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CONTENTS

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The Fibonacci Quarterly

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*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

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A NOTE ON THE FIBONACCI SEQUENCE OF ORDER K AND THE MULTINOMIAL COEFFICIENTS

ANDREAS N. PHILIPPOU

University of Patras, Patras, Greece

(Submitted July 1980; revised May 1982)

In the sequel, k is a fixed integer greater than or equal to 2, and n is a nonnegative integer as specified. Recall the following definition [6]:

Definition

The sequence $\{f_n^{(k)}\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geqslant k+1. \end{cases}$$

Gabai [2] called $\{f_n^{(k)}\}_{n=-\infty}^{\infty}$ with $f_n^{(k)} = 0$ for $n \leqslant -1$ the Fibonacci k-sequence. See, also, [1], [4], and [5].

Recently, Philippou and Muwafi [6] obtained the following theorem, which provides a formula for the nth term of the Fibonacci sequence of order k in terms of the multinomial coefficients.

Theorem 1

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$.

Presently, a new proof of this theorem is given which is simpler and more direct. In addition, the following theorem is derived, which provides a new formula for the nth term of the Fibonacci sequence of order k in terms of the binomial coefficients.

Theorem 2

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{-(k+1)i} - 2^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{-(k+1)i}, \quad n \geqslant 1,$$

where, as usual, $[x]$ denotes the greatest integer in x.

The proofs of the above formulas are based on the following lemma.

Lemma

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and denote its generating function by $g_k(x)$. Then, for $|x| < 1/2$,

$$g_k(x) = \frac{x - x^2}{1 - 2x + x^{k+1}} = \frac{x}{1 - x - x^2 - \cdots - x^k}.$$

Proof: We see from the definition that

$$f_2^{(k)} = 1,\ f_n^{(k)} - f_{n-1}^{(k)} = f_{n-1}^{(k)} \text{ for } 3 \leqslant n \leqslant k+1,$$

and

$$f_n^{(k)} - f_{n-1}^{(k)} = f_{n-1}^{(k)} - f_{n-1-k}^{(k)} \text{ for } n \geqslant k+1.$$

Therefore,

$$f_n^{(k)} = \begin{cases} 2^{n-2} & 2 \leqslant n \leqslant k \\ 2f_{n-1}^{(k)} - f_{n-1-k}^{(k)}, & n \geqslant k+1. \end{cases} \tag{1}$$

By induction on n, the above relation implies $f_n^{(k)} \leqslant 2^{n-2}$ $(n \geqslant 2)$ [5], which shows the convergence of $g_k(x)$ for $|x| < 1/2$. It follows that

$$g_k(x) = \sum_{n=0}^{\infty} x^n f_n^{(k)} = x + \sum_{n=2}^{k} x^n 2^{n-2} + \sum_{n=k+1}^{\infty} x^n f_n^{(k)}, \text{ by (1)}, \tag{2}$$

and

$$\begin{aligned} \sum_{n=k+1}^{\infty} x^n f_n^{(k)} &= 2 \sum_{n=k+1}^{\infty} x^n f_{n-1}^{(k)} - \sum_{n=k+1}^{\infty} x^n f_{n-1-k}^{(k)} \\ &= 2x\left(\sum_{n=0}^{\infty} x^n f_n^{(k)} - x - \sum_{n=2}^{k-1} x^n 2^{n-2}\right) - x^{k+1} g_k(x) \end{aligned} \tag{3}$$

$$= (2x - x^{k+1})g_k(x) - x^2 - \sum_{n=2}^{k} x^n 2^{n-2}.$$

The last two relations give $g_k(x) = x + (2x - x^{k+1})g_k(x) - x^2$, so that

$$g_k(x) = \frac{x - x^2}{1 - 2x + x^{k+1}} = \frac{x}{1 - x - x^2 - \cdots - x^k},$$

which shows the lemma.

Proof of Theorem 1: Let $|x| < 1/2$. Also let n_i $(1 \leqslant i \leqslant k)$ be nonnegative integers as specified. Then

$$\begin{aligned}\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} &= (1 - x - x^2 - \cdots - x^k)^{-1}, \text{ by the lemma,} \qquad (4)\\ &= \sum_{n=0}^{\infty} (x + x^2 + \cdots + x^k)^n, \text{ since } |x + x^2 + \cdots + x^k| < 1,\\ &= \sum_{n=0}^{\infty} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \ldots, n_k} x^{n_1 + 2n_2 + \cdots + kn_k},\end{aligned}$$

by the multinomial theorem. Now setting $n_i = m_i$ $(1 \leqslant i \leqslant k)$ and

$$n = m - \sum_{i=2}^{k} (i - 1)m_i,$$

we get

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \ldots, n_k} x^{n_1 + 2n_2 + \cdots + kn_k} \qquad (5)\\ = \sum_{m=0}^{\infty} x^m \sum_{\substack{m_1, \ldots, m_k \ni \\ m_1 + 2m_2 + \cdots + km_k = m}} \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k}.\end{aligned}$$

Equations (4) and (5) imply

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = \sum_{n=0}^{\infty} x^n \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}$$

from which the theorem follows.

Proof of Theorem 2: Set $S_k = \{x \in R;\ |x| < 1/2 \text{ and } |2x - x^{k+1}| < 1\}$, and let $x \in S_k$. Then

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = \frac{1 - x}{1 - 2x + x^{k+1}}, \text{ by the lemma,} \tag{6}$$
$$= (1 - x)\sum_{n=0}^{\infty}(2x - x^{k+1})^n$$
$$= (1 - x)\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(-1)^i x^{n+ki},$$

by the binomial theorem. Now setting $i = j$ and $n = m - kj$, and defining the sequence $\{b_n^{(k)}\}_{n=0}^{\infty}$ by

$$b_n^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]}(-1)^i\binom{n - ki}{i}2^{-(k+1)i}, \; n \geqslant 0, \tag{7}$$

we get

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(-1)^i x^{n+ki} = \sum_{m=0}^{\infty} x^m \sum_{j=0}^{[m/(k+1)]}\binom{m - kj}{j}2^{m-(k+1)j}(-1)^j \tag{8}$$
$$= \sum_{m=0}^{\infty} x^m b_m^{(k)} .$$

Relations (6) and (8) give

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = (1 - x)\sum_{n=0}^{\infty} x^n b_n^{(k)} = 1 + \sum_{n=1}^{\infty} x^n (b_n^{(k)} - b_{n-1}^{(k)}),$$

since $b_0^{(k)} = 1$ from (7). Therefore,

$$f_{n+1}^{(k)} = b_n^{(k)} - b_{n-1}^{(k)}, \; n \geqslant 1. \tag{9}$$

Relations (7) and (9) establish the theorem.

We note in ending that the above-mentioned same two relations imply

$$\sum_{i=1}^{n} f_i^{(k)} = 1 + \sum_{i=1}^{n-1}(b_n^{(k)} - b_{n-1}^{(k)}) = b_{n-1}^{(k)} \tag{10}$$
$$= 2^{n-1}\sum_{i=0}^{[(n-1)/(k+1)]}(-1)^i\binom{n - 1 - ki}{i}2^{-(k+1)i}, \; n \geqslant 1,$$

which reduces to

$$\sum_{i=1}^{n} F_i = 2^{n-1}\sum_{i=0}^{[(n-1)/3]}(-1)^i\binom{n - 1 - 2i}{i}2^{-3i}, \; n \geqslant 1, \tag{11}$$

since $F_i = f_i^{(2)}$ $(i \geqslant 0)$ from the definition. Also, observing that

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \ (n \geqslant 1),$$

see, for example, Hoggatt [3, (I_1), p. 52], we get, from (11), the following identity for the Fibonacci sequence:

$$F_{n+2} = 1 + 2^{n-1} \sum_{i=0}^{[(n-1)/3]} (-1)^i \binom{n-1-2i}{i} 2^{-3i}, \ n \geqslant 1. \tag{12}$$

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ASSOCIATED HYPERBOLIC AND FIBONACCI IDENTITIES

E. EHRHART

Institut de Recherche Mathématique Avancée
7 rue René Descartes, 67084 Strasbourg Cedex, France

(Submitted August 1980)

It is well known that by using the substitutions

$$\cos X = \cosh x, \ \sin X = -i \sinh x,$$

where $i = \sqrt{-1}$, trigonometric identities give rise to hyperbolic ones and conversely. This results from Euler's formulas

$$\cos X = \cosh iX \quad \text{and} \quad \sin X = -i \sinh iX.$$

For instance, we have the relations

$$\cos^2 X + \sin^2 X = 1, \ \cosh^2 x - \sinh^2 x = 1$$

and

$$\sin 2X = 2 \sin X \cos X, \ \sinh 2x = 2 \sinh x \cosh x.$$

Also, we shall see that a simple substitution *automatically* associates some Fibonacci identities to a class of hyperbolic ones.

This note is more original in its form than in its conclusions. Similar methods have been used by Lucas [1], Amson [2], and Hoggatt & Bicknell [3].

I. THE HYPERBOLIC-FIBONACCI ASSOCIATION

The following notation will be essential:*

$$[A, B]_n = \begin{cases} A \text{ if } n \text{ is odd,} \\ B \text{ if } n \text{ is even.} \end{cases}$$

We start from Binet's formulas:

$$F_n = \frac{a^n - b^n}{a - b}, \ L_n = a^n + b^n,$$

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*More generally $[u_1, u_2, \ldots, u_p]_n$ is equal to the u_i in the brackets such that $i = n$, modulo p [4].

where $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$ are the roots of the equation

$$X^2 - X - 1 = 0.$$

With $\alpha = \log a$, we have $a = e^{\alpha}$ and $b = -e^{-\alpha}$, and therefore

$$\frac{F_n}{2} = \frac{e^{\alpha n} - (-1)\ e^{-\alpha n}}{2\sqrt{5}},\ \frac{L_n}{2} = \frac{e^{\alpha n} + (-1)\ e^{-\alpha n}}{2}.$$

We now let $\alpha n = x$, then

$$\frac{\sqrt{5}F_n}{2} = [\cosh x,\ \sinh x]_n,\ \frac{L_n}{2} = [\sinh x,\ \cosh x]_n.$$

Substituting kn for n, k being an integer, we have

$$F_{kn} = \frac{2}{\sqrt{5}}[\cosh kx,\ \sinh kx]_{kn},\ L_{kn} = 2[\sinh kx,\ \cosh kx]_{kn}. \tag{1}$$

Substituting $n + m$ for n and putting $\alpha m = y$, we find that

$$\begin{aligned} F_{n+m} &= \frac{2}{\sqrt{5}}[\cosh(x+y),\ \sinh(x+y)]_{n+m}, \\ L_{n+m} &= 2[\sinh(x+y),\ \cosh(x+y)]_{n+m}. \end{aligned} \tag{2}$$

Equivalently, we have

$$\cosh kx = \frac{1}{2}[\sqrt{5}F_{kn},\ L_{kn}]_{kn},\ \sinh kx = \frac{1}{2}[L_{kn},\ \sqrt{5}F_{kn}]_{kn}, \tag{3}$$

and

$$\begin{aligned} \cosh(x+y) &= \frac{1}{2}[\sqrt{5}F_{n+m},\ L_{n+m}]_{n+m}, \\ \sinh(x+y) &= \frac{1}{2}[L_{n+m},\ \sqrt{5}F_{n+m}]_{n+m}. \end{aligned} \tag{4}$$

Formulas (2) and (4) also hold if we replace all the plus signs by minus signs.

Theorem 1

By substituting (1) and (2) on one side or (3) and (4) on the other, a Fibonacci identity gives one or several hyperbolic identities and conversely, provided that the indices or arguments have the form $kn \pm k'm$ or $kx \pm k'y$. The indices may be null or negative.

Remark

If we start with a Fibonacci identity, we must theoretically control the associated hyperbolic identity by other means, for then we pass from the particular to the general case. However, such an identity being true for $x = \alpha n$ and $y = \alpha m$ is probably true for all x and y, because

$$\alpha = \log \frac{1 + \sqrt{5}}{2}$$

is a transcendental number.

Since the hyperbolic identities are classic, we can easily establish some well-known Fibonacci identities.

II. DEVELOPMENT OF FIBONACCI IDENTITIES

Example 1

$$\sinh 2x = 2 \sinh x \cosh x$$

For all n, the substitution of (3) gives $F_{2n} = L_n F_n$.

Example 2

$$\cosh^2 x - \sinh^2 x = 1$$

The substitution of (3) gives:

$$\frac{5F_n^2}{4} - \frac{L_n^2}{4} = 1, \text{ if } n \text{ is odd,} \quad \text{and} \quad \frac{L_n^2}{4} - \frac{5F_n^2}{4} = 1, \text{ if } n \text{ is even.}$$

Thus for all n,

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{5}$$

Example 3

$$\sinh 5x = \sinh x \cosh\left(2x - \sqrt{5} \cosh x + \frac{3}{2}\right)\left(\cosh 2x + \sqrt{5} \cosh x + \frac{3}{2}\right).$$

By substitution

$$\frac{L_{5n}}{2} = 4 \frac{L_{2n}}{2}\left(\frac{L_{2n}}{2} - \sqrt{5} \frac{\sqrt{5}F_n}{2} + \frac{3}{2}\right)\left(\frac{L_{2n}}{2} + \sqrt{5} \frac{\sqrt{5}F_n}{2} + \frac{3}{2}\right), \text{ if } n \text{ is odd,}$$

and

$$\frac{\sqrt{5}F_{5n}}{2} = 4\,\frac{\sqrt{5}F_n}{2}\left(\frac{L_{2n}}{2} - \sqrt{5}\,\frac{L_n}{2} + \frac{3}{2}\right)\left(\frac{L_{2n}}{2} + \sqrt{5}\,\frac{L_n}{2} + \frac{3}{2}\right), \text{ if } n \text{ is even.}$$

Using $2n - 1$ in place of n if n is odd and $2n$ in place of n if n is even, we find the following two distinct identities, which are valid for every n:

$$L_{10n-5} = L_{2n-1}(L_{4n-2} - 5F_{2n-1} + 3)(L_{4n-2} + 5F_{2n-1} + 3)$$

and

$$F_{10n} = F_{2n}(L_{4n} - \sqrt{5}L_{2n} + 3)(L_{4n} + \sqrt{5}L_{2n} + 3).$$

Example 4

$$(\cosh x + \sinh x)^k = \cosh kx + \sinh kx \quad (k \text{ an integer} \geqslant 0)$$

Examining three cases (n even, n odd and k even, n odd and k odd), we find for all n and k that

$$\left(\frac{L_n + \sqrt{5}F_n}{2}\right)^k = \frac{L_{kn} + \sqrt{5}F_{kn}}{2}. \tag{6}$$

Application: Suppose we wish to express L_{kn} and F_{kn} as functions of L_n and F_n. We could do this by separating the expanded form of the left side of (6) into those terms with or without the factor $\sqrt{5}$.

Instead, we use the well-known fact that F_{kn} is divisible by F_n to show that the integer F_{kn}/F_n is a function of L_n of the form

$$P(L_n) + (-1)^n Q(L_n),$$

where $P(X)$ and $Q(X)$ are polynomials whose parities are opposite to that of k. By (6) we see that F_{kn} has the form

$$\sum c_i F_n^i L_n^{k-i},$$

where i takes the odd values equal to k or less. Therefore, F_{kn}/F_n has the form

$$\sum c_i (F_n^2)^{k'} L_n^{k-i},$$

but, according to (5),

$$F_n^2 = \frac{1}{5}(L_n^2 - 4(-1)^n).$$

Thus for $k = 2, 3, 4, 5, 6$, the values of F_{kn}/F_n are, respectively:

L_n, $L_n^2 - (-1)^n$, $L_n[L_n^2 - 2(-1)^n]$, $L_n^4 - 3(-1)^n L_n^2 + 1$, $L_n[L_n^4 - 4(-1)^n L_n^2 + 3]$.

Example 5

$$\frac{1}{2} + \cosh 2x + \cosh 4x + \cosh 6x + \cdots + \cosh 2kx = \frac{\sinh(2k+1)x}{\sinh x} \tag{7}$$

Therefore,

$$1 + L_{2n} + L_{4n} + L_{6n} + \cdots + L_{2kn} = \left[\frac{L_{(2k+1)n}}{L_n}, \frac{F_{(2k+1)n}}{F_n}\right]_n$$

If we replace n by $2n$, we get

$$1 + L_{4n} + L_{8n} + \cdots + L_{4kn} = \frac{F_{(4k+2)n}}{F_{2n}}.$$

If we substitute $X + (\pi/2)$ for X in the trigonometric identity associated with (7), we find a formula whose associated hyperbolic one is

$$\frac{1}{2} - \cosh 2x + \cosh 4x - \cosh 6x + \cdots + (-1)^k \cosh 2kx = \frac{(-1)^k \cosh(2x+1)x}{2 \cosh x}.$$

Hence,

$$1 - L_{2n} + L_{4n} - L_{6n} + \cdots + (-1)^k L_{2kn} = (-1)^k \left[\frac{F_{(2k+1)n}}{F_n}, \frac{L_{(2k+1)n}}{L_n}\right]_n.$$

Application: We can use these two Fibonacci identities to prove that *for any odd* k, L_{kn} *is divisible by* L_n.

Example 6

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Using (3) and (4), we see that for all n and m,

$$\begin{cases} 2F_{n+m} = F_n L_m + L_n F_m, \\ 2L_{n+m} = L_n L_m + 5F_n F_m. \end{cases} \tag{8}$$

Note that for $m = \pm 1$, (8) becomes

$$L_n + F_n = 2F_{n+1}, \quad L_n - F_n = 2F_{n-1}.$$

Example 7

$$\cosh(x + y) + \cosh(x - y) = 2 \cosh x \cosh y$$

$$\sinh(x + y) + \sinh(x - y) = 2 \sinh x \cosh y$$

Examining the four cases (according to the parities of n and m), we find that

$$L_{n+m} + L_{n-m} = [5F_n F_m, L_n L_m]_m, \tag{9}$$

$$F_{n+m} + F_{n-m} = [L_n F_m, F_n L_m]_m.$$

In particular, for $m = 1$, (9) becomes $L_{n-1} + L_{n+1} = 5F_n$. It can also be shown that

$$L_{n+m} - L_{n-m} = [5F_n F_m, L_n L_m]_{m-1}$$

and

$$F_{n+m} - F_{n-m} = [L_n F_m, F_n L_m]_{m-1}.$$

Application: We shall establish the following proposition using the preceding identities.

