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DUCCI PROCESSES

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1. Introduction

During the 1930s Professor E. Ducci of Italy [1] defined a function whose domain and range are the set of quadruples of nonnegative integers. Let

$$f(x_1, x_2, x_3, x_4) = (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, |x_4 - x_1|).$$

Let $f^n(x_1, x_2, x_3, x_4)$ be the nth iteration of f. Ducci showed that for any choice of x_1, x_2, x_3, x_4 there exists an integer N such that

$$f^m(x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \text{ for all } m > N.$$

We note the following properties of the function f of the previous paragraph:

(1) There exists a function $g(x, y)$ whose domain is the set of pairs of nonnegative integers and whose range is the set of nonnegative integers. [Here $g(x, y) = |x - y|$].

(2) $f(x_1, x_2, x_3, x_4) = \big(g(x_1, x_2), g(x_2, x_3), g(x_3, x_4), g(x_4, x_1)\big)$.

(3) The four entries of $f^n(x_1, x_2, x_3, x_4)$ are bounded for all n. The bound depends on the initial choice of x_1, x_2, x_3, x_4.

We call the successive iterations of a function satisfying these conditions a Ducci process. Condition (3) guarantees that a Ducci process is either periodic or that after a finite number of steps (say N)

$$f^{n+1}(x_1, x_2, x_3, x_4) = f^n(x_1, x_2, x_3, x_4) \text{ for all } n > N.$$

If a function g generates a Ducci process of the latter type, we say that g is Ducci stable (or simply stable).

2. Illustrations

(1) Let $g(x, y) = \overline{x+y} \pmod 3$, where $\overline{x} \pmod 3$ is the least nonnegative integer congruent to $x \pmod 3$. Then, an example shows that g is not stable. Set $x_1 = x_2 = x_3 = 0$ and $x_4 = 1$. We may tabulate the successive values of f as follows:

	(0, 0, 0, 1)
f^1:	(0, 0, 1, 1)
f^2:	(0, 1, 2, 1)
f^3:	(1, 0, 0, 1)
f^4:	(1, 0, 1, 2)
f^5:	(1, 1, 0, 0)
f^6:	(2, 1, 0, 1)
f^7:	(0, 1, 1, 0)
f^8:	(1, 2, 1, 0)
f^9:	(0, 0, 1, 1)

Since $f^9(0, 0, 0, 1) = f^1(0, 0, 0, 1) = (0, 0, 1, 1)$, the process is periodic with period 8 and g is not stable.

(2) Let $g(x, y) = \overline{x + y}$ (mod 8). We construct a similar table for the same initial values.

$$\begin{array}{rl} & (0, 0, 0, 1) \\ f^1: & (0, 0, 1, 1) \\ f^2: & (0, 1, 2, 1) \\ f^3: & (1, 3, 3, 1) \\ f^4: & (4, 6, 4, 2) \\ f^5: & (2, 2, 6, 6) \\ f^6: & (4, 0, 4, 0) \\ f^7: & (4, 4, 4, 4) \\ f^8: & (0, 0, 0, 0) \\ f^9: & (0, 0, 0, 0) \end{array}$$

We observe that for $n \geq 8$, $f^n(0, 0, 0, 1) = (0, 0, 0, 0)$. We prove below that g is stable, viz., that any choice of initial values leads to a similar result.

We now list a set of functions which can be proved to be stable. In some cases we prove the stability of the function and in others we leave the proof to the reader.

3. Theorem

The following functions are stable:

(1) $\overline{x + y}$ (mod 2^n), $n = 1, 2, 3, \ldots$.

(2) $\overline{x \cdot y}$ (mod 2^n), $n = 1, 2, 3, \ldots$.

(3) $\overline{x^t + y^t}$ (mod 2^n), $t = 2, 3, 4, \ldots$; $n = 1, 2, 3, \ldots$.

(4) $\overline{(x + y)^t}$ (mod 2^n), $t = 2, 3, 4, \ldots$; $n = 1, 2, 3, \ldots$.

(5) $|\overline{x^t - y^t}|$ (mod 2^n), $t = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$.

(6) $|\overline{(x - y)^t}|$ (mod 2^n), $t = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$.

(7) $\phi(x) + \phi(y)$, where ϕ is Euler's ϕ-function.

The notation $\overline{x}$ (mod 2^n) means the least nonnegative integer congruent to x modulo 2^n.

Proof of (1): We use f_i^n to denote the ith entry of the nth iteration of $f(x_1, x_2, x_3, x_4)$. The subscript $i + j$ of x will always represent $\overline{i + j}$ (mod 4). We first consider the function $g_1(x, y) = x + y$ and show that for any n:

$$f_i^{2(n+1)} = (2^n)[(2^n - 1)x_i + (2^n + 1)x_{i+2} + 2^n(x_{i+1} + x_{i+3})]. \tag{A}$$

We compute: $f_i^1 = x_i + x_{i+1}$. (B)

$$f_i^2 = x_i + 2x_{i+1} + x_{i+2}. \tag{C}$$

$$f_i^3 = x_i + 3x_{i+1} + 3x_{i+2} + x_{i+3}. \tag{D}$$

$$f_i^4 = 2x_i + 4x_{i+1} + 6x_{i+2} + 4x_{i+3}. \tag{E}$$

(A) is clearly true for $n = 1$ by (E). Suppose (A) is true for n. Then by (C)

$$f_i^{2(n+2)} = f_i^2(f_i^{2(n+1)}) = (2^{n+1})[(2^{n+1} - 1)x_{i+1} + (2^{n+1} + 1)x_{i+3} + (2^{n+1})(x_i + x_{i+2})]. \tag{F}$$

We note that for any iteration of $f(x_1, x_2, x_3, x_4)$ we can consider $(f_1^n, f_2^n, f_3^n, f_4^n)$ to be the same row as its transpositions $(f_4^n, f_1^n, f_2^n, f_3^n)$, $(f_3^n, f_4^n, f_1^n, f_2^n)$, and $(f_2^n, f_3^n, f_4^n, f_1^n)$. Therefore (F) indicates that (A) is also true for $n + 1$. It follows by finite induction that (A) is true for all n. Hence we conclude that

$$f_i^{2(n+1)} \equiv 0 \pmod{2^n}, \; n = 1, 2, 3, \ldots .$$

Since $f^n(0, 0, 0, 0) = (0, 0, 0, 0)$ for all n, the stability of the function $g(x, y) = \overline{x + y} \pmod{2^n}$ is established.

Proof of (3): For any initial numbers x_1, x_2, x_3, x_4, there are six ways to arrange even and odd numbers:

(i)	(e, e, e, e)	(iv)	(e, b, e, b)
(ii)	(e, e, e, b)	(v)	(e, b, b, b)
(iii)	(e, e, b, b)	(vi)	(b, b, b, b)

where e and b represent even and odd numbers, respectively. Since the sum of the tth powers of two even (or two odd) numbers is even, the sum of the tth powers of an even number and an odd number is odd. Therefore, when we consider the function $g_2(x, y) = x^t + y^t$, the initial arrangements (ii) and (v) yield the following:

$$\begin{array}{lccc}
 & (e, e, e, b) & & (e, b, b, b) \\
f^1: & (e, e, b, b) & & (b, e, e, b) \\
f^2: & (e, b, e, b) & \text{and} & (b, e, b, e) \\
f^3: & (b, b, b, b) & & (b, b, b, b) \\
f^4: & (e, e, e, e) & & (e, e, e, e)
\end{array} \tag{G}$$

The arrangements (i), (iii), (iv), and (vi) are included in the above operations. Thus there exists an integer $m \leq 4$ such that all numbers of $f^m(x_1, x_2, x_3, x_4)$ are even numbers for the arrangements (i)-(vi).

Let $f^m(x_1, x_2, x_3, x_4) = (2^i m_1, 2^j m_2, 2^u m_3, 2^v m_4)$, where $i, j, u, v, m_1, m_2, m_3, m_4$ are positive integers. Without loss of generality, we may assume that $i \leq j, u, v$. Then we have

$$(2^i m_1)^t + (2^j m_2)^t = 2^{it} m_1^t + 2^{jt} m_2^t = 2^{it}(m_1^t + 2^{(j-i)t} m_2^t).$$

This indicates that the value of i in f^m will increase by at least t times at the next step (where $t \geq 2$). After a finite number of steps, we can obtain an integer q such that $f^q(x_1, x_2, x_3, x_4) = (2^h q_1, 2^\ell q_2, 2^r q_3, 2^s q_4)$, where $h, \ell, r, s, q_1, q_2, q_3, q_4$ are positive integers and $h, \ell, r, s \geq n$, i.e., all numbers of f^q are the multiples of 2^n. Thus, the four numbers of $\underline{f^q \text{ are }}$ congruent to zero modulo 2^n. This shows that the function $g(x, y) = \overline{x^t + y^t} \pmod{2^n}$ is stable.

Before we prove the last statement of the theorem, let us recall a simple property of Euler's ϕ-function.

Lemma 1: For any even integer N, (i) if N is a power of 2, then $\phi(N) = (\frac{1}{2})N$; (ii) if N is not a power of 2, then $\phi(N) < (\frac{1}{2})N$.

Proof of (7): First, we consider only the initial numbers x_1, x_2, x_3, x_4 greater than 2. Since $\phi(x)$ is even for all $x > 2$. Therefore, we have

$$f^1(x_1, x_2, x_3, x_4) = (N_1, N_2, N_3, N_4),$$

where N_1, N_2, N_3, N_4 are even integers and $\min\{N_1, N_2, N_3, N_4\} \geq 4$. If $N_1 = N_2 = N_3 = N_4$, we can see below that statement (7) of the theorem is true in this case. If all four are not equal, by Lemma 1 it is clearly seen that the greatest integer (if two or three are equal and greater than the remaining, each of these may be called "the greatest") of $f^n(x_1, x_2, x_3, x_4)$ must get smaller within three steps for all n. Hence, after a finite number of steps (such as m), $f^m(x_1, x_2, x_3, x_4) = (N_5, N_6, N_7, N_8)$, where N_5, N_6, N_7, N_8 are even integers and either $N_5 = N_6 = N_7 = N_8 = 2^t$ for some integer $t \geq 3$, or $\max\{N_5, N_6, N_7, N_8\} = 4$. But, we also have $\min\{N_5, N_6, N_7, N_8\} \geq 4$ and

$$f^c(2^t, 2^t, 2^t, 2^t) = (2^t, 2^t, 2^t, 2^t) \text{ for all } c \text{ and } t.$$

This implies that the function $\phi(x) + \phi(y)$ is stable.

It remains only to show that the initial numbers x_1, x_2, x_3, x_4 contain some 2s or 1s [since $\phi(2) = \phi(1) = 1$, so we only need to consider either 1 or 2]. Suppose the initial numbers contain only one number 2, say $x_1 = 2$. Thus

$$f^1(x_1, x_2, x_3, x_4) = (1 + \phi(x_2), \phi(x_2) + \phi(x_3), \phi(x_3) + \phi(x_4), \phi(x_4) + 1).$$

Since $x_2, x_3, x_4 > 2$. Therefore, all four numbers of $f^1(x_1, x_2, x_3, x_4)$ are strictly greater than 2. Similarly, when the initial numbers contain two or three 2s, we can prove that there exists an integer $j \leq 3$ such that

$$f^j(x_1, x_2, x_3, x_4) = (J_1, J_2, J_3, J_4),$$

where J_1, J_2, J_3, J_4 are integers and $\min\{J_1, J_2, J_3, J_4\} > 2$. This completes the proof of (7).

4. Some More Ducci Processes

Let us denote the m-digit integer by

$$x = 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + 10a_2 + a_1$$

and

$$S_x^t = (a_m + a_{m-1} + \cdots + a_2 + a_1)^t,\ T_x^t = a_m^t + a_{m-1}^t + \cdots + a_2^t + a_1^t,$$

where $t = 1, 2, 3, \ldots$.

We now address the following problems:

(1) For what values of t is the function $|S_x^t - S_y^t|$ stable?

(2) For what values of t is the function $|T_x^t - T_y^t|$ stable?

(3) For what values of t and n is the function $\overline{T_x^t + T_y^t}$ (mod 2^n) stable?

Partial answers to these questions are given below.

Obviously, the function $|S_x^t - S_y^t|$ is stable for $t = 1$. In order to prove stability for $t = 2$, we need the following lemma.

Lemma 2: Let Z be the set of all nonnegative integers and let $H = \{3z\colon z \in Z\}$, $L = Z \backslash H$. Then for any $h, h_1, h_2 \in H$ and $\ell, \ell_1, \ell_2 \in L$ we have

(i) $|h_1^2 - h_2^2| \in H$ and $|\ell_1^2 - \ell_2^2| \in H$;

(ii) $|h^2 - \ell^2| \in L$.

Proof: For any $h \in H$, we have $h \equiv 0 \pmod 3$ and $h^2 \equiv 0 \pmod 3$. For any $\ell \in L$, we have either $\ell \equiv 1 \pmod 3$ or $\ell \equiv 2 \pmod 3$. But we see that $\ell^2 \equiv 1 \pmod 3$ for both cases. Therefore, we obtain

(i) $|h_1^2 - h_2^2| \equiv 0 \pmod 3$ and $|\ell_1^2 - \ell_2^2| \equiv 0 \pmod 3$, i.e.,

$|h_1^2 - h_2^2| \in H$ and $|\ell_1^2 - \ell_2^2| \in H$.

(ii) $|h^2 - \ell^2| = |1|$, i.e., $|h^2 - \ell^2| \in L$.

We may note that by division by three a nonnegative integer has the same remainder as the sum of its digits. Therefore, an immediate consequence of Lemma 2 is:

Lemma 3: Let Z be the set of all nonnegative integers and let $H = \{3z\colon z \in Z\}$, $L = Z \backslash H$. Then for any $h, h_1, h_2 \in H$ and $\ell, \ell_1, \ell_2 \in L$ we have

(i) $|S_{h_1}^2 - S_{h_2}^2| \in H$ and $|S_{\ell_1}^2 - S_{\ell_2}^2| \in H$;

(ii) $|S_h^2 - S_\ell^2| \in L$.

We now prove that the function $|S_x^t - S_y^t|$ is stable for $t = 2$. By Lemma 3 we see that e and b can play the same roles as shown in (G) if e represents the initial number which belongs to the set H and b represents the initial number which belongs to the set L. Thus we can find an integer $m \leq 4$ and four integers h_1, h_2, h_3, $h_4 \in H$ such that

$$f^m(x_1, x_2, x_3, x_4) = (h_1, h_2, h_3, h_4)$$

and $S_{h_1}, S_{h_2}, S_{h_3}, S_{h_4} \in H$. It follows that there exist four nonnegative integers h_5, h_6, h_7, h_8 such that

$$f^{m+1}(x_1, x_2, x_3, x_4) = (h_5, h_6, h_7, h_8)$$

and $S_{h_i} \equiv 0 \pmod 9$, $i = 5, 6, 7, 8$. On the other hand, if $\max\{h_5, h_6, h_7, h_8\}$ has four or more digits, then, after a finite number of steps (say d), we can find four nonnegative integers h_9, h_{10}, h_{11}, h_{12} such that

$$f^{m+d}(x_1, x_2, x_3, x_4) = (h_9, h_{10}, h_{11}, h_{12}),$$

where $S_{h_i} \equiv 0 \pmod 9$, $i = 9, 10, 11, 12$ and $\max\{h_9, h_{10}, h_{11}, h_{12}\} < 999$ (the proof is based on the same principle as shown in Steinhaus [2]). We know that $(9 + 9 + 9) = 27$. Therefore, $\max\{S_{h_9}, S_{h_{10}}, S_{h_{11}}, S_{h_{12}}\} < 27$. This indicates that the values of S_{h_i} ($i = 9, 10, 11, 12$) are either 0, 9, or 18. But we see that

$$18^2 - 0^2 = 324 \text{ and } 3 + 2 + 4 = 9;$$

$$18^2 - 9^2 = 234 \text{ and } 2 + 3 + 4 = 9;$$

$$18^2 - 18^2 = 0.$$

Thus, in the next step, we have

$$f^{m+d+1}(x_1, x_2, x_3, x_4) = (h_{13}, h_{14}, h_{15}, h_{16}),$$

where the values of S_{h_i} (i = 13, 14, 15, 16) are either 0 or 9. It is easily verified that $f^c(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ for all $c \geq m + d + 5$. This shows that the function $|S_x^2 - S_y^2|$ is stable.

The function $|S_x^t - S_y^t|$ is not stable for $t \geq 3$. For instance, letting

$$g(x, y) = |S_x^3 - S_y^3|$$

and

$$x_1 = 21951,\ x_2 = 21609,\ x_3 = 0,\ x_4 = 324,$$

we have

$$\begin{array}{rrrrr} & (21951, & 21609, & 0, & 324) \\ f^1: & (\ 0, & 5832, & 729, & 5103) \\ f^2: & (\ 5832, & 0, & 5103, & 729) \\ f^3: & (\ 5832, & 729, & 5103, & 0) \end{array}$$

Since

$$f^3(21951, 21609, 0, 324) = f^1(21951, 21609, 0, 324) = (5832, 729, 5103, 0),$$

the process is periodic with period 2. The same result is obtained if we take (531441, 0, 426465, 104976) as the initial entries for $t = 4$.

The reader is welcome to consider the stability of the function $|T_x^t - T_y^t|$ in problem (2). About 500 quadruples of two-digit numbers (x_1, x_2, x_3, x_4) have been tested for $t = 2$ and $t = 3$. In each case, the functions $|T_x^2 - T_y^2|$ and $|T_x^3 - T_y^3|$ stabilized after 80 steps.

With respect to problem (3), it is not difficult to get an example to show that the function $T_x^2 + T_y^2 \pmod{32}$ is not stable. Letting

$$x_1 = 10,\ x_2 = 22,\ x_3 = 6,\ x_4 = 26,$$

we have

$$\begin{array}{rrrrr} & (10, & 22, & 6, & 26) \\ f^1: & (\ 9, & 12, & 12, & 9) \\ f^2: & (22, & 10, & 22, & 2) \\ f^3: & (\ 9, & 9, & 12, & 12) \end{array}$$

Thus, the process is periodic.

5. Ducci Processes in k-Dimensions

By analogy with Section 1, we now consider a function f whose domain and range are the set of k-tuples of nonnegative integers. Suppose that there is a function $g(x, y)$ whose domain is the set of pairs of nonnegative integers, whose range is the set of nonnegative integers, and that

$$f(x_1, x_2, \ldots, x_k) = (g(x_1, x_2), g(x_2, x_3), \ldots, g(x_k, x_1)).$$

Let $f^m(x_1, x_2, \ldots, x_k)$ be the mth iteration of f. Assume that entries of $f^m(x_1, x_2, \ldots, x_k)$ are bounded for all m (as before the bound depends on the initial choice of entries).

A Ducci process is a sequence of iterations of f. We call a function g stable if g generates a Ducci process such that for any choice of entries

$$f^{m+1}(x_1, x_2, \ldots, x_k) = f^m(x_1, x_2, \ldots, x_k) \text{ for some } m.$$

All of the Ducci processes in Sections 1-4 can be generalized to an arbitrary dimension k, where k is any integer greater than 2. We propose to examine only two such generalizations.

B. Freedman [3] proved that function $g(x, y) = |x - y|$ is stable if the number of members of the initial entries k is a power of 2.

We now show that the following functions are stable if and only if k is a power of 2.

(I) $g(x, y) = \overline{x + y} \pmod{2^n}$, $n = 1, 2, 3, \ldots$.

(II) $g(x, y) = \overline{x + y} \pmod{k}$, where k is an arbitrary positive integer.

Proof of (I): Let ${}^k f_i^m$ be the ith entry of the mth iteration of $f(x_1, x_2, \ldots, x_k)$. The subscript $i + j$ of x will always represent $\overline{i + j} \pmod{k}$.

Consider the function $g(x, y) = x + y$. We can show by mathematical induction that for any m,

$$ {}^k f_i^m = \sum_{j=0}^{m} \binom{m}{j} x_{i+j}. \tag{H} $$

In fact, (H) is true for $m = 1$, because

$$ {}^k f_i^1 = x_i + x_{i+1}. $$

Suppose (H) is true for m. Then

$$ \begin{aligned} {}^k f_i^{m+1} &= {}^k f_i^1 ({}^k f_i^m) = \sum_{j=0}^{m} \binom{m}{j} x_{i+j} + \sum_{j=0}^{m} \binom{m}{j} x_{i+j+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \binom{m}{j} x_{i+j} + \sum_{j=0}^{m-1} \binom{m}{j} x_{i+j+1} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \binom{m}{j} x_{i+j} + \sum_{j=1}^{m} \binom{m}{j-1} x_{i+j} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \left[\binom{m}{j} + \binom{m}{j-1} \right] x_{i+j} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m+1}{0} x_i + \sum_{j=1}^{m} \binom{m+1}{j} x_{i+j} + \binom{m+1}{m+1} x_{i+m+1} \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} x_{i+j}. \end{aligned} $$

Therefore, (H) is true for all m.

In particular, if k is a power of 2 ($k = 2^r$), then from (H) we have

$$ \begin{aligned} {}^k f_i^k &= \sum_{j=0}^{2^r} \binom{2^r}{j} x_{i+j} = \binom{2^r}{0} x_i + \sum_{j=1}^{2^r - 1} \binom{2^r}{j} x_{i+j} + \binom{2^r}{2^r} x_{i+2^r} \\ &= 2x_i + \sum_{j=1}^{2^r - 1} \binom{2^r}{j} x_{i+j}. \end{aligned} $$

Adopting Freedman's technique, we see that $\binom{2^r}{j}$ is always even for $j = 1, 2, \ldots, 2^r - 1$. Hence

$$ {}^k f_i^k \equiv 0 \pmod{2},\ i = 1, 2, \ldots, k,\ k = 2^r $$

and

$$ {}^k f_i^{2k} \equiv 0 \pmod 4, \quad i = 1, 2, \ldots, k, \; k = 2^r. $$

In general

$$ {}^k f_i^{tk} \equiv 0 \pmod{2^t}, \quad i = 1, 2, \ldots, k, \; k = 2^r. $$

Thus, we conclude that for any n we have

$$ {}^k f_i^{nk} \equiv 0 \pmod{2^n}, \quad i = 1, 2, \ldots, k, \; k = 2^r. $$

This means the function $g(x, y) = \overline{x + y} \pmod{2^n}$ is stable if k is a power of 2.

Proof of (II): The function $g(x, y) = \overline{x + y} \pmod k$ is stable if and only if $k = 2^r$ for any r. That this condition is sufficient follows from the previous proof. We now show that it is necessary.

We prove first that $g(x, y)$ is not stable if k is an odd prime p. Let the initial entries be $x_1, x_2, \ldots, x_p$, and

$$ x_i = \begin{cases} 0 & \text{if } 0 < i < p \\ 1 & \text{if } i = p. \end{cases} $$

Then from (H) we have

$$ {}^p f_i^p = \begin{cases} \binom{p}{p - i} & \text{if } 0 < i < p \\ \binom{p}{p} + \binom{p}{0} & \text{if } i = p. \end{cases} $$

We know that $\binom{p}{p - i} = \binom{p}{i} \equiv 0 \pmod p$ for $0 < i < p$ when p is an odd prime. Hence,

$$ {}^p f_i^p = \begin{cases} 0 & \text{if } 0 < i < p \\ 2 & \text{if } i = p \end{cases} $$

and

$$ {}^p f_i^{pt} \equiv \begin{cases} 0 & \text{if } 0 < i < p \\ 2^t & \text{if } i = p \end{cases} \pmod p, $$

where t is a positive integer. Thus, by Fermat's theorem $2^{p-1} \equiv 1 \pmod p$, we obtain

$$ {}^p f_i^{p(p-1)} = \begin{cases} 0 & \text{if } 0 < i < p \\ 1 & \text{if } i = p. \end{cases} $$

Therefore, $g(x, y)$ is periodic.

Now let $k = ps$, where p is an odd prime and s is any integer greater than one. Let

$$ x_i = \begin{cases} s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases} \quad * $$

*For example, let $k = 6$; then $k = ps = 3 \times 2$. We set the initial entries as (0, 0, 2, 0, 0, 2). Thus we have

	(0, 0, 2, 0, 0, 2)
${}^6 f^1$:	(0, 2, 2, 0, 2, 2)
${}^6 f^2$:	(2, 4, 2, 2, 4, 2)
${}^6 f^3$:	(0, 0, 4, 0, 0, 4)
${}^6 f^4$:	(0, 4, 4, 0, 4, 4)
${}^6 f^5$:	(4, 2, 4, 4, 2, 4)
${}^6 f^6$:	(0, 0, 2, 0, 0, 2)

Then

$$ {}^k f_i^p = \begin{cases} 2s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise} \end{cases} $$

and

$$ {}^k f_i^{pt} \equiv \begin{cases} 2^t s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise} \end{cases} \pmod{k}, $$

where t is a positive integer. Hence

$$ {}^k f_i^{p(p-1)} = \begin{cases} s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases} $$

Thus, function $g(x, y)$ is periodic and the proof is complete.

We leave it to the reader to examine generalizations of the Ducci processes presented in Sections 1-4.

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MINIMUM PERIODS MODULO n FOR BERNOULLI POLYNOMIALS

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1. It is known that the sequence of the Bernoulli numbers b_m, defined by

$$b_0 = 1,$$

$$b_m = -\frac{1}{m+1}\sum_{i=0}^{m-1}\binom{m+1}{i}b_i \quad (m > 0),$$

is periodic after being reduced modulo n (where n is any positive integer), cf. [3]. [In this note, we use the symbols b_m for the Bernoulli numbers and $B_m(x)$ for the Bernoulli polynomials.] In [3] we proved

Theorem 1: Let $p \in \mathbb{P}$, $\mathbb{P}$ being the set of primes, $p \geq 3$, and $e,k,m \in \mathbb{N}$. For $k,m \geq e + 1$, we have:

$$b_k \ n\text{-integral and } k \equiv m \bmod p^e(p-1) \Rightarrow b_m \ n\text{-integral and } b_k \equiv b_m \bmod p^e.$$

In this note, we shall give some analogous results about the sequence of the Bernoulli polynomials $B_m(x)$ reduced modulo n (Theorem 6) and the polynomial functions over $\mathbb{Z}_n$ generated by the Bernoulli polynomials (Theorem 4). Here, $\mathbb{Z}_n$ is the ring of integers modulo n, where $n \in \mathbb{N}$, $n \geq 2$, and the Bernoulli polynomials in $\mathbb{Q}[x]$ are defined by

$$B_m(x) = \sum_{i=0}^{m}\binom{m}{i}b_i x^{m-i}, \ m \in \{0, 1, 2, \ldots\}.$$

Similar questions about Euler numbers and polynomials were asked by Professor L. Carlitz and Jack Levine in [2].

2. In [4] we discussed in which cases it is possible to define (in a natural way) analogs of Bernoulli polynomials in $\mathbb{Z}_n$. In this section, we shall prove the periodicity of the sequence of the polynomial functions B_m over $\mathbb{Z}_n$ generated by the Bernoulli polynomials. Each polynomial $F(x) \in \mathbb{Q}[x]$ generates a polynomial function $F : \mathbb{Z} \to \mathbb{Q}$ by

$$(1) \qquad x \mapsto F(x).$$

Now, considering (1) in $\mathbb{Z}_n$, we get a function $F : \mathbb{Z}_n \to \mathbb{Z}_n$ if and only if

(a) all values of F are interpretable mod n, and
(b) the relation (1) preserves congruence properties.

For this, it is useful to introduce the following notations ([4], p. 28).

Definition 1: A function $F : \mathbb{Z} \to \mathbb{Q}$ is said to be acceptable mod n, iff

(a) $\forall x$, $F(x)$ is n-integral,
(b) $x \equiv y \bmod n \Rightarrow F(x) \equiv F(y) \bmod n$.

A polynomial $F(x) \in \mathbb{Q}[x]$ is said to be acceptable mod n if this is true for its polynomial function.

Definition 2: Two functions $F,G : \mathbb{Z} \to \mathbb{Q}$ are said to be equivalent mod n iff

(a) F,G are acceptable mod n,
(b) $\forall\, x,\ F(x) \equiv G(x) \bmod n$.

Two polynomials $F(x), G(x) \in \mathbb{Q}[x]$ are said te be equivalent mod n if this is true for their polynomial functions. We write

$$F \sim G \bmod n \quad \text{and} \quad F(x) \sim G(x) \bmod n,$$

respectively.

From [4], p. 29, we have the following

Theorem 2: $B_m(x)$ is acceptable mod n

$$\Leftrightarrow b_m \text{ is } n\text{-integral and } mS_{m-1}(n) \equiv 0 \bmod n,$$

where

$$S_m(x) = \begin{cases} \sum_{k=0}^{x-1} k^m & \text{for } m \in \{0, 1, 2, \ldots\},\ x \in \mathbb{N} \\ 0 & \text{for } m = -1 \text{ or } x = 0. \end{cases}$$

A more explicit characterization of $B_m(x)$ acceptable mod n gives (cf. [4], p. 31)

Theorem 3:

(a) For $m > 1$ and $2 \nmid n$ we have: $B_m(x)$ acceptable mod n

$$\Leftrightarrow \forall\, p \in \mathbb{P} : (p \mid n \Rightarrow p - 1 \nmid m \text{ and } (p - 1 \nmid m - 1 \text{ or } p \mid m)).$$

(b) For $k \in \mathbb{N}$ we have: $B_m(x)$ acceptable mod $2k \Leftrightarrow m = 0$.

