathrm{S}_{\mathrm{n}-1}-\mathrm{S}_{\mathrm{n}}\right)(-1)^{\mathrm{n}}>\left(\mathrm{S}_{\mathrm{n}}-\mathrm{S}_{\mathrm{n}+1}\right)(-1)^{\mathrm{n}+1}>0$, for $\mathrm{n} \geq 2$,
3. $\lim _{\mathrm{n} \rightarrow \infty}\left(\mathrm{S}_{\mathrm{n}}-\mathrm{S}_{\mathrm{n}+1}\right)=0$,
then the sequence $\left\{S_{n}\right\}$ converges to a limit, $S$, such that $0<S<S_{1}$.

## 5. AN EXAMPLE OF AN APPLICATION OF THEOREM 2

For the following example a limit is known to exist by the application of Theorem 2 of Section 4.

Let $S_{n}=F_{n} / F_{n+1}$, where $\left\{F_{n}\right\}$ is the Fibonacci sequence, then $S_{n-1}$ $-S_{n}=(-1)^{n} /\left(F_{n} F_{n+1}\right)$. By Theorem 2 above, $\lim _{n \rightarrow \infty} S_{n}$ exists.

To find the limit, consider

$$
\frac{\mathrm{F}_{\mathrm{n}+1}}{\mathrm{~F}_{\mathrm{n}}}=1+\frac{\mathrm{F}_{\mathrm{n}-1}}{\mathrm{~F}_{\mathrm{n}}}
$$

which in terms of $\left\{S_{n}\right\}$ is $1 / S_{n}=1+S_{n-1}$. Let the limit of $S_{n}$ as $n$ tends to infinity be $S$, then $\quad \lim _{n \rightarrow \infty} S_{n}=\lim _{n \rightarrow \infty} S_{n-1}=S>0$. Applying the limit theorems of Section 2, it follows that $S$ satisfies

$$
S=\frac{1}{1+S} \text { or } S^{2}+S-1=0
$$

Thus $\mathrm{S}>0$ is given by

$$
S=\frac{\sqrt{5}-1}{2}
$$

the positive root of the quadratic equation $S^{2}+S-1=0$.

## 6. INFINITE SERIES

If we add together the members of a sequence $\left\{a_{n}\right\}$, we get the infinite series $a_{1}+a_{2}+\cdots+a_{n}+\cdots$. We now get another sequence from this infinite series.

Define a sequence $\left\{S_{n}\right\}$ in the following way. Let $S_{1}=\underset{n}{a_{1}}=\underset{i=1}{\sum} a_{i}, S_{2}$ $=a_{1}+a_{2}=\sum_{i=1} a_{i} \cdots$ or in general $S_{n}=a_{1}+a_{2}+\cdots+a_{n}=\sum_{i=1} a_{i}$. This is called the sequence of partial sums of the infinite series. The infinite series can also be denoted by

$$
A=a_{1}+a_{2}+a_{3}+\cdots+a_{n}+\cdots=\sum_{i=1}^{\infty} a_{i} .
$$

If the sequence $\left\{S_{n}\right\}$ converges to a limit, $S$, then the infinite series, A, is said to converge and converge to the limit $\underline{S}$; otherwise series $A$ is said to diverge.

## 7. SPECLAL RESULTS CONCERNING SERIES

1. If an infinite series $A=a_{1}+a_{2}+\cdots+a_{n}+\cdots$ converges, then $\lim _{n \rightarrow \infty} a_{n}=0$. This is immediate since $a_{n}=S_{n}-S_{n-1}$.
2. From Section 3 above, an infinite series of positive terms converges if the partial sums are bounded above since the partial sums form a monotone increasing sequence.
3. For the alternating series

$$
\sum_{n=1}^{\infty}(-1)^{n+1} a_{n} \text { such that } a_{n}>0, n \geq 1 ; a_{n+1} \leq a_{n}, n \geq 1 ; \lim _{n \rightarrow \infty} a_{n}=0
$$

then by Section 4, above, the infinite series converges; in the theorem

$$
S_{n}=\sum_{j=1}^{n}(-1)^{j} a_{j}
$$

An example of an alternating series was seen in Part IV, Section 8, of this Primer.

## 8. FIBONACCI NUMBERS, LUCAS NUMBERS AND $\Pi$

It is well known and easily verified that

$$
\frac{\Pi}{4}=\operatorname{Tan}^{-1} \frac{1}{1}=\operatorname{Tan}^{-1} \frac{1}{2}+\operatorname{Tan}^{-1} \frac{1}{3}
$$

Also one can verify

$$
\frac{\Pi}{4}=\operatorname{Tan}^{-1} \frac{1}{1}=\operatorname{Tan}^{-1} \frac{1}{2}+\operatorname{Tan}^{-1} \frac{1}{5}+\operatorname{Tan}^{-1} \frac{1}{8}
$$

or

$$
\frac{\Pi}{4}=\operatorname{Tan}^{-1} \frac{1}{3}+\operatorname{Tan}^{-1} \frac{1}{5}+\operatorname{Tan}^{-1} \frac{1}{7}+\operatorname{Tan}^{-1} \frac{1}{8}
$$

We note Fibonacci and Lucas numbers here, surely. We shall here easily extend these results in several ways.

In this section we shall use several new identities which are left as exercises for the reader and will be marked with an asterisk.
*Lemma 1: $\mathrm{L}_{2 \mathrm{n}} \mathrm{L}_{2 \mathrm{n}+2}-1=5 \mathrm{~F}_{2 \mathrm{n}+1}^{2}$. This is really a special case of a generalization of B-22, p. 76, Oct., 1963, Fibonacci Quarterly.