Theorem 2

A number of the form $F_n \pm F_m$ or $L_n \pm L_m$ is never prime if the indices have the same parity and a difference greater than 4.

Proof: The proof goes as follows: Let $a = n + m$ and $b = n - m$, then

$$F_a + F_b = \left[L_{\frac{a+b}{2}} F_{\frac{a-b}{2}}, F_{\frac{a+b}{2}} L_{\frac{a-b}{2}}\right]_{\frac{a-b}{2}}.$$

Since $a - b > 4$, we have $\frac{a-b}{2} > 2$, so that there is no term

$$L_1 = F_1 = F_2 = 1$$

in the brackets. Hence, $F_a + F_b$ is composite.

A similar demonstration exists for $F_a - F_b$ or $L_a \pm L_b$.

Example 8

$$\sinh(x + y)\sinh(x - y) = \sinh^2 x - \sinh^2 y$$

$$\cosh(x + y)\cosh(x - y) = \cosh^2 x + \sinh^2 y$$

By substitution, we have:

a) if n and m are even,

$$F_{n+m}F_{n-m} = F_n^2 - F_m^2$$

$$L_{n+m}L_{n-m} = L_n^2 + 5F_m^2$$

$$5F_{n+m}F_{n-m} = L_n^2 - L_m^2;$$

b) if n and m are odd,

$$L_{n+m}L_{n-m} = 5F_n^2 + L_m^2;$$

c) if n is even and m is odd,

$$5F_{n+m}F_{n-m} = L_n^2 + L_m^2$$

$$L_{n+m}L_{n-m} = 5F_n^2 - L_m^2;$$

d) if n is odd and m is even,

$$F_{n+m}F_{n-m} = F_n^2 + F_m^2$$

$$L_{n+m}L_{n-m} = L_n^2 - 5F_m^2.$$

With the help of (5) the four expressions thus obtained for $F_{n+m}F_{n-m}$ and $L_{n+m}L_{n-m}$ can be condensed into two identities:

$$F_{n+m}F_{n-m} - F_n^2 = (-1)^{n+m+1}F_m^2,$$

and

$$L_{n+m}L_{n-m} - L_n^2 = (-1)^{n+m}L_m^2 - 4(-1)^n.$$

The first is the Catalan formula.

Letting $n = 1$ and $m = 2$, we see that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad \text{(Simson's formula)}$$

$$F_{n+2}F_{n-2} - F_n^2 = (-1)^{n+1},$$

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1},$$

and

$$L_{n+2}L_{n-2} - L_n^2 = 5(-1)^n.$$

Example 9

Our last example is of a Fibonacci-trigonometric transposition. In [5], it is shown that if a, b, and c are even integers, then

$$L_a L_b L_c = L_{a+b+c} + L_{b+c-a} + L_{c+a-b} + L_{a+b-c},$$

and

$$5F_a F_b F_c = F_{a+b+c} + F_{a-b-c} + F_{b-c-a} + F_{c-a-b}.$$

Using the hyperbolic transposition, (substitutions (1) and (2), we obtain

$$4 \cosh x \cosh y \cosh z = \cosh(x + y + z) + \cosh(y + z - x) + \cosh(z + x - y) + \cosh(x + y - z),$$

and

$$4 \sinh x \sinh y \sinh z = \sinh(x + y + z) + \sinh(x - y - z) + \sinh(y - z - x) + \sinh(z - x - y).$$

Now, applying the trigonometric transposition, we have

$$4 \cos X \cos Y \cos Z = \cos(X + Y + Z) + \cos(Y + Z - X) + \cos(Z + X - Y) + \cos(X + Y - Z),$$

and

$$4 \sin X \sin Y \sin Z = -\sin(X + Y + Z) + \sin(Y + Z - X) + \sin(Z + X - Y) + \sin(X + Y - Z).$$

III. GENERALIZATION

Let s be a positive integer with a and b the roots of the equation

$$X^2 - sX - 1 = 0, \text{ where } a = \frac{s + \sqrt{s^2 + 4}}{2}.$$

Consider the two generalized Fibonacci sequences given by

$$f_n = \frac{a^n - b^n}{a - b}, \quad \ell_n = a^n + b^n. \tag{10}$$

Let

$$\Delta = s^2 + 4, \ \alpha = \log a, \ \alpha n = x, \ \alpha m = y,$$

then,

$$\frac{f_n}{2} = \frac{a^n - b^n}{2\sqrt{\Delta}} = \frac{e^{\alpha n} - (-1)\ e^{-\alpha n}}{2\sqrt{\Delta}} = \frac{1}{\sqrt{\Delta}}[\cosh x, \sinh x]_n,$$

and

$$\frac{\ell_n}{2} = \frac{a^n + b^n}{2} = \frac{e^{\alpha n} + (-1)\ e^{-\alpha n}}{2} = [\sinh x, \cosh x]_n.$$

Hence,

$$\begin{cases} \cosh kx = \frac{1}{2}[\sqrt{\Delta}f_{kn}, \ell_{kn}]_{kn}, \quad \sinh kx = \frac{1}{2}[\ell_{kn}, \sqrt{\Delta}f_{kn}]_{kn}, \\ \cosh(x+y) = \frac{1}{2}[\sqrt{\Delta}f_{n+m}, \ell_{n+m}]_{n+m}, \quad \sinh(x+y) = \frac{1}{2}[\ell_{n+m}, \sqrt{\Delta}f_{n+m}]_{n+m}. \end{cases} \tag{11}$$

Theorem 3

To a hyperbolic identity with arguments of the form $k \pm k'm$ (n and m integers), the substitution formulas of (11) associate one or several generalized Fibonacci identities (the same as for F_n and L_n, with the restriction that the factor 5 or $\sqrt{5}$ is replaced by Δ or $\sqrt{\Delta}$).

For instance,

$$f_{2n} = \ell_n f_n, \quad \ell_n^2 - \Delta f_n^2 = 4(-1)^n, \quad f_{n+m} + f_{n-m} = [\ell_n f_m, f_n \ell_m]_m.$$

Note that the neighborly relations,

$$L_n + F_n = 2F_{n+1}, \quad L_n - F_n = 2F_{n-1}, \quad L_{n-1} + L_{n+1} = 5F_n,$$

$$L_{n-1} - F_{n+1} = L_{n+1} - F_{n-1} = 3F_n, \quad L_n^2 - F_n^2 = 4F_{n-1}F_{n+1},$$

$$L_n^2 + F_n^2 = 2(F_{n-1}^2 + F_{n+1}^2),$$

do not hold for f_n and ℓ_n. However, for every s:

$$f_{n+1} + f_{n-1} = \ell_n.$$

The formulas of Simson and Catalan also hold for f_n and

$$\ell_{n+1}\ell_{n-1} - \ell_n^2 = \Delta(-1)^{n+1}.$$

Application: If we put $a = n + m$ and $b = n - m$, the formula

$$f_{n+m} + f_{n-m} = [\ell_n f_m, f_n \ell_m]_m$$

becomes

$$f_a + f_b = \left[\ell_{\frac{a+b}{2}} f_{\frac{a-b}{2}}, f_{\frac{a+b}{2}} \ell_{\frac{a-b}{2}}\right]_{\frac{a-b}{2}}$$

if $a - b$ is even. Therefore:

Theorem 4

A number $f_a + f_b$ is not prime if $a - b$ is even and other than 2, and $f_a + f_{a+2}$ is a prime only if ℓ_{a+1} is a prime.

Proof: Note that

$$f_a + f_{a+2} = \ell_{a+1} \quad \text{and} \quad f_a + f_{a+4} = 3f_{a+2},$$

so if $a - b > 4$, there is no factor $f_1 = 1$ or $f_2 = \ell_1 = s = 1$ in the brackets, and if $a - b = 4$, then

$$f_a + f_b = f_{\frac{a+b}{2}}\ell_2.$$

Remarks

1) Under the same conditions, $\ell_a \pm \ell_b$ is not prime. Furthermore, if $a - b$ is even other than 2 or 4, $f_a - f_b$ is not prime.
2) An integer $F_a \pm 1$ is not prime for $n > 6$. (See [6].)

The latter remark is true, since $F_a \pm 1$ can be considered as $F_a \pm F_1$ if a is odd and $F_a \pm F_2$ if a is even. For $a > 6$, the difference $a - 1$ or $a - 2$ exceeds 4.

Recurrence: The generalized Fibonacci sequences can also be defined by

$$f_{n+2} = sf_{n+1} + f_n,\ f_0 = 0,\ f_1 = 1,$$

and

$$\ell_{n+2} = s\ell_{n+1} + \ell_n,\ \ell_0 = 2,\ \ell_1 = s.$$

This results directly from Binet's formulas (10). Note that for $s = 2$, the f_n are the "Pell numbers."

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SEQUENCES GENERATED BY SELF-REPLICATING SYSTEMS

WESLEY A. DARBRO

Space Sciences Laboratory, Science and Engineering
NASA, Marshall Space Flight Center, AL 35812

and

GEORG VON TIESENHAUSEN

Advanced Systems Office, Program Development
NASA, Marshall Space Flight Center, AL 35812

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INTRODUCTION

In the late 1940's John von Neumann began to develop a theory of automa. His substantial and unique works covered a broad range of subjects from which one, self-replication, is of particular interest in this study.

A self-replicating system (SRS) is an organization of system elements that is capable of producing exact replicas of itself which, in turn, will produce exact replicas of themselves. The replication process uses materials or components from its environment and continues automatically until the process is terminated. Examples of potential space and terrestrial applications are in the areas of photoelectric cells, oxygen, planetary explorer rovers, ocean bottom mining, and desert irrigation. We will investigate an aspect of SRS's, that which concerns the number of replicas various systems would produce.

OUTLINE FOR A SELF-REPLICATING SYSTEM

For a description of self-replication, the reader sbould refer to [1]. The basic system elements of an SRS are:

- Mining and Materials Processing Plant
- Materials Depot
- Parts Production Plant
- Replication Parts Depot
- Production Parts Depot
- Production Facility
- Universal Constructor
- Product Depot
- Product Retrieval System
- Energy System

SEQUENCES GENERATED BY SELF-REPLICATING SYSTEMS

In the Mining and Materials Processing Plant, raw materials are gathered by mining, analyzed, separated, and processed into feedstock such as sheets, bars, ingots, and castings. The processed feedstock is then laid out and stored in the Materials Depot.

The Parts Production Plant selects and transports feedstock from the Materials Depot and produces all parts required for SRS replication and the products. The finished parts are laid out and stored in either the Replication Parts Depot or the Production Parts Depot. The Parts Production Plant includes material transport and distribution, production, control, and sub-assembly operations. All parts and sub-assemblies required for replication of complete SRS's are stored in the Replication Parts Depot in lots destined for specific facility construction. In the Production Parts Depot, parts are stored for use in manufacturing the desired products in the Production Facility.

The Production Facility produces the product. Parts and sub-assemblies are picked up from the Production Parts Depot, transported into the Production Facility, and undergo specific manufacturing and production processes depending on the specific product desired. The finished products are stored in the Product Depot to await pickup by the Product Retrieval System.

The Universal Constructor, in principle, is a system capable of constructing and system. The purpose of the Universal Constructor is to self-replicate a complete SRS a specified number of times in such a way that these replicas, in turn, construct replicas of themselves, and so on. The Universal Constructor has the overall control and command function for its own SRS as well as for the replicas until control and command functions have been replicated and transferred to the replicas. The Product Retrieval System collects the outputs of all units of an SRS field. Finally, the energy source generally considered practical is solar.

SELF-REPLICATING OPTIONS

There are several possible schemes that one must consider in designing a self-replicating system. One is to design each replica to reproduce simultaneously its n-replicas (Figure 1), and we will refer to this

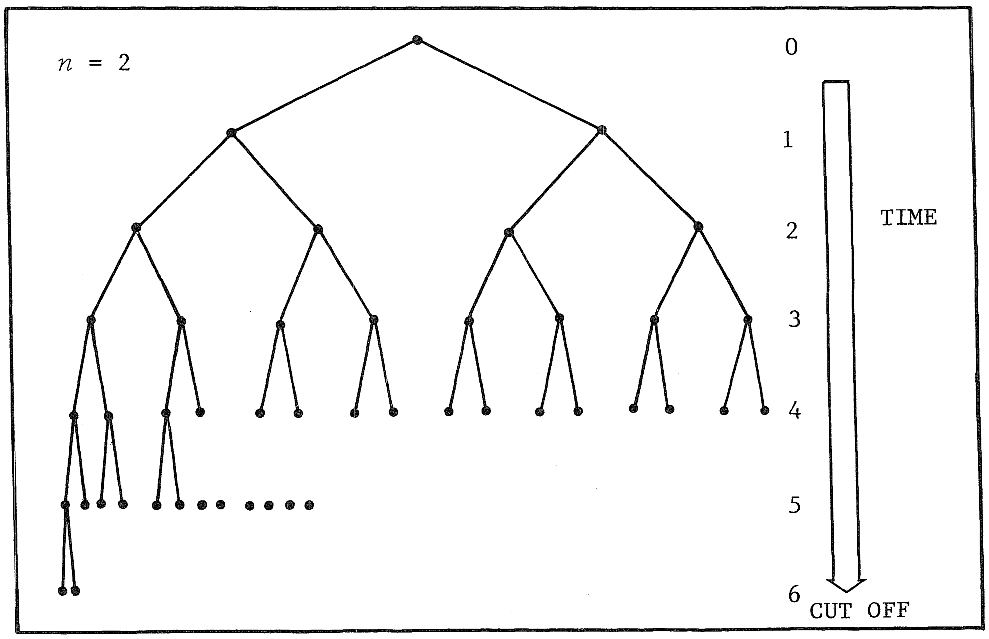

FIGURE 1. Option A, S = 127

case as Option A. Because of large mass flows and programming complexities, this option presently has little support. Another scheme, referred to as Option B, is to design each replica to produce its n-replicas sequentially (Figure 2). When a sufficient number have been obtained, reproduction is stopped and production begins. The main reason for limiting the number of replicas to, say n, is that with each replication a defective replica becomes more likely. An objection to Option B is that earlier branches will have reproduced more generations than later ones, which would result in some lower-quality replicas than necessary.

Therefore, a third scheme (Figure 3) is considered and referred to as Option C: a replica reproduces sequentially no more than n replicas and

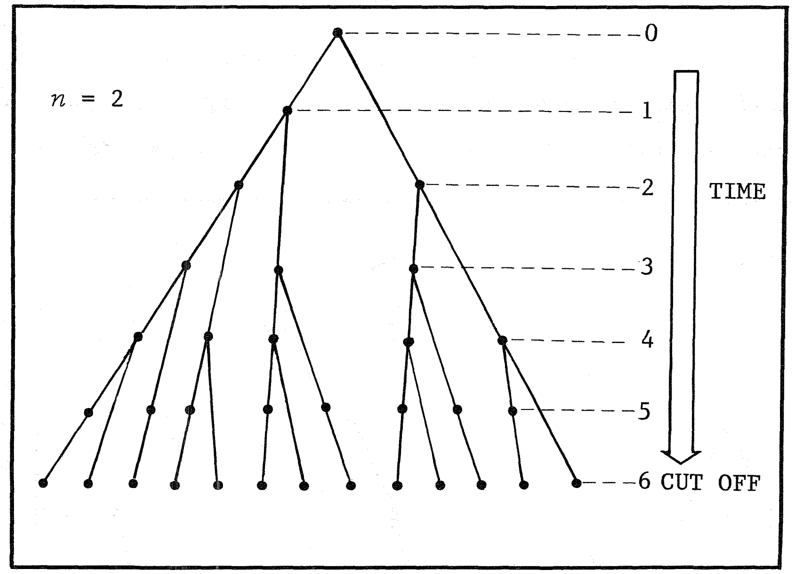

FIGURE 2. Option B, S = 33

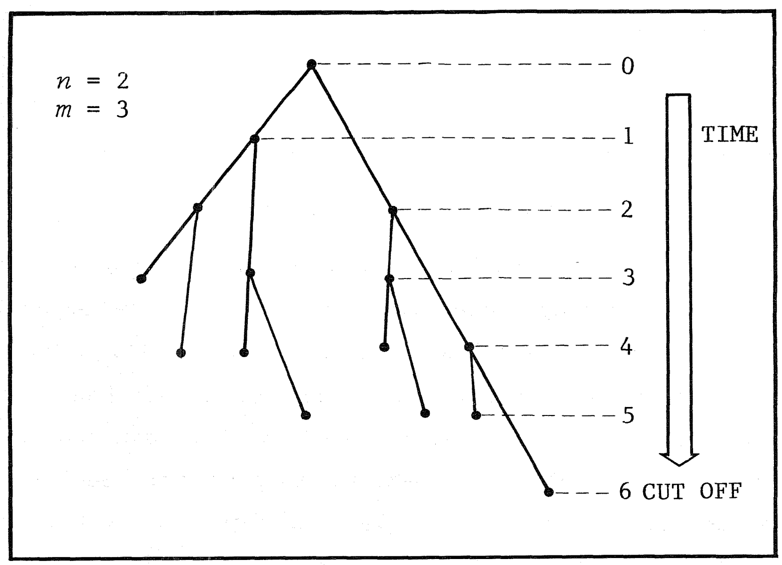

FIGURE 3. Option C, S = 15

in such a way that none will have more than m direct ancestors. We have given a comparison of growth rates between the three options (Figure 4) for the case of two replicas per primary and a limit of three ancestors in case of Option C.

