The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 21
NUMBER 1

FEBRUARY
1983

CONTENTS

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The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
Br. Alfred Brousseau, and I.D. Ruggles

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ACKNOWLEDGMENTS

In addition to the mebers of the Board of Directors and our Assistant Editors, the following mathematicians, engineers, and physicists have assisted **THE FIBONACCI QUARTERLY** by refereeing papers during the past year. Their special efforts are sincerely apreciated.

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ON FIBONACCI AND LUCAS NUMBERS WHICH ARE SUMS OF PRECISELY FOUR SQUARES

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(Submitted April 1981)

INTRODUCTION

A well-known theorem of Lagrange states that every positive integer is a sum of four squares [4, p. 302]. In this article we determine which Fibonacci and Lucas numbers are sums of not fewer than four positive squares. The nth Fibonacci and Lucas numbers are denoted $F(n), L(n)$, respectively, in order to avoid the need for subscripts that carry exponents.

PRELIMINARIES

(1) $m \neq a^2 + b^2 + c^2$ iff $m = 4^j k$, with $j \geqslant 0$ and $k \equiv 7 \pmod 8$
(2) $F(2n) = F(n)L(n)$
(3) $L(2n) = L(n)^2 - 2(-1)^n$
(4) $F(m + n) = F(m)F(n - 1) + F(m + 1)F(n)$
(5) $F(12n \pm 1) \equiv 1 \pmod 8$
(6) $F(n) \equiv 7 \pmod 8$ iff $n \equiv 10 \pmod{12}$
(7) $F(n) \equiv 0 \pmod 4$ implies $F(n) \equiv 0 \pmod 8$
(8) $L(n) \not\equiv 0 \pmod 8$
(9) $L(n) \equiv 7 \pmod 8$ iff $n \equiv 4, 8,$ or $11 \pmod{12}$
(10) $L(n) \equiv 28 \pmod{32}$ iff $n \equiv 21 \pmod{24}$
(11) $L(12n) \equiv 2 \pmod{32}$
(12) If $j \geqslant 2$, then $4^j \mid F(n)$ iff $n = 3(4^{j-1})m$, with $(6, m) = 1$.

Remarks: (1) is stated on p. 311 of [4]. (2) and (3) are 12b, d, and e on p. 101 of [1]. (4) is (1) on p. 289 of [2]. (5), (6), and (7) are established by observing the periodic residues of the Fibonacci sequence (mod 8), namely: 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, etc. (8) and (9) are established by observing the periodic residues of the Lucas sequence (mod 8), namely: 2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1, etc. (10) and (11) are established by observing the periodic residues of the Lucas sequence (mod 32), namely: 2, 1, 3, 4, 7, 11, 18, 29, 15, 12, 27, 7, 2, 9, 11, 20, 31, 19, 18, 5, 23, 28, 19, 15, 2, 17, 19, 4, 23, 27, 18, 13, 31, 12, 11, 23, 2, 25, 27, 20, 15, 3, 18, 21, 7, 28, 3, 31, 2, 1, etc. Finally, (12) follows from (37) on p. 225 of [3].

THE MAIN THEOREMS

Theorem 1

$L(n) \neq a^2 + b^2 + c^2$ iff $n \equiv 4, 8,$ or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$.

Proof: If $L(n) \neq a^2+b^2+c^2$, then (1) implies $L(n) = 4^j k$, with $j \geqslant 0$ and $k \equiv 7 \pmod 8$. (8) implies $j = 0$ or $j = 1$. Now (9) and (10) imply $n = 4, 8$, or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$. Conversely, if $n \equiv 4$, 8, or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$, then (9) and (10) imply $L(n) \equiv 7 \pmod 8$ or $L(n) \equiv 28 \pmod{32}$, i.e., $L(n) = 4^j k$, with $j = 0$ or $j = 1$, and $k \equiv 7 \pmod 8$. Therefore, (1) implies $L(n) \neq a^2 + b^2 + c^2$.

Lemma 1

$F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8$ for $j \geqslant 2$.

Proof: (Induction on j) If $j = 2$, then

$$F(12)/16 = 144/16 = 9 \equiv 1 \pmod 8.$$

Now let $j \geqslant 3$.

$$\frac{F(3 \star 4^j)}{4^{j+1}} = \frac{F(4 \star 3 \star 4^{j-1})}{4^{j+1}} = \frac{F(3 \star 4^{j-1})}{4^j}\;\frac{L(3 \star 4^{j-1})L(6 \star 4^{j-1})}{4}$$

by (2). (11) implies $L(3 \star 4^{j-1}) \equiv 2 \pmod{32}$; (3) implies $L(6 \star 4^{j-1}) \equiv 2 \pmod{32}$. Thus

$$L(3 \star 4^{j-1})L(6 \star 4^{j-1}) \equiv 4 \pmod{32},$$

which implies $L(3 \star 4^{j-1})L(6 \star 4^{j-1})/4 \equiv 1 \pmod 8$. By the induction hypothesis, $F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8$. Therefore,

$$F(3 \star 4^j)/4^{j+1} \equiv 1 \star 1 \equiv 1 \pmod 8.$$

Lemma 2

$F(3 \star 4^{j-1}m)/4^j \equiv m \pmod 8$ for $j \geqslant 2$ and $m \geqslant 0$.

Proof: (Induction on m) Since $F(0) = 0$, Lemma 2 holds for $m = 0$. (4) implies

$$\begin{aligned} F(3 \star 4^{j-1}(m+1))/4^j &= F(3 \star 4^{j-1}m + 3 \star 4^{j-1})/4^j \\ &= \big(F(3 \star 4^{j-1}m)/4^j\big)F(3 \star 4^{j-1} - 1) \\ &\qquad + F(3 \star 4^{j-1}m + 1)\big(F(3 \star 4^{j-1})/4^j\big); \end{aligned}$$

by the induction hypothesis, $F(3 \star 4^{j-1}m)/4^j \equiv m \pmod 8$; (5) implies

$$F(3 \star 4^{j-1} - 1) \equiv F(3 \star 4^{j-1}m + 1) \equiv 1 \pmod 8;$$

Lemma 1 implies

$$F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8.$$

Therefore,

$$F(3 \star 4^{j-1}(m+1))/4^j \equiv m \star 1 + 1 \star 1 \equiv m + 1 \pmod{8}.$$

Theorem 2

$F(n) \neq a^2 + b^2 + c^2$ iff $n \equiv 10 \pmod{12}$ or $n = 3 \star 4^{j-1}m$, with $j \geqslant 2$ and $m \equiv 7 \pmod{8}$.

Proof: If $F(n) \neq a^2 + b^2 + c^2$, then (1) implies $F(n) = 4^j t$ with $j \geqslant 0$ and $t \equiv 7 \pmod{8}$. (7) implies $j \neq 1$. If $j = 0$, then (6) implies $n \equiv 10 \pmod{12}$. If $j \geqslant 2$, then (12) implies $n = 3 \star 4^{j-1}m$. Now Lemma 2 implies $m \equiv t \equiv 7 \pmod{8}$. Conversely, if $n \equiv 10 \pmod{12}$, then (6) implies $F(n) \equiv 7 \pmod{8}$, hence (1) implies $F(n) \neq a^2 + b^2 + c^2$. If $n = 3 \star 4^{j-1}m$ with $j \geqslant 2$ and $m \equiv 7 \pmod{8}$, then (12) implies $F(n) = 4^j t$. Lemma 2 implies $t = F(n)/4^j \equiv m \pmod{8}$. Since $t \equiv 7 \pmod{8}$, (1) implies

$$F(n) \neq a^2 + b^2 + c^2.$$

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INTERSECTIONS OF SECOND-ORDER LINEAR RECURSIVE SEQUENCES

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1. INTRODUCTION

We consider here intersections of positive integer sequences

$$\{w_n(w_0, w_1; p, -q)\}$$

which satisfy the second-order linear recurrence relation

$$w_n = pw_{n-1} + qw_{n-2},$$

where p, q are positive integers, $p \geqslant q$, and which have initial terms w_0, w_1. Many properties of $\{w_n\}$ have been studied by Horadam [2; 3; 4] (and elsewhere), to whom some of the notation is due. We look at conditions for fewer than two intersections, exactly two intersections, and more than two intersections. This is a generalization of work of Stein [5] who applied it to his study of varieties and quasigroups [6] in which he constructed groupoids which satisfied the identity $a((a \cdot ba)a) = b$ but not $(a(ab \cdot a))a = b$.

2. FEWER THAN TWO INTERSECTIONS

We shall first establish some lemmas which will be used to show that two of these generalized Fibonacci sequences with the same p and q generally do not meet.

Suppose the integers a_0, a_1, a_2, a_3, b_0, and b_1 are such that

$$a_2 > b_0 > a_0 \quad \text{and} \quad a_3 > b_1 > a_1.$$

These conditions are not as restrictive as they might appear, although they may require the sequences being compared to be realigned by redefining the initial terms. We consider the sets

$$\{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad \{w_n(b_0, b_1; p, -q)\},$$

and we seek an upper bound L for the number of a_1's $(b_1 > a_1 \geqslant b_0)$ such that

$$\{w_n(a_0, a_1; p, -q)\} \cap \{w_n(b_0, b_1; p, -q)\} \neq \emptyset.$$

We shall show that if $A(b) = b - L$ $(b = b_1 - b_0)$ is the number of a_1's such that if this intersection is nonempty, then $\lim_{b \to \infty} A(b)/b = 1$; that is, these generalized sequences do not meet, because if $\lim_{n \to \infty} A(n)/n = 1$, then we can say that for the predicate P about positive integers n $\{n : P(n)$ is true$\}$ has density 1, which means that P holds "for almost all n."

We first examine where $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ might meet. Since $a_0 < b_0$ and $a_1 < b_1$, then $a_n < b_n$ for all n by induction. Thus, if $a_k \in \{w_n(b_0, b_1; p, -q)\}$ and $a_k = b_i$, then i must be less than k.

Now

$$a_2 > b_0, \text{ and } a_3 > b_1,$$

so that

$$a_4 = pa_3 + qa_2 > pb_1 + qb_0 = b_2, \text{ and so on;}$$

that is,

$$a_k > b_{k-2} \text{ for } k \geqslant 3.$$

Thus, if

$$a_k \in \{w_n(b_0, b_1; p, -q)\},$$

then

$$b_{k-2} < a_k < b_k; \text{ that is, } a_k = b_{k-1}.$$

We next examine the a_1 for which $a_k = b_{k-1}$. Since

$$a_k = a_1 u_{k-1} + qa_0 u_{k-2} \qquad \text{(from (3.14) of [2])}$$

where $\{u_n\} = \{w_n(1, p; p, -q)\}$ is related to Lucas' sequence, then

$$a_k = b_{k-1}$$

is equivalent to

$$b_{k-1} = a_1 u_{k-1} + qa_0 u_{k-2} \quad \text{or} \quad a_1 = (b_{k-1} - qa_0 u_{k-2})/u_{k-1}.$$

We now define

$$x_k = (b_{k-1} - qa_0 u_{k-2})/u_{k-1},$$

and we shall show that $x_1, x_2, x_3, \ldots$ has a limit X, that it approaches this limit in an oscillating fashion, and that $x_{k+1} - x_k$ approaches zero quickly.

<u>Lemma 1</u>

$$x_{k+1} - x_k = (-q)^{k-1}(b_1 - b_0 - qa_0)/u_k u_{k-1}.$$

<u>Proof:</u> $$x_{k+1} - x_k = \frac{b_k - qa_0 u_{k-1}}{u_k} - \frac{b_{k-1} - qa_0 u_{k-2}}{u_{k-1}}$$

$$= \frac{(b_k u_{k-1} - b_{k-1} u_k) + qa_0(u_k u_{k-2} - u_{k-1}^2)}{u_k u_{k-1}}$$

Now

$$(-q)^{k-1} = u_{k-1}^2 - u_k u_{k-2}, \quad \text{(from (27) of [3])}$$

$$b_k u_{k-1} = b_1 u_{k-1}^2 + q b_0 u_{k-1} u_{k-2}, \quad \text{(from (3.14) of [2])}$$

$$b_{k-1} u_k = b_1 u_k u_{k-2} + q b_0 u_k u_{k-3},$$

so that

$$b_k u_{k-1} - b_{k-1} u_k = b_1 (u_{k-1}^2 - u_k u_{k-2}) + q b_0 (u_{k-1} u_{k-2} - u_k u_{k-3})$$

$$= (-q)^{k-1} b_1 - (-q)^{k-1} b_0$$

since

$$(-q)^{k-2} = u_{k-1} u_{k-2} - u_k u_{k-3} \quad \text{(from 4.21) of [2]).}$$

This gives the required result.

Lemma 2

$|x_{k+1} - x_k| < |b_1 - b_0 - q a_0| / \alpha^{2k-4}$, where α, β, $|\alpha| > |\beta|$, are the roots, assumed distinct, of

$$x^2 - px - q = 0.$$

Proof: $u_k = p u_{k-1} + q u_{k-2} \geqslant p u_{k-1}$

$$\geqslant q u_{k-1} \quad (p \geqslant q)$$

$$\geqslant q^2 u_{k-2} \geqslant \cdots \geqslant q^k u_0 \geqslant q^{k-1}$$

and

$$u_k u_{k-1} > q^{2k-3}.$$

Thus

$$|x_{k+1} - x_k| < |(b_1 - b_0 - q a_0) / q^{k-2}|,$$

which implies that the x_k's converge to a limit X in an oscillating fashion. Now

$$|q|^{k-2} = |\alpha|^{k-2} |\beta|^{k-2} < \alpha^{2k-4},$$

and

$$|x_{k+1} - x_k| < |b_1 - b_0 - q a_0| / \alpha^{2k-4}.$$

Theorem 1

If a_0 is a positive integer and $\{w_n\}$ is a generalized Fibonacci sequence, then for almost all a_1, $\{w_n(a_0, a_1; p, -q)\} \cap \{w_n\}$ consists of at most the element a_0.

Proof: It follows from Lemma 2 that at most one x_k is an integer for those k which satisfy the inequality

$$(b_1 - b_0 - q a_0) / \alpha^{2k-4} < 1,$$

or, equivalently, the inequality

$$k > 2 + \underline{\log}(b_1 - b_0 - qa_0)^{1/2}$$

in which $\underline{\log}$ stands for logarithm to the base $|\alpha|$. Thus the total number of k's for which x_k is an integer (since a_1 must be an integer) is at most

$$L = 2 + \underline{\log}(b_1 - b_0 - qa_0)^{1/2}.$$

If we choose b_0 such that $b_0 = c_m$ and $b_1 = c_{m+1}$, $c_m \varepsilon \{w_n(c_0, c_1; p, -q)\}$, where $c_{m+1}/c_m < [1+\alpha]$, then L is small in comparison with $b - b_0$. There is such an integer m:

$$c_{m+1}/c_m < [1 + \alpha] \qquad \text{for all } k \geqslant m$$

since

$$\lim_{k\to\infty} c_{k+1}/c_k = \alpha. \qquad ((1.22) \text{ of } [4])$$

We could take $b_0 = c_{m+1}$ or c_{m+2} and still conclude that the total number of a_1's ($b_0 \leqslant a_1 < b_1$) for which $\{w_n(a_0, a_1; p, -q)\}$ meets $\{w_n(b_0, b_1; p, -q)\}$ is small in comparison with $b = b_1 - b_0$.

Thus

$$A(b) = b - L,$$

and since

$$\lim_{b\to\infty}(\underline{\log}\ b)/b = 0,$$

we have

$$\lim_{b\to\infty} A(b)/b = 1 - \lim_{b\to\infty}(2 + \underline{\log}(b - qa_0)^{1/2})/b$$

$$= 1, \text{ as required.}$$

Thus, for allmost all a_1, $\{w_n\} \cap \{w_n(a_0, a_1; p, -q)\}$ contains a_0 only or is empty.

3. EXACTLY TWO INTERSECTIONS

Lemma 3

If $a_i = b_j$ and $a_{i-1} \neq b_{j-1}$, then for $r \geqslant 1$

$$b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{i+r} \notin \{w_n(b_0, b_1; p, -q)\}.$$

Proof: If $a_{i-1} > b_{j-1}$, then $a_{i+1} > b_{j+1}$, and

$$a_{i+1} = pa_i + qa_{i-1} < pb_{j+1} + qb_j = b_{j+2},$$

since

$$a_{i-1} < a_i = b_j < b_{j+1}$$

Thus

$$a_i < b_{j+1} < a_{i+1} \quad \text{and} \quad a_{i+1} < b_{j+2} < a_{i+2},$$

and, by induction,

$$a_{i+r-1} < b_{j+r} < a_{i+r} \qquad (r \geqslant 1).$$

Hence, $b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\}$, $r \geqslant 1$, from which the lemma follows.

Theorem 2

If $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ meet exactly twice, then at least one of these statements holds:

$$a_0 \in \{w_n(b_0, b_1; p, -q)\}, \; b_0 \in \{w_n(a_0, a_1; p, -q)\}.$$

As an illustration of Theorem 2, consider the sequences

$$1, 4, 5, 9, 14, \ldots, \quad \text{and} \quad 1, 1, 2, 3, 5, 8, 13, \ldots;$$

the second of these is the sequence of ordinary Fibonacci numbers

$$\{w_n(1, 1; 1, -1)\}.$$

Proof of Theorem 2: If $a_i = b_j$, $i, j > 0$, and the sequences meet exactly twice, then $a_{i-1} \neq b_{j-1}$; otherwise the sequences would be identical from those terms on, as can be seen from Theorem 3. (We need $i, j > 0$, since we have not specified a_n, b_n for $n < 0$.) Thus, from Lemma 3,

$$b_{j+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{i+r} \notin \{w_n(b_0, b_1; p, -q)\}, \; r \geqslant 1.$$

So $a_n = b_m$, $0 < m < j$, $0 < n < i$, and, again, $a_{n-1} \neq b_{m-1}$; otherwise the sequences would be identical from those terms on. But from Lemma 3 this implies that

$$b_{m+r} \notin \{w_n(a_0, a_1; p, -q)\} \quad \text{and} \quad a_{n+r} \notin \{w_n(b_0, b_1; p, -q)\}, \; r \geqslant 1,$$

which contradicts the assumption that $a_i = b_j$. So the only other possibilities are that $a_0 = b_m$ for some m or $a_n = b_0$ for some n, as required. This establishes the theorem.

4. MORE THAN TWO INTERSECTIONS

Theorem 3

If $\{w_n(a_0, a_1; p, -q)\}$ and $\{w_n(b_0, b_1; p, -q)\}$ have two consecutive terms equal, then they are identical from those terms on.

Proof: If $a_i = b_j$ and $a_{i-1} = b_{j-1}$, then

$$a_{i+1} = pa_i + qa_{i-1} = pb_j + qb_{j-1} = b_{j+1}$$

and the result follows by induction.

5. REMARKS

A. It is of interest to note that the number of terms of $\{w_n(a_0, a_1; p, -q)\}$ not exceeding b_0 is asymptotic to

$$\underline{\log}(b_0(\alpha - \beta)/(a_1\alpha + a_0\alpha\beta)). \qquad \text{(Horadam [4])}$$

B. As an illustration of Theorem 1, if we consider the case where $p = q = 1$, and if we take $a_0 = 1$, $b_0 = 100$, $b_1 = 191$, then $b_2 = 291$, $b_3 = 392$, $b_4 = 683$. When:

$$a_1 = 100,\ a_1 = b_0; \quad a_1 = 190,\ a_2 = b_1; \quad a_1 = 145,\ a_3 = b_2;$$

$$a_1 = 130,\ a_4 = b_3; \quad a_1 = 136,\ a_5 = b_4.$$

Thereafter, there are no more integer values of a_1 that yield $a_k = b_{k-1}$. Thus 100, 130, 136, 145, and 190 are the only values of a_1 $(100 \leqslant a_1 < 191)$ for which

$$\{w_n(1, a_1; 1, -1)\} \cap \{w_n(100, 191; 1, -1)\} \neq \emptyset.$$

Also, $\left[\left(\frac{1}{2}(4 + \underline{\log}\ 90)\right)\right] = 6$, so the bound L is valid.

