# The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

VOLUME 21
NUMBER 3

AUGUST
1983



CONTENTS

## PURPOSE

The primary function of **THE FIBONACCI QUARTERLY** is to serve as a focal point for widespread interest in the Fibonacci and related numbers, especially with respect to new results, research proposals, challenging problems, and innovative proofs of old ideas.

## EDITORIAL POLICY

**THE FIBONACCI QUARTERLY** seeks articles that are intelligible yet stimulating to its readers, most of whom are university teachers and students. These articles should be lively and well motivated, with new ideas that develop enthusiasm for number sequences or the exploration of number facts. Illustrations and tables should be wisely used to clarify the ideas of the manuscript. Unanswered questions are encouraged, and a complete list of references is absolutely necessary.

## SUBMITTING AN ARTICLE

Articles should be submitted in the format of the current issues of the **THE FIBONACCI QUARTERLY.** They should be typewritten or reproduced typewritten copies, that are clearly readable, double spaced with wide margins and on only one side of the paper. The full name and address of the author must appear at the beginning of the paper directly under the title. Illustrations should be carefully drawn in India ink on separate sheets of bond paper or vellum, approximately twice the size they are to appear in print.

Two copies of the manuscript should be submitted to: **GERALD E. BERGUM, EDITOR, THE FIBONACCI QUARTERLY, DEPARTMENT OF MATHEMATICS, SOUTH DAKOTA STATE UNIVERSITY, BOX 2220, BROOKINGS, SD 57007-1297.**

Authors are encouraged to keep a copy of their manuscripts for their own files as protection against loss. The editor will give immediate acknowledgment of all manuscripts received.

## SUBSCRIPTIONS, ADDRESS CHANGE, AND REPRINT INFORMATION

Address all subscription correspondence, including notification of address change, to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, UNIVERSITY OF SANTA CLARA, SANTA CLARA, CA 95053.**

Requests for reprint permission should be directed to the editor. However, general permission is granted to members of The Fibonacci Association for noncommercial reproduction of a limited quantity of individual articles (in whole or in part) provided complete references is made to the source.

Annual domestic Fibonacci Association membership dues, which include a subscription to **THE FIBONACCI QUARTERLY** are \$20 for Regular Membership, \$28 for Sustaining Membership I, \$44 for Sustaining Membership II, and \$50 for Institutional Membership; foreign rates, which are based on international mailing rates, are somewhat higher than domestic rates; please write for details. **THE FIBONACCI QUARTERLY** is published each February, May, August and November.

All back issues of **THE FIBONACCI QUARTERLY** are available in microfilm or hard copy format from **UNIVERSITY MICROFILMS INTERNATIONAL, 300 NORTH ZEEB ROAD, DEPT P.R., ANN ARBOR, MI 48106.**

# The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
Br. Alfred Brousseau, and I.D. Ruggles

*THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES*

◆◇◆◇◆

# PROPERTIES OF SOME EXTENDED BERNOULLI AND EULER POLYNOMIALS

S. N. SINGH and B. K. RAI
*Banaras Hindu University, Varanasi - 221005, India*

(*Submitted September 1980*)

## 1. INTRODUCTION

The study of Bernoulli, Euler, and Eulerian polynomials has contributed much to our knowledge of the theory of numbers. These polynomials are of basic importance in several parts of analysis and calculus of finite differences, and have applications in various fields such as statistics, numerical analysis, and so on. In recent years, the Eulerian numbers and certain generalizations have been found in a number of combinatorial problems (see [1], [3], [4], [5], [6], for example). A study of the above polynomials led us to the consideration of the following extension (3.1) of the Bernoulli, Euler, and Eulerian numbers, as well as polynomials in the unified form from a different point of view just described.

## 2. PRELIMINARY RESULTS

It is well known that the formulas [2]

$$g(n) = \sum_{d|n} f(d) \qquad (n = 1, 2, 3, \ldots) \tag{2.1}$$

and

$$f(n) = \sum_{cd=n} \mu(c)g(d) \qquad (n = 1, 2, 3, \ldots), \tag{2.2}$$

where $\mu(n)$ is the Mobius function, are equivalent. If in (2.1) and (2.2) we take $n = e_1e_2 \ldots e_r$, where the $e_j$ are distinct primes, it is easily verified that (2.1) and (2.2) reduce to

$$g_r = \sum_{j=0}^{r} \binom{r}{j} f_j \qquad (r = 0, 1, 2, \ldots) \tag{2.3}$$

and

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g_j \qquad (r = 0, 1, 2, \ldots), \tag{2.4}$$

respectively, where for brevity we put

$$f_r = f(e_1e_2 \ldots e_r), \quad g_r = g(e_1e_2 \ldots e_r). \tag{2.5}$$

The equivalence of (2.3) and (2.4) is of course well known; the fact that the second equivalence is implied by the first is perhaps not quite so familiar. It should be emphasized that $f(n)$ and $g(n)$ are arbitrary arithmetic functions subject only to (2.1) or equivalently, (2.2); a like remark applies to $f_r$ and $g_r$.

Given a sequence

$$f_r \qquad (r = 0, 1, 2, \ldots), \tag{2.6}$$

we define an extended sequence

$$f(n) \qquad (n = 1, 2, 3, \ldots) \tag{2.7}$$

such that

$$f(e_1 e_2 \ldots e_r) = f_r, \tag{2.8}$$

where the $e_j$ are distinct primes. Clearly the extended sequence (2.7) is not uniquely determined by means of (2.8). If the sequence $g_r$ is related to $f_r$ by means of (2.3), then the sequence $g(n)$ defined by means of (2.1) furnishes an extension of the sequence $g_r$.

If we associate with the sequence $f_r$ the (formal) power series

$$F_t = \sum_{r=0}^{\infty} f_r \frac{t^r}{r!}, \tag{2.9}$$

then (2.3) is equivalent to

$$G_t = \exp t \cdot F_t, \tag{2.10}$$

where

$$G_t = \sum_{r=0}^{\infty} g_r \frac{t^r}{r!}.$$

We associate with the sequence $f(n)$ the (formal) Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{2.11}$$

Then (2.1) is equivalent to

$$G(s) = \zeta(s)F(x), \tag{2.12}$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

## 3. EXTENDED POLYNOMIAL

We now define the extended polynomial set $B(n, h, a, k; x)$ using the following formula:

$$\frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^h - a} = \sum_{n=1}^{\infty} \frac{B(n, h, a, k; x)}{n^s}, \tag{3.1}$$

where $a$ is a nonzero real number, $k$ is a nonnegative integer, and $h \neq 0$.

On specializing various parameters involved therein we find the following relationships between our polynomials $B(n, h, a, k; x)$ and the extended Bernoulli, Euler, and other polynomials:

(i) Extended Bernoulli polynomials

$$B(n, h, 1, 1; x) = \beta(n, h; x) \tag{3.2}$$

(ii) Extended Bernoulli numbers

$$B(n, h, 1, 1; 0) = \beta(n, h) \tag{3.3}$$

(iii) Extended Euler polynomials

$$B(n, h, -1, 0; x) = \varepsilon(n, h; x) \tag{3.4}$$

(iv) Extended Euler numbers

$$B(n, h, -1, 0; 0) = \varepsilon(n, h) \tag{3.5}$$

(v) Extended Eulerian polynomials

$$B(n, h, a, 0; x) = \frac{2}{1-a} H(n, h, a; x) \tag{3.6}$$

(vi) Extended Eulerian numbers

$$B(n, h, a, 0; 0) = \frac{a}{1-a} H(n, h, a), \tag{3.7}$$

where the extended Bernoulli, Euler, and Eulerian polynomials and numbers are those introduced by Carlitz [2].

In the present paper we obtain numerous properties of the polynomials and numbers defined above. These properties are of an algebraic nature, and for the most part are generalizations on the corresponding properties of the Bernoulli, Euler, and Eulerian polynomials and numbers.

## 4. COMPLEMENTARY ARGUMENT THEOREM

**Theorem 1**

$$B(n, h, a, k; 1-x) = \frac{(-1)^{k-1}}{a} B(n, -h, 1/a, k; x). \tag{4.1}$$

Proof: Consider the following:

$$\sum_{n=1}^{\infty} \frac{B(n, h, a, k; 1 - x)}{n^s} = \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{h(1-x)}}{(\zeta(s))^h - a}$$

$$= \frac{2\left(\frac{1}{2}h \log \zeta(x)\right)^k (\zeta(s))^{-hk}}{1 - a(\zeta(s))^{-h}}$$

$$= \frac{(-1)^{k-1}}{a} \cdot \frac{2\left(-\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{-hk}}{(\zeta(s))^{-h} - \frac{1}{a}}.$$

The theorem would follow if we interpret the above expression by (3.1).

Putting $x = 0$ in (4.1), we obtain

Corollary 1

$$B(n, h, a, k; 1) = \frac{(-1)^{k-1}}{a} B(n, -h, 1/a, k), \tag{4.2}$$

where (here and throughout this paper) $B(n, h, a, k; 0) = B(n, h, a, k)$.

## 5. RECURRENCE RELATIONS

To obtain some interesting results, we refer to [2] for the definition of $T_x(n)$:

$$(\zeta(s))^x = \sum_{n=1}^{\infty} \frac{T_x(n)}{n^s}, \tag{5.1}$$

where

$$T_x(n) = \prod_{e/h} \binom{j + x - 1}{j} \text{ with } n = \pi e^j,$$

and put

$$\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}, \tag{5.2}$$

where

$$\alpha(n) = \begin{cases} \frac{1}{r} & (n = e^r,\ r \geqslant 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{5.3}$$

We remark that $T_x(n)$ is a multiplicative function of $n$; that is,

$$T_x(mn) = T_x(m) \cdot T_x(n) \qquad [(m, n) = 1], \tag{5.4}$$

where $(m, n)$ denotes the highest common divisor of two numbers $m$ and $n$.

It is evident from (3.1) that

$$\sum_{n=1}^{\infty} \frac{B(n, h, a, k; x + y)}{n^s} = \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{h(x+y)}}{(\zeta(s))^h - a}$$

$$= \sum_{n=1}^{\infty} \frac{T_{hx}(n)}{n^s} \sum_{n=1}^{\infty} \frac{B(n, h, a, k; y)}{n^s},$$

which yields

Corollary 2

$$B(n, h, a, k; x + y) = \sum_{cd=n} T_{hx}(c) B(d, h, a, k; y). \tag{5.5}$$

In particular,

$$B(n, h, a, k; x) = \sum_{cd=n} T_{hx}(c) B(d, h, a, k). \tag{5.6}$$

From (3.1), it is easy to deduce the result:

$$\frac{d}{dx} B(n, h, a, k; x) = h \sum_{cd=n} \alpha(c) B(d, h, a, k; x). \tag{5.7}$$

Again, we may write (3.1) as

$$\sum_{n=1}^{\infty} \frac{B(n, 2h, a^2, k; x)}{n^s} = \frac{2(h \log \zeta(s))^k (\zeta(s))^{2hx}}{(\zeta(s))^{2h} - a^2}$$

$$= 2^{k-1} \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^h - a} \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^{k-1} (\zeta(s))^{hx}}{(\zeta(s))^h + a}$$

which gives a recurrence relation:

$$B(n, 2h, a^2, k; x) = 2^{k-1} \sum_{cd=n} B(c, h, a, 1; x) B(d, h, -a, k - 1; x). \tag{5.8}$$

Let us now consider the identity

$$\frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx/2}}{(\zeta(s))^{h/2} + a} = \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{h(x+1)/2}}{(\zeta(s))^h - a^2} - a \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx/2}}{(\zeta(s))^h - a^2}.$$

Because of the generating relation (3.1), we obtain:

$$2^k B(n, h/2, -a, k; x) = B\left(n, h, a^2, k; \frac{x + 1}{2}\right) - aB(n, h, a^2, k; x/2). \tag{5.9}$$

It follows from (3.1) that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} (B(n, h, a, k; x + 1) - aB(n, h, a, k; x)) = 2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx},$$

which implies:

$$B(n,\ h,\ a,\ k;\ x+1) - aB(n,\ h,\ a,\ k;\ x) = \frac{h^k}{2^{k-1}} \sum_{cd=n} \alpha_k(c) T_{hx}(d). \tag{5.10}$$

This leads to the summation formula:

$$\frac{h^k}{2^{k-1}} \sum_{cd=n} \alpha_k(c) \sum_{j=0}^{m-1} \frac{1}{a^j} T_{h(x+y)}(d)$$
$$= B(n,\ h,\ a,\ k;\ x+m) - aB(n,\ h,\ a,\ k;\ x). \tag{5.11}$$

It is easily verified that when $h = 1$ and $n = e_1 e_2 \ldots e_r$, (5.11) reduces to the familiar formula

$$D_n(x+m;\ a,\ k) - aD_n(x;\ a,\ k) = \frac{(n)_k}{2^{k-1}} \sum_{j=0}^{m-1} (x+j)^{n-k},$$

where

$(n)_k = n(n-1) \cdots (n-k+1)$ and $D_n(x;\ a,\ k)$ is defined in [7].

## 6. ADDITION THEOREMS

It may be of interest to deduce some addition theorems that are satisfied by $B(n,\ h,\ a,\ k;\ x)$.

Since

$$\frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{2hx}}{(\zeta(s))^h - a} \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{2hy}}{(\zeta(s))^h + a}$$
$$= 2 \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^{2k} (\zeta(s))^{2h(x+y)}}{(\zeta(s))^{2h} - a^2},$$

there follows at once:

Theorem 2

$$2^{2k-1} \sum_{cd=n} B(c,\ h,\ a,\ k;\ 2x) B(d,\ h,\ -a,\ k;\ 2y)$$
$$= B(n,\ 2h,\ a^2,\ 2k;\ x+y). \tag{6.1}$$

If we note the identity,

$$\frac{1}{(\zeta(s))^h - a} - \frac{1}{(\zeta(s))^h + a} = \frac{2a}{(\zeta(s))^{2h} - a^2}$$

then, as a consequence of (3.1), we arrive at:

Theorem 3

$$B(n,\ h,\ a,\ k;\ x) - B(n,\ h,\ -a,\ k,\ x) = \frac{a}{2^{k-1}}B(n,\ 2h,\ a^2,\ k;\ x/2). \tag{6.2}$$

On the other hand, since

$$\frac{1}{(\zeta(s))^h - a} - \frac{1}{(\zeta(s))^h - 1 + a} = \frac{2a - 1}{((\zeta(s))^h - a)((\zeta(s))^h - 1 + a)},$$

we get, from (3.1):

Theorem 4

$$\left(a - \frac{1}{2}\right)\sum_{cd=n} B(c,\ h,\ a,\ k;\ x)B(d,\ h,\ 1 - a,\ k;\ y)$$
$$= B(n,\ h,\ a,\ 2k;\ x + y) - B(n,\ h,\ 1 - a,\ 2k;\ x + y). \tag{6.3}$$

## 7. MULTIPLICATION THEOREMS

We establish the following multiplication theorems, in which $m$ stands for a positive integer.

Theorem 5

$$\sum_{r=0}^{m-1} \frac{1}{a^r} B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right) = \frac{m}{a^{m-1}} B(n,\ h,\ a,\ k;\ mx). \tag{7.1}$$

Proof: In order to obtain (7.1), we have, from (3.1), the relation,

$$\sum_{n=1}^{\infty} \frac{B(n,\ h,\ a,\ k;\ mx)}{n^s} = \frac{2\left(\frac{1}{2}\ h\ \log\ \zeta(s)\right)^k (\zeta(s))^{mhx}}{(\zeta(s))^h - a}$$
$$= a^{m-1} \cdot \frac{2\left(\frac{1}{2} h\ \log\ \zeta(s)\right)^k (\zeta(s))^{mhx}}{(\zeta(s))^{mh} - a^m} \cdot \sum_{r=0}^{m-1} \frac{(\zeta(s))^{rh}}{a^r}$$
$$= \frac{a^{m-1}}{m^k} \sum_{r=0}^{m-1} \frac{1}{a^r} \sum_{n=1}^{\infty} \frac{B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right)}{n^s},$$

which completes the proof.

Proceeding exactly as in the proof of Theorem 5, and recalling (3.1), we obtain:

Theorem 6

$$\sum_{r=0}^{m-1} \frac{1}{a^r} B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right) = \frac{m^k h}{2a^{m-1}} \sum_{cd=n} (c)B(d,\ h,\ a,\ k-1;\ mx). \tag{7.2}$$

Theorem 7

$$a^m p^k \sum_{r=0}^{m-1} \frac{1}{a^{rp}} B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)$$

$$= a^p m^k \sum_{q=0}^{p-1} \frac{1}{a^{qm}} B\left(n,\ ph,\ a^m,\ k;\ \frac{x}{p} + \frac{qm}{p}\right). \tag{7.3}$$

Proof: From (3.1), we have:

$$\sum_{n=1}^{\infty} \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)}{n^s}$$

$$= \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{2\left(\frac{1}{2}mh \log \zeta(s)\right)^k (\zeta(s))^{\left(\frac{x}{m} + \frac{rp}{m}\right)mh}}{(\zeta(s))^{mh} - a^m}$$

$$= m^k \cdot 2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx} \sum_{r=0}^{m-1} \frac{(\zeta(s))^{rph}}{a^{rp}}$$

$$= \frac{m}{a^{p(m-1)}} \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^{mh} - a^m} \cdot \frac{(\zeta(s))^{mph} - a^{mp}}{(\zeta(s))^{ph} - a^p}. \tag{7.4}$$

From (7.4), and using symmetry in $m$ and $p$, we obtain the required result.

Theorem 8

$$\sum_{r=0}^{m-1} \frac{1}{a^{rp}} B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)$$

$$= \frac{m^k h \cdot a^{p-m}}{2p^{k-1}} \sum_{q=0}^{p-1} \frac{1}{a^{qm}} \cdot \sum_{cd} \alpha(c) B\left(d,\ ph,\ a^m,\ k-1;\ \frac{x}{p} + \frac{qm}{p}\right). \tag{7.5}$$

Proof: It follows from (3.1) that

$$\sum_{n=1}^{\infty} \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)}{n^s}$$

$$= \frac{m^k h(\log \zeta(s))}{2p^{k-1} a^{p(m-1)}} \cdot \frac{2\left(\frac{1}{2}ph \log \zeta(s)\right)^{k-1} (\zeta(s))^{hx}}{(\zeta(s))^{ph} - a^p} \cdot \frac{(\zeta(s))^{mph} - a^{mp}}{(\zeta(s))^{ph} - a^p}$$

$$= \frac{m\ h \cdot a^{m(p-1)}}{2p^{k-1} a^{p(m-1)}} \cdot \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \sum_{n=1}^{\infty} \sum_{q=0}^{p-1} \frac{1}{a^{qm}} \cdot \frac{B\left(n,\ ph,\ a^p,\ k-1;\ \frac{x}{p} + \frac{qm}{p}\right)}{n^s},$$

which completes the proof.

Theorems 7 and 8 are elegant generalizations of Theorems 5 and 6, respectively.

## 8. ANOTHER MULTIPLICATION THEOREM

If we define the function $\overline{B}(n, h, a, k; x)$ by means of

$$\overline{B}(n, h, a, k; x) = B(n, h, a, k; x) \qquad (0 \leqslant x < 1),$$

$$\overline{B}(n, h, a, k; x + 1) = a\overline{B}(n, h, a, k; x),$$

then it is easily seen that multiplication formula (7.1) also holds for the barred function.

In this section we obtain an interesting generalization of (7.1) suggested by a recent result of Mordell [9]. In extending some results of Mikolas [8], Mordell proves the following theorem:

Let $u_1(x), \ldots, u_p(x)$ denote functions of $x$ of period 1 that satisfy the relations

$$\sum_{r=0}^{m-1} u_i\left(x + \frac{r}{m}\right) = C_i^{(m)} u_i(mx) \qquad (i = 1, \ldots, p), \tag{8.1}$$

where $C_i^{(m)}$ is independent of $x$, let $a_1, \ldots, a_p$ be positive integers that are prime in pairs. Then, if the integrals exist and $A = a_1 a_2 \ldots a_p$,

$$\begin{aligned}
&\int_0^A u_1(x/a_1)u_2(x/a_2) \ldots u_p(x/a_p)\,dx \\
&= A\int_0^1 u_1(Ax/a_1)u_2(Ax/a_2) \ldots u_p(Ax/a_p)\,dx \\
&= C_1^{(a_1)}C_2^{(a_2)} \ldots C_p^{(a_p)} \int_0^1 u_1(x)u_2(x) \ldots u_p(x)\,dx.
\end{aligned} \tag{8.2}$$

We prove:

### Theorem 8

Let $p \geqslant 1$; $n_1, \ldots, n_1 \geqslant 1$; $a_1, \ldots, a_p$ be positive integers that are relatively prime in pairs; $A = a_1 a_2 \ldots a_p$. Then,

$$\sum_{r=0}^{mA-1} \frac{1}{a^r} \overline{B}\left(n_1, ma_1h, a^{ma_1}, k; x_1 + \frac{r}{ma_1}\right) \cdot \overline{B}\left(n_2, ma_2h, a^{ma_2}, k; x_2 + \frac{r}{ma_2}\right)$$

$$\cdot \ldots \cdot \overline{B}\left(n_p, ma_ph, a^{ma_p}, k; x_p + \frac{r}{ma_p}\right)$$

*(continued)*

$$= C\sum_{r=0}^{m-1}\frac{1}{a^r}\overline{B}\left(n_1,\ mh,\ a^m,\ k;\ a_1x_1 + \frac{r}{m}\right)\cdot\overline{B}\left(n_2,\ mh,\ a^m,\ k;\ a_2x_2 + \frac{r}{m}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a\ x\ + \frac{r}{m}\right), \tag{8.3}$$

where

$$C = \frac{a_1^k a_2^k \ldots a_p^k}{a^{m(a_1-1)}a^{m(a_2-1)}\ \ldots\ a^{m(a_p-1)}}. \tag{8.4}$$

Proof: From (7.1), it is not difficult to show for arbitrary $a^* \geqslant 1$ that

$$\sum_{r=0}^{ma^*-1}\frac{1}{a}\overline{B}\left(n,\ ma^*h,\ a^{ma^*},\ k;\ x + \frac{r}{ma^*}\right)$$

$$= \sum_{r=0}^{m-1}\sum_{s=0}^{a^*-1}\frac{1}{a^{r+sm}}\overline{B}\left(n,\ ma^*h,\ a^{ma^*},\ k;\ x + \frac{s}{a^*} + \frac{r}{ma^*}\right)$$

$$= \frac{(a^*)^k}{a^{m(a^*-1)}}\cdot\sum_{r=0}^{m-1}\frac{1}{a^r}\overline{B}\left(n,\ mh,\ a^m,\ k;\ a^*x + \frac{r}{m}\right),$$

which agrees with (8.3) for $p = 1$.

