SOME DETERMINANTS INVOLVING POWERS OF FIBONACCI NUMBERS

BROTHER U. ALFRED
St. Mary's College, California

In the October, 1963 issue of this journal [1], the author discussed some of the periodic properties of Fibonacci summations. It was noted that a certain determinant was basic to these considerations. Its main characteristics and value were indicated and a promise was given of additional explanation in some later issue of the Fibonacci Quarterly. The purpose of this article is to set forth the manner of evaluating these determinants on an empirical basis. The proof of the general validity of the results obtained is to be found in an article by Terry Brennan in this issue of the Quarterly [2].

To fix ideas the determinant of the sixth order will be used. Written in a form that brings out its Fibonacci characteristics it would be:

$$\begin{vmatrix} 1^6+1^6-1 & 1^5 1 & 1^4 1^2 & 1^3 1^3 & 1^2 1^4 & 1\,1^5 \\ 2^6+1^6-1 & 2^5 1 & 2^4 1^2 & 2^3 1^3 & 2^2 1^4 & 2\,1^5 \\ 3^6+2^6-1 & 3^5 2 & 3^4 2^2 & 3^3 2^3 & 3^2 2^4 & 3\,2^5 \\ 5^6+3^6-1 & 5^5 3 & 5^4 3^2 & 5^3 3^3 & 5^2 3^4 & 5\,3^5 \\ 8^6+5^6-1 & 8^5 5 & 8^4 5^2 & 8^3 5^3 & 8^2 5^4 & 8\,5^5 \\ 13^6+8^6-1 & 13^5 8 & 13^4 8^2 & 13^3 8^3 & 13^2 8^4 & 13\,8^5 \end{vmatrix}.$$

A certain subtlety should be noted in the first line as the first "1" stands for F_2 and the second for F_1.

By separating the terms of the first column into groups, the problem can be changed to that of evaluating three determinants with first columns as indicated below:

(1)	(2)	(3)
1	1	-1
2^6	1	-1
3^6	2^6	-1
5^6	3^6	-1
8^6	5^6	-1
13^6	8^6	-1

Determinants (1) and (2) can be evaluated in terms of what shall be called the BASIC POWER DETERMINANT. Determinant (3) will be developed in terms of the cofactors of the first column which involve the basic power determinant minus one of its rows.

BASIC POWER DETERMINANT

The first determinant has a common factor in each of its rows (the factors are 1, 2, 3, 5, 8, 13 respectively). If these factors be taken out of the determinant, we have what will be called the basic power determinant. For the sixth order, it is as shown below:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2^5 & 2^4 & 2^3 & 2^2 & 2 & 1 \\ 3^5 & 3^4 2 & 3^3 2^2 & 3^2 2^3 & 3 \cdot 2^4 & 2^5 \\ 5^5 & 5^4 3 & 5^3 3^2 & 5^2 3^3 & 5 \cdot 3^4 & 3^5 \\ 8^5 & 8^4 5 & 8^3 5^2 & 8^2 5^3 & 8 \cdot 5^4 & 5^5 \\ 13^5 & 13^4 8 & 13^3 8^2 & 13^2 8^3 & 13 \cdot 8^4 & 8^5 \end{vmatrix} .$$

This determinant is a special case of the more general determinant in which the first row starts with any Fibonacci number whatsoever

$$\begin{vmatrix} F_i^5 & F_i^4F_{i-1} & F_i^3F_{i-1}^2 & F_i^2F_{i-1}^3 & F_iF_{i-1}^4 & F_{i-1}^5 \\ F_{i+1}^5 & F_{i+1}^4F_i & F_{i+1}^3F_i^2 & F_{i+1}^2F_i^3 & F_{i+1}F_i^4 & F_i^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{i+5}^5 & F_{i+5}^4F_{i+4} & F_{i+5}^3F_{i+4}^2 & F_{i+5}^2F_{i+4}^3 & F_{i+5}F_{i+4}^4 & F_{i+4}^5 \end{vmatrix} .$$

The basis for evaluating this determinant is the relation

$$F_nF_{n+k+1} - F_{n+1}F_{n+k} = (-1)^{n+1}F_k .$$

To evaluate the determinant we proceed to produce zeros in the first row. This is done by multiplying the first column by F_{i-1} and subtracting from this F_i times the second column; then multiplying the second column by F_{i-1} and subtracting from this F_i times the third column; etc. The operations for the first and second columns would be as follows:

$$F_{i+1}^4(F_{i-1}F_{i+1} - F_iF_i) = (-1)^iF_1F_{i+1}^4 ,$$

$$F_{i+2}^4(F_{i-1}F_{i+2} - F_iF_{i+1}) = (-1)^iF_2F_{i+2}^4 ,$$

$$F_{i+3}^4(F_{i-1}F_{i+3} - F_iF_{i+2}) = (-1)^iF_3F^4_{i+3} ,$$

and so on. It is clear that the second row would have a factor F_1, the third a factor F_2, etc. Thus, after eliminating the common factors and expanding by the non-zero term in the last column of the row, the absolute value of the resulting determinant would be $F_1F_2 \cdot F_3F_4F_5$ multiplied by the basic power determinant of the fifth order. If we adopt the notation Δn

to represent the basic power determinant of the nth order, this result may be expressed as

$$|\Delta_6| = \prod_{i=1}^{5} (F_i) \cdot |\Delta_5| .$$

In general, for a determinant of order n,

$$|\Delta_n| = \prod_{i=1}^{n-1} (F_i) \cdot |\Delta_{n-1}| .$$

Since the process may be repeated, it is not difficult to arrive at the final result:

$$|\Delta_n| = \prod_{i=1}^{n-1} F_i^{n-i} .$$

In the particular case of order six,

$$|\Delta_6| = F_1^5 F_2^4 F_3^3 F_4^2 F_5 = 2^3 3^2 \cdot 5 .$$

It is interesting to note that the values of these basic power determinants are independent of where we start in the Fibonacci sequence.

SIGN OF THE BASIC POWER DETERMINANT

It is important to be able to determine the sign of the basic determinant value inasmuch as we shall combine the values of determinants (1) and (2) with the values of the cofactors of determinant (3). The considerations involved are a bit tedious. We distinguish four cases according as n is of the form 4k, 4k+1, 4k+2, or 4k+3. The following three factors determine the outcome:

(i) The sign introduced by expanding from the last element in the first row;

(ii) The signs of the terms of the determinant resulting after each of the steps indicated above. These terms will be either all plus or all minus.

(iii) The sign of Δ_2 which is the second-order determinant of the first powers of the Fibonacci numbers in the last two rows. Thus, for the sixth order determinant we have been considering, Δ_2 is

$$\begin{vmatrix} 8 & 5 \\ 13 & 8 \end{vmatrix} = -1 \quad .$$

The final outcome is as follows:

(i) For order $4k$ or $4k+1$, the sign is always plus;

(ii) For order $4k+2$ or $4k+3$, the sign agrees with that of Δ_2.

As noted previously, the basic power determinant enables us to evaluate determinants (1) and (2). The latter can be brought to this form by shifting the first column so that it becomes the last column.

BASIC POWER DETERMINANTS WITH ONE ROW MISSING

To evaluate the third determinant we find the cofactors of the elements in the first column. For the element in the first row, this cofactor is a basic power determinant after removing common factors, but for all the others it is essentially a basic power determinant with one row missing. The absolute value of such a determinant of order n with a missing row between the k and $(k+1)$st row will be represented by

$$A_n(k \mid k+1)$$

the implication being that the absolute value does not depend on the particular Fibonacci number with which it starts. When developing such a determinant the procedure is the same as for the development of the basic power determinant, only in this case there is a gap. The calculation for a determinant of order n with a row missing between the third and fourth rows can be summarized schematically in the following manner. The column headings are Fibonacci numbers. A table entry is the power to which the Fibonacci number at the head of the column is being raised. The quantities in any one row are multiplied together. In the first row we have the result of the first step in the evaluation in which the order is changed from 9 to 8; in the second, the factors resulting in reducing the determinant from order 8 to order 7; etc.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1	1	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1	
0	1	1	1	1	1	1		
1	1	1	1	1	0			
1	1	1	1	0				
1	1	1	0					
1	1	0						
1	0							
0								

The sum of the quantities in any column gives the power of the Fibonacci number in the determinant. In the above case

$$A_9(3 \mid 4) = F_1^7 F_2^6 F_3^5 F_4^5 F_5^4 F_6^3 F_7^3 F_8^2 F_9 \quad .$$

The same result would have been obtained if the gap had been after the sixth row. In general, if the determinant is of order n, a gap after the kth row or the (n-k)th row gives the same result.

The pattern observed is as follows: (1) A reduction of 2 in the powers of F_1 to F_k inclusive (if k is less than n-k); (2) A reduction of 1 from F_{k+1} to F_{n-k} inclusive; (3) No reduction thereafter. If n-k is less than k, the roles of k and n-k are reversed. Finally, if n-k equals k (even n), there would be a reduction of 2 from 1 to k and no reduction thereafter.

These results may be summarized in the following formulas.

FORMULA FOR k LESS THAN n-k

$$A_n(k \mid k+1) = \prod_{i=1}^{k} F_i^{n-i-1} \prod_{i=k+1}^{n-k} F_i^{n-i} \prod_{i=n-k+1}^{n} F_i^{n-i+1} \quad ,$$

FORMULA FOR n-k LESS THAN k

$$A_n(k \mid k+1) = \prod_{i=1}^{n-k} F_i^{n-i-1} \prod_{i=n-k+1}^{k} F_i^{n-i} \prod_{i=k+1}^{n} F_i^{n-i+1} \quad ,$$

FORMULA FOR k EQUAL TO n-k

$$A_n(n/2 \mid n/2+1) = \prod_{i=1}^{n/2} F_i^{n-i-1} \prod_{i=\frac{n}{2}+1}^{n} F_i^{n-i+1} .$$

These formulas are not difficult of application. However, for the sake of convenience (in view of future considerations) and as a possible guide to readers the results for orders 12 and 13 are set down in detail. Since, however, there is symmetry in k and n-k only the first half need be given in each case.

TABLE OF $A_{12}(k \mid k+1)$

k	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	10	10	9	8	7	6	5	4	3	2	1	1
2	10	9	9	8	7	6	5	4	3	2	2	1
3	10	9	8	8	7	6	5	4	3	3	2	1
4	10	9	8	7	7	6	5	4	4	3	2	1
5	10	9	8	7	6	6	5	5	4	3	2	1
6	10	9	8	7	6	5	6	5	4	3	2	1

TABLE OF $A_{13}(k \mid k+1)$

k	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
1	11	11	10	9	8	7	6	5	4	3	2	1	1
2	11	10	10	9	8	7	6	5	4	3	2	2	1
3	11	10	9	9	8	7	6	5	4	3	3	2	1
4	11	10	9	8	8	7	6	5	4	4	3	2	1
5	11	10	9	8	7	7	6	5	5	4	3	2	1
6	11	10	9	8	7	6	6	6	5	4	3	2	1

SIGN OF POWER DETERMINANT WITH ONE LINE MISSING

The considerations leading to the determination of the sign of power determinants with a line missing are involved. The approach is precisely the same as for the power determinant. The results for all values of n, i (subscript of the leading Fibonacci number in the determinant) and k (as defined for the break point but taken modulo 4) are listed in the following table.

k	4r	4r+1	4r+2 i odd	4r+2 i even	4r+3 i odd	4r+3 i even
1	-	+	+	-	+	-
2	-	-	+	-	-	+
3	+	-	-	+	-	+
4	+	+	-	+	+	-

EVALUATION OF THE ORIGINAL DETERMINANT

We noted previously that the original determinant could be represented as the sum of three separate determinants (1), (2), and (3). Determinant (1) is simply the product of Fibonacci numbers (one from each row) by the basic power determinant. Thus for the sixth order, the situation would be as follows:

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
BPD	5	4	3	2	1		
F's		1	1	1	1	1	1
(1)	5	5	4	3	2	1	1

The sign would be negative.

Determinant (2) can be related to the basic power determinant by moving the first column into the last position. For the sixth order, this involves a change of sign. Again factors can be taken out leaving a basic power determinant. The pattern is as follows:

	F_1	F_2	F_3	F_4	F_5	F_6
BPD	5	4	3	2	1	
F's	1	1	1	1	1	1
(2)	6	5	4	3	2	1

The sign would be positive.

To evaluate (3) we expand by the first column. We shall designate successive elements of the expansion, due account being taken of all signs including the negative quantities in the first column, by successive capital letters: A, B, C, D, ... A gives rise to a simple basic power determinant; B to one with a line missing (k = 1); C with the second line missing (k = 2); etc. However, there are factors that have to be multiplied in each case. It should be noted too that we are referring to determinants of the fifth order and not of the sixth.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	4	3	2	1				
			2	2	2	2	1	
A	4	3	4	3	2	2	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	3	2	1	1			
			1	2	2	2	1	
B	3	3	3	3	3	2	1	(positive)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	2	2	2	1			
			1	1	2	2	1	
C	3	2	3	3	3	2	1	(positive)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	2	2	2	1			
			2	1	1	2	1	
D	3	2	4	3	2	2	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	3	2	1	1			
			2	2	1	1	1	
E	3	3	4	3	2	1	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	4	3	2	1				
			2	2	2	1		
F	4	3	4	3	2	1		(positive)

Summarizing in one table (omitting the first and second Fibonacci number factors as they are both unity) we have the following for the evaluation of the determinant of the sixth order.

	Sign	F_3	F_4	F_5	F_6	F_7
(1)	-	4	3	2	1	1
(2)	+	4	3	2	1	
A	-	4	3	2	2	1
B	+	3	3	3	2	1
C	+	3	3	3	2	1
D	-	4	3	2	2	1
E	-	4	3	2	1	1
F	+	4	3	2	1	

The following pairs of terms combine: E and (1); F and (2); A and D; B and C. The resulting sums have a common factor of $2^8\ 3^3\ 5^2$, the adjoint factor being 144. Thus finally the value of the sixth order determinant is found to be $2^{12}\ 3^5\ 5^2$.

DETERMINANT OF ORDER 12

Without justifying all the intermediate steps, the summation table for order 12 is shown below.

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
(1)	+	10	9	8	7	6	5	4	3	2	1	1
(2)	-	10	9	8	7	6	5	4	3	2	1	
A	+	10	9	8	7	6	5	4	3	2	2	1
B	-	9	9	8	7	6	5	4	3	3	2	1
C	-	9	8	8	7	6	5	4	4	3	2	1
D	+	9	8	7	7	6	5	5	4	3	2	1

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
E	+	9	8	7	6	6	6	5	4	3	2	1
F	-	9	8	7	6	6	6	5	4	3	2	1
G	-	9	8	7	7	6	5	5	4	3	2	1
H	+	9	8	8	7	6	5	4	4	3	2	1
I	+	9	9	8	7	6	5	4	3	3	2	1
J	-	10	9	8	7	6	5	4	3	2	2	1
K	-	10	9	8	7	6	5	4	3	2	1	1
L	+	10	9	8	7	6	5	4	3	2	1	

It will be noted that the following pairs add up to zero: E and F; D and G; C and H; B and I; A and J; (1) and K; (2) and L. Therefore, the value of the determinant is zero. The same result was found for orders 4, 8, and 16.