C. It is not apparent how Theorem 1 can be elegantly generalized to arbitrary order sequences. If $\{w_n^{(r)}\}$ satisfies the recurrence relation

$$w_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n-j}^{(r)} \qquad n \geqslant r$$

with suitable initial values, where the P_{rj} are arbitrary integers, and if $\{u_n^{(r)}\}$ satisfies the same recurrence relation, but has initial values given by

$$u_0^{(r)} = u_1^{(r)} = \cdots = u_{r-2}^{(r)} = 0,\ u_{r-1}^{(r)} = 1,$$

then it can be proved that

$$w_n^{(r)} = \sum_{j=0}^{r-1} \left(\sum_{k=0}^{j} (-1)^{j-k} P_{rj}\, w_k^{(r)} \right) u_{n-j+1}^{(r)},$$

where $P_{r0} = 1$. When $r = 2$, this becomes

$$\begin{aligned} w_n^{(2)} &= w_1^{(2)} u_n^{(2)} + w_0^{(2)}\ u_{n+1}^{(2)} - P_{21} u_n^{(2)} \\ &= w_1^{(2)} u_n^{(2)} - P_{22} w_0^{(2)} u_{n-1}^{(2)} \end{aligned}$$

which is Eq. (3.14) of [2] for the sequences

$$\{w_n^{(2)}\} = \{w_n(w_0^{(2)}, w_1^{(2)}; P_{21}, P_{22})\}$$

and

$$\{u_{n+1}^{(2)}\} = \{w_n(1, P_{21}; P_{21}, P_{22})\}.$$

Thus, one of the key equations in Theorem 1 generalizes to

$$w_{r-1}^{(r)} = \left(w_n^{(r)} - \sum_{j=0}^{r-2} (-1)^{j-r-1} P_{r,\, r-j-1} w_j^{(r)} u_{n-r+2}^{(r)} \right.$$
$$\left. + \sum_{k=0}^{j} (-1)^{j-k} P_{r,j-k}\, w_k^{(r)} u_{n-j+1}^{(r)} \right) \Big/ u_{n-r+2}^{(r)},$$

which is rather cumbersome.

Thanks are expressed to the referee for several useful suggestions.

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A PROPERTY OF FIBONACCI AND TRIBONACCI NUMBERS

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1. INTRODUCTION

The Fibonacci numbers are defined by setting

$$a_1 = a_2 = 1 \text{ and } a_{n+1} = a_n + a_{n-1} \text{ for } n \geqslant 2.$$

A related family of sequences are the t-bonacci numbers (where $t \geqslant 2$ is an integer). These are defined by setting

$$a_1 = 1,\ a_n = 2^{n-2} \text{ for } 2 \leqslant n \leqslant t$$

and

$$a_{n+1} = a_n + \cdots + a_{n-t+1} \text{ for } n \geqslant t.$$

Thus, for $t = 2$ we obtain the Fibonacci numbers again, and for $t = 3$ we obtain the so-called Tribonacci numbers.

The Fibonacci numbers have many interesting properties. The property of interest to us here is that this sequence satisfies the equation

$$\Delta a_n = a_{n-1} \qquad (n \geqslant 2),$$

where Δ denotes the forward difference operator. The Tribonacci numbers satisfy

$$\Delta^3 a_n = 2a_{n-2} \qquad (n \geqslant 3).$$

We call a sequence (a_n) that satisfies an equation of the form

$$\Delta^k a_n = ma_{n-r} \qquad (n > r), \tag{1}$$

a *self-generating sequence with parameters* (k, m, r). We abbreviate this to SGS(k, m, r). [We will work under the convenient assumption that k, m, and r are integers and that $k \geqslant 1$. Similarly, our sequences (a_n) will be integral.]

Thus, the Fibonacci numbers are an SGS(1, 1, 1) and the Tribonacci numbers form an SGS(3, 2, 2). This immediately suggests the question of whether, for any $t \geqslant 4$, the t-bonacci numbers form a self-generating sequence. The main result of this paper is as follows.

Theorem 1

The Fibonacci sequence is an SGS(1, 1, 1); the Tribonacci sequence is an SGS(3, 2, 2). For $t \geqslant 4$, no t-bonacci sequence is self-generating.

2. PROOF OF THEOREM 1

Let

$$F(x) = \sum_{n=1}^{\infty} a_n x^n$$

denote the generating function (G.F.) of the sequence (a_n) and let $\Delta^j F(x)$ denote the G.F. of the sequence of jth forward differences $(\Delta^j a_n)$.

Lemma 1

For $j \geqslant 1$, we have

$$\Delta^j F(x) = \frac{1}{x^j}[(1 - x)^j F(x) - xp_{j-1}(x)], \tag{2}$$

where $p_{j-1}(x)$ denotes a polynomial of degree at most $j - 1$.

Lemma 1 can be proved by induction on j. We leave the details as an exercise.

Now let (a_n) be an SGS(k, m, r). In order to satisfy (1), we have to subtract from $\Delta^k F(x)$ its first r terms [i.e., a polynomial $q_r(x)$ of degree at most r] and equate the rest with $mx^r F(x)$:

$$\frac{1}{x^k}[(1 - x)^k F(x) - xp_{k-1}(x)] - q_r(x) = mx^r F(x).$$

From this equation, we immediately obtain:

Theorem 2

The generating function of an SGS(k, m, r) is of the form

$$F(x) = \frac{p_{k+r}(x)}{(1 - x)^k - mx^{k+r}}, \tag{3}$$

where $p_{k+r}(x)$ is a polynomial of degree at most $k + r$ with zero constant term. □

Remark 1: It can be shown that any sequence with generating function of the form given in (3) is an SGS(k, m, r). We will not prove this because we will not make use of it here.

The G.F. for the t-bonacci sequence is

$$F(x) = \frac{x}{1 - x - \cdots - x^t};$$

hence, a necessary condition for this sequence to be self-generating is that the zeros $\alpha, \beta, \gamma, \ldots$ of $1 - x - \cdots - x^t$ are also zeros of the polynomial $(1 - x)^k - mx^{k+r}$ appearing in the form of $F(x)$ given in Theorem 1. Our aim is to show that for $t \geqslant 4$ we can find three zeros α, β, γ for which this necessary condition is violated. Thus, it will be useful to list some facts about the roots of

$$1 - x - \cdots - x^t = 0. \tag{4}$$

Remark 2: We observe that no root of (4) equals 1. Now, multiplying (4) by $1 - x$ and collecting terms transforms (4) into

$$x^{t+1} - 2x + 1 = 0. \tag{5}$$

Remark 3: A geometrical argument about the curves $y = x^{t+1}$ and $y = 2x - 1$ shows that for odd t there is exactly one, for even t there are exactly two, real roots of (5) not equal to 1. For all t, one of these tends monotonically to -1 from the left as t increases. In [1], the positive real roots have been calculated. For $t = 6$ this root is $\alpha = 0.504138\ldots$; hence, for $t \geqslant 6$ we have $\alpha < 0.505$.

Remark 4: In [2], it was proved that (5) has exactly one root z with $|z| < 1$ and one with $|z| = 1$; all other roots satisfy $|z| > 1$. We shall now give an upper bound for the absolute values of these roots.

Lemma 2

The roots of (5) with $|z| > 1$ satisfy $|z| <$ 3.

Proof: Let z be a root of (5) with $|z| > 1$. Then, since

$$|z|\,|z^t - 2| = |-1| = 1,$$

we have $|z^t - 2| < 1$, which implies $|z^t| < 3$ and, therefore, $|z| < \sqrt[t]{3}$. □

Combined with the previous lemma, our next result approximately determines the positions of the roots of (5).

Lemma 3

For each j with $1 \leqslant j \leqslant \dfrac{t-1}{2}$, Eq. (5) has a root z_j with

$$\arg z_j \in I_j = \left(\frac{2j\pi}{t}, \frac{2j\pi}{t-1}\right].$$

Proof: We use Gauss's method for trinomial equations (see e.g. [3], pp. 397–398). Write $z = \rho(\cos\varphi + i\sin\varphi)$. Then, if $z^{t+1} - 2z + 1$ is zero, we must have

$$\rho^{t+1}\cos(t+1)\varphi - 2\rho\cos\varphi + 1 = 0; \tag{6}$$

$$\rho^{t+1}\sin(t+1)\varphi - 2\rho\sin\varphi = 0. \tag{7}$$

From (7), we get

$$\rho^{t} = \frac{2\sin\varphi}{\sin(t+1)\varphi}.$$

Substituting this into (6) and using the trigonometric addition formulas, we obtain

$$\rho = \frac{\sin(t+1)\varphi}{2\sin t\varphi}. \tag{8}$$

Upon substitution into (7), this yields

$$2^{t+1}\sin^{t}t\varphi\sin\varphi - \sin^{t+1}(t+1)\varphi = 0, \tag{9}$$

which determines φ. Denote the left-hand side of (9) by $f(\varphi)$. Then

$$f\left(\frac{2j\pi}{t}\right) < 0$$

whereas

$$f\left(\frac{2j\pi}{t-1}\right) \geqslant 0.$$

By the continuity of f, the lemma follows. □

Now let $t \geqslant 4$ and let α, β, and γ denote three nonconjugate distinct roots of (4). If the t-bonacci sequence was self-generating, we would have $(1-\alpha)^k = m\alpha^{k+r}$ as well as $(1-\beta)^k = m\beta^{k+r}$ for some k, m, and r; hence,

$$\left(\frac{1-\alpha}{1-\beta}\right)^k = \left(\frac{\alpha}{\beta}\right)^{k+r}.$$

An analogous equation holds for α and γ. Taking logarithms, we get

$$k\log\frac{1-\alpha}{1-\beta} - (k+r)\log\frac{\alpha}{\beta} = 0$$

and

$$k\log\frac{1-\alpha}{1-\gamma} - (k+r)\log\frac{\alpha}{\gamma} = 0.$$

To obtain nontrivial solutions for given k and r, the two equations must be linearly dependent. Therefore, considering the absolute values, we must have

$$\log\left|\frac{1-\alpha}{1-\beta}\right| \log\left|\frac{\alpha}{\gamma}\right| = \log\left|\frac{1-\alpha}{1-\gamma}\right| \log\left|\frac{\alpha}{\beta}\right|. \tag{10}$$

Denote the left- and right-hand sides of (10) by L and R, respectively. Our aim is to find roots α, β, and γ for which $L \neq R$.

Let t be even, $t \geqslant 6$. Take as α the positive real root, as β the negative real root, and as γ a root with $0 < \arg \gamma < 2\pi/5$. (Such a exists, by Lemma 3.) Then the following inequalities hold, by virtue of Remark 3 and Lemma 2:

$$\begin{array}{rlrl} 0.5 < |\alpha| & < 0.505 & 0.495 < |1-\alpha| & < 0.5 \\ |\beta| & < 1.201 & 2 < |1-\beta| & \\ 1 < |\gamma| & & |1-\gamma| & < 1.304 \end{array}$$

From these, we calculate $L > 0.947$ and $R < 0.849$.

Now let t be odd, $t \geqslant 7$. As α we take the positive real root, as β a root with $6\pi/7 < \arg \beta < \pi$, and as γ a root with $0 < \arg \gamma < 2\pi/6$. The resulting inequalities are

$$\begin{array}{rlrl} 0.5 < |\alpha| & < 0.505 & 0.495 < |1-\alpha| & < 0.5 \\ |\beta| & < 1.17 & 1.94 < |1-\beta| & \\ 1 < |\gamma| & & |1-\gamma| & < 1.094 \end{array}$$

and we obtain $L > 0.926$ and $R < 0.675$.

The remaining cases, $t = 4$ and $t = 5$, can be settled by approximate calculation of φ and ρ using (8) and (9); again, roots can be found for which $L \neq R$. The details will be omitted here. □

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UNITARY HARMONIC NUMBERS

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1. INTRODUCTION

Ore [2] investigated the harmonic mean $H(n)$ of the divisors of n, and showed that

$$H(n) = n\tau(n)/\sigma(n),$$

where, as usual, $\tau(n)$ and $\sigma(n)$ denote, respectively, the number and sum of the divisors of n. An integer n is said to be *harmonic* if $H(n)$ is an integer. For example, 6 and 140 are harmonic, since

$$H(6) = 2 \quad \text{and} \quad H(140) = 5.$$

Ore proved that any perfect number (even or odd) is harmonic, and that no prime power is harmonic. Pomerance [3] proved that any harmonic number of the form $p^a q^b$, with p and q prime, must be an even perfect number. Ore also conjectured that there is no odd $n > 1$ which is harmonic, and Garcia [1] verified Ore's conjecture for $n < 10^7$; however, since Ore's conjecture implies that there are no odd perfect numbers, any proof must be quite deep.

A divisor d of an integer n is a *unitary divisor* if g.c.d. $(d, n/d) = 1$, in which case we write $d \| n$. Let $\tau^*(n)$ and $\sigma^*(n)$ be, respectively, the number and sum of the unitary divisors of n. If n has $\omega(n)$ distinct prime factors, it is easy to show that

$$\tau^*(n) = 2^{\omega(n)} \quad \text{and} \quad \sigma^*(n) = \prod_{p^e \| n} (1 + p^e),$$

both functions being multiplicative.

Let $H^*(n)$ be the harmonic mean of the unitary divisors of n. It follows that

$$H^*(n) = n\tau^*(n)/\sigma^*(n) = \prod_{p^e \| n} \frac{2p^e}{1 + p^e}$$

We say that n is *unitary harmonic* if $H^*(n)$ is an integer.

In this paper we outline the proofs of two results:

Theorem 1

There are 23 unitary harmonic numbers n with $\omega(n) \leqslant 4$ (see Table 1).

Theorem 2

There are 43 unitary harmonic numbers $n \leqslant 10^6$. These numbers, which include all but one of those in Theorem 1, are given in Table 2.

TABLE 1

$\omega(n)$	$H^*(n)$	n	$\omega(n)$	$H^*(n)$	n
0	1	1	4	10	$9{,}100 = 2^2 5^2 7 \cdot 13$
2	2	$6 = 2 \cdot 3$	4	10	$31{,}500 = 2^2 3^2 5^3 7$
2	3	$45 = 3^2 5$	4	10	$330{,}750 = 2 \cdot 3^3 5^3 7$
3	4	$60 = 2^2 3 \cdot 5$	4	11	$16{,}632 = 2^3 3^3 7 \cdot 11$
3	4	$90 = 2 \cdot 3^2 5$	4	12	$51{,}408 = 2^4 3^3 7 \cdot 17$
3	7	$15{,}925 = 5^2 7^2 13$	4	12	$66{,}528 = 2^5 3^3 7 \cdot 11$
3	7	$55{,}125 = 3^2 5^3 7^2$	4	12	$185{,}976 = 2^3 3^4 7 \cdot 41$
4	7	$420 = 2^2 3 \cdot 5 \cdot 7$	4	12	$661{,}500 = 2^2 3^3 5^3 7^2$
4	7	$630 = 2 \cdot 3^2 5 \cdot 7$	4	13	$646{,}425 = 3^2 5^2 13^2 17$
4	9	$3{,}780 = 2^3 3^3 5 \cdot 7$	4	13	$716{,}625 = 3^2 5^3 7^2 13$
4	9	$46{,}494 = 2 \cdot 3^4 7 \cdot 41$	4	15	$20{,}341{,}125 = 3^4 5^3 7^2 41$
4	10	$7{,}560 = 2^3 3^3 5 \cdot 7$			

TABLE 2

$H^*(n)$	n	$H^*(n)$	n
1	1	9	$3{,}780 = 2^2 3^3 5 \cdot 7$
2	$6 = 2 \cdot 3$	13	$5{,}460 = 2^2 3 \cdot 5 \cdot 7 \cdot 13$
2	$45 = 3^2 5$	10	$7{,}560 = 2^3 3^3 5 \cdot 7$
3	$60 = 2^2 3 \cdot 5$	13	$8{,}190 = 2 \cdot 3^2 5 \cdot 7 \cdot 13$
3	$90 = 2 \cdot 3^2 5$	10	$9{,}100 = 2^2 5^2 7 \cdot 13$
7	$420 = 2^2 3 \cdot 5 \cdot 7$	7	$15{,}925 = 5^2 7^2 13$
7	$630 = 2 \cdot 3^2 5 \cdot 7$		

(continued)

TABLE 2 (continued)

$H^*(n)$	n	$H^*(n)$	n
11	$16{,}632 = 2^3 3^3 7 \cdot 11$	12	$185{,}976 = 2^3 3^4 7 \cdot 41$
15	$27{,}300 = 2^2 3 \cdot 5^2 7 \cdot 13$	15	$232{,}470 = 2 \cdot 3^4 5 \cdot 7 \cdot 41$
10	$31{,}500 = 2^2 3^2 5^3 7$	20	$257{,}040 = 2^4 3^3 5 \cdot 7 \cdot 17$
15	$40{,}950 = 2 \cdot 3^2 5^2 7 \cdot 13$	10	$330{,}750 = 2 \cdot 3^3 5^3 7^2$
9	$46{,}494 = 2 \cdot 3^4 7 \cdot 41$	20	$332{,}640 = 2^5 3^3 5 \cdot 7 \cdot 11$
12	$51{,}408 = 2^4 3^3 7 \cdot 17$	18	$464{,}940 = 2^2 3^4 4 \cdot 7 \cdot 41$
7	$55{,}125 = 3^2 5^3 7^2$	22	$565{,}448 = 2^4 3^3 7 \cdot 11 \cdot 17$
17	$64{,}260 = 2^2 3^3 5 \cdot 7 \cdot 17$	19	$598{,}500 = 2^2 3^2 5^3 7 \cdot 19$
12	$66{,}528 = 2^2 3^3 7 \cdot 11$	13	$646{,}425 = 3^2 5^2 13^2 17$
18	$81{,}900 = 2^2 3^2 5^2 7 \cdot 13$	12	$661{,}500 = 2^2 3^3 5^3 7^2$
16	$87{,}360 = 2^6 3 \cdot 5 \cdot 7 \cdot 13$	13	$716{,}625 = 3^2 5^3 7^2 13$
14	$95{,}550 = 2 \cdot 3 \cdot 5^2 7^2 13$	17	$790{,}398 = 2 \cdot 3^4 7 \cdot 17 \cdot 41$
19	$143{,}640 = 2^3 3^3 5 \cdot 7 \cdot 19$	18	$859{,}950 = 2 \cdot 3^3 5^2 7^2 13$
20	$163{,}800 = 2^3 3^2 5^2 7 \cdot 13$	33	$900{,}900 = 2^2 3^2 5^2 7 \cdot 11 \cdot 13$
19	$172{,}900 = 2^2 5^2 7 \cdot 13 \cdot 19$	20	$929{,}880 = 2^2 3^4 5 \cdot 7 \cdot 41$

The complete proofs of Theorems 1 and 2 are quite tedious, requiring many cases and subcases. However, the techniques are quite simple, and are adequately illustrated by the cases discussed here.

2. TECHNIQUES FOR THEOREM 1

If p and q are (not necessarily distinct) primes and $p^a < q^b$, then it is easy to show that $H^*(p^a) > H^*(q^b)$. This fact can be used, once $\omega(n)$ and $H^*(n)$ are specified, to find an upper bound for the smallest prime power unitary divisor of n; for each choice, the process is repeated to find choices for the next smallest prime power unitary divisor, and the process continues until all but one of the prime power unitary divisors is found; the largest prime power can then be solved for directly, without a search. Of course, this procedure is interrupted any time it becomes obvious that the as yet unknown portion of n must have more prime divisors than allowed by the prespecified size of $\omega(n)$.

With $\omega(n)$ and $H^*(n)$ given, the problem is to find n with

$$n/\sigma^*(n) = H^*(n)/\tau^*(n)$$

being a prespecified fraction, which in turn requires that any odd prime that divides $\sigma^*(n)$ must also divide n. Also, since $\tau^*(n)$ is a power of 2, any odd prime that divides $H^*(n)$ must also divide n. Several of the cases are shortened by using results of Subbarao and Warren [4] for the special case $\sigma^*(n) = 2n$ (i.e., for n being unitary perfect).

We present here the proof for the case $\omega(n) = 4$, $H^*(n) = 15$, one of the longer and subtler cases of Theorem 1. Throughout, let $n = pqrs$ with $p < q < r < s$ and p, q, r, and s powers of distinct primes (though not necessarily prime). Note that because $n/\sigma^*(n) = 15/16$, $3 \cdot 5 | pqrs$. Also, if n has a prime power unitary divisor which is congruent to 3 (mod 4), then n must be even.

If $p \geqslant 59$, then $n/\sigma^*(n) > 15/16$, so $p \leqslant 53$.

$p = 53$: $q < 61$, so $q = 59$, which requires that $2 \cdot 3 \cdot 5 | rs$, a contradiction.

$p = 49$: $q < 64$. But $q = 61$ implies $3 \cdot 5 \cdot 31 | rs$, and $q = 59$ requires $2 \cdot 3 \cdot 5 | rs$; both of these are impossible. If $q = 53$, then $r < 79$, but there are no powers of 3 or 5 between 53 and 79.

$p = 47$: $q < 67$ and $2 \cdot 3 \cdot 5 | qrs$, so $q = 64$, from which follows the impossibility $3 \cdot 5 \cdot 13 | rs$.

$p = 43$: $2 \cdot 3 \cdot 5 \cdot 11 | qrs$, a contradiction.