Lemma 2:

$$
L_{\mathrm{n}}^{2}=\mathrm{L}_{2 \mathrm{n}}+2(-1)^{\mathrm{n}}
$$

Lemma 3:

$$
\mathrm{L}_{\mathrm{n}}^{2}-5 \mathrm{~F}_{\mathrm{n}}^{2}=4(-1)^{\mathrm{n}}
$$

*Lemma 4:

$$
\mathrm{L}_{\mathrm{n}} \mathrm{~L}_{\mathrm{n}+1}=\mathrm{L}_{2 \mathrm{n}+1}+(-1)^{\mathrm{n}}
$$

We now discuss
Theorem 3: If $\tan \psi_{\mathrm{n}}=1 / L_{\mathrm{n}}$, then

$$
\begin{array}{r}
\operatorname{Tan}\left(\psi_{2 n}+\psi_{2 n+2}\right)=1 / F_{2 n+1} \text { or } \operatorname{Tan}^{-1} \frac{1}{\mathrm{~F}_{2 n+1}}=\operatorname{Tan}^{-1} \frac{1}{\mathrm{~L}_{2 n}} \\
+\operatorname{Tan}^{-1} \frac{1}{\mathrm{~L}_{2 n+2}}
\end{array}
$$

Proof:

$$
\operatorname{Tan}\left(\psi_{2 n}+\psi_{2 n+2}\right)=\frac{L_{2 n}+L_{2 n+2}}{L_{2 n} L_{2 n+2}-1}=\frac{1}{F_{2 n+1}}
$$

since

$$
\mathrm{L}_{2 \mathrm{n}+2}+\mathrm{L}_{2 \mathrm{n}}=5 \mathrm{~F}_{2 \mathrm{n}+1} \text { and } \mathrm{L}_{2 \mathrm{n}} \mathrm{~L}_{2 \mathrm{n}+2}-1=5 \mathrm{~F}_{2 \mathrm{n}+1}^{2}
$$

by Lemma 1 above.
Theorem 4: If $\tan \theta_{\mathrm{n}}=1 / \mathrm{F}_{\mathrm{n}}$, then $\operatorname{Tan}\left(\theta_{2 \mathrm{n}}-\theta_{2 \mathrm{n}+2}\right)=1 / \mathrm{F}_{2 \mathrm{n}+1}$,
or

$$
\operatorname{Tan}^{-1} \frac{1}{\mathrm{~F}_{2 \mathrm{n}+1}}=\operatorname{Tan}^{-1} \frac{1}{\mathrm{~F}_{2 \mathrm{n}}}-\operatorname{Tan}^{-1} \frac{1}{\mathrm{~F}_{2 \mathrm{n}+2}}
$$

Proof:

$$
\operatorname{Tan}\left(\theta_{2 n}-\theta_{2 n+2}\right)=\frac{F_{2 n+2}-F_{2 n}}{F_{2 n} F_{2 n+2}+1}=\frac{1}{F_{2 n+1}}
$$

since

$$
\mathrm{F}_{2 \mathrm{n}+2}-\mathrm{F}_{2 \mathrm{n}}=\mathrm{F}_{2 \mathrm{n}+1} \quad \text { and } \quad \mathrm{F}_{2 \mathrm{n}} \mathrm{~F}_{2 \mathrm{n}+2}-\mathrm{F}_{2 \mathrm{n}+1}^{2}=(-1)^{2 \mathrm{n}+1}
$$

From Theorem 4,

$$
\begin{array}{r}
\sum_{n=1}^{M} \operatorname{Tan}^{-1} \frac{1}{F_{2 n+1}}=\sum_{n=1}^{M}\left(\operatorname{Tan}^{-1} \frac{1}{F_{2 n}}-\operatorname{Tan}^{-1} \frac{1}{F_{2 n+2}}\right) \\
=\operatorname{Tan}^{-1} \frac{1}{F_{2}}-\operatorname{Tan}^{-1} \frac{1}{F_{2 M+2}}
\end{array}
$$

Since $\lim _{\mathrm{M} \rightarrow \infty} \operatorname{Tan}^{-1} \frac{1}{\mathrm{~F}_{2 \mathrm{M}+2}}=0$ by continuity of $\operatorname{Tan}^{-1} \mathrm{x}$ at $\mathrm{x}=0$ we may write

Theorem 5:

$$
\frac{\Pi}{4}=\operatorname{Tan}^{-1} 1=\sum_{n=1}^{\infty} \operatorname{Tan}^{-1} \frac{1}{F_{2 n+1}}
$$

This is the celebrated result of D. H. Lehmer, Nov. 1936, American Mathematical Monthly, p. 632, Problem 3801.