Now, we may state our first new assertion. (By Theorem 3, it suffices to discuss the case $n = p^e$, p a prime, $p \geq 3$.)

Theorem 4: Let $p \in \mathbb{P}$, $p \geq 3$, and $e,k,m \in \mathbb{N}$.

(a) For $k,m \geq e + 1$, we have:

$$B_k \text{ acceptable mod } p \text{ and } k \equiv m \bmod p^e(p-1)$$
$$\Rightarrow B_m \text{ acceptable mod } p \text{ and } B_k \sim B_m \bmod p^e.$$

(b) $p^e(p - 1)$ is the smallest period length of the sequence of the Bernoulli polynomials in the sense of (a).

For the proof of this theorem, we need the following

Lemma: Let $p \in \mathbb{P}$, $p \geq 3$, $e \in \mathbb{N}$. Then

$$x^{\lambda(p^e) + V(p^e)} \equiv x^{V(p^e)} \bmod p^e \text{ for all } x,$$

where both $V(p^e) = e$ and $\lambda(p^e) = p^{e-1}(p - 1)$ are minimal for this property.

For the proof of this lemma, see [5], Theorem 1.

Proof of Theorem 4(a): Let $k,m > e$, $k \equiv m \bmod p^e(p-1)$ and B_k be acceptable mod p, so that b_k is p-integral and $kS_{k-1}(p) \equiv 0 \bmod p$. By Theorem 1, b_m is p-integral with $b_k \equiv b_m \bmod p^e$. Furthermore, from $k \equiv m \bmod p^e(p-1)$, and $k,m > e$, we have $k \cdot i^{k-1} \equiv m \cdot i^{m-1} \bmod p^e$ for all i, by the lemma above. Then

$$kS_{k-1}(x) = k\sum_{i=0}^{x-1} i^{k-1} \equiv m\sum_{i=0}^{x-1} i^{m-1} = mS_{m-1}(x) \bmod p^e$$

for all x. Now we use (5) from [4]:

$$B_m(x) = mS_{m-1}(x) + b_m.$$

Thus B_m is p-integral, too, and $B_k(x) \equiv B_m(x) \bmod p^e$ for all x; i.e.,

$$B_k \sim B_m \bmod p^e.$$

Proof of Theorem 4(b): Let B_k and B_m be acceptable mod p, let $k,m \geq e+1$, and let $B_k \sim B_m \bmod p^e$. Then $B_k(x) \equiv B_m(x) \bmod p^e$ for all x. We shall show that $k \equiv m \bmod p^e(p-1)$ if $p^e \nmid m$. Obviously this would prove the assertion. First, we get $b_k = B_k(0) \equiv B_m(0) = b_m \bmod p^e$, hence $kS_{k-1}(x) \equiv mS_{m-1}(x) \bmod p^e$ for all x; and moreover,

$$kx^{k-1} \equiv mx^{m-1} \bmod p^e \text{ for all } x, \tag{2}$$

since

$$kx^{k-1} = kS_{k-1}(x+1) - kS_{k-1}(x).$$

Putting $x = 1$ in (2) shows $k \equiv m \bmod p^e$. Let $d = \text{g.c.d.}(k, p^e)$. We know that $\text{g.c.d.}(m, p^e) = d$, and $d = p^i$ with $0 \leq i < e$, since $p^e \nmid k$. Thus (2) implies

$$x^{k-1} \equiv x^{m-1} \bmod p^{e-i} \text{ for all } x.$$

But this is possible only if $k-1 \equiv m-1 \bmod (p-1)$; i.e., $k \equiv m \bmod (p-1)$. Together with $k \equiv m \bmod p^e$, we have $k \equiv m \bmod p^e(p-1)$, and the theorem is proved.

Remark 1: The minimum period length of the Bernoulli polynomial functions mod n is the same as that of the Bernoulli numbers mod n.

Remark 2: By a very similar argument one may prove that when B_m is acceptable mod p, $m \equiv 0 \bmod p^e \Leftrightarrow B_m \sim 0 \bmod p^e$. For this, notice that $m \equiv 0 \bmod p^e$ implies $b_m \equiv 0 \bmod p^e$ ([1], p. 78, Theorem 5).

Remark 3: Let $v(p^e)$ denote the preperiod length of $B_m \bmod p^e$. Then Theorem 4 implies $v(p^e) \leq e+1$. Using Remark 2 one may slightly improve this inequality for special cases with $e \geq p$. For instance, $v(3^3) = 3$.

3. In this section we shall discuss the periodicity of Bernoulli polynomials reduced modulo n.

Definition 3: A polynomial $F(x) = a_0 + a_1x + \cdots + a_rx^r \in \mathbb{Q}[x]$ is said to be n-integral if and only if the coefficients $a_0, a_1, \ldots, a_r$ are all n-integral.

From [4], p. 32, we have, for the Bernoulli polynomials,

Theorem 5: Let $p \in \mathbb{P}$, $e \in \mathbb{N}$, and $m \in \mathbb{N} \cup \{0\}$ with p-adic representation

$$m = \sum_{k=0}^{s} m_k p^k.$$

Then

$$B_m(x) \in \mathbb{Q}[x] \text{ is } p^e\text{-integral if and only if } \sum_{k=0}^{s} m_k < p - 1.$$

Remark 4: Each n-integral polynomial is acceptable mod n, but there are polynomials acceptable mod n that are not n-integral (cf. [4], pp. 32-33). If we reduce the coefficients of any n-integral $B_m(x)$, we still get a polynomial of degree m, since the coefficient of x^m is 1. Consequently, no periodicity appears. But by the lemma above we have

$$x^{p^{e-1}(p-1)+e} \equiv x^e \bmod p^e \text{ for all } x.$$

Hence, any p-integral polynomial $F(x)$ is equivalent to a reduced polynomial with degree $< p^{e-1}(p - 1) + e$ having coefficients in $\{0, 1, \ldots, p^e - 1\}$. We shall denote such a polynomial $F(x)$, reduced mod n, by $\tilde{F}(x)$.

Remark 5: If $\tilde{F}_1(x)$ and $\tilde{F}_2(x)$ are reduced polynomials of $F(x)$ mod n, then

$$\tilde{F}_1(x) \sim \tilde{F}_2(x) \sim F(x) \bmod n.$$

We conjecture that the sequence of the Bernoulli polynomials, reduced mod n, is periodic in a strong sense too, with a proof here only for $n = p$, $p \in \mathbb{P}$.

Theorem 6: Let $p \in \mathbb{P}$, $k,m \geq 2$, and suppose $B_k(x)$, $B_m(x)$ are p-integral. If $k \equiv m \bmod p(p - 1)$, then

$$\tilde{B}_k(x) = \tilde{B}_m(x) \text{ in } \mathbb{Z}_p[x].$$

Proof: $B_k(x)$, $B_m(x)$ p-integral implies $B_k(x)$, $B_m(x)$ acceptable mod p (Remark 4). By Theorem 4 we get

$$B_k(x) \sim B_m(x) \bmod p, \text{ hence}$$
$$\tilde{B}_k(x) \sim \tilde{B}_m(x) \bmod p, \text{ i.e.,}$$
$$\tilde{B}_k(x) - \tilde{B}_m(x) \equiv 0 \bmod p \text{ for all } x.$$

The degree of this difference polynomial is $<\lambda(p) + V(p) = p - 1 + 1 = p$, but it has p zeros in $\mathbb{Z}_p$, hence it must be the zero polynomial, and we have

$$\tilde{B}_k(x) = \tilde{B}_m(x) \text{ in } \mathbb{Z}_p[x].$$

Remark 6: The question, whether Theorem 6 holds for arbitrary modulus n, remains open. The proof above fails in $\mathbb{Z}_n$ when $n \notin \mathbb{P}$, since $\tilde{B}_k(x) \sim \tilde{B}_m(x) \bmod n$ does not imply $\tilde{B}_k(x) = \tilde{B}_m(x)$ in $\mathbb{Z}_n[x]$. For example, let $e > 1$ and

$$F(x) = p^{e-1} \prod_{i=0}^{p-1} (x - i),$$

$$G(x) = \prod_{i=0}^{p^e-1} (x - i).$$

Then $F(x) \sim G(x)$ (~ 0) mod p^e, but $F(x) \neq G(x)$ in $\mathbb{Z}_{p^e}[x]$. Or, if $n = p_1p_2$, where $p_1,p_2 \in \mathbb{P}$ and $p_1 \neq p_2$, then one may consider the polynomials

$$p_2 \prod_{i=0}^{p_1-1} (x - i) \quad \text{and} \quad p_1 \prod_{i=0}^{p_2-1} (x - i)$$

for a counterexample.

Remark 7: The assumption in Theorem 6 that both $B_k(x)$ and $B_m(x)$ are p-integral cannot be weakened, since $B_k(x)$ p-integral and $k \equiv m \bmod p(p-1)$ does not imply B_m p-integral. For example

$$B_2(x) = x^2 - x + \frac{1}{6}$$

is 5-integral, while $B_{22}(x)$ is not so by Theorem 2, even though $22 \equiv 2 \bmod 5 \cdot 4$.

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A NOTE ON FIBONACCI PRIMITIVE ROOTS

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A prime p possesses a Fibonacci primitive root (FPR) g if g is a primitive root (mod p) satisfying

$$g^2 \equiv g + 1 \pmod{p}.$$

This definition was given in [2], and properties of FPRs were worked out in [2] and [3]. A good discussion of FPRs is contained in [4].

In [2] an asymptotic density for the set of primes having a FPR in the set of all primes was conjectured, and that this density is correct subject to a generalized Riemann Hypothesis was shown in [1], but it is still an open question as to whether or not infinitely many primes possess FPRs. The purpose of this note is to provide a sufficient condition that a prime should possess a FPR.

Theorem: If $p = 60k - 1$ and $q = 30k - 1$ are both prime, then p has a FPR.

Proof: $p \equiv 3 \pmod{4}$ implies that at most one of $\{a, -a\}$ is a primitive root of p for any a such that $2 \leq a \leq (p-1)/2 = q$; q prime implies that there are $q - 1$ primitive roots of p in all, so exactly one of $\{a, -a\}$ is a primitive root of p.

$p \equiv -1 \pmod{10}$ implies that two solutions to the congruence

$$x^2 - x - 1 \equiv 0 \pmod{p}$$

exist. These solutions may be written as g and $1 - g$.

Shanks points out that since $g^2 - g - 1 \equiv 0 \pmod{p}$,

$$g(g - 1) \equiv 1 \pmod{p},$$

so that g is a primitive root iff $g - 1$ is a primitive root. $g - 1$ is a primitive root iff $-(g - 1) = 1 - g$ is not a primitive root. Thus exactly one of the solutions to the congruence is a FPR of p.

Conditions similar to that in this theorem occur frequently in theorems in the literature about existence or ordering of primitive roots. Theorems 38-40 in [4] are well-known instances of this. In [3] it is observed that primes p satisfying sufficient conditions to have two sets of three consecutive primitive roots (a FPR g, $g - 1$, and $g - 2$, and -2, -3, and -4) must be of the form $120k - 1$, with $60k - 1$ also prime. Using the theorem above, it is not necessary to presuppose that p has a FPR.

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A PROPERTY OF THE FIBONACCI SEQUENCE (F_m), $m = 0, 1, \ldots$

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It is well known that the sequence of the (natural) logarithms reduced mod 1 of the terms F_m of the Fibonacci sequence are dense in the unit interval. See [1], [2]. This is also the case when the logarithms are taken with respect to a base b, where b is a positive integer ≥ 2. In order to see this, we start from the fact that

$$\log F_{n+1} - \log F_n \to \log \frac{1+\sqrt{5}}{2} \text{ as } n \to \infty.$$

Now $\log \frac{1+\sqrt{5}}{2}/\log b$ is an irrational number, for if we suppose that

$$\log \frac{1+\sqrt{5}}{2}/\log b = r/s,$$

where r and s are natural numbers, then we would have

$$b^r = ((1+\sqrt{5})/2)^s,$$

obviously a contradiction. Hence, $\log_b F_{n+1} - \log_b F_n$ tends to an irrational number as $n \to \infty$. This implies that the fractional parts of the sequence

$$(\log_b F_m),\ m = 1, 2, \ldots$$

is dense in the unit interval.

We assume that the Fibonacci numbers F_m, $m \geq 1$, are written in base b, that is,

$$F_m = a_0 b^n + a_1 b^{n-1} + \cdots + a_n,$$

where $a_0 \geq 1$, $0 \leq a_j \leq b - 1$, $j = 0, 1, \ldots, n$, $m = 1, 2, \ldots$, or to any m a set of digits $\{a_0, a_1, \ldots, a_n\}$ is associated.

Now, given an arbitrary sequence of digits $\{a_0, a_1, \ldots, a_r\}$, one may ask whether there exists an F_m which possesses this set as *initial digits*. The question can be answered in the affirmative.

We associate to the sequence $\{a_0, a_1, \ldots, a_r\}$ the value

$$a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r},$$

which is a point on the interval $[1, b)$. This value is the left endpoint of the interval

$$T = T(r) = \left[a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r},\ a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \frac{a_r + 1}{b^r}\right).$$

The function $\log_b x$, mapping $[1, b)$ onto $[0, 1)$, maps this interval $T(r)$ onto the interval

$$T^* = T^*(r) = \left[\log_b\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r}\right),\ \log_b\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \frac{1}{b^r}\right)\right),$$

a subinterval of $[0, 1)$.

Since the fractional parts of the logarithms to base b of the numbers F_m are dense in the unit interval, there is an m such that $\log_b F_m \pmod 1 \in T^*$. It follows that there exists a positive integer $n \geq r$ such that

$$\log_b F_m \pmod 1 = \log_b\left(a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right).$$

Hence, there exists an integer $k \geq n$ such that

$$\log_b F_m = k + \log_b\left(a_0 + \frac{a_1}{b_1} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right),$$

or

$$F_m = b^k\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right)$$

$$= a_0 b^k + a_1 b^{k-1} + \cdots + a_r b^{k-r} + \cdots + a_n^{k-n}.$$

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NOTES ON SUMS OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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The infinite sequence $\{s_n\}$ is a sequence of *generalized Fibonacci numbers* (also called a *generalized Fibonacci sequence* or simply *Fibonacci sequence*) if $s = s_{n-1} + s_{n-2}$ for all n. A particular Fibonacci sequence is completely specified by any two consecutive terms. In this paper we let $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, and $\{k_n\}$ represent generalized Fibonacci sequences, and we let $\{F_n\}$ represent the sequence of *Fibonacci numbers* defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

Theorem: If $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ are Fibonacci sequences, then the following summations hold for $y \geq x$.

$$\sum_{x \leq i \leq y} f_{i+r} = [f_{n+r+2}]_{n=x-1}^{n=y} \tag{1}$$

$$2\sum_{x \leq i \leq y} f_{i+r}g_{i+s} = [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y} \tag{2}$$

$$\begin{aligned} 2\sum_{x \leq i \leq y} f_{i+r}g_{i+s}h_{i+t} = [&f_{n+r+1}g_{n+s+1}h_{n+t} + f_{n+r+1}g_{n+s}h_{n+t+1} \\ &+ f_{n+r}g_{n+s+1}h_{n+t+1} - f_{n+r+1}g_{n+s+1}h_{n+t+1} \\ &- f_{n+r}g_{n+s}h_{n+t}]_{n=x-1}^{n=y}. \end{aligned} \tag{3}$$

Proof: The proofs are by induction on y. As base cases we take $y = x - 1$; the summations are empty, and the right-hand sides vanish identically. The induction steps are as follows:

1.
$$\begin{aligned} \sum_{x \leq i \leq y+1} f_{i+r} &= \sum_{x \leq i \leq y} f_{i+r} + f_{y+r+1} \\ &= [f_{n+r+2}]_{n=x-1}^{n=y} + f_{y+r+1} \\ &= [f_{y+r+2} + f_{y+r+1}] - [f_{(x-1)+r+2}] \\ &= [f_{y+r+3}] - [f_{(x-1)+r+2}] \\ &= [f_{n+r+2}]_{n=x-1}^{n=y+1}. \end{aligned}$$

2.
$$\begin{aligned} 2\sum_{x \leq i \leq y+1} f_{i+r}g_{i+s} &= 2\sum_{x \leq i \leq y} f_{i+r}g_{i+s} + 2f_{y+r+1}g_{y+s+1} \\ &= [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y} + 2f_{y+r+1}g_{y+s+1} \end{aligned}$$

(continued)

$$= [(f_{y+r}g_{y+s+1} + f_{y+r+1}g_{y+s+1}) + (f_{y+r+1}g_{y+s} + f_{y+r+1}g_{y+s+1})]$$
$$- [f_{(x-1)+r}g_{(x-1)+s+1} + f_{(x-1)+r+1}g_{(x-1)+s}]$$
$$= [f_{y+r+2}g_{y+s+1} + f_{y+r+1}g_{y+s+2}]$$
$$- [f_{(x-1)+r}g_{(x-1)+s+1} + f_{(x-1)+r+1}g_{(x-1)+s}]$$
$$= [f_{n+r}g_{n+s+1} + f_{n+r+1}g_{n+s}]_{n=x-1}^{n=y+1}.$$

3. We note that, as in the proofs of (1) and (2), the bottom limit of the right-hand side is unchanged, and we need only prove the following identity related to the top limit y:

$$(*)\quad [f_{y+r+1}g_{y+s+1}h_{y+t} + f_{y+r+1}g_{y+s}h_{y+t+1} + f_{y+r}g_{y+s+1}h_{y+t+1}$$
$$- f_{y+r+1}g_{y+s+1}h_{y+t+1} - f_{y+r}g_{y+s}h_{y+t}] + 2f_{y+r+1}g_{y+s+1}h_{y+t+1}$$
$$= [f_{y+r+2}g_{y+s+2}g_{y+t+1} + f_{y+r+2}g_{y+s+1}h_{y+t+2} + f_{y+r+1}g_{y+s+2}h_{y+t+2}$$
$$- f_{y+r+2}g_{y+s+2}h_{y+t+2} - f_{y+r+1}g_{y+s+1}h_{y+t+1}].$$

As a shorthand notation we let (abc) stand for $f_{y+r+a}g_{y+s+b}h_{y+t+c}$, where a, b, c are 0, 1, or 2. Then the identity (*) can be written as follows:

$$(**)\qquad (110) + (101) + (011) - (111) - (000) + 2(111)$$
$$= (221) + (212) + (122) - (222) - (111).$$

The following identities are easily verified, and the validity of (**), and therefore of (3), follows immediately:

$$(221) = (001) + (011) + (101) + (111),$$

$$(212) = (010) + (110) + (011) + (111),$$

$$(122) = (100) + (110) + (101) + (111),$$

$$(222) = (000) + (001) + (010) + (011) + (100) + (101) + (110) + (111). \square$$

Identity (1) is well known, although it is usually stated in terms of Fibonacci or Lucas numbers with limits of summations 1 to n or 0 to n.

Identity (2) is a generalization of the identities of Berzsenyi [1]. This is easily shown using the following identity, which is easily verified, where $\{f_n\}$ and $\{g_n\}$ are generalized Fibonacci sequences:

$$(4)\qquad f_{n+k}g_{m-k} - f_n g_m = (-1)^n (f_k g_{m-n-k} - f_0 g_{m-n}).$$

We have the following:

$$2\sum_{0\le i\le n} f_i g_{i+2m+b} = [f_{i+1}g_{i+2m+b} + f_i g_{i+2m+b+1}]_{i=-1}^{i=n}$$

$$= [f_{m+b+i}g_{m+i+1} - (-1)^{i+1}(f_{m+b-1}g_m - f_0g_{2m+b-1})$$
$$+ f_{m+b+i}g_{m+i+1} - (-1)^i(f_{m+b}g_{m+1} - f_0g_{2m+b+1})]_{i=-1}^{i=n}$$
$$= [2f_{m+b+i}g_{m+i+1} + (-1)^i(f_{m+b-1}g_m - f_{m+b}g_{m+1} + f_0g_{2m+b})]_{i=-1}^{i=n}.$$

Applying the limits, we obtain the following expression:

$$\sum_{0 \le i \le n} f_i g_{i+2m+b} = f_{n+m+b}g_{n+m+1} - f_{m+b-1}g_m + \left(\frac{1+(-1)^n}{2}\right)(f_{m+b-1}g_m - f_{m+b}g_{m+1} + f_0g_{2m+b}),$$

which is exactly the expression obtained in [1] for even or odd n and $b = 0$ or 1.

The advantages of identity (2), besides its attractive symmetry, are that (a) only a single case is needed instead of four separate cases and (b) it is applicable to general limits, not just the sum from 0 to n. In addition, (2) applies to the sum of the product of terms from *different* generalized Fibonacci sequences, as opposed to the original form in [1].

It should be noted that (4) can also be applied directly to (2), leading to the summation

$$\sum_{x \le i \le y} f_{i+r}g_{i+s} = \left[f_{n+r}g_{n+s+1} + \frac{(-1)^{n+r}}{2}(f_1g_{s-r} - f_0g_{s-r+1})\right]_{n=x-1}^{n=y}.$$

The summation of (2) has also been considered by Pond [1], whose result is valid for the sum of the products of terms from *identical* generalized Fibonacci sequences:

$$\sum_{x \le i \le y} f_i f_{i+s} = \left[\frac{1}{2}(F_{s-3}f_n f_{n+1} + F_s f_{n+2}^2)\right]_{n=x-1}^{n=y}.$$

Recall that $\{F_n\}$ is the sequence of Fibonacci numbers. This result is easily derived from (2) by use of the identity $f_{n+r} = F_{r-1}f_n + F_r f_{n+1}$.

Identity (3) has been considered by Pond [2], again in a simpler context; he requires all *three* generalized Fibonacci sequences to be identical and derives the following expression:

$$2\sum_{x \le i \le y} f_i f_{i+r} f_{i+s} = \left[(F_s F_r - F_{s-1}F_{r-1})D(-1)^n f_{n-1} + f_{s+r+n+1}f_n f_{n+1}\right]_{n=x-1}^{n=y},$$

where $D(-1)^n = f_{n-1}f_{n+1} - f_n^2 = (-1)^n(f_{-1}f_1 - f_0^2)$. It is not hard to show that this summation is a consequence of (3).

The advantages of identity (3) again lie in its pleasing symmetry, its applicability to general limits, and the fact that the summation is valid for products of *different* Fibonacci sequences.

A general methodology for finding summations of the form of identities (1), (2), or (3) has been discussed elsewhere [3]. This methodology expresses the sum of the products of terms from several sequences, each defined by a linear recurrence, as a *standard sum*, defined below.

If the sum to be found is

$$\sum_{x \le n \le y} f_{1,\,n+r_1} f_{2,\,n+r_2} \cdots f_{m,\,n+r_m}$$

where the m sequences $\{f_1\}$, ..., $\{f_m\}$ are defined by linear recurrence relations (i.e., not necessarily Fibonacci sequences), the standard sum is a linear combination

$$\left[\sum_{(i_1,\,\ldots,\,i_m)\,\in\, I^m} a_{i_1,\,i_2,\,\ldots,\,i_m} f_{1,\,n+r_1+i_1} f_{2,\,n+r_2+i_2} \cdots f_{m,\,n+r_m+i_m}\right]_{n=x-1}^{n=y}$$

with the following important properties:

1. each term of the standard sum is the product of m terms, one from each of the original sequences in the product to be summed;

2. the m-tuples $(i_1, \ldots, i_m)$ have constant integer components;

3. the coefficients $a_{i_1,\,\ldots,\,i_m}$ are constant and only a bounded number of the coefficients are nonzero.

Of interest is the result that a standard sum for the sum of the products of terms from recurrence sequences does not always exist. In particular, the sum

$$\sum f_{n+r} g_{n+s} h_{n+t} k_{n+u}$$

(with $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, and $\{k_n\}$ Fibonacci sequences) cannot be expressed as a standard sum. For details, the interested reader is referred to [3].

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BINET'S FORMULA FOR THE TRIBONACCI SEQUENCE

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1. Introduction

The terms of a recursive sequence are usually defined by a recurrence procedure; that is, any term is the sum of preceding terms. Such a definition might not be entirely satisfactory, because the computation of any term could require the computation of all of its predecessors. An alternative definition gives any term of a recursive sequence as a function of the index of the term. For the simplest nontrivial recursive sequence, the Fibonacci sequence, Binet's formula [1]

$$u_n = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of its index and the constants

$$\alpha = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \beta = \frac{1}{2}(1 - \sqrt{5}).$$

In this paper, an analog of Binet's formula for the Tribonacci sequence

$$1,\ 1,\ 2,\ 4,\ 7,\ \ldots,\ u_{n+1} = u_n + u_{n-1} + u_{n-2},\ \ldots$$

(see [2]), is derived. Binet's formula defines any term of the Tribonacci sequence as a function of the index of the term and three constants, ρ, σ, and τ.

2. Binet's Formula for the Tribonacci Sequence

Binet's formula is derived by determining the generating function for the difference equation

$$u_0 = u_1 = 1,\ u_2 = 2$$

$$u_{n+1} = u_n + u_{n-1} + u_{n-2} \qquad n \geq 2.$$

Let $f(x) = u_0 + u_1x + u_2x^2 + \cdots + u_nx^n + \cdots = \sum_{i=0}^{\infty} u_ix^i$ be the generating function; then

$$(1 - x - x^2 - x^3)f(x) = 1,$$

so

$$f(x) = \frac{1}{1 - x - x^2 - x^3} = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \tau x)} = \frac{1}{p(x)}.$$

The roots of $p(x) = 0$ are $1/\rho$, $1/\sigma$, and $1/\tau$, where ρ, σ, and τ are the roots of

$$p\left(\frac{1}{x}\right) = x^3 - x^2 - x - 1 = 0.$$

Applying Cardan's formulas to $p\left(\frac{1}{x}\right) = 0$ yields

$$\rho = \frac{1}{3}\left(\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1\right),$$

$$\sigma = \frac{1}{6}\left(\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} + \sqrt{3}i\ \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right]\right),$$

and

$\tau = \overline{\sigma}$, the complex conjugate of σ.

Approximate numerical values for ρ, σ, and $\overline{\sigma}$ are:

$$\rho = 1.8393,\ \sigma = -0.4196 + 0.6063i,\ \overline{\sigma} = -0.4196 - 0.6063i.$$

Since the roots of $p(x) = 0$ are distinct, by partial fractions

$$f(x) = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \overline{\sigma}x)} = \frac{A}{1 - \rho x} + \frac{B}{1 - \sigma x} + \frac{C}{1 - \overline{\sigma}x}.$$

Here

$$A = \frac{1}{\left(1 - \frac{\sigma}{\rho}\right)\left(1 - \frac{\overline{\sigma}}{\rho}\right)} = \frac{\rho^2}{(\rho - \sigma)(\rho - \overline{\sigma})},$$

$$B = \frac{1}{\left(1 - \frac{\rho}{\sigma}\right)\left(1 - \frac{\overline{\sigma}}{\sigma}\right)} = \frac{\sigma^2}{(\sigma - \rho)(\sigma - \overline{\sigma})},$$

and

$$C = \frac{1}{\left(1 - \frac{\rho}{\overline{\sigma}}\right)\left(1 - \frac{\sigma}{\overline{\sigma}}\right)} = \frac{\overline{\sigma}^2}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}.$$

Consequently,

$$f(x) = \frac{\rho^2}{(\rho - \sigma)(\rho - \overline{\sigma})}\sum_{i=0}^{\infty}\rho^i x^i + \frac{\sigma^2}{(\sigma - \rho)(\sigma - \overline{\sigma})}\sum_{i=0}^{\infty}\sigma^i x^i + \frac{\overline{\sigma}^2}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}\sum_{i=0}^{\infty}\overline{\sigma}^i x^i$$

$$= \sum_{i=0}^{\infty}\left(\frac{\rho^{i+2}}{(\rho - \sigma)(\rho - \overline{\sigma})} + \frac{\sigma^{i+2}}{(\sigma - \rho)(\sigma - \overline{\sigma})} + \frac{\overline{\sigma}^{i+2}}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}\right)x^i.$$

Thus, Binet's formula for the Tribonacci sequence is

$$u_n = \frac{\rho^{n+2}}{(\rho - \sigma)(\rho - \overline{\sigma})} + \frac{\sigma^{n+2}}{(\sigma - \rho)(\sigma - \overline{\sigma})} + \frac{\overline{\sigma}^{n+2}}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}.$$

Multiplying the numerators and denominators of the last two terms by $(\rho - \overline{\sigma})$ and $(\rho - \sigma)$, respectively, yields

$$u_n = \frac{\rho^{n+2}}{|\rho - \sigma|^2} + \frac{(\rho - \overline{\sigma})\sigma^{n+2}}{-2iI(\sigma)|\rho - \sigma|^2} + \frac{(\rho - \sigma)\overline{\sigma}^{n+2}}{2iI(\sigma)|\rho - \sigma|^2}.$$

Using the relations $\sigma = r(\cos\theta + i\sin\theta)$,

$$\sigma^n = r^n(\cos n\,\theta + i\sin n\,\theta),\ \theta = \tan^{-1}(I(\sigma)/R(\sigma))$$

and combining terms:

$$u_n = \frac{\rho^2}{|\rho - \sigma|^2}\rho^n + \frac{r(r - 2\rho\cos\theta)}{|\rho - \sigma|^2} r^n \cos n\,\theta$$
$$+ \frac{r^2\cos\theta - \rho r(1 - 2\sin^2\theta)}{\sin\theta|\rho - \sigma|^2} r^n \sin n\,\theta.$$

Denoting the coefficients of ρ^n, $r^n\cos n\,\theta$, and $r^n\sin n\,\theta$ by α, β, and γ, respectively, yields

$$u_n = \alpha\rho^n + r^n(\beta\cos n\,\theta + \gamma\sin n\,\theta).$$

Approximate values for the constants are:

$$\rho = 1.8393, \qquad \theta = 124.69°, \qquad r = 0.7374,$$
$$\alpha = 0.6184, \qquad \beta = 0.3816, \qquad \gamma = 0.0374.$$

3. An Application

Since $|r| = .7374 < 1$, the nth Tribonacci number is the integer nearest $\alpha\rho^n$ when

$$|r^n(\beta\cos n\,\theta + \gamma\sin n\,\theta)| < \frac{1}{2}.$$

Using calculus, the value of $|\beta\cos n\,\theta + \gamma\sin n\,\theta|$ is at a maximum when

$$n\theta = 5.60° + k\pi, \text{ for } k \text{ an integer.}$$

Consequently,

$$|r^n(\beta\cos n\,\theta + \gamma\sin n\,\theta| < \frac{1}{2} \text{ for } n \geq 1.$$

Since $[\alpha + .5] = 1$ (where [ ] is the greatest integer function), a short form of the formula that is suitable for calculating the terms of the Tribonacci sequence is

$$u_n = [\alpha\rho^n + .5] \text{ for } n \geq 0.$$

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PYTHAGOREAN TRIPLES

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Define the sequence $\{w_n\}$ by

(1) $$w_0 = a,\ w_1 = b,\ w_{n+2} = w_{n+1} + w_n$$
$$(n \text{ integer} \geq 0;\ a,\ b \text{ real and both not zero}).$$

Then [2],

(2) $$(w_n w_{n+3})^2 + (2w_{n+1}w_{n+2})^2 = (w_{n+1}^2 + w_{n+2}^2)^2.$$

Freitag [1] asks us to find a c_n, if it exists, for which

(3) $$(F_n F_{n+3},\ 2F_{n+1}F_{n+2},\ c_n)$$

is a Pythagorean triple, where F_n is the nth Fibonacci number. It is easy to show that $c_n = F_{2n+3}$.

