

FIBONACCI DRAWING BOARD

COLONEL ROBERT S. BEARD

It is in order to preface the work of the Fibonacci Drawing Board with something of the story of our emblem, The Fibonacci Star.

This star of 21 stars is a refinement of the beautiful star-pentagram commonly used to represent the heavenly bodies.

That star formed by the five diagonals of the regular pentagon was the emblem of the Pythagorean school of philosophy and mathematics. It was discovered by that organization somewhere around 500 B.C. It was probably the first emblem of any mathematical organization.

In time the pentagram or pentacle became credited with the magic power of warding off witches and demons and of insuring good health and good fortune somewhat akin to the power of the hex marks of the Pennsylvania Dutch.

Here are some pertinent quotations:

"They had their crystals I do know and rings and virgin parchment and their deadman's skulls. Their raven's wings, their lights and pentacles with characters," B. Johnson, Devil is an Ass.

"His shoes were marked with cross and shell. Upon his breast a pentacle," Scott Marmion.

"Sketching with slender pointed foot some figure like a wizard pentagram on garden gravel," Tennyson, The Brook.

The universal appeal of the pentagram to the imagination is demonstrated by its use in the design of the flags of many nations and also in its wide use as a symbol for hope, excellence, outstanding performance, rank and authority.

My United Nations flag design encircled a red cross with a wreath of pentagrams on a white field to symbolize world hope united around

our willingness to help each other. No comment on how things are going.

The rank of our top military commanders has been designated by pentagram insignia ever since the American Revolution.

"In 1780 Major Generals were ordered to wear 'two epaulettes with two stars each,' while Brigadier Generals had one star, and later when the rank of Lieutenant General was established for the Commander-in-Chief, Washington, three stars were prescribed for him. This was the commencement of our rank insignia." Orders, Decorations and Insignia, Military and Civil, Colonel Robert E. Wyllie, G. P. Putnam's Sons.

World War I produced our first four-star general, while World War II furnished our top admirals and generals with clusters of five pentagrams.

No doubt the foreign devils are being kept away from our shores by the pentacles flashing on the uniforms of our military leaders as they circulate around our great Pentagon Building on the banks of the Potomac.

The mathematics of the Fibonacci Star will be discussed in proper order at a later date. However, it might be mentioned here that it is proportioned in 10 successive terms of a Fibonacci series, the powers of the golden section or golden mean.

Perhaps you'll tolerate the winding up of this story with the second verse of Cowper's poem, "The Golden Mean," which is a translation from the Latin of Horace.

"He that holds fast the golden mean
And lives contentedly between
The little and the great,
Feels not the wants that pinch the poor
Nor plagues that haunt the rich man's door
Embittering all his state."

Have your pencils sharpened next time and get ready to go to work.

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ZECKENDORF'S THEOREM AND SOME APPLICATIONS

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1. INTRODUCTION

The subject theorem, due to E. Zeckendorf^[1], is one which deserves to be more widely known, particularly since the property involved in the theorem statement is a property which uniquely characterizes the Fibonacci numbers among all other sequences of positive integers. Our purpose in this paper is to give a brief exposition of theorem with its proof, and to examine several applications and consequences.

For the subsequent proof, it is convenient to define the Fibonacci numbers $\{u_n\}_1^\infty$ as follows: $u_1 = 1$, $u_2 = 2$, $u_{n+1} = u_n + u_{n-1}$ for $n \geq 2$. If we take $\{F_n\}_1^\infty$ according to the more common definition, $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, then $u_n = F_{n+1}$ for $n \geq 1$.

Zeckendorf's theorem essentially states that every positive integer can be represented uniquely as a finite sum of distinct Fibonacci numbers $\{u_n\}$, with the additional constraint that no two consecutive Fibonacci numbers appear in the representation of any particular integer. A formal statement of the theorem and its proof follow in section 2, while section 3 is concerned with applications and a converse.

2. ZECKENDORF'S THEOREM

Theorem: Every positive integer N has one and only one representation in the form

$$(1) \quad N = \sum_{i=1}^{\infty} \alpha_i u_i$$

where each α_i is a binary digit and

$$(2) \quad \alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1.$$

(In the following, we shall reserve the subscripted variables α and β for binary digits, that is, digits which have either the value zero or unit.)

The proof is accomplished with the aid of two lemmas:

Lemma 1: $u_n = 1 + u_{n-1} + u_{n-3} + \dots + u_{1,2}$,

where

$$u_{1,2} = \begin{cases} u_1 & \text{if } n \text{ is odd} \\ u_2 & \text{if } n \text{ is even.} \end{cases}$$

Proof: The elementary inductive verification of this identity is left to the reader.

Lemma 2: Representation of a positive integer in the form (1) with binary coefficients satisfying (2) is unique.

Proof: Assume \exists a positive integer N with two distinct representations of the required form, so that

$$(3) \quad N = \sum_{i=1}^{\infty} \alpha_i u_i = \sum_{i=1}^{\infty} \beta_i u_i$$

with $\alpha_i \alpha_{i+1} = \beta_i \beta_{i+1} = 0$ for $i \geq 1$, and

$$\sum_{i=1}^{\infty} |\alpha_i - \beta_i| \neq 0.$$

Let k be the largest integer i such that $\alpha_i \neq \beta_i$; then of the two quantities α_k and β_k , one must be unity and the other zero. Assume without loss of generality that $\alpha_k = 1$, $\beta_k = 0$, so that (3) becomes

$$(4) \quad \sum_{i=1}^k \alpha_i u_i = \sum_{i=1}^{k-1} \beta_i u_i.$$

But the left-hand side is $\geq u_k$ since $\alpha_k = 1$, while the right-hand side satisfies

$$\sum_{i=1}^{k-1} \beta_i u_i \leq u_{k-1} + u_{k-3} + \dots + u_{1,2} = u_k^{-1},$$

a contradiction. We conclude $\alpha_i = \beta_i$ for all $i \geq 1$; that is, the representation is unique.

Proof of Theorem:

It remains to be shown that every positive integer N has a representation in the form (1) with binary coefficients satisfying (2).

We will prove, by an induction on n , that $0 \leq N < u_n$ implies

$$N = \sum_{i=1}^{n-1} \alpha_i u_i \quad \text{with} \quad \alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1.$$

The statement is vacuously true for $n = 1$ and is verified by inspection for $n = 2$ and $n = 3$. Now, assume the proposition has been proved for $n = 1, 2, \dots, k$ where k is some integer ≥ 3 ; we wish to show the statement must necessarily be true for $n = k+1$, or equivalently, that $0 \leq N < u_{k+1}$ implies

$$(5) \quad N = \sum_{i=1}^k \alpha_i u_i \quad (\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1).$$

By the induction hypothesis, the result holds for N in the range $0 \leq N < u_k$, so that we need only consider the case $u_k \leq N < u_{k+1}$. For this latter case,

$$0 \leq N - u_k < u_{k+1} - u_k = u_{k-1},$$

and the induction hypothesis guarantees binary coefficient β_i such that

$$N - u_k = \sum_{i=1}^{k-2} \beta_i u_i. \quad (\beta_i \beta_{i+1} = 0 \quad \text{for } i \geq 1).$$

Transposing the u_k , we obtain

$$N = \sum_{i=1}^{k-2} \beta_i u_i + u_k,$$

so that the choices, $\alpha_i = \beta_i$ for $1 \leq i \leq k-2$, $\alpha_{k-1} = 0$ and $\alpha_k = 1$, yield a representation in the required form (5). q.e.d.

3. APPLICATIONS AND A RELATED PROBLEM

As our first application, we analyse the problem^[2] of determining the probability that at least two successive "heads" will occur in n flips of a fair coin. To investigate this problem, we consider the complementary situation and ask for the number of ways in which a coin can be tossed n times without ever getting two heads in sequence. Clearly, this number is equal to the number of distinct binary sequences $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of length n , where each α_i is either 1 (heads) or 0 (tails), with the additional constraint that a 1 is never followed immediately by another 1. This latter condition is concisely expressed by the requirement $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$, which, of course, is exactly the coefficient condition of the preceding section.

Let us term a sequence of n binary digits an "admissible" sequence if it satisfies the constraint $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$; then, we wish to determine, as a function of n , the number of admissible sequences.

To each admissible sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$, let us associate the number

$$\sum_{i=1}^n \alpha_i u_i$$

so that a one-to-one correspondence,

$$(\alpha_1, \alpha_2, \dots, \alpha_n) \leftrightarrow \sum_{i=1}^n \alpha_i u_i$$

is established between admissible sequences (of length n) and a subset of the positive integers. But, from the proof of Zeckendorf's theorem, we have that each integer N satisfying $0 \leq N < u_{n+1}$ has one and only one representation in the form

$$\sum_{i=1}^n \alpha_i u_i$$

with $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$, and clearly no integer $\geq u_{n+1}$ can be represented in this form. Hence, the number of different integers which can be represented is equal to the number of integers in the set $\{0, 1, 2, \dots, u_{n+1} - 1\}$, or u_{n+1} . By our correspondence, the number of admissible sequences of length n is therefore also u_{n+1} . Since the total number of binary sequences of length n is 2^n , the probability of not obtaining at least two successive heads in n throws is

$$\frac{u_{n+1}}{2^n},$$

or, equivalently, the required probability of having at least two successive heads in n tosses is

$$1 - \frac{u_{n+1}}{2^n}.$$

A second application may be found in Whinihan's recent paper [3] on determining an optimum strategy for the game of Fibonacci Nim. In developing the strategy, the author introduces a rule for representing an arbitrary integer as a unique sum of distinct Fibonacci numbers, so that in the sequence of expansion coefficients, it is "impossible for two 1's to appear...without at least one 0 separating them." As noted in an editorial comment, this unique representation property is precisely the content of the Zeckendorf theorem.

Lastly, we consider the unique representation property in Zeckendorf's theorem and ask what other integer sequences (if any), in addition to the Fibonacci sequence, enjoy the same property. For clarity, we define the property in question as follows:

Definition: A sequence of positive integers $\{v_n\}^\infty$ is said to possess the unique representation property (u. r. p.) if and only if every positive integer N has a unique representation in the form

$$(6) \quad N = \sum_1^{\infty} \alpha_i v_i,$$

where the α_i are binary digits satisfying

$$(7) \quad \alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1.$$

The main theorem concerning u. r. p. sequences is due to D. E. Daykin [4].

Theorem (Daykin): If $\{v_n\}$ is a sequence possessing the u.r.p., then v_n is necessarily increasing and $v_n = u_n$ for all $n \geq 1$.

Thus, the Fibonacci sequence is the only sequence, increasing or otherwise, for which unique representations in the form (6)-(7) are possible for every positive integer.

Daykin's theorem is easy to prove in the case of increasing v_n but is non-trivial for the general case in which the v_n 's may appear in any order. The general result provides a complete converse to Zeckendorf's theorem and also gives a concise characterization of the Fibonacci sequence as being the only sequence possessing the unique representation property.

A different, though related, characterization of the Fibonacci numbers in terms of "complete" sequences has been given earlier by the author^[5]. Any sequence possessing the u.r.p. is, a fortiori, complete; that is, every positive integer may be written as a sum of distinct members of the sequence. Moreover, it can be shown that the deletion of any single term from a u.r.p. sequence renders the remaining sequence incomplete. The definition of completeness, unlike that of the u.r.p., is invariant with respect to a reordering of the sequence and may provide an alternate method of proving Daykin's theorem. The connection between completeness and the unique representation property will be the subject of a future paper.

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SYMMETRIC SEQUENTIAL MINIMAX SEARCH FOR A MAXIMUM

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1. INTRODUCTION

Kiefer^[1] has given a sequential method for seeking the maximum of a unimodal (single-peaked) function of one variable in a finite interval. This procedure is minimax in the sense that no matter where the peak may happen to be, the final interval within which the peak will be known with certainty to lie will be as small as possible. In this technique the last experiment must be located as closely as possible to the experiment giving the greatest value among those previously run. If this distance ϵ is negligibly small, then Kiefer's procedure is indeed minimax. When on the other hand ϵ cannot be neglected, which is often the case in practical problems, then Kiefer's method can be modified to give a shorter final interval of uncertainty.

Kiefer's original technique is asymmetric in the sense that the last two experiments are not located symmetrically with respect to each other. The modified procedure is symmetric, since it permits the last experiment to be placed symmetrically with respect to the most effective previous experiment. In the extreme case when as many experiments as possible are run, the symmetric technique gives a final interval only two-thirds as long as for the asymmetric method. Formulae are given for the maximum number of experiments which can profitably be performed for a finite resolution ϵ . Analysis of them shows that the symmetric method can occasionally make use of at most one more experiment than the asymmetric procedure.

Problem: Let y be a single-valued function of x having a maximum y^* at the unknown point x^* somewhere in the interval $a \leq x \leq b$. Suppose that in this interval y is unimodal, i.e., that $a \leq x_1 < x_2 \leq x^*$ implies $y(x_1) < y(x_2)$, and $x^* \leq x_1 < x_2 \leq b$ implies $y(x_1) > y(x_2)$. If observations of y are taken at the k points $x_1 < x_2 < \dots < x_k$, and if the greatest value of y is found at x_j , then the unimodality implies that $x_{j-1} < x^* < x_{j+1}$, with the convention

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that $x_0 \equiv a$ and $x_{k+1} \equiv b$. Let

$$(1) \quad x_{j+1} - x_{j-1} \equiv L_k$$

the length of the interval of uncertainty after k observations ($L_0 = L_1 = b-a$). For $k > 1$, L_k will become smaller as more measurements are taken, and we wish to locate them in such a way that the length L_n of the final interval of uncertainty after n sequential observations will be as small as possible, no matter where x^* actually happens to be. If $\{x_n\}$ represents any sequence of n observations, then the minimax sequence $\{x_n^*\}$ is the one which gives this smallest interval L_n^* . Formally,

$$(2) \quad L_n^*/L_0 = \min_{(\{x_n\})} \max_{(a \leq x^* \leq b)} \{L_n/L_0\}$$

2. DISTINGUISHABILITY

Even when the function is known to be unimodal it may not be possible to detect, in a physical problem, the difference between the outcomes of two measurements that are too close together. When this happens, the experimenter is unable to reduce the interval of uncertainty, and one of the observations is useless. Thus in designing a sequential search technique one must take into account the minimum spacing ϵ for which two outcomes are distinguishable. The smallest interval of uncertainty obtainable practically is therefore 2ϵ .

$$(3) \quad L_n = x_{j+1} - x_{j-1} = (x_{j+1} - x_j) - (x_j - x_{j-1}) = 2\epsilon.$$

Although the resolution ϵ is usually negligible compared to the original interval of uncertainty L_0 , it is often a large fraction of the final interval L_n if the search is at all efficient.