We note in passing that the partial sums

$$
S_{M}=\sum_{n=1}^{M} \operatorname{Tan}^{-1} \frac{1}{F_{2 n+1}}=\operatorname{Tan}^{-1} \frac{1}{F_{2}}-\operatorname{Tan}^{-1} \frac{1}{F_{2 M+2}}
$$

are all bounded above by $\operatorname{Tan}^{-1} 1=\Pi / 4$ and $S_{M}$ is monotone. Thus Theorem 1 can be applied. From Theorem 3,

$$
\sum_{n=1}^{M} \operatorname{Tan}^{-1} \frac{1}{F_{2 n+1}}=\sum_{n=1}^{M}\left(\operatorname{Tan}^{-1} \frac{1}{L_{2 n}}+\operatorname{Tan}^{-1} \frac{1}{L_{2 n+2}}\right)
$$

so that

$$
\sum_{n=1}^{M} \operatorname{Tan}^{-1} \frac{1}{F_{2 n+1}}+\operatorname{Tan}^{-1} \frac{1}{3}=2 \sum_{n=1}^{M} \operatorname{Tan}^{-1} \frac{1}{L_{2 n}}+\operatorname{Tan}^{-1} \frac{1}{L_{2 M+2}}
$$

The limit on the left tends to $\operatorname{Tan}^{-1} 1+\operatorname{Tan}^{-1} 1 / 3=\operatorname{Tan}^{-1} 2$ and the right-hand side tends to this same limit and since $\operatorname{Tan}^{-1} 1 / L_{2 M+2} \rightarrow 0$, then Theorem 6:

$$
\sum_{n=1}^{\infty} \operatorname{Tan}^{-1} \frac{1}{\mathrm{~L}_{2 n}}=\operatorname{Tan}^{-1} \frac{\sqrt{5}-1}{2}=\frac{1}{2} \operatorname{Tan}^{-1} 2
$$

Compare with Theorem 5 in Part IV.
We shall continue this interesting discussion in the next issue.

## 

## CORRECTIONS FOR VOLUME 1, NO. 2

Page 45: In the tenth line up from the bottom, the subscripts on the Fibonacci numbers should be reversed.

Page 47: Replace "Lamda" by "Lambda" in the title.

Page 52: In line 6, replace ( $\mathrm{R}^{\mathrm{n}}$ ) with $\lambda\left(\mathrm{R}^{\mathrm{n}}\right)$.
In line 12, the author's name is Jekuthiel Ginsburg.

Page 55: In problem $\mathrm{H}-18$, part a, replace $=$ by $\doteqdot$.
Page 57: In E2, replace $\frac{a}{d}, \frac{b}{d}$ with $\left(\frac{a}{d}, \frac{b}{d}\right)$.
Page 58: Add three dots after the 4 on the last line.

Page 60: The title "Letters to the Editor" was omitted from Fibonacci Formulas, and, in that article, the "Correct Formula" due to the late Jekuthiel Ginsburg is $\mathrm{F}_{\mathrm{n}+2}^{3}-3 \mathrm{~F}_{\mathrm{h}}^{3}+\mathrm{F}_{\mathrm{n}-2}^{3}=3 \mathrm{~F}_{3 \mathrm{n}}$.

Page 68: The right side of identity xix should read

$$
\frac{1}{2}\left(F_{n+1}^{2}-F_{n} F_{n-1}-1\right)
$$

and in identity xx , the subscript $\mathrm{n}-1$ should be $\mathrm{n}-\mathrm{i}$.
The correct page number in reference 1 is 98 .

Page 75: Insert three dots after $\beta^{2}$, in line 15.

Page 80: In the last line, replace $p N$ by $p \mid N$ and $p\left(2 \cdot 3 \cdot 5 \cdots p_{n}\right)$ by $\mathrm{p} \mid\left(2 \cdot 3 \cdot 5 \cdots p_{\mathrm{n}}\right)$.

Page 81: Replace $T_{n}+1$ by $T_{n+1}$ in the left side of the first displayed equation.

Page 86: In $B-12, L_{n+1}=\left(a_{r s}\right), \quad a_{34}=i=\sqrt{-1}$ instead of zero.
Page 87: Change the equations in problem B-16 to read

$$
\begin{aligned}
R & =\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{array}\right) \\
R^{n} & =\left(\begin{array}{lrr}
F_{n-1}^{2} & F_{n-1} F_{n} & F_{n}^{2} \\
2 F_{n-1} F_{n} & F_{n+1}^{2}-F_{n-1} F_{n} & 2 F_{n} F_{n+1} \\
F_{n}^{2} & F_{n} F_{n+1} & F_{n+1}^{2}
\end{array}\right)
\end{aligned}
$$

See also solution in this issue.

Page 88: See the last written line for notational error due to exclamation point punctuation.

# ON THE GENERAL TERM OF A RECURSIVE SEQUENCE 

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## INTRODUCTION

It is often comforting and useful to obtain a specific formula for the general term of a recursive sequence. This paper reviews the Fibonacci and Lucas sequences, then presents a more general method which requires the solution of a set of linear equations. The solution may be effected by finding the inverse of a Vandermonde matrix, and a description of this inverse is included.

## THE SPECIAL CASE

As is well known, the Fibonacci sequence is completely defined by the difference equation $F(n)-F(n-1)-F(n-2)=0$ and the initial conditions $F(0)=0$ and $F(1)=1$. If we seek a solution of the difference equation of the form $F(n)=x^{n}$, we obtain $x^{n}-x^{n-1}-x^{n-2}=0$, or $x^{2}-x-1=0$. This has two solutions; $x_{1}=(1+\sqrt{5}) / 2$ and $x_{2}=(1-\sqrt{5}) / 2$. Now, the theory of homogeneous linear difference equations assures us that the most general solution is $F(n)=c_{1} x_{1}^{n}+c_{2} x_{2}^{n}$, where $c_{1}$ and $c_{2}$ are arbitrary constants. (The reader who encounters this result for the first time can verify it by substitution; the theory parallels quite nicely the theory of linear differential equations.)