Earlier, Wulczyn [3] had shown that

(4) $$(L_n L_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$$

is a Pythagorean triple, where L_n is the nth Lucas number.

Clearly, (3) and (4) are special cases of (2) in which $a = 0$, $b = 1$, and $a = 2$, $b = 1$, respectively. One would like to know whether (3) and (4) provide the only solutions of (2) in which the third element of the triple is a *single* term. Our feeling is that they do.

Now

(5) $$w_n = aF_{n-1} + bF_n,$$

so

(6) $$w_{n+1}^2 + w_{n+2}^2 = \begin{cases} (a^2 + b^2)F_{2n+3} + (2ab - a^2)F_{2n+2} \\ (b^2 + 2ab)F_{2n+3} + (a^2 - 2ab)F_{2n+1} \end{cases}$$

(7) $$= \begin{cases} \left.\begin{array}{ll} b^2F_{2n+3} & \text{if } a = 0 \\ 5b^2F_{2n+3} & \text{if } a = 2b \end{array}\right\} \mathrm{I} \\ \left.\begin{array}{ll} a^2F_{2n+1} & \text{if } b = 0 \\ 5a^2F_{2n+1} & \text{if } b = -2a \end{array}\right\} \mathrm{II} \end{cases}$$

whence

(8) $$w_n = \begin{cases} bF_n & \text{or} & bL_n & \text{by I} \\ aF_{n-1} & \text{or} & -aL_{n-1} & \text{by II,} \end{cases}$$

results which may be verified in (5).

Therefore, only the Fibonacci and Lucas sequences, and (real) multiples of them, satisfy our requirement that the right-hand side of (2) reduce to a *single* term.

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COMPOSITION ARRAYS GENERATED BY FIBONACCI NUMBERS

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The number of compositions of an integer n in terms of ones and twos [1] is F_{n+1}, the $(n + 1)$st Fibonacci number, defined by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_{n+2} = F_{n+1} + F_n.$$

Further, the Fibonacci numbers can be used to generate such composition arrays [2], leading to the sequences $A = \{a_n\}$ and $B = \{b_n\}$, where (a_n, b_n) is a safe pair in Wythoff's game [3], [4], [6].

We generalize to the Tribonacci numbers T_n, where

$$T_0 = 0,\ T_1 = T_2 = 1, \text{ and } T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

The Tribonacci numbers give the number of compositions of n in terms of ones, twos, and threes [5], and when Tribonacci numbers are used to generate a composition array, we find that the sequences $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$ arise, where A_n, B_n, and C_n are the sequences studied in [7].

1. The Fibonacci Composition Array

To form the Fibonacci composition array, we use the difference of the subscripts of Fibonacci numbers to obtain a listing of the compositions of n in terms of ones and twos, by using F_{n+1} in the rightmost column, and taking the Fibonacci numbers as placeholders. We index each composition in the order in which it was written in the array by assigning each to a natural number taken in order and, further, assign the index k to set A if the kth composition has a one in the first position, and to set B if the kth composition has a two in the first position. We illustrate for $n = 6$, using F_7 to write the rightmost column. Notice that every other column in the table is the subscript difference of the two adjacent Fibonacci numbers, and compare with the compositions of 6 in terms of ones and twos.

FIBONACCI SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS													INDEX: A or B
F_1	1	F_2	1	F_3	1	F_4	1	F_5	1	F_6	1	F_7	$1 = a_1$
		F_1	2	F_3	1	F_4	1	F_5	1	F_6	1	F_7	$2 = b_1$
		F_1	1	F_2	2	F_4	1	F_5	1	F_6	1	F_7	$3 = a_2$
		F_1	1	F_2	1	F_3	2	F_5	1	F_6	1	F_7	$4 = a_3$
				F_1	2	F_3	2	F_5	1	F_6	1	F_7	$5 = b_2$
		F_1	1	F_2	1	F_3	1	F_4	2	F_6	1	F_7	$6 = a_4$
				F_1	2	F_3	1	F_4	2	F_6	1	F_7	$7 = b_3$
				F_1	1	F_2	2	F_4	2	F_6	1	F_7	$8 = a_5$
		F_1	1	F_2	1	F_3	1	F_4	1	F_5	2	F_7	$9 = a_6$
				F_1	2	F_3	1	F_4	1	F_5	2	F_7	$10 = b_4$
				F_1	1	F_2	2	F_4	1	F_5	2	F_7	$11 = a_7$
				F_1	1	F_2	1	F_3	2	F_5	2	F_7	$12 = a_8$
						F_1	2	F_3	2	F_5	2	F_7	$13 = b_5$

One first writes the column of 13 F_7's, which is broken into 8 F_6's and 5 F_5's. The 8 F_6's are broken into 5 F_5's and 3 F_4's, and the 5 F_5's are broken into 3 F_4's and 2 F_3's. The pattern continues in each column until each F_2 is broken into F_1 and F_0, so ending with F_1. In each new column, one always replaces F_nF_n's with $F_{n-1}F_{n-1}$'s and $F_{n-2}F_{n-2}$'s. Note that the next level, representing all integers through $F_8 = 21$, would be formed by writing 21 F_8's in the right column, and the present array as the top $13 = F_7$ rows, and the array ending in 8 F_6's now in the top $8 = F_6$ rows would appear in the bottom 8 rows. Notice further that this scheme puts a one on the right of all compositions of $(n - 1)$ and a two on the right of all compositions of $(n - 2)$.

Now, we examine sets A and B.

n:	1	2	3	4	5	6	7	8	9	10 ...
a_n:	1	3	4	6	8	9	11	12	14	16 ...
b_n:	2	5	7	10	13	15	18	20	23	26 ...

Notice that A is characterized as being the set of smallest integers not yet used, while it appears that $b_n = a_n + n$. Indeed, it appears that, for small values of n, a_n and b_n are the numbers arising as the safe pairs in the solution of Wythoff's game, where it is known that [2]

$$a_n = [n\alpha], \qquad b_n = [n\alpha^2], \tag{1.1}$$

where $[x]$ is the greatest integer in x and $\alpha = (1 + \sqrt{5})/2$. Further, we can characterize A and B by

(1.2) $$a_m = 1 + \alpha_3 F_3 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

$$b_m = 2 + \alpha_4 F_4 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\}.$$

Any integer n has a unique Fibonacci Zeckendorf representation

(1.3) $$n = \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k,$$

where $\alpha_i \in \{0, 1\}$ and $\alpha_i \alpha_{i-1} = 0$, or, a representation as a sum of distinct Fibonacci numbers where no two consecutive Fibonacci numbers may be used. Now suppose 1 is the smallest term in the Zeckendorf representation of n. Then n is in the required form for a_m. Suppose that the smallest Fibonacci number used is F_k, where k is even. Replace F_k by $F_{k-1} + F_{k-2}$, F_{k-2} by $F_{k-3} + F_{k-4}$, F_{k-4} by $F_{k-5} + F_{k-6}$, ..., until one reaches $F_4 = F_3 + F_2$, so that we have smallest term 1, and the required form for a_m.

Similarly, if 2 is the smallest term in the representation of n, then n is in the required form for b_m. If the subscript of the smallest Fibonacci number used is odd, then we can replace F_k by $F_{k-1} + F_{k-2}$, F_{k-2} by $F_{k-3} + F_{k-4}$, ..., just as before, until we reach $F_5 = F_4 + F_3$, equivalent to ending in a 2 for the form of b_m.

Thus A is the set of numbers whose Zeckendorf representation has an even-subscripted smallest term, while elements of B have odd-subscripted smallest terms. Since the Zeckendorf representation is unique, A and B are disjoint and cover the set of positive integers. Also, the unique Zeckendorf representation allows us to modify the form to that given for a_m and b_m uniquely, by rewriting only the smallest term.

Now, we can prove that A and B do indeed contain the safe-pair sequences from Wythoff's game.

Theorem 1.1: Form the composition array for n in terms of ones and twos, using F_{n+1} on the right border. Number the compositions in order appearing. Then, if 1 appears as the first number in the kth composition,

$$k = a_m = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

and if 2 appears as the first number in the kth composition,

$$k = b_m = 2 + \alpha_4 F_4 + \alpha_5 F_5 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

where (a_m, b_m) is a safe pair in Wythoff's game.

Proof: We have seen this for $n = 6$ and $k = 1, 2, \ldots, 13 = F_7$, and by using subarrays found there, we could illustrate $n = 1, 2, 3, 4$, and 5. By the construction of the array, we can build a proof by induction.

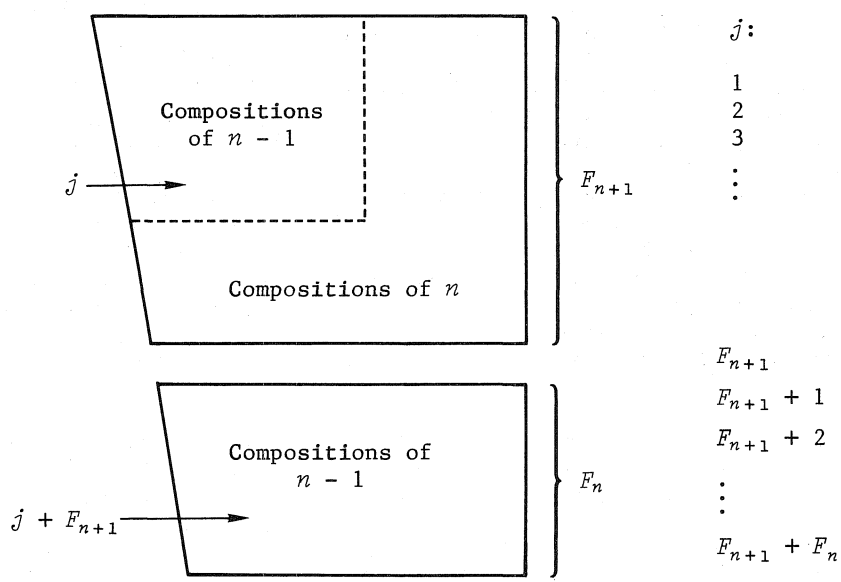
Assume we have the compositions of n using ones and twos made by our construction, using F_{n+1} in the rightmost column. We put the F_n compositions of $(n - 1)$ below. (See figure on page 125.)

Take $1 \le j \le F_n$. If $j \in A$, then $j + F_{n+1} \in A$ as the compositions starting with 1 go into A and those starting with 2 go into B, and addition of F_{n+1} will not affect earlier terms used. Note well that no matter how large the value of n becomes, the earlier compositions always start with the same number 1 or 2 as they did for the smaller value of n, within the range of the construction. Now, if j is of the form

$$j = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_n F_n, \; j \in A,$$

$$j + F_{n+1} = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_n F_n + F_{n+1},$$

and $(j + F_{n+1}) \in A$, $\alpha_i \in \{0, 1\}$. Since we know that all the integers from 1 to $F_{n+2} - 2$ can be represented with the Fibonacci numbers 1, 2, 3, ..., F_n, for the numbers through F_{n+1} we need only $F_2, F_3, \ldots, F_n$. Thus the numbers 1, 2, ..., F_{n+2} can be represented using 1, 2, ..., F_{n+1}, and we continue to build the sets A and B, having both completeness and uniqueness, recalling [1] that the number of compositions of n into ones and twos is F_{n+1}. Also, notice that there are F_{n-2} elements of B in the first F_n integers and F_{n-1} elements of A in the first F_n integers.



$1 \le j \le F_n$, $n \ge 2$.

2. The Tribonacci Composition Array

Normally, the Tribonacci numbers give rise to three sets A, B, C [8]:

$$\text{(2.1)} \quad \begin{aligned} A &= \{A_n : A_n = 1 + \alpha_3 T_3 + \alpha_4 T_4 + \cdots\}, \\ B &= \{B_n : B_n = 2 + \alpha_4 T_4 + \alpha_5 T_5 + \cdots\}, \\ C &= \{C_n : C_n = 4 + \alpha_5 T_5 + \alpha_6 T_6 + \cdots\}, \end{aligned}$$

where $\alpha_i \in \{0, 1\}$. Equivalently, see [7], if T_k is the smallest term appearing in the unique Zeckendorf representation of an integer N, then

$$N \in A \text{ if } k \equiv 2 \bmod 3, \; N \in B \text{ if } k \equiv 3 \bmod 3, \text{ and } N \in C \text{ if } k \equiv 1 \bmod 3, \; k > 3,$$

where we have suppressed $T_1 = 1$, but $T_2 = 1 = A_1$, and every positive integer belongs to A, B, or C, where A, B, and C are disjoint.

Also, recall that the compositions of a positive integer n using 1's, 2's, and 3's gives rise to the Tribonacci numbers, since T_{n+1} gives the number of such compositions [5].

Now, proceeding as in the Fibonacci case, we write a Tribonacci composition array. We illustrate for $T_6 = 13$ in the rightmost column, which is the number of compositions of 5 into 1's, 2's, and 3's. We put the index of those compositions which start on the left with a one into set A, those with a two into set B, and those with a three into set C, and compare with sets A, B, and C given in (2.1).

TRIBONACCI SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS											INDEX: A, B, or C
T_1	1	T_2	1	T_3	1	T_4	1	T_5	1	T_6	$1 = A_1$
		T_1	2	T_3	1	T_4	1	T_5	1	T_6	$2 = B_1$
		T_1	1	T_2	2	T_4	1	T_5	1	T_6	$3 = A_2$
				T_1	3	T_4	1	T_5	1	T_6	$4 = C_1$
		T_1	1	T_2	1	T_3	2	T_5	1	T_6	$5 = A_3$
				T_1	2	T_3	2	T_5	1	T_6	$6 = B_2$
				T_1	1	T_2	3	T_5	1	T_6	$7 = A_4$
		T_1	1	T_2	1	T_3	1	T_4	2	T_6	$8 = A_5$
				T_1	2	T_3	1	T_4	2	T_6	$9 = B_3$
				T_1	1	T_2	2	T_4	2	T_6	$10 = A_6$
						T_1	3	T_4	2	T_6	$11 = C_2$
				T_1	1	T_2	1	T_3	3	T_6	$12 = A_7$
						T_1	2	T_3	3	T_6	$13 = B_4$

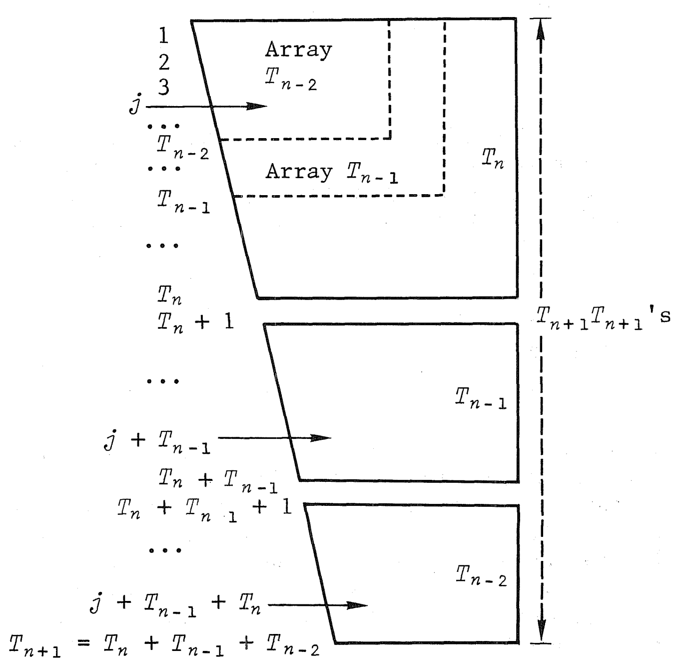
We note that thus far the splitting into sets agrees with these rules: A_n is the first positive integer not yet used; B_n is $2A_n$ decreased by the number of C_i's less than A_n; and C_n is $2B_n$ decreased by the number of C_i's less than B_n; where A_n, B_n, and C_n are the elements of sets A, B, and C of (2.1).

n:	1	2	3	4	5	6	7
A_n:	1	3	5	7	8	10	12
B_n:	2	6	9	13	15	19	22
C_n:	4	11	17	24	28	35	41

We next prove that this constructive array yields the same sets A, B, and C as characterized by (2.1) by mathematical induction.

We first study the array we have written, yielding the $T_6 = 13$ compositions of $n = 5$ using 1's, 2's, and 3's. We write 13 T_6's in the rightmost column. Then, write the preceding column on the left by dividing 13 T_6's into 7 T_5's, 4 T_4's, and 2 T_3's. In successive columns, replace the 7 T_5's by 4 T_4's, 2 T_3's, and 1 T_2, and the 4 T_4's by 2 T_3's, 1 T_2, and 1 T_1, then the 2 T_3's by 1 T_2 and 1 T_1. Any row that reaches T_1 stops. Continue until all the rows have reached T_1. Notice that the top left corner of the array, bordered by 7 T_5's on the right, is the array for the $T_5 = 7$ compositions of $n - 1 = 4$, and that the middle group bordered on the right by 4 T_4's is the $4 = T_4$ compositions of $n - 2 = 3$, and the bottom group bordered by 2 T_3's on the right is the $2 = T_3$ compositions of $n - 3 = 2$. The successive subscript differences give the compositions of n using 1's, 2's, and 3's.

If we write $T_{n+1}T_{n+1}$'s in the right-hand column, then we will have in the preceding column the arrays formed from T_nT_n's on the right, $T_{n-1}T_{n-1}$'s on the right, and $T_{n-2}T_{n-2}$'s on the right. All the integers from 1 through T_{n+1} will appear as indices because there are T_{n+1} compositions of n into 1's, 2's, and 3's. The subscript differences will give the compositions of n into 1's, 2's, and 3's, and we can make a correspondence between the natural numbers, the compositions of n, and the representative form of the appropriate set. Those T_{n+1} compositions are ordered with indices from the natural numbers. Each composition whose leftmost digit is one is cast into set A; those whose leftmost digit is two are cast into set B; and those whose leftmost digit is three are cast into set C. Descending the list, we then call the first A, A_1, the second A, A_2, ..., the first B, B_1, the second B, B_2, and so on. We have now listed the elements of A, B, and C in natural order. Since the representations of A_n, B_n, and C_n from (2.1) are unique, see [7], and since this expansion is constructively derived from the Zeckendorf representation so that the largest term used remains intact (by the lexicographic ordering theorem [7]), every integer $m < T_n$ uses only $T_2, T_3, \ldots, T_{n-1}$ in the representation, and T_n can itself be written such that the largest term used is T_{n-1}. Let j by any integer, $1 \leq j \leq T_n$. Assume that j can be expressed as in (2.1). Then $j' = j + T_{n+1}$ will be in the same set as j, since all early terms of j and j' will be the same. Further, if the leftmost digit of the jth composition is a, where $1 \leq j \leq T_n$, then the leftmost digit of the j'th composition, $j' = j + T_{n+1}$ will be a, since the leftmost digits are not changed in construction of the array.



Recall that set elements are not characterized by the composition, but only by its leading 1, 2, or 3. Each number in the jth position in the original gives rise to one in the $(j + T_{n+1})$st position in the same set A, B, or C. Also, $j + T_n + T_{n+1}$ belongs to the same set as j.

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THE CONGRUENCE $x^n \equiv a \pmod{m}$, WHERE $(n, \phi(m)) = 1$

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(Submitted January 1980)

Craig M. Cordes [2] and Charles Small [4] proved Theorem 1, a result that W. Sierpinski [3] proved, using elementary group theoretic considerations, for n being a prime, and J. H. E. Cohn [1, Theorem 7] proved for $n = m$. Moreover, Theorem 1 is implicit in some of the solutions to Problem E2446 in the *American Mathematics Monthly* (January 1975).

Throughout this paper, m and n will denote positive integers with $m > 1$.

__Theorem 1__: Let n be greater than 1. The congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a if and only if $(n, \phi(m)) = 1$ and m is a product of distinct primes.

Let $a_1, a_2, \ldots, a_m$ be a complete residue system modulo m. It follows from Theorem 1 that $a_1^n, a_2^n, \ldots, a_m^n$, where $n > 1$, is a complete residue system modulo m if and only if $(n, \phi(m)) = 1$ and m is a product of distinct primes.

We shall give a simple proof of Theorem 1 and, in addition, prove the following two related results.

__Theorem 2__: The following three conditions are equivalent.

I. The congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a with
$$\left(a, \frac{m}{(a, m)}\right) = 1.$$

II. The congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a relatively prime to m.

III. $(n, \phi(m)) = 1$.

From Theorem 2, it follows that for $a_1, a_2, \ldots, a_{\phi(m)}$ a reduced residue system modulo m, $a_1^n, a_2^n, \ldots, a_{\phi(m)}^n$ is a reduced residue system modulo m if and only if $(n, \phi(m)) = 1$.

The following result tightens the equivalence of Theorem 2.

__Theorem 3__: Conditions I and II are equivalent.

I. The congruence $x^n \equiv a \pmod{m}$ has a solution if and only if
$$\left(a, \frac{m}{(a, m)}\right) = 1.$$

II. $(n, \phi(m)) = 1$ and $p^{n+1} \nmid m$ for all primes p.

By Theorem 3, we can, with only the simplest of calculations, write down the nth-power residues modulo m if $(n, \phi(m)) = 1$ and $p^{n+1} \nmid m$ for all primes p.

We shall now state and prove several results needed for the proofs of these three theorems.

Lemma 4: Let a and n be positive integers. If $\left(\ , \frac{m}{(a, m)}\right) = 1$, then there is a positive integer t such that

$$a^{nt} \equiv a^{(n, \phi(m))} \pmod{m}.$$

Proof: Assume $\left(a, \frac{m}{(a, m)}\right) = 1$ and, for convenience, let $d = (a, m)$. Since

$$\left(a, \frac{m}{d}\right) = 1 \quad \text{and} \quad \phi\left(\frac{m}{d}\right) \mid \phi(m),$$

by the Euler-Fermat theorem,

$$a^{\phi(m)} \equiv 1 \left(\bmod \frac{m}{d}\right).$$

There are positive integers c and t such that $nt - (n, \phi(m)) = \phi(m)c$. Thus

$$a^{nt-(n, \phi(m))} \equiv a^{\phi(m)c} \equiv 1 \left(\bmod \frac{m}{d}\right).$$

Hence

$$a^{nt} \equiv a^{(n, \phi(m))} \pmod{m}.$$

Corollary 5: If $(n, \phi(m)) = 1$, then the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a with $\left(a, \frac{m}{(a, m)}\right) = 1$.

Corollary 6: If $(n, \phi(m)) = 1$ and m is a product of distinct primes, then the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a.

Corollary 6 follows directly from Lemma 4 since m being a product of distinct primes implies $\left(a, \frac{m}{(a, m)}\right) = 1$ for every integer a.

Lemma 7: If the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a relatively prime to m, then $(n, \phi(m)) = 1$.

Proof: Assume $(n, \phi(m)) \neq 1$. Thus, there is a prime q such that $q \mid n$ and $q \mid \phi(p^e)$, where $p^e \| m$ and p is a prime. We shall show that the assumption $p = 2$ leads to a contradiction and that the assumption $p > 2$ also leads to a contradiction.

First, assume $p = 2$. Thus, q divides $\phi(2^e) = 2^{e-1}$ so $q = 2$ and $e \geq 2$. Choose a such that $a \equiv 3 \pmod{2^e}$ and $a \equiv 1 \pmod{m/2^e}$. Thus $(a, m) = 1$; so, by assumption, the congruence $x^n \equiv a \pmod{m}$ has a solution. Since $4 \mid 2^e$ and $2^e \mid m$, we have $4 \mid m$. Hence, the congruence $x^n \equiv a \equiv 3 \pmod{4}$ has a solution. But $x^n \equiv 3 \pmod{4}$ is impossible, since n is divisible by $q = 2$.

Now assume $p > 2$. Choose a such that a is a primitive root modulo p^e and $a \equiv 1 \pmod{m/p^e}$. Thus $(a, m) = 1$, so there is an integer x such that $x^n \equiv a \pmod{m}$. Since $p^e \mid m$, $x^n \equiv a \pmod{p^e}$. For $k = \phi(p^e)/q$, $a^k \equiv x^{nk} \equiv 1 \pmod{p^e}$.

The last congruence is true because $\phi(p^e) = qk$, which divides nk. But $a^k \equiv 1 \pmod{p^e}$ is impossible, since a is a primitive root modulo p^e and

$$0 < k < \phi(p^e).$$

We shall now prove Theorem 1. First, assume that the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a. Thus $0, 1, 2, \ldots, (m - 1)$ must be incongruent modulo m. Now if there is a prime p such that $p^2 | m$ then, since $n > 1$, we would have the contradiction

$$0^n \equiv 0 \equiv \left(\frac{m}{p}\right)^n \pmod{m}.$$

Therefore, m must be a product of distinct primes. By Lemma 7, we have that $(n, \phi(m)) = 1$.

Conversely, assume $(n, \phi(m)) = 1$ and m is a product of distinct primes. By Corollary 6, the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a.

We shall now prove Theorem 2. Since $(a, m) = 1$ implies $\left(a, \dfrac{m}{(a, m)}\right) = 1$, II follows from I. The remaining implications—II implies III and III implies I—follow from Lemma 7 and Corollary 5, respectively.

To prove Theorem 3, we need

Lemma 8: Let a be an integer. If $p^{n+1} \nmid m$ for all primes p and the congruence $x^n \equiv a \pmod{m}$ has a solution, then $\left(a, \dfrac{m}{(a, m)}\right) = 1$.

Proof: Assume the congruence $x^n \equiv a \pmod{m}$ has a solution and there is a prime p such that $p | a$ and $p \,\Big|\, \dfrac{m}{(a, m)}$. Choose e such that $p^e \| m$; clearly $e \leq n$. Since $p | a$ and $p | m$, $p | x^n$; so $p^e | x^n$. From $p^e | m$ and $p^e | x^n$, we have that $p^e | a$, so $p^e | (a, m)$. But since $p \,\Big|\, \dfrac{m}{(a, m)}$, too, we have the contradiction $p^{e+1} | m$.

Finally, we prove Theorem 3. First, assume condition I. Thus, in particular, the congruence $x^n \equiv a \pmod{m}$ has a solution for every integer a relatively prime to m. Hence, by Lemma 7, $(n, \phi(m)) = 1$. To prove that $p^{n+1} \nmid m$ for all primes p, assume there is a prime p such that $p^{n+1} | m$. Thus

$$\left(p^n, \frac{m}{(p^n, m)}\right) = \left(p^n, \frac{m}{p^n}\right) \geq p > 1.$$

Therefore, by condition I, the congruence $x^n \equiv p^n \pmod{m}$ has no solution. But clearly $x = p$ is a solution to the congruence $x^n \equiv p^n \pmod{m}$.