DETERMINANT OF ORDER 13

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
(1)	+	11	10	9	8	7	6	5	4	3	2	1	1
(2)	+	11	10	9	8	7	6	5	4	3	2	1	
A	-	11	10	9	8	7	6	5	4	3	2	2	1
B	-	10	10	9	8	7	6	5	4	3	3	2	1
C	+	10	9	9	8	7	6	5	4	4	3	2	1
D	+	10	9	8	8	7	6	5	5	4	3	2	1
E	-	10	9	8	7	7	6	6	5	4	3	2	1
F	-	10	9	8	7	6	7	6	5	4	3	2	1
G	+	10	9	8	7	7	6	6	5	4	3	2	1
H	+	10	9	8	8	7	6	5	5	4	3	2	1
I	-	10	9	9	8	7	6	5	4	4	3	2	1
J	-	10	10	9	8	7	6	5	4	3	3	2	1
K	+	11	10	9	8	7	6	5	4	3	2	2	1
L	+	11	10	9	8	7	6	5	4	3	2	1	1
M	-	11	10	9	8	7	6	5	4	3	2	1	

It will be noted that the following pairs add up to zero: E and G; C and I; A and K; (2) and M. The others can be combined to give the following table.

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
(1), L	+	12	10	9	8	7	6	5	4	3	2	1	1
B, J	-	11	10	9	8	7	6	5	4	3	3	2	1
D, H	+	11	9	8	8	7	6	5	5	4	3	2	1
F	-	10	9	8	7	6	7	6	5	4	3	2	1

After taking out the common factor

$$F_3^{10}F_4^9F_5^8F_6^7F_7^6F_8^6F_9^5F_{10}^4F_{11}^3F_{12}^2F_{13}F_{14}$$

the following remains for evaluation:

Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
+	2	1	1	1	1							
-	1	1	1	1	1					1	1	
+	1			1	1			1	1	1	1	
-						1	1	1	1	1	1	

No easy method was found for evaluating the sum of these quantities. Essentially it was a matter of evaluating them, combining them and then factoring. Fortunately, as the numbers to be factored increased in size going up to 23 digits in one instance, a pattern involving Fibonacci and Lucas numbers was discovered with the result that the formulas (1) and (2) on page 38 of [1] were discovered.

The matter can be allowed to rest here. The path pursued has been illustrated in sufficient detail to allow others to explore these interesting determinants. The formulas obtained as well as the determinant values to the twentieth order are set forth in the paper [1] and need not be repeated.

REFERENCES

1. Brother U. Alfred, "Periodic Properties of Fibonacci Summations, The Fibonacci Quarterly, 1(1963), No. 3, pp. 33-42.
2. T. L. Brennan "Fibonacci Powers and Pascal's Triangle in a Matrix" The Fibonacci Quarterly, 2(1964), No. 2, pp. 93-103.

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FIBONACCI POWERS AND PASCAL'S TRIANGLE IN A MATRIX – PART I*

TERRENCE A. BRENNAN
Lockheed Missiles and Space Co., Sunnyvale, California

1. INTRODUCTION

The main point of this paper is to display some interesting properties of the $(n+1) \times (n+1)$ matrix P_n defined by imbedding Pascal's triangle in a square matrix:

$$(1.1) \qquad P_n = \begin{bmatrix} \cdots & 0 & 0 & 0 & 1 \\ \cdots & 0 & 0 & 1 & 1 \\ \cdots & 0 & 1 & 2 & 1 \\ \cdots & 1 & 3 & 3 & 1 \\ & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The matrix P_n was originally constructed by the author in order to evaluate a determinant presented by Brother U. Alfred. The determinant, and its origin, has subsequently been published in [1] and, for the sake of completeness, its evaluation will be presented here.

2. THE PROBLEM AND ITS SOLUTION

THE PROBLEM:

Evaluate the fifth order determinant

$$(2.1) \qquad \begin{vmatrix} 1^5 + 1^5 - 1 & 1^4\,1 & 1^3\,1^2 & 1^2\,1^3 & 1 \cdot 1^4 \\ 2^5 + 1^5 - 1 & 2^4\,1 & 2^3\,1^2 & 2^2\,1^3 & 2 \cdot 1^4 \\ 3^5 + 2^5 - 1 & 3^4\,2 & 3^3\,2^2 & 3^2\,2^3 & 3 \cdot 2^4 \\ 5^5 + 3^5 - 1 & 5^4\,3 & 5^3\,3^2 & 5^2\,3^3 & 5 \cdot 3^4 \\ 8^5 + 5^5 - 1 & 8^4\,5 & 8^3\,5^2 & 8^2\,5^3 & 8 \cdot 5^4 \end{vmatrix}$$

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and its n-th order generalization. For the n-th order the powers in the first column would be n and the determinant would extend to u_{n+1} and u_n in the last row, where u_n is the n-th Fibonacci number:

$$u_{n+1} = u_n + u_{n-1} \quad \text{with } u_0 = 0,\ u_1 = 1 . \tag{2.2}$$

The determinant (2.1), which we will call D_5 (D_n in general), will be evaluated as an expansion of cofactors of the first column. In order to keep track of terms in the expansion it is convenient to define D_n for an arbitrary sequence $a_0, a_1, a_2, \ldots, a_{n+1}$ by appropriately placing the members of this sequence in the first column of D_n:

$$D_5\{a\} = \begin{vmatrix} a_2 - 1^5 a_1 - 1^5 a_0 & 1^4 1 & \\ a_3 - 2^5 a_1 - 1^5 a_0 & 2^4 1 & \\ a_4 - 3^5 a_1 - 2^5 a_0 & 3^4 2 & \cdots \\ a_5 - 5^5 a_1 - 3^5 a_0 & 5^4 3 & \\ a_6 - 8^5 a_1 - 5^5 a_0 & 8^4 5 & \end{vmatrix} . \tag{2.3}$$

Clearly (2.1) is $D_5\{a\}$ with $a_0 = a_1 = a_2 = \ldots = a_6 = -1$.

For simplicity we will content ourselves with the reduction of the fifth order determinant (2.3) while mentioning the corresponding results for the general case. The reduction rests on the groundwork of Brother U. Alfred.

THE SOLUTION:

Basic to the reduction is the determinant of the following matrix:

$$B_{5,1} = \begin{bmatrix} 1^4 & 1^3 1 & 1^2 1^2 & 1 \cdot 1^3 & 1^4 \\ 2^4 & 2^3 1 & 2^2 1^2 & 2 \cdot 1^3 & 1^4 \\ 3^4 & 3^3 2 & 3^2 2^2 & 3 \cdot 2^3 & 2^4 \\ 5^4 & 5^3 3 & 5^2 3^2 & 5 \cdot 3^3 & 3^4 \\ 8^4 & 8^3 5 & 8^2 5^2 & 8 \cdot 5^3 & 5^4 \end{bmatrix} \tag{2.4}$$

and in general B_{ni} where n is the order of the matrix and i denotes the first row entries as u_{i+1} and u_i.

An interesting property of the determinant of $B_{5,i}$ is that its magnitude is independent of the index, or starting point, i. This fact is evident when we multiply the two matrices

$$B_{5,i}\, Q_4 = B_{5,i+1} \tag{2.5}$$

where

$$Q_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.6}$$

is the matrix of (1.1) "transposed" about its counter diagonal. Since the determinant of Q_4 is ± 1 we have

$$|B_{5,i}| = \pm\, |B_{5,i+1}| \, .$$

More precisely we can develop

$$\begin{gathered} |Q_0| = 1, \quad |Q_1| = -1, \quad |Q_2| = -1, \; \ldots, \\ |Q_{n-1}| = (-1)^{n(n-1)/2} \quad . \end{gathered} \tag{2.7}$$

We can start, then, with $B_{5,0}$ and shift indices on each row to obtain

$$B_{5,0}\, Q_4^i = B_{5,i} \quad . \tag{2.8}$$

But

$$B_{5,0} = \begin{bmatrix} 1^4 & 0 & 0 & 0 & 0 \\ 1^4 & 1^3 1 & 1^2\,1^2 & 1\cdot 1^3 & 1^4 \\ 2^4 & 2^3\,1 & 2^2\,1^2 & 2\cdot 1^3 & 1^4 \\ 3^4 & 3^3\,2 & 3^2\,2^2 & 3\cdot 2^3 & 2^4 \\ 5^4 & 5^3\,3 & 5^2\,3^2 & 5\cdot 3^3 & 3^4 \end{bmatrix}$$

Passing to determinants we have

$$|B_{5,i}| = |Q_4|^i \; |B_{5,0}| = (-1)^{10i}\, 1\cdot 2\cdot 3\cdot 5\, |B_{4,i}|$$

where $|B_{5,0}|$ has been expanded by cofactors of its first row and common row factors have been removed. Having established a recursive process for evaluating $|B_{ni}|$ the general formula may be shown:

$$|B_{ni}| = (-1)^s\, u_{n-1}\, u_{n-2}^2\, u_{n-3}^3 \cdots u_2^{n-2}$$

where $s = n(n-1)(3i + n-2)/6$.

In a notation which will be more convenient to use, we define

$$S_o = 1, \quad S_{n+1} = (-1)^n S_n \quad \text{for } n \ge 0$$

$$F_0(x) = 1, \quad F_{n+1}(x) = x_{n+1}\, F_n(x) \quad \text{for any sequence } \{x_n\}$$

$$W_n(x) = F_n(F(x)) .$$

Then

$$(2.9) \qquad |Q_{n-1}| = S_n \quad \text{and} \quad |B_{ni}| = S_n^{i-1} F_n(S)\, W_{n-1}(u) .$$

Let us see what progress can be made with $|D_n|$. Writing (2.3) as three separate determinants on the first column we have, symbolically,

$$(2.10)\quad D_5\{a\} = \begin{vmatrix} a_2 \\ a_3 \\ a_4 \ldots \\ a_5 \\ a_6 \end{vmatrix} - a_1 \begin{vmatrix} 1^5 \\ 2^5 \\ 3^5 \ldots \\ 5^5 \\ 8^5 \end{vmatrix} - a_0 \begin{vmatrix} 1^5 \\ 1^5 \\ 2^5 \ldots \\ 3^5 \\ 5^5 \end{vmatrix} .$$

The first determinant is the hard one. The second compares nicely with $|B_{5,1}|$ after a common factor is removed from each row. The third becomes $|B_{5,1}|$ when the first column is moved to the last (a change in sign for an even order determinant) and common factors are removed from each row. Hence

$$D_5\{a\} = \begin{vmatrix} a_2 \\ a_3 \\ a_4 \ldots \\ a_5 \\ a_6 \end{vmatrix} - a_1 F_6(u) |B_{5,1}| - a_0 F_5(u) |B_{5,1}|$$

and using (2.9)

$$(2.11)\qquad D_5\{a\} = \begin{vmatrix} a_2 \\ a_3 \\ a_4 \ldots \\ a_5 \\ a_6 \end{vmatrix} - a_1 F_5(S) F_6(u) W_4(u) - a_0 F_5(u) W_5(u) .$$

Expansion of the first determinant by cofactors of its first column gives rise to determinants of the form

$$(2.12)\qquad (1\cdot 2\cdot 5\cdot 8)\cdot(1\cdot 1\cdot 3\cdot 5)a_4 \begin{vmatrix} 1^3 & 1^2\,1 & 1\cdot 1^2 & 1^3 \\ 2^3 & 2^2\,1 & 2\cdot 1^2 & 1^3 \\ 5^3 & 5^2\,3 & 5\cdot 3^2 & 3^3 \\ 8^3 & 8^2\,5 & 8\cdot 5^2 & 5^3 \end{vmatrix}$$

the minor being similar to $|B_{4,1}|$ except that the third row is missing. Our task is to evaluate these minors. Starting with $B_{5,1}$ in (2.4) we form the product

$$(2.13)\qquad B_{5,1}\,Q_4^{-3} = \begin{bmatrix} u_{-1}^4 & u_{-1}^3 u_{-2} & u_{-1}^2 u_{-2}^2 & u_{-1} u_{-2}^3 & u_{-2}^4 \\ 0^4 & 0^3 u_{-1} & 0^2 u_{-1}^2 & 0\cdot u_{-1}^3 & u_{-1}^4 \\ 1^4 & 1^3\,0 & 1^2\,0^2 & 1\cdot 0^3 & 0^4 \\ 1^4 & 1^3\,1 & 1^2\,1^2 & 1\cdot 1^3 & 1^4 \\ 2^4 & 2^3\,1 & 2^2\,1^2 & 2\cdot 1^3 & 1^4 \end{bmatrix}$$

Here, as in (2.8), the matrix Q shifts indices on each row and, overapplying this shift, introduces the negative side of the Fibonacci sequence by way of the relation $u_{n-1} = u_{n+1} - u_n$. Expanding the determinant of (2.13) by the third row we have

$$(2.14)\qquad |B_{5,1}|\;|Q_4|^{-3} = (-1)^2(u_{-2}\,u_{-1})(u_1\,u_2)\,|B_{4,-2}^3|$$

where $B_{4,-2}^3$ is $B_{4,-2}$ (i.e., the fourth order matrix of (2.4) with u_{-1} and u_{-2} in its first row) and where the superscript 3 denotes that the third row is missing.

$$(2.15)\qquad B_{4,-2}^3 = \begin{bmatrix} u_{-1}^3 & u_{-1}^2 u_{-2} & u_{-1} u_{-2}^2 & u_{-2}^3 \\ 0^3 & 0^2 u_{-1} & 0\cdot u_{-1}^2 & u_{-1}^3 \\ 1^3 & 1^2\,0 & 1\cdot 0^2 & 0^3 \\ 1^3 & 1^2\,1 & 1\cdot 1^2 & 1^3 \end{bmatrix}$$

We transform (2.15) to the desired matrix of (2.12) by

$$B_{4,-2}^{3}\, Q_3^3 = B_{4,1}^{3} \tag{2.16}$$

Passing to determinants, and combining (2.16) with (2.14) we have

$$|B_{4,1}^{3}| = \frac{(-1)^2}{(u_{-2}\, u_{-1})(u_1\, u_2)}\; \frac{|Q_3|^3}{|Q_4|^3}\; |B_{5,1}| \; .$$

Using $u_{-n} = -(-1)^n u_n$ for the negative half of the Fibonacci sequence, and evaluating the known determinants,

$$|B_{4,1}^{3}| = (-1)^{12}\, S_5\, S_3\, F_4(S)\; \frac{W_4(u)}{F_2(u)\, F_2(u)}$$

The general case, using this technique, may be formulated as

$$|B_{n,1}^{r}| = (-1)^{nr}\, S_{n+1}\, S_r\, F_n(S)\; \frac{W_n(u)}{F_{r-1}(u)\, F_{n+1-r(u)}} \; . \tag{2.17}$$

Two simplifications to (2.17) are in order:

$$(-1)^{nr}\, S_{n+1}\, S_r = S_{n+1-r}$$

and

$$\frac{W_n(u)}{F_{r-1}(u)\, F_{n+1-r}(u)} = W_{n-1}(u)\; \frac{F_n(u)}{F_{r-1}(u)\, F_{n+1-r}(u)}$$

It seems appropriate, since $F_n(u) = u_n\, u_{n-1}\, u_{n-2} \ldots u_2 u_1$ is a factorial type product for the sequence $\{u_n\}$, that we define the specialized "binomial coefficient"

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_n(u)}{F_r(u)F_{n-4}(u)} \; ; \quad F_0(u) = 1, \begin{bmatrix} n \\ r \end{bmatrix} = 1 \ .$$

We then have

$$|B^r_{n,1}| = F_n(S)\ W_{n-1}(u)\ S_{n+1-r} \begin{bmatrix} n \\ r-1 \end{bmatrix} \ .$$

The remaining determinant of (2.10) may now be expanded, and the general case has the form

$$\sum_{r=2}^{n+1} (-1)^r\ a_r\ \frac{F_{n+1}(u)}{u_r}\ \frac{F_n(u)}{u_{r-1}}\ |B^r_{n-1,1}|$$

or

$$F_{n-1}(S)\ W_n(u) \sum_{r=2}^{n+1} (-1)^r\ S_{n+1-r} \begin{bmatrix} n+1 \\ r \end{bmatrix} a_r \ .$$