$p = 41$: $q < 71$ and $3 \cdot 5 \cdot 7 | qrs$. The only possibility is $q = 49$, which requires $r < 103$ and $3 \cdot 5 | rs$. This in turn forces $r = 81$, which implies $s = 125$. Thus we have a unitary harmonic number, since

$$H^*(3^4 5^3 7^2 41) = 15.$$

$p = 37$: $q < 79$ and $3 \cdot 5 \cdot 19 | qrs$, a contradiction.

$p = 32$: $q < 83$ and $3 \cdot 5 \cdot 11 | qrs$, so $q = 81$. But then $5 \cdot 11 \cdot 41 | rs$, a contradiction.

$p = 31$: $q < 89$ and $2 \cdot 3 \cdot 5 | qrs$. There are three unpalatable choices: $q = 81$ requires that $2 \cdot 5 \cdot 41 | rs$, and $q = 64$ implies $3 \cdot 5 \cdot 13 | rs$, while $q = 32$ forces $3 \cdot 5 \cdot 11 | rs$.

$p = 29$: $31 < q < 97$, and rs is divisible by at least three primes unless q is 89, 81, 59, or 49. If $q = 89$, then $r < 103$ and there are no powers of 3 or 5 between 89 and 103. If $q = 81$, then $r < 109$ and $5 \cdot 41 | rs$, a contradiction. If $q = 59$, then $r < 167$ and the only possible cases are $r = 125$, which implies $3 \cdot 7 | s$, and $r = 81$, which forces $5 \cdot 41 | s$. If $q = 49$, then $r < 193$, so either $r = 125$, which does not leave the required 5 in the numerator of $n/\sigma^*(n)$, or $r = 81$, which forces $5 \cdot 41 | s$. Thus $p = 29$ is impossible.

$p = 27$: $35 < q < 107$ and $2 \cdot 5 \cdot 7 | qrs$, so the only possible values for q are 64 and 49. If $q = 64$, then $5 \cdot 7 \cdot 13 | rs$. If $q = 49$, then $125 < r < 251$ and $2 \cdot 5 | rs$, so $r = 138$, whence $5 \cdot 43 | s$, a contradiction.

$p = 25$: $39 < q < 121$ and rs is divisible by three or more primes except when q is 107, 103, 89, 81, 64, or 53. If $q = 107$, then $r < 125$, while $q = 103$ implies $r < 128$, and $q = 89$ forces $r < 149$; in each case, $3 \cdot 13 | rs$, a contradiction. If $q = 81$, then $r < 157$ and $13 \cdot 41 | rs$, which is impossible. If $q = 64$, then $r < 211$ and $3 \cdot 13 | rs$; thus $r = 169$, which forces $3 \cdot 17 | s$, or $r = 81$, in which case $13 \cdot 41 | s$. If $q = 53$, then $r < 307$ and $3 \cdot 13 | rs$, so r is 243, 169, or 81; each of these possibilities forces s to be divisible by two distinct primes.

$p = 23$: $45 < q < 137$ and $2 \cdot 3 \cdot 5 | qrs$. The possible values for q are 128, 125, 81, and 64, but each of these forces rs to be divisible by three or more primes, a contradiction in any event.

$p = 19$: $75 < q < 227$ and $2 \cdot 3 \cdot 5 | qrs$. Thus, q is 128, 125, or 81. Each of these possibilities is ruled out since rs cannot be divisible by three primes.

$p = 17$: $135 < q < 407$ and $3 \cdot 5 | qrs$. To be within the interval, q cannot be a power of 5, and $q = 243$ forces $r < 611$ and $5 \cdot 61 | rs$, a contradiction. Therefore, q is a prime power between 135 and 407, congruent to 1 (mod 4), and such that $q + 1$ has no odd prime factor other than 3's, 5's, and at most one 17. There are but two possibilities: $q = 269$ and $q = 149$. If $q = 269$, then $r < 544$ and $3 \cdot 5 | rs$, a contradiction. If $q = 149$, then $1446 < r < 283$ and $3 \cdot 5 | rs$, so $r = 2187$, whence $5 \cdot 547 | s$, a contradiction.

$p = 16$: $255 < q < 765$ and $3 \cdot 5 \cdot 17 | qrs$, so q is 729, 625, or 289, each of which would require that rs be divisible by three distinct primes.

Finally, if $q < 16$, then $n/\sigma^*(n) < 15/16$.

3. TECHNIQUES FOR THEOREM 2

Suppose that n is unitary harmonic, i.e., that

$$H^*(n) = n\tau^*(n)/\sigma^*(n)$$

is an integer. Suppose also that $n \leqslant 10^6$ and that $2^a \| n$. Since $\tau^*(n)$ is a power of 2, any odd prime that divides $\sigma^*(n)$ must also divide n. For $a > 0$, $\sigma^*(2^a) = 1 + 2^a$, so $2^a \| n$ implies $2^a(1 + 2^a) | n$, and hence $a < 10$.

Except for $a = 0$, the supposition that $2^a \| n$ requires that n be divisible by the largest prime dividing $1 + 2^a$, and the restriction that $n \leqslant 10^6$ can be used to determine how many times this prime divides n. This gives rise to newly known unitary divisors of n, and therefore (usually) newly known odd primes dividing $\sigma^*(n)$ and hence n. The procedure is repeated until all the possibilities are exhausted.

No particular difficulty arises with this procedure, except when one runs out of primes with which to work, and then the procedure breaks down completely. In such a case we write $n = Nk$, where $N \| n$ and k is unknown. In light of Theorem 1, we may require $\omega(n) > 4$, which imposes a lower bound on $\omega(k)$; and $n \leqslant 10^6$ imposes an upper bound on k and hence on $\omega(k)$. There are also divisibility restrictions on k and $\sigma^*(k)$ from N and $\sigma^*(N)$. See the $2 \cdot 3^2 5 \| n$ and $2 \cdot 3 \cdot 7 \| n$ cases in the discussion below.

The n odd ($a = 0$) case of Theorem 2 is somewhat easier to handle than the others since $p^b \| n$ implies $p^b \equiv 1 \pmod 4$ in order to avoid having too many 2's in the denominator of $H^*(n)$.

We present here the $a = 1$ (i.e., $2n$) case of Theorem 2:

Immediate size contradictions result if $3^{12} | n$ or if $3^b \| n$ for $6 \leqslant b \leqslant 11$. If $3^3 \| n$, then $61 | n$, so either $61^2 | n$ or $61 \| n$, in which case $31 | n$; both possibilities make $n > 10^6$.

If $3^4 \| n$, then $41 | n$. If $41^3 | n$ or $41^2 \| n$, then $n > 10^6$, so $41 \| n$. Then $7 | n$, and $n > 10^6$ if $7^3 | n$ or $7^2 \| n$. If $n = 2 \cdot 3 \cdot 7 \cdot 41k$, then $1 < k \leqslant 21$, $(2 \cdot 3 \cdot 7,\ k) = 1$ and $\sigma^*(k)$ 18, so k is 5 or 17. Thus we have located two unitary harmonic numbers:

$$H^*(2 \cdot 3^4 5 \cdot 7 \cdot 41) = 15,$$

$$H^*(2 \cdot 3^4 7 \cdot 17 \cdot 41) = 17.$$

If $3^3 \| n$, then $7 | n$. Size contradictions easily result if $7^6 | n$ or $7^5 \| n$ or $7^4 \| n$ or $7^3 \| n$. If $7^2 \| n$, then $5^2 | n$, and $n > 10^6$ if $5^4 | n$. If $5^3 \| n$, then $n = 2 \cdot 3^3 5^3 7^2$ since $n < 10^6$, but $\omega(n) = 4$. Therefore, $5^2 \| n$, so $13 | n$ and hence $13 \| n$, and another unitary harmonic number is found:

$$H^*(2 \cdot 3^3 5^2 7^2 13) = 18.$$

If $3^3 7 \| n$, then $n = 2 \cdot 3^3 7k$. It follows that $H^*(n) = 9H^*(k)/2$. But $H^*(k)$ does not have an even numerator after reduction, so $H^*(n)$ is not an integer.

If $3^2 \| n$, then $5 | n$. Size contradictions occur if $5^7 | n$ or $5^6 \| n$ or $5^4 \| n$, while there are too many 3's in the denominator of $H^*(n)$ if $5^5 \| n$ or $5^3 \| n$. Therefore, $5^2 \| n$ or $5 \| n$.

If $3^2 5^2 \| n$, then $13 | n$, and $n > 10^6$ if $13^4 | n$ or $13^3 \| n$ or $13^2 \| n$. Thus, $13 \| n$, so $7 | n$, but $n > 10^6$ if $7^3 | n$, and if $7^2 \| n$ there are too many 5's in the denominator of $H^*(n)$, so $7 \| n$. Therefore,

$$n = 2 \cdot 3^2 5^2 7 \cdot 13 \cdot k,$$

where $k \leqslant 24$, $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 13,\ k) = 1$ and $\sigma^*(k) | 30$. This locates another unitary harmonic number:

$$H^*(2 \cdot 3^2 5^2 7 \cdot 13) = 15.$$

If $2 \cdot 3^2 5 \| n$, then $n = 2 \cdot 3^2 5 \cdot k$ with $(2 \cdot 3 \cdot 5, k) = 1$, $k \leqslant 11{,}111$ and $(\sigma^*(k), 3 \cdot 5) = 1$, so k is composed of prime powers from the set

$$\{7, 13, 31, 37, 43, 61, 67, 73, 97, 103, 121, \ldots\}.$$

Since $\omega(n) \geqslant 5$, $\omega(k) \geqslant 2$. However, $\omega(k) \leqslant 3$ since

$$7 \cdot 13 \cdot 31 \cdot 37 > 11{,}111.$$

If $\omega(k) = 3$, then the smallest prime dividing k is 7, since

$$13 \cdot 31 \cdot 37 > 11{,}111.$$

Also, $37 \nmid k$ or else $19 | k$, which is impossible if $k \leqslant 11{,}111$. Thus, the only possibility with $\omega(k) = 3$ is $k = 7 \cdot 13 \cdot 31$, which forces $H^*(n)$ to be nonintegral. If $\omega(k) = 2$, then write $n = 2 \cdot 3^2 5 \cdot p \cdot q$. Now, $p < 103$, since $103 \cdot 121 > 11{,}111$ and $\sigma^*(q) | 16p$, so the only possibility is $p = 7$ and $q = 13$, and another unitary harmonic number is found:

$$H^*(2 \cdot 3^2 5 \cdot 7 \cdot 13) = 13.$$

If $3 \| n$, then $n = 2 \cdot 3 \cdot k$ with $k \leqslant 166{,}666$, $(2 \cdot 3, k) = 1$, $(\sigma^*(k), 3) = 1$ and $\omega(k) \geqslant 3$. But $\omega(k) \leqslant 4$, since

$$7 \cdot 13 \cdot 19 \cdot 25 \cdot 31 > 166{,}666.$$

If $\omega(k) = 4$, the smallest possible next prime power is 7, since

$$13 \cdot 19 \cdot 25 \cdot 31 > 166{,}666.$$

But if $3 \cdot 7 \| n$, then $H^*(n)$ has at least one excess 2 in its denominator. Therefore, $\omega(k) = 3$, so let $k = pqr$ with $p < q < r$. Now, $p < 49$, since $49 \cdot 61 \cdot 67 > 166{,}666$. We have the following possibilities:

$p = 43$ forces $11 | n$. But $11 \nmid n$, so $n > 2 \cdot 3 \cdot 7^2 11^2 43 > 10^6$.

$p = 37$ implies $19 | n$. But $19 \nmid n$, so $n > 2 \cdot 3 \cdot 19^2 37 \cdot 43 > 10^6$.

$p = 31$ leaves extra 2's in the denominator of $H^*(n)$.

$p = 25$ requires $13 | n$, but $13 \nmid n$. If $13^4 | n$, then $n > 10^6$, and the same is true if $13^3 \| n$, because then $157 | n$. Then $13^2 \| n$, so $17 | n$ and $17 \| n$, so $n > 2 \cdot 3 \cdot 5^2 13^2 17^2 > 10^6$.

$p = 19$ forces $5 | n$, but $5 \nmid n$. But $n > 10^6$ if $5^6 | n$ or $5^4 \| n$, and there are extra 3's in the denominator of $H^*(n)$ if $5^5 \| n$ or $5^3 \| n$. Therefore, $5^2 \| n$, so $13 | n$ and $13 \nmid n$, but $n > 10^6$ if $13^3 | n$, and hence $13^2 \| n$, whence $17 | n$ and $n > 10^6$.

$p = 13$ requires $7|n$, and $7\nmid n$. If $7^5|n$ or $7^4\|n$ or $7^3\|n$, then $n > 10^6$. Thus, $7^2\|n$, so $5^2|n$. Then $n = 2 \cdot 3 \cdot 5^2 7^2 13 \cdot k$ with $k \leqslant 10$. The only value of k that checks out is $k = 1$:

$$H^*(2 \cdot 3 \cdot 5^2 7^2 13) = 14.$$

$p = 7$ leaves extra 2's in the denominator of $H^*(n)$.

Since $2\|n$, $3|n$ and the $3\nmid n$ subcase is eliminated. Thus, the $2\|n$ case of the theorem is proved.

4. LARGE INTEGRAL VALUES OF $H^*(n)$

It is not at all hard to construct n with $H^*(n)$ a large integer. For example, one may start with the fifth unitary perfect number [5],

$$2^{18} 3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

and have $H^*(n) = 2^{11} = 2048$. However, substituting for various blocks of unitary divisors yields the related number

$$n = 2^{18} 3^4 5^4 7^4 11^2 13^2 17 \cdot 19^2 31 \cdot 37 \cdot 41 \cdot 43 \cdot 61 \cdot 79 \cdot 109 \cdot 157 \cdot 181 \cdot 313 \cdot 601 \cdot 1201,$$

for which $H^*(n) = 2^{11} 3 \cdot 7 \cdot 19 = 817{,}152$.

The author conjectures that there are infinitely many unitary harmonic numbers, including infinitely many odd ones, but that there are only finitely many unitary harmonic numbers with $\omega(n)$ fixed.

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A GENERALIZATION OF EULER'S Φ-FUNCTION

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Euler's ϕ-function, $\phi(n)$, denotes the number of positive integers less than n and relatively prime to it. There are many generalizations of this famous function; for example, see [1; 2; 3]. In this note, we extend the ϕ-function to an arithmetic progression

$$D(s, d, n) = \{s, s + d, \ldots, s + (n - 1)d\},$$

where $(s, d) = 1$. A formula will be established giving the number of elements in $D(s, d, n)$ that are relatively prime to n. Observe that $\phi(n)$ is the number of elements in the progression $D(1, 1, n)$ that are relatively prime to n.

Before we establish the formula, we begin with some preliminary remarks. Let

$$P(x, d, n) = \{x, x + d, \ldots, x + (n - 1)d\}$$

be an arbitrary progression of nonnegative integers. Note that if $(x, d) = 1$, then $P(x, d, n) = D(x, d, n)$.

Lemma 1

Let $P(x, d, n)$ be an arbitrary progression with $(d, n) = g$. Suppose that $n = gk$ and $d = gk_1$. Then no two elements in each of the g blocks of k consecutive elements are congruent (mod n). Furthermore, every block contains the same residues (mod n).

Proof: $x + rd \equiv x + td \pmod{n}$ if and only if $r \equiv t \pmod{k}$.

Definition: Let $\phi(s, d, n)$ denote the number of elements in the arithmetic progression $D(s, d, n)$ that are relatively prime to n.

Remark: $\phi(1, 1, n) = \phi(n) = \phi(s, 1, n)$.

Theorem 1

Suppose $(m, n) = 1$. Then

$$\phi(s, d, mn) = \phi(s, d, m)\phi(s, d, n).$$

Proof: Write the elements of $D(s, d, mn)$ as follows:

$$\begin{array}{lllll}
s & s + d & s + 2d & \ldots & s + (m - 1)d \\
s + md & s + (m + 1)d & s + (m + 2)d & \ldots & s + (2m - 1)d \\
\vdots & & & & \\
s + (n - 1)md & & & \ldots & s + (nm - 1)d.
\end{array}$$

Since the elements in the first row are elements of the progression $D(s, d, m)$, the number of elements in it that are relatively prime to m is $\phi(s, d, m)$. Let C_i denote the column headed by $s + id$. If $(s + id, m) > 1$, no element of C_i is relatively prime to m. If $(s + id, m) = 1$, every elements of C_i is prime to m. So to complete the proof, we need to show that $\phi(s, d, n)$ of the elements in each column of C_i are prime to n.

Let $(d, n) = g$. Since $(m, n) = 1$, it follows that $(md, n) = g$, and by Lemma 1, there are g blocks of k consecutive elements in which no two of them are congruent $\pmod{n}$. Thus, all we need to show is that each element in the first block of C_i is congruent modulo n to an element in the first block of $D(s, d, n)$. This would imply that there are $\phi(s, d, n)$ elements in C_i that are relatively prime to n.

Suppose $(s + id) + jmd$, $0 \leqslant j \leqslant k - 1$, is an arbitrary element in the first block of C_i. Then there is an integer q such that

$$(i + jm) = qk + r, \quad 0 \leqslant r < k.$$

Thus

$$(s + id) + jmd \equiv s + rd \pmod{n},$$

where $s + rd$ is an element of $D(s, d, k)$.

Lemma 2

Let p be a prime and k a positive integer. Then

$$\phi(s, d, p^k) = \begin{cases} p^k\left(1 - \dfrac{1}{p}\right), & \text{if } p \nmid d, \\ p^k, & \text{if } p \mid d. \end{cases}$$

Proof: If $p \mid d$, then $(s, d) = 1$ implies that $(s + id, p^k) = 1$ and hence every element in $D(s, d, p^k)$ is relatively prime to p. If $p \nmid d$, then all p-consecutive elements in $D(s, d, p^k)$ form a complete residue system $\pmod{p}$. Thus, each has $(p - 1)$ elements relatively prime to p. Since there are p^{k-1} blocks of p-consecutive elements in $D(s, d, p^k)$, it follows that

$$\phi(s, d, p^k) = p^{k-1}(p - 1) = p^k\left(1 - \frac{1}{p}\right), \text{ if } p \nmid d.$$

Now combining Theorem 1 and Lemma 2, we have a formula for $\phi(s, d, n)$.

Theorem 2

Let $D(s, d, n)$ be an arithmetic progression with $n = p_1^{a_1} p_2^{a_2} \ldots p_j^{a_j}$. Then, for $n > 1$,

$$\phi(s, d, n) = \begin{cases} n, \text{ if } p_i \mid d \text{ for all } i, \\ n\Pi\left(1 - \dfrac{1}{p_i}\right) \text{ for all } p_i \nmid d. \end{cases}$$

Remark: $\phi(s, d, n)$ is independent of the first element in the progression $D(s, d, n)$.

The following corollaries are immediate.

Corollary 1

$$\phi(n) = \phi(1, 1, n) = n\Pi\left(1 - \frac{1}{p}\right).$$

Corollary 2

If $(n, d) = 1$, then $\phi(s, d, n) = \phi(n)$.

Corollary 3

Let a and b be any two positive integers. Then

$$\phi(ab) = \phi(a)\phi(s, a, b) = \phi(b)\phi(s, b, a).$$

Now we return to the arbitrary progression $P(x, d, n)$. Let $\Phi(x, d, n)$ denote the number of elements in $P(x, d, n)$ that are relatively prime to n. The proof of the following result is immediate.

Theorem 3

Suppose $P(x, d, n)$ is an arbitrary progression with $(x, d) = g$. Then

(i) If $(g, n) \neq 1$, then $\Phi(x, d, n) = 0$,

(ii) If $(g, n) = 1$, then $\Phi(x, d, n) = \Phi\left(\dfrac{x}{g}, \dfrac{d}{g}, n\right)$.

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HARMONIC SUMS AND THE ZETA FUNCTION

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1. SUMMARY

Consider the harmonic sequence

$$H_n = \sum_{k=1}^{n} k^{-1},\ n \geqslant 1,$$

and the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s},\ \operatorname{Re}(s) > 1.$$

Recently, Bruckman [2] proposed the problem of showing

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3).$$

See also Klamkin [3] and Steinberg [4]. Presently, we establish the following generalization.

Theorem

Let H_n and $\zeta(s)$ be as above. Then

$$\text{(i)}\quad \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+1}} = \frac{1}{2}\sum_{j=2}^{2n}(-1)^j\zeta(j)\zeta(2n+2-j),\ n \geqslant 1,$$

and

$$\text{(ii)}\quad \sum_{k=1}^{\infty} \frac{H_k}{k^n} = \left(1+\frac{n}{2}\right)\zeta(n+1) - \frac{1}{2}\sum_{j=2}^{n-1}\zeta(j)\zeta(n+1-j),\ n \geqslant 2.$$

Here and in the sequel, as usual,

$$\sum_{j=j_0}^{n} c_j = 0 \text{ if } n < j_0.$$

The series which will be manipulated are readily shown to be absolutely convergent, so that summation signs may be reversed.