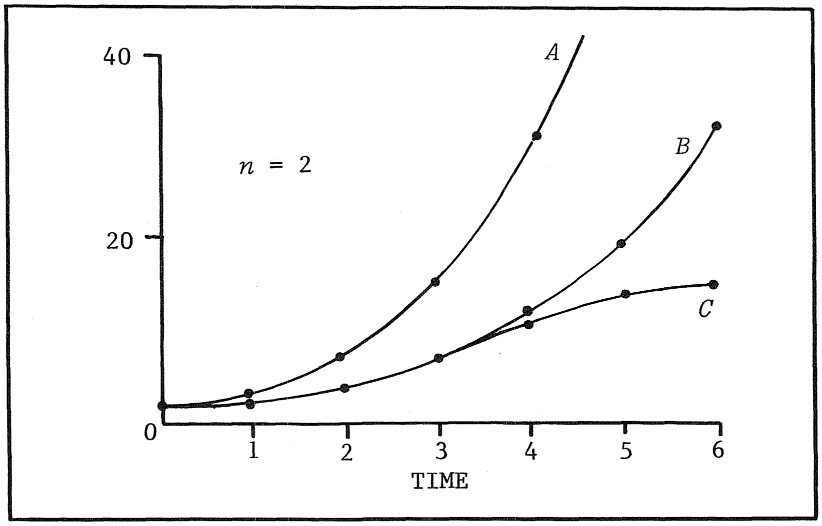

FIGURE 4. Growth Rate Comparison

COMPUTATIONAL ASPECTS OF SELF-REPLICATING SYSTEMS

There are multitudes of novel relationships that one may discover hidden in replicating sequences. We begin by looking at Option A, where replication continues throughout the system until cutoff. The number of replicas s_k generated in the kth time interval is clearly $s_k = n^k$ and accumulates to

$$S_k = \sum_{j=0}^{k} s_k = \frac{n^{k+1} - 1}{n - 1}, \tag{1}$$

so that this option triggers little mathematical curiosity.

In consideration of Option B, we begin with the case of two replications per primary, $n = 2$, and refer to Figure 3. Because each replica produces two offspring, one in each of the two time frames immediately following its own existence, any replica must have come from one of the two previous time frames. This means that the number of replicas s_i produced in the ith time interval equals that produced in the previous two,

$$s_i = s_{i-1} + s_{i-2}. \tag{2}$$

This recursion relation, with $s_0 = s_1 = 1$, gives precisely the Fibonacci numbers. By addition, one computes the total number of replicas S_k in k time intervals to be

$$S_k = s_{k+2} - 1. \tag{3}$$

An explicit formula for s_k is well known, since equation (2) holds:

$$s_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]. \tag{4}$$

Equations (3) and (4) give a formula for the cumulative replicas:

$$S_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+3}\right] - 1. \tag{5}$$

For n replicas per primary

$$s_k = \sum_{i=1}^{n} s_{k-i},$$

s_k and S_k may be calculated by division [2],

$$\frac{1}{1 - \sum_{k=1}^{n} x^k} = \sum_{k=0}^{\infty} s_k x^k \quad \text{and} \quad \frac{1}{(1-x)\left(1 - \sum_{k=1}^{n} x^k\right)} = \sum_{k=0}^{\infty} S_k x^k. \tag{6}$$

Because of linearity, a matrix method can be applied to this problem. To cast Option B with n replicas per primary in the framework of [3], we consider $n + 1$ types of individuals (replicas) denoted by $0, 1, \ldots, n$; the index referring to the number of offsprings this individual has reproduced. One then sets up an $n+1$ by $n+1$ matrix $F = (f_{ij})$, where each individual of type i in the kth time frame gives rise to f_{ij} individuals of type j in the $(k + 1)$th time frame $(1 \leqslant i, j \leqslant n + 1)$ and $k = 0, 1, \ldots$. If the vector $\mathbf{f}(k)$ is the state of the replicas at time k, then $\mathbf{f}(k)F = \mathbf{f}(k+1)$ and by induction $\mathbf{f}(0)F^k = \mathbf{f}(k)$. This means that once we have the matrix F, we can determine the replica state at any future time by matrix multiplication.

For example, let the number of offsprings per replica be 3, $n = 3$. An individual of type i produces a type $i + 1$ and a type 0 if it has reproduced less than 3 and remains a type i if $i = 3$. So if $i < 3$, $f_{i0} = 1$ and $f_{i\,i+1} = 1$, if $i = 3$, $f_{33} = 1$; and $f_{ij} = 0$ otherwise.

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and after four time frames,

$$\mathbf{f}(4) = (1,\ 0,\ 0,\ 0)\quad F^4 = (7,\ 4,\ 2,\ 2);$$

starting with one new replica we have seven with no offspring, four with 1, two with 2, and two with 3 for a total of fifteen.

We now turn to Option C, where the number of replicas is restricted to a fixed number m of generations. In the case where $n = 2$, $m = 3$ (see Figure 2), one observes that the diagram is the same as Option B until the limited number of generations begins to curtail replication; equality ceases after $k = 3$. One observes also that adding one more generation would add two replicas for each with maximal m ancestors; this would add a total of 2^m replicas and, in general, n^m replicas. We find the sum for m generations by adding the terms:

$$S = \sum_{j=0}^{m} n^j = \frac{n^{m+1} - 1}{n - 1}. \tag{7}$$

Again, we are able to use the matrix method [3] to find the state of the replicas at any time. Two indices, a and b, are used to denote the type of replica; the first for the number of offsprings, the second for its ancestors. This, of course, increases the dimension of the matrix by a factor of $m + 1$. A replica of type a, b results in two replicas, one of type $a + 1, b$ and one of type $0, b + 1$ unless $a = n$ or $b = m$, in which case a, b goes into a, b. For $n = 2$ and $m = 3$, one calculates

$$F = \begin{pmatrix} 0&1&0&1&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

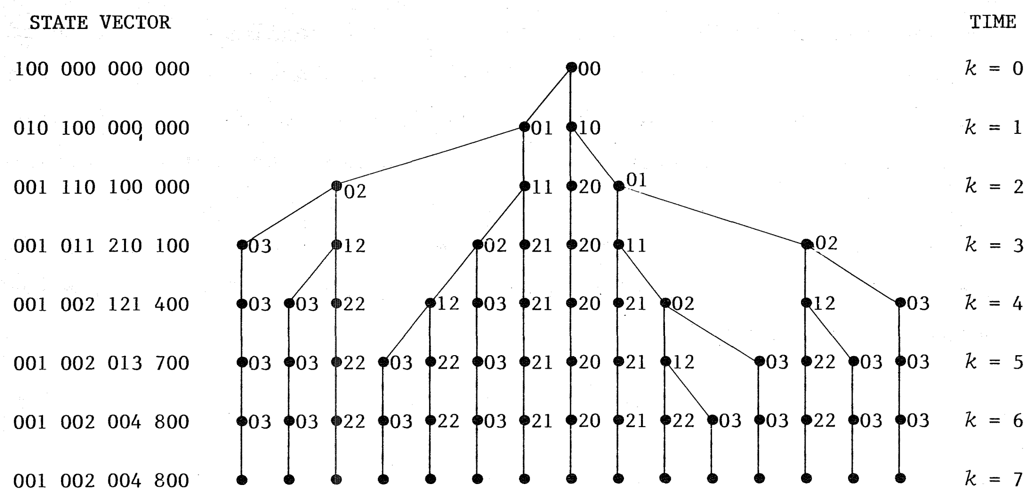

FIGURE 5. Cumulative Diagram

The diagram has a lack of symmetry which cannot be helped; it does, however, place the final replicas equidistant on a straight line and does not move them after they are first placed. The strategy for positioning the replicas is another problem and one we are not going to address. We would like to point out that the matrix is in a form

$$\begin{pmatrix} A & B & 0 & 0 \\ 0 & A & B & 0 \\ 0 & 0 & A & B \\ 0 & 0 & 0 & I \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The A matrix is due to the renaming of existing replicas, while the B matrix is due to replication. The obvious extension of this observation is useful in both setting up the F matrix and its subsequent calculations. Since for Option C there is a limit to the number of generations as well as offsprings, the state vector must eventually be constant. So, for some k and for all integers greater, $\mathbf{f}(0)F^k = \mathbf{f}(0)F^{k+1}$. The minimal such k^* is mn and, further, we note that the sum of the $\mathbf{f}(k)$ coordinates is given by equation (7) as is the sum of the first row of F^{k^*}, since $\mathbf{f}(0) = (1, 0, \ldots, 0)$.

Using the definitions, one can write relationships where complete tables can be generated to show various totals at any time. For n replicas per primary, $s_{m,k}$ denotes the number of replicas produced in the kth time frame under the m generation restriction and $S_{m,k}$ the cumulative number. Similarly, $p_{m,k}$ refers to those coming into production during the kth time frame and $P_{m,k}$ the cumulative number (see Table 1):

$$s_{m,k} = \sum_{i=1}^{n} s_{m-1,k-i}$$

$$s_{m,k+1} = S_{m,k} - p_{m,k}$$

$$p_{m,k} = 2s_{m,k} - s_{m,k+1}$$

$$p_{m,k} = \sum_{i-1}^{n} p_{m-1,k-i}$$

$$p_{m,k} = 2S_{m,k} - S_{m,k+1}$$

For Option B, a replica begins production when it has completed its n replications. Therefore, $p_k = s_{k-n}$ for k less than cutoff; at cutoff, the remaining replicas begin production. Finally, in Option A, since replication is simultaneous, $p_k = s_{k-1}$.

TABLE 1

Results for four replicas per primary with m = 2 and m = 3

k	$s_{2,k}$	$p_{2,k}$	$S_{2,k}$	$P_{2,k}$	$s_{3,k}$	$p_{3,k}$	$S_{3,k}$	$P_{3,k}$
0	1	0	1	0	1	0	1	0
1	1	0	2	0	1	0	2	0
2	2	1	4	1	2	0	4	0
3	3	2	7	3	4	1	8	1
4	4	4	11	7	7	4	15	5
5	4	5	15	12	10	7	25	12
6	3	4	18	16	13	12	38	24
7	2	3	20	19	14	15	52	39
8	1	2	21	21	13	16	65	55
9	0	0	21	21	10	14	75	69
10	0	0	21	21	6	9	81	78
11	0	0	21	21	3	5	84	83
12	0	0	21	21	1	2	85	85
13	0	0	21	21	0	0	85	85
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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WHY ARE 8:18 AND 10:09 SUCH PLEASANT TIMES?

M. G. MONZINGO

Mathematics Department, S.M.U., Dallas, TX 75275

To rephrase this facetious question: Why does a watch or a clock appear most pleasing when its hands are set at approximately 8:18 or 10:09? In case the reader has not noticed, nondigital watches and clocks (not running) on display in stores, or photographs of them in catalogs, often are set very nearly at one of these two times. One common myth concerning the time 8:18 (or 8:17) is that this is precisely the time at which President Abraham Lincoln died. In [1, p. 394], this myth is discussed. In reference to clock faces painted on signs, it is suggested that 8:17 is used for the setting of the hands to allow more space on the clock face for advertising.

The purpose of this note is to investigate the aforementioned question. In the process, interesting relationships between these two times and the golden ratio will be discovered.

First, one observes that at both 8:18 and 10:09 the angle between 12 o'clock and the hour hand is approximately equal to the angle between 12 o'clock and the minute hand. Of course, 8:20 and 10:10 would be "equal-angled" if the hour hnad moved in discrete hourly jumps rather than moving continuously. Certainly, then, symmetry plays a key role.

Theorem

For the times listed in the table, the clock hands are approximately "equal-angled."

Proof: The conclusion can be drawn by observing a clock or by using the following analysis. Let α be the angle formed by 12 o'clock and the minute hand and β the angle formed by 12 o'clock and the hour hand. Then, since each hour yields 30°,

$$\beta = (\alpha/360°)30° = \alpha/12. \qquad (*)$$

For equal angles,

$360° - \beta$ differs from α by some integral multiple of $360°$;

hence

$$\alpha + \beta = 360°k. \tag{**}$$

From (*) and (**), $\alpha = 12 \cdot 360°k/13$.

Now, the hour, h, is $[\beta/30°]$ (30° per hr.), the minute, m, is $[5\alpha/30°] - 60h$ (30° per 5 min.), and the second is 60 times the "decimal part" of m. The following table, generated by varying k, lists hours, minutes, seconds and, most importantly, the angle β. Note that all of the angles have been reduced to <90° so that, for half of the listed times, the angle is measured with respect to 6 o'clock, e.g., 5:32.

TABLE

Hour	Minute	Second	Angle (in degrees)
12	55	23	27.7
1	50	46	55.4
2	46	9	83.1
3	41	32	69.2*
4	36	55	41.5*
5	32	18	13.8*
6	27	42	13.8*
7	23	5	41.5*
8	18	28	69.2*
9	13	51	83.1
10	9	14	55.4
11	4	37	27.7

*Measured with respect to 6 o'clock.

The time 8:18 will be investigated first; for this, the following lemma will be useful.

Lemma

Let φ be the golden ratio, $a = \text{Arctan } \varphi$, and $b = 90° - a$; then,

$$\tan 2b = 2.$$

Proof: Since $\tan a = \varphi$, $\tan^2 a - \tan a - 1 = 0$; hence, division by $\tan a - \cot a = 1$. Then,

$$\tan 2b = \frac{2 \tan b}{1 - \tan^2 b} = \frac{2}{\cot b - \tan b} = \frac{2}{\tan a - \cot a} = \frac{2}{1} = 2.$$

Note: $a \approx 58.3°$.

Now, if one were to visualize a rectangle (see Figure 1) superimposed on a clock face at the time 8:18 (or at 3:41 when the hands are reversed) using the hour and the minute hands to form semidiagonals, one would see a rectangle whose corners were approximately at minutes 12, 18, 42, and 48. At these particular times, β ($\approx 69.2°$) is very nearly $2b$ ($\approx 63.4°$); in fact, the relative error,

$$\frac{69.2° - 63.4°}{63.4°}$$

is less than 10%. From the lemma, it follows that the imagined rectangle is approximately proportioned 2 to 1. That is, the rectangle would (almost) be formed by two squares. By checking the table, one can see that 8:18 (and 3:41) give the "equal-angle" times for which the imagined rectanble most closely approximates such a rectangle.

The imagined rectangle at 10:09 (or 1:50) is even more significant. If one were to visualize a rectangle (see Figure 2) at these times, one would see a rectangle whose corners were approximately at the minutes 9, 21, 39, and 51. At these particular times, β ($\approx 55.4°$) is very nearly Arctan φ ($\approx 58.3°$); in fact, the relative error,

$$\frac{58.3° - 55.4°}{58.3°}$$

is less than 5%. Therefore, at 10:09 (and 1:50), the imagined rectangle is approximately a golden rectangle. A check of the table shows that these times give the "equal-angle" times for which the imagined rectangle most closely approximates the golden rectangle.

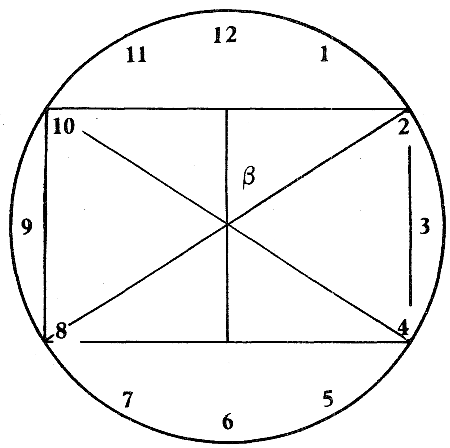

FIGURE 1

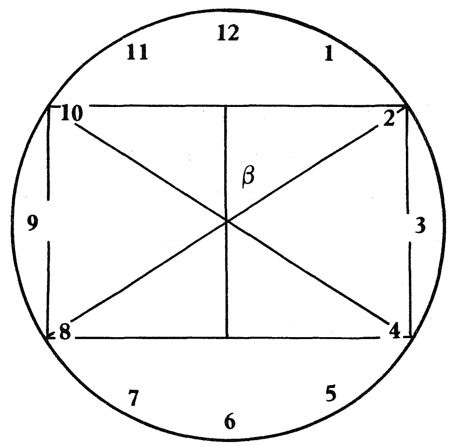

FIGURE 2

Perhaps the close association with the golden ratio for 8:18 and the good approximation to the golden rectangle for 10:09 are the reasons why these two times are chosen for display purposes.

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ON EXPONENTIAL SERIES EXPANSIONS AND CONVOLUTIONS

M. E. COHEN

California State University, Fresno, CA 93740

and

D. L. HUDSON

University of California, Davis, CA 95616

(Submitted January 1982)

1. INTRODUCTION

With the aid of the Lagrange Theorem, Pólya and Szegö [10, pp. 301, 302, Problems 210, 214] deduced the very important expansions

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(w)^n (\ell n + 1)^{n-1}}{n!}, \quad w = -ze^{z\ell}, \tag{1.1}$$

and

$$\frac{e^{-z}}{1 + z\ell} = \sum_{n=0}^{\infty} \frac{(w)^n (\ell n + 1)^n}{n!}. \tag{1.2}$$

For applications of the above equations, see Cohen [4], Knuth [8, Section 2.3.4.4], Riordan [12, Section 4.5]. In fact, (1.1) was of interest to Ramanujan [11, p. 332, Question 738]. The higher-dimensional extensions and their ramifications were studied by Carlitz [1], [2], Cohen [5], and others.

A two-dimensional generalization of (1.2) is one result presented in this paper:

For α, λ, a, c real or complex,

$$\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)^p (-y)^k \exp\left[\dfrac{x(\lambda + cp)}{(\alpha + ak)} + \dfrac{y(\alpha + ak)}{(\lambda + cp)}\right] (\alpha + ak)^{k-p} (\lambda + cp)^{p-k}}{p!k!}$$

$$= \frac{1 - \dfrac{acxy}{\alpha\lambda}}{\left(1 + \dfrac{ay}{\lambda}\right)\left(1 + \dfrac{cx}{\alpha}\right)} \tag{1.3}$$

where the double series is assumed convergent.

$x = 0$, along with other appropriate substitutions, reduces (1.3) to (1.2). For other similar two-dimensional exponential series, see Carlitz [2, Equations (1.4) and (1.9)] and Cohen [5, Equation 2.28].