The first two determinants (2.11) round out the summation nicely for k=1 and 2, so that we can state

$$D_n\{a\} = F_{n-1}(S)\ W_n(u) \sum_{r=0}^{n+1} (-1)^r\ S_{n+1-r} \begin{bmatrix} n+1 \\ r \end{bmatrix} a_r$$

or, summing backwards,

$$(2.18) \quad D_n\{a\} = (-1)^{n-1}\ F_{n-1}(S)\ W_n(u) \sum_{r=0}^{n+1} (-1)^r\ S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} a_{n+1-r} \ .$$

At this point we consider the summation

$$(2.19) \qquad \phi_n\{a\} = \sum_{r=0}^{n+1} (-1)^r\ S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} a_{n+1-r}$$

and the associated polynomial

$$(2.20) \qquad \phi_n(x) = \sum_{r=0}^{n+1} (-1)^r\ S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n+1-r}$$

For the first few values of n we have

$$
(2.21)\qquad
\begin{aligned}
\phi_0(x) &= x-1,\\
\phi_1(x) &= x^2-x-1,\\
\phi_2(x) &= x^3-2x^2-2x+1 = (x+1)(x^2-3x+1),\\
\phi_3(x) &= x^4-3x^3-6x^2+3x+1 = (x^2+x-1)(x^2-4x-1),\\
\phi_4(x) &= x^5-5x^4-15x^3+15x^2+5x-1 = (x-1)(x^2+3x+1)(x^2-7x+1) \quad .
\end{aligned}
$$

The factorizations suggest the relation

$$
(2.22)\qquad \phi_n(x) = (-1)^{n-1}\,(x^2 - v_n x + (-1)^n)\,\phi_{n-2}(-x)
$$

where v_n is a Lucas number, and $v_n = u_{n+1} + u_{n-1}$. (2.22) may be proved by induction, and the complete factorization of ϕ_n comes from the identity

$$
v_n = a^n + b^n\,, \quad \text{where } a^2 - a - 1 = b^2 - b - 1 = 0 \quad .
$$

Thus (2.22) becomes

$$
\begin{aligned}
\phi_n(x) = (-1)^{n-1}\,(x-a^n)(x-b^n)\;\phi_{n-2}(-x) = (ab)^{n-1}\\
\cdot\,(x-a^n)(x-b^n)\;\phi_{n-2}\,(x/ab)\,,
\end{aligned}
$$

and we can construct

$$
\phi_n(x) = \prod_{r=0}^{n} (x-a^r b^{n-r}) \quad .
$$

The evaluation of $D_n\{a\}$ for $a_0 = a_1 = \ldots = -1$ becomes, from (2.18) and (2.19), (2.24) $$D_n\{a\} = (-1)^n F_{n-1}(S)\, W_n(u)\, \phi_n(1) .$$

The evaluation of $\phi_n(1)$ requires the investigation of four separate cases. Using (2.23) with $b = -1/a$

(2.25) $$\phi_n(1) = \prod_{r=0}^{n} (1-(-1)^{n-r} a^{2r-n}) = 0 \quad \text{if and only if} \quad n = 4k .$$

When $n = 4k + 2$ the quadratic factorization (2.22) becomes

$$\phi_n(x) = (1+x) \prod_{r=1}^{n/2} (x^2 + (-1)^r v_{2r} + 1)$$

and $$\phi_n(1) = 2 \prod_{r=1}^{n/2} (v_{2r} + 2(-1)^r) .$$

Using the well known relation $v_r^2 = v_{2r} + 2(-1)^r$ we have

(2.26) $$\phi_n(1) = 2 \prod_{r=1}^{n/2} v_r^2 \quad \text{when} \quad n = 4k + 2 .$$

For $n = 4k \pm 1$ we have, from (2.22)

$$\phi_n(x) = \prod_{r=0}^{\frac{n-1}{2}} (x^2 - (-1)^{r+1} v_{2r+1}\, x - 1) , \quad \text{when} \quad n = 4k - 1$$

$$\phi_n(x) = \prod_{r=0}^{\frac{n-1}{2}} (x^2 + (-1)^{r+1} v_{2r+1}\, x - 1) , \quad \text{when} \quad n = 4k + 1$$

so that

(2.27) $$\phi_n(1) = \prod_{r=0}^{\frac{n-1}{2}} (-1)^r v_{2r-1} = S_{2k-1} \prod_{r=0}^{2k-1} v_{2r+1} ,$$

when $n = 4k - 1$

$$(2.28) \qquad \phi_n(1) = \prod_{r=0}^{\frac{n-1}{2}} (-1)^{r+1} v_{2r+1} = S_{2k+1} \prod_{r=0}^{2k} v_{2r+1} ,$$

$$\text{when} \quad n = 4k + 1 .$$

Combining (2.25), (2.26), (2.27), (2.28) with (2.23), and using the sign convention

$$S_n = (-1)^{n(n-1)/2} \quad \text{and} \quad F_n(S) = -1 \quad \text{only when} \quad n = 4k + 2 ,$$

we have

$$D_{4k} = 0 ,$$

$$D_{4k+1} = (-1)^k W_{4k+1}(u) \prod_{r=0}^{2k} v_{2r+1} ,$$

$$D_{4k-1} = (-1)^k W_{4k-1}(u) \prod_{r=0}^{2k-1} v_{2r+1} ,$$

$$D_{4k+2} = 2 W_{4k+2}(u) \prod_{r=0}^{2k-1} v_{r+1}^2 ;$$

$$W_n(u) = u_1^n u_2^{n-1} u_3^{n-2} \ldots u_{n-1}^2 u_n = \prod_{r=1}^{n} u_r^{n+1-r} .$$

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Continued next issue.

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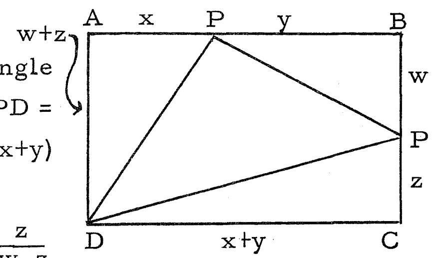
FIBONACCI GEOMETRY

H. E. Huntley

Below are some additional observations about Hunter's [1] article.



If the rectangle ABCD has a triangle DPP' inscribed within it so that $\Delta APD = \Delta BPP' = \Delta P'DC$ then $x(w+z) = wy = z(x+y)$ whence

(i) $$\frac{y}{x} = \frac{w+z}{w} = \frac{z}{w-z}$$

(ii) $$\therefore w^2 - z^2 = wz, \text{ i.e., } w^2 - wz - z^2 = 0 \quad w = \frac{z \pm \sqrt{z^2+4z^2}}{2} = \varphi z$$

(iii) $$\text{From (i) } \frac{y}{x} = \varphi \text{ or } y = \varphi x$$

Thus P, P' divide their sides in the Golden Section.

Now, suppose ABCD is the Golden Rectangle, beloved of the Greek architects, i.e. $AB/BC = \varphi$, then $\frac{x+y}{w+z} = \varphi$. Hence, from (ii) and (iii) $\frac{x(1+\varphi)}{z(1+\varphi)} = \varphi$, i.e. $x = \varphi z$ whence $x = w$. From (i) $y = w+z = \varphi^2 z$. Since $\sphericalangle A = \sphericalangle B = rt\sphericalangle$ and $x = w$, $y = w+z$, triangles PAD, P'BP are congruent. It follows that PD = PP', that $\sphericalangle APD$ is the complement of $\sphericalangle BPP'$, whence $\sphericalangle DPP'$ is a right angle.

The area of the right triangle is

$$\frac{1}{2}(w^2+y^2) = \frac{1}{2}(\varphi^2 z^2 + \varphi^4 z^2) = \frac{1}{2}\varphi^2 z^2(\varphi^2+1) = \frac{1}{2}z^2(\varphi+1)(\varphi+2) = \frac{1}{2}z^2(4\varphi+3).$$

we may conclude, therefore, that if the rectangle is the Golden Rectangle, that is, if its adjacent sides are in the Golden Ratio, φ, then the inscribed triangle is right-angled and isosceles, the length of the equal sides being $z\sqrt{4\varphi+3}$.

Editorial Note: PP' || AC

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ON SUMMATION FORMULAS FOR FIBONACCI AND LUCAS NUMBERS

DAVID ZEITLIN, Honeywell,
Minneapolis, Minnesota

Recently, Siler [1] gave a closed form for $\sum_{k=1}^{n} F_{ak-b}$, where $a > b$ are positive integers and F_k are Fibonacci numbers with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, $n = 0, 1, \ldots$. In this note, we will establish a more general summation formula which yields the result of [1] as a special case. General summation formulas for Fibonacci and Lucas numbers will be obtained as special cases of our general result.

Theorem. Let p, q, u_o, and u_1 be arbitrary real numbers, and let

(1) $$u_{n+2} = q\,u_{n+1} - pu_n \qquad (n = 0, 1, \ldots),$$

(2) $$S_n = r_1^n + r_2^n \qquad (n = 0, 1, \ldots),$$

where $r_1 \neq r_2$ are roots of $x^2 - qx + p = 0$ (i.e., $q^2 - 4p \neq 0$). We define

(3) $$u_{-n} = (u_0 S_n - u_n)/p^n \qquad (n = 1, 2, \ldots),$$

(4) $$S_{-n} = S_n/p^n \qquad (n = 1, 2, \ldots).$$

Let $a = 0, 1, \ldots$; $d = 0, \pm 1, \pm 2, \ldots$, and let x be a real number. Then

(5) $$(1 - S_a x + p^a x^2) \sum_{k=0}^{n} u_{ak+d} x^k = p^a x^{n+2} u_{an+d} - x^{n+1} u_{an+a+d} + x u_{a+d} + (1 - xS_a) u_d \ .$$

Moreover, in the region of convergence, we have

(6) $$(1 - S_a x + p^a x^2) \sum_{k=0}^{\infty} u_{ak+d} x^k = u_d + (u_{a+d} - u_d S_a)\, x \ .$$

Proof. If C_i, $i = 1, 2$, are arbitrary constants, then $u_n = C_1 r_1^n + C_2 r_2^n$, $n = 0, 1, \ldots$, is the general solution of (1). Then

$$v_k = u_{ak+d} = (C_1 r_1^d)(r_1^a)^k + (C_2 r_2^d)(r_2^a)^k, \quad k = 0, 1, \ldots,$$

satisfies the linear difference equation

$$v_{k+2} = S_a v_{k+1} - p^a v_k \qquad (k = 0, 1, \ldots) \tag{7}$$

since $(x^2 - S_a x + p^a) \equiv (x - r_1^a)(x - r_2^a)$. Let

$$g(x) = \sum_{k=0}^{n} v_k x^k .$$

Multiplying both sides of (7) by x^{k+2} and then summing both sides with respect to k, we obtain

$$\sum_{k=0}^{n} v_{k+2} x^{k+2} = xS_a \sum_{k=0}^{n} v_{k+1} x^{k+1} - p^a x^2 \sum_{k=0}^{n} v_k x^k . \tag{8}$$

We note that

$$\sum_{k=0}^{n} v_{k+2} x^{k+2} = g(x) + v_{n+2} x^{n+2} + v_{n+1} x^{n+1} - v_1 x - v_0 , \tag{9}$$

$$\sum_{k=0}^{n} v_{k+1} x^{k+1} = g(x) + v_{n+1} x^{n+1} - v_0 . \tag{10}$$

If we substitute (9) and (10) into (8), use (7) to eliminate v_{n+2}, and solve for $g(x)$, we obtain our principal result, (5).

The generating function for v_k is readily obtained from (5). Let $R > 0$ and suppose that $\sum_{k=0}^{\infty} v_k x^k$ converges for $|x| < R$. Then, for $|x| < R$, $v_n x^n \to 0$ as $n \to \infty$. Thus, for $|x| < R$, (5) yields (6) as $n \to \infty$.

Remarks. Let $q = 1$ and $p = -1$ in (1). Then, for $u_0 = 0$ and $u_1 = 1$, we have $u_n \equiv F_n$, $F_{-n} = (-1)^{n+1} F_n$, and $S_n \equiv L_n$, the well-known Lucas sequence, where $L_0 = 2$ and $L_1 = 1$. Thus, (5) and (6), for $u_n \equiv F_n$, become, respectively,

$$\begin{aligned}(1 - L_a x + (-1)^a x^2) \sum_{k=0}^{n} F_{ak+d} x^k &= (-1)^a x^{n+2} F_{an+d} \\ &\quad - x^{n+1} F_{an+a+d} + xF_{a+d} + (1 - xL_a) F_d ,\end{aligned} \tag{11}$$

$$(1 - L_a x + (-1)^a x^2) \sum_{k=0}^{\infty} F_{ak+d} x^k = F_d + (F_{a+d} - F_d L_a) x . \tag{12}$$

The main result of [1] is obtained from (11) for $x = 1$ and $d = -b$.

For $x = -1$, (11) yields the interesting result

$$(13) \quad \begin{aligned} (1+(-1)^a+L_a) \sum_{k=0}^{n} (-1)^k F_{ak+d} &= (-1)^{n+a} F_{an+d} \\ &+ (-1)^n F_{an+a+d} - F_{a+d} + (L_a+1)F_d \ . \end{aligned}$$

For $d = 0$, (12) yields

$$(14) \qquad (1-L_a x + (-1)^a x^2) \sum_{k=0}^{\infty} F_{ak} x^k = F_a x \ , \qquad (a = 0, 1, \ldots) \ .$$

Again, let $q = 1$ and $p = -1$ in (1). Then, for $u_0 = 2$, and $u_1 = 1$, we now have $u_n \equiv L_n$, $S_n \equiv L_n$, and $L_{-n} = (-1)^n L_n$. Thus, with $u_n \equiv L_n$, (5) and (6) become, respectively,

$$(15) \quad \begin{aligned} (1-L_a x+(-1)^a x^2) \sum_{k=0}^{n} L_{ak+d} x^k &= (-1)^a x^{n+2} L_{an+d} - x^{n+1} L_{an+a+d} \\ &+ x L_{a+d} + (1-xL_a) L_d , \end{aligned}$$

$$(16) \quad (1-L_a x + (-1)^a x^2) \sum_{k=0}^{\infty} L_{ak+d} x^k = L_d + (L_{a+d} - L_a L_d) x \ .$$

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1. Ken Siler, Fibonacci summations, Fibonacci Quarterly, Vol. 1, No. 3, October 1963, pp. 67-69.

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LETTER TO THE EDITOR

The Editor,
Fibonacci Quarterly.

Dear Dr. Hoggatt,

I refer to the article, "Dying Rabbit Problem Revived" in the December 1963 issue. The solution given there is patently wrong — if only because the alleged number of rabbits tends to minus infinity as n tend to infinity. It may easily be shown that the correct answer, X_n, is given by the recurrence relation

$$X_{n+13} = X_{n+12} + X_{n+11} - X_n \ , \quad n \quad 0$$

together with the initial conditions

$$X_n = F_{n+1} \quad \text{for } n = 0, 1, \ldots, 11; \quad X_{12} = 232 \ .$$

In view of the fact that the two equations $z^2 - z - 1 = 0$ and $z^{13} - z^{12} - z^{11} + 1 = 0$ have no common root, it is clear that the answer can never be expressed simply as a linear expression in Fibonacci and Lucas numbers whose coefficients are merely polynomials in n. For, any such expression, Y, where the highest power of n which occurs is n^m, satisfies

$$(E^2 - E - 1)^{m+1} \, Y = 0 \ .$$

In particular the expression found by Bro. Alfred satisfies

$$(E^2 - E - 1)^2 \, Y = 0 \ .$$

The error made by Bro. Alfred stems from his table on p. 54 where the number of dying rabbits in the (n+13)th month is seen to be F_n for $n = 1, 2, \ldots 11$ and it is then assumed without proof that this is true for other values of n. In fact the very next but one value on n, namely $n = 13$ shows that this is false. In fact of course the number of dying rabbits in the (n+13)th month equals the number of bred rabbits in the (n+1)th month, and this will be less than F_n for all n exceeding 12.