The fact that condition II implies condition I follows from Lemma 8 and Corollary 5.

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ON THE ENUMERATION OF CERTAIN COMPOSITIONS AND RELATED SEQUENCES OF NUMBERS

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Abstract

The numbers

$$A(m,\ k,\ s,\ r) = [\nabla^{m+1}\mathsf{E}^k\underline{(sx + r)_m}]_{x=0},$$

where $\nabla = 1 - \mathsf{E}^{-1}$, $\mathsf{E}^j f(x) = f(x + j)$, $\underline{u_x} = u_x$ when $0 \leq x \leq k$ and $\underline{u_x} = 0$ otherwise, $(y)_m = y(y - 1)\ \ldots\ (y - m + 1)$, are the subject of this paper. Recurrence relations, generating functions, and certain other properties of these numbers are obtained. They have many similarities with the Eulerian numbers

$$A_{m,k} = \frac{1}{m!}[\nabla^{m+1}\mathsf{E}^k\underline{x^m}]_{x=0},$$

and give in particular (i) the number $C_{m,n,s}$ of compositions of n with exactly m parts, no one of which is greater than s, (ii) the number $Q_{s,m}(k)$ of sets $\{i_1,\ i_2,\ \ldots,\ i_m\}$ with $i_n \in \{1,\ 2,\ \ldots,\ s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements, and (iii) the number $Q_{s,m}(r,\ k)$ of those sets which have $i_1 = r$. Also, they are related to the numbers

$$G(m,\ n,\ s,\ r) = \frac{1}{n!}[\Delta^n(sx + r)_m]_{x=0},\quad \Delta = \mathsf{E} - 1,$$

used by Gould and Hopper [11] as coefficients in a generalization of the Hermite polynomials, and to the Euler numbers and the tangent-coefficients T_m. Moreover, $\lim\limits_{s\to\pm\infty} s^{-m}m!A(m,\ k,\ s,\ su) = A_{m,k,u}$, where

$$A_{m,k,u} = \frac{1}{m!}[\nabla^{m+1}\mathsf{E}^k\underline{(x + u)^m}]_{x=0}$$

is the Dwyer [8, 9] cumulative numbers; in particular,

$$\lim_{s\to\pm\infty} s^{-m}m!A(m,\ k,\ s) = A_{m,k},\quad A(m,\ k,\ s) \equiv A(m,\ k,\ s,\ 0).$$

Finally, some applications in statistics are briefly discussed.

1. Introduction

A partition of a positive integer n is a collection of positive integers, without regard to order, whose sum is equal to n. The corresponding ordered collections are called "compositions" of n. The integers collected to form a partition (or composition) are called its "parts" (cf. MacMahon [14, Vol. I, p. 150] and Riordan [16, p. 124]). The compositions with exactly m parts, no one of which is greater than s, have generating function

$$C_{m,s}(t) = \sum C_{m,n,s}t^n = t^m(1-t)^{-m}(1-t^s)^m,$$

and therefore the number $C_{m,n,s}$ of compositions of n with exactly m parts, no one of which is greater than s, is given by the sum

$$C_{m,n,s} = \sum_{j=1}^{k}(-1)^j\binom{m}{j}\binom{n-1-sj}{m-1}, \tag{1.1}$$

where $k = [(n-m)/s]$, the integral part of $(n-m)/s$.

Compositions of this type arose in the following Montmort-Moivre problem (cf. Jordan [12, p. 140] and [13, p. 449]): Consider m urns each with s balls bearing the numbers 1, 2, ..., s. Suppose that one ball is drawn from each urn and let

$$Z = \sum_{i=1}^{m} X_i$$

be the sum of the selected numbers. Then the probability $p(n;\, m,\, s)$ that Z is equal to n is given by

$$p(n;\, m,\, s) = s^{-m}C_{m,n,s},\quad n = m,\, m+1,\, \ldots,\, sm. \tag{1.2}$$

Carlitz, Roselle, and Scoville [4] proved that the number $Q_{s,m}(k)$, of sets $\{i_1, i_2, \ldots, i_m\}$ with $i_n \in \{1, 2, \ldots, s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements, is given by

$$Q_{s,m}(k) = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+m-1}{m}, \tag{1.3}$$

and the number $Q_{s,m}(r,\, k)$ of those sets which have $i_1 = r$ is given by

$$Q_{s,m}(r,\, k) = \sum_{j=0}^{k-1}(-1)^j\binom{m}{j}\binom{s(k-j-1)+r+m-2}{m-1}. \tag{1.4}$$

The next problem is from applied statistics: Dwyer [8,9] studied the problem of computing the ordinary moments of a frequency distribution with the use of the cumulative totals and certain sequences of numbers. These numbers are the coefficients $A_{m,k,r}$ of the expansion of $(x+r)^m$ into a series of factorials $(x+k)_m$, $k = 0, 1, 2, \ldots, m$; that is,

$$(x+r)^m = \sum_{k=0}^{m} A_{m,k,r}(x+m-k)_m/m!$$

Using the notation $\underline{u}_x = u_x$ with $0 \le x \le k$ and $\underline{u}_x = 0$ otherwise, he proved that

$$(1.5)\qquad A_{m,k,r} = [\nabla^{m+1}E^k\underline{(x+r)^m}]_{x=0} = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}(k-j+r)^m.$$

These numbers for $r = 0$ reduce to the Eulerian numbers

$$(1.6)\qquad A_{m,k} = [\nabla^{m+1}E^k\underline{x^m}]_{x=0} = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}(k-j)^m.$$

In the present paper, starting from the problem of computing the factorial moments of a frequency distribution with the use of cumulative totals, we introduce the numbers

$$(1.7)\qquad A(m,\ k,\ s,\ r) = \frac{1}{m!}[\nabla^{m+1}E^k\underline{(sx+r)}_m]_{x=0}$$

so that

$$(1.8)\qquad (sx+r)_m = \sum_{k=0}^{m} A(m,\ k,\ s,\ r)(x+m-k)_m.$$

These numbers have many similarities with the Eulerian numbers (cf. Carlitz [1]). They are related to the numbers $C_{m,n,s}$ of the title of this paper by

$$C_{m,n,s} = A(m-1,\ k-1,\ s,\ r+m-1) = (-1)^{m-1}A(m-1,\ k-1,\ -s,\ -r-1),$$
$$k = [(n-m)/s],$$
$$r = (n-m) - s[(n-m)/s],$$

and to the numbers $Q_{s,m}(r,\ k)$ by

$$Q_{s,m}(r,\ k) = A(m-1,\ k-1,\ s,\ r+m-2) = (-1)^{m-1}A(m-1,\ k-1,\ -s,\ -r),$$

and their properties are discussed in Sections 2 and 3 below. Since

$$Q_{s,m}(k) = Q_{s,m+1}(s,\ k),$$

it follows that

$$Q_{s,m}(k) = A(m,\ k-1,\ s,\ s+m-1) = (-1)^m A(m,\ k,\ -s),$$

where $A(m,\ k,\ s) \equiv A(m,\ k,\ s,\ 0)$. Section 4 is devoted to the discussion of certain statistical applications of the numbers $A(m,\ k,\ s,\ r)$.

2. The Composition Numbers $A(m,\ k,\ s,\ r)$

Let $(x)_{m,b} = x(x-b)\ldots(x-mb+b)$ denote the generalized falling factorial of degree m with increment b; the usual falling factorial of degree m will be denoted by $(x)_m = (x)_{m,1}$. The problem of expressing the generalized factorial $(x)_{m,b}$ in terms of the generalized factorials $(x+ka)_{m,a}$, $k = 0,\ 1,\ 2,\ \ldots,\ m$ of the same degree arises in statistics in connection with the problem of expressing the generalized factorial moments in terms of the cumulations (see Dwyer [8, 9] and Section 4 below). More generally, let

$$(2.1)\qquad (x+rb)_{m,b} = \sum_{k=0}^{m} C_{m,k,r}(a,\ b)(x+(m-k)a)_{m,a}.$$

Following Dwyer, define $\underline{u_x} = u_x$ when $0 \le x \le k$ and $\underline{u_x} = 0$ otherwise. Moreover, let E_a denote the displacement operator defined by $\mathrm{E}_a f(x) = f(x + a)$ and $\nabla_a = 1 - \mathrm{E}_a^{-1}$, the receding difference operator; when $a = 1$, we write $\mathrm{E}_1 \equiv \mathrm{E}$ and $\nabla_1 \equiv \nabla$. Then, from (2.1), we have

$$(2.2) \qquad \underline{(x + rb)_{m,b}} = \sum_{k=0}^{m} C_{m,k,r}(a, b)\underline{(x + (m - k)a)_{m,a}}.$$

Since

$$[\nabla^{m+1}\mathrm{E}_a^n \underline{(x + (m - k)a)_{m,a}}]_{x=0} = \begin{cases} a^m m!, & k = n \\ 0, & 0 \le k < n \text{ or } n < k \le m, \end{cases}$$

we get, from (2.2)

$$C_{m,k,r}(a, b) = \frac{a^{-m}}{m!}[\nabla_a^{m+1}\mathrm{E}_a^k \underline{(x + rb)_{m,b}}]_{x=0}.$$

These coefficients may be expressed in terms of the operators ∇ and E and the usual falling factorials by using the relations

$$\nabla_a^{m+1}\mathrm{E}_a^k f(x) = \nabla^{m+1}\mathrm{E}^k f(ax), \quad (ax + rb)_{m,b} = b^m(sx + r)_m, \quad s = a/b.$$

We find

$$C_{m,k,r}(a, b) = s^{-m}A(m, k, s, r), \quad s = a/b,$$

where

$$(2.3) \qquad A(m, k, s, r) = \frac{1}{m!}[\nabla^{m+1}\mathrm{E}^k \underline{(sx + r)_m}]_{x=0}, \quad k = 0, 1, \ldots, m, \quad m = 0, 1, 2, \ldots .$$

Hence

$$(2.4) \qquad (x + rb)_{m,b} = \sum_{k=0}^{m} s^{-m}A(m, k, s, r)(x + (m - k)a)_{m,a}, \quad s = a/b$$

or

$$(2.5) \qquad (ax + r)_m = \sum_{k=0}^{m} A(m, k, s, r)(bx + m - k)_m.$$

Using the symbolic formula

$$\nabla^{m+1} = \sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\mathrm{E}^{-j},$$

we get, for the numbers (2.3), the explicit expression

$$(2.6) \qquad A(m, k, s, r) = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m}.$$

It is easily seen that

$$(2.7) \qquad A(m, k, -s, -r) = (-1)^m A(m, k, s, r + m - 1),$$

and, also, that the numbers $A(m, k, s, r)$ are integers when s and r are integers. Moreover, $A(m, k, s, r) = 0$ when $k > m$.

Remarks 2.1: As we have noted in the introduction, the number $C_{m,n,s}$ of compositions of n with exactly m parts, none of which is greater than s is given by

$$C_{m,n,s} = \sum_{j=0}^{k} (-1)^j \binom{m}{j}\binom{n - sj - 1}{m - 1}, \quad k = [(n - m)/s]. \tag{2.8}$$

Comparing (2.8) with (2.6) and using (2.7), we get the relation

$$\begin{aligned} C_{m,n,s} &= A(m - 1, k, s, r + m - 1) \\ &= (-1)^{m-1}A(m - 1, k, -s, -r - 1), \quad r = (n - m) - s[(n - m)/s], \end{aligned} \tag{2.9}$$

which justifies the title of this section.

Since the number $Q_{s,m}(r, k)$, of sets $\{i_1, i_2, \ldots, i_m\}$ with $i_n \in \{1, 2, \ldots, s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements which have $i_1 = r$, is given by (see [4])

$$Q_{s,m}(r, k) = \sum_{j=0}^{k-1} (-1)^j \binom{m}{j}\binom{s(k - 1 - j) + r + m - 2}{m - 1}, \tag{2.10}$$

we get, by virtue of (2.6), the relation

$$\begin{aligned} Q_{s,m}(r, k) &= A(m - 1, k - 1, s, r + m - 2) \\ &= (-1)^{m-1}A(m - 1, k - 1, -s, -r). \end{aligned} \tag{2.11}$$

These numbers give in particular the numbers

$$Q_{s,m}(k) = \sum_{j=0}^{k} (-1)^j \binom{m + 1}{j}\binom{s(k - j) + m - 1}{m} \tag{2.12}$$

of the above sets without the restriction $i_1 = r$. We have

$$Q_{s,m}(k) = Q_{s,m+1}(s, k),$$

and hence

$$Q_{s,m}(k) = A(m, k - 1, s, s + m - 1) = (-1)^m A(m, k, -s), \tag{2.13}$$

where

$$A(m, k, s) = \frac{1}{m!}[\nabla^{m+1}E^k(sx)_m]_{x=0} = \sum_{j=0}^{k} (-1)^j \binom{m + 1}{j}\binom{s(k - j)}{m}. \tag{2.14}$$

Since

$$\sum_{j=0}^{m+1} (-1)^j \binom{m + 1}{j}\binom{s(k - j) + r}{m} = \left[\Delta^{m+1}\binom{-sx + r}{m}\right]_{x=-k} = 0,$$

it follows that

$$\sum_{j=0}^{k} (-1)^j \binom{m + 1}{j}\binom{s(k - j) + r}{m} = \sum_{j=k+1}^{m+1} (-1)^{j+1}\binom{m + 1}{j}\binom{s(k - j) + r}{m}$$

and (2.6) may be rewritten as follows:

$$\begin{aligned}A(m, k, s, r) &= \sum_{j=k+1}^{m+1}(-1)^{j+1}\binom{m+1}{j}\binom{s(k-j)+r}{m}\\ &= \sum_{i=0}^{m-k}(-1)^{m+i}\binom{m+1}{i}\binom{-s(m-k+1-i)+r}{m}\\ &= \sum_{i=0}^{m-k+1}(-1)^{i}\binom{m+1}{i}\binom{s(m-k+1-i)+m-r-1}{m}\\ &\qquad + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m}.\end{aligned}$$

Hence

$$\begin{aligned}A(m, k, s, r) &= A(m, m-k+1, s, m-r-1) + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m}\\ &= (-1)^m A(m, m-k+1, -s, r) + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m}.\end{aligned} \tag{2.15}$$

In particular

$$A(m, k, s) = (-1)^m A(m, m-k+1, -s), \tag{2.16}$$

which should be compared with the symmetric property of the Eulerian numbers $A_{m,k} = A_{m,m-k+1}$.

Using the relation

$$\binom{m+2}{j}(s(k-j)+r-m) = (sk-m+r)\binom{m+1}{j} - (s(m-k+2)+m-r)\binom{m+1}{j-1},$$

we get, from (2.6), the recurrence relation

$$\begin{aligned}&(m+1)A(m+1, k, s, r)\\ &\quad= (sk-m+r)A(m, k, s, r) + (s(m-k+1)+m-r)A(m, k-1, s, r)\end{aligned} \tag{2.17}$$

with initial conditions

$$A(0, 0, s, r) = 1,\ A(m, 0, s, r) = \binom{r}{m},\ m > 0.$$

From (2.5), we have

$$(s(k-j)+r)_m = \sum_{i=0}^{m} A(m, m-i, k-j, r)(s+i)_m.$$

Hence (2.6) may be rewritten as

$$\begin{aligned}A(m, k, s, r) &= \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m}\\ &= \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\sum_{i=0}^{m}\binom{s+i}{m}A(m, m-i, k-j, r)\\ &= \sum_{i=0}^{m}\binom{s+i}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}A(m, m-i, k-j, r),\end{aligned}$$

and putting

$$(2.18)\qquad B(m,\ n,\ k,\ r) = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} A(m,\ n,\ k-j,\ r)$$

we get

$$(2.19)\qquad A(m,\ k,\ s,\ r) = \sum_{i=0}^{m} \binom{s+i}{m} B(m,\ m-i,\ k,\ r),$$

and

$$(2.20)\qquad B(m,\ n,\ k,\ r) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)+r}{m}.$$

It is clear from (2.20) that

$$(2.21)\qquad B(m,\ n,\ k,\ r) = B(m,\ k,\ n,\ r).$$

Since

$$\begin{aligned}
&\binom{m+2}{i}\binom{m+2}{j}((n-i)(k-j)+r-m) \\
&= (nk-m+r)\binom{m+1}{i}\binom{m+1}{j} - (n(m-k+2)+m-r)\binom{m+1}{i}\binom{m+1}{j-1} \\
&\quad - (k(m-n+2)+m-r)\binom{m+1}{i-1}\binom{m+1}{j} \\
&\quad + ((m-n+2)(m-k+2)-m+r)\binom{m+1}{i-1}\binom{m+1}{j-1},
\end{aligned}$$

it follows, from (2.20), that

$$\begin{aligned}
(2.22)\quad &(m+1)B(m+1,\ n,\ k,\ r) \\
&= (nk-m+r)B(m,\ n,\ k,\ r) + (n(m-k+2)+m-r)B(m,\ n,\ k-1,\ r) \\
&\quad + (k(m-n+2)+m-r)B(m,\ n-1,\ k,\ r) \\
&\quad + ((m-n+2)(m-k+2)-m+r)B(m,\ n-1,\ k-1,\ r).
\end{aligned}$$

with

$$B(0,\ 0,\ 0,\ r) = 1,\ B(0,\ 0,\ k,\ r) = B(0,\ k,\ 0,\ r) = 0.$$

Remark 2.2: Comparing (2.20) with the formula

$$R_m(n,\ k) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)+m-1}{m}$$

giving the number of permutations on m letters which have n jumps and require k readings (cf. [4]), we find

$$\begin{aligned}
(2.23)\qquad R_m(n,\ k) &= B(m,\ n,\ k,\ m-1) = B(m,\ m-n+1,\ k) \\
&= B(m,\ n,\ m-k+1),
\end{aligned}$$

where

$$(2.24)\quad B(m,\ n,\ k) \equiv B(m,\ n,\ k,\ 0) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)}{m}.$$

Using the relation

$$\sum_{j=0}^{k}\sum_{i=0}^{n}(-1)^{i+j}\binom{m+1}{i}\binom{m+1}{j}\binom{(n-i)(k-j)}{m}$$
$$=\sum_{j=k+1}^{m+1}\sum_{i=n+1}^{m+1}(-1)^{i+j}\binom{m+1}{i}\binom{m+1}{j}\binom{(n-i)(k-j)}{m},$$

it can be easily shown that

$$B(m,\ n,\ k)=B(m,\ m-n+1,\ m-k+1). \tag{2.25}$$

Expanding the generalized factorial $(x+rb)_{m,b}$ in terms of the generalized factorials $(x+ka)_{m,a}$, $k = 0, 1, 2, \ldots, m$ and then these factorials in terms of the factorials $(x+jb)_{m,b}$, $j = 0, 1, 2, \ldots, m$, by using (2.4), we get

$$\begin{aligned}(x+rb)_{m,b} &= \sum_{k=0}^{m} a^{-m}b^{m}A(m,\ m-k,\ a/b,\ r)(x+ka)_{m,a}\\ &= \sum_{k=0}^{m}\sum_{j=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ k)(x+jb)_{m,b}\\ &= \sum_{j=0}^{m}\sum_{k=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ r)(x+jb)_{m,b},\end{aligned}$$

which implies

$$\sum_{k=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ r) = \delta_{rj}, \tag{2.26}$$

with δ_{rj} the Kronecker delta: $\delta_{rr} = 1$, $\delta_{rj} = 0$, $j \neq r$. Hence, we have the pair of inverse relations

$$\alpha_r = \sum A(m,\ m-k,\ a/b,\ r)\beta_k,\quad \beta_k = \sum A(m,\ m-k,\ b/a,\ r)\alpha_r. \tag{2.27}$$

3. Generating Functions and Connection with Other Sequences of Numbers

Consider first the generating function

$$A_{m,s,r}(t) = \sum A(m,\ k,\ s,\ r)t^k, \tag{3.1}$$

where the summation is over all possible values of k which are 0 to m and can be left indefinite because $A(m,\ k,\ s,\ r)$ is zero elsewhere. Then, from (2.6), it follows that

$$A_{m,s,r}(t) = (1-t)^{m+1}\sum_{k=0}^{\infty}\binom{sk+r}{m}t^k. \tag{3.2}$$

In a generalization of the Hermite polynomials, Gould and Hopper [11] used as coefficients the numbers

$$G(m,\ n,\ s,\ r) = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(sj+r)_m, \tag{3.3}$$

which may be equivalently defined by

$$G(m,\ n,\ s,\ r) = \frac{1}{n!}[\Delta^n (sx + r)_m]_{x=0}.$$

Using the symbolic formula

$$\mathrm{E}^k = \sum_{n=0}^{\infty} \binom{k}{n} \Delta^n$$

and since $[\mathrm{E}^k (sx + r)_m]_{x=0} = (sk + r)_m$, we get

$$(sk + r)_m = \sum_{n=0}^{m} G(m,\ n,\ s,\ r)(k)_n. \tag{3.4}$$

The generating function (3.2) may then be rewritten as

$$A_{m,\,s,\,r}(t) = \sum_{n=0}^{m} \frac{n!}{m!} G(m,\ n,\ s,\ r) t^m (1 - t)^{m-n},$$

so that

$$A(m,\ k,\ s,\ r) = \sum_{n=0}^{m} (-1)^{k-n} \frac{n!}{m!} \binom{m - n}{m - k} G(m,\ n,\ s,\ r). \tag{3.5}$$

Since for $r = 0$ the numbers $G(m,\ n,\ s,\ r)$ reduce to the numbers

$$C(m,\ n,\ s) = \frac{1}{n!}[\Delta^n (sx)_m]_{x=0}$$

studied by the author [5, 6, 7] and also by Carlitz [2] as degenerate Stirling numbers, we have, in particular,

$$A(m,\ k,\ s) = \sum_{n=0}^{m} (-1)^{k-n} \frac{n!}{m!} \binom{m - n}{m - k} C(m,\ n,\ s). \tag{3.6}$$

The generating functions

$$A_{s,\,r}(t,\ x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A(m,\ k,\ s,\ r) t^k x^m \tag{3.7}$$

and

$$A_s(t,\ x,\ y) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} A(m,\ k,\ s,\ r) t^k y^r x^m, \tag{3.8}$$

using (3.2), may be obtained as

$$A_{s,\,r}(t,\ x) = \frac{(1 - t)[1 + (1 - t)x]^r}{1 - t[1 + (1 - t)x]^s}, \tag{3.9}$$

$$A_s(t,\ x,\ y) = \frac{(1 - t)}{\{1 - t[1 + (1 - t)x]^s\}\{1 - y[1 + (1 - t)x]\}}. \tag{3.10}$$

Since $\lim_{t \to 1} A_{s,\,r}(t,\ x) = (1 - sx)^{-1}$, we get

$$A_{m,\,s,\,r}(1) = \sum_{k=0}^{m} A(m,\ k,\ s,\ r) = s^m. \tag{3.11}$$

Using (2.19) and (2.21), (3.11) may be rewritten in the form

$$\begin{aligned} s^m &= \sum_{k=0}^{m} \sum_{i=1}^{m+1} \binom{s+i-1}{m} B(m,\ m-i+1,\ k,\ r) \\ &= \sum_{k=0}^{m} \sum_{i=1}^{m+1} \binom{s+i-1}{m} B(m,\ i,\ m-k+1,\ r) \\ &= \sum_{i=1}^{m+1} \binom{s+i-1}{m} \sum_{k=0}^{m} B(m,\ i,\ m-k+1,\ r). \end{aligned}$$

It is known that the Eulerian numbers $A_{m,i}$ satisfy the relation (see [19] or [1])

$$\sum_{i=1}^{m} \binom{s+i-1}{m} A_{m,i} = s^m.$$

Therefore

$$\sum_{k=1}^{m+1} B(m,\ i,\ k,\ r) = A_{m,i}. \tag{3.12}$$

The generating function

$$B_{m,n,r}(t) = \sum_{k=0}^{m} B(m,\ n,\ k,\ r)t^k \tag{3.13}$$

is connected with $A_{m,s,r}(t)$ by the relations

$$A_{m,s,r}(t) = \sum_{i=0}^{m} \binom{s+i}{m} B_{m,m-i,r}(t) \tag{3.14}$$

and

$$B_{m,n,r}(t) = \sum_{j=0}^{n} (-1)^j \binom{m+1}{j} A_{m,n-j,r}(t). \tag{3.15}$$

Returning to (2.6), let us put $r = su$. Then

$$\lim_{s \to \pm\infty} s^{-m} m! A(m,\ k,\ s,\ r) = A_{m,k,u}, \tag{3.16}$$

where

$$A_{m,k,u} = [\nabla^{m+1} \mathrm{E}^k \underline{(x+u)^m}]_{x=0} = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} (k+u-j)^m$$

are the numbers used by Dwyer [8] for computing the ordinary moments of a frequency distribution. In particular,

$$\lim_{s \to \pm\infty} s^{-m} m! A(m,\ k,\ s) = A_{m,k}. \tag{3.17}$$

Consider the function

$$\begin{aligned} H_m(t;\ s,\ r) &= (1-t)^{-m} A_{m,s,r}(t) \\ &= \sum_{n=0}^{m} \frac{n!}{m!} G(m,\ n,\ s,\ r) t^n (1-t)^{-n}. \end{aligned} \tag{3.18}$$

Then, using (3.3), we get

$$H(x;\ t,\ s,\ r) = \sum_{m=0}^{\infty} H_m(t;\ s,\ r)x^m = (1-t)(1+x)^r[1-t(1+x)^s]^{-1}. \tag{3.19}$$

Since $\lim_{s \to \pm\infty} H(x/s;\ -1,\ s,\ su) = E(x;\ u)$, where

$$E(x;\ u) = \sum_{m=0}^{\infty} E_m(u)x^m = 2e^{xu}/(1 + e^x) \tag{3.20}$$

is the generating function of the Euler polynomials ([12, p. 309]), it follows that for the polynomials

$$\zeta_m(u;\ s) \equiv H_m(-1;\ s,\ su) = 2^{-m} \sum_{k=0}^{m} (-1)^k A(m,\ k,\ s,\ su)$$

we have

$$\lim_{s \to \pm\infty} s^{-m}\zeta_m(u,\ s) = E_m(u),$$

which, on using (3.16), gives

$$\sum_{k=0}^{m} (-1)^k A_{m,\,k,\,u} = 2^m m! E_m(u) \tag{3.21}$$

and, in particular,

$$\sum_{k=0}^{m} (-1)^k A_{m,\,k,\,1/2} = E_m, \tag{3.22}$$

where $E_m = 2^m m! E_m(1/2)$ is the Euler number ([12, p. 300]).

Putting $u = 0$ in (3.21), we get

$$\sum_{k=0}^{m} (-1)^k A_{m,\,k} = T_m, \tag{3.23}$$

where $T_m = 2^m m! E_m(0)$ is the tangent-coefficient ([12, p. 298]).

Remark 3.1: The degenerate Eulerian numbers $A_{m,\,k}(\lambda)$ introduced by Carlitz [2, 3] by their generating function

$$1 + \sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{k=1}^{m} A_{m,\,k}(\lambda)t^k = \frac{1 - t}{1 - t[1 + \lambda x(1 - t)]^{1/\lambda}} \tag{3.24}$$

are related to the numbers

$$A(m,\ k,\ s) \equiv A(m,\ k,\ s,\ 0).$$

Indeed, comparing (3.24) with (3.9), we get

$$A(m,\ k,\ s) = \frac{s^m}{m!} A_{m,\,k}(s^{-1}).$$

4. Applications in Statistics

The numbers $A(m,\ k,\ s,\ r)$ like the Eulerian numbers $A_{m,\,k}$ seem to have many applications in combinatorics and statistics. Special cases of these numbers have already occurred in certain combinatorial problems, as was noted in the introduction. In this section, we briefly discuss three applications in statistics. The first is in the computation of the factorial moments of a frequency distribution with the use of cumulative totals. This method was suggested by Dwyer [8, 9] for the computation of the ordinary moments, as an alternative to the usual elementary method and, therefore, for details, the reader is referred to this work. We only note that the main advantage of this method is that the many multiplications involved in the usual process are replaced by continued addition. Let f_x denote the frequency distribution and

$$C^{m+1}f_x = C(C^m f_x),\ m = 1, 2, 3, \ldots,\ Cf_x = \sum_{j=x}^{r+k} f_j,$$

the successive frequency cumulations. Then, from the successive cumulation theorem of Dwyer, we get, for the factorial moments,

$$\sum_{x=0}^{k} (sx + r)_m f_{sx+r} = \sum_{n=0}^{m} m!A(m, n, s, r)C^{m+1}f_{sn+r}. \tag{4.1}$$

When $r = 0$, i.e., when the factorial moments are measured about the smallest variate, (4.1) reduces to

$$\sum_{x=0}^{k} (sx)_m f_{sx} = \sum_{n=0}^{m} m!A(m, n, s)C^{m+1}f_{sn}, \tag{4.2}$$

which for $s = 1$, i.e., when the distance between successive variates (class marks) is unity, gives ([8, §9])

$$\sum_{x=0}^{k} (x)_m f_x = \sum_{n=0}^{m} m!A(m, n, 1)C^{m+1}f_n = m!C_{m+1}^{m+1}, \tag{4.3}$$

since $A(m, m, 1) = 1$, $A(m, n, 1) = 0$ if $n \neq m$.