The initial conditions give us two linear equations,

$$
\begin{gather*}
c_{1}+c_{2}=0 \\
c_{1} x_{1}+c_{2} x_{2}=1 \tag{1}
\end{gather*}
$$

The solutions are $c_{1}=1 / \sqrt{5}$ and $c_{2}=-1 / \sqrt{5}$, and we obtain the well-known formula $F(n)=1 / \sqrt{5}\left(x_{1}^{n}-x_{2}^{n}\right)$.

The Lucas series is obtained from the same difference equation with different initial conditions. In this case, $F(0)=2, F(1)=1$, and equations (1) become

$$
\begin{gathered}
c_{1}+c_{2}=2 \\
c_{1} x_{1}+c_{2} x_{2}=1
\end{gathered}
$$

Then the general term of the Lucas sequence is

$$
\mathrm{L}(\mathrm{n})=\left[\frac{1+\sqrt{5}}{2}\right]^{\mathrm{n}}+\left[\frac{1-\sqrt{5}}{2}\right]^{\mathrm{n}} .
$$

From these formulas it is possible to prove such identities as $L(n)-F(n)=$ $2 F(n-1)$ and $L(n)+F(n)=2 F(n+1)$.

## THE GENERAL CASE

We might: solve all equations of the form of equations (1) by writing them in matrix form
(2)

$$
\left[\begin{array}{ll}
1 & 1 \\
x_{1} & x_{2}
\end{array}\right]\left[\begin{array}{l}
c_{1} \\
c_{2}
\end{array}\right]=\left[\begin{array}{l}
F(0) \\
F(1)
\end{array}\right]
$$

and then find the inverse of the first matrix, so that

$$
\left[\begin{array}{l}
c_{1} \\
c_{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & 1 \\
x_{1} & x_{2}
\end{array}\right]^{-1}\left[\begin{array}{l}
F(0) \\
F(1)
\end{array}\right]
$$

The matrix

$$
\left[\begin{array}{cc}
1 & 1 \\
& \\
\mathrm{x}_{1} & \\
\mathrm{x}_{2}
\end{array}\right]
$$

is a simple case of a Vandermonde matrix, and the determination of the constants is possible if the matrix can be inverted. Fortunately it can be easily inverted, even if the order exceeds two.

Let us suppose that a recursion relation gives birth to a linear homogenous difference equation with constant coefficients, say,
$F(n)+a_{1} F(n-1)+\cdots a_{k} F(n-k)=0$, and that $F(0)=b_{0}, F(1)=b_{1}, \cdots$, $F(k-1)=b_{k-1}$.

If we seek again a solution of the form $F(n)=x^{n}$, then we are led to the equation

$$
\begin{equation*}
f(x)=x^{k}+a_{1} x^{k-1}+\cdots+a_{k}=0 \tag{3}
\end{equation*}
$$

Then assuming that the roots of (3), $\mathrm{x}_{1}, \mathrm{x}_{2}, \cdots, \mathrm{x}_{\mathrm{k}}$ are all different*, the theory of difference equations assures us that $F(n)=c_{1} x_{1}^{n}+c_{2} x_{2}^{n}+\cdots+$ $c_{k^{2}} \mathrm{X}_{\mathrm{k}}^{\mathrm{n}}$. The subsequent equations corresponding to (1), but written in matrix form are
(4) $\left[\begin{array}{cccc}1 & 1 & \cdots & 1 \\ x_{1} & x_{2} & \cdots & x_{k} \\ \cdot & & & \\ \cdot & & & \\ \cdot & k-1 & & k-1 \\ k-1 & x_{2} & & x_{k} \\ x_{1} & x_{2}\end{array}\right]\left[\begin{array}{c}c_{1} \\ c_{2} \\ \cdot \\ \cdot \\ c_{k}\end{array}\right]=\left[\begin{array}{l}b_{0} \\ b_{1} \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1}\end{array}\right]$
or $\mathrm{V}_{\mathrm{k}} \mathrm{C}=\mathrm{B}$.
Now the polynomial $f_{1}(x)=f(x) /\left(x-x_{1}\right)$ has $k$ coefficients, and moreover $f_{1}\left(x_{2}\right)=f_{1}\left(x_{3}\right)=\cdots=f_{1}\left(x_{k}\right)=0$. Consequently if we form a row vector, (written in reverse order) of these coefficients, then this row vector will be orthogonal to every column of $\mathrm{V}_{\mathrm{k}}$ except the first. We need now only a normalizing factor, so that the scalar product of this row vector with the first column of $\mathrm{V}_{\mathrm{k}}$ is unity. Investigation shows that this scalar product (before normalizing) is $f_{1}\left(x_{1}\right)=f^{9}\left(x_{1}\right)$, the first derivative of $f(x)$ at $x=x_{1}$. Moreover, the fact that $f^{\prime}\left(x_{1}\right)=\lim _{x \rightarrow x_{1}} \frac{f(x)}{x-x_{1}}$ makes this scalar product easy to calculate by synthetic division.

This procedure is now continued; the coefficients of $f_{2}(x) / f^{g}\left(x_{2}\right)$ provide us with the second row of $V_{k}^{-1}$, and in general the coefficients of $f_{i}(x) / f^{\prime}\left(x_{i}\right)$ provide the $\mathrm{i}^{\text {th }}$ row of $\mathrm{V}_{\mathrm{k}}^{-1}$.