For the general case, let $S$ denote the left member of (8.3), and

$$A_s = a_1a_2\ \ldots\ a_s \qquad (1 \leqslant s \leqslant p).$$

If we replace $r$ by $smA_{p-1} + r$, we have

$$S = \sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1 + \frac{r}{ma_1}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1} + \frac{r}{ma_{p-1}}\right)$$

$$\cdot\sum_{s=0}^{a_{p-1}}\frac{1}{a^{sm}}\overline{B}\left(n_p,\ ma_ph,\ a^{ma_p},\ k;\ x_p + \frac{A_{p-1}s}{a_p} + \frac{r}{ma_p}\right)$$

$$= \sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1 + \frac{r}{ma_1}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1} + \frac{r}{ma_{p-1}}\right)$$

$$\cdot\sum_{s=0}^{a_p-1}\frac{1}{a^{sm}}\overline{B}\left(n_p,\ ma_ph,\ a^{ma_p},\ k;\ x_p + \frac{s}{a_p} + \frac{r}{ma_p}\right)$$

$$= \frac{a_p^k}{a^{m(a_p-1)}}\sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1 + \frac{r}{ma_1}\right)$$

*(continued)*

$$\cdot \cdots \cdot \overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1} + \frac{r}{ma_{p-1}}\right)$$

$$\cdot \overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a_p x_p + \frac{r}{m}\right).$$

Continuing the same process, we get:

$$S = \frac{a_1^k a_2^k \cdots a_p^k}{a^{m(a_1-1)}a^{m(a_2-1)} \cdots a^{m(a_p-1)}}$$

$$\cdot \sum_{r=0}^{m-1} \frac{1}{a^r} \overline{B}\left(n_1,\ mh,\ a^m,\ k;\ a_1 x_1 + \frac{r}{m}\right) \cdot \overline{B}\left(n_2,\ mh,\ a^m,\ k;\ a_2 x_2 + \frac{r}{m}\right)$$

$$\cdot \cdots \cdot \overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a_p x_p + \frac{r}{m}\right),$$

which completes the proof.

We remark that for $m = 1$, (8.3) reduces to

$$\sum_{r=0}^{A-1} \frac{1}{a^r} \overline{B}\left(n_1,\ a_1 h,\ a^{a_1},\ k;\ x_1 + \frac{r}{a_1}\right) \cdot \cdots \cdot \overline{B}\left(n_p,\ a_p h,\ a^{a_p},\ k;\ x_p + \frac{r}{a_p}\right)$$

$$= C^*\overline{B}(n_1,\ h,\ a,\ k;\ a_1 x_1) \cdot \cdots \cdot \overline{B}(n_p,\ h,\ a,\ k;\ a_p x_p), \tag{8.5}$$

where

$$C^* = \frac{a_1^k \cdots a_p^k}{a^{a_1-1} \cdots a^{a_p-1}}.$$

## ACKNOWLEDGMENT

We are grateful to the referee for suggestions which led to a better presentation of this paper.

## REFERENCES

1. L. Carlitz. "Eulerian Numbers and Polynomials." *Math. Magazine* 33 (1959): 247-60.
2. L. Carlitz. "Extended Bernoulli and Eulerian Numbers." *Duke Math. J.* 31 (1964):667-89.
3. L. Carlitz. "Enumeration of Sequences by Rises and Falls: A Refinement of the Simon Newcomb's Problem." *Duke Math. J.* 39 (1972):267-80.
4. L. Carlitz & R. Scoville. "Generalized Eulerian Numbers: Combinatorial Applications." *J. für die reine und angewandte Mathematik* 265 (1974);110-37.
5. J. F. Dillon & D. P. Roselle. "Simon Newcomb's Problems." *SIAM J. Appl. Math* 17 (1969):1086-93.

6. D. Foata & M. P. Schützenberger. *Théorie Géométrique des polynômes Eulériens*. Lecture Notes in Math 138. Berlin, Heidelberg, New York: Springer-Verlag, 1970.
7. B. K. Karande & N. K. Thakare. "On the Unification of Bernoulli and Euler Polynomials." *Indian J. Pure Appl. Math.* 6 (1975):98-107.
8. M. Mikolas. "Integral Formulas of Arithmetical Characteristics Relating to the Zeta-Function of Hurwitz." *Publicationes Mathematicae* 5 (1957):44-53.
9. L. J. Mordell. "Integral Formulas of Arithmetical Character." *J. London Math. Soc.* 33 (1959):371-75.

◆◇◆◇◆

# COMMENT ON PROBLEM H-315

WILHELM WERNER
*Johannes Gutenberg-Universitat in Mainz*

Recently I came to know Problem H-315 of *The Fibonacci Quarterly* (Vol. 18, 1980) which deals with "Kerner's method" for the simultaneous determination of polynomial roots. I want to comment on two aspects of the problem and its solution.

1. The method was already used by K. Wierstrass for a constructive proof of the fundamental theorem of algebra (cf. [1]). Kerner [2] realized that the method can be interpreted as a Newton method for the elementary symmetric functions; this fact is also observed in the textbook of Durand ([3], pp. 279-80) which appeared several years before Kerner's publication.

2. It is remarkable that the assumption

$$\sum_{i=1}^{n} z_i = -a_{n-1}$$

is *not* necessary for the validity of the assertion! This fact is mentioned by Byrnev and Dochev [4] where further references are given. The proof of the assertion

$$\sum_{i=1}^{n} \hat{z}_i = -a_{n-1}$$

is easy: following Kerner's derivation of the method, one must apply Newton's method to the system of elementary symmetric functions. Hence, one of the equations reads:

$$\sum_{i=1}^{n} x_i = -a_{n-1} \quad (x_1, x_2, \ldots, x_n \text{ denote the unknowns}).$$

*[Please turn to page 188]*

◆◇◆◇◆

# FIBONACCI GRACEFUL GRAPHS

DAVID W. BANGE and ANTHONY E. BARKAUSKAS*
*University of Wisconsin-La Crosse, La Crosse, WI 53601*

*(Submitted November 1980)*

## 1. INTRODUCTION

A simple graph $G(p, n)$ with $p$ vertices and $n$ edges is *graceful* if there is a labeling $\ell$ of its vertices with distinct integers from the set

$$\{0, 1, 2, \ldots, n\}$$

so that the induced edge labeling $\ell'$, defined by

$$\ell'(uv) = |\ell(u) - \ell(v)|$$

assigns each edge a different label. The problem of characterizing all graceful graphs remains open (Golomb [3]), and in particular the Ringel-Kotzig-Rosa conjecture that all trees are graceful is still unproved after fifteen years. (For a summary of the status of this conjecture, see Bloom [2].) Other classes of graphs that are known to be graceful include complete bipartite graphs (Rosa [7]), wheels (Höede & Kuiper [5]), and cycles on $n$ vertices where $n \equiv 0$ or 3 (mod 4) (Hebbare [4]).

A natural extension of the idea of a graceful graph is to have the induced edge labeling $\ell'$ of $G(p, n)$ be a bijection onto the first $n$ terms of an arbitrary sequence of positive integers $\{a_i\}$. In a recent paper, Koh, Lee, & Tan [6] chose the sequence $\{a_i\}$ to be the Fibonacci numbers $\{F_i\}$ where

$$F_n = F_{n-1} + F_{n-2}; \; F_1 = F_2 = 1.$$

They defined a *Fibonacci tree* to be a tree $T(n + 1, n)$ in which the vertices can be labeled with the first $n + 1$ Fibonacci numbers so that the induced edge numbers will be the first $n$ Fibonacci numbers. Koh, Lee, & Tan gave a systematic way to obtain all Fibonacci trees as subgraphs of a certain class of graphs and showed that the number of (labeled) Fibonacci trees on $n + 1$ vertices is equal to $F_n$. The only graphs other than trees which can be labeled in this fashion are certain unicyclic graphs where the cycle is of length three.

In this paper, we modify the definition of Koh, Lee, & Tan so that the vertex labels of $G(p, n)$ are allowed to be distinct integers (not necessarily Fibonacci numbers) from the set $\{0, 1, 2, 3, 4, \ldots, F_n\}$. Formally, we make the following:

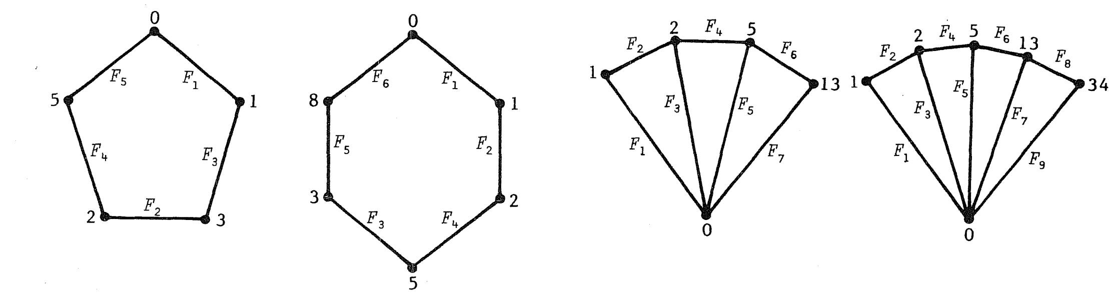
---

**This work was supported by a grant from the Faculty Research Committee, University of Wisconsin-La Crosse.*

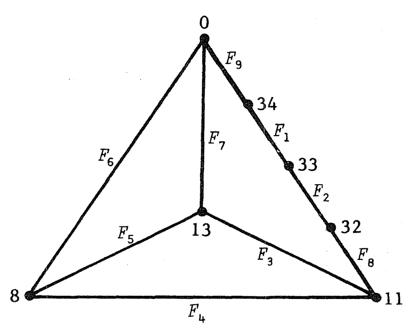
## Definition

A graph $G(p, n)$ will be called *Fibonacci graceful* if there is a labeling $\ell$ of its vertices with distinct integers from the set $\{0, 1, 2, 3, 4, \ldots, F_n\}$ so that the induced edge labeling $\ell'$, defined by $\ell'(uv) = |\ell(u) - \ell(v)|$, is a bijection onto the set $\{F_1, F_2, F_3, \ldots, F_n\}$.

This definition gives rise to an extensive class of graphs that are Fibonacci graceful; several examples appear in Figure 1. In Sections 2 and 3, we shall show how the cycle structure of Fibonacci graceful graphs is determined by the properties of the Fibonacci numbers. In Sections 4 and 5, we shall prove that several classes of graphs are Fibonacci graceful, including almost all trees. The general question of characterizing all Fibonacci graceful graphs will remain open.




a. Cycles $C_5$ and $C_6$ b. Fans




c. A graph homeomorphic to $K_4$

FIGURE 1. SOME FIBONACCI GRACEFUL GRAPHS

## 2. SOME PROPERTIES OF FIBONACCI GRACEFUL GRAPHS

From the definition of a Fibonacci graceful graph, it is apparent that the edge numbered $F_n$ must have 0 and $F_n$ as the vertex numbers of its endpoints.

Furthermore, any vertex adjacent to the vertex labeled 0 must be labeled with a Fibonacci number. The remaining vertices receive integer labels between 0 and $F_n$, but these need not be Fibonacci numbers.

It is easy to see that if a graph is Fibonacci graceful, then it may have several distinct labelings. In fact, we have the standard "inverse node numbering" ([3], p. 27).

Observation 1: If $\{0 = a_1, a_2, a_3, \ldots, a_n = F_n\}$ is a set of vertex labels of a Fibonacci graceful graph, then changing each label $a_i$ to $F_n - a_i$ also gives a Fibonacci graceful labeling of the graph.

We also have the following theorem which demonstrates that the cycle structure of Fibonacci graceful graphs is dependent on Fibonacci identities.

## Theorem 1

Let $G(p, n)$ be a graph with a Fibonacci graceful labeling and let $C_i$ be a cycle of length $k$ in $G$. Then there exists a sequence $\{\delta_{ij}\}_{j=1}^{k}$ with $\delta_{ij} = \pm 1$ for all $j = 1, 2, \ldots, k$ such that

$$\sum_{j=1}^{k} \delta_{ij} F_{ij} = 0$$

where $\{F_{ij}\}_{j=1}^{k}$ are the Fibonacci numbers for the edges in $C_i$.

Proof: Let $a_1, a_2, \ldots, a_k$ be the vertex labels for $C_i$. Clearly,

$$\sum_{j=1}^{k-1} (a_{j+1} - a_j) + (a_1 - a_k) = 0.$$

Since each difference $a_{j+1} - a_j$ equals either an edge label on $C_i$ or its negative, the theorem follows. ∎

## Corollary 1.1

If graph $G$ has a Fibonacci graceful labeling, then the edges of any cycle of length 3 in $G$ must be numbered with 3 consecutive Fibonacci numbers (note that $F_1, F_3, F_4$ is equivalent to $F_2, F_3, F_4$).

## Corollary 1.2

If graph $G$ has a Fibonacci graceful labeling, then the edges of any cycle of length 4 in $G$ must be numbered with a sequence of the form $F_i, F_{i+1}, F_{i+3}, F_{i+4}$.

## Corollary 1.3

If graph $G$ has a Fibonacci graceful labeling, then the edges of any cycle of length 5 must be numbered with either a sequence of the form $F_i, F_{i+1}, F_{i+3}, F_{i+5}, F_{i+6}$ or $F_1, F_2, F_i, F_{i+1}, F_{i+2}$.

Corollary 1.4

Let graph $G$ have a Fibonacci graceful numbering. Suppose that in cycle $C_i$ of length $k$ the three largest edge labels are consecutive Fibonacci numbers, $F_{ik-2}$, $F_{ik-1}$, $F_{ik}$. Then for the remaining labels on $C_i$ we have

$$\sum_{j=1}^{k-3} \delta_{ij} F_{ij} = 0.$$

Proof: Both $\delta_{ik-2}$ and $\delta_{ik-1}$ must be opposite in sign to $\delta_{ik}$ for, otherwise, the sum of $F_{ik}$ and either of $F_{ik-2}$ or $F_{ik-1}$ would exceed the sum of all the remaining edge labels on $C_i$, violating Theorem 1. [See Identity (2) below]. ■

For convenience, we list some of the basic Fibonacci identities that are useful later:

(1) $F_n = F_{n-1} + F_{n-2}$; $F_1 = F_2 = 1$.

(2) $F_1 + F_2 + F_3 + \cdots + F_{n-2} = F_n - 1$.

(3) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

(4) $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n} - 1$.

A variation of Identities (3) and (4) may be obtained by once omitting a pair of consecutive Fibonacci numbers:

(5) $F_n - 1 > F_{n-2} + F_{n-4} + F_{n-6} + \cdots + F_{j+2} + F_j + F_{j-3} + F_{j-5} + \cdots$, $(j \geqslant 3)$.

The next result, stated as a lemma, is useful both in seeking Fibonacci graceful labelings and in developing a theory of the structure of Fibonacci graceful graphs.

Lemma 1

Suppose $G(p, n)$ has a Fibonacci graceful labeling and $C$ is a cycle of $G$.

a. If $F_k$ is the largest Fibonacci number appearing as an edge label of $C$, then $F_{k-1}$ also appears on $C$. In particular, the edge labeled $F_{n-1}$ must be in every cycle that contains the edge labeled $F_n$.

b. The cycle $C$ whose largest edge number is $F_k$ must contain either the edge labeled $F_{k-2}$ or $F_{k-3}$.

Proof:

a. By Theorem 1, some linear combination of the edge numbers on $C$ must sum to 0. By Identity (2):

$$F_1 + F_2 + F_3 + \cdots + F_{k-2} = F_k - 1 < F_k.$$

Thus, $F_{k-1}$ must appear as an edge label of $C$.

b. Since $F_k - F_{k-1} = F_{k-2}$, some combination of the remaining labels on $C$ must equal $F_{k-2}$. But, since $F_1 + F_2 + \cdots + F_{k-4} < F_{k-2}$, there must be an edge labeled $F_{k-3}$ unless there is one labeled with $F_{k-2}$ itself. ∎

We also have the following theorem, which corresponds to a well-known result for graceful graphs [3, p. 26].

Theorem 2

If $G(p, n)$ is Eulerian and Fibonacci graceful, then $n \equiv 0$ or $2 \pmod 3$.

Proof: If $G$ is Eulerian, then it can be decomposed into edge-disjoint cycles. By Theorem 1, the sum of the edge numbers around any cycle must be even and, hence,

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

must also be even. Thus, $F_{n+2}$ must be odd, and this occurs if and only if $n \equiv 0$ or $2 \pmod 3$. ∎

## 3. FORBIDDEN SUBGRAPHS

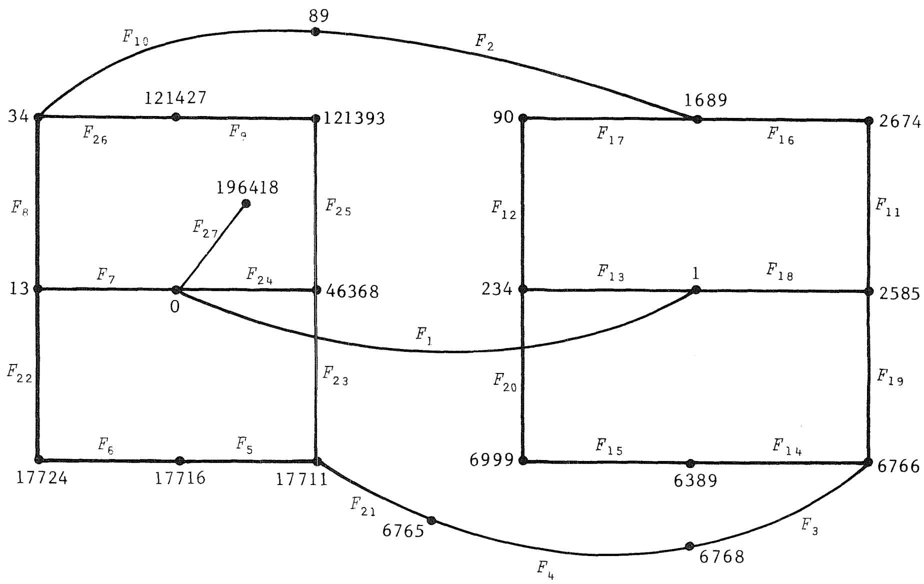
One possible way to characterize Fibonacci graceful graphs would be to find a complete list of graphs such that $G$ would be Fibonacci graceful if and only if it did not contain a subgraph isomorphic to one on this list. This approach seems difficult because gracefulness is a global rather than a local condition. Nevertheless, the following theorems do limit the structure of Fibonacci graceful graphs considerably.

Theorem 3

If $G(p, n)$ contains a 3-edge-connected subgraph, then $G$ is not Fibonacci graceful.

Proof: Suppose $G(p, n)$ is Fibonacci graceful, and $G'$ is a 3-edge connected subgraph. Let $F_k$ be the largest edge number appearing in $G'$, and let $v_1$ and $v_2$ be the endpoints of that edge. Since $G'$ is 3-edge connected, there is a path joining $v_1$ and $v_2$ which does not contain either the edge numbered $F_k$ or the edge numbered $F_{k-1}$. This path, together with the edge $(v_1, v_2)$ forms a cycle which contains the edge labeled $F_k$, but not the one labeled $F_{k-1}$. This violates Lemma 1. ∎

It is interesting to note that a graph $G$ which is not Fibonacci graceful may have homeomorphic copies thich are. For example, although $K_4$ is not Fibonacci graceful by Theorem 3, the graph in Figure 1(c), a homeomorphic copy of $K_4$, is Fibonacci graceful. Perhaps a more striking example is the nonplanar graph shown in Figure 2, which is Fibonacci graceful even though the complete graph $K_5$ and the complete bipartite graph $K_{3,3}$ are not. The graph in Figure 2 contains a subgraph homeomorphic to $K_{3,3}$. A consequence of the next theorem is that it is impossible for a nonplanar graph to contain a subgraph which is homeomorphic to $K_5$ and have a Fibonacci graceful labeling.




($K_{3,3}$ is homeomorphic with a subgraph containing the vertices labeled 24, 17711, 0 and 13, 46368, 1.)

FIGURE 2. A NONPLANAR FIBONACCI GRACEFUL GRAPH

## Theorem 4

If there is a pair of vertices joined by 4 edge-disjoint paths in $G(p, n)$, then $G$ is not Fibonacci graceful.

Proof: Let $v_1$ and $v_2$ be two vertices of $G$ joined by 4 edge-disjoint paths $P_1, P_2, P_3$, and $P_4$. Suppose $G$ has a Fibonacci graceful labeling. With no loss of generality, assume that $F_k$ is the largest Fibonacci number on these paths and that it lies on an edge of $P_1$. By Lemma 1(a), $F_{k-1}$ must also lie on $P_1$, since otherwise there are cycles containing edge $F_k$, but not $F_{k-1}$. Additionally, either $F_{k-2}$ or $F_{k-3}$ must also be an edge label on $P_1$, for if they were on other paths, say $P_2$ and/or $P_3$, then paths $P_1$ and $P_4$ would form a cycle violating Lemma 1(b).

Suppose that it is $F_{k-2}$ that appears as an edge label on $P_1$. Then Corollary 1.4 permits us to ignore $F_k$, $F_{k-1}$, and $F_{k-2}$ and tells us that some linear combination of the remaining Fibonacci numbers on any cycle must sum to 0. Repeat this process, beginning with the largest of the remaining edge labels, to discard or ignore the presence of three consecutive edge numbers on any of the paths. This repetition cannot discard all of the edge numbers along any path, for then vertices $v_1$ and $v_2$ would necessarily have the same vertex label. Thus, the process terminates at a time where $F_j$ is the largest remaining edge label and $F_j$, $F_{j-1}$, and $F_{j-3}$ appear on some path, say $P_2$, but $F_{j-2}$ appears on another path, say $P_4$. Then there is a cycle, $P_3$ and $P_4$ for example, on which $F_{j-2}$ is the largest Fibonacci number, but $F_{j-3}$ does not appear, violating Lemma 1.

## 4. CLASSES OF FIBONACCI GRACEFUL GRAPHS

We begin with easy observations that any Fibonacci graceful graph may be embedded in larger ones.

Observation 2: Let $G(p, n)$ have a Fibonacci graceful labeling. Then the graph $G_1(p+1, n+1)$ formed from $G$ by attaching a vertex $v$ of degree 1 at the vertex labeled 0 can be given a Fibonacci graceful labeling by labeling $v$ with $F_{n+1}$.

Observation 3: Let $G(p, n)$ have a Fibonacci graceful labeling. Then the graph $G_2(p+1, n+2)$ formed from $G$ by attaching a vertex $v$ of degree 2 to the vertices labeled 0 and $F_n$ can be given a Fibonacci graceful labeling by numbering $v$ with $F_{n+2}$.

### Theorem 5

The cycle $C_n$ is Fibonacci graceful if and only if $n \equiv 0$ or 2 (mod 3).

Proof: Since $C_n$ is Eulerian, it is not Fibonacci graceful for $n \equiv 1$ (mod 3) by Theorem 3.

If $n \equiv 0$ (mod 3), the following labeling sequence on the vertices is a Fibonacci graceful labeling:

$$0,\ F_n,\ F_{n-2},\ F_{n-1},\ \ldots,\ F_{n-3j},\ F_{n-3j-2},\ F_{n-3j-1},\ \ldots,\ F_6,\ F_4,\ F_5,\ F_3,\ F_1.$$

If $n \equiv 2$ (mod 3), the following numbering sequence on the vertices is a Fibonacci graceful labeling:

$$0,\ F_n,\ F_{n-2},\ F_{n-1},\ \ldots,\ F_{n-3j},\ F_{n-3j-2},\ F_{n-3j-1},\ \ldots,\ F_5,\ F_3,\ F_4,\ F_1. \ \blacksquare$$

### Theorem 6

A maximal outerplanar graph $G$ with at least four vertices is a Fibonacci graceful graph if and only if it has exactly two vertices of degree 2.

Proof: Let $G$ be a maximal outerplanar graph with more than two vertices of degree 2. Then $G$ must contain a subgraph isomorphic to the graph shown in Figure 3. Since there are 4 edge-disjoint paths between vertices $v_1$ and $v_2$ in this graph, $G$ cannot be Fibonacci graceful by Theorem 4.