3. RESULTS OBTAINED BY NEGLECTING RESOLUTION

Kiefer [1] has given a search procedure based on the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...), where the nth Fibonacci number is given by

$$(4) \quad F_0 = F_1 = 1; \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n = 2, 3, \dots$$

One places the first two experiments at distances $L_o F_{n-1}/F_n$ from one end of the original interval. By equations (1) and (4) the better observation will be a distance $L_o F_{n-3}/F_n$ from one end of the new interval of uncertainty, whose length will be $L_2 = L_o F_{n-2}/F_n$. The third observation is made symmetrically with respect to the one already in the interval, i.e., a distance $L_o F_{n-3}/F_n$ from the other end. This procedure is continued until all but one experiment has been run and the interval of uncertainty has length $L_{n-1} = L_o F_2/F_n = 2L_o/F_n$. The best observation will be exactly in the center of this interval, because $L_o F_1/F_n = L_o/F_n = L_{n-1}/2$. Thus if the final observation were placed symmetrically it would be completely indistinguishable from the one already in the interval. It must therefore be located a distance ϵ to one side or the other of the midpoint. For this reason we shall call this asymmetric minimax method.

If the experimenter's luck is bad he will be left with an interval of uncertainty of length

$$(5) \quad L_n^* = L_o/F_n + \epsilon$$

The asterisk has been added to L_n because Kiefer has shown that this interval is ϵ -minimax among all non-randomized procedures. If one randomizes the placement of the last experiment, the expected final interval is slightly less

$$(6) \quad E \{ L_n^* \} = L_o F_n + \epsilon/2$$

These results were obtained essentially by neglecting the resolution and minimaxing the other term. Thus as ϵ approaches zero L_n^* approaches the true minimax length.

A SHORTER INTERVAL

By taking proper account of the resolution ϵ we can obtain a shorter interval of uncertainty L_n^{**} . In establishing this result we can avoid a long proof by using an intermediate result of Johnson reported in [3]. Johnson showed, in an independent alternate proof of Kiefer's result, that the minimax procedure must be such that after k trials,

$$(6') \quad L_k^{**} = L_{k-2}^{**} - L_{k-1}^{**}; \quad k = 2, 3, \dots, n$$

Both Kiefer and Johnson have demonstrated that the final two experiments should be a distance ϵ apart in the center of the remaining interval, whose length is L_{n-1}^* . Our procedure will be called symmetric because it preserves this symmetry. With this spacing, the final interval is

$$(7) \quad L_n^{**} = L_{n-1}^{**}/2 + \epsilon/2$$

Equations (6') and (7) together give

$$(8) \quad L_{n-2}^{**} = L_n^{**} + L_{n-1}^{**} = L_n^{**} + (2L_n^{**} - \epsilon) = 3L_n^{**} - \epsilon$$

By iterating the recursion relation (6) we obtain

$$(9) \quad L_k^{**} = F_{n-k+1} L_n^{**} - F_{n-k-1} \epsilon$$

which can be proven readily by mathematical induction on the indices.

When in particular $k = 1$, then

$$L_1 = F_n L_n^{**} - F_{n-2} \epsilon$$

whence, since $L_0 = L_1$,

$$(10) \quad L_n^{**} = L_0/F_n + F_{n-2} \epsilon/F_n$$

This interval is shorter than that of the asymmetric technique by an amount

$$(11) \quad L_n^* - L_n^{**} = (1 - F_{n-2}/F_n) \epsilon = F_{n-1} \epsilon/F_n$$

As n becomes large, the ratio F_{n-1}/F_n approaches $(\sqrt{5}-1)/2 = 0.618033989\dots$ [1], [2], [3], and so the resolution term in the symmetric method is only about 38% as large as for the asymmetric procedure.

4. PLACEMENT OF THE EXPERIMENTS

Although we have given the final interval obtainable by the symmetric minimax method, we have not yet described how to locate the experiments. The symmetric procedure is similar to the asymmetric one in that each new experiment is placed symmetrically with respect to the observation already in the remaining interval of uncertainty.

Hence the technique is completely defined when the location of the first two experiments is specified. This is accomplished by noting that the interval remaining after these two experiments will be L_2^{**} , which is, from equation (9),

$$(12) \quad L_2^{**} = F_{n-1} L_n^{**} - F_{n-3} \epsilon$$

Equations (10) and (12) together give this length in terms of L_o .

$$(13) \quad L_2^{**} = [F_{n-1} L_o + (F_{n-2} F_{n-1} - F_n F_{n-3}) \epsilon] / F_n$$

The coefficient of ϵ can be rearranged

$$(14) \quad F_{n-2}^2 - F_{n-1} F_{n-3} = (F_{n-2} + F_{n-3}) F_{n-2} - (F_{n-1} + F_{n-2}) F_{n-3} =$$

$$F_{n-2}^2 - F_{n-1} F_{n-3}$$

so that it can be simplified by a result of Simson^{[3][4]}

$$(15) \quad F_{n-2}^2 - F_{n-1} F_{n-3} = (-1)^n$$

Equations (13), (14), and (15) together give the optimal placement of the first two experiments

$$(16) \quad L_2^{**} = F_{n-1} L_o / F_n + (-1)^n \epsilon / F_n$$

Thus for an odd number of experiments the first pair is slightly closer together than for an asymmetric search. Conversely when n is even they are slightly farther apart.

5. MAXIMUM NUMBER OF EXPERIMENTS

The need for distinguishability puts an upper bound on the number of experiments that can be performed profitably. Let m be this maximum number for a symmetric search. Equations (3) and (10) together give

$$L_m^{**} = L_o / F_m + F_{m-2} / F_m \geq 2\epsilon,$$

from which one can show that

$$(17) \quad F_{m+1} \leq L_o / \epsilon < F_{m+2}$$

Thus if ϵ is only one percent of L_0 , there is no advantage in performing more than nine experiments because $89 = F_{10} < 100 < F_{11} = 144$. When n is large, Lucas' relation [3] [5] gives approximately

$$F_{m+1} \approx (1.618)^{m+2} / \sqrt{5},$$

which can be used to obtain, from equation (17),

$$(18) \quad m \leq 4.785 \log(L_0/\epsilon) - 0.328$$

For an asymmetric search the final observation, which is a distance ϵ from the center, can be no closer than ϵ to the end of the interval. Hence the final asymmetric interval L_n^* can be no shorter than 3ϵ

$$(19) \quad L_n^* \geq 3\epsilon,$$

which is 50% longer than the limit on L_n^{**} for symmetric search. Equations (5) and (19) together give a limit on the number m' of asymmetric experiments that can be performed.

$$(20) \quad F_{m'} \leq L_0/2\epsilon < F_{m'+1}$$

When $L_0 = 100\epsilon$, $m' = 8$, one less experiment than for symmetric search.

It is not always possible for the symmetric search to employ more experiments than the asymmetric scheme (when $L_0 = 12\epsilon$, $m = m' = 4$). Moreover, the difference will never be more than one experiment, as can be seen by combining the equalities (17) and (20) with the definition (4) of the Fibonacci sequence.

$$2F_{m-1} < F_m + F_{m-1} = F_{m+1} \leq \frac{L_0}{\epsilon} < 2F_{m'+1},$$

whence

$$F_{m-1} < F_{m'+1},$$

or

$$F_{m-1} \leq F_{m'}.$$

It follows that

$$(21) \quad m - m' \leq 1 .$$

6. ACKNOWLEDGMENT

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FIBONACCI NUMBERS FROM A DIFFERENTIAL EQUATION

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In a course in differential equations, solving

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0 \quad (y = 0; y' = 1, x = 0)$$

leads to

$$(1) \quad y = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{x^n}{n!},$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$

both satisfy the auxiliary equation $m^2 - m - 1 = 0$.

On the other hand, solving this same problem directly in infinite series of the form

$$(2) \quad y = \sum_{n=0}^{\infty} a_n x^n$$

leads to the recurrence relation

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$$

with $a_0 = 0, a_1 = 1$.

If we set $a_n \equiv u_n/n!$ this becomes

$$(n+2)(n+1) \frac{u_{n+2}}{(n+2)!} - \frac{(n+2)u_{n+1}}{(n+1)!} - \frac{u_n}{n!} = 0$$

or

$$u_{n+2} - u_{n+1} - u_n = 0,$$

with $u_0 = 0$ and $u_1 = 1$.

$$\text{Thus } a_n = \frac{u_n}{n!} = F_n/n!,$$

where F_n is the nth Fibonacci number.

Substituting these values of $u_n/n! = a_n$ into (2) yields

$$(3) \quad y = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$

FIBONACCI POWERS AND PASCAL'S TRIANGLE IN A MATRIX - PART II

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3. THE P MATRIX — RECURSION RELATIONSHIPS FOR PRODUCTS AND POWERS OF u_n .

A convenient technique [2] for generating several basic Fibonacci identities lies in the use of the second order matrix

$$(3.1) \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The technique is based upon the fact that the characteristic polynomial of P is the characteristic polynomial of the second-order recurrent relation $u_{n+1} = u_n + u_{n-1}$ defining the Fibonacci sequence, i.e.,

$$(3.2) \quad |xI - P| = x^2 - x - 1 .$$

From (3.1) and (3.2) we have at once

$$P^2 = P + I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$P^n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{pmatrix}$$

We shall show that the matrix P_n of (1.1) provides a generalization of (3.1) relative to the n -th powers of u_1 . Indeed, (3.1) is Q_1 of (1.1), and $\phi_1(x)$ in (2.20) compares with (3.2).

Theorem I (due originally to Jarden [3])

Let

$$b_r = \prod_{j=1}^n h_r^j$$

be the element by element product of n (not necessarily distinct)

sequences $\{h_r^j\}$ each of which satisfy the relation

$$(3.3) \quad h_{r+1}^j = h_r^j + h_{r-1}^j .$$

Then $\{b_r\}$ satisfies the recurrence relation

$$\phi_n(b) = 0$$

for ϕ_n defined in (2.19).

Proof.

By virtue of (2.18) it is sufficient to show that the determinant $D_n(b)$ vanishes for $n+1$ consecutive members of the sequence $\{b_r\}$. Examining $D_n(a)$ we note that we can express the element in the r -th row and s -th column by

$$a_{r+1} - u_{r+1}^n a_1 - u_r^n a_0 , \quad \text{if } s = 1$$

$$u_{r+1}^{n+1-s} u_r^{s-1} , \quad \text{if } s \neq 1 .$$

Hence the determinant is zero for the sequence $\{a\}$ if we can find a solution $\{A_s\}$ which is independent of r and satisfies

$$(3.4) \quad a_{r+1} = u_{r+1}^n a_1 + u_r^n a_0 + \sum_{s=2}^n A_s u_{r+1}^{n+1-s} u_r^{s-1} ;$$

that is to say, some method of annihilating the first column by adding a linear combination of the remaining columns. We take

$$(3.5) \quad a_{r+1} = b_{r+1} = \prod_{j=1}^n h_{r+1}^j .$$

Using the well known formula for general sequences of the type (3.3)

$$h_{r+1} = u_{r+1} h_1 + u_r h_0$$

in (3.5) and expanding, we have

$$a_{r+1} = \prod_{j=1}^n (u_{r+1} h_1^j + u_r h_0^j) ,$$

$$a_{r+1} = u_{r+1}^n \prod_{j=1}^n h_1^j + u_r^n \prod_{j=1}^n h_0^j + \sum_{s=2}^n H_s u_{r+1}^{n+1-s} u_r^{s-1} .$$

Clearly, H_s is a combination of the h_0^j and h_1^j , and independent of r . We have satisfied (3.4) and the proof is complete.

Theorem I establishes the recurrence formulae $\phi_n(a) = 0$ of (2.19) as generators for the n -th powers and n -th order products of the sequence $\{h_r\}$ of (3.3), and in particular, products of the Fibonacci sequence $\{u_r\}$ of (2.2).

There remains to be constructed the link between P_n and these recurrence formulae. We prove

Theorem II

$$\phi_n(P_n) = 0 .$$

Proof.

Since P_n of (1.1) is related to Q_n of (2.6) by $P_n = E Q_n^T E^{-1}$ with $E = E^{-1}$ being a matrix with ones on the counter diagonal and zeros elsewhere, P_n and Q_n are similar and hence satisfy the same polynomial equations. It is sufficient to show that $\phi_n(Q_n) = 0$.

First, each element of the matrix $B_{n+1,i}$ (2.4) is an element of a sequence of the type

$$b_r = \prod_{j=1}^n h_r^j , \text{ defined in (2.8).}$$

Construct the sequence $b_r, b_{r+1}, \dots, b_{r+n+1}$ by choosing the corresponding elements from the matrix sequence $B_{n+1,i}, B_{n+1,i+1}, \dots, B_{n+1,i+n+1}$. By Theorem I, $\phi_n(b) = 0$. Since this is true for any element of $B_{n+1,i}$ it is true for the entire matrix. We have

$$(3.6) \quad \phi_n(B) = 0 \quad \text{identically.}$$

Writing out the summation in (3.6),

$$(3.7) \quad \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} B_{n+1, n+1-r-i} = 0 .$$

The matrix Q_n may be used (as in (2.5)) to shift the index of B so that

$$(3.8) \quad B_{n+1, n+1-r-i} = B_{n+1, 0} Q_n^{n+1-r-i} = B_{n+1, 0} Q^{-i} Q^{n+1-r} .$$

Using (3.8) in (3.7) we have

$$B_{n+1, 0} Q_n^{-i} \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} Q^{n+1-r} = 0 .$$

Now B is never singular, (2.9), nor is Q , (2.7), so that

$$\sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} Q^{n+1-r} = 0$$

which is to say, by (2.20),

$$\phi_n(Q_n) = 0 .$$

Theorem II is implied more directly by Theorem I after having established the following representations for Q_n^r :

$$Q_1^r = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1} & u_r \\ u_r & u_{r-1} \end{bmatrix}$$

$$\begin{aligned}
 Q_2^r &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1}^2 & u_{r+1}u_r & u_r^2 \\ 2u_{r+1}u_r & u_{r+1}u_{r-1} + u_r^2 & 2u_ru_{r-1} \\ u_r^2 & u_ru_{r-1} & u_{r-1}^2 \end{bmatrix} \\
 Q_3^r &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1}^3 & u_{r+1}^2u_r & u_{r+1}u_r^2 & u_r^3 \\ 3u_{r+1}^2u_r & \dots & \dots & 3u_ru_{r-1} \\ 3u_{r+1}u_r^2 & \dots & \dots & 3u_r^2u_{r-1} \\ u_r^3 & u_r^2u_{r-1} & u_ru_{r-1}^2 & u_{r-1}^3 \end{bmatrix}
 \end{aligned}$$

etc., where the bordering elements of Q_n^r build up in the manner suggested by these cases and the internal elements, while being more complicated in structure, nevertheless are sums of n -th order products of u 's.