A particular example makes the procedure clear. Suppose the recurrence relation is

[^1]\[

$$
\begin{aligned}
F(n)-3 F(n-1)-5 F(n-2)+15 F(n-3)+4 F(n-4)- & 12 F(n-5) \\
& =0
\end{aligned}
$$
\]

and the initial conditions are

$$
\begin{aligned}
F(0)= & 1, F(1)=1, F(2)=1, F(3)=2 \\
& F(4)=3 .
\end{aligned}
$$

The difference equation yields the poiynomial

$$
x^{5}-3 x^{4}-5 x^{3}+15 x^{2}+4 x-12=0
$$

whose roots are $[-2,-1,1,2,3]$ The coefficients of $f_{1}(x)$ are easily found by synthetic division, the normalizing factor by repeated synthetic division.

$$
-2 \begin{array}{|c}
-2 \underline{1-3-5+15+4-12} \\
-2 \begin{array}{|c|}
\hline 1-5+5+5-6 \\
\frac{-2+14-10+12}{1-7+19+66} \\
1-7+33+60
\end{array}
\end{array}
$$

The vector $(-6,5,5,-5,1)$ is orthogonal to all the columns of $V_{5}$ except the first, the normalizing factor is $1 / 60$, and the first row of $V_{5}^{-1}$ is

$$
\left(\frac{-1}{10}, \frac{1}{12}, \frac{1}{12}, \frac{-1}{12}, \frac{1}{60}\right)
$$

Synthetic division may be continued for the other roots until we obtain the desired inverse.

$$
\mathrm{V}_{5}^{-1}=\left[\begin{array}{rrrrr}
-\frac{1}{10} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{60} \\
\frac{1}{2} & -\frac{2}{3} & \frac{1}{24} & \frac{1}{6} & -\frac{1}{24} \\
1 & \frac{2}{3} & -\frac{7}{12} & \frac{1}{6} & \frac{1}{12} \\
-\frac{1}{2} & -\frac{1}{12} & \frac{7}{12} & \frac{1}{12} & -\frac{1}{12} \\
\frac{1}{10} & 0 & -\frac{1}{8} & 0 & \frac{1}{40}
\end{array}\right]
$$

$$
\left[\begin{array}{l}
c_{1} \\
c_{2} \\
c_{3} \\
c_{4} \\
c_{5}
\end{array}\right]=\left[\begin{array}{ccccc}
-\frac{1}{10} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{60} \\
\frac{1}{2} & -\frac{2}{3} & \frac{1}{24} & \frac{1}{6} & -\frac{1}{24} \\
1 & \frac{2}{3} & \frac{7}{12} & \frac{1}{6} & \frac{1}{12} \\
-\frac{1}{2} & -\frac{1}{12} & \frac{7}{12} & \frac{1}{12} & -\frac{1}{12} \\
\frac{1}{10} & 0 & -\frac{1}{8} & 0 & \frac{1}{40}
\end{array}\right]\left[\begin{array}{l}
1 \\
1 \\
1 \\
2 \\
3
\end{array}\right]=\left[\begin{array}{c}
\frac{1}{20} \\
\frac{1}{12} \\
1 \\
\frac{1}{12} \\
\frac{1}{20}
\end{array}\right]
$$

Hence the general term is given by

$$
\mathrm{F}(\mathrm{n})=-\frac{1}{20}(-2)^{\mathrm{n}}+\frac{1}{12}(-1)^{\mathrm{n}}+1(1)^{\mathrm{n}}-\frac{1}{12}(2)^{\mathrm{n}}+\frac{1}{20}(3)^{\mathrm{n}} .
$$

## 

## CORRECTIONS FOR VOLUME 1, NO. 2

Page 4: Equation (2.8) should read

$$
(a-b)^{p} \sum_{k=0}^{p}(-1)^{k}\binom{p}{k} \sum_{j=0}^{q}\binom{q}{j} F\left(a^{p+q-k-j} b^{k+j} x\right)=\sum_{n=0}^{\infty} A_{n} x^{n} F_{n}^{p} L_{n}^{p}
$$

Page 23: The fifth line up from the bottom should read:

$$
D_{0}=0, D_{1}=x+y, D_{2}=(x+y)^{2}
$$

Page 30: In Line 10, replace $m\left(u_{n+1}-1\right)$ by $m \mid\left(u_{n+1}-1\right)$.

Page 33: The $=$ signs in lines 10 and 11 should be replaced by $\equiv$ signs.

Page 37: The first line of the title should end in a lower case "m."

# ELEMENTARY PROBLEMS AND SOLUTIONS 

Edited by A. P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer should submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated.

Solutions to problems listed below should be submitted within two months of publication.

B-30 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.
Find the millionth term of the sequence $a_{n}$ given that
$a_{1}=1, a_{2}=1$, and $a_{n+2}=a_{n+1}-a_{n}$ for $n \geq 1$.

## B-31 Proposed by Douglas Lind, Falls Cburch, Virginia

If $n$ is even, show that the sum of $2 n$ consecutive Fibonacci numbers is divisible by $\mathrm{F}_{\mathrm{n}}$.
B-32 Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas.
Show that $\mathrm{nL}_{\mathrm{n}} \equiv \mathrm{F}_{\mathrm{n}}(\bmod 5)$.
B-33 Proposed by John A. Fuchs, University of Santa Clara, Santa Clara, California
Let $u_{n}, v_{n}, \cdots, w_{n}$ be sequences each satisfying the second order recurrence formula

$$
y_{n+2}=g y_{n+1}+h y_{n} \quad(n \geq 1)
$$

where $g$ and $h$ are constants. Let $a, b, \ldots, c$ be constants. Show that
[Feb. 1964]

$$
a u_{n}+b v_{n}+\cdots+c w_{n}=0
$$

is true for all positive integral values of $n$ if it is true for $n=1$ and $n=2$. B-34 Proposed by G. L. Alexanderson, University of Santa Clara, Santa Clara, California

Let $u_{n}$ and $v_{n}$ be any two sequences satisfying the second order recurrence formula

$$
y_{n+2}=g y_{n+1}+h y_{n}
$$

where $g$ and $h$ are constants. Show that the sequence of products $w_{n}=u_{n} v_{n}$ satisfies a third-order recurrence formula

$$
y_{n+3}=a y_{n+2}+b y_{n+1}+c y_{n}
$$

and find $a, b$, and $c$ as functions of $g$ and $h$.
B-35 Proposed by J. L. Brown, Jr., Pennsylvania State University, University Park, Pa.
Prove that

$$
\sum_{k=1}^{r-1}(-1)^{k}\binom{x}{k} F_{k}=0
$$

for $x$ an odd positive integer and generalize.

