The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 20
NUMBER 3

AUGUST
1982

CONTENTS

The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980),
Br. Alfred Brousseau, and I.D. Ruggles

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

EDITORIAL POLICY

The principal purpose of *The Fibonacci Quarterly* is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, and challenging problems.

The *Quarterly* seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with innovative ideas that develop enthusiasm for number sequences or the exploration of number facts.

Articles should be submitted in the format of the current issues of the *Quarterly*. They should be typewritten or reproduced typewritten copies, double spaced with wide margins and on only one side of the paper. Articles should be no longer than twenty-five pages. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in the *Quarterly*. Authors who pay page charges will receive two free copies of the issue in which their article appears.

Two copies of the manuscript should be submitted to GERALD E. BERGUM, DEPARTMENT OF MATHEMATICS, SOUTH DAKOTA STATE UNIVERSITY, BROOKINGS, SD 57007. The author is encouraged to keep a copy for his own file as protection against loss.

Address all subscription correspondence, including notification of address changes, to SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, UNIVERSITY OF SANTA CLARA, SANTA CLARA, CA 95053.

Annual domestic Fibonacci Association membership dues, which include a subscription to *The Fibonacci Quarterly,* are $20 for Regular Membership, $28 for Sustaining Membership I, $44 for Sustaining Membership II, and $50 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates (please write for details). *The Quarterly* is published each February, May, August and November.

All back issues of *The Fibonacci Quarterly* are available in microfilm or hard copy format from UNIVERSITY MICROFILMS INTERNATIONAL, 300 North Zeeb Road, Dept P.R., ANN ARBOR, MI 48106.

LEXICOGRAPHIC ORDERING AND FIBONACCI REPRESENTATIONS

(Submitted June 1980)

V. E. HOGGATT, JR.
(Deceased)
and
MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, CA 95192

The Zeckendorf theorem [1], which essentially states that every positive integer can be represented uniquely as a finite sum of distinct Fibonacci numbers 1, 2, 3, 5, ..., 8, where no two consecutive Fibonacci numbers appear, led to so much new work that the entire January 1972 issue of the *Fibonacci Quarterly* was devoted to representations.

Now, through consideration of the ordering of the terms in a representation and the ordering of the integers, we study mappings of one integer into another by increasing the subscripts of the terms in a representation. We are led to number sequences related to the solutions of Wythoff's game [2], [3], and the generalized Wythoff's game [4]. We investigate representations using Fibonacci numbers, Pell numbers, generalized Fibonacci numbers arising from the Fibonacci polynomials, Lucas numbers, and Tribonacci numbers.

1. The Fibonacci Numbers

If we define the Fibonacci numbers in the usual way,

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1},\ n \geq 1,$$

then every positive integer N can be written in its Fibonacci-Zeckendorf representation as

$$N = \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k, \tag{1.1}$$

where $\alpha_i \in \{0, 1\}$, $\alpha_i \alpha_{i-1} = 0$, or a representation as a sum of distinct Fibonacci numbers where no two consecutive Fibonacci numbers may be used. Such a representation is unique [5] and is also called the first canonical form of N.

If, instead, we write the Fibonacci representation of N in the second canonical form, we replace F_2 with F_1, and

$$N = \alpha_1 F_1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k, \tag{1.2}$$

where $\alpha_i \in \{0, 1\}$, $\alpha_2 = 0$, $\alpha_i \alpha_{i-1} = 0$. Such a representation is also unique.

Notice that, if the smallest Fibonacci number used in the representation has an odd subscript, the two forms are the same, but if the smallest Fibonacci number used has an even subscript, it can be written in either form. For example, the Zeckendorf representation of $8 = F_6$ becomes $8 = F_5 + F_3 + F_1$, and $11 = F_6 + F_4 = F_6 + F_3 + F_1$.

We next need some results on the ordering of the terms in a representation. A lexicographic ordering was earlier considered by Silber [7]. We define a lexicographic ordering as follows:

Let positive integers M and N each be represented in terms of a strictly increasing sequence of integers $\{a_n\}$ so that

$$M = \sum_{i=1}^{k} \alpha_i a_i, \qquad N = \sum_{i=1}^{k^*} \beta_i a_i, \tag{1.3}$$

where $\alpha_i, \beta_i \in \{0, 1, \ldots, p\}$. Let $\alpha_i = \beta_i$ for all $i > m$. If $\alpha_m > \beta_m$ only if $M > N$, then we say that the representation is a *lexicographic ordering.*

Theorem 1.1

The Zeckendorf representation of the positive integers in terms of Fibonacci numbers is a lexicographic ordering.

Proof: Let M and N be the two positive integers given in (1.3), where $a_n = F_{n+1}$, $p = 1$, and $\alpha_i\alpha_{i-1} = 0$, $\beta_i\beta_{i-1} = 0$. If $\alpha_i = \beta_i$ for all $i > m$, and if $\alpha_m > \beta_m$, then $\alpha_m = 1$ and $\beta_m = 0$, and we compare the truncated parts of the numbers.

$$M^* = \alpha_2 F_2 + \alpha_3 F_3 + \cdots + \alpha_{m-1}F_{m-1} + F_m \geq F_m$$

$$N^* = \beta_2 F_2 + \beta_3 F_3 + \cdots + \beta_{m-1}F_{m-1} \leq F_{m-1} + F_{m-3} + F_{m-5} + \cdots \leq F_m - 1,$$

so that $M^* > N^*$ and $M > N$, since it is well known that

$$F_{2k} + F_{2k-2} + \cdots + F_2 = F_{2k-1} - 1,$$

$$F_{2k-1} + F_{2k-3} + \cdots + F_3 = F_{2k} - 1.$$

Application: Let f^* be the transformation that advances by one the subscripts on each Fibonacci number used in the Zeckendorf representation of the positive integers M and N. If

$$M \xrightarrow{f^*} M' \quad \text{and} \quad N \xrightarrow{f^*} N',$$

and if $M > N$, then $M' > N'$.

Theorem 1.2

The Fibonacci representation of integers in the second canonical form is a lexicographic ordering.

The proof of Theorem 1.2 is very similar to that of Theorem 1.1. Next, we let f be the transformation that advances by one the subscripts of the Fibonacci numbers used in the representation in the second canonical form of the positive integers M and N. If

$$M \xrightarrow{f} M' \quad \text{and} \quad N \xrightarrow{f} N',$$

and if $M > N$, then $M' > N'$.

Let $A = \{A_n\}$ and $B = \{B_n\}$ be the sets of positive integers for which the smallest Fibonacci number used in the Zeckendorf representation occurred respectively with an even or with an odd subscript. Since the Zeckendorf representation is unique, sets A and B cover the set of positive integers and are disjoint.

Notice that, if the smallest subscript for a Fibonacci number used in the Zeckendorf representation for a number is odd, then the first and second canonical forms are the same. Thus, under f or f^*, every element of B is mapped into an element of A. But every element of A can be written in either canonical form, and under f every element of A is mapped into an element of A. Thus, every positive integer n is mapped into an element of A, or, aided by the lexicographic ordering theorems,

$$A_n \xrightarrow{f} A_{A_n}$$

$$B_n \xrightarrow{f} A_{B_n}$$

$$n \xrightarrow{f} A_n$$

$$A_n \xrightarrow{f^*} B_n$$

so that

$$A_{A_n} + 1 = B_n \tag{1.4}$$

follows, as well as

$$A_n + n = B_n. \tag{1.5}$$

Compare to the numbers a_n and b_n, where (a_n, b_n) is a safe pair for Wythoff's game [3], [4]. If one uses the Zeckendorf representation of positive integers using the Lucas numbers 2, 1, 3, 4, 7, ..., since the Lucas numbers are complete and have a unique Zeckendorf representation, we could make similar mappings. This is essentially developed in [4] but in a different way. For later comparison, we recall [3], [4], that

$$A_n = [n\alpha], \tag{1.6}$$

where $[x]$ is the greatest integer in x and $\alpha = (1 + \sqrt{5})/2$ is the positive root of $y^2 - y - 1 = 0$.

2. The Pell Numbers

Let us go to the Pell sequence $\{P_n\}$, defined by

$$P_1 = 1,\ P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n,\ n \geq 1.$$

The Pell sequence boasts of a unique Zeckendorf representation [6]. Consider the positive integers and the three sets $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$, where $A_n = B_n - 1$ and $C_n = 2B_n + n$, and A, B, and C contain numbers in their natural order of the form

$$A_n = 1 + \alpha_2 P_2 + \alpha_3 P_3 + \cdots + \alpha_k P_k,$$

(2.1) $$B_n = \alpha_2 P_2 + \alpha_3 P_3 + \cdots + \alpha_k P_k,\ \alpha_2 \neq 0,$$

$$C_n = \alpha_3 P_3 + \cdots + \alpha_k P_k,\ \alpha_3 \neq 0,$$

where $\alpha_i \varepsilon \{0, 1, 2\}$, and if $\alpha_i = 2$, then $\alpha_{i-1} = 0$.

Since we next wish to map the positive integers into set B, we will need a lexicographic ordering theorem for the Pell numbers.

Theorem 2.1

The Zeckendorf representation of the positive integers, in terms of Pell numbers, is a lexicographic ordering.

Proof: Let M and N be two positive integers given by

$$M = \sum_{i=1}^{k} \alpha_i P_i, \qquad N = \sum_{i=1}^{k} \beta_i P_i,$$

where α_i, $\beta_i \varepsilon \{0, 1, 2\}$ except α_1, $\beta_1 \neq 2$; and if $\alpha_i = 2$, then $\alpha_{i-1} = 0$, or if $\beta_i = 2$, then $\beta_{i-1} = 0$. If $\alpha_i = \beta_i$ for all $i > m$, and if $\alpha_m > \beta_m$, then $\alpha_m = 2$ and $\beta_m = 1$, or $\alpha_m = 1$ and $\beta_m = 0$, or $\alpha_m = 1$ and $\beta_m = 0$. We compare the truncated parts of the numbers when $\alpha_m = 2$ and $\beta_m = 1$:

$$M^* = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + 2P_m \geq 2P_m$$

$$N^* = \beta_1 P_1 + \beta_2 P_2 + \cdots + P_m \leq P_m + P_m - 2 < 2P_m,$$

Since, if β_1, β_2, β_3, ..., β_{m-1} are taken as large as possible, whether m is even or odd,

$$2(P_{2k-1} + \cdots + P_3) + P_1 = P_{2k} - 1 = P_m - 1,$$

$$2(P_{2k} + P_{2k-2} + \cdots + P_2) = P_{2k+1} - 1 = P_m - 1,$$

so that $M^* > N^*$ and $M > N$. If $\alpha_m = 2$ and $\beta_m = 0$, then N^* is even smaller. If $\alpha_m = 1$ and $\beta_m = 0$, then $M^* \geq P_m$, but notice that, if the coefficients β_i are taken as large as possible, we can only reach $N^* = P_m - 1$, and again $M^* > N^*$, making $M > N$. By definition (1.3), we have proved Theorem 2.1.

In an entirely similar manner, we could prove Theorem 2.2, where we write the second canonical form by replacing P_2 by $2P_1$ and $2P_2$ by $P_2 + 2P_1$ in the Zeckendorf representation, where again if $2P_k$ appears, then P_{k-1} is not used in that representation. This second canonical form is again unique [6]. We write:

Theorem 2.2

The Pell number representation of integers in the second canonical form is a lexicographic ordering.

Let f be the transformation that advances by one the subscripts of each Pell number used in the representation in the second canonical form of the positive integers M and N, and let f^* be the transformation that is used for the Zeckendorf form. Then, as before, if

$$M \xrightarrow{f} M' \quad \text{and} \quad N \xrightarrow{f} N',$$

and if $M > N$, then $M' > N'$, and the same for transformation f^*.

Now, we consider A_n, B_n, and C_n of (2.1), and mappings of the integers under f and f^*. We must first put B_n into the second canonical form. In the representation for B_n, replace P_2 by $2P_1$, or replace $2P_2$ by $P_2 + 2P_1$, since the smallest term of B_n is either P_2 or $2P_2$. Now, under f, B_n is mapped into B_{B_n}, while under f^*, B_n goes into C_n, applying the lexicographic theorems for Pell numbers.

$$P_2 \xrightarrow{f^*} P_3, \text{ or } 2 \xrightarrow{f^*} 5;$$

$$2P_1 \xrightarrow{f} 2P_2, \text{ or } 2 \xrightarrow{f} 4;$$

$$2P_2 \xrightarrow{f^*} 2P_3, \text{ or } 4 \xrightarrow{f^*} 10;$$

$$P_2 + 2P_1 \xrightarrow{f} P_3 + 2P_2, \text{ or } 4 \xrightarrow{f} 5 + 2 \cdot 2 = 9.$$

Thus, the image of B_n under f is one less than the image of B_n under f^*, and

$$B_{B_n} + 1 = C_n. \tag{2.2}$$

We know where the A_n's go under f: into B_n, since the A_n's start with a one, while their images start with a P_2. The B_n's (second form) have $2P_1$, so their images start with $2P_2$, clearly a B_n. Now, where do the C_n's go? Each C_n begins with 5 or 10. Replace $2P_3 = 10$ by $5 + 2 \cdot 2 + 1 = P_3 + 2P_2 + P_1$, and replace $1P_3 = 5$ by $2P_2 + P_1 = 2 \cdot 2 + 1$ and under f,

$$10 \rightarrow P_4 + 2P_3 + P_2 = 12 + 2 \cdot 5 + 2 = 24,$$

and

$$5 \rightarrow 2P_3 + P_2 = 2 \cdot 5 + 2 = 12.$$

Thus A_n, modified B_n, and modified C_n are all carried into B_n by f and

$$B_n \xrightarrow{f^*} C_n.$$

For later comparison, we note that

$$B_n = [n(1 + \sqrt{2})], \tag{2.3}$$

where $[x]$ is the greatest integer in x, and $(1 + \sqrt{2})$ is the positive root of $y^2 - 2y - 1 = 0$.

3. Generalized Fibonacci Numbers (Arising from Fibonacci Polynomials)

Next, consider the sequence of generalized Fibonacci numbers $\{u_n\}$,

$$u_0 = 0,\ u_1 = 1, \text{ and } u_{n+1} = ku_n + u_{n-1},\ n \geq 1.$$

[Note that, if the Fibonacci polynomials are given by $f_0(x) = 0$, $f_1(x) = 1$, and $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$, $n \geq 1$, then $u_n = f_n(k)$.] Let set B be the set of positive integers whose Zeckendorf representation has the smallest u_n used with an even subscript, and set 0 the set of integers whose Zeckendorf representation has the smallest u_n used with an odd subscript. We know from [6] that N has a unique representation of the form

$$N = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m, \tag{3.1}$$

where

$$\alpha_1 \in \{0, 1, \ldots, k - 1\},$$

$$\alpha_i \in \{0, 1, 2, \ldots, k\},\ i > 1,$$

$$\alpha_i = k \Longrightarrow \alpha_{i-1} = 0,$$

so that sets B and 0 cover the positive integers without overlapping.

We wish to demonstrate a second canonical form for elements of set B. We do this in two parts: Let α_{2k} be the coefficient of the least u_{2k} used; then $\alpha_{2k} = 1, 2, 3, \ldots, k$. Take $1u_{2k}$ and replace it by $ku_{2k-1} + u_{2k-2}$, and continue until you obtain $1u_2$, and replace that by ku_1,

$$u_{2k} = k(u_{2k-1} + u_{2k-3} + \cdots + u_3 + u_1).$$

Thus,

$$B_n = R + ku_{2k-1} + ku_{2k-3} + \cdots + ku_3 + ku_1.$$

If f is again the transformation that increases the subscripts by one for integers written in the second canonical form, and f^* the transformation for the Zeckendorf form, then, if we can again use lexicographic ordering,

$$B_n \xrightarrow{f} R' + ku_{2k} + ku_{2k-2} + \cdots + ku_2$$

$$B_n \xrightarrow{f^*} R' + u_{2k+1},$$

but from [6],

$$u_{2k+1} - 1 = k(u_{2k} + u_{2k-2} + \cdots + u_4 + u_2),$$

so that the images differ by 1. Now, under f, we see that all of the elements of 0 are mapped into set B and set B in second canonical form is also mapped into set B. Thus, provided we have lexicographic ordering, *the positive integers* n map into B_n under f. If we split set 0 into sets A whose elements use $1u_1$ in their representations and $C = \{C_n\}$, where C_n does not use $1u_1$ in its representation, then

$$B_n \xrightarrow{f^*} C_n, \qquad \text{and} \qquad B_n \xrightarrow{f} B_{B_n},$$

and since the images differ by 1,

$$B_{B_n} + 1 = C_n,\ n > 0. \tag{3.2}$$

The general lexicographic theorm should not be difficult.

Theorem 3.1

The Zeckendorf representation of the positive integers in terms of the generalized Fibonacci numbers $\{u_n\}$ is a lexicographic ordering.

Proof: Let M and N be positive integers which have Zeckendorf representations

$$M = \sum_{j=1}^{n} M_j u_j \qquad \text{and} \qquad N = \sum_{j=1}^{n} N_j u_j.$$

Compare the higher-ordered terms from highest to lowest. If $M_j = N_j$ for all $j > m$, and $M_m > N_m$, then we prove that $M > N$. It suffices to let $M = M_m u_m$ and $M_m \geq N_m + 1$.

$$N \leq N^* = ku_{2j-1} + ku_{2j-3} + ku_{2j-5} + \cdots + ku_3 + (k-1)u_1 = u_{2j} - 1$$

or

$$N \leq N^* = ku_{2j} + ku_{2j-2} + \cdots + ku_2 = u_{2j+1} - 1.$$

Thus $M \geq M^* > N^* \geq N$, so that $M > N$, proving Theorem 3.1.

This shows that, if two numbers M and N in Zeckendorf form are compared, then the one with the larger coefficient in the first place that they differ, coming down from the higher side, is larger. Now, what need be said about the second canonical form? If both M and N are in the second canonical form, and they differ in the jth place, whereas their smallest nonzero coefficient occurs in a position smaller than the jth place, then the original test suffices. If they both differ in the smallest position, then again the one with the larger coefficient there is larger, as their second canonical extensions are identical.

Theorem 3.2

The representation of positive integers in the second canonical form using generalized Fibonacci numbers $\{u_n\}$ is a lexicographic ordering.

Under transformation f, using the second canonical form, if $M = N + 1$, then

$$M \xrightarrow{f} M', \quad \text{and} \quad N \xrightarrow{f} N',$$

such that $M' > N' + k - 1$. For example,

$$u_1 = 1 \xrightarrow{f} u_2 = k, \; 2 = 2u_1 \xrightarrow{f} 2u_2 = 2k, \text{ and } 2k > k + k - 1,$$

taking $M = 2$ and $N = 1$.

We now return to sets A and C which made up set 0 and with set B covered the positive integers. Sets A, B, and C can be characterized as the positive integers written, in natural order, in the form

$$\begin{aligned} A_n &= \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m, \; \alpha_1 \neq 0, \; k, \\ (3.3) \quad B_n &= \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m, \; \alpha_2 \neq 0, \\ C_n &= \alpha_3 u_3 + \alpha_4 u_4 + \cdots + \alpha_m u_m, \; \alpha_3 \neq 0, \; \alpha_i \in \{0, 1, 2, 3, \ldots, k\}. \end{aligned}$$

For the numbers B_n, we can write:

Theorem 3.3

$$B_{B_n+1} - B_{B_n} = k + 1, \text{ and if } m \neq B_n,$$

$$B_{m+1} - B_m = k.$$

Also, it was proved by Molly Olds [18] that

Theorem 3.4

$$C_n = kB_n + n.$$

4. The Tribonacci Numbers

The Tribonacci numbers $\{T_n\}$ are

$$T_0 = 0, \; T_1 = 1, \; T_2 = 1, \; T_{n+3} = T_{n+2} + T_{n+1} + T_n, \; n \geq 0.$$

The Tribonacci numbers are complete with respect to the positive integers, and the positive integers again have a unique Zeckendorf representation in terms of Tribonacci numbers (see [8]). That is, a positive integer N has a unique representation in the form

$$(4.1) \qquad N = \alpha_2 T_2 + \alpha_3 T_3 + \cdots + \alpha_k T_k,$$

where $\alpha_i \in \{0, 1\}$, $\alpha_i \alpha_{i-1} \alpha_{i-2} = 0$.

Now, consider the numbers A_n, B_n, and C_n listed in Table 4.1. Here, because we want completeness in the array, we take A_n as the smallest positive integer not yet used, and we define Δ_n as the number of C_k's less than A_n, and φ_n as the number of C_k's less than B_n. Then, we compute B_n and C_n as

$$B_n = 2A_n - \Delta_n, \tag{4.2}$$

$$C_n = 2B_n - \varphi_n. \tag{4.3}$$

We write the Tribonacci recurrence relation:

$$n + A_n + B_n = C_n. \tag{4.4}$$

TABLE 4.1

n	A_n	B_n	C_n
1	1	2	4
2	3	6	11
3	5	9	17
4	7	13	24
5	8	15	28
6	10	19	35
7	12	22	41
8	14	26	48
9	16	30	55
10	18	33	61

Now, $A = \{A_n\}$ is the set of positive integers whose Zeckendorf representation has smallest term T_k, where $k \equiv 2 \bmod 3$; $B = \{B_n\}$ contains those positive integers using smallest term T_k, where $k \equiv 3 \bmod 3$; and $C = \{C_n\}$ has smallest term T_k, where $k \equiv 1 \bmod 3$, $k > 3$. We have suppressed $T_1 = 1$ in the above; thus, every positive integer belongs to A, B, or C by completeness, where A, B, and C are disjoint.

We write a second canonical form by rewriting each A_n by replacing T_2 by T_1; replacing $T_3 = 2$ in each B_n by $T_2 + T_1$; and leaving the numbers C_n alone.

Note that, instead of saying "A_n has smallest term T_{3m+2}," we could say "A_n has $3m + 1$ leading zeros."

Theorem 4.1

Each A_n has $k \equiv 1 \bmod 3$ leading zeros in the Zeckendorf representation and can be written so that

$$A_n = T_2 + \alpha_3 T_3 + \alpha_4 T_4 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Each B_n has $k \equiv 2 \bmod 3$ leading zeros and can be written as

$$B_n = T_3 + \alpha_4 T_4 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Each C_n has $k \equiv 0 \bmod 3$ leading zeros, $k \geq 3$, and can also be written

$$C_n = T_4 + \alpha_5 T_5 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Proof: Let T_{3m+2} have a nonzero coefficient. Replace T_{3m+2} by

$$T_{3m+1} + T_{3m} + T_{3m-1} = T_{3m+1} + T_{3m} + T_{3(m-1)+2}.$$

Continue until the right member ultimately lands in slot 2. The similar replacement for T_{3m} in B_n and T_{3m+1} in C_n will establish the forms given above.

Theorem 4.2

The Zeckendorf representation of the positive integers in terms of the Tribonacci numbers $\{T_n\}$ is a lexicographic ordering. The representation in the second canonical form is also a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representations,

$$M = \sum_{j=2}^{n} M_j T_j \qquad \text{and} \qquad N = \sum_{j=2}^{n} N_j T_j.$$

If $M_j = N_j$ for all $j > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. We let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_2 T_2 + M_3 T_3 + \cdots + M_m T_m \geq T_m,$$

$$N^* = N_2 T_2 + N_3 T_3 + \cdots + N_{m-1} T_{m-1}.$$

Since $N_i N_{i-1} N_{i-2} = 0$, N^* is as large as possible when both N_{m-1} and N_{m-2} are nonzero. Either $m = 3k$ or $m = 3k + 1$ or $m = 3k - 1$. We use three summation formulas given by Waddill and Sacks [9].

If $m = 3k$, then

$$N^* \leq \sum_{i=1}^{k} (T_{3i-1} + T_{3i-2}) - T_1 = T_{3k} - 1 < T_m \leq M^*.$$

If $m = 3k + 1$,

$$N^* \leq \sum_{i=1}^{k} (T_{3i} + T_{3i-1}) = T_{3k+1} - 1 < T_m \leq M^*.$$

If $m = 3k - 1$,

$$N^* \leq \sum_{i=1}^{k} (T_{3i-2} + T_{3i-3}) - T_1 = T_{3k-1} - 1 < T_m \leq M^*.$$

Thus in all three cases, $M^* > N^*$ so that $M > N$, and the Zeckendorf representation is a lexicographic ordering. The same summation identities would show that the second canonical form is also lexicographic.

Next, let f be the transformation that increases the subscripts by one for integers written in the second canonical form, and f^* the similar transformation for the Zeckendorf form. Now, the numbers in set A are ordered, and since we have lexicographic ordering for the second canonical form,

$$n \xrightarrow{f} A_n, \; A_n \xrightarrow{f} A_{A_n}, \; B_n \xrightarrow{f} A_{B_n}, \; C_n \xrightarrow{f} A_{C_n}.$$

Since we have lexicographic ordering for the Zeckendorf form,

$$A_n \xrightarrow{f^*} B_n, \; B_n \xrightarrow{f^*} C_n, \; C_n \xrightarrow{f^*} A_{C_n}.$$

But each A_{A_n} is one less than B_n, and each A_{B_n} is one less than C_n, so that

$$A_{A_n} + 1 = B_n, \quad \text{and} \quad A_{B_n} + 1 = C_n. \tag{4.5}$$

(4.5) reminds one of $a_{a_n} + 1 = b_n$ from Wythoff's game [3], [4]. Note that $\{C_n\}$ clearly maps into $\{A_n\}$ because they were of the form whose least term had subscript $k \equiv 2 \bmod 3$, so that an upward shift of one yields $k \equiv 3 \bmod 3$ and, hence, A_{C_n}.

Comments: Under f, A_n maps to A_{A_n}, and under f^*, A_n maps to B_n. If A_n is in second canonical form, then $A_n + 1 = A_n + T_2$ is also in second canonical form. Thus, using the Zeckendorf and then the second form for A_n,

$$A_n + T_1 \xrightarrow{f^*} B_n + T_2 = B_n + 1,$$

$$A_n + T_2 \xrightarrow{f} A_{A_n} + T_3 = A_{A_n} + 2,$$

so that $A_{A_n} + 1 = B_n$. Clearly $B_n + 1$ is an A_j since the B_n's have T_3 as the lowest nonzero Tribonacci number, but $B_n + 1$ has T_2. Thus,

$$A_{A_n} + 1 = B_n \quad \text{and} \quad B_n + 1 = A_{A_n+1} \tag{4.6}$$

so that

$$A_{A_n+1} - A_{A_n} = 2.$$

We also have shown that there are A_n of the A_j's less than B_n.

Under f, B_n maps to A_{B_n}, and under f^*, B_n maps to C_n. Therefore,

$$A_{B_n} + 1 = C_n,$$

which shows that there are B_n of the A_j's less than C_n. Also, $C_n + 1$ is an A_j since each C_n can be written with the least summand T_4. Therefore,

$$C_n + 1 = A_{B_n+1},$$

and

$$A_{B_n} + 1 = C_n \quad \text{and} \quad C_n + 1 = A_{B_n+1} \tag{4.7}$$

give us

$$A_{B_n+1} - A_{B_n} = 2.$$

Next, we look at C_n and $C_n + 1$.

$$C_n \xrightarrow{f} A_{C_n} \quad \text{and} \quad C_n + 1 = C_n + T_1 \xrightarrow{f} A_{C_n+1} = A_{C_n} + 1.$$

Since $C_n + 1$ is A_{B_n+1}, the one is T_1 in $C_n + 1$. We conclude that

$$A_{C_n+1} - A_{C_n} = 1.$$

This gives all the recurrent differences for the A sequence.