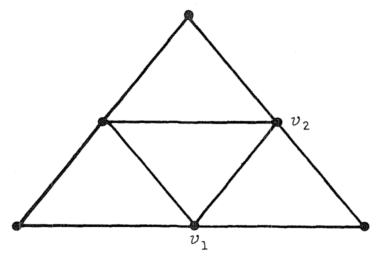



FIGURE 3. A FORBIDDEN SUBGRAPH

We next use induction to show that a maximal outerplanar graph $G(p,\ 2p-3)$ with exactly two vertices of degree 2 has a Fibonacci graceful labeling. Moreover, this labeling can be given so that the 0 label appears on either vertex of degree 2, say $v_0$, and that $F_{2p-3}$ may be the label of either neighbor of $v_0$. Since all the maximal outerplanar graphs with two vertices of degree 2 can be generated by repeatedly adjoining a new vertex of degree 2 to a previous vertex of degree 2 and one of its neighbors ([1], p. 607), Observation 3 will complete the proof.

To begin the induction and to illustrate the labeling, Figure 4 shows all the maximal outerplanar graphs with exactly two vertices of degree 2 for $p =$ 4, 5, and 6. Assume the inductive hypothesis is valid for $p = k$ and consider a maximal outerplanar graph $G(p+1,\ 2p-1)$ with exactly two vertices of degree 2. Let $v_0$ be a vertex of degree 2 in $G$ with neighbors $v_1$ and $v_2$. When $v_0$ is removed, one of its neighbors, say $v_1$, will become a vertex of degree 2 in $G - v_0$. By induction, $G - v_0$ may be given a vertex labeling $\ell$ such that

$$\ell(v_1) = 0 \quad \text{and} \quad \ell(v_2) = F_{2p-3}.$$

By Observation 3, $G$ can be made Fibonacci graceful by labeling $v_0$ with $F_{2p-1}$. By Observation 1, the transformation $F_{2p-1} - a_i$ applied to the vertex labels gives $G$ a Fibonacci graceful labeling $\ell_1$ with

$$\ell_1(v_0) = 0 \quad \text{and} \quad \ell_1(v_1) = F_{2p-1}.$$

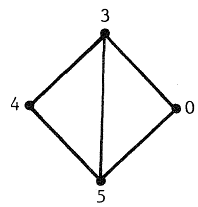
To show that $G$ has a second labeling $\ell_2$ in which

$$\ell_2(v_2) = F_{2p-1},$$

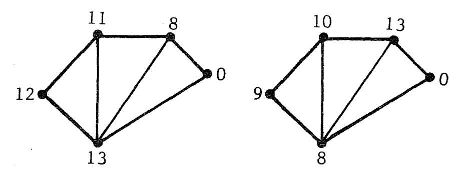
apply the transformation $F_{2p} - a_i$ to the vertex labels of $G - v_0$. This gives an induced edge labeling $\ell_2'$ to $G$ for which

$$\ell_2'(v_0v_1) = F_{2p-2} \quad \text{and} \quad \ell_2'(v_0v_1) = F_{2p-1}$$

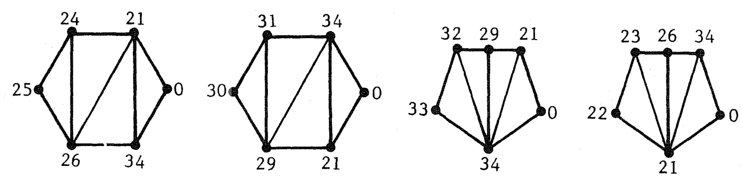
with all other edge labels unchanged. ■




a. $p = 4$




b. $p = 5$




c. $p = 6$

FIGURE 4. FIBONACCI GRACEFUL LABELINGS OF MAXIMAL OUTERPLANAR GRAPHS WITH SIX OR FEWER VERTICES AND EXACTLY TWO VERTICES OF DEGREE 2

## 5. FIBONACCI GRACEFUL TREES

In this section we will present an algorithm that will enable one to find a Fibonacci graceful labeling for nearly all trees. The trees which do not have such a labeling are easily characterized. Except for $K_1$ and $K_2$, which are trivially labeled, any tree $T(p, n)$ with five or fewer vertices cannot be Fibonacci graceful since with $n \leqslant 4$ edges there are not enough distinct integers between 0 and $F_n$ to label the $p = n+1$ vertices of $T$. It is also apparent that $K_{1,n}$ is not Fibonacci graceful for $n \geqslant 2$. That this is so follows from the fact that all the edges have a vertex in common and if the remaining vertices are distinctly labeled, there cannot be two edges with the label 1. The only other tree that is not Fibonacci graceful is shown in Figure 5.

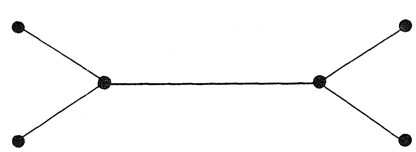


FIGURE 5. A TREE THAT IS NOT FIBONACCI GRACEFUL

It is generally easy to provide a labeling for any other tree, especially one with a large number of vertices, because for $n$ large, $F_n$ is considerably larger then $n + 1$ and there are many distinct integers from which to choose the vertex labels. In Figure 6 we show a Fibonacci graceful labeling for the remaining trees with six vertices.

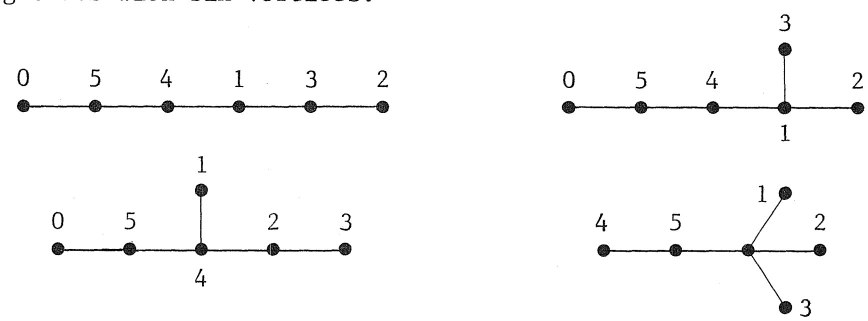



FIGURE 6. THE FIBONACCI GRACEFUL TREES $T(6, 5)$

The trees in Figure 6 are examples of a class of trees called "caterpillars"—trees which become paths when all of their endpoints are removed. (It is known that all caterpillars are graceful trees [8].) The *length* of a caterpillar will be the number of edges in the remaining path.

Theorem 7

All trees $T(n+1, n)$ with $n \geqslant 6$, except for $K_{1,n}$, are Fibonacci graceful.

Proof: We divide the proof into cases, and provide a labeling $\ell$ for each case. The cases are:

a. caterpillars of length 1;

b. caterpillars of length 2 or more;

c. noncaterpillars.

We begin with caterpillars of length 1. Since $T$ has at least six edges, there is a vertex $v_0$ of $T$ with degree 4 or more. Let $v_1$ denote the neighbor of $v_0$ which is not an endpoint. Let $\ell$ label $v_0$ with 0; $v_1$ with $F_n$; the $k + 1 \geqslant 3$ endpoints adjacent to $v_0$ with $1, F_{n-1}, F_{n-2}, \ldots, F_{n-k}$; and the endpoints adjacent to $v_1$ with $F_n - F_{n-k-1}, F_n - F_{n-k-2}, \ldots, F_n - 3, F_n - 2, F_n - 1$. Figure 7 gives an example of the results of this procedure. Clearly the algorithm gives a proper edge labeling; thus, it remains only to verify that the vertex labels are distinct. Note that, if $v_i$ is a neighbor of $v_0$ and $v_j$ is a neighbor of $v_1$, then $\ell(v_j) > \ell(v_i)$ since for $n \geqslant 6$ and $2 \leqslant k \leqslant n - 3$ we have:

$$\min\{\ell(v_j)\} = F_n - F_{n-k-1} > F_{n-k} = \max\{\ell(v_i)\}.$$

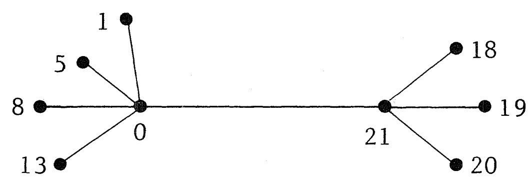



FIGURE 7. A FIBONACCI GRACEFUL CATERPILLAR OF LENGTH 1

For a caterpillar $T$ of length 2 or more, choose a longest path in $T$ and call its vertices $v_0, v_1, v_2, \ldots, v_k$. Denote the endpoints adjacent to $v_i$ by $v_{i1}, v_{i2}, \ldots, v_{ij}$, $i = 1, 2, \ldots, k$. We consider two subcases depending on the degree of $v_1$. If $v_1$ is of degree 2, define $\ell$ as follows. Let

$$\ell(v_0) = 0,\ \ell(v_1) = F_n,\ \ell(v_2) = F_n - 1.$$

Then label the neighbors of $v_2$ by

$$\ell(v_{21}) = \ell(v_2) - F_{n-1},\ \ell(v_{22}) = \ell(v_2) - F_{n-2},\ \ldots,\ \ell(v_{2j}) = \ell(v_2) - F_{n-j},$$

and, finally,

$$\ell(v_3) = \ell(v_2) - F_{n-j-1}.$$

Proceed to define for the $r + 1$ neighbors of $v_3$,

$$\ell(v_{31}) = \ell(v_3) + F_{n-j-2},\ \ell(v_{32}) = \ell(v_3) + F_{n-j-3},\ \ldots,$$

$$\ell(v_{3r}) = \ell(v_3) + F_{n-j-r-1},$$

ending with

$$\ell(v_4) = \ell(v_3) + F_{n-j-r-2}.$$

Notice that each neighbor of $v_3$ has been distinctly labeled with positive integers strictly between $\ell(v_2)$ and $\max\{\ell(v_3), \ell(v_{2i})\}$. For the neighbors of $v_4$ label each vertex with

$$\ell(v_4) - (\text{the appropriate Fibonacci number}).$$

Again each of these will be distinctly labeled with positive integers between $\ell(v_3)$ and $\min\{\ell(v_4), \ell(v_{3i})\}$. Continue in this manner, *adding* the continuing sequence of Fibonacci numbers to the neighbors of $v_5, v_7, v_9, \ldots$ and *subtracting* them from the neighbors of $v_6, v_8, v_{10}, \ldots$. An example of the resulting labels is shown in Figure 8(a).

If vertex $v_1$ is of degree more than 2, let

$$\ell(v_0) = 0 \quad \text{and} \quad \ell(v_1) = F_n$$

as before. For the neighbors of $v_1$, define

$$\ell(v_{11}) = F_n - 1,\ \ell(v_{12}) = \ell(v_1) - F_{n-1},\ \ell(v_{13}) = \ell(v_1) - F_{n-2},\ \ldots,$$

$$\ell(v_{1j}) = \ell(v_1) - F_{n-j-2},$$

ending with

$$\ell(v_2) = \ell(v_1) - F_{n-j-2}.$$

Proceed to label the neighbors of $v_2$ by *adding* the appropriate sequence of Fibonacci numbers to $\ell(v_2)$. In this instance, the vertex labels for these vertices will lie between $\ell(v_{11})$ and $\ell(v_2)$, the two largest vertex labels appearing on the neighbors of $v_1$. From here, proceed in a fashion analogous to that above. An example of such a caterpillar is shown in Figure 8(b).

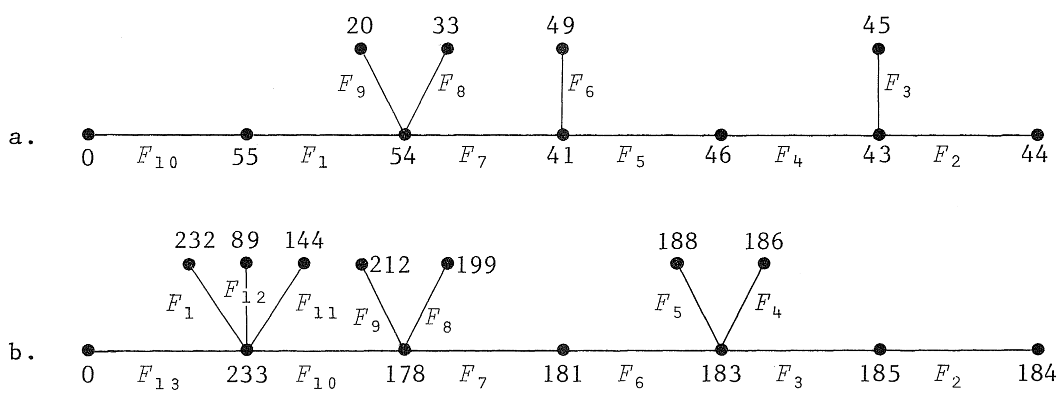



FIGURE 8. TWO LABELED CATERPILLARS OF LENGTH 4

Finally, we consider a tree $T$ which is not a caterpillar. Remove the two endpoints of a longest path in $T$ to form a subtree $T'$ that is not a path. $T'$ has either one or two centers, both lying on some longest path $P'$ in $T'$. Select one of the centers, denoted $v_0$, and root $T'$ at $v_0$. If $v_0$ is a vertex of degree $k \geqslant 2$, denote the neighbors of $v_0$ by $v_{11}, v_{12}, \ldots, v_{1k}$ in such a way that $v_{11}$ and $v_{1k}$ lie on $P'$ and $v_{1k}$ is the other center of $T'$ if there are two centers. Denote the "half" of $P'$ containing $v_0$ and $v_{11}$ by $P'_L$ (the "left half") and the section containing $v_0$ and $v_{1k}$ by $P'_R$ (the "right half"). (Thus, the vertices at the first level are labeled from left to right.) Also denote the $k$ subtrees rooted at $v_{11}, v_{12}, \ldots, v_{1k}$ by $T_1, T_2, \ldots, T_k$, respectively. Next call the vertices at a distance of 2 from $v_0$ by $v_{21}, v_{22}, \ldots, v_{2j}$ in such a way that $v_{21}$ is on $P'_R$ and $v_{2j}$ is on $P'_L$; that is, name the vertices from right to left. Proceed to name the vertices at distance 3, $v_{31}, v_{32}, \ldots, v_{3r}$ again from right to left. Continue from right to left at each level until all the vertices of $T'$ have been named. Note that there will be at least two vertices at each distance or level (except perhaps at the final level, where there may be only a single vertex on $P'_R$), since $v_0$ was a center. Also, there must be a level with at least three vertices, since $T'$ is not a path.

We define the Fibonacci graceful labeling $\ell$ on $T'$ as follows:

$$\ell(v_0) = 0;$$

$$\ell(v_{11}) = F_n,\ \ell(v_{12}) = F_{n-1},\ \ldots,\ \ell(v_{1k}) = F_{n-k-1};$$

$$\ell(v_{21}) = \ell(v_{1k}) - F_{n-k-2};$$

$$\ell(v_{22}) = \ell(\text{parent vertex of } v_{22}) - F_{n-k-3},\ \ldots;$$

that is, for any subsequent vertex in $T'$, its label will be the difference between the label of its parent vertex and the next smaller Fibonacci number. Note that the edges of $T'$ receive the labels $F_n, F_{n-1}, \ldots, F_3$ in decreasing order from left to right on the first level, and from right to left on all subsequent levels. To extend $\ell$ to the original tree $T$, label each of the two endpoints which were removed by $\ell$ (its neighbor) - 1. Figure 9 presents two applications of this algorithm.

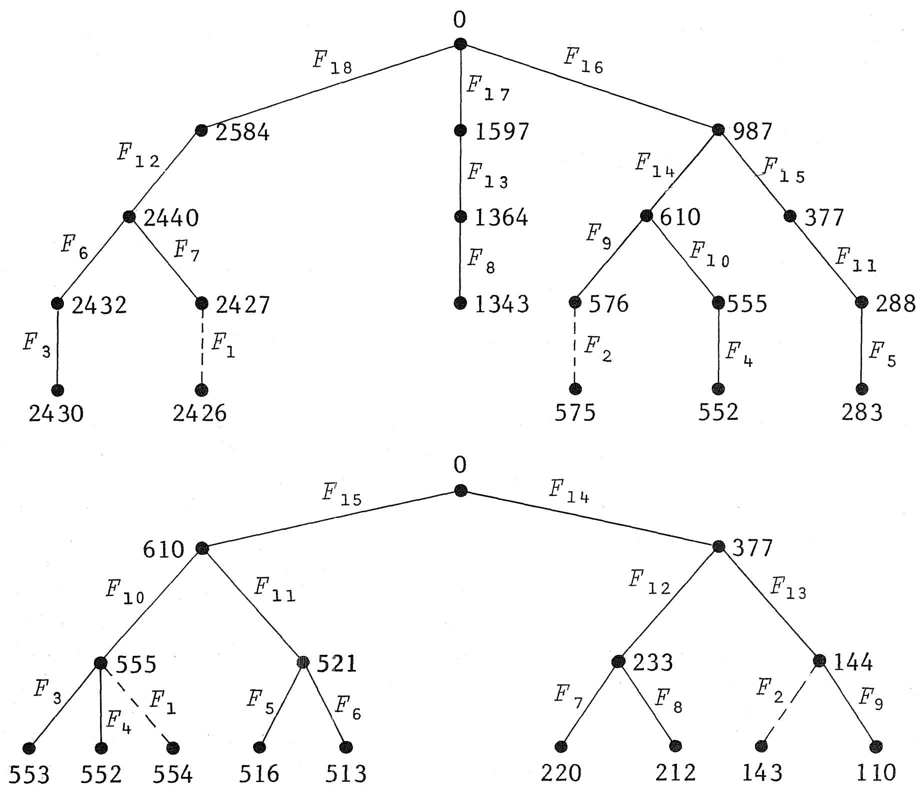



FIGURE 9. TREES WITH FIBONACCI GRACEFUL LABELINGS

It is clear that this procedure will properly label all the edges, so it remains only to observe that the vertex labels are distinct and nonnegative.

First, we note that within any of the rooted subtrees $T_i$, $i = 1, \ldots, k$, the vertex labels decrease as the distance from $v_0$ increases. Finally, we claim that for $i < j$, every vertex label in $T_i$ exceeds those in $T_j$. Note that the vertex labels in $T_1$ all equal

$$F_n - \text{(a sum of Fibonacci numbers)},$$

where the terms in this sum include *at most*

$$F_{n-3},\ F_{n-5},\ F_{n-7},\ \ldots,\ F_{n-r},\ F_{n-r-3},\ F_{n-r-5},\ \ldots,$$

for some $r$, since at each level there is at least one edge in $P_R$, and at some level there is at least some other edge not on $P$. Thus, by Identity (5), the smallest vertex number in $T_1$ is greater than

$$F_n - (F_{n-2} - 1) > F_{n-1}.$$

Thus, every vertex number in $T_1$ exceeds any vertex number in $T_2$. A similar argument will show that if $v \in T_2$ $(\neq T_k)$, then

$$F_{n-2} < \ell(v_2) \leqslant F_{n-1},$$

and that if $v \in T_k$, then

$$0 < \ell(v) \leqslant F_{n-k}.$$

This concludes the proof of the theorem. ∎

## 6. SUMMARY AND CONCLUSION

In this paper, we have extended the idea of graceful graphs to numberings where the vertex labels are distinct integers but the edge labels are members of the Fibonacci sequence. We investigated the cycle structure of Fibonacci graceful graphs and used this to find forbidden subgraphs. We found infinite classes of Fibonacci gracegul graphs, including almost all trees. It is interesting to note that, if we had required the edge numbers of $T(n + 1, n)$ to come from the set $\{F_2, F_3, \ldots, F_{n+1}\}$ in order to eliminate the problem with duplicate vertex labels in $K_{1,n}$, then all trees could be labeled eadily. This is due to the large size of $F_n$ relative to $n$, which leaves many possible distinct integers available for the vertex labels. Thus, in a certain sense, the Ringel-Kotzig-Rosa conjecture is a limiting case for this type of tree labeling problem, since to produce the edge labels $\{1, 2, 3, \ldots, n\}$ it is required to use every integer in $\{0, 1, 2, \ldots, n\}$.

For the Fibonacci graceful graphs, the problem remains to characterize all of them, perhaps by forbidden subgraphs, although this appears difficult in view of Observations 2 and 3. Further classes of Fibonacci graceful graphs can certainly be discovered. For example, we conjecture that all unicyclic graphs with at least one endpoint are Fibonacci graceful graphs.

## REFERENCES


1. D. W. Bange, A. E. Barkauskas, & P. J. Slater. "Using Associated Trees To Count the Spanning Trees of Labeled Maximal Outerplanar Graphs." *Proc. 8 S.E. Conference on Combinatorics, Graph Theory and Computing.* New York: Utilitas Press, 1977, pp. 605-14.
2. G. S. Bloom. "A Chronology of the Ringel-Kotzig Conjecture and the Continuing Quest To Call All Trees Graceful." *Topics in Graph Theory.* Scientist in Residence Program, ed. F. Harary. New York: Academy of Sciences, 1977.
3. S. W. Golomb. "How To Number a Graph." *Graph Theory and Computing.* Ed. R. C. Read. New York: Academic Press, 1972, pp. 23-27.
4. S. P. R. Hebbare. "Graceful Cycles." *Utilitas Mathematics* 10 (1976):307-317.
5. C. Höede & H. Kuiper. "All Wheels Are Graceful." *Utilitas Mathematica* 14 (1978):311.
6. K. M. Koh, D. G. Lee, & T. Tan. "Fibonacci Trees." *SEA Bull. Math.* 2, no. 1 (March 1978):45-47.
7. A. Rosa. "On Certain Valuations of the Vertices of a Graph." *Theory of Graphs.* Proc. Internat. Symposium Rome, 1966, ed. P. Rosenstiehl. Paris: Duond, 1968, pp. 349-55.
8. A. Rosa. "Labeling Snakes." *Ars Combinatoria* 3 (1977):67-74.


◆◇◆◇◆

*[continued from page 173]*

Since the Newton iterates always fulfill the *linear* equations which belong to the system of nonlinear equations that is to be solved (with the exception, of course, of the starting value), the conclusion follows at once.

References


1. K. Weierstrass. "Neuer Beweis des Satzes, dass jede ganze Funktion einer Veränderlichen dargestellt werden kann als Produce aus linearen Funktionen derselben Veränderlichen." *Ges. Werke* 3, pp. 251-69.
2. I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen." *Numer. Math.* 8 (1966):290-94.
3. E. Durand. *Solutions numériques des équations algébriques.* Paris: Masson, 1960.
4. P. Byrnev & K. Dochev. "Certain Modifications of Newton's Method for the Approximate Solution of Algebraic Equations." *Zh. Vych. Mat.* 4 (1964):915-20.


◆◇◆◇◆

◆◇◆◇◆

# EQUIPROBABILITY IN THE FIBONACCI SEQUENCE

LEE ERLEBACH

*Michigan Technological University, Houghton, MI 49931*

and

WILLIAM YSLAS VÉLEZ

*University of Arizona, Tuscon, AZ 85721*

*(Submitted April 1981)*

For any positive integer $m$, the Fibonacci sequence is clearly periodic modulo $m$. Many moduli $m$, characterized in [1], have the property that *every* residue modulo $m$ occurs in each period. (Indeed, 8 and 11 are the smallest moduli which do *not* have this property.) However, moduli $m$ with the property that all $m$ residues modulo $m$ appear in one period *the same number of times* occur very infrequently, as the following theorem from [2] shows.