Before stating the final theorem we will examine the special case used earlier in terms of what we now know. We have the two matrices

$$B_1 = \begin{bmatrix} u_2^4 & u_2^3u_1 & u_2^2u_1^2 & u_2u_1^3 & u_1^4 \\ u_3^4 & u_3^3u_2 & u_3^2u_2^2 & u_3u_2^3 & u_2^4 \\ u_4^4 & u_4^3u_3 & u_4^2u_3^2 & u_4u_3^3 & u_3^4 \\ u_5^4 & u_5^3u_4 & u_5^2u_4^2 & u_5u_4^3 & u_4^4 \\ u_6^4 & u_6^3u_5 & u_6^2u_5^2 & u_6u_5^3 & u_5^4 \end{bmatrix}$$

(where the index 1 on B indicates the indices of the first row) and

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have the polynomial

$$(3.9) \quad \phi(x) = x^5 - (5x^4 + 15x^3 - 15x^2 - 5x + 1)$$

from (2.21) with $n = 4$ and the corresponding recursion relation

$$(3.10) \quad b_{n+5} = 5b_{n+4} + 15b_{n+3} - 15b_{n+2} - 5b_{n+1} + b_n$$

which is satisfied by any sequence whose members are the element by element product of four Fibonacci sequences — in particular it is satisfied by the sequences formed by extending each column of B_1 ad infinitum, the index of each sequence increasing downward. In view of this fact we construct the matrix

$$(3.11) \quad E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -5 & -15 & 15 & 5 \end{bmatrix}$$

whose obvious property is that of transforming any column vector

$$\begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \\ b_{n+3} \\ b_{n+4} \end{bmatrix} \text{ into } \begin{bmatrix} b_{n+1} \\ b_{n+2} \\ b_{n+3} \\ b_{n+4} \\ b_{n+5} \end{bmatrix}$$

if the elements of the vector satisfy the relationship (3.10). E has the property, then, that

$$(3.12) \quad E B_1 = B_2 .$$

It is not difficult to show that the characteristic polynomial of (3.11) is

$$|xI - E| = \phi_4(x)$$

for $\phi_4(x)$ defined in (3.9). Combining (3.12) with the property (2.5) of Q

$$B_2 = E B_1 = B_1 Q ,$$

and B_1 is not singular, hence Q , and therefore P , is similar to, and has the same characteristic polynomial as E .

The preceding example illustrates the proof of the final

Theorem III

The $(n+1) \times (n+1)$ matrix P_n of (1.1), formed by imbedding Pascal's triangle in a square matrix, has the characteristic polynomial

$$(3.13) \quad |xI - Q_n| = \sum_{r=0}^{n+1} (-1)^r (-1)^{r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n+1-r}$$

where $\begin{bmatrix} n \\ r \end{bmatrix}$ is a generalized "binomial coefficient" defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{u_n \cdot u_{n-1} \cdots u_{n-r+1}}{u_r \cdot u_{r-1} \cdots u_1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1 .$$

Furthermore, the polynomial (3.13) is the same polynomial which characterizes the recursion relation for the element by element product sequence of any n sequences each of which satisfies the Fibonacci recurrence relation $u_{n+1} = u_n + u_{n-1}$.

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THE GOLDEN CUBOID

H. E. HUNTLEY

The problem of finding the dimensions of a cuboid (rectangular parallelopiped) of unit volume, having a diagonal 2 units in length leads to an interesting result.

Suppose the lengths of the edges are \underline{a} , \underline{b} and \underline{c} . Then

$$(1) \quad \underline{a} \cdot \underline{b} \cdot \underline{c} = 1 \quad \text{and} \quad (2) \sqrt{(\underline{a}^2 + \underline{b}^2 + \underline{c}^2)} = 2$$

If only the ratios of these lengths are required, we may, without loss of generality, write $\underline{b} = 1$, provided that $\underline{a} \cdot \underline{c}$ can have the value unity and that $\underline{a}^2 + \underline{c}^2 = 3$. Now it is evident from Fig. 1, which represents the base of the cuboid, that the maximum value of $\underline{a} \cdot \underline{c}$ occurs when $\underline{a} = \underline{c} = \sqrt{3}/2$, so that $\underline{a} \cdot \underline{c}$ may have any value from zero to $3/2$.

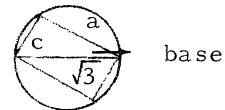


Fig. 1

Substituting $c = 1/a$ from (1) in (2), we have

$$\underline{a}^2 + \frac{1}{\underline{a}^2} = 3 \quad \text{i.e.,} \quad \underline{a}^4 - 3\underline{a}^2 + 1 = 0, \quad \text{whence}$$

$$\underline{a}^2 = \frac{3 + \sqrt{5}}{2} = 1 + \varphi = \varphi^2,$$

so that $a = \varphi$, the Golden Section. The positive solution of the equation $x^2 - x - 1 = 0$ and the value of u_n/u_{n-1} as $n \rightarrow \infty$, where u_n is a member of the Fibonacci Series.

From (1) it follows that $c = \varphi^{-1}$, so that the required ratios are $a:b:c = \varphi:1:\varphi^{-1}$. It is easily verified that $\varphi^2 + 1 + \varphi^{-2} = 4$.

Continued on page 240.

A PARTIAL DIFFERENCE EQUATION RELATED TO THE FIBONACCI NUMBERS

L. CARLITZ

1. Consider the equation

$$(1.1) \quad u_{mn} - u_{m-1,n} - u_{m,n-1} + u_{m-2,n} + 3u_{m-1,n-1} - u_{m,n-2} = 0 \quad (m \geq 2, n \geq 2).$$

If we put

$$(1.2) \quad G(x, y) = \sum_{m,n=0}^{\infty} u_{mn} x^m y^n$$

and

$$(1.3) \quad f(x, y) = 1 - x - y - x^2 + 3xy - y^2,$$

it follows from (1.1) that

$$(1.4) \quad f(x, y)G(x, y) = a + bx + cy,$$

where a, b, c are constants. Indeed it is evident that

$$(1.5) \quad a = u_{00}, \quad b = u_{10} - u_{00}, \quad c = u_{01} - u_{00}.$$

Thus if u_{00}, u_{10}, u_{01} , or equivalently a, b, c , are assigned u_{mn} is uniquely determined for all non-negative integers m, n . We shall show that the general solution of (1.1) can be expressed in terms of Fibonacci numbers.

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2. If we put

$$(2.1) \quad \alpha = \frac{1}{2} (1 + \sqrt{5}), \quad \beta = \frac{1}{2} (1 - \sqrt{5}),$$

it is easily verified that

$$\begin{aligned} (1 - \alpha x - \beta y)(1 - \beta x - \alpha y) &= 1 - (\alpha + \beta)(x + y) + \alpha\beta(x^2 + y^2) + (\alpha^2 + \beta^2)xy \\ &= 1 - x - y - x^2 + 3xy - y^2, \end{aligned}$$

so that

$$(2.2) \quad f(x, y) = (1 - \alpha x - \beta y)(1 - \beta x - \alpha y).$$

We now consider the case

$$(2.3) \quad a = 0, \quad b = 1, \quad c = -1.$$

Then

$$\begin{aligned} \frac{x-y}{f(x, y)} &= \frac{1}{\alpha - \beta} \left[\frac{1}{1 - \alpha x - \beta y} - \frac{1}{1 - \beta x - \alpha y} \right] \\ &= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left\{ (\alpha x + \beta y)^n - (\beta x + \alpha y)^n \right\} \\ &= \frac{1}{\alpha - \beta} \sum_{m, n=0}^{\infty} \binom{m+n}{n} (\alpha^m \beta^n - \alpha^n \beta^m) x^m y^n. \end{aligned}$$

If F_{mn} denotes the solution of (1.1) and (2.3) holds, we have therefore

$$(2.4) \quad F_{mn} = \binom{m+n}{m} \frac{\alpha^m \beta^n - \alpha^n \beta^m}{\alpha - \beta}$$

Now it is evident from (1.4) and (2.3) that

$$(2.5) \quad F_{mn} = -F_{nm}, \quad F_{nn} = 0,$$

so that it will suffice to determine F_{mn} when $m > n$.

If as usual we put

$$(2.6) \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

then it follows from (2.4) that

$$(2.7) \quad F_{mn} = (-1)^n \binom{m+n}{m} F_{m-n} \quad (m \geq n).$$

In view of (2.5), this result can be expressed in the following form:

$$(2.8) \quad \frac{x-y}{f(x,y)} = \sum_{m>n} (-1)^n \binom{m+n}{n} F_{m-n} (x^m y^n - x^n y^m).$$

We can also evaluate

$$(2.9) \quad \Phi(x, y) = \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} F_{mn} x^m y^n.$$

Indeed, by (2.7), we have

$$\begin{aligned} \Phi(x, y) &= \sum_{n=0}^{\infty} (-1)^n x^n y^n \sum_{k=0}^{\infty} \binom{k+2n}{n} F_k x^k \\ &= \sum_{k=0}^{\infty} F_k x^k \sum_{n=0}^{\infty} (-1)^n \binom{k+2n}{n} x^n y^n. \end{aligned}$$

Now it is known that

$$\sum_{n=0}^{\infty} \binom{k+2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \left(\frac{2}{1+\sqrt{1-4x}} \right)^k ,$$

so that

$$\Phi(x, y) = \frac{1}{\sqrt{1+4xy}} \sum_{k=0}^{\infty} F_k \left(\frac{2x}{1+\sqrt{1-4x}} \right)^k$$

This reduces to

$$(2.10) \quad \Phi(x, y) = \frac{z}{\sqrt{1+4xy}(1-z-z^2)}, \quad z = \frac{2x}{1+\sqrt{1-4xy}} .$$

We have also

$$(2.11) \quad \Phi(x, y) - \Phi(y, x) = \frac{x-y}{f(x, y)} .$$

It is not difficult to verify that

$$\frac{1}{1-\alpha z} = (1+\sqrt{1+4xy}) \frac{1-\sqrt{1+4xy}-2\alpha x}{-4\alpha x(1-\alpha x-\beta y)} ,$$

so that

$$\begin{aligned} \frac{z}{1-z-z^2} &= (1+\sqrt{1-4xy}) \frac{-1+x+2x^2-2xy+(1-x)\sqrt{1-4xy}}{4xf(x, y)} \\ &= \frac{x+y-2xy+(x-y)\sqrt{1-4xy}}{2f(x, y)} . \end{aligned}$$

It follows that

$$\Phi(x, y) - \Phi(y, x) = \frac{2(x-y)}{2f(x, y)} = \frac{x-y}{f(x, y)} ,$$

in agreement with (2.11).

3. We next take the case

$$(3.1) \quad a = 2, \quad b = c = -1 .$$

Then

$$\begin{aligned} \frac{2-x-y}{f(x, y)} &= \frac{1}{1-\alpha x - \beta y} + \frac{1}{1-\beta x - \alpha y} = \sum_{n=0}^{\infty} \left\{ (\alpha x + \beta y)^n + (\beta x + \alpha y)^n \right\} \\ &= \sum_{m, n=0}^{\infty} \binom{m+n}{m} (\alpha^m \beta^n + \alpha^n \beta^m) x^m y^n . \end{aligned}$$

Thus, if L_{mn} denotes the solution of (1.1) when (3.1) holds, we have

$$(3.2) \quad L_{mn} = \binom{m+n}{m} (\alpha^m \beta^n + \alpha^n \beta^m) .$$

Also it is evident from (1.4) and (3.1) that

$$(3.3) \quad L_{mn} = L_{nm} ,$$

so it will suffice to evaluate L_{mn} when $m \geq n$. If we put

$$(3.4) \quad L_n = \alpha^n + \beta^n$$

it follows from (3.2) that

$$(3.5) \quad L_{mn} = (-1)^n \binom{m+n}{m} L_{m-n} \quad (m \geq n) .$$

By (3.3) this result can be stated in the form

$$(3.6) \quad \begin{aligned} \frac{2-x-y}{f(x, y)} &= 2 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} x^n y^n \\ &\quad \sum_{m > n} (-1)^n \binom{m+n}{m} L_{m-n} (x^m y^n + x^n y^m) . \end{aligned}$$

4. We now take

$$(4.1) \quad a = 1, \quad b = c = 0$$

and let G_{mn} denote the solution of (1.1) in this case. Thus it is clear that

$$(4.2) \quad \frac{1}{f(x, y)} = \sum_{m, n=0}^{\infty} G_{mn} x^m y^n .$$

Comparing this with

$$\frac{x-y}{f(x, y)} = \sum_{m, n=0}^{\infty} F_{mn} x^m y^n$$

we get

$$(x-y) \sum_{m, n=0}^{\infty} G_{mn} x^m y^n = \sum_{m, n=0}^{\infty} F_{mn} x^m y^n ,$$

so that

$$(4.3) \quad G_{m-1, n} - G_{m, n-1} = F_{mn} \quad (m \geq 1, n \geq 1) .$$

It is evident from (4.2) that

$$(4.4) \quad G_{mn} = G_{nm}$$

and

$$(4.5) \quad G_{mo} = G_{om} = F_{m+1} .$$

If $m \geq n$ it follows from (4.3) and (2.7) that

$$G_{mn} = F_{m+1, n+1} + (-1)^n F_{m-n+1}$$

Repeated application of this formula leads to

$$(4.6) \quad G_{mn} = \sum_{r=0}^n (-1)^r \binom{m+n+1}{r} F_{m+n-2r+1} \quad (m \geq n)$$

By (4.4) this result can be stated in the following form,

$$(4.7) \quad \frac{1}{f(x, y)} = \sum_{n=0}^{\infty} \sum_{r=0}^n (-1)^r \binom{2n+1}{r} F_{2n-2r+1} x^n y^n + \sum_{m > n} \sum_{r=0}^n (-1)^r \binom{m+n+1}{r} F_{m+n-2r+1} (x^m y^n + x^n y^m)$$

5. It is now easy to express the general solution of (1.1) in terms of F_{mn} , L_{mn} , G_{mn} and therefore in terms of F_k and L_k . As we have seen above, if the numbers u_{00} , u_{10} , u_{01} are assigned, u_{mn} is uniquely determined for all $m, n \geq 0$. Indeed we may put

$$(5.1) \quad u_{mn} = AF_{mn} + BL_{mn} + CG_{mn},$$

where A, B, C are independent of m, n . Then

$$(5.2) \quad \left\{ \begin{array}{l} u_{00} = AF_{00} + BL_{00} + CG_{00} \\ u_{10} = AF_{10} + BL_{10} + CG_{10} \\ u_{01} = AF_{01} + BL_{01} + CG_{01} \end{array} \right.$$

But by (2.7), (3.5), (4.4) and (4.5)

$$\begin{aligned} F_{00} &= 0, & L_{00} &= 2, & G_{00} &= 1 \\ F_{10} &= 1, & L_{10} &= 1, & G_{10} &= 1 \\ F_{01} &= -1, & L_{01} &= 1, & G_{01} &= 1 \end{aligned}$$

Substituting these values in (5.2) we find that

$$(5.3) \quad \left\{ \begin{array}{l} A = \frac{1}{2}(u_{10} - u_{01}) \\ B = u_{00} - \frac{1}{2}(u_{10} + u_{01}) \\ C = -u_{00} + u_{10} + u_{01} \end{array} \right.$$

Thus (5.1) becomes

$$(5.4) \quad \begin{aligned} u_{mn} &= \frac{1}{2}(u_{10} - u_{01})F_{mn} + (u_{00} - \frac{1}{2}u_{10} - \frac{1}{2}u_{01})L_{mn} \\ &\quad + (-u_{00} + u_{10} + u_{01})G_{mn} . \end{aligned}$$

Finally, making use of (2.7), (3.5) and (4.6), we can express u_{mn} explicitly in terms of F_k and G_k .