The second statistical application of the numbers $A(m, k, s, r)$ is in the following problem: Let $X_1, X_2, \ldots, X_m$ be a random sample (that is, m independent and identically distributed random variables) from a population with a discrete uniform distribution

$$p(n; s) = P(X = n) = s^{-1},\ n = 0, 1, 2, \ldots, s - 1.$$

Then the probability function of the sum $Z_m = \sum_{i=1}^{m} X_i$ may be obtained as

$$\begin{aligned} p(n; m, s) &= s^{-m} \sum_{j=0}^{[n/s]} (-1)^j \binom{m}{j}\binom{n + m - 1 - sj}{m - 1} \\ &= s^{-m}A(m - 1, [n/s], s, r + m - 1), \\ &\qquad n = sk + r, \\ &\qquad 0 \le r < s. \end{aligned} \tag{4.4}$$

Note that the distribution function

$$F_{m,s}(w) = \sum_{n=0}^{[w]} p(n/s; m, s)$$

of the sum

$$W_m = \sum_{i=1}^{m} Y_i,\ Y_i = s^{-1}X_i,\ i = 1, 2, \ldots, m$$

approaches, for $s \to \infty$, the distribution function

$$F_m(u) = \frac{1}{m!} \sum_{j=0}^{[u]} (-1)^j \binom{m}{j} (u - j)^m$$

of the sum

$$U_m = \sum_{i=1}^{m} V_i$$

of m independent continuous uniform random variables on [0, 1) (Feller [10] and Tanny [18]).

Since

$$E(Z_m) = mE(X) = m(s - 1)/2,$$
$$\mathrm{Var}(Z_m) = m\mathrm{Var}(X) = m(s^2 - 1)/12,$$

it follows from the central limit theorem (see, e.g., Feller [10]) that the sequence

$$\frac{Z_m - m(s - 1)/2}{\sqrt{m(s^2 - 1)/12}}$$

converges in distribution to the standard normal. Hence

$$\lim_{m\to\infty} \sum_{k=0}^{[z_m]} s^{-m} A(m - 1, k, s, r) = \Phi(z),$$

(4.5)

$$z_m = z\sqrt{m(s^2 - 1)/12} + m(s - 1)/2.$$

and

(4.6) $$\lim_{m\to\infty} \sqrt{m(s^2 - 1)/12}\, s^{-m} A(m - 1, [z_m], s, r + m - 1) = \varphi(z),$$

where $\varphi(z)$ and $\Phi(z)$ are the density and the cumulative distribution functions of the standard normal.

Finally, consider a random variable X with the logarithmic series distribution

$$p(k;\ \theta) = P(X = k) = \alpha\theta^k/k,\ k = 1, 2, \ldots, \alpha^{-1} = -\log(1 - \theta),\ 0 < \theta < 1.$$

Patil and Wani [15] proved the following property of the moments $\mu_m(\theta) = E(X^m)$:

$$\mu_m(\theta) = \alpha(1 - \theta)^{-m} \sum_{k=0}^{\infty} c(m - 2, k)\theta^{k+1},$$

where the coefficients satisfy the recurrence relation

$$c(m, k) = (k + 1)c(m - 1, k) + (m - k + 1)c(m - 1, k - 1),$$
$$c(0, 0) = 1, \quad c(m, k) = 0,\ k > m.$$

It is not difficult to see that

$$c(m, k) = A_{m+1,\, k+1}$$

with the latter a Eulerian number. Hence

$$\mu_m(\theta) = \alpha(1 - \theta)^{-m} \sum_{k=0}^{\infty} A_{m-1,\, k}\theta^k = \alpha(1 - \theta)^{-m} A_{m-1}(\theta).$$

A similar result can be obtained for the generalized factorial moments

$$\mu_{(m;b)}(\theta) = E[(X)_{m,b}]$$

in terms of the numbers $A(m, k, s, r)$. Indeed, we have

$$\mu_{(m;b)}(\theta) = \alpha \sum_{k=1}^{\infty} (k)_{m,b}\theta^k/k = \frac{\alpha}{m} \sum_{k=1}^{\infty} (k - b)_{m,b}\theta^k$$

$$= \alpha s^{-m+1} \sum_{k=1}^{\infty} (sk - 1)_{m-1}\theta^k, \; s = b^{-1},$$

and since, by (2.17) and (2.18),

$$\sum_{k=1}^{\infty} (sk + r)_{m-1}t^k = (m - 1)!(1 - t)^{-m}A_{m-1,s,r}(t),$$

it follows that

$$\mu_{(m;b)}(\theta) = \alpha s^{-m+1}(1 - \theta)^{-m}(m - 1)!A_{m-1,s,-1}(\theta), \tag{4.7}$$

which, in particular, gives

$$\mu_{(m;1)}(\theta) = \alpha(1 - \theta)^{-m}(m - 1)! \sum_{k=1}^{m-1} A(m - 1, k, 1, -1)\theta^k$$

$$= \alpha(m - 1)!\theta^m(1 - \theta)^{-m}.$$

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A GENERALIZATION OF THE GOLDEN SECTION

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Introduction

It may surprise some people to find that the name "golden section," or, more precisely, *goldener Schnitt*, for the division of a line AB at a point C such that $AB \cdot CB = AC^2$, seems to appear in print for the first time in 1835 in the book *Die reine Elementar-Mathematik* by Martin Ohm, the younger brother of the physicist Georg Simon Ohm. By 1849, it had reached the title of a book: *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harminischen Theilung* by A. Wiegang. The first use in English appears to have been in the ninth edition of the *Encyclopaedia Britannica* (1875), in an article on Aesthetics by James Sully, in which he refers to the "interesting experimental enquiry . . . instituted by Fechner into the alleged superiority of 'the golden section' as a visible proportion. Zeising, the author of this theory, asserts that the most pleasing division of a line, say in a cross, is the golden section . . . ." The first English use in a purely mathematical context appears to be in G. Chrystal's *Introduction to Algebra* (1898).

The question of when the name first appeared, in any language, was raised by G. Sarton [11] in 1951, who specifically asked if any medieval references are known. The *Oxford English Dictionary* extends Sarton's list of names and references and, by implication, answers this question in the negative. (The 1933 edition of the *OED* is a reissue of the *New English Dictionary*, which appeared in parts between 1897 and 1928, together with a Supplement. The main dictionary entry "Golden," in a volume which appeared in 1900, makes no reference to the golden section, though it does cite mathematical references that will be noted later; the entry "Section" (1910) contains a reference to "medial section" (Leslie, *Elementary Geometry and Plane Trigonometry*, fourth edition, 1820) and to Chrystal's use of "golden section" noted above. The 1933

Supplement does not appear to contain any further references. A further Supplement, which started publication in 1972, has a long and detailed entry under "Golden" which is clearly based on and extends, but does not answer, Sarton's question.) Among the other names are: the Italian *divina proportione* (Luca Pacioli, in his book of that name, published in Venice in 1509) or Latin *proportio divina* (in a letter from Johannes Kepler to Joachim Tanck on May 12, 1608; then in Kepler's book *De Nive Sexangula*, 1611); the golden medial; the medial section; and the golden mean. This last term "golden mean" is credited by the *OED* to D'Arcy W. Thompson. (Further complications! The *OED*—1972 Supplement entry "Golden"—cites p. 643 of *On Growth and Form* [12]: "This celebrated series, which . . . is closely connected with the *Sectio aurea* or Golden Mean, is commonly called the Fibonacci series." The reference is to the now rare first edition of 1917; the second edition has an expanded and elaborately erudite version of this footnote on pp. 923 and 924, which starts differently: "This celebrated series corresponds to the continued fraction 1 + 1/1+ 1/1+ etc., [though Thompson, who uses a slightly different layout of the fraction, omits the first term in both versions of the footnote] and converges to 1.618..., the numerical equivalent of the *sectio divina*, or 'Golden Mean.'" This same dictionary entry later assigns the first use of the Latinized *sectio aurea* to J. Helemes, in 1844, in a heading in the *Archiv für Mathematik und Physik*, IV, 15: "*Eine . . . Auflösung der sectio aurea*.") Unfortunately, the same expression "golden mean" is usually applied to the Aristotelian principle of moderation: avoid extremes. Other quite different things with similar names are the golden rule (the rule of three; see the *OED* 1933 edition entry "Golden" for references) and the golden number (the astronomical index of Meton's lunar cycle of nineteen years). Also E. T. Bell, in "The Golden and Platinum Proportions" [2], refers to "the so-called golden proportion 6:9::8:12," but I cannot decide whether this article is meant as a serious contribution or not. If confusion and misapprehension were confined to nomenclature, that would, it is evident, be bad enough; alas, more is to be described, after a paragraph of sanity.

The mathematical theory of the golden section can be found in many places. I would cite Chapter 11 of H. S. M. Coxeter's *Introduction to Geometry* [4] as both the best and most accessible reference, and further developments can be found in other of Coxeter's works. The briefest acquaintance with any treatment of the Fibonacci series will indicate why many accounts of that topic will tend to the golden section, and *The Fibonacci Quarterly* is a rich source of articles and references on this subject. That there appears to be a connection between the Fibonacci numbers (and hence the golden section) and phyllotaxis (i.e., the arrangement of leaves on a stem, scales on a pine cone, florets on a sunflower, infloresences on a cauliflower, etc.) is an old and tantalizing observation. The subject is introduced in Coxeter [4], a brief historical survey is included in a comprehensive paper by Adler [1], and Coxeter [5] gives a short and authoritative statement.

The application of the golden section to other fields has, however, created a vast and generally romantic or unreliable literature. For instance, the application to aesthetics is, by its nature, subjective and controversial; a good brief survey with references is given in Wittkower [13]. For a comprehensive example of the genre, see the rival explanation and critical view of the role of the golden section in literature, art, and architecture in Brunés, *The Secrets of Ancient Geometry* [3]. (Lest I be incorrectly understood to be dismissing the scientific and experimental study of aesthetics as worthless, let me cite H. L. F. von Helmhlotz's *On the Sensations of Tone* [10] as an impressively successful example of this type of investigation, the very acme of science, mathematics, scholarship, and sensibility. In particular this book contains the first explanation of the ancient Greek observation that harmony seems to be connected with small integral ratios. But it is precisely Helmholtz's masterly blend of acoustics, physiology, physics, and mathematics that establishes firmly a standard which so few other writers on scientific aesthetice approach.)

With this outline of the recent history of the golden section behind us, my objective here is to treat the construction as it is described in Euclid's *Elements* under the name of "the line divided in extreme and mean ratio" and to develop and explore beyond the propositions we find proved there. My covert purpose is historical: to pose implicitly the question of whether the generalizations to be described here might have had any part, now lost, in the development of early Greek mathematics. To isolate this discussion of the ancient period from the later convoluted ramifications sketched in this introduction, I would like to finish with what is, I hope, an accurate description of the surviving evidence about the Greek period: the propositions to be found in Euclid's *Elements* constitute the only direct, explicit, and unambiguous surviving references to the construction in early Greek mathematics, philosophy, and literature; and the only other surviving Greek references are to be found in mathematical contexts, in Ptolemy's *Syntaxis*, Pappus' *Collectio*, Hypsicles' "Book XIV" of the *Elements*, and an anonymous Scholion on Book II of the *Elements*.

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The Definition in Euclid's *Elements*

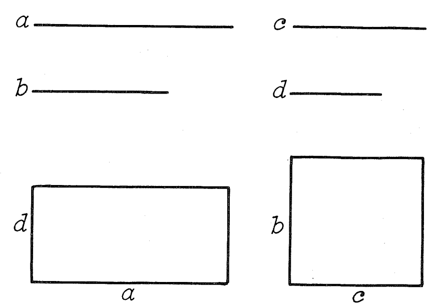
The golden ratio is defined at the beginning of Book VI of the *Elements*:

> A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.

Book VI applies the abstract proportion theory of Book V to geometrical magnitudes, and Proposition 16 describes how to manipulate the proportion in the definition above into a geometrical statement:

> If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes is equal to the rectangle contained by the means, the four straight lines will be proportional.

Otherwise said, if a, b, c, and d are four lines such that $a:b::c:d$, then rectangle (a, d) = rectangle (b, c) and conversely. Hence, if C divides the line AB in the golden section, the rectangle with sides AB and BC is equal to the square with side AC.



This is meant literally. An elaborate theory, now generally called the "application of areas," is developed in the *Elements*, and this describes how, for example, to manipulate any rectilinear plane area into another area equal to the original area and similar to a third figure. Our arithmetical definition of area ("base × height") is not needed and is never used; indeed, this theory of application of areas, together with the Book V theory of proportions, provides a completely adequate alternative to the construction of the real numbers and their use in plane rectilinear geometry. It merits considerable respect, and gets it: the same (probably equally unreliable) story is found about Pythagoras sacrificing an ox to the discovery of a result on the application of areas as is also told about the theorem on right angle triangles.

The golden section is constructed in Proposition 30:

> To cut a given finite straight line in extreme and mean ratio,

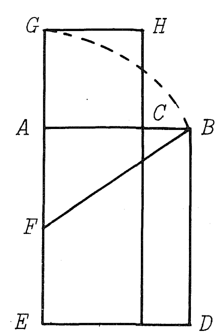
and the method used there involves an elaboration of the theory called "application with excess." Fortunately, an easier construction is possible and has already been given in Book II; and the manuscripts that we possess of the *Elements* contain a second, possibly interpolated, proof of VI, 30, referring back to this earlier construction. Using this method, it is possible to bypass the use of proportion theory, and the elaborations of the theory of application of areas, and to give a direct definition and construction of the golden section. This is now we shall proceed.

The Construction of the Line Divided in Extreme and Mean Ratio, and Its Generalization

Book II, Proposition 11, describes how:

> To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment,

and we shall hereinafter adopt this as the definition of the extreme and mean ratio. The construction is straightforward:



To construct the required point C on AB, draw the square $ABDE$; take F to be the midpoint of AE, and G on EA produced such that $FG = FB$. If $ACHG$ is the square with side AG, then C cuts AB in mean and extreme ratio.

The verification of this is easy:

$$\begin{aligned} FG^2 &= (AF + AG)^2 \\ &= AF^2 + AC^2 + 2AF \cdot AC \quad \text{(Since } AG = AC.\text{)} \end{aligned}$$

But $\quad FG^2 = FB^2 = AF^2 + AB^2$. (By Pythagoras' theorem.)

Therefore, $AC^2 + 2AF \cdot AG = AB^2$.
Subtract $2AF \cdot AG = AE \cdot AC$ from both sides,
then $AC^2 = AB \cdot CB = CB \cdot BD$.

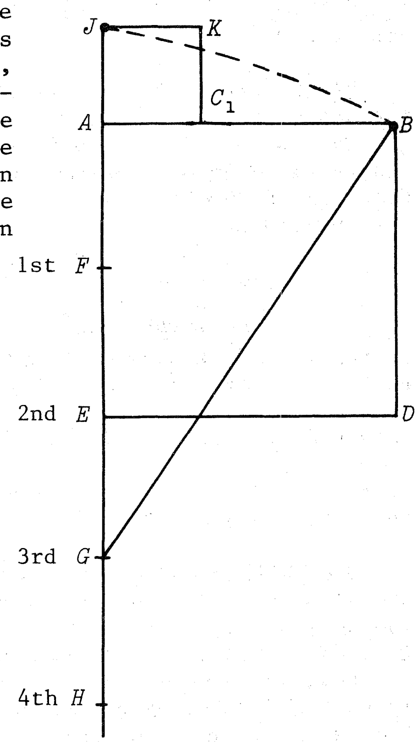
Q.E.F.

This proof can be read as if $AF \cdot AC$, for example, represented the product of two numbers, the lengths of AF and AC; or the purist can interpret $AF \cdot AC$ as a rectangle with sides equal to the lines AF and AC and, using some obvious manipulations, check that the proof makes sense and is correct. This latter method is in the spirit of the techniques of application of areas, though none of the subtle manipulations of that theory are needed.

It is clear that it must be the rectangle contained by the whole and the *lesser* segment that will be equal to the square on the *greater* segment, since the square on the lesser segment will fit inside the rectangle contained by the whole and the greater segment and so it has smaller area. (The common notions at the beginning of Book I set out what are, in effect, the axioms of a theory of equality and inequality of area or, more strictly, of content; and Common Notion 5 states: The whole is greater than the part.)

We now describe the generalization that we call the *nth order extreme and mean ratio*, abbreviated to the *noem ratio*. There is one such construction for each integer n, and the golden section corresponds to the case $n = 1$; the implications of the construction are somewhat simpler for the case of even values of n, and therefore we shall always illustrate the case of $n = 3$; and we shall shortly introduce and use a consistent and general notation and terminology to describe the resulting configuration.

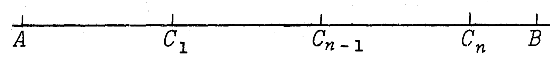
Start with the square $ABDE$ on the given line AB, and on AE produced as necessary, take points F, G, H, as shown, with $\frac{1}{2}AB = AF = FE = EG = GH$, etc.; then these points will be used in the construction of the 1st, 2nd, 3rd, 4th, etc., extreme and mean ratios. We always illustrate the case of $n = 3$ and so, here, work from the point G. On EA produced, take J such that $GJ = GB$; then the square $AJKC_1$ defines the point C_1 dividing AB in the 3rd extreme and mean ratio.



The Definition and Properties of the Noem Ratio

We start with the basic defining property of the generalization, and show that it is possessed by our constructed point.

Definition: The point C_1 is said to divide AB in the *noem ratio* (read: nth-order extreme and mean ratio) if, taking points $C_2, \ldots, C_{n-1}, C_n$ on AB such that

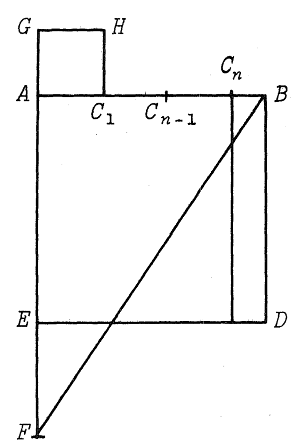


$AC_1 = C_1C_2 = \cdots = C_{n-1}C_n$, then C_n lies between A and B and $AB \cdot C_nB = AC_1^2$.

Note that the latter condition implies that C_nB is less than AC_1; it will be called the "lesser segment" of the noem ratio. The greater segment of the golden ratio generalizes two ways: to AC_1, which we call the initial segment of the noem ratio; and to AC_n, which we again call the greater segment of the noem ratio. Care must be exercised in generalizing the results on the golden section to make the appropriate choice. As remarked earlier, we shall always illustrate the case of $n = 3$, and will always use the same letters to label the points, calling the three division points C_1, C_{n-1}, C_n, so that their roles will be clear. Proofs will be given for the general case, sometimes referring to a phantom point C_2 and adding a few dots "+ $\cdots$ +."

Proposition: The point C_1, described in the construction, divides AB in the noem ratio.

Proof: The figure illustrates the construction for the case $n = 3$. The proof is a straightforward generalization of the proof given in the case $n = 1$, and we can even use the same letters to identify the vertices of the figure.



As before,

$$FG^2 = (AF + AG)^2 = AF^2 + AC_1^2 + 2AF \cdot AC_1.$$

But $FG^2 = FB^2 = AF^2 + AB^2$.

Therefore, $AC_1^2 + 2AF \cdot AC_1 = AB^2$.

But $2AF \cdot AC_1 = nAE \cdot AC_1 = AE \cdot AC_n$.

$\left(\text{Since } AF = \frac{n}{2}AE, \text{ and } nAC_1 = AC_n.\right)$

Hence $AC_n < AB$ and, subtracting $AE \cdot AC_n$ from both sides, we see that $AC_1^2 = AB \cdot C_nB$. Q.E.F.

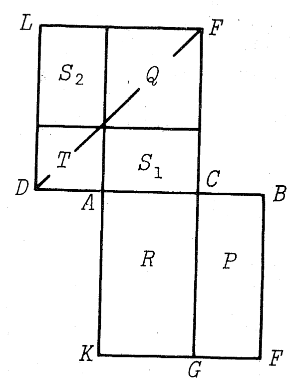
Book XIII of Euclid's *Elements* contains the details of the construction of the five regular "Platonic" solids, and a proof that these are the only regular solids; but it contains a lot more material besides that. In particular, it starts with six propositions on the extreme and mean ratio, together with alternative proofs of these results illustrating a method of "analysis and synthesis." These propositions follow on in the style of Book II—in particular, they do not explicitly need to use any more than the rudiments of the theory of application of areas—and they can easily be generalized to apply to the noem ratios. We now alternate the enunciations of these Euclidean propositions with their generalizations, interposing some general remarks. (Later propositions of Book XIII describe relationships between the extreme and mean ratio and pentagons, hexagons, decagons, icosahedra, and dodecahedra; we shall not consider them here.)

> XIII, Proposition 1. If a straight line be cut in extreme and mean ratio, the square on the greater segment added to half of the whole is five times the square on the half.

Paraphrase of Euclid's Proof:

If AB is cut in extreme and mean ratio at C, and $DA = \frac{1}{2}AB$, then we prove $CD^2 = 5AD^2$.

Draw the squares on DC and AB, and complete the figure as shown. (In addition to the Euclidean labelling of the vertices, we have also labelled the regions of the figure.)



We know that $AB \cdot CB = AC^2$ (Definition of mean and extreme ratio)

i.e., $P = Q$,

and $AB \cdot AC = 2AD \cdot AC$ (Since $AB = 2AD$)

i.e., $R = 2S_1 = S_1 + S_2$;

hence $P + R = Q + S_1 + S_2$.

Adding $AD^2 = T$, and assembling the result into squares,

$$DC^2 = AB^2 + AD^2.$$

But $AB^2 = 4AD^2$, so $DC^2 = 5AD^2$. Q.E.D.

Remark: Our way, today, of considering the golden ratio is almost always to identify it with the real number $\frac{1}{2}(\sqrt{5}+1)$; this and the following propositions represent the closest approach we find in surviving Greek texts to this evaluation. For instance, this proposition implies that if $AB = 2$, then $CD = \sqrt{5}$ (i.e., the side of a square of area equal to the rectangle with sides AB and $5AB$) so $AC = \sqrt{5} - 1$, and the ratio is $2:(\sqrt{5} - 1)$ $[= \frac{1}{2}(\sqrt{5} + 1):1]$.

Proposition 1': If a straight line be cut in the noem ratio, the square on the initial segment added to n times half of the whole is $n^2 + 4$ times the square on the half.

Remark: It is standard Euclidean practice to handle such a general proof by choosing a particular small value of n, typically $n = 2, 3,$ or 4. The figures for our proofs differ very slightly according as n is even or odd (a consequence of the occurrence of halves in the construction), with the case of even

n being slightly simpler. It is also standard Euclidean practice when there are several different cases to a proposition only to consider the most complicated one. Therefore, our choice of $n = 3$ is in line with the Euclidean procedure. We shall, however, use a general labelling system, writing C_1, C_{n-1}, C_n, rather than C_1, C_2, C_3, and develop further the practice of labelling regions of the figure, using letters P, Q, R, etc., and suffixing to denote equal regions, so $P_1 = P_2 = P_3$ etc. Euclidean practice appears to be to label only the vertices of the figure, working through the alphabet strictly in order of occurrence in the setting-out and construction of the figure.

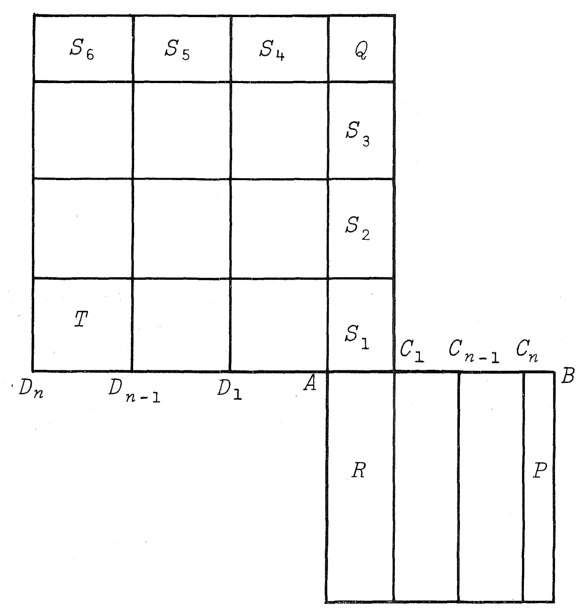
A final point in which our enunciation differs from Euclidean practice is in referring to the chosen parameter n. A more idiomatic expression, as rendered in English, might read:

Proposition 1'': If a straight line be cut in the general extreme and mean ratio to some number, the square on the initial segment added to that number of segments each equal to half of the whole is the square of that number increased by four times the square on the half.

Purists might like to try a similar rephrasing of later generalizations!

Proof: If AB is cut in the noem ratio at C_1, and $D_nA = \frac{n}{2}AB$, then we prove that $D_nC_1^2 = (n^2 + 4)AD_1^2$.

Draw the squares on D_nC_1 and AB, and complete the figure as shown:



We know that $AB \cdot C_nB = AC_1^2$ (Definition of the noem ratio)

i.e., $P = Q$,

and $AB \cdot AC_1 = 2AD_1 \cdot AC_1$ (Since $AB = 2AD_1$)

i.e., $R = 2S_1$,

and hence, $$P + nR = Q + 2nS_1.$$

Adding $AD_n^2 = n^2T$ and assembling the result into squares, we get

$$\begin{aligned} D_nC_1^2 &= AB^2 + n^2T \\ &= (n^2 + 4)T \qquad \text{(Since } AB^2 = 4T) \\ &= (n^2 + 4)AD_1^2. \end{aligned}$$

Q.E.D.

The next propositions give the converses to these results. We start with Euclid's enunciation:

> XIII, Proposition 2. If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

The Euclidean practice of never referring to a particular figure can make the enunciations of propositions very cumbersome, and these propositions, together with the propositions of Book II contain some particularly awkward examples. In these cases, it is best to ignore the enunciation and proceed directly into Euclid's proof of the proposition, where the setting-out will give a more accessible explanation of the result. In this case we find, paraphrasing and adjusting the labelling to accord with our convention, that if C and B are taken on a line DA produced with $DC^2 = 5DA^2$ and $AB = 2DA$, then C cuts AB in the extreme and mean ratio with AC the greater segment.

Proposition 2': If a line D_nA is divided equally into

$$D_nD_{n-1} = D_{n-1}D_{n-2} = \cdots = D_1A,$$

and C_1 and B are taken on D_nA produced with

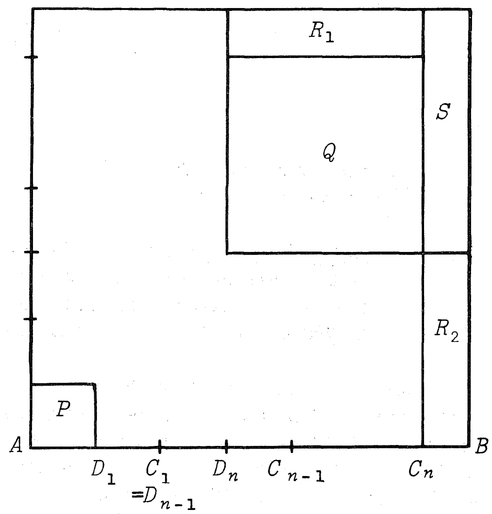
$$D_nC_1^2 = (n^2 + 4)D_nD_{n-1}^2 \qquad \text{and} \qquad AB = \frac{2}{n}D_nA = 2D_nD_{n-1},$$

then C_1 cuts AB in the noem ratio with AC_1 the initial segment.

Proof: For both propositions we can construct the same figures as for the preceding propositions and then read the previous arguments backwards. Q.E.D.

> XIII, Proposition 3. If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to half of the greater segment is five times the square on half of the greater segment.

Proposition 3': If C_1 cuts AB in the noem ratio, and $AD_n = \frac{n}{2}AC_1$ (as shown), then $D_nB^2 = (n^2+4)AD_1^2$; i.e., the square on (the lesser segment C_nB added to half of the greater segment AC_n) is equal to (n^2+4) times the square on (half of the initial segment AC_1).



Proof: Construct the figure shown, where A, B, C_1, C_{n-1}, C_n are as usual, and $AD_1 = D_1D_{n-1} = D_{n-1}D_n = \frac{1}{2}AC_1$.