Yours sincerely, (John H. E. Cohn)

BEDFORD COLLEGE
(University of London)

SQUARE FIBONACCI NUMBERS, ETC.

JOHN H. E. COHN
Bedford College, University of London, London, N.W.1.

INTRODUCTION

An old conjecture about Fibonacci numbers is that 0, 1 and 144 are the only perfect squares. Recently there appeared a report that computation had revealed that among the first million numbers in the sequence there are no further squares [1]. This is not surprising, as I have managed to prove the truth of the conjecture, and this short note is written by invitation of the editors to report my proof. The original proof will appear shortly in [2] and the reader is referred there for details. However, the proof given there is fairly long, and although the same method gives similar results for the Lucas numbers, I have recently discovered a rather neater method, which starts with the Lucas numbers, and it is of this method that an account appears below. It is hoped that the full proof together with its consequences for Diophantine equations will appear later this year. I might add that the same method seems to work for more general sequences of integers, thus enabling equations like $y^2 = Dx^4 + 1$ to be completely solved at least for certain values of D. Of course the Fibonacci case is simply $D = 5$.

PRELIMINARIES

In the first place, in accordance with the practice of the Fibonacci Quarterly, I here use the symbols F_n and L_n to denote the n-th. Fibonacci and Lucas number respectively; in other papers I use the more widely accepted, if less logical, notation u_n and v_n [3]. Throughout the following n, m, k will denote integers, not necessarily positive, and r will denote a non-negative integer. Also, wherever it occurs, <u>k will denote an even integer, not divisible by 3.</u> We shall then require the following formulae, all of which are elementary

(1) $$2F_{m+n} = F_mL_n + F_nL_m$$

(2) $$2L_{m+n} = 5F_mF_n + L_mL_n$$

(3) $$L_{2m} = L_m^2 + (-1)^{m-1}\,2$$

(4) $$(F_{3m}, L_{3m}) = 2$$

(5) $$(F_n, L_n) = 1 \quad \text{if} \quad 3 \nmid n$$

(6) $$2 \mid L_m \quad \text{if and only if} \quad 3 \mid m$$

(7) $$3 \mid L_m \quad \text{if and only if } m \equiv 2 \pmod 4$$

(8) $$F_{-n} = (-1)^{n-1}F_n$$

(9) $$L_{-n} = (-1)^nL_n$$

(10) $$L_k \equiv 3 \pmod 4 \quad \text{if} \quad 2 \mid k,\ 3 \nmid k$$

(11) $$L_{m+2k} \equiv -L_m \pmod{L_k}$$

(12) $$F_{m+2k} \equiv -F_m \pmod{L_k}$$

(13) $$L_{m+12} \equiv L_m \pmod 8$$

THE MAIN THEOREMS

Theorem 1.

If $L_n = x^2$, then $n = 1$ or 3.

Proof.

If n is even, (3) gives

$$L_n = y^2 \pm 2 \neq x^2 .$$

If $n \equiv 1$ (mod 4), then $L_1 = 1$, whereas if $n \neq 1$ we can write $n = 1 + 2\cdot 3^r\cdot k$ where k has the required properties, and then obtain by (11)

$$L_n \equiv -L_1 = -1 \pmod{L_k}$$

and so $L_n \neq x^2$ since -1 is a non-residue of L_k by (10). Finally, if $n \equiv 3 \pmod 4$ then $n = 3$ gives $L_3 = 2^2$, whereas if $n \not\equiv 3$, we write as before $n = 3 + 2 \cdot 3^r \cdot k$ and obtain

$$L_n \equiv -L_3 = -4 \pmod{L_k}$$

and again $L_n \neq x^2$.

This concludes the proof of Theorem 1.

Theorem 2.

If $L_n = 2x^2$, then $n = 0$ or ± 6.

Proof.

If n is odd and L_n is even, then by (6) $n \equiv \pm 3 \pmod{12}$ and so, using (13) and (9),

$$L_n \equiv 4 \pmod 8$$

and so $L_n \neq 2x^2$.

Secondly, if $n \equiv 0 \pmod 4$, then $n = 0$ gives $L_n = 2$, whereas if $n \not\equiv 0$, $n = 2 \cdot 3^r \cdot k$ and so

$$2L_n \equiv -2L_0 = -4 \pmod{L_k}$$

whence $2L_n \neq y^2$, i.e. $L_n \neq 2x^2$

Thirdly, if $n \equiv 6 \pmod 8$ then $n = 6$ gives $L_6 = 2 \cdot 3^2$ whereas if $n \not\equiv 6$, $n = 6 + 2 \cdot 3^r \cdot k$ where now $4 | k$, $3 \nmid k$ and so

$$2L_n \equiv -2L_6 = -36 \pmod{L_k}$$

and again, -36 is a non-residue of L_k using (7) and (10). Thus as before $L_n \neq 2x^2$.

Finally, if $n \equiv 2 \pmod 8$, then by (9) $L_{-n} = L_n$ where now $-n \equiv 6 \pmod 8$ and so the only admissible value is $-n = 6$, i.e. $n = -6$.

This concludes the proof of Theorem 2.

Theorem 3.

If $F_n = x^2$, then $n = 0, \pm 1$, 2 or 12.

Proof.

If $n \equiv 1 \pmod 4$, then $n = 1$ gives $F_1 = 1$, whereas if $n \neq 1$, $n = 1 + 2 \cdot 3^r \cdot k$ and so

$$F_n \equiv - F_1 = - 1 \pmod{L_k}$$

whence $F_n \neq x^2$. If $n \equiv 3 \pmod 4$, then by (8) $F_{-n} = F_n$ and $-n \equiv 1 \pmod 4$ and as before we get only $n = -1$. If n is even, then by (1) $F_n = F_{\frac{1}{2}n} L_{\frac{1}{2}n}$ and so, using (4) and (5) we obtain, if $F_n = x^2$

either $3 | n$, $F_{\frac{1}{2}n} = 2y^2$, $L_{\frac{1}{2}n} = 2z^2$. By Theorem 2, the latter is possible only for $\frac{1}{2}n = 0$, 6 or -6. The first two values also satisfy the former, while the last must be rejected since it does not.

or $3 \nmid n$, $F_{\frac{1}{2}n} = y^2$, $L_{\frac{1}{2}n} = z^2$. By Theorem 1, the latter is possible only for $\frac{1}{2}n = 1$ or 3, and again the second value must be rejected.

This concludes the proof of Theorem 3.

Theorem 4.

If $F_n = 2x^2$, then $n = 0, \pm 3$ or 6.

Proof.

If $n \equiv 3 \pmod 4$, then $n = 3$ gives $F_3 = 2$, whereas if $n \neq 3$, $n = 3 + 2 \cdot 3^r \cdot k$ and so

$$2F_n \equiv - 2F_3 = -4 \pmod{L_k}$$

and so $F_n \neq 2x^2$. If $n \equiv 1 \pmod 4$ then as before $F_{-n} = F_n$ and we get only $n = -3$. If n is even, then since $F_n = F_{\frac{1}{2}n} L_{\frac{1}{2}n}$ we must have if $F_{\frac{1}{2}n} = 2x^2$

either $F_{\frac{1}{2}n} = y^2$, $L_{\frac{1}{2}n} = 2z^2$; then by Theorems 2 and 3 we see that the only value which satisfies both of these is $\frac{1}{2}n = 0$

or $F_{\frac{1}{2}n} = 2y^2$, $L_{\frac{1}{2}n} = z^2$; then by Theorem 1, the second of these is satisfied only for $\frac{1}{2}n = 1$ or 3. But the former of these does not satisfy the first equation.

This concludes the proof of the theorem.

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1. M. Wunderlich, On the non-existence of Fibonacci Squares, Maths. of Computation, 17 (1963) p. 455.
2. J. H. E. Cohn, On Square Fibonacci Numbers, Proc. Lond. Maths. Soc. 39 (1964) to appear.
3. G. H. Hardy and E. M. Wright, Introduction to Theory of Numbers, 3rd. Edition, O. U. P. 1954, p. 148 et seq.

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EDITORIAL NOTE

Brother U. Alfred cheerfully acknowledges the priority of the essential method, used in "Lucas Squares" in the last issue of the Fibonacci Quarterly Journal, rest solely with J. H. E. Cohn. This was written at the request of the Editor and the unintentional omission of due credit rest solely with the Editor.

EXPLORING THE FIBONACCI REPRESENTATION OF INTEGERS

Proposed by Brother U. Alfred on page 72, Dec. 1963, "The Fibonacci Quarterly"

The completion of the Theorem stated in the article is: The Maximum number of different Fibonacci numbers required to represent an integer N for which $[N]^* = F_n$ is given by $\left[\frac{n}{2}\right]$.

This is a corollary of the following theorem. For $F_n < N \le F_{n+1}$ the number N can be represented as a sum of Fibonacci numbers, the largest which is F_n and the smallest greater than or equal to F_2. Moreover, the sum never contains two consecutive Fibonacci numbers. We therefore have at most the alternating terms of indices from 2 to n which gives us $\left[\frac{n-2}{2}+1\right] = \left[\frac{n}{2}\right]$, as claimed.

The proof of this theorem depends upon a Lemma which is a well known Fibonacci Identity that $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$ and that $F_3 + F_5 + F_7 + \ldots + F_{2n-1} = F_{2n} - 1$. The proof of the first part of this is given by induction and the second part is similarly proved.

Proof.

For n = 1, we have $F_2 = F_3 - 1$

n = 2, we have $F_2 + F_4 = F_5 - 1$ which clearly shows the Lemma holds for n = 1, 2.

Now assume that it holds for all $n \le K$, where K is a fixed but unspecified positive integer greater than or equal to 3.

i.e. $F_2 + F_4 + \ldots + F_{2K} = F_{2K+1} - 1$, therefore by addition to both sides we have that

$$F_2 + F_4 + \ldots + F_{2K} + F_{2K+2} = F_{2K+1} + F_{2K+2} - 1 = F_{2K+3} - 1$$

which implies the Lemma holds for all positive n.

Using this Lemma which we shall call Lemma 1, part A for the first part which was just proved, and part B for the second part with the odd indices; we can now prove the general theorem that for $F_n < N \le F_{n+1}$, we can represent N as a sum of at least alternating Fibonacci numbers where the largest is F_n for $N < F_{n+1}$ and which trivially is just F_{n+1} itself when $N = F_{n+1}$.

Proof. For N = 1, we have $1 = F_2$, and for N = 2, we have $2 = F_3$. Now assume the theorem true for all $N \le k$, where k is a fixed but unspecified positive integer and n is such that $F_n < k \le F_{n+1}$, $n \ge 3$. Now if

Continued on Page 134

PARTITION ENUMERATION BY MEANS OF SIMPLER PARTITIONS

DANIEL C. FIELDER,
Georgia Institute of Technology, Atlanta, Ga.

1. INTRODUCTION

Netto [1] illustrates a method for enumerating all partitions of n having exactly p members, all non-zero. It is shown herein that Netto's procedure can be reduced to an algorithmic form through use of simpler partitions which are limited in range (size and number of members) but otherwise unrestricted. Properties of these range-limited partitions *per se* and a means of adapting them to Netto's enumeration procedure are discussed. An obvious application of the algorithmic procedure is the digital computation of both types of partitions.

2. LIMITED-RANGE, UNRESTRICTED PARTITIONS

Chrystal's [2] partition terminology, suitably modified, is used throughout. Thus, $P(\geqq n_1, \leqq n_2 | \geqq p_1, \leqq p_2 | \geqq q_1, \leqq q_2)$ specifies the enumeration of partitions of all positive integers from n_1 to n_2, inclusive, no partition having less than p_1 nor more than p_2 members, each member being not less than q_1 nor more than q_2. However, the set rather than the enumeration of range-limited partitions is of immediate interest in this paper, and to specify a set a V is appended to the enumeration notation. Accordingly, $PV(\geqq n_1, \leqq n_2 | \geqq p_1, \leqq p_2 | \geqq q_1, \leqq q_2)$ denotes the set of partitions having the properties of the enumeration counterpart.

The existence conditions are $n_2 \geqq n_1$, $p_2 \geqq p_1$, $q_2 \geqq q_1$, $q_2 p_2 \geqq n_1$, $q_1 p_1 \leqq n_2$, simultaneously. Moreover, for fixed n_1, n_2, q_1, q_2, there are optimum extreme values for p_1 and p_2. These are*

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*Brackets [ ] except where obvious for references are used in the customary manner with real numbers to indicate the greatest integer less than or equal to the number bracketed. See Uspensky and Heaslet [3].

(1) $$p_{1\ \text{opt.}} = -\left[-(n_1/q_2)\right] ,$$

(2) $$p_{2\ \text{opt.}} = \left[n_2/q_1 \right] .$$

If $p_1 \leq p_{1\ \text{opt.}}$, p_1 can be changed to $p_{1\ \text{opt.}}$, but if $p_1 > p_{1\ \text{opt.}}$, p_1 cannot be changed. However, if $p_2 \geq p_{2\ \text{opt.}}$, p_2 can be changed to $p_{2\ \text{opt.}}$, but if p_2 $p_{2\ \text{opt.}}$, p_2 cannot be changed.

In generating the partitions, the p_1-member partitions are found first, then the $(p_1 + 1)$-member partitions, etc., until the p_2-member partitions are found. The procedure used herein for the partitions of a typical p-member set is as follows:

A trial "first" partition is formed from p q_1's. If the sum of the p members is equal to or greater than n_1 but less than or equal to n_2, the partition initiates the set. If such is not so, the right-hand member is augmented so that the sum of the p-members in n_1. To form new partitions, the right-hand member is successively increased by one until either it equals q_2 or the sum of the p members equals n_2 (or both). The next p-member trial partition is found by adding one to the member second from the right and replacing all members to the right with the new value of the changed member. The desired reinitiating partition is found from the sum of the p members, as before. The right-hand member is successively increased by one to form new partitions. When the possibilities of the particular second member from the right are exhausted, one is added to the third member from the right and the process repeated all over again. Eventually, all p-member partitions will be accounted for. An example for PV(≥8, ≤10 |≥2, ≤5 |≥2, ≤7) follows:

2, 6	2, 2, 4	2, 2, 2, 2	2, 2, 2, 2, 2
2, 7	2, 2, 5	2, 2, 2, 3	
3, 5	2, 2, 6	2, 2, 2, 4	
3, 6	2, 3, 3	2, 2, 3, 3	
3, 7	2, 3, 4		
4, 4	2, 3, 5		
4, 5	2, 4, 4		
4, 6	3, 3, 3		
5, 5	3, 3, 4		

3. APPLICATION OF NETTO'S METHOD

Netto [1] considers the enumeration $P(n|p|\leq q)$ of the partitions of n having exactly p members with no member greater than q. Netto's method is limited to $q \geq (n+1-p)$ with the existence conditions being $p \leq n$ and $qp \geq n$, simultaneously. In the terminology of this paper,

$$(3) \qquad P(n|p|\leq q) = \sum_t \left[\tfrac{1}{2}(n-p+2-3t_1-4t_2- \ldots -pt_{p-2})\right] ,$$

where $t_\alpha = 0, 1, \ldots, \left[\frac{n-p+2}{\alpha+2}\right]$. Inspection of (3) reveals that the typical term is

$$(4) \qquad \left[\frac{n-p+2-w}{2}\right] ,$$

in which w is always zero for $t_\alpha = 0$, always 3 for $t_\alpha = 1$, and always greater than 3 for all other t_α's. It can be observed that except for the zero value of w, each w in the enumeration $P(n|p|\leq q)$ is the sum of the members of each partition included in the set

$$(5) \qquad PV(\geq 3, \leq n-p \,|\, \geq 1, \leq [\tfrac{n-p}{3}] \,|\, \geq 3, \leq p) .$$

Thus, except for $\rho = 1$,

$$(6) \qquad P(n|p|\leq q) = \left[\frac{n-p+2}{2}\right] + \sum_i \left[\frac{n-p+2-w_i}{2}\right] .$$

It should be noted that (5) does not exist for $p = 2$, and/or $(n-p) < 3$. There are no w_i's under these conditions, and the summation term of (6) is accordingly zero. The special case of $p = 1$ is

$$(7) \qquad P(n|1|\leq q) = 1 .$$

As was stated earlier, the methods described herein are particularly adaptable to digital computations. To this end, the author can supply a limited number of ALGOL language programs and test examples for enumerating partitions with the Burroughs 220 digital computer.