6. It is of some interest to extend the solutions of (1.1) to arbitrary integral values of m and n . In the first place we define F_{mn} by means of

$$(6.1) \quad F_{mn} = \left(\frac{m+n}{m} \right) \frac{\alpha^m \beta^n - \alpha^n \beta^m}{\alpha - \beta}$$

for all integral m, n . Now since

$$\left(\frac{m-n}{m} \right) = 0 \quad (0 < n \leq m)$$

it follows that

(6.2)

$$F_{m,-n} = 0 \quad (0 < n \leq m) ;$$

similarly we have

(6.3)

$$F_{-m,n} = 0 \quad (0 < m \leq n) .$$

Also since, by definition,

$$\binom{-m-n}{-m} = 0 \quad (m > 0, n > 0)$$

we have

(6.4)

$$F_{-m,-n} = 0 \quad (m > 0, n > 0) .$$

On the other hand, since

$$\binom{m-n}{m} = (-1)^m \binom{n-1}{m} \quad (n > m) ,$$

it follows that

(6.5)

$$F_{m,-n} = (-1)^{m+n} \binom{n-1}{m} F_{m+n} \quad (n > m) ;$$

similarly

(6.6)

$$F_{-m,n} = -(-1)^{m+n} \binom{m-1}{n} F_{m+n} \quad (m > n) .$$

Note that in all cases we have

(6.7)

$$F_{mn} = -F_{nm} .$$

We remark that if we define

(6.8)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for all integral n , then (6.1) becomes

$$(6.9) \quad F_{mn} = (-1)^n \binom{m+n}{m} F_{m-n} = -(-1)^m \binom{m+n}{m} F_{n-m}.$$

It remains to show that F_{mn} as defined by (6.1) or (6.9) does satisfy (1.1) for all m, n . We have

$$\begin{aligned} & F_{mn} - F_{m-1, n} - F_{m, n-1} + F_{m-2, n} + 3F_{m-1, n-1} - F_{m, n-2} \\ &= (-1)^n \binom{m+n}{m} F_{m-n} - (-1)^n \binom{m+n-1}{m-1} F_{m-n-1} \\ &+ (-1)^n \binom{m+n-1}{m} F_{m-n+1} - (-1)^n \binom{m+n-2}{m-2} F_{m-n-2} \\ &- 3(-1)^n \binom{m+n-2}{m-1} F_{m-n} - (-1)^n \binom{m+n-2}{m} F_{m-n+2}. \end{aligned}$$

Now making use of

$$F_{n+1} = F_n + F_{n-1},$$

which holds for all integral n , we find that F_{mn} satisfies (1.1).

The extension of L_{mn} can be carried out in exactly the same way. We define

$$(6.10) \quad L_{mn} = (-1)^n \binom{m+n}{m} L_{m-n}$$

for all integral m, n , where

$$(6.11) \quad L_n = \alpha^n + \beta^n$$

for all integral n .

As for G_{mn} , we require that

$$(6.12) \quad G_{m-1, n} - G_{m, n-1} = F_{mn}$$

for all m, n . If n is negative we replace n by $-n$, so that (6.12) becomes

$$G_{m-1, -n} - G_{m, -n-1} = F_{m, -n} .$$

This may be written as

$$G_{m, -n-1} = G_{m-1, -n} - F_{m, -n} ,$$

which implies

$$G_{m, -n} = G_{m-n, 0} - \sum_{r=0}^{n-1} F_{m-r, -n+r-1} .$$

We put (compare (4.5))

$$(6.13) \quad G_{m0} = G_{0m} = F_{m+1}$$

for all m ; it follows that

$$(6.14) \quad G_{m, -n} = F_{m-n+1} - \sum_{r=0}^{n-1} F_{m-r, -n+r+1} \quad (n \geq 1) .$$

Similarly if m is negative we get

$$(6.15) \quad G_{-m, n} = F_{n-m+1} + \sum_{r=0}^{m-1} F_{r-m+1, n-r} \quad (m \geq 1) .$$

Indeed we find that $G_{-m, n}$ as defined by (6.15) satisfies (6.12) for all n . It can be verified easily that

$$(6.16) \quad G_{mn} = G_{nm}$$

for all m, n .

Finally we can show that G_{mn} as defined by (4.6), (6.14) and (6.15) satisfies (1.1). We omit the details of this verification.

7. We remark that the difference equation (1.1) can be generalized in an obvious way. Let α , β be roots of the quadratic equation

$$(7.1) \quad x^2 - px + q = 0,$$

where p, q are arbitrary numbers, and put

$$f(x, y) = (1 - \alpha x - \beta y)(1 - \beta x - \alpha y) = 1 - p(x+q) + qx^2 + (p^2 - 2q)xy + qy^2.$$

Then the generalized equation is

$$(7.2) \quad u_{m,n} - pu_{m-1,n} - pu_{m,n-1} + qu_{m-2,n} \\ + (p^2 - 2q)u_{m-1,n-1} + qu_{m,n-2} = 0.$$

The results obtained above for (1.1) can be carried over without difficulty to the more general equation (7.2).

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Continued from page 176.

Equating coefficients in (1) and (3), one obtains, the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If, on the other hand we let $y = 2$, $y' = 1$; $x = 0$, equation (1) becomes

$$y = e^{\alpha x} + e^{\beta x} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n) \frac{x^n}{n!}.$$

The series solution yields $u_0 = 2$ and $u_1 = 1$ so that equation (3) becomes

$$y = \sum_{n=0}^{\infty} \frac{L_n x^n}{n!},$$

and one obtains

$$L_n = \alpha^n + \beta^n.$$

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LINEAR RECURRENCE RELATIONS - PART III

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1. INTRODUCTION

We continue our study to acquaint the beginner with linear recurrence relations and the method of generating functions for solving them (see articles in [1] and [2]). In this concluding article we shall consider recurrence relations in which there is more than one independent variable.

2. DEFINITION

A partial linear recurrence relation in two independent variables m, n is an equation of the form

$$(2.1) \quad \sum_{i=0}^k \sum_{j=0}^p a_{ij}(m, n) y(m+i, n+j) = b(m, n)$$

where a_{ij} and b are given functions of the discrete variables m and n over the set of non-negative integers. Partial recurrence relations in three or more independent variables may be defined in a similar way.

If $b(m, n) = 0$, relation (2.1) is called homogeneous. The equation contains $(k+1)(p+1)$ possible terms and is said to be of order k with respect to m and of order p with respect to n . To solve certain recurrence relations we find it convenient to apply a generating function transform.

3. A SERIES TRANSFORM

The exponential generating function for the sequence $\{y(m, n)\}$, $(m, n = 0, 1, 2, \dots)$ is defined by the double infinite series

$$(3.1) \quad Y(s, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m, n) \frac{s^m}{m!} \frac{t^n}{n!}$$

If the series (3.1) converges when $|s| < \alpha$ and $|t| < \beta$ simultaneously, then all of the derived series of (3.1) will also converge in the same region. Thus, we have

$$(3.2) \quad \begin{aligned} \frac{\partial Y}{\partial s} &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m y(m, n) \frac{s^{m-1}}{m!} \frac{t^n}{n!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+1, n) \frac{s^m}{m!} \frac{t^n}{n!} \end{aligned}$$

and it is seen that $(\partial Y / \partial s)$ is the exponential generating function of the sequence $y(m+1, n)$. Similarly, one easily obtains the equations

$$(3.3) \quad \frac{\partial Y}{\partial t} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m, n+1) \frac{s^m}{m!} \frac{t^n}{n!}$$

and

$$(3.4) \quad \frac{\partial^2 Y}{\partial s \partial t} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+1, n+1) \frac{s^m}{m!} \frac{t^n}{n!},$$

which furnish exponential generating functions for the sequences $\{y(m, n+1)\}$ and $\{y(m+1, n+1)\}$ respectively. In general, the relation

$$(3.5) \quad \frac{\partial^{i+j} Y}{\partial s^i \partial t^j} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+i, n+j) \frac{s^m}{m!} \frac{t^n}{n!}$$

is the exponential generating function of the sequence $\{y(m+i, n+j)\}$. This equation permits us to transform linear partial recurrence relations (2.1) where the coefficients $a_{ij}(m, n)$ are all assumed to be constants (i. e., not a function of m and n). If we then multiply both sides of (2.1) by $\frac{s^m t^n}{m! n!}$ and sum on m and n from zero to infinity, we get the transformed equation

$$(3.6) \quad \sum_{i=0}^k \sum_{j=0}^p a_{ij} \frac{\partial^{i+j} Y}{\partial s^i \partial t^j} = B(s, t),$$

where

$$(3.7) \quad B(s, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b(m, n) \frac{s^m}{m!} \frac{t^n}{n!}.$$

After the transformed equation is solved for Y , we then obtain the sequence $\{y(m, n)\}$ either from the relation

$$(3.8) \quad y(m, n) = \left. \frac{\partial^{m+n} Y}{\partial s^m \partial t^n} \right|_{\begin{array}{l} s=0 \\ t=0 \end{array}},$$

or by expanding the function $Y(s, t)$ in the form (3.1).

We illustrate the procedure with two simple examples in which $b(m, n) = 0$.

4. EXAMPLES

Consider, for instance, the partial recurrence relation

$$(4.1) \quad y(m+1, n+1) - y(m+1, n) - y(m, n) = 0$$

with the given conditions

$$(4.2) \quad \begin{cases} y(m, 0) = 0 & \text{if } m \neq 0 \\ y(0, n) = 1 \\ y(m, n) = 0 & \text{if } m < n \end{cases}$$

The transformed equation for (4.1) is then

$$(4.3) \quad \frac{\partial^2 Y}{\partial s \partial t} - \frac{\partial Y}{\partial s} - Y = 0$$

with the conditions

$$(4.4) \quad Y(s, 0) = 1, \quad Y(0, t) = e^t.$$

Now, a particular solution of (4.3) is

$$(4.5) \quad Y(s, t) = e^y I_0(2\sqrt{st}),$$

where $I_0(z)$ is the modified Bessel function of the first kind defined by

$$(4.6) \quad I_0(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{(m!)^2}.$$

We can obtain the sequence $\{y(m, n)\}$ by expanding (4.5). Thus,

$$(4.7) \quad \begin{aligned} Y(s, t) &= e^t \sum_{m=0}^{\infty} \frac{s^m t^n}{(m!)^2} \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{m=0}^{\infty} \frac{s^m t^n}{(m!)^2}. \end{aligned}$$

Letting $n = m+j$, we then have

$$(4.8) \quad \begin{aligned} Y(s, t) &= \sum_{m=0}^{\infty} \frac{s^m t^m}{(m!)^2} \sum_{n=m}^{\infty} \frac{t^{n-m}}{(n-m)!} \frac{n!}{n!} \\ &= \sum_{m=0}^{\infty} \frac{s^m}{m!} \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^n}{n!}. \end{aligned}$$

Hence, from (3.1) it is clear that

$$(4.9) \quad \begin{aligned} y(m, n) &= \binom{n}{m}, \quad (m = 0, 1, \dots, n) \\ &= 0, \quad n < m, \end{aligned}$$

which simply represents the elements of Pascal's triangle (that is, the binomial coefficients).

As a second example, we take the partial recurrence relation

$$(4.10) \quad y(m+1, n+1) - y(m, n+1) - y(m, n) = 0$$

with the conditions

$$(4.11) \quad y(0, n) = F_n; \quad y(m, 0) = F_m,$$

where F_n denotes the n th Fibonacci number. Transformation of equation (4.10) yields

$$(4.12) \quad \frac{\partial^2 Y}{\partial s \partial t} - \frac{\partial Y}{\partial t} - Y = 0$$

with the conditions

$$(4.13) \quad Y(s, 0) = \frac{1}{\sqrt{5}} \begin{bmatrix} e^{sa_1} & e^{sa_2} \end{bmatrix}$$

$$Y(0, t) = \frac{1}{\sqrt{5}} \begin{bmatrix} e^{ta_1} & e^{ta_2} \end{bmatrix}$$

where $a_1 = \frac{1}{2} (1 + \sqrt{5})$, $a_2 = \frac{1}{2} (1 - \sqrt{5})$.

The solution of equation (4.12) is

$$(4.15) \quad Y(s, t) = \frac{1}{\sqrt{5}} \begin{bmatrix} e^{a_1(t+s)} & e^{a_2(t+s)} \end{bmatrix}.$$

Now, employing the inverse transform (3.8) yields

$$(4.16) \quad y(m, n) = \left. \frac{\partial^{m+n} Y}{\partial s^m \partial t^n} \right|_{s=0, t=0} = \frac{1}{\sqrt{5}} (a_1^{m+n} - a_2^{m+n})$$

which is the solution of (4.10) and represents a Fibonacci array shown in the following table

$\begin{array}{c} m \\ n \end{array}$	0	1	2	3	4	5	...
0	0	1	1	2	3	5	...
1	1	1	2	3	5	8	
2	1	2	3	5	8	13	
3	2	3	5	8	13	21	
4	3	5	8	13	21	34	
5	5	8	13	21	34	55	
...
...	
...	

Fibonacci arrays of higher dimension can also be obtained. These involve the solutions of partial recurrence relations in three or more independent variables.

5. CONCLUDING REMARKS

The above examples involved the solution of two partial recurrence relations having only constant coefficients. Recurrence relations with polynomial coefficients may also be transformed by the method of generating functions. For instance, it is easy to show that the recurrence relation

$$(5.1) \quad \sum_{i=0}^k \sum_{j=0}^p (\alpha_{ij} + m\beta_{ij} + n\gamma_{ij}) y(m+i, n+j) = b(m, n),$$

having linear coefficients, can be transformed to the equation

$$(5.2) \quad \sum_{i=0}^k \sum_{j=0}^p (\alpha_{ij} + \beta_{ij}\phi + \gamma_{ij}\psi) \cdot \frac{\partial^{i+j} Y}{\partial s^i \partial t^j} = B(s, t),$$

where $B(s, t)$ is given by (3.7), and ϕ and ψ are the differential operators

$$(5.3) \quad \phi = s \frac{\partial}{\partial s}, \quad \psi = t \frac{\partial}{\partial t}.$$

I wish to thank Prof. Paul F. Byrd for his many helpful suggestions during the preparation of this article and the two previous ones.

REFERENCES

1. J. A. Jeske, "Linear Recurrence Relations — Part I," The Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74.
2. _____, "Linear Recurrence Relations — Part II," The Fibonacci Quarterly, Vol. 1, No. 4, pp. 35-39.

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ASSOCIATIVITY AND THE GOLDEN SECTION

H. W. GOULD
West Virginia University, Morgantown, West Virginia

E. T. Bell, A functional equation in arithmetic, Trans. Amer. Math. Soc., 39(1936), 341-344, gave a discussion of some matters suggested by the functional equation of associativity

$$\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z).$$

As a prelude, Bell noted the following theorem.