We now turn to the B sequence.

$$1 = (A_{C_n+1} - A_{C_n}) \xrightarrow{f^*} (B_{C_n+1} - B_{C_n}) = 2,$$

$$2 = (A_{A_n+1} - A_{A_n}) \xrightarrow{f^*} (B_{A_n+1} - B_{A_n}) = 4,$$

$$1 + 1 = 2 = (A_{B_n+1} - A_{B_n}) \xrightarrow{f^*} (B_{B_n+1} - B_{B_n}) = 1 + 2 = 3.$$

We look first at

$$C_n \xrightarrow{f} A_{C_n} \quad \text{and} \quad C_n + 1 \xrightarrow{f} A_{C_n+1}$$

because $C_n + 1$ is an A_j so 1 in it is T_1. Thus

$$A_{C_n+1} = A_{C_n} + T_2 \xrightarrow{f^*} B_{C_n+1} = B_{C_n} + T_3 = B_{C_n} + 2,$$

and

$$B_{C_n+1} - B_{C_n} = 2.$$

Now, in second canonical form, A_n has T_1 but no T_2, but $A_n + 1$ has T_1 and T_2, or, $A_n + 1 = A_n + T_2$.

$$A_n \xrightarrow{f} A_{A_n} \xrightarrow{f^*} B_{A_n}$$

$$A_n + 1 = A_n + T_2 \xrightarrow{f} A_{A_n} + T_3 \xrightarrow{f^*} B_{A_n} + T_4 = B_{A_n} + 4,$$

$$A_n + 1 \xrightarrow{f} A_{A_n+1} \xrightarrow{f^*} B_{A_n+1} = B_{A_n} + 4.$$

Thus,

$$B_{A_n+1} - B_{A_n} = 4.$$

Next, let $B_n = R_n + T_3 = R_n + T_2 + T_1$ be in second canonical form.

$$B_n \xrightarrow{f} A_{B_n} = R_n' + T_3 + T_2 \xrightarrow{f^*} R_n'' + T_4 + T_3,$$

$$B_n + 1 = R_n + T_3 + T_1 \xrightarrow{f} R_n' + T_4 + T_2 \xrightarrow{f^*} R_n'' + T_5 + T_3,$$

$$B_n + 1 \xrightarrow{f} A_{B_n+1} \xrightarrow{f^*} B_{B_n+1} = R_n'' + T_5 + T_3.$$

Therefore,

$$B_{B_n+1} - B_{B_n} = (R''_n + T_5 + T_3) - (R''_n + T_4 + T_3) = T_5 - T_4 = 3.$$

Finally, for the third difference of B numbers,

$$C_n \xrightarrow{f} A_{C_n} \xrightarrow{f^*} B_{C_n} ,$$

$$C_n + 1 = C_n + T_1 \xrightarrow{f} A_{C_n} + T_2 \xrightarrow{f^*} B_{C_n} + T_3 ,$$

$$C_n + 1 \xrightarrow{f} A_{C_n+1} \xrightarrow{f^*} B_{C_n+1} .$$

Therefore,

$$B_{C_n+1} - B_{C_n} = T_3 = 2.$$

Lastly, the three differences of consecutive C_j's are found by using the above differences of A_j's and B_j's and (4.4).

$$\begin{aligned} C_{A_n+1} - C_{A_n} &= (A_n + 1 + A_{A_n+1} + B_{A_n+1}) - (A_n + A_{A_n} + B_{A_n}) \\ &= (A_n + 1 - A_n) + (A_{A_n+1} - A_{A_n}) + (B_{A_n+1} - B_{A_n}) \\ &= 1 + 2 + 4 = 7. \\ C_{B_n+1} - C_{B_n} &= (B_n + 1 - B_n) + (A_{B_n+1} - A_{B_n}) + (B_{B_n+1} - B_{B_n}) \\ &= 1 + 2 + 3 = 6. \\ C_{C_n+1} - C_{C_n} &= (C_n + 1 - C_n) + (A_{C_n+1} - A_{C_n}) + (B_{C_n+1} - B_{C_n}) \\ &= 1 + 1 + 2 = 4. \end{aligned}$$

We summarize all the possible differences of successive members of the A, B, and C sequences as:

Theorem 4.3

$$A_{A_n+1} - A_{A_n} = 2,\ A_{B_n+1} - A_{B_n} = 2,\ A_{C_n+1} - A_{C_n} = 1;$$

$$B_{A_n+1} - B_{A_n} = 4,\ B_{B_n+1} - B_{B_n} = 3,\ B_{C_n+1} - B_{C_n} = 2;$$

$$C_{A_n+1} - C_{A_n} = 7,\ C_{B_n+1} - C_{B_n} = 6,\ C_{C_n+1} - C_{C_n} = 4.$$

Returning to (4.6), we know that there are A_n of the A_j's less than B_n. Then, B_n is n plus the number of A_j's less than B_n, plus the number of C_k's less than B_n, or,

$$B_n = n + A_n + \varphi_n.$$

Then

$$C_n = 2B_n - \varphi_n = 2B_n - (B_n - n - A_n) = B_n + A_n + n,$$

a consistency proof that the C_n's are properly defined by the array of Table 4.1.

Theorem 4.4

The number of C_j's less than A_n is

$$\Delta_n = 2A_n - B_n.$$

Proof: We show that $2A_n - B_n$ increments by 1 if and only if $n = B_m$, and zero otherwise, applying Theorem 4.3:

$$2(A_{A_n+1} - A_{A_n}) - (B_{A_n+1} - B_{A_n}) = 2(2) - 4 = 0,$$

$$2(A_{B_n+1} - A_{B_n}) - (B_{B_n+1} - B_{B_n}) = 2(2) - 3 = 1,$$

$$2(A_{C_n+1} - A_{C_n}) - (B_{C_n+1} - B_{C_n}) = 2(1) - 2 = 0.$$

Note well that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are sets whose disjoint union is the set of positive integers. From (4.7), we see that

$$A_{B_n} + 1 = C_n = A_{B_n+1} - 1,$$

$$A_{B_n} < C_n < A_{B_n+1}.$$

From $A_{C_n+1} - A_{C_n} = 1$, there are no C_j's between those two A_k's. From (4.6), we see that

$$A_{A_n} + 1 = B_n = A_{\quad 1} - 1,$$

$$A_{A_n} < B_n < A_{A_n+1}.$$

Thus, $2A_n - B_n$ counts the number of C_j's less than A_n.

Theorem 4.4 shows that B_n is properly defined in the array of Table 4.1. We know from earlier work that $(B_n - A_n - n)$ counts the number of C_j's less than B_n and agrees with the definition of C_n in the array. Since each B_n and C_n is followed by some A_k, the choice of A_n as the first positive integer not yet used guarantees that the sets in the array cover the positive integers.

Nota bene: If $(2A_n - B_n)$ counts the number of C_j's less than A_n, it also counts the number of B_j's less than n. Further, $(B_n - A_n - n)$ counts the number of C_j's less than B_n; it also counts the number of B_j's less than A_n, and the number of A_j's less than n. These follow immediately from the lexicographic ordering by moving backward. Summarizing:

Theorem 4.5

(a) $(2n - 1 - A_n)$ counts the number of C_j's less than n;

(b) $(2A_n - B_n)$ counts the number of B_j's less than n;

(c) $(B_n - A_n - n)$ counts the number of A_j's less than n.

Next, we make application of a theorem of Moser and Lamdek [11];

Theorem (Leo Moser and J. Lamdek, 1954)

Let $f(n)$ be a nondecreasing function of nonnegative integers defined on the positive integers,

$$F(n) = f(n) + n, \qquad G(n) = f^*(n) + n, \tag{A}$$

where $f^*(n)$ is the number of positive integers x satisfying $0 \leq f(x) < n$. Then, $F(n)$ and $G(n)$ are complementary sequences. Conversely, every two increasing complementary sequences $F(n)$ and $G(n)$ decompose into form (A), with $f(n)$ nondecreasing.

Let $f^*(n) = B_n - A_n - n$; then

$$G(n) = B_n - A_n \quad \text{and} \quad F(n) = A_n + n = C_n - B_n,$$

since $C_n = B_n + A_n + n$. Thus, $(B_n - A_n)$ and $(C_n - B_n)$ are complementary sequences.

Let $f^*(n) = 2A_n - B_n$; then

$$G(n) = 2A_n - B_n + n = C_n - 2B_n + A_n = (C_n - B_n) - (B_n - A_n)$$

and

$$F(n) = B_n + n = C_n - A_n = (C_n - B_n) + (B_n - A_n).$$

Thus, $G(n) = (C_n - B_n) - (B_n - A_n)$ and $F(n) = (C_n - B_n) + (B_n - A_n)$ are complementary sets.

Let $f^*(n) = 2n - 1 - A_n$; then

$$G(n) = 3n - 1 - A_n \quad \text{and} \quad F(n) = C_n + n.$$

Thus, $F(n)$ and $G(n)$ are complementary sets. We have just proved:

Theorem 4.6

The three sequences $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are such that their disjoint union is the set of positive integers. That is, they form a triple of complementary sequences. Further, their differences $(B_n - A_n)$ and $(C_n - B_n)$ form a pair of complementary sequences, *and* the sum and differences of this pair of complementary sequences form another pair of complementary sequences:

$$(C_n - A_n) \quad \text{and} \quad (C_n - 2B_n + A_n = 2A_n - B_n + n).$$

5. The r-nacci Numbers

The r-nacci numbers $\{R_n\}$ are given by [14]

$$R_0 = 0,\ R_1 = 1,\ R_j = 2^{j-2},\ j = 2, 3, \ldots, r + 1,$$

and

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n. \tag{5.1}$$

The Fibonacci numbers $\{F_n\}$ are the case $r = 2$, while the Tribonacci numbers $\{T_n\}$ have $r = 3$, and the Quadranacci numbers $\{Q_n\}$ have $r = 4$.

We have the sequence of identities

$$(5.2)\quad r = 2: \quad \begin{aligned} F_2 + F_4 + F_6 + \cdots + F_{2n} &= F_{2n+1} - 1, \\ F_3 + F_5 + F_7 + \cdots + F_{2n+1} &= F_{2n+2} - 1. \end{aligned}$$

$$(5.3)\quad r = 3: \quad \begin{aligned} (T_2 + T_3) + (T_5 + T_6) + \cdots + (T_{3n-1} + T_{3n}) &= T_{3n+1} - 1, \\ (T_3 + T_4) + (T_6 + T_7) + \cdots + (T_{3n} + T_{3n+1}) &= T_{3n+2} - 1, \\ T_2 + (T_4 + T_5) + (T_7 + T_8) + \cdots + (T_{3n+1} + T_{3n+2}) &= T_{3n+3} - 1. \end{aligned}$$

$$(5.4)\quad r = 4: \quad \begin{aligned} &(Q_2 + Q_3 + Q_4) + (Q_6 + Q_7 + Q_8) + \cdots \\ &\qquad + (Q_{4n-2} + Q_{4n-1} + Q_{4n}) = Q_{4n+1} - 1, \\ &(Q_3 + Q_4 + Q_5) + (Q_7 + Q_8 + Q_9) + \cdots \\ &\qquad + (Q_{4n-1} + Q_{4n} + Q_{4n+1}) = Q_{4n+2} - 1, \\ &Q_2 + (Q_4 + Q_5 + Q_6) + (Q_8 + Q_9 + Q_{10}) + \cdots \\ &\qquad + (Q_{4n} + Q_{4n+1} + Q_{4n+2}) = Q_{4n+3} - 1, \\ &Q_2 + Q_3 + (Q_5 + Q_6 + Q_7) + (Q_9 + Q_{10} + Q_{11}) + \cdots \\ &\qquad + (Q_{4n+1} + Q_{4n+2} + Q_{4n+3}) = Q_{4n+4} - 1. \end{aligned}$$

Note that R_1 is never used on the left. Generalizing to the r-nacci numbers, we make groups of $(r - 1)$ terms, writing r equations:

$$(5.5)\quad \begin{aligned} &(R_2 + R_3 + \cdots + R_r) + (R_{r+2} + \cdots + R_{2r}) + \cdots \\ &\qquad + (R_{(k-1)r+2} + \cdots + R_{kr}) = R_{kr+1} - 1, \\ &\qquad r \\ &(R_3 + R_4 + \cdots + R_{r+1}) + (R_{r+3} + \cdots + R_{2r+1}) + \cdots \\ &\qquad + (R_{(k-1)r+3} + \cdots + R_{kr+1}) = R_{kr+2} - 1, \\ &R_2 + (R_4 + \cdots + R_{r+2}) + (R_{r+4} + \cdots + R_{2r+2}) + \cdots \\ &\qquad + (R_{(k-1)r+4} + \cdots + R_{kr+2}) = R_{kr+3} - 1, \\ &R_2 + R_3 + (R_5 + \cdots + R_{r+3}) + (R_{r+5} + \cdots + R_{2r+3}) + \cdots \\ &\qquad + (R_{(k-1)r+5} + \cdots + R_{kr+3}) = R_{kr+4} - 1, \\ &\cdots \qquad \cdots \qquad \cdots \qquad \cdots \end{aligned}$$

(5.5)—continued

$$R_2 + R_3 + \cdots + R_{r-1} + (R_{r+1} + \cdots + R_{2r}) + (R_{2r+2} + \cdots + R_{3r+2}) + \cdots$$
$$+ (R_{kr+1} + \cdots + R_{kr+(r-1)}) = R_{kr+r} - 1.$$

Notice that the proof of Eqs. (5.5) is very simple. In any of the equations, add $1 = R_1$ to the left, and observe that

$$R_1 + R_2 + R_3 + \cdots + R_i = R_{i+1} \text{ for } i = 1, 2, \ldots, r - 1,$$

and that R_{i+1} can be added to the next group of $(r - 1)$ consecutive terms to get R_{i+r+1}, which can be added to the next group of $(r - 1)$ consecutive terms. Repeat until reaching R_{kr+i}.

The r-nacci numbers, which are the generalized Fibonacci polynomials of [13] evaluated at $x = k = 1$, again give a unique Zeckendorf representation for each positive integer N,

$$N = \alpha_2 R_2 + \alpha_3 R_3 + \cdots + \alpha_k R_k, \tag{5.6}$$

where $\alpha_i \in \{0, 1\}$, and $\alpha_i \alpha_{i-1} \alpha_{i-2} \ldots \alpha_{i-r+1} = 0$.

Now let $A_i = \{a_{i,n}\}$ be the set of positive integers whose unique Zeckendorf representation has smallest term R_k, $k \geq 2$ (we have suppressed R_1), where $k \equiv i \bmod r$, $i = 2, 3, \ldots, r + 1$. Thus, every positive integer belongs to one of the sets A_i by completeness, where the sets A_i are disjoint.

Theorem 5.1

Each $a_{i,n}$ can be written so that

$$a_{i,n} = R_i + \alpha_{i+1} R_{i+1} + \alpha_{i+2} R_{i+2} + \cdots + \alpha_p R_p,$$

where $\alpha_i \in \{0, 1\}$ and $i = 2, 3, \ldots, r + 1$.

Proof: Let $N = a_{i,n}$ have R_{mr+i} as the smallest term used in its unique Zeckendorf representation. Write R_{mr+i} as

$$R_{mr+i-1} + R_{mr+i-2} + \cdots + R_{mr+i-r}.$$

Then rewrite $R_{(m-1)r+i}$ as

$$R_{(m-1)r+i-1} + R_{(m-1)r+i-2} + \cdots + R_{(m-1)r+i-r},$$

and continue replacing the smallest term used until the smallest term obtained is R_i, which is one of $R_2, R_3, \ldots, R_{r+1}$.

Theorem 5.2

The Zeckendorf representation of the positive integers in terms of the r-nacci numbers $\{R_n\}$ is a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representations,

$$M = \sum_{j=2}^{n} M_j R_j \quad \text{and} \quad N = \sum_{j=2}^{n} N_j R_j,$$

where M_j, $N_j \varepsilon \{0, 1\}$. If $M_j = N_j$ for all $j > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. Let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_2R_2 + M_3R_3 + \cdots + M_mR_m \geq R_m,$$

$$N^* = N_2R_2 + N_3R_3 + \cdots + N_{m-1}R_{m-1}.$$

Since $N_i N_{i-1} \ldots N_{i-r+1} = 0$, N^* is as large as possible when N_{m-1}, N_{m-2}, ..., N_{m-r+1} are nonzero. Then $m = rk + i$ for some $i = 1, 2, \ldots, r$. But Eqs. (5.5) show that N^* at its largest is $R_m - 1$, so that $N^* < R_m \leq M^*$, and thus $M > N$, so that the Zeckendorf representation is a lexicographic ordering.

6. The Rising Diagonals of Pascal's Triangle

The numbers $u(n;\, p,\, 1)$ of Harris and Styles [15] lie on the rising diagonals of Pascal's triangle with characteristic equation

$$x^{p+1} - x^p - 1 = 0.$$

We define $u(n;\, p,\, 1) = u_n$, where $n \geq 0$ and $p \geq 0$ are integers, by

$$(6.1) \qquad u_n = u(n;\, p,\, 1) = \sum_{i=0}^{[n/(p+1)]} \binom{n - ip}{i}, \; n \geq 1, \; u(0;\, p,\, 1) = 1,$$

where $[x]$ is the greatest integer function, and $\binom{n}{k}$ is a binomial coefficient. We note that, if $p = 1$,

$$u(n - 1;\, 1,\, 1) = F_n,$$

and if $p = 0$,

$$u(n;\, 0,\, 1) = 2^n.$$

Also,

$$u_0 = u_1 = u_2 = \cdots = u_p = 1, \; u_{p+1} = 2.$$

We write Pascal's triangle in left-justified form. Then $u(n;\, p,\, 1)$ is the sum of the term in the leftmost column and nth row (the top row is the zeroth row) and the terms obtained by starting at this term and moving p units up and one unit right throughout the array. We also have

(6.2) $$u_n = u_{n-1} + u_{n-p-1}$$

with the useful identity, for any given value of p,

(6.3) $$\sum_{i=0}^{n} u_i = u_{n+p+1} - 1.$$

Now, each positive integer N has a unique Zeckendorf representation in terms of $\{u(n;\ p,\ 1)\}$ for each given p, as developed by Mohanty [16]:

(6.4) $$N = \sum_{i=p}^{s} a_i u(i;\ p,\ 1),$$

with $a_s = 1$ and $a_i = 1$ or 0, $p \leq i < s$. Here, s is the largest integer such that F_s is involved in the sum, and $u_1 = u_2 = \cdots = u_{p-1} = 1$ are not used in any sum. If $a_i a_{i+j} = 0$ for all $i \geq p$ and $j = 1, 2, \ldots, p - 1$, then we have the unique Zeckendorf representation using the least number of terms. If $a_i + a_{i+j} \geq 1$ for all $i \geq p$ and $j = 1, 2, \ldots, p - 1$, then we have a third form, which also is a unique representation.

The results of Mohanty can be restated. Let A_i be the set of positive integers whose unique Zeckendorf representation in terms of $u(n;\ p,\ 1)$ has smallest term u_n, $n \geq p$, where $n \equiv i \bmod (p + 1)$, $i = 0, 1, 2, \ldots, p$. Then every positive integer belongs to one of the sets A_i, where the sets A_i are disjoint. Further, every element in set A_i can be rewritten uniquely so that the smallest term used is u_{p+i}, $i = 0, 1, 2, \ldots, p$, by replacing the smallest term repeatedly, as,

$$\begin{aligned} u_n &= u_{n-1} + u_{n-1-p} = u_{n-1} + u_{n-p-2} + u_{n-2p-2} \\ &= u_{n-1} + u_{n-(p+1)} + u_{n-2(p+1)} + \cdots + u_{p+i}. \end{aligned}$$

We write a second canonical form by replacing $u_p = 1$ by $u_{p-1} = 1$ whenever it occurs, but notice that only set A_p is affected.

We can establish the identity

(6.5) $$\sum_{k=1}^{n} u_{(p+1)k+i} = u_{(p+1)n+i+1} - 1$$

for each integer i, $0 \leq i \leq p$, by mathematical induction. For each value of p, when $n = 1$, we have, by (6.2):

$$u_{(p+1)\cdot 1+i} = u_{(p+1)\cdot 1+i+1} - u_p = u_{(p+1)\cdot 1+i+1} - 1.$$

If (6.5) holds for all integers $n \leq t$, then

$$\sum_{k=1}^{t+1} u_{(p+1)k+i} = \sum_{k=1}^{t} u_{(p+1)k+i} + u_{(p+1)(t+1)+i}$$

$$= (u_{(p+1)t+i+1} - 1) + u_{(p+1)t+p+i+1} - 1$$

$$= u_{(p+1)t+(p+1)+i+1} - 1$$

$$= u_{(p+1)(t+1)+i+1} - 1,$$

the form of (6.5) when $n = t + 1$, so that (6.5) holds for all integers n by mathematical induction.

We are now ready for our main theorem.

Theorem 6.1

The Zeckendorf representation of the positive integers in terms of

$$\{u(n;\ p,\ 1)\}$$

is a lexicographic ordering. The representation in second canonical form is also a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representation using the least number of terms,

$$M = \sum_{i=p}^{n} M_i u_i \quad \text{and} \quad N = \sum_{i=p}^{n} N_i u_i,$$

where $M_i,\ N_i \in \{0,\ 1\}$ and $M_i M_{i+j} = 0$ for all $i \geq p$ and $j = 1,\ 2,\ \ldots,\ p - 1$. If $M_i = N_i$ for all $i > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. Let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_p u_p + M_{p+1} u_{p+1} + \cdots + M_m u_m \geq u_m,$$

$$N^* = N_p u_p + N_{p+1} u_{p+1} + \cdots + N_{m-1} u_{m-1}.$$

Since $N_i N_{i+j} = 0$ for $j = 1,\ 2,\ \ldots,\ p - 1$, N^* is as large as possible when N_{m-1} is nonzero, but then $N_{m-2} = N_{m-3} = \cdots = N_{m-p} = 0$. The next largest possible u_i used is u_{m-p-1}, then u_{m-2p-1}, etc. Now, we can represent $(m - 1)$ as

$$m - 1 = (p + 1)k + i,$$

where $0 \leq i \leq p$. By (6.5), for any value of $(m - 1)$, we always have

$$N^* \leq \sum_{k=1}^{[(m-1-i)/(p+1)]} u_{(p+1)k+i} = u_m - 1 < M^*.$$

Thus, $M > N$, and the Zeckendorf representation is a lexicographic ordering.

Note that the same proof can be used in the second canonical form because only the smallest term in the Zeckendorf representation is changed.

7. Applications to the Generalized Fibonacci Numbers $u(n;\ 2,\ 1)$

Let us concentrate now on the sequence $u(n-1;\ 2,\ 1) = u_n$, where we take $p = 2$ in Section 6. We write

(7.1) $$u_1 = 1,\ u_2 = 2,\ u_3 = 3, \text{ and } u_{n+3} = u_{n+2} + u_n.$$

Theorem 7.1

Each positive integer N enjoys a unique Zeckendorf representation in the form

$$N = \sum_{i=1}^{k} \alpha_i u_i,\ \alpha_i\alpha_{i+1} = 0,\ \alpha_i\alpha_{i+2} = 0,$$

where $\alpha_i \in \{0,\ 1\}$.

Each positive integer N can be put into one of three sets A, B, or C according to the smallest u_k used in the unique Zeckendorf representation of N, by whether $k \equiv 1 \bmod 3$ for A, $k \equiv 2 \bmod 3$ for B, or $k \equiv 3 \bmod 3$ for C. Let $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$ be the listing of the elements of A, B, and C in natural order. Note that we can rewrite each unique Zeckendorf representation by changing only the smallest term used to make a *new form* where the smallest term appearing is u_1, u_2, or u_3. If the smallest term appearing is u_k, we replace the smallest term repeatedly:

$$\begin{aligned} u_k = u_{3m} = u_{3m-1} + u_{3m-3} &= u_{3m-1} + u_{3m-4} + u_{3m-6} = \cdots \\ &= u_{3m-1} + u_{3m-4} + \cdots + u_3, \\ u_k = u_{3m+1} = u_{3m} + u_{3m-2} &= u_{3m} + u_{3m-3} + u_{3m-5} = \cdots \\ &= u_{3m} + u_{3m-3} + \cdots + u_1, \\ u_k = u_{3m+2} = u_{3m+1} + u_{3m-1} &= u_{3m+1} + u_{3m-2} + u_{3m-4} \\ &= u_{3m+1} + u_{3m-2} + \cdots + u_2. \end{aligned}$$

We can summarize as

Theorem 7.2

Each member of set A has a representation in the form

$$A_n = 1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\ 1\};$$

each member of set B has a representation in the form

$$B_n = 2 + \alpha_3 u_3 + \alpha_4 u_4 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\ 1\};$$

and each member of set C has a representation in the form

$$C_n = 3 + \alpha_4 u_4 + \alpha_5 u_5 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\ 1\}.$$

There are some instant results:

(7.2) $$B_n - 1 = A_j, \qquad C_n - 1 = B_j.$$

Now, let $H = \{H_n\} = A \cup C$, where the elements of H are listed in natural order. We write the second canonical representation for sets A, B, and C, by replacing $u_1 = 1$ by $u_0 = 1$ in the representation of A_n but leaving B_n and C_n represented as in Theorem 7.2. Let f be the transformation that advances by one the subscripts of each of the summands u_n for each representation that is in *second canonical form.* Let f^* be the transformation that advances the subscripts by one of each summand u_n used in the Zeckendorf representation of N.

Theorem 7.3

$$N \xrightarrow{f} H \xrightarrow{f} A \xrightarrow{f^*} B \xrightarrow{f^*} C$$

Proof: It is clear that $A_n \xrightarrow{f^*} B_n \xrightarrow{f^*} C_n$ by the lexicographic ordering theorem (Theorem 6.1). Consider the sequence 1, 2, 3, ..., H_n; then, since H and B are complementary sets, we have

$$H_n = n + (\text{number of } B_j\text{'s less than } H_n).$$

Thus, by Theorem 6.1,

$$\begin{aligned}(\text{number of } B_j\text{'s less than } H_n) &= (\text{number of } A_j\text{'s less than } n)\\ &= C_n - B_n - n.\end{aligned}$$

Here we have assumed the equivalence of the definitions of A_n, B_n, and C_n and the following (see [17]):

A_n = smallest positive integer not yet used,

$B_n = A_n + n,$

$C_n = B_n + H_n.$

We now consider the sequence 1, 2, 3, ..., B_n; then

$$B_n = n + (\text{number of } H_j\text{'s less than } B_n).$$

From $j = B_n - n = A_n$ and Theorem 6.1, we conclude

(7.3) $$H_{A_n} + 1 = B_n,$$

but we also get that

$$A_n = (\text{number of } A_j\text{'s less than } C_n)$$

from Theorem 6.1. From 1, 2, ..., C_n, then

$$C_n = n + (\text{number of } A_j\text{'s less than } C_n) + (\text{number of } B_j\text{'s less than } C_n)$$

$$= n + A_n + (\text{number of } B_j\text{'s less than } C_n),$$

or

$$(\text{number of } B_j\text{'s less than } C_n) = C_n - (A_n + n) = C_n - B_n = H_n.$$

We therefore conclude from $C_n - 1 = B_j$ that

$$B_{H_n} + 1 = C_n. \tag{7.4}$$

From Theorem 6.1,

$$(\text{number of } B_j\text{'s less than } C_n) = (\text{number of } A_j\text{'s less than } B_n) = H_n.$$

Therefore, since $B_n - 1 = A_j$, we conclude

$$A_{H_n} + 1 = B_n. \tag{7.5}$$

From (7.5) and (7.3), we conclude

$$H_{A_n} = A_{H_n}. \tag{7.6}$$

We would normally have that $A_n \xrightarrow{f^*} B_n$ and $A_n \xrightarrow{f} B_n - 1 = H_{A_n} = A_{H_n}$. Also, $B_n \xrightarrow{f} C_n = H_{B_n}$. But, $C_n \xrightarrow{f} A_j$ for some j, so that set N under f goes into set H. From Theorem 6.1, $A_n \xrightarrow{f} H_{A_n} = B_n - 1 = A_{H_n}$ and $B_n \xrightarrow{f} H_{B_n} = C_n$, and $C_n \xrightarrow{f} H_{C_n}$. Now, $H_{C_n} = A_{B_n}$ as B and H are complementary, and these are the only elements left.