## Theorem 1

If all $m$ residues appear in one period of the Fibonacci sequence modulo $m$ the same number of times, then $m$ is a power of 5.

The converse of this theorem is also true [3]. Since (see [4]) for $k > 0$ the Fibonacci sequence modulo $5^k$ has period $4 \cdot 5^k$, it follows that if $m > 1$ is a power of 5, and $(u_n)$ is the Fibonacci sequence, then every residue modulo $m$ appears exactly four times in each sequence

$$u_s,\ u_{s+1},\ u_{s+2},\ \ldots,\ u_{s+4m-1}.$$

This result can be strengthened considerably.

## Theorem 2

Denote the Fibonacci sequence by $(u_n)$. If $m > 1$ is a power of 5, then every residue modulo $m$ appears exactly once in each sequence

$$u_s,\ u_{s+4},\ u_{s+8},\ \ldots,\ u_{s+4(m-1)}.$$

Proof: Write $m = 5^k$, and denote the greatest integer function by [ ]. The Fibonacci sequence $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, ... satisfies the well-known formula

$$u_n = \left(((1 + \sqrt{5})2^{-1})^n - ((1 - \sqrt{5})2^{-1})^n\right)/\sqrt{5}.$$

Apply the binomial expansion to this formula to obtain

$$u_n = (2^{-1})^{n-1}\left(\binom{n}{1} + \binom{n}{3}5 + \binom{n}{5}5^2 + \cdots\right),$$

where all terms after $\binom{n}{2\ell + 1}5^{\ell}$ vanish and $\ell = [(n - 1)/2]$. Fix $s$, and let

$S_k = \{0, 1, \ldots, 5^k - 1\}$. Then, for $n = s + 4a$, $a \in S_k$, we have

$$u_n = (2^{-1})^{s-1}(2^{-1})^{4a}\left(\binom{n}{1} + \binom{n}{3}5 + \cdots\right)$$

and it is obvious that $u_n$ represents every residue modulo $5^k$ if and only if

$$t_n = (2^{-1})^{4a}\left(\binom{n}{1} + \binom{n}{3}5 + \cdots\right)$$

represents every residue modulo $5^k$, since $s$ is fixed and $(2^{-1})^{s-1}$ is a unit modulo $5^k$. Thus, we shall only consider $t_n$ and prove the theorem by induction on $k$.

If $k = 1$, then $a \in \{0, 1, 2, 3, 4\}$ and $t_n \equiv s + 4a \pmod 5$, since $2^{-4} \equiv 1 \pmod 5$. Thus, the theorem is true for $k = 1$. Assume the theorem is true for $k$, and consider $k + 1$. For $a \in S_{k+1}$, write $a = b + c5^k$, where $b \in S_k$ and $c \in \{0, 1, 2, 3, 4\}$. Then,

$$\begin{aligned} t_n &= (2^{-1})^{4b}(2^{-1})^{4c5^k}\left(\binom{s + 4b + 4c5^k}{1} + \binom{s + 4b + 4c5^k}{3}5 + \cdots\right) \\ &\equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) + (2^{-1})^{4b}4c5^k \pmod{5^{k+1}}, \end{aligned}$$

since

$$(2^{-1})^{4 \cdot 5^k} \equiv 1 \pmod{5^{k+1}}$$

and

$$\binom{s + 4b + 4c5^k}{2j + 1}5^j \equiv \binom{s + 4b}{2j + 1}5^j \pmod{5^{k+1}} \text{ for } j \geqslant 1.$$

[To prove the last congruence, note first that it is equivalent to

$$\binom{s + 4b + 4c5^k}{2j + 1}5^{j-1} \equiv \binom{s + 4b}{2j + 1}5^{j-1} \pmod{5^k}.$$

Then, observe that the power of 5 dividing $(2j + 1)!$ is exactly $j - 1$ for $j = 1, 2$, and is

$$\sum_{\ell=1}^{\infty} [(2j + 1)/5^\ell] \leqslant \sum_{\ell=1}^{\infty} (2j + 1)/5^\ell = (2j + 1)/4 \leqslant j - 1 \text{ for } j \geqslant 3.$$

Hence, $5^{j-1}/(2j + 1)!$ is integral at 5, and this implies the congruence.]

Let us now consider the congruence modulo $5^k$. We obtain

$$t_n \equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) \pmod{5^k},$$

and, by the induction hypothesis, $t_n$ represents the complete residue system modulo $5^k$, for $n = s + 4b$, $b \in S_k$.

If we hold $b$ fixed in $S_k$ and let $c$ run through the set $\{0, 1, 2, 3, 4\}$, we obtain

$$t_n \equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) + (2^{-1})^{4b}4c5^k \pmod{5^{k+1}},$$

which are all distinct residues modulo $5^{k+1}$ since $(2^{-1})^{4b}4c$ takes on distinct values modulo 5. Since the five $t_n$ are all congruent to

$$(2^{-1})^{4b}\left(\binom{s+4b}{1}+\binom{s+4b}{3}5+\cdots\right) \pmod{5^k},$$

the induction is complete.

## ACKNOWLEDGMENT

We wish to thank the referee for a number of helpful suggestions and references.

## REFERENCES

1. S. A. Burr. "On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues." *The Fibonacci Quarterly* 9 (1971):497-504.
2. L. Kuipers & Jau-Shyong Shiue. "A Distribution Property of the Fibonacci Numbers." *The Fibonacci Quarterly* 10 (1972):375-76, 392.
3. H. Niederreiter. "Distribution of Fibonacci Numbers Mod $5^k$." *The Fibonacci Quarterly* 10 (1972):373-74.
4. D. D. Wall. "Fibonacci Series Modulo $m$." *Amer. Math. Monthly* 67 (1960): 525-32.

◆◇◆◇◆

◆◇◆◇◆

# LUCAS TRIANGLE

ŠÁNA JOSEF
*Technical University, Brno, Czechoslovakia*

*(Submitted May 1981)*

## 1. INTRODUCTION

Throughout this paper we let $\{F_n\}$ denote the Fibonacci sequence as defined in [1] by

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 1,\ F_1 = 1$$

and $\{L_n\}$ denote the Lucas sequence which is defined by

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1.$$

Furthermore, as in [2], we define the set of integers $\{g_{m,n}\}$ by the two relations

$$g_{m,n} = g_{m-1,n} + g_{m-2,n} \qquad (m \geqslant 2,\ m \geqslant n \geqslant 0) \tag{1}$$

$$g_{m,n} = g_{m-1,n-1} + g_{m-2,n-2} \tag{2}$$

where $g_{0,0} = 2$, $g_{1,0} = 1$, $g_{1,1} = 1$, and $g_{2,1} = 2$.

When we arrange this sequence in triangular form, like that of Pascal's triangle, we obtain what shall be called the Lucas triangle where the numbers in the same row have the same index $m$ and as we go from left to right the index $n$ changes from zero to $m$. See Figure 1.

```
                         2
                      1     1
                   3     2     3
                4     3     3     4
             7     5     6     5     7
         11    8     9     9     8    11
      18    13    15    14    15    13    18
   29    21    24    23    23    24    21    29
47    34    39    39    38    37    39    34    37
76    55    63    60    61    61    60    63    55    76
123   89   102   97    99    98    99    97   102   89   123
                         ⋮
```

FIGURE 1. LUCAS TRIANGLE

## 2. CHARACTERISTICS OF THE LUCAS TRIANGLE

Examining the Lucas triangle associated with $\{g_{m,n}\}$, we see that there are two Lucas sequences and two Fibonacci sequences in the triangle:

$$g_{m,0} = g_{m,m} = L_m, \qquad m \geqslant 0 \tag{3}$$

and

$$g_{m,1} = g_{m,m-1} = F_m, \qquad m \geqslant 1. \tag{4}$$

In other words, the first and second "roofs" of the triangle are formed by the familiar sequences $\{F_n\}$ and $\{L_n\}$. Because of the recursive definition for the set $\{g_{m,n}\}$, it is obvious that

$$g_{m,n} = F_{k+1} \cdot g_{m-k,n} + F_k \cdot g_{m-k-1,n} \text{ for any } 1 \leqslant k \leqslant m - n - 1$$

and

$$g_{m,n} = F_{k+1} \cdot g_{m-k,n-k} + F_k \cdot g_{m-k-1,n-k-1} \text{ for any } 1 \leqslant k \leqslant n - 1.$$

Furthermore, the Lucas triangle is symmetrical. That is,

$$g_{m,n} = g_{m,m-n}.$$

In forming the Lucas triangle, we used the following four numbers

$$\{g_{0,0},\ g_{1,0},\ g_{1,1},\ g_{2,1}\}$$

Because of the recursive relations defining $\{g_{m,n}\}$, it is obvious that we could start with any four numbers

$$\{g_{m,n},\ g_{m-1,n},\ g_{m-1,n-1},\ g_{m-2,n-1}\}$$

and, by using (1) and (2), working forward as well as backward, obtain the entire Lucas triangle.

There are also many relations that we could establish for the Lucas triangle, as was done for the Fibonacci triangle in [2]. We mention only a few, since they are so similar in form. First, note that

$$g_{m+2,n+1} = g_{m,n} + g_{m-1,n} + g_{m-1,n-1} + g_{m-2,n-1},$$

$$g_{m-4,n-2} = g_{m,n} - g_{m-1,n} - g_{m-1,n-1} + g_{m-2,n-1},$$

$$g_{m-1,n-2} = g_{m,n} + g_{m-1,n} - g_{m-1,n-1} - g_{m-2,n-1},$$

and

$$g_{m-1,n+1} = g_{m,n} - g_{m-1,n} + g_{m-1,n-1} - g_{m-2,n-1}.$$

Next, consider the three numbers

$$\{g_{m,n},\ g_{m-1,n},\ g_{m-1,n-1}\}$$

which form a triangle in the Lucas triangle with the peak of the triangle at $g_{m,n}$. Observe that the sum of these three numbers does not depend on their position with respect to $n$. That is, for a given $m$ and $n$,

$$g_{m,n} + g_{m-1,n} + g_{m-1,n-1} = g_{m,\ell} + g_{m-1,\ell} + g_{m-1,\ell-1},$$

for all $1 \leqslant \ell,\ n \leqslant m - 1$.

Furthermore, note that the sum of the three numbers forming such triangles for a given $m$ always equals the sum of a Lucas and Fibonacci number associated with the given $m$. That is,

$$g_{m,n} + g_{m-1,n} + g_{m-1,n-1} = L_{m-1} + F_{m+1}.$$

Finally, we examine the three numbers

$$\{g_{m,n},\ g_{m,n-1},\ g_{m-1,n-1}\}$$

which also form a triangle but with the peak at $g_{m-1,n-1}$. The sum of numbers forming the base minus the peak number is constant with regard to horizontal motion and it is again the sum of a Lucas and Fibonacci number. That is,

$$g_{m,n} + g_{m,n-1} - g_{m-1,n-1} = L_{m-2} + F_m = 4F_{m-2}.$$

## 3. GRAPHICAL EQUIVALENT OF LUCAS TRIANGLE

In [4], we find a nonadjacent number $P(G, k)$ for a graph defined as the number of ways in which $k$ disconnected lines are chosen from $G$. Furthermore, in [5], we find the definition of a topological index for nondirected graphs. This is a unique number associated with a given nondirected graph.

The topological index for a linear graph is given in [4], and it is shown to be a Fibonacci number (Table 1). Similarly, in the same manuscript, it is shown that the topological index for a cyclic graph is a Lucas number (Table 3). Using these concepts, Hosoya [2] defines a Fibonacci triangle for the sequence $\{F_{n,m}\}$ and constructs what is called *the graphical equivalent of the Fibonacci triangle* by letting $f_{n,m}$ be the index of a graph and then replacing $f_{n,m}$ with its graph. To do this, he also uses the composition principles defined in [4].

Adopting the procedures of Hosoya in [2], we replace each number $g_{m,n}$ in the Lucas triangle by its graph, obtaining *the graphical equivalent of the Lucas triangle* shown in Figure 2. Note that all the linear and cyclic graphs must occur in Figure 2.

The graphical equivalent of the Lucas triangle easily leads to its corresponding topological index, which can be used in the chemistry of organic substances [5] when dealing with the boiling point to determine the structure of saturated hydrocarbons.

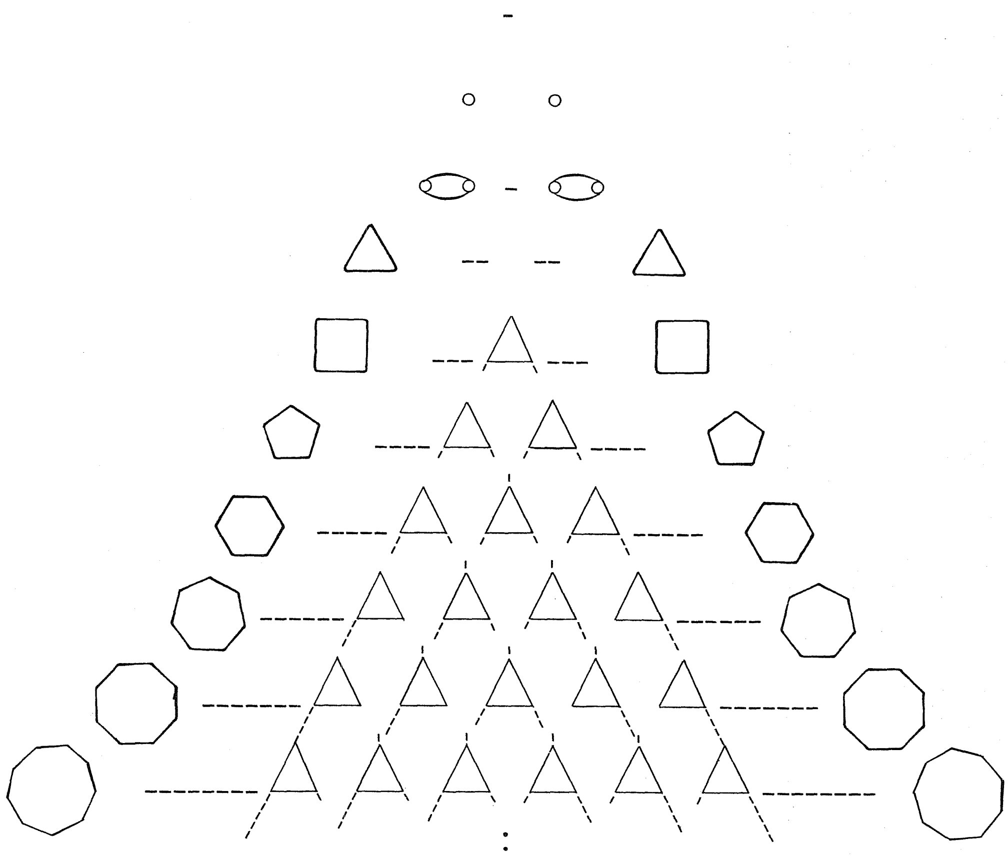


FIGURE 2. GRAPHICAL EQUIVALENT OF LUCAS TRIANGLE

## REFERENCES


1. N. N. Vorobjev. "Cisla Fibonacci." *Moskva* (1969).
2. Haruo Hosoya. "Fibonacci Triangle." *The Fibonacci Quarterly* 14 (1976): 173-79.
3. J. Sedlacek. *Uvod do teorie grafu.* Prague: Academia, 1977.
4. Haruo Hosoya. "Topological Index and Fibonacci Numbers with Relations to Chemistry." *The Fibonacci Quarterly* 11 (1973):255-66.
5. Haruo Hosoya. "Topological Index: A Newly Proposed Quantity Characterising the Topological Nature of Structural Isomers of Saturated Hydrocarbons." *Ball Chem. Soc.* (Japan) 44 (1971):2332-39.


◆◇◆◇◆

◆◇◆◇◆

# THE OCCUPATIONAL DEGENERACY FOR $\lambda$-BELL PARTICLES ON A SATURATED $\lambda \times N$ LATTICE SPACE*

J. L. HOCK
*Marquette University, Milwaukee, WI 53233*

and

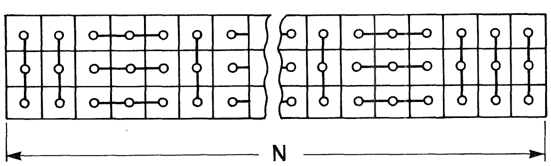
R. B. McQUISTAN
*University of Wisconsin-Milwaukee, Milwaukee, WI 53201*

*(Submitted March 1982)*

## 1. INTRODUCTION

A number of physical phenomena, e.g., adsorption, crystallization, magnetism, can be treated by considering the occupation statistics of lattice spaces. One of the interesting problems that arise from such an approach is that of determining the occupational degeneracy of $\lambda$-bell particles on lattice spaces of various dimensionalities. Exact solutions for one-dimensional spaces have been found for dumbbells [1, 2] ($\lambda = 2$) and for $\lambda$-bell particles [3] but exact solutions for spaces of higher-order dimensionality have only been obtained for dumbbells for very special cases [4, 5]. Consequently, approximation methods [6-8] have been used to attack this problem.

The present paper is concerned with a determination of the occupational degeneracy for indistinguishable $\lambda$-bell particles that completely fill a $\lambda \times N$ rectangular lattice space (see Fig. 1).




This figure shows one arrangement for $\lambda = 3$ particles that fill completely a $3 \times N$ lattice space.

FIGURE 1

We first derive a recursion relationship that describes exactly the multiplicity of arrangements when the $\lambda \times N$ lattice space is saturated.

In Sections 3 and 4, we derive an exact summation representation for the degeneracy and present the corresponding generating functions and continuous representation for large values of $N$.

*This work was supported by the Air Force Office of Scientific Research under Grant AFOSR 81-0192.*

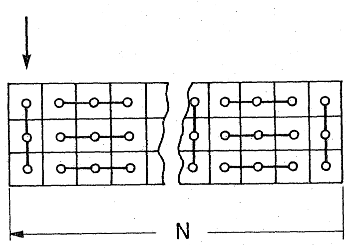
## 2. EXACT RECURSION RELATIONSHIP

Consider $A_N$ to be the set of all possible arrangements of indistinguishable λ-bell particles on a completely filled $\lambda \times N$ lattice space. $A_N$ can be considered to consist of two subsets, each of which is characterized by the state of occupation of the column of sites at one end of the space. One subset is identified by the occupation of the end column by a single (vertical) λ-bell particle (see Fig. 2a). In such a case, the remaining $(N - 1)$ λ-bell particles can be arranged on the remaining $\lambda \times (N - 1)$ lattice sites in $A_{N-1}$ independent ways. The other subset of which $A_N$ is composed consists of those arrangements in which the λ sites of the end column are occupied by λ (horizontal) λ-bell particles (see Fig. 2b). The remaining $(N - \lambda)$ λ-bell particles can be arranged in $A_{N-\lambda}$ independent ways. Thus

$$A_N = A_{N-1} + A_{N-\lambda} \qquad (\lambda > 1) \tag{1}$$

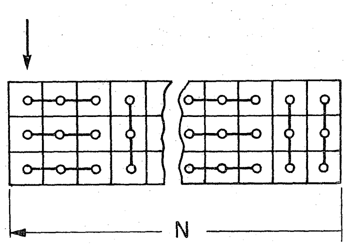
If $\lambda = 2$, Eq. 1 becomes the Fibonacci recursion [9] and

$$A_N = \frac{1}{2^{N+1}\sqrt{5}}\{[1 + \sqrt{5}]^{N+1} - [1 - \sqrt{5}]^{N+1}\}. \tag{2}$$




(a)

This arrangement is one member of the subset of $A_N$ that is characterized by the fact that all the compartments of the column at the left-hand end are occupied by a single, vertical $\lambda = 3$ particle.




(b)

This arrangement is one member of the subset of $A_N$ that is characterized by the fact that each compartment of the column on the left-hand end of the lattice space is occupied by three different $\lambda = 3$ particles.

FIGURE 2

For recursion relations of the kind given in Eq. 1, we may write [9]

$$A_N = C(R)^N \tag{3}$$

where $C$ and $R$ are not functions of $N$ but depend on $\lambda$. Substituting Eq. 3 into Eq. 1 yields

$$R^{\lambda} - R^{\lambda-1} - 1 = 0 \tag{4}$$

or

$$T^{\lambda} + T - 1 = 0 \tag{5}$$

where $T \equiv R^{-1}$.

If $\lambda$ is even, then by Descartes' rule of signs, we see that there are two real roots, one greater than unity and one less than unity; and $\left[\frac{\lambda - 2}{2}\right]$ pairs of complex roots, the largest absolute value of which is less than the largest real root. If $\lambda$ is odd, then there is one real root whose value is greater than the absolute value of any of the $\left[\frac{\lambda - 1}{2}\right]$ pairs of complex roots. Thus, for large values of $N$,

$$A_N = C(R_1)^N \tag{6}$$

where $R_1$ is the largest (real) root of Eq. 4.

Figure 3 shows $R_1^{-1} = T_1$ as a function of $\lambda$. Note that $T_1$ approaches unity for large values of $\lambda$.

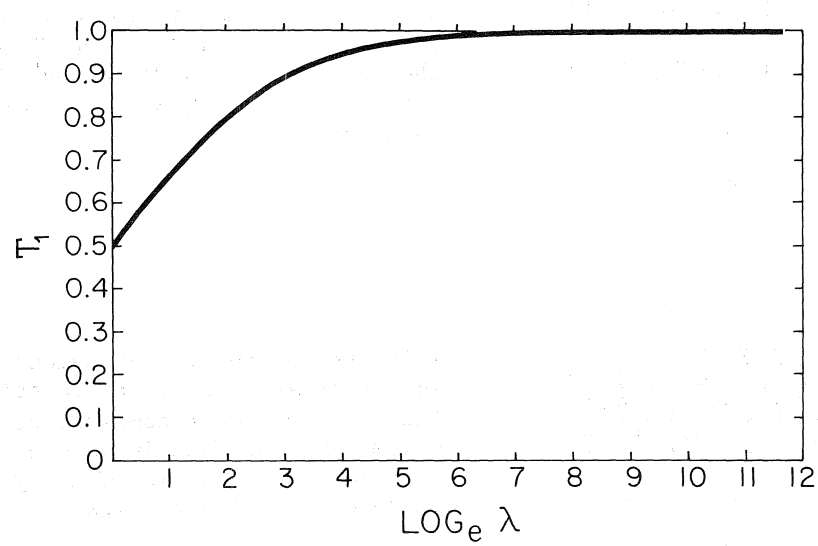



FIGURE 3. $T_1$ THE SMALLEST ROOT OF EQUATION 5 AS A FUNCTION OF $\lambda$

In Section 4, we will calculate the bivariant generating function which can be utilized to determine numerical values for $C$.

## 3. SUMMATION REPRESENTATION

Another representation of the occupational degeneracy can be developed through the following considerations. There are essentially two kinds of entities on the lattice space under consideration: vertical particles and groups of horizontal particles (each group consists of a block of $\lambda$ particles) (see Fig. 4). If there are $q_h' = q_h/\lambda$ groups of horizontal particles (where each group occupies $\lambda$ columns and $\lambda$ rows), then there are $N - \lambda q_h'$ vertical particles. Thus, there are a total of

$$q_h' + N - \lambda q_h' = N - q_h'(\lambda - 1)$$

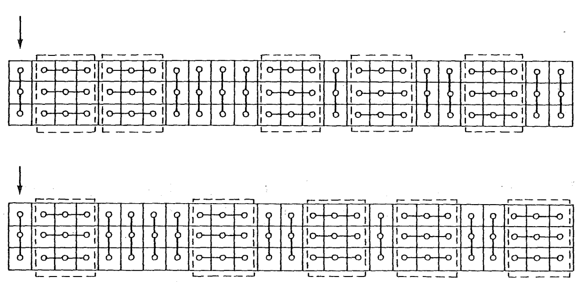
different individuals of which $q_h'$ are one kind (the blocks of horizontal particles) and $N - \lambda q_h'$ are another (the vertical particles). These may be permuted in

$$\binom{N - q_h'(\lambda - 1)}{q_h'}$$

independent ways [3]. Thus, the total degeneracy is obtained by summing over all values of $q_h'$,

$$A_N = \sum_{q_h'=0}^{[N/\lambda]} \binom{N - q_h'(\lambda - 1)}{q_h'}, \tag{7}$$

where $[N/\lambda]$ is the largest integer contained in $N/\lambda$.