The proof of the theorem will be given in Section 2 after some auxiliary results have been derived. Some further generalizations are given in Section 3, and an open problem is stated.

2. AUXILIARY RESULTS AND PROOF OF THE THEOREM

Define the generalized harmonic sequence

$$H_0^{(m)} = 0 \text{ and } H_n^{(m)} = \sum_{\ell=1}^{n} \ell^{-m}, \; m \geqslant 1, \; n \geqslant 1, \tag{2.1}$$

and set

$$\overline{H}_n^{(1)} = \gamma - H_n^{(1)} \quad \text{and} \quad \overline{H}_n^{(m)} = \zeta(m) - H_n^{(m)}, \; m \geqslant 2, \; n \geqslant 0, \tag{2.2}$$

where γ is Euler's constant. Note that

$$\sum_{\ell=1}^{N} \frac{n}{\ell(\ell+n)} = \sum_{\ell=1}^{N}\left(\frac{1}{\ell} - \frac{1}{\ell+n}\right) = H_N - \sum_{\ell=n+1}^{N+n} \frac{1}{\ell} = H_N - H_{N+n} + H_n$$

$$= H_n + (H_N - \log N) - [H_{N+n} - \log(N+n)] - \log\left(1 + \frac{n}{N}\right);$$

therefore, using the well-known limiting expression

$$\lim_{N\to\infty}(H_N - \log N) = \gamma, \tag{2.2a}$$

it follows that

$$H_n = H_n^{(1)} = \sum_{\ell=1}^{\infty} \frac{n}{\ell(\ell+n)}, \; n \geqslant 0; \tag{2.3}$$

it also follows from (2.1) and (2.2) that

$$\overline{H}_n^{(m)} = \sum_{\ell=1}^{\infty} (\ell+n)^{-m}, \; m \geqslant 2, \; n \geqslant 0. \tag{2.4}$$

Now define the sums

$$S_n^{(m)} = \sum_{k=1}^{\infty} \frac{H_k^{(m)}}{k^n}, \; m \geqslant 1, \; n \geqslant 2, \tag{2.5}$$

and

$$\overline{S}_n^{(m)} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^{(m)}}{k^n}, \; m \geqslant 2, \; n \geqslant 1, \tag{2.6}$$

which may be shown to exist. $S_n^{(1)}$ exists because $H_k = O(\log k)$ and

$$\sum_{k=1}^{\infty} \frac{\log k}{k^n}$$

exists for all $n \geqslant 2$. Also

$$\overline{S}_1^{(m)} = S_m^{(1)} - \zeta(m+1),$$

as will be shown in Lemma 2.1, so $\overline{S}_1^{(m)}$ exists for all $m \geqslant 2$. These sums are related to the zeta function as follows.

Lemma 2.1

Let $S_m^{(n)}$ and $\bar{S}_n^{(m)}$ be as in (2.5) and (2.6), respectively, and let $\zeta(\cdot)$ be the Riemann zeta function. Then

(i) $S_m^{(n)} = \bar{S}_n^{(m)} + \zeta(m+n)$, $m \geqslant 2$, $n \geqslant 1$,

and

(ii) $S_m^{(n)} + S_n^{(m)} = \zeta(m+n) + \zeta(m)\zeta(n)$, $m \geqslant 2$, $n \geqslant 2$.

Proof: (i) Clearly,

$$S_m^{(n)} = \sum_{k=1}^{\infty} \frac{H_k^{(n)}}{k^m} = \sum_{k=1}^{\infty} \frac{1}{k^{m+n}} + \sum_{k=1}^{\infty} \frac{H_{k-1}^{(n)}}{k^m}$$

$$= \zeta(m+n) + \sum_{k=1}^{\infty} \frac{H_k^{(n)}}{(k+1)^m}, \text{ by (2.5) and (2.1).}$$

Next,

$$\sum_{k=1}^{\infty} \frac{H_k^{(n)}}{(k+1)^m} = \sum_{k=1}^{\infty} (k+1)^{-m} \sum_{\ell=1}^{k} \ell^{-n}, \text{ by (2.1),}$$

$$= \sum_{\ell=1}^{\infty} \ell^{-n} \sum_{k=\ell}^{\infty} (k+1)^{-m}$$

$$= \sum_{\ell=1}^{\infty} \ell^{-n} \sum_{k=1}^{\infty} (k+\ell)^{-m}$$

$$= \bar{S}_n^{(m)}, \text{ by (2.4) and (2.6).}$$

The last two relations establish (i).

(ii) Relation (2.6) gives

$$\bar{S}_n^{(m)} = \zeta(m)\zeta(n) - S_n^{(m)}, \quad m \geqslant 2, \ n \geqslant 2,$$

by means of (2.2) and (2.5). This along with (i) establishes (ii).

Lemma 2.2

For each integer $m_1, m_2 \geqslant 1$, and $n_1 \neq n_2 \geqslant 0$, set

$$A_{1j} = A_{1j}(m_1, m_2, n_1, n_2)$$

$$= (-1)^{m_1+j} \binom{m_1 + m_2 - 1 - j}{m_2 - 1} (n_2 - n_1)^{-m_1-m_2+j},$$

and

$$A_{2j} = A_{2j}(m_1, m_2, n_1, n_2)$$
$$= (-1)^{m_2+j}\binom{m_1 + m_2 - 1 - j}{m_1 - 1}(n_1 - n_2)^{-m_1-m_2+j},$$

and let $\overline{H}_{n_1}^{(j)}$ and $\overline{H}_{n_2}^{(j)}$ be given by (2.2). Then

$$\sum_{k=1}^{\infty} \frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = \sum_{i=1}^{2}\sum_{j=1}^{m_i} A_{ij}\overline{H}_{n_i}^{(j)}.$$

Proof: Expanding $(k + n_1)^{-m_1}(k + n_2)^{-m_2}$ into partial fractions, we obtain (by residue theory or otherwise)

$$(k+n_1)^{-m_1}(k+n_2)^{-m_2} = \sum_{j=1}^{m_1}\frac{A_{1j}}{(k+n_1)^j} + \sum_{j=1}^{m_2}\frac{A_{2j}}{(k+n_2)^j} \tag{2.7}$$

with A_{1j} and A_{2j} as defined above. We see that $A_{21} = -A_{11}$. Then, summing in (2.7) over $k \geqslant 1$, and using (2.2) and (2.4), we obtain

$$\sum_{k=1}^{\infty}\frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = \sum_{k=1}^{\infty}\left\{\sum_{j=1}^{m_1}\frac{A_{1j}}{(k+n_1)^j} + \sum_{j=1}^{m_2}\frac{A_{2j}}{(k+n_2)^j}\right\}$$
$$= \sum_{k=1}^{\infty}\left\{\left(\frac{A_{11}}{k+n_1} + \frac{A_{21}}{k+n_2}\right) + \sum_{j=2}^{m_1}\frac{A_{1j}}{(k+n_1)^j}\right.$$
$$\left. + \sum_{j=2}^{m_2}\frac{A_{2j}}{(k+n_2)^j}\right\};$$

now

$$\sum_{k=1}^{\infty}\left(\frac{A_{11}}{k+n_1} + \frac{A_{21}}{k+n_2}\right) = A_{11}\sum_{k=1}^{\infty}\left(\frac{1}{k+n_1} - \frac{1}{k+n_2}\right)$$
$$= A_{11}\sum_{k=1+n_1}^{n_2}\frac{1}{k} \quad (\text{if } n_1 < n_2)$$
$$= A_{11}(H_{n_2} - H_{n_1}) = A_{11}(\overline{H}_{n_1}^{(1)} - \overline{H}_{n_2}^{(1)})$$
$$= A_{11}\overline{H}_{n_1}^{(1)} + A_{21}\overline{H}_{n_2}^{(1)}.$$

A similar conclusion follows if $n_1 \geqslant n_2$. Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{(k+n_1)^{m_1}(k+n_2)^{m_2}} = A_{11}\overline{H}_{n_1}^{(1)} + A_{21}\overline{H}_{n_2}^{(1)} + \sum_{j=2}^{m_1}A_{1j}\overline{H}_{n_1}^{(j)} + \sum_{j=2}^{m_2}A_{2j}\overline{H}_{n_2}^{(j)}$$

(continued)

$$= \sum_{i=1}^{2} \sum_{j=1}^{m_i} A_{ij} \overline{H}_{n_i}^{(j)},$$

which was to be shown.

Lemma 2.2 will be utilized to establish the following:

Lemma 2.3

Let $S_n^{(m)}$ and $\overline{S}_m^{(n)}$ be given by (2.5) and (2.6), respectively. Then

$$(-1)^{m+1}\overline{S}_m^{(n)} = \binom{m+n-2}{n-1} S_{m+n-1}^{(1)} - \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \overline{S}_{m+n-j}^{(j)}$$
$$- \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j)\zeta(m+n-j),$$
$$m \geqslant 1,\ n \geqslant 2.$$

Proof: We have

$$\overline{S}_m^{(n)} = \sum_{k=1}^{\infty} \frac{\overline{H}_k^{(n)}}{k^m} = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^m (k+\ell)^n}, \text{ by (2.4) and (2.6)},$$

$$= \sum_{\ell=1}^{\infty} \left\{ \sum_{j=1}^{m} A_{1j} \overline{H}_0^{(j)} + \sum_{j=1}^{n} A_{2j} \overline{H}_\ell^{(j)} \right\}, \text{ by Lemma 2,2,}$$

$$= \sum_{\ell=1}^{\infty} \left\{ A_{11} H_\ell^{(1)} + \sum_{j=2}^{m} A_{1j}\zeta(j) + \sum_{j=2}^{n} A_{2j} \overline{H}_\ell^{(j)} \right\}, \text{ by (2.1), (2.2) and } A_{21} = -A_{11},$$

$$= (-1)^{m+1} \binom{m+n-2}{n-1} \sum_{\ell=1}^{\infty} \frac{H_\ell^{(1)}}{\ell^{m+n-1}}$$
$$+ (-1)^m \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j) \sum_{\ell=1}^{\infty} \frac{1}{\ell^{m+n-j}}$$
$$+ (-1)^m \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \sum_{\ell=1}^{\infty} \frac{\overline{H}_\ell^{(j)}}{\ell^{m+n-j}}$$

$$= (-1)^{m+1} \binom{m+n-2}{n-1} S_{m+n-1}^{(1)}$$
$$+ (-1)^m \sum_{j=2}^{m} (-1)^j \binom{m+n-1-j}{n-1} \zeta(j)\zeta(m+n-j)$$

(continued)

$$+ (-1)^m \sum_{j=2}^{n} \binom{m+n-1-j}{m-1} \overline{S}_{m+n-j}^{(j)}, \text{ by (2.5) and (2.6)},$$

from which the lemma follows.

Proof of the Theorem

(i) Utilizing (2.3) and Lemma 2.2 with $m_1 = 2n$, $m_2 = 1$, $n_1 = 0$, and $n_2 = \ell$, we get

$$\sum_{k=1}^{\infty} \frac{H_k}{k^{2n+1}} = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{\ell=1}^{\infty} \frac{k}{\ell(k+\ell)} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{k=1}^{\infty} \frac{1}{k^{2n}(k+\ell)}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left\{ \sum_{j=1}^{2n} A_{1j} \overline{H}_0^{(j)} + A_{21} \overline{H}_\ell^{(1)} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left\{ \left(-\frac{\overline{H}_0^{(1)}}{\ell^{2n}} + \frac{\overline{H}_\ell^{(1)}}{\ell^{2n}} \right) + \sum_{j=2}^{2n} (-1)^j \frac{\overline{H}_0^{(j)}}{\ell^{2n+1-j}} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{-H_\ell^{(1)}}{\ell^{2n+1}} + \sum_{j=2}^{2n} (-1)^j \zeta(j) \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2n+2-j}}, \text{ by (2.1) and (2.2)},$$

$$= -\sum_{\ell=1}^{\infty} \frac{H_\ell^{(1)}}{\ell^{2n+1}} + \sum_{j=2}^{2n} (-1)^j \zeta(j)\zeta(2n+2-j),$$

from which (i) follows.

(ii) Setting $m = 1$ in Lemma 2.3, we get

$$\overline{S}_1^{(n)} = S_n^{(1)} - \sum_{j=2}^{n} \overline{S}_{n+1-j}^{(j)}, \; n \geqslant 2,$$

and from Lemma 2.1(i) we have

$$\overline{S}_{n+1-j}^{(j)} = S_j^{(n+1-j)} - \zeta(n+1), \; j \geqslant 2, \; n \geqslant 2.$$

In particular,

$$\overline{S}_1^{(n)} = S_n^{(1)} - \zeta(n+1), \; n \geqslant 2.$$

It follows that

$$\zeta(n+1) = \sum_{j=2}^{n} \overline{S}_{n+1-j}^{(j)} = \sum_{j=2}^{n} \left\{ S_j^{(n+1-j)} - \zeta(n+1) \right\}, \; n \geqslant 2,$$

or, equivalently,

$$S_n^{(1)} = n\zeta(n+1) - \sum_{j=2}^{n-1} S_j^{(n+1-j)}, \; n \geqslant 2. \tag{2.8}$$

Next, Lemma 2.1(ii) gives

$$S_j^{(n+1-j)} + S_{n+1-j}^{(j)} = \zeta(n+1) + \zeta(j)\zeta(n+1-j),\ j \geqslant 2,\ n \geqslant 3,$$

so that (by a change in variable from j to $n+1-j$)

$$\begin{aligned} 2\sum_{j=2}^{n-1} S_j^{(n+1-j)} &= \sum_{j=2}^{n-1}\left\{S_j^{(n+1-j)} + S_{n+1-j}^{(j)}\right\} \\ &= (n-2)\zeta(n+1) + \sum_{j=2}^{n-1}\zeta(j)\zeta(n+1-j),\ n \geqslant 2. \end{aligned} \tag{2.9}$$

Relations (2.8) and (2.9), along with (2.1) and (2.5), establish (ii).

As a byproduct of the theorem, we get the following interesting result, if we replace n by $2n+1$ in (ii) of the theorem, eliminate the series, then replace $n+1$ by n.

Corollary

$$\zeta(2n) = \frac{2}{2n+1}\sum_{j=1}^{n-1}\zeta(2j)\zeta(2n-2j),\ n \geqslant 2.$$

Remark: Taking into account that

$$\zeta(2n) = (-1)^{n-1}2^{2n-1}\pi^{2n}[(2n)!]^{-1}B_{2n},\ n \geqslant 1,$$

from [1], where B_n are the Bernoulli numbers, the above relation becomes

$$B_{2n} = -\frac{1}{2n+1}\sum_{j=1}^{n-1}\binom{2n}{2j}B_{2j}B_{2n-2j},\ n \geqslant 2.$$

3. FURTHER GENERALIZATIONS

In this section, we give the following additional results, which express generalized harmonic sums in terms of the zeta function.

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^{2n+1}} &= \zeta(2)\zeta(2n+1) - \frac{(n+2)(2n+1)}{2}\zeta(2n+3) \\ &\quad + 2\sum_{j=2}^{n+1}(j-1)\zeta(2j-1)\zeta(2n+4-2j),\ n \geqslant 1. \end{aligned} \tag{3.1}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k^n} = \frac{1}{2}[\zeta(2n) + \zeta(n)\zeta(n)],\ n \geqslant 2. \tag{3.2}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4} = -\frac{1}{3}\zeta(6) + \zeta(3)\zeta(3). \tag{3.3a}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^4} = 18\zeta(7) - 10\zeta(2)\zeta(5). \tag{3.3b}$$

Relation (3.1) follows from Lemma 2.3 (by setting $n = 2$ and replacing m by $2m + 1$), Lemma 2.1, and part (ii) of the theorem. Relation (3.2) follows immediately from Lemma 2.1(ii) by setting $m = n$. Finally, relations (3.3a) and (3.3b) can be derived from Lemma 2.3 by setting the appropriate values of m and n. We also note that the sum

$$\sum_{k=1}^{\infty} \frac{H_k^{(2\ell+1-n)}}{k^n} \quad \left(n \geqslant 5,\ \ell \geqslant \left[\frac{n+1}{2}\right]\right)$$

may be obtained from Lemma 2.3 by means of some algebra that becomes progressively cumbersome with increasing n.

It is still an open question to give a closed form of

$$\sum_{k=1}^{\infty} \frac{H_k^{(m)}}{k^n}$$

for any integers $m \geqslant 1$ and $n \geqslant 2$ in terms of the zeta function.

ACKNOWLEDGMENT

The authors express their thanks to the referee for his useful comments.

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INJECTIVITY OF EXTENDED GENERALIZED FIBONACCI SEQUENCES

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(Submitted December 1981)

INTRODUCTION

Let N denote the set of positive integers, and Z the set of all integers. The function $F: Z \to Z$ with $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-2) + F(n-1)$ for every $n \in Z$, constitutes the extention to the left of the original Fibonacci sequence, where the domain is restricted to N. With the arguments written as subscripts, the following table gives the "middle" section of this extended two-sided sequence:

...	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	...
	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	

Similarly, one obtains the extended Lucas sequence as $L: Z \to Z$ with $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-2} + L_{n-1}$ for every $n \in Z$:

...	L_{-7}	L_{-6}	L_{-5}	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	...
	-29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	

In general, $H: Z \to Z$ with $H_1 = a \in Z$, $H_2 = b \in Z$, and $H_n = H_{n-2} + H_{n-1}$ for every $n \in Z$, constitutes the extended generalized Fibonacci sequence generated by the ordered pair of integers (a, b):

...	H_{-4}	H_{-3}	H_{-2}	H_{-1}	H_0	H_1	H_2	H_3	H_4	...
	$-8a + 5b$	$5a - 3b$	$-3a + 2b$	$2a - b$	$-a + b$	a	b	$a + b$	$a + 2b$	

The functions F and L as defined above are not injective; i.e., there are different arguments having the same values, or, in the terminology of sequences, some terms with different indices are equal, or, simpler still,

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The authors of this paper cooperated as faculty advisor and student participant, respectively, in the NSF-supported Undergraduate Research Participation project in mathematics at the University of Santa Clara, Summer 1981, Grant SPI-8025433.

some terms occur more than once. An extended generalized Fibonacci sequence generated by (a, b) will be called *injective* or *noninjective* according as the function H is injective or not.

The problem posed and solved in this paper is:

> *What are the necessary and sufficient conditions for* a *and* b *to generate an* injective *extended generalized Fibonacci sequence?*

For the sake of brevity, in this paper the word "sequence" stands for an "extended generalized Fibonacci sequence." In general such a sequence will be denoted by H with the values of H_1 and H_2 given, while F is short for H with $H_1 = H_2 = 1$ and L is short for H with $H_1 = 1$, $H_2 = 3$. (Since F commemorates Fibonacci and L commemorates Lucas, perhaps H might commemorate Hoggatt.)

In Section 1, it is proved that injective sequences do *exist*, which makes the research meaningful.

In Section 2, the problem is reduced to the investigation of a certain subset of the set of all sequences, a subset which *represents* all "candidates" for injectivity.

In Section 3, the *solution* is given.

Without further reference, some well-known properties will be used, e.g.:

$H_n = aF_{n-2} + bF_{n-1}$ for every $n \in Z$, where $a = H_1$ and $b = H_2$;

$L_n = F_{n-1} + F_{n+1}$ for every $n \in Z$;

$F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ for every $n \in N \cup \{0\}$;

and finally, if two of any three consecutive terms of a sequence are known, then the whole sequence is known.

For these properties, see, e.g., [3; 5; 2].

1. EXISTENCE

The ordered pair (1, 1) generates F, but so does the ordered pair (1, 2), and any ordered pair of consecutive terms of F. The generated sequences are identical, the order-preserving shift of the indices is irrelevant. The pair (1, 3) generates L, and continuing along this line, one might consider the sequences generated by (1, 4), (1, 5), (1, 6), ... The following table shows that the first four sequences are noninjective, but from there on there seem to be candidates for injectivity, a conjecture strengthened by the use of a computer.

	... H_{-7}	H_{-6}	H_{-5}	H_{-4}	H_{-3}	H_{-2}	H_{-1}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7 ...
F	... 13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13 ...
L	... -29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29 ...
(1, 4)	... -50	31	-19	12	-7	5	-2	3	1	4	5	9	14	23	37 ...
(1, 5)	... -71	44	-27	17	-10	7	-3	4	1	5	6	11	17	28	45 ...
(1, 6)	... -92	57	-35	22	-13	9	-4	5	1	6	7	13	20	33	53 ...
(1, 7)	... -113	70	-43	27	-16	11	-5	6	1	7	8	15	23	38	61 ...
(1, 8)	... -134	83	-51	32	-19	13	-6	7	1	8	9	17	26	43	69 ...