From (7.5), we conclude that

$$A_{A_n} + 1 = B_{H_n} \quad \text{and} \quad A_{C_n} + 1 = B_{B_n}, \tag{7.7}$$

since $H_{B_n} = C_n$. Since $H_{A_n} + 1 = B_n$,

$$H_{A_{H_n}} + 1 = B_{H_n} = A_{H_{H_n}}. \tag{7.8}$$

Note that, if we remove all $H_{B_n} = C_n$ from the ordered sequence H_n, then all we have left are the A_n, and these are $H_{H_n} = A_n$. Thus,

$$A_{A_n} + 1 = B_{H_n}. \tag{7.9}$$

Putting it together, $A_{A_n} + 1 = B_{H_n}$ and $B_{H_n} + 1 = C_n$ imply that $A_{A_n+1} = C_n + 1$, since $C_n + 1 = A_j$ always. Thus,

$$A_{A_n+1} - A_{A_n} = 3. \tag{7.10}$$

From $B_{H_n} + 1 = C_n$, one concludes that, because H and B are complementary, $B_{B_n} + 1 \neq C_j$, and since no two B_j's are consecutive, $B_{B_n} + 1 = A_j$. From

$$A_{C_n} + 1 = B_{B_n} \quad \text{and} \quad B_{B_n} + 1 = A_j = A_{C_n+1},$$

we have

$$A_{C_n+1} - A_{C_n} = 2. \tag{7.11}$$

We consider 1, 2, 3, ..., H_n. Then

$$H_n = n + (\text{number of } B_j\text{'s less than } H_n),$$

and

$$C_n - B_n - n = (\text{number of } B_j\text{'s less than } H_n)$$

$$= (\text{number of } A_j\text{'s less than } n) = (\text{number of } C_j\text{'s less than } A_n).$$

Therefore,

$$C_{B_n} - B_{B_n} - B_n = (\text{number of } C_j\text{'s less than } A_{B_n}).$$

But, $H_{B_n} = C_n$, so $H_{B_n} - B_n = C_n - B_n = H_n$. Therefore, we conclude that

$$C_{H_n} + 1 = A_{B_n}. \tag{7.12}$$

No two C_j's have a difference of 2. Now, can $A_{B_n} + 1 = B_j$? The answer is no, since $A_{A_n} + 1 = B_n$ and H and B are complementary sequences. Then $A_{B_n+1} - A_{B_n} \geq 1$ so that $C_{B_n+1} - C_{B_n} \geq 3$, and (7.10) implies that $C_{A_n+1} - C_{A_n} = 6$, while (7.11) implies that $C_{C_n+1} - C_{C_n} = 4$.

By considering the mappings under f^*, we now conclude that:

Theorem 7.4

$$A_{A_n+1} - A_{A_n} = 3, \quad A_{B_n+1} - A_{B_n} = 1, \quad A_{C_n+1} - A_{C_n} = 2;$$

$$B_{A_n+1} - B_{A_n} = 4, \quad B_{B_n+1} - B_{B_n} = 2, \quad B_{C_n+1} - B_{C_n} = 3;$$

$$C_{A_n+1} - C_{A_n} = 6, \quad C_{B_n+1} - C_{B_n} = 3, \quad C_{C_n+1} - C_{C_n} = 4.$$

Finally, we list the first few members of A, B, C, and H in Table 7.1.

TABLE 7.1

n	A_n	B_n	H_n	C_n
1	1	2	1	3
2	4	6	3	9
3	5	8	4	12
4	7	11	5	16
5	10	15	7	22
6	13	19	9	28

Notice that we may extend the table with the recurrences:

$$C_n + A_n = A_{B_n},$$

$$B_n + H_n = C_n,$$

$$n + A_n = B_n,$$

$$A_n + 2C_n + B_n = C_{B_n},$$

$$A_n + C_n + B_n = B_{B_n}.$$

We have two corollaries to Theorem 7.4:

Corollary 7.4.1

$$(\text{Number of } A_j\text{'s less than } n) = C_n - B_n - n = f(n),$$

$$(\text{Number of } B_j\text{'s less than } n) = C_n - 2A_n - 1 = g(n),$$

$$(\text{Number of } C_j\text{'s less than } n) = 3B_n - 2C_n = h(n).$$

Proof: $f(1) = 0$ and

$$f(A_m + 1) - f(A_m) = 1,$$

$$f(B_m + 1) - f(B_m) = 0,$$

$$f(C_m + 1) - f(C_m) = 0.$$

Thus, $f(n)$ increments by one only when n passes A_m, so that $f(n)$ counts the number of A_j's less than n.

Next, $g(1) = 0$, and

$$g(A_m + 1) - g(A_m) = 0,$$

$$g(B_m + 1) - g(B_m) = 1,$$

$$g(C_m + 1) - g(C_m) = 0.$$

Thus, $g(n)$ increments by one only when n passes B_m, so that $g(n)$ counts the number of B_j's less than n.

Similarly, $h(1) = 0$, and

$$h(A_m + 1) - h(A_m) = 0,$$

$$h(B_m + 1) - h(B_m) = 0,$$

$$h(C_m + 1) - h(C_m) = 1.$$

Thus, $h(n)$ increments by one only when n passes C_m, so that $h(n)$ counts the number of C_j's less than n.

Corollary 7.4.2

Let $u_{m+1} - u_m = \begin{cases} p, & m \in A; \\ q, & m \in B; \\ r, & m \in C. \end{cases}$

Then

$$u_m = (C_m - B_m - m)p + (C_m - 2A_m - 1)q + (3B_m - 2C_m)r + u_1.$$

References

1. C. G. Lekkerkerker. "Voorstelling van natuurlÿke getallen door een som van Fibonacci." *Simon Stevin* 29 (1951-1952):190-95.
2. Michael J. Whinihan. "Fibonacci Nim." *The Fibonacci Quarterly* 1, no. 4 (Dec. 1963):9-14.
3. R. Silber. "Wythoff's Nim and Fibonacci Representations." *The Fibonacci Quarterly* 15, no. 1 (Feb. 1977):85-88.
4. V. E. Hoggatt, Jr., M. Bicknell-Johnson, and R. Sarsfield. "A Generalization of Wythoff's Game." *The Fibonacci Quarterly* 17 (1979):198-211.
5. L. Carlitz, Richard Scoville, and V. E. Hoggatt, Jr. "Fibonacci Representations." *The Fibonacci Quarterly* 10, no. 1 (Jan. 1972):1-28.
6. V. E. Hoggatt, Jr. "Generalized Zeckendorf Theorem." *The Fibonacci Quarterly* 10, no. 1 (Jan. 1972):89-94.
7. Robert Silber. "On the *N* Canonical Fibonacci Representations of Order *N*." *The Fibonacci Quarterly* 15, no. 1 (Feb. 1977):57-66.
8. L. Carlitz, Richard Scoville, and V. E. Hoggatt, Jr. "Fibonacci Representations of Higher Order." *The Fibonacci Quarterly* 10, no. 1 (Jan. 1972):43-70.
9. Marcellus E. Waddill and Louis Sacks. "Another Generalized Fibonacci Sequence." *The Fibonacci Quarterly* 5, no. 3 (Oct. 1967):209-22.
10. V. E. Hoggatt, Jr. and A. P. Hillman. "Nearly Linear Functions." *The Fibonacci Quarterly* 16, no. 1 (Feb. 1979):84-89.
11. L. Moser and J. Lamdek. "Inverse and Complementary Sequences of Natural Numbers." *Amer. Math. Monthly* 61 (1954):454-58.
12. A. S. Fraenkel. "Complementary Sequences of Integers." *Amer. Math. Monthly* 84 (Feb. 1977):114-15.
13. V. E. Hoggatt, Jr. and Marjorie Bicknell. "Generalized Fibonacci Polynomials and Zeckendorf's Theorem." *The Fibonacci Quarterly* 11, no. 4 (Nov. 1973):399-419.
14. V. E. Hoggatt, Jr. and Marjorie Bicknell. "Generalized Fibonacci Polynomials and Zeckendorf's Theorem." *The Fibonacci Quarterly* 11(1973):399.
15. V. C. Harris and Carolyn C. Styles. "A Generalization of Fibonacci Numbers." *The Fibonacci Quarterly* 2, no. 4 (Dec. 1964):277-89.
16. S. G. Mohanty. "On a Partition of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* 6, no. 1 (Feb. 1968):22-34.
17. V. E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. "A Class of Equivalent Schemes for Generating Arrays of Numbers." *The Fibonacci Quarterly* (to appear).
18. Molly Olds, in a private communication with V. E. Hoggatt, Jr., Spring, 1979.

*****

ROOTS OF RECURRENCE-GENERATED POLYNOMIALS

(Submitted July 1980)

A. F. HORADAM and E. M. HORADAM
University of Warwick, Coventry CV4 7AL, England
University of New England, Armidale 2351, Australia

1. Introduction

The object of this note is to synthesize information relating to certain polynomials forming the subject matter of [1], [2], [3], and [4], the notation of which will be used hereafter. In the process, a verification of the roots of the Fibonacci and Lucas polynomials obtained in [2] is effected.

Polynomials $A_n(x)$ were defined in [3] by

$$\left\{\begin{array}{l} A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1 \text{ and} \\ A_n(x) = xA_{n-2}(x) - A_{n-4}(x). \end{array}\right. \tag{1.1}$$

Squares of the roots of

$$\frac{A_{4n}(x)}{x} = 0 \tag{1.2}$$

(of degree $2n - 2$), associated with the Chebyshev polynomial of the second kind, $U_n(x)$, were shown in [4] to be given by

$$4\cos^2\frac{i\pi}{2n} \quad (i = 1, 2, \ldots, n - 1). \tag{1.3}$$

The actual roots may be written

$$\pm 2\sin\frac{(n - i)\pi}{2n} \quad (i = 1, 2, \ldots, n - 1) \tag{1.4}$$

or, what amounts to the same thing,

$$\pm 2\sin\frac{i\pi}{2n} \quad (i = 1, 2, \ldots, n - 1). \tag{1.5}$$

Proper divisors were defined in [4] as follows: "For any sequence $\{u_n\}$, $n \geq 1$, where $u_n \in \mathbf{Z}$ or $u_n(x) \in \mathbf{Z}(x)$, the *proper divisor* w_n is the quantity implicitly defined, for $n \geq 1$, by $w_1 = u_1$ and $w_n = \max\{d\colon d|u_n$ and g.c.d. $(d, w_m) = 1$ for every $m < n\}$."

For $\{A_n(x)\}$, the first few proper divisors are:

$w_1(x) = 1$, $w_2(x) = 1$, $w_3(x) = x + 1$, $w_4(x) = x$, $w_5(x) = x^2 + x - 1$,

$w_6(x) = x - 1$, $w_7(x) = x^3 + x^2 - 2x - 1$, $w_8(x) = x^2 - 2$,

$w_9(x) = x^3 - 3x + 1$, $w_{10}(x) = x^2 - x - 1$,

$w_{11}(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, $w_{12}(x) = x^2 - 3$.

From the definition of proper divisors, we obtain (see [3])

$$A_n(x) = \prod_{d|n} w_d(x). \tag{1.6}$$

2. Complex Fibonacci and Lucas Polynomials

Hoggatt and Bicknell [2] defined the Fibonacci polynomials $F_n(x)$ by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \tag{2.1}$$

and the Lucas polynomials $L_n(x)$ by

$$L_1(x) = x,\ L_2(x) = x^2 + 2,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x). \tag{2.2}$$

Table 1 in [2] sets out the Lucas polynomials for the values $n = 1, 2, \ldots, 9$ (while Table 2 of [7] gives the coefficients of the Lucas polynomials as far as $L_{11}(x)$). Using hyperbolic functions, Hoggatt and Bicknell ([2, p. 273]) then established complex solutions of the equations

$$F_{2n}(x) = 0,\ F_{2n+1}(x) = 0,\ L_{2n}(x) = 0, \text{ and } L_{2n+1}(x) = 0,$$

which are of degree $2n - 1$, $2n$, $2n$, and $2n + 1$, respectively.

Suppose we now replace x by ix ($i = \sqrt{-1}$) in (2.1) and (2.2). Designating the new polynomials by $F_n^*(x)$ and $L_n^*(x)$, we have, from $F_{2n}(x) = F_n(x)L_n(x)$:

$$F_{2n}^*(x) = F_n^*(x)L_n^*(x). \tag{2.3}$$

Referring to the details of Table 1 in [2], we can tabulate the ensuing information where, for visual ease, we have represented the polynomials $A_n(x)$ and the proper divisors $w_n(x)$ of $A_n(x)$ by A_n and w_n, respectively (see Table 1, p. 221).

Summarizing the tabulated data, we have

$$F_{2n}^*(x) = (-1)^{n-1}iA_{4n}(x) \quad (n \geq 1), \tag{2.4}$$

$$F_{2n+1}^*(x) = (-1)^n A_{4n+2}(x) = (-1)^n \Psi_{2n}(x) \quad (n \geq 0), \tag{2.5}$$

$$L_{2n}^*(x) = (-1)^n B_{8n}(x) \quad (n \geq 1), \tag{2.6}$$

$$L_{2n+1}^*(x) = (-1)^n ixB_{4(2n+1)}(x) = (-1)^n ix\Phi_{2n}(x) \quad (n \geq 1), \tag{2.7}$$

TABLE 1

n	$F_n^*(x)$ n even	$F_n^*(x)$ n odd	$L_n^*(x)$ n even	$L_n^*(x)$ n odd
1		A_2		ixw_1
2	iA_4		$-w_8$	
3		$-A_6$		$-ixw_{12}$
4	$-iA_8$		w_{16}	
5		A_{10}		ixw_{20}
6	iA_{12}		$-w_8w_{24}$	
7		$-A_{14}$		$-ixw_{28}$
8	$-iA_{16}$		w_{32}	
9		A_{18}		$ixw_{12}w_{36}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where the symbolism $\Psi_{2n}(x)$ and $\Phi_{2n}(x)$ of Hancock [1] has been introduced in (2.5) and (2.7). For the $B_{4n}(x)$ notation given in terms of proper divisors, see [4, p. 248]. Degrees of $F_n^*(x)$ and $L_n^*(x)$ are, of course, the same as those of the corresponding $F_n(x)$ and $L_n(x)$.

The results of (2.4) and (2.5) follow directly from (1.4) of [4] and the well-known fact:

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}.$$

To establish (2.6) and (2.7), we consider the evenness and oddness of n separately and invoke (4.1) of [4].

n even ($n = 2k$):

$$\begin{aligned} L_{2k}^*(x) &= \frac{F_{4k}^*(x)}{F_{2k}^*(x)} && \text{by (2.3)} \\ &= \frac{(-1)^{2k-1} iA_{8k}(x)}{(-1)^{k-1} iA_{4k}(x)} && \text{by (2.4)} \\ &= (-1)^k \frac{A_{8k}(x)}{A_{4k}(x)} = (-1)^k B_{8k}(x) && \text{by (4.1) of [4].} \end{aligned}$$

n odd ($n = 2k + 1$):

$$\begin{aligned} L_{2k+1}^*(x) &= \frac{F_{4k+2}^*(x)}{F_{2k+1}^*(x)} && \text{by (2.3)} \\ &= \frac{(-1)^{2k} iA_{8k+4}(x)}{(-1)^k A_{4k+2}(x)} && \text{by (2.4), (2.5)} \\ &= (-1)^k ixB_{8k+4}(x) && \text{by (4.1) of [4].} \end{aligned}$$

From (2.6) and (2.7), an explicit formula for $B_{4n}(x)$ may be obtained by appealing to the known expression for $L_n(x)$:

$$L_n(x) = \sum_{j=0}^{[n/2]} \frac{n}{n-j}\binom{n-j}{j}x^{n-2j} \qquad (n \geq 1).$$

Arguing for $A_{4n+2}(2x) = U_{2n}(x)$ (the Chebyshev polynomial of the second kind) as for $A_{4n}(x)$ in [4], we derive the $2n$ roots of

$$A_{4n+2}(x) = 0 \tag{2.8}$$

to be $\pm 2 \cos \dfrac{i\pi}{2n+1}$ $(i = 1, 2, \ldots, n)$ or, equivalently,

$$\pm 2 \sin \frac{(2i+1)}{(2n+1)} \cdot \frac{\pi}{2} \qquad (i = 0, 1, 2, \ldots, n-1). \tag{2.9}$$

Next, consider the roots of

$$B_{8n}(x) = 0. \tag{2.10}$$

From [4], these are the roots of $\dfrac{A_{8n}(x)}{x} = 0$ excluding those belonging to the set of roots of (1.2). Consequently, by (1.4), the roots of (2.10) are $\pm 2 \sin \dfrac{(2n-i)\pi}{4n}$ $(i = 1, 2, \ldots, 2n-1)$ diminished by $\pm 2 \sin \dfrac{2(n-i)\pi}{4n}$ $(i = 1, 2, \ldots, n-1)$. Calculation yields the remaining roots to be

$$\pm 2 \sin \frac{(2i+1)\pi}{4n} \qquad (i = 0, 1, 2, \ldots, n-1). \tag{2.11}$$

Finally, in our analysis of the roots of $F_n^*(x) = 0$ and $L_n^*(x) = 0$, we find from [1] that the $2n$ roots of

$$\Phi_{2n}(x) = 0 \tag{2.12}$$

are $\pm 2 \sin \dfrac{2i\pi}{2n+1} = \pm 2 \sin\left(\pi - \dfrac{2i\pi}{2n+1}\right)$, $i = 1, 2, \ldots, n$, that is, after manipulation,

$$\pm 2 \sin \frac{i\pi}{2n+1} \qquad (i = 1, 2, \ldots, n). \tag{2.13}$$

The roots of $F_{2n}^*(x) = 0$, $F_{2n+1}^*(x) = 0$, $L_{2n}^*(x) = 0$, and $L_{2n+1}^*(x) = 0$ are, respectively, those given in (1.5), (2.9), (2.11), and (2.13). See also [8]. It must be noted that the $2n - 2$ roots in (1.5) relate to $\dfrac{A_{4n}(x)}{x} = 0$ in (1.2), so $A_{4n}(x) = 0 = F_{2n}^*(x)$ in (2.4) has $(2n-2) + 1 = 2n - 1$ roots, one of these roots being $x = 0$. Also note the zero root associated with (2.7).

Verification of the Hoggatt-Bicknell roots is thus achieved by complex numbers in conjunction with the properties of the polynomials $A_n(x)$.

3. The Polynomials $A_{2n+1}(x)$

So far, the odd-subscript polynomials $A_{2n+1}(x)$ of degree n have not been featured. As mentioned in [4, pp. 245, 249],

$$A_{2n+1}(x) = \bar{f}_n(x) \tag{3.1}$$

in the notation of [1], where

$$f_n(x) = A_{2n+2}(x) - A_{2n}(x) = (-1)^n\bar{f}_n(-x) = (-1)^n A_{2n+1}(-x). \tag{3.2}$$

For instance,

$$\begin{aligned} f_5(x) &= A_{12}(x) - A_{10}(x) = x^5 - 4x^3 + 3x - (x^4 - 3x^2 + 1) \\ &= -(-x^5 + x^4 + 4x^3 - 3x^2 - 3x + 1) = -A_{11}(-x) \\ &= (-1)^5\bar{f}_5(-x). \end{aligned}$$

Using the information given in [1] for the n roots of $\bar{f}_n(x) = 0$, we have that the n roots of

$$A_{2n+1}(x) = 0 \tag{3.3}$$

are

$$2\cos\frac{2i\pi}{2n+1} \quad (i = 1, 2, \ldots, n). \tag{3.4}$$

Thus, the two roots of $A_5(x) = \bar{f}_2(x) = x^2 + x - 1 = 0$ are

$$2\cos\frac{2\pi}{5},\ 2\cos\frac{4\pi}{5}\left(= -2\cos\frac{\pi}{5}\right).$$

Following Legendre [6], Hancock [1] remarks that the equations

$$(-1)^n f_n(-x) = \bar{f}_n(x)$$

constitute a type of reciprocal equation obtained by substituting $z = x + \frac{1}{x}$ in $\frac{x^{2n+1} - 1}{x - 1} = 0$.

In [3, p. 55] it is shown that

$$A_{2n}(x) = \frac{s^{2n} - t^{2n}}{s^2 - t^2}, \tag{3.5}$$

where $s^2 = \frac{1}{2}(x + \sqrt{x^2 - 4})$ and $t^2 = \frac{1}{2}(x - \sqrt{x^2 - 4})$. Then

$$\begin{aligned} A_{4n+2}(x) &= A^2_{2n+2}(x) - A^2_{2n}(x) && \text{on using (3.5)} \\ &= (A_{2n+2}(x) - A_{2n}(x))(A_{2n+2}(x) + A_{2n}(x)) \\ &= f_n(x)\overline{f}_n(x) && \text{(Hancock [1])} \\ &= f_n(x)A_{2n+1}(x) && \text{by (3.1),} \end{aligned}$$

where we note as in [4, p. 248] that our $A_{2n}(x)$ is Hancock's $A_{n-1}(x)$. Thus,

(3.6) $$f_n(x) = A_{4n+2}(x)/A_{2n+1}(x),$$

so that the $f_n(x)$ are expressible in terms of proper divisors. As an example, $A_{18}(x) = w_6(x)w_{18}(x)A_9(x)$, i.e.,

(3.7) $$f_4(x) = A_{10}(x) - A_8(x) = w_6(x)w_{18}(x) = (x - 1)(x^3 - 3x - 1).$$

4. Concluding Comments

(a) The $2n$ roots of each equation

(4.1) $$A_{4n+2}(x) = (-1)^n \sec \frac{2i\pi}{2n + 1} \qquad (i = 1, 2, \ldots, n)$$

are shown in [1] to be

(4.2) $$\pm 2 \sin \frac{2i\pi}{2n + 1} \qquad (i = 1, 2, \ldots, n).$$

Combining (2.5), (2.7), (2.12), (2.13) (in the equivalent form), (4.1), and (4.2) we see that

(4.3) $$\frac{L^*_{2n+1}(x)}{ix} = 0 \text{ and } F^*_{2n+1}(x) - \sec \frac{2k\pi}{2n + 1} = 0 \qquad (k = 1, 2, \ldots, n)$$

of degree $2n$, for a given n and a given value of k have the roots

$$\pm 2 \sin \frac{2k\pi}{2n + 1} \qquad (k = 1, 2, \ldots, n)$$

in common. For example, if $n = 2$ we find that $\pm 2 \sin \alpha \left(\alpha = \frac{2\pi}{5}, \frac{4\pi}{5}\right)$ are roots of $\frac{L^*_5(x)}{ix} = 0$ and $F^*_5(x) - \sec \alpha = 0$.

(b) It is observed in [1] that the curves

$$y = f_n(x) = A_{2n+2}(x) - A_{2n}(x) \quad [(3.2)] \qquad (n = 1, 2, \ldots)$$

all pass through the point with coordinates (2, 1), and through one or the other of the points (0, 1), (0, -1). Examples for easy checking are $y = f_4(x)$ given in (3.7), and $y = f_5(x)$ appearing after (3.2).

(c) Mention must finally be made of the very recent article by Kimberling [5] on cyclotomic polynomials which impinges on some of the content herein. Among other matters, one may compare Table 2 of [5] with Table 1 of [2].

If the irreducible divisors of the Fibonacci polynomials $F_n(x)$ given by (2.1) are represented by $\mathcal{F}_d(x)$ where $d|n$, then by [5, p. 114],

$$F_n(x) = \prod_{d|n} \mathcal{F}_d(x). \tag{4.4}$$

Allowing x to be replaced by ix in the polynomials $\mathcal{F}_n(x)$ occurring in Kimberling's Table 2, and writing the polynomial corresponding to $\mathcal{F}_n(x)$ as $\mathcal{F}_n^*(x)$, we find using [8] that

$$\mathcal{F}_p^*(x) = F_p^*(x) \qquad p \text{ prime} \tag{4.5}$$

$$\mathcal{F}_{2n}^*(x) = (-1)^{\frac{1}{4}\phi(4n)} w_{4n}(x) \qquad (n > 1), \tag{4.6}$$

where $\phi(n)$ is Euler's function and, by [4],

$$\text{deg. } w_n(x) = \frac{1}{2}\phi(n). \tag{4.7}$$

While the proof of (4.5) is straightforward, that of (4.6) requires some amplification. Now

$$F_{2n}^*(x) = (-1)^{n-1} iA_{4n}(x) \qquad \text{which is (2.4)}$$

$$\prod_{d|2n} \mathcal{F}_d^*(x) = (-1)^{n-1} ix \prod_{d|4n} w_d(x) \quad \text{by (4.4) amended and [4, p. 244]}$$

$$\prod_{d|2n} \mathcal{F}_d^*(x) = (-1)^{n-1} i \prod_{d|4n} w_d(x) \qquad n \geq 1 \text{ since } w_4(x) = x. \tag{4.8}$$

Apart from the sign (+ or -), the highest factor $\mathcal{F}_{2n}^*(x)$ on the left-hand side of (4.8) must equal the highest factor $w_{4n}(x)$ on the right-hand side of (4.8). This sign must, on the authority of (4.7), be

$$i^{\frac{1}{2}\phi(4n)} = (-1)^{\frac{1}{4}\phi(4n)},$$

whence (4.6) follows.

For example,

$$F_6^* = i(x^4 - 4x^3 + 3x) = ix(x^2 - 1)(x^2 - 3) = \mathcal{F}_2^*(x)(-\mathcal{F}_3^*(x))(-\mathcal{F}_6^*(x))$$
$$= iA_{12} = ix(x + 1)(x - 1)(x^2 - 3) = iw_4(x)w_3(x)w_6(x)w_{12}(x),$$

whence

$$\mathcal{F}_6^*(x) = -w_{12}(x) = (-1)^{\frac{1}{2}\phi(12)} w_{12}(x).$$

Kimberling's article opens up many ideas which we do not pursue here.

This concludes the linking together of material from several sources. Consideration of the polynomials $A_n(x)$ does indeed enable us to encompass a wide range of results.

References

1. H. Hancock. "Trigonometric Realms of Rationality." *Rendiconti del Circolo Matematico di Palermo* 49 (1925):263-76.
2. V. E. Hoggatt, Jr. and Marjorie Bicknell. "Roots of Fibonacci Polynomials." *The Fibonacci Quarterly* 11, no. 3 (1973):271-74.
3. A. F. Horadam, R. P. Loh, and A. G. Shannon. "Divisibility Properties of Some Fibonacci-Type Sequences." *Combinatorial Mathematics VI: Proceedings, Armidale, August, 1978* (Springer-Verlag, 1979) [Lecture Notes in Mathematics, vol. 748], pp. 55-64.
4. A. F. Horadam and A. G. Shannon. "Irrational Sequence-Generated Factors of Integers." *The Fibonacci Quarterly* 19, no. 3 (1981):240-50.
5. C. Kimberling. "Generalized Cyclotomic Polynomials, Fibonacci Cyclotomic Polynomials, and Lucas Cyclotomic Polynomials." *The Fibonacci Quarterly* 18, no. 2 (1980):108-26.
6. A. M. Legendre. *Théorie des Nombres* II (Blanchard, 1955), 269.
7. A. G. Shannon and A. F. Horadam. "Combinatorial Aspects of an Infinte Pattern of Integers." *The Fibonacci Quarterly* 20, no. 1 (1982):44-51.
8. W. A. Webb and E. A. Parberry. "Divisibility Properties of Fibonacci Polynomials." *The Fibonacci Quarterly* 7, no. 5 (1969):457-63.

*****

HOGGATT READING ROOM DEDICATION

On April 30, 1982, the Department of Mathematics at San Jose State University dedicated the

VERNER E. HOGGATT, JR. READING ROOM.

The room, opposite the offices of the Department of Mathematics, houses a splendid research library and various mathematical memorabilia. At the ceremony, Dean L. H. Lange of the School of Sciences talked of his long association with Professor Hoggatt and about Fibonacci numbers. A reception followed for faculty members and guests. Among the guests were various friends and associates of Professor Hoggatt, a number of whom are active in carrying on the work that Professor Hoggatt started with *The Fibonacci Quarterly*. Mrs. Hoggatt and her daughters attended the dedication ceremony, and Mrs. Hoggatt was presented with a portrait of her late husband.

PRIMITIVE PYTHAGOREAN TRIPLES

(Submitted August 1980)

LEON BERNSTEIN
Illinois Institute of Technology, Chicago, IL 60616

Dedicated to Carl Menger, on the occasion of his 80th birthday.