This figure shows, for $\lambda = 3$ particles, two arrangements out of a total of $\binom{15}{5} = \binom{15}{10}$ arrangements that are possible when ten vertical particles and fifteen horizontal particles [which must be arranged in five groups] are distributed on a $3 \times 25$ lattice space.

FIGURE 4

## 4. GENERATING FUNCTIONS

According to Eq. 7, we form the polynomials [10]

$$u_N(x) = \sum_{k=0}^{[N/\lambda]} \binom{N - k(\lambda - 1)}{k} x^k; \tag{8}$$

then, utilizing Eq. 1, we see that

$$u_N(x) = u_{N-1}(x) + xu_{N-\lambda}(x), \tag{9}$$

with the initial conditions

$$u_0(x) = u_1(x) = \cdots = u_{\lambda-1}(x) = 1, \tag{10}$$

which reflect the fact that, if $N \leqslant \lambda - 1$, there is only one way in which the space can be completely filled; i.e., all the particles must be vertical.

The so-called bivariant generating function can be obtained as follows:

$$\begin{aligned} u(x, y) &= \sum_{N=0}^{\infty} u_N(x)y^N \\ &= \left[\frac{y^\lambda - 1}{-1}\right] + y\left\{u(x, y) - \frac{y^{\lambda-1} - 1}{y - 1}\right\} + xy^\lambda u(x, y) \end{aligned} \tag{11}$$

or

$$u(x, y) = [1 - y - xy^\lambda]^{-1}. \tag{12}$$

On the basis of Eq. 12, we may write

$$\begin{aligned} u(1, y) &= [1 - y - y^\lambda]^{-1} \\ &= \sum_{N=0}^{\infty} u_N(1)y^N \\ &= \sum_{N=0}^{\infty} A_N y^N. \end{aligned} \tag{13}$$

But, by partial fraction expansion,

$$\frac{1}{1 - y - y^\lambda} = \sum_{j=1}^{\lambda} \frac{C_j}{1 - S_j y} = \sum_{j=1}^{\lambda} \sum_{\ell=0}^{\infty} C_j [S_j y]^\ell, \tag{14}$$

where the $C_j$'s are constants (not functions of $N$), and $S_j$ are the reciprocals of the roots of $1 - y - y^\lambda$; i.e., $S_j$ are the roots of Eq. 4. Thus, $S_j \equiv R_j$. By comparing Eqs. 13 and 14, we obtain

$$A_N = \sum_{j=1}^{\lambda} C_j [R_j]^N. \tag{15}$$

To determine the $C_j$'s, we let $y \to R_j^{-1}$, then the dominant term in the partial fraction expansion, Eq. 14, is $\dfrac{C_j}{1 - R_j y}$. We may then write

$$\lim_{y \to R_j^{-1}} \left\{\frac{1}{1 - y - y^\lambda} - \frac{C_j}{1 - R_j y}\right\} = 0. \tag{16}$$

Applying L'Hôpital's rule yields

$$C_j = \lim_{y \to R_j^{-1}} \left[\frac{1 - R_j y}{1 - y - y^\lambda}\right] = \lim_{y \to R_j^{-1}} \left[\frac{R_j}{1 + \lambda y^{\lambda-1}}\right] = \frac{R_j}{1 + \lambda R_j^{1-\lambda}}, \tag{17}$$

and Eq. 15 becomes

$$A_N = \sum_{j=1}^{\lambda} \frac{R_j}{1 + \lambda R_j^{1-\lambda}} R_j^N \tag{18}$$

If $R_1$ is the dominant root, then as $N \to \infty$,

$$A_N = \frac{R_1^{N+1}}{1 + \lambda R_1^{1-\lambda}} = \frac{R_1}{1 + \lambda R_1^{1-\lambda}} R_1^N \tag{19}$$

so that the $C$ in Eq. 6 is given by

$$C = \frac{R_1}{1 + \lambda R_1^{1-\lambda}}. \tag{20}$$

As an example, for $\lambda = 3$, $R_1 = 1.46557123$ and $C = 0.611491992$, so that

$$A_N = 0.611491992(1.46557123)^N \tag{21}$$

For $N = 10$, Eq. 21 yields a value of 27.96. This is compared to an actual value of 28 or an error of 0.14%.

Note that as $\lambda$ becomes large, $C \to R_1$, so that (see Eq. 6)

$$A_N = R_1^{N+1} \quad \text{(for large values of } N \text{ and } \lambda\text{)}. \tag{22}$$

## CONCLUSION

The occupational degeneracy for a $\lambda \times N$ lattice space completely covered with λ-bell particles can be represented exactly by a two-term recursion relationship and by the summation of certain binomial coefficients. The appropriate generating functions have been derived and utilized to develop a continuous representation for the degeneracy as $N \to \infty$.

## ACKNOWLEDGMENTS

The authors would like to thank the reviewer for a number of very useful comments, and Professor D. Snider for several valuable discussions. We also wish to acknowledge Mr. R. Ristic for the drawings for this paper.

## REFERENCES

1. D. Lichtman & R. B. McQuistan. "Exact Occupation Statistics for One-Dimensional Arrays of Dumbbells." *J. Math. Phys.* 8 (1967):2441-44.
2. R. B. McQuistan & S. J. Lichtman. "Exact Recursion Relation for $2 \times N$ Arrays of Dumbbells." *J. Math. Phys.* 10 (1970):3095-99.

3. R. B. McQuistan. "Exact Occupation Statistics for One-Dimensional Arrays of $\lambda$-Bells." *Nuovo Cimento* 58B (1968):86-92.
4. P. W. Kasteleyn. "The Statistics of Dimers on a Lattice." *Physica* 27 (1961):1209-25.
5. H. N. V. Temperley & M. E. Fisher. "Dimer Problem in Statistical Mechanics —An Exact Result." *Phil. Mag.* 6 (1961):1061-63.
6. H. A. Bethe. "Statistical Theory of Superlattices." *Proc. Roy. Soc.* (London) A150 (1935):552-75.
7. R. H. Fowler & G. S. Rushbrooke. "An Attempt to Extend the Statistical Theory of Perfect Solutions." *Trans. Faraday Soc.* 33 (1937):1272-94.
8. J. K. Roberts & A. R. Miller. "The Application of Statistical Methods to Immobile Adsorbed Films." *Proc. Cam. Phil. Soc.* 35 (1939):293-97.
9. A. J. Phares & K. H. Simmons. "Fibonacci Numbers Revisited: A Combinatorics Function Point of View." *Amer. J. Phys.* 48 (1980):159-62.
10. J. Riordan. *Combinatorial Identities*. New York: Wiley & Sons, 1968.

♦◊♦◊♦

◆◇◆◇◆

# PASCAL GRAPHS AND THEIR PROPERTIES*

NARSINGH DEO and MICHAEL J. QUINN
*Washington State University, Pullman, WA 99164*

*(Submitted March 1982)*

## 1. INTRODUCTION

While searching for a class of graphs with certain desired properties to be used as computer networks, we have found graphs that come close to being optimal. One of the desired properties is that the design be simple and recursive, so that when a new node is added, the entire network does not have to be reconfigured. Another property is that one central vertex be adjacent to all others. The third requirement is that there exist several paths between each pair of vertices (for reliability) and that some of these paths be of short lengths (to reduce communication delays). Finally, the graphs should have good cohesion and connectivity [1]. Complete graphs $K_n$ satisfy all these properties, but are ruled out because of the expense.

This paper introduces a set of adjacency matrices called Pascal matrices, which are constructed using Pascal's triangle modulo 2. We also define Pascal graphs, the set of graphs corresponding to the Pascal matrices. We begin by showing that the Pascal graphs have the properties described above. In the second part of the paper we explore the properties of the determinants of the Pascal matrices. It appears that every Pascal matrix of order $\geqslant 3$ has a determinant of either 0 or 2. We indicate the sequence of matrix orders for which the determinant is 2. The third part of our report lists unexplored ideas and presents attributes of Pascal graphs which we have not been able to exploit in our proofs.

Standard graph theoretic terms are used throughout this paper. The reader seeking a reference should consult Deo [3] or Harary [6].

## 2. DEFINITIONS

### Definition 1

An $n \times n$ symmetric binary matrix is called the *Pascal matrix* $PM(n)$ of order $n$ if its main diagonal entries are all 0's and its lower triangle (and therefore the upper also) consists of the first $n - 1$ rows of the Pascal triangle modulo 2. Let $pm_{i,j}$ denote the element in the $i$th row and the $j$th column of the Pascal matrix.

---

**This work was supported by NSF grant MCS 78-25851 and by U.S. D.O.T. contract DTRS-5681-c-00033.*

(This definition should not be confused with another definition of Pascal matrix by Lunnon [8]. Note, however, that the matrix he defines as the Pascal matrix has been defined previously as *Tartaglia's rectangle* [9].)

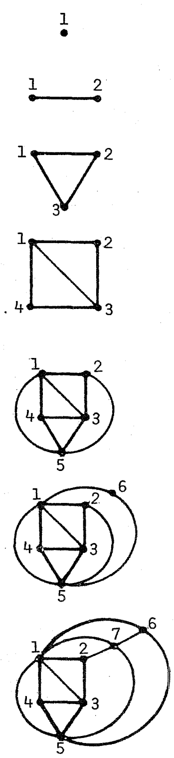
## Definition 2

An undirected graph with $n$ vertices corresponding to $PM(n)$ as its adjacency matrix is called the *Pascal graph* $PG(n)$ of order $n$.

The first seven Pascal graphs along with associated Pascal matrices are shown in Figure 1.

## Definition 3

Let $pt_{i,j}$ refer to the $j$th element of the $i$th row of Pascal's triangle, where rows and their elements are numbered beginning with 0.



$PG(1)$ $PM(1)$ $[0]$

$PG(2)$ $PM(2)$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$PG(3)$ $PM(3)$ $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$PG(4)$ $PM(4)$ $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$PG(5)$ $PM(5)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

$PG(6)$ $PM(6)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$PG(7)$ $PM(7)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

FIGURE 1

## 3. CONNECTIVITY PROPERTIES OF THE PASCAL GRAPHS

Lemma 1

$PG(n)$ is a subgraph of $PG(n + 1)$ for all $n \geqslant 1$.

Proof: This property is a direct consequence of the definition of the Pascal matrix

Theorem 1

All $PG(i)$ for $1 \leqslant i \leqslant 7$ are planar; all Pascal graphs of higher order are nonplanar.

Proof: Figure 1 clearly indicates that all $PG(i)$ for $1 \leqslant i \leqslant 7$ are planar. $K_{3,3}$ is a subgraph of $PG(8)$. Thus, by Lemma 1, all graphs of order 8 and higher are nonplanar.

Theorem 2

Vertex $v_1$ is adjacent to all other vertices in the Pascal graph. Vertex $v_i$ is adjacent to $v_{i+1}$ in the Pascal graph for $i \geqslant 1$.

Proof: $pm_{i,j} = pt_{i-2,j-1} \pmod 2$, $i > j \geqslant 1$ (Definition of Pascal matrix).

For all $i \geqslant 2$, $pm_{i,1} = pt_{i-2,0} \pmod 2 = \binom{i-2}{0} \pmod 2 = 1$.

Thus, $v_1$ is adjacent to all $v_i$, $i \geqslant 2$.

For all $i \geqslant 1$, $pm_{i+1,i} = pt_{i-1,i-1} \pmod 2 = \binom{i-1}{i-1} \pmod 2 = 1$.

Thus, $v_i$ is adjacent to $v_{i+1}$ for all $i \geqslant 1$.

Corollary 1

$PG(n)$ contains a startree for all $n \geqslant 1$.

Corollary 2

$PG(n)$ contains a Hamiltonian circuit $[1, 2, \ldots, n - 1, n, 1]$.

Corollary 3

$PG(n)$ contains $W_n - x$ (wheel of order $n$ minus an edge).

Lemma 2

If $k = 2^n + 1$, $n$ a positive integer, then $v_k$ is adjacent to all $v_i$, $1 \leqslant i < 2k$ and $i \neq k$.

Proof: Let $k = 2^n + 1$, where $n$ is a positive integer.

Case 1. $1 \leqslant i < k$

$$pm_{k,i} = pt_{k-2,i-1} \pmod 2 = \binom{2^n - 1}{i - 1} \pmod 2 = 1 \qquad [4].$$

Case 2. $k < i < 2k$

$$pm_{k,i} = pm_{i,k} = pt_{i-2,k-1} \pmod 2 = \binom{i - 2}{2^n} \pmod 2.$$

We may factor $i - 2$ into its binomial coefficients:

$$i - 2 = m_0 + m_1 \times 2^1 + \cdots + m_{n-1} \times 2^{n-1} + 1 \times 2^n.$$

Thus,

$$\binom{i - 2}{2^n} \pmod 2 = \binom{m_0}{0}\binom{m_1}{0} \cdots \binom{m_{n-1}}{0}\binom{1}{1} \pmod 2 = 1 \qquad [4].$$

Since for all $v_i$, $1 \leqslant i < 2k$ and $i \neq k$, $pm_{k,i} = 1$, $v_k$ is adjacent to all such $v_i$.

The following connectivity property is useful in the design of reliable communication and computer networks.

Theorem 3

There are at least two edge-disjoint paths of length $\leqslant 2$ between any two distinct vertices in $PG(n)$, $n \geqslant 3$.

Proof: Let $v_i$, $v_j$ be two vertices of $PG(n)$, $n \geqslant 3$, $i < j$.

Case 1. $i = 1$, $j = 2$

Two edge-disjoint paths are $[v_1, v_2]$ and $[v_1, v_3, v_2]$.

Case 2. $i = 1$, $j > 2$

Two edge-disjoint paths are $[v_1, v_i]$ and $[v_1, v_{i-1}, v_i]$ (Lemma 2).

Case 3. $i > 1$

By Theorem 2, we know that one path is $[v_i, v_1, v_j]$. Let

$$k = 1 + 2^{\lfloor \log_2(j) \rfloor}.$$

Lemma 2 indicates that $v_k$ is adjacent to all $v_m$ where $1 \leqslant m < 2k$ and $m \neq k$. If $i = k$ or $j = k$, then a second path is $[v_i, v_j]$; otherwise, a second path is $[v_i, v_k, v_j]$.

Corollary 4

All Pascal graphs of order $\geqslant 3$ are 2-connected.

Lemma 3

No two even-numbered vertices of a Pascal graph are adjacent.

Proof: Let $i$, $j$ be even integers, $i > j$.

$$pm_{i,j} = pt_{i-2,j-1} \pmod 2 = \binom{i-2}{j-1} \pmod 2.$$

Since $i - 2$ is even and $j - 1$ is odd, $\binom{i-2}{j-1} \pmod 2 = 0$ [4].

Theorem 4

If $v_i$ is adjacent to $v_j$, where $j$ is even and $|i - j| > 1$, then $i$ is odd and $v_i$ is adjacent to $v_{j-1}$.

Proof: Assume $v_i$ is adjacent to $v_j$, where $j$ is even and $|i - j| > 1$. By Lemma 3, we know that $i$ is odd.

Case 1. $i > j$

$1 = pm_{i,j} = pm_{i-1,i} + pm_{i-1,j-1} \pmod 2$ (Definition of Pascal triangle)

$= 0 + pm_{i-1,j-1}$ (Lemma 3)

$= pm_{i-1,j-1}$.

Thus,

$pm_{i,j-1} = pm_{i-1,j-1} + pm_{i-1,j-2} \pmod 2$ (Definition of Pascal triangle)

$= pm_{i-1,j-1} + 0$ (Lemma 3)

$= 1$ (Above).

Case 2. $i < j$

The proof proceeds similarly to Case 1. Thus, since $pm_{i,j-1} = 1$, $v_i$ is adjacent to $v_{j-1}$.

Although the set of complete graphs $K_n$ has maximal connectivity and cohesion properties, the fact that the number of edges in $K_n$ increases at a rate of $n^2$ makes it too costly to consider. The following theorem shows that the number of edges in the Pascal graphs increases at a much lower rate.

Theorem 5

Define $e(PG(n))$ to be the number of edges in $PG(n)$. Then

$$e(PG(n)) \leqslant \lfloor (n-1)^{\log_2 3} \rfloor.$$

Proof by Induction:

Basis

$$e(PG(1)) = 0 \leqslant 1 = \lfloor 0^{\log_2 3} \rfloor.$$

$$e(PG(2)) = 1 \leqslant 1 = \lfloor 1^{\log_2 3} \rfloor.$$

$$e(PG(3)) = 3 \leqslant 3 = \lfloor 2^{\log_2 3} \rfloor.$$

Induction

Assume true for all $PG(n)$, $1 \leqslant n \leqslant 2^k + 1$, $k > 0$.

Prove true for $PG(n)$, $2^k + 2 \leqslant n \leqslant 2^{k+1} + 1$.

Let $r$ be the positive integer such that $n - 1 = 2^k + r$.

$$\begin{aligned} e(PG(n)) &= e(PG(2^k + 1)) + 2e(PG(r + 1)) && [7] \\ &\leqslant \lfloor (2^k)^{\log_2 3} \rfloor + 2\lfloor r^{\log_2 3} \rfloor && \text{(Induction Hypothesis)} \\ &\leqslant \lfloor (2^k + r)^{\log_2 3} \rfloor = \lfloor (n - 1)^{\log_2 3} \rfloor. \end{aligned}$$

Pascal graphs are not the graphs with the fewest possible edges satisfying the preceding structural properties (which are useful in designing practical networks). For example, in $PG(7)$, the edge from $v_2$ to $v_7$ is redundant. There is a possibility that for some set of connectivity requirements, the Pascal graphs may exhibit optimal connectivity; i.e., they have no redundant edges. We have not found such a set of requirements, however.

## 4. DETERMINANTS OF THE PASCAL MATRICES

Theorem 6

Let $\det(PM(n))$ refer to the determinant of the Pascal matrix of order $n$. Then $\det(PM(n)) = 0$ for all even $n \geqslant 4$.

Proof: Given $PM(n)$ satisfying the conditions on $n$, we show that the even-numbered rows of $PM(n)$ are linearly dependent.

No two even-numbered vertices of a Pascal graph are adjacent (Lemma 3). Since even-numbered vertices are only adjacent to odd-numbered vertices, and since we desire to show that the even-numbered rows are linearly dependent, we may create a *reduced Pascal matrix* by removing the odd-numbered rows and even-numbered columns from $PM(n)$ (see Figure 2).

To show that the even-numbered rows are linearly dependent, it is sufficient to show that the determinant of the reduced Pascal matrix is 0. The reduced Pascal matrix contains two columns of 1's. Vertex $v_1$ is adjacent to all the other vertices (Theorem 2). Let $k = 2^{\lfloor \log_2 (n-1) \rfloor} + 1$. Vertex $v_k$ of $PG(n)$ is adjacent to all other $v_i$, $1 \leqslant i < k$ or $k + 1 < i < 2k$ (see Lemma 2). Thus, columns 1 and $(\lfloor k/2 \rfloor + 1)$ of the reduced Pascal matrix consist only of 1's.

Since the reduced Pascal matrix contains two identical columns, its determinant is 0. Thus $\det(PM(n)) = 0$.

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
4 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
5 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
7 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
8 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}
$$

$$
\begin{array}{cc|ccccl}
 & & 1 & 3 & 5 & 7 & \text{Original column} \\
 & & 1 & 2 & 3 & 4 & \text{New column} \\
\hline
2 & 1 & 1 & 1 & 1 & 1 & \\
4 & 2 & 1 & 1 & 1 & 0 & \\
6 & 3 & 1 & 0 & 1 & 1 & \\
8 & 4 & 1 & 1 & 1 & 1 & \\
\text{Old row} & \text{New row} & & & & &
\end{array}
$$

FIGURE 2. THE REDUCED PASCAL MATRIX

Theorem 7

$\operatorname{Det}(PM(n))$ is even for all odd $n \geqslant 3$.

Proof: Let $n$ be an odd integer $\geqslant 3$; $G_i$ be one of the $m$ linear subgraphs of $PG(n)$; $e_i$ be the number of components of $G_i$ which have an even number of vertices; and $c_i$ be the number of cycles in $G_i$.

$$\operatorname{Det}(PM(n)) = \sum_{i=1}^{m} (-1)^{e_i} \times 2^{c_i} \qquad [5].$$

Since there are an odd number of vertices, each linear subgraph of $PG(n)$ must contain at least one cycle. Thus, $\det(PM(n))$ is a sum of even integers, and therefore $\det(PM(n))$ is even.

Observations

$$\operatorname{Det}(PM(n)) = \begin{cases} 2, & \text{for } n = 3, 7, 11, 23, 43, 87 \\ 0, & \text{for all other } n,\ 4 \leqslant n \leqslant 86. \end{cases}$$

Let $t_0, t_1, t_2, \ldots$ be the sequence of integers such that $\det(PM(t_i)) = 2$. Then the sequence of $t_i$'s is conjectured to be:

$$t_0 = t_1 = 3$$

$$t_i = 2^i + t_{i-2},\ i \geqslant 2.$$

$\operatorname{Det}(PM(t_i + 1)) = 0$ for all $i$, since $t_i + 1$ is even. This implies that row $t_i + 1$ is linearly dependent upon other even-numbered rows in the Pascal matrix (Theorem 6). It appears that the first of these rows whose linear combination yields row $t_i + 1$ is row $t_{i-1} - 1$. This linear combination of rows must always break down at column $2^{i+1} + t_{i-1}$, since this column has a 1 in row $t_{i-1} - 1$ and 0's in rows $t_{i-1} + 1$ through $t_i + 1$. Note that it is precisely at this point, when the linear dependence must break down, that the Pascal matrix again has determinant 2. Figure 3 illustrates this phenomenon.

$$\begin{array}{r|ccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & 2^{i+1} & & & & & & & 2^{i+1}+t_{i-1} \\ \hline
 & 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
 & 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
 & 1&1&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\
 & 1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\
 & 1&1&1&1&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1 \\
t_{i-1}-1 & 1&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1 \\
 & 1&1&0&0&1&1&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0 \\
 & 1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
 & 1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0 \\
 & 1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0 \\
t_i & 1&1&0&0&0&0&0&0&1&1&0&1&1&0&0&1&1&0&0&0&0&0&0 \\
t_i+1 & 1&0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&0
\end{array}$$

FIGURE 3

Thus discussion leaves several questions unanswered. We just described why the linear combination of rows breaks down when it does. Why does it fail to break down sooner? When it does break down, why is there not another combination of linearly dependent rows? Why is the determinant of $PM(t_i)$, $i \geqslant 0$, equal to 2?