First observe that

$$\begin{aligned} D_nC_n &= AC_n - AD_n \\ &= nAC_1 - nAD_1 \\ &= nAD_1 \quad (\text{Since } AC_1 = 2AD_1) \\ &= AD_n. \end{aligned}$$

Hence,

$$\begin{aligned} D_nB^2 &= Q + R_1 + S \\ &= Q + R_2 + S \\ &= Q + AB \cdot C_nB \\ &= Q + AC_1^2 \quad (\text{Since } C_1 \text{ divides } AB \text{ in the noem ratio.}) \\ &= (n^2+4)AD_1^2. \quad (\text{Since } Q = D_nC_n^2 = n^2AD_1^2 \text{ and } AC_1 = 2AD_1.) \end{aligned}$$

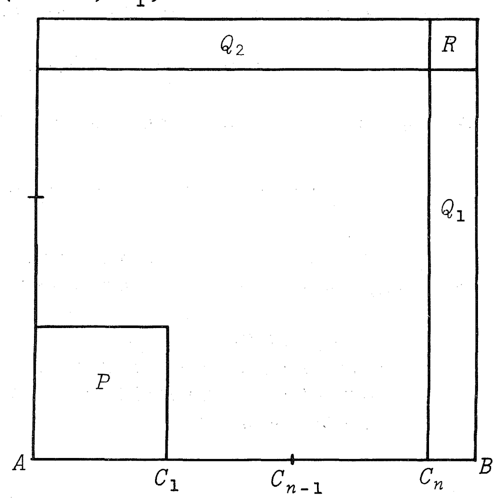
Q.E.D.

XIII, Proposition 4. If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Proposition 4': Let AB be cut in the noem ratio at C_1, then

$$AB^2 + C_nB^2 = (n^2+2)AC_1^2,$$

i.e., the square on the whole and the square on the lesser segment together are (n^2+2) times the square on the initial segment.



Proof: We have that $C_nB \cdot AB = AC_1^2$,

i.e., $Q_1 + R = P$.

Hence, $Q_1 + R + Q_2 + R = 2P$.

Adding $AC_n^2 = n^2P$ to each side, and assembling into squares, we see that $AB^2 + C_nB^2 = (n^2+2)AC_1^2$. Q.E.D.

> XIII, Proposition 5. If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in mean and extreme ratio, and the original straight line is the greater segment.

Proposition 5': If C_1 cuts AB in the noem ratio, and A_n is taken on BA produced with $A_nB = nAB$, and D on BA_n produced with $DA_n = AC_1$, then DB is cut in the noem ratio by A.

Note: In the Euclidean proposition, $n = 1$ and $A_n = A$; therefore, there is no need to mention the first step of constructing the point A_n. After this step the generalization states that, if there be added to A_nB a line equal to the initial segment, the whole BD has then been cut in the noem ratio, with the line BA_n being the greater segment, and so the original line BA the initial segment.

Proof: Complete the figure as shown; we want to show that $DA_n \cdot DB = AB^2$. Now,

$$DA_n \cdot DA = P + Q_1 + Q_{n-1} + Q_n$$

and

$$AB^2 = Q_n + Q_{n-1} + Q_1 + R$$

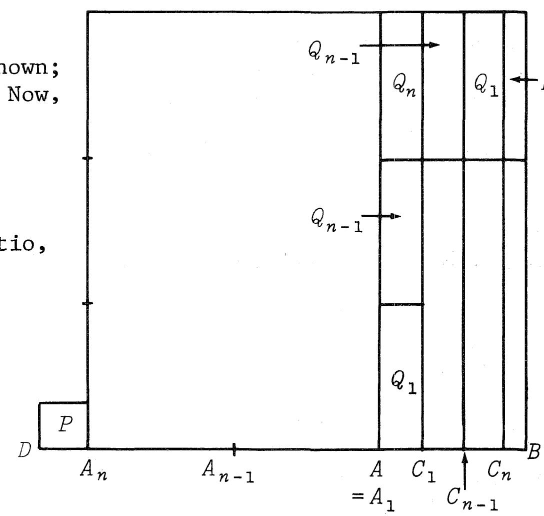
and, since C_1 cuts AB in the noem ratio,

$$AB \cdot C_nB = AC_1^2,$$

i.e.,

$$P = R.$$

Hence $DA_n \cdot DB = AB^2$.



> XIII, Proposition 6. If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome.

This result, together with its proof, generalizes directly to the noem ratio, but an explanation of what it means depends on a knowledge of the long and difficult Book X. It is perhaps worth noting that Euclid uses the words "rational" and "irrational" here in completely different sense from our modern usage: a short, though oversimplified explanation is that when a unit line ρ has been chosen, then anything of the form $\sqrt{\frac{p}{q}} \cdot \rho$ (where p and q are integers) is called rational; anything not of that form is an irrational; and an apotome is an irrational line that can be expressed as a difference of two rational lines, $\sqrt{\frac{p}{q}} \cdot \rho - \sqrt{\frac{r}{s}} \cdot \rho$.

Ratio in Eudlid's *Elements*

It is a curious and remarkable fact that ratio is not defined either in Eudlid's *Elements*, or anywhere else in the surviving corpus of Greek mathematics. All that we have is a vague description of the word at Book V, Definition 3:

> A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

What *is* defined (at Book V, Definition 5) is proportion, which is a relation that may or may not hold among four magnitudes, $a{:}b{::}c{:}d$; and we can think of it, and appear to be encouraged in this by Euclid, in terms of the equality of two "ratios." An examination of the scanty surviving evidence of pre-Eudlidean mathematics, and a reinterpretation of some of the books of the *Elements* has led me to suggest that ratio might have been defined, in the period before the development of the abstract proportion theory that we find in Book V of the *Elements*, by a process based on the "Euclidean" subtraction algorithm. (Actually, what little evidence we have indicates that the person who realized the importance of the procedure might have been Theaetetus, a colleague and friend of Plato, so the "Theaetetan subtraction algorism" might be a more appropriate name; here, I have also corrected what the *OED* calls a "pseudo-etymological perversion . . . in which algorithm is learnedly confused with Greek αριθμός.") Let me illustrate this by describing the operation of the procedure on two lines a_0 and a_1. Suppose that a_1 goes into a_0 some number n_0 of times, leaving a remainder a_2 less than a_1; and then a_2 goes into a_1 some number n_1 of times, leaving a remainder a_3; etc. Then the ratio $a_0{:}a_1$ will be *defined* by the sequence of integers $[n_0, n_1, n_2, \ldots]$.

If, at any stage, a remainder is zero, the process terminates, and this is characteristic of commensurable ratios. Among incommensurable ratios, with nonterminating expansions, the simplest will be the ratio in which, at each step, the smaller magnitude goes once into the larger magnitude, leaving a remainder for the next step, thus giving the ratio [1, 1, 1, ...]. This is the golden ratio, as can immediately be deduced from the figure of the regular pentagon of which the diagonals, which form an inscribed pentagon, cut each other in the golden ratio (this is explicitly proved at XIII, 8, but the result is implicit in the construction of the pentagon given at IV, 11); or it can easily be deduced from the defining property of the ratio. What we have been constructing here are the next simplest incommensurable ratios, of the form $[n, n, n, \ldots]$, in which, at each stage, the smaller magnitude goes n times with a remainder into the larger magnitude. By using a bit of algebra we can easily work out the numerical value θ of this ratio, since

$$\theta = n + \cfrac{1}{n + \cfrac{1}{n + \cdots}} = n + \frac{1}{\theta},$$

so $\theta^2 - n\theta - 1 = 0$, and, taking the positive root,

$$\theta = \tfrac{1}{2}(\sqrt{(n^2 + 4)} + n).$$

[Alternatively, we can read off from the construction that

$$\theta = AB/AC_1 = 2/(\sqrt{(n^2 + 4)} - n) = \tfrac{1}{2}(\sqrt{(n^2 + 4)} + n).]$$

This explains the occurrence of the number 5, generalizing to $n^2 + 4$, the halves, and the addition and subtraction of segments in the propositions that we have been proving.

It is possible to extend the construction, and thus describe a procedure for constructing any ratio that eventually becomes periodic, though the longer the period, the more involved becomes a perliminary calculation of two parameters needed in the construction. (One of these parameters describes the location of the initial point on the left-hand edge of the square on AB, in our diagram; the other describes the position of an auxiliary point B' on AB; the construction then continues from these two points as before.) Further details of these constructions, together with details of the historical and mathematical ideas that fill out, explain, and set in context these remarks, are given in the papers [7], [8], and [9].

I do not know whether any of the noem ratios, with $n \geq 3$, occur in any regular or semiregular figure, generalizing the appearance of the golden section in the pentagon and other figures, and the ratio [1, 2, 2, 2, ...] of the diagonal and side of a square.

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THE FIBONACCI SEQUENCE IN SUCCESSIVE PARTITIONS OF A GOLDEN TRIANGLE

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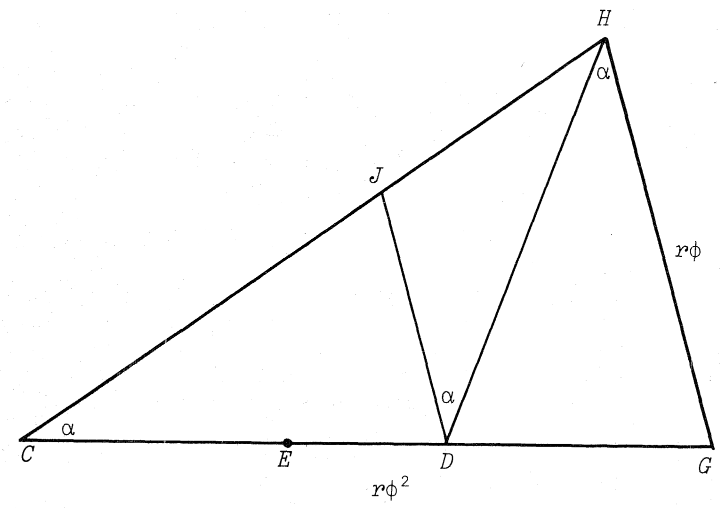
A Golden Triangle is a triangle with two of its sides in the ratio $\phi:1$, where ϕ is the Fibonacci Ratio, i.e., $\phi = \frac{1}{2}(1+\sqrt{5}) \cong 1.618$. Let ΔABC be a triangle whose sides are a, b, and c and let $a/b = k > 1$. Bicknell and Hoggatt [1] have shown that (1) a triangle with a side equal to b can be removed from ΔABC to leave a triangle similar to ΔABC if and only if $k = \phi$, and (2) a triangle similar to ΔABC can be removed from ΔABC to leave a triangle such that the ratios of the areas of ΔABC and the triangle remaining is k if and only if $k = \phi$.

Unlike the Golden Rectangle whose adjacent sides are in the ratio $\phi:1$ (or $1:\phi$), the Golden Triangle does not have a single shape. The diagonal of a Golden Rectangle divides it into two Golden Triangles whose sides are in the ratio $1:\phi:\sqrt{\phi^2+1}$. The most celebrated Golden Triangle, which can be found in the regular pentagon and regular decagon, has angles of 36°, 72°, and 72° and sides in the ratio $1:\phi:\phi$. In general, Bicknell and Hoggatt demonstrated that a Golden Triangle can be constructed with sides in the ratio $1:\phi:G$, where $\phi^{-1} < G < \phi^2$. Figure 1, adapted from their presentation, shows Golden Triangle CGH. Line GH is constructed to be of length $r\phi$ $(r > 0)$ and line CG to be of length $r\phi^2$. Line CG is twice divided in the Golden Section by points E and D, with $CE = DG = r$ and $ED = r/\phi$. A Golden Triangle is formed whenever H is a point on the circle whose center is G and whose radius is EG. Line DH produces $\Delta DGH \sim \Delta CGH$, and ΔCDH whose area is $1/\phi$ times the area of ΔCGH. In general, ΔCDH is not similar to ΔCGH. Nonetheless, ΔCDH is also a Golden Triangle, as $CH/DH = \phi$ [1].

The present paper will explore the consequences of successively partitioning Golden Triangles. To begin, let us show that ΔCDH can be partitioned into two triangles, one similar to itself and the other having an area $1/\phi$ times its own area. If line DJ is drawn parallel to line GH, one can readily verify that ΔDHJ is similar to ΔCDH. (Alternatively, we could have chosen point J so that $CH/CJ = \phi$. Lines DJ and GH would then be parallel, because $CH/CJ = CG/CD$.) We now need to show that the ratio of the area of ΔCDH to the area of ΔCDJ is ϕ. If we designate the area of ΔCGH by S, the area of ΔCDH is S/ϕ [1]. Since DJ is parallel to GH, $\Delta CDJ \sim \Delta CGH$. The ratio $CG/CD = \phi$, hence the area of ΔCDJ is S/ϕ^2. Accordingly, the ratio of ΔCDH to ΔCDJ is S/ϕ divided by S/ϕ^2, or ϕ. Since $S/\phi - S/\phi^2 = S/\phi^3$, we find that the area of ΔDHJ is S/ϕ^3.

We can note several other relationships. Two additional Golden Triangles, ΔCDJ and ΔDHJ, are produced so that ΔCGH is partitioned into three mutually exclusive Golden Triangles. Moreover, ΔCDJ is congruent to ΔDGH. They are similar, as both are similar to ΔCGH and both have areas equal to S/ϕ^2.

Moving beyond the Bicknell-Hoggatt demonstration and its immediate implications, we can show how successive partitions of Golden Triangles generate Fibonacci sequences. Let us repeat the above partitioning, subdividing all of the larger triangles produced in the previous partition. The partitions can be carried out in a manner analogous to the way in which ΔCDH was partitioned.



$$\phi = \frac{CG}{GH} = \frac{CH}{DH} = \frac{GH}{DG} = \frac{CD}{DJ} = \frac{DH}{HJ}$$

$$\alpha = \angle DCH = \angle DHG = \angle HDJ$$

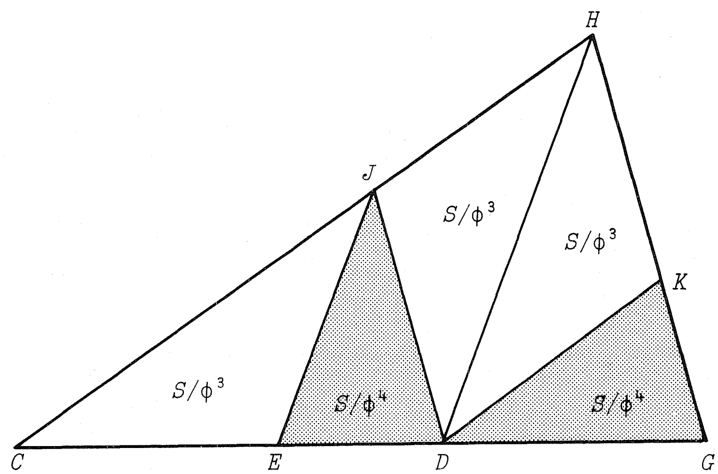
$$\Delta CDJ \cong \Delta DGH$$

FIGURE 1. The General Golden Triangle

For example, ΔCDJ can be split into two Golden Triangles by a line through point J parallel to HD. The resultant line is JE, which has a length equal to DH/ϕ and divides line CD in the Golden Section. As we proceed, the number of triangles and their areas are as follows:

Partition Number (n)	Fibonacci Number (F)	Number of Triangles	Area of Triangles
0	---	1	S
1	1	2	S/ϕ, S/ϕ^2
2	1	3	S/ϕ^2, S/ϕ^2, S/ϕ^3
3	2	5	S/ϕ^3, S/ϕ^3, S/ϕ^3, S/ϕ^4, S/ϕ^4
4	3	8	S/ϕ^4 (5 triangles), S/ϕ^5 (3 triangles)
5	5	13	S/ϕ^5 (8 triangles), S/ϕ^6 (5 triangles)
⋮	⋮	⋮	⋮
n	F_n	F_{n+2}	S/ϕ^n (F_{n+1} triangles), S/ϕ^{n+1} (F_n triangles)

The total number of triangles and the number of larger and smaller triangles increase in Fibonacci sequence. Figure 2 shows the five triangles produced by the third partition. The eight triangles produced by the fourth partition result from subdividing the three larger triangles in the figure. Because partitioning produced triangles whose areas, relative to the area of the partitioned triangle, are $1/\phi$ and $1/\phi^2$, the pattern is perpetuated.



S = Area ΔCGH

$\Delta CEJ \cong \Delta DHJ \cong \Delta DHK$

$\Delta DEJ \cong \Delta DGK$

FIGURE 2. The Partition of a Golden Triangle into Five Golden Triangles

By a repetition of the earlier demonstrations, it can be seen that every triangle that results from the partitioning is similar to one of the two Golden Triangles produced by the first partition (i.e., the partition effected by line DH), and that all triangles of the same area are congruent. Every triangle has an area equal to S/ϕ^i, for some integer i. For triangles similar to ΔCGH, i is even, while for triangles similar to ΔCDH, i is odd. Corresponding sides of triangles with areas S/ϕ^i and S/ϕ^{i+2} are in the ratio ϕ:1. The total area of the larger triangles relative to the total area of the smaller is $\phi F_{n+1}/F_n$ after the nth partition, and that ratio approaches ϕ^2 as n becomes large.

Each partition illustrates the equation for powers of ϕ, i.e.,

$$\phi^n = F_n\phi + F_{n-1}. \tag{1}$$

Dividing (1) through by ϕ^n, we have

$$1 = \frac{F_n}{\phi^{n-1}} + \frac{F_{n-1}}{\phi^n}, \tag{2}$$

which expresses the area of a unit Golden Triangle as the sum of the areas of partitioned triangles. For example, with $n = 4$, we have the situation before the fourth partition, shown in Figure 2, where

(3) $$S = \frac{3S}{\phi^3} + \frac{2S}{\phi^4}.$$

Multiplying (3) by ϕ^4/S gives

(4) $$\phi^4 = 3\phi + 2.$$

Let us move from the general case to the special case where the two Golden Triangles formed by the first partition are similar to ΔCGH, and hence to one another. In that case, the triangles must be "Fibonacci Right Triangles," with sides in the ratio $\phi:\phi^{3/2}:\phi^2$. To demonstrate that, consider triangles CDH and DGH in Figure 1. From (1) we know $\angle DCH = \angle DHG$. If triangles CDH and CGH are similar, $\angle CHD$ must equal $\angle DGH$ because $\angle CHD \neq \angle CHG$. Since $\Delta CDH \sim \Delta DGH$ and we have established equalities between two of their three angles, we must have $\angle CDH = \angle GDH$. As $\angle CDH$ and $\angle GDH$ sum to 180°, both of those angles equal 90° and line DH is an altitude. With $\angle CHG = \angle CHD + \angle DHG$ and $\angle DHG = \angle DCH$, we have $\angle CHG = \angle CHD + \angle DCH$. In right triangle CDH, $\angle CHD$ and $\angle DCH$ sum to 90°, hence $\angle CHG$ must be 90°. As Figure 1 was constructed with $GH = r\phi$ and $CG = r\phi^2$, applying the Pythagorean Theorem yields $r^2\phi^4 = r^2\phi^2 + CH^2$, and thus we find $CH = r\phi^{3/2}$.

The Fibonacci Right Triangle has been examined by a number of writers. Ghyka [2] identified it as one of the three most significant nonequilateral triangles. He noted that it was sometimes called the "Great Pyramid" triangle because its proportions are found in the Great Pyramid of Cheops, or the triangle of Price, after W. A. Price, who proved that it is the only right triangle whose sides are in geometric progression (i.e., if the sides of a triangle are 1, k, and k^2, $k = \sqrt{\phi}$ is the only positive real solution that satisfies the Pythagorean equation $1 + k^2 = k^4$). Hoggatt [3] noted that the altitude of a Fibonacci Right Triangle produced two Fibonacci Right Triangles that were "five parts congruent," that is, were similar and had two (but not three) sides of equal length. The Fibonacci Right Triangle is related to mean values, in that the harmonic, geometric, and arithmetic means of two positive numbers form a right triangle (the Fibonacci Right Triangle) if and only if those numbers are in the ratio $\phi^3:1$ [4]. In successive partitions of Fibonacci Right Triangles, all line segments are in Fibonacci proportions, as they are all multiples of $\phi^{i/2}$, with i an integer [5]. A multiply partitioned Fibonacci Right Triangle thus presents a striking geometric pattern. An example is given in Figure 3, which shows the 13 Fibonacci Right Triangles that result from five partitions of the original triangle.

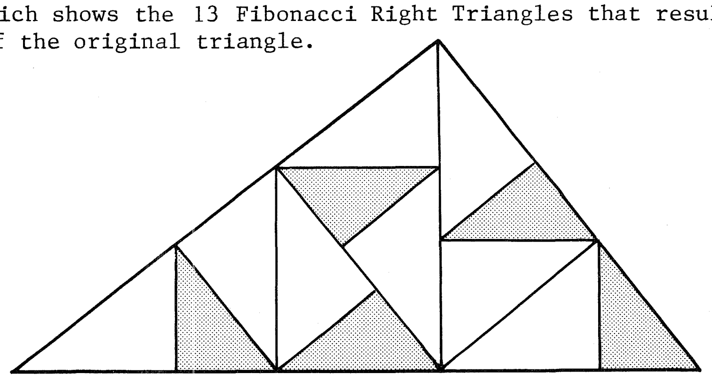


FIGURE 3. Five Partitions of a Fibonacci Right Triangle

In summary, successive partitions of a Golden Triangle provide a multifaceted geometric representation of the Fibonacci sequence. The triangles described above are Fibonacci in three different ways because they are in Fibonacci proportions with regard to their numbers, their areas, and the lengths of their sides. Golden Triangles not only embody the Fibonacci ratio, they also carry within them the ability to generate Fibonacci sequences.

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GEOMETRY OF A GENERALIZED SIMSON'S FORMULA

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1. Introduction

Articles of a geometrical nature relating to recurrence sequences have appeared in recent years in this journal (e.g. [1], [2], [6]).

The purpose of the present paper is to consider the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of successive numbers in recurrence sequences of a certain type. Readers might plot some points on the resulting noncontinuous curves (conics).

Extension to higher-dimensional space is briefly discussed.

2. The General Conic

Begin by defining [4] the general term of the sequence $\{w_n(a, b; p, q)\}$ as

$$w_{n+2} = pw_{n+1} - qw_n, \; w_0 = a, \; w_1 = b, \tag{1}$$

where a, b, p, and q belong to some number system, but are usually thought of as integers. Write [4]

$$e = pab - qa^2 - b^2. \tag{2}$$

Now [4]

$$w_n w_{n+2} - w_{n+1}^2 = eq^n, \tag{3}$$

which is a generalization of Simson's formula

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1} \tag{4}$$

occurring in the Fibonacci sequence $\{F_n\} = \{w_n(0, 1; 1, -1)\}$.

Equation (3) generalizes the famous geometrical paradox associated with (4). For the details, see [5].

From (1) and (3), we obtain

$$qw_n^2 + w_{n+1}^2 - pw_n w_{n+1} + eq^n = 0. \tag{5}$$

Next, put $w_n = x$, $w_{n+1} = y$. Then, by (5),

$$qx^2 + y^2 - pxy + eq^n = 0. \tag{6}$$

This equation represents a conic in rectangular Cartesian coordinates (x, y). Anticlockwise rotation of axes through an angle

$$\frac{1}{2} \tan^{-1}\left(\frac{-p}{q - 1}\right)$$

eliminates the xy term and produces the canonical form of the conic (an ellipse if $p^2 < 4q$, a hyperbola if $p^2 > 4q$, where the degenerate cases are excluded). Equation (6) is also obtainable by laborious reduction of the general equation of a conic using the uniqueness of a conic through 5 given points.

3. Some Particular Cases

I. $q < 0$ (Hyperbolas)

(a) $p = 1$, $q = -1$: Substituting in (6) yields the two systems (n even, n odd) of rectangular hyperbolas

(7) $$x^2 - y^2 + xy = e_1(-1)^n \qquad (e_1 = a^2 - b^2 + ab),$$

asymptotes of which are the perpendicular lines

(8) $$y = \alpha x,\ y = -\frac{1}{\alpha}x,$$

in which $\alpha = \dfrac{1 + \sqrt{5}}{2}$, the positive root of $t^2 - t - 1 = 0$. For the *Fibonacci sequence* ($a = 0$, $b = 1$) and the *Lucas sequence* ($a = 2$, $b = 1$), it follows that $e_1 = -1$ and 5, respectively. These *Fibonacci-type* curves (7) approach their asymptotes remarkably quickly.

With a fixed e_1 in (7), a hyperbola for which n is odd (even) may be transformed into the corresponding hyperbola for which n is even (odd), by a reflection in $y = x$ followed by a reflection in the y-axis (x-axis).

(b) $p = 2$, $q = -1$: For the *Pell sequence* ($a = 0$, $b = 1$), (6) gives

(9) $$x^2 - y^2 + 2xy = (-1)^{n+1},$$

rectangular hyperbolas with perpendicular asymptotes $y = kx$, $y = -\frac{1}{k}x$, where $k = 1 + \sqrt{2}$ is the positive root of $t^2 - 2t - 1 = 0$.

Gradients of the perpendicular asymptotes of the hyperbolas (6) for which $p > 0$, $q = -1$ are given by the roots of $t^2 - pt - 1 = 0$.

II. $q > 0$

Equation (6) now represents ellipses if $4q > p^2$ and hyperbolas if $4q < p^2$. For example, the loci for the *Fermat sequences*

$$\{w_n(0,\ 1;\ 3,\ 2)\} \quad \text{and} \quad \left\{w_n\left(\frac{3}{2},\ 2;\ 3,\ 2\right)\right\}$$

are hyperbolas (one point for each n)

(10) $$2x^2 + y^2 - 3xy = 2^n$$

and

(11) $$2x^2 + y^2 - 3xy = -2^{n-1}.$$

Further, for the *Chebyshev sequences*

$$\{w_n(1,\ 2\lambda;\ 2\lambda,\ 1)\} \quad \text{and} \quad \{w_n(2,\ 2\lambda;\ 2\lambda,\ 1)\},$$

where $\lambda = \cos\theta$, we obtain the ellipses

(12) $$x^2 + y^2 - 2\lambda xy = 1$$

and

(13) $$x^2 + y^2 - 2\lambda xy = 4 - 4\lambda^2.$$

III. Degenerate Case

When $\lambda = 1$ in (12), i.e., for the sequence $\{w\ (1,\ 2;\ 2,\ 1)\}$, of *integers*, we have the degenerate curve $x^2 + y^2 - 2xy = 1$, i.e., the line

$$x - y = -1.$$

No values of $x + y$, as defined, satisfy the equation $x - y = 1$. Successive pairs of *odd integers* and of *even integers*, generated by

$$\{w_n(1,\ 3;\ 2,\ 1)\} \quad \text{and} \quad \{w_n(2,\ 4;\ 2,\ 1)\},$$

respectively, satisfy the line

(15) $$x - y = -2.$$

4. Extension to Higher Space

Equations of the third, fourth, and higher degrees that are based on second-order recurrences like (1) (see, e.g. [3], [4]) cannot yield any nondegenerate loci in spaces of dimension greater than two.

For three-dimensional (nonprojective) space, it is necessary to consider third-order recurrence relations, of which the simplest is

(16) $$P_{n+3} = P_{n+2} + P_{n+1} + P_n \qquad (n \geq 0).$$

Waddill and Sacks [8] have established the following relation for $\{P_n\}$ corresponding to the Simson formula (4) for $\{F_n\}$:

(17) $$\begin{aligned} &P^2_{n+3}P_n + P^3_{n+2} + P^2_{n+1}P_{n+4} - P_{n+4}P_{n+2}P_n - 2P_{n+3}P_{n+2}P_{n+1} \\ &= P^3_0 + 2P^3_1 + P^3_2 + 2P^2_0P_1 + 2P_0P^2_1 + P^2_0P_2 - 2P_1P^2_2 - 2P_0P_1P_2 - P_0P^2_2. \end{aligned}$$

Putting $P_0 = 0$, $P_1 = P_2 = 1$ and $P_0 = 1$, $P_1 = 0$, $P_2 = 1$ they obtained their sequences $\{K_n\}$ and $\{Q_n\}$, respectively:

(18) $\{K_n\}$: 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ...;

(19) $\{Q_n\}$: 1, 0, 1, 2, 3, 6, 11, 20, 37, 68, 125, 230, ...

Letting $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$ in (17), we derive, after some algebraic manipulation,

(20) $$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = A,$$

where $A = 1$ for $\{K_n\}$ and $A = 2$ for $\{Q_n\}$. Equations (20) represent cubic surfaces in Euclidean space of three dimensions.

More general forms of (16) would lead to extremely cumbersome equations.

Observe that if we label any three successive numbers in (18) as x, y, z, and the corresponding three numbers in (19) as X, Y, Z, then we perceive that $X = y - x$, $Y = z - y$, $Z = x + y$.

Fourth- and higher-order recurrences should produce equations corresponding to (17) which are generalizations of Simson's formula (4). While (17) is not a pretty sight, the mind boggles at the prospect of further extensions, which we accordingly do not investigate. But the general pattern seems clear: a recurrence of the nth order ought to lead to a hypersurface (of dimension $n - 1$) in Euclidean n-space.