Further inspection of the table suggests the following lemma.

Lemma 1

If $H_1 = 1$ and $H_2 = b \geqslant 3$, then $H_n = L_n + (b - 3)F_{n-1}$ for every $n \in Z$.

Proof: Every $n \in Z$ yields the identity

$$\begin{aligned} H_n &= F_{n-2} + bF_{n-1} \\ &= F_{n-2} + 3F_{n-1} + (b - 3)F_{n-1} \\ &= F_n + 2F_{n-1} + (b - 3)F_{n-1} \\ &= F_{n+1} + F_{n-1} + (b - 3)F_{n-1} \\ &= L_n + (b - 3)F_{n-1}. \end{aligned}$$

Another fact revealed by the table is the importance of the terms with even negative index. These are the terms that might be equal to terms with positive index.

Lemma 2

If $H_1 = 1$ and $H_2 = b \geqslant 3$, then $H_{-n} = H_n + (b - 3)F_n$ for every even $n \in Z$.

Proof: Let $n \in N$ be even. By Lemma 1, $H_{-n} = L_{-n} + (b - 3)F_{-n-1}$. Since n is even, $L_{-n} = L_n$, and $-n - 1$ is odd, so that $F_{-n-1} = F_{n+1}$. Hence

$$H_{-n} = L_n + (b - 3)F_{n+1}$$

or

$$H_{-n} = L_n + (b - 3)F_{n-1} + (b - 3)F_n.$$

Hence, by Lemma 1,

$$H_{-n} = H_n + (b - 3)F_n.$$

Since the sequence H generated by (1, 6) is the first candidate for being injective, a close inspection of this sequence is helpful. It is obvious that the sequence consists exhaustively of three one-sided sequences:

(i) the sequence H_n, $n \in N$, the strictly increasing sequence

$$1, 6, 7, 13, \ldots;$$

(ii) the sequence H_{-2n}, $n \in N \cup \{0\}$, the strictly increasing sequence

$$5, 9, 22, 57, \ldots;$$

(iii) the sequence $H_{-(2n-1)}$, $n \in N$, the strictly decreasing sequence

$$-4, -13, -35, -92, \ldots .$$

The only possibility for H to be noninjective is that the sequences (i) and (ii) have a common term.

Theorem 1

If $H_1 = 1$ and $H_2 = 6$, then H is injective.

Proof: Assume that H is noninjective. Then, by the introductory remarks above and by Lemma 2, there are $n \in N$ and $p \in N$, where $0 < n < p$ and n even, such that $H_{-n} = H_p$ with $H_p = H_n + 3F_n$. Since

$$H_p = F_{p-2} + 6F_{p-1} \quad \text{and} \quad H_n = F_{n-2} + 6F_{n-1},$$

one obtains

$$F_{n-2} + 6F_{n-1} + 3F_n = F_{p-2} + 6F_{p-1},$$

and hence

$$5F_{n-1} + 4F_n = F_p + 5F_{p-1},$$

which yields

$$4F_{n+1} + F_{n-1} = 4F_{p-1} + F_{p+1},$$

which gives

$$3F_{n+1} + L_n = 3F_{p-1} + L_p,$$

which finally results in

$$3(F_{n+1} - F_{p-1}) = L_p - L_n. \tag{1}$$

Since $0 < n < p$, one obtains $L_p > L_n$ and $L_p - L_n$ is positive. Therefore, $F_{n+1} - F_{p-1}$ is also positive, and hence $F_{n+1} > F_{p-1}$ and $n + 1 > p - 1$ or $p < n + 2$, which combined with $n < p$ yields $p = n + 1$. Rewriting the identity (1) with $n + 1$ for p, one obtains

$$3(F_{n+1} - F_n) = L_{n+1} - L_n,$$

or

$$3F_{n-1} = L_{n-1}. \tag{2}$$

Since $n \geqslant 2$ and $n = 2$ gives the contradiction $3F_1 = L_1$ or $3 = 1$, one obtains $n > 2$.

Next, it is proved by induction on n that $L_n < 3F_n$ for *every* $n > 2$.

Base step: $n = 3$ yields $L_3 = 4 < 6 = 3F_3$.

Induction step: Assume $L_k < 3F_k$ for every $k \in N$, $3 \leqslant k < m$, $m \in N$. Then $L_{m-2} + L_{m-1} < 3F_{m-2} + 3F_{m-1}$, by assumption; or $L_m < 3F_m$. Hence, by induction, $L_n < 3F_n$ for every $n \in N$, $n > 2$. Thus, for every even $n > 2$, certainly $L_{n-1} < 3F_{n-1}$, contrary to (2). Hence, by *reductio ad absurdum*, H is injective.

Corollary

If $H_1 = 1$ and $H_2 = b \geqslant 6$, then H is injective.

Proof: Assuming again that H is noninjective, one obtains the identity:

$$(b - 3)(F_{n+1} - F_{p-1}) = L_p - L_n, \tag{3}$$

which again yields $p = n + 1$, because $(b - 3) > 0$. Substituting $n + 1$ for p in (3), one obtains

$$(b - 3)F_{n-1} = L_{n-1}. \tag{4}$$

Again, $n = 2$ is contradictory, and for $n > 2$, the proof of Theorem 1 arrived at $L_{n-1} < 3F_{n-1}$, and therefore, since $b \geqslant 6$, certainly

$$L_{n-1} < (b - 3)F_{n-1},$$

contrary to (4).

2. REPRESENTATION

The search for the necessary and sufficient conditions for (a, b) to generate an injective sequence is simplified in two ways: (i) by elimination of classes of sequences which are obviously or can be proved to be noninjective; (ii) by representation of the remaining set of sequences by a proper subset so that the investigation may be restricted to that subset.

Trivially, all sequences generated by a pair of equal integers are noninjective. Moreover, if $a \neq b$, but either $a = 0$ or $b = 0$, then the sequences generated by (a, b) are noninjective; if $a = 0$, then $H_2 = H_3 = b$

and if $b = 0$, then $H_1 = a = H_3$. As a matter of fact, all sequences containing 0 can be discarded, since the successor and predecessor of that 0 are equal.

The sequence obtained from the sequence H by multiplication of all its terms by $c \in Z$ may be called a multiple of H and denoted by cH. If H is generated by (a, b), then cH is generated by (ca, cb). Clearly, cH is injective if and only if H is injective. If a and b are relatively prime, then the sequence generated by (a, b) represents all its multiples with respect to injectivity. This implies that the search can be restricted to generating pairs (a, b), where a and b are relatively prime. Moreover, the sequence $-H$, short for $(-1)H$, can be seen as the negative of H and clearly only one of H and $-H$ has to be considered.

As is well known (see [1], and also [4]), every sequence has two parts: a right-hand part where all the terms have the same sign (the *monotonic* portion) and a left-hand part where the signs of the terms alternate (the *alternating* portion). Let a sequence be called positive or negative according as the monotonic portion has positive or negative signs. Since any sequence can be generated by any successive pair of its terms, the search for injectivity can be restricted to pairs (a, b) where a and b have the same sign. Moreover, since a negative sequence is the negative of a positive sequence, a further restriction can be made to pairs (a, b) where a and b are both positive. In a positive sequence there is a *last* alternating pair; namely, the pair (H_{i-2}, H_{i-1}), $i \in Z$, where $H_{i-2} < 0$, $H_{i-1} > 0$, and $H_i > 0$ (in general, $H_i = 0$ is possible, as in F, but these sequences have already been discarded as noninjective). If the pair (H_{i-2}, H_{i-1}) is the last alternating pair of the sequence, the pair (H_i, H_{i+1}) may be called the *characteristic pair* of the sequence. It is the *unique* pair of successive terms of a positive sequence such that:

(i) $H_i > 0$ is the smallest term of the monotonic portion of the sequence;
(ii) H_i is the only term of the monotonic portion that is smaller than its predecessor;
(iii) H_i is the unique term of the monotonic portion that is smaller than half its successor.

As to (i), $H_i < H_{i-1}$ because $H_i - H_{i-1} = H_{i-2} < 0$, and $H_i < H_j$, for every $j > i$, because $H_j = H_i$ + one or more positive numbers. As to (ii), if there is another term in the monotonic portion smaller than its predecessor, say H_k, then $k > i$, since $k = i - 1$ does not qualify; but then $H_{k-2} = H_k - H_{k-1}$ would be negative and (H_{i-2}, H_{i-1}) would not be the last alternating pair. As to (iii), in general, for every $m \in Z$, $2H_m < H_{m+1}$ if and only if $H_m < H_{m-1}$. The argument is as follows: since

$$2H_n = 2aF_{n-2} + 2bF_{n-1} \quad \text{and} \quad H_{n+1} = aF_{n-1} + bF_n,$$

one obtains $2H_n < H_{n+1}$ if and only if

$$2aF_{n-2} + 2bF_{n-1} < aF_{n-1} + bF_n,$$

which holds if and only if

$$aF_{n-2} + bF_{n-1} < a(F_{n-1} - F_{n-2}) + b(F_n - F_{n-1}),$$

which is the same as

$$H_n < aF_{n-3} + bF_{n-2} \quad \text{or} \quad H_n < H_{n-1}.$$

It follows that H_i is the unique term of the monotonic portion such that $2H_i < H_{i+1}$, since H_i is the only term of the monotonic portion smaller than its predecessor.

Since every positive sequence has a characteristic pair, this pair can be seen to generate the sequence, and the investigation may be restricted further to pairs (a, b) where $2a < b$.

Summarizing, the investigation may be restricted to ordered pairs of integers (a, b), where $a \neq b$, both $a > 0$ and $b > 0$, a and b relatively prime and, finally, $2a < b$.

3. CONCLUSIONS

The following lemma is strongly suggested, of course, by the table in Section 1.

Lemma 3

Let $H_1 = a$, $H_2 = b$, and $0 < 2a < b$. Then $H_{-n} > 0$ for every even $n \in N$ and $H_{-n} < 0$ for every odd $n \in N$.

Proof: By induction on n.

Base step: $H_{-1} = 2a - b = -(b - 2a) < 0$;

$$H_{-2} = 2b - 3a = (b - 2a) + (b - a) > 0.$$

Induction step: Assume the lemma holds for all $k < m$, $k \in N$, $m \in N$, $m > 2$. If m is odd, then $m - 1$ is even and $m - 2$ odd; hence, by assumption, $H_{-(m-1)} > 0$ and $H_{-(m-2)} < 0$, or, $H_{-m+1} > 0$ and $H_{-m+2} < 0$. Since $H_{-m} = H_{-m+2} - H_{-m+1}$, it follows that $H_{-m} < 0$. Similarly, if m is even, $H_{-m+1} < 0$ and $H_{-m+2} > 0$, so that $H_{-m} = H_{-m+2} - H_{-m+1}$ yields $H_{-m} > 0$.

Conclusion: The lemma holds for every $n \in N$, by induction.

Corollary

Let H be as in Lemma 3. Then H is noninjective if and only if there exist $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$.

Proof: Consider the set $\{H_m : m \in Z\}$ and its subsets $A = \{H_m : m > 0\}$, $B = \{H_m : m < 0,\ m \text{ odd}\}$, and $C = \{H_m : m \leqslant 0,\ m \text{ even}\}$. It can be readily shown, by considerations similar to those preceding Theorem 1, that within each of A, B, and C, one has $H_i = H_j$ if and only if $i = j$. Since A and B are clearly disjoint, and B and C also, for H to be noninjective, it is necessary that $A \cap C \neq \emptyset$. Since $H_0 = b - a$ is clearly not in A, it is then necessary that there exist an even $n \in N$ and $p \in N$ such that $H_{-n} = H_p$. Obviously, $H_{-n} = H_p$, $n \in N$, n even, $p \in N$, is sufficient for H to be noninjective.

Theorem 2

Let $H_1 = a$, $H_2 = b$, and $0 < 2a < b$. Moreover, let $H_{-n} = H_p$ for some even $n \in N$ and $p \in N$. Then $n - 2 < p < n + 2$.

Proof: If $H_{-n} = H_p$, then

$$aF_{-n-2} + bF_{-n-1} = aF_{p-2} + bF_{p-1},$$

or, since $n + 2$ is even and $n + 1$ is odd,

$$bF_{n+1} = a(F_{p-2} + F_{n+2}) + bF_{p-1}.$$

Further, $n > 0$ yields $F_{n+2} > 0$, and $p > 0$ yields $F_{p-2} \geqslant 0$, so that

$$a(F_{p-2} + F_{n+2}) > 0,$$

and hence

$$bF_{n+1} > bF_{p-1} \quad \text{or} \quad F_{n+1} > F_{p-1}.$$

Thus $n + 1 > p - 1$ or $p < n + 2$. On the other hand,

$$aF_{-n-2} + bF_{-n-1} = aF_{p-2} + bF_{p-1}$$

yields

$$b(F_{n+1} - F_{p-1}) = a(F_{p-2} + F_{n+2})$$

and, since $2a < b$, one obtains

$$a(F_{p-2} + F_{n+2}) > 2a(F_{n+1} - F_{p-1}).$$

Hence,

$$a(F_{p-2} + F_{p-1}) + aF_{n+2} + aF_{p-1} > 2aF_{n+1},$$

which yields

$$a(F_p + F_{p-1}) + aF_{n+2} > 2aF_{n+1}$$

or

$$aF_{p+1} + aF_{n+2} > 2aF_{n+1},$$

or

$$a(F_{n+2} - F_{n+1}) - aF_{n+1} > -aF_{p+1},$$

or

$$aF_n - aF_{n+1} > -aF_{p+1},$$

or

$$-aF_{n-1} > -aF_{p+1},$$

or

$$F_{n-1} < F_{p+1},$$

which in its turn yields $n - 1 < p + 1$ or $n - 2 < p$.

Remark: The next proof uses a little lemma of number theory which can be formulated as follows: let k and m be fixed positive integers generating the set of pairs $\{\{k, m+sk\} : s \in Z\}$; if one of these pairs is relatively prime, then each of them is. (The proof can be seen readily by consideration of the contrapositive: if one of the pairs has a common factor other than 1, then each pair has.) This lemma is applied four times in the proof of the next theorem, where a and b are fixed positive integers which are relatively prime:

1. If b is odd, then $2a$ and b are relatively prime also and hence, by the lemma, so are $2a$ and $b - 2a$.

2. If b is even, then $b - 2a$ is even and $2a$ and $b - 2a$ are not relatively prime, but a and $\frac{1}{2}b$ are, and hence, by the lemma, a and $\frac{1}{2}b - a$ are.

3. If a is odd, then a and $2b$ are relatively prime and hence, by the lemma, so are a and $2b - 3a$.

4. If a is even, then a and $2b - 3a$ have 2 as a common factor, but a and b are relatively prime and hence, by the lemma, so are $\frac{1}{2}a$ and $b - \frac{3}{2}a$.

Theorem 3

Let $H_1 = a$, $H_2 = b$, $0 < 2a < b$ and, moreover, let a and b be relatively prime. Then H is noninjective if and only if one of the following alternatives holds:

a. $a = 1$ and $b = 3$.

b. For some even $n > 0$, $2a = F_{n-1}$ and $b = F_{n+1}$, where b is odd.

c. For some even $n > 0$, $a = F_{n-1}$ and $\frac{1}{2}b = F_{n+1}$, where b is even (and hence a is odd).

d. For some even $n > 0$, $a = F_{n-1}$ and $2(b - a) = F_{n+1}$, where a is odd.

e. For some even $n > 0$, $\frac{1}{2}a = F_{n-1}$ and $b - a = F_{n+1}$, where a is even (and hence b is odd).

Proof: (i) If H is noninjective, then, by the corollary to Lemma 3, there exist $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$. Then, by the previous theorem, $n - 2 < p < n + 2$. In the proof of the latter theorem, the identity

$$b(F_{n+1} - F_{p-1}) = a(F_{p-2} + F_{n+2}) \tag{5}$$

did appear, which will be used again here.

Case 1. $p = n$ yields for (5) the identity:

$$b(F_{n+1} - F_{n-1}) = a(F_{n-2} + F_{n+2}),$$

which transforms into

$$b(F_{n+1} - F_n + F_n - F_{n-1}) = a(F_n + F_{n+1} + F_{n-2}),$$

and hence

$$bF_n = a(F_{n-1} + F_{n+1} + 2F_{n-2}),$$

which yields

$$bF_n = a(2F_{n-1} + F_n + 2F_{n-2}),$$

and hence, finally,

$$bF_n = 3aF_n,$$

or, since $n > 0$ and hence $F_n \neq 0$,

$$b = 3a.$$

Since a and b are relatively prime, $a = 1$ and $b = 3$, which is alternative a.

Case 2. $p = n + 1$ yields for (5) the identity:

$$b(F_{n+1} - F_n) = a(F_{n-1} + F_{n+2}),$$

which gives

$$bF_{n-1} = a(F_{n-1} + F_n + F_{n+1}),$$

and hence

$$bF_{n-1} = a(2F_{n-1} + 2F_n),$$

which transforms into

$$(b - 2a)F_{n-1} = 2aF_n,$$

or, since $n \geqslant 2$ and hence $F_{n-1} \neq 0$,

$$\frac{F_n}{F_{n-1}} = \frac{b - 2a}{2a}.$$

Any two successive Fibonacci numbers are known to be relatively prime (cf, e.g. [3], p. 40). If b is odd, $2a$ and $b - 2a$ are relatively prime (remark 1), and hence, for some even $n > 0$,

$$2a = F_{n-1} \quad \text{and} \quad b - 2a = F_n,$$

or

$$2a = F_{n-1} \quad \text{and} \quad b = F_{n+1},$$

which is alternative b. If b is even, then

$$\frac{F_n}{F_{n-1}} = \frac{\frac{1}{2}b - a}{a}$$

and, since a and $\frac{1}{2}b - a$ are relatively prime (remark 2), for some even $n > 0$,

$$a = F_{n-1} \quad \text{and} \quad F_n = \tfrac{1}{2}b - a,$$

or

$$a = F_{n-1} \quad \text{and} \quad \tfrac{1}{2}b = F_{n+1},$$

which is alternative c.

Case 3. $p = n - 1$ yields for (5) the relation

$$b(F_{n+1} - F_{n-2}) = a(F_{n-3} + F_{n+2}),$$

which, by some manipulations similar to those in the previous cases and left to the reader, can be transformed into

$$(2b - 3a)F_{n-1} = aF_n,$$

or, since $F_{n-1} \neq 0$,

$$\frac{2b - 3a}{a} = \frac{F_n}{F_{n-1}}.$$

If a is odd, then a and $2b - 3a$ are relatively prime (remark 3); hence, for some even $n > 0$,

$$a = F_{n-1} \quad \text{and} \quad 2b - 3a = F_n,$$

or

$$a = F_{n-1} \quad \text{and} \quad 2(b - a) = F_{n+1},$$

which is alternative d. Finally, if a is even, then $\frac{1}{2}a$ and $b - \frac{3}{2}a$ are relatively prime (remark 4); hence, for some even $n > 0$,

$$\tfrac{1}{2}a = F_{n-1} \quad \text{and} \quad b - \tfrac{3}{2}a = F_n,$$

or

$$\tfrac{1}{2}a = F_{n-1} \quad \text{and} \quad b - a = F_{n+1},$$

which is alternative e.

(ii) As to the converse, the first alternative with $a = 1$ and $b = 3$ generates the Lucas sequence, which is well known to be noninjective, as $L_{-n} = L_n$ for even $n \in N$. The second alternative, with $2a = F_{n-1}$ and $b = F_{n+1}$, where $n \in N$, n even and b odd, generates a sequence H with

$$H_{n+1} = aF_{n-1} + bF_n = 2a^2 + b(b - 2a) = 2a^2 - 2ab + b^2,$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(2b - 2a) + b^2 = 2a^2 - 2ab + b^2.$$

Hence $H_{-n} = H_{n+1}$ and, since obviously $-n \neq n + 1$, H is noninjective. The third alternative, with $a = F_{n-1}$ and $\frac{1}{2}b = F_{n+1}$, where $n \in N$, n even, b even, and a odd, generates a sequence H with

$$H_{n+1} = aF_{n-1} + bF_n = a^2 + b(\tfrac{1}{2}b - a) = a^2 - ab + \tfrac{1}{2}b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(b - a) + \tfrac{1}{2}b^2 = a^2 - ab + \tfrac{1}{2}b^2.$$

Hence, again, $H_{-n} = H_{n+1}$ and H is noninjective. The fourth alternative, with $a = F_{n-1}$ and $2(b - a) = F_{n+1}$, where $n \in N$, n is even and a is odd, generates a sequence H with

$$H_{n-1} = aF_{n-3} + bF_{n-2} = a(5a - 2b) + b(2b - 4a) = 5a^2 - 6ab + 2b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(4b - 5a) + b(2b - 2a) = 5a^2 - 6ab + 2b^2.$$

Hence $H_{-n} = H_{n-1}$ and H is noninjective. Finally, the fifth alternative, with $\frac{1}{2}a = F_{n-1}$ and $b - a = F_{n+1}$, where $n \in N$, n even, a even, and b odd, generates a sequence H with

$$H_{n-1} = aF_{n-3} + bF_{n-2} = a(\tfrac{5}{2}a - b) + b(b - 2a) = \tfrac{5}{2}a^2 - 3ab + b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(2b - \tfrac{5}{2}a) + b(b - a) = \tfrac{5}{2}a^2 - 3ab + b^2.$$

Hence, again, $H_{-n} = H_{n-1}$ and H is noninjective, which completes the proof.