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1. E. Netto, Lehrbuch der Combinatorik, Leipsiz, 1901, pp. 127, 128.
2. G. Chrystal, Textbook of Algebra, Vol. 2, (Reprint) Chelsea Publishing Co., New York, 1952.
3. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill Book Co., New York, 1939, pp. 94-99.

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CORRECTIONS FOR VOLUME 1, NO. 3

Page 44: On line 4 read "$0 \le k \le 2^r - 1$" for "$0 \le k \le 2^r$"

Page 49: On line 8 read $[mF_n]/F_m$ for $[mF_n]\,F_m$

Page 80: In B-7 line 2 $x = 1/4$ and $\sum_{i=0}^{\infty} F_i^2/4^i = \frac{12}{25}$?

FURTHER CORRECTIONS FOR VOLUME 1, NO. 4

Reference 4 The first author is IVAN NIVEN .

In H-25 $(i, j = 1, 2, 3, 4)$

A NOTE ON WARING'S FORMULA FOR SUMS OF LIKE POWERS OF ROOTS

S.L. BASIN
Sylvania Electronic Systems, Mountain View, California

Sums of powers $S_k = x_1^k + x_2^k + \ldots + x_n^k$ may be expressed in terms of elementary symmetric functions or in terms of the coefficients of:

$$f(x) = (x-x_1)(x-x_2) \ldots (x-x_n) = x^n + p_1 x^{n-1} + \ldots + p_n$$

by Newton's formulas, usually introduced in a course in the theory of equations, for example, J. V. Uspensky [1].

The relationship between Waring's formula for sums of like powers of the roots of a quadratic and Lucas numbers is quite obvious although perhaps a little too specialized for L. E. Dickson [2] to have pointed this out in his text, First Course in the Theory of Equations.

In order to obtain an explicit expression for S_k where $k = 1, 2, 3 \ldots$, first consider the quadratic

$$x^2 + px + q = 0 \quad . \tag{1}$$

If we denote the roots by α and β then (1) may be rewritten as

$$x^2 + px + q = (x-\alpha)(x-\beta) \quad . \tag{2}$$

After making the transformation $x = 1/y$ and multiplying by y^2 we obtain,

$$1 + py + qy^2 = (1-\alpha y)(1-\beta y) \quad . \tag{3}$$

Differentiating both sides of (3) with respect to y and dividing both members of the differentiated equation by the corresponding members of (3) we arrive at

$$(4) \qquad \frac{-p-2qy}{1+py+qy^2} = \frac{\alpha}{1-\alpha y} + \frac{\beta}{1-\beta y} .$$

Equations,

$$(5) \qquad \frac{\alpha}{1-\alpha y} = \alpha + \alpha^2 y + \ldots + \alpha^k y^{k-1} + \frac{\alpha^{k+1}}{1-\alpha y} y^k$$

and

$$(6) \qquad \frac{\beta}{1-\beta y} = \beta + \beta^2 y + \ldots + \beta^k y^{k-1} + \frac{\beta^{k+1}}{1-\beta y} y^k$$

are both obtained from the geometric series

$$(7) \qquad \frac{1}{1-r} = \sum_{j=0}^{k-1} r^j + \frac{r^k}{1-r} \quad ;$$

for example, let $r = \alpha y$ and multiply by α. Addition of (5) and (6) results in,

$$(8) \quad \frac{\alpha}{1-\alpha y} + \frac{\beta}{1-\beta y} = S_1 + S_2 y + \ldots + S_k y^{k-1} + \frac{\alpha^{k+1}(1-\beta y) + \beta^{k+1}(1-\alpha y)}{(1-\alpha y)(1-\beta y)}$$

where $$S_k = \alpha^k + \beta^k .$$

In order to expand the left-hand member of (4) using (7), let $r = -py -qy^2$, then

$$(9) \qquad \frac{1}{1+py+qy^2} = \sum_{j=0}^{k-1} (-1)^j (py+qy^2)^j + \frac{(-p-qy)^k y^k}{1+py-qy^2}$$

Employing the binomial theorem we may write

$$(py+qy^2)^j = \Sigma \frac{(g+h)!}{g!\,h!} (py)^g (qy^2)^h$$

where the summation is taken over all two-part partitions of j, i.e., for all $g \geq 0$ and $h \geq 0$ such that $g + h = j$. Therefore,

$$\frac{-p-2qy}{1+py-qy^2} = (p+2qy)\,\Sigma\,(-1)^{g+h+1}\,\frac{(g+h)!}{g!\,h!}\,p^g q^h y^{g+2h} + \frac{(-p-2qy)(-p-qy)^k y^k}{1+py+qy^2}\,. \tag{10}$$

Now the left-hand members of (8) and (10) are equal as shown by equation (4); therefore we may equate coefficients of like powers of y. Specifically, equating coefficients of y^{k-1} we arrive at

$$S_k = \Sigma\,(-1)^{i+j}\,\frac{(i+j-1)!}{(i-1)!\,j!}\,p^i q^i + 2\,\Sigma\,(-1)^{i+j} \cdot \frac{(i+j-1)!}{i!\,(j-1)!}\,p^i q^j \tag{11}$$

where we have replaced i for g + 1 and j for h in (10). The summations in (11) now extend over all $i \geq 0$ and $j \geq 0$ such that $i + 2j = k$. Combining both summations in (11) we have,

$$S_k = k\,\Sigma\,(-1)^{i+j}\,\frac{(i+j-1)!}{i!\,j!}\,p^i q^j \tag{12}$$

summed over all $i \geq 0$, $j \geq 0$ such that $i + 2j = k$. Clearly, for $p = q = -1$ we have $S_k = L_k$, the kth Lucas number; and (12) becomes

$$L_k = k \sum_{\substack{i \geq 0 \\ j \geq 0 \\ i+2j=k}} \frac{(i+j-1)!}{i!\,j!}\,. \tag{13}$$

Equation (12) of Waring's formula, published in 1762, which can be extended to include the sum of kth powers of the roots of n th degree polynomials.

The main point to observe is not necessarily the meager result given by (13) but the fact that implicit in the development of Waring's formula lies the generating function (4) for the Lucas numbers.

REFERENCES

1. J. V. Uspensky, Theory of Equations, McGraw-Hill, 1948.
2. L. E. Dickson, First Course in the Theory of Equations, John Wiley, 1922.

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OMISSIONS

H-3 and H-8 were also solved by John L. Brown, Jr., The Pennsylvania State University, State College, Penn.

The solution to H-16 given in the last issue was compounded from solutions given by L. Carlitz and John L. Brown, Jr. The varitypist omitted the credit line.

H-13 was also solved by John H. Halton, University of Colorado at Boulder, Colorado.

H-15 was also solved by L. Carlitz, Duke University, Durham, N. C.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, Jr.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-34 *Proposed by Paul F. Byrd, San Jose State College, San Jose, California*

Derive the series expansions

$$J_{2k}(\alpha) = I_k^2(\alpha) + \sum_{m=1}^{\infty} (-1)^{m+k} I_{m+k}(\alpha) I_{m-k}(\alpha) L_{2m}$$

($k = 0, 1, 2, 3, \ldots$) for the Bessel functions J_{2k} of all even orders, where L_n are Lucas numbers and I_n are modified Bessel functions.

H-35 *Proposed by Walter W. Horner, Pittsburgh, Pa.*

Select any nine consecutive terms of the Fibonacci sequence and form the magic square

u_8	u_1	u_6
u_3	u_5	u_7
u_4	u_9	u_2

show

$$u_8u_1u_6 + u_3u_5u_7 + u_4u_9u_2 =$$

$$u_8u_3u_4 + u_1u_5u_9 + u_6u_7u_2 .$$

Generalize.

H-36 *Proposed by J.D.E. Konhauser, State College, Pa.*

Consider a rectangle R. From the upper right corner of R remove a rectangle S (similar to R and with sides parallel to the sides of R. Determine the linear ratio $K = L_R/L_S$ if the centroid of the remaining L shaped region is where the lower left corner of the removed rectangle was.

H-37 *Proposed by H.W. Gould, West Virginia University, Morgantown, West. Va.*

Find a triangle with sides n+1, n, n-1 having integral area. The first two examples appear to be 3, 4, 5 with area 6; and 13, 14, 15 with area 84.

H-38 *Proposed by R.G. Buschman, Suny, Buffalo, N.Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations Vol. 1, No. 4, Dec. 1963, pp. 1-7.)

Show

$$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r$$

H-39 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the difference equation in closed form

$$C_{n+2} = C_{n+1} + C_n + F_{n+2} ,$$

where $C_1 = 1$, $C_2 = 2$, and F_n is the nth Fibonacci number. Give two separate characterizations of these numbers.

H-40 *Proposed by Walter Blumberg, New Hyde Park, L.I., N.Y.*

Let U, V, A and B be integers, subject to the following conditions

$$\text{(i)}\ \ U > 1, \qquad \text{(ii)}\ \ (U, 3) = 1; \qquad \text{(iii)}\ \ (A, V) = 1;$$

$$\text{(iv)}\ \ V = \sqrt{(u^2-1)/5} .$$

Show A^2U+BV is not a square.

SOLUTIONS

EXPANSIONS OF BESSEL FUNCTIONS IN TERMS OF FIBONACCI NUMBERS

P-2 *Proposed by P.F. Byrd, San Jose State College, San Jose, California.*

Derive the series expansions

$$J_0(\alpha) = \sum_{k=0}^{\infty} (-1)^k \quad I_k^2(\alpha) - I_{k+1}^2(\alpha) \quad F_{2k+1} \ ,$$

where J_0 and I_k are Bessel functions, with F_{2k+1} being Fibonacci numbers.

Solution by the proposer, P.F. Byrd, San Jose State College, San Jose, California

Note: This is a corrected version. Initially this read $J_0(x) = \ldots$ which does not make sense.

It is just as easy to derive the more general series expansions of $J_{2p}(\alpha)$, $(p=0, 1, 2, \ldots)$, for Bessel functions of all even orders, and then to obtain the desired result as a special case upon setting $p = 0$. We make principal use of formulas (6.1) and (6.5) presented in [1]. Since $J_{2p}(\alpha)$ is an even function, we first seek a polynomial expansion of the form

$$J_{2p}(\alpha) = \sum_{k=0}^{\infty} \beta_{2k,\,p}(\alpha)\ \varphi_{2k+1}(x) \ ,$$

where from equation (6.5) and [1] the coefficients are given by

$$\beta_{2k,\,p}(\alpha) = \frac{(i)^{2k}}{\pi} \int_0^{\pi} J_{2p}(-2i\,\alpha\cos v)\left[\cos 2kv - \cos(2k+2)v\right] dv \ ,$$

with $_{2k+1}$ being the Fibonacci polynomials defined in [1]. Now it is known (e.g., see [2]) that

$$\int_0^{\pi} J_{2p}(-2i\alpha \cos v) \cos 2mv\, dv = \pi\ J_{p+m}(-i\alpha)\, J_{p-m}(-i\alpha) \ ,$$

and also that $J_n(iz) = i^n I_n(-z) = (-i)^n I_n(z)$. Hence we easily obtain

$$\beta_{2k,\,p} = (-1)^{p+k}\left[I_{k+p}(\alpha) I_{k-p}(\alpha) - I_{k+p+1}(\alpha) I_{k-p+1}(\alpha)\right] \ .$$

Finally, taking $x = 1/2$, and thus with $\varphi_{2k+1}(1/2) = F_{2k+1}$, we have the formal expansions

$$J_{2p}(\alpha) = \sum_{k=0}^{\infty} (-1)^{k+p} \left[I_{k+p}(\alpha) I_{k-p}(\alpha) - I_{k+p+1}(\alpha) I_{k-p+1}(\alpha) \right] F_{2k+1} \;.$$

which in particular yield the solution to the proposed problem upon setting $p = 0$.

REFERENCES

1. P. F. Byrd, Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers, The Fibonacci Quarterly, Vol. 1, No. 1, pp. 16-29.

2. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 2nd Edition, 1944, p. 151.

Note: Corrected statements to P-1.

Verify that the polynomials $\varphi_{k+1}(x)$ satisfy the differential equation

$$(1+x^2)y'' + 3xy' - k(k+2)y = 0 \;,$$

$(k = 0, 1, 2, 3, \ldots)$.

Readers are requested to submit solutions to the problems in the above mentioned reference [1].

CORRECTIONS IN SAME PAPER

Page 19 Replace 2] by [2] in line 11.

Page 20 In line 5 read (3) as relation[3] referring to footnote 3 In view of ... .

Page 21 Read $\varphi_{j+1(x)}$ as $\varphi_{j+1}(x)$ in (4.7)

Page 23 In (5.6) place absolute value bars around the quantity approaching the limit zero.

SYMBOLIC RELATIONS

H-18 *Proposed by R.G. Buschman, Oregon State University, Corvallis, Oregon*

"Symbolic relations" are sometimes used to express identities. For example, if F_n and L_n denote, respectively, Fibonacci and

Lucas numbers, then

$$(1 + L)^n \doteq L_{2n}, \qquad (1 + F)^n \doteq F_{2n}$$

are known identities, where $\doteq$ denotes that the exponents on the symbols are to be lowered to subscripts after the expansion is made.

(a) Prove $(L + F)^n \doteq (2F)^n$.

(b) Evaluate $(L + L)^n$.

(c) Evaluate $(F + F)^n$.

(d) How can this be suitably generalized?

Solution by the proposer (now at Suny, Buffalo, N.Y.)

Consider the generating functions:

$$2\, e^{u/2} \sinh(u\sqrt{5}/2) = \sqrt{5} \sum_{n=0}^{\infty} F_n\, u^n/n! \quad ,$$

$$2\, e^{u/2} \cosh(u\sqrt{5}/2) = \sum_{n=0}^{\infty} L_n\, u^n/n! \quad .$$

From these and the product formula for power series we can write

(a) $$\sqrt{5} \sum_{n=0}^{\infty} F_n\, 2^n u^n/n! = 2\, e^u \sinh(u\sqrt{5}) = \sqrt{5} \sum_{n=0}^{\infty} \sum_{k=0} \frac{F_k\, L_{n-k}}{k!\,(n-k)!}\, u^n \quad ,$$

(b) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{L_k\, L_{n-1}}{k!\,(n-k)!}\, u^n = 2\, e^u (\cosh u\sqrt{5} + 1) = \sum_{n=0}^{\infty} \frac{L_n\, 2^n + 2}{n!}\, u^n \quad ,$$

(c) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{F_k\, F_{n-k}}{k!\,(n-k)!}\, u^n = 2\, e^u (\cosh u\sqrt{5} - 1)/5 = \sum_{n=0}^{\infty} \frac{L_n\, 2^n - 2}{n!}\, u^n \quad .$$

Equating coefficients and multiplying by $n!$ then gives

(a) $$\sum_{k=0}^{n} \binom{n}{k} F_k\, L_{n-k} = 2^n\, F_n \qquad \text{or} \quad (F + L)^n = (2F)^n \,,$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} L_k\, L_{n-k} = 2^n\, L_n + 2 \qquad \text{or} \quad (L + L)^n = (2L)^n + 2 \,,$$

(c) $$\sum_{k=0}^{n} \binom{n}{k} F_k\, F_{n-k} = (2^n\, L_n - 2)/5 \qquad \text{or} \quad (F + F)^n = (2L)^n - 2)/5 \,.$$

Solution by L. Carlitz, Duke University, Durham, N.C.