5. Concluding Comments

a. For the sequence $\{w_n(1, \alpha; 1, -1)\}$, $e = 0$ [see (1), (2), (7)] and the curve (7) degenerates to the line-pair $x^2 + xy - y^2 = 0$.

b. Graphing the Fibonacci numbers F_n against n reveals that they asymptotically approach the exponential values

$$\lim_{n \to \infty} F_n = \frac{\alpha^n}{\sqrt{5}} \qquad \left(\text{where } F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},\ \beta = -\frac{1}{\alpha}\right).$$

c. M. H. Eggar, in "Applications of Fibonacci Numbers" [*The Mathematical Gazette* 63 (1979):36-39], refers to (7), in the case where $e_1 = 1$, though his context is nongeometrical.

d. Interest in the theme of this article was stimulated by a private communication to the author in 1980 by L. G. Wilson, who determined the vertex of the hyperbola (7) for the Fibonacci sequence, but only in the case where n is odd, namely, $x = 0.920442065\ldots$, $y = 0.217286896\ldots$. He also calculated the angle of inclination of the axis of this hyperbola to the x-axis, namely,

$$13.28252259\ldots \text{ degrees } [(\doteqdot 13°17') = \tan^{-1}(\sqrt{5} - 2)].$$

Furthermore, Wilson briefly investigated the geometry of the third-order sequence $\{T_n\}$:

$$0,\ 2,\ 3,\ 6,\ 10,\ 20,\ 35,\ 66,\ \ldots, \tag{21}$$

defined in Neumann-Wilson [7] by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n + (-1)^n \qquad (n \geq 0). \tag{22}$$

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AN ENTROPY VIEW OF FIBONACCI TREES*

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Abstract

In a binary tree with n terminal nodes weighted by probabilities $p_1, \ldots, p_n$, $\Sigma p_i = 1$, it is assumed that each left branch has cost 1 and each right branch has cost 2. The cost a_i of terminal node p_i is defined to be the sum of costs of branches that form the path from the root to this node. The sum $\Sigma p_i a_i$ is called the average cost of the tree. As a top-down tree-building rule we consider *ψ-weight-balancing* which constructs a binary tree by successive dichotomies of the ordered set $p_1, \ldots, p_n$ according to a certain weight ratio closely approximating the golden ratio. Let $H = H(p_1, \ldots, p_n) = -\Sigma p_i \log p_i$ be the Shannon entropy of these probabilities. The ψ-weight-balancing rule is motivated by the fact that the entropy per unit of cost

$$H(x, 1 - x)/(1 \cdot x + 2 \cdot (1 - x))$$

for the division $x : (1 - x)$ of the unit interval is maximized when

$$x = \psi = (\sqrt{5} - 1)/2,$$

the golden cut point. It is then shown that the average cost of the tree built by ψ-weight-balancing is bounded above by $H/(-\log \psi) + 1$, if the terminal nodes have probabilities $p_1, \ldots, p_n$, $p_1 \geq \cdots \geq p_n$, from left to right in this order in the tree. If $p_{j+1}/p_j \geq (1/2)\psi$ for each j, the above bound can be improved to $H/(-\log \psi) + \psi$. For the case $p_1 = \cdots = p_n$, we obtain the following results. The ψ-weight-balancing constructs an optimal tree in the sense of minimum average cost and constructs the Fibonacci tree of order k when $n = F_k$, the kth Fibonacci number. The average cost of the optimal tree is given exactly. Furthermore, for an arbitrarily given number of terminal nodes, the ψ-weight-balanced tree is also "balanced" in the sense of Adelson-Velskii and Landis, and is the highest of all balanced trees.

We will discuss some properties of Fibonacci (Fibonaccian) trees in view of their construction by an entropic weight-balancing, beginning with the following preparatory section:

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1. Binary Tree with Branch Cost

Let us consider a binary tree (rooted and ordered) with $n - 1$ internal nodes (branch nodes) and n terminal nodes (leaves) [6]. An internal node has two sons, while a terminal node has no sons. A node is at level ℓ if the path from the root to this node has ℓ branches. The terminal nodes are assumed to be associated, from left to right, with probabilities or weights $p_1, \ldots, p_n$, $\Sigma p_i = 1$. We assume, furthermore, that every left branch has unit cost 1, and every right branch has cost c (≥ 1). A node is then associated with two numbers, probability and cost; the probability of an internal node is defined to be the sum of probabilities of its descendant terminal nodes, and the cost of a node is defined to be the sum of costs of branches that form the path from the root to this node. The root, then, has probability 1 and cost 0. Sometimes, for simplicity, a node will be named by the associated probability. We define the *average cost* of a tree as

$$C = \sum_{i=1}^{n} p_i a_i,$$

where a_i is the cost of the terminal node p_i. Since we interpret C as the average cost required to get to a terminal node by tracing the corresponding path from the root, C measures a global goodness of the tree: for fixed n, c, $p_1, \ldots, p_n$, the smaller C is, the more economical the tree is. If we view the binary tree, for example, as representing a binary code consisting of n codewords with code symbols 0 of duration 1 (corresponding to the left branch) and 1 of duration c (to the right) for the given source alphabet having letter-probabilities $p_1, \ldots, p_n$, then C is the average time needed to send one source letter.

An internal node will be called internal node j, $1 \leq j \leq n - 1$, if its left subtree has p_j as the rightmost terminal node. (The leftmost terminal node of its right subtree is then p_{j+1}.) Let us denote by L_j and R_j the probabilities of the left and the right sons of the internal node j, respectively. Put

$$T_j = L_j + R_j,$$

which is, of course, the probability of the internal node j.

We give here three general relations—(1), (2), and (3)—for use in later sections. First we have

$$C = \sum_{j=1}^{n-1} (L_j + cR_j). \tag{1}$$

This is seen by observing that the cost 1 [resp. c] of the left [right] branch that connects the internal node j and its left [right] son contributes $1 \cdot L_j$ [$c \cdot R_j$] to C.

Second, let

$$H \equiv H(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log p_i$$

be the Shannon entropy. (Logarithms will always be to the base 2.) We have

$$H = \sum_{j=1}^{n-1} T_j H\left(\frac{L_j}{T_j}, \frac{R_j}{T_j}\right). \tag{2}$$

This is the well-known binary branching property of the entropy [4, 9]. The entropy is known as a very appropriate function to measure the uncertainty, the uniformness, or the randomness, of the probability distribution $p_1, \ldots, p_n$. It is this aspect and the branching property that relates the entropy to the economical structures of the trees having weighted terminal nodes, as will be seen in the following sections.

Third, letting b_j be the cost of the internal node j, we have

Lemma 1:

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n-1} b_j + (n-1)(1+c). \tag{3}$$

Proof (by induction on n): When $n = 1$, (3) is trivially true. Consider an arbitrary tree with $n+1$ terminal nodes. At the maximum level there exist two terminal nodes that are sons of the same internal node, say k. Merge these nodes into k to obtain a tree having n terminal nodes. The decrease in the total cost of terminal nodes due to this merging is given by

$$(b_k + 1) + (b_k + c) - b_k = b_k + (1 + c).$$

On the other hand, the decrease in the total cost of internal nodes is b_k. This completes the proof.

2. Weight-Balancing and "Discrete" Golden Cut

A binary tree can be viewed as a pattern of successive choices between the left and the right branches started from the root in order to look for a terminal node. The uncertainty per unit of cost, removed by the choice at the internal node j, is measured by

$$\frac{H\left(\frac{L_j}{T_j}, \frac{R_j}{T_j}\right)}{1 \cdot \left(\frac{L_j}{T_j}\right) + c \cdot \left(\frac{R_j}{T_j}\right)}.$$

So the tree that maximizes this quantity at each step can be expected to have a small average cost C. Of course, the successive local optimizations of this type will not necessarily lead to a global minimization of the average cost. Nevertheless, we will be concerned with this process because it is interesting in its own right.

In the case $c = 1$, the above quantity reduces to $H(L_j/T_j, R_j/T_j)$, which becomes maximum when $|L_j - R_j|$ is minimum, i.e., when

$$L_j - p_j/2 < T_j/2 \leq L_j + p_{j+1}/2,$$

for fixed T_j. The rule for constructing a tree in a top-down, level-by-level manner, such that at each step L_j and R_j are made as equal as possible, is called "weight-balancing." The binary code corresponding to the tree thus built by weight-balancing under the monotonocity condition $p_1 \geq \cdots \geq p_n$ is known as the Shannon-Fano code ([3], [9], see also [11]). This code is not necessarily optimal (in the sense of minimum average cost), but it is almost optimal, and satisfies

$$C \le H + (1 - 2p_n)$$

(see [4]). *Henceforth, we assume* $p_1 \ge \cdots \ge p_n$.

In order to generalize the above weight-balancing rule to the general c, we naturally maximize the function

$$\frac{H(x,\ 1 - x)}{x + c(1 - x)},\ 0 \le x \le 1. \tag{4}$$

Let λ be the maximizing value of x. By differentiating, λ is the unique positive root of $x^c = 1 - x$. The maximum value of the function is $-\log \lambda$. Considering $\lambda = \lambda(c)$ as a function of c, we have $\lambda(1) = 1/2$, $\lambda(c)$ is strictly monotone increasing, and $\lambda(c) \to 1$ as $c \to \infty$. Now define λ-*weight-balancing* as a rule for constructing a tree satisfying

$$L_j - (1 - \lambda)p_j < \lambda T_j \le L_j + \lambda p_{j+1} \tag{5}$$

for each internal node $j = 1, \ldots, n - 1$. Recently, K. Mehlhorn has taken up a similar rule to study search trees [8]. *We shall be confined especially to the case* $c = 2$, where the "λ-cut" $\lambda : (1 - \lambda)$ of the unit interval becomes the golden cut, since we have $\lambda(2) = (\sqrt{5} - 1)/2 = 0.618\ldots$. We denote this number by ψ, its inverse $\psi^{-1} = \phi$ being commonly called the golden ratio, and $\psi^2 = 1 - \psi$, $-\log \psi = 0.694\ldots$. [Conversely, if $x = \psi$ maximizes (4), then c must be 2.]

3. Bounds on the Average Cost

For a reason that will be clear in the next section, trees constructed by ψ-weight-balancing may be called "Fibonaccian trees." In this section we find entropic bounds on the average cost of Fibonaccian trees. Since we are treating $c = 2$, and $-\log \psi$ is the maximum value of $H(x, 1 - x)/(2 - x)$, the function

$$f(x) = (-\log \psi)(2 - x) - H(x,\ 1 - x),\ 0 \le x \le 1,$$

is nonnegative.

Theorem 1: $\dfrac{H}{-\log \psi} \le C \le \dfrac{H}{-\log \psi} + (1 - p_n)$. [Note that $H/(-\log \psi)$ is the entropy with respect to the log-base ϕ, i.e., $H/(-\log \psi) = -\Sigma p_i \log_\phi p_i$.]

Proof: The proof technique is that used in [5]. Consider the difference $(-\log \psi)C - H$. From (1) and (2) in Section 1, we have

$$(-\log \psi)C - H = \sum_{j=1}^{n-1} d_j,$$

where

$$d_j = T_j\left\{(-\log \psi)\left(\frac{L_j}{T_j} + 2 \cdot \frac{R_j}{T_j}\right) - H\left(\frac{L_j}{T_j},\ \frac{R_j}{T_j}\right)\right\} = T_j f\left(\frac{L_j}{T_j}\right).$$

The fact $f(x) \geq 0$ implies the left-side inequality to be proved. (This lower bound is well known, see [2], and is valid for any tree, as we see from the proof.) To prove the upper bound, split (5) with $\lambda = \psi$ into two cases, for each j:

Case 1:

$$L_j - (1 - \psi)p_j < \psi T_j \leq L_j. \tag{6}$$

Equation (6) leads us to $\psi \leq L_j/T_j < 1$. The function $f(x)$ is clearly convex downward, and $f(\psi) = 0$, $f(1) = -\log \psi$. Hence,

$$f(x) \leq (-\log \psi)\frac{x - \psi}{1 - \psi} \quad \text{if } \psi \leq x \leq 1.$$

Therefore,

$$d_j = T_j f\left(\frac{L_j}{T_j}\right) \leq (-\log \psi)\frac{L_j - \psi T_j}{1 - \psi}.$$

But by the left-side inequality of (6), we have $L_j - \psi T_j < (1 - \psi)p_j$. Hence, $d_j < (-\log \psi)p_j$.

Case 2:

$$L_j < \psi T_j \leq L_j + \psi p_{j+1}. \tag{7}$$

The right-side inequality of Eq. (7), the obvious $p_j \leq L_j$, and the assumption $p_1 \geq \cdots \geq p_n$ imply $\psi T_j \leq L_j + \psi p_{j+1} \leq L_j + \psi p_j \leq L_j + \psi L_j$. Hence,

$$1 - \psi = \frac{\psi}{1 + \psi} \leq \frac{L_j}{T_j}.$$

This and the left-side inequality of (7) give

$$1 - \psi \leq \frac{L_j}{T_j} < \psi.$$

Now we have $f(\psi) = 0$ and

$$\begin{aligned} f(1 - \psi) &= (-\log \psi)(1 + \psi) - H(\psi, 1 - \psi) \\ &= (-\log \psi)(1 + \psi) - (-\log \psi)(2 - \psi) \\ &= (-\log \psi)(2\psi - 1). \end{aligned}$$

Therefore, by the downward convexity of $f(x)$, we have

$$f(x) \leq (-\log \psi)(\psi - x) \quad \text{if } 1 - \psi \leq x \leq \psi.$$

Hence,

$$d_j = T_j f\left(\frac{L_j}{T_j}\right) \leq (-\log \psi)(\psi T_j - L_j).$$

But by the right-side inequality of (7), we have $\psi T_j - L_j \leq \psi p_{j+1}$. Hence,

$$d_j \leq (-\log \psi)\psi p_{j+1} \leq (-\log \psi)p_{j+1}.$$

In either case, we have

$$d_j \le (-\log \psi)p_j, \quad j = 1, \ldots, n - 1,$$

since $p_j \ge p_{j+1}$. This finishes the proof.

For example, take the English alphabet including "space" ($n = 27$) with letter-frequencies given in [7]. If we construct a tree by ψ-weight-balancing for this source, we obtain $H/(-\log \psi) = 5.885$, $C = 5.958$.

Remarks: The above proof can be modified to prove the same inequalities (with ψ replaced by λ) for the average cost of the tree built by λ-weight-balancing whenever $1/2 \le \lambda \le \psi$, i.e., $1 \le c \le 2$.

If we impose an appropriate condition on $p_1, \ldots, p_n$, we may somewhat improve the upper bound on C.

Theorem 2: If $\dfrac{p_{j+1}}{p_j} \ge \dfrac{1}{2}\psi$, $j = 1, \ldots, n - 1$, then

$$C \le \frac{H}{-\log \psi} + \psi(1 - p_n).$$

Proof: It is sufficient to show that for Case 1 in the proof of Theorem 1 we have $d_j \le (-\log \psi)\psi p_j$, because we have shown $d_j \le (-\log \psi)\psi p_{j+1}$ for Case 2. From (6) and the assumption, we see that

$$\psi \le \frac{L_j}{T_j} < \psi + (1 - \psi)\frac{p_j}{T_j} \le \psi + (1 - \psi)\frac{p_j}{p_j + p_{j+1}} \le \psi + (1 - \psi)\frac{1}{1 + \psi/2} = \frac{3 - \psi}{2 + \psi}.$$

The downward convexity of $f(x)$ and a direct numerical check show

$$f(x) \le (-\log \psi)\psi\frac{x - \psi}{1 - \psi} \quad \text{if} \quad \psi \le x \le \frac{3 - \psi}{2 + \psi},$$

from which it follows that

$$d_j = T_j f\left(\frac{L_j}{T_j}\right) \le (-\log \psi)\psi\frac{L_j - \psi T_j}{1 - \psi} \le (-\log \psi)\psi p_j, \text{ using (6).}$$

This completes the proof.

4. The Case $p_1 = \cdots = p_n$ and Fibonacci Trees

In this section, we shall restrict ourselves to the special but important case $p_1 = \cdots = p_n$, i.e., all terminal nodes have equal weight. Let us first define the *Fibonacci tree of order k* according to [7].

Let

$$(F_0, F_1, F_2, F_3, F_4, F_5, \ldots) = (0, 1, 1, 2, 3, 5, \ldots),$$

$$F_k = F_{k-1} + F_{k-2},$$

be the Fibonacci sequence. The Fibonacci tree of order k has F_k terminal nodes, and it is constructed as follows: If $k = 1$ or 2, the tree is simply the "terminal" root only. If $k \geq 3$, the left subtree is the Fibonacci tree of order $k - 1$; and the right subtree is the Fibonacci tree of order $k - 2$.

Remark: The Fibonacci tree we assign order k is called in [7] the "Fibonacci tree of order $k - 1$." We choose this indexing for its neatness in our argument.

Figure 1 is the Fibonacci tree of order 7.

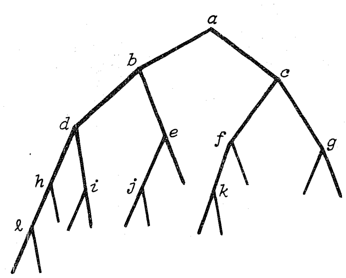

FIGURE 1. The Fibonacci tree of order 7

Lemma 2: The Fibonacci tree of order k, $k \geq 2$, has F_{k-1} terminal nodes of cost $k - 2$ and F_{k-2} terminal nodes of cost $k - 1$.

Proof (Induction on k): Trivially true when $k = 2$. The Fibonacci tree of order 3 obviously has one ($= F_2$) terminal node of cost 1 and one ($= F_1$) terminal node of cost 2. Suppose the lemma is true for each Fibonacci tree of order less than k, $k \geq 4$. By the construction of the Fibonacci tree of order k it has, in the left subtree, F_{k-2} terminal nodes of cost $(k - 3) + 1 = k - 2$ and F_{k-3} terminal nodes of cost $(k - 2) + 1 = k - 1$, and, in the right subtree, F_{k-3} terminal nodes of cost $(k - 4) + 2 = k - 2$ and F_{k-4} terminal nodes of cost $(k - 3) + 2 = k - 1$. Hence, the Fibonacci tree of order k has, in all, $F_{k-2} + F_{k-3} = F_{k-1}$ terminal nodes of cost $k - 2$ and $F_{k-3} + F_{k-4} = F_{k-2}$ terminal nodes of cost $k - 1$. This completes the proof.

Theorem 3: The average cost of the Fibonacci tree of order k is given by

$$C = \frac{F_{k-2}}{F_k} + (k - 2).$$

Proof: By Lemma 2, we have

$$C = \frac{1}{F_k}\{(k - 2)F_{k-1} + (k - 1)F_{k-2}\} = \frac{1}{F_k}\{F_{k-2} + (k - 2)F_k\}.$$

Since

$$F_k = \frac{1}{\sqrt{5}}\left\{\psi^{-k} - (-\psi)^k\right\}, \text{ by [6]},$$

we have $F_k \sim \frac{1}{\sqrt{5}}\psi^{-k}$ when k becomes large. Therefore, for large k,

$$C \sim (k - 2) + \psi^2 = k - 1 - \psi.$$

The following procedure, due to Varn [12], constructs an optimal tree (in the sense of minimum average cost) for general c: Suppose an optimal tree with $n - 1$ terminal modes has already been constructed. Split, in this tree, any one terminal node of minimum cost to produce two new terminal nodes. The resulting tree with n terminal nodes will be optimal. The validity of this procedure is an immediate consequence of Lemma 1:

$$\frac{1}{n}\sum_{i=1}^{n} a_i = \frac{1}{n}\sum_{j=1}^{n-1} b_j + \left(1 - \frac{1}{n}\right)(1 + c).$$

The left-hand side is the average cost to be minimized when $p_1 = \cdots = p_n$. To minimize the left-hand side is to minimize the sum of costs of $n - 1$ internal nodes; i.e., to minimize

$$\sum_{j=1}^{n-1} b_j.$$

Consider the infinite complete binary tree, and use the "greedy" procedure to pick the $n - 1$ cheapest nodes to be internal. It is easy to see that this grows a tree optimal at each step, the same tree as grown by Varn's procedure.

Returning to our case $c = 2$, we have the following:

Theorem 4: The Fibonacci tree of order k is optimal for each $k \geq 2$.

Proof: From Lemma 2, the Fibonacci tree of order $k \geq 2$ has F_{k-1} terminal nodes of cost $k - 2$ and F_{k-2} terminal nodes of cost $k - 1$. Hence, by Varn's procedure, it is sufficient to prove that if we split all terminal nodes of cost $k - 2$, then the resulting tree, which then has $F_{k-2} + 2F_{k-1} = F_{k-1} + F_k = F_{k+1}$ terminal nodes, is the Fibonacci tree of order $k + 1$. To prove this by induction on k, suppose the assertion is true for the Fibonacci trees of order less than k, $k \geq 3$. (When $k = 2$, the assertion is trivially true.) The left subtree of the Fibonacci tree of order k is the Fibonacci tree of order $k - 1$ with F_{k-2} terminal nodes of cost $(k - 3) + 1$. Splitting these nodes produces the Fibonacci tree of order k by the induction hypothesis. Similarly, the right subtree is the Fibonacci tree of order $k - 2$ with F_{k-3} terminal nodes of cost $(k - 4) + 2$. Splitting these nodes produces the Fibonacci tree of order $k - 1$ by the induction hypothesis. Therefore, splitting all terminal nodes of cost $k - 2$ of the Fibonacci tree of order k produces the Fibonacci tree of order $k + 1$.

Theorem 5: Express the number of terminal nodes by $n = F_k + r$ for some $k \geq 2$ and $0 \leq r < F_{k-1}$. The tree built according to ψ-weight-balancing is optimal, with the average cost given by

$$\frac{F_{k-2} + 3r}{F_k + r} + (k - 2).$$

When $r = 0$, the tree is the Fibonacci tree of order k.

Proof: When $k = 2$ or 3, the theorem is trivially true. Suppose $k \geq 4$. We prove by induction on the number of terminal nodes that the tree having $F_k + r$ terminal nodes and built by ψ-weight-balancing is the same as a tree constructed from the Fibonacci tree of order k by splitting r of its F_{k-1} terminal nodes of cost $k - 2$. Varn's procedure and Theorem 4, then, prove the optimality part of the theorem. When $n = 3$, we have $k = 4$ and $r = 0$. So the assertion is true, since the ψ-weight-balancing for $n = 3$ produces the Fibonacci tree of order 4. Suppose the assertion is true for each number of terminal nodes less than $n = F_k + r$, $k \geq 4$. By Lemma 2 and the construction of Fibonacci trees, there are F_{k-2} terminal nodes of cost $k - 2$ in the left subtree of the Fibonacci tree of order k, and there are F_{k-3} terminal nodes of cost $k - 2$ in the right subtree. Hence, we need only show that the ψ-weight-balancing "divides" $F_k + r$ into $F_{k-1} + s$, $0 \leq s < F_{k-2}$ for the left subtree, and $F_{k-2} + t$, $0 \leq t \leq F_{k-3}$ for the right subtree, with $s + t = r$. If this is true, then we can apply the induction hypothesis and incorporate the cost of the initial branch to find that the tree built by ψ-weight-balancing on the left is obtained by splitting s of its terminal nodes of cost $(k - 3) + 1$ and that on the right by splitting t of its terminal nodes of cost $(k - 4) + 2$.

Let us show, therefore, for the left, that the integer m given by

$$m - (1 - \psi) < \psi(F_k + r) \leq m + \psi,$$

corresponding to (5), satisfies $m = F_{k-1} + s$, $0 \leq s < F_{k-2}$. Using

$$F_{k-1} - \psi F_k = \frac{1}{\sqrt{5}}\{\psi^{-k+1} - (-\psi)^{k-1}\} - \frac{\psi}{\sqrt{5}}\{\psi^{-k} - (-\psi)^k\}$$

$$= (-\psi)^{k-1}\frac{1}{\sqrt{5}}(-1 - \psi^2) = (-\psi)^k,$$

the above inequalities may be written as

$$F_{k-1} + \psi r - \psi - (-\psi)^k \leq m < F_{k-1} + \psi r - \psi - (-\psi)^k + 1.$$

Since $(-\psi)^k < \psi^2 = 1 - \psi$, we have

$$-1 < -\psi - (-\psi)^k \leq \psi r - \psi - (-\psi)^k,$$

and, on the other hand,

$$\begin{aligned}\psi r - \psi - (-\psi)^k + 1 &\leq \psi(F_{k-1} - 1) - \psi - (-\psi)^k + 1\\ &= F_{k-2} - (-\psi)^{k-1} - \psi - (-\psi)^k + \psi^2\\ &= F_{k-2} - \psi\{\psi^2 - (-\psi)^k\} < F_{k-2}.\end{aligned}$$

Therefore, $F_{k-1} - 1 < m < F_{k-1} + F_{k-2}$; thus, $m = F_{k-1} + s$ for some s such that $0 \leq s < F_{k-2}$.

Similarly, we can show, for the right, using $F_{k-2} - (1 - \psi)F_k = -(-\psi)^k$, that the integer m given by

$$m - \psi \leq (1 - \psi)(F_k + r) < m + (1 - \psi)$$

satisfies $m = F_{k-2} + t$, $0 \leq t \leq F_{k-3}$.

The average cost, then, using Lemma 2, is given by

$$C = \frac{1}{F_k + r}\{(k - 2)(F_{k-1} - r) + (k - 1)(F_{k-2} + r) + kr\}$$
$$= \frac{F_{k-2} + 3r}{F_k + r} + (k - 2).$$

This completes the proof.

The *height* of a tree is defined as its maximum level, the length (the number of branches) of the longest path from the root to a terminal node. A binary tree is called *balanced* (the concept due to Adelson-Velskii and Landis [1]) if the height of the left subtree of every internal node never differs by more than 1 from the height of its right subtree.

Theorem 6: When $p_1 = \cdots = p_n$, the tree built by ψ-weight-balancing is balanced.

Proof: Let the number of terminal nodes be $F_k + r$, $0 \leq r < F_{k-1}$. It is easily seen, by induction on k, that the Fibonacci tree of order $k \geq 2$ is of height $k - 2$ and hence balanced, and, if $k \geq 4$, has only the leftmost two terminal nodes at the maximum level, with cost $k - 2$ (for the left node) and $k - 1$ (for the right node). From the proof of Theorem 5, the ψ-weight-balanced tree having $F_k + r$ terminal nodes is made by splitting $r = s + t$ $(0 \leq s < F_{k-2})$ terminal nodes of cost $k - 2$ of the Fibonacci tree of order k with s from the left subtree and t from the right subtree. In this splitting process, the leftmost terminal node at the maximum level is, however, never split as long as $0 \leq r < F_{k-1}$, and the tree remains balanced. Since $s < F_{k-2}$, $t \leq F_{k-3}$, this assertion is readily seen by induction.

Theorem 7: For $p_1 = \cdots = p_n$ and an arbitrarily given number of terminal nodes (or branch nodes), the tree built by ψ-weight-balancing is the highest of all balanced trees.

Proof: It is easily seen by induction on height that the balanced tree of height h with a minimum number of terminal nodes is the Fibonacci tree of order $h + 2$ [8]. Now suppose that there exists a balanced tree of height h with $F_k + r$ $(0 \leq r < F_{k-1})$ terminal nodes, then

$$F_{h+2} \leq F_k + r < F_k + F_{k-1} = F_{k+1},$$

hence, $h \leq k - 2$. But from the proof of Theorem 6 we know that the ψ-weight-balanced tree on $F_k + r$ nodes has height $k - 2$.

A Hypothetical Class of "Natural Trees"

It is amusing to draw (suggested by [10]) the Fibonacci trees upside down so that they look like real trees or shrubs, with each branch of cost 2 about twice (relatively) as long as its brother branch of cost 1 (i.e., bifurcation ratio 1:2; asparagus, as the author observed, seems to grow in this way). There may be variations in drawing. Figure 2 is a corresponding sketch of the Fibonacci tree of order 7 shown in Figure 1. As we saw in the last section, the simple repeating pattern (Fibonacci recursive rule) in the Fibonacci tree implies, and is implied by, the entropic balancing of the tree. This, along with the properties given in Theorems 5 and 7, might be of morphological interest for a class of mathematical "natural trees."

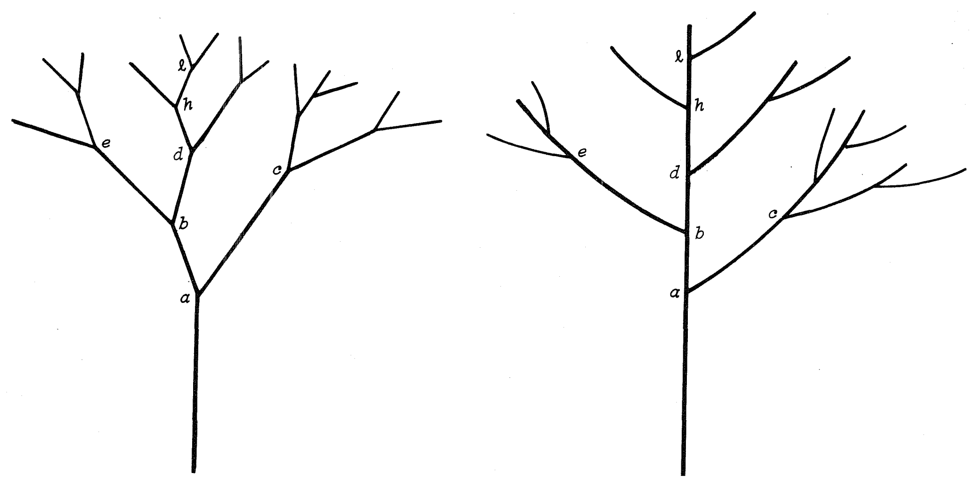

FIGURE 2. Sketch of a "natural tree"

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-472 *Proposed by Gerald E. Bergum, S. Dakota State University, Brookings, SD*

Find a sequence $\{T_n\}$ satisfying a second-order linear homogeneous recurrence $T_n = aT_{n-1} + bT_{n-2}$ such that every even perfect number is a term in $\{T_n\}$.