0. Introduction

This paper investigates some problems concerning PRIMITIVE PYTHAGOREAN TRIPLES (PPT) and succeeds in solving, completely or partially, some of these problems while leaving open others. Dickson [2], in his three-volume history of number theory has given a twenty-five-page account of what was achieved in the field of Pythagorean triangles during more than two millenia and up to Euler and modern times. Therefore, it is surprising that still more questions can be asked which, in their intriguing simplicity, do not lag behind anything the human mind has been occupied with since the times of Hamurabi. The author thinks that, in spite of the accelerated speed with which the modern mathematical creativeness is advancing in the era of Godel and Matajasevich, some of his unanswered questions will remain enigmatic for many decades to come.

1. Definition

There are a variety of definitions on the subject of PPTs. The author thinks that he was able to come up with some of his results thanks to a simplification of on such definition, which is as follows:

Definition 1

A triple (x, y, z) of natural numbers if a PPT iff there exists a pair (u, v) of natural numbers such that

$$x = u^2 - v^2,\ y = 2uv,\ z = u^2 + v^2, \qquad (1.1)$$
$$(u, v) = 1,\ u + v \equiv 1 \pmod 2.$$

The pair of numbers (u, v) as introduced in Definition 1 is called a *generator* of the PPT (x, y, z). We shall use the chain of inequalities

$$2u > u + v > u, \qquad (1.2)$$

which follows from Definition 1.

All small italic letters appearing in this paper denote natural numbers, 1, 2, 3, ..., if not stated otherwise.

By virtue of Definition 1, a countability of all PPTs has been established, namely,

$$(u, v) = (2, 1) \Rightarrow (x, y, z) = (3, 4, 5);$$

$$(u, v) = (3, 2) \Rightarrow (x, y, z) = (5, 12, 13);$$

$$(u, v) = (4, 1) \Rightarrow (x, y, z) = (15, 8, 17);$$

$$(u, v) = (4, 3) \Rightarrow (x, y, z) = (7, 24, 25);$$

etc.

If we drop the condition $(u, v) = 1$ in (1.1), then the resulting triple (x, y, z) is a Nonprimitive Pythagorean Triple. They are of no interest to us.

2. Pythagorean Frequency Indicator

We introduce the interesting

Definition 2

The number of times the integer n appears in some PPT, excluding order, is called the PYTHAGOREAN FREQUENCY INDICATOR (PFI) of n. The PFI of n is denoted by $\underline{f(n)}$. We write $\underline{f(n) = 2^{-\infty}}$, if n does not appear in any PPT. As we shall see later,

$$f(1) = 2^{-\infty},\ f(2) = 2^{-\infty},\ f(3) = 1,$$

$$f(4) = 1,\ f(5) = 2,\ \ldots,\ f(84) = 4, \text{ etc.}$$

The following result is due to Landau [4]:

Theorem 1

The number of positive solutions $L(n)$ of $x^2 + y^2 = n$ (excluding order), with $(x, y) = 1$ and

(2.1) $$x^2 + y^2 = n = \prod_{i=1}^{k} p_i^{s_i},\ p_i \text{ an odd prime,}$$

is given by

(2.2) $$L(n) = 2^{k-1} \text{ if each } p_i \equiv 1 \pmod 4$$

and

(2.3) $$L(n) = 0, \text{ if at least one } p_i \equiv 3 \pmod 4.$$

Landau's theorem also elaborates on such numbers n which are not of the form (2.1) with (2.2) or (2.3), but that is not relevant for us. To state the main theorem of this chapter, it is useful to introduce the following.

Let n be as in (2.1). If all primes are as in (2.2), we let $n = 0(k, 1)$; otherwise, we let $n = 0(k, 3)$.

Theorem 2

The PFI of any number n equals

$$\begin{aligned} & f(1) = f(2) = f(2\ 0(k,\ 1)\ ) = f(2\ 0(k,\ 3)\ ) = 2^{-\infty} \\ & f(\ 0(k + 1,\ 3)\ ) = f(\ 0(k,\ 1)\ ) = f(2^{s+1}\ 0(k,\ 1)\ ) \\ & \qquad = f(2^{s+1}\ 0(k,\ 3)\ ) = 2^{k},\ s \geq 1. \end{aligned} \tag{2.4}$$

Proof: We have, by Definition 1, $x \equiv z \equiv 1 \pmod 2$, $y \equiv 0 \pmod 4$, and $x, z \neq 1$. This proves the first line of (2.4). When $n = 0(k + 1, 3)$, then only $n = x$ is possible by Theorem 1. Let $n = fg$ with $(f, g) = 1$ and $f > g$. Since $(u, v) = 1$ and $n = (u - v)(u + v)$, we have

$$u = \frac{1}{2}(f + g) \qquad \text{and} \qquad v = \frac{1}{2}(f - g).$$

But,

$$\binom{k + 1}{0} + \binom{k + 1}{1} + \cdots + \binom{k + 1}{k + 1} = 2^{k+1}$$

is the total number of pairs $\{f, g\}$ with $(f, g) = 1$. Hence, we have only 2^k pairs with $f > g$. Now let $n = 0(k, 1)$. When $n = z$ there are, by Theorem 2, 2^{k-1} pairs (u, v) such that $u^2 + v^2 = n$; when $n = x$ there are 2^{k-1} pairs, by the same argument given for $n = 0(k + 1, 3)$. Hence, $f(\ 0(k, 1)\ ) = 2^k$. Let $n = y = 2^{s+1}0(k, 0)$ or $n = 2^{s+1}0(k, 3)$. Let $n = 2^{s+1}fg$, where $(f, g) = 1$. Since there are only 2^{k-1} pairs (f, g), excluding order, with $(f, g) = 1$, we can choose $u = 2^s f$ and $v = g$ or $u = f$ and $v = 2^s g$. Hence, there are 2^k possibilities. This proves the second line of Theorem 2, and proves the theorem completely.

Theorem 2 also holds for $n = 2^{s+1}$ with the symbolism $n = 2^{s+1}p$, since $f(2^{s+1}) = 2^0 = 1$. The following examples illustrate the use of Theorem 2:

$f(2^{s_1+1}) = f(p^{s_2}) = 2^0 = 1$, p any odd prime, $p \equiv 3 \pmod 4$.

$f(2^{s_1+1}p\ \ ) = 2^1 = 2$, p any odd prime.

$f(q^s) = 2^1 = 2$, $q \equiv 1 \pmod 4$, q prime.

$f(p_1^{s_1}p_2^{s_2}) = 2^1$, not both $p_1, p_2 \equiv 1 \pmod 4$, p_1, p_2 odd primes.

$f(p_1^{s_1}p_2^{s_2}p_3^{s_3}) = 2^2 = 4$, p_1, p_2, p_3 odd primes not all contruent to 1 modulo 4.

$f(q^{t_1}q^{t_2}) = 2^2 = 4$, q_1, q_2 odd primes congruent to 1 modulo 4.

$f(2^{s+1}p^tq^r) = 2^2 = 4$, p, q any odd primes, etc.

We have $f(60) = f(4 \cdot 3 \cdot 5) = 2^2 = 4$. The corresponding PPTs are (899, 60, 901), (91, 60, 109), (11, 60, 61), (221, 60, 229). Also, $f(16) = f(2^4) = 1$. The corresponding PPT is (63, 16, 65).

We let the smallest integer n such that $f(n) = 2^k$ ($k = -\infty, 0, 1, 2, \ldots$) be denoted by $M(n, k)$.

It is easily seen that $M(n, -\infty) = 1$, $M(n, 0) = 3$, $M(n, 1) = 5$. An interesting result is stated in Theorem 3, but first we let $p_1, p_2, \ldots$ denote the successive odd primes and we denote the product of k successive odd primes by $\Pi_k = p_1p_2 \cdots p_k$.

Theorem 3

If $k \geq 2$ then $M(n, k) = 4\Pi_k$.

Proof: The reader can easily verify the relations

$$2^{s+1}O(k, 1) > O(k, 1) > 4\Pi_k,$$

$$2^{s+1}O(k, 3) > 4\Pi_k$$

and

$$O(k + 1, 3) > 4\Pi_k,$$

if $k \geq 2$, while all have the same value of $f(n) = 2^k$. This proves the theorem. We thus have $M(n, 2) = 4\Pi_2 = 60$, $M(n, 3) = 4 \cdot 3 \cdot 5 \cdot 7 = 420$, etc. Hence, 420 is the smallest number which appears exactly eight times in PPTs.

3. Perimeters

This is the most important part of our paper. It contains problems never investigated previously. To clarify them, we start with:

Definition 3

Let (x, y, z) be a PPT and (u, v) be its generator. We call the sum $x + y + z$ the PERIMETER of PPT.

We denote the perimeter of a PPT with generator (u, v) by

$$\Pi(u, v) = x + y + z = 2u(u + v) \equiv \Pi. \tag{3.1}$$

Thus $\Pi(2, 1) = 12$, $\Pi(3, 2) = 30$, $\Pi(4, 1) = 40$, etc. Different PPTs may have the same Π for different generators. An example of this will be given in Theorem 5. (No two different generators can lead to the same PPT.) Accordingly, we introduce:

Definition 4

The (exact) number of different PPTs having the same perimeter is called the DOMAIN of this perimeter. In symbols, we write $D(\Pi) = k$ if the number of generator pairs in the set $\{(u, v) | \Pi(u, v) = \Pi\}$ is k. Since a number n may not be a perimeter, we introduce the notation $n \neq \Pi$ and write $D(\Pi) = 0$.

By (3.1), every perimeter is even. Hence $D(2t + 1) = 0$. Let $m \equiv 1$ (mod 2) and p be an odd prime such that $p^t > 2^s m$ for some s with $(p, m) = 1$. It is easy to prove that $D(2^s mp^t) = 0$. The method of proving this will emerge from the sequel.

Theorem 4

Let p be an odd prime.

a) If $2^{s+1}p = \Pi$ and $2^{s+1} > p > 2^s$, then $D(2^{s+1}p) = 1$.

(3.2) b) If $2p^t(p^t + 1) = \Pi$, then $D(2p^t(p^t + 2)) = 1$.

c) If $p^t(p^t + 1) = \Pi$, then $D(p^t(p^t + 1)) = 1$.

Proof: Generally, in order to investigate whether a given n is or is not a perimeter, it suffices to write n in the form $2u(u + v)$, where (u, v) is a generator. Then make use of the relation (1.2).

To prove [(3.2), a], we proceed as follows. Let $2^{s+1}p = 2u(u + v)$, then $2^s p = u(u + v)$. Since $u + v \equiv 1$ (mod 2), we have $2^s | u$. There are therefore two cases $p | u$ or $p | (u + v)$. If $p | u$, we have $u = 2^s p$ and $u + v = 1$, which is impossible because $u + v < u$. If $p | (u + v)$, we have $u = 2^s$, $u + v = p$, and $v = p - 2^s$. By hypothesis, $u < u + v < 2u$. Obviously $(u, v) = 1$ and $u + v \equiv 1$ (mod 2), so (u, v) is a generator and $D(2^{s+1}p) = 1$.

To prove [(3.2), b], we let $p^t(p^t + 2) = u(u + v)$. Since $p^t + 2$ may factor, we assume $p^t + 2 = fg$ with $f > g$. With $p^t fg = u(u + v)$, there are two obvious cases to consider. They are:

$$u = p^t,\ v = fg - p^t \quad \text{and} \quad u = fg,\ v = p^t - fg.$$

The latter case is out, since we need $v > 0$. The former case yields a solution since $(u, v) = 1$, $u + v \equiv 1$ (mod 2), and $u < u + v < 2u$. With $(f, g) = 1$, $g \neq 1$, there are six more possibilities, all of which can be ruled out, since the relations

$$p^t fg < 1,\ p^t fg < 2,\ g < p^t f < 2g,\ p^t f < g < 2fp^t,$$

$$f < p^t g < 2f, \text{ and } gp^t < f < 2gp^t$$

are impossible. Therefore, $D(2p^t(p^t + 2)) = 1$.

An argument similar to that of [(3.2), b] will show that the only solution for [(3.2), c] is $u = (p^t + 1)/2$, $v = (p^t - 1)/2$, so that $D(p^t(p^t + 1)) = 1$, completing the proof of the theorem.

The following is an immediate consequence of Theorem 4.

Corollary

Let $2^{p-1}(2^p - 1)$, p prime, be a perfect number. Then $2^{p-1}(2^p - 1)$ cannot be a perimeter, while $2^p(2^p - 1)$ can be a perimeter only once.

Theorem 4 also shows that there are infinitely many PPTs of domain 1. We prove the following interesting result.

Theorem 5

Let be an odd prime.

(3.3)
a) When $p > 6$, $12p(p + 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv 1 \pmod 3$ and $D(\Pi) = 2$ if $p \equiv -1 \pmod 3$.
b) When $p > 8$, $12p(p - 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv -1 \pmod 3$ and $D(\Pi) = 2$ if $p \equiv 1 \pmod 3$.

Proof: Since $6p(p + 2) = u(u + v)$, where u is even, $u + v \equiv 1 \pmod 2$ and $(u, u + v) = 1$, we have eight possible cases for the choices of the factors of u and $u + v$. However, we need $u < u + v < 2u$, so six of these cases can be eliminated immediately leaving only

$$u = 2p,\ v = 3p + 6 - 2p = p + 6 \tag{3.4}$$

and

$$u = 2(p + 1),\ v = p - 4. \tag{3.5}$$

When $p \equiv 1 \pmod 3$, then (3.5) is not a valid generator, since $(u, v) \neq 1$. However, (3.4) is a generator with perimeter $12p(p + 2)$. When $p \equiv -1 \pmod 3$ both (3.4) and (3.5) are valid generators of $12p(p + 2)$, since $(u, v) = 1$, $u + v \equiv 1 \pmod 2$, and $u < u + v < 2u$.

Let $6p(p - 2) = \Pi$. A similar argument to that of part (a) shows that

$$u = 2p,\ v = p - 6 \tag{3.6}$$

and

$$u = 2(p - 2),\ v = p + 4 \tag{3.7}$$

are generators of $12p(p - 2)$ if $p \equiv 1 \pmod 3$, while only (3.6) is a valid generator if $p \equiv -1 \pmod 3$.

By Dirichlet's theorem and Theorem 5, we know there are infinitely many PPTs with $D(\Pi) = 2$.

Actually, Theorem 5 is a special case of the following more general theorem whose proof we omit because of its similarity to that of Theorem 5.

Theorem 5a

Let p be an odd prime. Let q be a prime such that $2^q - 1$ is a prime.

(i) When $p > 2(2^q - 1)$, $2^q(2^q - 1)p(p + 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv -2 \pmod{2^q - 1}$ and $D(\Pi) = 2$ if $p \not\equiv -2 \pmod{2^q - 1}$. The solutions are

$$u = 2^{q-1}p,\ v = (2^{q-1} - 1)p + 2(2^q - 1)$$

and

$$u = 2^{q-1}(p + 2),\ v = (2^{q-1} - 1)p - 2^q$$

if $p \not\equiv -2 \pmod{2^q - 1}$. If $p \equiv 2 \pmod{2^q - 1}$, only the first solution is a valid generator.

(ii) When $p > 2^{q+1}$, $2^q(2^q - 1)p(p - 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv 2 \pmod{2^q - 1}$ and $D(\Pi) = 2$ if $p \not\equiv 2 \pmod{2^q - 1}$. The solutions are

$$u = 2^{q-1}p,\ v = (2^{q-1} - 1)p - 2(2^q - 1)$$

and

$$u = 2^{q-1}(p - 2),\ v = (2^{q-1} - 1)p + 2^q$$

if $p \not\equiv 2 \pmod{2^q - 1}$. If $p \equiv 2 \pmod{2^q - 1}$, only the first solution is a valid generator.

When $p \equiv -1 \pmod 3$ and $p + 2$ is also a prime, the two solutions of parts (a) and (b) of Theorem 5 are the same. Hence, twin primes enter into our analysis of the perimeter problem.

It is easy to show that the smallest value of Π with $D(\Pi) = 2$ is

$$12 \cdot 11 \cdot 13 = 1716.$$

The generators are $\Pi = \Pi(22, 17) = \Pi(26, 7)$, whose Pythagorean triples are, respectively, (195, 748, 773) and (627, 364, 725).

4. More on Domains

The following two theorems state the most important results of this paper. In the sequel, it will be convenient to denote the two numbers $T = p^s$ and $T + 2 = q^t$, where p, q are odd primes, by *prime power twins*. We state:

Theorem 6

Let

$$\Pi = 2u(u + v),\ T \text{ and } T + 2 \text{ be prime power twins,}$$

(4.1)

$$T > \Pi,\ D(\Pi) = k, \text{ and } (\Pi, T(T + 2)) = 1.$$

Then

(4.2) $\Pi' = \Pi T(T + 2)$ is a perimeter with $D(\Pi') = 2k$.

Proof: We prove that any generator for Π leads to exactly two generators for Π'. Since $T > \Pi > 2(u + v)$, we see that

$$T > 2(u + v)/(u - v) \quad \text{and} \quad T > 2u/v. \tag{4.3}$$

But, (4.3) implies that $T(u - v)/(u + v) > 2$, so $2uT/(u + v) > (T + 2)$ or

$$2uT > (u + v)(T + 2) > uT. \tag{4.4}$$

Furthermore, from (4.3) we obtain $2/T < v/u$, so

$$(T + 2)/T < (u + v)/u \quad \text{or} \quad (u + v)T > u(T + 2).$$

Hence, by (1.2),

$$2u(T + 1) > (u + v)T > u(T + 2). \tag{4.5}$$

Since we want $\Pi' = \Pi T(T + 2) = 2u(u + v)T(T + 2) = 2x(x + y)$, where (x, y) is a generator, there are sixteen possible ways of choosing the factors of $u(u + v)T(T + 2)$ for x and $x + y$. However, we need $x < x + y < 2x$. Therefore, fourteen of these possibilities can be easily eliminated. For example, if $x = u(u + v)$ and $x + y = T(T + 2)$, then

$$T(T + 2) > 2(u + v)(T + 2) > 2(u + v)T > 4(u + v)^2 > 2u(u + v),$$

so $x + y > 2x$. As another example, let $x = T(T + 2)$ and $x + y = u(u + v)$. Then

$$T(T + 2) > u(u + v),$$

so $x > x + y$. The only two cases that satisfy $x < x + y < 2x$, by (4.4) and (4.5), are

$$x = uT,\ x + y = (u + v)(T + 2),\ y = (u + v)(T + 2) - uT \tag{4.6}$$

and

$$x = u(T + 2),\ x + y = (u + v)T,\ y = vT - 2u. \tag{4.7}$$

In both of these cases, it is easy to show that $(x, y) = 1$, $x + y \equiv 1 \pmod 2$ and $2x(x + y) = \Pi'$.

Because $u(u + v) = f \cdot g$ with $(f, g) = 1$, where $f > g$ is possible, since $u(u + v) = p^s$, p a prime, is impossible, we need to show that these factorizations do not lead to any new generators of Π'. We let $f > 2g$, then $2g > f > g$ and $2f > g > f$ are both impossible, so that (f, g) is not a generator of Π.

With $2u(u + v)T(T + 2) = 2fgT(T + 2) = 2x(x + y)$, where (x, y) is a generator, there are, again, sixteen possible ways of choosing the factors of $fgT(T + 2)$ for x and $x + y$. All of these cases are easily eliminated. For example, if $x + y = g(T + 2)$ and $x = fT$, then $fT > 2gt > g(T + 2)$, so $x > x + y$, which contradicts (x, y) being a generator; as another example, let $x + y = gT$ and $x = f(T + 2)$, then $f(T + 2) > 2g(T + 2) > gT$ and again $x > x + y$, which is a contradiction. As our final example, we choose $x + y = fT$ and $x = g(T + 2)$. Then $4g > (f - 2g)T > T > 2u(u + v) = 2fg$, so that $2 > f$, which is a contradiction. We leave the other cases to the reader.

Hence, we have proved that the only generators for Π' are the generators for Π, each of which leads to exactly two generators for Π'. This proves the theorem.

Example

Let $\Pi = \Pi(22, 17) = \Pi(26, 7) = 1716$. We choose $T = 1721$ and $T + 2 = 1723$, where 1721 and 1723 are primes with $T > \Pi$. Hence,

$$\Pi' = 1716 \cdot 1721 \cdot 1723,\ D(\Pi') = 4,$$

and

$$\Pi' = \Pi'(37862, 29335) = \Pi'(37906, 29213)$$

$$= \Pi'(44746, 12113) = \Pi'(44798, 11995).$$

With doubling the $D(\Pi)$, the PPTs grow enormously, since $T > \Pi$. Hence, if the T's are finite in number, there may be an upper bound for $D(\Pi)$. The following modification of Theorem 6 may somehow be helpful.

Theorem 6a

Let

$$\Pi = 2u(u + v),\ (T, T + 2) \text{ be prime power twins, } D(\Pi) = k,$$
$$(\Pi, T(T + 2)) = 1, \text{ and } T > 2(u + v).$$

(4.8) Let the number of pairs $(f, g) = 1$, such that

$$u(u + v) = f \cdot g,\ f > 2g, \text{ and } 2g(T + 2) > fT > g(T + 2)$$

with f odd be m, where $m = 0, 1, 2, \ldots$.

Then $D(\Pi') = D(\Pi T(T + 2)) = 2k + m$.

Proof: With $T > 2(u + v) \geq 2(u + v)/(u - v)$ and $T > 2(u + v) > 2u/v$, we prove, as in Theorem 6, that each generator for Π leads to exactly two generators of $\Pi' = 2u(u + v)T(T + 2)$. Since $2g(T + 2) > fT > g(T + 2)$, (4.5) would account for another solution, so $D(\Pi') = 2k + m$, $m \geq 0$. The author was unable to find an example where $m \neq 0$.

Example

Let $\Pi = \Pi(22, 17) = \Pi(26, 7) = 1716$, then $2(u + v)$ equals 78 or 66. For $T > 78$, we choose $T = 101$ and $T + 2 = 103$. We then have

$$g \cdot f = 6(11 \cdot 13) = 2(3 \cdot 11 \cdot 13)$$

with $2g < f$ and f odd. But in neither of these cases does the relation

$$2g(T + 2) > fT > g(T + 2)$$

hold, as can be easily verified. When $T > 66$, we choose $T = 71$ and $T + 2 = 73$. Then

$$g \cdot f = 6(11 \cdot 13) = (3 \cdot 11 \cdot 13).$$

Again, in neither case, is

$$2g(T + 2) > fT > g(T + 2).$$

Thus, $m = 0$ and $\Pi' = 1716 \cdot 101 \cdot 103$ has $D(\Pi') = 4$.

Theorem 7

Let $\Pi = 2u(u + v)$. Let $(T, T + 2)$ be prime power twins with

$$(\Pi, T(T + 2)) = 1.$$

Let $D(\Pi) = k$. Further, let

$$\begin{aligned} &\left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 < T < (\Pi + 1)^{1/2} - 1,\ u - v \geq 5, \text{ and } v \geq 3, \text{ or} \\ &\left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 < T < \left(\frac{\Pi}{2} + 1\right)^{1/2} - 1,\ u - v \geq 6, \text{ and } v \geq 5 \text{ with } u \text{ odd.} \end{aligned} \tag{4.9}$$

Then

$$\begin{aligned} &\Pi' = 2u(u + v)T(T + 2) \text{ has } D(\Pi') = 2k + 1 + m, \\ &m \text{ as in Theorem 6a.} \end{aligned} \tag{4.10}$$

Proof: With the restrictions on $u - v$, and v from (4.9), we can easily prove that

$$\left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 > 2(u + v)/(u - v) \quad \text{and} \quad \left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 > 2u/v.$$

Also

$$\left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 > 2(u + v)/(u - v) \quad \text{and} \quad \left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 > 2u/v.$$

Thus, $T > 2(u + v)/(u - v)$ and $T > 2u/v$. From these last two relations, it is then proved, as before, that every generator (u, v) for Π leads to exactly two generators for $\Pi' = 2u(u + v)T(T + 2)$. We further have, from part (b) of (4.9), that

$$2T(T + 1) > u(u + v) > T(T + 2),$$

and from part (a) of (4.9) that

$$2u(u + v) > T(T + 2) > u(u + v) \text{ for a fixed } T.$$

This would account for the additional generator for Π'. The meaning of the possible m generators for Π' is the same as in Theorem 6a. This completes the proof of Theorem 7.

The reader may ask whether in the intervals given by (4.9) there is always a prime power (or prime) T. This fundamental question is answered affirmatively by a famous theorem by Chebyshev [1] which states that in the interval $(y, (1 + e)y)$, $e > 1/5$ there is always, from a certain point on, one prime, from a further point on, two primes, etc. The reader will easily verify that the intervals (4.9) satisfy the conditions of Chebyshev's theorem.

As our first example, we choose $\Pi = \Pi(40, 3) = 3440$, so that $\sqrt{1721} - 1 < T < \sqrt{3440} - 1$ or $40 < T < 57$. With $T = 41$ and $T + 2 = 43$, we have

$$(\Pi, T(T + 2)) = 43,$$

so that Theorems 6, 6a, and 7 do not apply. We choose $T = 47$ and $T + 2 = 7^2$. Note that $D(\Pi) = 1$ and $\Pi' = 3440 \cdot 47 \cdot 49 = 7922320$. Since $1720 = 8(5 \cdot 43) = gf$ with $f > 2g$ and f odd does not yield

$$2g(T + 2) > fT > g(T + 2),$$

we have $m = 0$ and $D(\Pi') = 3$.

As another example, we choose $\Pi(46, 29) = \Pi(50, 19) = 4 \cdot 3 \cdot 23 \cdot 25 = 6900$ so that $D(\Pi) = 2 = k$. We have $u - v = 17$, $v = 29$ and $u - v = 31$, $v = 19$. Further

$$\sqrt{\frac{1}{2} \cdot 6900 + 1} - 1 < T < \sqrt{6900 + 1} - 1,$$

and we choose $T = 59$, $T + 2 = 61$, so that $\Pi' = 6900 \cdot 59 \cdot 61 = 24833100$. We also have $u(u + v) = 6(23 \cdot 25) = 2(3 \cdot 23 \cdot 25) = fg$ with $f > 2$ and f odd. But the condition

$$2g(T + 2) > fT > g(T + 2)$$

is not satisfied here. Thus, by Theorem 7, $D(\Pi') = D(24833100) = 5$. The author leaves it to the reader to find the value of $D(\Pi')$ when $T = 71$, $T + 2 = 73$ and $T = 79$, $T + 2 = 3^4$.

5. n-Periadic Numbers

We introduce

Definition 5

A number t is called n-PERIADIC if t^n is a perimeter but t^{n+1} is not.

If t^n is a perimeter, then there exist x and y relatively prime such that $x + y \equiv 1 \pmod 2$, $2x(x + y) = t^n$, and $x < x + t < 2x$. Hence, there exist u and v relatively prime such that $x = 2^{n-1}u^n$ and $x + y = v^n$. Furthermore, $2^{(n-1)/n}u < v < 2u$. If t^{n+1} is not a perimeter, then $v < 2^{n/n+1}u$. This proves the necessary part of the following theorem.

Theorem 8

The number t is n-periodic iff there exists $(u, v) = 1$ such that

$$2^{(n-1)/n}u < v < 2u \quad \text{and} \quad v < 2^{n/(n+1)}u. \tag{5.1}$$

We leave a proof of the sufficiency part to the reader.

From (5.1), we see that $v > 2^{(n-1)/n}u > \left(1 + \frac{n-1}{n}\ln 2\right)u$, so

$$\frac{n-1}{n} < \frac{v}{u} < 2. \tag{5.2}$$

When $n = 2$, (5.2) yields $\frac{1}{2} < \frac{v}{u} < 2$. Choose $v = 6s + 1$ and $u = 4s + 1$ with $s \geq 2$. Then $(u, v) = 1$ and $v^2 > 2u^2$. Furthermore, $v^3 < 4u^3$. Let

$$x = 2(4s + 1)^2 \quad \text{and} \quad x + y = (6x + 1)^2,$$

as in the proof of the theorem. Then

$$t = \Pi(4s + 1, 2s) \tag{5.3}$$

is 2-periadic. In particular, with $s = 2$, we have that $\Pi(9, 4) = 18 \cdot 13$ is 2-periadic with generator $x = 162$, $y = 7$.