THEOREM 1. The only polynomial solutions of $\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$ in the domain of complex numbers are the unsymmetric solutions $\varphi(x, y) = x$, $\varphi(x, y) = y$, and the symmetric solution

$$\varphi(x, y) = a + b(x + y) + cxy,$$

in which a, b, c , are any constants such that $b^2 - b - ac = 0$.

It is amusing to note a special case. The operation defined by

$$x * y = 1 + b(x + y) + xy$$

is associative only if $b = \frac{1}{2} (\pm \sqrt{5})$.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-41 *Proposed by Robert A. Laird, New Orleans, La.*

Find rational integers, x , and positive integers, m , so that

$$N = x^2 - m \quad \text{and} \quad M = x^2 + m$$

are rational squares. There are no solutions for $m = 1, 2, 3, 4$ but $m = 5$ is historically interesting.

H-42 *Proposed by J.D.E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

H-43 *Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Let

$$\varphi(x) = \sum_{n=1}^{\infty} x^{mn},$$

where F_j is the j -th Fibonacci number, find

$$\lim_{x \rightarrow 1^-} \frac{\varphi(x)}{-\log(1-x)}.$$

See special case $m = 2$ in Revista Matematica Hispano-Americanica (2)

9 (1934) 223-225 problem 115.

H-44 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_0 = q$ and $u_1 = p$, and $u_{n+2} = u_{n+1} + u_n$, then the u_n are called generalized Fibonacci numbers.

$$(1) \text{ Show } u_n = pF_n + qF_{n-1}$$

(2) Show that if

$$V_{2n+1} = u_n^2 + u_{n+1}^2 \text{ and } V_{2n} = u_{n+1}^2 - u_{n-1}^2 ,$$

then V_n are also generalized Fibonacci numbers.

H-45 Proposed by R.L. Graham, Bell Telephone Labs., Murray Hill, N.J.

Prove

$$\sum_{p=0}^n \sum_{q=0}^p \sum_{r=0}^q \sum_{s=0}^r F_s^2 = F_{n+2}^2 - \frac{1}{8} (2n^2 + 8n + 11 - 3(-1)^n) ,$$

where F_n is the n th Fibonacci number.

SOLUTIONS WARD'S LAST THEOREM

H-24 Proposed by the late Morgan Ward, California Institute of Technology, Pasadena, California

Let $\phi_n(x) = x + x^2/2 + \dots + x^n/n$, and let $k(x) = k_p(x) = (x^{p-1} - 1)/p$, where p is an odd prime greater than 5. (The function $k(x)$ is called the "quotient of Fermat" in the literature.) Let $P = P_p$ be the rank of apparition of p in the sequence $0, 1, 1, 2, 3, 5, \dots, F_n$, (so $P_{13} = 7$, $P_7 = 8$ and so on).

Then

$$F_P \equiv 0 \pmod{p^2}$$

if and only if

$$\phi_{(p-1)/2}(5/9) = 2k(3/2) \pmod{p} .$$

Solution by L. Carlitz, Duke University, Durham, N.C.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

Then the condition $F_P \equiv 0 \pmod{p}$, $p > 5$, is equivalent to $\alpha^P = \beta^P \pmod{p}$, or what is the same thing

$$(*) \quad (-\alpha^2)^{\frac{P}{2}} = 1 \pmod{p} .$$

We now treat separately the two cases

$$(i) \quad \left(\frac{5}{p}\right) = +1, \quad (ii) \quad \left(\frac{5}{p}\right) = -1 ,$$

where $(5/p)$ is the Legendre symbol.

In the first case we have $5 = \pi \pi'$, where π, π' are primes of the quadratic field $R(\sqrt{5})$. It follows that $P \mid p-1$. Then clearly

$$(-\alpha^2)^{\frac{P}{2}} = 1 \pmod{p^2} \iff (-\alpha^2)^{p-1} = 1 \pmod{p^2} .$$

We therefore consider $\alpha^{2p} - \alpha^2 \pmod{p^2}$. Since

$$\alpha^{2p} = \alpha^2 \pmod{p^2} \iff \beta^{2p} = \beta^2 \pmod{p^2} ,$$

it will suffice to consider

$$\alpha^{2p} + \beta^{2p} - 3 \pmod{p^2} .$$

Now

$$\begin{aligned} \alpha^{2p} + \beta^{2p} &= \frac{2}{2^p} \sum_{r=0}^{\frac{1}{2}(p-1)} \binom{p}{2r} 3^{p-2r} 5^r \\ &= \frac{2}{2^p} \cdot 3^p + \frac{2}{2^p} \sum_{r=1}^{\frac{1}{2}(p-1)} \binom{p}{2r} 3^{p-2r} 5^r . \end{aligned}$$

Since

$$\binom{p}{2r} \equiv -\frac{p}{2r} \pmod{p^2} ,$$

we get

$$\begin{aligned} \alpha^{2p} + \beta^{2p} &= 2\left(\frac{3}{2}\right)^p - p\left(\frac{3}{2}\right)^p \sum_{r=1}^{\frac{1}{2}(p-1)} \frac{1}{r} \left(\frac{5}{9}\right)^r \\ &\equiv 2\left(\frac{3}{2}\right)^p - \frac{3p}{2} \not\equiv \frac{1}{\frac{1}{2}(p-1)} \left(\frac{5}{9}\right) \pmod{p^2} . \end{aligned}$$

Therefore $\alpha^{2p} + \beta^{2p} \equiv 3 \pmod{p^2}$ is equivalent to

$$\begin{aligned} \frac{3}{2} \phi_1 \frac{(5)}{\frac{1}{2}(p-1)} &\equiv 2\left(\frac{3}{2}\right)^p - 3 \\ &\equiv 3 \left[\left(\frac{3}{2}\right)^{p-1} - 1 \right] \\ &\equiv 3k\left(\frac{3}{2}\right) \pmod{p} . \end{aligned}$$

This completes the proof in case (i).

In case (ii) we have $\alpha^p \equiv \beta \pmod{p}$ so that $\alpha^{p+1} \equiv -1$. This implies

$$(-\alpha^2)^{p+1} = 1 \pmod{p} .$$

Since $(-\alpha^2)^{p+1} \not\equiv 1$, comparison with (*) shows that $p \mid p+1$, $p \nmid (p-1)$. Thus we may consider $\alpha^{2p} - \beta^2 \pmod{p^2}$. Since

$$\alpha^{2p} = \beta^2 \pmod{p^2} \iff \beta^{2p} = \alpha^2 \pmod{p^2}$$

it suffices to consider

$$\alpha^{2p} + \beta^{2p} - \alpha^2 - \beta^2 \pmod{p^2} .$$

Hence the proof is completed as in case (i).

Also solved by John Halton whose solution will appear in a paper to be published later in the Fibonacci Quarterly Journal

CORRECTED PROBLEM AND SOLUTION

H-25 Proposed by Joseph Erbacker and John A. Fuchs, University of Santa Clara, and F.D. Parker, SUNY, Buffalo, N.Y.

Prove:

$$D_n = |a_{ij}| = 36, \text{ for all } n ,$$

where

$$a_{ij} = F_{n+i+j-2}^3 \quad (i, j = 1, 2, 3, 4) .$$

Solution by C.R. Wall, Ft. Worth, Texas

$$\text{Since } F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0 ,$$

it follows that D_{n+1} can be obtained from D_n by column operations that leave the determinant invariant. Thus $D_n = D_{n+1}$. To evaluate set $n = 0$.

Also solved by the proposers

NO SOLUTIONS

H-26 *Proposed by Leonard Carlitz, Duke University, Durham, N.C.*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-s}$, then show $R_k^n = (a_{rs})$ such that

$$a_{rs} = \sum_{j=0}^{s-1} \binom{r-1}{j} \binom{k+1-r}{s-1-j} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^j .$$

GENERATING FUNCTIONS AND CONVOLUTION

H-27 *Proposed by Harlan L. Umansky, Emerson High School, Union City, N.J.*

Show that

$$F_k^3 = \sum_{j=1}^{k-2} (-1)^{j+1} F_j F_{3k-3j} + (-1)^k F_{k-3}, \quad k \geq 4 .$$

Solution by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

The generating functions below are easily verified

$$\frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n^3 x^n$$

$$\frac{2x}{1 - 4x - x^2} = \sum_{n=0}^{\infty} F_{3n} x^n$$

$$\frac{x}{1 + x - x^2} = \sum_{n=0}^{\infty} F_n x^n$$

$$\frac{x}{1 + x - x^2} = \sum_{n=0}^{\infty} F_n (-1)^{n+1} x^n$$

$$\frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \left[\frac{2x}{(1 - 4x - x^2)} \right] \left[\frac{x}{1 + x - x^2} \right] + \frac{x}{1 + x - x^2}$$

Thus

$$\sum_{k=0}^{\infty} F_k^3 x^k = \left(\sum_{k=0}^{\infty} F_{3k} x^k \right) \left(\sum_{k=0}^{\infty} F_k (-1)^{k+1} x^k \right) + \sum_{k=0}^{\infty} F_k (-1)^{k+1} x^k$$

so that, using Cauchy product of series, and equating coefficients, one obtains

$$\begin{aligned} F_k^3 &= \sum_{i=0}^k (-1)^{i+1} F_i F_{3k-3i} + (-1)^k F_k \\ &= \sum_{i=1}^{k-2} (-1)^{i+1} F_i F_{3k-3i} + (-1)^k \{2F_{k-1} - F_k\} \\ &= \sum_{i=1}^{k-2} (-1)^{i+1} F_i F_{3k-3i} + (-1)^k F_{k-3}. \end{aligned}$$

Also solved by C.R. Wall and the proposer.

A FIBONACCI BEAUTY

H-28 Proposed by H.W. Gould, West Virginia University, Morgantown, W. Va.

Let $C_j(r, n)$ be the number of numbers, to the base r ($r \geq 2$) with at most n digits, and the sum of the digits equal to j .

Sum the series:

$$\sum_{j=0}^{\infty} C_j(r, n) a^j b^{rn-n-j}.$$

Solution by John H. Halton, University of Colorado, Boulder, Colorado, and Leonard Carlitz, Duke University, Durham, N.C.

Let

$$S_n(r, a, b) = \sum_{j=0}^{\infty} C_j(r, n) a^j b^{rn-n-j} = b^{(r-1)n} \sum_{n=0}^{r^{n-1}} \left(\frac{a}{b}\right)^{N_0+N_1+\dots+N_{n-1}}$$

where

$$N = N_0 + N_1 r + N_2 r^2 + \dots + N_{n-1} r^{n-1}, \quad 0 \leq N_i \leq r-1$$

Then

$$\begin{aligned} S_n &= b^{(r-1)n} \sum_{N_0=0}^{r-1} \sum_{N_1=0}^{r-1} \dots \sum_{N_{n-1}=0}^{r-1} \left(\frac{a}{b}\right)^{N_0+N_1+\dots+N_{n-1}} \\ &= \prod_{m=0}^{n-1} \left\{ b^{r-1} \sum_{N_m=0}^{r-1} \left(\frac{a}{b}\right)^{N_m} \right\} = \prod_{m=1}^n \left\{ \frac{a^r - b^r}{a - b} \right\} = \left(\frac{a^r - b^r}{a - b}\right)^n \end{aligned}$$

If

$$a = \frac{1}{2}(1 + \sqrt{5}), \quad b = \frac{1}{2}(1 - \sqrt{5}), \quad \text{then } \frac{a^r - b^r}{a - b} = F_r,$$

the Fibonacci number. In that case,

$$S_n(r, \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 - \sqrt{5})) = F_r^n.$$

Also solved by the proposer.

XXXXXXXXXXXXXX

A DIGIT MUSES*

Oh!

4

2B

No zero

In the world of math!

Would that I were like that great

Built into the structure of the universe and art

The ideal of ideals dividing all things in proportions of gold — a paragon!

Brother U. Alfred

* This poem has the distinction that the number of syllables in each line proceeds by the sequence: 1, 1, 2, 3, 5, 8, 13, 21.

XXXXXXXXXXXXXX

FURTHER COMMENTS ON THE PERIODICITY OF THE DIGITS
OF THE FIBONACCI SEQUENCE

RICHARD L. HEIMER

Airborne Instrument Laboratory, Deer Park, New York

In the Fibonacci Quarterly, Volume 1, Number 4, Dov Jarden showed that the last $d \geq 3$ digits of the Fibonacci numbers repeat every $15 \cdot 10^{d-1}$ times. He also commented on Stephen P. Geller's announcement of the periodicity of the first three digits. Summarizing these results, we find that the digits of the Fibonacci series, 0, 1, 1, 2, 3, 5 repeat as follows:

Table I
Units Tens

Repetition Period	1	2	3	4	5	d
	60	300	1500	1.5×10^4	1.5×10^5	1.5×10^d

What aroused my interest in the sequence of repetition periods was that the repetition periods of the tens and hundreds digits were five times the value of the repetition periods of the units and tens digit respectively, while the remaining repetition periods increased by a factor of ten. Also I was interested in possible discovering any governing relationships which caused 60 to be the periodicity of the units digit as well as a factor of subsequent periods. Believing that number systems with different bases (i.e., Binary, Ternary, etc.) would also display these periodic qualities, I proceeded to generate Fibonacci series in these bases, and in the course of so doing, discovered further significant periodic properties of these sequences.

The units and tens digits were generated from Base 2 thru Base 16 while the third and fourth(hundreds and thousands) were derived only where it was practical to do so. These results are shown in Table II which tabulates the periodicity of the digits in the various bases as well as the corresponding multiplying factors which show the relationship between the repetition periods of the digits in a given base.

Table II

BASE	REPETITION PERIOD OF DIGIT				MULTIPLYING FACTOR		
	1	2	3	4	$\frac{R_{D_2}}{R_{D_1}}$	$\frac{R_{D_3}}{R_{D_2}}$	$\frac{R_{D_4}}{R_{D_3}}$
2	3	6	12	24	2	2	2
3	8	24	72		3	3	
4	6	24	96		4	4	
5	20	100			5		
6	24	144			6		
7	16	112			7		
8	12	96			8		
9	24	216			9		
10	60	300	1500	15,000	5	5	10
11	10	110			11		
12	24	24	288		1	12	
13	28	364			13		
14	48	336			7		
15	40	600			15		
16	24	96			4		

The first conclusion which may be drawn from this table is that the multiplying factors are either the number base or a rational fraction thereof and that:

$$(1) \quad R_D = A_B \cdot F \cdot B^D$$

Where: R_D is the periodicity of the D^{th} digit.

A_B is the repetition period of the units digit divided by the base B.

F is a rational fraction.

A further observation reveals that for a given digit:

The product of two bases is a third base whose periodicity is the product of the repetition periods of the original two bases or a rational fraction thereof.