With the aid of (1.3), we obtain a new convolution:

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left[-\frac{(\lambda+cp)}{(\alpha+ak)}\right]^{p}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\left[-\frac{(\alpha+ak)}{(\lambda+cp)}\right]^{k}\left[\frac{(\alpha+ak)}{(\lambda+cp)}+t\right]^{n-k}}{p!(m-p)!k!(n-k)!}$$
$$=\frac{s^m}{m!}\sum_{j=0}^{n}\frac{(-a/\lambda)^j t^{n-j}}{(n-j)!}-\frac{s^m t^n}{m!n!}+\frac{t^n}{n!}\sum_{i=0}^{m}\frac{(-c/\alpha)^i s^{m-i}}{(m-i)!}. \tag{1.4}$$

(1.4) may be considered as a two-dimensional extension of the Abel-type Gould [7] convolution. See also Carlitz [3] and, for another type of two-dimensional generalization, refer to Cohen [6]. Letting $m = 0$ in (1.4) and simplifying, one obtains the expressions (2) and (4) given in [6]. For an excellent discussion of convolutions, see Riordan [12, Sections 1.5 and 1.6].

A two-dimensional generalization of both (1.1) and (1.2) is also presented here:

For α, λ, μ, a, c, d real or complex,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-x)^p(-y)^k\exp\left[\frac{x(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]}{k!p!}$$
$$\cdot(\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k=\frac{1}{(\lambda+ya\mu)}, \tag{1.5}$$

where the double series is assumed convergent.

$y = 0$ and simplification gives (1.1), and $x = 0$ and reduction yields (1.2).

(1.5) is employed in the proof of the new expression:

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left[\frac{-(\lambda+cp)}{(\alpha+ak)}\right]^{p}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\left[\frac{-(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]^{k}\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^{n-k}}{p!(m-p)!k!(n-k)!(\lambda+cp)}$$
$$=\frac{s^m}{\lambda m!}\sum_{j=0}^{n}\frac{(-a\mu/\lambda)^j t^{n-j}}{(n-j)!}. \tag{1.6}$$

(1.6) may be regarded as a two-dimensional extension of the Abel-type Gould convolution to which it reduces for $m = 0$.

Another generalization of (1.1) is the expression,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-x)^p(-y)^k}{k!p!}\exp\left\{\frac{x(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right\}$$

$$\cdot\,(\alpha+ak)^{k-p-1}(\beta+bk)^p(\lambda+cp)^{p-k-1}(\mu+dp)^k$$

$$=\frac{1}{\alpha\lambda}\,{}_1F_2\left[1;\ (\alpha/a)+1,\ (\lambda/c)+1;\ \left(\frac{\alpha b}{a}-\beta\right)\left(\frac{\lambda d}{c}-\mu\right)xy\right], \qquad (1.7)$$

where α, β, λ, μ, a, b, c, d are real or complex, and the double series is assumed to be convergent.

The ${}_1F_2$ hypergeometric function is defined in Luke [9, p. 155]. In fact, this particular function is called the "Lommel function," given by [9, p. 413, Equation 1]. Letting $x = 0$ in (1.7) gives (1.1).

With the aid of (1.7), we are able to prove the expansion,

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left(\left[\frac{-(\beta+bk)(\lambda+cp)}{(\alpha+ak)}\right]^p\left[\frac{(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\cdot\left[\frac{-(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]^k\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^{n-k}\right)}{p!(m-p)!k!(n-k)!(\alpha+ak)(\lambda+cp)}$$

$$=\frac{1}{\alpha\lambda}\sum_{i=0}^{\min(m,n)}\frac{s^{m-i}t^{n-i}\left(\frac{\alpha b}{a}-\beta\right)^i\left(\frac{\lambda d}{c}-\mu\right)^i}{(m-i)!(n-i)!\left(\frac{\alpha}{a}+1\right)_i\left(\frac{\lambda}{c}+1\right)_i}, \qquad (1.8)$$

where $(a)_n = (a)(a+1)\cdots(a+n-1)$ for $n > 0$,
$\qquad\qquad = 1$ for $n = 0$.

The proofs of Equations (1.3) through (1.8) are given in the following section.

2. PROOFS OF EQUATIONS (1.3) THROUGH (1.8)

Proof of (1.3)

Consider the expression

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^my^n}{m!n!}(xD)^{n-m}[x^{\alpha}(1-x^a)^n](xD)^{m-n}[x^{\lambda}(1-x^c)^m]. \qquad (2.1)$$

At $x = 1$, it may be expanded to give

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m y^n}{m!n!}\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{(-n)_k(\alpha + ak)^{n-m}(-m)_p(\lambda + cp)^{m-n}}{k!p!} \tag{2.2}$$

$$= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-k)^p(-y)^k}{p!k!}\exp\left[\frac{x(\lambda + cp)}{(\alpha + ak)} + \frac{y(\alpha + ak)}{(\lambda + cp)}\right](\alpha + ak)^{k-p}(\lambda + cp)^{p-k}. \tag{2.3}$$

The double series transformation,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(n,\ k) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f(n + k,\ k) \text{ and } (-n)_k = \frac{(-1)^k n!}{(n - k)!}, \tag{2.4}$$

is used over k, n and p, m in going from (2.2) to (2.3). Also, after employing the transformation, the series over m and n are summed to give the exponentials.

Returning to (2.1), it may be observed that the only contributions in that expression give

$$\sum_{n=0}^{\infty}\frac{y^n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}(\alpha + ak)^n\lambda^{-n} + \sum_{m=1}^{\infty} x^m\alpha^m(-c)^m \tag{2.5}$$

$$= \frac{1 - \dfrac{acxy}{\alpha\lambda}}{\left(1 + \dfrac{ay}{\lambda}\right)\left(1 + \dfrac{cx}{\alpha}\right)} \tag{2.6}$$

(2.5) reduces to (2.6) with the aid of (2.4) and series simplification. Equating (2.3) and (2.6) gives the result (1.3).

Proof of (1.4)

Assuming (1.3), multiply both sides of that equation by $\exp[sx + ty]$. The exponentials may be expanded, and the left-hand side assumes the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p(-1)^k x^p y^k(\alpha + ak)^{k-p}(\lambda + cp)^{p-k}}{p!k!}$$
$$\frac{x^m}{m!}\left[\frac{(\lambda + cp)}{(\alpha + ak)} + s\right]^m\frac{y^n}{n!}\left[\frac{(\alpha + ak)}{(\lambda + cp)} + t\right]^n. \tag{2.7}$$

The right-hand side may be expanded to give

$$\left(1 - \frac{ac}{\alpha\lambda}xy\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x^m y^n \cdot \left\{\frac{s^m}{m!}\sum_{j=0}^{n}\frac{(-a/\lambda)^j t^{n-j}}{(n - j)!} - \frac{s^m t^n}{m!n!} + \frac{t^n}{n!}\sum_{i=0}^{m}\frac{(-c/\alpha)^i s^{m-i}}{(m - i)!}\right\}. \tag{2.8}$$

(2.7) may be expressed as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}x^m y^n \sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left[\frac{-(\lambda+cp)}{(\alpha+ak)}\right]^p\left[\frac{-(\alpha+ak)}{(\lambda+cp)}\right]^k\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\left[\frac{(\alpha+ak)}{(\lambda+cp)}+t\right]^{n-k}}{p!k!(m-p)!(n-k)!} \tag{2.9}$$

Comparing coefficients between Equations (2.8) and (2.9) gives the result (1.4).

Proof of (1.5)

Consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^m y^n}{m!n!}(xD)^{n-m}[x^{\alpha}(1-x^a)^n](y\delta)^n\left[x^{-\lambda+\frac{\mu c}{d}}(xD)^{m-n-1}[x^{\lambda}(1-x^c)^m]\right], \tag{2.10}$$

where $y = x^{c/d}$, $D \equiv \frac{d}{dx}$, $\delta \equiv \frac{d}{dy}$.

Following the procedure adopted in the proof of (1.3), (2.10) assumes the form

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-x)^p(-y)^k}{p!k!}\exp\left[\frac{x(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]$$
$$\cdot\ (\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k. \tag{2.11}$$

Referring to (2.10), it may be seen that at $x = 1$ for $n \geqslant m$, only $m = 0$ contributes and for $n < m$, the expression is zero. Hence, we have

$$\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{-ay\mu}{\lambda}\right)^n. \tag{2.12}$$

Equating (2.11) and (2.12) gives the result (1.5).

Proof of (1.6)

Following the procedure given in the proof of (1.4), the left-hand side of (1.5) multiplied by $\exp[sx+ty]$ may be expanded as

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^p(-1)^k x^p y^k}{p!k!}(\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k$$
$$\frac{x^m}{m!}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^m\frac{y^n}{n!}\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^n. \tag{2.13}$$

The right-hand side reduces to

$$\frac{1}{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m y^n s^m}{m!}\sum_{j=0}^{n}\frac{(-a\mu/\lambda)^j t^{n-j}}{(n-j)!}$$

Equating coefficients in (2.13) and (2.14) gives Equation (1.6).

Proof of (1.7)

Consider the operators

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^m y^n}{m!n!}(y_1\delta_1)^n\left[x^{-\lambda+\frac{\mu c}{d}}(xD)^{m-n-1}[x^{\lambda}(1-x^c)^m]\right]$$
$$\cdot\ (y_2\delta_2)^m\left[x^{-\alpha+\frac{\beta a}{b}}(xD)^{n-m-1}[x^{\alpha}(1-x^a)^n]\right], \tag{2.15}$$

where $y_1 = x^{c/d}$, $y_2 = x^{a/b}$, $D \equiv \dfrac{d}{dx}$, $\delta_1 \equiv \dfrac{d}{dy_1}$, $\delta_2 \equiv \dfrac{d}{dy_2}$.

As in the proof of (1.3) and (1.5), (2.15) reduces to

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-x)^p(-y)^k}{p!k!}\exp\left\{\frac{x(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right\}$$
$$\cdot\ (\alpha+ak)^{k-p-1}(\beta+bk)^p(\lambda+cp)^{p-k-1}(\mu+dp)^k. \tag{2.16}$$

Now, looking at (2.15), at $x = 1$, and noting that

$$\sum_{k=0}^{n}\frac{(-n)_k(\beta+bk)^n}{k!(\lambda+ck)}=\frac{n!\left(\beta-\frac{\lambda b}{c}\right)^n\Gamma\left(\frac{\lambda}{c}\right)}{c\,\Gamma\left(\frac{\lambda}{c}+n+1\right)} \tag{2.17}$$

with the only contributions coming from $m = n$, one has the reduced expression

$$\frac{1}{\alpha\lambda}\sum_{n=0}^{\infty}\frac{(xy)^n\left(\frac{\alpha b}{a}-\beta\right)^n\left(\frac{\lambda d}{c}-\mu\right)^n}{\left(\frac{\alpha}{a}+1\right)_n\left(\frac{\lambda}{c}+1\right)_n} \tag{2.18}$$

Comparing (2.16) and (2.18) gives (1.7).

Proof of (1.8)

Assuming the expansion (1.7) and following the type of proof adopted for (1.6), with suitable modifications, Equation (1.8) is obtained.

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NOTES ON FIBONACCI TREES AND THEIR OPTIMALITY*

YASUICHI HORIBE

Shizuoka University, Hamamatsu, 432, Japan

(Submitted February 1982)

INTRODUCTION

Continuing a previous paper [3], some new observations on properties and optimality of Fibonacci trees will be given, beginning with a short review of some parts of [3] in the first section.

1. FIBONACCI TREES

Consider a binary tree (rooted and ordered) with $n-1$ internal nodes (each having two sons) and n terminal nodes or leaves. A node is at level ℓ if the path from the root to this node has ℓ branches. Assign unit cost 1 to each left branch and cost c ($\geqslant 1$) to each right branch. The cost of a node is defined to be the sum of costs of branches that form the path from the root to this node. Further, we define the *total cost* of a tree as the sum of costs of all terminal nodes. For a given number of terminal nodes, a tree with minimum total cost is called *optimal*. Suppose we have an optimal tree with n terminal nodes. Split in this tree any one terminal node of minimum cost to produce two new terminal nodes. Then the resulting tree with $n+1$ terminal nodes will be optimal. This growth procedure is due to Varn [6]. (For a simple proof of the validity of this procedure, see [3].)

A beautiful class of binary trees is the class of Fibonacci trees (for an account, see [5]). The *Fibonacci tree of order* k has F_k terminal nodes, where $\{F_k\}$ are the Fibonacci numbers

$$F_0 = 0,\ F_1 = 1,\ F_k = F_{k-1} + F_{k-2},$$

and is defined inductively as follows: If $k = 1$ or 2, the Fibonacci tree

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of order k is simply the root only. If $k \geqslant 3$, the left subtree of the Fibonacci tree of order k is the Fibonacci tree of order $k - 1$; and the right subtree is the Fibonacci tree of order $k - 2$. The Fibonacci tree of order k will be denoted by T_k for brevity.

Let us say that T_k is *c-optimal*, if it has the minimum total cost of all binary trees having F_k terminal nodes, when cost c is assigned to each right branch, and cost 1 to each left branch.

We have the following properties [3]:

(A) T_k, $k \geqslant 2$, with cost $c = 2$ has F_{k-1} terminal nodes of cost $k - 2$ and F_{k-2} terminal nodes of cost $k - 1$.

(B) Splitting all terminal nodes of cost $k - 2$ in T_k with $c = 2$ produces T_{k+1}.

(C) T_k is 2-optimal for every k.

By the properties (A) and (B), it may be natural to classify the terminal nodes of T_k into two types, α and β: A terminal node is of type α (α-*node* for short) [respectively, type β (β-*node* for short)], if this node becomes one of the lower [higher] cost nodes when $c = 2$.

See Figure 1. (T_1 and T_2 consist only of a root node. In order that the assignment of types to nodes will satisfy the inductive construction in Lemma 1 below, we take the convention that the node in T_1 is of type β and the node in T_2 is of type α.)

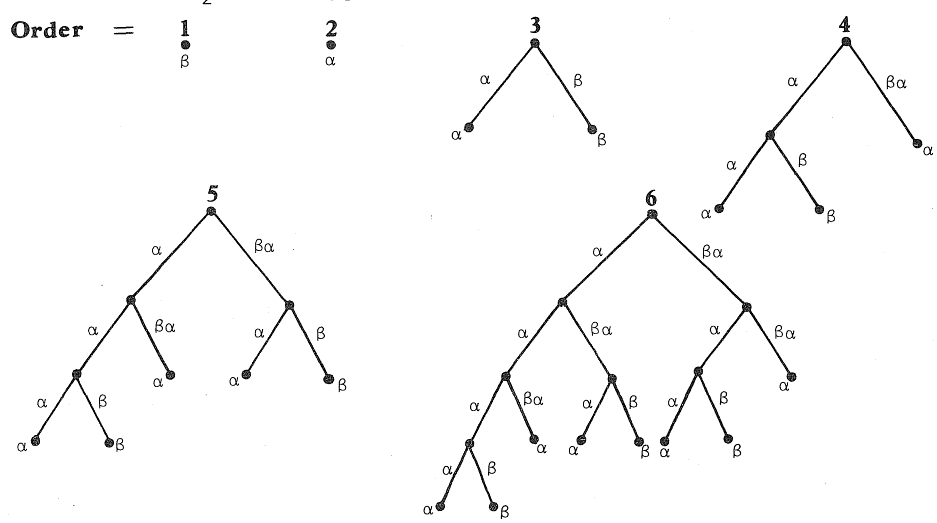

FIGURE 1. Fibonacci Trees (see Section 2 for branch labeling)

Lemma 1

The type determination within each of the left and right subtrees gives the correct type determination for the whole tree.

Proof (induction on order k): Trivially true for T_3. Consider T_k, $k \geqslant 4$, with $c = 2$. The left [right] subtree is T_{k-1} [T_{k-2}], so within this subtree, by (A), the α-nodes have cost $k - 3$ [$k - 4$] and the β-nodes have cost $k - 2$ [$k - 3$]. But in the whole tree, these α-nodes have cost

$$(k - 3) + 1 = k - 2 \quad [(k - 4) + 2 = k - 2],$$

hence, they are still of type α, and these β-nodes have cost

$$(k - 2) + 1 = k - 1 \quad [(k - 3) + 2 = k - 1],$$

hence, they are still of type β. This completes the proof.

Before going to the next section, we show two things. First, let us see that T_k with $c = 2$ has F_{j+1} internal nodes of cost j, $j = 0, 1, \ldots, k - 3$. In fact, T_{j+2} has F_{j+1} nodes of cost j, and they must all be terminal, by (A). Split all these α-nodes, then the resulting tree T_{j+3}, by (B), has F_{j+1} internal nodes of cost j, and so does every Fibonacci tree of order greater than $j + 3$.

Secondly, let us see what happens when we apply the operation "split all α-nodes" $n - 1$ times successively to T_{m+1}. The tree produced is, of course, the Fibonacci tree of order $(m + 1) + (n - 1) = m + n$, by (B). On the other hand, the β-nodes in the original tree of order $m + 1$ will change into α-nodes when the α-nodes in this tree are split to produce the tree of order $m + 2$. Hence, each of the F_m [resp. F_{m-1}] α-nodes [β-nodes] in the original tree of order $m + 1$ will become the root of T_{n+1} [T_n] when the whole process is completed. By counting the terminal nodes, we have obtained a "proof-by-tree" of the well-known relation [4]:

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n.$$

2. NUMBER OF TERMINAL NODES AT EACH LEVEL

In this section, we shall show the following:

Theorem 1

The number of α-nodes at level ℓ of the Fibonacci tree of order $k \geqslant 2$ is given by $\binom{\ell}{k-2-\ell}$, and the number of β-nodes is given by $\binom{\ell-1}{k-2-\ell}$, $\ell = 0, 1, \ldots, k-2$. [*Remark*: The height (the maximum level) of the Fibonacci tree of order $k \geqslant 2$ is $k-2$.]

Before proving this theorem, let us look at the Fibonacci trees more closely with the aid of the following branch labeling. We label (inductively on order k) each branch with one of the three signs, α, $\beta\alpha$, β, as follows: In T_3, the left branch is labeled α, and the right branch is labeled β. Suppose the labeling is already done for T_{k-1} and T_{k-2}. Let these labeled trees be the left and right subtrees of T_k, respectively, and let the left and right branches that are incident to the root of T_k be labeled α and $\beta\alpha$, respectively (see Figure 1). (The branch labeling may have the following "tree-growth" interpretation: Every branching occurs at discrete times $k = 3, 4, \ldots$, and produces two different types of branches α, β. Suppose a branching occurs at time k. The α-branch produced at this time is "ready" for similar branching at time $k+1$, but the β-branch must "mature" into a $\beta\alpha$-branch at time $k+1$ to branch at time $k+2$.) This labeling rule immediately implies that every left branch is labeled α and every right branch not incident to a terminal node of type β is labeled $\beta\alpha$.