As noted by Gould (this Quarterly, Vol. 1 (1963), p. 2) we have

$$\frac{e^{ax} - e^{bx}}{a-b} = \sum_{n=0}^{\infty} \frac{x^n}{n!} F_n , \qquad e^{ax} + e^{bx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n ,$$

$$a = \frac{1}{2}(1 + \sqrt{5}), \quad b = \frac{1}{2}(1 - \sqrt{5}) .$$

It follows at once that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} (L + F)^n = \frac{e^{2ax} - e^{2bx}}{a-b} ,$$

so that

(a) $$(L + F)^n = 2^n F_n .$$

Similarly

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} (L + L)^n = e^{2ax} + 2e^{(a+b)x} + e^{2bx}$$
$$= e^{2ax} + e^{2bx} + 2e^x ,$$

so that

(b) $$(L + L)^n = 2^n L_n + 2 ,$$

while

$$(\alpha - \beta)^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} (F + F)^n = e^{2ax} + e^{2bx} - 2e^x ,$$

so that

(c) $$5(F + F)^n = 2^n L_n - 2 .$$

To generalize these formulas consider

$$(e^{ax} + e^{bx})^r = \sum_{s=0}^{r} \binom{r}{s} e^{(r-s)ax+sbx} =$$

$$= \frac{1}{2} \sum_{s=0}^{r} \binom{r}{s} e^{sx} (e^{(r-2s)ax} + e^{(r-2s)bx})$$

$$= \frac{1}{2} \sum_{s=0}^{r} \binom{r}{s} e^{sx} \sum_{n=0}^{\infty} \frac{(r-2s)^n x^n}{n!} L_n .$$

Therefore

$$(L+L+\ldots+L)^n = \frac{1}{2}\sum_{s=0}^{r}\binom{r}{s}\sum_{k=0}^{n}\binom{n}{k}(r-2s)^k s^{n-k} L_k$$

$$= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} L_k \sum_{s=0}^{r}\binom{r}{s}(r-2s)^k s^{n-k} .$$

In particular

$$(L+L+L)^n = 3^n L_n + 3L_{2n} .$$

Similarly, since

$$5^r(F+\ldots+F)^n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} L_k \sum_{s=0}^{2r}(-1)^s\binom{2r}{s}(2r-2s)^k s^{n-k} ,$$

where the number of F's is 2r, and

$$5^r(F+\ldots+F)^n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} F_k \sum_{s=0}^{2r-1}(-1)^s\binom{2r+1}{s}(2r-2s+1)^k s^{n-k} ,$$

where the number of F's is 2r+1. In particular

$$5(F+F+F)^n = 3^n F_n + 3F_{2n} .$$

A formula for

$$(L+\ldots+L+F+\ldots+F)^n$$

can be obtained but it is very complicated. When the number of L's is equal to the number of F's it is less complicated. In particular

$$(L+L+F+F)^n = \frac{1}{5}(4^n L_n - 2^{n+1}) .$$

Indeed since

$$(e^{ax}+e^{bx})^r(e^{ax}-e^{bx})^r = (e^{2ax}-e^{2bx})^r$$

it follows that

$$(\underbrace{L+\ldots+L}_{r}+\underbrace{F+\ldots+F}_{r})^n = 2^n(\underbrace{F+\ldots+F}_{r})^n .$$

In particular

$$(L+L+L+F+F+F)^n = \frac{2^n}{5}(3\,F_n - 3F_{2n}) .$$

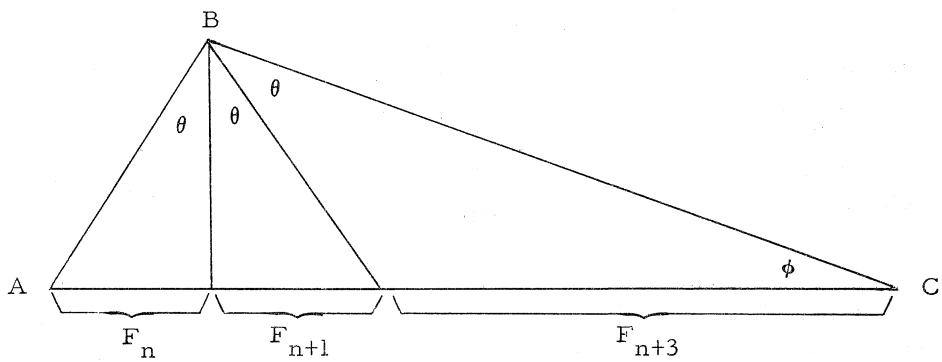
THE RACE

H-19 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle, B, divide side, AC, into segments of length F_n, F_{n+1}, F_{n+3}. Find:

(i) $\lim_{n \to \infty} \theta$

(ii) $\lim_{n \to \infty} \varphi$



Solution by Michael Goldberg, Washington, D.C.

As $n \to \infty$, the ratio F_{n+1}/F_n approaches $t = (\sqrt{5} + 1)/2$, and F_{n+3}/F_n approaches $t^3 = 2t + 1$. Hence, the limiting triangle ABC can be drawn by taking points D and E on AC so that AD = 1, DE = t and EC = $t^3 = 2t + 1$. Since BD is a bisector of angle ABE, the point B must lie on the circle which is the locus of points whose distances to A and E are in the ratio AD/DE = 1/t. The circle passes through D. If the diameter of the circle is $2r = x + 1$, then $x/(x + 1 + t) = 1/t$ from which

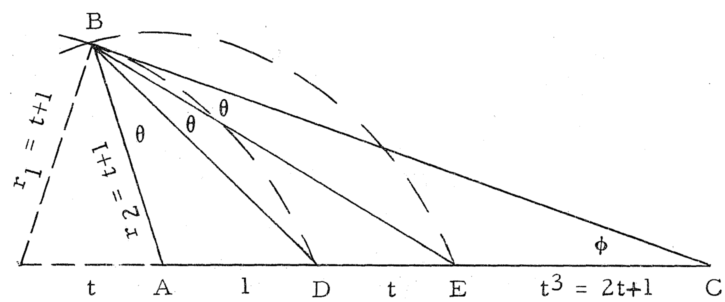
$$r_1 = t/(t - 1) = t^2 = t + 1 \quad .$$

Similarly, BE is a bisector of the angle DBC. The point B must lie on a circle which is the locus of points whose distances from D and C are in the ratio $DE/EC = t/t^3 = 1/t^2$. If the diameter of the circle is $2r_2 = y + t$, then $y/(y + t + t^2) = 1/t^2$ from which

$$r_2 = t_2^2 = t + 1 = r_1 \quad .$$

Hence, $\cos < BAE = -t/2(t+1) = -(\sqrt{5}-1)/4$ and $< BAE = 108^\circ$. From which $2\theta = 90^\circ - 108^\circ/2 = 36^\circ$; $\theta = 18^\circ$, $\phi = 180^\circ - 108^\circ - 3\theta = 18^\circ$

Also solved by the proposer and Raymond Whitney, Penn. State University, Hazelton, Penn.



FIBONACCI TO LUCAS

H-20 *Proposed by Verner E. Hoggatt, Jr., and Charles H. King, San Jose State College, San Jose, California.*

If

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{show} \quad D(e^{Q^n}) = e^{L_n},$$

where $D(A)$ is the determinant of matrix A and L_n is the n^{th} Lucas number.

Solution by John L. Brown, Jr., Penn. State University, State College, Penn.

Recall that

$$Q_n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

so that (by definition)

$$e^{Q^n} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} & \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} \\ \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} & \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} \end{pmatrix}$$

It is well-known that

$$\sum_{k=0}^{\infty} \frac{F_{nk}}{k!} x^k = \frac{e^{a^n x} - e^{b^n x}}{\sqrt{5}}$$

[e.g. equation (2.11), p. 5 of Gould's paper in Vol. 1, No. 2, April 1963], where

$$a = \frac{1+\sqrt{5}}{2}$$

and

$$b = \frac{1-\sqrt{5}}{2} \quad .$$

Similarly,

$$\sum_{k=0}^{\infty} \frac{L_{nk}}{k!} x^k = e^{a^n x} + e^{b^n x}$$

for the Lucas numbers.

But $L_{nk} = F_{nk+1} + F_{nk-1}$; therefore,

$$\sum_{k=0}^{\infty} \frac{F_{nk+1} + F_{nk-1}}{k!} = e^{a^n} + e^{b^n}, \qquad \text{or}$$

$$\text{(1)} \qquad \sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = (e^{a^n} + e^{b^n}) - \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} \quad .$$

Since $F_{nk+1} = F_{nk} + F_{nk-1}$ and $\displaystyle\sum_{k=0}^{\infty} \frac{F_{nk}}{k!} = \frac{e^{a^n} - e^{b^n}}{\sqrt{5}}$

from above, we also have

$$\text{(2)} \quad \sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} + \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} = \frac{e^{a^n} - e^{b^n}}{\sqrt{5}} + \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} \, .$$

Solving (1) and (2) simultaneously, we find

$$\text{(3)} \qquad \sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = \frac{1}{2}\left[\left(e^{a^n} + e^{b^n}\right) + \frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right]$$

and

$$\text{(4)} \qquad \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} = \frac{1}{2}\left[\left(e^{a^n} + e^{b^n}\right) - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)\right]$$

Now, $\displaystyle D(e^{Q^n}) = \left(\sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!}\right) - \left(\sum_{k=0}^{\infty} \frac{F_{nk}}{k!}\right)^2$

$$= \frac{1}{4}\left[\left(e^{a^n} + e^{b^n}\right)^2 - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)^2\right] - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)^2$$

$$= e^{a^n+b^n} = e^{L_n}, \text{ since } L_n = a^n+b^n \text{ for } n \geq 0. \quad \underline{\text{q.e.d.}}$$

FIBONACCI PROBABILITY

H-21 *Proposed by Francis D. Parker, University of Alaska, College, Alaska*

Find the probability, as n approaches infinity, that the n^{th} Fibonacci number, $F(n)$, is divisible by another Fibonacci number ($\neq F_1$ or F_2).

Solution by proposer

We use frequently the fact that $F(n)$ is divisible by $F(k)$ if k divides n. Then the probability that $F(n)$ is divisible by 2 is $1/3$; the probability that $F(n)$ is divisible by 3 but not 2 is $(\frac{1}{4})(\frac{2}{3})$; the probability that $F(n)$ is divisible by 5 but not by 2 or by 3 is $\frac{1}{5}\frac{2}{3}\frac{3}{4} = \frac{2}{4.5}$; and in general the probability that $F(n)$ is divisible by $F(k)$ but not any Fibonacci number of order less than k is $\frac{2}{k(k-1)}$. These probabilities are all independent, so that the probabilities that $F(n)$ is divisible by at least one Fibonacci number of order not exceeding k is

$$\frac{3}{3.2} + \frac{2}{4.3} + \frac{2}{5.4} + \cdots \frac{2}{k(k-1)} .$$

This sum is $\frac{k-2}{k}$, and as n approaches infinity, the probability approaches unity.

Also solved by J. L. Brown, Jr., Penn. State University, State College, Pennsylvania.

Continued from Page 114

$k = F_{n+1}$, we have that $k+1 = F_{n+1} + F_2$, and we are through. If $k = F_{n+1} - 1$, we have $k+1 = F_{n+1}$ and we are through. If $k = F_{n+1} - 2$ we have $k+1 = F_{n+1} - 1$, which by Lemma 1 (A or B), can be represented as claimed and we are through again. Therefore let us consider $k \le F_{n+1} - 3$.

Now the representation for k in this form can best be expressed as $k = F_n + a_{n-2} F_{n-2} + a_{n-3} F_{n-3} + a_{n-4} F_{n-4} + \cdots + a_3F_3 + a_2F_2$ where $a_i = 0$ or 1 for $2 \le i \le n-2$, and $a_i = 1$, implies that $a_i \pm 1 = 0$. Now there are only two possibilities for a_2 and a_3 in this representation. Either $a_2 = a_3 = 0$, or $a_2 \neq a_3$. If the first case is true for k, we can represent k+1 in the required manner, simply by adding 1 to k in the form of $a_2 = 1$. If the second case is true for k, we then claim that there exists at least one place in the representation where $a_i = a_{i+1} = 0$, since otherwise, $k = F_{n+1} - 1$ which we have already taken care of above.

Therefore we can represent k+1 by the following:

$$k+1 = F_n + a_{n-2}F_{n-2} + \cdots + a_{i+2}F_{i+2} + a_{i-1}F_{i-1} + \cdots + a_3F_3 + a_2F_2 + 1$$

Now consider the expression from $a_{i+2}F_{i+2}$ on and the resulting inequality.

$$a_{i+2}F_{i+2} + a_{i-1}F_{i-1} + \cdots + a_3F_3 + a_2F_2 + 1 \le F_{i+3} - 1 \le F_{n-1} - 1,$$

by our Inductive Assumption. Also by the Inductive Assumption, we can represent the expression from $a_{i+2}F_{i+2}$ on in the proper form which implies that we can then also represent k+1 in the proper form. This shows that the proof holds for all positive integers N. Q.E.D.

Phil Lafer, Oak Harbor, Ohio

BEGINNERS' CORNER

Edited by DMITRI THORO
San Jose State College

THE EUCLIDEAN ALGORITHM II

1. INTRODUCTION

In Part I [1] we saw that the greatest common divisor of two numbers could be conveniently computed via the famous Euclidean algorithm. Suppose that exactly n steps (divisions) are required to compute the g. c. d. of s and t $(s \geq t)$. We then have

$$
\begin{array}{lll}
(1) & s = t q_1 + r_1, & 0 < r_1 < t \\
(2) & t = r_1 q_2 + r_2, & 0 < r_2 < r_1 \\
(3) & r_1 = r_2 q_3 + r_3, & 0 < r_3 < r_2 \\
(4) & r_2 = r_3 q_4 + r_4, & 0 < r_4 < r_3 \\
(5) & r_3 = r_4 q_5 + r_5, & 0 < r_5 < r_4 \\
\vdots & & \\
(n-1) & r_{n-3} = r_{n-2} q_{n-1} + r_{n-1}, & 0 < r_{n-1} < r_{n-2} \\
(n) & r_{n-2} = r_{n-1} q_n + 0 \ . &
\end{array}
$$

Since each quotient $q_i \geq 1$, the above equations imply

$$
\begin{array}{ll}
(2') & t \geq r_1 + r_2 \\
(3') & r_1 \geq r_2 + r_3 \\
(4') & r_2 \geq r_3 + r_4 \\
(5') & r_3 \geq r_4 + r_5
\end{array}
$$

etc.

From (2') and (3'), $t \geq 2r_2 + r_3$; but from (4'), $2r_2 + r_3 \geq (2r_3 + 2r_4) + r_3$. Similarly, from (5'), $3r_3 + 2r_4 \geq (3r_4 + 3r_5) + 2r_4$, etc. Continuing in this manner we note the generous abundance of Fibonacci numbers. Thus

$$t \geq r_1 + r_2 \geq 2r_2 + r_3 \geq 3r_3 + 2r_4 \geq 5r_4 + 3r_5$$

$$\cdots \geq F_{n-1}r_{n-2} + F_{n-2}r_{n-1} .$$

2. A BASIC RESULT

Since the remainders form a strictly decreasing sequence with r_{n-1} the last non-zero remainder,

$$r_{n-2} > r_{n-1} \geq 1 .$$

Consequently,

$$t \geq F_{n-1}r_{n-2} + F_{n-2}r_{n-1} \geq 2F_{n-1} + F_{n-2} = F_{n+1} .$$

To summarize, if n divisions are required to compute the g.c.d. of s and t, then t is at least as large as the $(n+1)^{st}$ Fibonacci number!