Examples of noninjective sequences according to the alternatives of Theorem 3 are:

1. The Lucas sequence with characteristic pair (1, 3).

2. The sequence with characteristic pair (1, 5). Here

$$2a = 2 = F_3,\ b = 5 = F_5,\ H_{-4} = H_5 = 17.$$

3. The sequence with characteristic pair (1, 4). Here

$$a = 1 = F_1,\ \tfrac{1}{2}b = 2 = F_3,\ H_{-2} = H_3 = 5.$$

4. The sequence with characteristic pair (13, 30). Here

$$a = 13 = F_7,\ 2(b - a) = 34 = F_9,\ H_{-8} = H_7 = 305.$$

5. The sequence with characteristic pair (4, 9). Here

$$\tfrac{1}{2}a = 2 = F_3,\ (b - a) = 5 = F_5,\ H_{-4} = H_3 = 13.$$

The proof of the following corollary uses the fact that the ratios of successive Fibonacci numbers,

$$\frac{F_n}{F_{n-1}},\ n \in N,\ n > 1,$$

form a sequence which, for $n \to \infty$, converges to

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad (= 1.61803398875\ldots) \quad (\text{see [3], pp. 28, 29}).$$

In particular, the subsequence consisting of the ratios where the numerators have even indices, contains only terms $< \alpha$ and converges to α from below:

$$\frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \frac{55}{34}, \ldots .$$

This sequence is strictly increasing, i.e., if $\dfrac{F_n}{F_{n-1}} = \dfrac{F_m}{F_{m-1}}$, then $n = m$; moreover,

$$1 \leqslant \frac{F_n}{F_{n-1}} < 1.62.$$

Corollary 1

Let $H_1 = a$, $H_2 = b$, $0 < 2a < b$, a and b relatively prime, and $(a, b) \neq (1, 3)$. Moreover, let H be noninjective. Then there is a *unique* pair $n \in N$ and $p \in N$, where n is even and either $p = n - 1$ or $p = n + 1$, such that $H_{-n} = H_p$.

Proof: The hypothesis that H is noninjective implies that there is a pair $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$. The hypothesis that $(a, b) \neq (1, 3)$ implies that p is either $n + 1$ or $n - 1$. In case $p = n + 1$, the proof of the theorem arrives at

$$\frac{F_n}{F_{n-1}} = \frac{b - 2a}{2a}.$$

Assuming that n, p is not unique, one obtains a different pair, say $q \in N$ and even $m \in N$, such that $H_{-m} = H_q$. If $q = m + 1$, then

$$\frac{F_m}{F_{m-1}} = \frac{b - 2a}{2a}$$

also, and $m = n$ and $q = p$, since equal ratios imply equal indices (see the remarks preceding this corollary), contrary to the assumption that m, q was different from n, p. If $q = m - 1$, then the proof of the theorem yields

$$\frac{F_m}{F_{m-1}} = \frac{2b - 3a}{a} \quad \text{or} \quad \frac{F_m}{F_{m-1}} = 4 \cdot \frac{b - 2a}{2a} + 1 \quad \text{or} \quad \frac{F_m}{F_{m-1}} = 4 \cdot \frac{F_n}{F_{n-1}} + 1.$$

Even if F_n/F_{n-1} is as small as possible, namely $F_2/F_1 = 1$, then, still, $F_m/F_{m-1} = 5$ contrary to the fact that for even m, $F_m/F_{m-1} < 1.62$. Hence, in case $p = n + 1$, the pair n, p is the unique pair such that $H_{-n} = H_p$. In case $p = n - 1$, the argument is the same, be it in reversed order. In this case

$$\frac{F_n}{F_{n-1}} = \frac{2b - 3a}{a}$$

and a different pair, m, q with $q = m - 1$, would also yield

$$\frac{F_m}{F_{m-1}} = \frac{2b - 3a}{a}$$

and $n = m$, contrary to the assumption of different pairs; and a different pair, m, q with $q = m + 1$, would yield

$$\frac{F_m}{F_{m-1}} = \frac{b - 2a}{2a}$$

or

$$\frac{F_m}{F_{m-1}} = \frac{1}{4}\left(\frac{F_n}{F_{n-1}} - 1\right),$$

and since $F_n/F_{n-1} < 1.62$, this would yield $F_m/F_{m-1} < 1$, contrary to the remarks preceding the corollary.

The following table lists the first twenty noninjective sequences, ordered lexicographically by their generating characteristic pairs (a, b), where $0 < 2a < b$ and a and b are relatively prime.

Characteristic Pair	Equal Terms	Alternative of Theorem 3
(1, 3)	$H_{-2n} = H_{2n},\ n \in N$	1
(1, 4)	$H_{-2} = H_3 = 5$	3
(1, 5)	$H_{-4} = H_5 = 17$	2
(4, 9)	$H_{-4} = H_3 = 13$	5
(5, 26)	$H_{-6} = H_7 = 233$	3
(10, 23)	$H_{-6} = H_5 = 89$	5
(13, 30)	$H_{-8} = H_7 = 305$	4
(13, 68)	$H_{-8} = H_9 = 1597$	3
(17, 89)	$H_{-10} = H_{11} = 5473$	2
(68, 157)	$H_{-10} = H_9 = 4181$	5
(89, 466)	$H_{-12} = H_{13} = 75025$	3
(178, 411)	$H_{-12} = H_{11} = 28657$	5
(233, 538)	$H_{-14} = H_{13} = 98209$	4
(233, 1220)	$H_{-14} = H_{15} = 514229$	3
(305, 1597)	$H_{-16} = H_{17} = 1762289$	2
(1220, 2817)	$H_{-16} = H_{15} = 1346269$	5
(1597, 8362)	$H_{-18} = H_{19} = 24157817$	3
(3194, 7375)	$H_{-18} = H_{17} = 9227465$	5
(4181, 9654)	$H_{-20} = H_{19} = 31622993$	4
(4181, 21892)	$H_{-20} = H_{21} = 165580141$	3

The above table turns out to be considerably more than a list. It suggests several more corollaries to Theorem 3, only one of which will be mentioned here; the proof is left to the reader.

Corollary 2

Every even $n > 2$ determines exactly two ordered pairs of integers, (a, b) and (c, d), with $0 < 2a < b$, $0 < 2c < d$, $b > 3$, $d > 3$, $b \neq d$, a and b relatively prime, c and d relatively prime, and such that the sequence generated by one of the pairs has $H_{-n} = H_{n+1}$ and the sequence generated by the other pair has $H_{-n} = H_{n-1}$.

It should be noticed that $n = 2$ also determines two ordered pairs of integers, (a, b) and (c, d), generating noninjective sequences, but with slight modification that one of the pairs, say (a, b), has $0 < 2a = b$, namely the pair (1, 2) generating the sequence with $H_{-2} = H_1 = 1$ which is F, shifted one place.

Remark: If H is injective, then the terms of H form an abelian group under "multiplication" defined by $H_m H_n = H_{m+n}$, where $m \in Z$, $n \in Z$, with H_0 as multiplicative identity, and $H_n^{-1} = H_{-n}$. See also [2].

ACKNOWLEDGMENT

The authors are grateful to another faculty advisor in Santa Clara's URP project in Mathematics, Dr. D. Mugler, who provided them with a computer program generating sequences and flagging equal terms.

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ONE-FREE ZECKENDORF SUMS

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(Submitted December 1981)

The main theorem about representations of positive integers as sums of Fibonacci numbers, widely known as Zeckendorf's Theorem even before it was published [8], states that every positive integer is a sum of nonconsecutive Fibonacci numbers and that this representation is unique. Examples of such sums follow:

$$11 = 3 + 8,\ 12 = 1 + 3 + 8,\ 13 = 13,\ 70 = 2 + 13 + 55.$$

Zeckendorf's Theorem implies that the sums of distinct Fibonacci numbers form the sequence of all positive integers. It is the purpose of this note to prove that the sums of distinct terms of the *truncated* Fibonacci sequence (2, 3, 5, 8, ...) form the sequence

$$[(1 + \sqrt{5})n/2] - 1,\ n = 2, 3, 4, \ldots .$$

We shall use the usual notation for Fibonacci numbers, the greatest integer function, and fractional parts:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \text{ for } n = 1, 2, 3, \ldots;$$

$$[x] = \text{the greatest integer} \leqslant x; \text{ and}$$

$$\{x\} = x - [x].$$

A well-known connection between the number $\alpha = (1 + \sqrt{5})/2$ and F_n, to be used in the sequel, is that $[\alpha F_n] = F_{n+1}$ if n is odd and $= F_{n+1} - 1$ if n is even.

Lemma 1

Let n and c be positive integers satisfying $n \geqslant 2$ and $1 \leqslant c \leqslant F_n$. Let $S = \{\alpha c\} + \{\alpha F_n\}$. Then $S < 1$ for odd n and $S > 1$ for even n.

Proof: It is well known (e.g. [6, p. 101]) that

$$\frac{1}{F_{n+2}F_{n+4}} < \left|\alpha - \frac{F_{n+2}}{F_{n+1}}\right| < \frac{1}{F_{n+2}F_{n+3}}.$$

Shifting the index and multiplying by F_n gives

$$F_n/F_{n+1}F_{n+3} < \{\alpha F_n\} < F_n/F_{n+1}F_{n+2} \text{ for odd } n, \tag{1}$$

and

$$1 - F_n/F_{n+1}F_{n+2} < \{\alpha F_n\} < 1 - F_n/F_{n+1}F_{n+3} \text{ for even } n. \tag{2}$$

Now F_n/F_{n-1} is a best approximation of α, which means that

$$|\alpha F_{n-1} - F_n| \leqslant |\alpha e - d| \tag{3}$$

for all integers d and e satisfying $0 < e \leqslant F_n$.

Case 1. Suppose n is odd. Then (3) with $d = [\alpha c + 1]$ implies

$$F_n - \alpha F_{n-1} \leqslant [\alpha c + 1] - \alpha c,$$

so that $1 - \{\alpha F_{n-1}\} \leqslant 1 - \{\alpha c\}$, or equivalently, $\{\alpha c\} \leqslant \{\alpha F_{n-1}\}$. Thus

$$\begin{aligned} S &\leqslant \{\alpha F_{n-1}\} + \{\alpha F_n\} \\ &< 1 - F_{n-1}/F_n F_{n+2} + F_n/F_{n+1}F_{n+2} \text{ by (1) and (2)} \\ &= 1 - 1/F_n F_{n+1}F_{n+2} \\ &< 1. \end{aligned}$$

Case 2. Suppose n is even. Then (3) implies $\{\alpha F_{n-1}\} \leqslant 1 - \{\alpha c\}$, so that

$$\begin{aligned} S &\geqslant 1 - \{\alpha F_{n-1}\} + \{\alpha F_n\} \\ &> 1 - F_{n-1}/F_n F_{n+1} + 1 - F_n/F_{n+1}F_{n+2} \\ &= 2 - F_{n+1}/F_n F_{n+2} \\ &> 1. \end{aligned}$$

Lemma 2

Let n and c be positive integers satisfying $n \geqslant 2$ and $1 \leqslant c \leqslant F_n$. Then

$$[(\alpha + 1)(c + F_n) - 1] = [(\alpha + 1)c - 1] + F_{n+2}.$$

Proof: If n is odd and $\geqslant 3$, then

$$\begin{aligned} [(\alpha + 1)(c + F_n)] &= [(\alpha + 1)c] + [(\alpha + 1)F_n] \text{ by Lemma 1} \\ &= [(\alpha + 1)c] + F_n + [\alpha F_n] \\ &= [(\alpha + 1)c] + F_n + F_{n+1} \\ &= [(\alpha + 1)c] + F_{n+2}. \end{aligned}$$

If n is even, then

$$\begin{aligned}[(\alpha + 1)(c + F_n)] &= [(\alpha + 1)c] + [(\alpha + 1)F_n] + 1 \\ &= [(\alpha + 1)c] + F_n + [\alpha F_n] + 1 \\ &= [(\alpha + 1)c] + F_n + F_{n+1} \\ &= [(\alpha + 1)c] + F_{n+2}.\end{aligned}$$

Lemma 3

If M is a positive integer whose Zeckendorf sum uses 1, then there exists a positive integer C such that $M = [(\alpha + 1)C - 1]$. Explicitly, if

$$M = 1 + F_{n_1} + F_{n_2} + \cdots + F_{n_k} \text{ where } 4 \leqslant n_i \leqslant n_{i+2} - 1, \quad i = 1, 2, \ldots, k - 2, \tag{4}$$

then

$$C = 1 + F_{n_1 - 2} + F_{n_2 - 2} + \cdots + F_{n_k - 2}.$$

Proof: As a first step, $1 = [\alpha]$. Now, suppose $M > 1$ has Zeckendorf sum (4) and, as an induction hypothesis, that if m is any positive integer $< M$, then, in terms of its Zeckendorf sum

$$m = 1 + F_{u_1} + F_{u_2} + \cdots + F_{u_v},$$

we have $m = [(\alpha + 1)c - 1]$, where

$$c = 1 + F_{u_1 - 2} + F_{u_2 - 2} + \cdots + F_{u_v - 2}.$$

Let $c' = 1 + F_{n_1 - 2} + F_{n_2 - 2} + \cdots + F_{n_{k-1} - 2}$. Then

$$c' \leqslant \sum_{j=2}^{n_{k-1}-2} F_j = -2 + F_{n_{k-1}} < F_{n_k - 2}.$$

Lemma 2 therefore applies:

$$[(\alpha + 1)(c' + F_{n_k - 2}) - 1] = [(\alpha + 1)c' - 1] + F_{n_k},$$

and by the induction hypothesis, this equals

$$(1 + F_{n_1} + F_{n_2} + \cdots + F_{n_{k-1}}) + F_{n_k},$$

so that Lemma 3 is proved.

Lemma 4

The set of all positive integers C of the form

$$1 + F_{n_1 - 2} + F_{n_2 - 2} + \cdots + F_{n_k - 2},\ n \text{ as in (4)}, \tag{5}$$

together with 1, is the set of all positive integers.

Proof: Let C be any positive integer > 1 and let $C - 1$ have Zeckendorf sum

$$F_{u_1} + F_{u_2} + \cdots + F_{u_j}.$$

(If $F_{u_1} = 1$, it is understood that $u_1 = 2$.) Then C equals the sum (5) with $j = k$ and $n_i = u_i + 2$ for $i = 1, 2, \ldots, k$.

Theorem

The sums of distinct terms of the truncated Fibonacci sequence

$$(2, 3, 5, 8, \ldots)$$

form the sequence

$$[\alpha n - 1],\ n = 2, 3, 4, \ldots .$$

Proof: By Lemmas 3 and 4, the set of positive integers that are *not* such sums forms the sequence

$$[(\alpha + 1)n - 1],\ n = 1, 2, 3, \ldots .$$

Applying Beatty's method (based on a famous problem published in [1]) to the sequence $[(\alpha + 1)n]$, we conclude that the complement of this sequence is $[\alpha n]$. The complement of $[(\alpha + 1)n - 1]$ in the *positive* integers is therefore $[\alpha n - 1]$, $n = 2, 3, 4, \ldots$.

Remarks:

1. The first 360 terms of the sequence $[\alpha n - 1]$, i.e., the first 360 positive integers whose Zeckendorf sums do require 1, are listed in [2, pp. 62-64].

2. Fraenkel, Levitt, & Shimshoni [4] observe in their Corollary 1.3 that a certain property relating to Zeckendorf-type sums holds if and only if α has the form

$$\frac{1}{2}(2 - a + \sqrt{a^2 + 4})$$

for some positive integer a. When $a = 2$, we have $\alpha = \sqrt{2}$, and the sequence analogous to 1, 2, 3, 5, 8, 13, ... is 1, 3, 7, 17, 41, 99, ... The first few numbers expressible as Zeckendorf-type sums of the truncated sequence 3, 7, 17, 41, 99, ... (see [4, p. 337, item (i), for a precise definition of Zeckendorf-type sums in this setting) are 3, 6, 7, 10, 13, 14. Sequences of the form $[\gamma n + \delta]$ cannot yield 3, 6, 7, consecutively. Therefore, Corollary 1.3 of [4] offers no immediate generalization of the theorem on sums from the truncated Fibonacci sequence. Does any nontrivial generalization exist?

3. The interested reader should consult Fraenkel, Levitt, & Shimshoni [4]. Their Theorem 1 states that for $\alpha = (1 + \sqrt{5})/2$, the numbers $[n\alpha]$ are "even" P-system numbers (= Zeckendorf sums, although they are not so named in [4]) and the numbers $[n\beta]$ are "odd." The one-free Zeckendorf sums discussed in this present work are $[n\alpha - 1]$, some of which are even and some of which are odd in the sense of [4]. Being one-free is equivalent to ending in zero in [4]; however, the attention in [4] is on the number of terminal zeros—whether that number is even or odd, and no criterion is given in [4] for whether the terminal digit is zero.

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GENERALIZED PROFILE NUMBERS

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INTRODUCTION

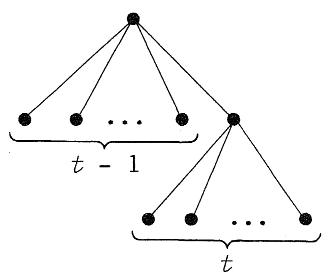
A family of binary trees $\{T_i\}$ is studied in [2]. The numbers $p(n, k)$ of internal nodes on level k in T_n (the root is considered to be on level 0) are called profile numbers, and they "enjoy a number of features that are strikingly similar to properties of binomial coefficients" (from [2]). We extend the results in [2] to t-ary trees.

DISCUSSION

We discuss t-ary trees (see Knuth [1]). A t-ary tree either consists of a single root, or a root that has t ordered sons, each being a root of another t-ary tree.

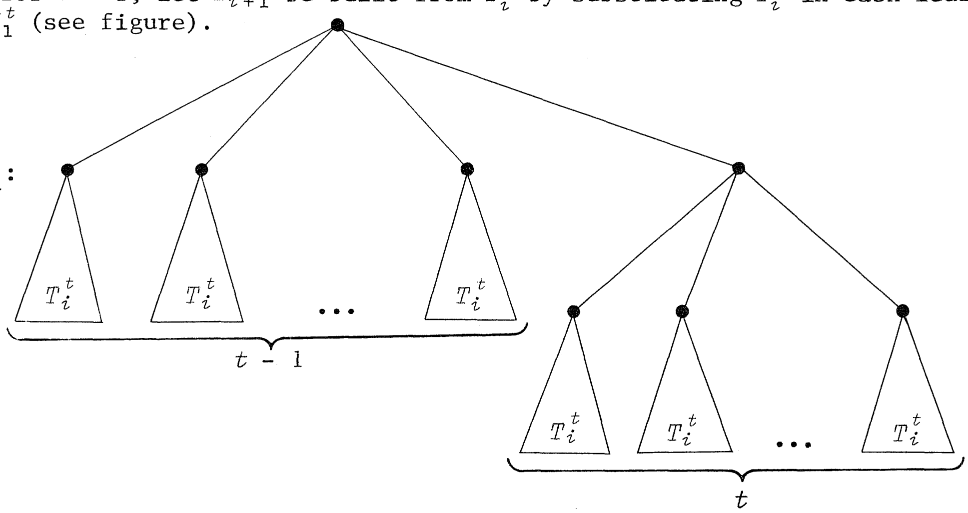
Let T_1^t be the tree

T_1^t:



and for $i \geq 1$, let T_{i+1}^t be built from T_i^t by substituting T_i^t in each leaf of T_1^t (see figure).

T_{i+1}^t:



Let $p_t(n, k)$ denote the number of internal nodes at level k in the tree T_n^t.