B-473 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $a = L_{1000}$, $b = L_{1001}$, $c = L_{1002}$, $d = L_{1003}$. Is $1 + x + x^2 + x^3 + x^4$ a factor of $1 + x^a + x^b + x^c + x^d$? Explain.

B-474 *Proposed by Philip L. Mana, Albuquerque, NM*

Are there an infinite number of positive integers n such that

$$L_n + 1 \equiv 0 \pmod{2n}?$$

Explain.

B-475 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k$. Prove that $|S_3(n) - S_1^2(n)|$ is $2[(n + 1)/2]$ times a triangular number. Here [ ] denotes the greatest integer function.

B-476 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k$. Prove that $|S_4(n) + S_2(n)|$ is twice the square of a triangular number.

B-477 *Proposed by Paul S. Bruckman, Sacramento, CA*

Prove that $\sum_{n=2}^{\infty} \text{Arctan } \frac{(-1)^n}{F_{2n}} = \frac{1}{2} \text{ Arctan } \frac{1}{2}$.

SOLUTIONS

Casting Out 27's

B-446 *Proposed by Jerry M. Metzger, University of N. Dakota, Grand Forks, ND*

It is familiar that a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3. The same is true for 9. For 27, this is false since, for example, 27 divides 1 + 8 + 9 + 9 but does not divide 1899. However, $27 | 1998$.

Prove that 27 divides the sum of the digits of n if and only if 27 divides one of the integers formed by permuting the digits of n.

Solution by Paul S. Bruckman, Concord, CA

Given

(1) $$N = \sum_{k=0}^{m} a_k 10^k,$$

where the a_k's are decimal digits, let the sum of the digits be given by

(2) $$s(N) = \sum_{k=0}^{m} a_k.$$

We begin by observing that the statement of the problem is false. The correct statement should read as follows: If 27 divides the sum of the digits of N, then 27 divides one of the integers formed by permuting the digits of N. The converse is clearly false, since, e.g., $27 | 27$ but $27 \nmid s(27) = 9$.

Suppose

(3) $$27 | s(N).$$

The smallest positive integer N satisfying (3) is 999. Since $27 | 999$, we see that the (modified) proposition is verified for $N = 999$. We may therefore suppose $m \geq 3$.

Since $9 | s(N)$, thus $9 | N$. Let $\mathcal{P}_N$ denote the set of all possible integers M formed by permuting the digits of N. Since $s(M) = s(N)$ for all $M \in \mathcal{P}_N$, we see that $9 | M$ for all $M \in \mathcal{P}_N$. We will assume that $M \equiv \pm 9 \pmod{27}$ for all $M \in \mathcal{P}_N$ and show that this leads to a contradiction.

Given k $(0 \leq k \leq m - 2)$, form $N_k^{(1)} \in \mathcal{P}_N$ and $N_k^{(2)} \in \mathcal{P}_N$ by merely permuting the triple (a_k, a_{k+1}, a_{k+2}) to (a_{k+1}, a_{k+2}, a_k) and (a_{k+2}, a_k, a_{k+1}), respectively. Then

$$\begin{aligned} N_k^{(1)} - N &= 10^k(a_{k+1} - a_k) + 10^{k+1}(a_{k+2} - a_{k+1}) + 10^{k+2}(a_k - a_{k+2}) \\ &\equiv 10^k\{a_{k+1} - a_k + 10(a_{k+2} - a_{k+1}) + 19(a_k - a_{k+2})\} \pmod{27} \\ &\equiv -9 \cdot 10^k(a_k + a_{k+1} + a_{k+2}) \pmod{27} \\ &\equiv -9(a_k + a_{k+1} + a_{k+2}) \pmod{27}. \end{aligned}$$

Similarly, we find that

$$N_k^{(2)} - N \equiv 9(a_k + a_{k+1} + a_{k+2}) \pmod{27}.$$

Having assumed that $N \equiv \pm 9 \pmod{27}$, we cannot have

$$a_k + a_{k+1} + a_{k+2} \equiv \pm 1 \pmod{3},$$

for we would then have either

$$N_k^{(1)} \equiv 0 \quad \text{or} \quad N_k^{(2)} \equiv 0 \pmod{27},$$

contradicting the assumption. Since k is arbitrary, we must therefore have

(4) $$a_k + a_{k+1} + a_{k+2} \equiv 0 \pmod{3}, \; k = 0, 1, \ldots, m - 2.$$

Thus the sum of any three consecutive digits of N must be divisible by 3. But we see by symmetry that this same property must hold for all $M \in \mathcal{P}_N$. This can only be true if all the a_k's are congruent (mod 3).

Suppose, therefore, that

(5) $$a_k = 3b_k + r, \text{ where } b_k = 0, 1, 2, \text{ or } 3, \quad r = 0, 1, \text{ or } 2, \quad \text{with } b_k = 3 \text{ only if } r = 0.$$

Let

(6) $$B = \sum_{k=0}^{m} b_k 10^k.$$

Then $N = \sum_{k=0}^{m}(3b_k + r)10^k = 3B + r\sum_{k=0}^{m}10^k$, or

(7) $$N = 3B + \frac{r}{9}(10^{m+1} - 1).$$

Also,

$$s(N) = \sum_{k=0}^{m}(3b_k + r), \text{ or}$$

(8) $$s(N) = 3s(B) + r(m + 1).$$

We consider two (*a priori*) possibilities:

(a) $m \equiv 0$ or $1 \pmod 3$. Since $3|s(N)$, we see from (8) that $r = 0$. Hence, $N = 3B$ and $s(N) = 3s(B)$. But

$$27|s(N) \Longrightarrow 9|s(B) \Longrightarrow 9|B \Longrightarrow 27|N,$$

which contradicts our assumption. This leaves the only remaining possibility:

(b) $m \equiv 2 \pmod 3$. Let $m = 3t - 1$, $t \geq 2$. Note that

$$\frac{r}{9}(10^{m+1} - 1) = \frac{r}{9}(10^{3t} - 1) = r\sum_{k=0}^{3t-1} 10^k = 111r\sum_{k=0}^{t-1} 10^{3k}$$

$$\equiv 3r\sum_{k=0}^{t-1} 1 \pmod{27} \equiv 3rt \pmod{27}.$$

Thus, using (7),

$$N \equiv 3B + 3rt \pmod{27} \Longrightarrow \frac{N}{3} \equiv B + rt \pmod 9.$$

Also, $\frac{s(N)}{3} = s(B) + rt \equiv B + rt \pmod 9$. Hence, $\frac{N}{3} \equiv \frac{s(N)}{3} \pmod 9$, which, together with $27|s(N)$ implies $27|N$, again contradicting our assumption.

Thus the assumption is false, establishing the (modified) proposition.

Also solved by the proposer.

Casting Out Eights

B-447 *Based on the previous proposal.*

Is there an analogue of B-446 in base 5?

Solution by Paul S. Bruckman, Concord, CA

Given

$$N = \sum_{k=0}^{m} a_k 5^k, \tag{1}$$

where the a_k's are digits in base 5 ($a_k = 0, 1, 2, 3,$ or 4), let the sum of the digits be given by

$$s(N) = \sum_{k=0}^{m} a_k. \tag{2}$$

We note that $N - s(N) = \sum_{k=0}^{m} a_k(5^k - 1) \equiv 0 \pmod 4$, since $5^k \equiv 1 \pmod 4$. Thus

$$4|N \text{ iff } 4|s(N). \tag{3}$$

The analogue suggested by B-446 would probably read as follows: If 8 divides the sum of the digits of N (in base 5), then 8 divides one of the integers formed by permuting the digits of N (in base 5).

Unfortunately, the above proposition is false, unless additional conditions on N are specified. A counterexample is

$$N = 3{,}908 = (111113)_5;$$

although $8|s(N) = 8$, we find that all six integers formed by permuting the digits (in base 5) of N are congruent to 4 (mod 8), and therefore not divisible by 8.

The following is the corrected (though more complicated) version of the proposition: If 8 divides the sum of the digits of N (in base 5), then 8 divides one of the integers formed by permuting the digits of N (in base 5), *unless* all the digits of N are odd (i.e., 1 or 3) and the number of such digits is congruent to 2 (mod 4). In the latter case, all the permutations are congruent to 4 (mod 8).

(The proof is similar to that of B-446 and was deleted by the Elementary Problems Editor.)

Sum of Products Modulo 5

B-448 *Proposed by Herta T. Freitag, Roanoke, Va*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} F_{5i+1}L_{5i} \equiv 0 \pmod{5}.$$

Solution by John Ivie, Glendale, AZ

It suffices to show that each pair $F_{5i+1}L_{5i} + F_{5i+6}L_{5i+5i}$ in the summation is divisible by 5. Using the Binet Formula, this pair equals

$$F_{10i+1} + F_{10i+11} = 5L_{10i+6}.$$

Also solved by Paul S. Bruckman, Bob Prielipp, Charles B. Shields, Sahib Singh, Lawrence Somer, Charles R. Wall, Stephen Worotynec, Gregory Wulczyn, and the proposer.

Sum of Products Modulo 7

B-449 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} (-1)^{i+1}F_{8i+1}L_{8i} \equiv 0 \pmod{7}.$$

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

It is easy to show that

$$(-1)^{i+1}F_{8i+1}L_{8i} = (-1)^{i+1}F_{16i+1} + (-1)^{i+1}.$$

The powers of -1 telescope, since the number of summands is even, and thus

$$\sum_{i=1}^{2t} (-1)^{i+1}F_{8i+1}L_{8i} = \sum_{i=1}^{2t} (-1)^{i+1}F_{16i+1}.$$

We group the $2t$ summands in the latter sum into t pairs, each of which is divisible by 21:

$$F_{16i+1} - F_{16i+17} = -F_8L_{16i+9} = -21L_{16i+9}.$$

The stronger result, that the original sum is divisible by 21 (rather than merely 7), follows at once.

Also solved by Paul S. Bruckman, John Ivie, Bob Prielipp, Sahib Singh, Lawrence Somer, Stephen Worotynec, Gregory Wulczyn, and the proposer.

Lucas Quadratic Residue

B-450 *Proposed by Lawrence Somer, Washington, D.C.*

Let the sequence $\{H_n\}_{n=0}^{\infty}$ be defined by $H_n = F_{2n} + F_{2n+2}$.

(a) Show that 5 is a quadratic residue modulo H_n for $n \geq 0$.
(b) Does H_n satisfy a recursion relation of the form $H_{n+2} = cH_{n+1} + dH_n$, with c and d constants? If so, what is the relation?

Solution by E. Primrose, University of Leicester, England

We prove (b) first, and use it to prove (a).

(b) Examination of the first few terms suggests that

$$H_{n+2} = 3H_{n+1} - H_n,$$

and this is easily verified by using the defining relation for H_n and the recurrence relation for F_n.

(a) We prove that $H_{n+1}^2 - H_nH_{n+2} = 5$, which gives the required result. Now

$$\begin{aligned} H_{n+1}^2 - H_nH_{n+2} &= H_{n+1}^2 - H_n(3H_{n+1} - H_n) \\ &= H_n^2 + H_{n+1}(H_{n+1} - 3H_n) = H_n^2 - H_{n-1}H_{n+1}. \end{aligned}$$

It follows by induction that $H_{n+1}^2 - H_nH_{n+2} = H_1^2 - H_0H_2 = 5$.

Also solved by Paul S. Bruckman, Herta T. Freitag, John Ivie, John W. Milsom, Sahib Singh, Bob Prielipp, A.G. Shannon, Charles R. Wall, Gregory Wulczyn, and the proposer.

Consequence of the Euler-Fermat Theorem

B-451 *Proposed by Keats A. Pullen, Jr., Kingsville, MD*

Let k, m, and p be positive integers with p an odd prime. Show that in base $2p$ the units digits of $m^{k(p-1)+1}$ is the same as the units digit of m.

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

Let ϕ be Euler's function. Since p is an odd prime, $\phi(2p) = p - 1$ and therefore by Euler's Theorem, $m^{p-1} \equiv 1 \pmod{2p}$. We take the kth power of both sides and then multiply both sides by m to obtain

$$m^{k(p-1)+1} \equiv m \pmod{2p}$$

as asserted. [This assumes $\gcd(m, p) = 1$ but also follows for $p|m$.]

Also solved by Paul S. Bruckman, Herta T. Freitag, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to: RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extensions of old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS

H-339 *Proposed by Charles R. Wall, Trident Technical College, Charleston, CA*

A dyadic rational is a proper fraction whose denominator is a power of 2. Prove that 1/4 and 3/4 are the *only* dyadic rationals in the classical Cantor ternary set of numbers representable in base three using only 0 and 2 as digits.

H-340 *Proposed by Verner E. Hoggatt, Jr. (Deceased.)*

Let $A_2 = B$, $A_4 = C$, and $A_{2n+4} = A_{2n} - A_{2n+2}$ $(n = 1, 2, 3, \ldots)$. Show:

a. $A_{2n} = (-1)^{n+1}(F_{n-2}B - F_{n-1}C)$.

b. If $A_{2n} > 0$ for all $n > 0$, then $B/C = (1 + \sqrt{5})/2$.

H-341 *Proposed by Paul S. Bruckman, Corcord, CA*

Find the real roots, in exact radicals, of the polynomial equation

$$p(x) \equiv x^6 - 4x^5 + 7x^4 - 9x^3 + 7x^2 - 4x + 1 = 0. \tag{1}$$

SOLUTIONS

Once Again

Professor M. S. Klamkin has pointed out that this problem was proposed previously by him (*Amer. Math. Monthly* 59 (1952):471]. It also appears in an article by W. E. Briggs, S. Chowla, A. J. Kempner, and W. E. Mientka entitled "On Some Infinite Series," *Scripta Math.* 21 (1955):28-30.

H-320 *Proposed by Paul S. Bruckman, Concord CA*
(Vol. 18, No. 4, December 1980)

Let

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},\ \text{Re}(s) > 1, \text{ the Riemann Zeta function.}$$

Also, let

$$H_n = \sum_{k=1}^{n} k^{-1},\ n = 1, 2, 3, \ldots, \text{ the harmonic sequence.}$$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

Method I: Clearly, the series converges; let S denote its sum. We note that

$$H_n = \sum_{k=1}^{n} k^{-1} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)} = -n\int_0^1 t^{n-1}\log(1-t)\,dy$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{x^{s-1}e^{-x}dx}{1-e^{-x}},\ \text{Re}(s) > 1,$$

where $\Gamma(s)$ is the Gamma function.

Then

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = -\sum_{n=1}^{\infty}\int_0^1 \frac{t^n}{n}\,\frac{\log(1-t)}{t}dt = -\int_0^1 \sum_{n=1}^{\infty}\left(\frac{t^n}{n}\right)\frac{\log(1-t)}{t}dt$$

$$= \int_0^1 \frac{\log^2(1-t)}{t}\,dt, \text{ since } \sum_{n=1}^{\infty}\frac{t^n}{n} = -\log(1-t) \text{ for } |t| < 1,$$

and the interchange of the summation and integration signs is permissible.

Setting $t = 1 - e^{-x}$ in the last integral, we get

$$S = \int_0^{\infty} \frac{x^2e^{-x}dx}{1-e^{-x}} = 2\zeta(3).$$

Method II: Clearly, the series converges; let S denote its sum. We note that

$$H_n = \sum_{k=1}^{n} k^{-1} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+n}\right) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)}.$$

Define also

$$H_n^2 = \sum_{k=1}^{n} k^{-2} \quad\text{and}\quad \overline{H}_n^2 = \zeta(2) - H_n^2.$$

Then

$\overline{H}_n^2 = \sum_{k=1}^{\infty} \frac{1}{(k+n)^2}$, the series $\sum_{n=1}^{\infty} \frac{\overline{H}_n^2}{n}$ converges, and we denote its sum by S.

The following hold true:

(i) $$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \overline{S}.$$

Indeed, by rearranging the above series, we get

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+k)^2} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^2}{k} = \overline{S};$$

(ii) $$S = \zeta(3) + \overline{S}.$$

Indeed, with $H_0 = 0$,

$$S = \sum_{n=1}^{\infty} \frac{\frac{1}{n} + H_{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} = \zeta(3) + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2},$$

which, by means of (i) establishes (ii).

(iii) $$2\overline{S} = S.$$

Indeed,

$$\overline{S} = \sum_{n=1}^{\infty} \frac{\overline{H}_n^2}{n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+k)^2} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{k^2}\left(\frac{k}{n(n+k)}\right) - \frac{1}{k}\left(\frac{1}{(k+n)^2}\right) \right\}$$
$$= \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \sum_{k=1}^{\infty} \frac{\overline{H}_k^2}{k} = S - \overline{S},$$

from which (iii) follows.

Combining (ii) and (iii), we have $S = 2\zeta(3)$.

Also solved by L. Carlitz, A. G. Shannon, and the proposer.

Big Deal

H-321 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA (Vol. 18, No. 4, December 1980)*

Establish the identity

$$\begin{aligned} F_{n+14r}^6 + F_n^6 &- (L_{12r} + L_{8r} + L_{4r} - 1)(F_{n+12r}^6 + F_{n+2r}^6) \\ &+ (L_{20r} + L_{16r} + L_{4r} + 3)(F_{n+10r}^6 + F_{n+4r}^6) \\ &- (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1)(F_{n+8r}^6 + F_{n+6r}^6) \\ &= 40(-1)^n \prod_{i=1}^{3} F_{2ri}^2. \end{aligned}$$

Solution by Paul S. Bruckman, Concord, CA

Let

(1) $$\phi(n, r) \equiv F^6_{n+14r} + F^6_n - A_r(F^6_{n+12r} + F^6_{n+2r}) + B\ (F^6_{n+10r} + F^6_{n+4r}) - C_r(F^6_{n+8r} + F^6_{n+6r}),$$

where

(2) $$A_r = L_{12r} + L_{8r} + L_{4r} - 1;$$

(3) $$B_r = L_{20r} + L_{16r} + L_{4r} + 3;$$

(4) $$C_r = L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1.$$

We make repeated use of the following identities:

(5) $$L_{2u}L_{2v} = L_{2u+2v} + L_{2u-2v};$$

(6) $$F^6_m = 5^{-3}\{L_{6m} - 6(-1)^m L_{4m} + 15L_{2m} - 20(-1)^m\}.$$

It is a tedious but straightforward exercise to prove the following identities, by means of (5):

(7) $$L_{14kr} - A_rL_{10kr} + B_rL_{6kr} - C_rL_{2kr} = 0,\ k = 1, 2, 3.$$

Let

(8) $$U_r = C_r - B_r + A_r - 1.$$

We note that

$$\begin{aligned}125F^2_{2r}F^2_{4r}F^2_{6r} &= (L_{4r} - 2)(L_{8r} - 2)(L_{12r} - 2)\\ &= (L_{4r} - 2)(L_{20r} + L_{4r} - 2L_{12r} - 2L_{8r} + 4)\\ &= L_{24r} - 2L_{20r} - L_{16r} + 2L_{12r} + 3L_{8r} - 6\end{aligned}$$

(after simplification)

$$\begin{aligned}&= (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1) - (L_{20r} + L_{16r} + L_{4r} + 3)\\ &\qquad + (L_{12r} + L_{8r} + L_{4r} - 1) - 1\\ &= C_r - B_r + A_r - 1,\end{aligned}$$

or

(9) $$U_r = 125F^2_{2r}F^2_{4r}F^2_{6r}.$$

Now

$$\begin{aligned}125\phi(n, r) = {} & L_{6n+84r} + L_{6n} - 6(-1)^nL_{4n+56r} - 6(-1)^nL_{4n} + 15L_{2n+28r} + 15L_{2n}\\ & - 40(-1)^n\\ & - A_r\{L_{6n+72r} + L_{6n+12r} - 6(-1)^nL_{4n+48r} - 6(-1)^nL_{4n+8r}\\ & + 15L_{2n+24r} + 15L_{2n+4r} - 40(-1)^n\}\\ & + B_r\{L_{6n+60r} + L_{6n+24r} - 6(-1)^nL_{4n+40r} - 6(-1)^nL_{4n+16r}\\ & + 15L_{2n+20r} + 15L_{2n+8r} - 40(-1)^n\}\end{aligned}$$

$$- C_r\{L_{6n+48r} + L_{6n+36r} - 6(-1)^n L_{4n+32r} - 6(-1)^n L_{4n+24r} + 15L_{2n+16r} + 15L_{2n+12r} - 40(-1)^n\}$$

[using (6)]

$$= L_{6n+42r}(L_{42r} - A_r L_{30r} + B_r L_{18r} - C_r L_{6r}) - 6(-1)^n L_{4n+28r}(L_{28r} - A_r L_{20r} + B_r L_{12r} - C_r L_{4r}) + 15L_{2n+14r}(L_{14r} - A_r L_{10r} + B_r L_{6r} - C_r L_{2r}) + 40(-1)^n(-1 + A_r - B_r + C_r)$$

[using (5) repeatedly once again, and factoring]

$= 40(-1)^n U_r$ [using (7) and (8)], or as a result of (9),

(10) $$\phi(n, r) = 40(-1)^n F_{2r}^2 F_{4r}^2 F_{6r}^2. \quad \text{Q.E.D.}$$

Also solved by the proposer.

Two Much

H-322 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon (Vol. 19, No. 1, February 1981)*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $f_n^{(k)}$ by

$$f_0^{(k)} = 0,\ f_1^{(k)} = 1,$$

and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k+1. \end{cases}$$

Show the following:

(a) $f_n^{(k)} = 2^{n-2}$ if $2 \leq n \leq k+1$;

(b) $f_n^{(k)} < 2^{n-2}$ if $n \geq k+2$;

(c) $\sum_{n=1}^{\infty} (f_n^{(k)}/2^n) = 2^{k-1}$.

Solution by the proposer.

For $2 \leq n \leq k$,

$$f_n^{(k)} = 2^{n-2} f_2^{(k)} = 2^{n-2} \quad \text{and} \quad f_{(k+1)}^{(k+1)} = f_k^{(k)} + \cdots + f_1^{(k)} = 2f_k^{(k)} = 2^{k-1},$$

which establish (a). Next, for $n \geq k+1$,

$$f_n^{(k)} = f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} \quad \text{and} \quad f_{n-1}^{(k)} = f_{n-2}^{(k)} + \cdots + f_{n-1-k}^{(k)},$$

so that

(1) $$f_n^{(k)} = 2f_{n-1}^{(k)} - f_{n-1-k}^{(k)} \quad (n \geq k+1).$$

Taking $n = k+2$ in (1), (a) implies $f_{k+2}^{(k)} = 2^k - 1 < 2^k$, which verifies (b) for $n = k+2$. Assume now $f_m^{(k)} < 2^m$ for some integer m ($\geq k+3$). It then follows by (1) and the positivity of $f_i^{(k)}$ ($i \geq 1$), that

$$f_{m+1}^{(k)} = 2f_m^{(k)} - f_{m-k}^{(k)} < 2^m,$$

and this proves (b). Using (1) again, we get

$$(f_n^{(k)}/2^n) - (f_{n+1}^{(k)}/2^{n+1}) = (f_{n-k}^{(k)}/2^{n+1}) > 0 \qquad (n \geq k+1).$$

Therefore,

$$\lim_{n\to\infty} (f_n^{(k)}/2^n) = 0. \tag{2}$$

Setting $s_m^{(k)} = \sum_{n=1}^{m} (f_n^{(k)}/2^n)$ ($m \geq 1$), and using (1) and (a), we get, after some algebra,

$$s_m^{(k)} = 2^{k-1} - (2^{k+1} - 1)(f_m^{(k)}/2^m) + \sum_{i=1}^{k} (f_{m-i}^{(k)}/2^{m-i}) \qquad (m \geq k+2). \tag{3}$$

Relations (2) and (3) give $\lim_{m\to\infty} s_m^{(k)} = 2^{k-1}$, and this shows (c).

Remark 1: For $k = 2$, (b) reduces to $F_n < 2^{n-2}$ if $n \geq 4$. (Fuchs [2] proposed and Scott [4] proved $F_n < 2^{n-2}$ if $n \geq 5$).

Remark 2: For $k = 2$, (c) reduces to $\sum_{n=1}^{\infty}(F_n/2^n) = 2$, a result obtained by Lind [3] in order to solve a problem of Brown [1].

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3. D. Lind. Solution of Problem B-118. *The Fibonacci Quarterly* 6, no. 2 (1968):186.
4. B. Scott. Solution of Problem B-39. *The Fibonacci Quarterly* 2, no. 3 (1964):327.

Also solved by P. Bruckman and L. Somer.

A Common Recurrence

H-323 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 19, No. 1, February 1981)

Let $(x_n)_0^\infty$ and $(y_n)_0^\infty$ be two sequences satisfying the common recurrence

$$p(E)z_n = 0, \tag{1}$$

where p is a monic polynomial of degree 2, and $E = 1 + \Delta$ is the unit right-shift operator of finite difference theory. Show that

$$x_n y_{n+1} - x_{n+1} y_n = (p(0))^n (x_0 y_1 - x_1 y_0),\ n = 0, 1, 2, \ldots . \tag{2}$$

Generalize to the case where p is of degree $e \geq 1$.

Solution by the proposer.

We solve the general case, with p any monic polynomial of degree $e \geq 1$. Suppose

$$(z_n^{(1)})_0^\infty,\ (z_n^{(2)})_0^\infty,\ \ldots,\ (z_n^{(e)})_0^\infty$$

are sequences satisfying the common recursion (1). We seek to evaluate Casorati's Determinant

$$D_n = \begin{vmatrix} z_n^{(1)} & z_n^{(2)} & \cdots & z_n^{(e)} \\ z_{n+1}^{(1)} & z_{n+1}^{(2)} & \cdots & z_{n+1}^{(e)} \\ \vdots & \vdots & & \vdots \\ z_{n+e-1}^{(1)} & z_{n+e-1}^{(2)} & \cdots & z_{n+e-1}^{(e)} \end{vmatrix} \tag{3}$$

Let $U_n = ((z_{n+i-1}^{(j)}))_{e \times e}$ be the matrix whose determinant is D_n; also, define the $e \times e$ matrix J as follows:

$$J = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p(0) & -p'(0)/1! & -p''(0)/2! & -p'''(0)/3! & \cdots & -p^{(e-1)}(0)/(e-1)! \end{vmatrix}. \tag{4}$$

Note that p has the Maclaurin Series expansion

$$p(z) = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} z^r. \tag{5}$$

Therefore, the sequences $(z_n^{(k)})_0^\infty$ $(k = 1, 2, \ldots, e)$ satisfy the common recursion

$$p(E)z_n = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} E^r z_n = \sum_{r=0}^{e} \frac{p^{(r)}(0)}{r!} z_{n+r} = 0;$$

since p is monic, $p^{(e)}(0)/e! = 1$, and hence

$$z_{n+e} = -\sum_{r=0}^{e-1} \frac{p^{(r)}(0)}{r!} z_{n+r} \quad (n = 0, 1, 2, \ldots). \tag{6}$$

We now observe, using (3), (4), and (6), that

$$J \cdot U_n = U_{n+1}, \; n = 0, 1, 2, \ldots . \tag{7}$$

It follows by an easy induction that

$$U_n = J^n U_0, \; n = 0, 1, 2, \ldots . \tag{8}$$

We may evaluate $|J|$ along the first column of J, and we find readily that $|J| = (-1)^{e-1}(-p(0)) = (-1)^e p(0)$. Therefore, taking determinants in (8) yields:

$$D_n = |U_n| = |J^n| \cdot |U_0| = |J|^n \cdot D_0,$$

or

(9) $$D_n = \{(-1)^e p(0)\}^n D_0,\ n = 0, 1, 2, \ldots .$$

This is the desired generalization of (2). Many interesting identities arise by specializing further. For example, taking

$$p(z) = z^2 - z - 1,\ (x_n) = (F_n),\ \text{and}\ (y_n) = (L_n),$$

yields:

(10) $$F_n L_{n+1} - F_{n+1} L_n = 2(-1)^{n-1},\ n = 0, 1, 2, \ldots .$$

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ERRATA

In the article "On the Fibonacci Numbers Minus One" by G. Geldenhuys, Volume 19, no. 5, the following two errors appear on pages 456 and 457:

1. The recurrence relation (1), which appears as

$$D_1 = 1 + \mu,\ D_2 = (1 - \mu)^2,\ \text{and}\ D_n = (1 + \mu)D_{n-1} - \mu D_{n-3}\ \text{for}\ n \geq 3$$

should read

$$D_1 = 1 + \mu,\ D_2 = (1 + \mu)^2,\ \text{and}\ D_n = (1 + \mu)D_{n-1} - \mu D_{n-3}\ \text{for}\ n \geq 3;$$

2. The alternative recurrence relation (4), which appears as

$$D_m - D_{m-1} - D_{m-2} = 1\ \text{for}\ m \geq 3$$

should read

$$D_m - \mu D_{m-1} - \mu D_{m-2} = 1\ \text{for}\ m \geq 3.$$

We thank Professor Geldenhuys for bringing this to our attention.

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