3. LAMÉ'S THEOREM

Although the Euclidean algorithm is over 2,000 years old, the following result was established by Gabriel Lamé in 1844.

Theorem

The number of divisions required to find the g.c.d. of two numbers is never greater than five times the number of digits in the smaller number.

Proof.

Let ϕ designate the golden ratio. In [2] it was shown that

$$\phi^n = F_n\phi + F_{n-1}, \qquad n = 1, 2, 3, \ldots$$

Now since $2 > \phi = (1 + \sqrt{5})/2$, we see that

$$2F_n + F_{n-1} > F_n\phi + F_{n-1} \quad \text{or}$$

$$F_{n+2} > \phi^n .$$

Replacing n by n-1 and using the "basic result" of the preceding section yields

$$t > \phi^{n-1} .$$

To complete the proof note that

(i) if t has d digits then $d > \log t$

(ii) $\log t > (n-1) \log \phi$

(iii) $\log \phi > 1/5$.

Thus $d > (n-1)/5$ or $n \leq 5d$.

REFERENCES

1. D. E. Thoro, "The Euclidean Algorithm I," Fibonacci Quarterly, Vol. 2, No. 1, February 1964.
(Note that in excercises E8 and E10, (F_{n+1}, F_n) and $\max_n (n, F-1)$ should be replaced by $N(F_{n+1}, F_n)$ and $\max_n N(n, F-1)$ respectively.)

2. D. E. Thoro, "The Golden Ratio: Computational Considerations," Fibonacci Quarterly, Vol. 1, No. 3, October 1963, pp. 53-59.

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EXPLORING FIBONACCI NUMBERS WITH A CALCULATOR

BROTHER U. ALFRED
St. Mary's College, California

It has often been noted that the study of numbers is both experimental and theoretical in character. Even in the days before the calculator, to say nothing of the computer, the truly great mathematicians often arrived at beautiful results on the basis of observation and numerical work before they proceeded to proof and theoretical justification. If we find ourselves enjoying calculation and seeing tangible results, we are in very good company and need not worry about the attitude of the theorist who is afraid to soil his hands with numbers.

In this vein, the following exploration is proposed. It may be observed by looking at a list of Fibonacci numbers that in certain cases F_n has the n corresponding to the terminal digits of the number. Thus F_5 is 5; F_{29} is 514229; F_{61} is 2504730781961. As long as we have a table of Fibonacci numbers on hand we can proceed to make such verifications. But suppose we set out to find all these coincidences for Fibonacci numbers up to a certain level such as $F_{10,000}$. In the absence of these numbers we now have an interesting mathematical problem involving computation.

One very simple way to proceed would be to take the successive Fibonacci numbers modulo 10,000. In other words we would consider only the last five digits and forget about all those that go before. This is a straightforward procedure but it would be long and tedious and subject to error. In fact, once a mistake in introduced, all results thereafter would be vitiated. There must be a better way. Perhaps there are several ways. We shall look forward to both the numerical results and the method employed in arriving at them.

Address all communications regarding this problem to: Brother U. Alfred, St. Mary's College, California. The solution will appear in the issue of December, 1964.

AMATEUR INTERESTS IN THE FIBONACCI SERIES – PRIME NUMBERS

JOSEPH MANDELSON
U.S. Army Edgewood Arsenal, Maryland

My interest in the Fibonacci series was born in 1959 when it was noticed that the preferred ratios developed in the research of my colleague, H. Ellner, and later included in Department of Defense Handbook H109 [1], were 1, 2, 3, 5 and 8. From recollection of a brief mention in college algebra, this was recognized as the first few terms of the Fibonacci. To test the supposition that the preferred ratios would all be from this series, the next one was calculated and, sure enough, it was 13. Then it was noted that the sample sizes, Acceptable Quality Levels (AQL's) and lot size ranges of all sampling standards since Dodge and Romig [2] were series approximately of the type:

$$u_{n+2} = u_{n+1} + u_n \tag{1}$$

In fact the latest version of Military Standard Mil Std 105 [3] shows sample sizes which are almost exactly the Fibonacci series itself. These occurrences were too remarkable to be ascribed to mere coincidence and my interest led me to examine the series empirically. According to Dickson [4], the literature on this subject is rich, extending as it does from the year 1202 to the present. However, it is almost completely unavailable to me and, I suspect, to most others.

On developing the series u_n from $n = 0$ to $n = 25$ or so, inspection soon revealed that two thirds of the series comprised odd numbers and exactly every third u_n was even. It did not take much to ascertain why this is so. In this way I found that n, the ordinal of u_n in the series was, in a manner of speaking, the determinant of the properties of u_n. Thus, if z is a factor of u_n it will infallibly be a factor of u_{2n}, u_{3n}, etc. Therefore, in general, if n is composite, so is u_n (except for the case $n = 4$, $u_n = 3$), but if n is prime, u_n may be prime. My first guess that, since the density of odd numbers in the Fibonacci is twice that of the even numbers, the density of primes would be greater than in the cardinal number domain was proven wrong when the primality of u_n was found dependent on n being prime. The next supposition of equal density was shown to be wrong when $u_{31} = 1346269$ was found to be a composite of 557 and 2417. When u_{37} and u_{41} were also determined to be composite it became obvious that the density of primes in u_n was less than that of the cardinal domain.

Several other interesting details were elucidated after extending and examining the series, first down to $n = 50$ then to $n = 100$ and finally to $n = 130$. No u_n is divisible by n except when $n = 5$ or powers of 5. For example $u_5 = 5$ and $u_{25} = 75025$. Except for u_6, every u_n seems to have at least one prime factor which has not been a factor of any previous u_n; some have two or three such new prime factors. Surely, any theory of prime numbers might profit from Fibonacci considerations.

However, the first gain from the extension of study of the series to $n = 100$ was a remarkable regularity found from the fact that if P_n is a prime factor of u_n it will also factor, more generally, u_{jn} where j goes from 1 to 00. Consider the multiple j and let this be expressed as a sum of multiples of powers of P_n, reduced to a minimum of terms, and provided that no multiples of the powers of $P_n \geq P_n$. Thus:

$$j = aP_n^0 + bP_n^1 + cP_n^2 + \ldots qP_n^r \tag{2}$$

where a, b, c ... q may be zero but must always be less than P_n. Then u_{jn} will be divisible by P_n^{x+1} where P_n^x is the lowest power term of P_n in the sum of multiples of powers of $P_n = j$.

Example 1.

The first prime to divide u_n is 2 ($P_n = 2$) and it divides the third number ($n = 3$) in the series: $u_3 = 2$. From the above lemma we have:

Ordinal jn	j	Sum of P_n* terms = j	P_n^x x	u_{jn} is divisible by P_n^{x+1}	u_{jn}
3	1	P_n^0	0	$P_n^{0+1} = P_n^1 = 2$	2
6	2	P_n^1	1	$P_n^{1+1} = P_n^2 = 4$	8
9	3	$P_n^0 + P_n^1$	0	$P_n^{0+1} = P_n^1 = 2$	34
24	8	P_n^3	3	$P_n^{3+1} = P_n^4 = 16$	46368
30	10	$P_n^1 + P_n^3$	1	$P_n^{1+1} = P_n^2 = 4$	832040
33	11	$P_n^0 + P_n^1 + P_n^3$	0	$P_n^{0+1} = P_n^1 = 2$	3524578

*Since $P_n = 2$, no multiples other than 0 or 1 appear in the sum of powers of $P_n = j$. Actually the sum of multiples of power terms for $j = 11$ should read:

$$11 = 1P_n^0 + 1P_n^1 + 0P_n^2 + 1P_n^3 = P_n^0 + P_n^1 + P_n^3 = 2^0 + 2^1 + 2^3 = 1 + 2 + 8.$$

Example 2.

Another prime dividing u_n is 5 ($P_n = 5$) and, as already mentioned, it divides the fifth number in the series: $u_5 = 5$. Again we make a table:

Ordinal jn	j	Sum of P_n terms = j	P_n^x x	u_{jn} is divisible by P_n^{x+1}	u_{jn}
5	1	P_n^0	0	$P_n^{0+1} = P_n^1 = 5$	5
10	2	$2P_n^0$*	0	$P_n^{0+1} = P_n^1 = 5$	55
20	4	$4P_n^0$	0	$P_n^{0+1} = P_n^1 = 5$	6765
25	5	P_n^1	1	$P_n^{1+1} = P_n^2 = 25$	75025
30	6	$P_n^0 + P_n^1$	0	$P_n^{0+1} = P_n^1 = 5$	832040
35	7	$2P_n^0 + P_n^1$	0	$P_n^{0+1} = P_n^1 = 5$	9227465
50	10	$2P_n^1$	1	$P_n^{1+1} = P_n^2 = 25$	12586269025
125	25	P_n^2	2	$P_n^{2+1} = P_n^3 = 125$	59425114757512643212875125

*The multiple 2 of $2P_n^0$ plays no part, only the power of P_n (zero in this case) is used.

At a later time, in a private communication, Dr. S. M. Ulam recommended Dickson [4] as a reference to the literature. In this I discovered that these findings were known to Lucas [5]. In particular, according to Dickson, the above was stated by Lucas as Theorem V of eight in the following form:

"If n is the rank of the first term u_n containing the prime factor p to the power λ, then u_{pn} is the first term divisible by $p^{\lambda+1}$ and not by $p^{\lambda+2}$; this is called the law of repetition of primes in the recurring series of u_n."

On reading this it is clear that precedence in this finding lay with Lucas who had, moreover, stated it more clearly and economically. Far from being discouraged, however, I continued my search, listing all prime numbers up to 10009 and laboriously testing the primality of most u_n's up to $n = 130$. Of course, primes up to 10009 are sufficient only to test u_n up to $n = 40$ directly but the fact that if z divides u_n it will divide u_{jn} helped greatly. Nevertheless it speedily became apparent that repeated division of u_n greater than u_{45} on a desk calculator was not only laborious but increasingly prone to error as the number of digits in u_n rose above 10. If only there were some way to eliminate some of the trial divisions!

A study of the primes, P_n, which divide u_n revealed that they were all of the form

$$P_n = an + 1 \tag{3}$$

Since P_n is prime it is obvious that an had to be even so that $an \pm 1$ could be odd. Therefore either a or n or both had to be even. Closer study of the primes indicated that, when a and n were both even, it was always necessary to add one to an to get P_n, i.e. with a and n even, $P_n = an + 1$, never $an - 1$. I cannot explain this but, empirically, it turns out this way. Now it was possible to cut down on the number of divisions required to determine the P_n which would divide u_n. Thus:

a. Calculate $2n + 1$ (If n is even, determine only $2n + 1$).

b. Determine whether $2n + 1$ and/or $2n - 1$ are prime.

c. Divide u_n by any prime number determined in a and b.

d. If u_n is not divided in c, calculate $3n + 1$.

e. Repeat setps c. and d.

f. If u_n is not divided in step e, calculate $4n + 1$ If n is even, determine $4n + 1$ only).

g. Continue until the P_n which divides u_n is found.

The relationship found above may be expressed as follows:

If P is any prime there exists an n such that $P_n = an + 1$ or $an - 1$ will divide u_n without remainder (a being some whole number > 0.). The only exception is $P_n = 5$ which divides $u_5 = 5$.

It is possible that the above relationship would repay investigation in prime number theory. In the past, a number of formulas have been proposed for the purpose of generating prime numbers. In every case the formulas have been found faulty in one or more of the following respects:

a. The density of primes generated has been much lower than the true density of primes.

b. They have generated composite numbers.

c. They have rarely been capable of generating paired primes (two consecutive primes which differ by 2, e.g. 11 and 13).

The formula given in (3) suffers only in generating very many composites. However, the procedure clearly furnishes a criterion whereby (empirically) it has been found that, if n is prime, u_n will be divided by P_n only when P_n, determined as in (3), is prime. If this can be proven, new light may be shed by the proof on this age-old problem.

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A MOTIVATION FOR CONTINUED FRACTIONS

A.P. HILLMAN and G.L. ALEXANDERSON
University of Santa Clara, Santa Clara, California

This Quarterly is devoted to the study of properties of integers, especially to the study of recurrent sequences of integers. We show below how such sequences and continued fractions arise naturally in the problem of approximating an irrational number to any desired closeness by rational numbers.

We begin with the equation

$$x^2 - x - 1 = 0 \quad . \tag{1}$$

One can easily see that there is a negative root between -1 and 0 and a positive root between 1 and 2, for example by graphing $y = x^2 - x - 1$. We call the positive root r. This number has been known since antiquity as the "golden mean." We now look for a sequence of rational approximations to r.

A rational number is of the form p/q with p and q integers (and $q \neq 0$). We therefore wish two sequences

$$\begin{array}{c} p_1,\ p_2,\ p_3,\ \cdots \\ q_1,\ q_2,\ q_3,\ \cdots \end{array} \tag{2}$$

of integers such that the quotients p_n/q_n are approximations which get arbitrarily close to r. It would also be helpful if each new approximation were obtainable simply from previous ones.

We go back to equation (1) and rewrite it as

$$x = 1 + \frac{1}{x} \quad . \tag{3}$$

This states that if we replace x by r in

$$1 + \frac{1}{x} \tag{4}$$

the result is r and suggests that if we replace x in (4) by an approximation to r we will get another approximation. We now change (3) into the form

(5) $$x_2 = 1 + \frac{1}{x_1}$$

and consider x_1 to be an approximation to r. The relative error of $1/x_1$ is the same as that of x_1 and, if x_1 is positive, the relative error of x_2 (i.e., $1 + 1/x_1$) is lower than that of x_1, since adding 1 increases the number but not the error. It can be shown that x_2 in (5) is a better approximation to r that x_1, if $x_1 > 0$.

We now let our first approximation x_1 be a rational number p_1/q_1 and substitute this in (5) obtaining

$$x_2 = 1 + \frac{1}{(p_1/q_1)} = 1 + \frac{q_1}{p_1} = \frac{p_1 + q_1}{p_1} \quad .$$

We therefore choose p_2 to be $p_1 + q_1$ and q_2 to be p_1. Similarly, our third approximation is p_3/q_3 with $p_3 = p_2 + q_2$ and $q_3 = p_2$. In general, the $(n + 1)$-st approximation p_{n+1}/q_{n+1} has

(6) $$p_{n+1} = p_n + q_n$$

(7) $$q_{n+1} = p_n \; .$$

It follows from (7) that $q_n = p_{n-1}$; substituting this in (6) gives

(8) $$p_{n+1} = p_n + p_{n-1} \quad .$$

Since r is between 1 and 2 we use 1 as the first approximation, i.e., we let $p_1 = q_1 = 1$. This means that $p_2 = 2$ and it now follows from (8) that p_n is the Fibonacci number F_{n+1}. Then (7) implies that $q_n = F_n$ and we see that the sequence of quotients F_{n+1}/F_n of consecutive Fibonacci numbers furnishes the desired approximations to the root r of (1). It can be shown that this sequence converges to r in the calculus sense.