There is a pattern to the rows that are linearly dependent on each other, causing the determinants of the matrices to be 0. Relationships among these rows are illustrated in Figure 4.

| This combination of rows... | yields row... | for Pascal matrices of size... |
|---|---|---|
| 2 | 4 | 4-6 |
| 2 | 8 | 8-10 |
| -6 +8 +10 | 12 | 12-22 |
| -10 +16 +18 | 24 | 24-42 |
| 22 -24 -26 +28 -38 +40 +42 | 44 | 44-86 |

For example, row 8 plus row 10 minus row 6 equals row 12 in all Pascal matrices of sizes 12 through 22. Thus, since row 12 is linearly dependent upon other rows, the determinant of each Pascal matrix of order 12 through 22 is zero.

Note that 6 = 4 + 2, 10 = 8 + 2, 22 = 12 + 10, 42 = 24 + 18, and 86 = 44 + 42.

Arranging the rows that are linearly dependent on each other in increasing order:

```
            +2  -4
            +2  -8
        -6  +8 +10 -12
       -10 +16 +18 -24
+22 -24 -26 +28 -38 +40 +42 -44
```

Looking at the differences between the rows:

```
             2
             6
         2   2   2
         6   2   6
 2   2   2  10   2   2   2
```

FIGURE 4

## 5. UNEXPLORED IDEAS AND UNUSED DATA

A necessary and sufficient condition for a matrix to have a zero determinant is that it have at least one eigenvalue that is zero. Unfortunately, deciding whether or not a matrix has a zero eigenvalue is no easier than deciding if it has a zero determinant. The only method not requiring direct calculation of the determinant involves finding linear subgraphs [5].

Figure 5 summarizes what we have discovered about the number of linear subgraphs of various types for the first few Pascal graphs. The number of linear subgraphs of $PG(n)$ grows very rapidly as $n$ increases, limiting our pursuit of additional data. We have not yet discovered a pattern in these data that would point to a proof showing those Pascal matrices that have 0 determinants and those that have determinant 2.

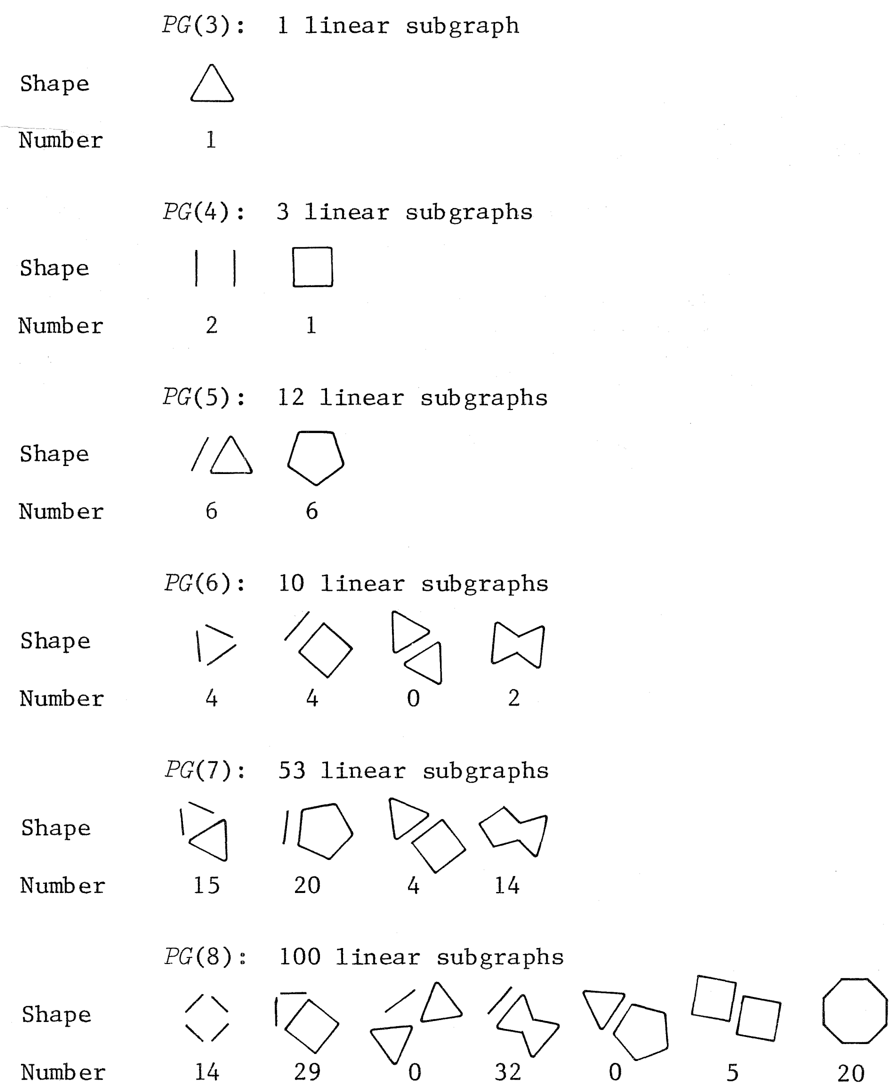



FIGURE 5

A topic that we have not explored is the eigenvalue spectra of the Pascal matrices. Since the matrices are symmetric, their eigenvalues are real. Perhaps a pattern in these spectra could be found. Several facts concerning the eigenvalue spectra may be useful. Let $\lambda_i$ be one of the $n$ eigenvalues of $PM(n)$; $\bar{d}_n$ be the mean valence of the vertices in $PM(n)$; $r_n$ be the greatest eigenvalue of $PM(n)$. Then the number of edges in $PG(n)$ is

$$\sum_{i=1}^{n} \lambda^2/2;$$

the number of triangles in $PG(n)$ is

$$\sum_{i=1}^{n} \lambda^3/2;$$

and $\bar{d}_n \leqslant r_n \leqslant n - 1$ [2].

Table 1 lists the number of edges in the Pascal graphs of small order and Table 2 shows the vertex valency spectra of Pascal graphs of small order.

TABLE 1

| $n$ | Edges in $PG(n)$ | $n$ | Edges in $PG(n)$ | $n$ | Edges in $PG(n)$ |
|---|---|---|---|---|---|
| 1 | 0 | 8 | 19 | 15 | 57 |
| 2 | 1 | 9 | 27 | 16 | 65 |
| 3 | 3 | 10 | 29 | 17 | 81 |
| 4 | 5 | 11 | 33 | 18 | 83 |
| 5 | 9 | 12 | 37 | 19 | 87 |
| 6 | 11 | 13 | 45 | 20 | 91 |
| 7 | 15 | 14 | 49 | | |

TABLE 2

| $n$ | Valency Spectrum for $PG(n)$ | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | | | | | | | | | | | | | | | | | | |
| 3 | 2 | 2 | 2 | | | | | | | | | | | | | | | | | |
| 4 | 2 | 2 | 3 | 3 | | | | | | | | | | | | | | | | |
| 5 | 3 | 3 | 4 | 4 | 4 | | | | | | | | | | | | | | | |
| 6 | 2 | 3 | 3 | 4 | 5 | 5 | | | | | | | | | | | | | | |
| 7 | 3 | 3 | 4 | 4 | 4 | 6 | 6 | | | | | | | | | | | | | |
| 8 | 3 | 3 | 4 | 4 | 5 | 5 | 7 | 7 | | | | | | | | | | | | |
| 9 | 4 | 4 | 5 | 5 | 6 | 6 | 8 | 8 | 8 | | | | | | | | | | | |
| 10 | 2 | 4 | 4 | 5 | 5 | 6 | 6 | 8 | 9 | 9 | | | | | | | | | | |
| 11 | 3 | 4 | 4 | 4 | 5 | 6 | 6 | 6 | 8 | 10 | 10 | | | | | | | | | |
| 12 | 3 | 4 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 8 | 11 | 11 | | | | | | | | |
| 13 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 7 | 8 | 8 | 8 | 12 | 12 | | | | | | | |
| 14 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 7 | 8 | 9 | 9 | 13 | 13 | | | | | | |
| 15 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 8 | 8 | 8 | 10 | 10 | 14 | 14 | | | | | |
| 16 | 5 | 5 | 5 | 5 | 5 | 5 | 7 | 7 | 8 | 8 | 9 | 9 | 11 | 11 | 15 | 15 | | | | |
| 17 | 6 | 6 | 6 | 6 | 6 | 6 | 8 | 8 | 9 | 9 | 10 | 10 | 12 | 12 | 16 | 16 | 16 | | | |
| 18 | 2 | 6 | 6 | 6 | 6 | 6 | 6 | 8 | 8 | 9 | 9 | 10 | 10 | 12 | 12 | 16 | 17 | 17 | | |
| 19 | 3 | 4 | 6 | 6 | 6 | 6 | 6 | 6 | 8 | 8 | 9 | 10 | 10 | 10 | 12 | 12 | 16 | 18 | 18 | |
| 20 | 3 | 4 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 8 | 8 | 9 | 10 | 10 | 11 | 12 | 12 | 16 | 19 | 19 |

## ACKNOWLEDGMENTS

The authors gratefully acknowledge the contributions of Professors Calvin Long and George Marsaglia through helpful and stimulating discussions.

## REFERENCES

1. F. T. Boesch & A. P. Felzer. "A General Class of Invulnerable Graphs." In *Large Scale Networks: Theory and Design*. New York: IEEE Press, 1976.
2. D. Cvetkovic; M. Doob; & H. Sachs. *Spectra of Graphs: Theory and Application*. New York: Academic Press, 1980.
3. N. Deo. *Graph Theory with Applications to Engineering and Computer Science*. Englewood Cliffs, N.J.: Prentice-Hall, 1974.
4. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* 54 (1947):589-92.
5. F. Harary. "The Determinant of the Adjacency Matrix of a Graph." *SIAM Review* 4 (1962):202-10.
6. F. Harary. *Graph Theory*. Reading, Mass.: Addison-Wesley, 1969.
7. C. T. Long. "Pascal's Triangle Modulo *p*." *The Fibonacci Quarterly* 19 (1981):458-63.
8. W. F. Lunnon. "The Pascal Matrix." *The Fibonacci Quarterly* 15 (1977): 201-04.
9. V. A. Uspenskii. *Pascal's Triangle*. Trans. and adapted from the Russian by D. J. Sookne & T. McLarnan. Chicago: University of Chicago Press, 1974.

◆◇◆◇◆

◆◇◆◇◆

# ON FIBONACCI NUMBERS WHICH ARE POWERS: II

NEVILLE ROBBINS
*San Francisco State University, San Francisco, CA 94132*

(Submitted April 1982)

## INTRODUCTION

Consider the equation:

$$F_m = c^t \tag{*}$$

where $F_m$ denotes the $m$th Fibonacci number, and $c^t > 1$. Without loss of generality, we may require that $t$ be prime. The unique solution for $t = 2$, namely $(m, c) = (12, 12)$, was given by J. H. E. Cohn [2], and by O. Wyler [11]. The unique solution for $t = 3$, namely $(m, c) = (6, 2)$, was given by H. London and R. Finkelstein [5] and by J. C. Lagarias and D. P. Weisser [4]. A. Petho [6] showed that (*) has only finitely many solutions with $t > 1$, where $m, c, t$ all *vary*. In fact, he shows that all solutions of (*) can be effectively determined; that is, there is an effectively computable bound $B$ such that all solutions of (*) have

$$\max(|m|,\ |c|,\ t) < B. \tag{**}$$

Similar results were obtained independently by C. L. Stewart [10], see, also, T. N. Shorey and C. L. Stewart [9]. The proofs of these results use lower estimates on linear forms in the logarithms of algebraic numbers due to A. Baker [1], and the bounds obtained for $B$ in (**) are astronomical. In [7], A. Petho claims that (*) has no solutions for $t = 5$.

In [8], we showed that if $m = m(t)$ is the least natural number for which (*) holds for given $t$, then $m$ is odd. In this paper, our main result, which we obtained by elementary methods, is that $m$ must be prime. If (*) has solutions for $t > 5$, and if $q$ is a prime divisor of $F_m$, one would therefore have $z(q^t) = z(q) = m$, where $z(q)$ denotes the Fibonacci entry point of $q$. This requirement casts doubt on the existence of such solutions. For the sake of convenience, we occasionally write $F(m)$ instead of $F_m$.

## PRELIMINARIES

(1) If $t$ is a given prime, $t \geqslant 5$, and $m = m(t)$ is the least natural number such that (*) holds, then $m$ is odd.

(2) $F_j \mid F_{jk}$

(3) $(F_j,\ F_k) = F_{(j,k)}$

(4) $(F_j, F_{jk}/F_j) | k$

(5) $F_1 = 1$

(6) $5^j \| k$ iff $5^j \| F_k$

(7) If $p$ is an odd prime, then $p^2 \nmid F(p^j k)/F(p^{j-1}k)$

(8) If $xy = z^n$, $n$ is odd, and $(x, y) = 1$, then $x = u^n$, $y = v^n$, where $(u, v) = 1$ and $uv = z$.

(9) If $xy = z^n$, $n$ is odd, $p$ is prime, $(x, y) = p$, and $p^2 \nmid y$, then $x = p^{n-1}u^n$, $y = pv^n$, where $(u, v) = (p, v) = 1$.

(10) If $2^k | F_m$, where $k \geqslant 3$, then $3 * 2^{k-2} | m$

(11) If $p$ is prime, then $p | F_{p-e_p}$, where $e_p = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod{10}, \\ 0, & \text{if } p = 5, \\ -1, & \text{otherwise.} \end{cases}$

(12) $F_j < F_{jk}$ if $j \geqslant 2$ and $k \geqslant 2$

Remarks: All but (1) and (4) are elementary and/or well known. (1) is the Corollary to Theorem 1 in [8], and (4) is Lemma 16 in [3].

## THE MAIN RESULTS

### Theorem 1

If $t$ is a given prime, $t \geqslant 5$, and $m = m(t)$ is the least natural number such that $F_m = c^t > 1$, then $m$ is prime.

Proof: Let

$$m = \prod_{i=1}^{r} p_i^{e_i},$$

where the $p_i$ are primes and $p_1 < p_2 < \cdots < p_r$ if $r > 1$. Furthermore, assume $m$ is composite, so that $p_r < m$. (1) implies $2 < p_1$. Let

$$d = (F(p_r), F(m)/F(p_r)).$$

(4) implies $d | (m/p_r)$. If $d = 1$, then since hypothesis implies

$$F(p_r) * F(m)/F(p_r) = c^t,$$

(8) and (12) imply $F(p_r) = a^t$ with $1 < a < c$, contradicting the minimality of $m$. If $d > 1$, then $p_i | d$ for some $i$ such that $1 \leqslant i \leqslant r$. If $i < r$, then Lemma 1, which is proved below, implies $p_i = 2$, a contradiction. If $i = r$, then (11) implies $p_r = 5$, so $r = 1$ or 2. If $r = 2$, then $m = 3^a 5^b$. But $F_3 = 2$, so the

hypothesis and (2) imply $2|c^t$, hence $2^t|c^t$, and $2^t|F_m$. Now (10) implies that $3 * 2^{t-2}|3^a5^b$, so that $t = 2$, a contradiction. If $r = 1$, then $m = 5^e$, which is impossible by Lemma 3, which is proved below.

Lemma 1

If $p$, $q$ are primes such that $p < q$ and $p|F(q^k)$ for some $k$, then $p = 2$ and $q = 3$.

Proof: The hypothesis, (11), and (3) imply $p|F_d$, where $d = (q^k, p - e_p)$. (5) implies $d > 1$, so that $d = q^j$ for some $j$ such that $1 \leqslant j \leqslant k$. Therefore, $q^j|(p - e_p)$, so that $q \leqslant q^j \leqslant p + 1$. But the hypothesis implies $q \geqslant p + 1$. Therefore, $q = p + 1$, so that $p = 2$ and $q = 3$.

Lemma 2

If $F(5^j) = 5^j v_j^e$, where $5 \nmid v_j$, then $F(5^{j-1}) = 5^{j-1}v_{j-1}^e$, where $5 \nmid v_{j-1}$.

Proof: The hypothesis and (2) imply $F(5^{j-1}) * F(5^j)/F(5^{j-1}) = 5^j v_j^e$. (6) and (7) imply

$$(F(5^{j-1}), F(5^j)/F(5^{j-1})) = 5,$$

so that (9) implies $F(5^{j-1}) = 5^{j-1}v_{j-1}^e$, and (6) implies $5 \nmid v_{j-1}$.

Lemma 3

$F(5^j) \neq c^t$ for $t > 1$.

Proof: If $F(5^j) = c^t$, then (6) implies $5^j d = c^t$, where $5 \nmid d$. Now (8) implies $5^j = u^t$, $d = v_j^t$, so that $F(5^j) = 5^j v_j^t$. Applying Lemma 2 $j - 2$ times, one obtains $F(5^2) = 5^2 v_2^t$. But $F(5^2)/5^2 = 3001$, so that $v_2^t = 3001$, a contradiction, since 3001 is prime.

## REFERENCES

1. A. Baker. "A Sharpening of the Bounds for Linear Forms in Logarithms II." *Acta Arith.* 24 (1973):33-36.
2. J. H. E. Cohn. "Square Fibonacci Numbers, Etc." *The Fibonacci Quarterly* 2, no. 2 (1964):109-13.
3. J. H. Halton. "On the Divisibility Properties of Fibonacci Numbers." *The Fibonacci Quarterly* 4, no. 3 (1966):217-40.
4. J. C. Lagarias & D. P. Weisser. "Fibonacci and Lucas Cubes." *The Fibonacci Quarterly* 19 (1981):39-43.
5. H. London & R. Finkelstein. "On Fibonacci and Lucas Numbers Which Are Perfect Powers." *The Fibonacci Quarterly* 7 (1969):476-81.

6. A. Petho. "Perfect Powers in Linear Second-Order Recurrences." *J. Number Theory* 15 (1982):5-13.
7. A. Petho. "Perfect Powers in Second-Order Recurrences." (To appear.)
8. N. Robbins. "On Fibonacci Numbers Which Are Powers." *The Fibonacci Quarterly* 16 (1978):515-17.
9. T. N. Shorey & C. L. Stewart. "On the Diophantine Equation $ax^{2t} + ax^t y + cy^2 = d$ and Pure Powers in Recurrence Sequences." (To appear in *Math. Scand.*)
10. C. L. Stewart. "On Some Diophantine Equations and Related Linear Recurrence Sequences." Seminar Delange-Pisot-Poitou, 1981.
11. O. Wyler. *Amer. Math. Monthly* 71 (1964):221-22.

◆◇◆◇◆

◆◇◆◇◆

# FIBONACCI NUMBERS OF GRAPHS: II

PETER KIRSCHENHOFER, HELMUT PRODINGER, and ROBERT F. TICHY
*Technische Universität Wien, Gußhausstraße 27-29, A-1040 Wien, Austria*

*(Submitted April 1982)*

## 1. INTRODUCTION

In [9], the Fibonacci number $f(G)$ of a (simple) graph $G$ is introduced as the total number of all Fibonacci subsets $S$ of the vertex set $V(G)$ of $G$, where a Fibonacci subset $S$ is a (possibly empty) subset of $V(G)$ such that any two vertices of $S$ are not adjacent. In *Graph Theory* [6, p. 257] a Fibonacci subset is called an independent set of vertices. From [9] we have the elementary inequality

$$F_{n+1} \leqslant f(G) \leqslant 2^{n-1} + 1, \tag{1.1}$$

where $F_n$ denotes the usual Fibonacci numbers with

$$F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2},$$

and $G$ is a tree with $n$ vertices. Furthermore, several problems are formulated concerning the Fibonacci numbers of some special graphs. The present aim is to derive a formula for $f(T_n(t))$, where $T_n(t)$ is the full $t$-ary tree with height $n$: [$T_0(t)$ is the empty tree.]

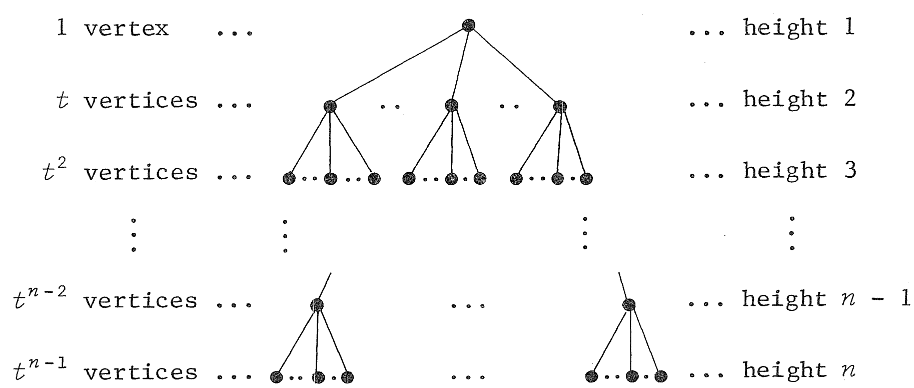



FIGURE 1

For $t = 1$, one can see immediately that $f(T_n(t)) = F_{n+1}$, so the interesting cases are $t \geqslant 2$. In Section 2, for $t = 2, 3, 4$, the asymptotic formula

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \qquad (n \to \infty) \tag{1.2}$$

is proved, where $A(t)$ and $k(t)$ denote constants (depending on $t$) with

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)}.$$

In Section 3, it is proved that for $t \geqslant 5$ such an asymptotic formula does not hold; we show that for $t \geqslant 5$:

$$\begin{aligned} f(T_{2m}(t)) &\sim B(t) \cdot k(t)^{t^{2m}} \\ f(T_{2m+1}(t)) &\sim C(t) \cdot k(t)^{t^{2m+1}} \end{aligned} \tag{1.3}$$

where $B(t) > C(t)$ are constants depending on $t$ with

$$\lim_{t\to\infty} B(t) = \lim_{t\to\infty} C(t) = 1.$$

In Section 4, we establish an asymptotic formula for the average value $S_n$ of the Fibonacci number of binary trees with $n$ vertices (where all such trees are regarded equally likely). For the sake of brevity, we restrict our considerations to the important case of binary trees; however, the methods would even be applicable in the very general case of so-called "simply generated families of trees" introduced by Meir and Moon [8].

By a version of Darboux's method (see Bender's survey [1]), we derive

$$S_n \sim G \cdot r^n \qquad (n \to \infty), \tag{1.4}$$

where $G = 1,\ 12928\ldots$ and $r = 1,\ 63742\ldots$ are numerical constants.

## 2. FIBONACCI NUMBERS OF $t$-ARY TREES ($t = 2, 3, 4$)

By a simple argument (compare [9]), the following recursion holds for the Fibonacci number $x_n: = f(T_n(t))$,

$$x_{n+1} = x_n^t + x_{n-1}^{t^2} \text{ with } x_0 = 1,\ x_1 = 2. \tag{2.1}$$

We proceed as in [4] and put $y_n = \log x_n$; by (2.1),

$$y_{n+1} = ty_n + \alpha_n \text{ with } \alpha_n = \log\left(1 + \frac{x_{n-1}^{t^2}}{x_n^t}\right). \tag{2.2}$$

Because of

$$x_{n-1}^{t^2} < (x_{n-1}^t + x_{n-2}^{t^2})^t = x_n^t,$$

the estimate

$$0 < \alpha_n < \log 2 \tag{2.3}$$

results. The solution of recursion (2.2) is given by

$$y_n = t^n\left(\frac{\alpha_0}{t} + \frac{\alpha_1}{t^2} + \cdots + \frac{\alpha_{n-1}}{t^n}\right).$$

It is now convenient to extend the series in $\alpha_i$ to infinity [because of (2.3) the series is convergent]:

$$Y_n : = \sum_{i=0}^{\infty} t^{n-1-i}\alpha_i. \tag{2.4}$$

For the difference

$$r_n : = Y_n - y_n = \sum_{i=n}^{\infty} t^{n-1-i}\alpha_i,$$

we have

$$0 < r_n \leqslant \frac{\log 2}{t-1}. \tag{2.5}$$

Therefore, we obtain

$$x_n = e^{Y_n - r_n} = e^{-r_n} \cdot k(t)^{t^n}, \tag{2.6}$$

where

$$k(t) = \exp\left(\sum_{i=0}^{\infty} t^{-i-1}\alpha_i\right) \tag{2.7}$$

and $1 < k(t) < 2^{1/(t-1)}$ by (2.3).