Formulating this, we have for a given digit:

$$(2) \quad R_{B_x} \cdot R_{B_y} = KR_{B_{xy}}$$

Where K is a rational fraction. As an example of this note that for units digit:

$$(3) \quad R_{B_2} = 3, \quad R_{B_3} = 8, \quad R_{B_2} \cdot R_{B_3} = 3 \cdot 8 = 24 = R_{B_{2 \cdot 3}}$$

It also may be seen that in the decimal system:

$$R_{B_{10}} = 60 = R_{B_{5 \cdot 2}} = R_{B_5} \times R_{B_2} = 20 \cdot 3 = 60$$

This of course only partially satisfies my understanding of the nature of the decimal base units digit repeating every 60 Fibonacci numbers.

While generating these sequences another significant result occurred which is the relationship of the digits within a given repetition cycle. This relationship is best indicated by way of illustration. Regard the units, tens, and hundreds digits of the Fibonacci sequence in the base 3: $\left[(000, 001, 001, 002, 010, 012, 022, 111) (210, 021, 001, 022, 100, 122, 222, 121) (120, 011, 201, 212, 120, 102, 222, 101) \right] \left[(100, 201, \dots) \right]$. The parentheses are drawn around the repeating units digit which occur at intervals of 8. The brackets are drawn after an interval of 24.

Note what happens if the tens digit only is written in 3 horizontal groups of 8 (8 being the periodicity of the digit preceding the tens digit, the units digit).

	0	0	0	0	1	1	2	1
→	1	2	0	2	0	2	2	2
→	2	1	0	1	2	0	2	0
Difference:	1	2	0	2	2	1	0	1

The difference between neighboring vertical terms is indicated next to the heading of "DIFFERENCE." Note that the differences are a Fibonacci type sequence where any term after the first two is the sum of the previous two in that base.

This relationship has held for all the sequences in all the bases that I have investigated. Assuming that it holds generally, the remaining hundreds digit sequence of the base 3 example may be generated as follows:

- Step 1: Calculate the digit desired (hundreds) for one period (24) of the preceding digit (tens). Now calculate two more terms (25 and 26). List the first group horizontally and the two subsequent terms on the 2nd line under the first two terms of the first line. (See Illustration I.)
- Step 2: On top of this calculate the difference of the first two sets of vertical terms (see Illustration I).
- Step 3: Beginning with these two differences generate a Fibonacci sequence in that base (see Illustration I).
- Step 4: Using the generated differences, vertically fill in the remaining terms of the digital sequence as follows:

Illustration I

Differences:	
Step 2 →	1 2 0 2 2 1 0 1 1 2 0 2 2 1 0 1 1 2 0 2 2 1 0 1
Step 1 →	0 0 0 0 0 0 0 1 2 0 0 0 1 1 2 1 1 0 2 2 1 1 2 1
	1 2 0 2 2 1 0 2 0 2 0 2 0 2 2 2 2 2 2 1 0 2 2 2
Step 4 →	2 1 0 1 1 2 0 0 1 1 0 1 2 0 2 0 0 1 2 0 2 0 2 0 2 0

Between Step 1 and Step 4 we have completely specified one period of the hundreds digit without making 72 three digit additions which this would normally require. This method also provides quick prediction of fractional multiplying factors mentioned earlier.

I am reporting these comments in hope that someone may further develop these thoughts in this fascinating sub-sub-field of Fibonacci sequences.

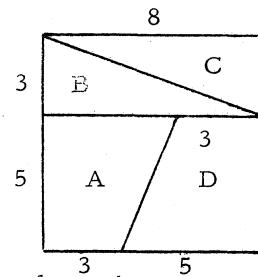
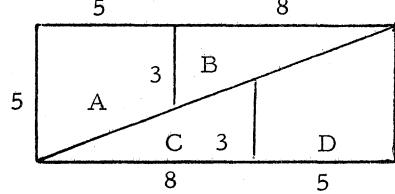
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THE VANISHING SQUARE

B.B. SHARPE

State University of New York at Buffalo

This well-known puzzle:



involves Fibonacci numbers and suggests a formula:

$$|F_{i-1}F_{i+1} - F_iF_i| = F_1$$

(Absolute value, since $3 \times 8 - 5 \times 5 = -1$ but $5 \times 13 - 8 \times 8 = +1$) Adding and subtracting $F_{i-1}F_i$:

$$F_{i-1}F_{i+1} - F_{i-1}F_i + F_{i-1}F_i - F_iF_i$$

$$F_{i-1}(F_{i+1} - F_i) + F_i(F_{i-1} - F_i)$$

$$F_{i-1}F_{i-1} - F_iF_{i-2}$$

Repeating the process with $F_{i-2}F_{i-1}$:

$$F_{i-1}F_{i-1} - F_{i-2}F_{i-1} + F_{i-2}F_{i-1} - F_iF_{i-2}$$

$$F_{i-1}(F_{i-1} - F_{i-2}) + F_{i-2}(F_{i-1} - F_i)$$

$$F_{i-1}F_{i-3} - F_{i-2}F_{i-2}$$

After a finite number of steps, the smallest subscript becomes 1, and:

$$F_3F_1 - F_2F_2 \quad \text{or} \quad F_2F_2 - F_3F_1 \quad \text{or} \quad \pm F_1$$

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EXPLORING FIBONACCI MAGIC SQUARES

BROTHER U. ALFRED
St. Mary's College, California

Magic squares have long had a strong appeal to people mathematically inclined — and so have Fibonacci numbers. When we put the two ideas together, what do we get: harmony or conflict?

Just to make the problem perfectly clear, the two concepts involved will be delimited as precisely as possible for the purpose of this investigation. A magic square will be considered as a square array of distinct positive integers such that the sums of all rows and columns as well as of the two main diagonals is the same. A common example of a three-by-three magic square is:

6	7	2
1	5	9
8	3	4

By Fibonacci numbers we shall understand the positive elements of the sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, Again let it be noted that the elements of the magic square must be distinct integers so that, for example, it would not be allowable to use 1 twice on the plea that it is two different Fibonacci numbers.

Two possibilities present themselves: (1) Either it will be possible to create one or more magic squares with the Fibonacci numbers; or (2) It will be possible to prove that no such magic squares may be formed.

The investigation may be generalized in various ways; (1) If we allow both positive and negative integers; (2) If we take as elements the terms of the generalized Fibonacci sequence: $a, b, a+b, a+2b, 2a+3b$, etc.

The results of this exploration will be published in the February 1963 issue of the Fibonacci Quarterly.

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ON FIBONACCI RESIDUES

JOHN H. HALTON
University of Colorado, Boulder, Colorado

In a recent note ("Exploring Fibonacci Residues" Fib. Quart. 2 (1964) 1: 42), Brother Alfred asks whether one or other of the least positive and negative residues, when one Fibonacci number is divided by another, is always itself a Fibonacci number.

The answer is YES, as is shown by the following somewhat more detailed result.

THEOREM. If $m \geq 1$ and $n \geq 3$ are integers, and if A and $-B$ are the least positive and negative residues when F_m is divided by F_n , then at least one of A and B is itself a Fibonacci number F_s , where k and s are unique integers such that $s = 0$ if n divides m , and otherwise

$$(1) \quad m = 2kn + r, \quad k \geq 0, \quad 0 < |r| < n, \quad s = |r| .$$

Proof. It is well-known that F_m is divisible by F_n if and only if either m is divisible by n or $n = 2$. Thus if $n \geq 3$ and n divides m , the theorem holds, since $F_0 = 0$. If n does not divide m , we can find k and r uniquely by (1). Well-known identities now show that

$$(2) \quad F_m = F_{2kn+r} = \sum_{h=0}^{2k} \binom{2k}{h} F_n^h F_{n-1}^{2k-h} F_{r+h} \equiv F_{n-1}^{2k} F_r \pmod{F_n}$$

and

$$(3) \quad F_{n-1}^2 = F_{n-2} F_n + (-1)^n \equiv (-1)^n \pmod{F_n} .$$

Therefore we see that

$$(4) \quad F_m \equiv (-1)^{kn} F_r \equiv (-1)^{kn+r-1} F_{-r} \equiv \pm F_s \pmod{F_n} .$$

Since the Fibonacci sequence is strictly increasing for values of the index greater than one, $F_s < F_n$, so that $\pm F_s$ is the least positive or negative residue of F_m modulo F_n ; that is, $F_s = A$ or B .

To complete the treatment of Brother Alfred's question, it must be noted that, if $n = 1$ or 2 , $F_n = 1$ and so divides F_m , yielding a residue of $F_0 = 0$. And if m or n is negative, the well-known relation

$$F_{-t} = (-1)^{t-1} F_t ,$$

which was used in the derivation of (4), shows that the residue is still $\pm F_s$.

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CORRECTED FACTORIZATIONS OF FIBONACCI NUMBERS

DAVID M. BLOOM
University of Massachusetts

Kraitchik's table of factors of the Fibonacci numbers (*Recherches sur la Theorie des Nombres*, " p. 77-79) contains at least two errors, as follows:

(u_n denotes n^{th} Fibonacci number, as in Kraitchik)

n	u_n	Kraitchik's Factorization	Correct Factorization
57	365, 435, 296, 162	$2 \cdot 37 \cdot 113 \cdot 4371901$	$2 \cdot 37 \cdot 113 \cdot 797 \cdot 54833$
67	44, 945, 570, 212, 853	prime	$269 \cdot 116849 \cdot 1429913$

Note: in the factorization of u_{57} , $797 \cdot 54833 = 43701901$, not 4371901)
Have these errors been pointed out elsewhere?

PROPORTIONS IN MUSIC

HUGO NORDEN

Boston University, Boston, Mass.

The systematic organization of a musical composition within a pre-determined time span by means of the lower numbers of the Fibonacci and Lucas series, singly or in combination, is common practice indeed. It seems that the more profound the composer, the stricter is his application of these proportions in the musical structure.

A neatly contrived example is found in the first fugue in The Art of the Fugue by Johann Sebastian Bach. The formal and thematic materials can be listed quite simply as follows:

$$\begin{aligned} \text{number of measures} &= 78 (13 \times 6) \\ \text{number of entries} &= 11 \end{aligned}$$

Of the 11 entries of the subject and answer, 9 begin on either "D" (the tonic) or "A" (the dominant), while 2 begin on "E". These 2 entries that begin on the note "E" define the form of the composition. The first, entry No. 8, occurs at measure 40, thereby beginning the latter half of the 78-measure time span; while the second, entry No. 9, comes at measure 49, thus announcing the start of the 5/13 portion of the $8/13 + 5/13$ (48 + 30) division. This formally significant pair of entries is assigned to the Tenor and Soprano parts, respectively.

The total number of 11 entries, however, is distributed within the time span as follows:

7	4
before the middle (measures 1-39)	after the middle (measures 40-78)
3 : 4	1 : 3
Answers Subjects	Subjects Answers
3 : 4	1 : 2
begin on "D"	begin on "A"
begin on "E"	begin on "E"

The fugue is given in full on the two following pages. Both the measures and entries are numbered and the type and starting note of each entry is indicated so that the reader can follow the plan of the composition. As several recordings of this music are available, it should be easy to experience this time span utilization audibly.

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The Fibonacci Association invites Educational Institutions to apply for Academic Membership in the Association. The minimum subscription fee is \$25 annually. (Academic Members will receive two copies of each issue and will have their names listed in the Journal.)

REMARKS ON A SECOND ORDER RECURRING SEQUENCE

JOHN BRILLHART
University of San Francisco, San Francisco, Calif.

Among the second order recurring sequences, the degenerate sequence $U_n = n^2 - n - 1$ is of some interest. In fact, we can observe the following special property among the more unusual properties such sequences have:

$$U_n U_{n+1} = U_{n^2-1} .$$

Proof:

$$\begin{aligned} U_n U_{n+1} &= [n^2 - n - 1] [(n+1)^2 - (n+1) - 1] \\ &= (n^2 - n - 1)(n^2 + n - 1) \\ &= (n^2 - 1)^2 - n^2 \\ &= (n^2 - 1)^2 - (n^2 - 1) - 1 \\ &= U_{n^2-1} . \end{aligned}$$

In what way this property can be generalized remains to be seen.

XXXXXXXXXXXXXXXXXXXX

FUGA I
a 4 voci.

J. S. Bach.

Andante con moto.
sempre legato

1 *Subject*

2 *Answer*

3 *Subject*

4 *Answer*

5 *Subject*

6 *Subject*

7 *Answer*

8 *Answer*

MIDDLE

42 43 44 45 46 47 48 49 50 51

cresc.

Answer

52 53 54 55 56 57 58 59 60 61

dim.

cresc.

62 63 64 65 66 67 68 69 70 71 72 73

f

dim.

74 75 76 77 78 79 80

cresc.

pallent. dim.

11 Answer

NEW SLANTS

MARK FEINBERG

"The Pyramid"

Pascal's Triangle is given by the coefficients of the binomial expansion $(a+b)^n$. The coefficients of a trinomial expansion $(a+b+c)^n$ take the shape of a three-dimensional pyramid:

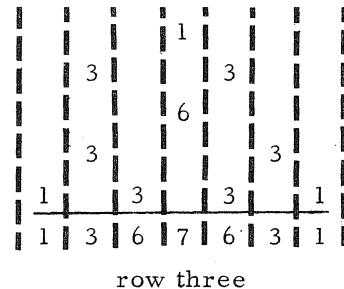
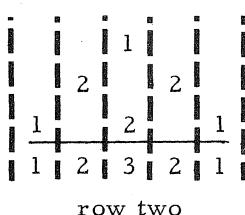
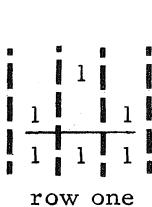
$$(1) \quad a + b + c$$

$$(2) \quad \frac{a + b + c}{a^2 + 2ab + 2ac + b^2 + 2bc + c^2}$$

$$(3) \quad \frac{a + b + c}{a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3}$$

$$\begin{array}{ccccccc}
 & & & & & 1c^2 & \\
 & & & & & 2ac & 2bc \\
 1a & & 1b & & & 1a^2 & 2ab & 1b^2 \\
 & & & & & & & \\
 & & & & & 1c^3 & \\
 & & & & & 3ac^2 & 3bc^2 \\
 & & & & & 6abc & \\
 & & & & & 3a^2c & 3b^2c \\
 & & & & & 1a^3 & 3a^2b & 3ab^2 & 1b^3
 \end{array}$$

Projecting this pyramid onto a plane:



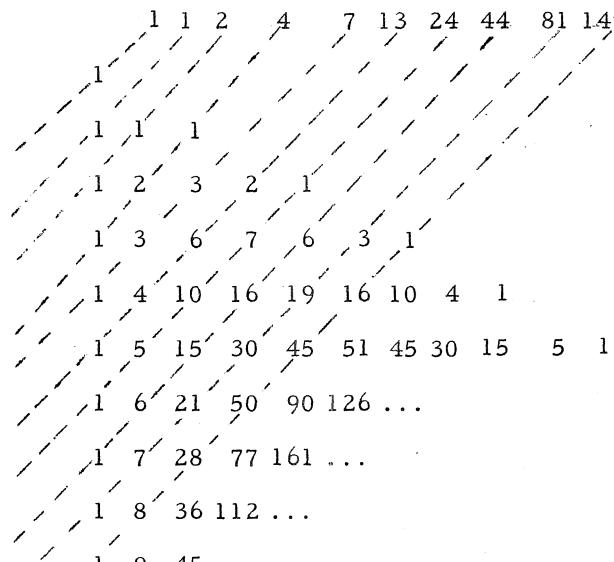
And arranging thus:

1								
1	1	1						
1	2	3	2	1				
1	3	6	7	6	3	1		
1	4	10	16	19	16	10	4	1

Each number is the sum of the one above it and two to the left of that.