The numbers $p_t(n, k)$ satisfy the recurrence relation

$$p_t(n+1, k+1) = (t-1)p_t(n, k) + t\,p_t(n, k-1) \tag{1}$$

together with the boundary conditions

$$\begin{aligned} p_t(n, 0) &= 1 \\ p_t(1, 1) &= 1 \\ p_t(n, 1) &= t \quad \text{for} \quad n > 1 \\ p_t(1, k) &= 0 \quad \text{for} \quad k > 1. \end{aligned} \tag{2}$$

The corresponding trees and sequences for the case of binary trees ($t = 2$) is studied in [2]. Thus, T_n and $p(n, k)$ in [2] are denoted here by T_n^2 and $p_2(n, k)$, respectively.

We first show that

$$p_t(n, k) = t^{k-n} \sum_{0 \leqslant i < 2n-k} (t-1)^i \binom{n}{i}, \tag{3}$$

where $n \geqslant 1$, $k \geqslant 0$, and the $\binom{n}{i}$'s are the binomial coefficients.

Note that when $k < n$ we have $p_t(n, k) = t^k$.

The expression in (3) is easily shown to satisfy the boundary conditions (2). To continue, we induct on n (and arbitrary k); using (1) and the inductive hypothesis, we get

$$\begin{aligned} p_t(n+1, k+1) &= (t-1)p_t(n, k) + t\,p_t(n, k-1) \\ &= t^{k-n} \sum_{0 \leqslant i < 2n-k} (t-1)^{i+1} \binom{n}{i} + t^{k-n} \sum_{0 \leqslant i < 2n-k+1} (t-1)^i \binom{n}{i} \\ &= t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n}{i-1} + t^{k-n} + t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n}{i} \\ &= t^{k-n} + t^{k-n} \sum_{0 < i < 2n-k+1} (t-1)^i \binom{n+1}{i} = t^{k-n} \sum_{0 \leqslant i < 2n-k+1} (t-1)^i \binom{n+1}{i} \end{aligned}$$

and this establishes (3).

Using (3), we get

$$p_t(n, k+1) = t\,p_t(n, k) - t^{k-n+1}(t-1)^{2n-k-1}\binom{n}{k-n+1} \tag{4}$$

and

$$p_t(n+1, k) = p_t(n, k) + t^{k-n-1}(t-1)^{2n-k}\left[\binom{n}{k-n} + (t-1)\binom{n+1}{k-n}\right] \quad (5)$$

where $n \geqslant 1$ and $k \geqslant 0$.

Let x_n^t be the number of internal nodes in T_n^t, namely

$$x_n^t = \sum_{0 \leqslant k < 2n} p_t(n, k). \quad (6)$$

Using (3), changing the order of summation, and applying the binomial theorem results in

$$x_n^t = \frac{(2t-1)^n - 1}{t-1}. \quad (7)$$

Note that, by their definition, the numbers x_n^t satisfy the recurrence relation

$$x_1^t = 2$$

$$x_{i+1}^t = (2t-1)x_i^t + 2 \quad \text{for} \quad i > 0, \quad (8)$$

which also implies (7).

Let ℓ_n^t denote the internal path length (see [1]) of T_n^t, namely

$$\ell_n^t = \sum_{0 \leqslant k < 2n} k\, p_t(n, k). \quad (9)$$

The numbers ℓ_n^t also satisfy the recurrence relation

$$\ell_1^t = 1$$

$$\ell_{i+1}^t = (2t-1)\ell_i + (3t-1)x_i + 1 \quad \text{for} \quad i > 0. \quad (10)$$

Using (9) and (3), or solving (10) with the use of (7), one gets

$$\ell_n^t = \frac{3t-1}{t-1}n(2t-1)^{n-1} - \frac{t}{(t-1)^2}((2t-1)^n - 1). \quad (11)$$

The average level e_n^t of a node in T_n^t is thus given by ℓ_n^t/x_n^t, and satisfies

$$e_n^t \approx \frac{3t-1}{2t-1}n + 0(1). \quad (12)$$

The results in (1), (2), (3), (4), (5), (7), and (11) are extensions of (1), (3), Theorems 1, 2a, 2b, 3, and 4 of [2], respectively.

If we denote

$$F_t(x, y) = \sum_{n \geq 1,\, k \geq 0} p_t(n, k) x^n y^k,$$

then, using (1) and (2), we get

$$F_t(x, y) = \frac{x(1 + y)}{(1 - x)(1 - txy + xy - txy^2)}. \tag{13}$$

Equations (1) and (7), for the case $t = 2$, were noted in [2] to be similar to the recurrence relation

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

and the summation formula

$$\sum_{0 \leq k < n} \binom{n}{k} = 2^n - 1.$$

The binomial coefficients also satisfy

$$\sum_k (-1)^k \binom{n}{k} = 0.$$

Using (3), one can show that the same identity holds for any t and n; namely,

$$\sum_{0 \leq k < 2n} (-1)^k p_t(n, k) = 0. \tag{14}$$

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PROPERTIES OF POLYNOMIALS HAVING FIBONACCI NUMBERS FOR COEFFICIENTS

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(Submitted January 1982)

In memory of Vern Hoggatt, Jr.

It is unusual when one comes across a sequence of polynomials whose coefficients, roots, and sums of powers can all be given explicitly. It is our purpose to expose such a sequence of polynomials involving Fibonacci numbers.

The general polynomial in question is of even degree, which it will be convenient to take as $2n - 2$. The coefficients are the first n Fibonacci numbers as follows:

$$P_n(x) = x^{2n-2} + x^{2n-3} + 2x^{2n-4} + \cdots + F_n x^{n-1} - F_{n-1}x^{n-2} + F_{n-2}x^{n-3} - F_{n-3}x^{n-4} + \cdots + (-1)^n x - (-1)^n.$$

In particular

$$\begin{aligned}
P_1(x) &= 1\\
P_2(x) &= x^2 + x - 1\\
P_3(x) &= x^4 + x^3 + 2x^2 - x + 1\\
P_4(x) &= x^6 + x^5 + 2x^4 + 3x^3 - 2x^2 + x - 1\\
P_5(x) &= x^8 + x^7 + 2x^6 + 3x^5 + 5x^4 - 3x^3 + 2x^2 - x + 1.
\end{aligned}$$

Thus the coefficients of $P\ (x)$ are the first n Fibonacci numbers followed by the reversed sequence with alternating signs.

We shall begin by showing that the roots of $P_n(x)$ lie on two concentric circles in the complex plane. More precisely, we have

Theorem A

The roots of $P_n(x)$ are given explicitly by

$$\alpha\zeta_n^\nu,\ \beta\zeta_n^\nu \quad (\nu = 1, 2, \ldots, n - 1),$$

where

$$\alpha = (1 + \sqrt{5})/2,\ \beta = (1 - \sqrt{5})/2$$

and ζ_n is the nth root of unity $e^{2\pi i/n}$.

Proof: If we multiply $P_n(x)$ by $x^2 - x - 1$, we find that, after collecting the coefficients of $1, x, x^2, \ldots, x^{2n}$, all these coefficients vanish except three, because

$$F_k = F_{k-1} + F_{k-2}.$$

The remaining trinomial is

$$x^{2n} - (F_n + 2F_{n-1})x^n + (-1)^n.$$

Since

$$F_n + 2F_{n-1} = F_{n-1} + F_{n+1} = L_n = \alpha^n + \beta^n,$$

we see at once that

$$(x^2 - x - 1)P_n(x) = x^{2n} - L_n x^n + (-1)^n = x^{2n} - (\alpha^n + \beta^n)x^n + (\alpha^n\beta^n).$$

It is obvious that the quadratic in y obtained by putting $x^n = y$ has for its roots α^n and β^n.

Hence $(x^2 - x - 1)P_n(x)$ has for its roots α, β times all the nth roots of unity. Omitting the extraneous roots α and β, we are left with the $2n - 2$ roots of $P_n(x)$ as specified by the theorem.

As for the sum $S_k(n)$ of the kth powers of the roots of $P_n(x)$, we have

Theorem B

$$S_k(n) = \begin{cases} (n-1)L_k & \text{if } n \text{ divides } k, \\ -L_k & \text{otherwise.} \end{cases}$$

Proof: Using Theorem A, we have

$$S_k(n) = (\alpha^k + \beta^k)\sum_{\nu=1}^{n-1}\zeta_n^{k\nu} = L_k\left(-1 + \sum_{\nu=0}^{n-1}\zeta_n^{k\nu}\right).$$

But if n divides k, then

$$\sum_{\nu=0}^{n-1}\zeta_n^{k\nu} = \sum_{\nu=0}^{n-1} 1 = n,$$

while if n does not divide k,

$$\sum_{\nu=0}^{n-1}\zeta_n^{k\nu} = (1 - (\zeta_n^k)^n)/(1 - \zeta_n^k) = 0.$$

We can make two statements about the factors of the discriminant D of $P_n(x)$, which is the product of all the (nonzero) differences of its roots, namely:

Theorem C

The discriminant D of $P_n(x)$ is divisible by $5^{n-1}n^{2n-4}$.

Proof: Among the differences there are three special types:

$$\alpha(\zeta_n^i - \zeta_n^j);\ \beta(\zeta_n^i - \zeta_n^j);\ \pm(\alpha - \beta)\zeta_n^i \quad (i \neq j = 1, 2, \ldots, n - 1).$$

The product of the last type is equal in absolute value to

$$(\alpha - \beta)^{2n-2} = 5^{n-1}.$$

If we allow i and j to be zero, the first two types contribute in absolute value the factor

$$\left[\prod_{i \neq j} |\zeta^i - \zeta^j|\right]^2,$$

which is the square of the discriminant of $x^n - 1$, which is well known to be n^n. If we now remove the product of those differences in which i or j equals zero, we remove

$$\prod_{j=1}^{n-1} (1 - \zeta_n^j)^2 = n^2$$

from the inner product. Hence the theorem.

We now present the following small table of the discriminant of P_n:

n	D
2	5
3	$2^2 \cdot 3^2 \cdot 5^2$
4	$2^8 \cdot 3^2 \cdot 5^3$
5	5^{16}
6	$2^{20} \cdot 3^8 \cdot 5^5$
7	$5^6 \cdot 7^{10} \cdot 13^{10}$

We note that Theorems A and B, as well as their proofs, remain valid if we replace F_n by U_n and L_n by V_n, where

$$U_0 = 0,\ U_1 = 1,\ U_n = Au_{n-1} + U_{n-2}$$
$$V_0 = 1,\ V_1 = A,\ V_n = AV_{n-1} + V_{n-2}$$

and α, β by $(A \pm \sqrt{A^2 + 4})/2$.

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THE PARITY OF THE CATALAN NUMBERS VIA LATTICE PATHS

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(Submitted April 1982)

The Catalan numbers

$$C_n = \binom{2n}{n} / (n + 1)$$

belong to the class of advanced counting numbers that appear as naturally and almost as frequently as the binomial coefficients, due to the extensive variety of combinatorial objects counted by them (see [1], [2]).

The purpose of this note is to give a combinatorial proof of the following property of the Catalan sequence using a lattice path interpretation.

Theorem

C_n is odd if and only if $n = 2^r - 1$ for some positive integer r.

Proof: The proof is based mainly on the following observation: If X is a finite set and α is an involution on X with fixed point set X^α, then $|X| \equiv |X^\alpha| \pmod 2$; i.e., $|X|$ and $|X^\alpha|$ have the same parity.

Now let D_n denote the set of lattice paths in the first quadrant from the origin to the point $(2n, 0)$ with the elementary steps

$$x: (a, b) \to (a + 1, b + 1)$$

$$\bar{x}: (a, b) \to (a + 1, b - 1).$$

It is well known that $|D_n| = C_n$ (see [2],[3]). Define $\alpha: D_n \to D_n$ by reflecting these paths about the line $x = n$. The fixed point set D_n^α of α consists of all paths in D_n symmetric with respect to the line $x = n$.

Now define an involution β on D_n^α as follows: for $w = w_1 u \bar{u} w_2 \in D_n^\alpha$ with $|w_1| = |w_2| = n - 1$ and $u \in \{x, \bar{x}\}$, set

$$\beta(w) = \begin{cases} w_1 \bar{u} u w_2 & \text{if } w_1 \notin D_{\frac{n-1}{2}} \\ w & \text{otherwise.} \end{cases}$$

Of course the set $D_{\frac{n-1}{2}}$ is empty unless n is odd. Hence, we can put

$$C_{\frac{n-1}{2}} = 0 \text{ for } n \text{ even.}$$

Note that

$$|D_n^{\alpha\beta}| = \left|D_{\frac{n-1}{2}}\right|,$$

since $w \to w_1$ is an obvious bijection between the sets $D_n^{\alpha\beta}$ and $D_{\frac{n-1}{2}}$. Thus we have

$$C_n \equiv C_{\frac{n-1}{2}} \pmod 2. \tag{1}$$

If C_n is odd, then induction on n gives $(n - 1)/2 = 2^r - 1$ for some r so that $n = 2^{r+1} - 1$ is of the required form. Of course, $C_{2-1} = C_1 = 1$.

The converse also follows immediately from (1) by a similar inductive argument.

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-490 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that the arithmetic mean of $L_{2n}L_{2n+3}$ and $5F_{2n}F_{2n+3}$ is always a Lucas number.

B-491 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_kF_{n+j} - F_jF_{n+k} = (L_jL_{n+k} - L_kL_{n+j})/5.$$

B-492 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers. Prove that

$$F_nF_{n+j+k} - F_{n+j}F_{n+k} = (L_{n+j}L_{n+k} - L_nL_{n+j+k})/5.$$

B-493 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Derive a formula for the largest integer $e = e(n)$ such that 2^e is an integral divisor of

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i},$$

where $\binom{n}{k} = 0$ for $k > n$.

B-494 *Proposed by Philip L. Mana, Albuquerque, NM*

For each positive integer n, find positive integers a_n and b_n such that $101n$ is the following sum of consecutive positive integers:

$$a_n + (a_n + 1) + (a_n + 2) + \cdots + (a_n + b_n).$$

B-495 *Proposed by Philip L. Mana, Albuquerque, NM*

Characterize an infinite sequence whose first 24 terms are given in the following:

1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55,
61, 64, 77, 81, 85, 86, 90, 91, ...

[Note that all perfect squares occur in the sequence.]

SOLUTIONS

Squares and Products of Consecutive Integers

B-466 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A_n = 1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \cdots + (-1)^{n-1}n(n + 1)$.

(a) Determine the values of n for which $2A_n$ is a perfect square.

(b) Determine the value of n for which $|A_n|/2$ is the product of two consecutive positive integers.

Solution by Graham Lord, Québec, Canada

$A_1 = 2$, $A_2 = -4$, $A_3 = 8$, $A_4 = -12$, and one can easily establish (by induction)

$$A_{2m-1} = 2m^2 \quad \text{and} \quad A_{2m} = -2m(m + 1).$$

Then $2A_n$ is a perfect square if n is odd and $|A_n|/2$ is the product of two consecutive positive integers if n is even. But since the equation

$$x^2 = y^2 + 1$$

has no solution in positive integers, $2|A_n|$ cannot be a perfect square when n is even and $|A_n|/2$ cannot be the product of two consecutive integers when n is odd.

Also solved by Paul S. Bruckman, H. Klauser, P. V. Satyanarayana Murty, Bob Prielipp, Sahib Singh, Gregory Wulczyn, a solver at the Madchengymnasium Essen-Borbeck, and the proposer.

A's into B's

B-467 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A_n be as in B-466 and let

$$B_n = \sum_{i=1}^{n} \sum_{k=1}^{i} k.$$

For which positive integers n is $|A_n|$ an integral divisor of B_n?

Solution by Graham Lord, Québec, Canada

Note that $2 = A_1$ does not divide $B_1 = 1$. As $B_n = n(n+1)(n+2)/6$, then

$$B_{2m-1} = m(4m^2 - 1)/3,$$

which is evidently not divisible by $A_{2m-1} = 2m^2$, for $m > 1$. And for n even,

$$B_{2m} = 2m(2m+1)/3,$$

which will be divisible by $|A_{2m}| = 2m(m+1)$ as long as $(2m+1)/3$ is an integer; that is, if $m \equiv 1 \pmod 3$ or, equivalently, $n \equiv 2 \pmod 6$.

Also solved by Paul S. Bruckman, H. Klauser, P. V. Satyanarayana Murty, Bob Prielipp, Sahib Singh, the solver at the Madchengymnasium Essen-Borbeck, and the proposer.

Fibonacci Sines

B-468 *Proposed by Miha'ly Bencze, Brasov, Romania*

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

Solution by Sahib Singh, Clarion State College, Clarion, PA

Let $a_1 = \text{Sin } A$, $a_2 = \text{Sin } B$, where A, B are in radian measure and belong to the open interval $(0, \pi/2)$. Thus

$$a_3 = \text{Sin}(A + B),\ a_4 = \text{Sin}(A + 2B),$$

and in general, by induction technique, we conclude that

$$a_n = \text{Sin}(AF_{n-2} + BF_{n-1}), \text{ where } n \geqslant 1.$$

Also solved by Paul S. Bruckman, L. Carlitz, and the proposer.

Base F_n Expansions

B-469 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

Describe the appearance in base F_n notation of:

(a) $1/F_{n-1}$ for $n \geqslant 5$; (b) $1/F_{n+1}$ for $n \geqslant 3$.

Solution by Graham Lord, Québec, Canada

Let $F_{2n-2} = u$, $F_{2n-1} - 1 = v$, $F_{2n} - 2 = w$, $F_{2n} - 1 = x$, $F_{2n} = y$, and $F_{2n+1} = z$. The identity $F_{m-1}F_{m+1} - F_m^2 = (-1)^m$ gives, for $m = 2n + 1$:

$$1/F_{2n} = F_{2n+2}/(z^2 - 1) = (z + y)(z^{-2} + z^{-4} + z^{-6} + \cdots)$$
$$= z^{-1} + yz^{-2} + z^{-3} + yz^{-4} + \cdots,$$

which is $.\overline{1y}$ in base F_{2n+1}. And

$$1/F_{2n+2} = F_{2n}/(z^2 - 1),$$

which is $.\overline{0y}$ in base F_{2n+1}. The same identity for $m = 2n$ yields:

$$1/F_{2n-1} = F_{2n+1}/(y^2 + 1) = (F_{2n} + F_{2n-1})(F_{2n}^2 - 1)/(y^4 - 1)$$
$$= [F_{2n}^3 + F_{2n}^2(F_{2n-1} - 1) + F_{2n}(F_{2n} - 2) + F_{2n-2}]/(y^4 - 1),$$

which is $.\overline{1vwu}$ in base F_{2n}. Similarly,

$$1/F_{2n+1} = F_{2n-1}(F_{2n}^2 - 1)/(y^4 - 1)$$
$$= [F_{2n}^2(F_{2n-1} - 1) + F_{2n}(F_{2n} - 1) + F_{2n-2}]/(y^4 - 1),$$

which is $.\overline{0vxu}$ in base F_{2n}. The lower bounds imposed on the subscripts guarantee the digits are nonnegative.

Also solved by Paul S. Bruckman, L. Carlitz, Bob Prielipp, J. O. Shallit, Sahib Singh, and the proposer.

3 Term A.P.

B-470 *Proposed by Larry Taylor, Rego Park, NY*

Find positive integers a, b, c, r, and s, and choose each of G_n, H_n, and I_n to be F_n or L_n, so that

$$aG_n,\ bH_{n+r},\ \text{and } cI_{n+s}$$

are in arithmetic progression for $n \geqslant 0$ and this progression is 6, 6, 6 for some n.

Solution by Paul S. Bruckman, Carmichael, CA

In order for the indicated quantities to equal 6, they must lie in the set

$$S = \{6F_1,\ 6F_2,\ 6L_1,\ 3F_3,\ 3L_0,\ 2F_4,\ 2L_2\}$$

for some n. This means that for all n, the indicated quantities must lie in the set T_n, defined as follows:

$$\{6F_n,\ 6F_{n+1},\ 6F_{n+2},\ 6L_n,\ 6L_{n+1},\ 3F_n,\ 3F_{n+1},\ 3F_{n+2},\ 3F_{n+3},\ 3L_n,$$
$$2F_n,\ 2F_{n+1},\ 2F_{n+2},\ 2F_{n+3},\ 2F_{n+4},\ 2L_n,\ 2L_{n+1},\ 2L_{n+2}\}.$$

Of the 18 elements of T_n, 3 are to be in arithmetic progression for all n. We may choose n sufficiently large so that no duplication of elements occurs in T_n, e.g., $n = 5$. Thus,

$$T_5 = \{10, 15, 16, 22, 24, 26, 30, 33, 36, 39, 42, 48, 58, 63, 66, 68, 78, 108\}.$$

Considering all possible combinations, we find that the only triplets which are subsets of T_5 in arithmetic progression are as follows:

(10,16,22), (10,26,42), (10,39,68), (15,24,33), (15,39,63), (16,26,36),
(16,42,68), (22,24,26), (22,26,30), (24,30,36), (24,33,42), (24,36,48),
(24,66,108), (26,42,58), (30,33,36), (30,36,42), (30,39,48), (30,48,66),
(33,36,39), (33,48,63), (36,39,42), (36,42,48), (48,63,78), (48,78,108),
(58,63,68), and (58,68,78).