In the following, we investigate the factor $e^{-r_n}$ of (2.6); we set

$$q_n = x_n^t / x_{n+1}$$

and obtain the recursion

$$q_{n+1} = \frac{1}{1 + q_n^t}, \quad q_0 = \frac{1}{2} \tag{2.8}$$

from (2.1). It is useful to split up the sequence $(q_n)$ into two complementary subsequences

$$\begin{aligned} (g_m) &: = (q_{2m}) = (q_0, q_2, \ldots), \\ (u_m) &: = (q_{2m+1}) = (q_1, q_3, \ldots). \end{aligned} \tag{2.9}$$

Lemma 1

The following inequalities hold for the subsequences $(g_m)$ and $(u_m)$ of $(q_n)$:

(i) $g_{m+1} > g_m$ for all $m = 0, 1, 2, 3, \ldots$

(ii) $u_{m+1} < u_m$ for all $m = 0, 1, 2, 3, \ldots$

(iii) $u_m > g_m$ for all $m = 0, 1, 2, 3, \ldots$ .

Proof: Let $q_{n-2} > q_n$; then $1 + q_{n-2}^t > 1 + q_n^t$, $1/(1 + q_{n-2}^t) < 1/(1 + q_n^t)$, and so

$$\frac{1}{1 + \left(\dfrac{1}{1 + q_{n-2}^t}\right)^t} > \frac{1}{1 + \left(\dfrac{1}{1 + q_n^t}\right)^t}.$$

Applying (2.8), we have proved:

$$\text{If } q_{n-2} > q_n, \text{ then } q_n > q_{n+2}. \tag{2.10}$$

Because of $g_1 > g_0$, (i) is proved by induction; (ii) and (iii) follow by a similar argument.

By Lemma 1, $(g_m)$ and $(u_m)$ are monotone sequences with the obvious bounds

$$\frac{1}{2} \leqslant g_m < u_m \leqslant 1. \tag{2.11}$$

So the sequences $(g_m)$ and $(u_m)$ must be convergent to limits $g$ and $u$ (depending on $t$). The following proposition shows that $g = u$ in the cases $t = 2, 3, 4$.

Proposition 1

For $t = 2, 3, 4$, the sequence $(q_n)$ is convergent to a limit $w(t)$, where $w(t)$ is the unique root of the equation $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} \leqslant w(t) \leqslant 1$.

Proof: By Lemma 1 we only have to show that $(g_m)$ and $(u_m)$ are convergent to the same limit. For $(g_m)$ and $(u_m)$ the following system of recursions holds:

$$\begin{aligned} u_m &= \frac{1}{1 + g_m^t} \\ g_{m+1} &= \frac{1}{1 + u_m^t}. \end{aligned} \tag{2.12}$$

Taking the limit $m \to \infty$, we obtain

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \text{ with } \frac{1}{2} \leqslant g \leqslant u \leqslant 1. \tag{2.13}$$

Let us start with the case $t = 2$. By (2.13), we have $ug^2 = 1 - u$, $gu^2 = 1 - g$, and therefore, $u - u^2 = g - g^2$. Because the function $x \mapsto x - x^2$ is strictly decreasing in the interval $\left[\frac{1}{2}, 1\right]$, $u = g$ follows immediately.

In the case $t = 3$, we derive in a similar way the relation $u^2 - u^3 = g^2 - g^3$. Since the function $x \mapsto x^2 - x^3$ is strictly decreasing in the interval $\left[\frac{2}{3}, 1\right]$ and $\frac{2}{3} < g_4 = 0,684\ldots$, we obtain $u = g$ again.

Since the function $x \mapsto x^3 - x^4$ is strictly decreasing in the interval $\left[\frac{3}{4}, 1\right]$ and $g_{73} = 0,7500138\ldots > \frac{3}{4}$, we obtain $u = g$ in the case $t = 4$, too.

So $u = g$ in all considered cases; therefore, a limit $w(t)$ of $(q_n)$ exists for $t = 2, 3, 4$, and $w(t)$ fulfills the equation

$$w = \frac{1}{1 + w^t}.$$

Since the function $f(w) = w^{t+1} + w - 1$ is strictly monotone in the interval $\left[\frac{1}{2}, 1\right]$ and $f\left(\frac{1}{2}\right) < 0$, $f(1) > 0$, there exists a unique root of this equation in the interval $\left[\frac{1}{2}, 1\right]$, which is the limit $w(t)$ from above.

By (2.2) we derive

$$\lim_{n\to\infty} \alpha_n = \log(1 + w(t)^t). \tag{2.14}$$

Because of

$$\left| r_n - \frac{1}{t-1} \log(1 + w(t)^t) \right| \leqslant \sum_{i=n}^{\infty} t^{n-1-i} |\alpha_i - \log(1 + w(t)^t)| < \frac{\varepsilon}{t-1}$$

[for all $\varepsilon > 0$, $n \geqslant n_0(\varepsilon)$], the sequence $(r_n)$ is convergent; so

$$\lim_{n\to\infty} e^{-r_n} = (1 + w(t)^t)^{-1/(t-1)} = w(t)^{1/(t-1)} = (1 - w(t))^{1/(t^2-1)} \tag{2.15}$$

results. Altogether we have established:

Theorem 1

Let $T_n(t)$ be the full $t$-ary tree ($t = 2, 3$, or $4$) with height $n$. Then, the Fibonacci number $f(T_n(t))$ fulfills the following asymptotic formula:

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \qquad (n \to \infty)$$

where $A(t) = w(t)^{1/(t-1)}$ and $k(t)$, defined by (2.7), are constants (only depending on $t$) bounded by

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)};$$

$w(t)$ is the unique root of $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} < w(t) < 1$.

Remark: The numerical values of $w(t)$ are

$w(2) = 0, 68233\ldots$, $w(3) = 0, 72449\ldots$, and $w(4) = 0, 75488\ldots$ .

## 3. FIBONACCI NUMBERS OF $t$-ARY TREES ($t \geqslant 5$)

In this section we consider $t$-ary trees with $t \geqslant 5$. Let $(g_m)$, $(u_m)$ be the subsequences of $(q_n)$ defined by (2.9). $(g_m)$ and $(u_m)$ are convergent to limits $g$ and $u$, respectively (depending on $t$). We shall prove that $g \neq u$; therefore, $(q_n)$ has two accumulation points. For $g$, $u$ the following system of equations holds,

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \tag{3.1}$$

and $g = u$ if and only if $u$ or $g$ is the unique solution of

$$w^{t+1} + w - 1 = 0 \tag{3.2}$$

in the interval $\left[\frac{1}{2}, 1\right]$. If $(u', g')$ and $(u'', g'')$ are two pairs fulfilling (3.1) with $u' < u''$, then $g' > g''$. Let $(\bar{u}, \bar{g})$ denote the pair of solutions with minimal $g$ and maximal $u$.

Lemma 2

The subsequence $(g_m)$ of $(q_n)$ is convergent to the limit $\bar{g}$ and the subsequence $(u_m)$ to the limit $\bar{u}$.

Proof: First we show that $g_m < \bar{g}$ implies $u_m > \bar{u}$ and $g_{m+1} < \bar{g}$.

Because of $g_m < \bar{g}$, we obtain $1 + g_m^t < 1 + \bar{g}^t = 1/\bar{u}$, and so $u_m > \bar{u}$. From $u_m > \bar{u}$, it follows that $1 + u_m^t > 1 + \bar{u}^t = 1/\bar{g}$, hence $g_{m+1} < \bar{g}$.

Using the fact that $g_0 = \frac{1}{2} < \bar{g}$, we obtain, by induction,

$$\lim_{m \to \infty} g_m \leqslant \bar{g} \quad \text{and} \quad \lim_{m \to \infty} u_m \geqslant \bar{u}.$$

By the definition of $(\bar{u}, \bar{g})$, it follows that

$$\lim_{m \to \infty} g_m = \bar{g} \quad \text{and} \quad \lim_{m \to \infty} u_m = \bar{u},$$

and the Lemma is proved.

Lemma 3

Let $t \geqslant 5$ be a positive integer; then there exists a solution $(u, g)$ of the system (3.1) with

$$\frac{1}{2} < g < \frac{1}{2} + \frac{1}{t}.$$

Proof: System (3.1) is equivalent to the equation

$$g = \cfrac{1}{1 + \cfrac{1}{(1 + g^t)^t}} \tag{3.3}$$

We consider the function

$$\varphi_t(g) = \frac{(1 + g^t)^t}{1 + (1 + g^t)^t} - g,$$

and obtain $\varphi_t\left(\frac{1}{2}\right) > 0$; in the sequel, we show $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$. For $t = 5$ or $6$, this inequality can be shown by direct computation:

$$\varphi_5\left(\frac{7}{10}\right) = -0,\ 01502 \quad \text{and} \quad \varphi_6\left(\frac{2}{3}\right) = -0,\ 04306.$$

Let us assume $t \geqslant 7$ in the sequel. By elementary manipulations, the inequality

$$\frac{1}{1 + \frac{1}{\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t}} - \frac{1}{2} - \frac{1}{t} < 0$$

is equivalent to $1 + \left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^{-t} > \frac{2t}{t + 2}$ or

$$\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t < \frac{t + 2}{t - 2}. \tag{3.4}$$

Because of $\left(1 + \frac{2}{t}\right)^{t/2} < e$, it is sufficient to prove

$$\left(1 + \frac{e^2}{2^t}\right)^t < \frac{t + 2}{t - 2}: \tag{3.5}$$

We have

$$\left(1 + \frac{e^2}{2^t}\right)^t \leqslant \exp\left(\frac{e^{2t}}{2^t}\right)$$

and

$$\exp\left(\frac{e^{2t}}{2^t}\right) \leqslant \frac{t + 2}{t - 2} \text{ for } t \geqslant 7.$$

So $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$, and the Lemma is proved, because the continuous function $\varphi$ has a root between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{t}$.

Equation (3.3) is equivalent to

$$g(g^t + 1)^t - (g^t + 1)^t + g = 0. \tag{3.6}$$

The polynomial on the left-hand side of (3.6) is divisible by $g^{t+1} + g - 1$. Because of $\left(\frac{3}{4}\right)^{t+1} + \frac{3}{4} - 1 < 0$ (for $t \geqslant 5$), the unique solution $w(t)$ of $g^{t+1} + g - 1 = 0$ is contained in the interval $\left[\frac{3}{4}, 1\right]$. By Lemma 3, we have found a pair of solutions $(u, g)$ with $u \neq g$ such that $\frac{1}{2} < g < \frac{3}{4} < w(t) < u < 1$. We denote by $(u(t), g(t))$, $t \geqslant 5$, the pair of solutions of (3.1) such that $g(t)$ is minimal and $u(t)$ is maximal. Because of $g(t) < \frac{1}{2} + \frac{1}{t}$ and $u(t) = 1 + g(t)^{-t}$ for $t \geqslant 5$, we obtain

$$\lim_{t \to \infty} g(t) = \frac{1}{2}, \quad \lim_{t \to \infty} u(t) = 1. \tag{3.7}$$

Altogether, we have proved:

Theorem 2

Let $T_n(t)$ be the full $t$-ary tree (for $t \geqslant 5$) with height $n$. Then the Fibonacci numbers fulfill the following asymptotic formulas, respectively:

$$f(T_{2m}(t)) \sim B(t) \cdot k(t)^{t^{2m}},$$

$$f(T_{2m+1}(t)) \sim C(t) \cdot k(t)^{t^{2m+1}},$$

where

$$C(t) = (g(t)^t u(t))^{1/(t^2-1)} = (1 - u(t))^{1/(t^2-1)},$$

$$B(t) = (g(t)u(t)^t)^{1/(t^2-1)} = (1 - g(t))^{1/(t^2-1)},$$

and $k(t)$, defined by (2.7), are constants (only depending on $t$) bounded by

$$2^{1/(1-t)} < C(t) < B(t) < 1 < k(t) < 2^{1/(t-1)};$$

$g(t)$ is the minimal root and $u(t)$ the maximal root of

$$x(x^t + 1)^t - (x^t + 1)^t + x = 0$$

in the interval $\left[\frac{1}{2}, 1\right]$; furthermore,

$$\lim_{t\to\infty} B(t) = \lim_{t\to\infty} C(t) = 1.$$

Remark: In [2], similar recurrences are treated by a slightly different method. The recursion for $(q_n)$ can be considered as a fixed-point problem and our results can be derived in principal by studying this fixed-point problem.

## 4. THE AVERAGE FIBONACCI NUMBER OF BINARY TREES

The family $\beta$ of all binary trees is defined by the following formal equation ($\square$ is the sumbol for a leaf and $\circ$ for an internal node):

$$\beta = \square + \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \beta \qquad \beta \end{array}; \tag{4.1}$$

(this notation is due to Ph. Flajolet [3]). The generating function

$$B(z) = \sum_{n \geqslant 0} b_n z^n$$

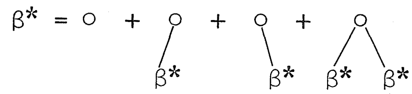
of the numbers of binary trees with $n$ internal nodes is given by

$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \tag{4.2}$$

and, therefore,

$$b_n = \frac{1}{n+1}\binom{2n}{n}. \tag{4.3}$$

For technical reasons, we consider the family $\beta^*$ of all binary trees with leaves removed; $\beta^*$ fulfills


(4.4)

Let $\beta_n$ be the family of binary trees $t$ with $n$ internal nodes, and let

$$f(z) = \sum_{n \geqslant 1} f_n z^n \quad \text{and} \quad g(z) = \sum_{n \geqslant 1} g_n z^n$$

be generating functions of

$$\begin{aligned} f_n &= \sum_{T \in \beta_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset } \textit{not} \text{ containing the root}\}, \\ g_n &= \sum_{T \in \beta_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset containing the root}\}. \end{aligned} \tag{4.5}$$

Obviously, the average value of the Fibonacci number of a binary tree with $n$ internal nodes is given by

$$S_n = \frac{h_n}{b_n} \quad \text{with} \quad h_n = f_n + g_n. \tag{4.6}$$

The remainder of this paper is devoted to the asymptotic evaluation of $S_n$. By Stirling's approximation of the factorials, the well-known formula

$$b_n \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-3/2} \qquad (n \to \infty) \tag{4.7}$$

holds and we can restrict our attention to $h_n$.

For the generating functions, we obtain

$$\begin{aligned} f &= z + z(f + g) + z(f + g) + z(f + g)^2 \\ g &= z + zf + zf + zf^2. \end{aligned} \tag{4.8}$$

[The contributions of (4.8) correspond to the terms in (4.4).] Setting

$$y(z) = 1 + f(z) + g(z),$$

we derive, by some elementary manipulations,

$$z^3y^4 + (2z^2 + z)y^2 - y + (z + 1) = 0. \tag{4.9}$$

Now we want to apply Theorem 5 of [1]; for this purpose, we have to determine the singularity $\rho$ of $y(z)$ nearest to the origin. (4.9) is an implicit representation of $y(z)$. Abbreviating the left-hand side of (4.9) by $F(z, y)$, the singularity $\rho$ (nearest to the origin) and $\sigma = y(\rho)$ are given as solutions of the following system of algebraic equations:

$$\begin{aligned} F(z, y) &= 0, \\ \frac{\partial F}{\partial y}(z, y) &= 0. \end{aligned} \tag{4.10}$$

Now $\rho$ and $\sigma$ are simple roots of the above equations. By a theorem of Pringsheim [7, p. 389], $\rho$ and $\sigma$ are positive (real) numbers. Using the two-dimensional version of Newton's algorithm (starting with $z_0 = 0, 2$ and $y_0 = 1$), we obtain the following numerical values:

$$\rho = 0,\ 15268\ldots \quad \text{and} \quad \sigma = 2,\ 15254\ldots\ . \tag{4.11}$$

Now Theorem 5 of [1] allows us to formulate the following:

<u>Proposition 2</u>

$$\begin{aligned} h_n &\sim \left(\frac{\rho \cdot F_z(\rho, \sigma)}{2\pi \cdot F_{yy}(\rho, \sigma)}\right)^{1/2} \cdot \rho^{-n} \cdot n^{-3/2}, \\ &\sim (0,\ 63713\ldots) \quad (0,\ 15268\ldots)^{-n} \cdot n^{-3/2}. \end{aligned} \qquad (n \to \infty) \tag{4.12}$$

Altogether, we have proved:

<u>Theorem 3</u>

The average value $S_n$ of the Fibonacci number of a binary tree with $n$ internal nodes fulfills asymptotically

$$S_n \sim G \cdot r^n \qquad (n \to \infty),$$

where $G = 1,\ 12928\ldots$ and $r = 1,\ 63742\ldots$ are numerical constants.

## ACKNOWLEDGMENT

We would like to express our thanks for helpful discussions to Professor G. Baron and Dr. Ch. Buchta.

## REFERENCES

1. E. A. Bender. "Asymptotic Methods in Enumeration." *SIAM Review* 16 (1974): 485-515.
2. N. G. de Bruijn. *Asymptotic Methods in Analysis*. Amsterdam: North Holland, 1958.
3. Ph. Flajolet. "Analyse d'algorithmes de manipulation d'arbres et de fichiers." Thèse Paris-Sud-Orsay, 1979.
4. D. H. Greene & D. E. Knuth. *Mathematics for the Analysis of Algorithms*. Basel: Birkhäuser Verlag, 1981.
5. D. E. Knuth. *The Art of Computer Programming: Fundamental Algorithms*. I. New York: Addison Wesley, 1968.
6. C. L. Liu. *Introduction to Combinatorial Mathematics*. New York, 1968.

7. A. I. Markushevich. *Theory of Functions of a Complex Variable.* Vol. I. Englewood Cliffs, N.J.: Prentice-Hall, 1965.
8. A. Meir & J. W. Moon. "On the Altitude of Nodes in Random Trees." *Canad. J. of Math.* 30 (1978):997-1015.
9. H. Prodinger & R. F. Tichy. "Fibonacci Numbers of Graphs." *The Fibonacci Quarterly* 20 (1982):16-21.

◆◇◆◇◆

# CORRIGENDA TO "SOME SEQUENCES LIKE FIBONACCI'S"

B. H. NEUMANN and L. G. WILSON
*The Fibonacci Quarterly, Vol. 17, No. 1, 1979, pp. 80-83*

The following changes should be made in the above article. These errors are the responsibility of the editorial staff and were recently brought to the editor's attention by the authors.

p. 80, at the end of formula (1), add superscript "$n$".
p. 81, in formula (7), replace the second "$y$" by "$t$".
p. 82, in the line following (8), add subscript "$d$" to the last "$a$".
p. 82, in the line following (10), add subscript "$d$" to the last "$\alpha$".
p. 83, line 3, insert "growth" between "slower" and "rate".
p. 83, end of text and reference, delete "t" from the name "Johnson".

Gerald E. Bergum

◆◇◆◇◆

◆◇◆◇◆

# ELEMENTARY PROBLEMS AND SOLUTIONS

*Edited by*
A. P. HILLMAN and G. C. PADILLA

*Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be submitted on a separate signed sheet, or sheets. Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.*

## DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha$ and $\beta$ designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

## PROBLEMS PROPOSED IN THIS ISSUE

B-502 *Proposed by Herta T. Freitag, Roanoke, VA*

Given that $h$ and $k$ are integers with $h + k$ an integral multiple of 3, prove that $F_k F_{k-h-1} + F_{k+1} F_{k-h}$ is even.

B-503 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that every even perfect number except 28 is congruent to 1 or -1 modulo 7.

B-504 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that if $n$ is an odd integer and $F_n$ is in the set

$$\{0, 1, 3, 6, 10, \ldots\}$$

of triangular numbers, then $n \equiv \pm 1 \pmod{24}$.

B-505 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$N = N(m, a) = L_{m-2a} L_m - L_{m+1-2a} L_{m-1},$$

where $m$ and $a$ are positive integers. Prove or disprove that $N$ is always (exactly) divisible by 5; never divisible by 3, 4, 6, 7, 8, 9, or 11; and is divisible by 10 only if $a \equiv 2 \pmod 3$.

B-506 *Proposed by Heinz-Jürgen Sieffert, student, Berlin, Germany*

Let $G_n = (n+1)F_n$ and $H_n = (n+1)L_n$. Prove that:

(a) $$\sum_{k=0}^{n} G_k G_{n-k} = \frac{(n+2)(n+3)}{30} H_n - \frac{2}{25} H_{n+2} + \frac{4}{25} F_{n+3};$$

(b) $$\sum_{k=0}^{n} H_k H_{n-k} = \frac{(n+2)(n+3)}{6} H_n + \frac{2}{5} H_{n+2} - \frac{4}{5} F_{n+3}.$$

B-507 *Proposed by Heinz-Jürgen Sieffert, Berlin, Germany*

Let $G_n$ and $H_n$ be as in B-506. Find a formula for $\sum_{k=0}^{n} G_k H_{n-k}$ similar to the formulas in B-506.

## SOLUTIONS

### Fibonacci Norm Identity

B-478 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

(a) Show that the congruence

$$x^2 \equiv -1 \pmod{4m^2 + 4m + 5}$$

has $x = \pm(2m^2 + m + 2)$ as a solution for $m$ in $N = \{0, 1, \ldots\}$.

(b) Show that the congruence

$$x^2 \equiv -1 \pmod{100m^2 + 156m + 61}$$

has a solution $x = am^2 + bm + c$ with fixed integers $a$, $b$, $c$ for $m$ in $N$.

*Solution by Paul S. Bruckman, Carmichael, CA*

The identities

(1) $$(2m^2 + m + 2)^2 + 1 = (m^2 + 1)(4m^2 + 4m + 5), \text{ and}$$

(2) $$(50m^2 + 53m + 11)^2 + 1 = (25m^2 + 14m + 2)(100m^2 + 156m + 61)$$

are particular instances of the more general identity (due to Fibonacci himself!)

(3) $$(pq - rs)^2 + (ps + qr)^2 = (p^2 + r^2)(q^2 + s^2).$$

Setting $p = 2m + 1$, $q = 1$, $r = 2$, and $s = m$ in (3) yields (1). Setting

$p = 3m + 1$, $q = 8m + 6$, $r = 4m + 1$, and $s = 6m + 5$ in (3) yields (2). This establishes parts (a) and (b) of the problem; in part (b), we have $a = 50$, $b = 53$, $c = 11$.

*Also solved by Herta T. Freitag, L. Kuipers, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, and the proposer.*

## Divisibility from a Lucas Sum

B-479 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a$ is an integral multiple of $L_d$ for positive integers $a$, $d$, and $n$ with $d$ odd.