When $n = 3$, (5.2) yields $\frac{2}{3} < \frac{v}{u} < 2$. Choose $v = 10s + 1$ and $u = 6s + 1$ with $s \geq 2$. Then $(u, v) = 1$, $v^3 > 4u^3$ and $v^4 < 8u^4$. Let

$$x = 4(6s + 1)^3 \quad \text{and} \quad y + x = (10s + 1)^3.$$

Then

$$t = \Pi(6s + 1, 4s) \tag{5.4}$$

is 3-periadic. In particular, for $s = 2$, we have

$$v = 21,\ u = 13,\ x = 4 \cdot 13^3 = 8788,\ y = 473$$

and $t = \Pi(13, 8)$ is 3-periadic.

By this method, we can obtain any n-periodic number. However, those obtained by (5.3) and (5.4) are by far not all of the infinitely many 2-periadic and 3-periadic numbers.

Conspicuously absent are the 1-periadic numbers. We have,

$$\Pi = \Pi(u, 1),\ u \geq 3 \tag{5.5}$$

is 1-periadic, since $2u > u + 1 > u$ and $(u + 1)^2 > 2u^2$.

The reader should not overlook the following trivial relation. Let $\Pi(u, v) = 2u(u + v)$, then

$$(\Pi(u, v))^n = \Pi(2^{n-1}u^n, (u + v)^n - 2^{n-1}u^n),\ n \geq 1. \tag{5.6}$$

Note that if $\Pi(u, v) = x + y + z$, then

$$\Pi(u, v)^2 = x\Pi(u, v) + y\Pi(u, v) + z\Pi(u, v),$$

but

$$(\Pi(u, v)x, \Pi(u, v)y, \Pi(u, v)z) \neq \Pi(u, v).$$

In this context, we prove

Theorem 9

For every perimeter $\Pi(u, v)$ there exists at least one prime p such that $p\Pi(u, v)$ is a perimeter.

Proof: Let $\Pi = 2u(u + v)$. By Bertrand's postulate, there is at least one prime p such that $2u(u + v) > p > u(u + v)$. Hence, $2u(u + v)p$ is a perimeter.

6. Associating with Fibonacci

We introduce

Definition 6

Let (x, y, z) be a PPT. It is called *associative* if $f(x) = f(y) = f(z)$, *nonassociative* if all PFIs of x, y, z are different, *quasi-associative* if the PFIs of exactly any two x, y, z are equal. If

$$f(x) = f(y) = f(z) = 2^k,\ k = 0, 1, \ldots,$$

we say the PPT = (x, y, z) is *k-associative*.

Examples

(3, 4, 5) is quasi-associative,

(5, 12, 13) is 1-associative,

(7, 24, 25) is quasi-associative,

(99, 100, 101) is 1-associative, since $f(99) = f(3^2 \cdot 11) = 2^1$, $f(100) = f(4 \cdot 5^2) = 2^1$ and $f(101) = 2^1$,

(675, 52, 677) is quasi-associative,

(11, 60, 61) is nonassociative, since $f(11) = 2^0$, $f(60) = f(4 \cdot 3 \cdot 5) = 2^2$, $f(61) = 2^1$,

(3477, 236, 3485) is nonassociative, since $f(3477) = f(3 \cdot 19 \cdot 61) = 2^2$, $f(236) = f(4 \cdot 59) = 2^1$, $f(3485) = f(5 \cdot 17 \cdot 41) = 2^3$.

The Fibonacci sequence

$$F_1 = F_2 = 1,\ F_{n+2} = F_n + F_{n+1} \quad (n = 1, 2, \ldots),$$

has solved and raised many puzzles. Every mathematician should have a copy of Hoggatt's precious booklet [3] on this subject. Since $F_{6k+3} \equiv 2 \pmod 4$, F_{6k+3} does not appear in any PPT; all other F_n, $n > 3$, do. $F_{12} = 144$ has $\Pi = \Pi(8, 1)$, with the PPT being (63, 16, 65). The only Fibonacci numbers known to appear in the same PPT are 3, 5 and 5, 13, see [5]. The Fibonacci number $F_8 = 21$ has $(21) = 2$ where the two PPTs are (21, 20, 29) and (21, 220, 221). Note that (21, 220, 221) is quasi-associative, since

$$f(21) = f(3 \cdot 7) = 2^1,\ f(220) = f(4 \cdot 5 \cdot 11) = 2^2,$$

and

$$f(221) = f(13 \cdot 17) = 2^2.$$

Observe that (21, 20, 29) is also quasi-associative. The Fibonacci number $F_{11} = 89$ appears in (89, 3960, 3961) with

$$f(89) = 2^1,\ f(3960) = f(8 \cdot 3^2 \cdot 5 \cdot 11) = 2^3,$$

and

$$f(3961) = f(17 \cdot 233) = 2^2.$$

Hence, the triple is nonassociative. The first Fibonacci number which is a perimeter is 144, the largest perfect square in the Fibonacci sequence. $\Pi(8, 1) = 144$ leads to the PPT (63, 16, 65), with $D(144) = 1$. This PPT is nonassociative with

$$f(63) = 2^1,\ f(16) = 2^0,\ f(65) = 2^2.$$

Concluding, we want to point out that apart from the riddle of associativity the most saddening unsolved problem in this paper is the question of whether or not there are infinitely many PPTs of any given domain. Since a solution seems to hinge on the unsolved problem of the number of prime twins, it seems to be a difficult problem.

Acknowledgment

The author is deeply indebted to the editor for improving important results and adding new ones, for refining concepts and definitions, and for correcting proofs and calculating errors. Without his contribution, this paper would not be complete.

References

1. Chebyshev. See Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, I. New York: Chelsea, 1953.
2. Leonard E. Dickson. *History of Number Theory*. 3 vols. New York: Chelsea, 1919.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969 (rpt. The Fibonacci Association, 1980).

4. Edmund Landau. *Vorlesungen uber Zahlentheorie.* New York: Chelsea, 1950.
5. Marjorie Bicknell-Johnson. "Pythagorean Triples Containing Fibonacci Numbers: Solutions for $F_n^2 \pm F_k^2 = K^2$." *The Fibonacci Quarterly* 17, no. 1 (1979):1-12.
6. Marjorie Bicknell-Johnson. "Addend to 'Pythagorean Triples Containing Fibonacci Numbers: Solutions for $F_n^2 \pm F_k^2 = K^2$.'" *The Fibonacci Quarterly* 17, no. 4 (1979):293.

*****

A NOTE ON THE FAREY-FIBONACCI SEQUENCE

(Submitted October 1980)

K. C. PRASAD
Ranchi University, India

1. Introduction

The Fibonacci sequence $\{F_n : n \geq 0\}$ is defined as

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

Let $r_{i,j} = F_i/F_j$. Alladi [1] defined a Farey-Fibonacci sequence f_n of order n as the sequence obtained by arranging the terms of the set

$$\sum_n = \{r_{i,j} \mid 1 \leq i < j \leq n\}$$

in ascending order and studied its properties in detail. Alladi [2] and Gupta [3] gave rapid methods to write out f_n. Finally, Alladi and Shannon [4] briefly considered certain special properties of consecutive members of f_n.

We now prescribe a different scheme to write out f_n, which is rapid, direct, and simpler than the earlier approaches. We not only obtain the term-number of a preassigned member of f_n as found by Gupta [3], but also a formula for the general term of f_n not explicitly obtained before.

2. Scheme

Let us write out the terms of $\sum_n$ in a triangular array as shown below:

$$\begin{array}{l}
r_{1,n};\ r_{1,n-1};\ r_{1,n-2};\ \ldots;\ r_{1,1+n-i};\ \ldots;\ r_{1,2} \\
\qquad r_{2,n};\ r_{2,n-1};\ \ldots;\ r_{2,2+n-i};\ \ldots;\ r_{2,3} \\
\qquad\qquad r_{3,n};\ \ldots;\ r_{3,3+n-i};\ \ldots;\ r_{3,4} \\
\qquad\qquad\quad \ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad r_{i,n};\ \ldots;\ r_{i,i+1} \\
\qquad\qquad\qquad\qquad\quad \ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad r_{n-1,n}
\end{array}$$

Next, we designate the terms of the ith column of this array by

$$x_1, x_2, \ldots, x_i.$$

Clearly, $x_j = r_{j,j+n-i}$ for $1 \leq j \leq i$. Observe that

(i) $x_1 < x_2$, an inequality equivalent to $F_{n-i} < F_{1+n-i}$

and

(ii) x_k lies between x_{k-1} and x_{k-2} for $3 \leq k \leq i$,

a consequence of the simple rule that the fraction

$$(h + h')/(k + k')$$

lies between h/k and h'/k'.

Let $a_{i,1}$; $a_{i,2}$; ...; $a_{i,i}$ denote the sequence obtained by arranging the x's in ascending order. Then the observations (i) and (ii) above imply

(A)
$$a_{i,1} = x_1;\ a_{i,i} = x_2$$
$$a_{i,2} = x_3;\ a_{i,i-1} = x_4$$
and so on.

In fact, the x's arranged in ascending order are

$$x_1, x_3, x_5, \ldots, x_6, x_4, x_2.$$

This reveals the scheme of writing, in ascending order, the members of any given column of the above array.

Now since $a_{i,i} < a_{i+1,1}$ for $1 \leq i \leq n - 1$ is equivalent to $F_{n-i} < F_{1+n-i}$ for $1 \leq i \leq n - 1$, we get f_n as follows:

$$a_{1,1};\ a_{2,1};\ a_{2,2};\ \ldots;\ a_{i,1};\ a_{i,2};\ \ldots$$
$$\ldots\ a_{i,i};\ a_{i+1,1};\ a_{i+2,2};\ \ldots,\ a_{i+1,i+1};\ \ldots;\ a_{n-1,1};\ \ldots;\ a_{n-1,n-1}.$$

3. Formulas

I. If F_q/F_m is the tth term (T_t) of f_n, then

$$t = \begin{cases} \frac{1}{2}(n - m + q)(n - m + q - 1) + \frac{q+1}{2}: & \text{if } q \text{ is odd,} \\ \frac{1}{2}(n - m + q)(n - m + q - 1) + n - m + \frac{q}{2} + 1: & \text{if } q \text{ is even.} \end{cases}$$

Proof: If F_q/F_m or $r_{q,m}$ appears in the ith column of the array, then obviously $m - q = n - i$, $t = \frac{1}{2}i(i - 1) + j$, and from (A) $j = M$ or $i - M + 1$ according as $q = 2M - 1$ or $2M$, respectively. Thus t is apparent.

II. The following is the formula for the tth term of f_n:

$$T_t = F_{i-2|k|+\delta(i,k)}/F_{n-2|k|+\delta(i,k)},$$

where

$$i = \begin{cases} [\sqrt{(2t-2)}] & \text{if } 2t \leq [\sqrt{(2t-2)}]([\sqrt{(2t-2)}] + 1) \\ [\sqrt{(2t-2)}] + 1 & \text{otherwise,} \end{cases}$$

$$k = t - i(i-1)/2 - [(i+1)/2],$$

and

$$\delta(i, k) = \begin{cases} -1 \text{ if } i \text{ is even and } k \leq 0 \\ 0 \text{ if } i \text{ is odd and } k \leq 0 \\ 1 \text{ if } i \text{ is odd and } k > 0 \\ 2 \text{ if } i \text{ is even and } k > 0. \end{cases}$$

Proof: If T_t appears in the ith column of the array, then

$$i(i-1)/2 + 1 \leq t \leq (i+1)_i/2$$

and consequently i is as described above. Furthermore, if

$$T_t = a_{i,j} = x_p = r_{p,p+n-i},$$

then

$$i(i-1)/2 + j = t.$$

To find p, we examine its dependence on k where $j = [(i+1)/2] + k$. From relations (A) it is clear that

$$\text{for even } i,\ p = \begin{cases} i + 2k - 1 & \text{if } k \leq 0 \\ i - 2k + 2 & \text{if } k > 0 \end{cases}$$

and

$$\text{for odd } i,\ p = \begin{cases} i + 2k & \text{if } k \leq 0 \\ i - 2k + 1 & \text{if } k > 0. \end{cases}$$

These observations suffice.

References

1. K. Alladi. "A Farey Sequence of Fibonacci Numbers." *The Fibonacci Quarterly* 13, no. 1 (1975):1-10.
2. K. Alladi. "A Rapid Method to Form Farey-Fibonacci Fractions." *The Fibonacci Quarterly* 13, no. 1 (1975):31-32.
3. H. Gupta. "A Direct Method of Obtaining Farey-Fibonacci Sequences." *The Fibonacci Quarterly* 14, no. 4 (1976):389-91.
4. A. G. Shannon and K. Alladi. "On a Property of Consecutive Farey-Fibonacci Fractions." *The Fibonacci Quarterly* 15, no. 2 (1977):153-55.

*****

THEVENIN EQUIVALENTS OF LADDER NETWORKS

(Submitted January 1981)

WILLIAM P. RISK
Tempe, AZ 85282

An electrical network of considerable importance in applications is known as the *ladder network*. A common form of this circuit consists of resistive elements connected together as shown in Figure 1. It is often used as an attenuator to reduce the applied input voltage to various other values which are made available to subsequent loads through the m taps shown in the same figure.

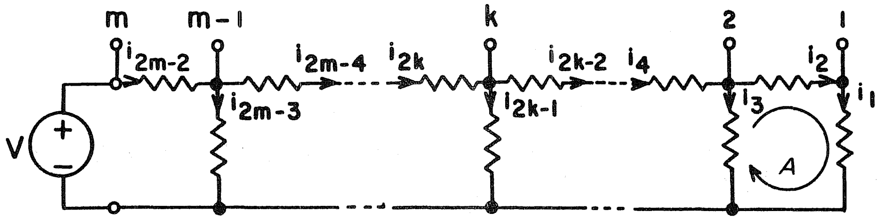


Figure 1

A basic result of elementary circuit analysis is that any network of linear resistors and sources may be replaced by an equivalent circuit consisting of an ideal voltage source and a single series resistor. This configuration is known as the *Thevenin equivalent of the original network*. It is often desirable to find the Thevenin equivalent voltage and resistance of a ladder network as perceived by a load connected to one of its taps. The case in which all the resistors in the ladder network have identical values is of particular interest since the expressions for the Thevenin equivalents involve the Fibonacci sequence.

The derivation of these expressions requires the use of three basic rules of circuit analysis and one observation. The three rules are known as *Kirchoff's voltage law*, *Kirchoff's current law*, and *Ohm's law* (for a full discussion of these, see [1]). The observation is that in a ladder network such as that shown in Figure 1, the current in the jth resistor is related to that in the rightmost resistor by:

$$i_j = F_j i_1, \; j = 1, 2, \ldots, 2m - 2.$$

To derive an expression for the Thevenin equivalent voltage at the kth tap, with a given input voltage v, one must find the voltage appearing at that tap with the tap open-circuited. Under these conditions:

$$v_k = Ri_{2k-1} = F_{2k-1}Ri_1,$$

where R is the common value of all the resistors. Applying Kirchoff's voltage law to the leftmost loop yields:

$$v = (i_{2m-2} + i_{2m-3})R = F_{2m-1}Ri_1$$

or

$$i_1 = \frac{v}{RF_{2m-1}}.$$

Hence,

$$v_k = v\left(\frac{F_{2k-1}}{F_{2m-1}}\right), \text{ for } k = 1, 2, \ldots, m.$$

Derivation of an expression for the Thevenin equivalent resistance at the kth tap requires the application of a principle of circuit analysis which says that the Thevenin equivalent resistance of a network may be found by evaluating the effective resistance of the network after all independent sources have been set equal to zero. In this case, the mth tap must be shorted to ground to eliminate the voltage source supplying the input voltage v. To determine the effective resistance once this is done, a current source of unit value may be applied at the kth tap. If the voltage at the kth tap can be determined, the Thevenin equivalent resistance may then be found from Ohm's law.

Applying the unit current source to the kth tap, as shown in Figure 2, Kirchoff's current law at the kth tap becomes:

$$i_c + i_d + i_e = 1.$$

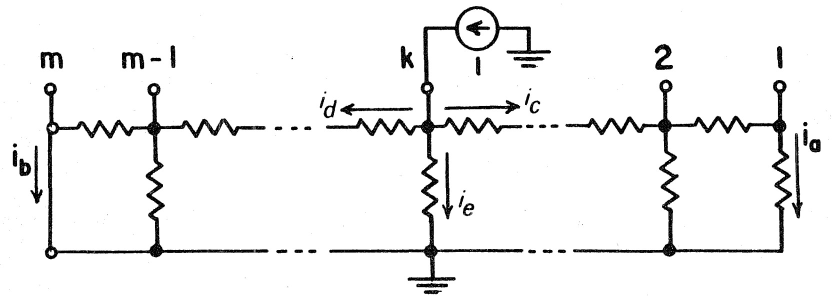


Figure 2

What is the relationship between i_c, i_d, and i_e? We previously cited the observation that, in the circuit of Figure 1, the current in the jth resistor was related to that in the rightmost resistor by:

$$i_j = F_j i_1, \; j = 1, 2, \ldots, 2m - 2.$$

This was obtained from examining the results of applying Kirchoff's voltage law to loop A in Figure 1, then applying Kirchoff's current law to node 2, and so forth, each time relating the currents and voltages back to i_1. Since these relationships depend on the way the resistors are connected, they are still valid for resistors to the right of the kth tap in Figure 2. Hence, we obtain

$$i_c = F_{2k-2} i_a.$$

If we start at the left end and work rightward, alternately writing loop and node equations and relating the voltages and currents back to i_b, we can similarly obtain

$$i_d = F_{2(m-k)-1} i_b.$$

Working again from the right end, we find

$$i_e = F_{2k-1} i_a.$$

Working from the left end, we find

$$i_e = F_{2(m-k)} i_b.$$

The current i_e can be eliminated to give a relationship between i_a and i_b, namely

$$F_{2k-1} i_a = F_{2(m-k)} i_b.$$

Now, we may replace i_c, i_d, and i_e in Kirchoff's current law at the kth tap. We obtain:

$$F_{2k-2} i_a + F_{2(m-k)-1} i_b + F_{2k-1} i_a = 1.$$

Eliminating i_b gives us:

$$(1) \qquad i_a \left(F_{2k-2} + \frac{[F_{2(m-k)-1}][F_{2k-1}]}{F_{2(m-k)}} + F_{2k-1} \right) = 1.$$

Now, the voltage at the kth tap is $v_k = Ri_e$. The Thevenin equivalent resistance is then

$$(2) \qquad R_k = v_k/1 = Ri_e = R[F_{2k-1} i_a].$$

Solving for i_a from Eq. (1) above and substituting it into the expression for R_k (2), we obtain:

$$(3) \qquad R_k = R \left(\frac{F_{2k-1}}{F_{2k} + \dfrac{[F_{2(m-k)-1}][F_{2k-1}]}{F_{2(m-k)}}} \right), \; k = 1, 2, \ldots, m.$$

The Fibonacci sequence is thus seen to insinuate itself into the expression for ladder network Thevenin equivalents, chiefly as a result of the manner in which currents are related in the network. These results may be of some practical value in affording a simple means of analyzing a particular ladder network. If nothing else, they provide an interesting example of the occurrence of the Fibonacci sequence in an applied situation.

References

1. Donald A. Calahan, Alan B. Macnee, and E. Lawrence McMahon. *Introduction to Modern Circuit Analyais*. New York: Holt, Rinehart and Winston, 1974.
2. S. L. Basin. "The Fibonacci Sequence as it Appears in Nature." *The Fibonacci Quarterly* 1, no. 1 (Feb. 1963):53-66.
3. S. L. Basin. "The Appearance of the Fibonacci Numbers and the *Q* Matrix in Electrical Network Theory." *Math. Mag.*, March-April, 1963, pp. 84-97.

*****

FIBONACCI RESEARCH CONFERENCE

October 9, 1982

San Jose City College
San Jose, California 95128

Two sessions are scheduled: 9:00 A.M. to noon and 1:30 P.M. to 4:30 P.M.

Following the morning session, there will be a luncheon available for conference participants.

For detailed information concerning the names of the conference speakers and their topics, and about the luncheon, please write to:

Gerald E. Bergum
Department of Mathematics
South Dakota State University
P.O. Box 2220
Brookings SD 57007

A PROPERTY OF BINOMIAL COEFFICIENTS

(Submitted February 1981)

MAURO BOSCAROL
Libera Università di Trento, I-38050, Povo (Trento), Italy

The purpose of this paper is to prove identity (1), related to the binomial coefficients.

For each pair of integers n, $m \geq 0$, the following identity holds:

$$\sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{h=0}^{n} \frac{\binom{n+m-h}{m}}{2^{n+m-h}} = 2. \tag{1}$$

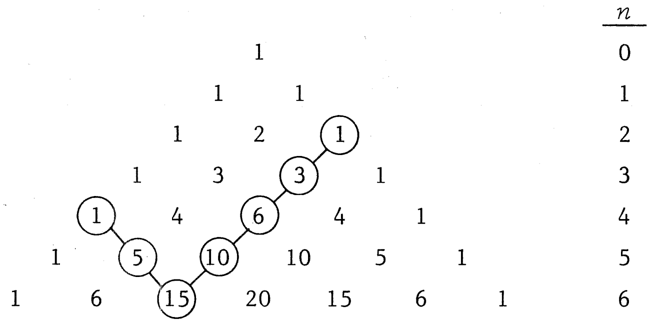
The meaning of this identity becomes more clear if one considers Pascal's triangle:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & & & \underline{n} \\
 & & & & & & 1 & & & & & & & & 0 \\
 & & & & & 1 & & 1 & & & & & & & 1 \\
 & & & & 1 & & 2 & & 1 & & & & & & 2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & & 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & & 4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & 5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & 6
\end{array}$$

Let us consider a path which starts from any point on the left side and goes down following a line parallel to the right side, then stops at any point and goes up again, following a line parallel to the left side, until it reaches the right side. If we add all the binomial coefficients we have met, each multiplied by 2^{-n}, the result is always 2. (The binomial coefficient at the turning point of the path being considered twice.) For example, the following path yields

$$\frac{\binom{4}{0}}{2^4} + \frac{\binom{5}{1}}{2^5} + \frac{\binom{6}{2}}{2^6} + \frac{\binom{6}{2}}{2^6} + \frac{\binom{5}{2}}{2^5} + \frac{\binom{4}{2}}{2^4} + \frac{\binom{3}{2}}{2^3} + \frac{\binom{2}{2}}{2^2} = 2.$$

The Pascal triangle is shown in the following figure.



To prove identity (1), we need:

Lemma 1

Let a, b, $c \in \mathbb{Z}$ with $a < b$ and $c \geq 0$. Then we have

$$2\sum_{k=a}^{b}\binom{c}{k} = \sum_{k=a+1}^{b}\binom{c+1}{k} + \binom{c}{a} + \binom{c}{b}. \tag{2}$$

Proof: This identity stems immediately from the fact that

$$\binom{c}{k} + \binom{c}{k+1} = \binom{c+1}{k+1}.$$

Now we can prove identity (1). This identity is true if $n = m = 0$. Let us assume that n is different from zero and change the index h to $j = n - h$. We obtain the following equivalent identity:

$$\sum_{i=0}^{m}\frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{j=0}^{n}\frac{\binom{m+j}{m}}{2^{m+j}} = 2, \tag{3}$$

which is symmetrical in n and m since $\binom{m+j}{m} = \binom{m+j}{j}$. We can thus assume that $m \leq n$ (and $n \neq 0$).

Let us now write the binomial theorem $(1+1)^m = 2^m = \sum_{k=0}^{m}\binom{m}{k}$ in the following form:

$$2 = \frac{\sum_{k=0}^{m}\binom{m}{k}}{2^{m-1}} = 2\frac{\sum_{k=0}^{m}\binom{m}{k}}{2^{m}}.$$

Since, for $k < 0$, $\binom{m}{k} = 0$, the sum may start at $k = m - n \leq 0$:

$$2 = 2\frac{\sum_{k=m-n}^{m}\binom{m}{k}}{2^{m}}.$$

Since $m > m - n$, applying Lemma 1 to the sum we get:

$$\frac{2\sum\limits_{k=m-n}^{m}\binom{m}{k}}{2^m} = \frac{\sum\limits_{k=m-n+1}^{m}\binom{m+1}{k} + \binom{m}{m-n} + \binom{m}{m}}{2^m}$$

$$= \frac{2\sum\limits_{k=m-n+1}^{m}\binom{m+1}{k}}{2^{m+1}} + \frac{\binom{m}{m-n} + \binom{m}{m}}{2^m}.$$

Again we have $m \geq m - n + 1$, and again we can apply Lemma 1. If we proceed in this way, after using the lemma r times we get:

$$\frac{2\sum\limits_{k=m-n+r}^{m}\binom{m+r}{k}}{2^{m+r}} \sum_{j=0}^{r-1} \frac{\binom{m+j}{m+n+j} + \binom{m+j}{m}}{2^{m+j}}.$$

So we can apply Lemma 1 until $m - n + r < m$, i.e., until $r < n$. At this point we get:

$$\frac{2\binom{m+n}{m}}{2^{m+n}} + \sum_{j=0}^{n-1} \frac{\binom{m+j}{m-n+j} + \binom{m+j}{m}}{2^{m+j}} = \sum_{j=0}^{n} \frac{\binom{m+j}{m-n+j} + \binom{m+j}{m}}{2^{m+j}}$$

$$= \sum_{s=0}^{n} \frac{\binom{m+s}{m-n+s}}{2^{m+s}} + \sum_{j=0}^{h} \frac{\binom{m+j}{m}}{2^{m+j}}.$$

If we select the index transformation $i = s + m - n$ and observe that, due to the fact that $m - n < 0$, we can restrict the range of i to nonnegative values, we obtain

$$2 = \sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{j=0}^{h} \frac{\binom{m+j}{m}}{2^{m+j}},$$

which is what we desired.

*****

CHARACTERIZATION OF A SEQUENCE

(Submitted June 1981)

JOSEPH McHUGH
La Salle College, Philadelphia, PA 19141

In [1], Hoggatt and Johnson characterize all integral sequences $\{u_n\}$ satisfying

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n. \tag{1}$$

The purpose of this paper is to characterize all sequences which satisfy the relation

$$s_n^2 - s_m^2 = s_{n+m}s_{n-m} \tag{2}$$

for all integers m and n. Of necessity, we see that

$$s_0 = 0, \tag{3}$$

while $m = -n$ yields

$$s_{-n} = \pm s_n \tag{4}$$

for all integers n. Let $n = 0$ in (2), then replace m by n. This gives

$$s_n(s_n + s_{-n}) = 0 \tag{5}$$

for all integers n. Replacing n by $n + 1$ and m by n in (2) yields

$$s_{n+1}^2 - s_n^2 = s_{2n+1}s_1 \tag{6}$$

for all integers n.

Letting $s_1 = 0$ in (6) and using mathematical inducation with (6) we see that $s_n = 0$ for all nonnegative integers. However, by (4) we than have $s_n = 0$ for all integers n. The sequence, all of whose terms are 0, obviously satisfies (2), so for the remainder of this paper we assume $s_1 = a \neq 0$. By (5), we than have

$$s_{-n} = -s_n. \tag{7}$$

Using (2) with $n = 2k + 1$, $m = 2k - 1$ and $n = 2k + 2$, $m = 2k$, we obtain $s_{2k+1}^2 - s_{2k-1}^2 = s_{4k}s_2$ and $s_{2k+2}^2 - s_{2k}^2 = s_{4k+2}s_2$, so that when $s_2 = 0$ we have

$$s_{2k+1} = \pm s_{2k-1} \tag{8}$$

and

$$s_{2k+2} = \pm s_{2k}. \tag{9}$$

Mathematical induction and (9) together with (7) imply that $s_{2n} = 0$ for all integers n. Furthermore, (8) and mathematical induction together with (7) tell us that $s_{2n+1} = \pm a$ for all integers . However, $s_{2n}^2 - s_1^2 = s_{2n+1}s_{2n-1}$, so $-s_1^2 = s_{2n+1}s_{2n-1}$ showing that s_{2n+1} and s_{2n-1} have opposite signs. Therefore, with $s_1 = a \neq 0$ and $s_2 = 0$, we have

$$s_n = \begin{cases} a, & n \equiv 1 \pmod 4 \\ -a, & n \equiv -1 \pmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The sequence just calculated in (10) is a solution to the problem at hand because, if n and m are of the same parity, then $m + n$ and $m - n$ are even, and $s_n^2 = s_m^2$ so $s_n^2 - s_m^2 = 0 = s_{n+m}s_{n-m}$. If n is odd and m is even, then $n + m$ and $n - m$ are odd and separated by $2m$, which is a multiple of 4. Hence,

$$s_{n+m}s_{n-m} = a^2 = s_n^2 - s_m^2.$$

Similarly, if n is even and m is odd.