Adding diagonals of this triangle gives the Tribonacci series $(1, 1, 2, 4, 7, 13, 24, 44, \dots)$. The convergent of this sequence fits the equation

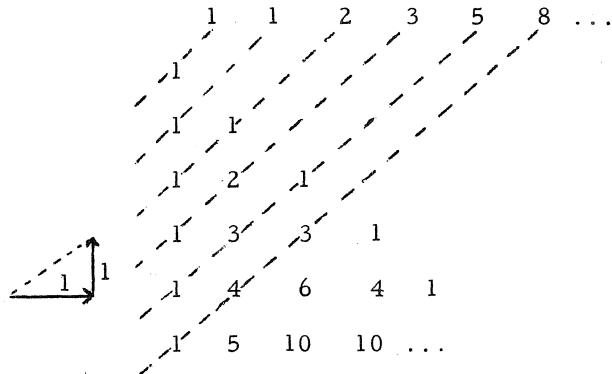
$$x = 1 + \frac{1}{x} + \frac{1}{x^2}$$



"New Slants"

Summing diagonals of Pascal's Triangle obtained by going across one column and up one row gives the Fibonacci series.² Its convergent, "phi," fits the equation

$$x = 1 + \frac{1}{x}$$

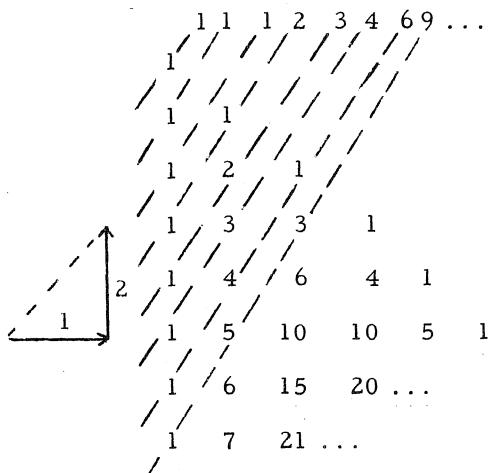


Going across one column and up two rows on the same triangle gives

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88 \dots$$

This series' convergent, $1.46\dots$, fits

$$x = 1 + \frac{1}{x^2}$$



Going across one column and up three rows gives

$$1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26 \dots$$

Its convergent, $1.38\dots$, fits

$$x = 1 + \frac{1}{x^3}$$

In the triangle of the 3-D expansion, going across one column and up one row gives the Tribonacci series.

Going across one column and up two rows gives

$$1, 1, 1, 2, 3, 5, 8, 12, 19, 30, 47, 74 \dots$$

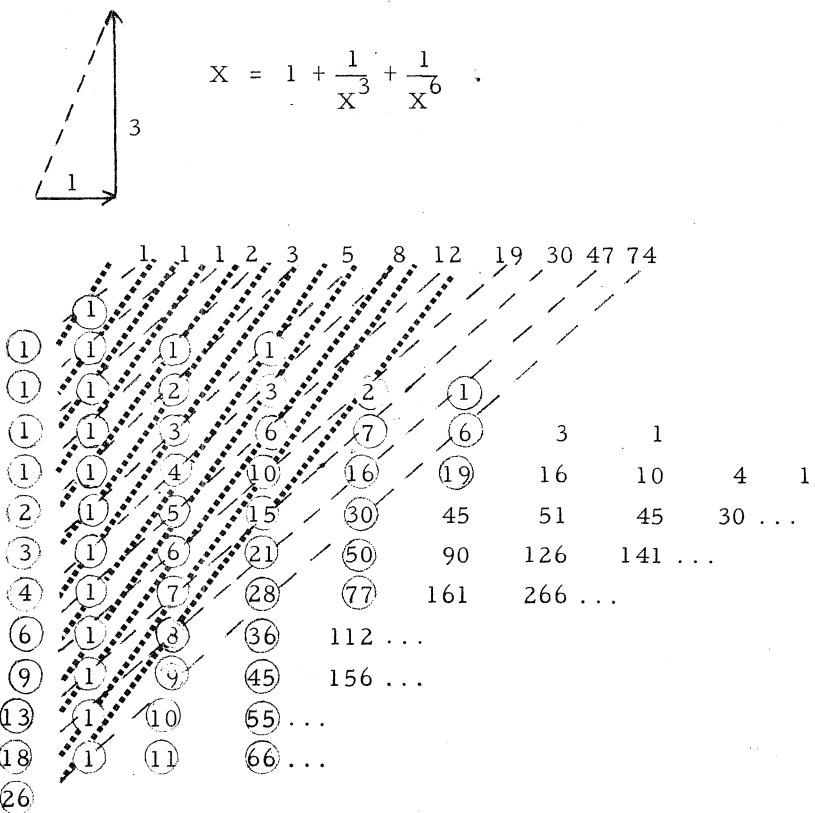
which converges upon 1.57 . . . and fits

$$X = 1 + \frac{1}{X^2} + \frac{1}{X^4}$$

Going across one column and up three rows gives

$$1, \ 1, \ 1, \ 1, \ 2, \ 3, \ 4, \ 6, \ 9, \ 13, \ 18, \ 26, \ 38, \ 55 \dots$$

which converges upon 1.44... and fits



A table can be made from the expansion $(a+b+c+d)^n$ so that each number is the sum of the one above it and the three to the left of that.

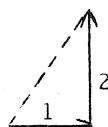
1										
1	1	1	1							
1	2	3	4	3	2	1				
1	3	6	10	12	12	10	6	3	1	

Going across one column and up one row gives a series whose convergent fits



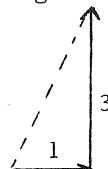
$$X = 1 + \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^3}$$

Going across one column and up two rows gives



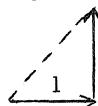
$$X = 1 + \frac{1}{X^2} + \frac{1}{X^4} + \frac{1}{X^6}$$

Going across one column and up three rows gives



$$X = 1 + \frac{1}{X^3} + \frac{1}{X^6} + \frac{1}{X^9}$$

Thus a general convergent formula is derived for diagonal series by letting "n" equal the number of terms in the expansion, and letting "u" equal the number of rows up:



$$X = 1 + \frac{1}{X_1^u} + \frac{1}{X_2^{2u}} + \dots + \frac{1}{X_{n-1}^{(n-1)u}}$$

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FIBONACCI AND PASCAL

WALTER W. HORNER
Pittsburg, Pa.

The purpose of this note is to point out a connection between the Fibonacci sequence and rows of Pascal's triangle. It is known that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Expanding by the binomial theorem and collecting terms we get

$$F_n = \frac{\binom{n}{1} + \binom{n}{3} 5 + \binom{n}{5} 5^2 + \binom{n}{7} 5^3 + \binom{n}{9} 5^4 + \dots}{2^{n-1}}$$

$$\text{But } 2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \dots$$

$$\text{Therefore } F_n = \frac{\binom{n}{1} + \binom{n}{3} 5 + \binom{n}{5} 5^2 + \binom{n}{7} 5^3 + \binom{n}{9} 5^4 + \dots}{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \dots}$$

As for the Lucas sequence it is known that

$$L_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Expanding as above and collecting terms and remembering that 2^{n-1} is also equal to $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \binom{n}{9} + \dots$ we get

$$L_n = \frac{\binom{n}{0} + \binom{n}{2} 5 + \binom{n}{4} 5^2 + \binom{n}{6} 5^3 + \binom{n}{8} 5^4 + \dots}{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \binom{n}{9} + \dots}$$

Note the exchange of binomial coefficients in the two formulas. Numerical examples: To find F_7 we look in row 7 of Pascal's Triangle and find

$$\frac{7 + 35 \cdot 5 + 21 \cdot 5^2 + 1 \cdot 5^3}{1 + 21 + 35 + 7} = \frac{832}{64} = 13$$

Similarly for the 7th Lucas number

$$\frac{1 + 21 \cdot 5 + 35 \cdot 5^2 + 7 \cdot 5^3}{7 + 35 + 21 + 1} = \frac{1856}{64} = 29$$

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ON THE INFINITUDE OF FIBONACCI PSEUDO-PRIMES

EMMA LEHMER

University of California; Berkeley, California

A Fermat pseudo-prime is usually defined to be a composite number m which satisfies the Fermat congruence

$$(1) \quad a^{m-1} - 1 \equiv 0 \pmod{m}, \quad (a, m) = 1$$

thus showing that the converse of Fermat's theorem does not hold without further conditions on m . Hardy and Wright (Theory of Numbers, p. 72) show that there are infinitely many composite numbers m which satisfy (1).

The Lucas congruence for Fibonacci numbers which can be thought of as a generalization of the Fermat's congruence (1) states

$$(2) \quad U_{m-\epsilon_m} = 0 \pmod{m}, \quad (m, 10) = 1$$

where $U_n = U_{n-1} + U_{n-2}$, $U_0 = 0$, $U_1 = 1$ are the Fibonacci numbers and $\epsilon_m = 1$ if $m = 10n \pm 1$, while $\epsilon_m = -1$ if $m = 10m \pm 3$. Congruence (2) holds for m a prime. We next show that it also holds for an infinitude of composite numbers, which we call Fibonacci pseudo-primes. Let $p > 5$ be a prime, and let $m = U_{2p} = U_p V_p$, where V_n is the series $V_0 = 2$, $V_1 = 1$, $V_n = V_{n-1} + V_{n-2}$. Hence m is composite. Also m is odd since the only even Fibonacci numbers have subscripts which are multiples of 3 and $p \neq 3$. From the known expansions

$$(3) \quad \left\{ \begin{array}{l} 2^{p-1} U_p = \sum_{k=0}^{(p-1)/2} \binom{p}{2k+1} 5^k \\ 2^{p-1} V_p = \sum_{k=0}^{(p-1)/2} \binom{p}{2k} 5^k \end{array} \right.$$

it follows, since all the binomial coefficients are divisible by p , that

$$U_p \equiv 5^{\frac{p-1}{2}} \equiv \epsilon_p \pmod{p}$$

$$V_p \equiv 1 \pmod{p}$$

Hence $m = U_{2p} \equiv \epsilon_p \pmod{p}$ and, since U_{2p} is odd, $2p$ divides $m - \epsilon_p$, hence U_{2p} divides $U_{m-\epsilon_p}$. It remains to show that $\epsilon_m = \epsilon_p$ or that $m \equiv p \pmod{10}$. Taking (3) modulo 5 we have

$$4^{p-1} U_p V_p \equiv (-1)^{p-1} U_{2p} = U_{2p} \equiv p \pmod{5}, \text{ and}$$

since $m = U_{2p}$ and p are both odd ($p \neq 3$) we have $m = p \pmod{10}$ or $\epsilon_p = \epsilon_m$ and the result follows.

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TWO VERY SPECIAL NUMBERS

J. A. H. HUNTER
Toronto, Ontario

Stimulated by my derivation of the two 17-digit automorphic numbers (Recreational Mathematics Magazine, No. 14), Mr. R. A. Fairbairn of Willowdale, Ontario, has derived the two,100-digit automorphics.

The labor involved in this tremendous task would deter most enthusiasts, since the results were achieved (and of course checked) using no help other than a simple desk adding machine.

An automorphic number is distinguished by having its square end with the number itself.

The two 100-digit automorphic numbers, never before published so far as I know, are:

3, 953, 007, 319, 108, 169, 802, 938, 509, 890, 062, 166,
509, 580, 863, 811, 000, 557, 423, 423, 230, 896, 109,
004, 106, 619, 977, 392, 256, 259, 918, 212, 890, 625

and

6, 046, 992, 680, 891, 830, 197, 061, 490, 109, 937, 833,
490, 419, 136, 188, 999, 442, 576, 576, 769, 103, 890,
995, 893, 380, 022, 607, 743, 740, 081, 787, 109, 376

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-44 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that for every positive integer k there are no more than n^k Fibonacci numbers between n^k and n^{k+1} .

B-45 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let H_n be the n -th generalized Fibonacci number, i.e., let H_1 and H_2 be arbitrary and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Show that $nH_1 + (n-1)H_2 + (n-2)H_3 + \dots + H_n = H_{n+4} - (n+2)H_2 - H_1$.

B-46 *Proposed by C.A. Church, Jr., Duke University, Durham, North Carolina*

Evaluate the n -th order determinant

$$D_n = \begin{vmatrix} a+b & ab & 0 & 0 & \dots \\ 1 & a+b & ab & 0 & \dots \\ 0 & 1 & a+b & ab & \dots \\ 0 & 0 & 1 & a+b & \dots \\ \dots & & & & \\ \dots & & & & \\ \dots & & & & \end{vmatrix}$$

B-47 *Proposed by Barry Litvack, University of Michigan, Ann Arbor, Michigan*

Prove that for every positive integer k there are k consecutive Fibonacci numbers each of which is composite.

B-48 *Proposed by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada*

Prove that

$$\sum_{k=1}^{r-1} (-2)^k \binom{r}{k} F_k = \begin{cases} -2^r F_r & \text{if } r \text{ is an even positive integer} \\ 2^r F_r - 2(5)^{(r-1)/2} & \text{if } r \text{ is an odd positive integer,} \end{cases}$$

where $F_{n+2} = F_{n+1} + F_n$ ($F_1 = F_2 = 1$) and find the corresponding sum in which the F_k are replaced by the Lucas numbers L_k .

B-49 *Proposed by Anton Glaser, Pennsylvania State University, Abington, Pennsylvania*

Let ϕ represent the letter "oh".

TW ϕ

Given that T, W, ϕ , L, V, P, and TW ϕ are
Fibonacci numbers, solve the cryptarithm
in the base 14, introducing the digits
 α , β , δ , and δ in base 14 for 10, 11,
12, and 13 in base 10.

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EVEN

PRIME

B-50 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that

$$\sum_{j=0}^n \left[2F_j^2 - \binom{n}{j} F_j \right] = F_n^2 .$$

B-51 *Proposed by Douglas Lind, Falls Church, Virginia*

Let $\phi(n)$ be the Euler totient and let $\phi^k(n)$ be defined by $\phi^1(n) = \phi(n)$, $\phi^{k+1}(n) = \phi[\phi^k(n)]$. Prove that $\phi^n(F_n) = 1$, where F_n is the n -th Fibonacci number.