Now, by *F-sequence* (called *PM* sequence in [2]), we mean a sequence of α and β with no two β's adjacent. It is easy to see, by induction on order k, that paths (by which we always mean paths from the root to terminal nodes) in T_k correspond, in one-to-one manner, to F-sequences of length $k-2$ obtained by concatenating branch labels along paths, and that all possible F-sequences of length $k-2$ appear in T_k; hence, there are F_k F-sequences of length $k-2$ in all. For example, if we enumerate all paths in T_6 (see Figure 1) "from left to right," we have eight ($=F_6$) F-sequences of length 4: $\alpha\alpha\alpha\alpha$, $\alpha\alpha\alpha\beta$, $\alpha\alpha\beta\alpha$, $\alpha\beta\alpha\alpha$, $\alpha\beta\alpha\beta$, $\beta\alpha\alpha\alpha$, $\beta\alpha\alpha\beta$, $\beta\alpha\beta\alpha$.

Proof of Theorem 1: It is also easy to show, using Lemma 1 and by induction on order k, that any path leading to an α-node [resp. a β-node] corresponds to an F-sequence ending with α [β]. Therefore, the number of

α-nodes at level ℓ of T_k is the number of F-sequences of length $k - 2$ ending with α and composed of ℓ α's and $k - 2 - \ell$ β's. The number of such F-sequences is the number of ways to choose $k - 2 - \ell$ positions to receive a β from the ℓ starred positions in the alternating sequence $*\alpha*\alpha \ldots *\alpha$. This is $\binom{\ell}{k-2-\ell}$. Similarly, the number of β nodes at level ℓ of T_k is the number of F-sequences of length $k - 2$ ending with β and composed of $\ell - 1$ α's and $k - 1 - \ell$ β's. The number of such F-sequences is the number of ways to choose $k - 2 - \ell$ positions to receive a β from the $\ell - 1$ starred positions in the (almost) alternating sequence $*\alpha*\alpha \ldots *\alpha\beta$. This is $\binom{\ell-1}{k-2-\ell}$. This completes the proof.

Note that, since

$$\binom{\ell - 1}{k - 2 - \ell} = \binom{\ell - 1}{k - 3 - (\ell - 1)},$$

the number of β-nodes at level $\ell \geqslant 1$ of the Fibonacci tree of order $k \geqslant 3$ equals the number of α-nodes at level $\ell - 1$ of the Fibonacci tree of order $k - 1$.

Now, let us look at a relation between the numbers of the terminal nodes of each type and some sequences of binomial coefficients appearing in the Pascal triangle. Draw diagonals in the Pascal triangle as shown in Figure 2. It is well known ([2], [4]) that, if we add up the numbers between the parallel lines, the sums are precisely the Fibonacci numbers.

F_1	1								
F_2	1	1							
F_3	1	2	1						
F_4	1	3	3	1					
F_5	1	4	6	4	1				
F_6	1	5	10	10	5	1			
F_7	1	6	15	20	15	6	1		
F_8	1	7	21	35	35	21	7	1	
F_9	1	8	28	56	70	56	28	8	1

. . . .

FIGURE 2. Pascal Triangle

We observe that the sequences totalling F_{k-2} and F_{k-1} in the triangle

$$F_{k-2}\colon \quad \binom{k-3}{0}, \binom{k-3}{1}, \binom{k-3}{2}, \ldots, \binom{\ell-1}{k-2-\ell}, \ldots$$

$$F_{k-1}\colon \quad \binom{k-2}{0}, \binom{k-2}{1}, \binom{k-2}{2}, \ldots, \binom{\ell}{k-2-\ell}, \ldots$$

$$\text{level} = k-2, \quad k-3, \quad k-4, \quad \ldots, \quad \ell, \quad \ldots$$

display the numbers of the β-nodes and the α-nodes, respectively, at decreasing levels of T_k. For example, we find in Figure 2 that T_{10} has 15 α-nodes and 10 β-nodes at level 6. In [1], the total number of terminal nodes at level ℓ of T_k is also given (with a slightly different interpretation) but not in the form of the sum of two meaningful numbers:

$$\binom{\ell}{k-2-\ell} + \binom{\ell-1}{k-2-\ell}.$$

3. c-OPTIMALITY OF FIBONACCI TREES

Property (C) above states that T_k is 2-optimal for every k. In this section we prove the following.

Theorem 2

When $1 \leqslant c < 2$, the Fibonacci tree of order $k \geqslant 3$ is c-optimal if and only if

$$k \leqslant 2\left\lfloor \frac{1}{2-c} \right\rfloor + 3.$$

When $c > 2$, the Fibonacci tree of order $k \geqslant 3$ is c-optimal if and only if

$$k \leqslant 2\left\lfloor \frac{1}{c-2} \right\rfloor + 4.$$

($\lfloor x \rfloor$ is the largest integer $\leqslant x$.)

To prove the theorem, we first note the following: T_k, $k \geqslant 5$, has the shape shown in Figure 3 and Figure 4, and $k-2$ $(k \geqslant 3)$ is the maximum level of T_k, where both α- and β-nodes exist, because from Theorem 1 the maximum level of T_k must be $\leqslant k-2$ and $\ell = k-2$ gives

$$\binom{\ell}{k-2-\ell} = \binom{\ell-1}{k-2-\ell} = 1 \text{ if } k \geqslant 3.$$

The minimum level where a terminal α-node [resp. β-node] exists is given by

$$\left\lfloor \frac{k-1}{2} \right\rfloor \left[\left\lfloor \frac{k}{2} \right\rfloor \right],$$

the smallest integer ℓ satisfying $k - 2 - \ell \leqslant \ell$ $[k - 2 - \ell \leqslant \ell - 1]$, from Theorem 1 (see Figures 3 and 4).

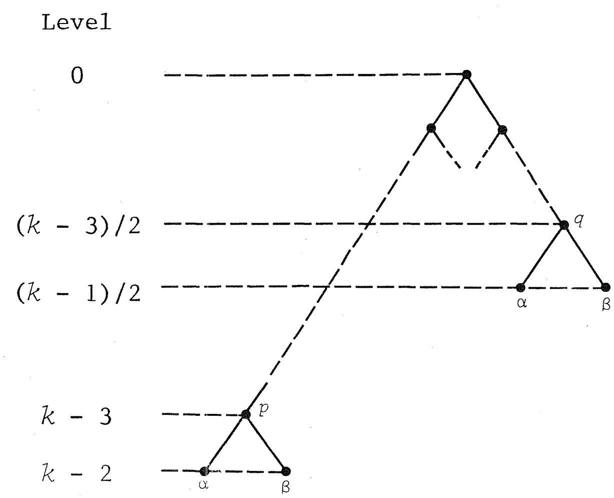

FIGURE 3. Fibonacci Tree of Odd Order $k \geqslant 5$

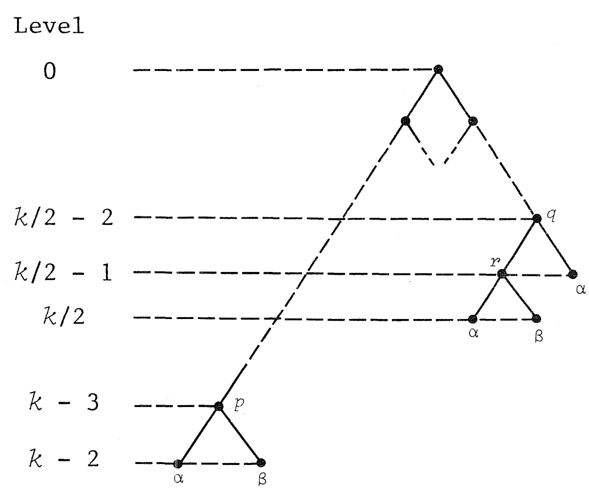

FIGURE 4. Fibonacci Tree of Even Order $k \geqslant 6$

Proof of the "only if" part of Theorem 2

Trivial for $k = 3, 4$.

Case $1 \leqslant c < 2$, odd $k \geqslant 5$: See Figure 3. Change T_k into a non-Fibonacci tree having F_k terminal nodes by deleting the two sons of the node p and by splitting the left son of the node q. Let us compute the change in the total cost by this transformation. Deletion of the old vertices saves $(k - 3) + (1 + c) = s$. The new vertices add cost

$$1 + c\left(\frac{k - 3}{2}\right) + (1 + c) = t.$$

The net change in cost is

$$t - s = 1 + (c - 2)\left(\frac{k - 3}{2}\right).$$

If T is c-optimal, we must have $t - s \geqslant 0$, so

$$\frac{k - 3}{2} \leqslant \frac{1}{2 - c} \qquad \text{or} \qquad k \leqslant \frac{2}{2 - c} + 3.$$

Case $1 \leqslant c < 2$, even $k \geqslant 6$: See Figure 4. Change T_k into a non-Fibonacci tree having F_k terminal nodes by deleting the two sons of the node p and by splitting the right son of the node q. Again, if t is the added cost of the new vertices and s the savings from deleting old vertices, we have $s = (k - 3) + (1 + c)$, $t = 1 + c(k/2)$, so

$$t - s = 1 + (c - 2)\left(\frac{k - 2}{2}\right).$$

If T_k is c-optimal, we must have $t - s \geqslant 0$, so

$$\frac{k - 2}{2} \leqslant \frac{1}{2 - c} \qquad \text{or} \qquad k \leqslant \frac{2}{2 - c} + 2.$$

The conditions $k \leqslant \frac{2}{2 - c} + 3$ for k odd and $k \leqslant \frac{2}{2 - c} + 2$ for k even can be combined to get

$$k \leqslant 2\left\lfloor \frac{1}{2 - c} \right\rfloor + 3.$$

Case $c > 2$, odd $k \geqslant 5$: See Figure 3. Change T_k into a non-Fibonacci tree having F_k terminal nodes by deleting the two sons of the node q and by splitting the left son of the node p. Here

$$s = 1 + c\left(\frac{k-1}{2}\right),\ t = 1 + (k - 2 + c),\ t - s = (2 - c)\left(\frac{k-3}{2}\right) + 1.$$

Fibonacci c-optimality requires $t - s \geqslant 0$, so

$$\frac{k-3}{2} \leqslant \frac{1}{c-2} \qquad \text{or} \qquad k \leqslant \frac{2}{c-2} + 3.$$

Case $c > 2$, even $k \geqslant 6$: See Figure 4. Change T_k into a non-Fibonacci tree having F_k terminal nodes by deleting the two sons of the node r and by splitting the left son of the node p. Here

$$s = 1 + c\left(\frac{k}{2} - 2\right) + (1 + c),\ t = 1 + (k - 2) + c,$$

$$t - s = (2 - c)\left(\frac{k-4}{2}\right) + 1.$$

Fibonacci c-optimality requires $t - s \geqslant 0$, so

$$\frac{k-4}{2} \leqslant \frac{1}{c-2} \qquad \text{or} \qquad k \leqslant \frac{2}{c-2} + 4.$$

The conditions $k \leqslant \dfrac{2}{c-2} + 3$ for k odd and $k \leqslant \dfrac{2}{c-2} + 4$ for k even can be combined to get

$$k \leqslant 2\left\lfloor\frac{1}{c-2}\right\rfloor + 4.$$

Our proof of the "if" part of the theorem will be based on the next lemma.

Lemma 2

Denote by $\alpha(k, \ell, c)$ and $\beta(k, \ell, c)$ the costs of the α-nodes and the β-nodes at level ℓ of the Fibonacci tree of order $k \geqslant 3$ with cost c for right branches. Then we have:

$$\alpha(k, \ell, c) = (2 - c)\ell + (c - 1)(k - 2),$$

$$\beta(k, \ell, c) = (2 - c)\ell + (c - 1)(k - 1).$$

Proof: Obviously, $\alpha(k, \ell, 1) = \beta(k, \ell, 1) = \ell$. By (A), we have

$$\alpha(k, \ell, 2) = k - 2,\ \beta(k, \ell, 2) = k - 1.$$

Since $(2 - c)(1, 1) + (c - 1)(1, 2) = (1, c)$, i.e., the cost assignment $(1, c)$ to (left branch, right branch) may be written as this linear combination of two cost assignments $(1, 1)$ and $(1, 2)$, the proof is finished.

Proof of the "if" part of Theorem 2

Case $1 \leqslant c < 2$: Put

$$k^* = 2\left\lfloor \frac{1}{2 - c} \right\rfloor + 2.$$

We show that, for every $k \leqslant k^*$,

$$(1) \quad \alpha(k, k - 2, c) \leqslant \beta\left(k, \left\lfloor \frac{k}{2} \right\rfloor, c\right),$$

$$(2) \quad \alpha(k, k - 2, c) \leqslant \beta\left(k, \left\lfloor \frac{k - 1}{2} \right\rfloor, c\right) + 1.$$

To show (1) [(2) and (3) and (4) below can be verified similarly), consider the difference:

$$D = \beta\left(k, \left\lfloor \frac{k}{2} \right\rfloor, c\right) - \alpha(k, k - 2, c).$$

If k is even, we have, using Lemma 2 and $k \leqslant k^*$,

$$D = (2 - c)\left(\frac{k}{2}\right) + (c - 1)(k - 1) - (k - 2)$$

$$= -(2 - c)\left(\frac{k - 2}{2}\right) + 1 \geqslant -(2 - c)\left\lfloor \frac{1}{2 - c} \right\rfloor + 1 \geqslant 0.$$

If k is odd, we have, using Lemma 2 and $k \leqslant k^* - 1$ (note that k^* is even),

$$D = (2 - c)\left(\frac{k - 1}{2}\right) + (c - 1)(k - 1) - (k - 2)$$

$$= -(2 - c)\left(\frac{k - 1}{2}\right) + 1 \geqslant -(2 - c)\left\lfloor \frac{1}{2 - c} \right\rfloor + 1 \geqslant 0.$$

Now, let us remember the remarks given just before the proof of the "only if" part. By Lemma 2, $\alpha(k, \ell, c)$ and $\beta(k, \ell, c)$ increase linearly in ℓ, so (1) implies that all α-nodes in T_k, $k \leqslant k^*$, are the cheapest of all terminal nodes. The inequality (2) implies that, if the cheapest α-node—its cost is $\alpha\left(k, \left\lfloor \frac{k-1}{2} \right\rfloor, c\right)$—is split, the cost $\alpha\left(k, \left\lfloor \frac{k-1}{2} \right\rfloor, c\right) + 1$

of its left son will never be less than the highest cost $\alpha(k,\ k-2,\ c)$ of all α-nodes. This means that the successive applications (F_{k-1} times) of Varn's procedure mentioned in the first section will result in splitting all α-nodes of T_k. Hence, if this tree of order k is c-optimal, the resulting tree, which is T_{k+1} by (B), is also c-optimal. Since T_3 is c-optimal and $k^* \geqslant 3$, we conclude, inductively, that T_k is c-optimal for every $k \leqslant k^* + 1$.

Case $c > 2$: Put $k^* = 2\left\lfloor \dfrac{1}{c-2} \right\rfloor + 3$. We have, for every $k \leqslant k^*$,

$$\text{(3)}\quad \alpha\left(k,\ \left\lfloor \frac{k-1}{2} \right\rfloor,\ c\right) \leqslant \beta(k,\ k-2,\ c),$$

$$\text{(4)}\quad \alpha\left(k,\ \left\lfloor \frac{k-1}{2} \right\rfloor,\ c\right) \leqslant \alpha(k,\ k-2,\ c) + 1.$$

The remainder of the proof is similar to Case $1 \leqslant c < 2$. Note in this case that $\alpha(k,\ \ell,\ c)$ and $\beta(k,\ \ell,\ c)$ decrease linearly in ℓ by Lemma 2. ∎

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A NOTE ON FIBONACCI CUBATURE

ROBERT D. GIRSE

Idaho State University, Pocatello, ID 83209

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Zaremba [3] considered the two-dimensional cubature formula

$$\int_0^1\int_0^1 f(x,\ y)\,dx\,dy \ = \frac{1}{F_N}\sum_{k=1}^{F_N} f(x_k,\ y_k),$$

where F_N is the Nth Fibonacci number and the nodes $(x_k,\ y_k)$ are defined as follows: $x_k = k/F_N$ and $y_k = \{F_{N-1}x_k\}$, where { } denotes the fractional part. Thus, $y_k = F_{N-1}x_k - [F_{N-1}x_k]$, where [ ] denotes the greatest integer function. The purpose of this paper is to prove the conjecture stated by Squire in [2]; that is,

Theorem

If $(x_k,\ y_k)$ is a node for $1 \leqslant k \leqslant F_N - 1$ and if N is $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$, then

$$\begin{pmatrix}(y_k,\ x_k)\\ (y_k,\ 1-x_k)\end{pmatrix}$$

is also a node.

We will assume throughout that $1 \leqslant k \leqslant F_N - 1$, $N > 2$, and will show:

(i) Each y_k is equal to some x_m, $1 \leqslant m \leqslant F_N - 1$.

(ii) The y_k's are distinct.

By definition, the x_k's are distinct, and so (i) and (ii) imply that for every node $(x_k,\ y_k)$ there is a unique node $(x_m,\ y_m)$ with $x_m = y_k$.

Finally, we show:

(iii) If $(x_m,\ y_m)$ is the node with $x_m = y_k$, then

$$y_m = \begin{cases} x_k & \text{if } N \text{ is even,}\\ 1 - x_k & \text{if } N \text{ is odd.}\end{cases}$$

Proof of (i): We have

$$
\begin{aligned}
y_k &= \{F_{N-1}x_k\} = \left\{k\frac{F_{N-1}}{F_N}\right\} \\
&= k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] \qquad (1)\\
&= \left(k\,F_{N-1} - F_N\left[k\frac{F_{N-1}}{F_N}\right]\right)\bigg/F_N.
\end{aligned}
$$

Now from [1, p. 288], gcd $(F_{N-1}, F_N) = 1$, and so

$$0 < k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] < 1.$$

Thus

$$0 < k\,F_{N-1} - F_N\left[k\frac{F_{N-1}}{F_N}\right] < F_N,$$

where the middle quantity in this inequality is an integer and is also the numerator of the right-hand side of (1). Hence, y_k is equal to some x_m, $1 \leqslant m \leqslant F_N - 1$.