Throughout the remainder of this paper, we assume that $s_2 = b \neq 0$ and $s_1 = a \neq 0$. From (6) and (2),

$$as_{2n+1} = s_{n+1}^2 - s_n^2 = (s_{n+1}^2 - s_{n-1}^2) - (s_n^2 - s_{n-1}^2) = bs_{2n} - as_{2n-1},$$

so that

$$s_{2n+1} = \frac{bs_{2n} - as_{2n-1}}{a}, \text{ for all } n \tag{11}$$

or, equivalently,

$$a(s_{2n+1} + s_{2n-1}) = bs_{2n}. \tag{12}$$

Now

$$\begin{aligned} s_{n+2}^2 - s_n^2 = bs_{2n+2} &= (s_{n+2}^2 - s_{n-1}^2) + (s_{n-1}^2 - s_n^2) \\ &= s_3s_{2n+1} + s_{2n-1}s_{-1}. \end{aligned}$$

Furthermore, by (11), $s_3 = (b^2 - a^2)/a$ and by (7), $s_{-1} = -a$. Hence, substitution and (12) yield

$$\begin{aligned} bs_{2n+2} &= \frac{b^2 - a^2}{a} s_{2n+1} - as_{2n-1} \\ &= \frac{b^2}{a} s_{2n+1} - a(s_{2n+1} + s_{2n-1}) \\ &= \frac{b^2}{a} s_{2n+1} - bs_{2n}. \end{aligned} \tag{13}$$

Hence,

$$s_{2n+2} = \frac{bs_{2n+1} - as_{2n}}{a}, \text{ for all } n. \tag{14}$$

Combining (11) and (14), we have

$$s_{k+1} = \frac{bs_k - as_{k-1}}{a}, \text{ for all } k. \tag{15}$$

Therefore, the only sequences other than the two exceptions which might satisfy (2) for all n and m must be second-order linear recurrences of the form (15), where $s_1 = a \neq 0$ and $s_2 = b \neq 0$.

Using standard techniques with

$$\alpha = \frac{b + \sqrt{b^2 - 4a^2}}{2a} \quad \text{and} \quad \beta = \frac{b - \sqrt{b^2 - 4a^2}}{2a}$$

as the roots of $ax^2 - bx + a = 0$, we see that, for all integers n,

$$s_n = \begin{cases} \dfrac{a(\alpha^n - \beta^n)}{\alpha - \beta}, & b \neq \pm 2a \\ na, & b = 2a \\ (-1)^{n+1}na, & b = -2a \end{cases} \tag{16}$$

If $s_n = na$ or $s_n = (-1)^{n+1}na$ for all n, then it is easy to verify the truth of (2). Hence, we assume $b \neq \pm 2a$; then, with $\alpha\beta = 1$, we have

$$\begin{aligned} s_n^2 - s_m^2 &= \left(\frac{a}{\alpha - \beta}\right)^2 [(\alpha^{2n} - 2 + \beta^{2n}) - (\alpha^{2m} - 2 + \beta^{2m})] \\ &= \left(\frac{a}{\alpha - \beta}\right)^2 (\alpha^{2n} - \alpha^{2m} + \beta^{2n} - \beta^{2m}). \end{aligned}$$

Furthermore,

$$s_{n+m}s_{n-m} = \left(\frac{a}{\alpha - \beta}\right)^2 (\alpha^{2n} - \alpha^{2m} + \beta^{2n} - \beta^{2m}),$$

and again (2) is true for all integers n and m. Thus, we have found all sequences satisfying (2) for all integers n and m.

It is interesting to note that the Fibonacci and Lucas sequences do not satisfy (15). However, the sequence of Fibonacci numbers $\{F_{2n}\}_{n=1}^{\infty}$ does, if we let $a = 1$ and $b = 3$, for then

$$\alpha = \frac{3 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^2, \quad \beta = \left(\frac{1 - \sqrt{5}}{2}\right)^2, \text{ and } s_n = F_{2n}.$$

Another interesting example of such a sequence is found by letting $s_1 = 1$ and $s_2 = i$, then

$$s_3 = -2,\ s_4 = -3i,\ s_5 = 5,\ s_6 = 8i,\ s_7 = -13,\ s_8 = -21i, \text{ etc.}$$

It should be noted that s_n is an integer for all integers n if and only if a and b are integers and a divides b. This follows directly by using the recursive formula in the form

$$s_{n+1} = \frac{b}{a} s_n - s_{n-1},$$

for then, by induction,

$$s_k = \frac{b^{k-1}}{a^{k-2}} + (\text{integer}) \frac{b^{k-3}}{a^{k-4}} + \cdots + (\text{integer}) \frac{b^3}{a^2} + (\text{integer})\, b, \; k \text{ even}$$

and

$$s_k = \frac{b^{k-1}}{a^{k-2}} + (\text{integer}) \frac{b^{k-3}}{a^{k-4}} + \cdots + (\text{integer}) \frac{b^2}{a} + (\text{integer})\, a, \; k \text{ odd.}$$

Hence, by induction, $s_n \in Z$ if and only if a^n divides b^{n+1} for all $n \geq 3$, but then a must divide b.

Also note that if a divides b, then a divides s_n for all integers n. Hence, the only integral solutions to the problem are multiples of those generated by letting $s_1 = 1$ and $s_2 = b$, where b is an integer.

Acknowledgment

I am greatly indebted to the editor for his streamlined approach to this article.

Reference

V. E. Hoggatt, Jr. and Marjorie Bicknell Johnson. "A Primer for the Fibonacci Numbers XVII: Generalized Fibonacci Numbers Satisfying $u_{n+1}u_{n-1} - u_n^2 = \pm 1$." *The Fibonacci Quarterly* 16, no. 2 (1978):130-37.

*****

CONSEQUENCES OF WATSON'S QUINTUPLE-PRODUCT IDENTITY

(Submitted June 1981)

JOHN A. EWELL
Northern Illinois University, DeKalb, IL 60115

1. Introduction

In this investigation, the leading role is played by the following identity:

$$(1)\qquad \prod_{n=1}^{\infty}(1-x^n)(1-ax^n)(1-a^{-1}x^{n-1})(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$
$$=\sum_{-\infty}^{\infty}x^{n(3n+1)/2}(a^{3n}-a^{-3n-1}),$$

which is valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. As presently expressed, identity (1) was first presented by Basil Gordon [2, p. 286]. However, as observed by M. V. Subbarao and M. Vidyasagar [5, p. 23], Gordon was anticipated some 32 years earlier by G. N. Watson [6, pp. 44-45], who stated and proved a fivefold-product identity easily shown to be equivalent to (1). We are here concerned about several applications of (1). Our first result is:

Theorem 1

For each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$,

$$(2)\qquad \prod_{n=1}^{\infty}(1-x^n)^2(1-ax^n)(1-a^{-1}x^n)(1-ax^{n-1})(1-a^{-1}x^{n-1})(1-a^2x^{2n-1})^2$$
$$\cdot(1-a^{-2}x^{2n-1})^2$$
$$=P(x)\sum_{-\infty}^{\infty}x^{3m^2}a^{6m}+Q(x)\sum_{0}^{\infty}x^{m(3m+1)}(a^{6m+1}+a^{-6m-1})$$
$$+R(x)\sum_{0}^{\infty}x^{m(3m+2)}(a^{6m+2}+a^{-6m-2})+S(x)\sum_{0}^{\infty}x^{3m(m+1)}(a^{6m+3}+a^{-6m-3})$$
$$+T(x)\sum_{0}^{\infty}x^{m(3m+4)}(a^{6m+4}+a^{-6m-4})+U(x)\sum_{0}^{\infty}x^{m(3m+5)}(a^{6m+5}+a^{-6m-5}),$$

where

$$P(x)=2\sum_{-\infty}^{\infty}x^{k(3k+1)},\quad Q(x)=-\sum_{-\infty}^{\infty}x^{3k^2},\quad R(x)=-x\sum_{-\infty}^{\infty}x^{3k(k+1)},$$

$$S(x) = 2x\sum_{-\infty}^{\infty} x^{k(3k+2)}, \quad T(x) = -x^2\sum_{-\infty}^{\infty} x^{3k(k+1)}, \quad U(x) = -x^2\sum_{-\infty}^{\infty} x^{3k^2}.$$

The details of the proof are given in Section 2. As a corollary of Theorem 1, we then represent the decuple infinite product

$$\prod(1 - x^n)^6(1 - x^{2n-1})^4$$

by a double series in the single variable x. In Section 3 we shall need the following identity:

$$\prod_{n=1}^{\infty}(1 - x^n)^3(1 - x^{2n-1})^2 = \sum_{-\infty}^{\infty}(6n + 1)x^{n(3n+1)/2}, \tag{3}$$

shown by Gordon to be a fairly straightforward consequence of (1). On the strength of (3) and two other well-known identities, we then derive a recursive formula for the number-theoretic function $r_2(n)$, which for a given nonnegative integer n counts the number of representations of n as a sum of two squares.

2. Proof of Theorem 1

For given a, x let $G(a, x)$ be defined by:

$$G(a, x) = \prod_{n=1}^{\infty}(1 - ax^n)(1 - a^{-1}x^n)(1 - ax^{n-1})(1 - a^{-1}x^{n-1}) \cdot (1 - a^2x^{2n-1})^2(1 - a^{-2}x^{2n-1})^2.$$

Then, for each pair of positive real numbers A, X, with $X < 1$, $G(a, x)$ converges absolutely and uniformly on the set of all pairs a, x such that

$$A^{-1} \le |a| \le A \quad \text{and} \quad |x| \le X.$$

Hence, for a fixed choice of x, $|x| < 1$, $G(a, x)$ defines a unique function of a, which is analytic at all points of the finite complex plane except $a = 0$, where it has an essential singularity. Accordingly,

$$G(a, x) = C_0(x) + \sum_{n=1}^{\infty}[C_n(x)a^n + C_{-n}(x)a^{-n}],$$

where the coefficients $C_n(x)$, $C_{-n}(x)$ are uniquely determined by the chosen x.

Now, $G(a, x) = G(a^{-1}, x)$, whence $C_n(x) = C_{-n}(x)$, for each positive integer n. Hence,

$$G(a, x) = C_0(x) + \sum_{n=1}^{\infty}C_n(x)(a^n + a^{-n}). \tag{4}$$

An easy calculation then establishes the following identity:

$$G(ax, x) = a^{-6}x^{-3}G(a, x).$$

With the help of (4) we expand both sides of this identity in powers of a, and subsequently equate coefficients of like powers to obtain the following recurrence:

$$C_n(x) = C_{n-6}(x)x^{n-3}.$$

The coefficients $C_0(x)$, $C_1(x)$, $C_2(x)$, $C_3(x)$, $C_4(x)$, $C_5(x)$ are here undetermined, but for all $n > 5$, we distinguish six cases,

(i) $n = 6m$, (ii) $n = 6m + 1$, (iii) $n = 6m + 2$,

(iv) $n = 6m + 3$, (v) $n = 6m + 4$, (vi) $n = 6m + 5$,

$m \geq 0$, and iterate the recurrence to obtain:

$$C_{6m}(x) = x^{3m^2}C_0(x),\ C_{6m+1}(x) = x^{m(3m+1)}C_1(x),\ C_{6m+2}(x) = x^{m(3m+2)}C_2(x),$$

$$C_{6m+3}(x) = x^{3m(m+1)}C_3(x),\ C_{6m+4}(x) = x^{m(3m+4)}C_4(x),\ C_{6m+5}(x) = x^{m(3m+5)}C_5(x).$$

Hence,

$$(5)\quad G(a, x) = C_0(x)\sum_{-\infty}^{\infty} x^{3m^2}a^{6m} + C_1(x)\sum_{0}^{\infty} x^{m(3m+1)}(a^{6m+1} + a^{-6m-1})$$

$$+ C_2(x)\sum_{0}^{\infty} x^{m(3m+2)}(a^{6m+2} + a^{-6m-2})$$

$$+ C_3(x)\sum_{0}^{\infty} x^{3m(m+1)}(a^{6m+3} + a^{-6m-3})$$

$$+ C_4(x)\sum_{0}^{\infty} x^{m(3m+4)}(a^{6m+4} + a^{-6m-4})$$

$$+ C_5(x)\sum_{0}^{\infty} x^{m(3m+5)}(a^{6m+5} + a^{-6m-5}).$$

To evaluate C_0, C_1, C_2, C_3, C_4, and C_5, we multiply identity (1) and the identity which results from (1) under the substitution $a \to a^{-1}$ to get

$$\prod_{n=1}^{\infty}(1 - x^n)^2G(a, x) = P(x)a^0 + Q(x)(a + a^{-1}) + R(x)(a^2 + a^{-2})$$

$$+ S(x)(a^3 + a^{-3}) + T(x)(a^4 + a^{-4})$$

$$+ U(x)(a^5 + a^{-5}) + \text{a series in } a^n, a^{-n}, n > 5.$$

Between identity (5) and the foregoing identity, we eliminate the product $G(a, x)$ and, thereafter, equate coefficients of a^0, $a + a^{-1}$, ..., $a^5 + a^{-5}$ to get

$$C_0 = P(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_1 = Q(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_2 = R(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},$$

$$C_3 = S(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_4 = T(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_5 = U(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2}.$$

Substituting these values of C_i ($i = 0, 1, \ldots, 5$) into (5) we thus prove our theorem.

Corollary

For each complex number x such that $|x| < 1$,

$$(6)\qquad \prod_{n=1}^{\infty}(1 - x^n)^6(1 - x^{2n-1})^4 = -\sum_{-\infty}^{\infty}x^{k(3k+1)}\sum_{-\infty}^{\infty}(6m)^2x^{3m^2}$$

$$+\sum_{-\infty}^{\infty}x^{3k^2}\sum_{-\infty}^{\infty}(6m + 1)^2x^{m(3m+1)}$$

$$+\,x\sum_{-\infty}^{\infty}x^{3k(k+1)}\sum_{-\infty}^{\infty}(6m + 2)^2x^{m(3m+2)}$$

$$-\,x\sum_{-\infty}^{\infty}x^{k(3k+2)}\sum_{-\infty}^{\infty}(6m + 3)^2x^{3m(m+1)}.$$

Proof: For given a, x, let $F(a, x)$ be defined by

$$(1 - a)(1 - a^{-1})F(a, x) = \prod_{n=1}^{\infty}(1 - x^n)^2G(a, x),$$

which is the left side of (2). Now, put $a = e^{2it}$, and for brevity

$$f(t) = F(e^{2it}, x).$$

Identity (2) is hereby transformed into a new identity, the left side of which is $4f(t)\sin^2 t$. Hence, we multiply both sides of this new identity by 4^{-1} to get

$$f(t)\sin^2 t = \frac{P(x)}{4}\left[1 + 2\sum_{n=1}^{\infty}x^{3m^2}\cos(12mt)\right] + \frac{Q(x)}{2}\sum_{0}^{\infty}x^{m(3m+1)}\cos(12m + 2)t$$

$$+\frac{R(x)}{2}\sum_{0}^{\infty}x^{m(3m+2)}\cos(12m + 4)t + \frac{S(x)}{2}\sum_{0}^{\infty}x^{3m(m+1)}\cos(12m + 6)t$$

$$+\frac{T(x)}{2}\sum_{0}^{\infty}x^{m(3m+4)}\cos(12m + 8)t + \frac{U(x)}{2}\sum_{0}^{\infty}x^{m(3m+5)}\cos(12m + 10)t.$$

We now differentiate the foregoing identity twice with respect to t to get

$$2f(t)\cos^2 t + 2\sin t\, D_t[f(t)\cos t] + D_t[f'(t)\sin^2 t]$$

$$= -2P(x)\sum_{1}^{\infty}x^{3m^2}(6m)^2\cos(12mt) - 2Q(x)\sum_{0}^{\infty}x^{m(3m+1)}(6m + 1)^2\cos(12m + 2)t$$

$$- 2R(x)\sum_0^\infty x^{m(3m+2)}(6m+2)^2\cos(12m+4)t$$

$$- 2S(x)\sum_0^\infty x^{3m(m+1)}(6m+3)^2\cos(12m+6)t$$

$$- 2T(x)\sum_0^\infty x^{m(3m+4)}(6m+4)^2\cos(12m+8)t$$

$$- 2U(x)\sum_0^\infty x^{m(3m+5)}(6m+5)^2\cos(12m+10)t.$$

In the foregoing we first put $t = 0$ and cancel a factor of 2 from both sides of the resulting identity. Of course, $f(0)$ is the left side of (6). To get the right side, we then combine the 2nd and 6th, and the 3rd and 5th sums on the right side of the last-mentioned identity, while effecting some fairly obvious transformations along the way.

3. Recurrences for $r_2(n)$

In order to carry out our present assignment, we also need the following well-known identities:

$$\prod_{n=1}^\infty (1 - x^n) = \sum_{-\infty}^\infty (-1)^n x^{n(3n+1)/2}, \tag{7}$$

$$\prod_{n=1}^\infty (1 - x^n)(1 - x^{2n-1}) = \sum_{-\infty}^\infty (-x)^{n^2}. \tag{8}$$

(7) is a famous result due to Euler, and both identities are easy consequences of the celebrated Gauss-Jacobi triple-product identity [3, pp. 282-284].

For convenience, put $r(n) = r_2(n)$.

Theorem 2

For each nonnegative integer n,

$$r(n) + \sum_{j=1} [(-1)^{j(3j-1)/2}r(n - (3j+1)/2) + (-1)^{j(j-1)/2}r(n - (3j-1)/2)] \tag{9}$$

$$= \begin{cases} (-1)^n[6(\pm m) + 1], \text{ if } n = m(3m \pm 1)/2, \\ 0, \text{ otherwise,} \end{cases}$$

where summation extends as far as the arguments of r remain nonnegative.

Proof: First of all, we recall that the generating function of $r(n)$ is given by:

$$\left(\sum_{-\infty}^{\infty} x^{n^2}\right)^2 = \sum_{n=0}^{\infty} r(n)x^n.$$

We now realize that (3) is equivalent to

$$\prod_{1}^{\infty}(1 - x^n)\prod_{1}^{\infty}(1 - x^n)^2(1 - x^{2n-1})^2 = \sum_{-\infty}^{\infty}(6n + 1)x^{n(3n+1)/2},$$

whence [owing to (7) and (8)]

$$\sum_{-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}\sum_{0}^{\infty} r(n)(-x)^n = \sum_{-\infty}^{\infty}(6n + 1)x^{n(3n+1)/2}.$$

Expanding the left side of the foregoing identity and thereafter equating coefficients of like powers of x, we obtain the desired conclusion.

Remarks

It is of interest to compare the recursive determination (9) of the arithmetical function r with similar ones for the partition function p and the sum-of-divisors function σ. Accordingly, let us briefly recall that for a given positive integer n, $p(n)$ denotes the number of unrestricted partitions of n, while $\sigma(n)$ denotes the sum of the positive divisors of n; conventionally, $p(0) = 1$. From his identity, Euler derived the following recursive formulas for p and σ.

$$(10) \qquad p(n) + \sum_{j=1}(-1)^j[p(n - j(3j + 1)/2) + p(n - j(3j - 1)/2)] = 0,$$

where $n > 0$ and summation extends as far as the arguments of p remain nonnegative.

$$(11) \qquad \sigma(n) + \sum_{j=1}(-1)^j[\sigma(n - j(3j + 1)/2) + \sigma(n - j(3j - 1)/2)] = \begin{cases} (-1)^{m+1}n, & \text{if } n = m(3m \pm 1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

where $n > 0$ and summation extends as far as the arguments of σ remain positive.

For proofs of (10) and (11), see [4, pp. 235-237].

Thus, for these three important arithmetical functions r, p, and σ, we have pentagonal-number recursive formulas for each of them. And for each of them one needs about $2\sqrt{(2/3)n}$ of the earlier values to compute a given value for large n.

In [1] the author has also derived the following triangular-number recursive formula for r:

$$(12)\qquad \sum_{j=0} (-1)^{j(j+1)/2} r(n - j(j+1)/2) = \begin{cases} (-1)^{m(m+3)/2}(2m+1), & \text{if } n = m(m+1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

where $n \geq 0$ and summation extends as far as the arguments of r remain nonnegative.

We now observe that recursive formula (12) is more efficient than (9). For with (12) one needs about $\sqrt{2n}$ of the earlier values in order to compute $r(n)$ for large n.

References

1. J. A. Ewell. "On the Counting Function for Sums of Two Squares." *Acta Arithmetica* (to appear).
2. B. Gordon. "Some Identities in Combinatorial Analysis." *Quart. J. Math.* (Oxford), Ser. (2), 12 (1961):285-90.
3. G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 4th ed. Oxford: Clarendon Press, 1960.
4. I. Niven and H. S. Zuckerman. *An Introduction to the Theory of Numbers*, 2nd ed. New York: Wiley, 1966.
5. M. V. Subbarao and M. Vidyasagar. "On Watson's Quintuple-Product Identity." *Proc. Amer. Math. Soc.* 26 (1970):23-27.
6. G. N. Watson. "Theorems Stated by Ramanujan (VII): Theorems on Continued Fractions." *J. London Math. Soc.* 4 (1929):39-48.

*****

ANALYSIS OF A BETTING SYSTEM

(Submitted October 1981)

JOHN RABUNG
Virginia Commonwealth University, Richmond, VA 23284
and
JIM HYLAND
TRW, Inc., Redondo Beach, CA 90278

1. Introduction

A friend of ours, on hearing about a "new" system for betting in roulette, did some initial investigating with pencil and paper, thought it looked good, and proceeded to try it out in Las Vegas. With a set goal and a capital he was willing to risk, he played the system religiously ... and won! This was the incentive for our more thorough investigation of the system. The outcome of the investigation may be guessed in advance; if not from a mathematical standpoint, surely from the facts that:

1. we have decided to publish the findings, and
2. neither of us is yet wealthy.

Since roulette is a game of (presumably) independent trials and since the house holds an edge on each trial, it is a foregone conclusion (see, for example, [1]) that there can be no betting scheme which gives the bettor a positive expectation. Nonetheless, there is a certain enticement to a scheme which is designed for use in a nearly "even" game of independent trials and which promises, by its nature, to leave the bettor ahead by a certain amount after the completion of a little routine which seems unavoidably destined for completion. Betting on red or black in roulette (probability of success with an American wheel is 18/38 since there are 18 red numbers, 18 black numbers, and, yes, two green numbers—0 and 00) provides the nearly "even" game. The scheme for betting in the game begins with a prechosen but arbitrary sequence of numbers $b_1, b_2, \ldots, b_n$, which we shall call the *betting sequence*. The algorithm to be followed is then:

```
1.   (Make bet $b$)  $b = b_1 + b_n$ if $n \geq 2$.  $b = b_1$ if $n = 1$.
2.   (Decrease betting sequence after a win)  If win, then
     2.1.  (Scratch outer numbers)  Delete the values $b_1$ and $b_n$ from the betting sequence.
     2.2.  If sequence is exhausted, then halt.  (Completion of a betting cycle)
     2.3.  Decrease $n$ by 2.
     2.4.  (Relabel sequence numbers)  Renumber remaining betting sequence to $b_1, \ldots, b_n$.
```

```
3. (Increase betting sequence after a loss) If lose, then
    3.1. Increase $n$ by 1.
    3.2. (Attach current bet to sequence) Set $b_n = b$.
4. Repeat, starting at step 1.
```

As an example, suppose a bettor begins with the sequence 1, 2, 3, 4, 5. His first bet would be 6 units (1 + 5). If he wins that bet, his betting sequence becomes 2, 3, 4, and his next bet would be 6 units again (2 + 4). Given a loss of this second bet, his betting sequence would become 2, 3, 4, 6, and 8 units would be bet next. A complete betting cycle is illustrated below:

Trial No.	Betting Sequence	Bet	Outcome	Financial Status
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Lose	Even
3	2, 3, 4, 6	8	Lose	-8
4	2, 3, 4, 6, 8	10	Win	+2
5	3, 4, 6	9	Win	+11
6	4	4	Lose	+7
7	4, 4	8	Win	+15

cycle complete (betting sequence exhausted)

Now the invitation to wealth is clear. With a nearly even chance of winning any bet and with the system scratching two numbers from the betting sequence on every win while adding only one number to the sequence on a loss, how can we fail eventually to exhaust the betting sequence? And sequence exhaustion beings with it a reward equal in monetary units to the sum of the numbers in the original betting sequence (easily proved). The only hitch in this otherwise wonderful plan is that there may come a time when we cannot carry a betting cycle through to completion simply because we do not have the resources to do so; i.e., we cannot cover the bet required by the system. (House limits on bets may also impose on our scheme, but these are not considered here.)

It turns out that this system is an old one called either *Labouchere* or *the cancellation system*. Mention is made of it (in a dismissing way) in the writings of professional gamblers (see [4], [6], [7], and [8]) and (in a promotional way) in one book [5], where the author claims to have won $163,000 using an anti-Labouchere system (turn around the win and lose actions) in a French casino in 1966.

Here we investigate Labouchere by first probing (in Section 2) a system which is somewhat similar to Labouchere, but more amenable to mathematical analysis. This gives a forecast of results to come. Next (Section 3) we look at Labouchere in a setting where there is no limit on the bettor's capital. Here the probabilities of cycle completion become clear. Finally, in Section 4, we simulate (mathemetical analysis seems very difficult) various situations under which Labouchere is applied with finite working capital. The intent is to display how the control of certain parameters (initial capital, goal, length of initial betting sequence, size and order of values in the original sequence) can impact the outcome statistics (frequency of goal achievement, mean bet size, mean number of bets to a win, mean earnings).

2. Analysis of a Simpler Scheme

Consider for a moment the popular *double-up* or *Martingale* betting system wherein the bettor doubles his wager after each loss and returns to his original bet after each win. This can be considered somewhat close to a Labouchere scheme by viewing the Martingale bettor as starting with a single number in his betting sequence, adding to the sequence any bet that he loses, betting the sum of the whole sequence, and deleting the whole sequence after any win. Thus, any win completes a betting cycle.

What "control" does the Martingale bettor have over his fortunes? Suppose the probability of success on any trial is p and let $q = 1 - p$. For simplicity, we let the gambler's initial capital be

$$C_0 = (2^k - 1)b$$

for some positive integers b and k, where b is the amount to be bet initially. We shall also say that the gambler's profit goal is G and, again for simplicity, set $G = mb$, where $0 < m < 2^k$. Under this arrangement, the bettor must experience m successful trials (complete betting cycles) to achieve his profit goal, while k consecutive losses will ruin him (i.e., leave him with insufficient resources to continue with the Martingale scheme). Thus,

$$\text{Prob[achieve } G] = (1 - q^k)^m.$$

We note immediately that, given the same capital, the greedier gambler (the one with a larger profit goal) has a smaller probability of achieving his objective, but that the amount by which this probability diminishes with increased ambition depends on the bettor's initial capital.

Now assume that the gambler will achieve his profit goal $G = mb$, and let $X_1, X_2, \ldots, X_m$ be random variables whose values are determined by the number of trials needed to complete cycle 1, cycle 2, ..., cycle m, respectively. Then the expected number of trials to achieve G (given that G will be achieved) is

$$E[X_1 + X_2 + \cdots + X_m] = \frac{1}{1 - q^k} \sum_{j=1}^{m} \sum_{i=1}^{k} i\, pq^{i-1} = \frac{mp}{1 - q^k} \sum_{i=1}^{k} i\, q^{i-1}.$$

To get a feeling for these numbers, we present some examples in Table 1 where we assume that $p = 18/38$ (as in roulette) and that in each case shown the initial amount wagered is 10 units (i.e., $b = 10$).

The main observations that we wish to make from Table 1 are that under a Martingale system a bettor has the following "controls":

1. He can adjust his probability of achievement of the profit goal, G, by adjusting his goal-to-initial capital ratio (G/C_0). This apparent dependence is shown dramatically in Figure 1, a plot of Table 1 data.