B-36 Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California

The sequence $1,2,5,12,29,70, \cdots$ is defined by $c_{1}=1, c_{2}=2$, and $c_{n+2}=2 c_{n+1}+c_{n}$ for all $n \geq 1$. Prove that $c_{5 m}$ is an integral multiple of 29 for all positive integers $m$.

B-37. Proposed by Brother U. Alfred, St. Mary's College, California

Given a line with a point of origin O and four positive positions $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and $D$ with respect to $O$. If the line segments $O A, O B, O C$, and $O D$ correspond respectively to four consecutive Fibonacci numbers $F_{n}, F_{n+1}, F_{n+2}, F_{n+3}$, determine for which set(s) of Fibonacci numbers the points $A, B, C$, and D are in simple harmonic ratio, i.e.,

$$
\frac{A B}{B C} \frac{A D}{D C}=-1
$$

DIFFERENCES MADE INTO PRODUCTS
B-17 Proposed by Charles R. Wall, Ft. Worth, Texas
If m is an integer, prove that

$$
F_{n+4 m+2}-F_{n}=L_{2 m+1} F_{n+2 m+1}
$$

where $F_{p}$ and $L_{p}$ are the $p^{\text {th }}$ Fibonacci and Lucas numbers, respectively. Solution by I. D. Ruggles, San Jose State College, San Jose, California

In "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2, p. 77, it is shown that

$$
F_{q+p}-F_{q-p}=L_{p} F_{q}, \quad \text { for } p \text { odd. }
$$

If $q=n+2 m+1$ and $p=2 m+1$, then this becomes the desired formula. Also solved by Douglas Lind, Falls Church, Virginia, and the proposer.

## A TRIGONOMETRIC SUM

B-18 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Show that

$$
\mathrm{F}_{\mathrm{n}}=2^{\mathrm{n}-1} \sum_{\mathrm{k}=0}^{\mathrm{n}-1}(-1)^{\mathrm{k}} \cos ^{\mathrm{n}-\mathrm{k}-1} \frac{\pi}{5} \sin ^{\mathrm{k}} \frac{\pi}{10} \text { for } \mathrm{n} \geq 0
$$

(It should be "for $n \geq 1$ " instead of "for $n \geq 0 . "$ )
Solution by the proposer
It is well known (e. g. , I. J. Schwatt, "An Introduction to the Operations with Series," Chelsea Pub. Co., p. 177) that

$$
\begin{aligned}
& \cos \frac{\pi}{5}=\frac{1+\sqrt{5}}{4} \\
& \cos \frac{\pi}{10}=\frac{\sqrt{5}-1}{4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathrm{a}=\frac{1+\sqrt{5}}{2}=2 \cos \frac{\pi}{5} \\
& \mathrm{~b}=\frac{1-\sqrt{5}}{2}=-2 \sin \frac{\pi}{10}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{F}_{\mathrm{n}} & =\frac{\mathrm{a}^{\mathrm{n}}-\mathrm{b}^{\mathrm{n}}}{\mathrm{a}-\mathrm{b}}=2^{\mathrm{n}-1}\left[\frac{\cos ^{\mathrm{n}} \frac{\pi}{5}-(-1)^{\mathrm{n}} \sin ^{\mathrm{n}} \frac{\pi}{10}}{\cos \frac{\pi}{5}+\sin \frac{\pi}{10}}\right] \\
& =2^{\mathrm{n}-1} \frac{\cos ^{\mathrm{n}} \frac{\pi}{5}-\sin ^{\mathrm{n}}\left(-\frac{\pi}{10}\right)}{\cos \frac{\pi}{5}-\sin \left(-\frac{\pi}{10}\right)}=2^{\mathrm{n}-1} \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \cos ^{\mathrm{n}-\mathrm{k}-1} \frac{\pi}{5} \sin ^{\mathrm{k}}\left(-\frac{\pi}{10}\right) \\
& =2^{\mathrm{n}-1} \sum_{\mathrm{k}=0}^{\mathrm{n}-1}(-1)^{\mathrm{k}} \cos ^{\mathrm{n}-\mathrm{k}-1} \frac{\pi}{5} \sin ^{\mathrm{k}} \frac{\pi}{10}
\end{aligned}
$$

as stated. We have made use of the algebraic identity

$$
\frac{x^{n}-y^{n}}{x-y}=\sum_{k=0}^{n-1} x^{n-k-1} y^{k}
$$

Also solved by Charles R. Wall, Texas Cb̈ristian University, who pointed out that the identity' does not hold for $n=0$.

## A TELESCOPING SUM

B-19 Proposed by L. Carlitz, Duke University, Durham, N.C.