Proof of (ii): To show the y_k's are distinct, we will prove $y_k = y_m$ if and only if $k = m$. Assume, without loss of generality, that $1 \leqslant m \leqslant k$. If $y_k = y_m$, we have

$$
\begin{aligned}
\left\{k\frac{F_{N-1}}{F_N}\right\} &= \left\{m\frac{F_{N-1}}{F_N}\right\}, \\
k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] &= m\frac{F_{N-1}}{F_N} - \left[m\frac{F_{N-1}}{F_N}\right], \\
(k-m)\frac{F_{N-1}}{F_N} &= \left[k\frac{F_{N-1}}{F_N}\right] - \left[m\frac{F_{N-1}}{F_N}\right]. \qquad (2)
\end{aligned}
$$

Now recalling gcd $(F_{N-1}, F_N) = 1$ and since $0 \leqslant k - m < F_N$, $(k - m)F_{N-1}/F_N$ is never an integer unless $k - m = 0$. However, the right-hand side of (2) is always an integer, and so $y_k = y_m$ if and only if $k = m$.

Proof of (iii): Assume that (x_m, y_m) is the node with $x_m = y_k$. Then

$$y_m = \{F_{N-1}x_m\} = \{F_{N-1}y_k\}$$

$$= \left\{ F_{N-1}\left(k\,\frac{F_{N-1}}{F_N} - \left[k\,\frac{F_{N-1}}{F_N} \right] \right) \right\}$$

$$= \left\{ k\,\frac{F_{N-1}^2}{F_N} - F_{N-1}\left[k\,\frac{F_{N-1}}{F_N} \right] \right\}.$$

From [1, p. 294], we have $F_{N-1}^2 = F_N F_{N-2} + (-1)^{N-2}$ for $N \geqslant 3$, and so

$$y_m = \left\{ k\,F_{N-2} + (-1)^{N-2}k/F_N - F_{N-1}\left[k\,\frac{F_{N-1}}{F_N} \right] \right\}.$$

Now if n is any integer $\{n + x\} = x - [x]$, and since

$$k\,F_{N-2} - F_{N-1}[k\,F_{N-1}/F_N]$$

is an integer, we have

$$\begin{aligned} y_m &= (-1)^{N-2}k/F_N - [(-1)^{N-2}k/F_N] \\ &= \begin{cases} k/F_N - 0 = x_k & \text{if } N \text{ is even,} \\ -k/F_N - (-1) = 1 - x_k & \text{if } N \text{ is odd.} \end{cases} \end{aligned}$$

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A FAMILY OF POLYNOMIALS AND POWERS OF THE SECANT

O. R. AINSWORTH
and
J. NEGGERS

The University of Alabama, University, AL 35486
(Submitted March 1982)

In this paper, we discuss a family of polynomials $A_n(z)$, defined by the conditions

$$A_0(z) = 1 \quad \text{and} \quad A_k(z) = z(z + 1)A_{k-1}(z + 2) - z^2A_{k-1}(z).$$

Using these polynomials, we may express complex powers of the secant and cosine functions as infinite series. These polynomials provide ways to obtain numerous relations among Euler numbers and Bell numbers. They appear to be unrelated to other functions which arise in this context.

Suppose that we consider the family of polynomials $A_n(z)$, $n = 0, 1, 2, \ldots$, defined as follows: $A_0(z) = 1$ and if $A_0(z), \ldots, A_{k-1}(z)$ have already been defined, then $A_k(z)$ is given by the recursion formula:

$$A_k(z) = z(z + 1)A_{k-1}(z + 2) - z^2A_{k-1}(z). \tag{1}$$

It follows immediately that if $A_\ell(z)$ is a polynomial of degree ℓ for $0 \leqslant \ell \leqslant k - 1$, then $A_k(z)$ has leading coefficient $(2k - 1)a_{k-1}$, where a_{k-1} is the leading coefficient of $A_{k-1}(z)$, and where $(2k - 1)a_{k-1}$ is the coefficient of z_k. Thus, we generate a series of leading terms:

$$1,\ z,\ 3z^2,\ 15z^3,\ 105z^4,\ \ldots,\ [(2k - 1)!/2^{k-1}(k - 1)!]z^k,\ \ldots, \tag{2}$$

so that $A_k(z)$ is a polynomial of degree precisely k. It also follows immediately from (1) that $A_k(0) = 0$ for all $k > 1$. We note further that if $A_k^*(z) = A_k(-z)$, then $A_0^*(z) = A_0(-z) = 1$, and from (1), $A_k^*(z) = A_k(-z) = -z(-z + 1)A_{k-1}^*(z - 2) - z^2A_{k-1}^*(z)$ so that we have a corresponding family of polynomials $A_n^*(z)$, $n = 0, 1, 2, \ldots$, given by the recursion formula:

$$A_k^*(z) = z(z - 1)A_{k-1}^*(z - 2) - z^2A_{k-1}^*(z). \tag{3}$$

It follows immediately that for the sequence $A_n^*(z)$ we have a corresponding sequence of leading terms:

$$1,\ -z,\ 3z^2,\ -15z^3,\ 105z^4,\ \ldots\ [(-1)^k(2k-1)!/2^{k-1}(k-1)!]z^k,\ \ldots \tag{4}$$

It is our purpose in this note to prove that

$$\sec^z x = \sum_{n=0}^{\infty} [A_n(z)/(2n)!]x^{2n} \tag{5}$$

and

$$\cos^z x = \sum_{n=0}^{\infty} [A_n^*(z)/(2n)!]x^{2n}, \tag{6}$$

as well as derive some consequences of these facts.

In particular, if $z = 1$, then we obtain the corresponding formulas,

$$\sec x = \sum_{n=0}^{\infty} [A_n(1)/(2n)!]x^{2n} \tag{7}$$

and

$$\cos x = \sum_{n=0}^{\infty} [A_n^*(1)/(2n)!]x^{2n}, \tag{8}$$

so that we obtain the results: $A_n(1) = E_{2n}$, the usual Euler number; and $A_n^*(1) = A_n(-1) = (-1)^n$, so that we are able to evaluate these polynomials at these values by use of the definitions.

Given that formulas (5) and (6) hold, we obtain from

$$\sec^{z_1} x \cdot \sec^{z_2} x = \sec^{z_1+z_2} x$$

the relation

$$\begin{aligned}\left(\sum_{m=0}^{\infty} [A_m(z_1)/(2m)!]x^{2m}\right)&\left(\sum_{\ell=0}^{\infty} [A_\ell(z_2)/(2\ell)!]x^{2\ell}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{m+\ell=k} A_m(z_1)A_\ell(z_2)/(2m)!(2\ell)!\right)x^{2k};\end{aligned} \tag{9}$$

whence,

$$\sum_{m+\ell=k} A_m(z_1)A_\ell(z_2)/(2m)!(2\ell)! = A_k(z_1+z_2)/(2k)!, \tag{10}$$

so that we obtain finally the addition formula:

$$A_k(z_1+z_2) = \sum_{j=0}^{k}\binom{2k}{2j}A_j(z_1)A_{k-j}(z_2). \tag{11}$$

From (11) we have the consequence

$$A_k(z-z) = \sum_{j=0}^{k}\binom{2k}{2j}A_j(z)A_{k-j}^*(z) = 0,\ k > 0; \tag{12}$$

whence, since $\sec^z x \cdot \cos^z x = 1$, it follow that for $k \geqslant 1$,

$$\sum_{j=0}^{k} \binom{2k}{2j} A_j(z) A^*_{k-j}(z) = 0. \tag{13}$$

In particular, if $z = 1$, then we obtain the formula for $k \geqslant 1$, using $A^*_{k-j}(-1) = (-1)^{k-j}$,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} E_{2j} = 0. \tag{14}$$

To generate sample polynomials, we use the original relations (1) and take consecutive values of k,

$$k = 1,\ A_1(z) = z(z+1) - z^2 = z$$

$$k = 2,\ A_2(z) = z(z+1)(z+2) - z^3 = 3z^2 + 2z$$

$$\begin{aligned} k = 3,\ A_3(z) &= z(z+1)[3(z+2)^2 + 2(z+2) - z^2(3z^2 + 2z) \\ &= 15z^3 + 30z^2 + 16z, \text{ etc.,} \end{aligned}$$

with $A_1(1) = E_2 = 1$, $A_2(1) = E_4 = 5$, $A_3(1) = E_6 = 61$.

From Equation (1) we find, taking $z = 1$, that

$$E_{2k} = 2A_{k-1}(3) - E_{2k-2},\ E_0 = 1 \tag{15}$$

and

$$A_{k-1}(3) = 1/2[E_{2k} + E_{2k-2}],\ k \geqslant 1, \tag{16}$$

so that we have an immediate expansion for $\sec^3 z$ in terms of the Euler numbers:

$$\begin{aligned} \sec^3 x &= \sum_{n=1}^{\infty} ([E_{2n} + E_{2n-2}]/2(2n-2)!)x^{2n-2} \\ &= \sum_{n=0}^{\infty} [E_{2n+2} + E_{2n}]/2(2n)!x^{2n}. \end{aligned} \tag{17}$$

By repeated use of (1) in this fashion, we may generate expressions for $\sec^5 z$, $\sec^7 z$, ..., $\sec^{2n+1} z$, which are expressed in terms of the standard Euler numbers only.

To prove the formulas (5) and (6), we proceed as follows:

$$(\sec^m x)' = m \sec^m x \tan x,$$

$$(\sec^m x)'' = (m^2 + m)\sec^{m+2} x - m^2 \sec^m x,$$

and thus, if we write (formally)

$$\sec^m x = \sum_{n=0}^{\infty} [A_n(m)/(2n)!]x^{2n},$$

then

$$(m^2 + m)\sec^{m+2}x - m^2\sec^m x = \sum_{n=0}^{\infty} [((m^2 + m)A_n(m + 2) - m^2 A_n(m))/(2n)!]x^{2n}$$

$$= \sum_{n=0}^{\infty} [A_{n+1}(m)/(2n)!]x^{2n}, \tag{18}$$

so that upon equating coefficients, we find:

$$A_{n+1}(m) = m(m + 1)A_n(m + 2) - m^2 A_n(m). \tag{19}$$

From (19), it is immediate that $A_k(m)$ is a polynomial in the variable m, where we consider m a real number $m > 1$, and such that $\sec^m x$ has the appropriate expression.

If we fix x so that $\sec^z x > 1$, then $f(z) = \sec x$ yields

$$f'(z) = f(z) \cdot \log(\sec x),$$

and thus $f(z)$ is an analytic function of z which agrees with the series given in (5) for the real variable $m > 1$. Since $g(z)$ given by the series in z is also analytic and since $f(m) = g(m)$ for the real variable $m > 1$, it follows that $f(z) = g(z)$, or what amounts to the same thing, equation (5) holds for all z. Equation (6) is now a consequence of equation (5) if we replace z by $-z$.

Making use of what we have derived above, we may also analyze other functions in this way, as the examples below indicate.

Suppose we write

$$\tan x = \sum_{n=0}^{\infty} (T_n/n!)x^n, \text{ where } T_{2n} = 0 \text{ and}$$
$$T_{2n-1} = \frac{(-1)^{n-1}2^{2n}(2^{n-1} - 1)B_{2n}}{(2n)!}. \tag{20}$$

Then from $\frac{d}{dx}(\sec x) = z \sec^z x \tan^z x$ we obtain the relation

$$z\left[\sum_{n=0}^{\infty}\left\{\frac{A_n(z)/z}{(2n - 1)!}\right\}x^{2n-1}\right] = z\left[\sum_{m=0}^{\infty}\frac{A_m(z)}{(2m)!}x^{2m}\right]\left[\sum_{\ell=0}^{\infty} T_\ell/\ell!x^{2\ell}\right], \tag{21}$$

whence it follows that:

$$\frac{A_n(z)/z}{(2n-1)!} = \sum_{2m+\ell=2n-1} \frac{A_m(z)T_\ell}{(2m)!\,\ell!} \tag{22}$$

or

$$A_n(z)/z = \sum_{2m+\ell=2n-1} \binom{2n-1}{\ell} A_m(z)T_\ell. \tag{23}$$

In particular, we conclude that if ℓ is even, then $T_\ell = 0$, which is of course known, and the corresponding expression is

$$A_n(z) = \sum_{m=0}^{n-1} \binom{2n-1}{2m} T_{(2(n-m)-1)} zA_m(z). \tag{24}$$

Hence we may derive a variety of formulas. For example, by taking $z = 1$, (24) yields

$$E_{2n} = \sum_{m=0}^{n-1} \binom{2n-1}{2m} T_{2(n-m)-1} E_{2m}; \tag{25}$$

or, since the coefficients T_ℓ are vastly more complicated:

$$T_{2n-1} = E_{2m} - \sum_{m=1}^{n-1} \binom{2n-1}{2m} T_{2(n-m)-1} E_{2m}, \tag{26}$$

which yields a recursion formula involving the Euler numbers.

Similarly, from $z = -1$, $A_n(-1) = (-1)^n$, we obtain

$$(-1)^n = \sum_{m=0}^{n-1} (-1)^{m+1} \binom{2n-1}{2m} T_{2(n-m)-1}, \tag{27}$$

or, once again, for $m = 0$,

$$T_{2n-1} = \sum_{m=1}^{n-1} (-1)^{m+1} \binom{2n-1}{2m} T_{2(n-m)-1} + (-1)^{n+1}. \tag{28}$$

Using the fact that $1 + \tan^2 x = \sec^2 x$, we obtain the relation

$$\begin{aligned} &\left[\sum_{m=0}^{\infty} \frac{T_{2m+1}}{(2m+1)!} x^{2m+1}\right]\left[\sum_{\ell=0}^{\infty} \frac{T_{2\ell+1}}{(2\ell+1)!} x^{2\ell+1}\right] \\ &= \sum_{k=1}^{\infty}\left(\sum_{\ell+m=k-1} \frac{T_{2m+1}}{(2m+1)!} \frac{T_{2\ell+1}}{(2\ell+1)!}\right) x^{2k}; \end{aligned} \tag{29}$$

so that

$$\sum_{\ell+m=k-1} \frac{T_{2m+1}}{(2m+1)!} \frac{T_{2\ell+1}}{(2\ell+1)!} = \frac{A_k(2)}{(2k)!} \tag{30}$$

and

$$A_k(2) = \sum_{m=0}^{k-1} \binom{2k}{2m+1} T_{2m+1} T_{2(k-m)-1}. \tag{31}$$

If we use the fact that $(\tan x)' = \sec^2 x$, then

$$\sum_{m=0}^{\infty} \frac{T_{2m+1}}{(2m)!} x^{2m} = \sum_{m=0}^{\infty} \frac{A_m(2)}{(2m)!} x^{2m}, \tag{32}$$

so that immediately:

$$A_m(2) = T_{2m+1}. \tag{33}$$

Hence, by using (31), we have the relation:

$$T_{2k+1} = \sum_{m=0}^{k-1} \binom{2k}{2m+1} T_{2m+1} \cdot T_{2(k-m)-1}. \tag{34}$$

Having these relations at hand, we use the fact that

$$\tan x \cdot \cos x = \sin x$$

to obtain

$$\begin{aligned} \sin x &= \left[\sum_{m=0}^{\infty} (A_m(2)/(2m+1)!)x^{2m+1}\right]\left[\sum_{\ell=0}^{\infty} A_\ell(-1)/(2\ell)!x^{2\ell}\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{\ell+m=k} \frac{A_m(2) \cdot A_\ell(-1)}{(2m+1)!(2\ell)!}\right]x^{2k+1} \end{aligned} \tag{35}$$

so that

$$\sum_{\ell+m=k}^{\infty} \frac{A_m(2) \cdot A_\ell(-1)}{(2m+1)!(2\ell)!} = \frac{(-1)^k}{(2k+1)!}. \tag{36}$$

Hence:

$$\sum_{m=0}^{\infty} \binom{2k+1}{2m+1} A_m(2) A_{k-m}(-1) = (-1)^k. \tag{37}$$

Using the fact that $A_\ell(-1) = (-1)^\ell$, it follows that:

$$\sum_{n=0}^{k} (-1)^{2k-m} \binom{2k+1}{2m+1} A_m(2) = 1. \tag{38}$$

From these examples, it should be clear that the polynomials $A_n(z)$, $n = 0, 1, 2, \ldots$ are a family closely related to the trigonometric functions and, hence, they should prove interesting. The sampling of such properties given here seems to indicate that this is indeed the case.

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A VARIANT OF NIM AND A FUNCTION DEFINED BY FIBONACCI REPRESENTATION

DAVID R. HALE

University of Wisconsin-Milwaukee, Milwaukee, WI 53201

(Submitted March 1982)

The game Spite Nim I was introduced by Jesse Croach in [2] and discussed further briefly in [3]. No solution was given for the same in these references, and some questions were raised about a partial solution for certain simple cases of the game. This note will solve part of one of these questions, and will show that the solution is closely related to the golden ratio $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Spite Nim is played in the following way: Two players pick from several rows of counters. On a player's turn to move, he announces a positive number of counters. This number must be less than or equal to the number of counters in the longest row. His opponent then indicates from which row these counters are to be taken. (This is the "spite" option.) This row must have at least as many counters as the call. The players alternate moves. The player who takes the last counter wins.

In this note only the case of two rows will be considered. A configuration of two rows of lengths n and r will be denoted by (n, r). This actually should be considered an unordered pair.

Given any pair, a person receiving such a pair can either make a call which with best play on both sides will give him a win, or he loses, no matter what call he makes. In the first case, the position is called unsafe (it is unsafe to leave it to your opponent); in the second case, it is called safe.

It will be shown that for each n there is an $r \leqslant n$ for which (r, n) is safe, and if $s < r$, (n, s) is unsafe. The number r will be shown to be equal to a function of n which has been previously studied.

Define a function f on the natural numbers by $f(1) = 1$, and for $n > 1$, $f(n) = r$, where r is the smallest number for which $r + f(r) \geqslant n$. Since $f(r) \geqslant 1$, such an r clearly exists.