We then relate each triplet above to the appropriate multiple of a Fibonacci or Lucas number, e.g., $(10,16,22) = (2F_5,\ 2F_6,\ 2L_5)$. From the resulting set of 26 triplets, we exclude those where the *smallest* subscript is repeated (which is a consequence of the requirement that r and s be positive); thus, we would not count (10,16,22), since the subscript 5 is repeated. We thus reduce the foregoing set of triplets to the following set:

$(2F_5,\ 2F_7,\ 2F_8)$, $(2F_5,\ 3F_7,\ 2F_9)$, $(3F_5,\ 3F_7,\ 3F_8)$, $(2F_6,\ 2F_8,\ 2F_9)$,
$(2L_5,\ 3F_6,\ 2F_7)$, $(3F_6,\ 6F_5,\ 2L_6)$, $(3F_6,\ 3L_5,\ 2F_8)$, $(3F_6,\ 6L_5,\ 6L_6)$,
$(6F_5,\ 2L_6,\ 2F_8)$, $(6F_5,\ 3F_7,\ 6F_6)$, $(3L_5,\ 2L_6,\ 3F_7)$, $(3L_5,\ 6F_6,\ 3F_8)$,
$(2L_6,\ 3F_7,\ 2F_8)$, $(6F_6,\ 3F_8,\ 7F_7)$, and $(2L_7,\ 3F_8,\ 2F_9)$.

Each of the foregoing triplets corresponds in the general case to a triplet which is a subset of T_n; if we form these corresponding triplets, however, with the smallest subscript in the triplets from T_5 replaced by n, we obtain some triplets which must be rejected, since they do not reduce to (6, 6, 6) for any value of n. To illustrate, the triplet ($2F_5$, $2F_7$, $2F_8$) suggests the possible triplet ($2F_n$, $2F_{n+2}$, $2F_{n+3}$) in the general case; however, the latter triplet clearly can never equal (6, 6, 6) for any n. This further restriction reduces the total set of possible triplets to four possibilities, and these turn out to be acceptable solutions:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3}),\ (6F_n,\ 3F_{n+2},\ 6F_{n+1}),$$
$$(2L_n,\ 3F_{n+1},\ 2F_{n+2}),\ (3L_n,\ 6F_{n+1},\ 3F_{n+3}).$$

The above triplets assume the values (6, 6, 6) for n = 1, 1, 2, and 0, respectively. It is an easy exercise to verify that the above triplets are in arithmetic progression for all n, and the proof is omitted here.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

4 Term A.P.

B-471 *Proposed by Larry Taylor, Rego Park, NY*

Do there exist positive integers d and t such that

$$aG_n,\ bH_{n+r},\ cI_{n+s},\ dJ_{n+t}$$

are in arithmetic progression, with J_n equal to F_n or L_n and everything else as in B-470?

Solution by Paul S. Bruckman, Carmichael, CA

Any quadruplet consisting of the indicated quantities must contain a solution of B-470 as its first three elements. Referring to that solution, if we set $n = 5$, for example, we obtain the triplets:

$$(30,36,42),\ (30,39,48),\ (22,24,26),\text{ and }(33,48,63).$$

Therefore, any solution of this problem must reduce, for $n = 5$, to the quadruplets:

$$(30,36,42,48),\ (30,39,48,57),\ (22,24,26,28),\text{ or }(33,48,63,78).$$

Each element of any quadruplet must be of the form kU_m, where k = 2, 3, or 6, U is F or L, and m is a nonnegative integer. However, 57 and 28 are not of this form ($57 = 3 \cdot 19$, and 19 is neither a Fibonacci nor a Lucas number; $28 = 2 \cdot 14$, and 14 is neither a Fibonacci nor a Lucas number). We must therefore eliminate the second and third of the above indicated quadruplets. This leaves the following two triplets as possibly generating acceptable solutions of this problem:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3}) \quad \text{or} \quad (3L_n,\ 6F_{n+1},\ 3F_{n+3}).$$

If these *do* generate acceptable solutions to this problem, the fourth element of the desired quadruplet must equal twice the third element, less the second element. Thus, if x_i denotes the missing fourth element corresponding to the ith triplet above (i = 1 or 2), then

$$\begin{aligned} x_1 &= 4F_{n+3} - 2L_{n+1} = 4F_{n+2} + 4F_{n+1} - 2F_{n+2} - 2F_n \\ &= 2F_{n+2} - 2F_n + 4F_{n+1} = 6F_{n+1}; \end{aligned}$$

also,

$$x_2 = 6F_{n+3} - 6F_{n+1} = 6F_{n+2}.$$

This suggests the possible solutions:

$$(6F_n,\ 2L_{n+1},\ 2F_{n+3},\ 6F_{n+1}) \quad \text{and} \quad (3L_n,\ 6F_{n+1},\ 3F_{n+3},\ 6F_{n+2}).$$

It only remains to verify that these quadruplets assume the values (6, 6, 6, 6) for the same values of n which generated the triplets (6, 6, 6) in B-470, i.e., for $n = 1$ and $n = 0$, respectively. Obviously, this is the case. Therefore, the above two solutions are the only solutions to this problem.

Also solved by Bob Prielipp, Sahib Singh, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-349 *Proposed by Paul S. Bruckman, Carmichael, CA*

Define S_n as follows:

$$S_n \equiv \sum_{k=1}^{n-1} \csc^2 \pi k/n,\ n = 2, 3, \ldots .$$

Prove $S_n = \dfrac{n^2-1}{3}$.

H-350 *Proposed by M. Wachtel, Zürich, Switzerland*

There exist an infinite number of sequences, each of which has an infinite number of solutions of the form:

$$\begin{array}{lll} A \cdot x_1^2 + 1 = 5 \cdot y_1^2 & \underline{A} = 5 \cdot (a^2 + a) + 1 & \underline{a} = 0, 1, 2, 3, \ldots \\ A \cdot x_2^2 + 1 = 5 \cdot y_2^2 & \underline{x_1} = 2;\ \underline{x_2} = 40(2a+1)^2 - 2 & \\ A \cdot x_3^2 + 1 = 5 \cdot y_3^2 & \underline{y_1} = 2a + 1;\ \underline{y_2} = (2a+1)\cdot(16A+1) & \\ \ldots & \ldots & \\ A \cdot x_n^2 + 1 = 5 \cdot y_n^2 & & \end{array}$$

Find a recurrence formula for x_3/y_3, x_4/y_4, $\ldots$, x_n/y_n. (y_n is dependent on x_n.)

Examples

$\underline{a = 0}$ $\quad 1 \cdot \left(\dfrac{L_3}{2}\right)^2 + 1 = 5 \cdot \left(\dfrac{F_3}{2}\right)^2$ $\qquad \underline{a = 1}$ $\quad 11 \cdot 2^2 + 1 = 5 \cdot 3^2$

$\underline{a = 0}$ $\quad 1 \cdot \left(\frac{L_9}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_9}{2}\right)^2$ $\qquad \underline{a = 1}$ $\quad 11 \cdot 358^2 + 1 = 5 \cdot 531^2$

$\quad 1 \cdot \left(\frac{L_{15}}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_{15}}{2}\right)^2$ $\qquad 11 \cdot 63722^2 + 1 = 5 \cdot 94515^2$

$\quad 1 \cdot \ldots \quad + 1 = 5 \cdot \ldots$ $\qquad 11 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 5}$ $\quad 151 \cdot 2^2 + 1 = 5 \cdot 11^2$

$\quad 151 \cdot 4{,}838^2 + 1 = 5 \cdot 26{,}587^2$

$\quad 151 \cdot 11{,}698{,}282^2 + 1 = 5 \cdot 64{,}287{,}355^2$

$\quad 151 \quad \ldots \qquad + 1 = 5 \quad \ldots$

H-351 *Proposed by Verner E. Hoggatt, Jr. (deceased)*

Solve the following system of equations:

$$\begin{aligned}
U_1 &= 1 \\
V_1 &= 1 \\
U_2 &= U_1 + V_1 + F_2 = 3 \\
V_2 &= U_2 + V_1 = 4 \\
&\vdots \\
U_{n+1} &= U_n + V_n + F_{n+1} \qquad (n \geqslant 1) \\
V_{n+1} &= U_{n+1} + V_n \qquad (n \geqslant 1)
\end{aligned}$$

SOLUTIONS

Eventually

H-332 *Proposed by David Zeitlin, Minneapolis, MN (Vol. 19, No. 4, October 1981)*

Let $\alpha = (1 + \sqrt{5})/2$. Let $[x]$ denote the greatest integer function. Show that after k iterations $(k \geqslant 1)$, we obtain the identity

$$[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]] = F_{(2p+1)(2k+1)}/F_{2p+1} \qquad (p = 0, 1, \ldots).$$

Remarks: The special case $p = 0$ appears as line 1 in Theorem 2, p. 309, in the paper by Hoggatt & Bicknell-Johnson, this Quarterly, Vol. 17, No. 4, pp. 306-318. For $k = 2$, the above identity gives

$$[\alpha^{4p+2}[\alpha^{4p+2}]] = F_{5(2p+1)}/F_{2p+1} = L_{4(2p+1)} - L_{2(2p+1)} + 1.$$

Solution by Paul S. Bruckman, Carmichael, CA

We may proceed by induction on k. For brevity, let Φ_k denote

$$\underbrace{[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]]}_{k \text{ pairs of brackets}}, \text{ considering } p \text{ fixed;}$$

we seek to prove that

$$\Phi_k = \frac{F_{(2p+1)(2k+1)}}{F_{2p+1}}, \quad k = 1, 2, 3, \ldots . \tag{1}$$

Let S denote the set of natural numbers k for which (1) holds. Note that

$$\Phi_1 = [\alpha^{4p+2}] = [L_{4p+2} - \beta^{4p+2}] = L_{4p+2} - 1,$$

since $0 < \beta^{4p+2} < 1$. Thus, $1 \in S$.

Suppose $k \in S$. Then

$$\Phi_{k+1} = \left[\frac{\alpha^{4p+2}F_{(2p+1)(2k+1)}}{F_{2p+1}}\right],$$

under the inductive hypothesis.

Now if m and n are odd, with $n \geqslant 3$, then

$$\alpha^{2m}F_{mn}/F_m = \alpha^{2m}(\alpha^{mn} - \beta^{mn})/F_m\sqrt{5} = \frac{\alpha^{m(n+2)} - \beta^{m(n-2)}}{\sqrt{5}F_m}$$

$$= \frac{\alpha^{m(n+2)} - \beta^{m(n+2)} - \beta^{mn}(\alpha^{2m} - \beta^{2m})}{\sqrt{5}F_m} = \frac{F_{m(n+2)}}{F_m} - \beta^{mn}L_m.$$

Since $-1 < \beta^{mn} < 0$, $\beta^{mn}L_m < 0$. Also,

$$-\beta^{mn}L_m = \alpha^{-mn}(\alpha^m - \alpha^{-m}) = \alpha^{-m(n-1)} - \alpha^{-m(n+1)} < \alpha^{-m(n-1)} \leqslant \alpha^{-2} < 1.$$

Therefore, $0 < -\beta^{mn}L_m < 1$, which implies

$$\left[\frac{\alpha^{2m}F_{mn}}{F_m}\right] = \frac{F_{m(n+2)}}{F_m}. \tag{2}$$

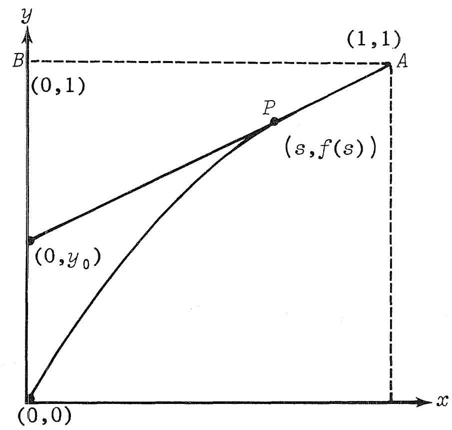
Setting $m = 2p + 1$, $n = 2k + 1$ in (2), this is equivalent to the assertion of (1) for $k + 1$. Since $k \in S \rightarrow (k + 1) \in S$, the proof by induction follows at once.

Also solved by the proposer.

Nab That Pig

H-333 *Proposed by Paul S. Bruckman, Carmichael, CA*
(Vol. 19, No. 5, December 1981)

The following problem was suggested by Problem 307 of *536 Puzzles & Curious Problems*, by Henry Ernest Dudeney, edited by Martin Gardner (New York: Charles Scribner's Sons, 1967).



Leonardo and the pig he wishes to catch are at points A and B, respectively, one unit apart (which we may consider some convenient distance, e.g., 100 yards). The pig runs straight for the gateway at the origin, at uniform speed. Leonardo, on the other hand, goes directly toward the pig at all times, also at a uniform speed, thus taking a curved course. What must be the ratio r of Leonardo's speed to the pig's, so that Leonardo may catch the pig just as they both reach the gate?

Solution by the proposer

Let the curve along which Leonardo runs be represented by the equation

$$y = f(x). \tag{1}$$

We note that f must be continuously differentiable in $(0, 1)$ and that the following additional conditions are to be satisfied:

$$f(1) = 1; \tag{2}$$

$$f'(1) = 0; \tag{3}$$

$$f(0) = 0. \tag{4}$$

The tangent line at any point $P \equiv (s, f(s))$ of the curve has the equation: $y - f(s) = f'(s)(x - s)$, with y-intercept $y_0 = f(s) - sf'(s)$. Thus, the distance the pig has traveled when Leonardo is at point P is equal to $1 - y_0 = 1 - f(s) + sf'(s)$. On the other hand, the distance Leonardo has traveled at that point is equal to

$$\int_s^1 \sqrt{1 + (f'(t))^2}\, dt,$$

as is well known from the calculus.

With a change of notation, this implies the relationship:

$$\int_x^1 \sqrt{1 + (f'(t))^2}\, dt = r(1 - f(x) + xf'(x)), \tag{5}$$

which is to be satisfied, along with (2), (3), and (4).

We may differentiate each side of (5) with respect to x (assuming this to be legitimate), thereby obtaining

$$-\sqrt{1 + (f'(x))^2} = rxf''(x),$$

or equivalently:

$$\frac{f''(x)}{\sqrt{1 + (f'(x))^2}} = -\frac{1}{rx}. \tag{6}$$

Integrating each side of (6) and using (3), we find that

$$\log\left\{f'(x) + \sqrt{1 + (f'(x))^2}\right\} = -\frac{1}{r}\log x,$$

or

$$\sqrt{1 + (f'(x))^2} + f'(x) = x^{-1/r}. \tag{7}$$

Solving for $f'(x)$ in (7) (by transposing and squaring), we obtain:

$$f'(x) = \tfrac{1}{2}(x^{-1/r} - x^{1/r}). \tag{8}$$

Now integrating (8) and using (2), this yields:

$$\begin{aligned} f(x) &= \frac{1}{2}\left\{\frac{x^{1-1/r}}{1-1/r} - \frac{x^{1+1/r}}{1+1/r}\right\} + C \\ &= \frac{r}{2(r^2 - 1)}\left\{(r + 1)x^{1-1/r} - (r - 1)x^{1+1/r}\right\} + C, \end{aligned}$$

where $f(1) = 1 = \dfrac{r}{r^2 - 1} + C$; hence, $C = (r^2 - r - 1)/(r^2 - 1)$, and

$$f(x) = \frac{2(r^2 - r - 1) + r(r + 1)x^{1-1/r} - r(r - 1)x^{1+1/r}}{2(r^2 - 1)}. \tag{10}$$

In order for Leonardo to catch his pig, it is clearly necessary that $r > 1$. We need to determine the particular value(s) of r satisfying (4), with $r > 1$. Setting $x = 0$ in (10), and assuming $f(0) = 0$ and $r > 1$, we obtain the equation $r^2 - r - 1 = 0$, whose only admissible solution is

$$r = \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \text{ the Golden Mean.} \tag{11}$$

If $r > \alpha$, Leonardo will catch the pig before reaching the gate, while if $r < \alpha$, the pig will escape.

NOTE: In the original problem Dudeney gives the value $r = 2$ and asks for $f(0)$, which turns out to be $1/3$.

CHECK: Substituting the value $r = \alpha$ in (10), we obtain:

$$f(x) = \frac{\alpha^3 x^{1-1/\alpha} + \alpha\beta x^{1+1/\alpha}}{2\alpha}$$

or equivalently:

$$f(x) = \frac{\alpha^2 x^{\beta^2} + \beta x^{\alpha}}{2}, \text{ where } \beta = \tfrac{1}{2}(1 - \sqrt{5}). \tag{12}$$

The distance that the pig runs to the gate is, of course, 1. We should thus find that the length of the curve from (0, 0) to (1, 1) (call this distance d) is equal to α. Now

$$d = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx.$$

Differentiating (12), we obtain:

$$f'(x) = \tfrac{1}{2}\left\{\alpha^2\beta^2 x^{\beta^2-1} + \alpha\beta x^{\alpha-1}\right\} = \tfrac{1}{2}(x^{\beta} - x^{-\beta});$$

$$1 + (f'(x))^2 = \left\{\tfrac{1}{2}(x^{\beta} + x^{-\beta})\right\}^2;$$

and

$$d = \tfrac{1}{2}\int_0^1 (x^{\beta} + x^{-\beta})dx = \tfrac{1}{2}\left(\frac{x^{1+\beta}}{1+\beta} + \frac{x^{1-\beta}}{1-\beta}\right)\Bigg|_0^1 = \tfrac{1}{2}(\alpha^2 x^{\beta^2} - \beta x^{\alpha})\Bigg|_0^1$$

$$= \tfrac{1}{2}(\alpha^2 - \beta) = \tfrac{1}{2}(\alpha + 1 - \beta) = \alpha,$$

as expected. The other conditions on f are readily verified for the function given by (12).

Also solved by B. Cheng.

Little Residue

H-334 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 19, No. 5, December 1981)

Let the Fibonacci-like sequence $\{H_n\}_{n=0}^{\infty}$ be defined by the relation

$$H_{n+2} = aH_{n+1} + bH_n,$$

where a and b are integers, $(a, b) = 1$, and $H_0 = 0$, $H_1 = 1$. Show that if p is an odd prime such that $-b$ is a quadratic nonresidue of p, then

$$p \nmid H_{2n+1} \text{ for any } n \geqslant 0.$$

(This is a generalization of Problem B-224, which appeared in the December 1971 issue of this Quarterly.)

Solution by the proposer

I offer three solutions.

First Solution: It can be shown by induction or by the Binet formula that

$$H_{2n+1} = bH_n^2 + H_{n+1}^2.$$

Suppose that $p|H_{2n+1}$ and $(-b/p) = -1$. Since

$$(n,\ 2n + 1) = (n + 1,\ 2n + 1) = 1,$$

$p \nmid H_n$ and $p \nmid H_{n+1}$. This follows because $\{H_n\}$ is periodic modulo p and because $H_0 = 0$. Thus,

$$bH_n^2 + H_{n+1}^2 \equiv 0 \pmod{p}$$

and

$$H_{n+1}^2 \equiv -bH_n^2 \pmod{p}.$$

Since neither H_n nor $H_{n+1} \equiv 0 \pmod{p}$ and since $(-b/p) = -1$, this is a contradiction.

Second Solution: It can be shown by the Binet formula or by induction that

$$H_n^2 - H_{n-1}H_{n+1} = (-b)^{n-1}.$$

Suppose $p|H_{2n+1}$ and $(-b/p) = -1$. Then it follows that

$$H_{2n+2}^2 - H_{2n+1}H_{2n+3} \equiv H_{2n+2}^2 \equiv (-b)^{2n+1} \pmod{p}.$$

Since $(-b/p) = -1$, this is a contradiction.

Third Solution: Let $\left\{J_n\right\}_{n=0}^{\infty}$ be defined by

$$J_{n+2} = aJ_{n+1} + bJ_n,$$

with $J_0 = 2$ and $J_1 = a$. It can be shown by the Binet formulas that

$$J_n^2 - (a^2 + 4b)H_n^2 = 4(-b)^n.$$

Suppose that $p|H_{2n+1}$ and $(-b/p) = -1$. Then

$$J_{2n+1}^2 - (a^2 + 4b)H_{n+1}^2 \equiv J_{2n+1}^2 \equiv 4(-b)^{2n+1}.$$

Since $(-b/p) = -1$, this is a contradiction.

Also solved by A. Shannon and P. Bruckman.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

A Primer for the Fibonacci Numbers. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

Fibonacci's Problem Book. Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971.

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972.

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973.

Recurring Sequences by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965.

A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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