Show that

$$
\sum_{n=1}^{\infty} \frac{1}{F_{n} F_{n+2}^{2} F_{n+3}}+\sum_{n=1}^{\infty} \frac{1}{F_{n} F_{n+1}^{2} F_{n+3}}=\frac{1}{2}
$$

Solution by Jobn H. Avila, University of Maryland, College, Park Maryland

Our solution is similar to that by Francis D. Parker for B-9. Let $\mathrm{a}=$ $a(n)=F_{n}, b=F_{n+1}, c=F_{n+2}$, and $d=F_{n+3}$. Then $a+b=c, b+c=d$, and the left side of the desired formula is

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{a c^{2} d}+\frac{1}{a b^{2} d}\right) & =\sum_{n=1}^{\infty}\left(\frac{b}{a b c^{2} d}+\frac{c}{a b^{2} c d}\right) \\
& =\sum_{n=1}^{\infty}\left(\frac{c-a}{a b c^{2} d}+\frac{d-b}{a b^{2} c d}\right) \\
& =\sum_{n=1}^{\infty}\left(\frac{1}{a b c d}-\frac{1}{b c^{2} d}+\frac{1}{a b^{2} c}-\frac{1}{a b c d}\right) \\
& =\sum_{n=1}^{\infty}\left(\frac{1}{a b^{2} c}-\frac{1}{b c^{2} d}\right)
\end{aligned}
$$

The last sum is the telescoping series

$$
\left(\frac{1}{F_{1} F_{2}^{2} F_{3}}-\frac{1}{F_{2} F_{3}^{2} F_{4}}\right)+\left(\frac{1}{F_{2} F_{3}^{2} F_{4}}-\frac{1}{F_{3} F_{4}^{2} F_{5}}\right)+\cdots
$$

whose sum is

$$
\frac{1}{F_{1} F_{2}^{2} F_{3}}=\frac{1}{1 \cdot 1^{2} \cdot 2}=\frac{1}{2}
$$

Also solved by the proposer.

## SUMIMING GENERALIZED FIBONACCI NUMBERS

B-20 Proposed by Louis G. Brökling, Redwood City, California
Generalize the well-known identities,
(i) $\quad F_{1}+F_{2}+F_{3}+\cdots+F_{n}=F_{n+2}-1$
(ii) $L_{1}+L_{2}+L_{3}+\cdots+L_{n}=L_{n+2}-3$.

Solution by Charles R. Wall, Texas Christian University, Fi. Worth, Texas

$$
\text { If } H_{0}=q, H_{1}=p \text {, and } H_{n+2}=H_{n+1}+H_{n} \text {, then } H_{n}=p F_{n}+q F_{n-1}
$$ so that

$$
\begin{aligned}
\sum_{i=1}^{n} H_{i} & =p \sum_{i=1}^{n} F_{i}+q \sum_{i=0}^{n-1} F_{i}=p\left(F_{n+2}-1\right)+q\left(F_{n+1}-1\right) \\
& =p F_{n+2}+q F_{n+1}-(p+q)=H_{n+2}-(p+q)=H_{n+2}-H_{2} .
\end{aligned}
$$

This identity is also obtained from Horadam's "A Generalized Fibonacci Sequence," American Mathematical Monthly, Vol. 68 (1961), p. 456.

Also solved by Fern Grayson, Lockheed Missiles and Space Company, Sunnyvale California and the proposer.

## EVENS AND ODDS

B-21 Proposed By L. Carlitz, Duke University, Durham, N. C.

If

$$
u_{n}=\frac{1}{2}\left[(x+1)^{2^{n}}+(x-1)^{2^{n}}\right]
$$

show that

$$
u_{n+1}=u_{n}^{2}+2^{2 n} u_{0}^{2} u_{1}^{2} \cdots u_{n-1}^{2}
$$

Solution by Robert Means, University of Michigan
Let Let $v_{n}=(x+1)^{2^{n}}-u_{n^{\circ}}$ Then $u_{0}=x, v_{0}=1$ and for $n \geq 1 u_{n}$ and $v_{n}$ are the terms of even and of odd degree respectively in $(x+1)^{2}$. Now $u_{n+1}+v_{n+1}=\left(u_{n}+v_{n}\right)^{2}=u_{n}^{2}+2 u_{n} v_{n}+v_{n}^{2}$ and equating sums of terms of even and of odd degree respectively we have for $n \geq 0$,
(a)
(b)

$$
\begin{aligned}
& u_{n+1}=u_{n}^{2}+v_{n}^{2} \\
& v_{n+1}=2 u_{n} v_{n}
\end{aligned}
$$

Repeated use of (b) leads to $v_{n}=2 u_{n-1} v_{n-1}=2^{2} u_{n-1} u_{n-2} v_{n-2}=\cdots=$ $2^{n} u_{n-1} u_{n-2} \cdots u_{0} v_{0}$. . Since $v_{0}=1$, the desired result is obtained by substituting the last expression for $\mathrm{v}_{\mathrm{n}}$ in (a).