Theorem 1

For all natural numbers n, $(n, f(n))$ is safe, and if $s < f(n)$, (n, s) is unsafe.

Proof: Use induction. (1, 0) is clearly unsafe, while (1, 1) is safe. So the theorem holds for $n = 1$.

Assume Theorem 1 holds for all $s < n$. Then, first, if $s < f(n)$, then (n, s) is unsafe.

To show this, suppose a player is given (n, s). Since $s < f(n)$, $s + f(s) < n$, by definition of f. Therefore, $n - f(s) > s$. So, on the call $n - f(s)$, the resulting pair is $(f(s), s)$, which by hypothesis is safe.

Secondly, $(n, f(n))$ is safe. On a call of $r \leqslant f(n)$, take from the second row to get $(n, f(n) - r)$. This has just been shown to be unsafe.

On a call $r > f(n)$, the result is $(n - r, f(n))$. But since $n \leqslant f(n) + f(f(n))$, $n - r < f(f(n))$. So by hypothesis, $(n - r, f(n))$ is unsafe; thus, Theorem 1 is proved.

Now, reexamine f. $f(n)$ is in fact the same as $e(n)$, defined in [1].

To show $f(n) = e(n)$, we will show $e(n)$ satisfies the recursion $f(n)$ does. Since $f(1) = 1 = e(1)$, this will show the functions are identical.

First, write n in Fibonacci notation. Let F_m be the mth Fibonacci number. Then $n = F_{r_1} + F_{r_2} + \cdots + F_{r_k}$, where $r_i - r_{i+1} \geqslant 2$, and $r_k \geqslant 2$. By definition, $e(n) = F_{r_1 - 1} + F_{r_2 - 1} + \cdots + F_{r_k - 1}$.

If $r_k \neq 2$, then

$$e(e(n)) = F_{r_1 - 2} + F_{r_2 - 2} + \cdots + F_{r_k - 2}.$$

So $e(n) + e(e(n)) = n$. Also, since $e(n)$ is nondecreasing, if $s < e(n)$, then $s + e(s) < e(n) + e(e(n)) = n$. So $e(n)$ satisfies the recursion here.

If $r_k = 2$, again $e(n) = F_{r_1 - 1} + \cdots + F_{r_k - 1}$. However, since $r_k - 1 = 1$, this no longer expresses $e(n)$ in correct Fibonacci representation, so the preceding argument requires modification. We can say, however, that $e(n) > F_{r_1 - 1} + \cdots + F_{r_{k-1} - 1}$, so $e(e(n)) \geqslant F_{r_1 - 2} + \cdots + F_{r_{k-1} - 2}$, and $e(n) + e(e(n)) \geqslant n$.

Also, if $s < e(n)$, then $s \leqslant F_{r_1 - 1} + \cdots + F_{r_{k-1} - 1}$. Thus

$$e(s) + e(e(s)) \leqslant F_{r_1} + \cdots + F_{r_{k-1}} \leqslant n - 1 < n.$$

So here, also, $e(n)$ satisfies the recursion. Therefore, $e(n) = f(n)$ for all natural numbers n.

Alternate formulas for $e(n)$ are given in [1] that indicate how close $e(n)$ is to $\alpha^{-1}n$. Let $\{x\}$ be the integer nearest x, and let $[x]$ be the greatest integer $< x$. Then if $n = F_{r_1} + \cdots + F_{r_k}$ is the Fibonacci representation for n,

$$e(n) = \{\alpha^{-1}n\} \qquad \text{if } r_k \neq 2,$$

$$e(n) = [\alpha^{-1}n] + 1 \quad \text{if } r_k = 2.$$

Deeper inspection of Fibonacci notation might possibly solve the two-row game, but I have been unable to do so.

To close this note, here are two weak results regarding safe (n, s) with $s > e(n)$.

Theorem 2

Exactly one of the pairs $(n, e(n)+1)$ and $(n-1, e(n)+1)$ is safe.

Proof: If $(n, e(n)+1)$ is unsafe, the only call must be 1. But then, $(n-1, e(n)+1)$ must be safe. The converse follows in the same way.

Consider for any natural number n the number

$$h(n) = \#\{s : s \leqslant n,\ (s, e(s) + 1) \text{ is safe}\}.$$

Since $e(s) = e(s - 1)$ for approximately $(1 - \alpha^{-1})n$ numbers $s \leqslant n$, this gives, with Theorem 2,

$$\frac{1}{2}(1 - \alpha^{-1}) \leqslant \frac{h(n)}{n} \leqslant \frac{1}{2}(n - \alpha^{-1}) + \alpha^{-1}.$$

Theorem 3

If $e(n) < s < n$, (n, s) is unsafe, and r is a winning call, then $(n, n - r)$ is unsafe and $n - s$ is a winning call.

Proof: If (n, s) is unsafe and r is a winning call, then $(n - r, s)$ and $(n, s - r)$ are both safe. But the call $n - s$ on $(n, n - r)$ gives rise to the identical results.

Perhaps these may help determine for what $s > e(n)$ is (n, s) safe. Results for the three- or more-row game would also be interesting.

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KRONECKER'S THEOREM AND RATIONAL APPROXIMATION OF ALGEBRAIC NUMBERS

WAYNE M. LAWTON

Jet Propulsion Laboratory, Pasadena, CA 91109

(Submitted March 1982)

Let the circle T be parametrized by the real numbers modulo the integers. When a real number is used to denote a point in T, it is implied that the fractional part of the number is being considered. If $a, b \in T$ with $a - b \neq .5$, then (a, b) will denote the shortest open arc in T whose endpoints are a and b.

Fix an irrational number x. For any positive integer n let S_n denote the set of n open arcs in T formed by removing the points $x, \ldots, nx$ from T, and let L_n be the length of the longest arc in S_n. Then, the result of Kronecker in [1, p. 363, Theorem 438] implies that $L_n \to 0$ as $n \to \infty$. Without further restrictions on x it is not possible to characterize the rate of convergence of L_n. However, if x is an algebraic number of degree d (that is, if x satisfies a polynomial equation having degree d and integer coefficients), then the following result gives an upper bound for the rate of convergence of L_n.

Theorem 1

If x is an irrational algebraic number of degree d, there exists $c(x) > 0$ such that for all $n > 3$

$$L_n < c(x)/n^{1/(d-1)}. \tag{1}$$

The proof of this theorem is based on the following three lemmas.

Lemma 1

If x is an irrational algebraic number of degree d, there exists $k(x) > 0$ such that, if $(x, px + x)$ is an arc in S_n, then

$$\text{Length } (x, px + x) > k(x)/p^{(d-1)}. \tag{2}$$

Proof: This inequality follows from Liouville's theorem [1, p. 160, Theorem 191].

Lemma 2

If x is irrational and $n > 3$, choose $p < q$ such that $(x, px + x)$ and $(x, qx + x)$ are arcs in S_n. Then the set S_n can be partitioned into two or three subsets as follows:

$$A_p = \{(kx, px + kx)\}: 1 \leqslant k \leqslant n - p \tag{3}$$

$$A_q = \{(kx, qx + kx)\}: 1 \leqslant k \leqslant n - q \tag{4}$$

$$A_r = \{(nx - qx + kx, nx - px + kx)\}: 1 \leqslant k \leqslant p + q - n. \tag{5}$$

Proof: Let (a, b) be any arc in S_n with $a < b$. Then $(a + x, b + x)$ is an arc in S_n or $b = nx$ or $(a + x, b + x)$ contains the point x. In the latter case, $a = px$ and $b = qx$. Hence, letting $a = x$, $b = px + x$, and successively translating the arc (a, b) by x yields the $n - p$ arcs in set A_p. Similarly, set A_q is formed if $a = x$ and $b = qx + x$. Finally, if (a, b) is an arc not contained in A_p or A_q, then successive translation by x must terminate at the arc (px, qx). Since there are

$$n - (n - p) - (n - q) = p + q - n$$

arcs in S_n that are not in A_p or in A_q, the proof is complete.

Lemma 3

Assume the hypothesis and notation of Lemma 2. Let I_p and I_q denote the lengths of the arcs in sets A_p and A_q, respectively. Then the arcs in set A_r have length $I_r = I_p + I_q$. Furthermore, the following relations are valid:

$$p + q \geqslant n; \tag{6}$$

$$pI_q + qI_p = 1. \tag{7}$$

Proof: Clearly $I_r = I_p + I_q$, since

$$\begin{aligned} I_r &= \text{length } (px, qx) = \text{length } (px + x, qx + x) \\ &= \text{length } (px + x, x) + \text{length } (x, qx + x) \\ &= I_p + I_q. \end{aligned}$$

Also, since the total number of arcs in A_p and A_q does not exceed n,

$$(n - p) + (n - q) \leqslant n;$$

hence, $p + q \geqslant n$, which is inequality (6). Finally, since the sum of the lengths of the arcs in S_n is 1, it follows that

$$1 = (n - p)I_p + (n - q)I_q + (p + q - n)(I_p + I_q) = pI_q + qI_p,$$

which is equality (7). The proof is finished.

Proof of Theorem 1: Assume x is an irrational algebraic number of degree d and that $k(x) > 0$ is chosen as in Lemma 1 so that inequality (2) is valid. Then, for any $n > 3$, choose $p < q$ as in Lemma 2. Therefore, combining inequality (2) with equality (7) yields the following inequality:

$$1 > k(x)[p/q^{d-1} + q/p^{d-1}] > k(x)q/p^{d-1}. \tag{8}$$

This combines with inequality (6) to yield

$$p^{d-1} > k(x)q \geqslant k(x)(n - p). \tag{9}$$

Therefore,

$$p^{d-1} + k(x)p > k(x)n. \tag{10}$$

Clearly, there exists a number $g(x) > 0$ which depends only on $k(x)$ and d such that for every $n > 3$

$$p > g(x)n^{1/d-1}. \tag{11}$$

Substituting inequality (11) into equation (7) yields

$$1 = pI_q + qI_p > p(I_q + I_p) > g(x)n^{1/(d-1)}I_r. \tag{12}$$

Since $L_n \leqslant I_r$, if $c(x) = 1/g(x)$, then inequality (12) implies inequality (1). This completes the proof of Theorem 1.

If in Lemma 1, $d = 2$ and x is irrational and satisfies the equation $ax^2 + bx + c = 0$ and $k(x) < (b^2 - 4ac)^{-1/2}$, then inequality (2) is valid for all except a finite number of values for p.

Clearly, as $n \to \infty$, both $p \to \infty$ and $q \to \infty$; hence, it follows from inequality (10) that inequality (11) is valid for all except a finite number of values for n if

$$g(x) = k(x)/(1 + k(x)).$$

Hence, the inequality (1) in Theorem 1 is valid for all except a finite number of values for n if

$$c(x) = 1/g(x) = 1 + 1/k(x) > 1 + (b^2 - 4ac)^{1/2}.$$

The smallest value of the right side of this inequality occurs for $a = 1$, $b = -1$, $c = -1$ in which case $x = (1 + \sqrt{5})/2$ (the classical "golden ratio"), or $x = (1 - \sqrt{5})/2$.

Remark

The referee has noted that, for algebraic numbers of degree three or more, the bound in Theorem 1 is not the best possible. If Roth's theorem [2, p. 104] is used in place of Liouville's in Lemma 1, then one obtains a bound of the form

$$L_n < c(\varepsilon)/n^{1-\varepsilon} \tag{13}$$

for any $\varepsilon > 0$, where $c(\varepsilon)$ is a constant depending on ε.

ACKNOWLEDGMENT

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. HILLMAN

University of New Mexico, Albuquerque, NM 87131

Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be submitted on a separate signed sheet, or sheets. Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-496 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

Show that the centroid of the triangle whose vertices have coordinates (F_n, L_n), (F_{n+1}, L_{n+1}), (F_{n+6}, L_{n+6}) is (F_{n+4}, L_{n+4}).

B-497 *Proposed by Stanley Rabinowitz, Digital Equip. Corp., Merrimack, NH*

For d an odd positive integer, find the area of the triangle with vertices (F_n, L_n), (F_{n+d}, L_{n+d}), and (F_{n+2d}, L_{n+2d}).

B-498 *Proposed by Herta T. Freitag, Roanoke, VA*

Characterize the positive integers k such that, for all positive integers n, $F_n + F_{n+k} \equiv F_{n+2k} \pmod{10}$.

B-499 *Proposed by Herta T. Freitag, Roanoke, VA*

Do the Lucas numbers analogue of B-498.

B-500 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $A(n)$ and $B(n)$ be polynomials of positive degree with integer coefficients such that $B(k)|A(k)$ for all integers k. Must there exist a nonzero integer h and a polynomial $C(n)$ with integer coefficients such that $hA(n) = B(n)C(n)$?

B-501 *Proposed by J. O. Shallit & J. P. Yamron, U.C., Berkeley, CA*

Let α be the mapping that sends a sequence $X = (x_1, x_2, \ldots, x_{2k})$ of length $2k$ to the sequence of length k

$$\alpha(X) = (x_1x_{2k}, x_2x_{2k-1}, x_3x_{2k-2}, \ldots, x_kx_{k+1}).$$

Let $V = (1, 2, 3, \ldots, 2^h)$, $\alpha^2(V) = \alpha(\alpha(V))$, $\alpha^3(V) = \alpha(\alpha^2(V))$, etc. Prove that $\alpha(V)$, $\alpha^2(V)$, $\ldots$, $\alpha^{h-1}(V)$ are all strictly increasing sequences.

SOLUTIONS

Where To Find Perfect Numbers

B-472 *Proposed by Gerald E.Bergum, S. Dakota State Univ., Brookings, SD*

Find a sequence $\{T_n\}$ satisfying a second-order linear homogeneous recurrence $T_n = aT_{n-1} + bT_{n-2}$ such that every even perfect number is a term in $\{T_n\}$.

Solution by Graham Lord, Université Laval, Québec

A (trivial) solution to this problem is the sequence of even integers $a = 2$ and $b = -1$, with seeds $T_1 = 2$ and $T_2 = 4$. With $a = 6$ and $b = -8$, the sequence T_n is $2^{n-1}(2^n - 1)$ if $T_1 = 1$ and $T_2 = 6$. The proof is immediate:

$$\begin{aligned} T_n &= 6T_{n-1} - 8T_{n-2} \\ &= 6(2^{2n-3} - 2^{n-2}) - 8(2^{2n-5} - 2^{n-3}) \\ &= 2^{2n-1} - 2^{n-1}. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Edgar Krogt, Bob Prielipp, Sahib Singh, Paul Smith, J. Suck, Gregory Wulczyn, and the proposer.

Primitive Fifth Roots of Unity

B-473 *Proposed by Philip L. Mana, Albuquerque, NM*

Let

$$a = L_{1000},\ b = L_{1001},\ c = L_{1002},\ d = L_{1003}.$$

Is $1 + x + x^2 + x^3 + x^4$ a factor of $1 + x^a + x^b + x^c + x^d$? Explain.

Solution by Paul S. Bruckman, Carmichael, CA

It is easy to verify that $\{L_n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 4. Specifically,

$$L_{4k} \equiv L_0 = 2,\ L_{4k+1} \equiv L_1 = 1,\ L_{4k+2} \equiv L_2 = 3$$

and

$$L_{4k+3} \equiv L_3 = 4 \pmod 5,\ k = 0, 1, 2, \ldots .$$

Therefore, $a \equiv 2$, $b \equiv 1$, $c \equiv 3$, and $d \equiv 4 \pmod 5$.

A polynomial $p(x)$ divides another polynomial $q(x)$ if $q(x_0) = 0$ for all x_0 such that $p(x_0) = 0$. Letting $p(x) = 1 + x + x^2 + x^3 + x^4$, we see that $p(x)$ is the cyclotomic polynomial $(x^5 - 1)/(x - 1)$, which has four complex zeros equal to the complex fifth roots of unity. Let θ denote any of these roots. Since $p(\theta) = 0$, it suffices to show that $q(\theta) = 0$, where $q(x) \equiv 1 + x^a + x^b + x^c + x^d$.

Now $\theta^5 = 1$, and it follows from this and the congruences satisfied by a, b, c, and d, that

$$q(\theta) = 1 + \theta^2 + \theta + \theta^3 + \theta^4 = p(\theta) = 0.$$

This shows that the answer to the problem is affirmative.

Also solved by C. Georghiou, Walther Janous, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Sequence of Congruences

B-474 *Proposed by Philip L. Mana, Albuquerque, NM*

Are there an infinite number of positive integers n such that

$$L_n + 1 \equiv 0 \pmod{2n}?$$

Explain.

Solution by Bob Prielipp, Univ. of Wisconsin-Oshkosh, WI

Induction will be used to show that

$$L_{2^k} + 1 \equiv 0 \pmod{2^{k+1}}$$

for each nonnegative integer k. Clearly, the desired result holds when $k = 0$ and when $k = 1$. Assume that

$$L_{2^j} + 1 \equiv 0 \pmod{2^{j+1}},$$

where j is an arbitrary positive integer. Then

$$L_{2^j} = q \cdot 2^{j+1} - 1$$

for some integer q. It is known that if m is even, $L_m^2 = L_{2m} + 2$ [see p. 189 of "Divisibility and Congruence Relations" by Verner E. Hoggatt, Jr. and Gerald E. Bergum in the April 1974 issue of this journal]. Thus,

$$\begin{aligned} L_{2^j} + 1 = L_{2(2^j)} + 1 &= (L_{2^j})^2 - 2 + 1 \\ &= (q \cdot 2^{j+1} - 1)^2 - 1 \\ &= (q^2 \cdot 2^{2j+2} - q \cdot 2^{j+2}) + (1 - 1) \\ &\equiv 0 \pmod{2^{j+2}}. \end{aligned}$$

Also solved by Paul S. Bruckman, C. Georghiou, Graham Lord, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Wrong Sign

B-475 *Proposed by Herta T. Freitag, Roanoke, VA*

The problem should read: "Prove that $|S_3(n)| - S_1^2(n)$ is $2[(n + 1)/2]$ times a triangular number."