*Solution by J. Suck, Essen, Germany*

For positive integers $a$, $d$, and $n$ with $d$ odd, we have

$$L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a = (L_{a+d} + L_{a+2d} + \cdots + L_{a+(n-1)d})L_d.$$

Proof by induction on $n$: For $n = 1$, both sides equal 0 (the empty sum on the right-hand side). For the step $n \to n + 1$, we have to show that

$$L_{a+(n+1)d} = L_{a+(n-1)d} + L_{a+nd}L_d.$$

This is clear from the identities $I_8$ and $I_{23}$ of Hoggatt's list; namely,

$$L_k = F_{k-1} + F_{k+1} \quad \text{and} \quad F_{k+p} - F_{k-p} = F_kL_p \text{ for } p \text{ odd.}$$

Thus,

$$\begin{aligned} L_{a+(n-1)d} + L_{a+nd}L_d &= F_{a+(n-1)d-1} + F_{a+(n-1)d+1} + F_{a+(n+1)d-1} \\ &\quad - F_{a+(n-1)d-1} + F_{a+(n+1)d+1} - F_{a+(n-1)d+1} \\ &= L_{a+(n+1)d}. \end{aligned}$$

*Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.*

## Even Case

B-480 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a$ is an integral multiple of $L_d - 2$ for positive integers $a$, $d$, and $n$ with $d$ even.

*Solution by Sahib Singh, Clarion State College, Clarion, PA*

This result is true. The proof is based on applying induction on $n$. The result is obvious when $n = 1$. For $n = 2$, with $d$ even, it is easy to verify that

$$L_{a+2d} - 2L_{a+d} + L_a = (L_d - 2)L_{a+d}.$$

Using the pattern for $n = 3$, we assume the validity of the result:

$$L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a = (L_{a+d} + L_{a+2d} + \cdots + L_{a+(n-1)d})(L_d - 2).$$

Also, with $d$ even, we have

$$L_{a+(n+1)d} - 2L_{a+nd} + L_{a+(n-1)d} = L_{a+nd}(L_d - 2).$$

By addition, we get the confirmation that the result is true for $(n + 1)$. Thus, the proof is complete.

*Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, J. Suck, and the proposer.*

## Matching Pennies

B-481 *Proposed by Jerry Metzger, Univ. of North Dakota, Grand Forks, ND*

A and B compare pennies with A winning when there is a match. During an unusual sequence of $mn$ comparisons, A produced $m$ heads followed by $m$ tails followed by $m$ heads, etc., while B produced $n$ heads followed by $n$ tails followed by $n$ heads, etc. By how much did A's wins exceed his losses? (For example, with $m = 3$ and $n = 5$, one has

A: HHHTTTHHHTTTHHH
B: HHHHHTTTTTHHHHH

and A's 8 wins exceed his 7 losses by 1.)

*Solution by J. Suck, Essen, Germany*

Let $d = \gcd(m, n)$. The excess is 0 if $m/d$ or $n/d$ is even, and $d^2$ if both are odd.

Proof: The whole double sequence can be split into $d$ blocks of length $mn/d = \operatorname{lcm}(m, n)$. If $m/d$ or $n/d$ is even, then the first block ends in a non-match. Let there be $M$ matches and $N$ non-matches in the first block. Interchanging heads and tails in the row which ends with a T means writing down the block in reverse order, leaving the number of matches unaffected. But matches have become non-matches and vice versa, so that $M = N$. Interchanging heads and tails in both rows of the second block (if $d > 1$) produces the first block in reverse order, and so, again, there is no excess of matches over non-matches. The third block is equal to the first, etc.

If $m/d$ and $n/d$ are odd, odd-numbered blocks are identical, even-numbered blocks equal the first block when heads and tails are interchanged. Now, we may assume that $m$ and $n$ are relatively prime: replacing each symbol by a run of $d$ of the same sort produces the first block of the general case. Furthermore, assume $m < n$.

Insert a bar after every $n$ symbols of the double sequence. Let $a_\mu$ resp. $-a_\mu$ ($b_\mu$ resp. $-b_\mu$) be the length of the string of successive matches resp.

non-matches preceding (following) the $\mu$th bar, $\mu = 1, \ldots, m - 1$. For example, with $m = 3$ and $n = 5$, one has $a_1 = -2$, $b_1 = 1$, $a_2 = 1$, and $b_2 = -2$. Since $d = 1$, $0 < |a_\mu|, |b_\mu| < m$. We have $|a_\mu| + |b_\mu| = m$, and so the excess of matches over non-matches is

$$e := \sum_{\nu=1}^{n-(m-1)} (-1)^{\nu+1} m + \sum_{\mu=1}^{m-1} a_\mu + \sum_{\mu=1}^{m-1} b_\mu.$$

Now, $|a_1|$ won't recur among $|a_2|, \ldots, |a_{m-1}|$ since $d = 1$. Also, $|b_1|$ won't recur among $|b_2|, \ldots, |b_{m-1}|$ since, otherwise, $|a_1|$ would have had to recur, etc. Thus, $|a_1|, \ldots, |a_{m-1}|$ and $|b_1|, \ldots, |b_{m-1}|$ are permutations of $1, \ldots, m - 1$. Setting $b_0 := 0$, let $q_\mu$ be defined by

$$n - |b_{\mu-1}| = q_\mu m + |a_\mu|, \quad \mu = 1, \ldots, m - 1.$$

It is clear that

$$\text{if } b_{\mu-1} \text{ is } \begin{cases} \text{even}, < 0 \\ \text{even}, < 0 \\ \text{odd}, > 0 \\ \text{odd}, > 0 \end{cases} \text{ and } q_\mu \text{ is } \begin{cases} \text{even} \\ \text{odd} \\ \text{even} \\ \text{odd} \end{cases}, \text{ then } a_\mu \text{ is } \begin{cases} \text{odd}, > 0 \\ \text{even}, < 0 \\ \text{even}, < 0 \\ \text{odd}, > 0 \end{cases}$$

and that

$$\text{if } a_\mu \text{ is } \begin{cases} \text{even}, < 0 \\ \text{odd}, > 0 \end{cases}, \text{ then } b_\mu \text{ is } \begin{cases} \text{odd}, > 0 \\ \text{even}, < 0 \end{cases}.$$

This implies that $e = m + (1 - 2 + 3 - \cdots - (m-1))2 = 1$, which had to be shown.

*Also solved by Paul S. Bruckman and the proposer.*

## Distinct Limits

**B-482** *Proposed by John Hughes and Jeff Shallit, Univ. of California, Berkeley, CA*

Find an infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \quad \text{and} \quad \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right]$$

both exist but are unequal.

In the following tabulation of the solutions,

$$L = \lim_{n\to\infty}(a_n)^{1/n} \quad \text{and} \quad L' = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right].$$

| SOLVER: | $a_{2m-1}$ | $a_{2m}$ | $L$ | $L'$ |
|---|---|---|---|---|
| *Paul S. Bruckman* *Carmichael, CA* | 1 | 3 | 1 | 5/3 |
| *Walther Janous* *Univ. Innsbruck, Austria* | 1 | 2 | 1 | 5/4 |
| *L. Kuipers,* *Sierre, Switzerland* | 2 | 4 | 1 | 5/4 |
| *J. Suck* *Essen, Germany* | $a/a^{1/(2m-1)}$ | $a \cdot a^{1/(2m)}$ | $a$ | $\frac{1}{2}\left(a^3 + \frac{1}{a}\right)$ |
| *S. Uchiyama* *Univ. of Tsukuba, Japan* | $2^{m-1}$ | $2^m$ | $\sqrt{2}$ | 3/2 |
| *Proposers* *Hughes & Shallit* | Same as that of S. Uchiyama | | | |

## Limit, No Limit

B-483 *Proposed by John Hughes and Jeff Shallit*
*Univ. of California, Berkeley, CA*

Find an infinite sequence $a_1$, $a_2$, ... of positive integers such that

$$\lim_{n\to\infty}(a_n)^{1/n} \text{ exists and } \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right] \text{ does not exist.}$$

*Solution by Walther Janous, Universitaet Innsbruck, Innsbruck, Austria*

Let $a_{2m-1} = 1$ and $a_{2m} = m$. Then $\lim(a_n)^{1/n} = 1$. Also

$$\{a_{n+1}/a_n\} = 1,\ 1,\ 2,\ 1/2,\ 3,\ 1/3,\ 4,\ 1/4,\ \ldots,$$

and thus

$$\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k) > (1 + 2 + 3 + \cdots + [n/2])/n \to \infty \text{ as } n \to \infty.$$

*Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, S. Uchiyama, and the proposers.*

◆◇◆◇◆

◆◇◆◇◆

# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
RAYMOND E. WHITNEY
*Lock Haven State College, Lock Haven, PA 17745*

*Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or any other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

H-356 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Consider a set of $r$ types of letter with $n_i$ occurrences of letter $i$. How many words can we form, using some or all of these letters?

If we use $k_i$ of letter $i$, then there are obviously $\binom{\Sigma k_i}{k_1, \ldots, k_r}$ ways to form a word, and the desired number is $\sum_{k_i \leqslant n_i} \binom{\Sigma k_i}{k_1, \ldots, k_r}$. When $r = 2$, this can be readily evaluated using properties of Pascal's triangle and we get $\binom{n_1 + n_2 + 2}{n_1 + 1} - 1$. W. O. J. Moser has found a nice combinatorial derivation of this result, but neither approach works for $r > 2$.

Moser's solution for $r = 2$ is as follows: In the case $r = 2$,

$$\sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \binom{i + j}{i} \tag{**}$$

is the number of ways of forming words with some of $m$ A's and $n$ B's. Any such word with $i$ A's and $j$ B's can be extended to a word of $m + 1$ A's and $n + 1$ B's by appending $m + 1 - i$ A's and $n + 1 - j$ B's to it. If our original word begins with an A, we append a block of $m + 1 - i$ A's followed by a block of $n + 1 - j$ B's at the beginning. If the original word begins with a B, we append the block of B's followed by the block of A's at the beginning. The empty word can be extended in two ways: AA ... ABB ... A or BB ... BAA ... A. Otherwise, we have a one-to one correspondence between our original words and words formed from all of $m + 1$ A's and $n + 1$ B's. The reverse correspondence is to take any word of $m + 1$ A's and $n + 1$ B's and delete its first two blocks (i.e., constant subintervals). Since the empty word arises from two extended words, we have $\binom{m+n+2}{m+1} - 1$ of our original words.

As an illustration, let $m = n = 1$.

| Original Word | Extended Word |
|---|---|
| - | AABB or BBAA |
| A | ABBA |
| B | BAAB |
| AB | ABAB |

H-357 *Proposed by Clark Kimberling, Univ. of Evansville, Evansville, IN*

For any positive integer $N$, arrange the fractional parts of the first $N$ integral multiples of $\alpha = (1 + \sqrt{5})/2$ in increasing order:

$$\{k_1\alpha\} < \{k_2\alpha\} < \cdots < \{k_N\alpha\}.$$

Is $k_n + k_{N+1-n}$ a sum of two Fibonacci numbers for $n = 1, 2, 3, \ldots, N$?

I have not been able to prove that $k_n + k_{N+1-n}$ is always a sum of two Fibonacci numbers. However, a computer has verified that it is so for $N = 1, 2, \ldots, 666$.

The following table may be helpful:

| $N$ | $\alpha N$ | $\{\alpha N\}$ | $k_1, k_2, \ldots, k_N$ | $k_1 + k_N, k_2 + k_{N-2}, \ldots, k_N + k_1$ |
|---|---|---|---|---|
| 1 | 1.618 | .618 | 1 | 2 |
| 2 | 3.236 | .236 | 2 1 | 3 3 |
| 3 | 4.854 | .854 | 2 1 3 | 5 2 5 |
| 4 | 6.472 | .472 | 2 4 1 3 | 5 5 5 5 |
| 5 | 8.090 | .090 | 5 2 4 1 3 | 8 3 8 3 8 |
| 6 | 9.708 | .708 | 5 2 4 1 6 3 | 8 8 5 5 8 8 |
| 7 | 11.326 | .326 | 5 2 7 4 1 6 3 | 8 8 8 8 8 8 8 |
| 8 | 12.944 | .944 | 5 2 7 4 1 6 3 8 | 13 5 13 5 5 13 5 13 |
| 9 | 13.562 | .562 | 5 2 7 4 9 1 6 3 8 | 13 5 13 5 18 5 13 5 13 |

As you see, all numbers in the fifth column are sums of two Fibonacci numbers. For $N = 662$, for example, there are six (and only six) different numbers $k_n + k_{N+1-n}$ as $n$ ranges from 1 to 662; they are:

$$\begin{aligned}
144 &= 89 + 55\\
377 &= 233 + 144\\
521 &= 377 + 144\\
754 &= 377 + 377\\
987 &= 610 + 377\\
1131 &= 987 + 144
\end{aligned}$$

H-358 *Proposed by Andreas N. Philippou, Univ. of Patras, Patras, Greece*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{n_1,\ldots,n_k} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1}, \quad n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$. Show that

$$\sum_{n=0}^{\infty} (f_{n+1,r}^{(k)}/2^n) = 2^{rk}.$$

You may note that the present problem reduces to H-322(c) for $r = 1$ (and $k \geqslant 2$), because of Theorem 2.1 of Philippou and Muwafi [1]. In addition, the present problem includes as special cases [for $k = 1$, $r = 1$, and $k = 1$, $r(\geqslant 1)$] the following infinite sums; namely,

$$\sum_{n=0}^{\infty} (1/2^n) = 2 \quad \text{and} \quad \sum_{n=0}^{\infty} \left[\binom{n + r - 1}{n}/2^n\right] = 2^r.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the $k$th Consecutive Success and the Fibonacci Sequence of Order $K$." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

H-359 *Proposed by Paul S. Bruckman, Carmichael, CA*

Define the "Zetanacci" numbers $Z(n)$ as follows:

$$Z(n) = \prod_{p^e \| n} F_{e+1}, \quad n = 1, 2, 3, \ldots \text{ [with } Z(1) = 1]. \tag{1}$$

For example, $Z(n) = 1$, $n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, \ldots$; $Z(n) = 2$, $n = 4, 9, 12, 18, 20, \ldots$; $Z(8) = 3$, $Z(16) = 5$, $Z(135{,}000) = Z(2^3 3^3 5^4) = 45$, etc.

(A) Show that the (Dirichlet) generating function of the Zetanacci numbers is given by:

$$\sum_{n=1}^{\infty} Z(n)n^{-s} = \prod_{p} (1 - p^{-s} - p^{-2s})^{-1}, \tag{2}$$

the product taken over all primes.

(B) Show that

$$\prod_{p} (1 - p^{-s} - p^{-2s}) = \sum_{n=1}^{\infty} \mu(P(n)) \cdot |\mu(n/P(n))| \cdot n^{-s},$$

where $\mu$ is the Möbius function and

$$P(n) = \prod_{p|n} p \text{ [with } P(1) = 1].$$

## SOLUTIONS

### Rational Thirds

H-339 *Proposed by Charles R. Wall, Trident Technical College, Charleston, CA (Vol. 20, No. 2, May 1982)*

A dyadic rational is a proper fraction whose denominator is a power of 2. Prove that 1/4 and 3/4 are the *only* dyadic rationals in the classical Cantor ternary set of numbers representable in base three using only 0 and 2 as digits.

*Solution by the proposer*

Clearly $1/2 = .\overline{1}$ (base three) is not in the set, but $1/4 = .\overline{02}$ and $3/4 = .\overline{20}$ are. The other cases require a lemma:

If $k \geqslant 3$ and $0 \leqslant a < 2^{k-2}$, the numbers $\pm 3^a$ are distinct modulo $2^k$.

This assertion is true for $k = 3$ by observation: $3^0 \equiv 1$, $-3^0 \equiv 7$, $3^1 \equiv 3$, and $-3^1 \equiv 5$ (all mod 8). Thus, we may assume $k \geqslant 4$. That the numbers $3^a$ are distinct (mod $2^k$) rests on the congruence

$$3^{2^{k-3}} \equiv 1 + 2^{k-1} \pmod{2^k},$$

which is easily proved by induction for $k \geqslant 4$, and its corollary

$$3^{2^{k-2}} \equiv 1 \pmod{2^k}.$$

To show that the numbers $3^a$ are distinct from their negatives, note that $3^x \equiv (-1)^x \pmod 4$. If $k \geqslant 4$ and $0 \leqslant b < a < 2^{k-2}$ and $3^a \equiv -3^b \pmod{2^k}$, then $3^{a-b} \equiv -1 \pmod{2^k}$, so $a - b$ is odd. Then $3^{2(a-b)} \equiv 1 \pmod{2^k}$, so $2^{k-2}$ divides $2(a-b)$, and thus $2^{k-3}$ divides the odd number $a - b$, which is impossible if $k \quad 4$.

Let $f(t)$ be the fractional part of $t$: $f(t) = t - [t]$, where the brackets denote the greatest integer function. For $k \geqslant 3$, by the lemma, each dyadic rational with denominator $2^k$ can be written uniquely as $f(\pm 3^a/2^k)$, $0 \leqslant a < 2^{k-2}$. If a fraction $x = f(\pm 3^a/2^k)$ is in the Cantor set, so (by shifting the ternary point) is $f(3x) = f(\pm 3^{a+1}/2^k)$, and so is the 2's complement $1 - x = f(\mp 3^a/2^k)$. Thus, if any dyadic rational $x = f(\pm 3^a/2^k)$ is in the set, all such fractions with the same denominator are. However, the two fractions closest to 1/2 are forbidden, so all are.

*Also solved by P. Bruckman.*

### Making a Difference

H-340 *Proposed by Verner E. Hoggatt, Jr. (deceased) (Vol. 20, No. 2, May 1982)*

Let $A_2 = B$, $A_4 = C$, and $A_{2n+4} = A_{2n} - A_{2n+2}$ $(n = 1, 2, 3, \ldots)$. Show:

a. $A_{2n} = (-1)^{n+1}(F_{n-2}B - F_{n-1}C)$

b. If $A_{2n} > 0$ for all $n > 0$, then $B/C = (1 + \sqrt{5})/2$

c. $A_{2n} = C^{n-1}/B^{n-2}$

*Solution by Paul Bruckman, Carmichael, CA*

For all $n \geqslant 1$, let

$$G_n = A_{2n}. \tag{1}$$

The given recursion is then transformed to the following recursion:

$$G_{n+2} + G_{n+1} - G_n = 0,\ n = 1, 2, 3, \ldots, \tag{2}$$

with initial conditions

$$G_1 = B,\ G_2 = C. \tag{3}$$

The characteristic polynomial $p(z)$ of (2) is given by

$$p(z) = z^2 + z - 1 = (z + \alpha)(z + \beta), \tag{4}$$

where $\alpha$ and $\beta$ are the usual Fibonacci constants. Hence, there exist constants $p$ and $q$ such that, for all $n$,

$$G_n = p(-\alpha)^n + q(-\beta)^n. \tag{5}$$

We find $p$ and $q$ by setting $n = 1$ and $n = 2$ in (5) and using (3). After simplification, we find the following expression (which is readily verifiable):

$$G_n = (-1)^{n+1}(F_{n-2}B - F_{n-1}C),\ n = 1, 2, 3, \ldots . \tag{6}$$

Note that the expression in (6) is of the same form as given in (5), and moreover satisfies (3). Hence, $A_{2n}$ is given by (6).

Thus,

$$G_{2n} = F_{2n-1}C - F_{2n-2}B \text{ for } n \geqslant 1$$

and

$$G_{2n+1} = F_{2n-1}B - F_{2n}C \quad \text{for } n \geqslant 0.$$

Since $G_n > 0$ for all $n > 0$, we have $B > C > 0$ and

$$F_{2n}/F_{2n-1} < B/C < F_{2n-1}/F_{2n-2},\ n = 2, 3, 4, \ldots . \tag{7}$$

Taking limits in (7) as $n \to \infty$, each extreme expression approaches $\alpha$, which implies $B/C = \alpha$. Q.E.D.

*Also solved by H. Freitag, C. Georghiou, W. Janous, G. Lord, A. Shannon, and the proposer.*

Late Acknowledgment: *G. Wulczyn solved* H-332.

◆◇◆◇◆

## SUSTAINING MEMBERS

*Alder, H.L.
*Arkin, J.
Arthur, B.I., Jr.
Bankoff, Leon
Berg, Murray
Bergart, J.G.
Bergum, G.E.
Berzsenyi George
*Bicknell-Johnson M.
Bridger, Clyde
Bruckman, P.S.
Byrd, P.F.
Chakerian G.D.
Chamberlain, R.M., Jr.
Commodity Advisory Corp. of Texas
Creely, J.W.
DeCaux, P.A.
DeLeon, M.J.
Desmond, James
Diehl, Harvey
Ercolano, J.L.
Farmer, D.R.
Frey, F.F., Jr.
Gardner, C.L.
Gioia, A.A.
Giuli, R.M.
*Gould, H.W.
Greig, W.E.
Harris, V.C.
Heatherly, H.E.
Hillman, A.P.
*Horadam, A.F.
Howell, R.J.
Kelisky, R.P.
Kimberling, C.H.
*Long, C.T.
*Maxwell, James
McConnell, R.K., Jr.
*NcNabb, Sr. M. DeSales
Miller, Leslie
Monzingo, M.G.
Ossiander, F.J.
Robinson, E.D.
Ross, R.J.
Schneifer, H.C.C.
Schumaker, J.A.
Somer, L.
Taylor, L.
*Thoro, D.
Umansky, H.L.
Vogel, R.
Volpe, C.C.
Waddill, Marcellus
*Walker, L.A.
Weekly, G.
Whitney, R.E.
Williams, B.E.
Yang, E.L.

*Charter Members

## INSTITUTIONAL MEMBERS

THE BAKER STORE EQUIPMENT COMPANY
*Cleveland, Ohio*

CALIFORNIA STATE POLYTECHNIC UNIVERSITY, POMONA
*Pomona, California*

CALIFORNIA STATE UNIVERSITY, SACRAMENTO
*Sacramento, California*

REED COLLEGE
*Portland, Oregon*

SAN JOSE STATE UNIVERSITY
*San Jose, California*

IBM, THOMAS J. WATSON RESEARCH CENTER
*Yorktown Heights, New York*

SCIENTIFIC ENGINEERING INSTRUMENTS, INC.
*Sparks, Nevada*

TRI STATE UNIVERSITY
*Angola, Indiana*

UNIVERSITY OF REDLANDS
*Redlands, California*

UNIVERSITY OF SANTA CLARA
*Santa Clara, California*

UNIVERSITY OF TORONTO
*Toronto, Canada*

WASHINGTON STATE UNIVERSITY
*Pullman, Washington*

JOVE STATISTICAL TYPING SERVICE
2088 Orestes Way
Campbell, California 95008

**BOOKS AVAILABLE THROUGH THE FIBONACCI ASSOCIATION**

*Introduction to Fibonacci Discovery* by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

*Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. FA, 1972.

*A Primer for the Fibonacci Numbers.* Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972.

*Fibonacci's Problem Book.* Edited by Majorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974.

*The Theory of Simply Periodic Numerical Functions* by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969.

*Linear Recursion and Fibonacci Sequences* by Brother Alfred Brousseau. FA, 1971.

*Fibonacci and Related Number Theoretic Tables.* Edited by Brother Alfred Brousseau. FA, 1972.

*Number Theory Tables.* Edited by Brother Alfred Brousseau. FA, 1973.

*Recurring Sequences* by Dov Jarden. Third and enlarged edition. Riveon Lematematika, Israel, 1973.

*Tables of Fibonacci Entry Points, Part One.* Edited and annotated by Brother Alfred Brousseau. FA, 1965.

*Tables of Fibonacci Entry Points, Part Two.* Edited and annotated by Brother Alfred Brousseau. FA, 1965.

*A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume.* Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.