Also solved by Charles R. Wall, Texas Christian University and the proposer

## LUCAS ANALOGUES

B-22 Proposed by Brother U. Alfred, St. Mary's College, California
Prove the Fibonacci identity

$$
F_{2 k} F_{2 k^{\prime}}=F_{k+k^{\prime}}^{2}-F_{k-k^{\prime}}^{2}
$$

and find the analogous Lucas identity。
(Editor's Note: The Fibonacci identity here is proved by I. D. Ruggles in "Some Fibonacci Results Using Fibonacci-Type Sequences," this Quarterly, Vol. 1, Issue 2, p. 77.) Proofs were submitted by Douglas Lind, Falls Church, Virginia; V. E. Hoggatt, Jr., San Jose State College; and Charles R. Wall, Texas Christian University, Ft. Worth, Texas. Lind and Hoggatt gave

$$
L_{2 k} L_{2 j}=L_{k+j}^{2}+L_{k-j}^{2}-4(-1)^{k-j}
$$

as the analogous Lucas identity and Wall gave it as

$$
L_{2 k} L_{2 j}=L_{k+j}^{2}+5 F_{k-j}^{2}=5 F_{k+j}^{2}+L_{k-j}^{2}
$$

Proofs of these are left to the readers.

## TELESCOPING PRODUCTS AND SUMS

B-23 Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif.

Prove the identities

$$
\begin{equation*}
F_{n+1}=\prod_{i=1}^{n}\left(1+\frac{F_{i-1}}{F_{i}}\right) \tag{i}
\end{equation*}
$$

(ii)
(iii)

$$
\begin{aligned}
& \frac{F_{n+1}}{F_{n}}=1+\sum_{i=2}^{n} \frac{(-1)^{i}}{F_{i} F_{i-1}} \\
& \frac{1+\sqrt{5}}{2}=1+\sum_{i=2}^{\infty} \frac{(-1)^{i}}{F_{i} F_{i-1}}
\end{aligned}
$$



Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.
(i) $\quad F_{n+1}=\frac{F_{n+1} F_{n} \cdots F_{2}}{F_{n} F_{n-1} \cdots F_{1}}=\prod_{i=1}^{n} \frac{F_{i+1}}{F_{i}}=\prod_{i=1}^{n} \frac{F_{i}+F_{i-1}}{F_{i}}=\prod_{i=1}^{n}\left(1+\frac{F_{i-1}}{F_{i}}\right)$.
(ii)

$$
\begin{aligned}
\frac{F_{n+1}}{F_{n}} & =\left(\frac{F_{n+1}}{F_{n}}-\frac{F_{n}}{F_{n-1}}\right)+\left(\frac{F_{n}}{F_{n-1}}-\frac{F_{n-1}}{F_{n-2}}\right)+\cdots+\left(\frac{F_{3}}{F_{2}}-\frac{F_{2}}{F_{1}}\right)+1 \\
& =1+\sum_{i=2}^{n}\left(\frac{F_{i+1}}{F_{i}}-\frac{F_{i}}{F_{i-1}}\right)=1+\sum_{i=2}^{n} \frac{F_{i+1} F_{i-1}-F_{i}^{2}}{F_{i} F_{i-1}} \\
& =1+\sum_{i=2}^{n} \frac{(-1)^{i}}{F_{i} F_{i-1}}
\end{aligned}
$$

using the well-known identity,

$$
F_{i+1} F_{i-1}-F_{i}^{2}=(-1)^{i}
$$

(iii) In (ii) take the limit as $n \rightarrow \infty$ and recall that $\lim _{\mathrm{n} \rightarrow \infty} \frac{\mathrm{F}_{\mathrm{n}+1}}{\mathrm{~F}_{\mathrm{n}}}=\frac{1+\sqrt{5}}{2}$.

Also solved by Dermott A. Breault, SylvaniamARL, Waltham, Mass.; Douglas Lind, Falls Cburch, Va; Charles R. Wall, Texas Cbristian University, Ft. Worth, Texas; and the proposer

## A CORRECTED SOLUTION

B-4 Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, California, and Vladimir Ivanoff, San Carlos, California
Show that $\quad \sum_{i=0}^{n}\binom{n}{i} F_{i}=F_{2 n}$
Generalize.
(Readers: Can you find the errors in the previously published solution?)
Solution by Joseph Erbacher, University of Santa Clara, Santa Clara, Calif., and J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania

Using the Binet formula,

$$
F_{2 n+j}=\frac{\left(a^{2}\right)^{n} a^{j}-\left(b^{2}\right)^{n} b^{j}}{a-b}=\frac{(1+a)^{n} a^{j}-(1+b)^{n} b^{j}}{a-b}
$$

since

$$
\mathrm{a}^{2}=\mathrm{a}+1, \quad \mathrm{~b}^{2}=\mathrm{b}+1 \quad \text { when } \mathrm{a}=\frac{1+\sqrt{5}}{2}, \quad \mathrm{~b}=\frac{1-\sqrt{5}}{2}
$$

we have

$$
\begin{aligned}
& F_{2 n+j}=\frac{1}{a-b}\left[\sum_{i=0}^{n}\binom{n}{i} a^{i+j}-\sum_{i=0}^{n}\binom{n}{i} b^{i+j}\right]=\sum_{i=0}^{n}\binom{n}{i} \frac{a^{i+j}-b^{i+j}}{a-b}= \\
& \sum_{i=0}^{n}\binom{n}{i} F_{i+j}
\end{aligned}
$$

Therefore, for arbitrary integral $j$,

$$
F_{2 n+j}=\sum_{i=0}^{n}\binom{n}{i} F_{i+j}
$$

If $\mathrm{j}=0$, we have the original problem. The identity also holds, with arbitrary $j$, for Lucas numbers since $L_{n}=F_{n+1}+F_{n-1}$.

## 

CORRECTION TO VOLUME 1, NO. 1
See Vol. 1, No. 2, p. 46 for correction to last two references on page 42 .


[^0]:    *This paper stems from a talk prepared under the guidance of Professor C. T.

[^1]:    * When there are multiple roots, the matrix takes a different form; the inverse for this case is not presented here.