Solution by Paul Smith, Univ. of Victoria, B.C., Canada

It is easily shown that if $n = 2m$,

(i) $S_3(n) = -m^2(4m + 3)$

(ii) $S_1^2(n) = m^2$

(iii) $2[(n + 1)/2] = 2m.$

Thus

$$|S_3(n)| - S_1^2(n) = m^2(4m + 2) = 2m \cdot \frac{2m(2m + 1)}{2} = 2[(n + 1)/2] \cdot T_n.$$

If $n = 2m + 1$,

$$S_3(n) = -m^2(4m + 3) + (2m + 1)^3,$$

$$S_1^2(n) = (m + 1)^2$$

and

$$2[(n + 1)/2] = 2(m + 1).$$

And now

$$|S_3(n)| - S_1(n)^2 = 2(2m^3 + 5m^2 + 4m + 1) = 2(m + 1)(2m + 1)(m + 1)$$

(continued)

$$= 2(m+1) \cdot \frac{(2m+1)(2m+2)}{2} = 2[(n+1)/2] \cdot T_n.$$

Also solved by Paul S. Bruckman, Graham Lord, Bob Prielipp, Sahib Singh, J. Suck, Gregory Wulczyn, and the proposer.

Multiples of Triangular Numbers

B-476 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k.$$

Prove that $|S_4(n) + S_2(n)|$ is twice the square of a triangular number.

Solution by Graham Lord, Université Laval, Québec

As $(k+1)^4 - k^4 + (k+1)^2 - k^2 = 2(k+1)^3 + 2k^3$, then

$$S_4(2m) + S_2(2m) = -2(1^3 + 2^3 + \cdots + (2m)^3)$$
$$= -2\{2m(2m+1)/2\}^2.$$

And

$$S_4(2m+1) + S_2(2m+1) = S_4(2m) + S_2(2m) + (2m+1)^4 + (2m+1)^2$$
$$= 2\{(2m+1)(2m+2)/2\}^2.$$

Also solved by Paul S. Bruckman, Walther Janous, H. Klauser, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, Gregory Wulczyn, and the proposer.

Telescoping Series

B-477 *Proposed by Paul S. Bruckman, Sacramento, CA*

Prove that

$$\sum_{n=2}^{\infty} \text{Arctan}\,\frac{(-1)^n}{F_{2n}} = \frac{1}{2}\,\text{Arctan}\,\frac{1}{2}.$$

Solution by C. Georghiou, Univ. of Patras, Patras, Greece

It is known [see, e.g., Theorem 5 of "A Primer for the Fibonacci Numbers—Part IV" by V. E. Hoggatt, Jr. and I. D. Ruggles, this Quarterly, Vol. 1, no. 4 (1963):71] that

$$\sum_{m=1}^{\infty} (-1)^{m+1}\text{Arctan}\,\frac{1}{F_{2m}} = \text{Arctan}\,\frac{\sqrt{5}-1}{2}.$$

The problem is readily solved by noting that

$$(-1)^m \text{Arctan } x = \text{Arctan}(-1)^m x$$

and that

$$\text{Arctan } 1 - \text{Arctan } \frac{\sqrt{5} - 1}{2} = \frac{1}{2} \text{ Arctan } \frac{1}{2}.$$

Also solved by John Spraggon, J. Suck, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

RAYMOND E. WHITNEY

Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-352 *Proposed by Stephen Turner, Babson College, Babson Park, Mass.*

One night during a national mathematical society convention, n mathematicians decided to gather in a suite at the convention hotel for an "after hours chat." The people in this group share the habit of wearing the same kind of hats, and each brought his hat to the suite. However, the chat was so engaging that at the end of the evening each (being deep in thought and oblivious to the practical side of matters) simply grabbed a hat at random and carried it away by hand to his room.

Use a variation of the Fibonacci sequence for calculating the probability that none of the mathematicians carried his own hat back to his room.

H-353 *Proposed by Jerry Metzger, Univ. of North Dakota, Grand Forks, ND*

For a positive integer n, describe all two-element sets $\{a, b\}$ for which there is a polynomial $f(x)$ such that $f(x) \equiv 0 \pmod{n}$ has solution set exactly $\{a, b\}$.

H-354 *Proposed by Paul Bruckman, Concord, CA*

Find necessary and sufficient conditions so that a solution in relatively prime integers x and y can exist for the Diophantine equation:

$$ax^2 - by^2 = c,$$

given that a, b, and c are pairwise relatively prime positive integers, and, moreover, a and b are not both perfect squares.

H-355 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA*

Solve the second-order finite difference equation:

$$n(n-1)a_n - \{2rn - r(r+1)\}a_{n-r} + r^2 a_{n-2r} = 0.$$

r and n are integers. If $n - kr < 0$, $a_{n-kr} = 0$.

SOLUTIONS

Al Gebra

H-335 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 20, no. 1, February 1982)

Find the roots, in exact radicals, of the polynomial equation:

$$p(x) = x^5 - 5x^3 + 5x - 1 = 0. \tag{1}$$

Solution by M. Wachtel, Zürich, Switzerland

It is easy to see that one of the solutions is: $x = 1$.

Step 1: Dividing the original equation by $x - 1$, we obtain

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0.$$

Step 2: To eliminate x^3, we set $x = z - \frac{1}{4}$, which yields:

$$z^4 - \frac{35}{8}z^2 - \frac{15}{8}z + \frac{445}{256} = 0.$$

Step 3: Using the formula $t^3 + \frac{p}{2}t^2 + \frac{1}{4}\left[\left(\frac{p}{2}\right)^2 - r\right]t - \left(\frac{q}{8}\right)^2 = 0$, and setting $p = -\frac{35}{8}$, $q = -\frac{15}{8}$, $r = \frac{445}{256}$, the above equation is transformed into a cubic equation:

$$t^3 - \frac{35}{16}t^2 + \frac{195}{256}t - \frac{225}{4096} = 0.$$

Step 4: To eliminate t^2, we set $t = u + \frac{35}{48}$, which yields:

$$u^3 - \frac{5}{6}u - \frac{475}{1728} = 0.$$

Step 5: Using the Cardano formula, we obtain the "Casus irreduzibilus" ($\cos 3\alpha$) with three real solutions:

$$u_1 = -\frac{5}{12}; \quad u_2 = \frac{5 + 9\sqrt{5}}{24}; \quad u_3 = \frac{5 - 9\sqrt{5}}{24};$$

and it follows $\left(t = u + \dfrac{35}{48}\right)$:

$$t_1 = \frac{5}{16}; \quad t_2 = \frac{15 + 6\sqrt{5}}{16}; \quad t_3 = \frac{15 - 6\sqrt{5}}{16}.$$

Further:

$$U = \sqrt{t_1} = \frac{\sqrt{5}}{4}; \quad V = \sqrt{t_2} = \frac{\sqrt{15 + 6\sqrt{5}}}{4}; \quad W = \sqrt{t_3} = \frac{\sqrt{15 - 6\sqrt{5}}}{4}.$$

Step 6: Considering $x = z - \dfrac{1}{4}$ and the identities

$$\frac{\sqrt{15 + 6\sqrt{5}}}{4} \pm \frac{\sqrt{15 - 6\sqrt{5}}}{4} = \frac{\sqrt{30 \pm 6\sqrt{5}}}{4}$$

we obtain the following solutions:

$$x_0 = 1$$

$$x_1 = U + V + W = \frac{\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{4}$$

$$x_2 = U - V - W = \frac{-\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{4} \qquad x_1 \cdot x_2 = -\left(\frac{\sqrt{5} + 1}{2}\right)^2$$

$$x_3 = -U + V - W = \frac{\sqrt{30 - 6\sqrt{5}} - \sqrt{5} + 1}{4} \qquad x_3 \cdot x_4 = -\left(\frac{\sqrt{5} - 1}{2}\right)^2$$

$$x_4 = -U - V + W = \frac{-\sqrt{30 - 6\sqrt{5}} - \sqrt{5} + 1}{4} \qquad x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 1$$

The proofs of Steps 3 and 5 are tedious, but the respective formulas can be found in formula registers of algebra.

Also solved by the proposer. (One incorrect solution was received.)

Mod Ern

H-336 (Corrected) *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 20, no. 1, February 1982)

Let p be an odd prime.

(a) Prove that if $p \equiv 3$ or $7 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv -4 \pmod{p}.$$

(b) Prove that if $p \equiv 11$ or $19 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv 4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv 1 \pmod{p}.$$

(c) Prove that if $p \equiv 13$ or $17 \pmod{20}$, then

$$F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 0 \pmod{p}.$$

(d) Prove that if $p \equiv 1$ or $9 \pmod{20}$, then

$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv \pm 1 \pmod{p}.$$

Show that both the cases $F_{(p+1)/2} \equiv -1 \pmod{p}$ and $F_{(p+1)/2} \equiv 1 \pmod{p}$ do in fact occur.

Solution by the proposer

It is known that $F_p \equiv (5/p) \pmod{p}$ and $F_{p-(5/p)} \equiv 0 \pmod{p}$, where $(5/p)$ is the Legendre symbol. It is further known that

$$F_{(\frac{1}{2}(p-(5/p)))} \equiv 0 \pmod{p}$$

if and only if $(-1/p) = 1$. (See [1] or [3]). We also make use of the following identities:

$$F_{2n} = F_n(F_{n-1} + F_{n+1}). \tag{1}$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2. \tag{2}$$

Letting $k = (p - 1)/2$, we are now ready to prove parts (a)-(d).

(a) In this case $(5/p) = (-1/p) = -1$. Then, by (1) and (2),

$$F^2_{p+1} = F_{k+1}(F_k + F_{k+2}) \equiv 0 \pmod{p}. \tag{3}$$

and

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv -1 \pmod{p}. \tag{4}$$

Since $(-1/p) = -1$, $F_k \not\equiv 0 \pmod{p}$. Thus, by (3), $F_{k+2} \equiv -F_k \pmod{p}$. Hence,

$$F_k = F_{k+2} - F_{k+1} \equiv -F_k - F_{k+1} \pmod{p}.$$

Thus, $2F_k \equiv -F_{k+1} \pmod{p}$ and $4F_k^2 \equiv F_{k+1}^2 \pmod{p}$. Thus, by (4),

$$F_p^2 \equiv F_k^2 + 4F_k^2 = 5F^2_{(p-1)/2} \equiv -1 \pmod{p}.$$

Since $F_{k+1}^2 \equiv 4F_k^2$, $5F^2_{(p+1)/2} \equiv 4(5F^2_{(p-1)/2}) \equiv -4 \pmod{p}$.

(b) In this case $(5/p) = 1$ and $(-1/p) = -1$. Then

$$F^2_{p-1} = F_k(F_{k-1} + F_{k+1}) \equiv 0 \pmod{p}$$

and

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv 1 \pmod{p}.$$

Making use of the fact that $F_k \not\equiv 0 \pmod{p}$ and solving as in the solution of part (a), we find that

$$5F^2_{(p+1)/2} \equiv 1 \pmod p \text{ and } 5F^2_{(p-1)/2} \equiv 4 \pmod p.$$

(c) In this case $(5/p) = -1$ and $(-1/p) = 1$. Thus, $F_k \equiv 0 \pmod p$. Also,

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv F_{k+1}^2 \equiv -1 \pmod p.$$

(d) In this case $(5/p) = (-1/p) = 1$. Thus, $F_k \equiv 0 \pmod p$. Also,

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv F_{k+1}^2 \equiv 1 \pmod p.$$

Thus, $F_{k+1} \equiv \pm 1 \pmod p$. For $p = 29, 89, 101$, or 281, $F_{(p+1)/2} \equiv 1 \pmod p$ and for $p = 41, 61, 109$, or 409, $F_{(p+1)/2} \equiv -1 \pmod p$. These examples were obtained from [2].

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1. Robert P. Backstrom. "On the Determination of the Zeros of the Fibonacci Sequence." *The Fibonacci Quarterly* 4, no. 4 (1966):313-22.
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3. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Annals of Mathematics*, Second Series 31 (1910:419-48.

Also solved by P. Bruckman and G. Wulczyn.

Pivot

H-337 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA (Vol. 20, no. 1, February 1982)*

(a) Evaluate the determinant

det A

$$\begin{array}{cc}
\begin{vmatrix}
1 & -4L_{2r} & 6L_{4r}+16 & -(4L_{6r}+24L_{2r}) & L_{8r}+16L_{4r}+36 \\
L_{2r} & -(3L_{4r}+10) & 3L_{6r}+25L_{2r} & -(L_{8r}+25L_{4r}+60) & 10L_{6r}+60L_{2r} \\
L_{4r} & -(2L_{6r}+6L_{2r}) & L_{8r}+12L_{4r}+30 & -(6L_{6r}+50L_{2r}) & 30L_{4r}+80 \\
L_{6r} & -(L_{8r}+7L_{4r}) & 7L_{6r}+21L_{2r} & -(21L_{4r}+70) & 70L_{2r} \\
L_{8r} & -8L_{6r} & 28L_{4r} & -56L_{2r} & 140
\end{vmatrix} &
\begin{array}{c}(e)\\(d)\\(c)\\(b)\\(a)\end{array}
\end{array}$$

(b) Show that

$$\begin{aligned}
625F^2_{2r} &= L^2_{8r} - 8L^2_{6r} + 28L^2_{4r} - 56L^2_{2r} + 140 \\
&= -8L^2_{6r} + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r+10})^2 - 280L^2_{2r} \\
&= 28L^2_{4r} - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2 \\
&\qquad + 20(3L_{4r} + 8)^2 \\
&= -56L^2_{2r} + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2 \\
&\qquad - 40(L_{6r} + 6L_{2r})^2.
\end{aligned}$$

Grace Note: If the elements of this determinant are the coefficients of a 5 x 5 linear homogeneous system, then the solution to the 4 x 5 system represented by Equations (b), (c), (d), and (e) is given by the elements of the first column. The solution to (a), (c), (d), and (e) is given by the elements of the second column. And so on.

Solution by the proposer

(a) Using a Chio pivot reduction and taking out $L_{4r} - 2$ as a common factor from each term

$$\text{Det } A = (L_{4r} - 2)^4 \begin{vmatrix} 1 & -3L_{2r} & 3L_{4r} + 9 & -(L_{6r} + 9L_{2r}) \\ 2L_{2r} & -(5L_{4r} + 14) & 4L_{6r} + 26L_{2r} & -(L_{8r} + 18L_{4r} + 42) \\ 3L_{4r} + 3 & -(6L_{6r} + 21L_{2r}) & 4L_{8r} + 32L_{4r} + 63 & -(L_{10r} + 18L_{6r} + 71L_{2r}) \\ 4L_{6r} + 4L_{2r} & -(6L_{8r} + 28L_{4r} + 28) & 4L_{10r} + 32L_{6r} + 84L_{2r} & -(L_{12r} + 18L_{8r} + 71L_{4r} + 140) \end{vmatrix}$$

$$= (L_{4r} - 2)^7 \begin{vmatrix} 1 & -2L_{2r} & L_{4r} + 4 \\ 3L_{2r} & -(5L_{4r} + 14) & 2L_{6r} + 16L_{2r} \\ 6L_{4r} + 8 & -(8L_{6r} + 32L_{2r}) & 3L_{8r} + 28L_{4r} + 58 \end{vmatrix}$$

$$= (L_{4r} - 2)^9 \begin{vmatrix} 1 & -L_{2r} \\ 4L_{2r} & -(3L_{4r} + 10) \end{vmatrix} = (L_{4r} - 2)^{10}$$

$$= (5F_{2r}^2)^{10} = 5^{10}F_{2r}^{20}.$$

(b) $$L_{8r}^2 - 8L_{6r}^2 + 28L_{4r}^2 - 56L_{2r}^2 + 140 = L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70$$
$$= 625F_{2r}^8.$$

$$-8L_{6r}^2 + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r} + 10)^2 - 280L_{2r}^2$$

$$= L_{16r} + L_{12r}(14 - 8 - 14) + L_{8r}(49 - 84 + 63) + L_{4r}(14 - 84 - 126 + 420 - 280)$$
$$- 16 + 2 + 98 - 28 - 252 + 126 + 700 - 560$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8$$

$$28L_{4r}^2 - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2 + 20(3L_{4r} + 8)^2$$

$$= L_{16r} + L_{12r}(-14 + 24 - 18) + L_{8r}(28 - 84 + 144 + 60 - 300 + 180) + L_{4r}(-84 - 126 + 24 + 720 - 300 - 1250 + 960) + 56 - 28 - 252 + 2 + 288 + 900 - 36 - 2500 + 360 + 1280$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8$$

$$-56L_{2r}^2 + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2 - 40(L_{6r} + 6L_{2r})^2$$

$$= L_{16r} + L_{12r}(-18 + 50 - 40) + L_{8r}(63 - 300 + 120 + 625 - 480) + L_{4r}(-56 + 420 - 300 - 1250 + 50 + 3000 - 480 - 1440) - 112 + 126 + 700 - 36 - 2500 + 2 + 1250 + 3600 - 80 - 2880$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8.$$

Some Abundance

H-338 *Proposed by Charles R. Wall, Trident Tech. Coll., Charleston, SD (Vol. 20, no. 1, February 1982)*

An integer n is abundant if $\sigma(n) > 2n$, where $\sigma(n)$ is the sum of the divisors of n. Show that there is a probability of at least:

(a) 0.15 that a Fibonacci number is abundant;

(b) 0.10 that a Lucas number is abundant.

Solution by the proposer

Three well-known background facts are needed:

1. Any multiple of an abundant number is abundant.

2. F_{nm} is a multiple of F_n for all m.

3. L_{nm} is a multiple of L_n if m is odd.

From published tables of factors of Fibonacci numbers, we see that F_n is abundant if n is 12, 18, 30, 40, 42, 140, 315, 525, or 725. Since none of these numbers is a multiple of any other, the probability that a Fibonacci number is abundant is at least

$$\frac{1}{6}\left(1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\right) + \frac{1}{140}\cdot\frac{2}{3} + \frac{1}{40}\cdot\frac{2}{3}\cdot\frac{6}{7} + \frac{1}{2}\cdot\frac{1}{105}\left(1 - \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\right) = \frac{184}{1255}$$

$$= 0.1502\ldots .$$

Also, L_n is abundant if n is 6, 45, 75, or 105, and so the probability that a Lucas number is abundant is at least

$$\frac{1}{2\cdot 6} + \frac{1}{2\cdot 15}\left(1 - \frac{2}{3}\ \frac{4}{5}\ \frac{6}{7}\right) = \frac{71}{700} = 0.1014\ldots .$$

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