TABLE 1

Expected Results Using a Martingale Betting Scheme with Initial Bet $10 and p = 18/38

m	k	Profit Goal (G) $G = 10m$	Initial Capital (C_0) $C_0 = (2^k - 1)10$	Prob[achieve G] $P_G = (1 - q^k)^m$	Expected Number of Trials to achieve G	Expected Earnings per Play $GP_G - C_0(1 - P_G)$
5	3	50	70	.455	7.995	-15.40
4	3	40	70	.532	6.396	-11.48
3	3	30	70	.623	4.797	-7.70
10	5	100	310	.662	21.350	-38.58
5	4	50	150	.671	9.612	-15.80
9	5	90	310	.690	19.217	-34.00
8	5	80	310	.719	17.082	-29.59
4	4	40	150	.727	7.690	-11.87
2	3	20	70	.730	3.198	-4.30
7	5	70	310	.749	14.947	-25.38
6	5	60	310	.781	12.811	-21.03
3	4	30	150	.787	5.767	-8.34
5	5	50	310	.814	10.676	-16.96
4	5	40	310	.848	8.541	-13.20
2	4	20	150	.852	3.845	-5.16
1	3	10	70	.854	1.599	-1.68
3	5	30	310	.883	6.406	-9.78
5	6	50	630	.898	11.348	-19.36
4	6	40	630	.918	9.078	-14.94
2	5	20	310	.921	4.270	-6.07
1	4	10	150	.923	1.922	-2.32
3	6	30	630	.938	6.809	-10.92
2	6	20	630	.958	4.539	-7.30
1	5	10	310	.960	2.135	-2.80
1	6	10	630	.979	2.270	-3.44

2. He can adjust his expected time for achieving his goal. The general rule here seems to be that a need for more cycles to achieve the goal increases expected achievement time as does having a larger initial capital. That is, increasing either k or m increases the expected number of trials for gaining G.
3. The more a gambler is willing to risk (C_0) and the greedier he is (G), the larger his expected loss.

We shall see shortly that the Labouchere bettor has the same sorts of controls over his fortunes, but that his setting provides for more controls, in that he also has a choice of betting sequence. This is the real complicating factor in the analysis of Labouchere. We turn now to some of the mechanics of the cancellation scheme before going into the full simulation of practical situations.

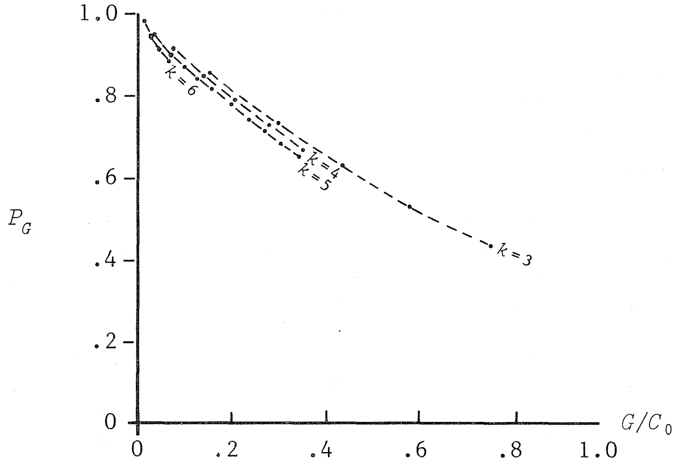


FIGURE 1

A plot of the Martingale results in Table 1 showing the relationship between probability of goal achievement and goal-to-initial-capital ratio. Dashes connect points with same initial capital.

3. The Case of Infinite Capital

Throughout this section we shall assume that our Labouchere bettor really does not care what sorts of temporary losses he incurs, for he has enough money to cover any loss. His only real concern is how long it will take him to recover the loss by completing his betting cycle. He asks then for the probability that he will complete a cycle in t or fewer trials.

Suppose the initial betting sequence consists of n numbers. Let w represent the number of wins in t trials and let $\ell = t - w$ represent the number of losses. In order for a betting cycle to be completed on trial t we must have

$$(1) \qquad 2w \geq \ell + n \quad \text{and} \quad 2(w - 1) < \ell + n.$$

That is, since two numbers are deleted from the sequence with each win and only one number added for each loss, the first inequality gives a condition for sequence exhaustion and the second assures that the sequence was not exhausted on the $(t-1)$st trial. Together, these yield

$$(2) \qquad \frac{1}{3}(t + n) \leq w < \frac{1}{3}(t + n + 2).$$

Because w is an integer and the two extremes of this inequality differ by only 2/3, there can be at most one solution w for given t and n. In fact, since t and w are also integers, the only situation in which no such w exists will be that for which

$$\frac{1}{3}(t + n) = m + \frac{1}{3},$$

where m is an integer; i.e.,

$$t \equiv 2n + 1 \pmod{3}.$$

Hence, if $t = 3k + i$ and $n = 3h + j$, where $0 \leq i,\ j \leq 2$, then

$$w = \begin{cases} k + h, & \text{if } i = j = 0 \\ k + h + 1, & \text{otherwise, unless } i + j \equiv 1 \pmod{3}, \\ & \text{where there is no solution.} \end{cases}$$

From the way we set up our conditions to find w, it is seen that not every permutation of w wins and $t - w$ losses will result in cycle completion on trial t. (Some will dictate earlier sequence exhaustion.) However, every t-trial cycle completion with an initial betting sequence of n numbers will involve exactly w wins where w is determined as above.

Our question now becomes: How many permutations of w wins in t trials result in cycle completion on trial t? To address this, we make our setting more definite, and note simply that other settings are similar. We take the case where there are five numbers in the original betting sequence ($n = 5$). In this case our analysis above shows that it requires exactly

$$w = k + 2$$

wins to complete a betting cycle in $t = 3k + i$ trials, providing $i = 0$ or 1. It is impossible to complete a cycle in t trials if $i = 2$.

Figure 2 shows a graph in the $w\ell$-plane of the inequalities (1), which here become

$$2w \geq \ell + 5 \quad \text{and} \quad 2(w - 1) < \ell + 5.$$

The lines $t = w + \ell$ are shown at various levels. We consider a random walk on this graph where each loss corresponds to a positive unit step vertically and each win corresponds to a positive unit step horizontally. Beginning at the origin, we hope to follow the determined path into the region described by the inequalities, since this corresponds to completing a cycle. Hence, we call this region the *completion zone*.

For the purpose of restating our question in this new context, let us say that a path in our random walk from the origin to some point (a, b) is *permissible* if it never enters the completion zone before reaching (a, b). Then our question asks how many different permissible paths lead to the point

$$(k + 2,\ 2k + i - 2),\ k \geq 1,\ i = 0 \text{ or } 1.$$

Now a recursion formula that answers the question is easily derived from noticing that any path leading to (a, b) in this random walk must have as its last step either the step from $(a - 1, b)$ to (a, b) or from $(a, b - 1)$ to (a, b). So denoting the number of permissible paths to (a, b) by $N(a, b)$, we have

$$N(a, b) = N(a - 1, b) + N(a, b - 1).$$

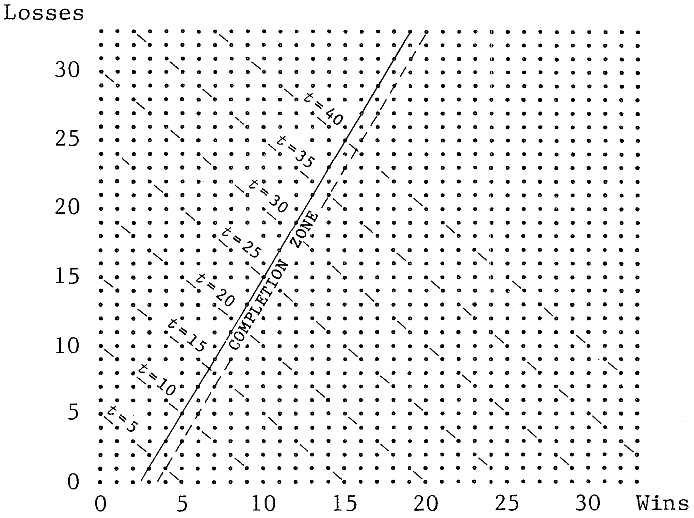


FIGURE 2

The Random Walk Setting with n = 5

This formula is, of course, subject to the provision that neither $(a - 1, b)$ nor $(a, b - 1)$ is in the completion zone, for then the path to (a, b) would not be permissible. So, for example, if $(a, b - 1)$ is in the completion zone, then $N(a, b) = N(a - 1, b)$.

Putting this in terms of t and w rather then w and ℓ and denoting the number of permissible ways to achieve w wins in t trials by $\left\{ t \atop w \right\}$, our basic formula becomes

$$\left\{ t \atop w \right\} = \left\{ t - 1 \atop w - 1 \right\} + \left\{ t - 1 \atop w \right\}$$

with the same provision that if either $t - 1$ and $w - 1$ or $t - 1$ and w determine a point in the completion zone, then the corresponding number is not added in the formula. These numbers clearly act somewhat like binomial coefficients and, in fact, we get a modification of Pascal's triangle as shown in Figure 3. There the circled items represent numbers corresponding to points in the completion zone and, consequently, are not added in the derivation of the succeeding row. Now we have, for example, that $\left\{ 9 \atop 4 \right\} = 83$; that is, there are 83 permissible paths to the point (4, 5) in Figure 2. This, in turn, is equivalent to saying that there are 83 sequences consisting of four wins and five losses which lead to completion of the betting cycle in exactly nine trials.

t													
0	1												
1	1	1											
2	1	2	1										
3	1	3	3	①									
4	1	4	6	③	0								
5	1	5	10	6	0	0							
6	1	6	15	16	⑥	0	0						
7	1	7	21	31	⑯	0	0	0					
8	1	8	28	52	31	0	0	0	0				
9	1	9	36	80	83	㉛	0	0	0	0			
10	1	10	45	116	163	(83)	0	0	0	0	0		
11	1	11	55	161	279	163	0	0	0	0	0	0	
12	1	12	66	216	440	442	(163)	0	0	0	0	0	0

FIGURE 3. $\left\{\begin{matrix} t \\ w \end{matrix}\right\}$

These counts of the number of ways to complete a cycle in exactly t trials can be written explicitly in terms of binomial coefficients and, somewhat more neatly, in terms of binomial coefficients and the analogous numbers associated with a three-number initial betting sequence. We sketch the derivation of this latter expression in the appendix to this paper.

Given these numbers, we have essentially answered the question posed by the infinitely wealthy gambler at the beginning of this section. For if the probability of a win on any turn is p, then the probability of completing a cycle on or before the tth trial (still assuming $n = 5$) is found by adding terms of the form

$$\left\{\begin{matrix} 3k + i \\ k + 2 \end{matrix}\right\} p^{k+2}(1 - p)^{2x+i-2}$$

with $i = 0$ or 1 and k ranging from 1 to $\left[\frac{t}{3}\right]$ with the restriction that

$$3k + i \leq t.$$

Some of these numbers are given in Table 2 below under the assumption that p is, again, 18/38.

The mean of such a distribution of the number of trials (bets) needed to complete a cycle with an initial betting sequence of length n can be found without too much trouble. We let X_i, for $i = 1, 2, 3, \ldots$, be a random variable which takes the value -2 if bet i is won and +1 if bet i is lost. Then after t bets we see that $S_t = X_1 + X_2 + \cdots + X_t$ gives the change in length

TABLE 2

Completion Probabilities with a Five-Member Initial Sequence and p = 18/38

t	Probability of Completion on Trial t	Probability of Completion within t Trials
3	.1063	.1063
4	.1678	.2741
6	.0837	.3578
7	.1174	.4752
9	.0567	.5319
10	.0799	.6119
12	.0391	.6510
13	.0559	.7069
15	.0278	.7347
16	.0401	.7748
18	.0202	.7950
19	.0295	.8246
21	.0151	.8396
22	.0221	.8617

of the betting sequence from its original length. For any t we have

$$E[S_t] = E[X_1]t,$$

since the X_t's are identically distributed and independent. Since

$$E[X_1] = 1 - 3p,$$

where p is the probability of success on any trial, then

$$E[S_t] = (1 - 3p)t.$$

We note that, in terms of wins and losses, if we combine the conditions

$$w + \ell = t$$

$$-2w + \ell = (1 - 3p)t,$$

we get the line

$$\ell = \frac{1 - p}{p}w .$$

Plotting this line of expected results on a graph like that in Figure 2 and extending it to meet the completion zone, we can get an idea of the expected number of trials to complete a cycle by computing the point of intersection with the completion zone boundary line. We get

$$E[T] = \frac{1}{3p - 1}n,$$

where T is the random variable whose value is the number of trials in the completion of a cycle. In fact, this is a geometric version of Wald's identity (see [3]) which relates $E[S_T]$ and $E[T]$.

Of course, this analysis addresses only the number of bets needed to complete a cycle and, like the infinitely wealthy gambler, ignores any consideration of the money involved in completing a cycle. In the next section, our gambler has finite capital and the difficult questions of financial impact of parameter adjustment become paramount.

4. A More Realistic Setting

Consider the following two betting cycles, each of which is completed on the 10th trial:

Trial	Bet Sequence	Bet	Outcome	Financial Status
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Win	+12
3	3	3	Lose	+9
4	3, 3	6	Lose	+3
5	3, 3, 6	9	Win	+12
6	3	3	Lose	+9
7	3, 3	6	Lose	+3
8	3, 3, 6	9	Win	+12
9	3	3	Lose	+9
10	3, 3	6	Win	+15
	Exhausted			
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Lose	Even
3	2, 3, 4, 6	8	Win	+8
4	3, 4	7	Lose	+1
5	3, 4, 7	10	Lose	-9
6	3, 4, 7, 10	13	Lose	-22
7	3, 4, 7, 10, 13	16	Lose	-38
8	3, 4, 7, 10, 13, 16	19	Win	-19
9	4, 7, 10, 13	17	Win	-2
10	7, 10	17	Win	+15
	Exhausted			

Notice that in each cycle there occurred five wins and five losses, as expected for a completion on trial ten. However, the money required of the bettor greatly differed between the two cycles. In the first, the bettor needed only enough money to cover his first bet (6 units). From there on he was always "ahead of the game." But in the second cycle, the bettor needed to have an initial capital of at least 57 units in order to be able to bet 19 units on the eighth trial, while being 38 units behind. So we see that the arrangement of the win-loss sequence in a cycle of fixed length can have great impact on the amount of money needed to survive the cycle. It is this dependence of monetary needs on both the bet sequence and the sequence of wins and losses that drives us to the computer in an effort to understand generally what can be expected from various situations. Using a random number generator we have simulated (naturally, a Monte Carlo simulation) a tournament of Labouchere gamblers. The rules of the tournament were:

1. Each player begins with $500 and tries to realize a profit of $60.
2. Each player must strictly follow a given Labouchere scheme until he either earns the $60 profit or cannot meet the bet level necessary to continue playing. At such time, he is given another $500 and begins another play of his system.
3. All players gamble simultaneously at the same American roulette wheel until they have completed at least 2,000 plays and at least 62,500 spins of the wheel.

The 24 simulated players who competed in this tournament (which took under 2 minutes of computer time) had various ideas about what constitutes a good betting sequence. The following fairly well characterize the two extremes in these ideas:

Claim of Gambler A: If I structure my sequence so that generally my bets are quite small relative to my capital, then chances are that I'll have sufficient capital to survive most streaks of misfortune.

Claim of Gambler U: I'll use a sequence which is short and requires only one cycle completion to achieve the profit goal. This way, on any given play I probably won't be around long enough to run into a disastrous win-loss sequence. Besides, with my bets being fairly large, chances are that not being able to cover a bet still leaves me with substantial capital (i.e., a ruin is not so bad).

The 24 simulated gamblers and a host of simulated officials gathered around the simulated wheel of fortune and watched it spin more than a quarter of a million times until the final player had completed his 2,000 plays. (A required 254,661 bets to complete 2,000 plays.) The results, as reported in Table 3 in ascending order of goal achievement rate, tend to support the notions of gambler U, up to a point. We do see that, initially, the sequences with fewer cycles needed to achieve the goal yield better returns in terms of both achievement percentage and mean earnings per play. However, toward the bottom on Table 3 some leanings toward player A's ideas can be noted. Where, under player U's philosophy, we would have expected his ultimately short sequence to have done better then the sequences of players V, W, or X, we see instead that, apparently on occasion, player U's bets built up a little too quickly for his $500 capital to withstand, while the sequences of players V, W, and X allowed for more moderate build-up and a better achievement percentage. Compare the mean bet size of players R, S, T, U, V, W, X to see this. Note, however, that the six 10s of player S allowed for a moderate bet size, too, but also required substantially more bets to complete a winning cycle. Note, too, that while players W and X achieved the profit goal most frequently, player U was correct about mean earnings and fared better than anyone in that category.

TABLE 3

Simulation Results for a Minimum of 2,000 Plays and 62,500 Bets at an American Roulette Wheel Using Various Betting Sequences to Attempt to Achieve a Profit Goal of $60 from an Initial Capital of $500

Player	Initial Bet Sequence	% of Plays Goal Achieved	Mean Bet Size	Mean # of Bets to Win	Mean Earnings per Play	# of Plays	# of Cycles to Achieve Goal
A	1,1,1,1,1,1	74.6	10.26	136.7	-64.86	2000	10
B	1,2,3	76.6	14.76	71.1	-54.16	2000	10
C	3,2,1	76.6	14.87	71.1	-53.97	2000	10
D	2,2,2	76.8	14.97	71.0	-52.32	2000	10
E	5,1	78.2	19.36	43.8	-45.65	2000	10
F	4,2	78.2	19.62	43.9	-45.04	2000	10
G	2,4	78.4	19.95	43.9	-44.47	2000	10
H	3,3	78.6	19.87	43.9	-43.76	2000	10
I	5,4,1,2,3	79.0	20.55	41.6	-42.71	2000	4
J	1,5	79.0	19.81	44.1	-41.68	2000	10
K	1,2,3,4,5	79.3	20.30	42.0	-41.89	2000	4
L	5,4,3,2,1	79.4	20.67	41.7	-41.26	2000	4
M	6	79.7	23.99	30.6	-38.78	2040	10
N	7,5,3	80.4	27.44	25.8	-35.81	2368	4
O	3,5,7	80.4	27.58	25.8	-35.55	2362	4
P	5,5,5	80.5	27.57	25.8	-35.16	2367	4
Q	5,10	80.8	36.42	15.3	-32.59	3809	4
R	10,5	81.0	36.08	15.5	-31.57	3802	4
S	10,10,10,10,10,10	82.2	46.73	10.2	-26.77	5470	1
T	50,10	82.2	87.43	2.7	-12.42	17018	1
U	60	83.2	115.36	1.8	-10.52	30117	1
V	10,50	83.5	93.19	2.6	-13.84	16380	1
W	10,20,30	83.6	68.55	4.8	-18.35	11160	1
X	30,20,10	84.3	68.66	5.0	-17.52	11143	1

To a good extent these results reflect what is generally the case in the classical setting where a constant amount is wagered on each trial. In that situation an increase in bet size (with initial capital held constant) brings a decrease in probability of ruin for a player whose probability of success on any trial is less than 1/2 (see [2, p. 347]). This principle needs modification under Labouchere only where bet sizes tend to grow too rapidly for underlying capital.

Table 4 gives results of another simulation which was run as a study of the effects of initial capital on relative frequency of goal achievement. Here 15 players stood around the same wheel (actually playing along with the 24 players in the first simulation) betting the same Labouchere system (bet sequence 1, 2, 3, 4, 5), aiming for a $60 profit, but starting with different capital amounts. The effects can be noted to be much like those under a Martingale scheme by comparing Figure 1 in Section 2 with Figure 4, which gives a visual presentation of Table 4 entries. Again, the wealthy, unambitious gambler has a high likelihood of goal achievement, but a worse expectation since he loses so much in the infrequent disasters he encounters.

TABLE 4

Simulation Results for a Minimum of 2,000 Plays and a Minimum of 62,500 Bets at an American Roulette Wheel in an Effort to Gain a Profit Goal of $60 Using Betting Sequence 1,2,3,4,5 from Various Initial Capital Values

Initial Capital	Percentage of Plays Goal Achieved	Mean Bet Size	Mean Number of Bets to a Win	Mean Earnings per Play	Number of Plays
60	33.3	10.97	30.3	-11.67	2809
80	39.6	11.82	32.1	-15.05	2440
100	45.0	12.47	33.3	-17.53	2217
120	49.6	13.17	34.4	-20.18	2062
150	54.9	14.11	36.0	-23.55	2000
180	59.5	14.97	37.0	-25.70	2000
240	65.7	16.20	38.6	-30.35	2000
300	70.6	17.53	39.8	-32.28	2000
500	79.3	20.30	42.0	-41.89	2000
600	81.6	21.58	42.5	-46.44	2000
1200	88.7	27.30	45.6	-63.77	2000
1800	91.2	30.98	46.7	-83.17	2000
2400	92.5	33.79	47.1	-99.61	2000
3000	93.8	36.60	47.9	-104.18	2000
6000	96.1	44.74	48.6	-148.92	2000

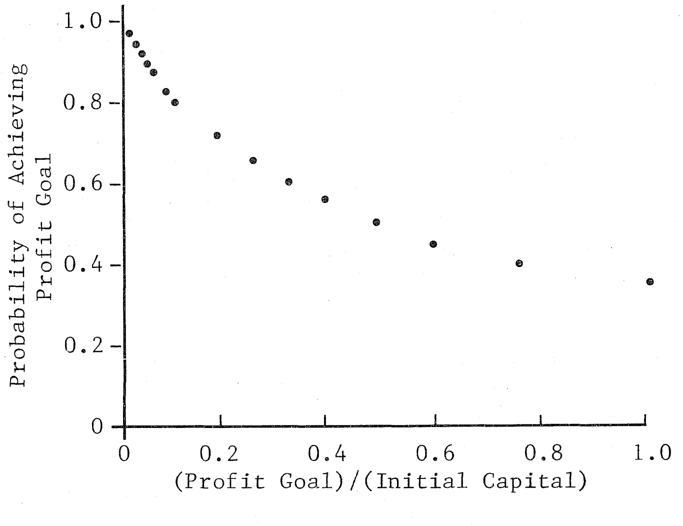


FIGURE 4. A Summary of Entries in Table 4

Just to see what sort of capital it would take to make it successfully through all 2,000 plays under the setting of Table 4, we simulated a gambler with \$9,000,000 initial capital rather modestly seeking the \$60 profit goal. He did win all 2,000 plays of the system, but needed to place some HUGE bets from time to time in order to complete a betting cycle. In five of the 8,000 cycles which he completed he was forced to lay down bets exceeding \$100,000. His moment of most concern occurred when a bet of \$751,440 was demanded by the system and his capital was down to \$6,926,517. This, of course, suggests that the gambler had to have a minimum of \$2,824,923 in working capital in order to survive all 2,000 plays. Consequently, without an unreasonably large capital relative to a given profit goal, we cannot expect to play a Labouchere scheme without occasional losses. It is, as with other schemes, possible to manipulate the probability of achieving the profit goal and the expected duration of the betting, but, alas, the losses more than cover the gains eventually. We are, as noted at the outset, victims of the house edge.

References

1. Richard A. Epstein. *The Theory of Gambling and Statistical Logic*. New York: Academic Press, 1977.
2. William Feller. *An Introduction to Probability Theory and Its Applications*, I, 3rd ed. New York: Wiley, 1957.
3. P. G. Hoel, S. C. Port and C. J. Stone. *Introduction to Probability Theory*. Boston: Houghton Mifflin, 1971.
4. Oswald Jacoby. *Oswald Jacoby on Gambling*. New York: Doubleday, 1963.
5. Norman Leigh. *Thirteen against the Bank*. New York: Wm. Morrow, 1976.
6. Clement McQuaid, ed. *Gambler's Digest*. New York: Digest Books, 1971.
7. John Scarne. *Scarne's Encyclopedia of Games*. New York: Harper & Row, 1973.
8. John Scarne. *Scarne's New Complete Guide to Gambling*. New York: Simon and Schuster, 1974.

Appendix

In this appendix we first investigate the number of ways in which a betting cycle can be completed in exactly t trials under the assumption that our initial betting sequence contains only three numbers. Having answered that question, we shall then apply the result to the now more familiar situation of a five-number initial sequence answering the same question in that setting. The result is easily generalized to fit any initial sequence.

Given a three-number initial sequence, let

$$C_j = \begin{cases} 1, \text{ if } j = 0 \text{ or } -1 \\ \text{the number of ways of completing a cycle in exactly } j \text{ trials,} \\ \text{if } j > 0. \end{cases}$$

In this new setting, the inequalities corresponding to (2) in Section 3 are

$$\frac{1}{3}(t + 3) \leq w < \frac{1}{3}(t + 5).$$

These, as before, give the conditions for completing a cycle. Consequently, in this setting a cycle can be completed only when $t \equiv 0$ or 2 (mod 3). This gives us $C_{3k+1} = 0$ for $k = 0, 1, 2, \ldots$.

With an argument like the one advanced in Section 3, we see that if $t = 3k + i$ where $i = -1$ or 0 and $k > 1$, exactly $k + 1$ wins are needed to complete a cycle in t trials. However, these wins must be distributed over the trials so that the cycle is not completed before trial t.

For example, exactly 10 wins are required to complete a cycle in 27 trials, but if 9 of these occurred in the first 24 trials, the cycle would have previously completed. Similarly, no cycle which completes in exactly 27 trials will have 8 wins in the first 21 trials, and so on. We are faced with the model shown in Figure 5,

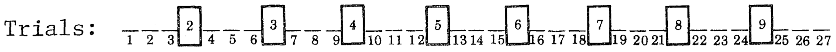


FIGURE 5

wherein we consider the number of ways to place 10 wins among the 26 trials, while respecting the barriers shown. The number on each barrier is meant as a strict upper bound to the number of wins which may fall to the left of the barrier.

To compute C_{27} we start with the observation that the last trial of any complete cycle must be a win. Consequently, we must determine in how many ways 9 wins may be appropriately distributed among the first 26 trials. First, $\binom{26}{9}$ gives the number of ways to do this distribution without regard to barriers. From this we first subtract the number of win-loss sequences in which the 9 wins occur in trials 1-24, $\binom{24}{9}$. Next we need to subtract out of of these $\binom{26}{9}-\binom{24}{9}$ remaining win-loss sequences those which do not respect the barrier at 21, and have not yet been subtracted; i.e., those with 8 wins in the first 21 trials and the 9th win in the 25th or 26th trial—$2\binom{21}{8}$.

Now we are left with $\binom{26}{9} - \binom{24}{9} - 2\binom{21}{8}$ win-loss sequences, all of which respect the rightmost two barriers of the figure. How many of these should be subtracted out for violating the restriction on the barrier at 18? Such a sequence would have 7 wins in the first 18 trials and two more wins in trials 22 to 26 respecting the barrier at 24. Notice this last condition of disstributing two wins appropriately among trials 22 to 26 with a fixed win in trial 27 is exactly the condition for completing a cycle in six trials (i.e., having a win in the 6th trial and distributing two more wins among trials 1-3). Consequently, the number of sequences to be subtracted out at this step is

$$C_6\binom{18}{7}.$$

Continuing this reasoning, we finally get

$$C_{27} = \binom{26}{9} - C_0\binom{24}{9} - C_3\binom{21}{8} - C_6\binom{18}{7} - C_9\binom{15}{6} - C_{12}\binom{12}{5}$$
$$- C_{15}\binom{9}{4} - C_{18}\binom{6}{3} - C_{21}\binom{3}{2}.$$

And a simple inductive argument gives us that

$$C_t = \begin{cases} 1, \text{ if } t = -1,\ 0 \\ 0, \text{ if } t = 3k + 1,\ k \geq 0 \\ \dbinom{3k + i - 1}{k} - \displaystyle\sum_{\ell=0}^{k-2} C_{3\ell+i}\dbinom{3(k - \ell - 1)}{k - \ell}, \text{ if } t = 3k + i,\ k \geq 1, \text{ and } i = 0 \text{ or } -1, \end{cases}$$

where we take a sum to be zero if its upper index limit is less than its lower index limit.