SOLUTIONS

A PERIODIC RECURRENT SEQUENCE

B-30 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California*

Find the millionth term of the sequence a_n given that

$$a_1 = 1, a_2 = 1, \text{ and } a_{n+2} = a_{n+1} - a_n \text{ for } n \geq 1.$$

Solution by J.A.H. Hunter, Toronto, Ontario, Canada

It is simple to show that a_n has a period of 6, with:

$$a_{6k+4} = a_{6k+5} = -1.$$

$10^6 \equiv 4 \pmod{6}$, hence the millionth term must be -1.

Also solved by Charles R. Wall, Texas Christian University, Ft. Worth, Texas; John H. Halton, University of Colorado, Boulder, Colorado; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; Vassili Daiev, Sea Cliff, L.I., N.Y.; George Ledin, Jr., San Francisco, California; Ronald Weinschenk, San Jose State College, San Jose, California; Dermott A. Breault, Sylvania-A.R.L., Waltham, Mass.; David E. Zitarelli, Temple University, Philadelphia, Pennsylvania; B. Litvack, University of Michigan, Ann Arbor, Michigan; and the proposer.

SUMS OF CONSECUTIVE FIBONACCI NUMBERS

B-31 *Proposed by Douglas Lind, Falls Church, Virginia*

If n is even, show that the sum of $2n$ consecutive Fibonacci numbers is divisible by F_n .

Solution by Roseanna Torretto, University of Santa Clara, Santa Clara, California

Let T be the sum $F_{a+1} + \dots + F_{a+2n}$ of $2n$ consecutive Fibonacci numbers. Let $S_n = F_1 + F_2 + \dots + F_n$. It is well known that $S_n = F_{n+2} - 1$. Hence

$$T = S_{a+2n} - S_n = F_{a+2n+2} - F_{a+2}.$$

Since $F_{q+p} - F_{q-p} = L_q F_p$ for p even (see I. D. Ruggles, Some Fibonacci Results using Fibonacci-Type Sequences, this Quarterly, Vol. 1, No. 2, p. 77), $T = L_{a+n+2} F_n$ as desired.

Also solved by J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; B. Litvack, University of Michigan, Ann Arbor, Michigan; John H. Halton, University of Colorado, Boulder, Colorado; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer.

A CONGRUENCE RELATION

B-32 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $nL_n \equiv F_n \pmod{5}$.

Solution by John Allen Fuchs, University of Santa Clara, California

It follows from basic results on homogeneous linear difference equations that the sequence $Y_n = nL_n - F_n$ satisfies

$$(1) \quad Y_{n+4} = 2Y_{n+3} + Y_{n+2} - 2Y_{n+1} - Y_n,$$

i.e., $(E^2 - E - 1)^2 Y = 0$ with the operator E defined as in James A. Jeske, Linear Recurrence Relations — Part I, this Quarterly, Vol. 1, No. 2. The desired result now follows by trial for $n = 1, 2, 3$, and 4 and mathematical induction using (1).

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; Douglas Lind, Falls Church, Virginia; and the proposer.

TERM BY TERM SUMS

B-33 *Proposed by John A. Fuchs, University of Santa Clara, Santa Clara, California*

Let u_n, v_n, \dots, w_n be sequences each satisfying the second order recurrence formula

$$y_{n+2} = gy_{n+1} + hy_n \quad (n \geq 1),$$

where g and h are constants. Let a, b, \dots, c be constants.

Show that

$$au_n + bv_n + \dots + cw_n = 0$$

is true for all positive integral values of n if it is true for $n = 1$ and $n = 2$.

Solution by B. Litvack, University of Michigan, Ann Arbor, Michigan; John H. Halton, University of Colorado, Boulder, Colorado; and the proposer.

Suppose that

$$(1) \quad au_n + bv_n + \dots + cw_n = 0$$

for $n = 1$ and $n = 2$. Multiplying the first case by h and the second by g , we see that

$$ahu_1 + bhv_1 + \dots + chw_1 = 0 ,$$

$$agu_2 + bgv_2 + \dots + cgw_2 = 0 .$$

Adding, we obtain

$$au_3 + bv_3 + \dots + cw_3 = 0$$

since u_n, v_n, \dots, w_n all satisfy

$$y_{n+2} = gy_{n+1} + hy_n .$$

Repeating the process (or, more formally, using mathematical induction) we verify that (1) holds for all n if it holds for $n = 1, 2$.

Also solved

JARDEN PRODUCTS

Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, California

Let u_n and v_n be any two sequences satisfying the second-order recurrence formula

$$(1) \quad y_{n+2} = gy_{n+1} + hy_n$$

where g and h are constants. Show that the sequence of products $w_n = u_n v_n$ satisfies a third-order recurrence formula

$$(2) \quad y_{n+3} = ay_{n+2} + by_{n+1} + cy_n$$

and find a , b , and c as functions of g and h .

Solution by the proposer.

Let r and s be the roots of the auxiliary polynomial $x^2 - gx - h$ of (1). We assume $r \neq s$; the case $r = s$ has the same result. Now $u_n = c_{11}r^n + c_{12}s^n$, $v_n = c_{21}r^n + c_{22}s^n$, and so $w_n = c_1(r^2)^n + c_2(rs)^n + c_3(s^2)^n$. Hence the auxiliary polynomial of (2) is

$$\begin{aligned} x^3 - ax^2 - 6x - c &= (x - r^2)(x - rs)(x - s^2) = [x^2 - (r^2 + s^2)x + (rs)^2] (x - rs) = \\ &[x^2 - (g^2 + 2h)x + h^2] (x + h) = x^3 - (g^2 + h)x^2 - (g^2 + h)hx + h^3. \end{aligned}$$

Now $a = g^2 + h$, $b = (g^2 + h)h$, and $c = -h^3$.

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; and Charles R. Wall, Texas Christian University, Ft. Worth, Texas. This problem is a special case of formulas of D. Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem (Israel), 1958, p. 43.

This problem is a special case of formulas of D. Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem (Israel), 1958, p. 43.

AN ALTERNATING BINOMIAL TRANSFORM

B-35 *Proposed by J.L. Brown, Jr., Pennsylvania State University, University Park, Pennsylvania*

Prove that

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = 0$$

for r an odd positive integer and generalize.

Solution by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada

We have the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}.$$

Thus

$$1 - \alpha = \beta \quad \text{or} \quad 1 - \beta = \alpha .$$

Now

$$\begin{aligned} \sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k &= -\binom{r}{1} F_1 + \binom{r}{2} F_2 - \binom{r}{3} F_3 + \dots + (-1)^{r-1} \binom{r}{r-1} F_{r-1} \\ &= -\binom{r}{1} \left(\frac{\alpha - \beta}{\sqrt{5}} \right) + \binom{r}{2} \left(\frac{\alpha^2 - \beta^2}{\sqrt{5}} \right) - \binom{r}{3} \left(\frac{\alpha^3 - \beta^3}{\sqrt{5}} \right) \\ &\quad + \dots + (-1)^{r-1} \binom{r}{r-1} \left(\frac{\alpha^{r-1} - \beta^{r-1}}{\sqrt{5}} \right) \\ &= \frac{1}{\sqrt{5}} - \binom{r}{1} \alpha + \binom{r}{2} \alpha^2 - \binom{r}{3} \alpha^3 + \dots + (-1)^{r-1} \binom{r}{r-1} \alpha^{r-1} \\ &\quad + \binom{r}{1} \beta - \binom{r}{2} \beta^2 + \binom{r}{3} \beta^3 + \dots + (-1)^{r-1} \binom{r}{r-1} \beta^{r-1} . \end{aligned}$$

Let r be an odd positive integer. Then

$$\begin{aligned} \sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k &= \frac{1}{\sqrt{5}} \left\{ [(1 - \alpha)^r - 1 + \alpha^r] + [1 - (1 - \beta)^r - \beta^r] \right\} \\ &= \frac{1}{\sqrt{5}} [\beta^r - 1 + \alpha^r + 1 - \alpha^r - \beta^r] \\ &= 0 . \end{aligned}$$

Let r be an even positive integer. Then

$$\begin{aligned} \sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k &= \frac{1}{\sqrt{5}} \left\{ [(1 - \alpha)^r - 1 + \alpha^r] + [1 - (1 - \beta)^r + \beta^r] \right\} \\ &= \frac{1}{\sqrt{5}} [\beta^r - 1 - \alpha^r + 1 - \alpha^r + \beta^r] \\ &= -2 \left(\frac{\alpha^r - \beta^r}{\sqrt{5}} \right) \\ &= -2 F_r . \end{aligned}$$

For Lucas numbers it can be shown by analogous methods that

$$\sum_{k=1}^{r-1} (-1)^{k+1} \binom{r}{k} L_k = \begin{cases} 2 & \text{if } r \text{ is even} \\ 2 - 2L_r & \text{if } r \text{ is odd.} \end{cases}$$

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; Douglas Lind, Falls Church, Virginia; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer.

THE PELL SEQUENCE

B-36 *Proposed by Roseanna Torreto, University of Santa Clara, Santa Clara, California*

The sequence 1, 2, 5, 12, 29, 70, ... is defined by $c_1 = 1$, $c_2 = 2$, and $c_{n+2} = 2c_{n+1} + c_n$ for all $n \geq 1$. Prove that c_{5m} is an integral multiple of 29 for all positive integers m .

Solution by Douglas Lind, Falls Church, Virginia

Since $c_5 = 29$, the solution follows at once from the more general fact that for the above defined sequence,

$$(1) \quad c_m \mid c_{nm} .$$

We shall prove this more general assertion following N. N. Vorobyov (*The Fibonacci Numbers*, Heath, 1963).

We need first establish that

$$(2) \quad c_{n+k} = c_{n-1}c_k + c_nc_{k+1} .$$

Proof is by induction on k . The cases $k = 1, k = 2$ are easily shown true. We then assume (2) true for k and $k + 1$. Hence

$$(3) \quad c_{n+k} = c_{n-1}c_k + c_nc_{k+1} ,$$

$$(4) \quad c_{n+k+1} = c_{n-1}c_{k+1} + c_nc_{k+2} .$$

Multiplying (4) by two and adding to (3), we obtain

$$c_{n+k+2} = c_{n-1}c_{k+2} + c_nc_{k+3} ,$$

completing the induction step and proving (2).

We now prove the general assertion (1) by induction using (2). (1) is obviously true for $n = 1$. Now assume c_{nm} is divisible by c_m , $n \geq 1$, and consider $c_{(n+1)m}$. By (2),

$$c_{(n+1)m} = c_{nm-1}c_m + c_{nm}c_{m+1}.$$

The first term on the right is divisible by c_m , and by the induction hypothesis so is the last term. Applying the fundamental theorem of arithmetic, so also must be $c_{(n+1)m}$. This completes the induction step and the proof of (1).

Also solved by B. Litvack, University of Michigan, Ann Arbor, Michigan; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; John H. Halton, University of Colorado, Boulder, Colorado; Dermott A. Breaule, Sylvania A.R.L., Waltham, Mass.; J.A.H. Hunter, Toronto, Ontario, Canada; H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; and the proposer.

HARMONIC DIVISION

B-37 *Proposed by Brother U. Alfred, St. Mary's College, California*

Given a line with a point of origin O and four positive positions A, B, C, and D with respect to O. If the line segments OA, OB, OC, and OD correspond respectively to four consecutive Fibonacci numbers $F_n, F_{n+1}, F_{n+2}, F_{n+3}$, determine for which set(s) of Fibonacci numbers the points A, B, C, and D are in simple harmonic ratio, i.e.,

$$\frac{AB}{BC} \frac{AD}{DC} = -1.$$

Solution by John H. Halton, University of Colorado, Boulder, Colorado

O, A, B, C, D are five consecutive points on a line, with $OA=F_n$, $OB=F_{n+1}$, $OC=F_{n+2}$, $OD=F_{n+3}$. Thus $AB=F_{n+1}-F_n=F_{n-1}$, $BC=F_{n+2}-F_{n+1}=F_n$, $AD=F_{n+3}-F_n=(F_{n+2}+F_{n+1})-(F_{n+2}-F_{n+1})=2F_{n+1}$, $DC=F_{n+2}-F_{n+3}=-F_{n+1}$. Thus $AD/DC=-2$, and $AB/BC=F_{n-1}/F_n$. If B and D divide A and C harmonically, $(AB/BC)(AD/DC) = -1$. That is, $F_{n-1}/F_n = \frac{1}{2}$. This occurs precisely once, for positive n , when $n=3$, and never for negative n . The only set of points is therefore that in which $OA=F_3=2$, $OB=F_4=3$, $OC=F_5=5$, $OD=F_6=8$.

Editorial note. Let $R_n = F_{n-1}/F_n$. It is well known and easily proved that $R_2 > R_4 > R_6 > \dots > R_7 > R_5 > R_3$. This shows that the n for which $R_n = \frac{1}{\varphi}$ is unique.

Also solved by Charles R. Wall, Texas Christian University, Ft. Worth, Texas and the proposer.

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Continued from page 184.

Moreover, these are the dimensions of the cuboid of unit volume, for $\varphi \times 1 \times \varphi^{-1} = 1$.

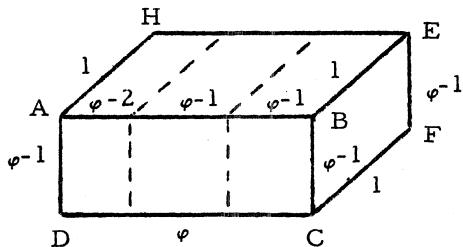


Fig. 2

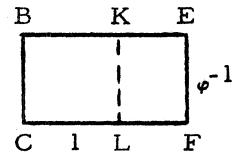


Fig. 3

Certain other properties of the Golden Cuboid may be noted.

1. It is clear from Fig. 2 that the ratios of the areas of the faces are: $AE:AC:CE = \varphi:1:\varphi^{-1}$.
2. The total surface area of the cuboid is $3(\varphi + 1 + \varphi^{-1}) = 6\varphi$
3. Four of the six faces of the cuboid are Gold Rectangles, e.g., CE (Fig. 3)
4. Each of the four diagonals of the cuboid is inclined to the base at an angle of 30° .
5. The ratio of the area of the sphere circumscribing the cuboid to that of the cuboid is $2\pi:3\varphi$.

One further point is of interest.

6. It is well known that, if a square CK is cut off from the Golden Rectangle CE (Fig. 3), the sides of the remaining rectangle LE are also in the ratio $\varphi:1$. And of course the dissection may be repeated until the rectangle size approaches that of a point, which is the intersection of BF and KE .

It is not so well known that, if two cuboids of square cross section ($\varphi^{-1} \times \varphi^{-1}$) are cut from the Golden Cuboid (broken lines, Fig. 2), the edge lengths of the remaining cuboid are in the same ratio as those of the original cuboid, viz., $1:\varphi^{-1}:\varphi^{-2} = \varphi:1:\varphi^{-1}$, so that this also is a Golden Cuboid, φ^{-3} times the size of the original.

The repetition of the decapitation process will lead to an indefinitely small Golden Cuboid located about a fixed point. The location of this point is left as an exercise to the reader.

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