We next consider the problem of approximating $s = \sqrt{10}$ in this way. The number s is the positive root of

$$x^2 - 10 = 0 \quad . \tag{9}$$

We write (9) in the forms

$$\begin{aligned} &x^2 - 9 = 1 \\ &(x - 3)(x + 3) = 1 \\ &(x - 3) = 1/(x + 3) \\ &x = 3 + 1/(x + 3) \end{aligned} \tag{10}$$

and change (10) into

$$x_{n+1} = 3 + \frac{1}{3 + x_n} \quad . \tag{11}$$

Again, if x_n is a positive approximation to s, it can be seen that x_{n+1} is an approximation with smaller relative error. There is a sequence of rational approximations p_n/q_n with

$$p_{n+1} = 3p_n + 10q_n, \qquad q_{n+1} = p_n + 3q_n \quad .$$

Letting the first approximation be 3, i. e., letting $p_1 = 3$ and $q_1 = 1$, we obtain the sequence

$$3/1,\ 19/6,\ 117/37,\ \ldots$$

which can be shown to converge to s.

Equation (11) contains the equations

$$x_2 = 3 + \frac{1}{3 + x_1}, \qquad x_3 + \frac{1}{3 + x_2} \quad .$$

Substituting the first of these into the second gives us

$$x_3 = 3 + \cfrac{1}{6 + \cfrac{1}{3 + x_1}}$$

If this is substituted into $x_4 = 3 + 1/(3 + x_3)$ and if we let x_1 be 3, we obtain

$$x_4 = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6}}} .$$

In this way we can write continued fraction expressions for any one of the x_n. Then it is natural to let the infinite continued fraction

$$3 + \cfrac{1}{6 + \cfrac{1}{6 + \dots}}$$

represent the limit s of the sequences x_n defined by (11) and $x_1 = 3$.

The infinite continued fraction for the root r of $x^2 - x - 1 = 0$ is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}} ,$$

whose elegant simplicity is worthy of the title "golden mean."

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FIBONACCI AND LUCAS NUMBER TABLES

Those interested may secure bound mimeographed tables of the first 1505 Fibonacci numbers, F_n, and the first 1506 Lucas numbers, L_n, by sending two dollars to Professor Jack K. Ward, Westminster College, Fulton, Missouri.

FIBONACCI NUMBERS: THEIR HISTORY THROUGH 1900

MAXEY BROOKE
Sweeny, Texas

In 1202, a remarkable man wrote a remarkable book. The man was Leonardo of Pisa, known as Fibonacci, a brilliant man in an intellectual wilderness. The book Liber Abacci (The Book of the Abacus) introduced Arabic numbers into Europe.

In the book was a seemingly simple little problem:

"A pair of rabbits are enclosed on all sides by a wall. To find out how many pairs of rabbits will be born in the course of one year, it being assumed that every month a pair of rabbits will produce another pair, and that rabbits begin to bear young two months after their own birth."

On the margin of the manuscript, Fibonacci gives the tabulation:

A pair
1
First
2
Second
3
Third
5
Fourth
8
Fifth
13
Sixth
21
Seventh
34
Eighth
55
Ninth
89
Tenth
144
Eleventh
233
Twelfth
377

He sums up his calculations

"...we see how we arrive at it. We add to the first number the second one, i. e., 1 and 2; the second to the third; the third to the fourth; the fourth to the fifth; and in this way, one after another, until we add together the tenth and eleventh and obtain the total number of rabbits — 377; and it is possible to do this in this order for an infinite number of months."

There the matter lay for 400 years. In 1611, Johann Kepler [1] of astronomy fame, arrived at the series 1, 1, 2, 3, 5, 8, 13, 21, ... There is no indication that he had access to one of Fibonacci's hand-written books (The Liber Abacci was not published until 1857 [2]). At any rate, in discussing the Golden Section and phyllotaxis, Kepler wrote:

"For we will always have as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost. I think that the seminal faculty is developed in a way analogous to this proportion which perpetuates itself, and so in the flower is displayed a pentagonal standard, so to speak. I let pass all other considerations which might be adduced by the most delightful study to establish this truth."

Simon Stevens (1548-1620) also wrote on the Golden Section. The editor of his works, A. Gerard [3] arrived at the formula for expressing the series in 1634

$$U_{n+2} = U_{n+1} + U_n$$

A hundred years must pass before the problem is again considered. In 1753, R. Simpson [4] derived a formula, implied by Kepler

$$U_{n-1} U_{n+1} - U_n^2 = (-1)^{n+1}$$

A second hundred years pass by and the series again comes under study. In 1843, J. P. M. Binet [5] derives an analytical function for determining the value of any Fibonacci number

$$2^n \sqrt{5}\, U_n = (1+\sqrt{5})^n - (1-\sqrt{5})^n$$

The following year, B. Lamé [6] first used the series to solve a problem in Theory of Numbers. He investigated the number of operations needed to find the GCD of two integers (it does not exceed 5 times the number of digits in the smaller number).

Two years later, E. Catalan [7] derived the important formula

$$2^{n-1}U_n = \frac{n}{1} + \frac{5n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{5^2 n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots$$

By now, the series had received enough attention to deserve a name. It was variously called the Braun Series, the Schimper-Braun series, the Lamé series and the Gerhardt series.

A. Braun [8], applied the series to the arrangement of the scales of pine cones. Schimper is completely unknown. Lamé has already been mentioned, but the name has been credited to Father Bernard Lami, a contemporary of Newton and the discoverer of the parallelogram of forces. Gerhardt is probably a mis-spelling of Girard.

Edouard Lucas [20], who dominated the field of recursive series during the period 1876-1891, first applied Fibonacci's name to the series and it has been known as Fibonacci series since then.

About this time, 1858, Sam Loyd claimed to have invented the checkerboard paradox [9]. It is first found in print in a German journal in 1868 [10]. Today it seems proper to call it the Carroll Paradox after Lewis Carroll [11] (Charles Dodgson, 1832-93) who was quite fond of it.

Before the century ended, a number of familiar relations were found. Among them: (V_n is the nth Lucas number.)

1876, E. Lucas [12]

$$U_{n-1}^2 + U_n^2 = U_{2n+1}$$

$$V_{4n} = U_{2n}^2 - 2$$

$$V_{4n+2} = U_{2n+1}^2 + 2$$

$$U^{n+p} = U^{n-p}(U+1)^p$$

$$U^{n-p} = U^n (U-1)^p$$

1886, E. Catalan [13] [14]

$$U_{n+1-p}\,U_{n+1+p} - U_{n+1}^2 = (-1)^{n+2-p}\,U_p$$

$$U_n^2 - U_{n-p}\,U_{n+p} = (-1)^{n-p+1}\,U_{n-1}$$

1899, E. Landau [15] related the series

$$\sum_{n=1}^{\infty} (1/U_{2n})$$

to Lambert's series and

$$\sum_{n=0}^{\infty}(1/U_{2n+1})$$

to the theta series.

A complete list can be found in Vol. 1 of Dickson's "History of the Theory of Numbers."

This small history ends arbitrarly at 1900 for the pregmatic reason that a mere listing of twentieth century developments would fill a moderately sized volume. It would be interesting to see Fibonacci's reaction to the application of his rabbit problem to such diverse subjects as musical composition [16], process optimization [17], electrical network theory [18], and genetics [19].

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer should submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated.

Solutions to problems listed below should be submitted within two months of publication.

B-38 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

Characterize simply all the sequences c_n satisfying

$$c_{n+2} = 2c_{n+1} - c_n \ .$$

B-39 *Proposed by John Allen Fuchs, University of Santa Clara, Santa Clara, California*

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Prove that

$$F_{n+2} < 2^n \text{ for } n \geq 3 \ .$$

B-40 *Proposed by Charles R. Wall, Texas Christian University, Fort Worth, Texas*

If H_n is the n-th term of the generalized Fibonacci sequence, i.e.,

$$H_1 = p,\ H_2 = p + q,\ H_{n+2} = H_{n+1} + H_n \text{ for } n \geq 1\ ,$$

show that

$$\sum_{k=1}^{n} kH_k = (n + 1)H_{n+2} - H_{n+4} + 2p + q\ .$$

B-41 *Proposed by David L. Silverman, Beverly Hills, California*

Do there exist four distinct positive Fibonacci numbers in arithmetic progression?

B-42 *Proposed by S.L. Basin, Sylvania Electronic Systems, Mountain View, California*

Express the $(n + 1)$-st Fibonacci number F_{n+1} as a function of F_n. Also solve the same problem for Lucas numbers.

B-43 *Proposed by Charles R. Wall, Texas Christian University, Fort Worth, Texas*

(a) Let $x_0 \geq 0$ and define a sequence x_k by $x_{k+1} = f(x_k)$ for $k \geq 0$, where $f(x) = \sqrt{1 + x}$. Find the limit of x_k as $k \to \infty$.

(b) Solve the same problem for $f(x) = \sqrt[3]{1 + 2x}$.

(c) Solve the same problem for $f(x) = \sqrt[4]{2 + 3x}$.

(d) Generalize.

SOLUTIONS

FIBONACCI AND PASCAL AGAIN

B-16 *Proposed by Marjorie Bicknell, San Jose State College, San Jose, California, and Terry Brennan, Lockheed Missiles and Space Co., Sunnyvale, California*

Show that if

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

then

$$R^n = \begin{pmatrix} F_{n-1}^2 & 2F_{n-1}F_n & F_n^2 \\ F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & F_nF_{n+1} \\ F_n^2 & 2F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} .$$

(There are some misprints in the original statement.)

Solution by L. Carlitz, Duke University, Durham, N.C.

Put

$$R_k = \left[\binom{r}{k-s} \right] \qquad (r,\ s = 0,\ 1,\ \ldots,\ k) ,$$

a matrix of order $k + 1$; for example

$$R_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

It is easily verified that

$$(1) \qquad R_1^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \qquad (n = 1,\ 2,\ \ldots) .$$

Indeed this is obviously true for $n = 1$. Assuming that the formula holds for n, we have

$$R_1^{n+1} = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix} .$$

In the next place we notice that the transformation

$$T_1: \begin{cases} x' = y \\ y' = x + y \end{cases}$$

induces the transformations

$$T_2: \begin{cases} x'^2 = y^2 \\ x'y' = xy + y^2 \\ y'^2 = x^2 + 2xy + y^2 \end{cases},$$

$$T_3: \begin{cases} x'^3 = y^3 \\ x'^2y' = xy^2 + y^3 \\ x'y'^2 = x^2y + 2xy^2 + y^3 \\ y'^3 = x^3 + 3x^2y + 3xy^2 + y^3 \end{cases}$$

and so on. Also it is evident from (1) that T_1^n is given by

$$T_1^n: \begin{cases} x^{(n)} = F_{n-1}x + F_ny \\ y^{(n)} = F_nx + F_{n+1}y \end{cases}.$$

We therefore get

$$T_1^n: \begin{cases} (x^{(n)})^2 = F_{n-1}^2x^2 + 2F_{n-1}F_nxy + F_n^2y^2 \\ x^{(n)}y^{(n)} = F_{n-1}F_nx^2 + (F_n^2 + F_{n-1}F_{n+1})xy + F_nF_{n+1}y^2 \\ (y^{(n)})^2 = F_n^2x^2 + 2F_nF_{n+1}xy + F_{n+1}^2y^2 \end{cases}.$$

Also solved by the proposers.

LAMBDA FUNCTION OF A MATRIX

B-24 *Proposed by Brother U. Alfred, St. Mary's College, California*

It is evident that the determinant

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$

has a value of zero. Prove that if the same quantity k is added to each element of the above determinant, the value becomes $(-1)^{n-1}k$.

Solution by Raymond Whitney, Pennsylvania State University, Hazelton Campus

Using the basic Fibonacci recursion formula $F_{n+2} = F_{n+1} + F_n$ and elementary row and column transformations we may reduce the determinant to:

$$= k \begin{vmatrix} F_n & F_{n+1} & -1 \\ F_{n+1} & F_{n+2} & -1 \\ 0 & 0 & 1 \end{vmatrix} = k(F_n F_{n+2} - F_{n+1}^2) ,$$

which is $(-1)^{n-1}k$ by a basic identity.

Also solved by Marjorie Bicknell, San Jose State College, San Jose, California who pointed out the relation to "Fibonacci Matrices and Lambda Functions," by M. Bicknell and V. E. Hoggatt, Jr., this Quarterly, Vol. 1, No. 2; R. M. Grassl, University of Santa Clara, California; *F.D. Parker, State University of New York, Buffalo, N.W., R.N. Vawter, St. Mary's College, California; H.L. Walton, Yorktown H.S., Arlington, Virginia; and the proposer.*

EXPONENTIALS OF FIBONACCI NUMBERS

B-25 *Proposed by Brother U. Alfred, St. Mary's College, California*

Find an expression for the general term(s) of the sequence $T_0 = 1$, $T_1 = a$, $T_2 = a$, ... where

$$T_{2n} = \frac{T_{2n-1}}{T_{2n-2}} \quad \text{and} \quad T_{2n+1} = T_{2n}T_{2n-1} \; .$$

Solution by Vassili Daiev, Sea Cliff, L.I., N.Y.

The first few terms are

$$a^{F_0},\ a^{F_2},\ a^{F_1},\ a^{F_3},\ a^{F_2},\ a^{F_4},\ \ldots\ .$$

It is easy to see that $T_n = a^k$ where $k = F_{(n/2)}$ if n is even and $k = F_{(n+3)/2}$ if n is odd.

Also solved by J.A.H. Hunter, Toronto, Ontario, Canada who suggested the consideration of $\log T_n$; Ralph Vawter, St. Mary's College, California, and the proposer.

Editorial Comment: The problem can be solved by showing that $\log T_{m+4} = \log T_{m+2} + \log T_m$.

MAXIMIZING A DETERMINANT

B-28 *Proposed by Brother U. Alfred, St. Mary's College, California*

Using the nine Fibonacci numbers F_2 to F_{10} (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

Solution by Marjorie Bicknell, San Jose State College, California

By considering combinations of Fibonacci numbers which give minimum and maximum values to sums of the form abc + def + ghi, the following determinant seems to have the maximum value obtainable with the nine Fibonacci numbers given:

$$\begin{vmatrix} F_{10} & F_4 & F_7 \\ F_6 & F_9 & F_3 \\ F_2 & F_5 & F_8 \end{vmatrix} = F_{10}F_9F_8 + F_7F_6F_5 + F_4F_3F_2 - (F_{10}F_3F_5 + F_9F_2F_7 + F_8F_4F_6)$$

$$= 39796 - 1496$$

$$= 38300.$$

Also solved by the proposer.

B-29 *Proposed by A.P. Boblett, U.S. Naval Ordnance Laboratory, Corona, California*

Define a general Fibonacci sequence such that

$$F_1 = a; \quad F_2 = b; \quad F_n = F_{n-2} + F_{n-1}, \quad n \geq 3$$
$$F_n = F_{n+2} - F_{n+1}, \quad n \leq 0 \quad .$$

Also define a characteristic number, C, for this sequence, where $C = (a + b)(a - b) + ab$.

Prove:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n C, \text{ for all } n \quad .$$

Solution by F.D. Parker, State University of New York, Buffalo, N.Y.

From $F(n) = F(n - 1) + F(n - 2)$, $F(1) = a$, $F(2) = b$, we get

$$F(n) = \frac{b - ar}{1 + s^2} s^n + \frac{b - as}{1 + r^2} r^n,$$

where r and s are solutions of the quadratic $x^2 - x - 1 = 0$. Using the fact that $r + s = -rs = 1$, direct calculation yields

$$F(n + 1)F(n - 1) - F^2(n) = \left[(a - b)(a + b) + ab\right] (-1)^n \quad .$$

The well known result $F(n + 1)F(n - 1) - F^2(n) = (-1)^n$ is the special case in which $a = b = 1$.

Also solved by Marjorie Bicknell, San Jose State College, San Jose, California; Donna J. Seaman, Sylvania Co.; R.N. Vawter, St. Mary's College, California; and the proposer, J.A.H. Hunter, of Toronto, Ontario.