Returning to our example with the five-number initial betting sequence, an argument entirely like the preceding one leads us to

$$\left\{\begin{matrix} 3k + i \\ k + 2 \end{matrix}\right\} = \binom{3k + 2i - 2}{k + i} - \sum_{\ell=0}^{k+i-3} C_{3\ell+2i}\binom{3k + 3i - 5 - 3\ell}{k + i - \ell},$$

where $k \geq 1$, $i = 0$ or 1, and the C_j are the same numbers as before (arising in the three-number initial sequence case). Here, again, we take sums whose upper index limit is less than the lower index limit to be zero.

This, of course, is easily generalized to the case where the length of the initial betting sequence is arbitrary.

*****

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, NM 87131

Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-478 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

(a) Show that the congruence

$$x^2 \equiv -1 \pmod{4m^2 + 4m + 5}$$

has $x = \pm(2m^2 + m + 2)$ as a solution for m in $N = \{0, 1, \ldots\}$.

(b) Show that the congruence

$$x^2 \equiv -1 \pmod{100m^2 + 156m + 61}$$

has a solution $x = am^2 + bm + c$ with fixed integers a, b, c for m in N.

B-479 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a$ is an integral multiple of L_d for positive integers a, d, n with d odd.

B-480 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a$ is an integral multiple of $L_d - 2$ for positive integers a, d, n with d even.

B-481 *Proposed by Jerry Metzger, University of N. Dakota, Grand Forks, ND*

A and B compare pennies with A winning when there is a match. During an unusual sequence of mn comparisons, A produced m heads followed by m tails followed by m heads, etc., while B produced n heads followed by n tails followed by n heads, etc. By how much did A's wins exceed his losses? [For example, with $m = 3$ and $n = 5$, one has

A: HHHTTTHHHTTTHHH
B: HHHHHTTTTTHHHHH

and A's 8 wins exceeds his 7 losses by 1.]

B-482 *Proposed by John Hughes and Jeff Shallit, U.C., Berkeley, CA*

Find an infinite sequence a_1, a_2, ..., of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \text{ and } \lim_{n\to\infty} \left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right]$$

both exist and are unequal.

B-483 *Proposed by John Hughes and Jeff Shallit, U.C., Berkeley, CA*

Find an infinite sequence a_1, a_2, ..., of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \text{ exists and } \lim_{n\to\infty} \left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right] \text{ does not exist.}$$

SOLUTIONS

Generating F_n^2 and L_n^2

B-452 *Proposed by P. L. Mana, Albuquerque, NM*

Let $c_0 + c_1x + c_2x^2 + \cdots$ be the Maclaurin expansion for

$$[(1 - ax)(1 - bx)]^{-1},$$

where $a \neq b$. Find the rational function whose Maclaurin expansion is

$$c_0^2 + c_1^2x + c_2^2x^2 + \cdots$$

and use this to obtain the generating functions for F_n^2 and L_n^2.

Solution by A. G. Shannon, New South Wales Inst. of Tech., Sydney, Australia

Let $U(x) = \sum_{n=0}^{\infty} x^n$, formally. Then $\sum_{n=0}^{\infty} c_nx^n = [(1-ax)(1-bx)]^{-1}$ yields $c_n = (a^{n+1} - b^{n+1})/(a - b)$ so that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n^2 x^n &= (a^2U(a^2x) + b^2U(b^2x) - 2abU(abx))/(a - b)^2 \\ &= (1 + abx)/(1 - a^2x)(1 - b^2x)(1 - abx).\end{aligned}$$

Thus, when $a = 1 - b = \frac{1}{2}(1 + \sqrt{5})$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^2 x^n = \sum_{n=1}^{\infty} F_n^2 x^n = x\sum_{n=1}^{\infty} c_{n-1}^2 x^{n-1} &= x(1 - x)/(1 - a^2x)(1 - b^2x)(1 + x) \\ &= (x - x^2)/(1 - 2x - 2x^2 + x^3).\end{aligned}$$

Also

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^2 x^n = U(a^2x) + U(b^2x) + 2U(abx) &= (4 + 7x - x^2)/(1 - a^2x)(1 - b^2x)(1 + x) \\ &= (4 - 7x - x^2)/(1 - 2x - 2x^2 + x^3).\end{aligned}$$

These results are particular cases of Eqs. (33) and (42) of A. F. Horadam's article, "Generating Functions for Powers of a Certain Generalised Sequence of Numbers," *Duke Math. J.* 32 (1965):437-46.

Also solved by Paul S. Bruckman, John Ivie, E. Primrose, Heinz-Jürgen Seiffert, Sahib Singh, John Spraggan, Gregory Wulczyn, and the proposer.

FiFibonacci and LuLucas Equations

B-453 *Proposed by Paul S. Bruckman, Concord, CA*

Solve in integers r, s, t with $0 \leq r < s < t$ the FiFibonacci Diophantine equation

$$F_{F_r} + F_{F_s} = F_{F_t}$$

and the analogous LuLucas equation in which each F if replaced by an L.

Solution by Sahib Singh, Clarion State College, Clarion, PA

It is easy to see that the FiFibonacci Diophantine equation has solutions:

$r = 0;\ s = 1,\ 2;\ t = 3$
$r = 1;\ s = 2,\ 3;\ t = 4$
$r = 2,\ s = 3;\quad t = 4$

and the LuLucas equation admits the following solutions:

$r = 0;\ s = 1;\ t = 2$
$r = 0;\ s = 2;\ t = 3$

We show that there is no other solution possible.

After considering the above values, we see that onward the values of F_r and F_s are neither equal nor consecutive. Hence $F_s - 1$ is greater than F_r. Thus $F_{F_s} < F_{F_r} + F_{F_s} < F_{(F_s - 1)} + F_{F_s} = F_{(F_s + 1)}$. Therefore, $F_{F_r} + F_{F_s}$ lies between two consecutive Fibonacci numbers F_{F_s} and $F_{(F_s + 1)}$ and cannot qualify as a Fibonacci number. Thus no other solution is possible. Similar arguments enable us to conclude that no other solution is possible for the LuLucas Diophantine equation.

Also solved by Herta T. Freitag, John Ivie, Lawrence Somer, and the proposer.

Magic Corners

B-454 *Proposed by Charles W. Trigg, San Diego, CA*

In the square array of the nine nonzero digits $\begin{matrix} 6 & 7 & 5 \\ 2 & 1 & 3 \\ 9 & 4 & 8 \end{matrix}$

the sum of the four digits in each 2-by-2 corner array is 16. Rearrange the nine digits so that the sum of the digits in each such corner array is *seven* times the central digit.

Solution by J. Suck, Essen, E. Germany

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 9 & 2 \\ 8 & 3 & 7 \\ 4 & 6 & 5 \end{pmatrix}$ is up to rotating and flipping the only solution:

$e = 1$ is impossible, since $9 > 7$ must be in some corner, $e = 2$ is impossible from the fact that, in the corner of the 9, the sum of the other two digits would have to be 3, $e \geq 4$ is impossible, since the sum of the remaining three digits in a corner is at most $7 + 8 + 9 = 24$ with no other corner reaching this.

Now, let $e = 3$. Then the sum of all corner sums is $4 \cdot 21 = 1 + 2 + \cdots + 9 + b + d + f + h + 3 \cdot 3$, i.e., $b + d + f + h = 30$, showing that

$$\{b, d, f, h\} = \{6, 7, 8, 9\}.$$

Let, say, $b = 9$, then $h = 8$ is impossible (since, say, $f = 7$ so that $i = 3$, and $d = 6$ so that $a = 3$, too). Also, $h = 7$ is impossible (since, say, $f = 6$ making $g = 3$, and $d = 8$ making $g = 3$, too). Thus, $h = 6$, which entails the given solution.

Also solved by Paul S. Bruckman, Karen S. Carter, Derek Chang, Frank Higgins, Walther Janous, Birgit Kober, John Milsom, Bob Prielipp, Sahib Singh, Lawrence Somer, W. R. Utz, Gregory Wulczyn, and the proposer.

Simplified Convolution

B-455 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i}$ and $T_m = 10S_m/(m + 2)$. Prove that T_m is a sum of two Lucas numbers for $m = 0, 1, 2, \ldots$.

Solution by Sahib Singh, Clarion State College, Clarion, PA

$$S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i} = \frac{1}{5}\sum_{i=0}^{m} (L_i + L_{i+2})L_{m-i}.$$

Using $L_i = a^i + b^i$, the above summation becomes

$$S_m = \frac{1}{5}\sum_{i=0}^{m} [(L_m + L_{m+2}) + (-1)^i\{L_{m-2i-2} + L_{m-2i}\}]$$

$$= \frac{1}{5}[(m + 1)(L_m + L_{m+2}) + (L_m + L_{m+2})]$$

$$= \frac{(m + 2)}{5}(L_m + L_{m+2}).$$

Thus,

$$T_m = 2(L_{m+2} + L_m) = L_{m+4} + L_{m-2}.$$

Also solved by Paul S. Bruckman, Frank Higgins, B. S. Popov, Bob Prielipp, Gregory Wulczyn, and the proposer.

Fibonacci Products of Two Primes

B-456 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that any two consecutive Fibonacci numbers are coprime (i.e., their g.c.d. is 1). Prove or disprove: two distinct Fibonacci numbers are coprime if each of them is the product of two distinct primes.

Solution by Lawrence Somer, Washington, D.C.

A counterexample is provided by the Fibonacci numbers

$$F_{22} = 17711 = 89 \cdot 199$$

and

$$F_{121} = 8670007398507948658051921 = 89 \cdot 97415813466381445596089.$$

These numbers were found with the help of Table 1 in [1]. However, the following result is true.

Theorem: Two distinct Fibonacci numbers, each the product of two distinct primes, can have a common factor greater than 1 only if one of the numbers is of the form F_{2p} and the other number is of the form F_{p^2}, where p is an odd prime such that F_p is prime.

Proof: If p is a prime, call p a primitive factor of F_n if $p | F_n$ but $p \nmid F_m$ for $0 < m < n$. R. D. Carmichael [2] proved that F_n has a primitive prime factor for every n except $n = 1, 2, 6$, or 12. In none of these cases is F_n a product of exactly two distinct primes. It is also known that if $m|n$, then $F_m | F_n$. Thus, if n has two or more distinct proper factors r and s which are not equal to 1, 2, 6, or 12, then F_n has at least three prime factors—the primitive prime factors of F_r, F_s, and F_{rs} respectively. Since $F_6 = 8 = 2^3$, it follows that if n is a multiple of 6, then F_n is not a product of two distinct primes. It thus follows that if F_n is a product of two distinct primes, then F_n is of the form F_p, F_{2p}, or F_{p^2}, where p is prime. Moreover, inspection shows that $n \neq 2$ or 4. However, if F_p is a product of two distinct primes, then $(F_n, F_p) > 1$ implies that n is a multiple of p. But then $n = p$ or F_n has at least three distinct prime factors. Further, if p and q are distinct primes, then

$$(F_{2p}, F_{2q}) = (F_{2p}, F_{q^2}) = (F_{p^2}, F_{2q}) = (F_{p^2}, F_{q^2}) = 1.$$

The theorem now follows.

References

1. Brother Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables.* Santa Clara, Calif: The Fibonacci Association, 1972.
2. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n + \beta^n$." *Annals of Mathematics*, 2nd Ser. 15 (1913):30-70.

Also solved by Paul S. Bruckman, Herta T. Freitag, and the proposer.

*****

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANDED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS

H-342 *Proposed by Paul S. Bruckman, Concord, CA*

Let

$$A_n = \sum_{k=0}^{[\frac{1}{2}n]} \binom{n}{k}\binom{2n-2k}{n}4^k,\ n = 0, 1, 2, \ldots . \tag{1}$$

Prove that

$$\sum_{k=0}^{n} A_k A_{n-k} = 4^n F_{n+1}. \tag{2}$$

H-343 *Proposed by Verner E. Hoggatt, Jr., deceased*

Show that every positive integer m has a unique representation in the form

$$m = [A_1[A_2[A_3[\ldots[A_n]\ldots],$$

where $A_j = \alpha$ or α^2 for $j = 1, 2, \ldots, n - 1$, and

$$A_n = \alpha^2, \text{ where } \alpha = (1 + \sqrt{5})/2.$$

H-344 *Proposed by M. D. Agrawal, Government College, Mandasaur, India*

Prove:

1. $L_k L_{k+3m}^2 - L_{k+4m} L_{k+m}^2 = (-1)^k 5^2 F_m^2 F_{2m} F_{k+2m}$, and
2. $L_k L_{k+3m}^2 - L_{k+2m}^3 = 5(-1)^k F_m^2 (L_{k+4m} + 2(-1)^m L_{k+2m})$.

SOLUTIONS

Say A

(Corrected)
H-324 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*
(Vol. 19, No. 1, February 1981)

Establish the identity

$$A \equiv F_{14r}(F^7_{n+4r} + F^7_n) - 7F_{10r}(F^6_{n+4r}F_n + F_{n+4r}F^6_n) + 21F_{6r}(F^5_{n+4r}F^2_n + F^2_{n+4r}F^5_n) - 35F_{2r}(F^4_{n+4r}F^3_n + F^3_{n+4r}F^4_n)$$

$$= F^7_{4r}F_{7n+14}.$$

Solution by Paul S. Bruckman, Concord, CA

We first observe that there is a misprint in the statement of the problem. The first quantity under the first parenthesis in the definition of A should be "F^7_{n+4r}," not "F^7_{n+14r}." For brevity, let

(1) $$u = F_{n+4r},\ v = F_n.$$

Using the extension to negative integers:

(2) $$F_{-m} = (-1)^{m-1}F_m,$$

we see that we may express A as follows:

$$A = \sum_{k=0}^{7}\binom{7}{k}u^{7-k}(-v)^kF_{(14-4k)r}.$$

Thus,

$$A\sqrt{5} = \sum_{k=0}^{7}\binom{7}{k}u^{7-k}(-v)^k\{a^{14r-4kr} - b^{14r-4kr}\},$$

where $a = \frac{1}{2}(1 + \sqrt{5})$, $b = \frac{1}{2}(1 - \sqrt{5})$; thus

$$A\sqrt{5} = a^{14r}\sum_{k=0}^{7}\binom{7}{k}u^{7-k}(-va^{-4r})^k - b^{14r}\sum_{k=0}^{7}\binom{7}{k}u^{7-k}(-vb^{-4r})^k$$

$$= a^{14r}(u - vb^{4r})^7 - b^{14r}(u - va^{4r})^7, \text{ or}$$

(3) $$A\sqrt{5} = (ua^{2r} - vb^{2r})^7 - (ub^{2r} - va^{2r})^7.$$

Now

$$ua^{2r} - vb^{2r} = 5^{-1/2}\{a^{2r}(a^{n+4r} - b^{n+4r}) - b^{2r}(a^n - b^n)\}$$

$$= 5^{-1/2}(a^{n+6r} - b^{n+2r} - a^nb^{2r} + b^{n+2r})$$

$$= 5^{-1/2}a^{n+2r}(a^{4r} - b^{4r}) = a^{n+2r}F_{4r}.$$

Also,

$$ub^{2r} - va^{2r} = 5^{-1/2}\{b^{2r}(a^{n+4r} - b^{n+4r}) - a^{2r}(a^n - b^n)\}$$

$$= 5^{-1/2}(a^{n+2r} - b^{n+6r} - a^{n+2r} + a^{2r}b^n) = 5^{-1/2}b^{n+2r}(a^{4r} - b^{4r}) = b^{n+2r}F_{4r}.$$

Therefore, $A\sqrt{5} = (a^{n+2r}F_{4r})^7 - (b^{n+2r}F_{4r})^7 = (a^{7n+14r} - b^{7n+14r})F_{4r}^7$, or

$$A = F_{4r}^7 F_{7n+14r}. \quad \text{Q.E.D.} \tag{4}$$

Also solved by the proposer.

Sum Fun

H-325 *Proposed by Leonard Carlitz, Duke University, Durham NC*
(Vol. 19, No. 1, February 1981)

For arbitrary a, b put

$$S_m(a, b) = \sum_{j+k=m} \binom{a}{j}\binom{b+k-1}{k} \qquad (m = 0, 1, 2, \ldots).$$

Show that

$$\sum_{m+n=p} S_m(a, b)S_n(c, d) = S_p(a + c, b + d) \tag{1}$$

$$\sum_{m+n=p} (-1)^n S_m(a, b)S_n(c, d) = S_p(a - d, b - c). \tag{2}$$

Solution by the proposer.

We have

$$\sum_{m=0}^{\infty} S_m(a, b)x^m = \sum_{j,k=0}^{\infty} \binom{a}{j}\binom{b+k-1}{k} x^{j+k} = (1 + x)^a(1 - x)^{-b}. \tag{3}$$

Thus

$$\begin{aligned}\sum_{p=0}^{\infty} x^p \sum_{m+n=p} S_m(a, b)S_n(c, d) &= \sum_{m=0}^{\infty} S_m(a, b)x^m \sum_{n=0}^{\infty} S_n(c, d)x^n \\ &= (1 + x)^a(1 - x)^{-b}(1 + x)^c(1 - x)^{-d} \\ &= (1 + x)^{a+c}(1 - x)^{-b-d} \\ &= \sum_{p=0}^{\infty} S_p(a + c, b + d)x^p.\end{aligned}$$

Equating coefficients of x^p, we get (1). By (3) we have

$$\sum_{n=0}^{\infty} (-1)^n S_n(c, d)x^n = (1 - x)^c(1 + x)^{-d}.$$

Hence

$$\begin{aligned}\sum_{p=0}^{\infty} x^p \sum_{m+n=p} (-1)^n S_m(a, b)S_n(c, d) &= (1 + x)^a(1 - x)^{-b}(1 - x)^c(1 + x)^{-d} \\ &= (1 + x)^{a-d}(1 - x)^{-(b-c)}\end{aligned}$$

and (2) follows immediately.

Also solved by P. Bruckman.

A Primitive Solution

H-326 *Proposed by Larry Taylor, Briarwood, NY*
(Vol. 19, No. 1, February 1981)

(A) If $p \equiv 7$ or $31 \pmod{36}$ is prime and $(p - 1)/6$ is also prime, prove that $32(1 \pm \sqrt{-3})$ is a primitive root of p.

(B) If $p \equiv 13$ or $61 \pmod{72}$ is prime and $(p-1)/12$ is also prime, prove that $32(\sqrt{-1} \pm \sqrt{3})$ is a primitive root of p.

For example, $11 \equiv \sqrt{-3} \pmod{31}$, 12 and 21 are primitive roots of 31; $11 \equiv \sqrt{-1} \pmod{61}$, $8 \equiv \sqrt{3} \pmod{61}$, 59 and 35 are primitive roots of 61.

Solution by Paul S. Bruckman, Concord, CA

Part (A): We must first show that $(-3/p) = 1$, so that we can indeed define $x \equiv 32(1 \pm \sqrt{-3}) \pmod{p}$. Since $(p/3) = (7/3) = (31/3) = 1$, thus $(3/p)(p/3) = (-1)^{1/2(p-1)} = -1$, or $(3/p) = -1$. Thus,

$$(-3/p) = (-1/p)(3/p) = (-1)^{1/2(p-1)}(3/p) = (-1)^2 = 1,$$

which shows that x exists.

Let $w \equiv 2^{-1}(1 \pm \sqrt{-3}) \pmod{p}$. Thus $x \equiv 2^6 w \pmod{p}$. Note that $p > 7$, since $q = (p-1)/6$ must be prime. Note also that $w^3 \equiv -1 \pmod{p}$. This implies that $w \not\equiv 1 \pmod{p}$. Also, $w \not\equiv -1 \pmod{p}$, for if we suppose $w \equiv -1 \pmod{p}$, then

$$1 \pm \sqrt{-3} \equiv -2 \pmod{p} \Rightarrow \pm\sqrt{-3} \equiv -3 \pmod{p} \Rightarrow -3 \equiv 9 \pmod{p} \Rightarrow p \mid 12,$$

a contradiction. We observe further that, whichever sign is taken with $\sqrt{-3}$ in the definition of w, the other sign must be taken to define w^{-1}, since

$$2^{-1}(1 + \sqrt{-3})2^{-1}(1 - \sqrt{-3}) \equiv 4^{-1} \cdot 4 \equiv 1 \pmod{p}.$$

But, since $w^3 \equiv -1 \pmod{p}$, thus $w^{-1} \equiv -w^2 \pmod{p}$. We conclude that $w \not\equiv \pm 1 \pmod{p}$ and $w^2 \not\equiv \pm 1 \pmod{p}$.

In order to show that x is a primitive root of p, it suffices to show that $x^m \not\equiv 1 \pmod{p}$ for all proper divisors m of $\varphi(p) = p - 1 = 6q$. Since all the proper divisors of $6q$ divide at least one of the exponents 6, $2q$, and $3q$, it suffices to show that x^6, x^{2q}, and x^{3q} are $\not\equiv 1 \pmod{p}$.

Now $x^6 \equiv 2^{36} w^6 \equiv 2^{36}(-1)^2 \equiv 2^{36} \pmod{p}$. Note that

$$2^{36} - 1 = 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109.$$

Since all the primes in this decomposition are $\not\equiv 7$ or $31 \pmod{36}$, with the exception of 7, which is excluded, the congruence $2^{36} \equiv 1 \pmod{p}$ is impossible. Thus $x^6 \not\equiv 1 \pmod{p}$.

Since $q = 6r \pm 1$ for some r, $w^q \equiv w^{6r \pm 1} \equiv w^{\pm 1} \equiv w$ or $-w^2 \not\equiv \pm 1 \pmod{p}$; similarly, $(w^{-1})^q \not\equiv \pm 1 \pmod{p}$. Thus, $x^q \equiv 2^{6q} w^q \equiv 2^{p-1} w^q \equiv w^q \not\equiv 1 \pmod{p}$.

Thus, $x^{2q} \equiv (w^2)^q \equiv (-w^{-1})^q \equiv -(w^{-1})^q \not\equiv 1 \pmod{p}$. Finally,

$$x^{3q} \equiv (w^3)^q \equiv (-1)^q \equiv -1 \not\equiv 1 \pmod{p}.$$

This completes the proof of (A).

Part (B): The proof of (B) is patterned after that for (A). Since

$$(p/3) = (13/3) = (61/3) = 1,$$

thus $(3/p)(p/3) = (-1)^{1/2(p-1)} = 1$, or $(3/p) = 1$. Also, $(-1/p) = (-1)^{1/2(p-1)} = 1$. Defining $y \equiv 32(\sqrt{-1} \pm \sqrt{3}) \pmod{p}$, we then see that y exists. Also, we see that $(-3/p) = 1$.

Let $\theta \equiv 2^{-1}(\sqrt{-1} \pm \sqrt{3}) \pmod{p}$. Then $y \equiv 2^6\theta \pmod{p}$. Note that $p > 13$, since $q = (p-1)/12$ must be prime. Note also that $\theta^2 \equiv 2^{-1}(1 \pm \sqrt{-3}) \pmod{p}$, and $\theta^6 \equiv -1 \pmod{p}$. This implies that θ and θ^3 are $\not\equiv \pm 1 \pmod{p}$ and $\theta^2 \not\equiv 1 \pmod{p}$. Moreover, $\theta^2 \not\equiv -1 \pmod{p}$, for the congruence $\theta^2 \equiv -1 \pmod{p}$ would, as in part (A), lead to a contradiction. Also, whichever sign is taken with $\sqrt{3}$ in the definition of θ, the other sign must be taken to define $-\theta^{-1}$, since

$$2^{-1}(\sqrt{-1} + \sqrt{3})2^{-1}(\sqrt{-1} - \sqrt{3}) \equiv 4^{-1}(-1 - 3) \equiv -1 \pmod{p}.$$

Therefore, $\theta^{-1} \not\equiv \pm 1 \pmod{p}$. Combining this with the congruences $\theta^2 \equiv -\theta^{-4} \pmod{p}$ and $\theta^3 \equiv -\theta^{-3} \pmod{p}$, we conclude that $\theta^k \not\equiv \pm 1 \pmod{p}$ if $k = \pm 1$, ± 2, ± 3, ± 4, or ± 5.

In order to show that y is a primitive root of p, it suffices to show that $y^m \not\equiv 1 \pmod{p}$ for all proper divisors m of $\varphi(p) = 12q$. Since all the proper divisors of $12q$ divide at least one of the exponents 12, $3q$, and $4q$, it suffices to show that y^{12}, y^{3q}, and y^{4q} are $\not\equiv 1 \pmod{p}$.

Now $y^{12} \equiv 2^{72}\theta^{12} \equiv 2^{72}(-1)^2 \equiv 2^{72} \pmod{p}$. We may verify that

$$2^{72} - 1 = 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 109 \cdot 241 \cdot 433 \cdot 38{,}737,$$

this being the prime decomposition. Since the only prime in this decomposition that is $\equiv$ 13 or 61 (mod 72) is 13, which is excluded, we see that $2^{72} \not\equiv 1 \pmod{p}$. Therefore, $y^{12} \not\equiv 1 \pmod{p}$.

Since $q = 6r \pm 1$ for some r, thus

$$y^q \equiv 2^{6q}\theta^q \equiv 2^{1/2(p-1)}\theta^{6r\pm 1} \equiv (2/p)(-1)^r\theta^{\pm 1} \equiv \pm\theta^{\pm 1} \not\equiv \pm 1 \pmod{p}.$$

Therefore,

$$y^{3q} \equiv \pm\theta^{\pm 3} \equiv \theta^{\pm 3} \not\equiv 1 \pmod{p},$$

and

$$y^{4q} \equiv \theta^{\pm 4} \not\equiv 1 \pmod{p}.$$

This completes the proof of part (B).

Also solved by the proposer.

Belated Acknowledgment

M. D. Agrawal solved H-294 and H-319.

*****

SUSTAINING MEMBERS

*H.L. Alder
J. Arkin
B.I. Arthur, Jr.
Leon Bankoff
Murray Berg
J.G. Bergart
G. Bergum
George Berzsenyi
*M. Bicknell-Johnson
Clyde Bridger
J.L. Brown, Jr.
P.S. Bruckman
P.F. Byrd
L. Carlitz
G.D. Chakerian
R.M Chamberlain, Jr.
P.J. Cocuzza
Commodity Advisory
  Corp. of Texas
J.W. Creely
P.A. DeCaux
M.J. DeLeon
James Desmond
Harvey Diehl
J.L. Ercolano
D.R. Farmer
F.F. Frey, Jr.
C.L. Gardner
R.M. Giuli
*H.W. Gould
W.E. Greig
V.C. Harris
H.E. Heatherly
A.P. Hillman
*A.F. Horadam
R.J. Howell
R.P. Kelisky
C.H. Kimberling
Joseph Lahr
*C.T. Long
*James Maxwell
R.K. McConnell, Jr.
*Sr. M. DeSales McNabb
Leslie Miller
M.G. Monzingo
F.J. Ossiander
E.D. Robinson
J.A. Schumaker
D. Singmaster
John Sjoberg
M.N.S. Swamy
L. Taylor
*D. Thoro
H.L. Umansky
R. Vogel
C.C. Volpe
Marcellus Waddill
*L.A. Walker
J.E. Walton
R.E. Whitney
B.E. Williams
E.L. Yang

---

*Charter Members

INSTITUTIONAL MEMBERS

THE BAKER STORE EQUIPMENT COMPANY
Cleveland, Ohio

CALIFORNIA STATE POLY UNIVERSITY, POMONA
Pomona, California

CALIFORNIA STATE UNIVERSITY, SACRAMENTO
Sacramento, California

INDIANA UNIVERSITY
Bloomington, Indiana

ORION ENTERPRISES
Cicero, Illinois

REED COLLEGE
Portland, Oregon

SAN JOSE STATE UNIVERSITY
San Jose, California

SCIENTIFIC ENGINEERING INSTRUMENTS, INC.
Sparks, Nevada

SONOMA STATE UNIVERSITY
Rohnert Park, California

TRI STATE UNIVERSITY
Angola, Indiana

UNIVERSITY OF SANTA CLARA
Santa Clara, California

UNIVERSITY OF SCRANTON
Scranton, Pennsylvania

UNIVERSITY OF TORONTO
Toronto, Canada

WASHINGTON STATE UNIVERSITY
Pullman, Washington

BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer For the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972

Fibonacci's Problem Book. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

Please write to the Fibonacci Association, University of Santa Clara, CA 95053, U.S.A., for current prices.